The Role of Pension Systems and Demographic Change for Asset Prices and Capital Formation

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Abstract

The paper develops a large-scale overlapping generations model with production and a stochastic asset market. The role of a pension system and the impact of demographic change on real and financial markets are analyzed. In the absence of demographic change a reduction in contribution rates increases the long-run levels of capital and asset prices while reducing interest rates. In addition a lower contribution rate may stabilize financial markets by reducing the volatility and avoiding crashes in asset prices. Demographic change due to a shrinking population induces a meltdown of capital and asset prices confirming results in the literature.

Introduction

One of the biggest challenges faced by virtually all industrialized countries is the accelerating demographic change within their populations. These structural changes are mainly due to low birth rates accompanied by a constant increase in people's life expectancy. Both effects put enormous pressure on the existing pay-as-you-go pension systems as they simultaneously increase the number of beneficiaries and decrease the number of contributors to the systems. Against this background the past years have seen a vivid political debate about the efficiency and sustainability of pension systems and numerous reform proposals have been suggested. However, any pay-as-you-go structure implies a fundamental trade-off between maintaining a sufficiently high level of pension incomes and keeping contributions at a reasonably low level. As a consequence a mere adjustment of contribution rates and/or pension payments can shift the demographic burden between contributors and beneficiaries but cannot solve the demographic problem.

To ameliorate this dilemma, many economists have suggested to supplement or even replace the public pension system by an increased share of private savings for retirement. The economic reasoning behind this measure to attenuate the demographic problem is that an increase in private savings potentially fosters the accumulation of capital, cf. Feldstein (1974). This in turn would enhance the production possibilities of the economy providing a potential way to overcome the loss in aggregated workforce induced by the demographic change. Opponents of such a reform have argued, however, that private savings are exposed to capital market risk. Hence, the proposed change towards a funded retirement system would necessarily increase the risk to which pension incomes are exposed due to the volatility and unpredictability of capital markets in general and stock markets in particular. This argument has to some extent been supported by the drastic decline in stock prices at the beginning of this century.

The latter argument apparently suggests a trade-off between the efficiency of a pension system and the risk to which pension incomes are subjected. It also stresses the importance to pay adequate respect to the role of uncertainty and financial risk when studying pension reforms. In this regard, it seems natural to assume that adjustments in the pension system affect consumers' savings behavior which in turn affect prices on financial markets. A comprehensive theoretical analysis therefore requires a framework which incorporates not only the issue of demographic change but also the mutual *interactions* between the pension system and asset/stock markets. Conceptually, this calls for a macroeconomic model which incorporates the following three building blocks: Firstly, a population model to study the impact of demographic changes in the population structure. Secondly, a description of the production side of the economy to analyze the consequences of pension reforms on real variables such as capital stock, real wages, etc. Thirdly, a stochastic asset market in order to study the role of financial risk and the impact of pension parameters on financial variables such as stock prices, interest rates, etc. The conception and study of such a model forms the core of this paper.

The literature on pension systems mostly confines itself to a deterministic framework. In this regard, the multi-period overlapping generations (OLG) model developed by

Auerbach & Kotlikoff (1987) has been employed by numerous authors to study pension reforms and the role of demographic changes in a deterministic world. Examples may be found in Altig, Auerbach, Kotlikoff, Smetters & Walliser (2001), Börsch-Supan, Heiss, Ludwig & Winter (2003, 2006) or İmrohoroğlu, İmrohoroğlu & Joines (1995). Models which incorporate randomness and stochastic asset markets typically treat asset returns and/or consumers' income as given stochastic processes which follow an exogenously determined probability law (cf. Demange & Laroque (1999), Demange (2002) and also Barbie, Hagedorn & Kaul (2007), or Farhi & Panageas (2007)). While this permits a study of risk to which pension incomes and savings for retirement are exposed, it does not incorporate the *interactions* between asset/stock markets and the pension system described above. Models which partly overcome this problem can be found, for instance, in Chattopadhyay & Gottardi (1999), Gottardi & Kübler (2006), and Krüger & Kübler (2006). Common to all these approaches is a particular stochastic setting where the underlying probability space is finite such that the theoretical framework of incomplete markets (see, e.g., Magill & Quinzii 1998) becomes applicable. While this permits the derivation of valuable analytical results on the efficiency of pension system and inter-generational risk-sharing, the proposed structure makes it difficult to characterize the evolution of the model on a time series level using tools and methods from (random) dynamical systems theory and time series analysis. As a consequence, a comparison of the long-run dynamic behavior of real and financial variables such as aggregate output or asset prices and their statistical properties depending on the population structure and/or the parameters of the pension system is not possible. In addition these models typically adopt an OLG structure with only two generations implying a relatively coarse time scale. Further studies which focus on the interactions between real capital markets (interpreted as stock markets) and the evolution of the population can be found in Abel (2001, 2003) and Geanakopolos, Magill & Quinzii (2004). Again these models employ a deterministic or simplified stochastic setting.

The intention of this paper is twofold. The first objective is to complement the existing approaches by developing a dynamic macroeconomic model which incorporates the three building blocks described above. In this regard, the explicit modeling strategy successfully applied in the asset market models by Böhm & Chiarella (2005) and Hillebrand & Wenzelburger (2006) is adopted. These models provide an explicit description of the formation of expectations and the dynamic evolution of prices and allocations on financial markets. The conceptual challenge is to join these financial models with a real sector describing the production and investment activities of firms and the income streams of consumers generated through the production process. In addition the OLG structure is extended by a population model and consumers with multi-period lives as in Hillebrand & Wenzelburger (2006). Compared to an OLG setting with only two generations this not only enhances the possibilities to study demographic changes in the population structure. It also permits a more detailed analysis of how a pension system affects the distribution of wealth and consumption and the savings behavior over the life cycle. Using this framework, the second goal of the paper is to study the macroeconomic consequences of pension reforms and demographic changes as motivated above.

The paper is organized as follows. Section 1 introduces the OLG model followed by a derivation of the demand behavior of consumers and the firm in Sections 2 and 3. The formation of prices and the sequential structure of the model is introduced in Section 4, followed by a description of the demographic model and the formation of expectations in Section 5. Sections 6 and 7 study the role of pension systems with a stationary population, while Section 8 considers the case with demographic change. Section 9 draws some conclusions, mathematical proofs are placed in the appendix.

1 The OLG model

Consider an economy with discrete time and a population consisting of overlapping generations (OLG) of homogeneous consumers who live for $J+1$ consecutive time periods. In each period $t \in \mathbb{N}_0$, each generation is identified by the index $j \in \{0, 1, \ldots, J\}$ describing the remaining lifetime of the consumers in this generation. In particular, $j = J$ refers to the young generation of consumers born at the beginning of period t and $j = 0$ identifies the old generation whose members die at the end of the current period. Let $N_t^{(j)} > 0$ denote the number of consumers in generation j at time t and define for each t the population vector $N_t := (N_t^{(j)})_{j=0}^J$. Since all generations live identically for $J+1$ periods one has $N_t^{(j)} = N_{t-1}^{(j+1)}$ for each $j = 0, 1, \ldots, J-1$. Assuming that the number $N_t^{(J)} > 0$ of consumers born at the beginning of period t is determined from the previous population vector N_{t-1} by some continuous mapping $\mathcal{N} : \mathbb{R}_{++}^{J+1} \longrightarrow \mathbb{R}_{++}$, the population evolves according to the population law

$$\begin{cases} N_t^{(j)} &= N_{t-1}^{(j+1)}, \qquad j = 0, 1, \ldots, J-1 \\ N_t^{(J)} &= \mathcal{N}(N_{t-1}). \end{cases} \tag{1}$$

Each consumer in generation $j \in \{j_L, \ldots, J\}$ supplies $\bar{L}^{(j)} > 0$ units of labor inelastically to the labor market where the threshold $j_L > 0$ defines the retirement age. The total amount of labor supplied at time t is given by

$$L_t^S := \sum_{j=j_L}^J \bar{L}^{(j)} N_t^{(j)}. \tag{2}$$

There is a single consumption good in the economy which serves as numeraire such that all prices and payments are denominated in terms of the consumption good. Let $\omega_t > 0$ denote the gross real wage per unit of labor at time t out of which a fraction $\tau_t \in [0, 1[$ has to be contributed to the public pension system. Then each working consumer in generation $j \in \{j_L, \ldots, J\}$ earns net labor income

$$e_t^{(j)} = (1 - \tau_t)\omega_t \bar{L}^{(j)} > 0 \tag{3}$$

at time t. Assuming that contributions to the pension system are divided up equally between current retirees and letting $N_t^R := \sum_{j=0}^{j_L-1} N_t^{(j)}$ denote the number of pensioners,

the non-capital income of each consumer in generation $j \in \{0, \ldots, j_L - 1\}$ at time t is

$$e_t^{(j)} = e_t^R := \tau_t \frac{\omega_t \, L_t^S}{N_t^R} \geq 0. \tag{4}$$

To transfer income between different periods there exist two investment possibilities available to each consumer. The first one is a one-period lived bond which is traded at a price of unity at time t and pays a non-random return $R_t > 0$ in the following period $t+1$. Since R_t is determined at time t, the bond provides a riskless investment possibility between any two consecutive periods. The second investment opportunity is given by retradeable shares of a firm which are traded at stock price $p_t > 0$ and pay a random dividend $d_t \geq 0$ (per share, prior to trading) in each period t. Dividend payments are generated endogenously from the production activities of the firm. The total number of shares in the market is constant and denoted as $\bar{x} > 0$. While consumers may short-sell bonds without bound short-selling of shares is not possible. The bond thus provides the sole possibility for consumers (and the firm) to obtain credit. The space $\mathbb{Z} := \mathbb{R} \times \mathbb{R}_+$ defines the set of feasible portfolios of bonds and stocks for each consumer.

Denote by $z_t^{(j)} := (y_t^{(j)}, x_t^{(j)}) \in \mathbb{Z}$ the portfolio purchased by a consumer in generation $j \in \{1, \ldots, J\}$ at time t consisting of a bond investment $y_t^{(j)}$ and a non-negative number $x_t^{(j)}$ defining the number of shares in the portfolio. The wealth of a consumer belonging to generation j at time t consists of his current *non-capital income* defined by (3) and (4), respectively, and his *capital income* corresponding to the return on his previous investment $z_{t-1}^{(j+1)} = (y_{t-1}^{(j+1)}, x_{t-1}^{(j+1)})$. The latter is given by the return R_{t-1} on the bond investment $y_{t-1}^{(j+1)}$ and the return on the stock portfolio $x_{t-1}^{(j+1)}$ consisting of dividend earnings and the selling revenue at time t. Since the capital income of young consumers is zero, we define the wealth of a consumer in generation j at time t as

$$w_t^{(j)} \; := \; \begin{cases} e_t^{(j)}, & j = J \\ e_t^{(j)} + R_{t-1} \, y_{t-1}^{(j+1)} + x_{t-1}^{(j+1)} \, (p_t + d_t) & j = 0, 1, \ldots, J - 1. \end{cases} \tag{5}$$

2 Demand behavior of consumers

To derive the consumption and investment behavior of consumers we consider a typical consumer belonging to generation $j > 0$ in an arbitrary period t who dies at the end of period $t + j$. A more detailed treatment may be found in Hillebrand & Wenzelburger (2007). To alleviate the time script notation we set $t = 0$ for the current period and use the index $n = 0, 1, \ldots, j$ to refer to periods within the consumer's remaining lifetime. The index j identifying the consumer's generation will be suppressed for convenience.

In each period $n = 0, \ldots, j$ the consumer can consume part of his wealth and use the investment possibilities described in the previous section to transfer wealth into future periods. Let $\mathbb{C} := \mathbb{R}_+$ denote the consumption set describing feasible consumption plans in each period. It is assumed that the decision in $t = 0$ is made *after* the dividend payment $d_0 \geq 0$ and the current non-capital income $e_0 \geq 0$ have been observed but *prior*

to trading, i.e., before the bond return R_0 and asset prices p_0 have been determined. Hence the consumer treats these variables as parameters $R > 0$ and $p > 0$. Likewise his current wealth defined by (5) is treated as parameter $w \in \mathbb{R}$ in the decision. Note that wealth may be negative if the consumer has taken credit in the previous period.

At time $t = 0$ the consumer holds expectations $\hat{e} := (\hat{e}_1, \dots, \hat{e}_j) \in \mathbb{R}_+^j$ for his future non-capital income and $\hat{R} := (\hat{R}_1, \dots, \hat{R}_{j-1}) \in \mathbb{R}_{++}^{j-1}$ for future bond returns. Here $\hat{e}_n \geq 0$ denotes the non-capital income expected to be received in period $n \in \{1, \dots, j\}$ while $\hat{R}_n > 0$ is the expected bond return between future periods n and $n+1$, $n \in \{1, \dots, j-1\}$. For each $n \in \mathbb{N}_0$ let $s_n := (p_n, d_n) \in \mathbb{S} := \mathbb{R}_{++} \times \mathbb{R}_+$ denote the asset price and dividend payment in period n. At time $t = 0$ there is uncertainty about all future s_n, $n > 0$. These are treated as an $\mathbb{S}$-valued stochastic process $\{s_n\}_{n>0}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is adapted to a suitable filtration $\{\mathcal{F}_n\}_{n \geq 1}$ of sub-σ-algebras of $\mathcal{F}$. The consumer's expectations for prices and dividends within his remaining lifetime are characterized in the following assumption. In what follows $\mathcal{B}(\mathbb{A})$ denotes the Borel σ- algebra on a given topological space $\mathbb{A}$.

Assumption 1

Given the planning horizon $j > 0$ the consumer's subjective expectations for future asset prices and dividends are given by a probability measure ν on $(\mathbb{S}^j, \mathcal{B}(\mathbb{S}^j))$ defining a joint distribution of the random variables $s_1, \dots, s_j$. The measure ν is supported on some compact set $\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \dots \times \bar{\mathbb{S}}_j \in \mathcal{B}(\mathbb{S}^j)$ with each $\bar{\mathbb{S}}_n$, $n = 1, \dots, j$, being compact.

The behavior implied by Assumption 1 suggests that the consumer perceives future asset prices and dividends to be the primary source of uncertainty while his non-capital income and future bond returns can be relatively precisely predicted. It will be shown later that this assumption is actually consistent with the behavior of the model.

To derive the consumer's decision problem the following notion of a strategy is adopted. For ease of notation we define $\hat{R}_0 := R$ and write $s_1^n := (s_1, \dots, s_n) \in \mathbb{S}^n$, $n \geq 1$.

Definition 1

(i) A strategy (C, Z) consists of a decision $(c_0, z_0) \in \mathbb{C} \times \mathbb{Z}$ and a list of $\mathcal{B}(\mathbb{S}^n) - \mathcal{B}(\mathbb{C} \times \mathbb{Z})$ measurable functions $(c_n, z_n) : \mathbb{S}^n \longrightarrow \mathbb{C} \times \mathbb{Z}$, $z_n = (y_n, x_n)$, $n = 1, \dots, j$.

(ii) Given the current bond return $R > 0$, prices $p > 0$ and initial wealth w defined by (5) as well as expectations $\hat{e}$ and $\hat{R}$ at time $t = 0$, a strategy (C, Z) is called feasible if

    (a) $c_0 + y_0 + x_0\, p = w$

    (b) for each $n = 1, \dots, j$ and all $s_1^n \in \mathbb{S}^n$

$$c_n(s_1^n) + y_n(s_1^n) + x_n(s_1^n)\, p_n = \hat{e}_n + \hat{R}_{n-1}\, y_{n-1}(s_1^{n-1}) + x_{n-1}(s_1^{n-1})\, (p_n + d_n).$$

(iii) The set of feasible strategies is denoted by $\mathscr{B}(R, p, w; \hat{e}, \hat{R})$.

A strategy thus specifies current consumption c_0 and investment $z_0 = (y_0, x_0)$ and mutually consistent plans for consumption and investment in future periods $n = 1, \dots, j$

within the consumer's remaining lifetime. These plans are made conditional on the random variables $s_1, \ldots, s_n$ observed up to time n.[1] Since the consumer's planning horizon ends in period j, no portfolio is carried over to period $j+1$ such that $z_j \equiv 0$. It can be shown (cf. Hillebrand & Wenzelburger (2007, Lemma 1)) that $\mathscr{B}(R, p, w; \hat{e}, \hat{R})$ is non-empty, if and only if $w \geq -\hat{\epsilon}_0/R$ where

$$\hat{\epsilon}_0 := \hat{e}_1 + \frac{\hat{e}_2}{\hat{R}_1} + \ldots + \frac{\hat{e}_j}{\hat{R}_1 \cdots \hat{R}_{j-1}} \geq 0 \tag{6}$$

denotes the discounted non-capital income stream derived from expectations $\hat{e}$ and $\hat{R}$.[2] Next the consumer's preferences over alternative consumption strategies are specified.

Assumption 2
Given the planning horizon $j > 0$, the consumer's preferences over consumption within his remaining lifetime can be represented by the utility function

$$(c_0, c_1, \ldots, c_j) \longmapsto \ln(c_0) + \sum_{n=1}^{j} \beta^n \ln(c_n), \quad \beta > 0.$$

For each $(C, Z) \in \mathscr{B}(R, p, w, \hat{e}, \hat{R})$ define expected utility over the remaining lifetime as

$$\mathbb{E}_\nu \left[U_0(C, \cdot) \right] = \ln(c_0) + \int_{\mathbb{S}^j} \sum_{n=1}^{j} \beta^n \ln(c_n(s_1^n)) \, \nu(ds_1^j). \tag{7}$$

The consumer's objective is to choose a strategy $(C^\star, Z^\star) \in \mathscr{B}(R, p, w; \hat{e}, \hat{R})$ which maximizes the expected utility (7). His decision problem at time $t = 0$ reads

$$\max_{(C,Z)} \left\{ \mathbb{E}_\nu \left[U_0(C, \cdot) \right] \middle| (C, Z) \in \mathscr{B}(R, p, w, \hat{e}, \hat{R}) \right\}. \tag{8}$$

The existence of a solution $(C^\star, Z^\star) \in \mathscr{B}(R, p, w; \hat{e}, \hat{R})$ to (8) for general expectations and preferences is studied in Hillebrand & Wenzelburger (2007). Associated with such a solution is an optimal decision $(c_0^\star, z_0^\star) \in \mathbb{C} \times \mathbb{Z}$ for $t = 0$ which determines the consumer's demand behavior in the decision period. Proposition 1 below establishes the existence and functional form of demand functions which determine the optimal consumption-investment decision for alternative asset prices (R, p) and wealth w determined by (5). In this regard, note that the distribution ν from Assumption 1 induces a distribution ν_q of the random variable $q := p_1 + d_1$ defining the cum-dividend price of the following period. Proposition 1 below shows that the derived distribution ν_q is sufficient to characterize the consumer's demand behavior in the decision period. By virtue of Assumption 1

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[1] The literature often defines a strategy as an adapted stochastic process defined on a probability space representing uncertain future states of the world. While the definition given here is equivalent from a mathematical point of view, the uncertainty here rests on future asset prices and dividends rather than states of the world. This formulation appears more suitable from an economic point of view since prices and dividends are the relevant quantities which are directly observable.

