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An Axiomatic Approach to Composite Solutions*

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Abstract

We investigate a situation in which gains from cooperation are represented by a cooperative TU-game and a solution proposes a division of coalitional worths. In addition, asymmetries among players outside the game are captured by a vector of exogenous weights. If a solution measures players' payoffs inherent in the game, and a coalition has formed, then the question is how to measure players' overall payoffs in that coalition. For this we introduce the notion of a composite solution. We provide an axiomatic characterization of a specific composite solution, in which exogenous weights enter in a proportional fashion.

JEL Classification: C71

Keywords: composite solution, external weights

1 Introduction

A cooperative game with transferable utility describes the possibilities for any coalition of players to obtain a payoff. A solution concept for TU games is based on this information and proposes a (set of) distribution(s) of coalitional worth. As a solution only uses the data of the game itself, it has to ignore *exogenous asymmetries* among the players that are not captured by the characteristic function. However, it is undoubted that such external weights also have an influence on the final distribution of payoffs.

A common example for this fact is power measurement in a parliament. There a (simple) game describes the possibilities to form majorities and application of power indices

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determines a distribution of power within the parliament. A frequent criticism towards this approach is that parties' external strengths, such as distribution of seats, are faded out. In particular, power indices do not conform with ideas of proportionality with respect to exogenous asymmetries (see, e.g., Snyder et al. (2005) for a debate on power indices).

In this paper we tackle the question how to bring together a solution concept, which reflects players' weights inherent in the game, and their external weights. This leads to a notion of an *overall payoff*, that takes both sources into account. We adopt a normative point of view and ask for plausible requirements for merging internal and external weights.

To answer the question we introduce the concept of a *composite solution*. More precisely, a composite solution F takes each collection (α, S, v, φ) of an exogenous weights vector α, a coalition S, a cooperative TU-game v and a solution concept φ to a distribution of coalitional worth with the interpretation that $F_i(\alpha, S, v, \varphi)$ reflects player i's overall payoff within coalition S, when v describes the gains from cooperation. We present four axioms for composite solutions and show that they uniquely characterize a specific composite solution, in which exogenous weights enter in a proportional fashion.

The idea to express differences in players' external characteristics via (strictly positive) weights goes back to Shapley (1953a), where it was used to characterize a particular parametrized solution (the weighted Shapley value (cf. Shapley (1953b), Owen (1968), Kalai and Samet (1987), Haeringer (2006)). A *parametrized solution* assigns a payoff for each player to each pair (α, v). In contrast to the above papers, we take solution concepts φ defined on the set of all games rather than on the collection of all pairs (α, v). Hence, we let the players play the game as if there were no external asymmetries among them and, after that, we view external weights as having a redistributive power in a composite solution. This will be important when discussing the axioms we use for the characterization of our specific composite solution. Clearly, if we fix the solution concept φ and take only the grand coalition into account, then a parametrized solution can be seen as a composite solution as well.

The paper is organized as follows. Section 2 contains basic notions and definitions such as the one for composite solutions. Moreover, we present and discuss four axioms for composite solutions. In Section 3 we show that these four axioms, being independent, characterize a specific composite solution. Section 4 closes with some final remarks.

2 Composite solutions and axioms

A *cooperative game with transferable utility* - or simply a (TU-)game - is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of players and $v : 2^N \longrightarrow \mathbb{R}$ is a *characteristic function* on

N satisfying $v(\emptyset) = 0$. For any *coalition* $S \in 2^N$, $v(S)$ is the worth of coalition S, i.e., the members of S can obtain a total amount of $v(S)$ by agreeing to cooperate. In what follows we keep N fixed and identify a game (N, v) with its characteristic function v. For $S \in 2^N$ define the *subgame with respect to* S, (N, v_S), by $v_S(T) = v(S \cap T)$ for all $T \in 2^N$. Note that v_S is also an n-player game (possibly with $v_S(N) = v(S) = 0$). The set of all games on the player set N will be denoted by $\mathcal{G}$. Clearly, if a game v is in the set $\mathcal{G}$, then so is any of its subgames. A game $v \in \mathcal{G}$ is *monotonic* if $v(S) \geq v(T)$ for all $S, T \in 2^N$ with $S \supseteq T$. The set of all monotonic games on the player set N is denoted by $\mathcal{G}^m$, $\mathcal{G}^m \subset \mathcal{G}$.

