

INEQUALITY OF NATIONS AND ENDOGENOUS
FLUCTUATIONS IN A TWO COUNTRY MODEL

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Abstract

The aim of this paper is to investigate the spillover effects through an endogenous world interest rate on the inequality of nations. The model of K. Matsuyama (*Econometrica* 72 (2004)) is explored under the alternative assumption that the world economy consists of two countries, instead of a continuum of small open economies. The two countries possibly differ in their levels of capital stock *and* in their population sizes. The spillover effects on capital stock through the world interest rate may cause a poor country and a rich country to diverge. Furthermore, unequal population sizes reinforce the unequal distribution of incomes between the two countries. Interaction of countries with unequal population sizes may also cause endogenous fluctuations.

Keywords: credit market imperfection, endogenous cycles, symmetry-breaking, two-country model

JEL classification: E44, F43, O11

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1 Introduction

The debate on the “convergence controversy” does not seem to be resolved in the empirical literature. Some economists claim that the distribution of income per capita has become more unequal in the world while others claim the opposite. There is little doubt that the operation of and the transactions in any markets today are global. This being a commonly accepted fact, both sides claim a reinforcement of their own arguments. In other words, for one group globalization is responsible for the global inequality while for the other one it induces convergence of income per capita in the world.

Quah (1993, 1996a, 1997) was the first who revealed a robust tendency towards an endogenous formation of convergence clubs and polarization of distribution of income across countries. However, this literature uses countries as their unit of analysis. Jones (1997) showed that the emergence of the so called “twin peaked” distribution disappears once each country data point is weighted by population. More recently Sala-i-Martin (2006) merged survey data about the income distribution within individual countries with national account data to estimate the world income distribution. He concludes that there has been a reduction in global inequality during the 1980s and 1990s. This result is not surprising given the high growth trend of populous countries such as China and India during that period. However, poor countries still comprise a large part of the world population while rich countries represent only a small fraction (see Milanovic 2002). These observations of course do not imply immediately that the global financial market is to be blamed for the unequal distribution of income across nations. In fact, to the author’s knowledge there are no theoretical studies of how the world income distribution is influenced by the interaction among countries with different population sizes in a global financial market.

If countries are equipped with identical technologies and there are no operation costs in international financial market, the standard neoclassical theory would predict that per capita incomes are immediately equalized. This is because the international financial market would allocate the savings of the integrated economies to where it yields the highest return inducing conditional convergence of per capita income across countries even without international mobility of labor forces. The solution the literature offers for this paradoxical results of the integrated economies is to incorporate some kind of imperfections in financial markets.¹ In a one sector overlapping generations model mod-

¹Lucas (1990) discusses why capital does not flow from rich to poor countries to the extent that a

ified to incorporate capital market imperfections, Boyd & Smith (1997) and Matsuyama (2004) show that the interaction between competitive international financial trade can amplify the inequality of income across nations. In the two country model of Boyd & Smith (1997) capital investment in both countries is subject to a costly state verification (CSV) problem. The country with higher capital stock provides more internal financing mitigating the CSV problem. Higher internal financing in the rich country counteracts the higher marginal product in the poor country. As a consequence an initially poorer country may remain poorer in the long run by the operation of international financial markets. In Matsuyama (2004) domestic investment requires borrowing in the financial market, which is constrained by domestic wealth. Poorer countries with higher marginal productivity face borrowing constraints, which preclude countries from immediate global convergence. This endogenous borrowing constraint generates multiple steady states for the small open economy. In a world consisting of a continuum of small open economies, symmetry breaking occurs in the presence of an international financial market. That is, the symmetric steady state loses its stability and stable asymmetric steady states come to exist.

One aim of this paper is to identify the feedback effects of interacting economies integrated in international financial markets on world income distribution. These feedback effects are absent in the model by Matsuyama (2004) since the atomless economies do not influence the world interest rate. In their two country model by Boyd & Smith (1997) do not analyze these feedback effects in the international financial market explicitly and consider only two homogeneous countries. The present paper explores the Matsuyama model under the alternative assumption that the world economy consists of two countries which are possibly different in population size, instead of a continuum of homogeneous small open economies. This confines the state space of the dynamical system to two dimensions. As each country has a positive measure, the capital stock in each country has an impact on the world interest rate and vice versa. It is shown that new stable steady states emerge in the presence of the spillover effects of capital stock via an endogenous determination of the world interest rate. Thus the present paper identifies additional forces of international financial markets and characterizes all sets of steady states of the two country model.

The present paper also analyzes the dynamics of the model when two countries have heterogeneous population sizes. This may be justified on the ground that the relative

standard neoclassical model would predict.

population size might be one of the most persistent attributes in the world economy considering the immobile nature of population and the long time span needed for adjustment. It is shown that the heterogeneity in population sizes breaks the symmetric structure of the model. The model implies that, if the initial capital stocks of the two countries are sufficiently unequal, greater inequality in population size also induces greater inequality in income distribution. This result may be consistent with the situation in today's world, which consists for the large part of the world population of poor countries while rich countries represent only a small fraction. The model predicts that if two countries have sufficiently unequal initial capital stocks, the unequal population size of countries induces an unequalizing force through the international financial market.

Boyd & Smith (1997) motivate their paper by referring to cyclicity of credit allocation between developing and developed economies in empirical data (see United Nations (1992)). However, their theoretical findings are confined to a dynamical equilibrium paths displaying damped oscillation. The asymmetric steady state generated by heterogeneous population sizes of the two countries in the present paper induces endogenous cycles. This implies cycles in capital stock as well as international capital flows in the long run in contrast to the transitory feature in Boyd & Smith (1997). As opposed to the real business cycles models where fluctuation is viewed as a propagation mechanism of exogenous shocks, the model implies that endogenous fluctuation is inherent in the international financial market.

The remainder of the paper is organized as follows. Section 2 introduces the basic structure of the model. Section 3 and 4 reviews the autarky case and the small open economy case by Matsuyama (2004). Section 5 generalizes the model to a two country case. Section 6 then investigates the effect of a change in relative population size on the distribution of incomes between the two countries. Section 7 concludes.

2 The Model with Financial Market Imperfections

There are domestic markets for output, labor, capital, and an international credit market. It is assumed that production factors are nontradable and agents cannot start an investment project abroad. In other words we rule out foreign direct investment. This is to focus on the effects of financial market globalization not on factor market globalization. All markets operate under perfect competition implying that the respective agents are price takers.

2.1 The Production Sector

There exists a single firm that lives infinitely long in each country, which produces aggregate consumption goods Y_t in each period t using the total amount of labor L_t and physical capital K_t by use of a linear homogeneous production function $F(L_t, K_t)$. Then output per capita is given by

$$y_t = f(k_t),$$

where $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$ and $f(k_t) := F(1, k_t)$. We assume that factor markets are competitive meaning that the firm pays wages and returns on capital according to the marginal product rule, i.e., $W(k_t) := f(k_t) - k_t f'(k_t)$ and $r(k_t) := f'(k_t)$ respectively. We also assume that the capital stock depreciates fully after one period and there is no population growth.

Assumption 1 *The production function in intensive form $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is C^2 , and satisfies $f(0) = 0$, $f''(k) < 0 < f'(k)$, and the Inada conditions $\lim_{k \rightarrow \infty} f'(k) = 0$ and $\lim_{k \rightarrow 0} f'(k) = \infty$.*

To avoid multiple steady states that are not related to credit market imperfection we impose the following assumption.

Assumption 2 $\lim_{k \rightarrow 0} W'(k) = \infty$ and $W''(k) < 0$.

Many standard production functions satisfy Assumption 2. Especially, if we use the Cobb-Douglas production function $f(k) = Ak^\alpha$, $W(k) = (1 - \alpha)Ak^\alpha$ which satisfies Assumption 2.

2.2 The Consumption Sector

There are overlapping generations of two-period lived consumers, who supply one unit of labor inelastically in the first period and consume only in the second period. There are a continuum of young consumers indexed by $j \in [0, 1]$ with income $W(k_t)$ at the beginning of period t . They have two options to transfer their income to the next period. Firstly they may lend their income in the competitive credit market and receive $r_{t+1}W(k_t)$ in the next period. Secondly they may start an investment project which

comes in discrete, nondivisible units. The project the investor can start is restricted to one which requires one unit of consumption goods. There exists a homogeneous linear technology to transform one unit of consumption goods into R units of physical capital.

Assumption 3 $W(R) < 1$.

Assumption 3 ensures that $W(k_t) < 1$ as we will see later. This assumption is crucial for the results later since it means that young consumers always have to borrow an amount $1 - W(k_t)$ to start an investment project. The R units of physical capital are used as an input for production. Then, the investor's return in the next period will be the rate of return on the capital investment minus the debt repayment, $Rf'(k_{t+1}) - r_{t+1}(1 - W(k_t))$.

2.3 The Financial Market

There are two major assumptions which characterize the financial market. Firstly, the investor has to be willing to start a project. We call this condition the profitability constraint meaning the return from starting a project has to be at least equal to the return from saving. This requires

$$Rf'(k_{t+1}) \geq r_{t+1}. \quad (1)$$

Secondly, the borrower in the financial market cannot credibly commit to repay more than a fraction of the revenue of the investment project. Thus the borrowing constraint is written as

$$\lambda Rf'(k_{t+1}) \geq r_{t+1}(1 - W(k_t)), \quad (2)$$

where $\lambda \in (0, 1)$ can be interpreted as a measure of imperfection in the financial market. Note if the financial market were perfect, i.e. $\lambda = 1$, this constraint would never be binding. These two constraints have to always hold in the financial market. In other words, agents must be willing and able to start an investment project. The two constraints can be summarized as

$$r_{t+1} \leq R_t := \begin{cases} \frac{\lambda Rf'(k_{t+1})}{1 - W(k_t)} & \text{if } k_t < K(\lambda) \\ Rf'(k_{t+1}) & \text{if } k_t \geq K(\lambda), \end{cases} \quad (3)$$

where R_t may be interpreted as the project productivity required in order for the project to be undertaken in period t , and $K(\lambda)$ is defined implicitly by $W(K(\lambda)) = 1 - \lambda$. If

$k_t < K(\lambda)$, the borrowing constraint is the relevant constraint since the profitability constraint is always satisfied. If $k_t \geq K(\lambda)$, the profitability constraint is the relevant constraint since the borrowing constraint is always satisfied. Therefore it depends entirely on k_t as to which constraint has to be considered in the financial markets. The young consumers are price takers in the competitive financial markets and make investment decision to maximize their next period consumption. If $k_t \geq K(\lambda)$, the agents prefer starting the investment project to lending until the profitability constraint is binding. In other words at the market equilibrium where the profitability constraint is binding, young agents are indifferent between borrowing and lending. If $k_t < K(\lambda)$, the agents always prefer starting the investment project to lending. This implies that at the market equilibrium where the borrowing constraint is binding, some of young consumers will be denied to take credit. If \bar{j}_t denotes the measure of investors among young agents at time t , the aggregate capital investment is given by

$$k_{t+1} = \int_0^{\bar{j}_t} R dj.$$

Obviously, the proportion of young agents credit who are rationed in equilibrium will be $(\bar{j}_t, 1]$. In order to analyze the equilibrium we have to know how the interest rate is determined.

3 The Autarky Case

Without international lending and borrowing, saving must be equal to investment in the economy in equilibrium. From equation (3), investment is equal to zero if $r_{t+1} > R_t$, and to one if $r_{t+1} < R_t$, and may take any value between zero and one if $r_{t+1} = R_t$. Since the young agents receive wage income $W(k_t)$ and consume only when they are old, the aggregate saving is equal to $W(k_t)$, which is less than one from Assumption 3. The equilibrium interest rate is determined so that the aggregate investment is made equal to the aggregate saving. This requires $r_{t+1} = R_t$ in equilibrium.