[2] It is argued in Hillebrand & Wenzelburger (2007) that if expectations for future non-capital income and bond returns are sufficiently precise this requirement is automatically satisfied in each period of the life cycle due to the consumer's credit taking behavior as derived in Proposition 1 below.

the support of ν_q will be a compact subset $\bar{\mathbb{Q}} \subset \mathbb{R}_{++}$. The next assumption restricts expectations to be from the class of elliptical distributions which play a major role in portfolio theory (cf. Chamberlain (1983)). Here the equivalence relation $\stackrel{d}{=}$ indicates that two random variables have the same distribution.

Assumption 3
The subjective distribution ν_q of the random variable q is taken from a fixed class of elliptical distributions parameterized in $(\mu, \sigma) \in \mathbb{R}_{++}^2$. The class is generated by a random variable ε with non-degenerate ($\mathbb{E}_{\nu_\varepsilon}[\varepsilon^2] > 0$) and symmetric ($\varepsilon \stackrel{d}{=} -\varepsilon$) distribution ν_ε supported on the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$ such that q has the stochastic representation

$$q \stackrel{d}{=} \mu + \sigma\varepsilon$$

and $\nu_q = \nu_{\mu,\sigma} := \nu_\varepsilon(\frac{\bullet - \mu}{\sigma})$ is given by the corresponding image measure.

In the sequel we identify ν_q with the parameters (μ, σ) which define the mean and the dispersion of the distribution and which we call the consumer's beliefs about q. Note that the assumption $\mu > \sigma\,\bar{\varepsilon}$ is sufficient to ensure that ν_q has strictly positive support. Beliefs satisfying this requirement will be called feasible such that $\mathbb{B} := \{(\mu, \sigma) \in \mathbb{R}_{++}^2 \,|\, \mu > \sigma\,\bar{\varepsilon}\}$ defines the set of feasible beliefs. The following proposition is a version of the results obtained by Hakansson (1969). A proof for the present case may be found in Hillebrand & Wenzelburger (2007).

Proposition 1
Let $j > 0$ and expectations $\hat{e} = (\hat{e}_1, \ldots, \hat{e}_j) \in \mathbb{R}_+^j$ and $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_{j-1}) \in \mathbb{R}_{++}^{j-1}$ be given and beliefs $(\mu, \sigma) \in \mathbb{B}$ be feasible. Moreover, let Assumptions $1 - 3$ be satisfied and define $\hat{\epsilon}_0 \geq 0$ as in (6). Then for all $(R, p) \gg 0$ and $w > -\hat{\epsilon}_0/R$ the consumer's consumption and investment in stocks and bonds can be described by the functions

$$
\begin{aligned}
\varphi_c^{(j)}(R, p, w; \mu, \sigma, \hat{e}, \hat{R}) &= \bar{c}^{(j)}(w + \hat{\epsilon}_0/R) \\
\varphi_x^{(j)}(R, p, w; \mu, \sigma, \hat{e}, \hat{R}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_0/R)\,\theta(Rp; \mu, \sigma)/p \qquad (9) \\
\varphi_y^{(j)}(R, p, w; \mu, \sigma, \hat{e}, \hat{R}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_0/R)(1 - \theta(Rp; \mu, \sigma)) - \hat{\epsilon}_0/R.
\end{aligned}
$$

Here $\bar{c}^{(j)} := [1 + \beta + \ldots + \beta^j]^{-1}$ and the share of (lifetime) income invested in shares is determined by the map $\theta : \mathbb{R}_{++} \times \mathbb{B} \longrightarrow [0, 1]$

$$\theta(\pi; \mu, \sigma) := \arg\max_\vartheta \left\{ \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \ln(\pi + \vartheta\,(\mu + \sigma\varepsilon - \pi))\,\nu_\varepsilon(d\varepsilon) \mid \vartheta \in [0, 1] \right\}. \qquad (10)$$

Since $c^{(0)} = 1$ the demand functions (9) also describe the behavior of the old generation $j = 0$ whose members only consume and do not invest.

3   Demand behavior of the firm

Next consider the production and investment behavior of the firm. In each period $t \in \mathbb{N}_0$ the firm employs labor $L_t \geq 0$ and uses its capital stock $K_t \geq 0$ to produce

the consumption good. In addition, the production process at time t is subjected to a random shock η_t. More specifically, suppose that the production technology can be described by the production function[3] $F : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, \eta_{max}] \longrightarrow \mathbb{R}_+$

$$F(L_t, K_t, \eta_t) = \kappa\, L_t^\alpha K_t^{1-\alpha} + \eta_t, \quad \kappa > 0, \quad \alpha \in\,]0,1[. \tag{11}$$

The following assumption specifies the stochastic nature of the production shock in (11).

Assumption 4
The process $\{\eta_t\}_t$ consists of i.i.d. random variables. Each η_t has symmetric distribution ν_η supported on $[0, \eta_{max}]$. The expected value $\bar\eta := \mathbb{E}_{\nu_\eta}\left[\eta_t\right] = \frac{\eta_{max}}{2}$ is known to the firm.

The assumption of independent production shocks is made mainly for convenience and could be modified easily. Symmetry of the distribution ν_η will be required in Section 6 for consistency with Assumption 3.

At time t the firm takes its current capital stock $K_t > 0$ as given and decides about labor input $L_t \geq 0$ and investment $I_t \geq 0$. Assuming that capital depreciates at constant rate $\delta \in\,]0,1]$, any investment I_t made at time t determines next period's capital stock as

$$K_{t+1} = I_t + (1 - \delta)K_t. \tag{12}$$

To extend its capital stock the firm can transform consumption goods into capital goods. As in Abel (2003), suppose that given $K_t > 0$, the amount of consumption goods needed to produce $I_t \geq 0$ units of new capital is determined by the adjustment cost function $G : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_+$,

$$G(I, K) := K\, g(I/K). \tag{13}$$

The function G may be viewed as an input requirement function for a capital adjustment technology. Its properties are mainly determined by the function g which depends on the investment ratio $i := I/K$. The sequel assumes that g is of the form

$$g(i) = \begin{cases} 0 & i = 0 \\ \gamma_0 \exp\{\gamma_1\, i\} & i > 0 \end{cases}, \quad \gamma_0 > 0, \gamma_1 > \frac{1}{\delta}. \tag{14}$$

The function g has a discontinuity at zero implying that there are some fixed costs to investment which have to be incurred whenever the investment is positive as argued by Rotschild (1971). The restriction $\gamma_1 > \frac{1}{\delta}$ will be explained below.

Assume that the firm's investment decision I_t at time t is exclusively financed by issuing bonds $B_t \geq 0$ inducing the obligation to pay $R_t B_t$ units of output/consumption good at time $t + 1$. Recalling that the bond price is normalized to unity, one immediately finds that investment decision and bond supply at time t are related as

$$B_t = G(I_t, K_t) = K_t\, g(I_t/K_t). \tag{15}$$

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[3] The choice of a Cobb-Douglas production function with additive noise is made mainly for simplicity. While this structure will be convenient for the subsequent derivations, it is straightforward to replace the Cobb-Douglas form by more general, e.g., CES technologies. Likewise it should be possible to relax the specification with additive noise.

After paying for labor and the bond debt incurred in the previous period, the firm distributes all excess output to its shareholders. Letting as before $\omega_t > 0$ denote the real wage in period t and $\bar{x} > 0$ the constant number of shares in the market, the dividend (per share) at time t is given by

$$d_t = \frac{F(L_t, K_t, \eta_t) - \omega_t L_t - R_{t-1}B_{t-1}}{\bar{x}}. \tag{16}$$

To derive the firm's labor demand and investment behavior, consider an arbitrary period $t = 0$. Let the current capital stock $K_0 > 0$, the production shock $\eta_0 \in [0, \eta_{max}]$ as well as the bond debt $R_{-1}B_{-1} \geq 0$ resulting from the previous investment decision be given. The current real wage and the current bond return enter the decision problem as parameters $\omega > 0$ and $R > 0$, respectively. Suppose that the firm seeks to act in favor of its shareholders by maximizing dividend payments (a related setting has been adopted in Magill & Quinzii 2003). Given this objective, labor demand and investment may be determined separately. Suppose first that the firm chooses labor input at time $t = 0$ to maximize the current dividend payment. This implies that current labor demand is a solution to the optimization problem

$$\max_{L \geq 0} \left\{ \frac{F(L, K_0, \eta_0) - \omega L - R_{-1}B_{-1}}{\bar{x}} \right\}. \tag{17}$$

Second, suppose that investment at time $t = 0$ is chosen to maximize the expected dividend payment of the following period subject to the constraint that this payment is non-negative for any possible realization of next period's production shock. For this purpose, the firm holds expectations $\hat{\omega}_1 > 0$ for the real wage prevailing in the following period $t = 1$. The decision involves an investment decision I made at time $t = 0$ and planned labor demand L_1 for the following period. Using equations (12), (13), (15), and (16) the corresponding maximization problem reads

$$\max_{I \geq 0, L_1 \geq 0} \left\{ \frac{F(L_1, I + (1-\delta)K_0), \bar{\eta}) - \hat{\omega}_1 L_1 - RK_0 g(I/K_0)}{\bar{x}} \right. \tag{18}$$

$$\left. \text{s.t. } F(L_1, I + (1-\delta)K_0, 0) - \hat{\omega}_1 L_1 - RK_0 \, g(I/K_0) \geq 0 \right\}.$$

The following proposition characterizes the firm's demand behavior derived from (17) and (18). The proof may be found in the appendix.

Proposition 2
Let the firm's technologies be given by equations (11), (13) and (14) and let Assumption 4 be satisfied. Then given the values $R_{-1}B_{-1} \geq 0$, $\eta_0 \in [0, \eta_{max}]$ and $K_0 > 0$ the firm's demand behavior is as follows:

(i) For each $\omega > 0$ labor demand is determined by the function

$$L(\omega; K_0) := \left(\frac{\alpha \kappa}{\omega} \right)^{\frac{1}{1-\alpha}} K_0. \tag{19}$$

(ii) Given $\hat{\omega}_1 > 0$, positive investment requires that $R \leq \bar{\bar{R}}(\hat{\omega}_1) := \frac{(1-\alpha)\kappa}{e\,\gamma_0\,\gamma_1}\left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{\alpha}{1-\alpha}}$. In this case, an optimal investment decision is determined by the function

$$I(R;\hat{\omega}_1, K_0) \;=\; \frac{1}{\gamma_1}\ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\,\gamma_1\,R}\left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{\alpha}{1-\alpha}}\right)K_0. \tag{20}$$

Proposition 2 (ii) restricts attention to interior investment solutions by imposing an upper bound $\bar{\bar{R}}(\hat{\omega}_1)$ on the current bond return. Note that this does not imply that the firm necessarily increases its capital stock. In fact, we have in (20)

$$\lim_{R \nearrow \bar{\bar{R}}(\hat{\omega}_1)} \frac{1}{\gamma_1}\ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\,\gamma_1\,R}\left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{\alpha}{1-\alpha}}\right) = \frac{1}{\gamma_1} < \delta$$

due to our restriction $\gamma_1 > \frac{1}{\delta}$. Hence, for R sufficiently close to the boundary $\bar{\bar{R}}(\hat{\omega}_1)$, the firm will disinvest in the sense that the investment will not be sufficient to compensate depreciation and the capital stock will be smaller in the next period. Combining (15) and (20) the firm's bond supply may be written as a function

$$B(R;\hat{\omega}_1, K_0) := \frac{(1-\alpha)\kappa}{\gamma_1\,R}\left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{\alpha}{1-\alpha}} K_0. \tag{21}$$

The bond supply function satisfies $B(R;\hat{\omega}_1, K_0) = R^{-1}B(1;\hat{\omega}_1, K_0)$. This homogeneity property will become crucial in the following section to determine the equilibrium bond return. Its validity is exclusively due to the specification of the adjustment cost function (14) which is therefore indispensable to obtain the subsequent results.

4 Price formation and sequential structure

Based on the demand behavior of consumers and the firm the following part describes their interactions on real and financial markets and the sequential structure of the model in each period. It is shown that market-clearing prices are well-defined and can even be determined explicitly. For the following derivations let $t \in \mathbb{N}_0$ be arbitrary and the population vector N_{t-1}, the capital stock K_{t-1} as well as the firm's investment decision I_{t-1} from the previous period be given. Furthermore, let the initial distribution of shares and bonds be defined by the asset allocation $z_{t-1} := (x_{t-1}^{(j)}, y_{t-1}^{(j)})_{j=1}^{J}$ and B_{t-1} which is given together with the previous bond return $R_{t-1} > 0$. Suppose that initially all shares are distributed among (non-young) consumers such that $\sum_{j=1}^{J} N_{t-1}^{(j)} x_{t-1}^{(j)} = \bar{x}$ and that the previous bond allocation satisfies $\sum_{j=1}^{J} N_{t-1}^{(j)} y_{t-1}^{(j)} - B_{t-1} = 0$. Given these quantities, the following five steps describe the sequential structure of the model in period t.

Step 1: Population and labor force, capital stock. Given N_{t-1} the population vector $N_t = (N_t^{(j)})_{j=0}^{J}$ is determined at the beginning of the period according to the population law (1). Given the labor supply $\bar{L}^{(j)}$ of each consumer in generation $j \in \{j_L, \ldots, J\}$

this determines aggregate labor supply $L_t^S > 0$ according to (2). Moreover, the previous investment I_{t-1} together with K_{t-1} determines the capital stock K_t according to (12).

Step 2: Real wage, non-capital income, dividend payment. Utilizing the labor demand function (19) of the firm, suppose that the real wage ω_t at time t is determined such that market clearing on the labor market obtains, i.e. $L(\omega_t; K_t) = L_t^S$. Exploiting the functional form (19) this condition may be solved explicitly for ω_t to obtain

$$\omega_t = \mathcal{W}(L_t^S, K_t) := \kappa\alpha \left(\frac{K_t}{L_t^S} \right)^{1-\alpha}. \tag{22}$$

Given some contribution rate $\tau_t \in [0, 1[$ determined by the public pension system the equilibrium real wage ω_t defines the non-capital income distribution $e_t = (e_t^{(j)})_{j=0}^J$ among consumers according to (3) and (4). After the realization of the production shock $\eta_t \in [0, \eta_{max}]$ the dividend payment d_t of the firm is determined by (16).

Step 3: Expectations formation. Based on the values determined in the previous steps (and observations from previous periods $t - 1, t - 2, \ldots$) each consumer in generation $j \in \{1, \ldots, J\}$ forms expectations $\hat{e}_t^{(j)} = (\hat{e}_{t,t+n}^{(j)})_{n=1}^j$ and $\hat{R}_t^{(j)} = (\hat{R}_{t,t+n})_{n=2}^j$ for his future non-capital income stream and future bond returns. Here $\hat{e}_{t,t+n}^{(j)} \geq 0$ denotes the point forecast made by a member of generation j at time t for her non-capital income at time $t + n$ and $\hat{R}_{t,t+n} > 0$ denotes the homogeneous forecast made by all generations for the bond return R_{t+n}. To obtain a compact notation let $\hat{e}_t := (\hat{e}_t^{(j)})_{j=1}^J$ and $\hat{R}_t := (\hat{R}_t^{(j)})_{j=1}^J$. Likewise consumers determine their subjective beliefs $(\mu_t, \sigma_t) \in \mathbb{B}$ about the distribution of next period's cum-dividend price $q_{t+1} := p_{t+1} + d_{t+1}$ and the firm makes a forecast $\hat{\omega}_{t,t+1} > 0$ for next period's real wage. Based on these expectations consumers determine their asset demand (9) and the firm determines its bond supply (21).