A *solution* (for a TU-game) is a mapping $\varphi : \mathcal{G} \rightarrow \mathbb{R}^N$ taking each $v \in \mathcal{G}$ to a single vector in $\mathbb{R}^N$, i.e., it assigns a real number $\varphi_i(v)$ to each player $i \in N$. The set of all solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}$. A solution $\varphi \in \mathcal{S}$ is *positive* if $\varphi_i(v) \geq 0$ $(i \in N)$ holds for all $v \in \mathcal{G}^m$. The set of all positive solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}^+$, $\mathcal{S}^+ \subset \mathcal{S}$.

Composite solutions, as defined next, are designed to incorporate exogenous weights as well as endogenous distributions of coalitional worth. For this we combine a vector of weights with a solution. Thus, a composite solution does not only reflect players' opportunities within a cooperative game, but also respects asymmetries among the players outside the game.

Let $\mathcal{D} := \mathbb{R}^N_{++} \times 2^N \times \mathcal{G} \times \mathcal{S}$ and $\mathcal{D}^0 \subseteq \mathcal{D}$. Formally, a *composite solution* $F : \mathcal{D}^0 \longrightarrow \mathbb{R}^N$ on $\mathcal{D}^0$ assigns a vector of players' payoffs to each tuple (α, S, v, φ) consisting of a strictly positive weight vector, a coalition, a cooperative game, and a solution. We interpret a composite solution as follows: Suppose that the game v represents the possibilities for obtaining payoffs from cooperation. The vector α represents asymmetries outside v, and φ is the solution that measures players' payoffs inherent in v. Provided that a coalition $S \in 2^N$ has formed, we view the real number $F_i(\alpha, S, v, \varphi)$ as "player i's overall payoff" in S. Put in another way, taking external weights into account, a player's internal payoff is transformed to an overall one.

Next we ask for a "correct way" to combine players' internal and external weights within a composite solution. We propose the following four axioms:

Ignorance of Outsiders (IO): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i \in N \setminus S$,

$$F_i(\alpha, S, v, \varphi) = 0.$$

Sign Inheritance (SI): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i \in S$,

$$\varphi_i(v_S) > 0 \Rightarrow F_i(\alpha, S, v, \varphi) > 0 \text{ and } \varphi_i(v_S) = 0 \Rightarrow F_i(\alpha, S, v, \varphi) = 0.$$

Payoff Redistribution (PR): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$,

$$\sum_{k \in S} F_k(\alpha, S, v, \varphi) = \sum_{k \in S} \varphi_k(v_S).$$

Constant Transformation rates per Weight (CTW): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$,

$$\frac{\dfrac{F_i(\alpha, S, v, \varphi)}{\varphi_i(v_S)}}{\alpha_i} = \frac{\dfrac{F_j(\alpha, S, v, \varphi)}{\varphi_j(v_S)}}{\alpha_j}.$$

All axioms rely on the following situation: Suppose coalition S has formed and an overall payoff distribution (among all players in N) shall be determined by a composite solution.

First of all, it is natural that all "outsiders", i.e., players $i \notin S$ shall not obtain a payoff. That is the content of the IO axiom.

The remaining axioms concentrate on the distribution of payoffs within coalition S. In the same spirit that solutions are based on the data of the (unrestricted) game, the distribution in S should be based on that part of the game that S is concerned with, i.e., on the restricted game v_S.

