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On Core Stability and Extendability*

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Abstract

This paper investigates conditions under which the core of a TU cooperative game is stable. In particular the author extends the idea of extendability to find new conditions under which the core is stable. It is also shown that these new conditions are not necessary for core stability.
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1 Introduction

In this section well-known results concerning core stability will be discussed as well as an introduction to the question which forms the basis of the next sections. Definitions which are given in the next sections will not be given here. The reader is referred to either the original articles or the next sections for any unfamiliar notation or concepts.

The core was one of the first solution concept to be introduced in the field of transferrable utility n-person cooperative games. Since it's conception the core has been extensively study and characterised. It represents the set of payoffs where no coalition of players receive less than what they can achieve on their own. An open question relating to the core is the question, when is the core stable? The idea of stability arises when the core coincides with the von Neumman Morgernstern stable set. This means that every payoff not within the core can be dominated by a payoff within the core. Such a

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solution concept is a very convincing one as there is always a "better" payoff in the core than the payoffs outside the core (i.e. the payoff that dominates the payoff outside the core). So the question, under what conditions (i.e. what properties must the coalition function possess) is the core stable, is an important question that deserves the attention of game theorists.

The question of core stability is almost as old as the field of Game Theory itself. One of the earliest attempts to characterise when the core is stable was provided by Gillies ([5]) in 1959. In this article Gillies introduced the idea of majorization and proved that the core is stable if and only if any imputation not belonging to the core is "majorizable". However this solution only partially answered the question, as the conditions under which the core is stable were not given via the coalition function. A second noteworthy attempt was that of T.E. Kulakovskaja in 1971, ([9]). In this article the author claims to have found necessary and sufficient conditions, expressed via the coalition function, for core stability. Unfortunately this paper contains an error which will be expounded now. For all the necessary notation not given here the reader is referred to the article ([9]). However, some of the notation required to state the "Theorem" will be provided here. First of all, let $\mathcal{I}(v)$ be the set of imputations in $0-1$ normalized form and let

$$U_k(\{S_1, \ldots, S_k\})$$

be the following set

$$\{x \in \mathcal{I}(v) \mid \sum_{j \in S_i} x_j < v(S_i), 1 \leq i \leq k; \sum_{j \in T_i} x_j \geq v(T_i), T \neq S_i, 1 \leq i \leq k\}.$$

Secondly, the following definition will be necessary to state the "Theorem".

Definition 1.1. *A coalition S such that the following inequalities,*

$$y(T) \geq v(T) \,\forall\, T \text{ such that } T \nsubseteq S$$

$$y(S) \leq v(S)$$

is solvable is said to be a strong coalition.

The following "Theorem" is stated in the paper ([9]).

Theorem 1.2. *A game (N, v) has a stable core if and only if for any set of coalitions $\{S_1, \ldots, S_k\}$ with $U_k(\{S_1, \ldots, S_k\}) \neq \emptyset$ the following conditions are satisfied,*
1) the set $\{S_1, \ldots, S_k\}$ contains at least one strong coalition (without loss of generality, let $\{S_1, \ldots, S_p\}$ be the set of strong coalitions)
2) for any system of coverings $\{\mathcal{U}_i, \Lambda_i\}_{i=1}^p$ associated respectively the strong coalitions $\{S_1, \ldots, S_p\}$ there exists a covering $\{\mathcal{U}, \Lambda\}$ generated by it such that one of the following two relations hold
a) $N \backslash S_i \notin \mathcal{U}, 1 \leq i \leq k$ and

$$\sum_{i=1}^{m} \alpha_i \bar{v}(T_i) + \sum_{i=1}^{p} c_i [v(N) - \sum_{j=1}^{m_i} \beta_j^{(i)} v^{(i)}(T_j^{(i)})] > v(N) \tag{1.1}$$

$$\sum_{i=1}^{m} \alpha_i \bar{v}(T_i) + \sum_{i=1}^{p} c_i[v(N) - \sum_{j=1}^{m_i} \beta_j^{(i)} v^{(i)}(T_j^{(i)})] > v(N) \tag{1.2}$$

However when one considers the case $p = 1$, a plausible case for which examples can be easily found (eg. $k = 1$ in the above theorem such that $U_1(\{S_1\}) \neq \emptyset$ with S_1 a strong coalition suffices. I give an example below for this case.), then equation (1.1) reduces to the following

$$\sum_{i=1}^{m} \alpha_i \bar{v}(T_i) + v(I) - \sum_{i=1}^{m} \alpha_i \bar{v}(T_i) > v(I)$$

which is equivalent to $v(I) > v(I)$ which is absurd. Equation (1.2) then also reduces to the statement that $v(i) \geq v(I)$ which is also absurd. So the question concerning core stability was not solved by this article.

Example 1.3. An example will be given here showing the absurdity of the previous theorem. Consider the game with $N = \{1, 2, 3\}$ that satisfies $v(N) = 2$,

$$v(S) = 1 \ \forall \ S \subset N \text{ with } |S| = 2$$

and

$$v(i) = 0 \ \forall \ i \in N.$$

The core of this game is stable (see the comments below for symmetric games). Note that $U_1(\{12\}) \neq \emptyset$, as for example

$$x = (0, 0.5, 1.5) \in U_1(\{12\}).$$

However $\{12\}$ is a strong coalition, as the imputation $x = (0.5, 0.5, 1)$ demonstrates. Hence the core is stable, according to the previous "Theorem", if and only if either $2 > 2$ or $0 \geq 2$. Both these conditions are absurd hence the core cannot be stable contradicting the fact that the core is stable, showing that the theorem is false.

The main method of addressing the problem of core stability, which is utilised by Game Theorists today, can be traced back to a paper by W. Sharkey in 1982 ([15]). In this paper he introduced the idea of a *large core* and proved that if the core is *large* then it is stable. Following the ideas of this result, K. Kikuta and L. Shapley ([8]) invented a new concept, that of *extendability*, and showed that if a core is *large* then it is *extendable* and if the core is *extendable* then it is stable. Researchers have also found examples of cores which are *extendable* but not *large*, see J.R.G. van Gellekom et al. ([4]). Since then, Game Theorists have continued working in this direction in an attempt to generalise the results to find a 'largeness' concept that is both necessary and sufficient for core stability for all games in general.

That is not to say, however, that for specific classes of games core stability has not already been characterised. In the article of A.K. Biswas et al. ([2])), the authors provide necessary and sufficient conditions for core stability in the case of symmetric games. For the definition of a symmetric game the reader is referred to the aforementioned article. The theorem is as follows

Theorem 1.4. *For a symmetric game (N, v) the statements "(N, v) has a large core", "(N, v) has a stable core" and "(N, v) is exact" are equivalent.*

As well, in the case of Assignment games and totally balanced Min-colouring games (see the corresponding sections for the relevant definitions) core stability has also been fully characterised. For the relevant theorems the reader is referred to the section on Assignment games and Min-colouring games.

These results may be satisfactory for the special classes of games considered, however one still would like necessary and sufficient conditions for core stability for all games in general. So game theorists are working parallel by attempting to solve the problem for special classes of games and concomitantly hoping that these solutions will provide insights into the general case. In this thesis an attempt is made to contribute to this endeavour and the methods employed are mainly based on the aforementioned results by K. Kikuta and L. Shapley, namely extendability.

1.1 Cooperative Games

In this section the relevant ideas employed in this thesis and the development and theory of these ideas that pertains to the following work will be presented. The type of game analysed in this and the later sections is that of a cooperative game in normalised form as introduced by von Neumann and Morgernstern in "The Theory of Games and Economic Behaviour", ([18]). The cooperative game in normalised form is one of two of the major game types introduced in their book, the second style of game falling under the rubric of "non-cooperative". The major difference between a cooperative game and a non-cooperative game was elucidated by John Harsanyi in his paper "A General Theory of Rational Behavior in Game Situations", ([6]), published in 1966. The basic idea was that a game is cooperative if one is allowed to make agreements, promises and other such contracts between players such that an ascendence between the participants of the game is enforceable. Non-cooperative games treat the situation where such enforceable contracts are not available.

The games treated in this section will encompass games with a finite number of players and transferrable utility (TU). In the setting of an n-person cooperative TU game one has pair, (N, v), where N represents the set of players and is usually taken to be a finite subset of the natural numbers with numbers representing players (which is what will be adopted here), i.e. $N \subsetneqq \mathbb{N}$. For the sake of simple notation it will be assumed that $|N| = n$. Finally v is the coalition function, $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, which describes intuitively the worth of the coalition. A payoff to the players is generated by a vector x, $x \in \mathbb{R}^n$. To simplify the notation one often introduces the following convention for a vector $x \in \mathbb{R}^n$ and a set $S \subseteq N$:

$$x(C) := \sum_{i \in C} x_i,$$

where each x_i stands for $\langle x, e_i \rangle$, (the brackets, $\langle\ ,\ \rangle$, denote the scalar product) and e_i for the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i^{th} position. In addition, when comparing two vectors, $x, y \in \mathbb{R}^n$,

$$x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n),$$

if the notation

$$x \leq y$$

(or any other type of inequality for that matter, with the corresponding changes in the preceding and following equations) for two vectors is used then it means

$$x_i \leq y_i \ \forall\ i = 1, \ldots, n.$$

There are special classes of vectors in $\mathbb{R}^n$ for a given game (N, v) that will be relevant for later on and one of these germane classes is known as the class of imputations, denoted by $\mathcal{I}(v)$. In order to define the class of imputations one needs two concepts, that of Pareto optimality and individual rationality. The two definitions are as follows.

Definition 1.5. *A payoff vector, $x \in \mathbb{R}^n$, is said to be* ***Pareto optimal*** *if $x(N) = v(N)$.*

The second definition of individual rationality is

Definition 1.6. *A payoff vector, $x \in \mathbb{R}^n$, is said to be* ***individually rational*** *if $x_i \geq v(\{i\})$ for all $i = 1, \ldots, |N|$.*

For a given game v, the class of imputations, $\mathcal{I}(v)$, are the set of all vectors $x \in \mathbb{R}^n$ such that x is Pareto optimal and individually rational. That is,

$$\mathcal{I}(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(\{i\})\ \forall\ i \in N\}.$$

It is the relationship between imputations that defines the requisite solution concepts investigated in this thesis and one particularly important relationship between imputations is that of domination. This idea is encompassed by the following definition.