Step 4: Asset market clearing, wealth, asset holdings. Next the bond return R_t and asset prices p_t are determined simultaneously from the demand behavior of consumers and the firm such that the aggregated excess demand function for stocks and bonds is equal to zero. Using that $\sum_{j=1}^J N_{t-1}^{(j)} x_{t-1}^{(j)} = \sum_{j=0}^{J-1} N_t^{(j)} x_{t-1}^{(j+1)} = \bar{x}$ the market clearing conditions for the stock market and the bond market read

$$\sum_{j=1}^J N_t^{(j)} \varphi_x^{(j)} \left(R_t, p_t, w_t^{(j)}; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)} \right) = \bar{x} \tag{23}$$

$$\sum_{j=1}^J N_t^{(j)} \varphi_y^{(j)} \left(R_t, p_t, w_t^{(j)}; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)} \right) = B(R_t; \hat{\omega}_{t,t+1}, K_t) \tag{24}$$

where $w_t^{(j)}$ is determined by (5) (and thus also depends on p_t). Utilizing the demand functions (9) and (21) together with the definition of wealth (5) and defining the values

$$\hat{\epsilon}_t^{(j)} := \hat{e}_{t,t+1}^{(j)} + \frac{\hat{e}_{t,t+2}^{(j)}}{\hat{R}_{t,t+1}} + \ldots + \frac{\hat{e}_{t,t+j}^{(j)}}{\hat{R}_{t,t+1} \cdots \hat{R}_{t,t+j-1}}, \quad j = 1, \ldots, J \tag{25}$$

$$\hat{m}_t := \frac{1}{\bar{x}} \left[\sum_{j=1}^J N_t^{(j)} \hat{\epsilon}_t^{(j)} + B(1; \hat{\omega}_{t,t+1}, K_t) \right] \tag{26}$$

we show in Appendix A.2 that the conditions (23) and (24) possess a unique solution (R_t, p_t). These values are determined by the temporary equilibrium mappings[4]

$$R_t = \mathcal{R}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \tag{27}$$

$$:= \frac{B(1; \hat{\omega}_{t,t+1}, K_t) + \sum_{j=1}^{J} N_t^{(j)} \bar{c}^{(j)} \hat{e}_t^{(j)} + \pi(\mu_t, \sigma_t, \hat{m}_t) \sum_{j=0}^{J-1} N_t^{(j)} \bar{c}^{(j)} x_{t-1}^{(j+1)}}{\sum_{j=1}^{J} N_t^{(j)} (1 - \bar{c}^{(j)}) e_t^{(j)} + \sum_{j=1}^{J-1} N_t^{(j)} (1 - \bar{c}^{(j)}) \left[R_{t-1} y_{t-1}^{(j+1)} + d_t \, x_{t-1}^{(j+1)} \right]}$$

$$p_t = \mathcal{P}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \tag{28}$$

$$:= \frac{\pi(\mu_t, \sigma_t, \hat{m}_t)}{\mathcal{R}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1})}.$$

Here $\pi : \mathbb{B} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a C^1 map which is defined in equation (54) in Appendix A.2. The equilibrium asset prices determine the equilibrium wealth levels $w_t^{(j)}$, $j = 0, 1, \ldots, J$ according to (5) and define a new asset allocation as

$$x_t^{(j)} = \varphi_x^{(j)} \left(R_t, p_t, w_t^{(j)}; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)} \right), \quad j = 1, \ldots, J$$

$$y_t^{(j)} = \varphi_y^{(j)} \left(R_t, p_t, w_t^{(j)}; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)} \right), \quad j = 1, \ldots, J$$

$$B_t = B(R_t; \hat{\omega}_{t,t+1}, K_t).$$

Step 5: Consumption, capital formation. In the final step, consumers realize their consumption decision and the firm uses the consumption goods collected from its bond supply to adjust the capital stock. Given the consumption functions from (9) and the investment function (20), the final step is described by the equations

$$c_t^{(j)} = \varphi_c^{(j)} \left(R_t, p_t, w_t^{(j)}; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)} \right), \quad j = 0, 1, \ldots, J$$

$$I_t = I(R_t; \hat{\omega}_{t,t+1}, K_t).$$

The previous steps and (5), (9) and (16) imply $\sum_{j=0}^{J} N_t^{(j)} c_t^{(j)} + B_t = F(L_t^S, K_t, \eta_t)$. Hence, the model is indeed closed in the sense that the commodity market also clears.

5 Population dynamics and expectations formation

The previous section has derived the temporary structure of the economy in an arbitrary period t taking the population structure as given. To complete our demographic model

---

[4] Consistency between the solution R_t defined by (27) and the investment function (20) requires that $R_t < \bar{\bar{R}}(\hat{\omega}_{t,t+1})$. While it can be shown that this condition holds automatically at any rational expectations equilibrium studied in Section 6 due to the restriction $\delta > \gamma_1^{-1}$, it seems difficult to derive explicit conditions for the general multi-period case due to the complexity of the map $\mathcal{R}$. However, the numerical simulations presented in Sections 7 and 8 show that there exists a robust set of parameters such that the condition $R_t < \bar{\bar{R}}(\hat{\omega}_{t,t+1})$ is indeed satisfied for all times t.

we assume that the map $\mathcal{N}$ in (1) is of the following form

$$\mathcal{N}(N_{t-1}) = \sum_{j=0}^{J} N_{t-1}^{(j)} \, n^{(j)} \left(1 + \exp\left\{ -n_2 \sum_{i=0}^{J} N_{t-1}^{(i)} \right\} \right) \qquad (29)$$

where $n^{(j)} \geq 0$, $j = 0, \ldots, J$ and $n_2 \geq 0$. If $n_2 = 0$, the population law (29) is linear with constant fertilities of each generation $j = 0, \ldots, J$. If $n_2 > 0$, fertility is a decreasing function of the previous population size $\sum_{i=0}^{J} N_{t-1}^{(i)}$. If $J = 0$ (corresponding to the degenerate case where consumers live for one period only), equation (29) reduces to the well-known Ricker-type map which is widely-used in biological sciences to model the evolution of (human and non-human) populations, see, e.g., Henson et al. (1999). Apart from this justification the functional form (29) turns out to be convenient to model demographic transitions of the population due to its dynamic properties. None of the results derived in the sequel hinges on the particular functional form in (29).
Let $\hat{\mathcal{N}} : \mathbb{R}_+^{J+1} \longrightarrow \mathbb{R}_+^{J+1}$, $(N^{(j)})_{j=0}^{J} \longmapsto \hat{\mathcal{N}}((N^{(j)})_{j=0}^{J}) := ((N^{(j)})_{j=1}^{J}, \mathcal{N}((N^{(j)})_{j=0}^{J}))$ denote the corresponding time-one map of the population dynamics defined by equations (1) and (29), i.e., $N_t = \hat{\mathcal{N}}(N_{t-1})$ for each $t > 0$. Some immediate properties of the population dynamics are stated in the following lemma. The proof is straightforward.

Lemma 1
Suppose that the parameters in (29) satisfy $n_2 > 0$ and $n^{(j)} \geq 0$, $j = 0, \ldots, J$. Define $n := \sum_{j=0}^{J} n^{(j)}$ and assume, in addition, that $\frac{1}{2} < n < 1$. Then the following holds true:

(i) *The population dynamics defined by equations (1) and (29) possess a unique positive steady state $N^\star = (\bar{N})_{j=0}^{J}$ where $\bar{N} := \frac{1}{(J+1)\,n_2} \ln \frac{n}{1-n} > 0$.*

(ii) *The steady state in (i) is asymptotically stable if all eigenvalues of the Jacobian matrix $D\hat{\mathcal{N}}(N^\star)$ lie inside the complex unit disc. This matrix is of the form*

$$D\hat{\mathcal{N}}(N^\star) \;=\; \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ \delta^{(0)} & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(J)} \end{bmatrix}, \; \delta^{(j)} := \frac{n^{(j)}}{n} - \frac{1-n}{J+1} \ln \frac{n}{1-n}.$$

Observe that the parameter n_2 is crucial to determine the steady state value $\bar{N}$ in (i) but does not affect its asymptotic stability in (ii). Hence, given suitable parameter choices $n^{(j)}$, $j = 0, 1, \ldots, J$, *any* value $\bar{N} > 0$ can be induced as a stable steady state of the population dynamics by appropriately choosing a value for n_2. In the sequel this property will allow us to study the case with a stationary population where $N_t^{(j)} \equiv \bar{N}$ and to consider the comparative-static effect of alternative values for $\bar{N}$. In addition, demographic transition phases may be modeled as shifts in the parameter n_2.
To complete the description of the model we are left to specify the forecasting behavior of consumers and the firm. Recall that the expectations formation at time t involves

consumers' point forecasts $\hat{e}_t^{(j)} = (\hat{e}_{t,t+n}^{(j)})_{n=1}^j$ and $\hat{R}_t^{(j)} = (\hat{R}_{t,t+n})_{n=1}^{j-1}$, $j = 1, \ldots, J$ for non-capital income and bond returns, beliefs $(\mu_t, \sigma_t) \in \mathbb{B}$ about next period's cum-dividend price q_{t+1} as well as the firm's real wage prediction $\hat{\omega}_{t,t+1}$. In accordance with the sequential structure introduced in the previous section, the information set upon which expectations at time t are based contains the current non-capital income distribution e_t and the dividend payment d_t but not the current price p_t and the bond return R_t.

To obtain a first characterization of the dynamic behavior of the model the present paper refrains from the assumption of fully rational expectations but instead assumes that agents form expectations according to simple prediction rules. Integrating more sophisticated rules into the model is straightforward. To this end, suppose that the firm's wage prediction satisfies

$$\hat{\omega}_{t,t+1} = \omega_t \tag{30}$$

for all times $t \in \mathbb{N}_0$. Consumers in generation $j \in \{1, \ldots, J\}$ derive their non-capital income expectation $\hat{e}_{t,t+n}^{(j)}$ for period $t+n$ from the current income of generation $j-n$ (corresponding to their age at $t+n$) such that for each $t \in \mathbb{N}_0$

$$\hat{e}_{t,t+n}^{(j)} = e_t^{(j-n)}, \quad n = 1, \ldots, j, \ j = 1, \ldots, J. \tag{31}$$

The prediction for future bond returns is assumed to be uniformly equal to the last observed bond return which gives for each $t \in \mathbb{N}_0$

$$\hat{R}_{t,t+n} = R_{t-1}, \quad n = 1, \ldots, J-1. \tag{32}$$

As for consumers' beliefs about cum-dividend prices assume that second moment beliefs are constant such that $\sigma_t \equiv \bar{\sigma}$ for all times t and that first moment beliefs μ_t are updated using a simple error-correction principle of the from

$$\mu_t = \mu_{t-2} + \varrho(q_{t-1} - \mu_{t-2}), \quad 0 \le \varrho \le 1. \tag{33}$$

Note that (33) includes the cases of naive ($\varrho = 1$) and of static ($\varrho = 0$) expectations. Despite their simplicity the prediction rules (30)-(33) can be shown to induce a form of near-rational behavior with relatively small forecasting errors. In the two-period case studied in the following section they provide fully rational expectations in the long run as soon as the economy has converged to a steady state. In the general stochastic case studied in Section 7 they yield unbiased, i.e., on average correct forecasts since the respective variable follows a stationary stochastic process. Also note that point forecasts are made for future bond returns and non-capital income while consumers treat next period's cum-dividend price as a random variable and attempt to predict the first two moments of its hypothesized distribution. This kind of behavior suggests that, e.g., fluctuations in real wages are negligible compared to fluctuations in asset prices. The following sections will reveal that this behavior is consistent with the model.

The framework developed so far offers a convenient possibility to study the influence of a pension system (represented by different contribution rates τ_t) on production and financial markets as well as the impact of changes in the population structure. The study of these two influences forms the core of the remaining sections of this paper.

6 Rational expectations equilibria

Based on the model developed in the previous sections we seek to study the impact of alternative contribution rates as well as of changes in the population size on the long-run state of the economy. The present section studies the simplest possible case with one working and one retired generation setting $J = 1$ and $j_L = 0$. Proofs for all results are given in the appendix. The amount of labor supplied by young consumers is normalized to unity such that $\bar{L}^{(1)} \equiv 1$ and, therefore, $L_t^S = N_t^{(1)}$ in (2). In addition, suppose that consumers do not update their first moment beliefs about cum-dividend prices such that $\varrho = 0$ in (33) and $\mu_t \equiv \bar{\mu}$. Finally, let the contribution rate to the pension system be constant over time such that $\tau_t \equiv \tau$. In this case, it can be shown that the state dynamics of the model reduce to the following three building blocks:

(a) The exogenous *population dynamics.* Using (1) and (29) these take the form

$$N_t = \hat{\mathcal{N}}(N_{t-1}) := (N_{t-1}^{(1)}, \mathcal{N}(N_{t-1})). \tag{34}$$

(b) The *capital dynamics.* Employing equations (12) and (20) together with (2), (22) and (30) the evolution of capital may be written in the form

$$
\begin{aligned}
K_t = \hat{\mathcal{K}}(N_{t-1}, K_{t-1}, R_{t-1}) &:= I(R_{t-1}; \mathcal{W}(N_{t-1}^{(1)}, K_{t-1}), K_{t-1}) + (1-\delta)K_{t-1} \\
&= \left[\frac{1}{\gamma_1} \ln \left(\frac{(1-\alpha)\kappa}{\gamma_0 \gamma_1 R_{t-1}} \left(\frac{N_{t-1}^{(1)}}{K_{t-1}} \right)^\alpha \right) + 1 - \delta \right] K_{t-1}. \tag{35}
\end{aligned}
$$

(c) The *bond return dynamics.* First note that equations (2), (4), (21), (22) and (31) imply that in the present case the value $\hat{m}_t$ defined in (26) may be written as

$$\hat{m}_t = \hat{m}_\tau(N_t, K_t) := \frac{\kappa N_t^{(1)\,\alpha} K_t^{1-\alpha}}{\bar{x}} \left[\alpha\,\tau\,\frac{N_t^{(1)}}{N_t^{(0)}} + \frac{1-\alpha}{\gamma_1} \right]. \tag{36}$$

Combining (36) with (2), (22) and (31) and abusing notation, (27) becomes

$$R_t = \mathcal{R}_\tau(N_t, K_t; \bar{\mu}) := \frac{\hat{m}_\tau(N_t, K_t) + \pi(\bar{\mu}, \bar{\sigma}, \hat{m}_\tau(N_t, K_t))}{\alpha\kappa(1-\bar{c}^{(1)})(1-\tau)N_t^{(1)\,\alpha} K_t^{1-\alpha}/\bar{x}} - \frac{\tau}{1-\tau}\frac{N_t^{(1)}}{N_t^{(0)}}. \tag{37}$$

Using (34) and (35) in (37) imply that the dynamics of bond returns take the form

$$R_t = \hat{\mathcal{R}}_\tau(N_{t-1}, K_{t-1}, R_{t-1}; \bar{\mu}) := \mathcal{R}_\tau(\hat{\mathcal{N}}(N_{t-1}), \hat{\mathcal{K}}(K_{t-1}, R_{t-1}, N_{t-1}); \bar{\mu}). \tag{38}$$

Equations (34), (35) and (38) constitute a four-dimensional deterministic dynamical system where the exogenous population dynamics completely decouple from the other variables. Hence, in view of Lemma 1, to study the long-run dynamic behavior of the model one may restrict attention to the two-dimensional dynamics defined by (35) and (38) assuming that the population is constant, i.e., $N_t^{(j)} \equiv \bar{N}$, $j = 0, 1$. Letting

$$\lambda_0 := \frac{\alpha\,\kappa\bar{N}^\alpha}{\bar{x}}\left[\frac{1-\alpha}{\alpha\,\gamma_1} + \tau\right] > 0 \quad \text{and} \quad \lambda_1 := \frac{1+\beta}{\beta}\frac{1}{1-\tau}\left[\frac{1-\alpha}{\alpha\,\gamma_1} + \tau\right] > \frac{\tau}{1-\tau} \tag{39}$$

one has from (36) $\hat{m}_\tau(\bar{N}, K_t) = \lambda_0 K_t^{1-\alpha}$. Using this in (35) and (38), the dynamics may - by a slight abuse of notation - be written as the following two-dimensional system

$$
\begin{aligned}
K_t &= \hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N}) &:= \left[\frac{1}{\gamma_1} \ln \left(\frac{(1-\alpha)\kappa}{\gamma_0 \gamma_1 R_{t-1}} \left(\frac{\bar{N}}{K_{t-1}} \right)^\alpha \right) + 1 - \delta \right] K_{t-1} \\
R_t &= \hat{\mathcal{R}}_\tau(K_{t-1}, R_{t-1}; \bar{N}, \bar{\mu}) &:= \frac{\lambda_1 \pi(\bar{\mu}, \bar{\sigma}, \lambda_0 \hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N})^{1-\alpha})}{\lambda_0 \hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N})^{1-\alpha}} + \lambda_1 - \frac{\tau}{1-\tau}.
\end{aligned}
$$

(40)

Now suppose that the dynamical system (40) converges to a steady state $(\bar{K}, \bar{R}) \gg 0$. Let $\tilde{\eta}_t := \eta_t - \bar{\eta}$, $t \geq 0$ denote the centered noise process derived from Assumption 4 with induced distribution $\nu_{\tilde{\eta}}$. Then equations (16) and (30) imply a stationary dividend process $\{\bar{d}_t\}_{t \geq 0}$ along the steady state with

$$
\bar{d}_t := \frac{F(\bar{N}, \bar{K}, \eta_t) - \mathcal{W}(\bar{N}, \bar{K}) \bar{N} - B(1, \mathcal{W}(\bar{N}, \bar{K}), \bar{K})}{\bar{x}} = \frac{\bar{D} + \tilde{\eta}_t}{\bar{x}}
$$

(41)

where, using equations (11), (21) and (22) and recalling that $\mathbb{E}_{\nu_\eta}[\eta_t] = \bar{\eta}$

$$
\bar{D} := \kappa \bar{N}^\alpha \bar{K}^{1-\alpha}(1-\alpha)(1 - \frac{1}{\gamma_1}) + \bar{\eta} > 0.
$$

(42)

Letting as before $q_t := p_t + d_t$ for each t, one obtains the cum-dividend price process $\{\bar{q}_t\}_{t \geq 0}$ along the steady state which, using the price law (28) and (41) takes the form

$$
\bar{q}_t := \frac{\pi(\bar{\mu}, \bar{\sigma}, \hat{m}(\bar{K}, \bar{N}))}{\bar{R}} + \frac{\bar{D} + \tilde{\eta}_t}{\bar{x}}.
$$

(43)

Observe that in contrast to bond returns and the capital stock, dividends and cum-dividend prices are random variables along the steady state. It is clear from (43) that the process $\{\bar{q}_t\}_{t \geq 0}$ inherits the properties of the (centered) noise process $\{\tilde{\eta}_t\}_{t \geq 0}$ and is thus given by an i.i.d. process with time-invariant distribution $\nu_{\bar{q}}$ corresponding to the image measure of $\nu_{\tilde{\eta}}$ under the affine map (43). Note further that due to Assumption 4 the measure $\nu_{\tilde{\eta}}$ defines a symmetric probability distribution supported on the compact interval $[-\bar{\eta}, \bar{\eta}]$. On the other hand, Assumption 3 implies that the perceived distribution ν_q of q is constant over time and satisfies $q \stackrel{d}{=} \bar{\mu} + \bar{\sigma}\varepsilon$ for some random variable ε with symmetric distribution ν_ε supported on $[-\bar{\varepsilon}, \bar{\varepsilon}]$. This raises the question, under what conditions the *objective distribution* $\nu_{\bar{q}}$ along the steady state coincides with the *perceived distribution* ν_q. In the present case, this can be achieved by setting $\nu_\varepsilon = \nu_{\tilde{\eta}}$ and $\bar{\sigma} = \frac{1}{\bar{x}}$ and determining $\bar{\mu}$ according to the implicit condition $\bar{\mu} = \mathbb{E}_{\nu_{\bar{q}}}[q]$. As an immediate implication, note that in this case $\bar{\sigma}\bar{\varepsilon} = \bar{\eta}/\bar{x}$. The following definition of a rational expectations equilibrium is now straightforward.