Now, SI requires that the incorporation of external weights can neither create positive overall payoff, when the internal payoff was zero, nor can it destroy a positive internal payoff. In particular, this axiom rules out extreme cases in which only players' external characteristics (captured by α) determine their overall payoffs in a coalition. Note that the assumption of positive players' weights correctly supports the idea behind the SI axiom.

The PR axiom requires that total overall payoff in the coalition S is the same as total internal payoff (determined by φ applied to v_S). Put in other words, external weights do not enter the game itself and, hence, they are neither productive nor destructive: What was distributed among the players in a coalition according to φ should exactly be redistributed among them when external weights enter the scene.

CTW is the key axiom that specifies the role of external weights compared to internal payoffs. For example, suppose on the one hand that within coalition S players i, j obtain the same (internal) payoff $\varphi_i(v_S) = \varphi_j(v_S)$. But on the other hand, if player i's weight is twice as high as player j's weight, then i's overall payoff F_i should also be twice as high as j's overall payoff F_j. Hence, if internal payoffs are equal, the ratio of overall payoffs matches the ratio of players' weights.
Reading the CTW axiom in another way, suppose player i's internal payoff were three times as much as player j's payoff. However, to arrive at the same overall payoff, this can exactly be "compensated" by external weights, namely by α_j being three times higher than α_i.
In general, the CTW axiom describes, how overall payoffs should result from the internal payoff distribution. It is based *only* on exogenous weights, since *all* asymmetries among the players *outside* the game are captured by the weights vector α. Now, the ratio F_i/φ_i describes the rate with which player i's internal payoff is transformed. Then CTW requires that the

ratio of two players' transformation rate is given by the ratio of their external weights. Put in other words, the vector of transformation rates $(F_i/\varphi_i)_{i\in S,\varphi_i(v_S)>0}$ is proportional to the (corresponding) vector of external weights. Thus, the rate of transformation per unit of external weight is constant across players in a coalition. To sum, with CTW in place, α_i represents the rate with which player i's internal payoff transforms to the final (overall) one.

3 A characterization result

The concept of external weights is frequently associated with ideas of proportionality. For example, the distribution of seats in a parliament is supposed to have a direct effect on how responsibilities in a government are shared. Consequently, the proportional version of a composite solution, denoted Φ should read as follows:

As domain for Φ we take $\bar{\mathcal{D}} := \mathbb{R}^N_{++} \times 2^N \times \mathcal{G}^m \times \mathcal{S}^+ \subset \mathcal{D}$, that restricts attention to monotonic games and positive solutions. As an example, $\bar{\mathcal{D}}$ covers monotonic simple games and power indices as solutions. Then define the composite solution $\Phi : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ by[1]

$$(1) \qquad \Phi_i(\alpha, S, v, \varphi) = \begin{cases} \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k\in S} \alpha_k \varphi_k(v_S)} \cdot \displaystyle\sum_{k\in S} \varphi_k(v_S), & i \in S, \\ 0, & i \notin S \end{cases} \qquad (i \in N).$$

Thus, within a coalition S exactly the worth $\sigma := \sum_{k\in S} \varphi_k(v_S)$ is redistributed among the agents in S. Without external weights, player i's ($i \in S$) share of σ was $\varphi_i(v_S)/\sum_{k\in S} \varphi_k(v_S)$. This share is now multiplied with his external weight α_i. Normalization, i.e., the sum of shares equals 1, yields the form in (1). Note also that Φ is homogeneous of degree zero w.r.t. the weights vector, which means that only relative weights among agents matter.

Before we proceed with a characterization result for Φ on $\bar{\mathcal{D}}$ using the four axioms from the previous section, consider the following example:

Example: Let v be a monotonic simple game, and let φ be an efficient, symmetric solution that satisfies the null player property. Furthermore, let S be a minimal winning coalition in v. Then it is readily seen that φ allocates the payoff $v_S(S) = 1$ equally among the members in S and assigns zero to all players not in S. Now, with external weights[2] the composite solution proposes a share of $\alpha_i/\alpha(S)$ to player i, where $\alpha(S)$ is the sum of weights in S. Thus Φ allocates $v_S(S)$ within S proportional to players' external weights. Using the interpretation of a parliament, such an allocation is widely observed in problems of sharing responsibilities within a government. □

[1]We use the convention that $\frac{0}{0} = 0$ (for example, for the case that v_S is the zero game).
[2]Here, we can think of seats in the parliament or number of voters that voted for a party.

