Multi-period Consumption and Investment Decisions under Uncertainty Revisited *

Marten Hillebrand[†]
Department of Economics
Bielefeld University
P.O. Box 100 131
D-33501 Bielefeld, Germany
marten.hillebrand@uni-bielefeld.de

Jan Wenzelburger
Centre for Economic Research
Keele University
Keele, ST5 5BG, UK
j.wenzelburger@econ.keele.ac.uk

June 2007

Discussion Paper No. 567

Abstract

This paper studies the consumption-investment problem of a consumer with a multi-period planning horizon under uncertainty. Such a setting occurs naturally in stochastic OLG models with multi-period-lived consumers. We provide a rigorous characterization of consumption-investment strategies under general assumptions on preferences and expectations and derive conditions for the existence of an optimal strategy. The properties of the resulting demand functions are investigated in general as well as for the popular case with constant relative risk aversion (CRRA).

Keywords: Multi-period planning horizon, consumption investment strategies, portfolio decisions, stochastic dynamic programming, principle of optimality.
JEL Classification: G11, D11, D91

First version: Nov. 2006, this version: June 2007.

*Acknowledgment. We are indebted to Volker Böhm, Olaf Schmitz and Thorsten Pampel for many discussions.

†Corresponding author. Tel.: +49 521 106 4835.

Introduction

The study of multi-period consumption-investment problems has been the subject of numerous studies in the literature. While the employed settings and assumptions differ considerably, all contributions share the common objective to offer a sound microeconomic foundation of the decision behavior of investors with multi-period lives in the presence of uncertainty due to stochastic investment returns. Such a setting occurs naturally in many economic models such that the results open a wide array of applications in portfolio theory and financial and stochastic macroeconomic models.

One of the first theoretical studies of a multi-period consumption-investment problem under uncertainty dates back to Phelps (1965). He studies a multi-period expected utility maximization problem with a single risky asset where returns follow an i.i.d. process with a discrete probability distribution. The investor is endowed with additive separable preferences over lifetime consumption and receives a constant non-capital income stream in each period. Within this setting a recursive solution technique is applied to derive optimal consumption and investment in each period. The setting covers the case with a finite and an infinite planning horizon as a limiting case. Closed form solutions, however, are only derived in the absence of non-capital income.

An extension to the case with possibly non-additive preferences and more general and multivariate return distributions is provided in Fama (1970, 1976) . Utilizing a similar methodology the problem is solved recursively to obtain optimal decisions at each stage of the planning horizon. Although it is argued that at each stage the problem is well-defined and implies existence and uniqueness of a solution, several technical requirements like measurability and/or integrability as well as existence of conditional distributions needed for the solution to be well-defined are neglected there. Neither credit nor safe investment is possible in this setting nor does the investor receive any non-capital income.

Another extension into a somewhat different direction may be found in the articles by Hakansson (1969, 1970) , see also Ingersoll (1987) for a survey. These studies restrict attention to a particular class of utility functions and extend the work by Phelps (1965) by allowing for credit, a risk-less investment possibility and an arbitrary (possibly time-varying) number of risky assets. By adopting the same recursive solution technique the class of utility functions under scrutiny lead to closed form solutions which allow to obtain several insights into the consumption-investment behavior for the case with a finite and an infinite planning horizon.

One of the major drawbacks common to all these approaches is that the underlying decision problem is only formulated implicitly in a recursive fashion. Starting one period prior to the terminal period a sequence of one-period decision problems is solved to obtain solutions for each period of the planning horizon. It is then argued that these recursively defined decisions define an optimal strategy over the entire life cycle. This type of approach does not formulate the underlying one-shot decision problem involving

the choice of a consumption-investment strategy drawn from a suitably defined strategy space. As a consequence, a comparison with strategies which deviate from the potentially optimal one is not possible in this setting. While the recursive formulation solution technique is justified by stochastic dynamic programming and the renowned principle of optimality due to Bellman (1957), the prescribed setting does not offer the possibility to verify the validity of this principle since no appropriate strategy space has been defined. A more general treatment of decision problems under uncertainty may be found in Grandmont & Hildebrand (1974) and Grandmont (1982). These studies restrict attention to a three-period planning horizon and derive sufficient conditions for the existence of a solution to the problem. The intention of this paper is not to maintain the full generality of their setup but to instead restrict attention to the particular case of a multi-period consumption-investment problem as described above. In this regard, we amend the existing approaches by defining consumption-investment strategies as adapted stochastic processes defined in the appropriate space of random variables. This type of approach seems to be common sense in the finance literature (see Duffie (1992) or Pliska (1997) for the special case of a discrete probability space) and is therefore by no means new. Among other advantages, our approach allows us to verify the validity of the recursive solution approach and to prove the validity of the principle of optimality. A particular goal of the paper is to characterize the demand functions defining the optimal behavior in the decision period. For this purpose we adopt a slightly different setup compared to the literature which typically formulates the decision problem for given prices and expectations for asset returns. This formulation seems indeed favorable if one restricts attention to the decision problem on the individual level in a given period of time. If, however, the ultimate goal is to embed the individual decision problem into a model of market interaction where prices are determined from the demand behavior of investors, a formulation of the decision problem with respect to level asset prices rather than asset returns seems to be desirable. If, in addition, the goal is a sequential formulation of the model which captures the updating process of consumers' expectations about uncertain variables, it seems advantageous to parameterize the demand behavior in expectations as well. In particular, this captures the possibility that consumers update their expectations and revise their strategies over time as new and unexpected information become available (see Hillebrand & Wenzelburger (2006) for a study in this spirit).
The paper is organized as follows. Section 1 introduces the general setup of a multi-period consumption investment problem followed by a formulation of the decision problem in Section 2. Existence and properties of solutions and of demand functions are studied under general assumptions in Section 3 and for the much-studied class of CARA preferences in Section 4. Section 5 concludes,mathematical proofs and technical prerequisites are placed in Appendices A and B.

1 Consumption and investment strategies

Consider a consumer who takes decisions in discrete time. Let $t = 0$ denote the current period and $N > 0$ the consumer's planning horizon at the end of which he dies such that the set $\{0, 1, \ldots, N\}$ defines the consumer's remaining lifetime. In each period $n \in \{0, 1, \ldots, N\}$ the consumer receives a non-capital income $e_n \geq 0$. To transfer income between different periods there are $M+1$ investment possibilities corresponding to different assets $m = 0, 1, \ldots, M$ in each period. The first asset $m = 0$ is a one-period lived bond which is traded at a price of unity at time n and pays a non-random return $R_n > 0$ in the following period $n+1$. Since R_n is determined at time n, the bond provides a riskless investment possibility between any two consecutive periods. The remaining assets $m = 1, \ldots, M$ are retradeable shares which are traded at strictly positive asset prices $p_n = (p_n^{(1)}, \ldots, p_n^{(M)})^\top \in \mathbb{R}^M_{++}$ and pay a non-negative random dividend $d_n = (d_n^{(1)}, \ldots, d_n^{(M)})^\top \in \mathbb{R}^M_+$ (prior to trading) in each period $n \in \{0, 1, \ldots, N\}$. The bond may be sold short without bound but no short selling of shares is possible such that the sets $\mathbb{Y} = \mathbb{R}$ and $\mathbb{X} = \mathbb{R}^M_+$ describe feasible bond investments respectively feasible risky portfolios in each period. The space $\mathbb{Z} := \mathbb{Y} \times \mathbb{X}$ defines the set of feasible portfolios in each period.

