On the Role of Equity for the Dynamics of Capital Accumulation

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Abstract

The standard overlapping generations model is extended to include retradeable paper assets (shares) of firms. Two period lived consumers hold portfolios including paper assets and capital in order to transfer wealth over time. An infinitely lived firm produces a stochastic output using a Cobb-Douglas production function with an additive shock. The random output is paid as dividends to shares while factor prices receive their marginal products. Using the functional rational expectations approach, the paper derives the dynamic MSV solution for i. i. d. production shocks induced by the GAMMA distribution and for CARA utility functions for consumers. It is shown that the resulting dynamics is one dimensional, deterministic, and noncyclical. Multiple steady states may arise when the capital share is more than one half and the risk adjusted return in the asset market is high. In this case, an economy may suffer from a poverty trap which is purely endogenously generated without requiring externalities or non convexities.

Keywords: asset market, capital accumulation, rational expectations, poverty trap
JEL classification: C62, G11, O11

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1 Introduction

Economists seem to agree on the fact that the development of financial markets boosts overall economic growth. There are few who argue that the existence of financial markets could fuel an inflationary pressure on equity and asset prices, thereby encouraging a boom in consumption or unproductive investment with a negative effect on growth. Liberalization of financial markets is always considered as being accompanied by increasing opportunities for allocating risk where assets play a major role of transferring wealth over time. Therefore, it is usually assumed that an active equity market provides new funds for productive purposes stimulating accumulation whenever firms issue *new* equity in the market. In contrast, the role of a given equity market for the overall performance of an economy has not been studied systematically.

A principle issue addressed in this paper is whether the availability of equity/paper assets as a portfolio alternative to real capital could be an obstacle to growth. In other words, is it possible that liberalization of financial markets may lead to a situation in which the economy suffers from persistent underdevelopment, low capital stock, and consequently low income. In a development context, it is commonly argued that the existence of free market forces and the availability of equity capital in functioning financial markets are the major factors for poor (low capital) economies to revive and catalyze their convergence to the rich economies. The paper presents a dynamic model in which a poverty trap for a low capital economy may arise when financial investment possibilities (equity) are freely available, while this effect would not arise in the same economy without a market for paper assets.

The literature has identified several causes why an economy may suffer from a poverty trap (for a recent review see Matsuyama (2006)). Models explaining the reasons behind the poverty trap, can be divided into two groups, so called threshold effect model and externality type models. In threshold effect models, a poverty trap arises due to either borrowing constraints or investment indivisibility. These frictions induce a critical level in wealth, income, or other variables, which consumers cannot surpass once they are below it (see for example Azariadis & Drazen (1990), Lee (1996), Ciccone & Kiminori (1996)). In externality type models a poverty trap can arise either by increasing returns to scale, other kinds of externalities, or market imperfections (see Stokey (1988), Murphy, Shleifer & Vishny (1989), Matsuyama (2002)).

Among the threshold type models are those of Saint-Paul (1992), Zilibotti (1994), Acemoglu & Zilibotti (1997), which describe an economy with capital accumulation and financial markets. The authors argue that under-development of the financial market can be a major reason behind limited opportunities to diversify risk. The major reasons behind the existence of the poverty trap are 1) the inability to obtain external finance due to the weak credit and 2) the risk aversion of investors who have no incentive to self-finance a high return investment project due to its high risk inheritance. In Acemoglu & Zilibotti (1997) there are no explicit costs of financial relations. However, risk averse agents hold more of the safe assets initially due to investment indivisibility in highly productive but risky investment. Eventually, with sufficient capital accumulation the economy takes off and converges to its steady state where all the sectors are open and risk is completely diversified. In Saint-Paul (1992) and Zilibotti (1994) one can obtain non-convergence results in both levels and growth rates across countries which are identical in technology and preferences. Saint-Paul (1992) introduces a strategic complementarity between financial markets and technology. Due to fixed cost, individuals may be reluctant to open a financial market. Consequently, a transition to a faster growth path with financial markets can happen at any time when income is between two critical levels[1]. In other words, multiple equilibria can arise between these two critical values. Financial intermediaries have to internalize the externality of the impact of individual actions on technological choice to achieve a financial equilibrium. Similarly, in Zilibotti (1994) economies endowed with capital below some threshold level might not attain self-sustained growth with the first-best allocation. Each firm, when investing, adds to the total demand of intermediation services and causes a fall in the wedge between the price of internal and external capital. In a laissez-faire economy, firms ignore such external effects and cannot coordinate their demand. Multiple equilibria exist between certain levels of initial capital and the equilibrium depends on the coordination of demand. In other words, there exists strategic complementarity in investment demands of final producers in the presence of imperfect intermediation.

Instead of focusing on the role of financial intermediation in the presence of fixed costs or investment indivisibility, the present paper investigates the effects of portfolio decisions between productive capital investment and unproductive asset holdings in a fully competitive setting without market restrictions or externalities. Given a concave pro-

[1]Saint-Paul (1992) assume that the cost of trading in financial market is less than proportional to the volume traded whereas if technological diversification implies less productivity, its cost is proportional to output. Multiple equilibria arise in the range where the cost of trading and the cost of technological diversification are equal.

duction function, two effects have to be considered to possibly explain the causes of impediments to growth from equity. Firstly, the lower the capital stock the higher is the return from capital. High returns attract funds for productive investment and induce a convergence force. Secondly, if a market for unproductive equity investment is available as an portfolio alternative, the risk adjusted return from an unproductive investment might be higher than the return from the productive one. In this case, investors may choose to make only unproductive investment. As a consequence, the capital stock is pushed to a lower level by depreciation. If unproductive investment still remains more attractive at the lower level of capital, then the capital stock will accumulate to low (zero) levels, creating a so called poverty trap for an economy.

