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The Semistrict Core

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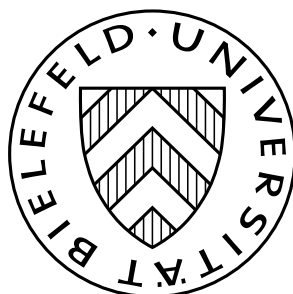
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March 2, 2006

Abstract

We consider the class of proper monotonic simple games and study coalition forma-
tion when an exogenous share vector and a solution concept are combined to guide the
distribution of coalitional worth. Using a multiplicative composite solution, we induce
players' preferences over coalitions in a hedonic game, and present conditions under
which the semistrict core of the game is nonempty.

JEL Classification: D72, C71
Keywords: coalition formation, paradox of smaller coalitions, semistrict core, simple
games, winning coalitions

1   Introduction

The analysis of election results is one of the most popular applications of cooperative game
theory. Thereby a game describes the parties' possibilities to form a winning coalition,
respectively a government. Application of a solution concept, such as the Shapley value
(or *Shapley-Shubik index* as it is often termed in this setup), is readily interpreted as the
(*endogenous*) *power* that a party exerts in the parliament. This notion of power is then often
used to distribute responsibilities in a government.

Two drawbacks of this approach are frequently criticized: First, this notion of power only takes into account the data of the game, i.e., it only takes into account, which coalitions may form a government. What it ignores is a party's total number of votes or its total number of seats in the parliament. So, it disregards ideas of proportionality or *exogenous power distributions*. It is undoubted that a government consisting of a "small" and a "large" party does not share responsibilities (e.g., offices) equally. Second, this approach does not answer the question, which government is likely to form or, regarded from a normative point of view, should form.

In this paper we are interested in tackling these two problems. For this, we consider the class of *proper monotonic simple games*. In a simple game any possible coalition is either winning or not. Monotonicity guarantees that supercoalitions of a winning coalition are also winning and properness requires that the complementary coalition of a winning coalition is not winning. As argued above there are two sources that should play a role, when describing a power distribution among parties. We bring these endogenous and exogenous impacts together by introducing the concept of a *composite solution*. More precisely, a composite solution F takes each collection (α, S, v, φ) of an exogenous *share vector* α, a coalition S, a simple game v and a cooperative solution concept φ to a distribution of power with the interpretation that $F_i(\alpha, S, v, \varphi)$ reflects player (party) i's (overall) power within coalition S, when v describes the possibilities for winning coalitions. Thereby, we will be interested in a specific composite solution, in which exogenous shares enter in a proportional fashion.

To come back to the government formation problem, we may assume that a player's incentive to take part in a (winning) coalition depends on how much power he has within this coalition according to a composite solution. In effect, we obtain preferences over coalitions. The collection of these preferences forms a *hedonic coalition formation game* (cf. Banerjee et. al. (2001) and Bogomolnaia and Jackson (2002)). A solution for this (and each) hedonic game proposes a (set of) partition(s) of the set of players into coalitions. In the context of simple games this in effect means which winning coalition forms. The focus in the context of solutions are *stability* considerations, meaning that the final partition should not provide incentives for a coalition to deviate and form instead. As it can be easily seen, it is not possible a coalition structure to be stable if it does not contain a winning coalition. Hence, the answer to the question which partitions are stable is at the same time an answer to the question which winning coalition (or government) should form with respect to stability concerns.

Depending on how restrictive conditions for coalitional deviations are formulated, we get different notions of stability. We have chosen the *semistrict core* as our stability concept for hedonic games. This stability notion is weaker than the strict core and stronger than the standard core notion, and the idea of it can already be found in the work of Kirchsteiger

and Puppe (1997) and, more definitive, in the works of Dimitrov and Haake (2005) and Dimitrov (2005). In order to state our main existence result with respect to the semistrict core, we require the solution concept φ to satisfy efficiency, symmetry, and the null player property. Then, as it turns out, if the simple game does not exhibit Shenoy's (1979) *paradox of smaller coalitions* w.r.t. the given cooperative solution, the corresponding hedonic game has a nonempty semistrict core. If we eliminate the influence of exogenous factors by requiring *equal shares*, then our semistrict core existence result can more clearly be seen as being stronger and more general than the corresponding core existence result of Shenoy (1979). On the other hand, if we take Farrell and Scotchmer's (1988) *partnership solution* as cooperative solution concept, then a full characterization of the semistrict core of the corresponding hedonic game can be provided.

The paper is organized as follows. Section 2 includes basic notions and solution concepts from the theory of simple games and hedonic games. We define a specific composite solution and use it to induce players' preferences over coalitions in a hedonic game. Our main result is presented in Section 3, while Section 4 contains the mentioned special cases in which we have either equal shares or fix the partnership solution. Section 5 closes with some final remarks.

2   Preliminaries

In this section we introduce the basic ingredients of our setup.