Denote by $z_{-1} = (y_{-1}, x_{-1}) \in \mathbb{Z}$ the portfolio purchased by the consumer during the previous period consisting of a bond investment $y_{-1} \in \mathbb{Y}$ and a non-negative vector $x_{-1} \in \mathbb{X}$ defining the number of shares in the portfolio. The investor's initial wealth at time $t = 0$ consists of his current non-capital income $e_0 \geq 0$ and his capital income corresponding to the return on his previous investment z_{-1} consisting of the return on the bond investment (which is negative if $y_{-1} < 0$) and the dividend earnings and selling revenue of the stock investment (which is positive). We therefore set

$$w_0 := e_0 + y_{-1} \cdot R_{-1} + x_{-1}^\top (p_0 + d_0) \tag{1}$$

for the consumer's initial wealth. It is assumed that the decision in $t = 0$ is made *after* the dividend payment $d_0 \in \mathbb{R}^M_+$ and the consumer's current non-capital income $e_0 \geq 0$ have been observed but *prior* to trading, i.e., before the bond return R_0 and asset prices p_0 have been determined. Hence the consumer treats these variables as parameters $R > 0$ and $p \in \mathbb{R}^M_{++}$. Likewise his current wealth position defined by (1) is treated as parameter $w \in \mathbb{R}$ in the decision problem. Although the latter value will generically (whenever $x_{-1} \neq 0$) depend on current asset prices, it will be convenient to treat current wealth as a separate parameter.

At time $t = 0$ the consumer holds expectations $\hat{e} := (\hat{e}_1, \ldots, \hat{e}_N) \in \mathbb{R}^N_+$ for his future non-capital income with $\hat{e}_n \geq 0$ denoting the non-capital income expected to receive in period $n \in \{1, \ldots, N\}$. Likewise he holds expectations $\hat{R} := (\hat{R}_1, \ldots, \hat{R}_{N-1}) \in \mathbb{R}^{N-1}_{++}$ for future bond returns where $\hat{R}_n$ is his point forecast for the bond return R_n between future periods n and $n+1$, $n \in \{1, \ldots, N-1\}$. For the following derivations the consumer's planning horizon N as well as his expectations will be assumed to be fixed quantities and

will therefore be suppressed as arguments of functions, etc. to alleviate the notation.
In each period $n \in \{0, 1, \ldots, N\}$ the consumer can consume part of his wealth and use the investment possibilities described above to transfer wealth into future periods. Restricting consumption to be non-negative the set $\mathbb{C} = \mathbb{R}_+$ defines feasible consumption plans in each period. For $n > 0$ denote the pair $s_n := (p_n, d_n) \in \mathbb{S}$ of prices and dividends in period n where $\mathbb{S} := \mathbb{R}^M_{++} \times \mathbb{R}^M_+$ denotes the corresponding price-dividend space. In principle, the vector s_n could contain other uncertain variables, however, the previous definition will turn out to be sufficient and contains all relevant quantities to formulate and solve the consumer's decision problem. At time $t = 0$ there is uncertainty about all future s_n, $n > 0$ which are treated as random variables in the decision problem. More specifically, the consumer considers future prices and dividends as an $\mathbb{S}$-valued stochastic process $\{s_n\}_{n \geq 1}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which are adapted to a suitable filtration $\{\mathcal{F}_n\}_{n \geq 1}$ of sub-σ-algebras of $\mathcal{F}$.[1]
The following notion of a strategy will be employed in the sequel. In this regard, $\mathcal{B}(\mathbb{A})$ denotes the Borel σ- algebra on a topological space $\mathbb{A}$.

Definition 1

(i) A consumption strategy is a list $C = (c_0, c_1(\cdot), \ldots, c_N(\cdot))$ consisting of a decision $c_0 \in \mathbb{C}$ and $\mathcal{B}(\mathbb{S}^n) - \mathcal{B}(\mathbb{C})$ measurable functions $c_n : \mathbb{S}^n \longrightarrow \mathbb{C}$ for each $n = 1, \ldots, N$.

(ii) An investment strategy is a list $Z = (z_0, z_1(\cdot), \ldots, z_N(\cdot))$ consisting of a decision $z_0 = (y_0, x_0) \in \mathbb{Z}$ and $\mathcal{B}(\mathbb{S}^n) - \mathcal{B}(\mathbb{Z})$ measurable functions $z_n = (y_n, x_n) : \mathbb{S}^n \longrightarrow \mathbb{Z}$ for each $n = 1, \ldots, N$.

(iii) The pair (C, Z) is called a consumption investment strategy *or simply a* strategy.

A consumption strategy C specifies a current consumption decision c_0 and consumption plans for all future periods $n = 1, \ldots, N$ within the consumer's planning horizon which are made conditional on the random variables $s_1, \ldots, s_n$ observed up to time n.[2] Likewise the investment strategy Z specifies a current investment $z_0 = (y_0, x_0)$ and planned investments in bonds and shares for all future periods. Since the consumer's economic life ends in period N, no portfolio is carried over to period $N+1$ such that $z_N \equiv 0$ for the investment plan in period N. In the sequel we shall adopt the notation $s_1^n := (s_1, \ldots, s_n) \in \mathbb{S}^n$, $n \geq 1$ and set $\hat{R}_0 := R$. Furthermore, we shall frequently suppress the arguments of a plan for period n, writing just c_n, y_n, and x_n instead of $c_n(s_1^n)$, $y_n(s_1^n)$,

[1] This kind of behavior suggests that the consumer perceives future asset prices and dividends to be the primary source of randomness and uncertainty while future non-capital incomes and bond returns are either known or can be relatively precisely predicted. This corresponds to the standard setup applied in most models in the literature.

[2] The literature often defines a strategy as an adapted stochastic process defined on a probability space representing uncertain future states of the world. While the definition given here is equivalent from a mathematical point of view, the uncertainty here rests on future asset prices and dividends rather than states of the world. This formulation appears more suitable from an economic point of view since prices and dividends are the relevant quantities which are directly observable.

and $x_n(s_1^n)$. Given these conventions, the following definition characterizes the strategies which are feasible from the initial situation at time $t = 0$.