Theorem: *A composite solution $F : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ satisfies IO, SI, PR, and CTW if and only if $F = \Phi$. Moreover, all four axioms are logically independent.*

Proof. The proof proceeds in three steps:

Step 1: Φ satisfies the four axioms on $\bar{\mathcal{D}}$

By construction, Φ clearly satisfies IO. On the domain $\bar{\mathcal{D}}$, Φ always assigns non-negative payoffs, since any considered game v is monotonic and φ is positive. From the construction and recalling that weights are strictly positive, is is readily seen that, $\Phi_i(\alpha, S, v, \varphi) > 0$ holds if and only if $\varphi_i(v_S) > 0$ is true. Hence, SI is satisfied.

Next, for all $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$ we have

$$\sum_{i\in S} \Phi_i (\alpha, S, v, \varphi) = \frac{\sum_{i\in S} \alpha_i \varphi_i (v_S)}{\sum_{k\in S} \alpha_k \varphi_k (v_S)} \sum_{k\in S} \varphi_k (v_S) = \sum_{k\in S} \varphi_k (v_S),$$

which shows that the PR axiom is also fulfilled.

Finally, we establish CTW. Take $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$, $i, j \in S$ with $\varphi_i (v_S) > 0$ and $\varphi_j (v) > 0$. Then we obtain

$$\begin{aligned} \frac{\frac{\Phi_i (\alpha, S, v, \varphi)}{\varphi_i (v_S)}}{\alpha_i} &= \frac{\alpha_i \varphi_i (v_S) \sum_{k\in S} \varphi_k (v_S)}{\alpha_i \varphi_i (v_S) \sum_{k\in S} \alpha_k \varphi_k (v_S)} = \frac{\sum_{k\in S} \varphi_k (v_S)}{\sum_{k\in S} \alpha_k \varphi_k (v_S)} \\ &= \frac{\alpha_j \varphi_j (v_S) \sum_{k\in S} \varphi_k (v_S)}{\alpha_j \varphi_j (v_S) \sum_{k\in S} \alpha_k \varphi_k (v_S)} = \frac{\frac{\Phi_j (\alpha, S, v, \varphi)}{\varphi_j (v_S)}}{\alpha_j}, \end{aligned}$$

which is the condition in CTW.

Step 2: Uniqueness

Let $F : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ satisfy the above four axioms. Take $i \in N$ and $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$, and consider the following four cases:

Case 1 ($i \in N \setminus S$): Then, by IO, $F_i (\alpha, S, v, \varphi) = 0 = \Phi_i (\alpha, S, v, \varphi)$.

Case 2 ($S = \{i\}$): By PR, $F_i (\alpha, S, v, \varphi) = \varphi_i (v_S)$, which is, by definition, equal to $\Phi_i (\alpha, S, v, \varphi)$.

Case 3 ($i \in S$, $|S| \geq 2$ and $\varphi_i (v_S) = 0$): By SI, $F_i (\alpha, S, v, \varphi) = 0$, which, again by definition, is equal to $\Phi_i (\alpha, S, v, \varphi)$.

Case 4 ($i \in S$, $|S| \geq 2$ and $\varphi_i (v_S) > 0$): We consider the following two subcases.

Subcase 4.1 ($\varphi_k (v_S) = 0$ for all $k \in S \setminus \{i\}$): We have

$$F_i (\alpha, S, v, \varphi) = \sum_{k\in S} \varphi_k (v_S) - \sum_{k\in S\setminus\{i\}} F_k (\alpha, S, v, \varphi) = \varphi_i (v_S) = \Phi_i (\alpha, S, v, \varphi),$$

where the first equation follows from PR and the second by SI.