Definition 1.7. *Let $(N, \mathbb{P}, v)$ be an n-person cooperative game. An imputation η is said to* ***dominate*** *an imputation ζ via the coalition $\emptyset \neq D \subsetneqq N$ if the imputation η satisfies*

$$\eta(D) \leq v(D)$$

as well as

$$\eta_i > \zeta_i\ \forall\ i \in D.$$

In the case that η dominates ζ through the coalition D one writes

$$\eta\ \mathrm{dom}_D\ \zeta.$$

The idea of domination is used to define two important solution concepts for n-person cooperative games. One solution concept is that of the vNM stable set. It was defined by von Neumann and Morgernstern in their book "The Theory of Games and Economic Behavior" using the ideas of internal and external stability which are as follows.

Definition 1.8. *A set of imputations $\mathcal{V}$ is* ***internally stable*** *if there does not exist η and $\zeta \in \mathcal{V}$ such that η dominates ζ.*

Definition 1.9. *A set of imputations $\mathcal{V}$ is* ***externally stable*** *if for all $\zeta \in \mathcal{I}\backslash\mathcal{V}$ it follows that there exists a $\eta \in \mathcal{V}$ such that η dominates ζ.*

Definition 1.10. *Let (N, v) be an n-person cooperative game. A set of imputations in $\mathcal{I}$, $\mathcal{V}$, is said to be a* ***vNM stable set*** *if $\mathcal{V}$ is both internally and externally stable.*

The second solution concept introduced by D. B. Gillies in 1952 in Contributions to the Theory of Games, Volume IV, ([5]), utilising the idea of domination is that of the core. The core is defined as the set of all undominated imputations that is,

Definition 1.11 (Core)**.** *Let (N, v) be an n-person cooperative game. The core, $\mathcal{C}$, is the set of all undominated imputations.*

The core is a popular solution concept because of a property that it possesses that the vNM stable sets tend not to possess, that is, a mathematical description of it for finite games is readily available. This description is provided through the following Lemma

Lemma 1.12. *Let (N, v) be an n-person cooperative game. The core of the game v is the set of all imputations satisfying*

$$\boldsymbol{x}(S) \geq v(S) \; \forall \; \emptyset \neq S \subseteq N.$$

The core is a solution concept that has been extensively investigated. One of the main results concerning the core is a characterisation of when the core is not empty. This characterisation was provided by Bondareva ([3]) in 1962 and independently by Shapley ([13]) in 1967. To be able to state the theorem the idea of a balanced collection needs to be introduced. The definition here follows that of eg. Peleg and Sudhölter ([10])

Definition 1.13. *A collection of sets $\mathcal{B} \subseteq 2^N$, $\emptyset \notin \mathcal{B}$, is called N-balanced (or balanced over N) if there exists numbers $\delta_S > 0$, $S \in \mathcal{B}$ such that*

$$\sum_{S \in \mathcal{B}} \delta_S \chi_S = \chi_N,$$

where $\chi_T(i) = 1$ for all $i \in T$ and $\chi_T(i) = 0$ for all $i \in N\backslash T$. The collection $(\delta_S)_{S\in\mathcal{B}}$ is called a system of balancing weights.

In what follows, if there is no chance of confusion then the type of balancedness in question (N-balanced, S-balanced, etc.) will just be referred to as balanced in stead of N-balanced, etc. To present the Bondareva-Shapley Theorem one requires the definition of a minimally balanced collection.

Definition 1.14. *A balanced collection is minimally balanced if it has a unique system of balancing weights.*

Theorem 1.15. *Let (N, v) be a game. Then $\mathcal{C} \neq \emptyset$ if and only if*

$$v(N) \geq \sum_{S \in \mathcal{B}} \delta_S v(S) \; \forall \text{ minimally balanced } \mathcal{B} \text{ with balancing } (\delta_S)_{S \in \mathcal{B}}$$

Hence a game with a nonempty core will be called balanced. The next definition forms the basis of the first part of this investigation, that of the stable core. The core is stable when the core coincides with the vNM stable set.

Definition 1.16. *The core is **stable** if for all imputations $x \in \mathcal{I}(v) \backslash \mathcal{C}$ there exists an imputation $y \in \mathcal{C}$ and a set $\emptyset \neq S \subsetneqq N$ such that $y \ \mathrm{dom}_S \ x$.*

It will also be necessary to consider subgames of the game (N, v). The game (N, v) restricted to the player set S will be denoted by v_S and the core of this subgame will be denoted by $\mathcal{C}_S = \mathcal{C}(v_S)$. v_S is defined via $v_S(T) = v(T)$ for all $T \subseteq S$. The following definition is also important for a number of results later on.

Definition 1.17 (Extendability)**.** *Let (N, v) be a game. An imputation $y \in \mathcal{C}_{\mathcal{S}}$, for $\emptyset \neq S \subsetneqq N$, is **extendable** if there is a core element x of $\mathcal{C}$, such that $x_i = y_i$ for all $i \in S$. A coalition $\emptyset \neq S \subsetneqq N$ is **extendable** if for every core element y of the subgame (S, v_S) there is a core element x of (N, v) such that $x_i = y_i$ for all $i \in S$. The game (N, v) is **extendable** if every core element of every subgame can be extended to an element of the core of the game (N, v).*

The idea of extendability can be viewed from a number of different perspectives. A definition equivalent to the previous definition of extendability which reveals, more clearly, the role extendability plays for subsets of N is the following. Such definitions are not new and can be found in articles such as ([1]). Denote by $\mathcal{C}_S(v)$ (note that $\mathcal{C}_S := \mathcal{C}(v_S)$ and hence $\mathcal{C}_S(v) :\neq \mathcal{C}_S$) the projection of $\mathcal{C}$ to the subspace corresponding to the coalition S. That is

$$\mathcal{C}_S(v) := \{x \in \mathbb{R}^S \mid \text{there is } y \in \mathcal{C} \text{ such that } y_j = x_j \; \forall \, j \in S\}.$$

Definition 1.18. *A coalition S is extendable if $\mathcal{C}(v_S) \subseteq \mathcal{C}_S(v)$. The game v is extendable if $\mathcal{C}(v_S) \subseteq \mathcal{C}_S(v)$ for all $S \subseteq N$.*

As extendability plays a significant role in what is to come it would be useful if one could find conditions under which a coalition is extendable. The following result provides a sufficient condition for extendability.

Proposition 1.19. *Let (N, v) be a balanced game and $S \subseteq N$. Define the following coalition function for all $T \subsetneqq N \backslash S$*

$$w(T) := \max_{R \subseteq S}\{v(T \cup R) - v(R)\} \text{ and } w(N\backslash S) = v(N) - v(S).$$

Then S is extendable if the core of the game $(N\backslash S, w)$ is nonempty.

Proof: If the core of the game $(N\backslash S, w)$ is nonempty then it follows that there exists an $y' \in \mathcal{C}(w)$ with $y'(T) \geq w(T)$ for all $T \subseteq N\backslash S$. Then define for $y \in \mathcal{C}_S$

$$\zeta = (y, y').$$

It will now be shown that $\zeta \in \mathcal{C}$. Note that

$$\zeta(N) = y(N \cap S) + y'(N \cap N\backslash S) = v(S) + v(N) - v(S) = v(N).$$

As well for all $T \subseteq N$

$$\zeta(T) = y(T \cap S) + y'(T \cap N\backslash S) \geq v(T \cap S) + \max_{R \subseteq S}\{v(T \cap N\backslash S \cup R) - v(R)\}.$$

Letting $T \cap S = Q$ it follows that

$$v(Q) + \max_{R \subseteq S}\{v(T \cap N\backslash S \cup R) - v(R)\} \geq v(Q) + v(T \cap N\backslash S \cup Q) - v(Q) = v(T).$$

Hence $\zeta \in \mathcal{C}$ and all elements $y \in \mathcal{C}_S$ are extendable. **q.e.d.**

Related to this definition is the following Theorem. The goal of this section is to attempt to generalise this Theorem and explore the possibilities of providing necessary and sufficient conditions for core stability via the definition of extendability.

Theorem 1.20 (Kikuta and Shapley (1986)([8]))**.** *Let (N, v) be balanced. If (N, v) is extendable then the core of (N, v) is stable.*

Now a question that has been posed is, is the extendability of the game (N, v) a necessary condition for core stability? This question was answered by J. R. G. van Gellekom et al. (1999) ([4]) with the following example.

Example 1.21. Let $\delta \in (0, 1]$, $N = \{1, \ldots, 6\}$ and v be given by the following: $v(12) = v(13) = v(45) = v(46) = 1$, $v(N) = 4 - \delta$ and $v(S) = 0$ otherwise. The coalition $\{1, 4\}$ is not extendable, as $v(14) = 0$ as there does not exist an element, z, of the core of (N, v) with $z_1 = z_4 = 0$. The core is also stable (see article).

As this type of extendibility is not necessary one could also consider other types of extendability such as the following.

Definition 1.22 (i-Extendability)**.** *Let (N, v) be a game. An imputation $y \in \mathcal{C}_S$, for $\emptyset \neq S \subsetneq N$, is* ***i-extendable*** *if there is a core element x of $\mathcal{C}_{S \cup i}$, for some $i \in N \backslash S$, such that $x_i = y_i$ for all $i \in S$. A coalition $\emptyset \neq S \subsetneq N$ is* ***i-extendable*** *if for every core element y of the subgame (S, v_S) there is a core element x of the subgame $(S \cup i, v_{S \cup i})$, for some $i \in N \backslash S$, such that $x_i = y_i$ for all $i \in S$. The game (N, v) is* ***i-extendable*** *if every core element of every subgame (S, v_S) can be extended to an element of the core of the game $(S \cup i, v_{S \cup i})$, for some $i \in N \backslash S$ and all $S \subseteq N$.*

However this type of extendability also is not necessary because of the following result.