The model developed in this paper provides a detailed analysis of the endogenous interplay of investment diversification between an equity market and a capital market and economic performance. The standard overlapping generations model is extended to include paper assets traded by consumers in addition to investment in real capital in order to transfer wealth over time. Thus, agents hold unproductive assets not as in Bencivenga & Smith (1991) and Greenwood & Smith (1997) to insure themselves against unpredictable liquidity needs, but simply to maximize expected future consumption under rational expectations. In other words, asset holdings are motivated in the spirit of the standard capital asset pricing model (CAPM) and not by the liquidity needs of consumers à la Diamond & Dybvig (1983). The asset market is modelled as in Böhm & Chiarella (2005) where asset prices are determined endogenously by the interaction of heterogeneous consumers.

For the real side of the economy, it is assumed that capital accumulation occurs with irreversible investment and depreciation. Factor prices are determined by their respective marginal products, so that consumer income and the price of the paper asset becomes endogenous. As a consequence, the returns in the asset market and in the capital market are endogenous, making portfolio choices and capital accumulation depend on the two rates of return *and* on disposable income of consumers. It is shown that for a Cobb-Douglas production function there is a universal threshold value of the capital share in production above which multiple equilibria with a poverty trap may occur depending on the risk adjusted return on unproductive investment and the production technology. Thus, there may be a spiralling effect on asset prices and their returns over the real rate on capital leading to zero levels of capital and income while asset prices diverge to infinity. The model here identifies a particular feed back mechanism between asset markets and real capital markets under perfect competition and rational expectations as an important cause for the existence of a poverty trap, while all of the other sources of a poverty trap as summarized in Matsuyama (2006) are missing.

The paper is organized as follows. Section 2 introduces the model and discusses some of its properties. Section 3 derives the minimum state representative solution under rational expectations. Section 4 analyzes the dynamics of the economy and Section 5 concludes.

2 The Model

Consider a market economy evolving in discrete time with consumption, production, and capital accumulation. In each period a single commodity is produced from labor and capital subject to a random productivity shock. The produced commodity can be either consumed or invested irreversibly into capital. There is no inventory holding and capital depreciates at a constant rate. In addition to the markets for output, labor, and capital there is a market for retradeable assets which are issued as equity of a single firm/producer. Consumers chose a portfolio consisting of real savings and of shares of the firm. All markets operate under perfect competition implying that the respective agents are price takers. There is neither strategic behavior nor any information asymmetry.

2.1 The Production Sector

For simplicity we assume that there is a single infinitely lived firm in the economy. Its random technology can be decomposed into a deterministic part described by a production function which is homogeneous of degree one in capital and labor *plus* a shock to labor productivity. More specifically, production in per capita terms is described by a standard (instantaneous) per capita production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with an additive shock ε,

$$y = f(k) + \varepsilon,$$

satisfying the following assumption.[2]

[2] In order to minimize notation, the index t will be suppressed as much as possible. Variables without time subscript refer to an arbitrary period t while subscripts 1 and -1 refer to periods $t+1$ and $t-1$ respectively.

Assumption 1 *The technology of the firm is described by an isoelastic production function, $f : \mathbb{R}_+ \to \mathbb{R}_+$, with scale parameter, $A > 0$, and elasticity parameter, $\theta \in [0, 1]$,*

$$f(k) = Ak^\theta.$$

Assumption 2 *The random perturbation, ε, is an i. i. d. random variable distributed according to the gamma distribution with parameters, $(\alpha, \beta) \in \mathbb{R}^2_{++}$, i.e., the distribution function, F, is given by*

$$F(x) := \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-\beta y} dy.$$

The firm pays wages and the rental rate on capital according to the marginal product rule,

$$w(k) := f(k) - kf'(k) \quad \text{and} \quad r(k) := f'(k),$$

while the remaining random output is paid as dividends to the share holders of the firm. Assumption 2 implies that the expected dividend and the variance of dividends are given by $\mathbb{E}\varepsilon := \mu = \frac{\alpha}{\beta}$ and $\mathbb{V}\varepsilon := \sigma^2 = \frac{\alpha}{\beta^2}$ respectively.

2.2 The Consumption Sector

The consumption sector in every period consists of overlapping generations of finitely many consumers. Each generation lives for two periods and, for simplicity, we assume that there is no population growth. The typical young agent of generation t supplies one unit of labor inelastically to the labor market in the first period of his lifetime and receives labor income w. His lifetime utility depends on old age consumption only. He can transfer his wage income as wealth to the next period either by real savings (investing directly into the firm) or by purchasing equity shares. Therefore, a young agent faces the budget constraint

$$b + xp \leq w,$$

where b denotes the amount of real savings/real investment, and x is the number of equity shares purchased at the price p. It is assumed that the young consumer can neither take credit in the capital market nor can he take a short position in the asset market in order to finance investment, implying $b \geq 0$ and $x \geq 0$. In the second period of his life time, an agent receives the rate of return r_1 on his productive investment. In addition, he receives $x(p_1 + \varepsilon_1)$ in consumption goods from selling his share holdings and as the random dividend payment. Old agents do not leave any bequest to future generations and consume their entire wealth. Therefore, final consumption c_1 is restricted to final wealth as

$$c_1 \leq br_1 + x(p_1 + \varepsilon_1). \tag{1}$$

The young agent's objective is to maximize the expected utility of second period consumption.

Assumption 3 *Young agents' preferences over the second period consumption are described by the CARA utility function with risk aversion parameter $a > 0$, i. e.*

$$u(c) = 1 - e^{-ac}.$$

When making the portfolio decision, next period's rate of return and equity price, (r_1, p_1) are unknown for young agents. It is assumed that agents make point forecasts (r^e, p^e) for both variables. In the subsequent analysis we will assume that agents will form rational expectations, i. e. , agents expectation about these quantities always coincides with actual realizations.