Subcase 4.2 ($\varphi_k(v_S) > 0$ for some $k \in S \setminus \{i\}$): Let $S' = \{l \in S \setminus \{i\} : \varphi_l(v_S) > 0\}$. Then,

$$\begin{aligned}
F_i(\alpha, S, v, \varphi) &= \sum_{k\in S} \varphi_k(v_S) - \sum_{k\in S\setminus\{i\}} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k\in S} \varphi_k(v_S) - \sum_{k\in S'} F_k(\alpha, S, v, \varphi) - \sum_{k\in (S\setminus\{i\})\setminus S'} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k\in S} \varphi_k(v_S) - \sum_{k\in S'} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k\in S} \varphi_k(v_S) - \sum_{k\in S'} \frac{F_i(\alpha, S, v, \varphi)\,\alpha_k \varphi_k(v_S)}{\alpha_i \varphi_i(v_S)} \\
&= \sum_{k\in S} \varphi_k(v_S) - \frac{F_i(\alpha, S, v, \varphi)}{\alpha_i \varphi_i(v_S)} \sum_{k\in S'} \alpha_k \varphi_k(v_S) \\
&= \sum_{k\in S} \varphi_k(v_S) - \frac{F_i(\alpha, S, v, \varphi)}{\alpha_i \varphi_i(v_S)} \sum_{k\in S\setminus\{i\}} \alpha_k \varphi_k(v_S), \qquad (2)
\end{aligned}$$

where the first equation follows from PR, the third one from $\varphi \in \mathcal{S}^+$ and SI, the fourth one by CTW, and the last one by $\varphi_k(v_S) = 0$ for each $k \in (S \setminus \{i\}) \setminus S'$. Rearranging (2) leads to

$$\begin{aligned}
F_i(\alpha, S, v, \varphi) &= \frac{\sum_{k\in S} \varphi_k(v_S)}{\left(1 + \frac{\sum_{k\in S\setminus\{i\}} \alpha_k\varphi_k(v_S)}{\alpha_i\varphi_i(v_S)}\right)} \\
&= \frac{\alpha_i \varphi_i(v_S)}{\sum_{k\in S} \alpha_k \varphi_k(v_S)} \sum_{k\in S} \varphi_k(v_S) \quad = \quad \Phi_i(\alpha, S, v, \varphi).
\end{aligned}$$

The four cases together establish $F = \Phi$.

Step 3: Logical independence of axioms

Finally, in order to show independence of the axioms used for the characterization of Φ, we present four composite solutions that satisfy exactly three of the four axioms and are different from Φ.

Example 1 Define the composite solution $F^1 : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ by

$$F_i^1(\alpha, S, v, \varphi) = \begin{cases} \sum_{k\in S} \varphi_k(v_S), & \text{if } S = \{i, j\},\ j \neq i, \text{ and} \\ & \varphi_j(v_S) > \varphi_i(v_S) = 0, \\ 0, & \text{if } S = \{i, j\},\ j \neq i, \text{ and} \\ & \varphi_i(v_S) > \varphi_j(v_S) = 0, \\ \Phi_i(\alpha, S, v, \varphi), & \text{otherwise} \end{cases} \qquad (i \in N).$$

It is easy to check that, by its construction, F^1 satisfies PR and IO. It also satisfies CTW since this axiom only states a requirement for coalitions S, in which there are $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$, and we have $F^1 = \Phi$ in such cases. However, F^1 violates SI since for $S = \{i, j\}$, $j \neq i$, with $\varphi_j(v_S) > \varphi_i(v_S) = 0$ we have $F_i^1(\alpha, S, v, \varphi) = \varphi_j(v_S) > 0$.

