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A note on the paradox of smaller coalitions*

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Abstract

We consider hedonic coalition formation games that are induced by a simple TU-game and a cooperative solution. For such models, Shenoy's (1979) absence of the paradox of smaller coalitions provides a sufficient condition for core existence. We present three different versions of his condition in order to compare it to the top coalition property of Banerjee et al. (2001) that guarantees nonemptiness of the core in more general models. As it turns out, the top coalition property implies a condition in which Shenoy's paradox is not present for at least one minimal winning coalition. Conversely, if for each non-null player Shenoy's paradox is not present for at least one minimal winning coalition containing that player, then the induced hedonic game satisfies the top coalition property.

JEL Classification: D72, C71

Keywords: coalition formation, core, paradox of smaller coalitions, simple games, top coalition property

1 Introduction

One of the most apparent applications of coalition formation games is the formation of governments after a political election. A simple (TU-) game thereby describes the possibilities to form majorities, whereas application of a cooperative solution concept (e.g. a power index) yields a party's *power* within a government. In fact, for each party one obtains preferences over possible coalitions (governments) it is a member of. That means, we induce a hedonic coalition formation game (see, e.g., Shenoy (1979)).

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Solutions for hedonic games, such as *core stability*, then propose a (set of) stable coalition structures that roughly partition the set of players into a winning coalition (the government) and a losing one; and therefore arrive at a notion of a *stable government*.

For this (specific) context, Shenoy (1979) on the one hand introduced a sufficient condition for nonemptiness of the core termed *absence of the paradox of smaller coalitions.* Essentially, it requires that a party should not suffer from a re-formation of the government by throwing out other parties. On the other hand, Banerjee et al. (2001) discuss a condition that ensures core existence for general hedonic games; the *top coalition property.* Each coalition is required to have a subcoalition that is favored most among all subcoalitions by any of its members.

In this paper, we investigate the relationship between these two sufficient conditions. More precisely, we first present three versions of Shenoy's condition, each of them being weaker than the previous one. However, as a generalization of Shenoy's result, even the weakest form is still sufficient to obtain core existence (Proposition 1). Second, we range in the top coalition property by showing which of the above versions it implies and by which it is implied (Propositions 2 and 3). In effect, we answer the following two questions: (1) To what extent is the paradox of smaller coalitions allowed to appear, when the induced hedonic game satisfies the top coalition property and (2) How much of the absence of the paradox of smaller coalitions needs to hold in order to have the induced hedonic game satisfy the top coalition property.

The next section contains preliminaries on simple games and hedonic games. In Section 3 we introduce two weaker versions of Shenoy's (1979) original condition and show that the weakest of them suffices for nonemptiness of the core of the induced hedonic game. The implications between the top coalition property and these two weaker conditions are presented in Section 4. We close by showing in two examples that the implications are not reversible.

2 Preliminaries

2.1 Simple games and solutions

Let N be a finite *set of players*, which we will keep fixed throughout the paper. A (*cooperative*) *simple game with transferable utility* (a simple game) is a pair (N, v), where $v : 2^N \to \{0, 1\}$ is called *characteristic function* and satisfies $v(\emptyset) = 0$. We refer to a coalition $S \subseteq N$ with $v(S) = 1$ as a *winning coalition.* In what follows we will identify a simple game (N, v) with its characteristic function v.

A simple game v is *monotonic* if $v(S) = 1$ implies $v(T) = 1$ for all $T \supseteq S$, and *proper* if $v(S) = 1$ implies $v(N \setminus S) = 0$. We denote by $\mathcal{W}^v = \{S \subseteq N \mid v(S) = 1\}$ the set of winning coalitions and by $\mathcal{MW}^v = \{S \subseteq N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for all } T \subset S\}$ the set of *minimal winning coalitions* in the simple game v (cf. Shapley (1962)). For $S \subseteq N$ define the *subgame* (N, v_S) by $v_S(T) = v(S \cap T)$ for all $T \in 2^N$. Note that v_S is also a simple game with player set N (possibly with $v_S(N) = v(S) = 0$). The set of all *proper monotonic simple games* on the player set N will be denoted by $\mathcal{G}$. Clearly, if a game v is in the set $\mathcal{G}$, then so is any of its subgames.