Simple games and solutions

Let N be a finite *set of players*, which we will keep fixed throughout the paper. A (*cooperative*) *simple game with transferable utility* (a simple TU-game) is a pair (N, v), where $v : 2^N \to \{0, 1\}$ is called *characteristic function* and satisfies $v(\emptyset) = 0$. We refer to a coalition $S \subseteq N$ with $v(S) = 1$ as a *winning coalition*. In what follows we will identify a simple game (N, v) with its characteristic function v.

A simple game v is *monotonic* if $v(S) = 1$ implies $v(T) = 1$ for all $T \supseteq S$, and *proper* if $v(S) = 1$ implies $v(N \setminus S) = 0$. A player $i \in N$ is a *null player* in v if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$. Players $i, j \in N$ are *symmetric* in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. We denote by $\mathcal{W}^v = \{S \subseteq N : v(S) = 1\}$ the set of winning coalitions and by $\mathcal{MW}^v = \{S \subseteq N : v(S) = 1 \text{ and } v(T) = 0 \text{ for all } T \subset S\}$ the set of *minimal winning coalitions* in the simple game v (cf. Shapley (1962)). For $S \subseteq N$ define the *subgame* (N, v_S) by $v_S(T) = v(S \cap T)$ for all $T \in 2^N$. Note that v_S is also an N player simple game (possibly

with $v_S(N) = v(S) = 0$). The set of all proper monotonic simple games on the player set N will be denoted by $\mathcal{G}$. Clearly, if a game v is in the set $\mathcal{G}$, then so is any of its subgames.

A *solution* (of a proper monotonic simple game) is a mapping $\varphi : \mathcal{G} \to \mathbb{R}_+^N$ taking each $v \in \mathcal{G}$ to a single vector in $\mathbb{R}_+^N$, i.e., it assigns a nonnegative real number $\varphi_i(v)$ to each player $i \in N$[1]. A solution φ satisfies *efficiency* if $\sum_{i \in N} \varphi_i(v) = v(N)$, and the *null player property* if $\varphi_i(v) = 0$ holds for all $i \in N$ who are null players in v. Finally, a solution φ is *symmetric* if $\varphi_i(v) = \varphi_j(v)$ for all $i, j \in N$ who are symmetric in v. The set of all solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}$.

Next, we recall two specific solutions: the *Shapley value* (cf. Shapley (1953) and Aumann and Dréze (1974)) and the *partnership solution* (cf. Farrell and Scotchmer (1988)). We provide here the exact form of the corresponding solution for a subgame.

The Shapley value $Sh : \mathcal{G} \longrightarrow \mathbb{R}_+^N$ applied to a (sub-)game v_S ($S \subseteq N$) is given by

$$Sh_i(v_S) := \begin{cases} \sum_{T \subseteq S} \frac{(|S|-|T|)!\,(|T|-1)!}{|S|!} \left(v_S(T) - v_S(T \setminus \{i\})\right), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \quad (i \in N).$$

The partnership solution $Pa : \mathcal{G} \longrightarrow \mathbb{R}_+^N$ applied to a (sub-)game v_S ($S \subseteq N$) is given by

$$Pa_i(v_S) := \begin{cases} \frac{v_S(S)}{|S|}, & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \quad (i \in N).$$

It is easy to check that both solutions satisfy efficiency and symmetry. In addition, the Shapley value also satisfies the null player property, while the partnership solution does not.

Composite solutions

In the context of simple games, solutions such as the Shapley value are often termed *power indices*. One frequent criticism to power indices is that they on the one hand measure "endogenous power", but on the other hand they cannot take "exogenous power distributions" into account. For instance, the distribution of seats in a parliament is completely ignored, when describing the corresponding majority voting game. Hence, it does not enter the solution, either.

Composite solutions, as defined below, are designed to incorporate exogenous shares as well as endogenous power. This is done by a combination of a share vector and a solution.

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[1] Requiring nonnegativity is in accordance with the interpretation that $\varphi_i(v)$ reflects the power of player i in the game v. It can be easily seen that our results also hold in the case when a solution assigns a negative real number to some players in the game.

Thus, a composite solution does not only reflect the players' opportunities to form winning coalitions, but also respects asymmetries among the players outside the game.

Formally, a *composite solution* $F : \mathbb{R}_{++}^N \times 2^N \times \mathcal{G} \times \mathcal{S} \to \mathbb{R}_+^N$ assigns a vector of players' payoffs to each tuple (α, S, v, φ) consisting of a share vector, a coalition, a simple game, and a solution.[2] We interpret a composite solution as follows: Suppose the game $v \in \mathcal{G}$ describes the possibilities to form winning coalitions. The vector α represents asymmetries outside the model and φ is the solution that measures players' power inherent in v. Then (the real number) $F_i(\alpha, S, v, \varphi)$ should be viewed as "player i's overall power" within a coalition $S \subseteq N$. In the following, we concentrate on a specific composite solution Φ, which is defined by

$$(1) \qquad \Phi_i(\alpha, S, v, \varphi) = \begin{cases} \frac{\alpha_i \varphi_i(v_S)}{\sum_{j \in S} \alpha_j \varphi_j(v_S)} \cdot v(S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

Suppose α, v, and φ are fixed and a winning coalition $S \in \mathcal{W}^v$ has formed. How is the worth $v(S) = 1$, i.e., how is power distributed among the players in S? First of all, any player not in S gets zero. For each player in S, we compute his share of $v(S)$ by weighing his internal share $\varphi_i(v_S)$ with his external power α_i. The denominator in (1) serves for normalization purposes.