Definition 2

Given the bond return $R > 0$, asset prices $p \gg 0$ and initial wealth w at time $t = 0$, a strategy (C, Z) is called feasible if

(i) $c_0 + y_0 + x_0^\top p = w$

(ii) for each $n = 1, \ldots, N$

$$c_n + y_n + x_n^\top p_n = \hat{e}_n + \hat{R}_{n-1}\, y_{n-1} + x_{n-1}^\top (p_n + d_n) \qquad \forall s_1^n \in \mathbb{S}^n.$$

The set of all strategies which are feasible from (R, p, w) is denoted by $\mathscr{B}(R, p, w)$.

Throughout we shall assume that the strategy set $\mathscr{B}(R, p, w)$ is non-empty. Conditions under which this is the case are stated in Lemma 1 below. Associated with the choice of an investment strategy Z is the induced wealth process $\{W_n(Z, s_1^n)\}_{n=1}^N$ where

$$W_n(Z, s_1^n) := \hat{e}_n + \hat{R}_{n-1}\, y_{n-1} + x_{n-1}^\top\, (p_n + d_n)\,. \tag{2}$$

The real number $W_n(Z, s_1^n)$ describes the consumer's future wealth at time $n \geq 1$ depending on the investment strategy $Z = (z_n)_{n=0}^N = (y_n, x_n)_{n=0}^N$ and the random variables $s_1, \ldots, s_n$ observed up to period n. Note that wealth may well become negative. However, the following Lemma 1 shows that there exist lower bounds on planned bond investments and on the wealth process which are essentially determined by the discounted non-capital income stream. To obtain a compact notation define the values

$$\begin{aligned} \hat{\epsilon}_0 &:= \hat{e}_1 + \frac{\hat{e}_2}{\hat{R}_1} + \ldots + \frac{\hat{e}_N}{\hat{R}_1 \cdots \hat{R}_{N-1}} \geq 0 \\ \hat{E}_n &:= \frac{\hat{e}_{n+1}}{\hat{R}_n} + \ldots + \frac{\hat{e}_N}{\hat{R}_n \cdots \hat{R}_{N-1}} \geq 0,\ n = 1, \ldots, N \end{aligned} \tag{3}$$

derived from the expectations $\hat{e}$ and $\hat{R}$ with the understanding that $\hat{E}_N := 0$. For each future period $n = 1, \ldots, N$, the value $\hat{E}_n \geq 0$ defines the discounted non-capital income stream expected to receive after period n. Likewise, given the current bond return $R > 0$, the value $\hat{\epsilon}_0 / R$ defines the expected future non-capital income stream discounted to the decision period $t = 0$. Utilizing these values, the following lemma establishes the desired properties of the consumer's investment behavior and the wealth process and provides conditions under which the strategy set is non-empty.

Lemma 1

Let $\hat{\epsilon}_0 \geq 0$ and $\hat{E}_n \geq 0$, $n = 1, \ldots, N$ be defined as in (3). Then for each strategy $(C, Z) \in \mathscr{B}(R, p, w)$ the following holds true:

(i) The bond investments $(y_0, y_1(\cdot), \dots, y_{N-1}(\cdot))$ associated with Z satisfy

$$y_0 \geq -\hat{\epsilon}_0/R \quad \text{and} \quad y_n(s_1^n) \geq -\hat{E}_n \quad \text{for all } s_1^n \in \mathbb{S}^n.$$

(ii) The associated wealth process $\{W_n(Z, s_1^n)\}_{n=1}^N$ defined as in (2) satisfies:

$$W_n(Z, s_1^n) \geq -\hat{E}_n \quad \text{for all } s_1^n \in \mathbb{S}^n.$$

(iii) The strategy set $\mathscr{B}(R, p, w)$ is non-empty if and only if $w \geq -\hat{\epsilon}_0/R$.

Proof: We show claim (i) for $n = N - 1$ and then apply an induction argument. Let $(C, Z) \in \mathscr{B}(R, p, w)$ be an arbitrary strategy. Since $\mathbb{C} = \mathbb{R}_+$ and there is no investment in the terminal period, the consumption plan for period N must satisfy

$$c_N = \hat{e}_N + \hat{R}_{N-1}\, y_{N-1} + x_{N-1}^\top (p_N + d_N) \geq 0 \tag{4}$$

for all $s_1^N \in \mathbb{S}^N$. If $N > 1$, let $s_1^{N-1} \in \mathbb{S}^{N-1}$ be arbitrary but fixed. Then (4) must hold for any $s_N \in \mathbb{S}$. Recalling that $\mathbb{S} = \mathbb{R}_{++}^M \times \mathbb{R}_+^M$, the last term on the r.h.s. of (4) is non-negative but may become arbitrarily small. It follows that (4) can only be satisfied for *all* $s_N \in \mathbb{S}$ if $\hat{R}_{N-1} y_{N-1} \geq -\hat{e}_N$. If $N > 1$ this requires $y_{N-1}(s_1^{N-1}) \geq -\hat{E}_{N-1}$ for all $s_1^{N-1} \in \mathbb{S}^{N-1}$ while for $N = 1$ one must have $y_0 \geq -\hat{\epsilon}_0/R$.

Now let $n \in \{0, 1, \dots, N-2\}$ be arbitrary and assume that the claim is true for $n+1$, i.e., $y_{n+1}(s_1^{n+1}) \geq -\hat{E}_{n+1}$ for all $s_1^{n+1} \in \mathbb{S}^{n+1}$. By Definition 2

$$\begin{aligned} c_{n+1} &= \hat{e}_{n+1} + \hat{R}_n\, y_n + x_n^\top (p_{n+1} + d_{n+1}) - y_{n+1} - x_{n+1}^\top p_{n+1} \\ &\leq \hat{e}_{n+1} + \hat{R}_n\, y_n + x_n^\top (p_{n+1} + d_{n+1}) + \hat{E}_{n+1}. \end{aligned} \tag{5}$$

Using a similar argument as in the first step equation (5) requires that for all $s_{n+1} \in \mathbb{S}$

$$\hat{e}_{n+1} + \hat{R}_n y_n + (p_{n+1} + d_{n+1})^\top x_n + \hat{E}_{n+1} \geq 0$$

and, therefore, $\hat{e}_{n+1} + \hat{R}_n y_n + \hat{E}_{n+1} \geq 0$. For $n > 0$ this is equivalent to $y_n(s_1^n) \geq -\hat{E}_n$ for all $s_1^n \in \mathbb{S}^n$. If $n = 0$ the above inequality requires $y_0 \geq -\hat{\epsilon}_0/R$. This proves claim (i). The assertion (ii) is an immediate consequence of (i) and equation (2). The 'only if' part in (iii) can be proved by using Definition 2 (i) and the result from (i) to see that $w < -\hat{\epsilon}_0/R$ implies $c_0 = w - y_0 - x^\top p \leq w - y_0 \leq w + \hat{\epsilon}_0/R < 0$ such that the condition $w \geq -\hat{\epsilon}_0/R$ is necessary. The 'if' part in (iii) follows from the fact that as soon as $w \geq -\hat{\epsilon}_0/R$ the set $\mathscr{B}(R, p, w)$ contains the pure-bond investment strategy which never invests in any risky assets $m > 0$ and consumes only in the terminal period of life. ∎

The results from Lemma 1 are direct consequences of the definition of a strategy and are thus independent of any preferences, etc. The following corollary is immediate from Definition 2 and Lemma 1.