Proposition 1.23. *Let (N, v) be a game. If (N, v) is i-extendable then (N, v) is extendable.*

The proof of this result is left to the reader. Naturally one could also consider $S - extendability$ with appropriate modifications for sets $T \subseteq N$ such that $N \subseteq T \cup S$ however they also inevitably lead to a result like the previous proposition.

To explore, more carefully, the exact meaning of this example the idea of exactness needs to be introduced.

Definition 1.24 (Exact)**.** *A coalition, $\emptyset \neq S \subsetneq N$, is* ***exact*** *if there exists a $y \in \mathcal{C}$ with $y(S) = v(S)$. A game is said to be exact if every coalition, $\emptyset \neq S \subsetneq N$, is exact.*

Note that proposition (1.19) can also be seen as a sufficient condition for exactness of a coalition S. Using the definition of exactness it seems apparent that to demand the extendability of every coalition would be unreasonable. The reason for this is, if one considers the example of J. R. G. van Gellekom et al. then one notices that the coalition $\{14\}$ is not exact. So it would seem absurd to require that the core of the subgame $v_{\{14\}}$ be extendable. However, although the extendability of every coalition is not a necessary condition for core stability, one has the following result.

Lemma 1.25. *Let (N, v) be a balanced game and let $\mathcal{C}$ be stable. Let S be a coalition such that $\mathcal{C}_S$ is not empty. If S satisfies*

$$v(S) + (|N \backslash S| - 1)v(N) = \sum_{i \in N \backslash S} v(N \backslash i) \tag{1.3}$$

then S is extendable.

Proof: For all $y \in \mathcal{C}$ the following statement is true for all $i \in N$,

$$y_i \leq v(N) - v(N \backslash i). \tag{1.4}$$

as $y(N\backslash i) \geq v(N\backslash i)$. Let S satisfy equation (1.3) and choose an element $y \in \mathcal{C}_S$ and let $|N\backslash S| = m$. Then define

$$z := (v(N) - v(N\backslash j_1), \ldots, v(N) - v(N\backslash j_m))$$

for all $j_i \in N\backslash S$ as well as the following vector

$$y' = (y, z).$$

Then y' satisfies

$$y'(N) = y(S) + z(N\backslash S) = v(S) + \sum_{i \in N\backslash S} (v(N) - v(N\backslash i)) =$$

$$v(S) + (|N\backslash S| - 1)v(N) - \sum_{i \in N\backslash S} v(N\backslash i) + v(N) = v(N)$$

by equation (1.3). Were y' not an element of the core then, because the core is stable, there must exist an imputation $x \in \mathcal{C}$ that dominates y'. However it is impossible to dominate y' via a coalition $T \subseteq S$ because

$$y'(T) \geq v(T).$$

This then implies that the dominating coalition must contain $i \in N\backslash S$. But that is impossible because for all $x \in \mathcal{C}$, x fulfills equation (1.4) for all $i \in N$. Hence all elements $y \in \mathcal{C}_S$ are extendable. **q.e.d.**

As a Corollary one has the following result.

Corollary 1.26. *Let (N, v) be a balanced game and let $\mathcal{C}$ be stable. Let $i \in N$. If $\mathcal{C}_{N\backslash\{i\}}$ is nonempty then it is extendable.*

This result is interesting because of the following result.

Proposition 1.27. *Let (N, v) be a balanced game and let for all $i \in N$, $\mathcal{C}_{N\backslash\{i\}}$ be stable and extendable. Then $\mathcal{C}$ is stable.*

Proof: First of all let x be an imputation, $\notin \mathcal{C}$ and let $S \subseteq N$ such that $x(S) < v(S)$ and for all $T \subsetneqq S$ it follows that

$$x(T) \geq v(T).$$

If $S = N\backslash i$ for $i \in N$ then define

$$\varepsilon = v(N\backslash i) - x(N\backslash i).$$

Let $|N\backslash i| = n - 1$. Now add to each element x_i, for $i \in N\backslash i$, the amount $\frac{\varepsilon}{n-1}$. That is define x' by

$$x' = x + \frac{\varepsilon}{n-1} \chi_{N\backslash i}.$$

Now $x' \in \mathcal{C}_{N\backslash i}$ and the coalition $N\backslash i$ is extendable and hence x' is extendable to an element $y \in \mathcal{C}$. This element y then satisfies

$$y \ \mathrm{dom}_{N\backslash i} \ x.$$

Hence the core $\mathcal{C}$ is stable. In the case that $S \neq N\backslash i$ then $S \subsetneqq N\backslash i$ for some $i \in N$. As $\mathcal{C}_{N\backslash i}$ is stable there exists an $y \in \mathcal{C}_{N\backslash i}$ and a coalition $S \subsetneqq N\backslash i$ such that

$$y \ \mathrm{dom}_S \ x$$

and this y is extendable to an element of $\mathcal{C}$. Hence the core $\mathcal{C}$ is stable. **q.e.d.**

Both these results are interesting as they show that necessary and sufficient conditions for core stability lie somewhere between the extendibility of all $N\backslash i$ coalitions and the stability and extendibility of all $N\backslash i$ coalitions.

So as has been demonstrated the extendability of every coalition implies core stability however the extendability of all $N\backslash i$ coalitions (when nonempty as well as all coalitions satisfying equation (1.3)) is implied by core stability. Based on these thoughts the next logical step seems to be

Conjecture 1.28. *Let (N, v) be a balanced game. If for all exact coalitions $\emptyset \neq S \subsetneqq N$, S is extendable then $\mathcal{C}$ is stable.*

To simplify the language the extendability of all exact coalitions will be referred to as *exact extendability*. The Conjecture will now be proven. To prove the Conjecture however the following two Lemmata will be required. To simplify the original method of proof presented by the author, two Lemmata proposed by Peter Sudhölter are presented here. To see the original proof by the author, the reader is referred to the Appendix.

Lemma 1.29. *Let (N, v) be a balanced TU game and let $x \in \mathbb{R}^N$, $x(N) = v(N)$, $x \notin \mathcal{C}$. Then there exists an exact $P \subsetneqq N$ such that $x(P) < v(P)$*

Proof:
Let $x^1 \notin \mathcal{C}$ be as in the statement of the Lemma and let $x^0 \in \mathcal{C}(N, v)$. Define

$$x^\lambda := \lambda x^1 + (1 - \lambda)x^0$$

for $0 \leq \lambda \leq 1$. As the core is convex and closed, there exists $0 < \hat{\lambda} < 1$ such that

$$\{x^\lambda \mid 0 \leq \lambda \leq 1\} \cap \mathcal{C}(N, v) = \{x^\lambda \mid 0 \leq \lambda \leq \hat{\lambda}\}.$$

Set $y = x^{\hat{\lambda}}$. Then there exists $P \subsetneqq N$ such that $y(P) = v(P)$ and $x^1(P) < v(P)$, because $x^1 \notin \mathcal{C}$. The coalition P has the desired properties. **q.e.d.**

A Corollary of this result which will be useful later on is the following. Let

$$E(x) := \{S \mid S \text{ is exact and } x(S) < v(S)\}. \tag{1.5}$$

Then it follows that

Corollary 1.30. *Let $x \in \mathbb{R}^N$, $x(N) = v(N)$, $x \notin \mathcal{C}$. Then $E(x) \neq \emptyset$.*

The following Lemma is the main tool used in the proof of the Conjecture. However before the Lemma is presented the following notation is required. Let x be an imputation and define

$$M(x) := \{S \mid S \subsetneqq N; S \text{ exact with } x(S) < v(S) \text{ and } x(T) \geq v(T) \,\forall\, T \subsetneqq S\}.$$

The set $M(x)$, for a given imputation x, represents the exact coalitions S such that there does not exist any exact subcoalition T with $x(T) < v(T)$. With this notation the following Lemma will be proven:

Lemma 1.31. *Let (N, v) be a balanced game and let every exact coalition in $\mathcal{C}$ be extendable. If $x \notin \mathcal{C}$ then $M(x) \neq \emptyset$.*

Proof:
The Lemma will be proven via contradiction. So assume that there exists an $x \notin \mathcal{C}$ such that $M(x) = \emptyset$. That is, for all exact $S \subsetneqq N$ with $x(S) < v(S)$ there exist $T \subsetneqq S$ with $x(T) < v(T)$ and T is not exact in $\mathcal{C}$. Now choose an exact $S \subsetneqq N$ such that there does not exist an exact $R \subsetneqq S$ with $x(R) < v(R)$. Then as before there exists a non-exact T with $x(T) < v(T)$. Now define the vector

$$y = x + \frac{v(S) - x(S)}{|S \setminus T|}\chi_{S\setminus T}$$

(where χ_R is the indicator function on $R \subseteq S$). Then $x \leq y$, $y \notin \mathcal{C}_S$ and $y(S) = v(S)$. Lemma (1.29) applied this to the imputation y with N in Lemma (1.29) replaced by S implies that there exists a Q, $Q \subsetneqq S$, exact in $\mathcal{C}_S$ such that $y(Q) < v(Q)$ and hence $x(Q) < v(Q)$. However as S is extendable, by assumption, Q must be exact in $\mathcal{C}$ contradicting the assumption that there did not exist an exact $R \subsetneqq S$ with $x(R) < v(R)$. **q.e.d.**

With this Lemma the main result can now be proven.