We assume that agents know the distribution function F and fully utilize this information when making their portfolio choice. For given values of $(p, w, p^e, r^e) \geq 0$, the young agent's asset demand, $\varphi(p, w, p^e, r^e)$, is defined as

$$\varphi(p, w, p^e, r^e) := \arg \max_{x \in B(p,w)} \int_{\mathbb{R}_+} u(c_1) dF(\varepsilon_1), \tag{2}$$

where

$$c_1 := r^e w + x(p^e + \varepsilon_1 - pr^e) \tag{3}$$

is the random future consumption, while

$$B(p, w) = \{x | x \geq 0, xp \leq w\} \tag{4}$$

is the young agent's current budget constraint. As shown in Böhm, Kikuchi & Vachadze (2005), the demand for equity shares can be written as

$$\varphi(p, w, p^e, r^e) = \begin{cases} 0 & \text{if} \quad p \in \left[\dfrac{p^e + \mathbb{E}(\varepsilon)}{r^e}, \infty \right) \\ \dfrac{1}{a} \left(\dfrac{\alpha}{pr^e - p^e} - \beta \right) & \text{if} \quad p \in \left[\dfrac{p^e + \chi(w, p^e, r^e)}{r^e}, \dfrac{p^e + \mathbb{E}(\varepsilon)}{r^e} \right) \\ \dfrac{w}{p} & \text{if} \quad p \in \left(0, \dfrac{p^e + \chi(w, p^e, r^e)}{r^e} \right) \end{cases} \tag{5}$$

where $\chi(w, p^e, r^e)$ is the unique positive solution for p, of the following quadratic equation $\alpha p = (\beta p + aw)(pr^e - p^e)$.

2.3 Temporary Equilibrium

Assume that firms do not issue new equity and that the total stock of equity is constant over time and normalized to one. Given the demographic structure and the preferences of consumers, it is clear that all assets are retraded in every period between young and old consumers. In this case, temporary asset market equilibrium is defined by an asset price, p, solving

$$\varphi(p, w(k), p^e, r^e) = 1. \tag{6}$$

For a given (k, p^e, r^e), let $p = S(k, p^e, r^e)$ denote the solution of (6), i. e.

$$\varphi(S(k, p^e, r^e), w(k), p^e, r^e) = 1, \qquad \text{for all} \qquad (k, p^e, r^e).$$

The mapping $S : \mathbb{R}^3_+ \to \mathbb{R}_+$ is usually referred to as the temporary equilibrium mapping defining current prices as a function of current expectations for the future. It is a price law in the sense of Böhm & Wenzelburger (1999, 2002) with an expectations feedback. The following proposition provides a full parametric characterization of this mapping.

Proposition 1 *If Assumptions (2) and (3) are satisfied. Then, the price law is defined by*

$$p = S(k, p^e, r^e) := \min\left(w(k), \frac{p^e + \pi}{r^e}\right) \tag{7}$$

where $\pi = \frac{\alpha}{a+\beta} = \frac{\mu^2}{a\sigma^2+\mu}$ *is the risk adjusted expected return on dividend.*

Proof: Suppose $pr^e \in (p^e + \chi(w, p^e, r^e), p^e + \mathbb{E}\varepsilon)$ then the expression (5) implies that the market clearing price should satisfy $pr^e - p^e = \pi$, where $\pi = \frac{\alpha}{a+\beta}$. If $pr^e \leq p^e + \chi(w, p^e, r^e)$, the market clearing price is $w(k)$. Since the equity price cannot exceed the wage income the claim of proposition follows immediately. $\square$

Finally, given a temporary equilibrium with an asset price, $p = S(k, p^e, r^e)$, aggregate savings in assets is equal to $S(k, p^e, r^e)$, implying that, $w(k) - S(k, p^e, r^e)$, is equal to investment in capital. Since investment is assumed to be irreversible, all non-depreciated capital remains in the firm. Therefore, the equation describing capital accumulation is given by

$$k_1 = A(k, p) := (1 - \delta)k + w(k) - p. \tag{8}$$

The two mappings (7) and (8) define the temporary equilibrium (k, p) for any given expected equity price and capital return of the next period. For a dynamic analysis, the description of expectations formation has to be added.

3 Rational Expectations Equilibrium

Since the asset price map (7) contains expectations regarding the asset price and the rate of return of the following period (a so called *expectational lead*), there exist two different principles for stationary forecasting rules according to which rational expectations or perfect foresight along a dynamic path may be obtained. The first principle relies only on the past observations and utilizes them in order to obtain future predictions.

The second principle to obtain perfect foresight solutions (orbits) leads to the notion of the so called minimum state variable solution (MSV). From a dynamical systems point of view this corresponds to the associated functional rational expectations equilibrium[3] discussed and used in the literature. The basic idea for an MSV for the model here is straightforward. If the asset price p could be determined in any period as a function of the level of capital alone and independent of expectations, the capital accumulation equation implies an explicit perfect predictor for next period's asset price and capital return. Suppose that there exists such a price function $p = P(k)$, determining prices in any period as a function of the level of capital alone. Then the capital accumulation function is given by

$$G(k) := A(k, P(k)) := (1 - \delta)k + w(k) - P(k).$$

Using the capital accumulation function G, one can derive a price predictor ψ and the perfect predictor ρ for the return on capital by

$$\psi(k) := P(G(k)), \qquad \text{and} \qquad \rho(k) := r(G(k)) := f'(G(k)).$$

In order for the *same* price predictor ψ to be perfect in every period, it must be consistent with the price law for each capital level, i. e. the unknown function P must satisfy the following functional equation

$$P(k) = S(k, P(G(k)), f'(G(k))).$$

[3]see McCallum (1998, 1999) , Spear (1988), Böhm & Wenzelburger (2004), Wenzelburger (2005)

Observe that in such a case, capital accumulation is one dimensional, and the predictors ρ and ψ guarantee perfect foresight in every period along any orbit of the economy. Based on the above arguments one obtains the following definition[4].