Definition 2
Let $\bar{N} > 0$ and $\tau \in [0, 1[$ be given and let Assumptions 3 and 4 be satisfied. In addition, suppose $\nu_\varepsilon = \nu_{\tilde{\eta}}$ and $\bar{\sigma} = \frac{1}{\bar{x}}$. A rational expectations equilibrium (REE) of the system (40) is a triple $(\bar{K}, \bar{R}, \bar{\mu}) \gg 0$ such that

(i) Given $\bar{\mu} > 0$ the pair $(\bar{K}, \bar{R})$ is a steady state of (40), i.e.,

$$\begin{aligned} \bar{K} &= \hat{\mathcal{K}}(\bar{K}, \bar{R}; \bar{N}) \\ \bar{R} &= \hat{\mathcal{R}}_\tau(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}). \end{aligned}$$

(ii) Beliefs $(\bar{\mu}, \bar{\sigma})$ are feasible and satisfy the consistency condition

$$\bar{\mu} = \mathbb{E}_{\nu_{\bar{q}}}[q] = \frac{\pi(\bar{\mu}, \bar{\sigma}, \hat{m}_\tau(\bar{K}, \bar{N}))}{\bar{R}} + \frac{\kappa \bar{N}^\alpha \bar{K}^{1-\alpha}(1-\alpha)(1-\frac{1}{\gamma_1})}{\bar{x}} + \frac{\bar{\eta}}{\bar{x}}.$$

(iii) The REE is said to be stable if the steady state in (i) is asymptotically stable.

Hence, if a stable REE exists, the economy will converge to a steady state at which the capital stock and bond returns are constant while dividends and (cum-dividend) asset prices follow stationary stochastic processes as determined by (41) and (43). As a consequence of (ii) and the prediction behavior introduced in the previous section, expectations are fully rational in the long run in the sense that all variables for which point predictions are made are correctly anticipated while the subjective distribution for the cum-dividend price coincides with the objective distribution. The following theorem states a sufficient condition under which a unique stable REE exists.

Theorem 1
Let $\bar{N} > 0$ and $\tau \in [0, 1[$ be given. In addition, suppose that the production parameter in (11) satisfies $\alpha > \frac{1}{2}$. Then there exists a unique stable REE $(\bar{K}, \bar{R}, \bar{\mu}) \gg 0$ of the dynamical system (40) which satisfies $\bar{\mu} > \bar{\sigma}\, \bar{\varepsilon}$ and $\bar{R} > 1$.

If $\alpha \leq \frac{1}{2}$ it can be shown that a REE may well fail to exist or to be unique. In the sequel we shall therefore assume that the requirement $\alpha > \frac{1}{2}$ is satisfied. In this case, Theorem 1 ensures the existence of mappings $\bar{K} : \mathbb{R}_{++} \times [0, 1[\longrightarrow \mathbb{R}_{++}$, $\bar{R} : \mathbb{R}_{++} \times [0, 1[\longrightarrow]1, \infty[$ and $\bar{\mu} : \mathbb{R}_{++} \times [0, 1[\longrightarrow]\bar{\sigma}\bar{\varepsilon}, \infty[$ which define the unique REE for alternative $\bar{N}$ and τ. The next theorem describes how changes in the contribution rate and in the population size affect the REE and, thus, the long-run properties of the economy.

Theorem 2
Under the hypotheses of Theorem 1 the following holds true:

(i) For each $N > 0$ the maps $\tau \longmapsto \bar{K}(\tau, N)$ and $\tau \longmapsto \bar{\mu}(\tau, N)$ are strictly decreasing while the map $\tau \longmapsto \bar{R}(\tau, N)$ is strictly increasing.

(ii) For each $\tau \in [0, 1[$ the maps $N \longmapsto \bar{K}(\tau, N)$ and $N \longmapsto \bar{\mu}(\tau, N)$ are strictly increasing while the map $N \longmapsto \bar{R}(\tau, N)$ is strictly decreasing.

The result in (i) asserts that any reduction in τ increases the long-run level of the capital stock while it decreases the interest rate. Similar results have been obtained by Conesa & Krüger (1999). This confirms the fundamental insight by Feldstein (1974)

that a reduction in the contribution rate fosters the accumulation of capital through an increase of private savings. By (11) and (22) it follows that in the presence of a constant population any decrease in τ implies higher production output and a higher real wage in the long run. In addition, since $\bar{\mu}$ defines the expected value of cum-dividend prices along the steady state, a lower contribution rate leads on average to higher asset prices. The result in (ii) claims that a shrinking population leads to lower levels of capital and asset prices. This emergence of a so-called asset market meltdown due to a shrinking population has been predicted by various deterministic models in the literature (cf. Abel (2003, 2001)). In addition, a smaller population leads to a higher interest rate and a smaller real wage. The later argument is due to the properties of the map $N \longmapsto \frac{N}{K(\tau,N)}$ which is shown in the appendix (cf. equation (88)) to be strictly decreasing. This implies that the real wage function $\bar{\omega}(\tau, N) := \mathcal{W}(N, \bar{K}(\tau, N))$ is strictly increasing in N. Hence, the direct effect of a smaller population and an increased scarcity of labor is overcompensated by the adjustment of capital resulting in a negative net effect.

The following Sections 7 and 8 will demonstrate that qualitatively all results from Theorem 2 carry over to the general stochastic case with multi-period lived consumers. One also observes from (43) that neither N nor τ affect the volatility of the cum-dividend price process. The following section indicates that this may change in the general case.

7 Dynamics with a constant population

Returning to the general stochastic case with multi-period-lived consumers the remainder of this paper presents results from numerical simulations using a calibrated parametrization of the model.[5] The study itself is divided into two scenarios. The present section studies the influence of alternative contribution rates under the assumption of a stationary population where $N_t^{(j)} \equiv \bar{N}$. This allows to isolate those effects which are entirely due to the presence of a pension system while abstracting from changes in the population structure. Assuming that the requirements of Lemma 1 are satisfied, the case of a stationary population may be viewed as a scenario to which the population adjusts in the long run. From this perspective, the present section deals with the long-run effects of a public pension system. The second scenario which is presented in the next section studies the case with demographic change due to a shrinking population.

The OLG structure is specified as follows. The parameters defining consumers' life expectancy and retirement age are set to $J = 14$ and $j_L = 6$ such that in each period there are fifteen generations nine of which work while six are retired. Assuming as in Börsch-Supan, Heiss, Ludwig & Winter (2003) that economic life starts at the age of 20 years and ends at the age of 80 years, each consumer lives for 60 years and one time unit in our simulations corresponds to four years. While the choice of $j_L = 6$ may seem quite large, it can be shown that qualitatively the subsequent results remain intact with

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[5] All simulations are carried out using the software package MACRODYN (see Böhm (2003) for a detailed introduction to the software). Further information on the simulations may be obtained from the web-site `http://www.marten-hillebrand.de/research/560/paper_560.htm`.

$j_L = 5$ or even $j_L = 4$. Each consumer in working age supplies one unit of labor such that $\bar{L}^{(j)} = 1$ for all $j \in \{j_L, \ldots, J\}$. The choice of the discount factor β is based on an empirical study by Hurd (1989) who reports a discount rate of 0.011 corresponding to an annual discount factor of $1/1.011$. Given a time unit of four years we therefore set $\beta = (1/1.011)^4 \approx 0.96$. The perceived distribution ν_ε of ε from Assumption 3 is taken to be a standard normal distribution which is truncated to the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Setting $\bar{\varepsilon} = 0.92$ implies a perceived variance $\mathbb{V}_{\nu_q}[q] = \bar{\sigma}^2 \mathbb{V}_{\nu_\varepsilon}[\varepsilon] = \bar{\sigma}^2/4$. Since it will be shown that cum-dividend prices follow a stationary process with constant variance, we choose $\bar{\sigma}$ to match this value for the intermediate case where $\tau = 0.1$.

The firm's production and capital adjustment technologies defined in equations (11) and (14) are specified as follows. The production parameter α is set to $\alpha = 0.66$ which is justified by most empirical studies suggesting a range $\alpha \in]0.6, 0.7[$. The choice of the depreciation rate δ follows İmrohoroğlu, İmrohoroğlu & Joines (1995) who use an annual value of 8%. With our four years time unit we therefore set $\delta = 1 - (1 - 0.08)^4 \approx 0.28$. The parameters of the adjustment cost function (14) are set to $\gamma_0 = 0.02$ and $\gamma_1 = 7.5$. While the theory of adjustment costs is widely used in the literature, there are many authors who suggest that the function g is of the quadratic form $g(i) = \frac{\psi}{2} \cdot i^2$ (Abel (2003)) or $g(i) = i + \frac{\psi}{2} \cdot i^2$ (Altig, Auerbach, Kotlikoff, Smetters & Walliser (2001)) for some parameter $\psi > 0$. To justify the functional form (14), recall that the argument i represents the fraction of the capital stock that is replaced which will typically take values within the unit interval and fluctuate around the depreciation rate $\delta = 0.28$. Comparing the functional form (14) to the quadratic specifications with $\psi = 10$ (as used by Altig, Auerbach, Kotlikoff, Smetters & Walliser (2001)) and, alternatively, to $\psi = 1.5$ (corresponding to the value used by Börsch-Supan, Ludwig & Winter (2006)) shows that the functions exhibit only minor differences on the interval $i \in]0, 0.5[$. In fact, as Gould (1968, p.49) states, the primary cause for using the quadratic adjustment cost function is its technical simplicity and not its justification on empirical grounds.

Finally, each random variable η_t of the stochastic process $\{\eta_t\}_t$ is uniformly distributed on the interval $[0, \eta_{max}]$ where $\eta_{max} = 2000$. Table 1 summarizes all parameter values.

Parameter	Value	Description	Parameter	Value	Description
J	14	Life expectancy	κ	2.5	Production parameter
j_L	6	Retired generations	γ_0	0.02	Adjustment cost parameter
$\bar{N}$	1000	Consumers per gen.	γ_1	7.5	Adjustment cost parameter
$\bar{L}^{(j)}$	1	Individual labor supply	δ	0.28	Rate of depreciation
β	0.96	Discount factor	$\bar{x}$	5000	Total number of shares
$\bar{\varepsilon}$	0.92	Expectations parameter	η_{max}	2000	Upper bound for real noise
ϱ	0.5	Expectations parameter	K_0	4,500	Initial capital stock
$\bar{\sigma}$	0.96	Expectations parameter	q_0	9.5	Initial cum-dividend price
α	0.66	Production parameter	R_0	1.12	Initial bond return

Table 1: Standard parameter set for the numerical simulations

It can be shown that all of the following results are robust against parameter changes and independent of initial choices[6] of state variables. The following figures study the impact of alternative contribution rates on real and financial markets.

Figure 1 exhibits the impact of contribution rates on the stock market showing time-windows of the cum-dividend price process $\{q_t\}_{t\geq 0}$ with a high ($\tau = 0.2$) and low ($\tau = 0$) contribution rate. In addition the sample mean $\hat{\mathbb{E}}[q_t]$ and the sample variance $\hat{\mathbb{V}}[q_t]$ depending on τ are displayed.[7] Both time series in Figure 1 fluctuate about stationary

(a) Cum-dividend prices, $\tau = 0$ (b) Cum-dividend prices, $\tau = 0.20$

(c) Level of cum-dividend prices   (d) Volatility of cum-div. prices
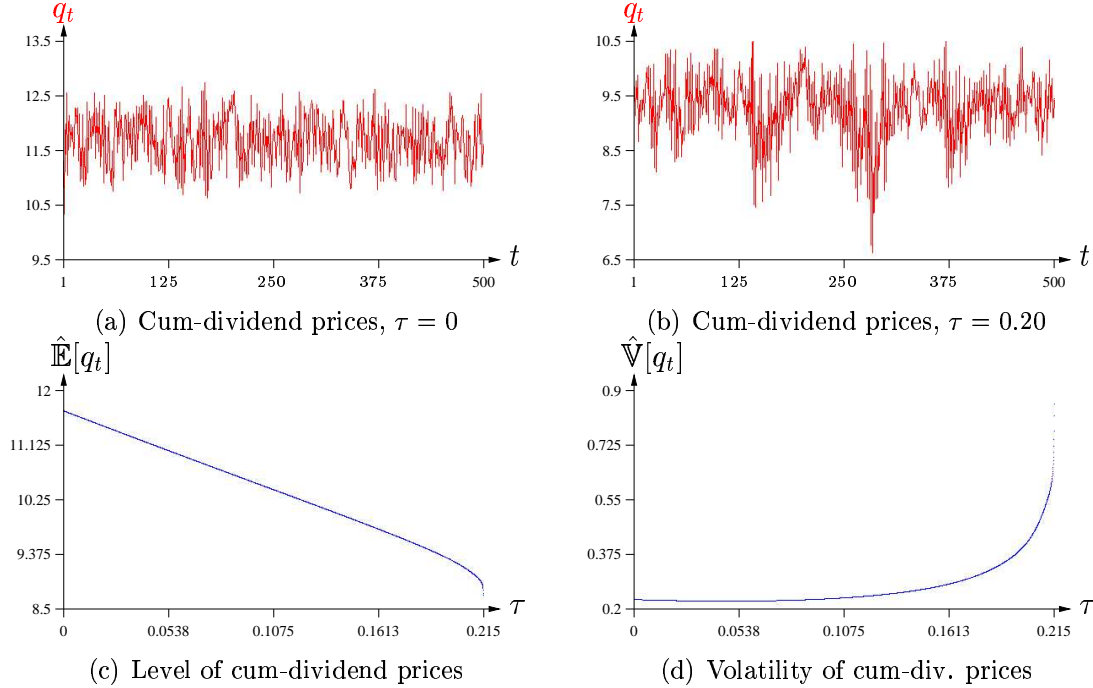
	q_t					
Contrib. rate	$\tau = 0$	$\tau = 0.05$	$\tau = 0.1$	$\tau = 0.15$	$\tau = 0.2$	$\tau = 0.215$
Mean	11.675	11.083	10.502	9.919	9.266	8.739
Variance	0.232	0.227	0.235	0.267	0.41	0.857

Figure 1: Impact of contribution rates on the stock market

levels which decrease as the contribution rate is increased (note the different scaling of the vertical axes). This observation is confirmed by the corresponding sample means and is in line with the assertion of Theorem 2 suggesting a strictly negative relationship between the level of asset prices and contributions to the public pension system. In addition, a comparison of Figures 1(a) and 1(b) reveals that the price process becomes considerably more volatile as the contribution rate is increased to $\tau = 0.2$. More importantly, a 'crash' is observable for $t \in \{250, 300\}$ showing a drastic decline in asset prices during that time window. A comparison of the sample variances reveals that the variance in case $\tau = 0.2$ has almost doubled compared to the case where $\tau = 0$ or $\tau = 0.1$.

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[6] Mathematically this is due to the existence of a stable random fixed point corresponding to a stationary stochastic process which governs the long run behavior of the model. This concept is used in Böhm & Hillebrand (2007) to study the long-run welfare properties of the model.

[7] To avoid possible dependence on initial conditions, only the realizations from $t = 51$ until $t = 500$ are used in the calculations of these quantities.

In this regard, note that all time series are based on the same realization of the noise process. Hence the additional volatility observed can fully be attributed to the public pension system and the larger contribution rate. As τ is further increased to $\tau = 0.215$, the volatility effect becomes even more dramatic resulting in a sample variance of 0.86, almost four times as large as with $\tau = 0.1$. Increasing τ beyond this critical value leads to bankruptcy problems on the part of consumers and is therefore not possible.

More insight into the structural cause for the additional volatility generated is provided by Figure 2 showing the corresponding time series of ex-dividend prices and dividends. A comparison of Figures 2(a) and 2(b) reveals that the additional volatility in cum-

(a) Asset prices and dividends, $\tau = 0$ (b) Asset prices and dividends, $\tau = 0.20$
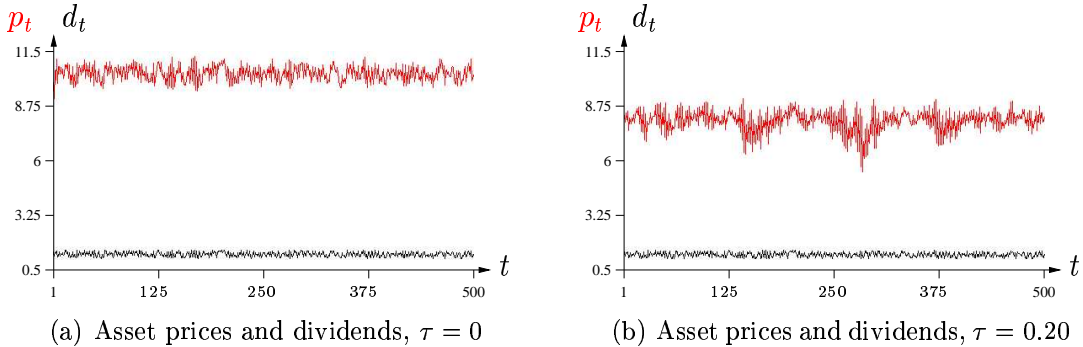
Contrib. rate	p_t				d_t			
	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.215$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.215$
Mean	10.391	9.225	7.999	7.478	1.284	1.277	1.266	1.26
Variance	0.141	0.151	0.333	0.783	0.013	0.013	0.013	0.013

Figure 2: Impact of contribution rates on asset prices and dividend

dividend prices as τ is exclusively due to an increased volatility in ex-dividend prices while no change in fluctuations of the dividend process can be observed. The model generates strong excess volatility in the sense that the volatility of asset prices exceeds the volatility of dividends by a factor of more than ten in the case $\tau = 0$ and by a factor of more than fifty (!) in the case where $\tau = 0.215$. Also observe that throughout the variance in cum-dividend prices exceeds the sum of variances in ex-dividend prices and dividends indicating - as one would expect - a positive correlation between ex-dividend prices and dividend payments. In addition, the levels of both series decrease as the contribution rate is increased.