Note that for fixed $\alpha \in \mathbb{R}_{++}^N$ and $\varphi \in \mathcal{S}$ the mapping $\Phi(\alpha, N, \bullet, \varphi) : \mathcal{G} \longrightarrow \mathbb{R}_+^N$ is a solution (in the above sense). Observe that $\Phi(\alpha, S, v, \varphi) = \Phi(\alpha, S, v_S, \varphi) = \Phi(\alpha, N, v_S, \varphi)$ is valid.

The paradox of smaller coalitions

Let $v \in \mathcal{G}$ and $\varphi \in \mathcal{S}$. We say that v *does not exhibit the paradox of smaller coalitions w.r.t.* φ, if for all $S, T \in \mathcal{W}^v$,

$$S \subseteq T \text{ implies } \varphi_i(v_S) \geq \varphi_i(v_T) \text{ for all } i \in S.$$

The absence of this paradox in simple games simply respects the fact that if players form a smaller winning coalition, then their (internal) power should not decrease since there are fewer players to share the same amount of power (cf. Shenoy (1979)[3]). Notice that if a simple game v does not exhibit the paradox w.r.t. φ, then for all $\alpha \in \mathbb{R}_{++}^N$ and all $S, T \in \mathcal{W}^v$ with $S \subseteq T$, we have $\Phi_i(\alpha, S, v, \varphi) \geq \Phi_i(\alpha, T, v, \varphi)$ for all $i \in S$.

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[2] $\alpha \in \mathbb{R}_{++}^N$ means $\alpha_i > 0$ for all $i \in N$.

[3] Table A.1 in this work lists all games in $\mathcal{G}$ with up to four players and verifies presence or absence of the paradox w.r.t. the Shapley value.

Hedonic games and stability notions

For each player $i \in N$ we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A partition Π of N is called a *coalition structure*. For each coalition structure Π and each player $i \in N$, we denote by $\Pi(i)$ the coalition in Π containing player i, i.e., $\Pi(i) \in \Pi$ and $i \in \Pi(i)$. The set of all coalition structures of N will be denoted by $\mathbf{C}^N$.

Further, we assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, i.e., a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. Denote by $\succ_i$ and $\sim_i$ the strict and indifference relation associated with $\succeq_i$ and by $\succeq := (\succeq_1, \succeq_2, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. A player's preference relation over coalitions canonically induces a preference relation over coalition structures in the following way:[4] For any two coalition structures Π and Π', player i weakly prefers Π to Π' if and only if he weakly prefers "his" coalition in Π to the one in Π', i.e., $\Pi \succeq_i \Pi'$ if and only if $\Pi(i) \succeq_i \Pi'(i)$. Hence, we assume that players' preferences over coalition structures are *purely hedonic*, i.e., they are completely characterized by their preferences over coalitions. Finally, a *hedonic game* $(N, \succeq)$ is a pair consisting of the set of players and a preference profile.

Unlike solution concepts for (simple) cooperative games do, there is no worth to distribute in hedonic games. The relevant question is rather, which coalition structure should form, taking players' preferences into account. The basic property that we require is *core stability*, which we define next in three versions.

Let $(N, \succeq)$ be a hedonic game. For any coalition $\emptyset \neq X \subseteq N$ and coalition structure Π of N, let $\mathcal{X}^\Pi(X) := \{X \cap P \mid P \in \Pi\}$. A partition Π is *strictly core stable* if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ holds for all $i \in X$ and $X \succ_j \Pi(j)$ is true for some player $j \in X$. Π is *semistrictly core stable* if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ for all $i \in X$ and for each $X' \in \mathcal{X}^\Pi(X)$ there is $j \in X'$ with $X \succ_j \Pi(j)$. Π is *core stable* if there does not exist a nonempty coalition X such that $X \succ_i \Pi(i)$ holds for each $i \in X$.

Put in other words, a coalition structure Π is strictly core stable if no group of players are willing to form a coalition, so that each player is at least as well off with this new coalition and some player is strictly better off compared to the corresponding coalitions in Π. For semistrict core stability we again want to exclude the case that a new coalition X forms. However, the requirement for some players being strictly better off is more subtle. For this we partition the deviating coalition X into groups that come from the same coalition in Π. Then, to make X a profitable deviation it is required that in each such group there

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[4]With slight abuse in notation, we use the same symbol to denote preferences over coalitions and preferences over coalition structures.

has to be some player who is better off in the new coalition. Clearly, the weakest notion of a coalitional deviation is incorporated in the definition of core stability - everyone in the deviating coalition should be strictly better off. Observe that strict core stability implies semistrict core stability that, in turn, implies core stability. In what follows, we denote by $SC(N, \succeq)$, $SSC(N, \succeq)$, and $C(N, \succeq)$ the sets of strict core stable, semistrict core stable, and core stable coalition structures, respectively, of a hedonic game $(N, \succeq)$. Alternatively, call $SC(N, \succeq)$, $SSC(N, \succeq)$, and $C(N, \succeq)$ the strict core, semistrict core, and core of $(N, \succeq)$.