Definition 1 *The minimum state variable solution (MSV) is defined by a pair of price and capital accumulation functions* (P, G)*,* $P : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for any* $k \geq 0$

(a) for a given capital accumulation function, G*, the price function satisfies*

$$P(k) = S(k, P(G(k)), r(G(k))) \tag{9}$$

(b) for a given price function, P*, the capital accumulation function satisfies*

$$G(k) = (1 - \delta)k + w(k) - P(k). \tag{10}$$

In the sequel, equations (9) and (10) will be referred to as the price and capital accumulation equations. In the next proposition we will construct a rational expectations equilibrium, i. e. by constructing an algorithm to approximate the functions of the equilibrium price, $P(.)$, and of the capital accumulation, $G(.)$. Define the set, $\mathcal{K} := \{k|P(k) < w(k)\}$, where $P(k)$ is the equilibrium price function. From Lemma (2), it follows that for k outside the set $\mathcal{K}$, the solution of the system (9) and (10) is $P(k) = w(k)$ and $G(k) = (1 - \delta)k$. Lemma (2) establishes bounds of the set $\mathcal{K}$, i.e., for given parameter values, we know the unique price and capital accumulation functions outside the set $\mathcal{K}$. The question remains a) whether there exists functions P and G defined on $\mathcal{K}$ and satisfying equations (9) and (10) and b) how to approximate them. The next proposition gives an answers to these questions.

Proposition 2 *Consider the sequence of price and capital accumulation functions,* $\{P_n, G_n\}_{n=1}^{\infty}$*, defined on the set* $\mathcal{K}$ *as follows*

(a) $P_0 = 0$*;*

(b) $k_1 = G_n(k)$ *solves the following equation*

$$k_1 + S(k, P_{n-1}(k_1), r(k_1)) = (1 - \delta)k + w(k); \tag{11}$$

[4]see McCallum (1999) and Spear (1988)

(c) $P_n(k)$ *is determined by*

$$P_n(k) = S(k, P_{n-1}(G_n(k)), r(G_n(k))); \tag{12}$$

then the sequence of $\{P_n, G_n\}_{n=1}^{\infty}$ *functions converges uniformly to the equilibrium price and capital accumulation functions,* $\{P, G\}$*;*

(d) $\{P, G\}$ *functions are both continuous and monotonically increasing.*

Proof: (see appendix) □

In the above proposition we established that the system of functional equations, (9) and (10), has at least one continuous and monotonically increasing solution, which we can approximate by forming a sequence of functions. In the next proposition we establish that the solution of the system (9) and (10) is unique.

Proposition 3 *The system of functional equations (9) and (10) has a unique continuous solution.*

Proof: (see appendix) □

It was shown above that the system of functional equations (9) and (10) has a unique continuous solution. This solution implies some special properties for the G map and consequently for the dynamics of the economy. In the next section we study the dynamics of the economy for all possible parameter configuration.

4 Rational Expectations Dynamics

In spite of the fact that the explicit functional form of the price function cannot be obtained, its implications for the time one map G can be determined which suffice to characterize completely the dynamics of the economy. The crucial parameter is whether the production elasticity θ is greater or less than, equal, or larger than one half. The three cases will be discussed separately.

4.1 Steady States and Stability

Proposition 4 *If* $\theta < 0.5$*, then the dynamical system* $(\mathbb{R}_+, G)$ *has a unique positive, stable steady state.*

Proof: If $\theta < 0.5$, Lemma (1) implies that there are two steady states. One of them is zero and the other one is positive. G is a monotonically increasing function. In order to show the claim of the this proposition, it is enough to show that $G'(0) = \infty$.

Let's start by contradiction. Suppose $G'(0) = a < \infty$. This implies,

$$\lim_{k\to 0} \frac{G(k)}{k} = a \Rightarrow \lim_{k\to 0} \frac{G(k)}{w(k)} = 0 \Rightarrow \lim_{k\to 0} \frac{P(k)}{w(k)} = \lim_{k\to 0} \frac{(1-\delta)k + w(k) - G(k)}{w(k)} = 1 \Rightarrow$$

$$\lim_{k\to 0} \frac{P(k)}{w(k)} = 1 \Rightarrow \lim_{k\to 0} \frac{\frac{P(G(k))+\pi}{A\theta}(G(k))^{1-\theta}}{w(k)} \geq 1 \Rightarrow \lim_{k\to 0} \frac{\pi}{A\theta} \frac{(G(k))^{1-\theta}}{w(k)} \geq 1.$$

This is a contradiction, because

$$\lim_{k\to 0} \frac{\pi}{A\theta} \frac{(G(k))^{1-\theta}}{w(k)} = \lim_{k\to 0} \frac{\pi}{A\theta} \frac{(ak)^{1-\theta}}{A(1-\theta)k^{\theta}} = \frac{\pi a^{1-\theta}}{A^2\theta(1-\theta)} \lim_{k\to 0} k^{1-2\theta} = 0.$$

Therefore, $G'(0) = \infty$. □

Proposition 5 *Suppose $\theta = 0.5$, then the dynamical system $(\mathbb{R}_+, G)$*

(a) has a unique positive, stable steady state if $A^2 > 4\pi$;

(b) has a unique stable zero steady state if $A^2 \leq 4\pi$;

Proof: If $A^2 \leq 4\pi$ then the dynamical system $(\mathbb{R}_+, G)$ has a unique steady state which is stable. If $A^2 > 4\pi$ then there are two steady states. In order to show that the zero steady state is an unstable one we use the same logic as in Proposition (4).