Corollary 1

If $w = -\hat{\epsilon}_0/R$ the set $\mathscr{B}(R, p, w)$ contains a single strategy (C, Z) defined by the decision $c_0 = 0$, $y_0 = -\hat{\epsilon}_0/R$, $x_0 = 0$ and plans $c_n \equiv 0$, $y_n \equiv -\hat{E}_n$ and $x_n \equiv 0$ for each $n = 1, \dots, N$.

2 The general decision problem

To derive the consumer's decision problem we make assumptions on his expectations for future asset prices and dividends as well as on his preferences over alternative consumption streams. For the following derivations assume that the strategy set $\mathscr{B}(R,p,w)$ is non-empty. Recall that at time $t=0$ the consumer treats future prices and dividends $s_1, s_2, \ldots$ as random variables. The following assumption characterizes his expectations for future asset prices and dividends within his planning horizon.

Assumption 1
Given the planning horizon $N>0$ the consumer's expectations at time $t=0$ for future asset prices and dividends are given by a probability measure ν on the measurable product space $\big(\mathbb{S}^N, \mathcal{B}(\mathbb{S}^N)\big)$ defining a subjective joint probability distribution of the random variables $s_1, \ldots, s_N$.

In the sequel we denote the class of all probability measures on $\big(\mathbb{S}^N, \mathcal{B}(\mathbb{S}^N)\big)$, $N \geq 1$, by $\mathrm{Prob}(\mathbb{S}^{\mathrm{N}})$. In the literature the joint distribution over future asset prices (respectively asset returns) is usually assumed to be of the product form $\nu = \nu_1 \otimes \ldots \otimes \nu_N$ such that the signals s_n and s_m are perceived to be independent whenever $n \neq m$. No such assumption is needed here.
The consumer's preferences over alternative consumption streams are characterized next.

Assumption 2
Given the planning horizon $N>0$, the consumer's preferences over consumption within his remaining lifetime can be represented by the utility function

$$(c_0, c_1, \ldots, c_N) \longmapsto u(c_0) + \sum_{n=1}^{N} \beta^n u(c_n) \tag{6}$$

defined on $\mathbb{C}^{N+1}$ with discount factor $\beta > 0$. The instantaneous utility function $u: \mathbb{C} \longrightarrow \mathbb{R}$ is continuous, strictly increasing, and strictly concave.

For each strategy $(C,Z) \in \mathscr{B}(R,p,w)$ and $s_1^N \in \mathbb{S}^N$, define the utility attained over the remaining lifetime

$$U_0(C, s_1^N) := u(c_0) + \sum_{n=1}^{N} \beta^n u(c_n(s_1^n)).$$

The expected utility induced by strategy $(C,Z) \in \mathscr{B}(R,p,w)$ is thus given by

$$\mathbb{E}_\nu \left[U_0(C,\cdot)\right] = \int_{\mathbb{S}^N} U_0(C, s_1^N)\ \nu(ds_1^N). \tag{7}$$

For each (R,p,w) for which $\mathscr{B}(R,p,w) \neq \emptyset$ let

$$V_0(R,p,w) := \sup \left\{ \mathbb{E}_\nu \left[U_0(C,\cdot)\right] \,\middle|\, (C,Z) \in \mathscr{B}(R,p,w) \right\}. \tag{8}$$

The definition of an optimal strategy is now straightforward.

Definition 3
Given a triple (R, p, w) with $\mathscr{B}(R, p, w) \neq \emptyset$, a strategy $(C^\star, Z^\star) \in \mathscr{B}(R, p, w)$ is termed an optimal strategy *if* $\mathbb{E}_\nu [U_0(C^\star, \cdot)] = V_0(R, p, w)$ *with* $V_0(R, p, w)$ *being defined in (8).*

The consumer's objective is to choose an optimal strategy $(C^\star, Z^\star) \in \mathscr{B}(R, p, w)$ in the sense of Definition 3. Formally his decision problem at time $t = 0$ may be stated as

$$\max_{(C,Z)} \left\{ \mathbb{E}_\nu [U_0(C, \cdot)] \,\middle|\, (C, Z) \in \mathscr{B}(R, p, w) \right\}. \tag{9}$$

It is clear that the problem (9) is only well-defined if the supremum in (8) is finite, i.e., if $V_0(R, p, w) < \infty$. While this is trivially satisfied for the case where the utility function U is bounded (as is often assumed in the literature), this requirement turns out to be too strong in many scenarios. This is for example the case with a power utility function which is studied in Section 4. The following assumption offers an alternative providing two sufficient conditions for the decision problem (9) to be well-defined.

Assumption 3
At least one of the following two conditions is satisfied:

(i) The utility function u in Assumption 2 is bounded.

(ii) The measure ν in Assumption 1 has compact support $\bar{\mathbb{S}}_1 \times \ldots \times \bar{\mathbb{S}}_N \in \mathcal{B}(\mathbb{S}^N)$ with each $\bar{\mathbb{S}}_n \subset \mathbb{S}$, $n = 1, \ldots, N$, being compact.

The requirement in (ii) that the support is the product of compact sets is made just for convenience. In principle, it suffices to assume that the support of ν is a compact set. The following lemma shows that (ii) in Assumption 3 is indeed sufficient for the decision problem to be well-defined even if U is not bounded. The proof is given in Section A.1 in Appendix A of this chapter.

Lemma 2
Let Assumptions 1, 2 and 3 be satisfied. Then the supremum defined in (8) satisfies $V_0(R, p, w) < \infty$ *for all* (R, p, w) *for which* $\mathscr{B}(R, p, w) \neq \emptyset$.

In the sequel we shall assume that Assumption 3 is satisfied such that the consumer's decision problem (9) is well-defined. Associated with a solution to (9) defining an optimal strategy $(C^\star, Z^\star)$ is an optimal decision $(c_0^\star, z_0^\star) \in \mathbb{C} \times \mathbb{Z}$ for $t = 0$. The main goal of the following section is to state conditions under which this optimal decision is well-defined and can be represented by a continuous function describing the consumer's demand behavior in the decision period.