Since the investment project requires one unit of consumption goods and the aggregate saving is less than one, the fraction of young agents who become borrowers and start the project, \bar{j}_t is equal to $W(k_t)$ while the rest become lenders.² If $k_t \geq K(\lambda)$, young

²This follows from “fraction of young agents \times one unit of consumption goods = own endowment + borrowing = investment” = $W(k_t) \times 1 = W(k_t) \times W(k_t) + (1 - W(k_t)) \times W(k_t) = W(k_t)$ ”.

agents are indifferent between borrowing and lending. When $k_t < K(\lambda)$, on the other hand, they strictly prefer borrowing to lending. Therefore, the equilibrium allocation necessarily involves credit rationing, where the fraction $1 - W(k_t)$ of young agents are denied credit when the borrowing constraint is binding. Since the measure of the young agents who start the project is equal to $W(k_t)$ and every one of them supplies R units of physical capital,

$$k_{t+1} = RW(k_t). \quad (4)$$

Equation (4) completely describes the dynamics of capital formation in autarky. Note that, if $k_t < R$, $k_{t+1} = RW(k_t) < RW(R) < R$ from Assumption 3. Therefore, $k_0 < R$ implies $k_t < R$ and $W(k_t) < 1$ for $t > 0$, as has been assumed. From equations (3), (4), and $r_{t+1} = R_t$, the equilibrium interest rate is given by

$$r_{t+1} = \begin{cases} \frac{\lambda R f'(RW(k_t))}{1 - W(k_t)} & \text{if } k_t < K(\lambda) \\ R f'(RW(k_t)) & \text{if } k_t \geq K(\lambda). \end{cases} \quad (5)$$

Assumptions 2 and 3 ensure that equation (4) has a unique steady state $k = K^*(R) \in (0, R)$ defined implicitly by $k = RW(k)$, and for $k_0 \in (0, R)$, k_t converges monotonically to $k = K^*(R)$. The function $K^*(R)$ is increasing and satisfies $K^*(0) = 0$ and $K^*(R^+) = R^+$ where R^+ is defined by $W(R^+) = 1$. It is worth mentioning that the dynamics of capital formation in autarky is unaffected by the degree of credit market imperfection λ . This is because domestic investment is made equal to domestic saving by the adjustment of the interest rate. Hence, the credit market imperfection has only an influence on whether the borrowing constraint will be binding or not in equilibrium.

4 The Small Open Economy

The world interest rate r is constant in the small open economy. This means that the small open economy does not have any influence on the world economy and the influence of the world economy on the small open economy is constant throughout time. From Section 2 we know that both constraints are binding in equilibrium. Then we obtain from equation (3) the following proposition.

Proposition 1 *There exists a temporary equilibrium of the small open economy defined by*

$$r = \begin{cases} \frac{\lambda Rf'(k_{t+1})}{1-W(k_t)} & \text{if } k_t < K(\lambda) \\ Rf'(k_{t+1}) & \text{if } k_t \geq K(\lambda). \end{cases} \quad (6)$$

if and only if $Rf'(R) \leq r$ where $k_{t+1} = \bar{j}_t R$.

Proof: Note that $j \in [0, 1]$ implies $k_{t+1} \in [0, R]$. Then $Rf'(k_{t+1}) \in [Rf'(R), \infty)$. Therefore $Rf'(R) \leq r$ guarantees the existence for $k_t \geq K(\lambda)$. $Rf'(R) \leq r$ also guarantees the existence for $k_t < K(\lambda)$ as $1 - W(k_t) > \lambda$. This proves the proposition. \square

Figure 1 visualizes the idea of the temporary equilibrium. Recall that there exists a continuum of young consumers with altogether, unit mass. If all young consumers start an investment project, $k_{t+1} = R$. Given a fixed level of technology R , the lowest possible revenue from the investment project is $Rf'(R)$. If the interest rate in the credit market is lower than the lowest possible revenue, i.e., $r < Rf'(R)$, the profitability constraint is always violated. If $r \geq Rf'(R)$, there will be more and more young consumers who start the investment project until $r = Rf'(k_{t+1})$. The proportion of young consumers who start the investment project is denoted by $\bar{j} \in [0, 1]$ in equilibrium.

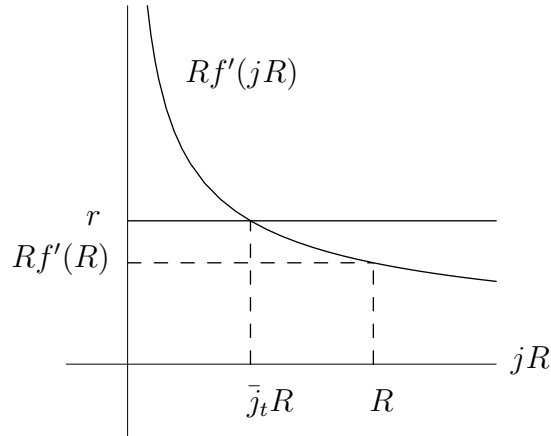


Figure 1: Temporary equilibrium in the international financial market

Solving equation (6) for k_{t+1} , the physical capital investment of any country subject to the two constraints is given by

$$k_{t+1} = \Psi(k_t, r) := \begin{cases} \Phi \left[\frac{r(1 - W(k_t))}{\lambda R} \right] & \text{if } k_t < K(\lambda) \\ \Phi \left(\frac{r}{R} \right) & \text{if } k_t \geq K(\lambda) \end{cases} \quad (7)$$

where $\Phi := (f')^{-1}$. The following lemma characterizes the steady states of the small open economy.

Lemma 1 (Lemma Matsuyama (2004))

- (a) Equation (7) has at least one steady state.
- (b) Equation (7) has at most one steady state above $K(\lambda)$. If it exists, it is stable and equal to $\Phi(r/R)$.
- (c) Equation (7) has at most two steady states below $K(\lambda)$. If there is only one, k_L , either it satisfies $0 < k_L < \lambda R/r$ and is stable, or $k_L = \lambda R/r$ at which Φ is tangent to the 45° line. If there are two, k_L and k_M , they satisfy $0 < k_L < \lambda R/r < k_M < K(\lambda)$, and k_L is stable and k_M is unstable.

For the exact condition for each of the three cases see Matsuyama (2004), Proposition 2. Figure 2 shows the case where there exist three steady states of the small open economy. Notice that there are two steady states below and one steady state above the critical value $K(\lambda)$. We denote these steady states by $k_L < k_M < k_H$. The steady states k_L and k_H are stable while the steady state k_M is unstable. In contrast to the autarky case, the credit market imperfection in the small open economy generates multiple steady states due to the borrowing constraint. This is because domestic saving is not necessarily equal to domestic investment in the small open economy. The world interest rate does not adjust to equate domestic saving to domestic investment in the small open economy. Instead, the fraction of young agents who start investment project changes so that either the borrowing constraint or the profitability constraint is binding in equilibrium.

Section 2 introduced the model by Matsuyama (2004) with financial market imperfection. Section 3 showed that if saving is equal to investment, the financial market imperfection has no influence on the dynamics of capital formation in autarky. Section

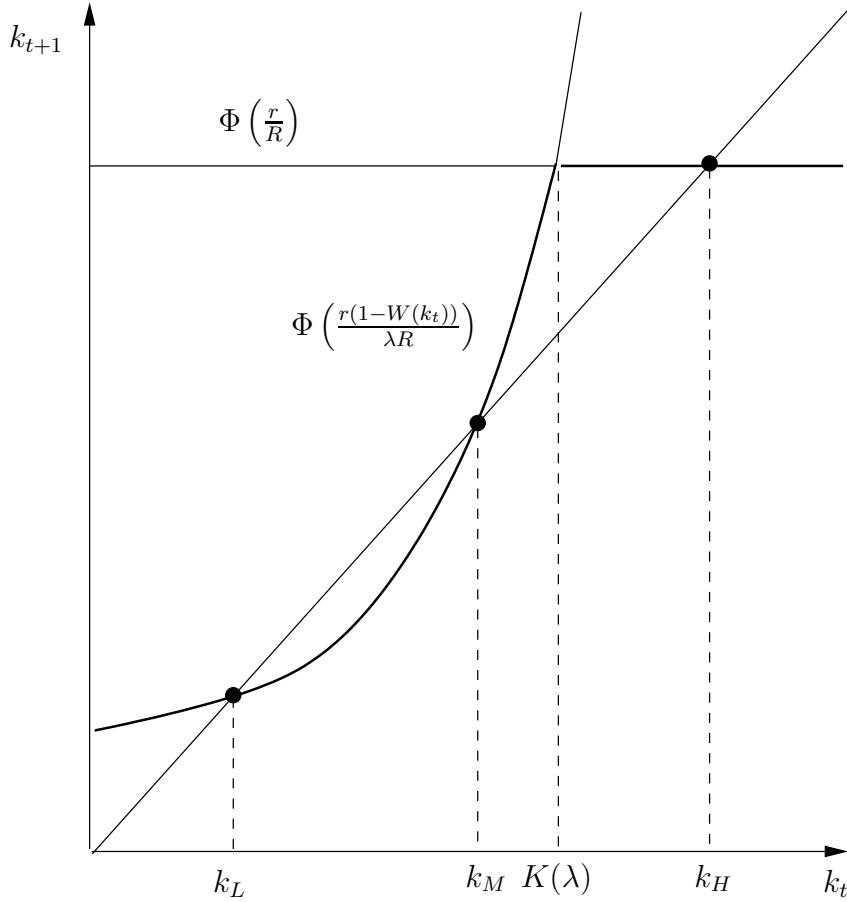


Figure 2: Time one map of the small open economy

4 showed that the result in the autarky case no longer holds in the small open economy since the interest rate does not adjust to equalize the investment and the saving in the domestic economy. Instead, the fraction of people who become lenders in the international credit market is adjusted so that either the profitability constraint or the borrowing constraint is binding in equilibrium. This has consequences on the dynamics of the economy. While the autarky has a unique steady state, which is globally stable, the small open economy may have multiple steady state, which are locally stable. The small open economy model shows that access to international financial markets may be detrimental to economies with low initial capital stock. This result can be interpreted as a poverty trap where the small open economy, caught in a vicious cycle, suffers from persistent underdevelopment. Section 5 generalizes the small open economy to a two country case. Allowing for interactions between two identical economies enables us to draw implications for inequality between countries.