Theorem 1.32. *Let (N, v) be a balanced game. If (N, v) is exact extendable then $\mathcal{C}$ is stable.*

Proof: By the previous Lemma it follows that for all imputations $x \notin \mathcal{C}$, $M(x) \neq \emptyset$. So for a given x, choose $S \in M(x)$ and define

$$\varepsilon = v(S) - x(S).$$

Let $s = |S|$. Now add to each element x_i, for $i \in S$, the amount $\frac{\varepsilon}{s}$. That is define x' by

$$x' = x + \frac{\varepsilon}{s}\chi_S.$$

Now $x' \in \mathcal{C}_S$ and the coalition S is exact and hence x' is extendable to an element $y \in \mathcal{C}$. This element y then satisfies

$$y \ \mathrm{dom}_S \ x.$$

Hence the core $\mathcal{C}$ is stable. **q.e.d.**

Although exact extendability is sufficient for core stability it is not necessary. An example, due to Peter Sudhölter, demonstrating this will be given in the next subsection on Assignment games. However before this is done a class of games will be presented here, for which exact extendability is a necessary condition for core stability. The class of games are known as simple games.

Definition 1.33. *A game (N, v) is a simple game if the coalition function v only takes on the values 0 or 1.*

Before the result can be stated for simple games the following definition is needed.

Definition 1.34. *A veto player is a player $i \in N$ such that $v(S - i) = 0$, for all coalitions $\emptyset \neq S \subseteq N$. Let $V = \{i \in N \,|\, i \text{ is a veto player}\}$ be the set of all veto players.*

One also has the following well known result.

Lemma 1.35. *Let (N, v) be a simple game. The core is non empty if and only if $V \neq \emptyset$.*

Using this result the following theorem can be proven.

Theorem 1.36. *Let (N, v) be a simple game then the following are equivalent,*
1) the core is stable,
2) $V \neq \emptyset$ and
3) (N, v) is exact extendable.

Proof: The direction 1) $\Rightarrow$ 2) follows directly from Lemma (1.35). The direction 2) $\Rightarrow$ 3) can be proven as follows. Note that all imputations in the core are of the form

$$\sum_{i \in V} x_i = 1 \text{ and } x_i \geq 0 \; \forall \, i \in V \text{ and } x_i = 0 \; \forall \, i \notin V. \tag{1.6}$$

One notices that if $V \neq \emptyset$ the game (N, v) is exact. Hence it will be shown how an arbitrary coalition $S \subsetneqq N$ can be extended. To do this a number of cases will be considered. Firstly consider the case that $S \cap V \neq \emptyset$ and $V \nsubseteq S$. Then the only imputation in $\mathcal{C}_S$ is the vector with zero in all the coordinates $i \in S$. To extend this vector one defines $y_i = 0$ for all $i \in N \backslash (V \cup S)$ and then simply divides 1 amongst the players $V \backslash S$. By equation (1.6) it follows that this new vector y is an element of the core. If $S \cap V \neq \emptyset$ and $V \subseteq S$ then one extends an element of $\mathcal{C}_S$ by giving all $i \in N \backslash S$ the value 0. Finally if $S \cap V = \emptyset$ then it is obvious how to extend the vector. Finally 3) $\Rightarrow$ 1) is implied by Theorem (1.32). **q.e.d.**

As was stated earlier, in general, exact extendability is not necessary for core stability. This requires one to look for new ideas relating to extendability that might prove to be necessary for core stability. The following is such.

Definition 1.37 (Full Coalition)**.** *Let (N, v) be a game and S be a coalition. Call $\emptyset \neq S \subsetneqq N$ a **full coalition** if there exists a core element, x, of the subgame (S, v_S) such that $x(T) > v(T)$ for all proper subcoalitions T of S; that is, S is a full coalition if the dimension of $\mathcal{C}_S$ is full, i.e. $(|S| - 1)$.*

Full coalitions are very interesting coalitions and play an important role in the stability of the core as the following Lemma and subsequent Corollary (1.41) demonstrate.

Lemma 1.38. *Let (N, v) be a game and an imputation $x \notin \mathcal{C}$ such that there exists an imputation y and a coalition S such that*

$$y \ \mathrm{dom}_S \ x.$$

Then there exists a full coalition $T \subseteq S$ such that

$$y \ \mathrm{dom}_T \ x.$$

Proof: Were S and all subcoalitions of S such that none of them were full then it follows that there exists $T_1 \subsetneqq S$ such that $y(T_1) = v(T_1)$ however $y_i > x_i$ for all $i \in T_1$. However T_1 is also not full hence there exists $T_2 \subsetneqq T_1$ such that $y(T_2) = v(T_2)$ with $y_i > x_i$ for all $i \in T_2$. This argument can be continued to the point where one is only left with singleton subsets of the original set T_n. However this set T_n is also not full hence there exists a $i \in T_n$ such that $y_i = v(i)$ and hence y cannot dominate x. **q.e.d.**

The goal is now to restrict the extendability of coalitions not only to exact coalitions but to exact coalitions which are also full. However there are two possible ways to do this. They are both defined here and called strong and weak fully exactness.

Definition 1.39 (Weak Fully Exact)**.** *Let (N, v) be a game. A coalition S is called **weak fully exact**, if S is both a full coalition and there exists a $y \in \mathcal{C}$ such that $y(S) = v(S)$.*

Weak fully exactness is due to Peter Sudhölter.

Definition 1.40 (Strong Fully Exact)**.** *Let (N, v) be a game. A coalition S is called **strong fully exact**, if S is both a full coalition and there exists a $y \in \mathcal{C}$ such that $y(S) = v(S)$ and $y(T) > v(T)$ for all $T \subsetneqq S$.*

Obviously strong fully exactness implies weak fully exactness.

Again if all weak/strong fully exact coalitions are extendable then the game will be referred to as *weak/strong fully exact extendable.*

As a corollary of Lemma (1.38) one has the following result.

Corollary 1.41. *Let (N, v) be a balanced game.*
Let $\Phi = \{S \subseteq N | S$ is strong fully exact$\}$. If the core is stable then $\Phi \neq \emptyset$ or $\mathcal{C} = \mathcal{I}(v)$.

Proof: If $\Phi = \emptyset$ and $\mathcal{C} \neq \mathcal{I}(v)$ then it follows be the previous lemma that there exists an imputation $x \notin \mathcal{C}$ such that x cannot be dominated. Hence if $\Phi = \emptyset$ the only possibility is that $\mathcal{C} = \mathcal{I}(v)$. **q.e.d.**

Before discussing weak/strong fully exact coalitions and results related to them, a few observations regarding the earlier set $M(x)$ are appropriate. First of all one notices that for all $S \in M(x)$, S is weak fully exact. This observation will be interesting when the set $FE_W(x)$ is introduced later on.

To state the next result the following idea of a minimal set is requisite.

Definition 1.42. *A coalition S is **minimal** in a collection of sets C, if there does not exist $T \subsetneqq S$ with $T \in C$.*

One now notices the following result regarding the elements of $E(x)$ and the elements of $M(x)$.

Lemma 1.43. *Let (N, v) be a balanced TU game and let x be an imputation and $x \notin \mathcal{C}$. Then all minimal $S \in E(x)$ satisfy $S \in M(x)$.*

Proof: So choose minimal $S \in E(x)$ and assume that $S \notin M(x)$. This implies that there exists a coalition $T \subsetneqq S$ such that $x(T) < v(T)$. A simple application of Lemma (1.29), exactly as was done in Lemma (1.31), shows now that there exists a $R \subsetneqq S$ such that R is exact and $x(R) < v(R)$. A contradiction. **q.e.d.**

This Lemma also provides an indirect proof of the coming Lemma (1.44). So as one can see the sets that play an important and interesting role are those S that are minimal in $E(x)$. By examining these minimal sets in $E(x)$, i.e. the sets in $M(x)$, this leads one to the definition of weak/strong fully exactness. This line of thought will again be taken up after Theorem (1.46) when the corresponding sets for weak/strong fully exactness are examined.

Weak and strong fully exact coalitions are a rather auspicious concept when it comes to proving results relating to full exact coalitions. This is due to the fact that fully exact coalitions are a rare species of coalitions. In a specific type of game either only a small number of coalitions turn out to be fully exact or fully exact coalitions do not exist at all. This can be seen as an advantage. For the classes of games, for which fully exact coalitions do exist, this makes proving that fully exact extendability is a necessary condition for core stability a lot easier because one doesn't have to consider all coalitions.

In order to prove that weak fully exact extendability is sufficient for core stability two Lemmata need to be proven. In the following Lemma let the set $E(x)$ be that defined by equation (1.5). The proof given here is due to Peter Sudhölter. To see a proof by the author the reader is referred to the appendix.

Lemma 1.44. *Let (N, v) be a balanced TU game and let x be an imputation and $x \notin \mathcal{C}$. Then for all minimal $S \in E(x)$ it follows that S is weak fully exact.*

Proof: By Corollary (1.30), $E(x) \neq \emptyset$. Let $S \in E(x)$ be minimal. It will now be shown that S is weak fully exact. Indeed, let $\beta \in \mathcal{C}$ such that $\beta(S) = v(S)$. By the minimality of S, $\beta(T) > v(T)$ for all $T \subsetneqq S$ with $x(T) < v(T)$. Hence, there exists $\varepsilon > 0$ such that $y(T) \geq v(T)$ for all $T \subsetneqq S$, where $y = (1-\varepsilon)\beta + \varepsilon x$. As $\varepsilon > 0$, $d = v(S) - y(S) > 0$. Hence it can be concluded that

$$z = y + \frac{d}{|S|}\chi_S \in \mathcal{C}_S$$

and $z(T) > v(T)$ for all $\emptyset \neq T \subsetneqq S$. **q.e.d.**

With this result a Lemma similar to Lemma (1.31) can be proven, which will then be applied in a similar fashion to prove the main result relating weak fully exact extendability and core stability. So as before, let x be an imputation and define

$$FE_W(x) := \{S \mid S \subsetneqq N; S \text{ is weak fully exact with } x(S) < v(S)$$

$$\text{and } x(T) \geq v(T)\ \forall\, T \subsetneqq S\}.$$

Lemma 1.45. *Let (N, v) be a balanced game and let every weak fully exact coalition in $\mathcal{C}$ be extendable. If $x \notin \mathcal{C}$ then $FE_W(x) \neq \emptyset$.*

Proof:
The Lemma will be proven via contradiction. So assume that there exists a $x \notin \mathcal{C}$ such that $FE_W(x) = \emptyset$. That is, for all weak fully exact $S \subsetneqq N$ with $x(S) < v(S)$ there exist $T \subsetneqq S$ with $x(T) < v(T)$ and T is not weak fully exact in $\mathcal{C}$. Now choose a weak fully exact $S \subsetneqq N$ such that there does not exist a weak fully exact $R \subsetneqq S$ with $x(R) < v(R)$. Then as before there exists a non-weak fully exact T with $x(T) < v(T)$. Define the vector

$$y = x + \frac{v(S) - x(S)}{|S \setminus T|}\chi_{S \setminus T}.$$

Then $x \leq y$ and $y(S) = v(S)$. As $y \notin \mathcal{C}_S$ it follows, from Lemma (1.44), that there exists a Q, $Q \subsetneqq S$, weak fully exact in $\mathcal{C}_S$ such that $y(Q) < v(Q)$ and hence $x(Q) < v(Q)$. However as S is extendable, by assumption, Q must be weak fully exact in $\mathcal{C}$ contradicting the assumption that there did not exist a weak fully exact $R \subsetneqq S$ with $x(R) < v(R)$. **q.e.d.**

With this result the main Theorem relating weak fully exact extendability and core stability can be proven.