3 Existence of demand functions

Consider next the derivation of an optimal strategy defining a solution to the consumer's decision problem. Since this task turns out to be trivial in the case where $w = -\hat{e}_0/R$ (see Corollary 1), assume for the following derivations that $w > -\hat{e}_0/R$. Employing a recursive solution technique from stochastic dynamic programming, it will be shown that under some mild additional restrictions a continuous demand function describing the optimal consumption and investment decision for $t = 0$ can be defined. To alleviate the subsequent notation we define for each $x \in \mathbb{R}^M$ and $s = (p,d) \in \mathbb{S}$ the operation $x \oplus s := x^\top(p+d)$ and the function $W : \mathbb{Z} \times \mathbb{S} \times \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$,

$$W\big((y,x), s, \hat{e}, \hat{R}\big) := \hat{e} + \hat{R}\,y + x \oplus s. \tag{10}$$

Given expectations $\hat{e}_{n+1} \geq 0$, $\hat{R}_n > 0$ and a portfolio $z_n \in \mathbb{Z}$ purchased at time n the value $W(z_n, s_{n+1}, \hat{e}_{n+1}, \hat{R}_n)$ describes the consumer's wealth in period $n+1$ depending on prices p_{n+1} and dividends d_{n+1}. The following definition introduces the concept of non-redundant assets that will become important in the sequel.

Definition 4
A probability measure $\hat{\nu}$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is said to induce non-redundant assets if for each $z', z'' \in \mathbb{Z}$, $z' \neq z''$ the set $A(z', z'') := \Big\{s \in \mathbb{S} | W(z', s, 0, 1) \neq W(z'', s, 0, 1)\Big\} \in \mathcal{B}(\mathbb{S})$ satisfies $\hat{\nu}(A(z', z'')) > 0$.

The property of non-redundancy ensures that two distinct portfolios can not induce the same return with probability one. This condition will turn out to be necessary in order to obtain demand functions. Intuitively, it is clear that otherwise the consumer's portfolio decision may not be uniquely determined. Note that the property of non-redundancy depends neither on the expected non-capital income nor on the bond return which is the reason why they have been set to zero and unity in the definition.

Lemma 3
Let $\hat{\nu}$ be a probability measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ and $\hat{s} : \mathbb{S} \longrightarrow \mathbb{S}$ be the identity. Suppose that the variance-covariance matrix $\hat{\Sigma} := \mathbb{E}_{\hat{\nu}}\left[(\hat{s} - \mathbb{E}_{\hat{\nu}}[\hat{s}])(\hat{s} - \mathbb{E}_{\hat{\nu}}[\hat{s}])^\top\right]$ exists and is positive definite. Then $\hat{\nu}$ induces non-redundant assets.

Proof: We show that conversely if the measure $\hat{\nu}$ does not induce non-redundant assets then the variance-covariance matrix $\hat{\Sigma}$ cannot be positive definite. To this end, suppose there exist portfolios $z' = (y', x'), z'' = (y'', x'') \in \mathbb{Z}$, $z' \neq z''$ such that $W(z', \hat{s}, 0, 1) - W(z'', \hat{s}, 0, 1) = 0$ $\hat{\nu}$ - a.s. Let $\Delta z := z' - z'' \neq 0$ and define $\hat{s}$ as above. Note that $x' \neq x''$ (otherwise $x' = x''$ would imply $y' \neq y''$ which is not possible). Let $\Delta x := x' - x'' \neq 0$ and $c := y' - y''$ and define the random variable $\Delta W : \mathbb{S} \longrightarrow \mathbb{R}$, $\Delta W(s) := \Delta x \oplus s$. Then $\Delta W(\hat{s}) = c$ $\hat{\nu}$-a.s. and, letting $H := [I_M, I_M] \in \mathbb{R}^{M \times 2M}$ where I_M is the $M \times M$ identity matrix $0 = \mathbb{V}_{\hat{\nu}}[\Delta W(\hat{s})] = \Delta x^\top H \mathbb{E}_{\hat{\nu}}\left[(\hat{s} - \mathbb{E}_{\hat{\nu}}[\hat{s}])(\hat{s} - \mathbb{E}_{\hat{\nu}}[\hat{s}])^\top\right] H \Delta x = \Delta x^\top H \hat{\Sigma} H \Delta x$. ■

The previous proof made obvious that the requirement of Lemma 3 that $\hat{\Sigma}$ be positive definite can be weakened as follows.

Corollary 2
Let $H := [I_M, I_M] \in \mathbb{R}^{M \times 2M}$. If under the hypotheses of Lemma 3 the matrix $H\hat{\Sigma}H^\top$ is positive definite, then $\hat{\nu}$ induces non-redundant assets.

Given the random variable $\hat{s}$ with distribution $\hat{\nu}$ Corollary 2 essentially requires only that the induced distribution of the cum-dividend prices $q := H\,\hat{s}$ has a positive definite variance-covariance matrix.
The subsequent recursive solution of the consumer's decision problem requires to characterize for each $n \geq 1$ the conditional distribution of the random variable s_n depending on the previous observations $s_1, \dots, s_{n-1}$. Clearly, if $N = 1$, this task is trivial. If $N > 1$, the following lemma describes how the joint probability distribution ν introduced in Assumption 1 can be factorized into conditional probabilities and a marginal probability. The proof is given in Section A.2 in Appendix A of this chapter.

Lemma 4
Let $N > 1$ and $\mathbb{S} = \mathbb{R}^M_{++} \times \mathbb{R}^M_{+}$. There exists a factorization of the measure ν into conditional probabilities $Q_n : \mathbb{S}^{n-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$, $n = 2, \dots, N$, and a marginal probability $\nu_1 : \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ such that for each $B \in \mathcal{B}(\mathbb{S}^N)$:

$$\nu(B) \quad = \quad \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \mathbf{1}_B(s_1^N)\, Q_N(s_1^{N-1}, ds_N) \cdots Q_2(s_1, ds_2)\nu_1(ds_1). \tag{11}$$

The factorization is ν-a.s. unique.

For each $n = 2, \dots, N$ the map $Q_n(s_1^{n-1}, \cdot) : \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ defines the conditional distribution of the random variable s_n depending on the previous observations $s_1, \dots, s_{n-1}$, while the measure ν_1 defines the marginal distribution of the random variable s_1. The following assumption imposes some additional restrictions on these distributions.

Assumption 4
The probability distributions $Q_n(s_1^{n-1}, \cdot)$, $n = 2, \dots, N$ and ν_1 defined on the measurable space $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ satisfy the following conditions:

(i) Each Q_n has the following Feller-property (for details, see Stokey & Lucas 1994): For each bounded and continuous function $h : \mathbb{H} \times \mathbb{S}^n \longrightarrow \mathbb{R}$ with $\mathbb{H} \subset \mathbb{R}^m$ the integral function $H(x, s_1^{n-1}) := \int_{\mathbb{S}} h(x, s_1^{n-1}, s) Q_n(s_1^{n-1}, ds)$ is again bounded and continuous on $\mathbb{H} \times \mathbb{S}^{n-1}$.