5 Two Homogeneous Countries

In Section 4 the world interest rate was assumed to be constant. In this section the world interest rate will be determined endogenously by the excess demand of both countries. Since we ruled out international factor movements, one country influences the other only through the world interest rate. From equation (7) the capital investment in country i is described by

$$k_{t+1}^i = \Psi(k_t^i, r_{t+1}). \quad (8)$$

In the present model the world economy consist of two countries $i = 1, 2$ with arbitrary initial conditions. Equating total credit demand and total credit supply, the equilibrium interest rate $r_{t+1} = \mathcal{R}(k_t^1, k_t^2)$ in the international financial market is implicitly defined by a solution of

$$L\Psi(k_t^1, r_{t+1}) + (1 - L)\Psi(k_t^2, r_{t+1}) = R(LW(k_t^1) + (1 - L)W(k_t^2)), \quad (9)$$

where $L \in (0, 1)$ is defined to be the relative population size of country 1.³ For a Cobb-Douglas production function of the form $f(k) := Ak^\alpha$, the equilibrium interest rate can be obtained explicitly (see the appendix). Equation (9) defines the temporary equilibrium of the two country model. By dropping the time index in (9) we obtain

$$L\Psi(k^1, r) + (1 - L)\Psi(k^2, r) = R [LW(k^1) + (1 - L)W(k^2)].$$

For any $k^1, k^2 > 0$ the right hand side of the above equation is a positive constant. The left hand side is monotonically decreasing in r since $\Psi(k, r)$ is monotonically decreasing in r . Since $\lim_{r \rightarrow 0} \Psi(k, r) = \infty$ and $\lim_{r \rightarrow \infty} \Psi(k, r) = 0$, there exists a unique solution $r = \mathcal{R}(k^1, k^2)$. Substituting the solution $r_{t+1} = \mathcal{R}(k_t^1, k_t^2)$ into equation (8) we can explicitly solve the two dimensional dynamical system

$$k_{t+1}^i = \Psi(k_t^i, \mathcal{R}(k_t^1, k_t^2)), \quad i = 1, 2. \quad (10)$$

In general, the spillover effect $\partial \mathcal{R}(k_t^1, k_t^2) / \partial k_t^i, \forall i = 1, 2$ is non-zero.⁴

³In the Matsuyama model, there is a continuum of small open homogeneous economies, hence the world interest rate is determined by the condition

$$\int_0^1 \Psi(k_t^i, r_{t+1}) di = R \int_0^1 W(k_t^i) di.$$

This equation with equation (8) defines the dynamical system which is infinite-dimensional.

⁴The spillover effect is zero when there is a continuum of small open economies as no country has a positive measure.

This section analyzes the dynamic behavior of the world economy with two homogeneous countries as the benchmark case. By homogeneous we mean that all characteristics of two economies are identical and they differ only in the stock of capital. In particular we set the relative population size L to one half. In Section 6 we relax this assumption and see how heterogeneous population sizes affect the existence and stability of the steady states.

5.1 Multiple Steady States

The symmetric steady state is identical to the steady state of autarky. It can be easily confirmed that setting $L = 1$ in equation (9) induces the same interest rate as setting $k^1 = k^2$ in steady state when $L = 1/2$. In other words, the world interest rate is identical to that of the autarky at the symmetric steady state. This implies that there is no transfer of capital across two countries. Note also that the symmetric steady state always exists.

Definition 1 *From equations (8) and (9), the steady state in the two country model is defined by a pair (k^1, k^2) satisfying for $i = 1, 2$*

$$r = \mathcal{R}(k^1, k^2) = \begin{cases} \frac{f'(k^i)\lambda R}{1 - W(k^i)} & \text{if } k^i < K(\lambda) \\ f'(k^i)R & \text{if } k^i \geq K(\lambda), \end{cases} \quad (11)$$

and

$$G(k^1, k^2) := L(k^1 - RW(k^1)) + (1 - L)(k^2 - RW(k^2)) = 0. \quad (12)$$

Proposition 2 *Suppose that Assumption 1 and 2 hold.*

1. *There exists a unique positive symmetric steady state value $K^*(R)$ which coincides with the steady state value of autarky defined implicitly by $k = RW(k)$.*
2. *The function $K^* : [0, R^+) \rightarrow \mathbb{R}_{++}$, $R \mapsto K^*(R)$ is increasing in R and satisfies $K^*(0) = 0$.*

Proof: The proof follows directly from equations (8) and (9). □

In the present model each country has a size of positive measure. This means that we have to take the spillover effects through the endogenously determined world interest

rate into account in order to analyze the stability properties of steady states. We cannot assume that in any stable steady state of the world economy each country must be at a stable steady state of the small open economy model as in Matsuyama (2004). However, we can still make a somewhat weaker statement that at any steady state of the two country model, each country must be at a steady state of the small open economy. Let us consider only asymmetric steady states. Since both countries face the same world interest rate, equation (11) for $k^1 \neq k^2$ can be rewritten as

$$H(k^1) = H(k^2) \quad \text{if } k^1, k^2 < K(\lambda) \quad (13)$$

$$H(k^1) = f'(k^2) \quad \text{if } k^1 < K(\lambda) < k^2 \quad (14)$$

$$H(k^2) = f'(k^1) \quad \text{if } k^2 < K(\lambda) < k^1, \quad (15)$$

where $H(k) := \frac{\lambda f'(k)}{1-W(k)}$, $\forall k \in [0, R^+)$. If $k^1, k^2 > K(\lambda)$, the steady state is a symmetric steady state. It can be seen from equation (6) that each country must be at a steady state of the small open economy to satisfy equation (13), (14) or (15). This implies that $(k^1, k^2) \in \{(k_L, k_M), (k_M, k_L), (k_L, k_H), (k_H, k_L), (k_M, k_H), (k_H, k_M)\}$ at any asymmetric steady state. Due to the symmetric structure of the model, these asymmetric steady states emerge pairwise along the diagonal in the (k^1, k^2) space. Before analyzing the exact condition for each asymmetric steady state to exist let us redefine the zero contour $G(k^1, k^2) = 0$ in (k^1, k^2) space to help the technical exposition later on.

Lemma 2 *Let $k \in [0, K^*(R)]$, $R > 0$ and $L = 1/2$.*

1. *There exists an implicit function*

$$g : [0, K^*(R)] \times \mathbb{R}_+ \rightarrow \mathbb{R}_{++}, (k; R) \mapsto g(k; R)$$

satisfying $G(k, g(k; R)) = 0$.

2. *Due to the symmetry of G , $G(g(k; R), k) = 0$ holds. The zero contour of $G(k^1, k^2)$ is defined by the union of the graphs $g(k^1; R)$ and $g(k^2; R)$ in (k^1, k^2) space.*
3. *The map $k \mapsto g(k; R)$ is increasing if and only if $k \in [0, (W')^{-1}(\frac{1}{R})]$. By the implicit function theorem $\frac{dg(K^*(R); R)}{dk} = -1$.*

Proof: The function $k - RW(k)$ is decreasing for $k \in [0, (W')^{-1}(\frac{1}{R})]$ and increasing afterwards. Also $0 - RW(0) = 0$ and $K^*(R) - RW(K^*(R)) = 0$. The zero contour

$G(k^1, k^2) = 0$ can be written as $\{(k^1, k^2) \in \mathbb{R}_+^2 : k^1 - RW(k^1) = -(k^2 - RW(k^2))\}$. The property of function g follows directly. \square

Figure 3 shows that the zero contour of $G(k^1, k^2)$ is the union of graphs $g(k^i)$ for $i = 1, 2$ defined on $[0, K^*(R)]$. More formally,

$$\begin{aligned} \mathcal{G}(R) &:= \{(k^1, k^2) \in \mathbb{R}_+^2 | G(k^1, k^2) = 0\} \\ &= \{(k, g(k; R)) | k \in [0, K^*(R)]\} \cup \{(g(k; R), k) | k \in [0, K^*(R)]\}. \end{aligned}$$

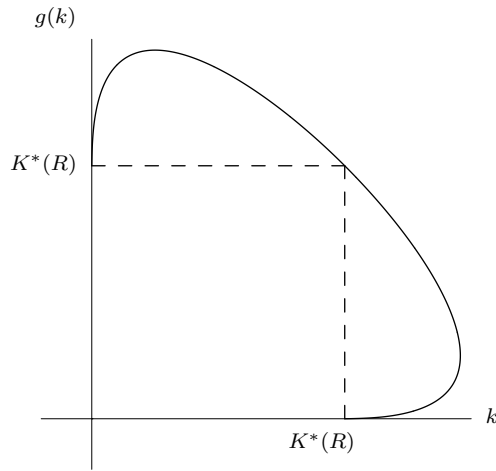


Figure 3: The graph of $g(k)$

To prove the existence of asymmetric steady states we have to show that the zero contour of $G(k^1, k^2)$ has an intersection with the set defined by equations (13), (14) or (15). Lemma 3 characterizes the property of equation (13).

Lemma 3

1. *There exists an implicit function*

$$h : [0, f^{-1}(1)] \rightarrow [f^{-1}(1), W^{-1}(1)], k \mapsto h(k)$$

such that $H(k) - H(h(k)) = 0$.

2. *The function h is decreasing in k and satisfies $h(f^{-1}(1)) = f^{-1}(1)$.*

See the appendix for a proof. \square

Given Lemma 3 we obtain the set

$$\begin{aligned}\mathcal{H} &:= \{(k^1, k^2) \in \mathbb{R}_+^2 \mid H(k^1) - H(k^2) = 0, k^1, k^2 < K(\lambda)\} \\ &= \{(k, h(k)) \mid k \in [0, f^{-1}(1)]\} \cup \{(h(k), k) \mid k \in [0, f^{-1}(1)]\}\end{aligned}$$

Figure 4 shows the graph of $h(k)$.

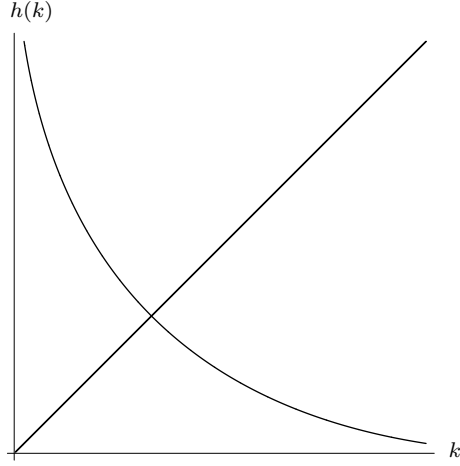


Figure 4: The graph of $h(k)$

In the following we consider only the asymmetric steady states which lie above the diagonal in the (k^1, k^2) space. In other words, we only consider points in the set $\mathcal{U} := \{(k^1, k^2) \in \mathbb{R}_+^2 \mid k^1 \geq k^2\}$. Due to the symmetric structure of the system, the asymmetric steady states in the set $\mathbb{R}_+^2 \setminus \mathcal{U}$ can be obtained analogously.

Proposition 3 shows the existence of the steady state where $k^1, k^2 < K(\lambda)$, i.e. $(k^1, k^2) = (k_M, k_L) \in \mathcal{U} \cap \mathcal{G}(R) \cap \mathcal{H}$ by the intersection of the graphs $h(k^2)$ and $g(k^2; R)$. For the following analysis we will use the Cobb-Douglas production function specified in the following assumption.