The impact of contribution rates on bond returns, wages and the capital stock are studied in Figure 3. The result suggests that any increase in τ decreases the levels of capital and the real wage while it increases the level of bond returns. Again these results confirm the insights obtained from Theorem 2 for the two-period case. Also observe that in all cases the average bond return exceeds unity. If the bond is interpreted as a safe asset with interest rate $r_t := R_t - 1$ one obtains on average an annual interest rate of $\approx 3\%$ in case $\tau = 0$ and of $\approx 4\%$ in case $\tau = 0.2$ which seems quite reasonable from an empirical point of view. Apart from that, the time series of bond returns and capital stocks exhibit the same volatility effect as before: As the contribution rate is increased, this results in significantly higher variances of both series which even seem to increase
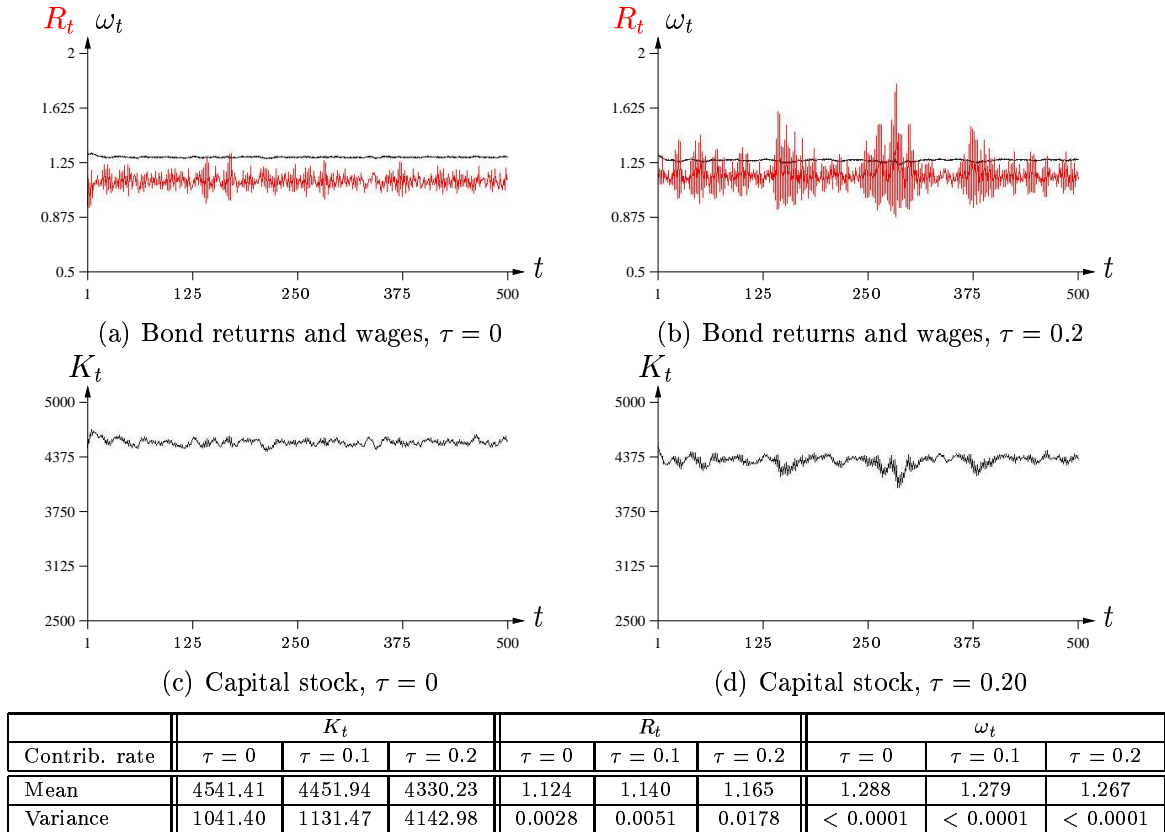
(a) Bond returns and wages, $\tau = 0$

(b) Bond returns and wages, $\tau = 0.2$

(c) Capital stock, $\tau = 0$

(d) Capital stock, $\tau = 0.20$

Contrib. rate	K_t			R_t			ω_t		
	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$
Mean	4541,41	4451,94	4330,23	1,124	1,140	1,165	1,288	1,279	1,267
Variance	1041.40	1131.47	4142.98	0.0028	0.0051	0.0178	< 0.0001	< 0.0001	< 0.0001

Figure 3: Impact of contribution rates on bond returns, real wages and the capital stock

monotonically with τ. Furthermore, for large τ the bond return series exhibits strong volatility clustering, a phenomenon typically observed in empirical financial time series. Surprisingly, fluctuations in the capital stock do not transmit to the real wage process whose fluctuations are negligible throughout all cases.

A comparison of the sample variances in Figures 1 and 3 further reveals that fluctuations in bond returns are still small compared to those of asset prices. This shows again that the model's dynamic behavior is consistent with the prediction behavior of consumers and the firm who form point-predictions for bond returns, non-capital income and real wages while treating asset prices as random variables (see Section 5).

We close this section by analyzing how changes in the contribution rate affect consumers' investment behavior and the distribution of wealth and consumption over the life cycle. The series depicted in Figure 4 pertain to the cases where $\tau = 0$ (red) and $\tau = 0.2$ (blue). Since it can be shown that in each case the processes $\{y_t^{(j)}\}_{t \geq 0}$, $\{x_t^{(j)}\}_{t \geq 0}$, $\{w_t^{(j)}\}_{t \geq 0}$ and $\{c_t^{(j)}\}_{t \geq 0}$ are stationary, the sample averages displayed in Figure 4 approximate the expected value of the respective process which does not depend on t. Figure 4(a) shows consumers' average bond investment over the life cycle depending on $\tau \in \{0, 0.2\}$. In both cases consumers take credit during their first periods of life by selling bonds. The amount of credit taken decreases with age and the bond investment becomes positive for $j \leq 8$. While this behavior is qualitatively the same in both cases, the absolute bond investment is larger with $\tau = 0$ in each period of the life cycle. This suggests

$\hat{\mathbb{E}}[y_t^{(j)}]$

$\hat{\mathbb{E}}[x_t^{(j)}]$

(a) Bond holdings

(b) Share holdings

$\hat{\mathbb{E}}[w_t^{(j)}]$

$\hat{\mathbb{E}}[c_t^{(j)}]$

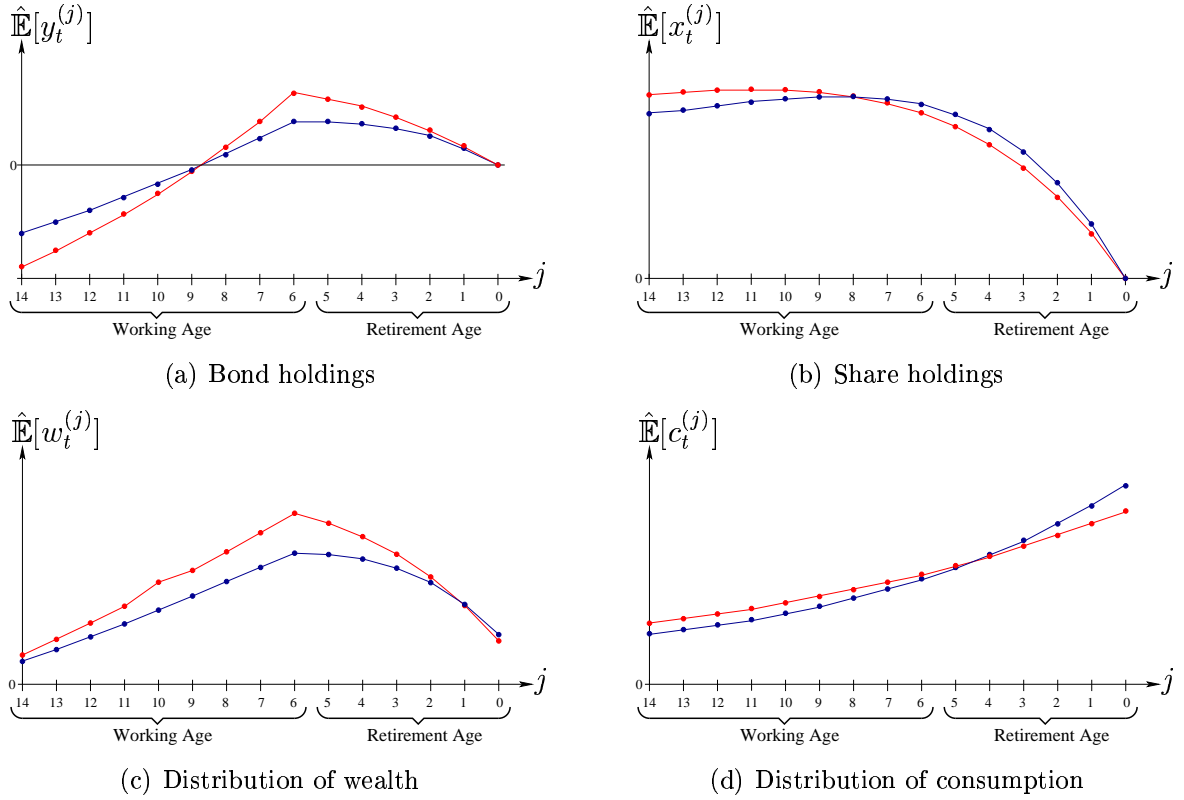(c) Distribution of wealth

(d) Distribution of consumption

Figure 4: Portfolio holdings, wealth and consumption over the life cycle; $\tau \in \{0, 0.2\}$

that the number of bonds traded and, hence, the trading volume on the bond market are negatively affected by a larger contribution rate and that the presence of a pension system leads to a crowding out of private bond investment.

Figure 4(b) provides evidence that the investment in risky shares is larger during the first periods of life and decreases sharply during the retirement age. This result is again true for both scenarios and may be viewed as an indication that consumers are more willing to take risks when young. One also observes that a lower τ increases share holdings during the working age and reduces them during the retirement age.

The average levels of wealth shown in Figure 4(c) are higher with $\tau = 0$ in almost all periods of life and, notably, even in most periods of the retirement age. Comparing the cases $\tau = 0$ and $\tau = 0.2$ this indicates that in the absence of a pension system consumers are able to compensate for the loss of non-capital income during their retirement age by a larger capital income. In both cases, wealth initially increases with age reaching a maximum in the terminal period of the working age ($j = 6$) after which it decreases again. In contrast to that, Figure 4(d) reveals that the level of consumption is a strictly increasing function of age. While this is again true of both scenarios, the increase over the life cycle is steeper with $\tau = 0.2$. This confirms the intuition that a lower contribution rate shifts consumption from the retirement to the working period. In the present case, this leads to a more uniform distribution of consumption over the life cycle. Also observe that with $\tau = 0$ consumption is larger in all periods of the working period while it is smaller in most periods of the retirement age.

8 Demographic change

Finally consider the case with demographic change which affects the age structure of the population. These structural changes are captured by the *economic dependency ratio*

$$\Delta_t := \frac{\sum_{j=0}^{j_L-1} N_t^{(j)}}{\sum_{j=j_L}^{J} N_t^{(j)}}$$

which relates the number of pensioners to the number of workers in each period and thus plays a pivotal role to describe and assess the problem of demographic change.[8] Population scenarios for Germany predict a significant increase in the dependency ratio over the next fifty years from currently $\approx 66\%$ to a value between 75% and 130% (cf. Börsch-Supan, Heiss, Ludwig & Winter (2003)). Similar forecasts apply for the U.S. and other European countries. To study the consequences of this development within our OLG framework we model demographic change as a transitory phenomenon due to downward shifts in the steady state value $\bar{N}$ of the population dynamics. The adjustment towards the new steady state defines a transition period during which the population shrinks and the dependency ratio increases. The parameters of the population law (29) used in the simulations are summarized in the following table.

Parameter	Value	Description	Parameter	Value	Description
J	14	Number of generations	$n^{(11)}$	0.1	Fertility ($\approx$ age 32-35)
$n^{(14)}$	0.275	Fertility ($\approx$ age 20-23)	$n^{(10)}$	0.05	Fertility ($\approx$ age 36-39)
$n^{(13)}$	0.25	Fertility ($\approx$ age 24-27)	$n^{(9)}$	0.01	Fertility ($\approx$ age 40-43)
$n^{(12)}$	0.2	Fertility ($\approx$ age 28-31)	$n^{(j)}, j \leq 8$	0	Fertility ($\approx$ age ≥ 44)

Table 2: Parameter values for the population dynamics.

The values in Table 2 satisfy the stability condition (ii) in Lemma 1. Recalling that a consumer's economic life starts at the age of 20 ($j = J$) and ends at the age of 80 years ($j = 0$) they reflect the plausible assumption that fertility is a decreasing function of age. Nevertheless, it should be noted that neither the employed population model nor the parameter choices are justified on empirical grounds. For our purpose the proposed specification offers a convenient way to study the issue of demographic change and the consequences for the pension system.

To model the demographic transition we assume in (29) an initial value $n_2 = 0.000067$ for $t \leq 50$ implying a steady state value $\bar{N} \approx 2000$. From $t = 51$ onwards we set $n_2 = 0.00013$ shifting the steady state to a lower level $\bar{N}' \approx 1000$. Since this corresponds to the value used in Section 7 the previous results remain valid in the long run as soon as the population has reached the new steady state. Figure 5 depicts the evolution of births as well as the dependency ratio in our demographic scenario. Figure 5(a) shows

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[8] Börsch-Supan & Miegel (2001) distinguish between the *demographic dependency ratio* (retired persons relative to persons in working age) and the *economic dependency ratio* (retired persons relative to employed persons). Since there is no unemployment in our model, both definitions coincide here.

e $N_t^{(J)}$

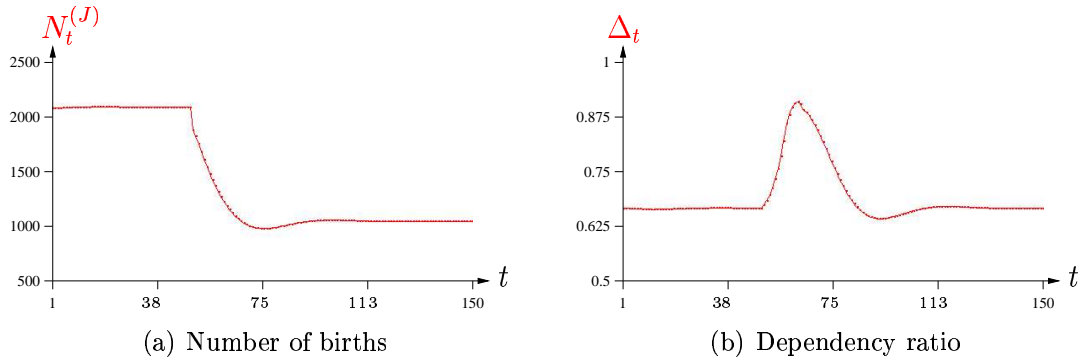(a) Number of births

Δ_t

(b) Dependency ratio

Figure 5: Demographic transition of the population

that the level of births converges to its new steady state value within slightly less than 50 periods. As a consequence the population is constant again for $t \geq 115$ and the dynamic behavior is as described in Section 7. Figure 5(b) shows that the transition is accompanied by a temporary increase in the dependency ratio which reaches a maximum of $\approx 90\%$ in $t = 63$, i.e., 12 periods (≈ 50 years) after the demographic shift before it eventually returns to its initial value of 66%. This range corresponds roughly to the predicted evolution of the German population over the next 50 years.

To study how this change affects real and financial markets we maintain the parameter set from Table 1.[9] While the time series in Figure 6 pertain to the case where $\tau = 0.1$ it can be shown that qualitatively the result remains unchanged with a higher and a lower contribution rate. The table in Figure 6 displays the long-run levels associated with the initial ($\bar{N} \approx 2000$) as well as with the shifted ($\bar{N}' \approx 1000$) steady state value of the population. The most striking phenomenon in Figure 6 is the drastic decline in the levels of asset prices and the capital stock. The table below reveals that the change of the population triggers a decline in these variables of approximately 50%! Although the respective level depends on the contribution rate, the percentage loss is roughly the same for all three scenarios and is approximately of the same magnitude as the percentage change of the population size. A comparison of Figures 6(a) and 6(b) reveals that the decline in cum-dividend prices is almost entirely due to a decrease in ex-dividend prices. This supports the emergence of a so-called asset market meltdown as predicted by Theorem 2 and by various models in the literature, see, e.g., Abel (2001, 2003). Due to the absence of technical progress the decline in birth rates translates directly into a decline in aggregated labor supply. Figure 6(c) shows that the capital stock mirrors this development with a slight delay resulting again in a loss of almost 50%. While the percentage loss is the same in all three scenarios, the initial as well as the shifted long-run level is -as one would expect from our previous results - higher with a lower contribution rate. In contrast to capital and asset prices Figure 6(d) shows that the levels of real wages and bond returns are much less affected by the decline in

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[9] To improve the model's behavior during the first simulation periods the initial values are adjusted to $K_0 = 10,000$ and $q_0 = 19.5$, respectively. Since the model's behavior (for $t \geq 50$) does not depend on initial conditions, this does not impact the following results.