(ii) Each conditional distribution $Q_n(s_1^{n-1}, \cdot)$ as well as the marginal distribution ν_1 induces non-redundant assets in the sense of Definition 4.

Consider now the existence of a solution to the consumer's decision problem (9). Using a well-known dynamic programming approach the original $N+1$-period decision problem is split into a sequence of $N+1$ one-stage problems. Since this is again trivial if $N=1$, assume for the following derivations that $N>1$. To alleviate the subsequent notation we set $\mathbb{B}_n(w,p) := \{(c,y,x) \in \mathbb{C}\times\mathbb{Y}\times\mathbb{X} \,|\, c+y+x^\top p = w,\ y \geq -\hat{E}_n\}$ for each $n=1,\dots,N$. Define the value functions $V_n : [-\hat{E}_n,\infty[\ \times \mathbb{S}^n \longrightarrow \mathbb{R}$ recursively by setting $V_N(w_N,s_1^N) := u(w_N)$ and for each $n=1,\dots,N-1$

$$V_n(w_n,s_1^n) := \max_{(c,z)\in\mathbb{B}_n(w_n,p_n)} \left\{ u(c) + \beta \int_{\mathbb{S}} V_{n+1}\left(W(z,s,\hat{e}_{n+1},\hat{R}_n),s_1^n,s\right) Q_{n+1}(s_1^n,ds) \right\}. \tag{12}$$

In the literature the recursion (12) is called Bellmann's equation. The following proposition ensures that the functions in (12) are indeed well-defined objects.

Proposition 1
Let Assumptions 1- 4 be satisfied. Then the following holds true:

(i) The value functions V_n, $n=1,\dots,N$ defined recursively by (12) are well-defined and continuous. If u is bounded, so is each V_n.

(ii) Each $V_n(\cdot,s_1^n)$ is strictly increasing and strictly concave for all $s_1^n \in \mathbb{S}^n$.

(iii) At each stage $n=1,\dots,N-1$ the solutions to the maximization problem (12) can be represented by a pair of continuous functions $(c_n^\star,z_n^\star) : [-\hat{E}_n,\infty[\times\mathbb{S}^n \longrightarrow \mathbb{C}\times\mathbb{Z}$.

Proof: Properties (i) and (ii) are obviously true for $n=N$. Hence assume by way of induction that there exists $n\in\{1,\dots,N-1\}$ such that V_{n+1} satisfies properties (i) and (ii). We show that this implies the properties (i) and (ii) for the function V_n and the solution to problem (12) satisfies (iii). The first part of the proof covers the case where Assumption 3 (i) is satisfied and the utility function u is bounded. The second part extends the argument to an unbounded utility function by assuming that the measure ν satisfies Assumption 3 (ii).
The first induction hypothesis is that V_{n+1} is well-defined, continuous, and bounded and $V_{n+1}(\cdot,s_1^{n+1}) : [-\hat{E}_{n+1},\infty[\longrightarrow\mathbb{R}$ is strictly increasing and strictly concave for each fixed $s_1^{n+1}\in\mathbb{S}^{n+1}$. We show by induction that this implies that the claim holds for V_n and the solution at stage n. To enhance readability the remainder of this proof is organized in five steps.
Step 1: Let $s_1^n\in\mathbb{S}^n$ and $w_n \geq -\hat{E}_n$ be arbitrary but fixed. Note that the set $\mathbb{B}_n(w_n,p_n)$ defined as above is non-empty since $w_n \geq -\hat{E}_n$. For each $(c,z)\in\mathbb{B}_n(w_n,p_n)$ define

$$U_n(c,z;s_1^n) := u(c) + \beta\int_{\mathbb{S}} V_{n+1}\left(W(z,s,\hat{e}_{n+1},\hat{R}_n),s_1^n,s\right) Q_{n+1}(s_1^n,ds) \tag{13}$$

and note that the integral in (13) is indeed well-defined since for all $(c,z)\in\mathbb{B}_n(w_n,p_n)$: $W(z,s,\hat{e}_{n+1},\hat{R}_n) = \hat{e}_{n+1} + \hat{R}_n y + x\oplus s \geq \hat{e}_{n+1} - \hat{R}_n\hat{E}_n = -\hat{E}_{n+1}$ for each $s\in\mathbb{S}$.

Moreover, Assumptions 2 and 4 (i) together with the induction hypothesis imply that U_n is continuous and bounded. Since $p_n \gg 0$ the set $\mathbb{B}_n(w_n, p_n)$ is compact implying that the maximization problem

$$\max_{(c,z)\in\mathbb{C}\times\mathbb{Z}} \left\{ U_n(c, z; s_1^n) \Big| (c, z) \in \mathbb{B}_n(w_n, p_n) \right\} \tag{14}$$

possesses a solution $(c^\star, z^\star)$.

Step 2: We show that the solution to (14) is unique. To this end, note that $\mathbb{B}_n(w_n, p_n)$ is convex, hence it suffices to show that the map $U_n(\cdot, s_1^n)$ is strictly concave. Let $(c', z'), (c'', z'') \in \mathbb{B}_n(w_n, p_n)$, $(c', z') \neq (c'', z'')$ and $\lambda \in]0, 1[$ be arbitrary and define $(c_\lambda, z_\lambda) := \lambda(c', z') + (1 - \lambda)(c'', z'')$. We show that $U_n(c_\lambda, z_\lambda, s_1^n) > \lambda U_n(c', z', s_1^n) + (1 - \lambda)U_n(c'', z'', s_1^n)$. If $z' = z''$, this is trivially satisfied, for in this case $c' \neq c''$ and the assertion follows immediately from the strict concavity of u. So assume that $z' \neq z''$. The non-redundancy condition (ii) from Assumption 4 implies that the measurable set $A_n(z', z'') := \left\{ s \in \mathbb{S} | W(z', s, \hat{e}_{n+1}, \hat{R}_n) \neq W(z'', s, \hat{e}_{n+1}, \hat{R}_n) \right\} \in \mathcal{B}(\mathbb{S})$ has positive measure, i.e., $Q_{n+1}(s_1^n, A_n(z', z'')) > 0$. The linearity of $W(\cdot, \cdot, \hat{e}_{n+1}, \hat{R}_n)$ implies $W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n) = \lambda W(z', s, \hat{e}_{n+1}, \hat{R}_n) + (1 - \lambda)W(z'', s, \hat{e}_{n+1}, \hat{R}_n)$ for all $s \in \mathbb{S}$. This together with the strict concavity of the function $V_{n+1}(\cdot, s_1^{n+1})$ gives

$$\begin{aligned} V_{n+1}(W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) &\geq \lambda V_{n+1}(W(z', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) \\ &+ (1 - \lambda)V_{n+1}(W(z'', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) \end{aligned} \tag{15}$$

for all $s \in \mathbb{S}$ whereas the inequality is strict for $s \in A_n(z', z'')$. Also note that the strict concavity of u implies that $u(c_\lambda) \geq \lambda u(c') + (1 - \lambda)u(c'')$. Integrating both sides of (15) and applying Lemma 7 from Appendix B yields the desired inequality

$$\begin{aligned} U_n(c_\lambda, z_\lambda; s_1^n) &= u(c_\lambda) + \beta \int_{\mathbb{S}} V_{n+1}(W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \\ &> \lambda \left(u(c') + \beta \int_{\mathbb{S}} V_{n+1}(W(z', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \right) \\ &+ (1 - \lambda) \left(u(c'') + \beta \int_{\mathbb{S}} V_{n+1}(W(z'', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \right) \\ &= \lambda U_n(c', z'; s_1^n) + (1 - \lambda)U_n(c'', z''; s_1^n). \end{aligned}$$