Assumption 4 *The production function is of the Cobb-Douglas form $f(k) := Ak^\alpha$.*

Proposition 3 *Suppose that Assumption 4 and $L = 1/2$ are satisfied. Let R_c be defined by $f(K^*(R_c)) = 1$. There exists the asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ if $R > R_c$.*

Proof: The asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ is defined by following equations

$$k^1 = h(k^2) \quad (16)$$

$$k^1 = g(k^2) \quad (17)$$

$$k^2 < k^1 < K(\lambda). \quad (18)$$

The graph of $g(k^2; R)$ defined on $[0, f^{-1}(1)]$ has a unique intersection with the graph of $h(k^2)$ if and only if $K^*(R) > f^{-1}(1)$ (see Figure 5). Due to the symmetric structure, the asymmetric steady state $(k^1, k^2) = (k_L, k_M)$ can be obtained analogously. \square

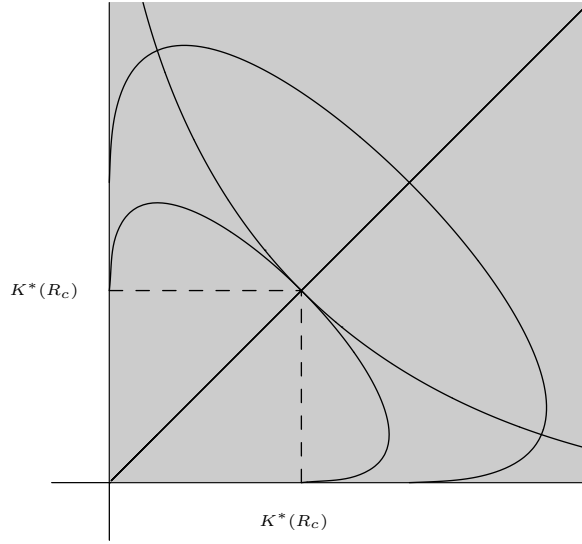


Figure 5: Existence of the asymmetric steady state $(k^1, k^2) = (k_M, k_L)$

To prove the existence of the steady state where $k^2 < K(\lambda) < k^1$, i.e., $(k^1, k^2) = (k_H, k_L)$, we have to show that the graph $g(k^2; R)$ has an intersection with the set defined by equation (15). Equation (15) defines k^1 as a function of k^2 . This function $\phi : k^2 \mapsto \phi(k^2) := (f')^{-1} \left(\frac{\lambda f'(k^2)}{1 - W(k)} \right)$ is increasing if and only if $k_2 < f^{-1}(1)$ and satisfies $\phi(0) = 0$ and $\phi(K(\lambda)) = K(\lambda)$. More formally,

$$\begin{aligned} \mathcal{F} &:= \{(k^1, k^2) \in \mathbb{R}_+^2 \mid H(k^1) - f'(k^2) = 0, k^2 < K(\lambda) < k^1\} \\ &= \{(k, \phi(k)) \mid k \in [0, K(\lambda)]\}. \end{aligned}$$

Hence, $(k^1, k^2) = (k_H, k_L) \in \mathcal{G}(R) \cap \mathcal{F}$. Figure 6 shows the graph of $\phi(k)$.

Let $\mathcal{R}_{LM} = \{R \in [0, R^+) \mid \exists k \in (0, K^*(R)) : \text{graph } g(k^2) \cap \text{graph } \phi(k^2) \cap \text{graph } K(\lambda) \neq \emptyset, k^2 \neq K(\lambda)\}$.

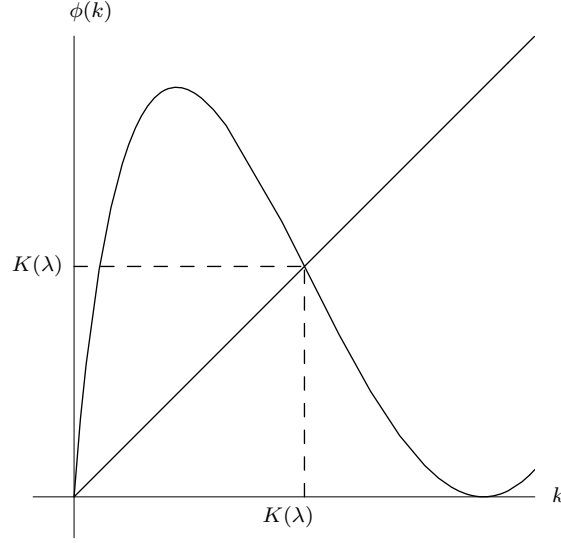


Figure 6: The graph of $\phi(k)$

Proposition 4 *Suppose that Assumption 4 and $L = 1/2$ are satisfied.*

1. *There exists no asymmetric steady state where $k^1, k^2 > K(\lambda)$ if $\phi'(K(\lambda)) > 0$, which is equivalent to $\lambda > \alpha$.*
2. $\mathcal{R}_{LM} = R_{LM}$
3. *For $-1 < \phi'(K(\lambda)) < 0$, the asymmetric steady state $(k^1, k^2) \in (k_H, k_L)$ exists if and only if $R \in [R_{LM}, (K^*)^{-1}(K(\lambda))]$.*
4. *The transition from $(k^1, k^2) = (k_M, k_L)$ to $(k^1, k^2) = (k_H, k_L)$ is continuous in R .*

See the appendix for a proof. □

Let $\mathcal{R}_{LH} = \{R \in [0, R^+) | \exists k \in (0, K^*(R)) : g'(k; R) = \phi'(k), g(k; R) = \phi(k)\}$.

Proposition 5 *Suppose that Assumption 4 and $L = 1/2$ are satisfied.*

1. $\mathcal{R}_{LH} = R_{LH}$
2. *For $\phi'(K(\lambda)) < -1$, the asymmetric steady state $(k^1, k^2) = (k_H, k_L)$*
 - (a) *exists if $R \in [R_{LM}, R_{LH})$*
 - (b) *coexists with $(k^1, k^2) = (k_H, k_M)$ if and only if $R \in [(K^*)^{-1}(K(\lambda)), R_{LH})$.*

See the appendix for a proof. \square

Figure 7 visualizes the results in Proposition 3, 4, and 5 where intersections of zero contours of the functions $\Delta k^1(k^1, k^2) := k^1 - \Psi^1(k^1, k^2)$ and $\Delta k^2(k^1, k^2) := k^2 - \Psi^2(k^1, k^2)$ for different values of R yield the set of steady states.⁵ In what follows we will use the standard parameter set in Table 1 as a benchmark case unless it is otherwise indicated.

A	α	λ	L	k_0^1	k_0^2	$K(\lambda)$	R_c
1	0.5	0.15	0.5	5	2	2.89	2

Table 1: Standard parameter set

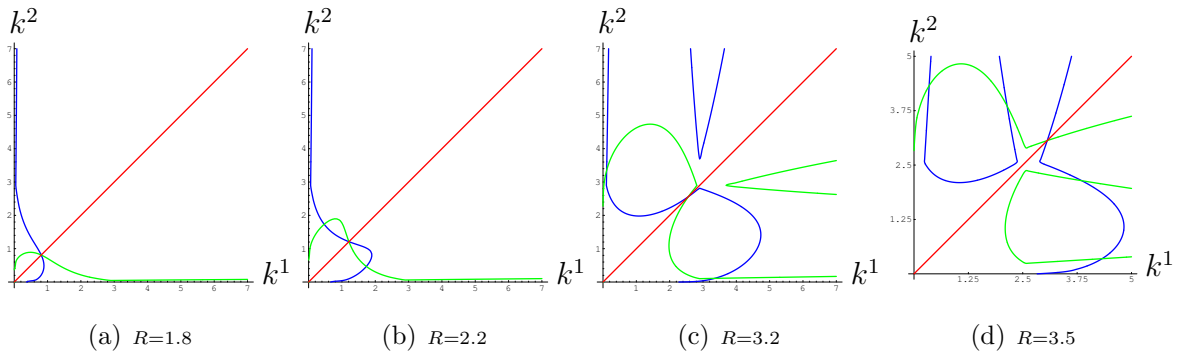


Figure 7: Existence of steady states: $K(\lambda) = 2.89$

One finds that the system has one fixed point for $0 < R \leq 2$ as in (a), three fixed points for $2 < R \leq 3.4$ as in (b) and (c) and five fixed points for $R > 3.4$ as in (d). Figure 7 (b),(c) and (d) corresponds to Proposition 3, 4, and 5 respectively.

5.2 Stability and Spillover Effects

First, we analyze the stability properties of symmetric steady states in Proposition 6. Note that the stability properties of the symmetric case are independent of the population size. Second, we analyze the stability properties of asymmetric steady states with the help of numerical methods.

Proposition 6 *Suppose that Assumption 4 and $L = 1/2$ are satisfied. Two homogeneous countries converge to the symmetric steady state $(K^*(R), K^*(R))$*

⁵Varying λ or α induces similar effects on steady states of the system through their influence on the productivity of the economy.

1. for all $k_0^1, k_0^2 > 0$ if $R < R_c$
2. if $k_0^1 = k_0^2 > 0$ or $k_0^1, k_0^2 \geq K(\lambda)$ for all $R \in \mathbb{R}_+$

See the appendix for a proof. □

Proposition 6 says that if the initial capital stocks of the two countries are high enough so that they do not face the borrowing constraint, they will converge to a symmetric steady state. This will happen even if the symmetric steady state is unstable, i.e. $K^*(R) \leq K(\lambda)$. This is because if $k_0^1, k_0^2 \geq K(\lambda)$, the capital stocks in both countries adjust to the same level in the following period. Once the capital stocks in both countries are the same, there is no trade between countries and both countries follow a convergence path of the autarky economy.

To analyze the stability of the steady states globally, basins of attraction are derived and shown by different colors in Figure 8 for different values of R . Figure 8 (b), (c) and (d) show that the asymmetric steady states which emerge for $R > 2$ are stable. The additional asymmetric steady states which emerge for $R > 3.4$ are unstable (Compare Figure 7 (d) and Figure 8 (d)). The small open economy predicts that the stability of steady states depends on whether the initial condition lies below or above $K(\lambda)$. This is no longer true in the two country model. The change in the stability property can be observed through the following two examples. Firstly, Figure 8 (b) corresponds to Proposition 3. If the world consisted of two small open economies, at this asymmetric steady state one small open economy would be at an unstable steady state while the other at a stable steady state. Nevertheless, this asymmetric steady state is stable. Secondly, Figure 8 (c) and (d) show that both initial conditions of two countries need not to be above $K(\lambda)$ for two countries to converge to the symmetric steady state (k_H, k_H) . Two countries also converge to the symmetric steady state (k_H, k_H) if the capital stock of one country is above $K(\lambda)$ and the other sufficiently close to $K(\lambda)$. These two examples reveal the spillover effects in the two country model which change the stability property of the world with small open economies. In other words, the endogenous world interest rate generates stable steady states which do not emerge in the world with small open economies.

The analysis above shows that the spillover effects can induce converging as well as diverging forces on the world income distribution. This result is robust since there exist open parameter sets in which either the symmetric steady states or the asymmetric steady states are the only asymptotically stable steady states. The spillover effects exert

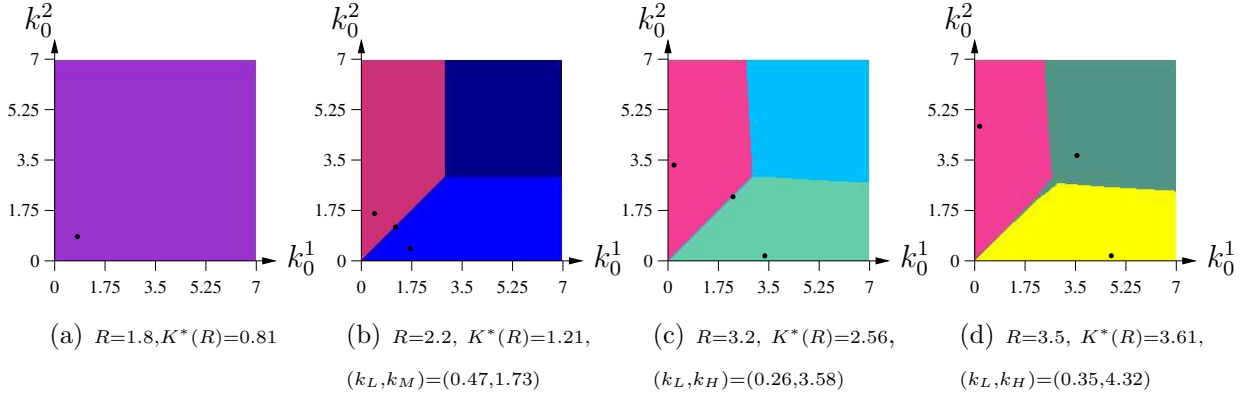


Figure 8: Stability of steady states

a converging force when the borrowing constraint is not binding. When the borrowing constraint is binding, global financial flows may not bring about convergence of income across countries. If both countries have identical initial conditions there is no transfer of physical capital between countries and they converge to the symmetric steady state.