A player $i \in N$ is a *null player* in $v \in \mathcal{G}$ if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$. For any $W \in \mathcal{W}^v$, we denote by $A^v(W)$ the set of all players from W who are *not* null players in v_W. For any $W \in \mathcal{W}^v$ and any $i \in A^v(W)$, we denote by $\mathcal{M}_i^v(W)$ the set of all minimal winning coalitions in W containing i.

A *(feasible) solution* (of a proper monotonic simple game) is a mapping $\varphi : \mathcal{G} \to \mathbb{R}_+^N$ satisfying $\sum_{i \in N} \varphi_i(v) \leq v(N)$ for all $v \in \mathcal{G}$. Thus, it takes each $v \in \mathcal{G}$ to a single vector in $\mathbb{R}_+^N$. That means, it assigns a nonnegative real number $\varphi_i(v)$ to each player $i \in N$, which we interpret as player i's *power* in the game v.[1] The set of all feasible solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}$.

In what follows, we make use of the solution φ in the following way. Suppose a simple game v describes the possibilities for players to form winning coalitions and suppose that such a winning coalition S has formed. When the members of S have to agree on how power is shared among them, then the subgame v_S naturally should be taken into account, as it reflects possibilities *given* S has already formed. For example, within a minimal winning coalition, all members should be treated equally, since all have the same possibilities to form other winning coalitions within S (see Dimitrov & Haake (2006) for further details).

It turns out that the following property of a solution is the main source to drive the results in the remainder of the paper. It says that within minimal winning coalitions, the total power of 1 is shared equally. Formally, a solution $\varphi \in \mathcal{S}$ satisfies *equal treatment in minimal winning coalitions*, if for all $v \in \mathcal{G}, S \in \mathcal{MW}^v$ and $i \in S$ we have

$$\text{(ETMW)} \qquad \varphi_i(v_S) = \frac{1}{|S|}.$$

Remark 1 *The (more familiar) conditions of coalitional efficiency and symmetry together are sufficient for the ETMW requirement to hold. Formally, a solution $\varphi \in \mathcal{S}$ satisfies coalitional efficiency if $\sum_{i \in S} \varphi_i(v_S) = v_S(S)$ holds for all $v \in \mathcal{G}$ and all $S \in 2^N$. Players $i, j \in N$ are symmetric in $v \in \mathcal{G}$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. A solution $\varphi \in \mathcal{S}$ is symmetric if $\varphi_i(v) = \varphi_j(v)$ for all $v \in \mathcal{G}$ and all $i, j \in N$ who are symmetric in v.*

[1] Requiring nonnegativity is in accordance with this interpretation of φ.

Finally, a solution $\varphi \in \mathcal{S}$ satisfies the *null player property* if $\varphi_i(v) = 0$ holds for all $v \in \mathcal{G}$ and $i \in N$ who are null players in v.

2.2 Hedonic games, core stability and the top coalition property

For each player $i \in N$ we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A partition Π of N is called a *coalition structure*. For each coalition structure Π and each player $i \in N$, we denote by $\Pi(i)$ the coalition in Π containing player i, i.e., $\Pi(i) \in \Pi$ and $i \in \Pi(i)$.

Further, we assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, i.e., a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. Denote by $\succ_i$ the associated strict relation and by $\succeq := (\succeq_1, \succeq_2, \ldots, \succeq_n)$ the corresponding profile of preferences. A player's preference relation over coalitions canonically induces a preference relation over coalition structures in the following way:[2] For any two coalition structures Π and Π', player i weakly prefers Π to Π' if and only if he weakly prefers "his" coalition in Π to the one in Π', i.e., $\Pi \succeq_i \Pi'$ if and only if $\Pi(i) \succeq_i \Pi'(i)$. Hence, we assume that players' preferences over coalition structures are *purely hedonic*, i.e., they are completely characterized by their preferences over coalitions. Finally, a *hedonic game* $(N, \succeq)$ is a pair consisting of the set of players and a preference profile (cf. Banerjee et al. (2001) and Bogomolnaia & Jackson (2002)).