Theorem 1.46. *Let (N, v) be a balanced game. If (N, v) is weak fully exact extendable then $\mathcal{C}$ is stable.*

Proof: By the previous Lemma it follows that for all imputations $x \notin \mathcal{C}$, $FE_W(x) \neq \emptyset$. So for a given x, choose $S \in FE_W(x)$ and define

$$\varepsilon = v(S) - x(S).$$

Let $s = |S|$. Now add to each element x_i, for $i \in S$, the amount $\frac{\varepsilon}{s}$, that is define x' by

$$x' = x + \frac{\varepsilon}{s}\chi_S$$

Now $x' \in \mathcal{C}_S$ and the coalition S is exact and hence x' is extendable to an element $y \in \mathcal{C}$. This element y then satisfies

$$y \ \mathrm{dom}_S \ x.$$

Hence the core, $\mathcal{C}$, is stable. **q.e.d.**

In addition one can replace 3) in Theorem (1.36) with *(N,v) is weak fully exact extendable* and the result remains valid. The proof of this claim is left to the reader.

However like exact extendability, weak fully exact extendability is not a necessary condition for core stability. The truth of this statement is demonstrated in the section on Orthogonal games.

A short analysis of the set $FE_W(x)$, $M(x)$, *inter alia* will be provided here, similar to that given after the definition of fully exactness. First of all if one examines the definition of $FE_W(x)$ one notices that in reality

$$FE_W(x) = M(x).$$

This claim follows directly from the definitions. In addition, by defining the following set

$$F_W(x) := \{S \mid S \text{ is weak fully exact and } x(S) < v(S)\}, \tag{1.7}$$

one can show the following result, analogous to Lemma (1.43).

Lemma 1.47. *Let (N, v) be a balanced TU game and let $x \in \mathbb{R}^N$, $x(N) = v(N)$, $x \notin \mathcal{C}$. Then all minimal $S \in F_W(x)$ are such that $S \in FE_W(x)$ and hence $S \in M(x)$.*

The reader may then ask him or herself, is it then true that the minimal coalitions in $E(x)$ and $F_W(x)$ are the same? If so, then the reader will have inferred correctly. This follows directly from Lemma (1.43). These considerations lead to the result that an analogous Lemma, like Lemma (1.44), would not be correct for strong fully exactness. That is, for $x \notin \mathcal{C}$ there does not necessarily exist a strong fully exact coalition S so that $x(S) < v(S)$. The following example demonstrates this observation.

Example 1.48. Define (N, v) as follows.

$$v(N) = 2, v(12) = 2, v(23) = 1, v(13) = 1 \text{ and } v(i) = 0 \;\forall\, i \in N$$

Then $\mathcal{C} = (1, 1, 0)$ and for the point $x = (0.8, 1.2, 0) \notin \mathcal{C}$ there does not exist a strong fully exact S so that $x(S) < v(S)$ as only $x(13) < v(13)$ and the coalition $S = \{1, 3\}$ is not strong fully exact.

Although strong fully exact extendability is, in general, not sufficient for core stability. It is sufficient for certain classes of games as the following results demonstrate.

Proposition 1.49. *Let (N, v) be a balanced TU game and and let the coalition N be full. Let x be an imputation and $x \notin \mathcal{C}$. Then there exists $S \in E(x)$ such that S is strong fully exact.*

Proof: Consider $x \notin \mathcal{C}$ and choose $y \in \mathcal{C}$ such that $y(N) = v(N)$ and $y(S) > v(S)$ for all $S \subsetneqq N$. Let $1 \geq \varepsilon > 0$ and consider the following vector

$$\beta = (1 - \varepsilon)y + \varepsilon x.$$

Then choose ε so that $\beta(T) \geq v(T)$ for all $T \subseteq N$ and that equality holds for at least one coalition $S \subsetneqq N$. Then take the smallest $R \subsetneqq N$ such that $\beta(R) = v(R)$ and there does not exist $Q \subsetneqq R$ with $\beta(Q) = v(Q)$ (this is possible as $x(i) \geq v(i)$ for all $i \in N$ and $y(i) > v(i)$ for all $i \in N$). Then the coalition R is strong fully exact and $x(R) < v(R)$. **q.e.d.**

Now one can define a set analogous FE_W as follows

$$FE_S(x) := \{S \mid S \subsetneqq N; S \text{ is strong fully exact with } x(S) < v(S)$$

$$\text{and } x(T) \geq v(T) \; \forall \, T \subsetneqq S\}.$$

and prove an analogous result to Lemma (1.45), that is,

Lemma 1.50. *Let (N, v) be a balanced game with the coalition N full and let every strong fully exact coalition in $\mathcal{C}$ be extendable. If $x \notin \mathcal{C}$ then $FE_S(x) \neq \emptyset$.*

Subsequently the following theorem follows from Lemma (1.50).

Theorem 1.51. *Let (N, v) be a balanced game such that the coalition N is full. If (N, v) is strong fully exact extendable then $\mathcal{C}$ is stable.*

The drawback of this result is that numerous TU cooperative games do not satisfy the condition that N is full, for example Assignment games and Orthogonal games.

Other ways of extending the idea of extendability have also been examined by numerous researches in order to try and find a type of extendability that is both sufficient and necessary for core stability. One such idea that extends the idea of extendability is the following given in ([17]) and uses the idea of an essential coalition.

Definition 1.52. *A coalition $S \subseteq N$ is essential if for all partitions P of S it follows that*

$$\sum_{T \in P} v(T) < v(S)$$

Definition 1.53. *A game (N, v) is called essential extendable if all essential coalitions are extendable.*

In ([17]) they prove the following result

Lemma 1.54. *Let (N, v) be a balanced game. If (N, v) is essential extendable then the core is stable.*

A natural extension of this result would be to not just consider partitions but also balanced collections.

Definition 1.55. *A coalition $S \subseteq N$ is minimally balanced essential if for all minimally balanced collections $\mathcal{B}$ of S with balancing coefficients $(\delta_T)_{T\in\mathcal{B}}$ it follows that*

$$\sum_{T\in\mathcal{B}} \delta_T v(T) < v(S)$$

Definition 1.56. *A game (N, v) is called min balanced essential extendable if all minimally balanced essential coalitions are extendable.*

Lemma 1.57. *Let (N, v) be a balanced game. If (N, v) is min balanced essential extendable then the core is stable.*

Proof: Let $x \notin \mathcal{C}$ be an imputation. Choose $S \subsetneqq N$ such that $x(S) < v(S)$ however for all $T \subsetneqq S$, $x(T) \geq v(T)$. It then follows that the coalition S is minimally balanced essential. If S were not min balanced essential then there would exist a minimally balanced coalition $\mathcal{B}$ such that

$$v(S) \leq \sum_{T\in\mathcal{B}} \delta_T v(T) \leq \sum_{T\in\mathcal{B}} \delta_T x(T) = x(S) < v(S)$$

a contradiction. So let

$$\varepsilon = v(S) - x(S)$$

Let $s = |S|$. Now add to each element x_i, for $i \in S$, the amount $\frac{\varepsilon}{s}$, that is define x' by

$$x' = x + \frac{\varepsilon}{s}\chi_S$$

Now $x' \in \mathcal{C}_S$ and the coalition S is min balanced essential and hence x' is extendable to an element $y \in \mathcal{C}$. This element y then satisfies

$$y \ \mathrm{dom}_S \ x.$$

Hence the core, $\mathcal{C}$, is stable. **q.e.d.**

The idea of a minimal balanced essential coalition is not only interesting because of the previous result but also because of the following.