The rich country is always better off at an asymmetric steady state than at a symmetric steady state and the poor country worse off. Suppose $k^1 > k^2$ at the asymmetric steady state. Then,

$$k^1 - RW(k^1) > 0 \quad (19)$$

$$k^2 - RW(k^2) < 0. \quad (20)$$

This implies that at the asymmetric steady state the country with the higher capital stock has an excess demand of physical capital and the country with lower capital stock an excess supply. Therefore, at the asymmetric steady state capital flows from the poor country to the rich country.

6 Population Size Effects

The analysis of the world economy with two homogeneous countries showed that the spillover effects change the stability property of the small open economy. This section introduces heterogeneity into the two country model by allowing for different population sizes. We investigate how robust the results for two homogeneous countries are with respect to a change in the relative population size of two countries. This means that we treat the relative population size as an exogenous parameter and investigate the

sensitivity of the dynamical system with respect to the parameter. The size effect of the population of one country is transmitted only through the world interest rate.

6.1 Multiple Steady States

It is obvious from equation (9) that the population size does not play any role if two countries have the same capital stock. Thus, the symmetric steady state always exists as in the case with two homogenous countries. Section 5.1 showed that the level of the technology that transforms consumption goods into physical capital is decisive for the existence of multiple steady states. This section considers the impact of the relative population size on the existence of multiple steady states. We only consider the case, $L < 1/2$, since the case, $L > 1/2$, can be analyzed analogously given the symmetric structure of the model.

Proposition 7 *Suppose that Assumption 4 is satisfied. There exists a critical value of relative population size L_c below which two asymmetric steady states emerge for $R < R_c$. Each of these asymmetric steady states is of the type $(k^1, k^2) = (k_M, k_L)$ and the country with the smaller population is richer at these steady states.*

See the appendix for a proof. □

Proposition 3 showed that there exists a unique steady state which is symmetric if two countries have the same population sizes and $R < R_c$. Proposition 7 says that if we change the relative population size of the two countries, two asymmetric steady states emerge even for $R < R_c$. The reason behind the emergence of the two coexisting asymmetric steady states of the type $(k^1, k^2) = (k_M, k_L)$ is the broken symmetric structure of the economy. Figure 9 shows the effect of the broken symmetry. This is shown that the zero contour of $G(k^1, k^2)$ loses its symmetric in (k^1, k^2) space in response to the change in the relative population size. Two asymmetric steady states emerge even for $R > R_c = 2$ (Compare with Figure 5).

Notice that the population size only shifts the market clearing condition (9) while the graph of h is unaffected. To show the full set of asymmetric steady states for $L < 1/2$, we have to proceed analogously to the case for $L = 1/2$. To avoid a taxonomic presentation of each case and further complications we proceed here instead to analyze the stability properties of those asymmetric steady states by means of numerical simulation.

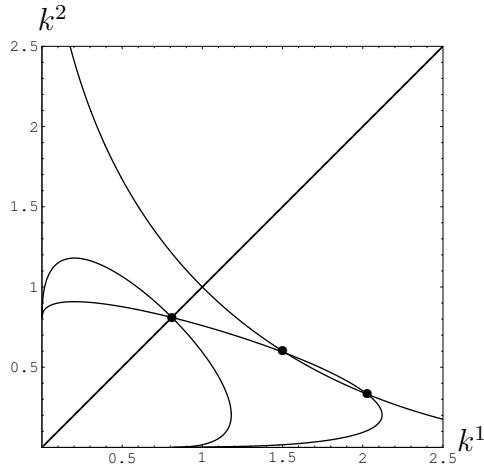


Figure 9: Broken symmetric structure: $R = 1.8$.

6.2 Inequality of Nations

Figure 10 (a) shows the zero contours of the functions $\Delta k^1(k^1, k^2)$ and $\Delta k^2(k^1, k^2)$ for $L = 0.19$ indicating that the zero contours are no longer symmetric along the diagonal. Note also that there are two asymmetric steady states of the type $(k^1, k^2) = (k_M, k_L)$ as stated in Proposition 7. This can be confirmed by noting that at both steady states $k^1, k^2 < K(\lambda) = 2.89$. The sensitivity of the behavior of the system on initial conditions is shown in Figure 10 (b). It shows that the asymmetric steady state which lies closer to the diagonal is unstable.

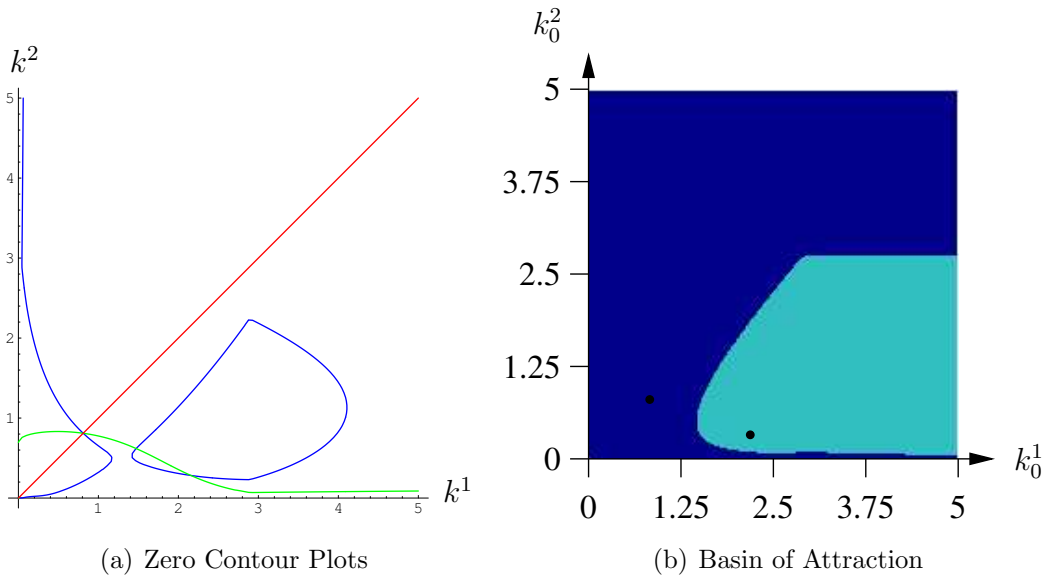


Figure 10: Broken symmetric structure and inequality: $L = 0.19, R = 1.8$.

Let us consider the following scenario. From Proposition 7 we know that the world economy with two homogeneous economies has a globally stable steady state when $R < R_c$. Two economies with arbitrary initial conditions would converge to this steady state in which no international financial transaction takes place. As we move from $L = 1/2$, to below L_c two asymmetric steady states emerge. Now it is crucial as far as the distribution of incomes is concerned, which of these asymmetric steady states is stable. The poor country who faces the borrowing constraint can only invest a fraction of its income into investment projects and is therefore forced to become a lender to the rich country in the international credit market. Figure 11 shows the population size effect on inequality of nations. From Figure 10 we know that the asymmetric steady state, which lies closer to the diagonal is unstable. For sufficiently unequal initial conditions, this causes the world economy to converge to the asymmetric steady state where the richer country remains richer while the poorer country remains poorer as the relative population of the rich country declines. It is the world interest rate that forces both countries to move together to adjust to the new situation.

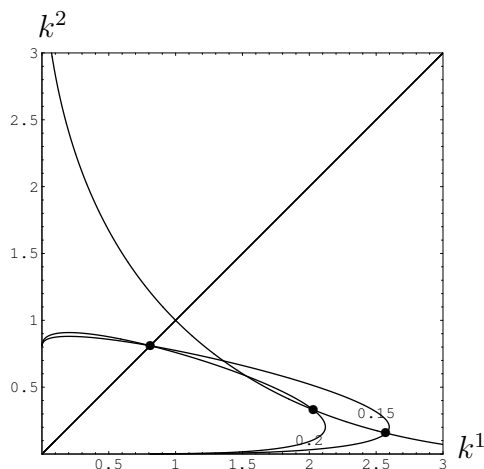


Figure 11: Population size effects on inequality: $R = 1.8, K(\lambda) = 2.89$.

If the relative population size of the rich country declines, the fraction of people who are denied credit in the poor country increases since the borrowing constraint remains the same. As a consequence, the supply in the international credit market increases while the demand decreases. The rich country becomes a bigger borrower and the poor country a bigger lender at a lower world interest rate.

Let us turn to empirical evidence. Milanovic (2002) found that the richest 25 percent of the world's population receives 75 percent of the world's income even when adjusting

for Purchasing Power Parity. The poorest 75 percent of the world’s population share just 25 percent. This occurs because a large proportion of the world’s population lives in the poorest countries, and within the poorest regions of those countries, particularly in the rural areas of China, rural and urban India and Africa. It is beyond the scope of this thesis, however, to conduct numerical calibration. Nevertheless, it is instructive to show a simple numerical example to think about the implications of the model. The results by Milanovic (2002) imply that the income per capita of the richest 25 percent of the world’s population is approximately 9.1 times more than the the poorest 75 percent. Figure 12 shows time series of the present model where two countries with very close initial conditions diverge in the long run.

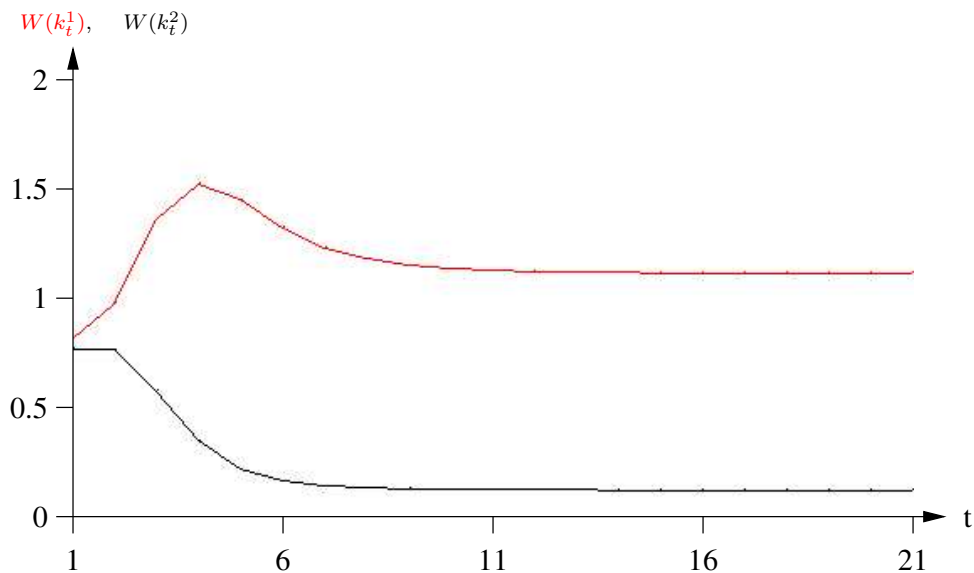


Figure 12: Divergence of identical economies: $L = 0.25, k_0^1 = 2.1, k_0^2 = 2, \alpha = 0.504, \lambda = 0.1, R = 3.5$.