Unlike solution concepts for (simple) cooperative games do, there is no worth to distribute in hedonic games. The relevant question is rather, which coalition structure should form, taking players' preferences into account. The basic property is the one of core stability.

Let $(N, \succeq)$ be a hedonic game. A partition Π of N is *core stable* if there does not exist a nonempty coalition X such that $X \succ_i \Pi(i)$ holds for each $i \in X$. Put in other words, a coalition structure Π is core stable if no group of players are willing to form a coalition, so that each player is better off with this new coalition compared to the corresponding coalitions in Π. In what follows, we denote by $C(N, \succeq)$ the set of core stable coalition structures of a hedonic game $(N, \succeq)$. Alternatively, we call $C(N, \succeq)$ the *core* of $(N, \succeq)$.

The top coalition property, introduced by Banerjee et al. (2001), is a sufficient condition for nonemptiness of the core and it is satisfied in many interesting economic applications. Given a hedonic game $(N, \succeq)$ and a player set $V \subseteq N$, a coalition $S \subseteq V$ is a *top coalition* of V if for any $i \in S$ and any $T \subseteq V$ with $i \in T$, one has $S \succeq_i T$. The game $(N, \succeq)$ satisfies the *top coalition property* if every player set has a top coalition.

[2]With slight abuse in notation, we use the same symbol to denote preferences over coalitions and preferences over coalition structures.

2.3 Hedonic games via solutions of simple games

Now, each pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ of a simple game and a solution naturally induces a hedonic game in the following way: For each player $i \in N$ define a preference relation $\succeq_i$ over $\mathcal{N}_i$ by

$$(1) \qquad S \succeq_i T \quad \text{if and only if} \quad \varphi_i(v_S) \geq \varphi_i(v_T) \qquad (S, T \in \mathcal{N}_i).$$

Put differently, $\varphi_i(v_\bullet)|_{\mathcal{N}_i}$ is a representation of i's preferences. Still put in another way, player i evaluates a coalition S that he is a member of by how much power he obtains within S according to the solution φ. Notice that paying attention to the corresponding coalitions is compatible with the very definition of a hedonic game; each player in such a game evaluates any two coalition structures based only on his preferences over the coalitions in the two partitions he belongs to (cf. Aumann and Dréze (1974) and Shenoy (1979)).

It is apparent that for any hedonic game $(N, \succeq)$ that is induced by (v, φ) as in (1), a coalition structure Π can only be stable if it contains a winning coalition. Hence, by properness of v, there is exactly one winning coalition in each core stable coalition structure. Therefore, choosing a core stable coalition structure may be interpreted as choosing a "stable" winning coalition (i.e., a government) in the underlying simple game.

3 The paradox of smaller coalitions and core stability

Let $v \in \mathcal{G}$ and $\varphi \in \mathcal{S}$. The main building blocks in our analysis will be pairs (W, S) of winning coalitions satisfying $S \subseteq W$. Then, the pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ of a game and a solution *does not exhibit the paradox of smaller coalitions on* (W, S), if $\varphi_i(v_S) \geq \varphi_i(v_W)$ holds for all $i \in S$. In words, if a smaller winning coalition S forms, then each player's power, measured by φ, should not decrease, since there are fewer players to share the same total amount of power.

Starting with the original definition in Shenoy (1979), we define three different extents to which a pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ does not exhibit the paradox of smaller coalitions. More precisely, we specify for which pairs (W, S) the paradox should not occur.

(C1) A pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ satisfies C1, if for all $W, S \in \mathcal{W}^v$ with $S \subseteq W$ the pair (v, φ) does not exhibit the paradox of smaller coalitions on (W, S).

(C2) A pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ satisfies C2, if for all $W \in \mathcal{W}^v$ and all $i \in A^v(W)$, there exists $S \in \mathcal{M}_i^v(W)$ such that (v, φ) does not exhibit the paradox of smaller coalitions on (W, S).