Lemma 1.58. *A coalition is full if and only if it is minimally balanced essential.*

Proof: To prove the if direction assume that the coalition S is not minimally balanced essential, that is there exists a minimally balanced collection $\mathcal{B}$ with balancing coefficients $(\delta_T)_{T\in\mathcal{B}}$ such that

$$v(S) \leq \sum_{T\in\mathcal{B}} \delta_T v(T).$$

As S is full there exists an imputation $x \in \mathcal{C}_S$ such that $x(T) > v(T)$ for all $T \subsetneqq S$. This then implies that

$$v(S) \leq \sum_{T\in\mathcal{B}} \delta_T v(T) < \sum_{T\in\mathcal{B}} \delta_T x(T) = x(S) = v(S)$$

which is a contradiction. To prove the other direction assume that S is minimally balanced essential. Then define

$$d = \max \sum_{T\in\mathcal{B}} \delta_T v(T)$$

for all minimally balanced coalitions $\mathcal{B}$. By Theorem (1.15) it follows that the following game defined by w has a nonempty core.

$$w(T) = v(T)\ \forall\ T \subsetneqq S \text{ and } w(S) = d$$

Choose $x \in \mathcal{C}_S$ for the game w and define

$$\delta = v(S) - d > 0$$

Let $s = |S|$. Now add to each element x_i, for $i \in S$, the amount $\frac{\delta}{s}$, that is define x' by

$$x' = x + \frac{\delta}{s}\chi_S$$

Now $x' \in \mathcal{C}_\mathcal{S}$ for the game v and satisfies $x'(T) > v(T)$ for all $T \subsetneqq S$ and $x'(S) = v(S)$ hence S is a full coalition. **q.e.d.**

As a corollary one has that

Corollary 1.59. *Let (N, v) be a balanced game. If every exact coalition S satisfying*

$$v(S) > \sum_{T\in\mathcal{B}} \delta_T v(T)\ \forall \textit{ minimally balanced } \mathcal{B} \subseteq 2^S \textit{ with balancing } (\delta_T)_{T\in\mathcal{B}}$$

is extendable then the core is stable.

Finally another question that one may ask is whether weak/strong fully exactness is a necessary or sufficient condition for core stability. As the following two examples demonstrate the condition is neither necessary nor sufficient. The first example demonstrates that it is not necessary.

Example 1.60. Let $N = \{1, \ldots, 6\}$ and v be given by the following: $v(14) = 0.5$, $v(12) = v(13) = v(45) = v(46) = 1$, $v(N) = 3.5$ and $v(S) = 0$ otherwise. The core is stable however the coalition $\{14\}$ is neither weak nor strong fully exact. Take $x \notin \mathcal{C}$, an imputation, and without loss of generality (see Lemma (1.29)) assume that $x(12) < v(12)$. Let

$$\varepsilon = v(12) - x(12) > 0.$$

Then define the following vector

$$y := (x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}, 1 - x_1 - \frac{\varepsilon}{2}, 0.5 - x_1 - \frac{\varepsilon}{2}, 0.5 + x_1 + \frac{\varepsilon}{2}, 1.5 - x_2 - \frac{\varepsilon}{2}).$$

Then $y \in \mathcal{C}$ and

$$y \ \mathrm{dom}_{12} \ x.$$

The second example demonstrates that weak/strong fully exactness is not sufficient for core stability.

Example 1.61. Let $N = \{1, 2, 3\}$ and define v as follows. $v(S) = 0.6$ for all $S \subsetneqq N$ with $|S| = 2$ and $v(N) = 1$ and $v(i) = 0$ for all $i \in N$. Then $\mathcal{C}$ is not stable (eg. $x = (0.5, 0.5, 0)$) however $(0.3, 0.3) \in \mathcal{C}_S$ for all $S \subsetneqq N$ with $|S| = 2$.

The goal of the following subsections is to investigate classes of games on which the idea of weak fully exact extendability could be shown to be a necessary condition as well as presenting an example, which shows that weak fully exact extendability is not necessary for core stability. In each subsection a class of games will be introduced and then, on this class of games, results relating to core stability and weak fully exact extendability will be presented. The subsections are as follows, Assignment games, Min-colouring games and finally Orthogonal games.

1.2 Assignment Games

In this subsection the concept of an Assignment game will be introduced and core stability in relation to fully exact extendability will be explored.

The concept of an Assignment game was introduced by Shapley and Shubik in ([14]) in 1972. The idea is that there exists a market for goods with buyers and sellers willing to trade a certain good. Each buyer has a value for each sellers good and the buyers and sellers want to know how to distribute the common gains when they all cooperate together and maximise the total amount that exchanges hands during the buying and selling process. In mathematical terms an Assignment game can be defined as follows. The notation here follows that of, eg. Raghavan and Sudhölter ([11]). For finite sets S and T, both subsets of N, an assignment is a bijection $b : S' \to T'$ such that $S' \subset S$ and $T' \subset T$ and $|S'| = |T'| = \min\{|S|, |T|\}$. To simplify summations later on, b will be identified with $\{(i, b(i)) \,|\, i \in S'\}$. Let $\mathcal{B}(S, T)$ denote the set of all assignments. A game (N, v) is an assignment game if there exists a partition $\{P, P'\}$ of N and a non-negative real valued matrix $A = (a_{ii'})_{i \in P, i' \in P'}$ such that

$$v(S) = \max_{b \in \mathcal{B}(S \cap P, S \cap P')} \sum_{(i,i') \in b} a_{ii'}.$$

In order to characterise when the core is stable the following concept of a dominant diagonal is needed.

Definition 1.62. *The matrix A is said to have a* ***dominant diagonal*** *if*

$$a_{ii'} = \max_{j' \in P'} a_{ij'} = \max_{j \in P} a_{ji'} \; \forall \; i \in P$$

In Solymosi and Raghavan, ([16]), they show that for assignment games characterized by such matrices, this condition of a "dominant diagonal" is necessary and sufficient for core stability. That is

Theorem 1.63. *Let (N, v) be an assignment game. Then A has a dominant diagonal if and only if the core is stable.*

As was mentioned in the previous section exact extendability is not necessary for core stability. The example that verifies this claim, due to Peter Sudhölter, is as follows.

Example 1.64. Let (N, v) be an assignment game associated to the following square matrix, $A = \begin{pmatrix} 6 & 4 & 0 \\ 0 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}$. Let 1, 2 and 3 be the row players and 4, 5 and 6 be the column players. Note that, e.g., with $S = \{1, 3, 4, 5\}$, $v(S) = \max\{6 + 0, 4 + 4\} = 8$. Hence, $x_S = (4, 0, 4, 0) \in \mathcal{C}(S, v_S)$. Let $y = (3, 5, 1, 3, 1, 5)$. Then $y \in \mathcal{C}(N, v)$ and $y_S \in \mathcal{C}(S, v_S)$ and, hence, S is exact. Clearly, x_S cannot be extended (because any core element must assign 6 to all pairs of optimally matched players, e.g., to the pair $\{1, 4\}$). Note that the diagonal entries of the matrix A are the maximal entries for their respective row and coloumn (A has a "dominant diagonal"). Hence by Theorem (1.63) the core of this example is stable however the exact coalition $S = \{1, 3, 4, 5\}$ is not extendable.

Although exact extendability is not necessary for core stability, weak fully exact extendability is. To prove this, a fact concerning the nature of elements in the core in Assignment games are needed. In ([14]), apart from introducing the idea of an assignment game, the authors show, inter alia, that assignment games are totally balanced and that elements of the core are characterised as follows. An element $x \in \mathcal{I}(v)$ is an element of the core if it satisfies the following three conditions.

$$x(N) = v(N)$$

$$x_i \geq 0 \,\forall\, i \in N$$
$$x(ij') \geq a_{ij'} \,\forall\, i \in P \text{ and } j' \in P'.$$

This characterisation will now be used to prove the following result. The following result was conjectured by Peter Sudhölter.

Theorem 1.65. *Let (N, v) be an assignment game. The following two conditions are equivalent:*
1) All weak fully exact coalitions $S \subseteq N$ are extendable,
2) The core $\mathcal{C}$ is stable.

Proof: Note that the direction 1) $\Rightarrow$ 2) is implied by Theorem (1.46). The direction 2) $\Rightarrow$ 1) is proven as follows. Let (N, v) be an assignment game and let S be an exact subset of N such that $|S| > 2$. Let $b \in \mathcal{B}(S \cap P, S \cap P')$ such that

$$v(S) = \sum_{(i,j') \in b} a_{ij'}.$$

Firstly notice that for all $x \in \mathcal{C}_S$, x satisfies $x(i, b(i)) \geq v(i, b(i))$. However as $x(S) = v(S)$ it follows from

$$x(S) = \sum_{(i,i') \in b} x_{ij}$$

that $x(i, b(i)) = v(i, b(i))$ and hence there do not exist fully exact coalitions S for all $|S| > 2$. So it just needs to be shown that for all fully exact coalitions S with $|S| = 2$ that S is extendable. To this end, let k, l be elements of P. It suffices to show the following: If x in $\mathcal{C}$ such that $x_k + x_{l'} = a_{kl'}$, then there exists y in $\mathcal{C}$ such that

$$y_k = 0 \tag{1.8}$$

and

$$y_{l'} = a_{kl'}. \tag{1.9}$$

Define $y_i = \max(0, x_i - x_k)$, $y_{i'} = \min(x_{i'} + x_k, a_{ii'})$ for all i in P. It is clear that y satisfies (1.8) and (1.9). From the definition of y it follows that $y_i \geq 0$. It will now be shown that for this y, $y(N) = v(N)$. Now

$$y(N) = \sum_{i \in P} \max(0, x_i - x_k) + \min(x_{i'} + x_k, a_{ii'}) \tag{1.10}$$

where $i' = b(i)$ for the b which defines $v(N)$. Now $v(N) = \sum_{i \in P} a_{ii'}$ so it just needs to be shown that $y_i + y_{i'} = a_{ii'}$. So consider the case that $x_i \geq x_k$ then $x_k + x_{i'} \leq a_{ii'}$ which implies that the first summand in equation (1.10) is 0 and the second summand is $a_{ii'}$, hence $y_i + y_{i'} = a_{ii'}$. Now for the case that $x_i < x_k$ then it follows that $x_k + x_{i'} > a_{ii'}$ and hence for this i the sum $y_i + y_{i'}$ becomes

$$x_i - x_k + x_{i'} + x_k = a_{ii'}.$$

Hence one has that $y(N) = v(N)$. The final step in showing that y is an element of the core is to show that

$$y(ij') \geq a_{ij'} \,\forall\, i, j \,\in N.$$

To this end consider the case that $x_i \geq x_k$ then

$$y(ij') = x_i - x_k + min(x_{j'} + x_k, a_{ij'}) \geq a(ij').$$

In the second case that $x_i < x_k$ then $y_i = 0$ and it follows that

$$y(j') \geq x_{j'} + x_k > x_i + x_{j'} \geq v(ij').$$

Hence all two player coalitions are extendable. **q.e.d.**

So on the class of Assignment games one finds that weak fully exact extendability is both necessary and sufficient for core stability.