3 Coalition formation via composite solutions

In this section we address the following question: Given a simple game that describes the incentives to forming coalitions, which (winning) coalition should form? Clearly, the preferences over winning coalitions that a player forms depend on how much influence or power he has within such a coalition. In effect preferences are based on the solution concept at hand as well as on the exogenous share vector. Once preferences are clear, the question arises, which coalition structures are stable. Clearly, the best one can get are strictly core stable partitions. However, as the following example demonstrates, this requirement is in fact too strict. Therefore, we concentrate our analysis on semistrict core stability for which we obtain positive results.

Example 1
Let $|N| = 4, \alpha := (4, 4, 1, 1)$ and let the simple game $v \in \mathcal{G}$ be given by its minimal winning coalitions, which are $\mathcal{MW}^v := \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Thus, we consider a scenario with two symmetric "large" players (1 and 2) and two symmetric "small" players (3 and 4). Also, v does not exhibit the paradox of smaller coalitions w.r.t. the Shapley value (see Shenoy (1979)). Now, using the Shapley value as (inherent) power index, each player forms preferences according to how much power the composite solution Φ assigns to him in a coalition. One readily computes:

$$\Phi(\alpha, N, v, Sh) = \left(\frac{4}{9}, \frac{4}{9}, \frac{1}{18}, \frac{1}{18}\right) \qquad \Phi(\alpha, N, v_{\{1,2\}}, Sh) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$

$$\Phi(\alpha, N, v_{\{1,3,4\}}, Sh) = \left(\frac{2}{3}, 0, \frac{1}{6}, \frac{1}{6}\right) \qquad \Phi(\alpha, N, v_{\{2,3,4\}}, Sh) = \left(0, \frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

It follows that player 1's power is largest within the coalition $\{1, 3, 4\}$. Taking this to extract preferences over coalitions, player 1 evaluates coalitions as follows:

$$\{1, 3, 4\} \succ_1 \{1, 2\} \sim_1 \{1, 2, 3\} \sim_1 \{1, 2, 4\} \succ_1 \{1, 2, 3, 4\} \succ_1 \{1\}.$$

Collecting all preferences, we obtain a hedonic game $(N, \succeq)$, the preferences of which are induced by the composite solution Φ. Inspecting $(N, \succeq)$, one finds that the strict core is empty, showing that strict core stability is too restrictive to answer the question, which coalition(s) should be formed here. $\square$

More precisely, here we consider solutions on $\mathcal{G}$ that satisfy efficiency, symmetry, and the null player property. The set of all such solutions will be denoted by $\mathcal{S}^*$, and let $\varphi \in \mathcal{S}^*$. Notice then that efficiency, symmetry and the null player property help us to know the payoffs if players are members of minimal winning coalitions, i.e., we have

$$\varphi_i(v_S) = \frac{1}{|S|} \qquad \text{for all } i \in S \in \mathcal{MW}^v$$

and hence,

$$(2) \qquad \Phi_i(\alpha, S, v, \varphi) = \frac{\alpha_i}{\alpha(S)} \qquad \text{for all } i \in S \in \mathcal{MW}^v,$$

where $\alpha(S) := \sum_{i \in S} \alpha_i$ is the *total share* of coalition S, $S \subseteq N$. Moreover, for all $S \subseteq N$, we have that

$$(3) \qquad \Phi_i(\alpha, S, v, \varphi) = \varphi_i(v_S) = 0 \qquad \text{if } i \in S \text{ is a null player in } v_S.$$

Let $\alpha \in \mathbb{R}_{++}^N$, $v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$ be fixed. To simplify notation, for all $S \subseteq N$ and all $i \in N$, we write $\Phi_i(S)$ instead of $\Phi_i(\alpha, S, v, \varphi)$ to denote i's payoff according to the composite solution Φ. We are now ready to define a hedonic coalition formation game by inducing players' preferences over coalitions in the following way. For each $i \in N$ define a preference relation $\succeq_i$ over $\mathcal{N}_i$ by

$$S \succeq_i T \qquad \text{if and only if} \qquad \Phi_i(S) \geq \Phi_i(T) \qquad (S, T \in \mathcal{N}_i),$$

i.e., $\Phi_i(\cdot)|_{\mathcal{N}_i}$ is a representation of i's preferences. In words, player i's preferences over any two coalitions S and T that he is a member of are induced via i's payoffs according to the composite solution Φ. Notice that paying attention to the corresponding coalitions is compatible with the very definition of a hedonic game - each player in such a game evaluates any two coalition structures based only on his preferences over the coalitions in the two partitions he belongs to (cf. Aumann and Dréze (1974) and Shenoy (1979)). In what follows, we shall use the notation $(N, \succeq)$ to denote the hedonic game induced via Φ as indicated above.