The income per capita of country 1 converges to 1,118 while that of country 2 converges to 0.123. This means that the income per capita of country 1, which composes 25 percent of the world’s population, is approximately 9.1 time more than that of country 2, which composes 75 percent of the world’s population. Thus, the model replicates the findings of Milanovic (2002). Of course, we have to be cautious to interpret the implications of the model. First of all, we separated the world into two units. This is, if at all, a very rough approximation Secondly, we assumed that the two countries have identical structural characteristics. We would typically expect the production elasticity α , the technology to produce physical capital R , and the degree of imperfection in financial

markets λ to be different across countries. However, the identical structure of the model rather strengthens the implications of the model. That is, even if identical technology were available to all the countries, the model predicts that a rich country would diverge from a populous poor country.

6.3 Endogenous Fluctuations

Figure 13 shows a bifurcation diagram with respect to L for k_t^1 and k_t^2 where $k_0^1 > k_0^2$. Firstly, it can be confirmed that the greater inequality in population sizes is associated with greater inequality in income. This means that if the initial conditions of both countries are sufficiently different, both countries converge to the asymmetric steady state that is associated with increasing inequality as the relative population size of the rich country declines.

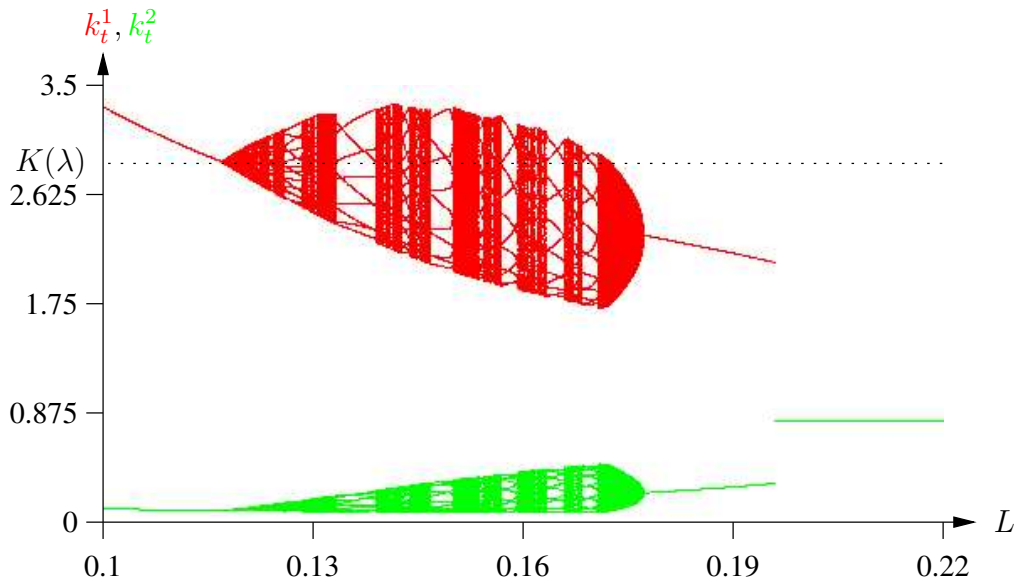


Figure 13: Bifurcation diagram: $R = 1.8, K(\lambda) = 2.89$.

We observe that the asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ loses its stability for sufficiently low L , undergoing a bifurcation. This means that if both countries are unequal initially and if population of the rich country is sufficiently small, fluctuations are an inherent feature of the international credit market. The asymmetric steady state loses its stability and a closed invariant curve appears (see Kuznetsov (1998) for details). This can be confirmed by looking at Figure 14.

The following proposition summarizes the observations.

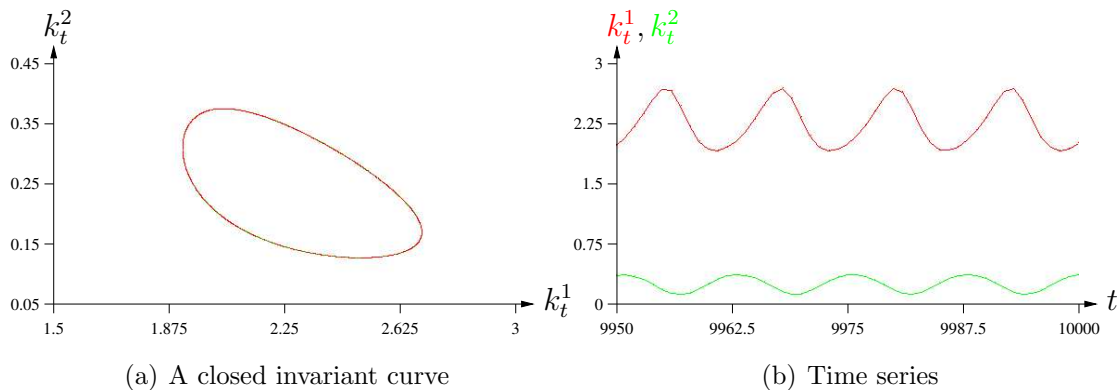


Figure 14: Endogenous cycles: $L = 0.175, R = 1.8$.

Proposition 8 *Suppose that Assumption 4 is satisfied. Consider the dynamics of the world economy by changing the bifurcation parameter L . There exists an open parameter set with coexisting asymmetric steady states $(k^1, k^2) = (k_H, k_L)$ and $(k^1, k^2) = (k_M, k_L)$. The asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ undergoes a supercritical Neimark-Sacker bifurcation.*

See the appendix for a proof. □

Note that bifurcations occur at two points. At the bifurcation point with lower L the steady state value of k^1 reaches $K(\lambda) = 2.89$. At this point the dynamical system is not differentiable, as it switches from the case in which one country faces the borrowing constraint to the case in which both countries face the borrowing constraint. As shown in the proof of Proposition 8, the asymmetric steady state $(k^1, k^2) = (k_H, k_M)$ is stable while $(k^1, k^2) = (k_M, k_L)$ is unstable after this bifurcation point. This implies that there are two forces which pull the world economy in opposite directions. One pulls the economy close to $k^2 < K(\lambda) < k^1$ and the other pulls the economy away from $k^2 < K(\lambda) < k^1$. These forces generate non-stationary orbits of k_t^1 around $K(\lambda)$. This can be confirmed by looking at Figure 15. The observed bifurcation phenomena imply that the spillover effects generated by two countries with different population size are the source of endogenous fluctuations of capital stocks in both countries.

7 Concluding Remarks

We have examined how the income distribution between the two countries is influenced by the spillover effects on capital stocks in both countries via an endogenous determi-

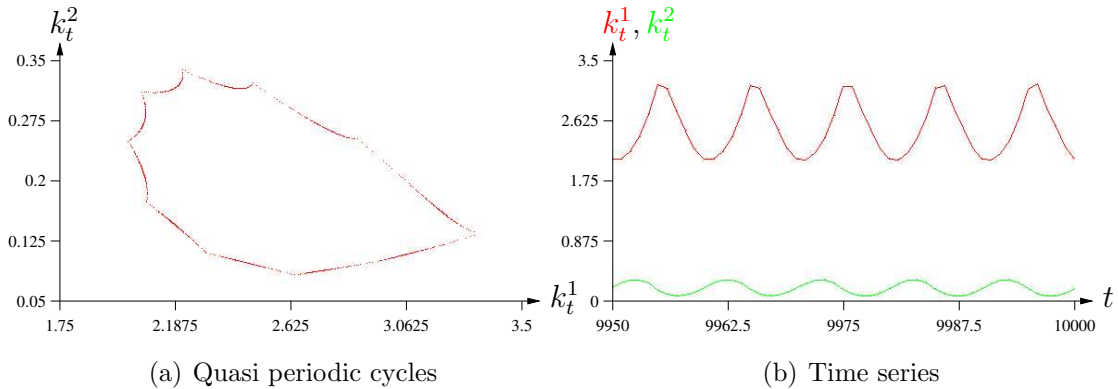


Figure 15: Endogenous cycles: $L = 0.15, R = 1.8$.

nation of the world interest rate. We singled out the spillover effects by comparing the results of the world economy model with small open economies to a two country model. The two country model allowed us to find a closed form of the dynamical system and rely on numerical simulations when analytical results could not be obtained. The symmetry breaking results in Boyd & Smith (1997) and Matsuyama (2004) hold in the present paper. There are some additional common features in Boyd & Smith (1997), Matsuyama (2004) and in the present paper. Firstly, an initially poorer country remain relatively poorer if it does not converge to a symmetric steady state. Secondly, the poor country is better off in a symmetric steady state than in an asymmetric steady state while the rich country is worse off. Thirdly, the aggregate wealth of the world economy is higher in a symmetric steady states than in an asymmetric steady state given homogeneous population sizes.

In addition, the present paper offers new insights into the nature of integrated economies. The spillover effects change the stability properties of steady states by inducing an equalizing as well as an unequalizing force depending on initial conditions of the two countries. While the spillover effect may prevent the two economies with different capital stocks from converging, it may also induce an equalizing force when the two countries are sufficiently rich. Heterogeneous population sizes are also identified as factors which preclude two countries from converging to each other. The smaller the relative population of the rich country, the more unequal is the income distribution between the two countries at the induced steady state. These theoretical findings have important policy implications as they formalize the “unfair” size effects, which are often propagated by globalization opponents and are subject to interpretation. The spillover effects, which are generated by the two countries with heterogeneous populations in the international financial mar-

ket have a striking influence on the world income distribution. However, the present paper does not necessarily try to argue against the financial market globalization as the spillover effects also exert an equalizing force. Endogenous fluctuations occur if the relative population size of the rich country is sufficiently small. This implies that fluctuations of capital stocks of two countries as well as international capital flows are inherent in the international financial market. The observation, that international financial markets can lead to enhanced economic volatility seems to have wide empirical support.

Appendix

The Equilibrium Interest Rate: The Cobb-Douglas Production Function Case

Let the production function be of the Cobb-Douglas form, $f(k) = Ak^\alpha$. Then, the equilibrium interest rate, which is defined by equation (9), is given by

$$r_{t+1} = \mathcal{R}(k_t^1, k_t^2) := \begin{cases} \frac{\alpha A \lambda R \left[L(1-W(k_t^1))^{\frac{1}{\alpha-1}} + (1-L)(1-W(k_t^2))^{\frac{1}{\alpha-1}} \right]^{1-\alpha}}{\left[R[LW(k_t^1) + (1-L)W(k_t^2)] \right]^{1-\alpha}} & \text{if } k_t^1, k_t^2 < K(\lambda) \\ \frac{\alpha A \lambda R \left[L[1-W(k_t^1)]^{\frac{1}{\alpha-1}} + (1-L)\lambda^{\frac{1}{\alpha-1}} \right]^{1-\alpha}}{\left[[LW(k_t^1) + (1-L)W(k_t^2)]R \right]^{1-\alpha}} & \text{if } k_t^1 < K(\lambda) \leq k_t^2 \\ \frac{\alpha A \lambda R \left[L\lambda^{\frac{1}{\alpha-1}} + (1-L)[1-W(k_t^2)]^{\frac{1}{\alpha-1}} \right]^{1-\alpha}}{\left[[LW(k_t^1) + (1-L)W(k_t^2)]R \right]^{1-\alpha}} & \text{if } k_t^2 < K(\lambda) \leq k_t^1 \\ \frac{\alpha A R}{\left[[LW(k_t^1) + (1-L)W(k_t^2)]R \right]^{1-\alpha}} & \text{if } k_t^1, k_t^2 \geq K(\lambda). \end{cases}$$