Example 2 Next, consider $F^2 : \bar{\mathcal{D}} \to \mathbb{R}^N$ defined by

$$F_i^2(\alpha, S, v, \varphi) = \begin{cases} \frac{\alpha_i \varphi_i(v_S)}{\alpha(N)} \cdot \sum_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N),$$

where $\alpha(N) := \sum_{i \in N} \alpha_i$ denotes the total sum of external weights. Clearly, this solution satisfies SI and IO, and it is easy to check that it also satisfies CTW. However, F^2 violates PR, since

$$\sum_{i \in S} F_i^2(\alpha, S, v, \varphi) = \frac{\sum_{i \in S} \alpha_i \varphi_i(v_S)}{\alpha(N)} \cdot \sum_{k \in S} \varphi_k(v_S) \neq \sum_{i \in S} \varphi_k(v_S).$$

Example 3 Let $F^3 : \bar{\mathcal{D}} \to \mathbb{R}^N$ be defined by

$$F_i^3(\alpha, S, v, \varphi) = \begin{cases} \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k \in S} \alpha_k \varphi_k(v_S)} \cdot \displaystyle\sum_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k \in N \setminus S} \alpha_k \varphi_k(v_S)} \cdot \displaystyle\sum_{k \in N \setminus S} \varphi_k(v_S), & \text{otherwise} \end{cases} \qquad (i \in N).$$

As F^3 coincides with Φ for all members in S, inspection of Step 1 reveals that this solution satisfies all axioms but IO.

Example 4 Define $F^4 : \bar{\mathcal{D}} \to \mathbb{R}^N$ by

$$F_i^4(\alpha, S, v, \varphi) = \begin{cases} \dfrac{(\varphi_i(v_S))^{\alpha_i}}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}} \cdot \displaystyle\sum_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N),$$

and notice that it does not satisfy CTW since, for $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$,

$$\begin{aligned} \frac{\frac{F_i^4(\alpha, S, v, \varphi)}{\varphi_i(v_S)}}{\alpha_i} &= \frac{(\varphi_i(v_S))^{\alpha_i}}{\alpha_i \varphi_i(v_S)} \cdot \frac{\sum_{k \in S} \varphi_k(v_S)}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}} \\ &\neq \frac{(\varphi_j(v_S))^{\alpha_j}}{\alpha_j \varphi_j(v_S)} \cdot \frac{\sum_{k \in S} \varphi_k(v_S)}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}} = \frac{\frac{F_j^4(\alpha, S, v, \varphi)}{\varphi_j(v_S)}}{\alpha_j}. \end{aligned}$$

It is easy to check that F^4 satisfies SI, PR, and IO. □

4 Concluding remarks

The concept of a composite solution measures players' overall payoff in a coalition when their internal payoff in a game is given by a cooperative solution concept, and a vector of positive weights describes asymmetries among the players outside the game. As an example, one may consider simple games which describe possibilities to form winning coalitions. Traditionally, power indices, such as the Shapley value, propose a distribution of power among the players. However, a frequent criticism towards power indices is that they ignore characteristics besides the data of the game (e.g., distribution of seats in a parliament). Composite solutions now provide a tool to incorporate both, external weights and the data of the game to arrive at an overall power distribution within a winning coalition, i.e., how to distribute responsibilities within a government (cf. Dimitrov and Haake (2006)).

In addition to the properties used for the characterization of Φ, this composite solution also satisfies other desirable properties we would like to mention. For example, Φ inherits standard properties from the solution φ such as efficiency or the null player property. Moreover, if φ is symmetric, then the ratio of overall payoffs of any two players who are symmetric in v is exactly the ratio of their external weights.

Finally, the reader may easily verify that if, on the one hand, external asymmetries among the players are faded out (i.e., if we assume equal weights) then, roughly spoken, the composite solution Φ coincides with the solution φ.[3] If, on the other hand, internal differences among the players in the game are ignored by the solution concept (e.g., if $\varphi_i(v_S) = \frac{v_S(S)}{|S|}$ for each $i \in S \subseteq N$) then, since all players are symmetric, the ratio of overall payoffs of any two players is the ratio of external weights. Hence Φ is in essence determined by the weights vector α.

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