G is an increasing function. Suppose $G'(0) = a < \infty$. This implies,

$$\lim_{k\to 0} \frac{G(k)}{k} = a \quad \text{and} \quad \lim_{k\to 0} \frac{G(k)}{\sqrt{k}} = 0. \tag{13}$$

The continuity of the price function P together with equation (13) implies

$$\lim_{k\to 0} \frac{P(k)}{\sqrt{k}} = \lim_{k\to 0} 2\frac{P(G(k)) + \pi}{A} \frac{\sqrt{G(k)}}{\sqrt{k}} = \frac{2\pi}{A}\sqrt{a}. \tag{14}$$

The capital accumulation equation can be rewritten as follows

$$\frac{G(k)}{\sqrt{k}} + \frac{P(k)}{\sqrt{k}} = (1-\delta)\sqrt{k} + \frac{w(k)}{\sqrt{k}}. \tag{15}$$

Taking the limit of both sides of equation (15) as $k \to 0$ and using equations (13) and (14), we obtain that $\frac{2\pi}{A}\sqrt{a} = \frac{A}{2}$. The last equation implies that $G'(0) = \left(\frac{A^2}{4\pi}\right)^2 > 1$, which guarantees the instability of the zero steady state. □

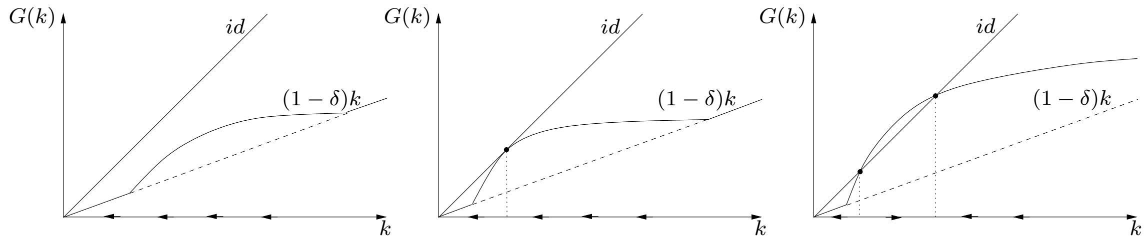
Proposition 6 *Suppose $\theta > 0.5$, then there exists a threshold value, $A^*(\theta, \delta, \pi)$, depending on which the dynamical system $(\mathbb{R}_+, G)$ can have either one, or two stable steady states.*

(a) If $A < A^*(\theta, \delta, \pi)$ then there is a unique, stable, zero steady state;

(b) If $A = A^*(\theta, \delta, \pi)$ then there are two, zero and positive steady states; zero is a stable, while the positive steady state is unstable;

(c) If $A > A^*(\theta, \delta, \pi)$ then there are three steady states. zero and the largest steady states are stable and the intermediate steady state is unstable.

Proof: If there are one or two steady states then the graph of G is below the identity line. This means that zero is the only stable steady state. If there are three steady states then from Lemma (2) we know that $G(0) = 0$ and $G'(0) = 1 - \delta$, and the graph of G crosses the identity line first from below (intermediate steady state) and then from above (high steady state). This implies the stability of the large steady state and the instability of the intermediate one. □



(a) $A < A^*(\theta, \delta, \pi)$ (b) $A = A^*(\theta, \delta, \pi)$ (c) $A > A^*(\theta, \delta, \pi)$

Figure 1: Time one map $G(k)$; $\theta > 0.5$

Figure 1 displays all possible shapes of the map G, when $\theta > 0.5$. When A is smaller than the threshold level $A^*(\theta, \delta, \pi)$, (for definition see Lemma 1) there is a unique, stable, zero steady state. If $A = A^*(\theta, \delta, \pi)$ there are two steady states and the zero steady state is the stable one. If $A > A^*(\theta, \delta, \pi)$ there are three steady states with zero and the highest steady state being stable.

4.2 Comparative Dynamics

The analysis above shows that the capital share in production has a critical impact on the dynamic path of the economy. The economy with an asset market converges to a unique positive steady state whatever the initial capital stock of the economy is if $\theta < 0.5$. Thus, given a relatively high labor share in production, the economy accumulates enough capital stock and consumers invest both in real capital and in equity in the steady state.

If $\theta = 0.5$, however, the economy with an asset market converges to a positive steady state only if $A^2 > 4\pi$ otherwise it converges to zero. In other words, if the productivity of the economy is relatively low compared to the risk adjusted expected dividend, it will eventually be trapped in poverty for any initial capital stock.

If $\theta > 0.5$, the economy with an asset market can only escape from the poverty trap for sufficiently high production technology level and high initial capital stock. Even for a high production technology level, multiple steady states arise with a low and a high positive steady state. The low steady state is unstable and the economy converges to zero if it's initial capital stock is smaller than the low steady state, otherwise it converges to the high steady state. In other words, the economy may still be trapped in poverty if the initial capital stock is not sufficiently high even for sufficiently high production technology level. The main reason for this poverty trap can be seen if we compare the return from capital investment, $r(k) = A\theta k^{\theta-1}$, with the risk adjusted return from unproductive investment, $R(k) = \frac{\pi}{A(1-\theta)}k^{-\theta}$. If $\theta > 0.5$, for sufficiently low capital stock, k, $R(k) > r(k)$, for any A. Due to the higher risk adjusted return from unproductive equity investment agents stop to invest in the productive technology. This leads to the gradual deprecation of the existing capital stock. Thus, the poverty trap in the economy with an asset market arises due to the existence of a market for unproductive investment. The condition for escaping the poverty trap is a high production technology level *and* a higher initial capital stock.