1.3 Orthogonal Games

In this section the relationship between Orthogonal games and fully exact extendability will be investigated. Orthogonal games are games that represent a subset of the games defined by the minimum of a finite number of additive measures. A major step in the studying the class of games defined by the minimum of a finite number of additive measures was taken by Kalai and Zemel in 1982 in ([7]). In this article they showed that a game (N, v) is totally balanced if and only if it is the minimum of finitely many additive measures. The subclass of these games, Orthogonal games, are such that the coalition function can be represented as the minimum of finitely many additive measures, whereby each of the measures has no player (the coordinates defining the values) in common with another measure that has a positive measure. This class of games has quite a long history within the field of Game theory and comes in a number of guises. Orthogonal games such as Glove market games have been well studied and have led to multifarious variations on this theme.

It is Orthogonal games that will be studied in this section and the main result presented here is that fully exact extendability is not necessary for core stability.

The mathematical definition of an Orthogonal game will now be given and the notation follows that of, eg. Raghavan and Sudhölter ([11]). Firstly a game (N, v) is the minimum of finitely many additive measures if v can be written as follows. Let λ^ρ be such that $\lambda^\rho \in \mathbb{R}^N_+$ for all $\rho \in \{1, \dots, r\}$,

$$v(S) = \min_{\rho=1,\dots,r} \lambda^\rho(S) \,\forall\, S \subseteq N.$$

A totally balanced game (N, v) is said to orthogonal if the carriers of the finite measures λ^ρ, $\{\rho = 1, \dots, r\}$, are mutually disjoint. This assumption allows one to present a more informative notation for orthogonal games. For each $\rho \in \{1, \ldots, r\}$ it follows that there exists a $N^\rho \subseteq N$ such that $\{i \in N^\rho \,|\, \lambda_i^\rho > 0\} \subseteq N$ and the resulting $\{N^\rho \,|\, \rho = 1, \ldots, r\}$ is a partition of N. One then normally writes $\lambda = \sum_{\rho=1}^{r} \lambda^\rho$ and with this notation one has $\lambda_i^\rho = \lambda_i$ for $i \in N^\rho$ and $\lambda_j^\rho = 0$ for $j \in N \backslash N^\rho$. So for orthogonal games one can write the coalition function as follows

$$v(S) = \min_{\rho=1,\ldots,r} \lambda(S \cap N^\rho)$$

for all $S \subseteq N$. The pair $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$ is called a representation of (N, v). One should note that the representation of an orthogonal game is practically unique. Hence the measure λ defining the game is uniquely determined. Provided that the game is not such that $v(S) = 0$ for all $S \subseteq N$ then the partition $\{N^\rho \,|\, \rho = 1, \ldots, r\}$ is uniquely determined up to dummy players.

To present a result concerning stable cores for Orthogonal games the following two Lemmata will be needed. The proofs of the following results can be found in ([11]).

Lemma 1.66. *An Orthogonal game (N, v) is exact if and only if $\lambda(N^\rho) = v(N)$ for every $\rho = 1, \ldots, r$.*

Lemma 1.67. *If an Orthogonal game has a stable core then it is exact.*

Within the class of Orthogonal games certain sets play a special role in determining the structure of the core. One such collection of sets is that of the diagonal sets, **D**, defined as follows.

$$\mathbf{D} := \{S \subseteq N \,|\, \lambda(S \cap N^\rho) = \lambda(S \cap N^\sigma)\ \forall\, \rho, \sigma \in \{\rho = 1, \ldots, r\}\}$$

The significance of these sets evinces when one considers the core. The diagonal sets can be used to define conditions under which the core takes a special form. Before the results can be stated, however, some notation needs to be introduced. This notation follows that of, eg. Rosenmüller ([12]). First of all, define

$$\mathbb{A}^c = \{\lambda^\rho \in \mathbb{R}_+^N \mid \lambda^\rho(N) = c\}.$$

That is, $\mathbb{A}^c$ is the set of all additive measures with a constant total measure equal to c. Then in order to introduce the definition of a weakly non degenerate system of measures, the following two sets need to be defined.

$$\mathbb{L} := \{x \in \mathbb{R}^N \mid \exists\, c \in \mathbb{R}^r,\ \sum_{i=1}^{r} c_i = 1,\ x = \sum_{i=1}^{r} c_i \lambda^i\}$$

and

$$\mathbb{X} := \{x \in \mathbb{R}^N \,|\, x(S) = v(S),\ (S \in \mathbb{D})\}.$$

Note that, for Orthogonal games, it is always the case that $\mathbb{L} \subseteq \mathbb{X}$. With these two sets the following definition of weakly non degenerate with respect to a set of measures $\lambda^1, \ldots, \lambda^r \in \mathbb{A}^c$ can be introduced.

Definition 1.68. *A set of measures $\lambda^1, \ldots, \lambda^r \in \mathbb{A}^c$ is said to be* ***weakly non degenerate*** *(weakly n.d.) if $\mathbb{L} = \mathbb{X}$.*

In ([12]) additional conditions are provided which are equivalent to the statement given here. The importance of a weakly n.d. set of measures is embodied in the following theorem proven in ([12]).

Theorem 1.69. *Let $\lambda^1, \ldots, \lambda^r \in \mathbb{A}^c$ be orthogonal and weakly n.d. Then $\lambda^1, \ldots, \lambda^r$ are all the extreme points of the core.*

As a result of this Theorem and Lemma (1.66) and (1.67), one has the following corollary.

Corollary 1.70. *Let (N, v) be a weakly n.d., Orthogonal game (represented by $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$) and let the core $\mathcal{C}$ be stable.*
Then $\mathcal{C} = convH\{\lambda^1, \ldots, \lambda^r\}$, where $convH$ means 'the convex hull of'.

However it is not this result that is quintessential here. The most important result in this section is the demonstration that weak fully exact extendability is not necessary for core stability. To expound this result some notation must be introduced. The notation introduced just modifies the way the games are written and make explicit certain 'types' that define the measures λ^ρ. As well, for the example it will only be required that $\rho \in \{1, 2\}$, that is there are two orthogonal measures, λ^1 and λ^2 defining v. It will also only be required that each measure is positive in a finite number of coordinates and hence there are a fixed set of values that the measures take, say g^1_α, $\alpha \in \{1, \ldots, n\} := G^1$, for λ^1 and similarly g^2_α, $\alpha \in \{1, \ldots, m\} := G^2$, for λ^2. One can then write, for a given measure λ^ρ,

$$\lambda^\rho(S) = |K^\rho_1 \cap S| g^\rho_1 + |K^\rho_2 \cap S| g^\rho_2 + \ldots + |K^\rho_k \cap S| g^\rho_k \tag{1.11}$$

($k = n$ for $\rho = 1$ and $k = m$ when $\rho = 2$) where K^ρ_i represent the coordinates (entries in λ^ρ) with the value g^ρ_i and satisfy

$$\bigcup_{i=1}^{n} K^\rho_i = N^\rho \text{ and } \bigcap_{i,j \in G^\rho} K^\rho_i = \emptyset \text{ for } \rho = \{1, 2\}.$$

By restricting $n = 2$ and defining,

$$M_\rho := \max(g^\rho_2 - 1) + g^\rho_2 \min(|K^\rho_2|, \frac{g^{3-\rho}}{d} - 1), \tag{1.12}$$

where $d := g.c.d.(g^\rho_1, g^\rho_2)$, one has the following result for the case where $g^\rho_1 = 1$ for $\rho \in \{1, 2\}$,

Lemma 1.71. *For $\rho = \{1, 2\}$, let λ^ρ be given by (1.11) and M_ρ by (1.12). Whenever $|K^\rho_1| \geq M_\rho$, $\rho = \{1, 2\}$, holds true, then the core is stable.*

The reader is referred to Rosenmüller ([12]) for a proof. This Lemma and Theorem (1.69) will now be used to show that weak fully exact extendability is not necessary for core stability.

Example 1.72. Let $\lambda = (2,1,1,2,1,1)$ and let $N^1 = \{1,2,3\}$ and $N^2 = \{4,5,6\}$, that is

$$v(S) = \lambda^1(S \cap N) + \lambda^2(S \cap N),$$

with $\lambda^1 = (2,1,1,0,0,0)$ and $\lambda^2 = (0,0,0,2,1,1)$. Note that $M_1 = M_2 = 1$ and $|K_1^1| = |K_1^2| = 2$, so that this game satisfies the conditions of Lemma (1.71) and hence has a stable core. It will now be shown that the game is weakly n.d. In order to do this it will be shown that $\mathbb{X} \subseteq \mathbb{L}$. So first of all

$$\mathbb{D} = \{\{1,5,6\},\{2,3,4\},\{1,4\},\{2,5\},\{2,6\},\{3,5\},\{3,6\},$$

$$\{1,2,4,5\},\{1,2,4,6\},\{1,3,4,5\},\{1,3,4,6\},\{2,3,5,6\}\}.$$

By applying the equalities $x(S) = v(S)$ for $S \in \mathbb{D}$ it follows that for all $x \in \mathbb{X}$

$$x_1 = 2x_2 = 2x_3 \text{ and } x_4 = 2x_5 = 2x_6 \text{ and } x(N) = 4.$$

(For the first equality above, one subtracts from $x(\{1,5,6\})$, $x(\{2,5\})$ and $x(\{2,6\})$ and one has that $x_1 = 2x_2$. Similar computations yield the other equalities.) Hence all points $x \in \mathbb{X}$ are of the form

$$x = \alpha(2,1,1,0,0,0) + (1-\alpha)(0,0,0,2,1,1)$$

for $\alpha \in [0,1]$. This then implies that $x \in \mathbb{L}$ hence $\mathbb{X} \subseteq \mathbb{L}$ and the game is weakly non degenerate. This implies that $\mathcal{C} = convH\{\lambda^1, \lambda^2\}$. However if one considers the subgame on the player set $S = \{1,5,6\}$ one sees that the core of this subgame is fully exact (e.g. $(1, 0.5, 0.5) \in \mathcal{C}_S$) however this subcore is not extendable (e.g. the point $(0.5, 1, 0.5) \in \mathcal{C}_S$ is not extendable as $\mathcal{C} = convH\{\lambda^1, \lambda^2\}$). This demonstrates that weak fully exact extendability is not a necessary condition for core stability.