(a) Cum-dividend prices

(b) Asset prices and dividends

(c) Capital and labor force

(d) Wages and bond returns
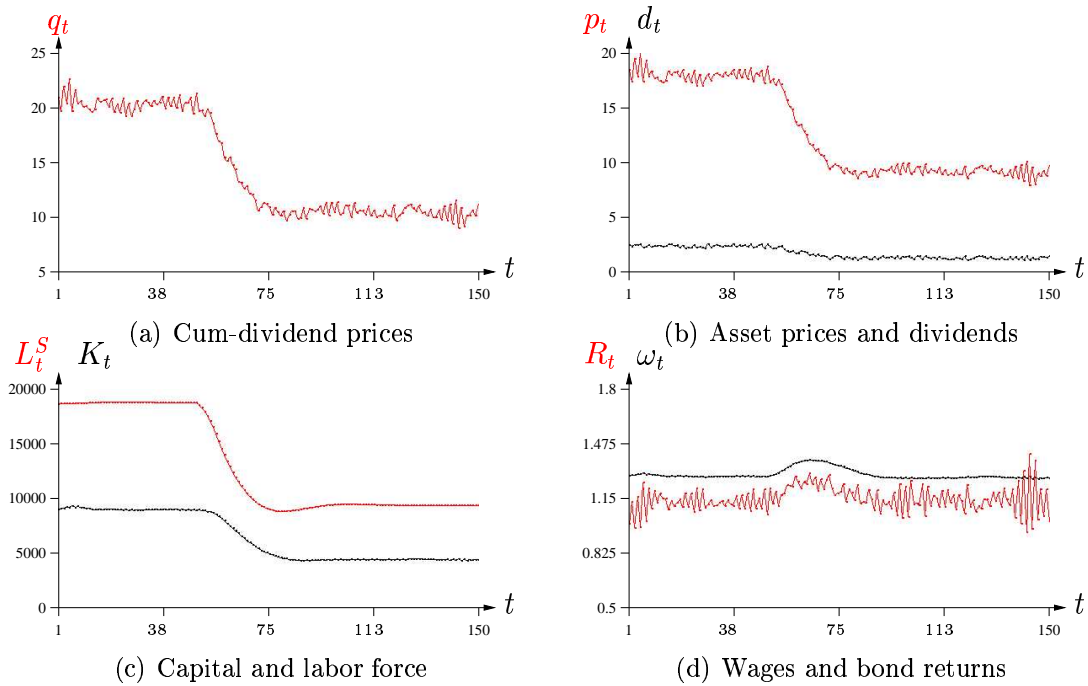
Level of Cont.rate	q_t			K_t			R_t		ω_t	
	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.2$
$\bar{N} \approx 2000$	21.97	19.74	17.50	8881.1	8712.7	8501.0	1.118	1.153	1.291	1.272
$\bar{N}' \approx 1000$	11.68	10.50	9.27	4541.4	4451.9	4330.2	1.124	1.165	1.288	1.267
% Change	- 46.8	- 46.8	- 47.0	-48.86	- 48.9	- 49.1	+ 0.54	+ 1.04	- 0.23	- 0.39

Figure 6: Impact of demographic change on real and financial markets

the population. The levels of bond returns slightly increase while those of real wages slightly decrease in all cases confirming again the insights from Theorem 2. In this regard the absolute percentage change is almost twice as large with $\tau = 0.2$ compared to $\tau = 0$. This indicates that a larger contribution rate increases the sensitivity of wages and interest rates to demographic changes. One also observes a temporary increase in both series during the transition period. For wages this is due to the increased scarcity of labor resulting in higher wages before the capital stock is adjusted.

9 Conclusions and outlook

The model developed in this paper offers a rich potential to study the macroeconomic consequences of pension reforms and the role of demographic change within a random environment. A first attempt to analyze these consequences theoretically and with the help of numerical simulations has been presented in this paper. The results indicate that the parameters of the pension system entail significant consequences for real and financial variables. On the real part, a reduction in contribution rates fosters the accumulation of capital and increases the long run levels of capital stock and production output showing that the results obtained by Feldstein (1974) continue to hold in a stochastic setting. This also confirms the fundamental mechanism described in the in-

troduction which makes a funded pension system potentially superior to a pay-as-you-go system. On the financial part a reduction in contributions increases the level of asset prices while decreasing the interest rate. In addition the numerical insights obtained for a calibrated parametrization of the model provide evidence that a lower contribution rate may stabilize asset markets by reducing the volatility and avoiding crashes in asset prices. Although an intuitive argument for this phenomenon is not available yet, it proved to be robust in the simulations. Referring to the discussion mentioned in the introduction, this result indicates that the risk associated with an increased share of private savings for retirement should not be considered independently of the parameters of the pension system. In fact, a reduction of public pension payments may reduce the capital market risk to which private savings are exposed. As for the role of demographic change the results show that a shrinking population may induce a meltdown of asset prices and capital confirming existing results in the literature.

Several issues remain for future research. The first one is to employ the model developed in this paper and perform a comprehensive study of the welfare effects induced by demographic change and adjustments of the pension system. This task has been undertaken in a related paper (Böhm & Hillebrand (2007)). Another interesting point concerns an extension of the closed economy model to a multi-country setting as studied, e.g., in Krüger & Ludwig (2007). Such an extended setting would permit to study, e.g., the capital flows between countries facing different demographic developments.

A   Mathematical Appendix

A.1   Proof of Proposition 2

Let $\omega > 0$, $\hat{\omega}_1 > 0$, $K_0 > 0$, $\eta_0 \in [0, \eta_{max}]$ and $R_{-1} B_{-1} \geq 0$ be arbitrary but fixed and define $\bar{\bar{R}}(\hat{\omega}_1)$ as in the proposition. Given the production technology (11) it is straightforward to show that (19) is the unique solution to (17). As for (18), note that the additive term $\bar{\eta}$ and the scaling factor $1/\bar{x}$ do not affect the solution and may be omitted. Defining the map $G : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$, $G(L_1, I) := F(L_1, I + (1-\delta)K_0, 0) - \hat{\omega}_1 L_1 - RK_0\, g(I/K_0)$ the problem (18) may be written as

$$\max_{L_1 \geq 0, I \geq 0} \Big\{ G(L_1, I) \quad \text{s.t.} \quad G(L_1, I) \geq 0 \Big\}. \tag{44}$$

Note that $G(0,0) = 0$, hence the constraint $G(L_1, I) \geq 0$ can be dispensed with since it will automatically be satisfied by any solution $(I^\star, L_1^\star)$ to (44). We claim that

$$G(L_1, I) \leq V(I) := G\Big(\Big(\frac{\alpha\kappa}{\hat{\omega}_1}\Big)^{\frac{1}{1-\alpha}} (I + (1-\delta)K_0), I \Big) \tag{45}$$

for any $(L_1, I) \in \mathbb{R}_+^2$ with strict inequality whenever $L_1 \neq \big(\frac{\alpha\kappa}{\hat{\omega}_1}\big)^{\frac{1}{1-\alpha}} (I + (1-\delta)K_0)$. To see this, note that for each fixed $I \geq 0$ the map $L_1 \longmapsto G(L_1, I)$, $L_1 \geq 0$ is strictly concave. Solving the necessary and sufficient first order condition $\frac{\partial G}{\partial L_1}(L_1, I) = 0$ yields

the unique maximizer of $G(\cdot, I)$ on $\mathbb{R}_+$ as $\tilde{L}_1^\star = \left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{1}{1-\alpha}} (I + (1-\delta)K_0)$. This proves (45) and shows that any solution $(L_1^\star, I^\star)$ to (44) has to satisfy the condition

$$L_1^\star = \left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{1}{1-\alpha}} (I^\star + (1-\delta)K_0) . \tag{46}$$

Define the map $V : \mathbb{R}_+ \longrightarrow \mathbb{R}$ as in (45) and consider the problem

$$\max_I \left\{ V(I) \quad \text{s.t.} \quad I \geq 0 \right\}. \tag{47}$$

Suppose $0 < R \leq \bar{\bar{R}}(\hat{\omega}_1)$. Note that the map $I \longmapsto V(I)$, $I > 0$ is strictly concave. Solving the necessary and sufficient condition $\frac{\partial V}{\partial I}(I) = 0$ yields the unique maximizer

$$I^\star = \ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\,\gamma_1\,R} \left(\frac{\alpha\kappa}{\hat{\omega}_1}\right)^{\frac{\alpha}{1-\alpha}}\right)\frac{K_0}{\gamma_1} > 0 \tag{48}$$

of V on $\mathbb{R}_{++}$. Since one easily shows that also $V(I^\star) \geq V(0)$, $I^\star$ is a maximizer of V on $\mathbb{R}_+$. Moreover, the latter inequality is strict whenever $R < \bar{\bar{R}}(\hat{\omega}_1)$ such that $I^\star$ is the unique solution to (47) whenever $R < \bar{\bar{R}}(\hat{\omega}_1)$ while if $R = \bar{\bar{R}}(\hat{\omega}_1)$, $I = 0$ is a second solution to (47). If $R > \bar{\bar{R}}(\hat{\omega}_1)$ it is straightforward to show that $V(I) < V(0)$ for all $I > 0$ such that $I^\star = 0$ is the unique solution to (47). Hence positive investment requires $R \leq \bar{\bar{R}}(\hat{\omega}_1)$. Since by (45) the pair $(L_1^\star, I^\star)$ defined by (46) and (48) is a solution to (44) this proves the claim in (ii) of Proposition 2. ∎

A.2   Derivation of temporary equilibrium maps

Let $\pi_t := R_t\, p_t > 0$ and $\theta_t := \theta(\pi_t; \mu_t, \sigma_t) \in [0,1]$ with θ being defined in (10). We seek a pair $(R_t, p_t) \gg 0$ which solves the market clearing conditions (23) and (24). Using (9) and the homogeneity of the function (21) these conditions may be written as

$$\theta_t \sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t) = \bar{x}\, p_t \tag{49}$$

$$(1 - \theta_t) \sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t) = \frac{\hat{m}_t\, \bar{x}}{R_t} \tag{50}$$

with $\hat{\epsilon}_t^{(j)}$ and $\hat{m}_t$ being defined as in (25) and (26). For given beliefs $(\mu_t, \sigma_t) \in \mathbb{B}$ and fixed $\pi > 0$, define the map $U(\cdot; \pi) : [0,1] \longrightarrow \mathbb{R}$,

$$U(\theta; \pi) := \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \ln\big(\pi + \theta\,(\mu_t + \sigma_t\varepsilon - \pi)\big)\, \nu_\varepsilon(d\,\varepsilon).$$

Applying Lemma 16.2 in Bauer (1992, pp.102) shows that $U(\cdot; \pi)$ is C^2 with derivatives

$$\frac{\partial U}{\partial \theta}(\theta; \pi) = \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \frac{\mu_t + \sigma_t\varepsilon - \pi}{\pi + \theta\,(\mu_t + \sigma_t\varepsilon - \pi)}\, \nu_\varepsilon(d\,\varepsilon), \qquad \frac{\partial^2 U}{(\partial\theta)^2}(\theta; \pi) < 0$$

implying that $U(\cdot; \pi)$ is strictly concave. Since the right-hand sides of equations (49) and (50) are strictly positive, an immediate observation is that $0 < \theta_t < 1$. The definition (10) implies that the pair (θ_t, π_t) has to satisfy the (necessary and sufficient) condition

$$\frac{\partial U}{\partial \theta}(\theta_t, \pi_t) = 0 \iff \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \frac{\pi_t}{\pi_t + \theta_t \left(\mu_t + \sigma_t \varepsilon - \pi_t\right)} \, \nu_\varepsilon(d\,\varepsilon) = 1. \tag{51}$$

On the other hand, by eliminating the common term $\sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t)$ from (49) and (50) one finds that π_t and θ_t have to satisfy the condition

$$\pi_t = \frac{\theta_t \, \hat{m}_t}{1 - \theta_t}. \tag{52}$$

Substituting (52) into (51) shows that θ_t is determined by the map $\vartheta : \mathbb{B} \times \mathbb{R}_{++} \longrightarrow]0, 1[$

$$\theta_t = \vartheta(\mu_t, \sigma_t, \hat{m}_t) := \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \frac{\mu_t + \sigma_t \varepsilon}{\mu_t + \sigma_t \varepsilon + \hat{m}_t} \, \nu_\varepsilon(d\,\varepsilon). \tag{53}$$

Using (53) in (52) shows that π_t is determined by the map $\pi : \mathbb{B} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$

$$\pi_t = \pi(\mu_t, \sigma_t, \hat{m}_t) := \frac{\vartheta(\mu_t, \sigma_t, \hat{m}_t) \, \hat{m}_t}{1 - \vartheta(\mu_t, \sigma_t, \hat{m}_t)}. \tag{54}$$

Note from (52) that $\theta_t = \frac{\pi_t}{\pi_t + \hat{m}_t}$ and, therefore, $\frac{\bar{x}\pi_t}{\theta_t} = \bar{x}\pi_t + \bar{x}\hat{m}_t$. Hence, multiplying (49) by R_t/θ_t and using (26) one obtains

$$R_t \sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})w_t^{(j)} = \bar{x}\pi_t + \bar{x}\hat{m}_t - \sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})\hat{\epsilon}_t^{(j)}.$$

Equation (26) implies $\bar{x}\hat{m}_t - \sum_{j=1}^{J} N_t^{(j)}(1 - \bar{c}^{(j)})\hat{\epsilon}_t^{(j)} = B(1; \hat{\omega}_{t,t+1}, K_t) + \sum_{j=1}^{J} N_t^{(j)}\bar{c}^{(j)}\hat{\epsilon}_t^{(j)}$. Furthermore, since $\bar{c}^{(0)} = 1$ and $\sum_{j=0}^{J-1} N_t^{(j)}x_{t-1}^{(j+1)} = \bar{x}$ one may write $\bar{x} - \sum_{j=1}^{J-1} N_t^{(j)}(1 - \bar{c}^{(j)})x_{t-1}^{(j+1)} = \sum_{j=0}^{J-1} N_t^{(j)}\bar{c}^{(j)}x_{t-1}^{(j+1)}$. Using this together with the definition (5) of $w_t^{(j)}$ one may solve (49) for R_t to obtain (27). This together with $\pi_t = R_t\,p_t$ gives (28). ∎

A.3   Proof of Theorem 1

Let $\bar{N} > 0$ and $\tau \in [0, 1[$ be arbitrary but fixed and, in addition to (39) define

$$\lambda_2 := \frac{(1 - \alpha)\kappa}{\gamma_0\,\gamma_1}e^{-\gamma_1\delta}\bar{N}^\alpha > 0, \qquad \lambda_3 := \kappa(1 - \alpha)(1 - \gamma_1^{-1})\frac{\bar{N}^\alpha}{\bar{x}} > 0. \tag{55}$$

Utilizing (i) and (ii) in Definition 2 we look for a triple $(\bar{K}, \bar{R}, \bar{\mu})$ which solves

$$\bar{K} = \bar{K}\left[\frac{1}{\gamma_1}\ln\left(\frac{(1 - \alpha)\kappa}{\gamma_0\gamma_1\bar{R}}\left(\frac{\bar{N}}{\bar{K}}\right)^\alpha\right) + 1 - \delta\right] \tag{56}$$

$$\bar{R} = \frac{\lambda_1\,\pi(\bar{\mu}, \bar{\sigma}, \lambda_0\bar{K}^{1-\alpha})}{\lambda_0\bar{K}^{1-\alpha}} + \lambda_1 - \frac{\tau}{1 - \tau} \tag{57}$$

$$\bar{\mu} = \frac{\pi(\bar{\mu}, \bar{\sigma}, \lambda_0\bar{K}^{1-\alpha})}{\bar{R}} + \lambda_3\,\bar{K}^{1-\alpha} + \frac{\bar{\eta}}{\bar{x}}. \tag{58}$$

Solving the first equation (56) shows that the equilibrium $\bar{R}$ is determined as

$$\bar{R} = \frac{\lambda_2}{\bar{K}^\alpha}. \tag{59}$$

For each $K \geq 0$, let

$$\hat{\mu}(K) := \frac{\bar{\eta}}{\bar{x}} + \frac{\lambda_0}{\lambda_1} \left[\frac{\lambda_0 + \lambda_3 \lambda_1}{\lambda_0} K^{1-\alpha} - \frac{\lambda_1 - \frac{\tau}{1-\tau}}{\lambda_2} K \right]. \tag{60}$$

Solving (57) for $\pi(\bar{\mu}, \bar{\sigma}, \lambda_0 \bar{K}^{1-\alpha})$ and using the result together with (59) in (58) shows that the equilibrium $\bar{\mu}$ is determined as

$$\bar{\mu} = \hat{\mu}(\bar{K}). \tag{61}$$

For $K > 0$ sufficiently small such that $\hat{\mu}(K) > \frac{\bar{\eta}}{\bar{x}}$ define the map

$$H(K) := \frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau} - \frac{\lambda_1}{I(K)} \quad \text{where} \quad I(K) := \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{\lambda_0 K^{1-\alpha}}{\lambda_0 K^{1-\alpha} + \hat{\mu}(K) + \sigma\varepsilon} \, \nu_\varepsilon(d\varepsilon). \tag{62}$$

Note from Bauer (1992, Lemma 16.1 & 16.2, pp. 102) that I and therefore H are both C^1 and differentiation and integration may be interchanged. Substituting (59) and (61) back into (57) and using the definition of the map π given in (54) solving (56)-(58) reduces to finding $\bar{K} > 0$ which satisfies

$$H(\bar{K}) = 0 \quad \text{and} \quad \hat{\mu}(\bar{K}) > \frac{\bar{\eta}}{\bar{x}}. \tag{63}$$

The following lemma describes some properties of solutions to (63).

Lemma 2
Let the maps $\hat{\mu}$ and H be defined by (60) and (62) and let $K_{max} := \left[\frac{\lambda_2}{\lambda_1 - \tau/(1-\tau)} \right]^{\frac{1}{\alpha}} > 0$. Then the following holds true:

(i) Any solution $\bar{K} > 0$ to (63) satisfies $\bar{K} \in]0, K_{max}[$

(ii) For all $K \in]0, K_{max}[$ we have $\hat{\mu}(K) > \frac{\bar{\eta}}{\bar{x}}$ and $\underline{H}(K) < H(K) < \bar{H}(K)$ where $\underline{H}(K) := \frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau} - \lambda_1 - \frac{\lambda_1 \hat{\mu}(K)}{\lambda_0 K^{1-\alpha}}$ and $\bar{H}(K) := \frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau} - \lambda_1 - \frac{\lambda_1 (\hat{\mu}(K) - \frac{\bar{\eta}}{\bar{x}})}{\lambda_0 K^{1-\alpha}}$

(iii) Any solution $\bar{K} > 0$ to (63) satisfies $\bar{K}^\alpha < \lambda_2$.