5 Summary and Conclusions

We have formulated a dynamic model of capital accumulation, with an asset market and overlapping generations of consumers. Consumers live for two period, work in the first period and consume in the second period only. Given their wage income, consumers decide how much to invest in productive capital and unproductive assets. The minimum state variable solution of the economy is found to be a deterministic one dimensional map, which is monotonically increasing. Thus, all rational expectations solutions are monotonically converging to a steady state without cycles or random shocks.

The paper shows that the capital share in production plays a crucial role for the existence of a poverty trap. Specifically, it is shown that there is a universal threshold level of the capital share in production, equal to one half, below which a poverty trap never occurs. Therefore, when the capital share in production is less than one half, the economy has a unique stable positive steady state. This means no matter how underdeveloped the economy is initially, it will converge to a unique steady state. It is shown in Böhm, Kikuchi & Vachadze (2005) that this steady steady is associated with a higher average consumer welfare than in the steady state of an economy without the asset market. Therefore, there is an incentive to open the market for an unproductive asset.

However, when the capital share in production is higher than one half, the economy converges to the zero steady state unless it is equipped with a sufficiently high production technology level *and* high initial capital stock. If the economy without a market for an unproductive asset opens such a market, then the capital stock has to be above the lower steady state value of the economy with the asset market in order to avoid convergence to the zero steady state.

6 Appendix

Lemma 1 *Let (P, G) is the solution of the system of functional equations (9) and (10). Then the non-negative steady states of the G map can be classified as follows*

- *if $\theta < 0.5$ then there are two steady states, with $k_1^* = 0$ and $k_2^* > 0$;*
- *if $\theta = 0.5$ and*
  - *$A^2 \leq 4\pi$ then there is a unique zero steady state;*
  - *$A^2 > 4\pi$ there are two steady states, with $k_1^* = 0$ and $k_2^* > 0$;*
- *if $\theta > 0.5$ then there exists a threshold value $A^*(\theta, \delta, \pi)$, such that if*
  - *$A < A^*(\theta, \delta, \pi)$ then there is a unique steady state with $k_1^* = 0$;*
  - *$A = A^*(\theta, \delta, \pi)$ then there are two steady states, with $k_1^* = 0$ and $k_2^* > 0$;*

– $A > A^*(\theta, \delta, \pi)$ *then there there are three steady states, with* $k_1^* = 0$*,* $k_2^* > 0$ *and* $k_3^* > 0$*;*

In all steady states $r(k) > 1$*.*

Proof: Let $k = G(k)$ be a steady state. Using this equilibrium condition in equations (9) and (10) we obtain that the equilibrium price and the capital stock level, (p^*, k^*), satisfy the following system of equations

$$\begin{aligned} p &= w(k) - \delta k \\ p &= \frac{\pi}{r(k) - 1}. \end{aligned} \tag{16}$$

It is clear that the pair $(k^*, p^*) = (0, 0)$ satisfies the above system of equation for any θ. By eliminating p from the system (16), we obtain the following equation with respect to k.

$$(w(k) - \delta k)(r(k) - 1) = \pi \tag{17}$$

Substituting

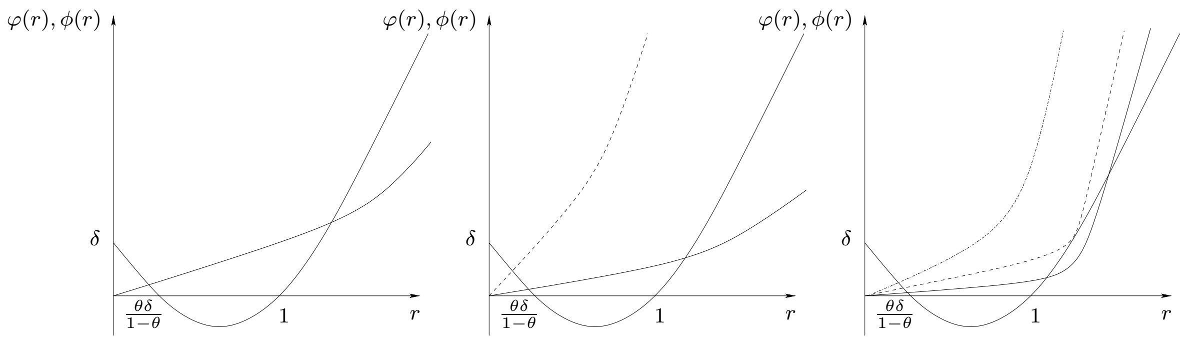
$$\frac{w(k)}{k} = \frac{1-\theta}{\theta} r(k), \qquad \text{and} \qquad k = \left[\frac{A\theta}{r(k)}\right]^{\frac{1}{1-\theta}}.$$

into equation (17), the equilibrium interest rate should satisfy the following equation

$$\varphi(r) = \phi(r), \tag{18}$$

where

$$\varphi(r) = \left(\frac{1-\theta}{\theta} r - \delta\right)(r - 1) \quad \text{and} \quad \phi(r) = \frac{\pi}{[A\theta]^{\frac{1}{1-\theta}}} r^{\frac{1}{1-\theta}} \tag{19}$$