(C3) A pair $(v, \varphi) \in \mathcal{G} \times \mathcal{S}$ satisfies C3, if for all $W \in \mathcal{W}^v$, there exists $S \in \mathcal{MW}^v$ with $S \subseteq W$ such that (v, φ) does not exhibit the paradox of smaller coalitions on (W, S).

Clearly, condition C2 is weaker than C1.[3] While C1 requires the absence of the paradox for all pairs (W, S) with $S \subseteq W$, the requirement in C2 is only that for each $W \in \mathcal{W}^v$ and each non-null player in v_W there has to be a minimal winning coalition S containing that player such that the paradox of smaller coalitions is not there on (W, S).

Condition C3 is even weaker than C2. Here we only claim that any winning coalition W contains some minimal winning coalition S so that no paradox shows up on (W, S). If C2 is satisfied one may take a non-null player's coalition $S \in \mathcal{M}_i^v(W)$ as in the formulation of C2 which shows C3.

Shenoy (1979, Theorem 7.4) uses condition C1 to derive a core existence result for hedonic games $(N, \succeq)$ that are induced as in (1). Thereby, he uses the Shapley value[4] as solution φ. The following proposition shows that the weaker condition C3 is sufficient for core existence, in case the solution satisfies the null player property and the equal treatment in minimal winning coalitions requirement.

Proposition 1 *Let $v \in \mathcal{G}$ and $\varphi \in \mathcal{S}$ satisfy the null player property and ETMW. Let $(N, \succeq)$ be induced as in (1). If (v, φ) satisfies C3, then $C(N, \succeq) \neq \emptyset$.*

Proof. Let $T \in \mathcal{MW}^v$ be with minimal cardinality and Π be a coalition structure containing T. We show that Π is core stable.

Suppose to the contrary that there is $X \subseteq N$ such that

$$X \succ_i \Pi(i) \text{ for each } i \in X. \tag{2}$$

Clearly, $X \in \mathcal{W}^v$. By C3, there exists a minimal winning coalition $Y \subseteq X$ such that

$$\varphi_i(v_Y) \geq \varphi_i(v_X) \text{ for each } i \in Y. \tag{3}$$

By the properness of v, $Y \cap T \neq \emptyset$. Hence, there is $i \in Y \cap T$ such that

$$\frac{1}{|Y|} = \varphi_i(v_Y) \geq \varphi_i(v_X) > \varphi_i\left(v_{\Pi(i)}\right) = \varphi_i(v_T) = \frac{1}{|T|},$$

where the first inequality is due to (3), the second due to (2), and the two outer equalities follow from ETMW. Notice, however, that we have a contradiction to $|T| \leq |Y|$. □

Note finally, that the *Bundestag game* discussed in Dimitrov & Haake (2006) shows that C3 is not a necessary condition for core existence.

[3]Examples are easily constructed, since C2 means no requirement for pairs (W, S) where both coalitions are winning but not minimal winning.

[4]or *Shapley-Shubik index* as it is often termed in this setup; see Shapley (1953) and Shapley and Shubik (1954).

4 Implications

Proposition 1 establishes a sufficient condition for nonemptiness of the core for a specific class of hedonic games. As mentioned above, the top coalition property also suffices to obtain core elements. In this section we range in the top coalition property into the versions of the absence of a paradox. We start by showing that condition C2 implies the top coalition property.

Proposition 2 *Let $v \in \mathcal{G}$ and $\varphi \in \mathcal{S}$ satisfy the null player property and ETMW. Let $(N, \succeq)$ be induced as in (1). If (v, φ) satisfies C2, then $(N, \succeq)$ satisfies the top coalition property.*

Proof. We have to show that there exists a top coalition for each player set $U \subseteq N$. Take $U \subseteq N$ and consider the following two possible cases:

Case 1 ($U \notin \mathcal{W}^v$): By monotonicity of v and the null player property of φ, each subset of U is a top coalition of U.

Case 2 ($U \in \mathcal{W}^v$): Let $S \subseteq U$ be a smallest (in terms of cardinality) minimal winning coalition contained in U. We show that S is a top coalition of U, i.e., we show that $S \succeq_i T$ for each $i \in S$ and each $T \subseteq U$ with $i \in T$. Take $T \subseteq U$ and consider the following three subcases:

Subcase 2.1 ($T \notin \mathcal{W}^v$): By non-negativity of φ and the null player property, we have $\varphi_i(v_S) \geq 0 = \varphi_i(v_T)$ for each $i \in S \cap T$ and thus, $S \succeq_i T$ for each $i \in S \cap T$.