Proof of Lemma 2.
(i) Equation (62) implies that all $K > 0$ for which $\hat{\mu}(K) > \frac{\bar{\eta}}{\bar{x}}$ satisfy $I(K) < 1$ and, therefore, $H(K) < \frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau} - \lambda_1$. Hence a necessary condition for $\bar{K}$ to be a solution to (63) is that $\frac{\lambda_2}{\bar{K}^\alpha} + \frac{\tau}{1-\tau} - \lambda_1 > 0$ which is equivalent to $\bar{K} < K_{max}$.
(ii) Let $K \in]0, K_{max}[$ be arbitrary. Equation (60) implies $\hat{\mu}(K) > \frac{\bar{\eta}}{\bar{x}}$ if and only if $K < \left[1 + \frac{\lambda_1 \lambda_3}{\lambda_0} \right]^{\frac{1}{\alpha}} K_{max}$ which is implied by the requirement $K < K_{max}$. Since $\bar{\sigma}\bar{\varepsilon} = \frac{\bar{\eta}}{\bar{x}}$ we have $\hat{\mu}(K) + \bar{\sigma}\varepsilon \geq \hat{\mu}(K) - \frac{\bar{\eta}}{\bar{x}} > 0$ for all $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$. Using this in (62) implies

$I(K) < \frac{\lambda_0 K^{1-\alpha}}{\lambda_0 K^{1-\alpha} + \hat{\mu}(K) - \frac{\bar{\eta}}{\bar{x}}}$. On the other hand, noting that $\varepsilon \longmapsto \frac{\lambda_0 K^{1-\alpha}}{\lambda_0 K^{1-\alpha} + \hat{\mu}(K) + \sigma \varepsilon}$ is strictly convex and that $\mathbb{E}_{\nu_\varepsilon}[\varepsilon] = 0$, by Jensen's inequality $I(K) > \frac{\lambda_0 K^{1-\alpha}}{\lambda_0 K^{1-\alpha} + \hat{\mu}(K)}$. Using both inequalities in (62) proves (ii).

(iii) The map $K \longmapsto \bar{H}(K)$, $K > 0$ has two zeros, say $0 < \hat{K}_1 < \hat{K}_2$ in $\mathbb{R}_{++}$. To see this, note that $\lim_{K \to 0} \bar{H}(K) = \lim_{K \to \infty} \bar{H}(K) = \infty$ and $\bar{H}(K_{max}) = -\lambda_1 \frac{\lambda_3}{\lambda_0} < 0$ implying by continuity that there are at least two zeros in $\mathbb{R}_{++}$. On the other hand, the functional form of $\bar{H}$ implies that its zeros are solutions to a quadratic equation showing that there are precisely two zeros which satisfy $0 < \hat{K}_1 < K_{max} < \hat{K}_2$. In particular, for all $K \in]0, \hat{K}_1[$ we have $\bar{H}(K) > 0$ while $\bar{H}(K) < 0$ if and only if $K \in]\hat{K}_1, \hat{K}_2[$. Since $\bar{H}(\bar{K}) > 0$ requires $\bar{K} \in]0, \hat{K}_1[$ it suffices to show that $\lambda_2^{\frac{1}{\alpha}} \in]\hat{K}_1, \hat{K}_2[$ or, equivalently, $\bar{H}(\lambda_2^{\frac{1}{\alpha}}) < 0$. Direct calculations give $\bar{H}(\lambda_2^{\frac{1}{\alpha}}) = -\lambda_1 \frac{\lambda_3}{\lambda_0} < 0$ completing the proof. $\qquad \square$

Note that the properties derived so far do not require restrictions on α and thus hold in general. In particular, by (59) and Lemma 2 (iii) $\bar{R} > 1$ at any REE. Assuming from now on $\alpha > \frac{1}{2}$ we successively establish existence and uniqueness of a solution to (63) and prove dynamic stability of the corresponding REE of (40).

Existence. By (ii) of Lemma 2, $H(K_{max}) < \bar{H}(K_{max}) = -\lambda_1 \frac{\lambda_3}{\lambda_0} < 0$. By continuity it suffices to show $\lim_{K \to 0} H(K) = \infty$ to prove existence of a solution $\bar{K} \in]0, K_{max}[$ to (63). By (ii) of Lemma 2 , it suffices to show $\lim_{K \to 0} \underline{H}(K) = \infty$. Since $\alpha > \frac{1}{2}$ one has

$$\lim_{K \to 0} \underline{H}(K) = \frac{\tau}{1-\tau} - \lambda_1 - \frac{\lambda_0 + \lambda_1 \lambda_3}{\lambda_0} + \lim_{K \to 0} \frac{1}{K^\alpha} \lim_{K \to 0} \left(\lambda_2 - \frac{\lambda_1 \frac{\bar{\eta}}{\bar{x}} K^{2\alpha-1}}{\lambda_0} \right) = \infty.$$

Uniqueness. Define $h : [0, K_{max}] \times [-\bar{\varepsilon}, \bar{\varepsilon}] \longrightarrow]0, 1[$ $h(K, \varepsilon) := \frac{\lambda_0 K^{1-\alpha}}{\lambda_0 K^{1-\alpha} + \hat{\mu}(K) + \sigma \varepsilon}$. Furthermore, to alleviate our notation define for each $K \in]0, K_{max}[$ from equation (62)

$$B(K) := \frac{1}{I(K)^2} \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} [h(K, \varepsilon)]^2 \; \nu_\varepsilon(d\,\varepsilon) > 1 > \frac{1}{I(K)} \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} [h(K, \varepsilon)]^2 \; \nu_\varepsilon(d\,\varepsilon) =: b(K) \tag{64}$$

where the inequalities are due to $0 < h^2 < h < 1$ and the well-known variance formula which imply $\mathbb{E}_{\nu_\varepsilon}[h]^2 < \mathbb{E}_{\nu_\varepsilon}[h^2] < \mathbb{E}_{\nu_\varepsilon}[h] < 1$. Noting from (64) that $B(K) = \frac{b(K)}{I(K)}$ it is straightforward to show that any solution $\bar{K} > 0$ to (63) satisfies

$$B(\bar{K}) = \frac{b(\bar{K})}{\lambda_1} \left[\frac{\lambda_2}{\bar{K}^\alpha} + \frac{\tau}{1-\tau} \right]. \tag{65}$$

Since H is C^1 uniqueness may be proved by showing that any solution $\bar{K}$ to (63) satisfies $\partial_K H(\bar{K}) := \frac{\partial H}{\partial K}(\bar{K}) < 0$ or, equivalently, $-\bar{K} \partial_K H(\bar{K}) > 0$. For each $K \in]0, K_{max}[$ the derivative of (62) computes as

$$\partial_K H(K) = -\frac{1}{K} \left[\frac{\alpha \lambda_2}{K^\alpha} - \lambda_1 \frac{\partial_K I(K) K}{I(K)^2} \right] \tag{66}$$

$$\text{where} \quad \partial_K I(K) K = I(K) \left[1 - \alpha - b(K) \left(1 - \alpha + \frac{\partial_K \hat{\mu}(K) K}{\lambda_0 K^{1-\alpha}} \right) \right] \tag{67}$$

and $b(K)$ is defined by (64). By Lemma 2 (iii) the derivative of (60) at any $\bar{K}$ satisfies

$$\lambda_1 \frac{\partial_K \hat{\mu}(\bar{K})\bar{K}}{\lambda_0 \bar{K}^{1-\alpha}} = \left[(1-\alpha)\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha} \right] \tag{68}$$

where K_{max} is defined as in Lemma 2. Using (68) in (67) gives, exploiting $B(\bar{K}) = \frac{b(\bar{K})}{I(\bar{K})}$

$$\lambda_1 \frac{\partial_K I(\bar{K})\bar{K}}{I(\bar{K})^2} = (1-\alpha)\left[\frac{\lambda_1}{I(\bar{K})} - B(\bar{K})\left(\lambda_1 + \frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right) \right] + \alpha \frac{B(\bar{K})\bar{K}^\alpha}{K_{max}^\alpha}. \tag{69}$$

Using (69) in (66) yields

$$-\partial_K H(\bar{K})\bar{K} = \alpha\left[\frac{\lambda_2}{K^\alpha} - \frac{B(\bar{K})\bar{K}^\alpha}{K_{max}^\alpha} \right] - (1-\alpha)\left[\frac{\lambda_1}{I(\bar{K})} - B(\bar{K})\left(\lambda_1 + \frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right) \right]. \tag{70}$$

Recall that $B(\bar{K}) > 1$ and by Lemma 2 (iii) $\frac{\bar{K}^\alpha}{K_{max}^\alpha} = \frac{\lambda_1 - \tau/(1-\tau)}{\lambda_2}\bar{K}^\alpha < \lambda_1 - \frac{\tau}{1-\tau}$. Using this in (70) together with $\frac{\lambda_1}{I(K)} = \frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau}$ implied by (63) gives the inequality

$$-\partial_K H(\bar{K})\bar{K} > \alpha\left[\frac{\lambda_2}{K^\alpha} - \frac{B(\bar{K})\bar{K}^\alpha}{K_{max}^\alpha} \right] - (1-\alpha)\left[\frac{\lambda_2}{\bar{K}^\alpha} - B(\bar{K})\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} \right]. \tag{71}$$

Since $\frac{\bar{K}^\alpha}{K_{max}^\alpha} < 1$ the first bracketed term will always be larger than the second one. Since $\alpha > \frac{1}{2}$ showing that $\frac{\lambda_2}{K^\alpha} > \frac{B(\bar{K})\bar{K}^\alpha}{K_{max}^\alpha}$ will therefore be sufficient for the right hand side in (71) to be positive. Using (64), (65) and the definition of K_{max} give

$$B(\bar{K})\frac{\bar{K}^\alpha}{K_{max}^\alpha} = b(\bar{K}) - \frac{\tau}{1-\tau}\left[B(\bar{K}) - b(\bar{K})\frac{\bar{K}^\alpha}{\lambda_2} \right] < b(\bar{K}) < 1 \tag{72}$$

which implies the desired inequality due to Lemma 2 (iii). Using this result in (71) shows that $-\partial_K H(\bar{K})\bar{K} > 0$ as claimed.

Stability. The partial derivatives of the dynamical system (40) at any REE satisfy

$$\frac{\partial \hat{\mathcal{K}}}{\partial K}(\bar{K}, \bar{R}; \bar{N}) = 1 - \frac{\alpha}{\gamma_1} \qquad \frac{\partial \hat{\mathcal{K}}}{\partial R}(\bar{K}, \bar{R}; \bar{N}) = -\frac{\bar{K}}{\gamma_1 \bar{R}}$$

$$\frac{\partial \hat{\mathcal{R}}_\tau}{\partial K}(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}) = (1 - \frac{\alpha}{\gamma_1})\bar{a} \qquad \frac{\partial \hat{\mathcal{R}}_\tau}{\partial R}(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}) = -\frac{\bar{K}}{\gamma_1 \bar{R}}\bar{a}$$

where, using that $H(\bar{K}) = 0$ together with (64) $\bar{a} := \frac{1-\alpha}{\bar{K}}\left[B(\bar{K})\lambda_1 - \frac{\tau}{1-\tau} - \frac{\lambda_2}{\bar{K}^\alpha} \right]$. The eigenvalues ι_i, $i = 1, 2$ of the Jacobian

$$D(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}) := \begin{bmatrix} \frac{\partial \hat{\mathcal{K}}}{\partial K}(\bar{K}, \bar{R}; \bar{N}) & \frac{\partial \hat{\mathcal{K}}}{\partial R}(\bar{K}, \bar{R}; \bar{N}) \\ \frac{\partial \hat{\mathcal{R}}_\tau}{\partial K}(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}) & \frac{\partial \hat{\mathcal{R}}_\tau}{\partial R}(\bar{K}, \bar{R}; \bar{N}, \bar{\mu}) \end{bmatrix} = \begin{bmatrix} 1 - \frac{\alpha}{\gamma_1} & -\frac{\bar{K}}{\gamma_1 \bar{R}} \\ (1 - \frac{\alpha}{\gamma_1})\bar{a} & -\frac{\bar{K}}{\gamma_1 \bar{R}}\bar{a} \end{bmatrix}$$

can be calculated explicitly as $\iota_1 = 0$ and $\iota_2 = \frac{\partial \hat{\mathcal{K}}}{\partial K}(\bar{K}, \bar{R}; \bar{N}) + \frac{\partial \hat{\mathcal{R}}_\tau}{\partial R}(\bar{K}, \bar{R}; \bar{N}, \bar{\mu})$. Using (56) and (59) the second one may be expanded as

$$\iota_2 = 1 - \frac{2\alpha - 1}{\gamma_1} - \frac{(1-\alpha)\bar{K}^\alpha}{\gamma_1 \lambda_2}\left[B(\bar{K})\lambda_1 - \frac{\tau}{1-\tau} \right].$$

Since $\alpha > \frac{1}{2}$ and $B(\bar{K}) > 1$, the last equation implies $\iota_2 < 1$. Using (65) gives

$$\iota_2 \quad = \quad 1 - \frac{\alpha}{\gamma_1} + \frac{1-\alpha}{\gamma_1}(1 - b(\bar{K})) + \frac{1-\alpha}{\gamma_1}\frac{\bar{K}^\alpha}{\lambda_2}\frac{\tau}{1-\tau}(1 - b(\bar{K})) > 0.$$

This proves that $|\iota_i| < 1$ for $i = 1, 2$ implying dynamic stability of (40). ∎

A.4 Proof of Theorem 2

In the sequel we will frequently highlight the dependence of the parameters defined in (39) and (55) on τ and N by writing $\lambda_0(\tau, N)$, $\lambda_1(\tau)$, $\lambda_2(N)$ and $\lambda_3(N)$. In addition we treat τ and N as arguments of the maps $\hat{\mu}$, H and I defined in (60) and (62) by writing $\hat{\mu}(K, \tau, N)$, $H(K, \tau, N)$ and $I(K, \tau, N)$. By a similar reasoning as above the maps I and therefore H are both C^1 on their extended domain and differentiation and integration may be interchanged. To alleviate the notation the parameters (τ, N) are suppressed as arguments whenever convenient. Note that using (14), (39) and (55) the map $\hat{\mu}$ takes the expanded form

$$\hat{\mu}(K, \tau, N) = \frac{\bar{\eta}}{\bar{x}} + \frac{\kappa\,\bar{N}^\alpha K^{1-\alpha}}{\bar{x}}\left[\frac{\alpha\beta(1-\tau)}{1+\beta} + (1-\alpha)(1 - \frac{1}{\gamma_1})\right] - \frac{g'(\delta)K}{\bar{x}}\left[\frac{1}{\gamma_1} + \frac{\alpha}{1-\alpha}\frac{\tau}{1+\beta}\right]. \tag{73}$$

The equilibrium value $\bar{K}$ is determined by the function $\bar{K} : [0, 1[\times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ defined implicitly by the condition $H(\bar{K}, \tau, N) = 0$. Since H is differentiable, an application of the implicit function theorem gives

$$\partial_\tau \bar{K}(\tau, N) = -\frac{\partial_\tau H(K, \tau, N)}{\partial_K H(K, \tau, N)}\Big|_{K=\bar{K}}. \tag{74}$$

(i) (a). We show that $\partial_\tau \bar{K}(\tau, N) < 0$. As shown in (70), the denominator in (74) satisfies $\partial_K H(\bar{K}, \tau, N) < 0$ such that it suffices to show $\partial_\tau H(K, \tau, N) < 0$. Using (62) the partial derivative $\partial_\tau H(K, \tau, N)$ computes as

$$\partial_\tau H(K, \tau, N) \quad = \quad \frac{1}{(1-\tau)^2} - \frac{\partial_\tau \lambda_1(\tau)}{I(K)} + \frac{\lambda_1}{I(K)}\frac{\partial_\tau I(K, \tau, N)}{I(K)} \tag{75}$$

$$\text{where} \quad \frac{\partial_\tau I(K, \tau, N)}{I(K)} \quad = \quad \frac{\partial_\tau \lambda_0(\tau, N)}{\lambda_0} - b(K)\left(\frac{\partial_\tau \lambda_0(\tau, N)}{\lambda_0} + \frac{\partial_\tau \hat{\mu}(K, \tau, N)}{\lambda_0 K^{1-\alpha}}\right). \tag{76}$$

By (39) one has $\frac{\partial_\tau \lambda_1(\tau)}{\lambda_1} - \frac{\partial_\tau \lambda_0(\tau, N)}{\lambda_0} = \frac{1}{1-\tau}$. Using this with (76) and (65) in (75) gives

$$\partial_\tau H(\bar{K}, \tau, N) = \frac{1}{(1-\tau)^2} - \frac{\lambda_1}{I(\bar{K})}\frac{1}{1-\tau} - B(\bar{K})\lambda_1\left[\frac{\partial_\tau \lambda_0(\tau, N)}{\lambda_0} + \frac{\partial_\tau \hat{\mu}(K, \tau, N)}{\lambda_0 K^{1-\alpha}}\right]. \tag{77}$$

Using (55) the partial derivative of (73) may be written as

$$\partial_\tau \hat{\mu}(K, \tau, N) = -\frac{\kappa\,\bar{N}^\alpha K^{1-\alpha}}{\bar{x}}\frac{\alpha}{1+\beta}\left(\beta + \frac{K^\alpha}{\lambda_2}\right) < 0. \tag{78}$$