Substituting the equilibrium interest rate into the capital accumulation law (8), we

obtain the dynamical system of the two country model in a closed form given by

$$k_{t+1}^1 = \tilde{\Psi}^1(k_t^1, k_t^2) := \begin{cases} \frac{R[LW(k_t^1) + (1-L^1)W(k_t^2)]}{L + (1-L) \left[\frac{1-W(k_t^1)}{1-W(k_t^2)} \right]^{\frac{1}{1-\alpha}}} & \text{if } k_t^1, k_t^2 < K(\lambda) \\ \frac{R[LW(k_t^1) + (1-L^1)W(k_t^2)]}{L + (1-L) \left[\frac{1-W(k_t^1)}{\lambda} \right]^{\frac{1}{1-\alpha}}} & \text{if } k_t^1 < K(\lambda) \leq k_t^2 \\ \frac{R[LW(k_t^1) + (1-L^1)W(k_t^2)]}{L + (1-L) \left[\frac{\lambda}{1-W(k_t^2)} \right]^{\frac{1}{1-\alpha}}} & \text{if } k_t^2 < K(\lambda) \leq k_t^1 \\ R[LW(k_t^1) + (1-L)W(k_t^2)] & \text{if } k_t^1, k_t^2 \geq K(\lambda). \end{cases}$$

and

$$k_{t+1}^2 = \tilde{\Psi}^2(k_t^1, k_t^2) := \begin{cases} \frac{R[LW(k_t^1) + (1-L^1)W(k_t^2)]}{L \left[\frac{1-W(k_t^2)}{1-W(k_t^1)} \right]^{\frac{1}{1-\alpha}} + (1-L)} & \text{if } k_t^1, k_t^2 < K(\lambda) \\ \frac{R[LW(k_t^1) + (1-L)W(k_t^2)]}{L \left[\frac{\lambda}{1-W(k_t^1)} \right]^{\frac{1}{1-\alpha}} + (1-L)} & \text{if } k_t^1 < K(\lambda) \leq k_t^2 \\ \frac{R[LW(k_t^1) + (1-L)W(k_t^2)]}{L \left[\frac{1-W(k_t^2)}{\lambda} \right]^{\frac{1}{1-\alpha}} + (1-L)} & \text{if } k_t^2 < K(\lambda) \leq k_t^1 \\ R[LW(k_t^1) + (1-L)W(k_t^2)] & \text{if } k_t^1, k_t^2 \geq K(\lambda). \end{cases}$$

Proof of Lemma 3

We first investigate the properties of the function $H(k)$ by looking at the first and second derivative. One has that

$$H'(k) = \frac{f''(k)(1-f(k))}{(1-W(k))^2} \underset{\leq}{\geq} 0 \Leftrightarrow f(k) \underset{\leq}{\geq} 1. \quad (\text{A.1})$$

The function H has its global minimum at $f^{-1}(1)$. In addition, $H(0) = \infty$ and $H(W^{-1}(1)) = \infty$. Moreover, for all $k > 0$

$$H''(k) = \frac{f'''(k)(1-f(k)) - f''(k)f'(k)}{(1-W(k))^2} + \frac{2f''(k)(1-f(k))W'(k)}{(1-W(k))^3} > 0. \quad (\text{A.2})$$

Hence, the function H is strictly convex and has a configuration as depicted in Figure 16.

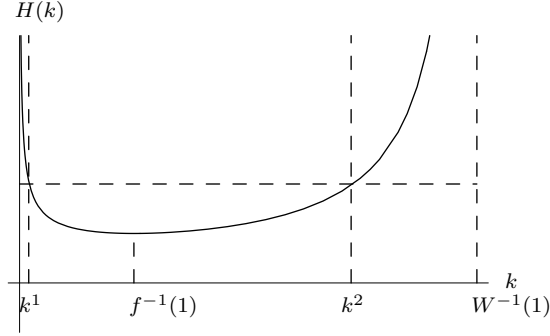


Figure 16: The graph of $H(k)$

It follows immediately that $(k^1, k^2) = (f^{-1}(1), f^{-1}(1))$ is the unique pair which solves equation (13). Then for $k^2 < f^{-1}(1)$, there exists a unique $k^1 > f^{-1}(1)$ which solves equation (13) (see Figure 16). Suppose that $k \in [0, f^{-1}(1)]$, then we obtain an implicit function $h : [0, f^{-1}(1)] \rightarrow [f^{-1}(1), W^{-1}(1)]$, $k \mapsto h(k)$ such that $H(k) - H(h(k)) = 0$. From the implicit function theorem,

$$h'(k) = \frac{H'(k)}{H'(h(k))} < 0. \quad (\text{A.3})$$

Hence, the function h is decreasing and satisfies $h(f^{-1}(1)) = f^{-1}(1)$.

Proof of Proposition 4

The asymmetric steady state $(k^1, k^2) = (k_H, k_L)$ is defined by the following equations

$$k^1 = g(k^2) \quad (\text{A.4})$$

$$k^1 = \phi(k^2) \quad (\text{A.5})$$

$$k^2 < K(\lambda) < k^1. \quad (\text{A.6})$$

Let us first introduce the expressions $\phi'(k)$, $\phi''(k)$, and $\phi'(K(\lambda))$ for the Cobb-Douglas function, which will be used below. One obtains that

$$\phi'(k) = \frac{1}{1-\alpha} C^{\frac{1}{1-\alpha}} f''(k) \left(\frac{k}{1-W(k)} - \frac{1}{f'(k)} \right) \quad (\text{A.7})$$

and

$$\begin{aligned}\phi''(k) = & \left(\frac{1}{1-\alpha}\right)^2 C^{\frac{1}{1-\alpha}} (f''(k))^2 \left(\frac{k}{1-W(k)} - \frac{1}{f'(k)}\right)^2 \\ & - \frac{1}{1-\alpha} C^{\frac{1}{1-\alpha}} f''(k) \frac{(2-\alpha)}{k} \left(\frac{k}{1-W(k)} - \frac{1}{f'(k)}\right) \\ & + \frac{1}{1-\alpha} C^{\frac{1}{1-\alpha}} f''(k) \left(\frac{1-(1-\alpha)W(k)}{(1-W(k))^2} - \frac{1-\alpha}{\alpha A k^\alpha}\right)\end{aligned}\quad (\text{A.8})$$

where $C = \left(\frac{\alpha A(1-W(k))}{\lambda f'(k)}\right)$. We can show that $\phi''(k) = 0$ is equivalent to $k = \left(\frac{1+\alpha}{A}\right)^2$. This implies that the function ϕ has a unique inflection point.

Moreover,

$$\phi'(K(\lambda)) = -f'(k) \left(\frac{k}{\lambda} - \frac{1}{f'(k)}\right).\quad (\text{A.9})$$

1) We show that there exists no asymmetric steady state for $\phi'(K(\lambda)) > 0$. Using equation (A.9) we can show that $\phi'(K(\lambda)) > 0$ is equivalent to $\lambda > \alpha$. This is also equivalent to $f^{-1}(1) > K(\lambda)$. Hence, the graph of $\phi(k)$ lies below $K(\lambda)$ for $k \in [0, K(\lambda))$. This proves that there exists no asymmetric steady state where $k^1, k^2 > K(\lambda)$.

2), 3) We show that $\phi''(k) < 0, \forall k \in [0, K(\lambda)]$ if $\phi'(K(\lambda)) > -1$ by contradiction. Suppose that $\phi''(k) > 0$. Then, the Cobb-Douglas production function implies that

$$k > \left(\frac{1+\alpha}{A}\right)^{\frac{1}{\alpha}}.\quad (\text{A.10})$$

Equation (A.10) and $k < K(\lambda)$ imply that

$$\left(\frac{1+\alpha}{A}\right)^{\frac{1}{\alpha}} < K(\lambda) := \left(\frac{1-\lambda}{A(1-\alpha)}\right)^{\frac{1}{\alpha}} \iff \lambda < \alpha^2.\quad (\text{A.11})$$

This means that the inflection point of $\phi(k)$ lies below $K(\lambda)$ if and only if $\lambda < \alpha^2$.

Our assumption was $\phi'(K(\lambda)) > -1$, which equivalent to

$$\lambda > \frac{\alpha}{2-\alpha}.\quad (\text{A.12})$$

However, equations (A.11) and (A.12) together imply

$$\alpha - \frac{1}{2-\alpha} > 0,\quad (\text{A.13})$$

which is never satisfied for $\alpha \in (0, 1)$. Hence, $\phi''(k) < 0, \forall k \in [0, K(\lambda)]$ if $\phi'(K(\lambda)) > -1$. This means that $\phi(k)$ is concave for all $k \in [0, K(\lambda)]$ if $\phi'(K(\lambda)) > -1$.

Now, suppose that $-1 < \phi'(K(\lambda)) < 0$. Then, the graph of $\phi(k)$ has a unique intersection with the graph of $g(k)$ for $k < K(\lambda) < \phi(k)$ if and only if $R \in [R_{LM}, (K^*)^{-1}(K(\lambda))]$. R_{LM} is the value of R for which the graph of $g(k)$ has a unique intersection point with the graphs of $\phi(k)$ and $K(\lambda)$ for $k \neq K(\lambda)$. Note that if $\phi'(K(\lambda)) > 0$, $(K(\lambda), K(\lambda))$ is the only point which satisfies $(K(\lambda), \phi^{-1}(K(\lambda)))$. Due to the symmetric structure, the asymmetric steady state $(k^1, k^2) = (k_L, k_H)$ can be obtained analogously.

4) To prove that the transition from $(k^1, k^2) = (k_M, k_L)$ to $(k^1, k^2) = (k_H, k_L)$ is continuous in R , observe that when $k = K(\lambda)$, $h(k) = \phi(k)$. The claim follows since the steady states $(k^1, k^2) = (k_M, k_L)$ and $(k^1, k^2) = (k_H, k_L)$ are continuous functions in R .

Proof of Proposition 5

Suppose that $\phi'(K(\lambda)) < -1$. Then, the graph of $g(k)$ has a unique intersection with the graph of $\phi(k)$ if $R \in [R_{LM}, (K^*)^{-1}(K(\lambda))]$ and two intersections if and only if $R \in [(K^*)^{-1}(K(\lambda)), R_{LH}]$. R_{LH} is the value of R for which the graph of $\phi(k)$ is tangent to the graph of $g(k)$. For $R > R_{LH}$ there is no intersection of the graphs $\phi(k)$ and $g(k)$ (see Figure 17).