It turns out that certain minimal winning coalitions are crucial w.r.t. semistrict core stability. Let $\mathcal{A}^v$ be the set of all minimal winning coalitions with minimal total share,

i.e., $\mathcal{A}^v := \{S \in \mathcal{MW}^v \mid \alpha(S) \leq \alpha(T) \text{ for all } T \in \mathcal{MW}^v\}$. The following theorem shows that coalition structures containing a coalition from $\mathcal{A}^v$ are semistrictly core stable.

Theorem 1 *Let $\alpha \in \mathbb{R}_{++}^N$, $v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$. If v does not exhibit the paradox of smaller coalitions w.r.t. φ, then $SSC\,(N, \succeq) \neq \emptyset$.*

Proof. Let $T \in \mathcal{A}^v$ and Π be a partition of N containing T. We show that $\Pi \in SSC\,(N, \succeq)$. Suppose to the contrary that there is $X \subseteq N$ such that

$$(4) \qquad\qquad \text{for all } i \in X : \Phi_i\,(X) \geq \Phi_i\,(\Pi(i))$$

and

$$(5) \qquad\qquad \text{for all } X' \in \mathcal{X}^\Pi : \Phi_j\,(X) > \Phi_j\,(\Pi(j)) \text{ for some } j \in X'.$$

Clearly, $X \in \mathcal{W}^v$. Let $Y \in \arg\min_{S \subseteq X, S \in \mathcal{MW}^v} \alpha(S)$. Since v does not exhibit the paradox of smaller coalitions w.r.t. φ,

$$(6) \qquad\qquad \Phi_i\,(Y) \geq \Phi_i\,(X) \text{ for all } i \in Y.$$

Since v is proper and $Y \in \mathcal{W}^v$, $Y \cap T \neq \emptyset$. Consider the following possible cases:

(a) $Y = X$. Then, in view of (2) and (5), there is $i \in Y \cap T$ such that

$$\frac{\alpha_i}{\alpha(Y)} = \Phi_i\,(Y) = \Phi_i\,(X) > \Phi_i\,(\Pi(i)) = \Phi_i(T) = \frac{\alpha_i}{\alpha(T)},$$

which is a contradiction to $\alpha(Y) \geq \alpha(T)$.

(b) $Y \subset X$ and $\alpha\,(Y) > \alpha\,(T)$. Then, by (2), (6), and (4), we have that for all $i \in Y \cap T$,

$$\frac{\alpha_i}{\alpha(Y)} = \Phi_i\,(Y) \geq \Phi_i\,(X) \geq \Phi_i\,(\Pi(i)) = \Phi_i(T) = \frac{\alpha_i}{\alpha(T)},$$

which is a contradiction to $\alpha(Y) > \alpha(T)$.

(c) $Y \subset X$ and $\alpha\,(Y) = \alpha\,(T)$. Then, $\Phi_i\,(Y) = \Phi_i\,(T)$ for all $i \in Y \cap T$. Thus, by (2), (6), and (4), we have that for all $i \in Y \cap T$,

$$\frac{\alpha_i}{\alpha(T)} = \Phi_i\,(Y) \geq \Phi_i\,(X) \geq \Phi_i\,(\Pi(i)) = \Phi_i(T) = \frac{\alpha_i}{\alpha(T)},$$

and hence, $\Phi_i\,(X) = \Phi_i\,(\Pi(i))$ for all $i \in Y \cap T$. By (5), there is $i' \in (X \setminus Y) \cap T$ such that

$$(7) \qquad\qquad \Phi_{i'}\,(X) > \Phi_{i'}\,(\Pi(i')) = \frac{\alpha_{i'}}{\alpha\,(T)} = \frac{\alpha_{i'}}{\alpha\,(Y)}.$$

On the other hand, by (6),

$$(8) \qquad\qquad \Phi_{i'}\left(Y \cup \{i'\}\right) \geq \Phi_{i'}\left(X\right).$$

Combining (8) and (7) we get

$$(9) \qquad\qquad \Phi_{i'}\left(Y \cup \{i'\}\right) > \frac{\alpha_{i'}}{\alpha\left(Y\right)}.$$

If Y is the only minimal winning coalition in $Y \cup \{i'\}$, then i' is a null player in the game $v_{Y \cup \{i'\}}$. By (3), we have $\Phi_{i'}\left(Y \cup \{i'\}\right) = 0$ in contradiction to (9).