Subcase 2.2 ($T \in \mathcal{W}^v$ and $T \supseteq S$): Let $i \in S$. Since i is not a null player in v_T, there exists by C2 a minimal winning coalition $P_i \subseteq T$ containing i for which

$$\varphi_k(v_{P_i}) \geq \varphi_k(v_T) \text{ for all } k \in P_i,$$

and in particular, by the properness of v,

$$\varphi_k(v_{P_i}) \geq \varphi_k(v_T) \text{ for all } k \in P_i \cap S.$$

Since S is a smallest minimal winning coalition contained in U, $i \in S$ and $S \subseteq T \subseteq U$, we have $|P_i| \geq |S|$. Thus, using ETMW,

$$\varphi_k(v_S) = \frac{1}{|S|} \geq \frac{1}{|P_i|} = \varphi_k(v_{P_i}) \geq \varphi_k(v_T) \text{ for all } k \in P_i \cap S$$

implying that $S \succeq_k T$ for each $k \in P_i \cap S$. Since *each player* from S is not a null player in v_T, the above argument can be repeatedly applied to conclude that $S \succeq_i T$ for each $i \in S \cap T = S$.

Subcase 2.3 ($T \in \mathcal{W}^v$ and $T \not\supseteq S$): Notice that $T \cap S \neq \emptyset$ by properness of v. Let $i \in S \cap T$. If i is a null player in v_T, then, by ETMW and the null player property, $\varphi_i(v_S) = \frac{1}{|S|} > 0 = \varphi_i(v_T)$. If i is not a null player in v_T, there exists by C2 a minimal wining coalition $R \subseteq T$ containing i such that $\varphi_k(v_R) \geq \varphi_k(v_T)$ for each $k \in R$. Observe then that $\varphi_i(v_S) = \frac{1}{|S|} \geq \frac{1}{|R|} = \varphi_i(v_R) \geq \varphi_i(v_T)$ where the first inequality follows from the minimality of S and the equalities from ETMW. We conclude that $S \succeq_i T$ for each $i \in S \cap T$.

The two cases together show that $S \succeq_i T$ for each $i \in S \cap T$ and each $T \subseteq U$; thus, S is a top coalition of U. □

The next result (Proposition 3) demonstrates that the top coalition property implies condition C3. This implication even holds, if we do not impose the null player property. In order to prove this result, we make use of the following lemma.

Lemma 1 *Let $v \in \mathcal{G}$, $\varphi \in \mathcal{S}$ satisfy ETMW and $(N, \succeq)$ be induced as in (1). Let $W \in \mathcal{W}^v$ and $T \subseteq W$ be a top coalition of W. Then, T contains only one minimal winning coalition.*

Proof. Let P and Q be two different minimal winning coalitions contained in T. Then, since T is a top coalition of W, $P \subset T$, and by ETMW, $\varphi_i(v_T) \geq \varphi_i(v_P) = \frac{1}{|P|} > 0$ for each $i \in P$. Using the feasibility constraint $\sum_{i \in N} \varphi_i(v_T) \leq v_T(N) = v(T) = 1$ and non-negativity of $\varphi(\cdot)$, this implies $\varphi_i(v_T) = 0$ for each $i \in T \setminus P$. With the same argument, $\varphi_i(v_T) = 0$ for each $i \in T \setminus Q$. Since there is at least one player who belongs to P but not to Q, we have a contradiction. □

Remark 2 *In view of the proof of Lemma 1, if $T \subseteq W$ is a top coalition of $W \in \mathcal{W}^v$, then the minimal winning coalition contained in T is a top coalition of W as well.*

Proposition 3 *Let $v \in \mathcal{G}$, $\varphi \in \mathcal{S}$ satisfy ETMW, and let $(N, \succeq)$ be induced as in (1). If $(N, \succeq)$ satisfies the top coalition property, then (v, φ) satisfies condition C3.*