(a) $\theta < 0.5$ (b) $\theta = 0.5$ (c) $\theta > 0.5$

Figure 2: All possible configurations of $\varphi(r)$ and $\phi(r)$ functions

Figure 2 visualizes all possible configurations of the $\varphi(r)$ and $\phi(r)$ functions. In particular if $\theta < 0.5$, functions $\varphi(r)$ and $\phi(r)$ functions cross each other only once at $r > 1$.[5] If $\theta = 0.5$, both functions $\varphi(r)$ and $\phi(r)$ are quadratic and depending on their slopes at $r = 1$, they can either cross only once or never at $r > 1$. c) If $\theta > 0.5$, three different situations may occur. A solution of the equation $\varphi(r) = \phi(r) = 0$, if it exists, determines the equilibrium interest rate $r^*(A, \theta, \delta, \pi)$. If in addition, $\varphi'(r^*(A, \theta, \delta, \pi)) = \phi'(r^*(A, \theta, \delta, \pi))$ holds, the solution determines a critical level $A^*(\theta, \delta, \pi)$ for which $\varphi(r)$ and $\phi(r)$ are tangent to each other. Since $\phi(r)$ is decreasing with respect to A, it follows that for $A < A^*(\theta, \delta, \pi)$ there is no solution of equation (18) and for $A > A^*(\theta, \delta, \pi)$ there are two solutions (see Figure 2(c)). $\square$

Lemma 2 *Let us define the set* $\mathcal{K} = \{k | P(k) < w(k)\}$*, where* $P(k)$ *solves the system of functional equations (9) and (10) then*
a) if $\theta < 0.5$ *then* $\mathcal{K} = [0, (1-\delta)\widehat{k}_2)$ *where* $\widehat{k}_2$ *is the unique solution of the following nonlinear equation*

$$A(1-\theta)(1-\delta)^\theta k^\theta = \frac{A^2\theta(1-\theta)}{(1-\delta)^{1-\theta}} k^{2\theta-1} - \pi; \tag{20}$$

b) if $\theta = 0.5$ *and* $A^2 > 4\pi\sqrt{1-\delta}$ *then* $\mathcal{K} = [0, (1-\delta)\widehat{k}_2)$*, where* $\widehat{k}_2 = \left(\frac{A^2 - 4\pi\sqrt{1-\delta}}{2A\sqrt{1-\delta}}\right)^2$*;*
c) if $\theta = 0.5$ *and* $A^2 \leq 4\pi\sqrt{1-\delta}$ *then* $\mathcal{K} = \emptyset$*;*
e) if $\theta > 0.5$ *then* $\mathcal{K} = \left(\widehat{k}_1, (1-\delta)\widehat{k}_2\right)$*, where* $0 < \widehat{k}_1 < \widehat{k}_2 < \infty$ *are solutions of the equation (20);*

Proof: Suppose $P(k) = w(k)$ and $P(G(k)) = w(G(k))$. Then the capital accumulation equation implies that, $G(k) = (1-\delta)k$, and the price equation implies that

$$\frac{P(G(k)) + \pi}{A\theta}[G(k)]^{1-\theta} \geq w(k). \tag{21}$$

Equation (21) can be rewritten as

$$A(1-\theta)(1-\delta)^\theta k^\theta + \pi \geq \frac{A^2\theta(1-\theta)}{(1-\delta)^{1-\theta}} k^{2\theta-1}. \tag{22}$$

Graphical analysis of the functions given in equation (22) implies the claim of the lemma. $\square$

[5] We are not interested with equilibrium in which $r < 1$, because in such equilibrium asset price is negative.

Lemma 3 *Suppose $f_{n-1} \leq f_n$ and $f_n(x)$ is monotonically increasing function. If $x + f_{n-1}(x) = y + f_n(y)$ then $x \geq y$.*

Proof: Let us start by contradiction, suppose $y > x$ then we have

$$0 < y - x = f_{n-1}(x) - f_n(y) < f_{n-1}(x) - f_n(x) \leq 0, \tag{23}$$

which is a contradiction, i.e., $x \geq y$. $\square$

Lemma 4 *Let us consider a sequence of functions $\{P_n, G_n\}_{n=1}^{\infty}$ defined in equations (11) and (12). Then*
a) $\{P_n, G_n\}_{n=1}^{\infty}$ is a monotonic sequence of functions, in particular, $0 \leq P_1 \leq P_2 \leq ...$ and $G_1 \geq G_2 \geq ... \geq 0$;
b) $P_n(k)$ and $G_n(k)$ are continuous, and monotonically increasing functions with respect to k, for each $n \in \mathbb{N}$;

Proof: We use the induction argument. Firstly we show that claims (a) and (b) are correct for $n = 1$ and then by assuming that both claims are correct for arbitrary n, we show that they are also correct for $n + 1$. By construction $G_1(k)$ solves the following equation

$$x + S(k, 0, r(x)) = (1 - \delta)k + w(k) \tag{24}$$

Using the continuity and the monotonicity property of the price law, from equation (24) it follows that $G_1(k)$ is positive, continuous and monotonically increasing function with respect to k. From the relation, $P_1(k) = S(k, 0, r(G_1(k)))$, we get that $P_1(k)$ is also positive, continuous and monotonically increasing function.

Now let us assume that claims (a) and (b) are correct for an arbitrary n. By construction $G_n(k)$ and $G_{n+1}(k)$ solve the following equations

$$x + S(k, P_{n-1}(x), r(x)) = (1 - \delta)k + w(k) \tag{25}$$

$$y + S(k, P_n(y), r(y)) = (1 - \delta)k + w(k). \tag{26}$$

Since $P_{n-1} \leq P_n$ and the price law is a monotonic transformation of the price function it follows that for any given k, $f_{n-1} \leq f_n$, where $f_{n-1}(x) = S(k, P_{n-1}(x), r(x))$ and $f_n(y) = S(k, P_n(y), r(y))$. In addition monotonicity of $P_n(y)$ implies the monotonicity of $f_n(y)$. Now we can apply Lemma (3) in order to get that $G_n \geq G_{n+1}$. The last inequality with the following equations

$$P_n(k) = S(k, P_{n-1}(G_n(k)), r(G_n(k))) \tag{27}$$

$$P_{n+1}(k) = S(k, P_n(G_{n+1}(k)), r(G_{n+1}(k))) \tag{28}$$

implies that $P_n \leq P_{n+1}$. This concludes the proof of claim (a). By applying the implicit function theorem and using the property of the price law it follows that the solution of equation (26) is a continuous and monotonically increasing function. From equation (28) follows the continuity and the monotonicity of $P_{n+1}(k)$ function. $\square$