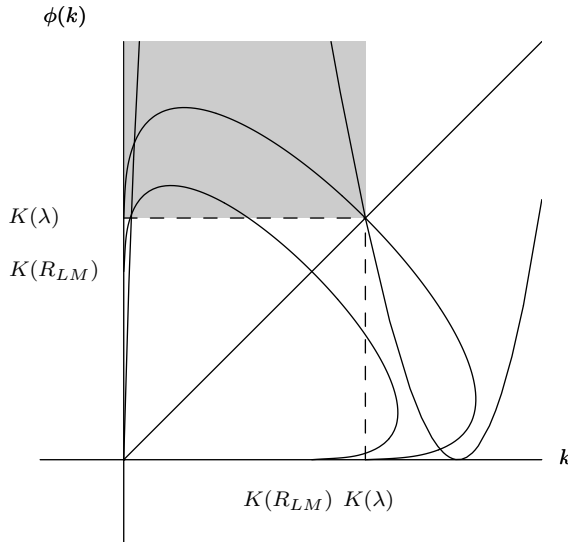


Figure 17: Existence of the asymmetric steady states $(k^1, k^2) \in \{(k_H, k_L), (k_H, k_M)\}$

For a third intersection to exist, the graph $\phi(k^2)$ has to cut the graph $g(k^2)$ from inside at the third intersection. This would imply that $\phi''(k^2)$ has to change its sign two times in $[0, K(\lambda)]$. This is a contradiction because we know for the Cobb-Douglas function

that there exist a unique inflection point, $k = (\frac{1+\alpha}{A})^{\frac{1}{\alpha}}$ where $\phi''(k)$ changes its sign. Hence, $\phi'(K(\lambda)) < -1$ guarantees that there are no more than two intersection points of the graphs $\phi(k)$ and $g(k)$. Due to the symmetric structure, the asymmetric steady state $(k^1, k^2) \in \{(k_L, k_H), (k_M, k_H)\}$ can be obtained analogously. \square

Proof of Proposition 6

Let us first prove that the asymmetric steady state is stable if $R < R_c$. For $k^1 < K(\lambda), k^2 < K(\lambda)$, we have

$$\begin{aligned} k^1 = \tilde{\Psi}^1(k^1, k^2) &= \frac{R[W(k^1) + W(k^2)]}{1 + \left[\frac{1-W(k^2)}{1-W(k^1)}\right]^{\frac{1}{\alpha-1}}} \\ k^2 = \tilde{\Psi}^2(k^1, k^2) &= \frac{R[W(k^1) + W(k^2)]}{1 + \left[\frac{1-W(k^1)}{1-W(k^2)}\right]^{\frac{1}{\alpha-1}}}. \end{aligned}$$

Let

$$J(k, k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{\Psi}^1(k, k)}{\partial k^1} & \frac{\partial \tilde{\Psi}^1(k, k)}{\partial k^2} \\ \frac{\partial \tilde{\Psi}^2(k, k)}{\partial k^1} & \frac{\partial \tilde{\Psi}^2(k, k)}{\partial k^2} \end{pmatrix}.$$

Observe that $a = d, b = c$. The characteristic polynomial reads $p(\mu) = \mu^2 - 2a\mu + a^2 - b^2$. The eigenvalues of the system are

$$\begin{aligned} \mu_1 &= a + b = \alpha \\ \mu_2 &= a - b = \alpha \left(\frac{Ak^\alpha}{1 - (1 - \alpha)Ak^\alpha} \right) \end{aligned}$$

where $k = K^*(R) := \left(\frac{1}{(1-\alpha)AR} \right)^{\frac{1}{\alpha-1}}$. It follows that $0 < \mu_1 < 1$ and $0 < \mu_2 < 1$ if and only if

$$R < R_c = \frac{1}{(1-\alpha)A^{\frac{1}{\alpha}}}.$$

Now, let us prove that countries converge to the symmetric steady state if $k_0^1 = k_0^2 =: k_0 > 0$ or $k_0^1, k_0^2 \geq K(\lambda)$. Let $\Psi^i(k_t^1, k_t^2) := \Psi(k_t^i, \mathcal{R}(k_t^1, k_t^2))$ for $i = 1, 2$. If $k_0^1 = k_0^2 =: k_0 > 0$, then $\Psi^1(k_0, k_0) = \Psi^2(k_0, k_0) = RW(k_0) = k_1^1 = k_1^2 =: k_1$. By induction, $(\Psi^1)^n(k_0, k_0) = (\Psi^2)^n(k_0, k_0) = \underbrace{RW \circ RW \cdots \circ RW}_{n\text{-times}}(k_0) = k_n^1 = k_n^2 =: k_n, \forall n \in \mathbb{N}$.

Given Assumption 1, the orbit $\lim_{n \rightarrow \infty} \underbrace{RW \circ RW \cdots \circ RW}_{n\text{-times}}(k_0)$ converges to $K^*(R)$ and

hence (k_n^1, k_n^2) converges to the symmetric steady states $(K^*(R), K^*(R))$. If $k_0^1, k_0^2 \geq K(\lambda)$, $\Psi^1(k_0^1, k_0^2) = \Psi^2(k_0^1, k_0^2) \implies k_1^1 = k_1^2 =: k_1 > 0$. Convergence to $(K^*(R), K^*(R))$ follows from the first part of this proof. \square

Proof of Proposition 7

The asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ requires

$$k^1 = h(k^2) \tag{A.14}$$

$$k^1 = g(k^2) \tag{A.15}$$

Let us first investigate how the change in L affects $k^1 = g(k^2)$, which is implicitly defined by $G(k^1, k^2) := L(k^1 - RW(k^1)) + (1 - L)(k^2 - RW(k^2))$. Figure 18 depicts how the function $L(k - RW(k))$ depends on L . The function $L(k - RW(k))$ implies

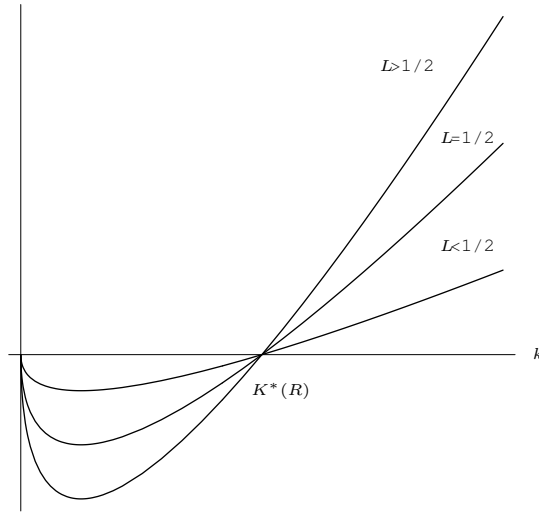


Figure 18: The graph of $L(k - RW(k))$

that the graph $g(k^1)$ which satisfies $G(k^1, k^2) = 0$ shifts downwards for lower L . On the other hand, $g(k^2)$ shifts upwards for lower L . Furthermore, $\frac{dg(K^*(R); R, L)}{dL} = -\frac{L}{1-L}$ by the implicit function theorem. If $R < R_c$, there exists always a unique $L_c < 1/2$ defined by $\frac{\partial g(k^2)}{\partial k^2} = \frac{\partial h(k^2)}{\partial k^2}$ and $g(k^2) = h(k^2)$. Analogously we can prove the existence of the asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ for $L_c > 1/2$. \square

Proof of Proposition 8

We will prove that the asymmetric steady state $(k^1, k^2) = (k_M, k_L)$ undergoes a supercritical Neimark-Sacker bifurcation by calculating the determinant and trace numerically. The determinant and the trace of the Jacobian matrix of the system (10) when $k^1, k^2 < K(\lambda)$ can be written as

$$\begin{aligned} \det = & \frac{W'(k^1)k^1W'(k^2)k^2}{(LW(k^1) + (1-L)W(k^2))^2} \frac{1}{R(1-\alpha)} \left(k^2L^2 \frac{(1-W(k^1))^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right. \\ & \left. + k^1(1-L)^2 \frac{(1-W(k^2))^{\frac{1}{\alpha-1}}}{(1-W(k^1))^{\frac{\alpha}{\alpha-1}}} \right. \\ & \left. + k^2(1-L)L \frac{(1-W(k^1))^{\frac{2-\alpha}{\alpha-1}}}{(1-W(k^2))^{\frac{1}{\alpha-1}}} + k^1(1-L)L \frac{(1-W(k^2))^{\frac{2-\alpha}{\alpha-1}}}{(1-W(k^1))^{\frac{1}{\alpha-1}}} \right) \end{aligned} \quad (\text{A.16})$$

and

$$\begin{aligned} \text{tr} = & \frac{1}{LW(k^1) + (1-L)W(k^2)} \left(W'(k^1)k^1 \left(L + \frac{k^1(1-L)}{R(1-\alpha)} \frac{(1-W(k^2))^{\frac{1}{\alpha-1}}}{(1-W(k^1))^{\frac{\alpha}{\alpha-1}}} \right) \right. \\ & \left. + W'(k^2)k^2 \left(1-L + \frac{k^2L}{R(1-\alpha)} \frac{(1-W(k^1))^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right) \right). \end{aligned} \quad (\text{A.17})$$

The determinant and the trace when $k^2 < K(\lambda) < k^1$ can be written as

$$\begin{aligned} \det = & \frac{W'(k^1)k^1W'(k^2)k^2}{(LW(k^1) + (1-L)W(k^2))^2} \frac{1}{R(1-\alpha)} \left(k^2L^2 \frac{\lambda^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right. \\ & \left. + k^1(1-L)L \frac{(1-W(k^2))^{\frac{2-\alpha}{\alpha-1}}}{\lambda^{\frac{1}{\alpha-1}}} \right) \end{aligned} \quad (\text{A.18})$$

and

$$\text{tr} = \frac{W'(k^1)k^1L + W'(k^2)k^2}{LW(k^1) + (1-L)W(k^2)} \left(1-L + \frac{k^2L}{R(1-\alpha)} \frac{\lambda^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right). \quad (\text{A.19})$$

If $k^1 = K(\lambda)$, equations (A.18) and (A.19) can be written as

$$\begin{aligned} \det = & \frac{W'(k^1)k^1W'(k^2)k^2}{(LW(k^1) + (1-L)W(k^2))^2} \frac{1}{R(1-\alpha)} \left(k^2L^2 \frac{(1-W(k^1))^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right. \\ & \left. + k^1(1-L)L \frac{(1-W(k^2))^{\frac{2-\alpha}{\alpha-1}}}{(1-W(k^1))^{\frac{1}{\alpha-1}}} \right) \end{aligned} \quad (\text{A.20})$$

and

$$\text{tr} = \frac{W'(k^1)k^1L + W'(k^2)k^2}{LW(k^1) + (1-L)W(k^2)} \left(1 - L + \frac{k^2L}{R(1-\alpha)} \frac{\lambda^{\frac{1}{\alpha-1}}}{(1-W(k^2))^{\frac{\alpha}{\alpha-1}}} \right). \quad (\text{A.21})$$

Comparing equations (A.16) and (A.17) with equations (A.18) and (A.19), we can see that the determinant and the trace are not equal respectively at $k^1 = K(\lambda)$. The dynamical system is not differentiable at this point. Figure 19 shows how the determinant and the trace of the system moves as we change the bifurcation parameter L . The points (a),(b),(c),(d),(e),(f) correspond to $L = (0.117, 0.117, 0.13, 0.16, 0.177, 0.19)$. The points (a) and (b) are defined by equations (A.16) and A.17, and equations (A.20) and (A.21) respectively. We can observe that at $L = 0.177$ when $k^1 = K(\lambda)$ the determinant and the trace jump from (a) to (b). As the value of L increases, the determinant crosses 1 at (e) which proves that the bifurcation is a Neimark-Sacker bifurcation.

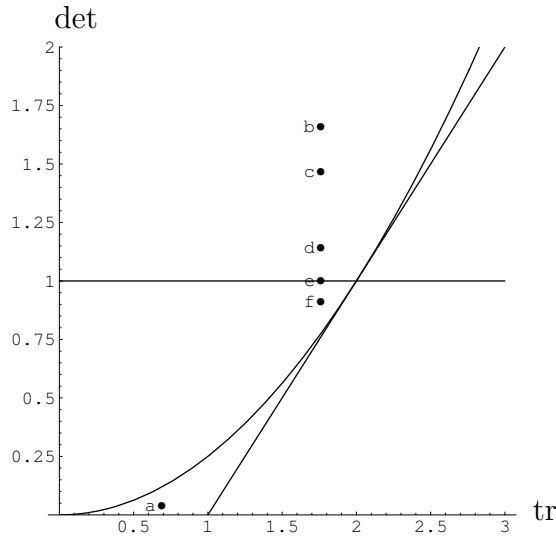


Figure 19: Stability triangle: $L = (0.117, 0.117, 0.13, 0.16, 0.177, 0.19)$

□

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