1.4 Appendix

In this section the author's original method for proving Theorem (1.32) as well as a second proof of Lemma (1.29), given by Peter Sudhölter, are provided. The author would like to thank Professor J. Rosenumüller for his helpful suggestions and comments in improving the perspicuousness of the first Lemma's proof.

Lemma 1.73. *Let (N, v) be a balanced game and let $T \subsetneqq N$ be non exact in $\mathcal{C}$. Let $x \in \mathbb{R}^N$ with $x(N) < v(N)$ and $x(T) \leq v(T)$. Then there exists an exact $P \subseteq N$ with*

$$x(P) < v(P)$$

Proof:
1$^{\text{st}}$STEP :
First of all the following claim will be proven. Let T be non exact and let $y \in \mathbb{R}^N$ satisfy

$$y(N) = v(N),\ y(T) \leq v(T) \tag{1.13}$$

hence $y \notin \mathcal{C}$. Then there exists an exact $S \in 2^N$ such that

$$y(S) < v(S). \tag{1.14}$$

To see this, assume 'per absurdum' that

$$y(S) \geq v(S),\ \forall\, S \in 2^N,\ S \text{ exact.} \tag{1.15}$$

holds true. As v is balanced one can choose $\beta \in \mathcal{C}$ and define

$$z^\lambda := \lambda y + (1-\lambda)\beta,\ 0 \leq \lambda \leq 1.$$

Let

$$\lambda^* := \max\{\lambda \,|\, z^\lambda \in \mathcal{C}\} < 1.$$

Then $z^* := z^{\lambda^*} \in \mathcal{C}$ and there exists $M^* \in 2^N$ with

$$z^*(M^*) = v(M^*) \tag{1.16}$$

$$z^{\lambda^*+\varepsilon} < v(M^*),\ \varepsilon > 0 \tag{1.17}$$

Now, (1.16) implies that M^* is exact (as $z^* \in \mathcal{C}$). Hence

$$y(M^*) \geq v(M^*)$$

by (1.15) and $\beta(M^*) \geq v(M^*)$ as $\beta \in \mathcal{C}$, which contradicts (1.17), hence (1.14) is true.
2$^{\text{nd}}$STEP :
To prove the Lemma, consider now $x \in \mathbb{R}^N$ such that $x(N) < v(N)$ and $x(T) \leq v(T)$. Define

$$\delta := v(N) - x(N)$$

and

$$\varepsilon := v(T) - x(T).$$

The proof will now be divided up into a number of cases.
Case 1 Assume that $\delta > \varepsilon$
Let $|T| := t < n$. In this case define a new vector x' by adding

$$\frac{\delta}{t}$$

to each x_i for $i \in T$, that is

$$x'_i = x_i + \frac{\delta}{t}\ \forall\, i \in T$$

and

$$x'_i = x_i\ \forall\, i \in N \backslash T.$$

One then has that

$$x'(N) = v(N) \text{ and } x'(T) < v(T)$$

and hence, from the first step, there exists an exact coalition P such that

$$x'(P) < v(P).$$

As $x \leq x'$ it follows that

$$x(P) < v(P).$$

Case 2 Assume that $\delta < \varepsilon$
Let $|T| := t < n$. In this case define a new vector x' by adding

$$\frac{\varepsilon}{t}$$

to each x_i for $i \in T$, that is

$$x'_i = x_i + \frac{\varepsilon}{t} \; \forall \, i \in T$$

and

$$x'_i = x_i \; \forall \, i \in N \backslash T.$$

One then has that

$$x'(N) < v(N) \text{ and } x'(T) = v(T)$$

Choose $j \in N \backslash T$ and define

$$\sigma := v(N) - x'(N)$$

and define a new vector $\hat{x}$ by adding σ to x'_j so that

$$\hat{x}_j = x_j + \sigma$$

and

$$\hat{x}_i = x'_i \; \forall \, i \in N \backslash j.$$

Then one has that

$$\hat{x}(N) = v(N) \text{ and } \hat{x}(T) = v(T)$$

and hence again, from the first step, there exists an exact coalition P such that

$$\hat{x}(P) < v(P).$$

As $x < x' < \hat{x}$ it follows that

$$x(P) < v(P).$$

Case 2 Assume that $\delta < \varepsilon$
Repeat what was done earlier in Case 1 and this time one has that

$$x'(N) = v(N) \text{ and } x'(T) = v(T)$$

and again the result follows from the first step. **q.e.d.**

The proof of this Lemma given by Peter Sudhölter was the following.

Proof:
As $\mathcal{C} \neq \emptyset$, $T \subsetneq N$. Define

$$x^1 = x + \frac{v(N) - x(N)}{|N \setminus T|} \chi_{N \setminus T}$$

(where χ_R is the indicator function on $R \subseteq N$). Then $x \leq x^1$, $x^1(T) \leq x(T)$ and $x^1(N) = v(N)$. As T is non-exact, $x^1 \notin \mathcal{C}$. Let $x^0 \in \mathcal{C}(N, v)$ and define

$$x^\lambda := \lambda x^1 + (1 - \lambda) x^0$$

for $0 \leq \lambda \leq 1$. As the core is convex and closed, there exists $0 < \hat{\lambda} < 1$ such that

$$\{x^\lambda \mid 0 \leq \lambda \leq 1\} \cap \mathcal{C}(N, v) = \{x^\lambda \mid 0 \leq \lambda \leq \hat{\lambda}\}.$$

Set $y = x^{\hat{\lambda}}$. Then there exists $P \subsetneq N$ such that $y(P) = v(P)$ and $x^1(P) < v(P)$, because $x^1 \notin \mathcal{C}$. The coalition P has the desired properties. **q.e.d.**

The author's original proof of Lemma (1.31) will now be given. However before that is done recall again the set $M(x)$. Let x be an imputation and define

$$M(x) := \{S \mid S \subset N; S \text{ exact with } x(S) < v(S) \text{ and } x(T) \geq v(T) \, \forall T \subset S\}.$$

With this notation the following Lemma will be proven:

Lemma 1.74. *Let (N, v) be a balanced game and let every exact coalition in $\mathcal{C}$ be extendable. If $x \notin \mathcal{C}$ then $M(x) \neq \emptyset$.*

Proof:
The Lemma will be proven via contradiction. So assume that there exists an $x \notin \mathcal{C}$ such that $M(x) = \emptyset$. That is, for all exact $S \subsetneq N$ with $x(S) < v(S)$ there exist $T \subsetneq S$ with $x(T) < v(T)$ and T is not exact in $\mathcal{C}$. Now choose an exact $S \subsetneq N$ such that there does not exist an exact $R \subsetneq S$ with $x(R) < v(R)$. Then as before there exists a non-exact T with $x(T) < v(T)$. Lemma (1.73) applied this to the imputation x with N in Lemma (1.73) replaced by S implies that there exists a Q, $Q \subsetneq S$, exact in $\mathcal{C}_\mathcal{S}$ such that $x(Q) < v(Q)$. However as S is extendable, by assumption, Q must be exact in $\mathcal{C}$ contradicting the assumption that there did not exist an exact $R \subsetneq S$ with $x(R) < v(R)$. **q.e.d.**

With this Lemma, Theorem (1.32) can now be proven as was proven in the section on Cooperative Games.

The following is the author's proof of Lemma (1.44).

Lemma 1.75. *Let (N, v) be a balanced TU game and let x be an imputation $x \notin \mathcal{C}$. Then for all minimal $S \in E(x)$ it follows that S is weak fully exact.*

Proof: Let x be an imputation $x \notin \mathcal{C}$. By Lemma (1.29) it follows that $E(x) \neq \emptyset$. Choose a $S \in E(x)$ such that there does not exist $T \subsetneqq S$ and $T \in E(x)$. As S is exact there exists $\beta \in \mathcal{C}$ with $\beta(S) = v(S)$. Let $P_i \subsetneqq S$ be the coalitions such that $\beta(P_i) = v(P_i)$, that is the P_i are exact (if any exist at all). As well, by assumption, $x(P_i) \geq v(P_i)$ as otherwise there would exist a $P_i \subsetneqq S$ such that $x(P_i) < v(P_i)$ contradicting the assumption that S was the smallest such exact coalition with $x(S) < v(S)$. Now consider the vector

$$x^\lambda := (1-\lambda)\beta + \lambda x$$

for $0 \leq \lambda \leq 1$. Then for all $0 < \lambda \leq 1$ it follows that $x^\lambda(S) < v(S)$. Let

$$L := 2^S \setminus (\bigcup_i P_i \cup \emptyset).$$

Then for all $M \subsetneqq S$ with $M \in L$ it follows that $\beta(M) > v(M)$. Now choose $0 < \varepsilon \ll 1$ such that the vector x^ε still satisfies $x^\varepsilon(M) > v(M)$ for all $M \subsetneqq S$ with $M \in L$. It now follows however that $x^\varepsilon(T) \geq v(T)$ for all $T \subsetneqq S$ and $x^\varepsilon(S) < v(S)$. Hence define

$$\delta := v(S) - x^\varepsilon(S)$$

and add $\dfrac{\delta}{s}$, where $s = |S|$, to each element x_i^ε for $i \in S$. That is, define

$$x' = x^\varepsilon + \frac{\delta}{s}\chi_S.$$

It then follows that $x'(T) > v(T)$ for all $T \subsetneqq S$ and that $x'(S) = v(S)$ and as S was exact, S is weak fully exact. **q.e.d.**

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