6.1 Proof of Proposition 2

Lemma (4) implies a point wise convergence of the sequence, $\{P_n(k), G_n(k)\}_{n=1}^{\infty}$. By contraction of this sequence it follows that

$$\|P - P_0\| = \sup_{k \in \mathcal{K}} |P(k) - P_0(k)| = \sup_{k \in \mathcal{K}} |P(k)| < \sup_{k \in \mathcal{K}} |w(k)| = M < \infty. \tag{29}$$

Price equation with Lemma (4) implies that

$$0 < P(k) - P_1(k) = \frac{P(G(k)) + \pi}{r(G(k))} - \frac{P_0(G_1(k)) + \pi}{r(G_1(k))} \leq \frac{P(G(k)) - P_0(G(k))}{r(G(k))}. \tag{30}$$

Equation (30) implies that

$$0 < P(k) - P_n(k) \leq \frac{P(G(k)) - P_{n-1}(G(k))}{r(G(k))} \leq ... \leq \frac{P(G^n(k)) - P_0(G^n(k))}{r(G(k))r(G^2(k))...r(G^n(k))}. \tag{31}$$

Since we know that in steady state $r(k) > 1$, it follows that for any $k \in \mathcal{K}$ and for sufficiently large n, $r(G(k))r(G^2(k))...r(G^n(k)) > 1$ and consequently from (31) it follows that

$$\|P - P_n\| = \sup_{k \in \mathcal{K}} |P(k) - P_n(k)| \leq \beta_n M, \tag{32}$$

where

$$\beta_n = \sup_{k \in \mathcal{K}} \frac{1}{r(G(k))r(G^2(k))...r(G^n(k))}, \tag{33}$$

and $\lim_{n \to \infty} \beta_n = 0$.

Equations (32) and (33) imply the uniform convergence of the P_n series. Uniform convergence of P_n implies the uniform convergence in G_n series and subsequently continuity and monotonicity of solution (P, G). $\square$

6.2 Proof of Proposition 3

Suppose there are two different continuous equilibrium maps P_1, G_1 and P_2, G_2. Then, there exists at least one k such that $P_1(k) \neq P_2(k)$. Let us assume without loss of generality that $P_1(k) > P_2(k)$. This inequality implies that $G_1(k) < G_2(k)$ and consequently the following inequality should be satisfied

$$\frac{P_1(G_1(k)) + \pi}{A\theta}[G_1(k)]^{1-\theta} > \frac{P_2(G_2(k)) + \pi}{A\theta}[G_2(k)]^{1-\theta}. \tag{34}$$

Inequality (34) implies that $P_1(G_1(k)) > P_2(G_2(k))$, from which it follows that $G_1^2(k) < G_2^2(k)$. Continuing this process we obtain that for any $n \in \mathbb{N}$,

$$P(G_1^n(k)) > P(G_2^n(k)) \quad \text{and} \quad G_1^n(k) < G_2^n(k). \tag{35}$$

G_i map, for $i = 1, 2$ can have two stable fixed points either 0 or $\widehat{k} > 0$. We consider three cases, either 1) $\lim_{n\to\infty} G_1^n(k) = \lim_{n\to\infty} G_2^n(k) = 0$; 2) $\lim_{n\to\infty} G_1^n(k) = 0$ and $\lim_{n\to\infty} G_2^n(k) = \widehat{k}$, or 3) $\lim_{n\to\infty} G_1^n(k) = \lim_{n\to\infty} G_2^n(k) = \widehat{k}$.

1) If $\lim_{n\to\infty} G_1^n(k) = \lim_{n\to\infty} G_2^n(k) = 0$, then there exists a constant, $N > 1$, such that $G_1^n(k) = (1-\delta)G_1^{n-1}(k)$, and $G_2^n(k) = (1-\delta)G_2^{n-1}(k)$ for $n > N$. This implies that the price function is the wage function which contradicts the inequalities in (35).

2) If $\lim_{n\to\infty} G_1^n(k) = 0$ and $\lim_{n\to\infty} G_2^n(k) = \widehat{k}_2$, this leads to a contradiction because

$$0 = P_1(0) = \lim_{n\to\infty} P_1(G_1^n(k)) \geq \lim_{n\to\infty} P_2(G_2^n(k)) = P_2(\widehat{k}) > 0. \tag{36}$$

3) If $\lim_{n\to\infty} G_1^n(k) = \lim_{n\to\infty} G_2^n(k) = \widehat{k}_2$, then for sufficiently large n, $P_1(G_1^n(k))$ and $P_2(G_2^n(k))$ can be expressed as

$$P_1(G_1^n(k)) = \frac{\pi}{A\theta}[G_1^n(k)]^{1-\theta} + \frac{\pi}{A^2\theta^2}[G_1^n(k)G_1^{n+1}(k)]^{1-\theta} + ... \tag{37}$$

$$P_2(G_2^n(k)) = \frac{\pi}{A\theta}[G_2^n(k)]^{1-\theta} + \frac{\pi}{A^2\theta^2}[G_2^n(k)G_2^{n+1}(k)]^{1-\theta} + ... \tag{38}$$

Equations (37) and (38) are in contradiction with the inequalities given in (35).

□

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