Note from equations (39) and (78) that $\frac{\partial_\tau \hat{\mu}(K,\tau,N)}{\lambda_0 \bar{K}^{1-\alpha}} = -\frac{\partial_\tau \lambda_0(\tau,N)}{\lambda_0} \frac{1}{1+\beta}\left(\beta + \frac{\bar{K}^\alpha}{\lambda_2}\right)$ implying that $\lambda_1\left[\frac{\partial_\tau \lambda_0(\tau,N)}{\lambda_0} + \frac{\partial_\tau \hat{\mu}(K,\tau,N)}{\lambda_0 K^{1-\alpha}}\right] = \frac{1}{\beta}\frac{1}{1-\tau}\left[1 - \frac{\bar{K}^\alpha}{\lambda_2}\right]$. Using this with $H(\bar{K},\tau,N) = 0$ in (77) and applying Lemma 2 (iii) yields the desired result

$$\partial_\tau H(\bar{K},\tau,N) = -\frac{1}{1-\tau}\frac{1}{\beta}\left(\frac{\lambda_2}{\bar{K}^\alpha} - 1\right)\left(\beta + B(\bar{K})\frac{\bar{K}^\alpha}{\lambda_2}\right) < 0. \qquad (79)$$

(i) (b). We show that $\partial_\tau \bar{R}(\tau,N) > 0$. Defining from (59) $\bar{R}(\tau,N) := \frac{\lambda_2(N)}{\bar{K}(\tau,N)^\alpha}$ the previous result implies

$$\partial_\tau \bar{R}(\tau,N) = -\frac{\alpha}{\bar{K}^{1+\alpha}}\lambda_2 \partial_\tau \bar{K}(\tau,N) > 0.$$

(i) (c). We show that $\partial_\tau \bar{\mu}(\tau,N) > 0$. Defining from (61) $\bar{\mu}(\tau,N) := \hat{\mu}(\bar{K}(\tau,N),\tau,N)$ the chain rule and (74) give

$$\partial_\tau \bar{\mu}(\tau,N) = \frac{\partial_\tau \hat{\mu}(\bar{K},\tau,N)\bar{K}\partial_K H(\bar{K},\tau,N) - \bar{K}\partial_K \hat{\mu}(\bar{K},\tau,N)\partial_\tau H(\bar{K},\tau,N)}{\bar{K}\partial_K H(\bar{K},\tau,N)}. \qquad (80)$$

We show that the numerator in (80) is strictly positive. If $\partial_K \hat{\mu}(\bar{K},\tau,N) \geq 0$ this result is immediate due to (71), (78) and (79). If $\partial_K \hat{\mu}(\bar{K},\tau,N) < 0$ equations (68), (79) and (80) together with $\beta + B(\bar{K})\frac{K^\alpha}{\lambda_2} < B(\bar{K})(\beta + \frac{K^\alpha}{\lambda_2})$ imply that the second term in (80) satisfies

$$-\bar{K}\partial_K \hat{\mu}(\bar{K},\tau,N)\partial_\tau H(\bar{K},\tau,N)$$

$$> \frac{\kappa \bar{N}^\alpha K^{1-\alpha}}{\bar{x}}\frac{\alpha}{1+\beta}\left(\beta + \frac{K^\alpha}{\lambda_2}\right)\overbrace{\left[(1-\alpha)\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right]}^{<0}\left(\frac{\lambda_2}{K^\alpha} - 1\right)B(\bar{K})$$

$$= -\partial_\tau \hat{\mu}(\bar{K},\tau,N)\left[(1-\alpha)\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right]\left(\frac{\lambda_2}{K^\alpha} - 1\right)B(\bar{K}). \qquad (81)$$

Using (81) in (80) the claim that the numerator is positive will follow if we show that $Z := -\bar{K}\partial_K H(\bar{K},\tau,N) + \left[(1-\alpha)\frac{\lambda_0+\lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right]\left(\frac{\lambda_2}{K^\alpha} - 1\right)B(\bar{K}) > 0$. Using (70) gives

$$Z = \alpha\frac{\lambda_2}{K^\alpha} - (1-\alpha)\left[\frac{\lambda_2}{K^\alpha} + \frac{\tau}{1-\tau} - B(\bar{K})\lambda_1\right] + \frac{B(\bar{K})\lambda_2}{K^\alpha}\left[(1-\alpha)\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - \frac{\bar{K}^\alpha}{K_{max}^\alpha}\right]$$

$$> \alpha\frac{\lambda_2}{K^\alpha}\left[1 - B(\bar{K})\frac{\bar{K}^\alpha}{K_{max}^\alpha}\right] + (1-\alpha)\frac{\lambda_2}{K^\alpha}\left[\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0} - 1\right] > 0$$

where we have used the definition of K_{max}, $B(\bar{K}) > 1$ and (72). This proves the claim.

(ii) (a) We show that $\partial_N \bar{K}(\tau,N) > 0$. Applying the implicit function theorem gives

$$\partial_N \bar{K}(\tau,N) = -\frac{\partial_N H(K,\tau,N)}{\partial_K H(K,\tau,N)}\bigg|_{K=\bar{K}}. \qquad (82)$$

By (71) it suffices to show that $\partial_N H(\bar{K},N,\tau) > 0$. The derivative of (62) satisfies

$$N\,\partial_N H(\bar{K},N,\tau) = \frac{\alpha\lambda_2}{\bar{K}^\alpha} + \lambda_1\frac{\partial_N I(\bar{K},\tau,N)\,N}{I(\bar{K})^2} \qquad (83)$$

where using (62), (64) and (60) resp. (73)

$$N\,\partial_N I(\bar{K},\tau,N) = I(\bar{K})\Big[\alpha - b(\bar{K})\Big(\alpha + \frac{N\,\partial_N\hat{\mu}(\bar{K},\tau,N)}{\lambda_0\bar{K}^{1-\alpha}}\Big)\Big] \tag{84}$$

$$\partial_N\hat{\mu}(\bar{K},\tau,N) = \frac{\alpha}{N}\frac{\lambda_0}{\lambda_1}\Big[\frac{\lambda_0 + \lambda_3\lambda_1}{\lambda_0}\Big]\bar{K}^{1-\alpha} > 0. \tag{85}$$

Recall from (71) that $-\bar{K}\partial_K H(\bar{K},N,\tau) > 0$. The claim will follow if we show that $N\partial_N H(\bar{K},N,\tau) > -\bar{K}\partial_K H(\bar{K},N,\tau)$. Using (70) and (83) gives

$$N\partial_N H(\bar{K},N,\tau) + \bar{K}\partial_K H(\bar{K},N,\tau) = \frac{\lambda_1}{I(\bar{K})}\big[N\partial_N I(\bar{K},\tau,N) + K\partial_K I(\bar{K},\tau,N)\big]. \tag{86}$$

From (68) and (85) we note that $\partial_N\hat{\mu}(\bar{K},\tau,N)N + \partial_K\hat{\mu}(\bar{K},\tau,N)\bar{K} = \hat{\mu}(\bar{K},\tau,N) - \bar{\eta}/\bar{x}$. Using this together with equations (67) and (84) yields, recalling that $\bar{\eta}/\bar{x} = \bar{\sigma}\bar{\varepsilon}$

$$N\partial_N I(\bar{K},\tau,N) + \bar{K}\partial_K I(\bar{K},\tau,N) = I(\bar{K})\Big[1 - b(\bar{K})\frac{\lambda_0\bar{K}^{1-\alpha} + \hat{\mu}(\bar{K},\tau,N) - \bar{\eta}/\bar{x}}{\lambda_0\bar{K}^{1-\alpha}}\Big]$$

$$= I(\bar{K})\Big[1 - \frac{b(\bar{K})}{h(\bar{K},-\bar{\varepsilon})}\Big]. \tag{87}$$

Note that the map $\varepsilon \longmapsto h(\bar{K},\varepsilon)$ is strictly decreasing and therefore $\frac{h(\bar{K},\varepsilon)}{h(\bar{K},-\varepsilon)} < 1$ for all $\varepsilon \in]-\bar{\varepsilon},\bar{\varepsilon}]$. Using this together with the definition (64) of $b(\bar{K})$ gives

$$\frac{b(\bar{K})}{h(\bar{K},-\bar{\varepsilon})} = \frac{1}{I(\bar{K})}\int_{[-\bar{\varepsilon},\bar{\varepsilon}]}\frac{\big[h(\bar{K},\varepsilon)\big]^2}{h(\bar{K},-\bar{\varepsilon})}\,\nu_\varepsilon(d\varepsilon) < \frac{1}{I(\bar{K})}\int_{[-\bar{\varepsilon},\bar{\varepsilon}]}\big[h(\bar{K},\varepsilon)\big]\,\nu_\varepsilon(d\varepsilon) = 1.$$

This proves that (87) is strictly positive which together with (86) gives the claim.
(ii) (b). We show that $\partial_N\bar{R}(\tau,N) < 0$. Recalling that $\bar{R}(\tau,N) = \frac{(1-\alpha)\kappa}{g'(\delta)}\big(\frac{N}{\bar{K}(\tau,N)}\big)^\alpha$ it suffices to show that the map $N \longmapsto \frac{N}{\bar{K}(\tau,N)}$ is decreasing. Using (82) the previous result in (86) and equation (71) imply

$$\partial_N\Big(\frac{N}{\bar{K}(\tau,N)}\Big) = \frac{1}{\bar{K}^2\partial_K H(\bar{K},\tau,N)}\big[N\partial_N H(\bar{K},N,\tau) + \bar{K}\partial_K H(\bar{K},N,\tau)\big] < 0. \tag{88}$$

(ii) (c). We show that $\partial_N\bar{\mu}(\tau,N) > 0$. Recall that $\bar{\mu}(\tau,N) = \hat{\mu}(\bar{K}(\tau,N),\tau,N)$. Using (82) the derivative takes the form

$$\partial_N\bar{\mu}(\tau,N) = \frac{\partial_N\hat{\mu}(\bar{K},\tau,N)\partial_K H(\bar{K},\tau,N) - \partial_K\hat{\mu}(\bar{K},\tau,N)\partial_N H(\bar{K},\tau,N)}{\partial_K H(\bar{K},\tau,N)}. \tag{89}$$

By (71) it suffices to show that the numerator in (89) is negative. It is clear from (71), (85) and the result in (ii) (a) that if $\partial_K\hat{\mu}(\bar{K},\tau,N) \geq 0$ the claim holds automatically. Hence suppose $\partial_K\hat{\mu}(\bar{K},\tau,N) < 0$. Rearranging terms and exploiting (85) the numerator in (89) is negative if and only if

$$Y := -\bar{K}\partial_K H(\bar{K},\tau,N) + N\partial_N H(\bar{K},\tau,N)\frac{\bar{K}\partial_K\hat{\mu}(\bar{K},\tau,N)}{N\partial_N\hat{\mu}(\bar{K},\tau,N)} > 0. \tag{90}$$

Combining (68) and (85) we see that

$$\frac{\bar{K} \partial_K \hat{\mu}(\bar{K}, \tau, N)}{N \partial_N \hat{\mu}(\bar{K}, \tau, N)} = \frac{1 - \alpha}{\alpha} - \frac{1}{\alpha} \frac{\bar{K}^\alpha}{K_{max}^\alpha} \frac{\lambda_0}{\lambda_0 + \lambda_3 \lambda_1}. \tag{91}$$

Using (84) and (85) the derivative (83) may be written as

$$N \, \partial_N H(\bar{K}, \tau, N) = \alpha \left[\frac{\lambda_2}{\bar{K}^\alpha} + (1 - b(\bar{K})) \left(\frac{\lambda_2}{\bar{K}^\alpha} + \frac{\tau}{1 - \tau} \right) - B(\bar{K}) \frac{\lambda_0 + \lambda_3 \lambda_1}{\lambda_0} \right]. \tag{92}$$

Using (70), (91) and (92) in (90) and exploiting (65) gives

$$Y = \frac{\lambda_2}{\bar{K}^\alpha} \left[1 - \frac{\lambda_0}{\lambda_0 + \lambda_3 \lambda_1} \frac{\bar{K}^\alpha}{K_{max}^\alpha} \right] - \frac{\lambda_0}{\lambda_0 + \lambda_3 \lambda_1} \frac{\bar{K}^\alpha}{K_{max}^\alpha} \left[\frac{\lambda_2}{\bar{K}^\alpha} + \frac{\tau}{1 - \tau} - B(\bar{K}) \lambda_1 \right].$$

Using the definition of K_{max} and $B(\bar{K}) > 1$ yields finally the desired result

$$Y > \left[\frac{\lambda_2}{\bar{K}^\alpha} - (\lambda_1 - \frac{\tau}{1 - \tau}) \right] \left[1 - \frac{\lambda_0}{\lambda_0 + \lambda_3 \lambda_1} \frac{\bar{K}^\alpha}{K_{max}^\alpha} \right] > 0.$$

References

ABEL, A. B. (2001): "Will Bequest Attenuate the Predicted Meltdown in Stock Prices When Baby Boomers Retire?", *Review of Economics and Statistics*, 84(3), 589–595.

————— (2003): "The Effects of a Baby Boom on Stock Prices and Capital Accumulation in the Presence of Social Security", *Econometrica*, 71(2), 551–578.

ALTIG, D., A. J. AUERBACH, L. J. KOTLIKOFF, K. A. SMETTERS & J. WALLISER (2001): "Simulating Fundamental Tax Reform in the United States", *American Economic Review*, 91(3), 574–595.

AUERBACH, A. J. & L. J. KOTLIKOFF (1987): *Dynamic Fiscal Policy*. Cambridge University Press, Cambridge.

BARBIE, M., M. HAGEDORN & A. KAUL (2007): "On the Interaction between Risk Sharing and Capital Accumulation in a Stochastic OLG Model with Production", *Journal of Economic Theory*, forthcoming.

BAUER, H. (1992): *Maß– und Integrationstheorie*. Walter de Gruyter Verlag, Berlin a.o., 2ndAuflage.

BÖHM, V. (2003): "MACRODYN – The Handbook", Discussion Paper 498, Department of Economics, University of Bielefeld.

BÖHM, V. & C. CHIARELLA (2005): "Mean Variance Preferences, Expectations Formation, and the Dynamics of Random Asset Prices", *Mathematical Finance*, 15(1), 61–97.

BÖHM, V. & M. HILLEBRAND (2007): "On the Inefficiency of Pay-As-You-Go Pension Systems in Stochastic Economies with Assets", Discussion Paper 565, Department of Economics, Bielefeld University.

BÖRSCH-SUPAN, A., F. HEISS, A. LUDWIG & J. WINTER (2003): "Pension Reform, Capital Markets and the Rate of Return", *German Economic Review*, 4(2), 151–181.

BÖRSCH-SUPAN, A., A. LUDWIG & J. WINTER (2006): "Aging, Pension Reform, and Capital Flows: A Multi-Country Simulation Model", *Economia*, 73(4), 625 – 658.

BÖRSCH-SUPAN, A. H. & M. MIEGEL (2001): *Pension Reform in Six Countries.* Springer Verlag, Berlin a.o.

CHAMBERLAIN, G. (1983): "A Characterization of the Distributions that Imply Mean-Variance Utility Functions", *Journal of Economic Theory*, 29, 185–201.

CHATTOPADHYAY, S. & P. GOTTARDI (1999): "Stochastic OLG Models, Market Structure and Optimality", *Journal of Economic Theory*, 89, 21–67.

CONESA, J. C. & D. KRÜGER (1999): "Social Security Reform with Heterogeneous Agents", *Review of Economic Dynamics*, 2, 757–795.

DEMANGE, G. (2002): "On Optimality in Intergenerational Risk Sharing", *Economic Theory*, 20, 1–27.

DEMANGE, G. & G. LAROQUE (1999): "Social Security and Demographic Shocks", *Econometrica*, 67(3), 527–542.

FARHI, E. & S. PANAGEAS (2007): "Saving and investing for early retirement: A theoretical analysis", *Journal of Financial Economics*, 83, 87–121.

FELDSTEIN, M. (1974): "Social Security, Induced Retirement and Aggregated Capital Accumulation", *Journal of Political Economy*, 82(5), 905–926.

GEANAKOPOLOS, J., M. MAGILL & M. QUINZII (2004): "Demography and the Long-Run Predictability of the Stock Market", *Brookings Papers on Economic Activity*, 1, 241–325.

GOTTARDI, P. & F. KÜBLER (2006): "Social Security and Risk Sharing", CESifo Working Paper 1705, Center for Economic Studies & Ifo Institute for Economic Research.

GOULD, J. P. (1968): "Adjustment Costs in the Theory of Investment of the Firm", *The Review of Economic Studies*, 35(1), 47–55.

HAKANSSON, N. (1969): "Optimal Investment and Consumption Strategies under Risk, an Uncertain Lifetime and Insurance", *International Economic Review*, 10, 443–466.

HENSON, S. M., R. COSTANTINO, J. CUSHING, B. DENNIS & R. A. DESHARNAIS (1999): "Multiple Attractors, Saddles and Population Dynamics in Periodic Habitats", *Bulletin of Mathematical Biology*, 61, 1121–1149.

HILLEBRAND, M. & J. WENZELBURGER (2006): "The Impact of Multiperiod Planning Horizons on Portfolios and Asset Prices in a Dynamic CAPM", *Journal of Mathematical Economics*, 42(4–5), 565–593.

————— (2007): "Multiperiod Consumption and Investment Decisions Revisited", Discussion Paper 567, Department of Economics, Bielefeld University, Downloadable at: http://www.marten-hillebrand.de/research/567/paper_567.htm.

HURD, M. (1989): "Mortality Risk and Bequests", *Econometrica*, 57, 173–209.

İMROHOROĞLU, A., S. İMROHOROĞLU & D. JOINES (1995): "A Life Cycle Analysis of Social Security", *Economic Theory*, 6, 83–114.

KRÜGER, D. & F. KÜBLER (2006): "Pareto-Improving Social Security Reform when Financial Markets Are Incomplete!?", *American Economic Review*, 96(3), 737–755.

KRÜGER, D. & A. LUDWIG (2007): "On the Consequences of Demographic Change for Rates of Returns to Capital and the Distribution of Wealth and Welfare", *Journal of Monetary Economics*, 54(1), 49–87.

MAGILL, M. & M. QUINZII (1998): *Theory of Incomplete Markets*. MIT Press, Cambridge.

MAGILL, M. & M. QUINZII (2003): "Nonshiftable Capital, Affine Price Expectations and Convergence to the Golden Rule", *Journal of Mathematical Economics*, 39, 239–272.

ROTSCHILD, M. (1971): "On the Cost of Adjustment", *Quarterly Journal of Economics*, 85(4), 605–622.