Suppose finally that Y is not the only minimal winning coalition in $Y \cup \{i'\}$, i.e., since $Y \in \arg\min_{S \subseteq X, S \in \mathcal{MW}^v} \alpha(S)$, there is $Y' \subset Y$ with $Y' \cup \{i'\} \in \arg\min_{S \subseteq X, S \in \mathcal{MW}^v} \alpha(S)$. Again, since v does not exhibit the paradox of smaller coalitions w.r.t. φ, and the coalitions $Y' \cup \{i'\}$ and $Y \cup \{i'\}$ are winning with $Y' \cup \{i'\} \subseteq Y \cup \{i'\}$, we have

$$\frac{\alpha_{i'}}{\alpha\left(Y\right)} = \Phi_{i'}\left(Y' \cup \{i'\}\right) \geq \Phi_{i'}\left(Y \cup \{i'\}\right),$$

which again contradicts (9). Hence, we conclude that $\Pi \in SSC\left(N, \succeq\right)$. $\qquad\square$

Theorem 1 says that, if the solution concept "behaves well" in the sense that a player's power increases with shrinking winning coalition, then there are coalition structures that are core stable in the semistrict sense. The reader may verify that the game in Example 1 satisfies the conditions of Theorem 1. The coalition structures $\{\{1,3,4\},\{2\}\}$ and $\{\{2,3,4\},\{1\}\}$ are indeed semistrictly core stable. In both cases the winning coalition W has a minimal total share of $\alpha(W) = 6$.

Next, we define the sets $\mathbf{P}^v$ and $\mathbf{P}^v_{\mathcal{A}}$ as follows:

$$\begin{aligned} \mathbf{P}^v &= \left\{\Pi \in \mathbf{C}^N \mid \Pi \cap \mathcal{MW}^v \neq \emptyset\right\}, \\ \mathbf{P}^v_{\mathcal{A}} &= \left\{\Pi \in \mathbf{C}^N \mid \Pi \cap \mathcal{A}^v \neq \emptyset\right\}. \end{aligned}$$

$\mathbf{P}^v$ is the set of all coalition structures containing a minimal winning coalition, whereas partitions in $\mathbf{P}^v_{\mathcal{A}}$ contain a winning coalition with minimal total share. Clearly, $\mathbf{P}^v_{\mathcal{A}} \subseteq \mathbf{P}^v$ holds.

Notice that, as shown in the proof of Theorem 1, $\mathbf{P}^v_{\mathcal{A}} \subseteq SSC\left(N, \succeq\right)$. Our next result (Theorem 2) provides more information about the structure of the semistrict core under the above circumstances. Basically, it says that the semistrict core does not include any partition containing a minimal winning coalition that does not have minimal total share. The reason

here is in the observation (stated in Lemma 1) that the core in fact does not contain such partitions.

Lemma 1 *Let $\alpha \in \mathbb{R}_{++}^N, v \in \mathcal{G}$, and let $\varphi \in \mathcal{S}^*$. Then,*

$$\mathbf{P}^v \cap C(N, \succeq) = \mathbf{P}_{\mathcal{A}}^v \cap C(N, \succeq).$$

Proof. Since $\mathbf{P}_{\mathcal{A}}^v \subseteq \mathbf{P}^v$, it is enough to show that there is no partition containing a minimal winning coalition that is not of minimal total share.

Suppose to the contrary that there is $\Pi \in \mathbf{P}^v \cap C(N, \succeq)$ and $S \in \Pi$ such that $S \in \mathcal{MW}^v \setminus \mathcal{A}^v$. Let $T \in \mathcal{A}^v$. Since v is proper, $T \cap S \neq \emptyset$. Since $\alpha(S) > \alpha(T)$, we have for all $i \in T \cap S$,

$$\Phi_i(\Pi(i)) = \Phi_i(S) = \frac{\alpha_i}{\alpha(S)} < \frac{\alpha_i}{\alpha(T)} = \Phi_i(T).$$

On the other hand, for all $i \in N \setminus S$, $\Phi_i(\Pi(i)) = 0$. Thus, for all $i \in T \setminus S$,

$$\frac{\alpha_i}{\alpha(T)} = \Phi_i(T) > \Phi_i(\Pi(i)) = 0.$$

Hence, we have $\Phi_i(T) > \Phi_i(\Pi(i))$ for all $i \in T$ implying that T is a deviation (in the sense of the core) from Π in contradiction to $\Pi \in C(N, \succeq)$. $\qquad \square$

Theorem 2 *Let $\alpha \in \mathbb{R}_{++}^N$, $v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$. If v does not exhibit the paradox of smaller coalitions w.r.t. φ, then*

$$(\mathbf{P}^v \cap C(N, \succeq)) \subseteq SSC(N, \succeq).$$

Proof. By Lemma 1, the set $\mathbf{P}^v \cap C(N, \succeq)$ consists only of partitions containing minimal winning coalitions with minimal total share. In view of Theorem 1, each such a partition is semistrictly core stable. $\qquad \square$

4 Special cases

In a composite solution asymmetries among the players can either be expressed by an unequal share vector, or by a solution concept φ that takes players' possibilities to form winning coalitions into account. In this section we analyze the two cases in which either source is ruled out: We first restrict our interest to share vectors with equal shares, i.e., we rule out asymmetries among the players that are based on external considerations and again consider hedonic games induced by the composite solution Φ as introduced in the previous section. The second part of this Section is devoted to the case, in which the solution concept is the

partnership solution Pa as defined in Section 2. Thus, the solution ignores asymmetries stemming from endogenous considerations. Here we obtain a full characterization of the semistrict core.