This result permits us to define the solution to the maximization problem (14) as a function $(c_n^\star, z_n^\star) : [-\hat{E}_n, \infty[\times \mathbb{S}^n \longrightarrow \mathbb{C} \times \mathbb{Z}$,

$$(c_n^\star, z_n^\star)(w_n, s_1^n) := \arg\max_{(c,z)\in\mathbb{C}\times\mathbb{Z}} \left\{ U_n(c, z; s_1^n) \Big| (c, z) \in \mathbb{B}_n(w_n, p_n) \right\}. \tag{16}$$

Step 3: We claim that the mappings $c_n^\star$, $z_n^\star$ and the function V_n are continuous. To see this, define the budget set for alternative (w_n, p_n) as a correspondence $\mathbb{B}_n$:

$[-\hat{E}_n, \infty[\times \mathbb{R}^M_{++} \rightrightarrows \mathbb{C} \times \mathbb{Z}$, which is non-empty-, compact- and convex-valued on its domain of definition. Furthermore, we show in Lemma 10 in Appendix B that $\mathbb{B}_n$ is continuous. This together with the continuity of the function U_n implies the continuity of the solution function $c_n^\star$ and $z_n^\star$ by virtue of the Theorem of the Maximum (cf. Stokey & Lucas 1994, p.57 and p.62). Substituting the solution (16) into (14) yields the value function defined for all $w_n \geq -\hat{E}_n$ and $s_1^n \in \mathbb{S}^n$ as

$$V_n(w_n, s_1^n) = U_n\left(c_n^\star(w_n, s_1^n), z_n^\star(w_n, s_1^n); s_1^n\right), \tag{17}$$

the continuity of which follows immediately from the continuity of U_n and $c_n^\star$, $z_n^\star$.
Step 4: We show that $V_n(\cdot, s_1^n)$ is strictly concave. To this end, let w' and w'' be arbitrary such that $w' \geq -\hat{E}_n$, $w'' \geq -\hat{E}_n$ and $w' \neq w''$. Fix $\lambda \in]0,1[$ and define $w_\lambda := \lambda w' + (1-\lambda)w''$. Furthermore, let $(c'^\star, z'^\star) := (c^\star, z^\star)(w', s_1^n) \in \mathbb{B}_n(w', p_n)$ and $(c''^\star, z''^\star) := (c^\star, z^\star)(w'', s_1^n) \in \mathbb{B}_n(w'', p_n)$ be the optimal solutions belonging to w' and w'' and let $(c_\lambda^\star, z_\lambda^\star) := \lambda(c'^\star, z'^\star) + (1-\lambda)(c''^\star, z''^\star)$ be their convex combination. Note that $(c_\lambda^\star, z_\lambda^\star) \in \mathbb{B}_n(w_\lambda, p_n)$ but possibly $(c_\lambda^\star, z_\lambda^\star) \neq (c^\star, z^\star)(w_\lambda, s_1^n)$, i.e., $(c_\lambda^\star, z_\lambda^\star)$ does not have to be the optimal solution at w_λ. Also note from the budget set that $w' \neq w''$ implies that $(c'^\star, z'^\star) \neq (c''^\star, z''^\star)$. This together with (17) and the strict concavity of the function $U_n(\cdot, s_1^n)$ therefore implies the desired inequality

$$\begin{aligned} V_n(w_\lambda, s_1^n) &\geq U_n\left(c_\lambda^\star, z_\lambda^\star; s_1^n\right). \\ &= U_n\left(\lambda(c'^\star, z'^\star) + (1-\lambda)(c''^\star, z''^\star); s_1^n\right) \\ &> \lambda U_n\left(c'^\star, z'^\star; s_1^n\right) + (1-\lambda) U_n\left(c''^\star, z''^\star; s_1^n\right) \\ &= \lambda V_n(w', s_1^n) + (1-\lambda) V_n(w'', s_1^n). \end{aligned}$$

Step 5: We are left to show that $V_n(\cdot, s_1^n)$ is strictly increasing. To this end, let $w' > w'' \geq -\hat{E}_n$ be arbitrary. Let $(c'^\star, z'^\star)$ and $(c''^\star, z''^\star)$ be defined as in the previous step and set $\delta := w' - w'' > 0$ and $(c_\delta^\star, z_\delta^\star) := (c''^\star + \delta, z''^\star)$. Note that $(c_\delta^\star, z_\delta^\star) \in \mathbb{B}_n(w', p_n)$ but possibly $(c'^\star, z'^\star) \neq (c_\delta^\star, z_\delta^\star)$. Hence, exploiting the strict monotonicity of u:

$$\begin{aligned} V_n(w', s_1^n) &\geq U_n\left(c_\delta^\star, z_\delta^\star; s_1^n\right) = U_n\left(c''^\star + \delta, z''^\star; s_1^n\right) \\ &> U_n\left(c''^\star, z''^\star; s_1^n\right) = V_n(w'', s_1^n). \end{aligned}$$

Consider now the second case where u is not necessarily bounded but ν satisfies Assumption 3 (ii). The previous induction proof may then be repeated under the induction hypothesis that V_{n+1} is well-defined and continuous and $V_{n+1}(\cdot, s_1^{n+1}) : [-\hat{E}_{n+1}, \infty[\longrightarrow \mathbb{R}$ is strictly increasing and strictly concave for each $s_1^{n+1} \in \mathbb{S}^{n+1}$. From Lemma 9 it follows that for each $s_1^n \in \mathbb{S}^n$ the measure $Q_{n+1}(s_1^n, \cdot)$ is supported on a subset of the compact set $\bar{\mathbb{S}}_{n+1}$. From Lemma 8 and the Feller-property of Q_{n+1} it then follows that for each $(c, z) \in \mathbb{B}_n(w_n, p_n)$ and $s_1^n \in \mathbb{S}^n$ the function

$$U_n(c, z; s_1^n) := u(c) + \beta \int_{\bar{\mathbb{S}}_{n+1}} V_{n+1}\left(W(z, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s\right) Q_{n+1}(s_1^n, ds) \tag{18}$$

is well-defined (i.e., $U_n(c, z; s_1^n) < \infty$) and continuous. Repeating steps 1-5 of the previous argument then shows the claim. ■

Utilizing the value function V_1 obtained in the final recursion step, consider the following one-stage decision problem defined for all $(R, p) \gg 0$ and $w > -\hat{\epsilon}_0/R$:

$$\max_{(c,z)\in\mathbb{C}\times\mathbb{Z}} \left\{ u(c) + \beta \int_{\mathbb{S}} V_1\left(W(z, s, \hat{e}_1, R), s\right) \nu_1(ds) \,\middle|\, c + y + x^\top p = w,\ y \geq -\hat{\epsilon}_0/R \right\}. \quad (19)$$

The following theorem shows that the solution to (19) is unique and can be used together with the functions defined by (16) to construct an optimal strategy.