Proof. We have to show that for each $W \in \mathcal{W}^v$ there exists $S \in \mathcal{MW}^v$ with $S \subseteq W$ such that $\varphi_i(v_S) \geq \varphi_i(v_W)$ for each $i \in S$. Let $W \in \mathcal{W}^v$, T be a top coalition of W, and let T' be the minimal winning coalition contained in T (see Lemma 1). By Remark 2, T' is a top coalition of W as well. By the definition of a top coalition, $\varphi_i(v_{T'}) \geq \varphi_i(v_W)$ for each $i \in T'$. Thus, we conclude that (v, φ) satisfies condition C3. □

So, Propositions 2 and 3 show where to range in the top coalition property in the present setup. Observe also that due to Proposition 1, we can confirm that the top coalition property is a sufficient condition for existence of core stable coalition structures. We close by showing that neither of the two implications in Propositions 2 and 3 can be reversed. This is done by the following examples.

Example 1 (Top coalition property does not imply C2)
Let $N = \{1,2,3,4\}$ and $v \in \mathcal{G}$ be given by $\mathcal{MW}^v := \{12, 134\}$[5]. Let the solution φ be defined as follows:

$$\varphi(v_{1234}) = \varphi(v_{123}) = \varphi(v_{124}) = \varphi(v_{12}) = (\frac{1}{2}, \frac{1}{2}, 0, 0), \quad \varphi(v_{134}) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}),$$

and $\varphi(v_S) = 0$ for all $S \notin \mathcal{W}^v$. Clearly, φ satisfies ETMW and the null player property.

Further, it is easy to check that the top coalition property is satisfied. Note for this, that the coalition 134 is a top coalition of itself and 12 is a top coalition of any other winning coalition. However, C2 is violated. To see this, take player $i = 4 \in A^v(N)$. The only minimal winning subcoalition of N that contains i is $S = 134$. But (v, φ) does show the paradox of smaller coalitions on (N, S), as $\varphi_1(v_S) = \frac{1}{3} < \frac{1}{2} = \varphi_1(v_N)$. □

Example 2 (C3 does not imply the top coalition property)
Let $N = \{1,2,3,4\}$ and $v \in \mathcal{G}$ be given by $\mathcal{MW}^v := \{12, 13, 14, 234\}$. Furthermore, $\varphi \in \mathcal{S}$ is defined by

$$\varphi(v_{1234}) = (\frac{2}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}), \quad \varphi(v_{123}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0), \; \varphi(v_{124}) = (\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}), \quad \varphi(v_{134}) = (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}),$$
$$\varphi(v_{12}) = (\frac{1}{2}, \frac{1}{2}, 0, 0), \quad \varphi(v_{13}) = (\frac{1}{2}, 0, \frac{1}{2}, 0), \; \varphi(v_{14}) = (\frac{1}{2}, 0, 0, \frac{1}{2}), \quad \varphi(v_{234}) = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$$

and $\varphi(v_S) = 0$ for all $S \notin \mathcal{W}^v$. Again, φ satisfies ETMW.

Then, on the one hand C3 is satisfied. For instance, for the winning coalition N the minimal winning coalition 234 satisfies $\varphi_i(v_{234}) \geq \varphi_i(v_N)$ for all $i = 2, 3, 4$. On the other hand, the top coalition property is not satisfied, as N does not possess a top coalition. To see this, N itself is not a top coalition, since all members of 234 are better off in 234. Coalition $S = 234$ is not a top coalition of N, either, since any of the players $2, 3, 4$ is better off in a coalition with player 1. However, player 1 strictly favors the grand coalition. Hence, the example shows that the top coalition property is not implied by C3. □

To sum the results, we have established the following implications, none of which is reversible:

$$\text{C1 (no paradox)} \Longrightarrow \text{C2} \Longrightarrow \text{top coalition property} \Longrightarrow \text{C3} \Longrightarrow \text{core existence.}$$

[5] With slight abuse in notation for coalitions, we write, e.g., 134 instead of $\{1, 3, 4\}$.

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