4.1 Equal shares and core existence

Let $v \in \mathcal{G}$ and let $\varphi \in \mathcal{S}$ satisfy efficiency and the null player property. Moreover, let $\alpha \in \mathbb{R}^N_{++}$ be a share vector with equal shares, i.e., $\alpha_i = \bar{\alpha}$ for all $i \in N$. Then, for all $S \subseteq N$ and all $i \in S$ we have,

$$\frac{\alpha_i \varphi_i(v_S)}{\sum_{j \in S} \alpha_j \varphi_j(v_S)} \cdot v(S) = \frac{\bar{\alpha} \varphi_i(v_S)}{\bar{\alpha} \sum_{j \in S} \varphi_j(v_S)} \cdot v(S) = \varphi_i(v_S),$$

and therefore

$$\Phi_i(S) = \begin{cases} \varphi_i(v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

Notice that if we take equal shares and $\varphi = Sh$ then Theorem 1 can be seen as being stronger and more general than the corresponding result of Shenoy (1979). In his Theorem 7.4, Shenoy (1979) shows that if players' preferences over coalitions are induced via the Shapley value of the corresponding subgames, and the simple game does not exhibit the paradox of smaller coalitions w.r.t. the Shapley value, then the core (in our terms: the core of the corresponding hedonic game) is nonempty.

As we show next, even if we do not assume equal shares, a direct generalization of Shenoy's core existence result is possible. In order to state it, we will require a solution $\varphi \in \mathcal{S}$ to satisfy *coalitional efficiency*, i.e., $\sum_{i \in S} \varphi_i(v_S) = v(S)$ should hold for all $S \subseteq N$. Notice that coalitional efficiency is implied by efficiency and the null player property, i.e., we consider a larger domain of solutions than the one for which Theorem 1 applies. Moreover, under coalitional efficiency and symmetry, (2) still holds. We have the following theorem.

Theorem 3 *Let $\alpha \in \mathbb{R}^N_{++}$, $v \in \mathcal{G}$, and let $\varphi \in \mathcal{S}$ satisfy coalitional efficiency and symmetry. If v does not exhibit the paradox of smaller coalitions w.r.t. φ, then $C(N, \succeq) \neq \emptyset$.*

Proof. Let $T \in \mathcal{A}^v$ and Π be a partition of N containing T. We show that $\Pi \in C(N, \succeq)$. Suppose to the contrary that there is $X \subseteq N$ such that

$$(10) \qquad \qquad \text{for all } i \in X : \Phi_i(X) > \Phi_i(\Pi(i)).$$

Clearly, $X \in \mathcal{W}^v$. Let $Y \in \arg\min_{S \subseteq X, S \in \mathcal{MW}^v} \alpha(S)$. Since v does not exhibit the paradox of smaller coalitions w.r.t. φ,

$$(11) \qquad \qquad \Phi_i(Y) \geq \Phi_i(X) \text{ for all } i \in Y.$$

12

Since v is proper and $Y \in \mathcal{W}^v$, $Y \cap T \neq \emptyset$. Then, in view of (2), (11) and (10), there is $i \in Y \cap T$ such that

$$\frac{\alpha_i}{\alpha(Y)} = \Phi_i(Y) \geq \Phi_i(X) > \Phi_i(\Pi(i)) = \Phi_i(T) = \frac{\alpha_i}{\alpha(T)},$$

which is a contradiction to $\alpha(Y) \geq \alpha(T)$. □

Notice finally that Theorem 1 shows that the absence of the paradox of smaller coalitions is in fact a sufficient condition for nonemptiness even of the semistrict core, provided that one replaces coalitional efficiency by efficiency and imposes in addition the null player property on φ. However, as shown by Dimitrov and Haake (2005), the absence of the paradox is not a necessary condition for an induced hedonic game to have a nonempty semistrict core.

4.2 Partnerships

According to Farrell and Scotchmer (1988), a partnership is a coalition that divides its output equally. In the following, we use $\varphi = Pa$ as cooperative solution and show next that semistrictly core stable partitions always exist. More precisely, it turns out that each core stable coalition structure is semistrictly core stable as well.

Consequently, with $\varphi = Pa$ we have for all $S \subseteq N$ and all $i \in N$,

$$\frac{\alpha_i Pa_i(v_S)}{\sum_{j \in S} \alpha_j Pa_j(v_S)} \cdot v(S) = \frac{\alpha_i}{\alpha(S)} \cdot v(S),$$

and hence

$$\Phi_i(S) = \begin{cases} \frac{\alpha_i}{\alpha(S)} \cdot v(S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

In other words, the worth of a coalition S (its power) is distributed in this case proportionally to the exogenously given weights and parties' possibilities to form winning coalitions are ignored. Three remarks are in order:

Remark 1 *It is always possible to order all (winning) coalitions according to their total shares and to state, for all $S, T \in 2^N$, that S is commonly preferred over T if and only if $\alpha(S) \leq \alpha(T)$. Hence, if $\varphi = Pa$ and players' preferences in a hedonic game are induced via Φ, then we would have, for all $i \in N$, that $S \succeq_i T$ if and only if S is commonly preferred over T. Hence, the corresponding hedonic game would satisfy the common ranking property of Farrell and Scotchmer (1988).*

Remark 2 *Clearly, each game $v \in \mathcal{G}$ does not exhibit the paradox of smaller coalitions w.r.t. the partnership solution.*

Remark 3 *Notice that the partnership solution satisfies efficiency and symmetry but not the null player property. Thus, we cannot directly apply Theorem 1 to deduce nonemptiness of the semistrict core.*

Hence, in view of Remark 3, we have to look for a different way when providing a positive result with respect to this stability concept.

Theorem 4 *Let $\alpha \in \mathbb{R}_{++}^N$, $v \in \mathcal{G}$, and $\varphi = Pa$. Then, $SSC(N, \succeq) = C(N, \succeq) = \mathbf{P}_{\mathcal{A}}^v$.*

Proof. By Remark 1, the game $(N, \succeq)$ satisfies the common ranking property of Farrell and Scotchmer (1988). Hence, $C(N, \succeq) = \mathbf{P}_{\mathcal{A}}^v$ follows easily from their core existence result by noticing that the common ranking is based on $\alpha(S)$ for each $S \subseteq N$. Thus, we only have to show that $SSC(N, \succeq) = C(N, \succeq)$.

Suppose to the contrary that there is $\Pi \in C(N, \succeq)$ and $X \subseteq N$ such that

(12) $$\text{for all } i \in X : \Phi_i(X) \geq \Phi_i(\Pi(i)),$$

and

(13) $$\text{for all } X' \in \mathcal{X}^\Pi(X) : \Phi_j(X) > \Phi_j(\Pi(j)) \text{ for some } j \in X'.$$

Since $\Pi \in C(N, \succeq)$, there is $i^* \in X$ such that $\Phi_{i^*}(\Pi(i^*)) \geq \Phi_{i^*}(X)$ which, in combination with (12), implies $\Phi_{i^*}(\Pi(i^*)) = \Phi_{i^*}(X)$. By the common ranking property, $\Phi_j(\Pi(i^*)) = \Phi_j(X)$ for all $j \in \Pi(i^*) \cap X$ in contradiction to (13). $\square$

Remark 4 *If $\varphi = Pa$, and $\alpha \in \mathbb{R}_{++}^N$ and $v \in \mathcal{G}$ allow for only one minimal winning coalition with minimal total share, then, clearly, $SC(N, \succeq) = SSC(N, \succeq) = C(N, \succeq) = \mathbf{P}_{\mathcal{A}}^v$.*

Remark 5 *As it can be easily seen, the proof that a core element for the induced hedonic game is also a semistrict core element only uses the fact that players' preferences are derived from a common ranking. Hence, we may conclude that the core of a general hedonic game in this case consists of semistrictly core stable partitions only (see also Dimitrov (2005)).*

In order to obtain a complete characterization of the core, Shenoy (1979) considers the class of symmetric monotonic simple games. A simple game is *symmetric* if the worth of a coalition, i.e., whether it is winning or not, only depends on its size. Notice then that if φ satisfies efficiency, symmetry and the null player property, then we will have $\varphi_i(v_S) = \frac{1}{|S|}$ for each player i who is a member of a winning coalition S (S needs not to be minimal winning). Thus, any such solution coincides with the partnership solution on this class and therefore the previous theorem tells us, how the semistrict core looks like.

Corollary 1 *Let $\alpha \in \mathbb{R}_{++}^N$ and v be a symmetric monotonic simple game. If $\varphi \in \mathcal{S}^*$, then $SSC(N, \succeq) = C(N, \succeq) = \mathbf{P}_{\mathcal{A}}^v$.*

We should not fail to mention that, if we restrict ourselves to equal shares and take $\varphi = Sh$, then Corollary 1 in fact restates Proposition 7.6 of Shenoy (1979).

5 Conclusion

In this paper we studied conditions that guarantee semistrict core stability in hedonic games, provided that players' preferences are derived from an underlying simple game. By considering a multiplicative composite solution we were able to generalize previous results in Shenoy (1979) by enlarging the domain of solution concepts applied to a simple game and by using a stronger stability notion. The use of the specific composite solution allowed us to incorporate the influence of both exogenous and endogenous factors on players' preferences over coalitions. The main insight from our analysis is that each partition containing a minimal winning coalition with minimal total share is semistrictly core stable. Moreover, in some interesting special cases, the semistrict core consists only of such partitions. Hence, our results with respect to the mentioned special cases can be seen as a formal proof of Riker's (1962) 'size principle' in a more general setting (see also Laver and Schofield (1990) for an extensive survey). Notice finally that nonemptiness of the semistrict core for the case of the partnership solution was already indicated by Kirchsteiger and Puppe (1997). However, to the best of our knowledge, our analysis is the first rigorous account using the semistrict core concept that takes into account both a large domain of solutions on simple games and exogenously given share vectors.

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