Theorem 1

Under the hypotheses of Proposition 1 let the functions $(c_n^\star, z_n^\star)$, $n = 1, \ldots, N$ be defined as in (16) where $c_N^\star(w_N, s_1^N) := w_N$ and $z_N^\star(s_1^N) := 0\ \forall s_1^N \in \mathbb{S}^N$. Then for each $(R, p) \gg 0$ and $w > -\hat{\epsilon}_0/R$ the following holds true:

(i) The problem (19) has a unique solution $(c_0^\star, z_0^\star) \in \mathbb{C} \times \mathbb{Z}$.

(ii) The strategy $(C^\star, Z^\star)$ defined by the optimal decision $(c_0^\star, z_0^\star)$ from (i) and plans $({c_n^\star}', {z_n^\star}') : \mathbb{S}^n \longrightarrow \mathbb{C} \times \mathbb{Z}$, $n = 1, \ldots, N$ defined recursively as[3]

$$\begin{aligned} {c_n^\star}'(s_1^n) &:= c_n^\star(W({z_{n-1}^\star}'(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}), s_1^n) \\ {z_n^\star}'(s_1^n) &:= z_n^\star(W({z_{n-1}^\star}'(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}), s_1^n) \end{aligned}$$

is an optimal strategy in the sense of Definition 3.

(iii) For any strategy $(C, Z) \in \mathscr{B}(R, p, w)$:

$$(c_0, z_0) \neq (c_0^\star, z_0^\star) \quad \Rightarrow \quad \mathbb{E}_\nu\left[U_0(C, \cdot)\right] < \mathbb{E}_\nu\left[U_0(C^\star, \cdot)\right].$$

The proof of Theorem 1 can be found in Section A.3 in Appendix A of this chapter. It asserts that an optimal strategy exists and can be constructed from the solutions obtained from the recursive definition (12). More importantly, however, it ensures that the optimal decision for $t = 0$ is uniquely defined by the solution to the one-stage problem (19). For alternative prices (R, p) and wealth w determined by (1) this optimal decision defines the demand behavior of the consumer in the decision period. The main result of this section is summarized in the following theorem.

Theorem 2

Let the consumer's planning horizon $N > 1$ and expectations $\hat{e} = (\hat{e}_1, \ldots, \hat{e}_N) \in \mathbb{R}_+^N$, $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_{N-1}) \in \mathbb{R}_{++}^{N-1}$ and $\nu \in \mathrm{Prob}(\mathbb{S}^N)$ be given and let Assumptions 1 - 4 be

[3] By abuse of notation we set ${z_{n-1}^\star}'(s_1^{n-1}) := z_0^\star$ if $n = 0$.

satisfied. Then there exists a continuous demand function $\varphi(\cdot;\nu,\hat{e},\hat{R})$ *defined for all* $(R,p) \gg 0$ *and* $w \geq -\hat{\epsilon}_0/R$ *as*

$$\varphi(R,p,w;\nu,\hat{e},\hat{R}) = \begin{pmatrix} \varphi_c(R,p,w;\nu,\hat{e},\hat{R}) \\ \varphi_y(R,p,w;\nu,\hat{e},\hat{R}) \\ \varphi_x(R,p,w;\nu,\hat{e},\hat{R}) \end{pmatrix} \tag{20}$$

$$:= \underset{(c,y,x)\in\mathbb{C}\times\mathbb{Y}\times\mathbb{X}}{\arg\max} \left\{ u(c) + \beta \int_{\mathbb{S}} V_1\left(W(y,x,s,\hat{e}_1,R),s\right)\nu_1(ds) \middle| c + y + x^\top p = w, y \geq -\hat{\epsilon}_0/R \right\}.$$

Proof: Existence of the demand function follows immediately from Theorem 1 (i). Continuity can be proved as in step 3 in the proof of Proposition 1. In this regard, continuity of the budget correspondence $(R,p,w) \Rightarrow \mathbb{B}_0(R,p,w) := \{(c,y,x) \in \mathbb{C}\times\mathbb{Y}\times\mathbb{X} | c + y + x^\top p = w, y \geq -\hat{\epsilon}_0/R\}$ follows by applying Lemma 10, noting that $\mathbb{B}_0(R,p,w) = \{(c,y,x) \in \mathbb{C}\times\mathbb{Y}\times\mathbb{X} | c + y + x^\top p = R\,w, y \geq -\hat{\epsilon}_0\}$ for each $R > 0$. ■

The function $\varphi(\cdot;\nu,\hat{e},\hat{R})$ describes the consumer's optimal consumption and investment in bonds and stocks in the decision period $t = 0$ for alternative prices $(R,p) \gg 0$ and wealth $w \geq -\hat{\epsilon}_0/R$ determined by (1). At this point, some remarks about the consumer's wealth w and the condition $w \geq -\hat{\epsilon}_0/R$ are in order which was required to obtain a non-empty strategy set. Using equation (1) this constraint can be written as

$$e_0 + R_{-1}y_{-1} + (p + d_0)^\top x_{-1} \geq -\hat{\epsilon}_0/R. \tag{21}$$

Since by (3) $\hat{\epsilon}_0 \geq 0$, it is clear that (21) can only be violated if $y_{-1} < 0$, i.e., if the consumer has taken credit by selling bonds in the previous period. We see that a sufficient condition for (21) to hold is that $y_{-1} \geq -(e_0/R_{-1} + \hat{\epsilon}_0/(R\,R_{-1}))$. However, Lemma 1 has shown that there exist lower bounds on the consumers credit taking behavior which are determined by the expectations for his discounted future non-capital income stream. Given these subjective expectations the consumer chooses a strategy which ensures his solvency at any point in time for any possible realization of prices and dividends. Hence we see that if during the previous $t = -1$ the consumer has correctly anticipated his non-capital income e_0 and the bond return $R > 0$ at time $t = 0$ (and continues to hold the same expectations for his discounted future non-capital income stream), the inequality $y_{-1} \geq -(e_0/R_{-1} + \hat{\epsilon}_0/(R\,R_{-1}))$ is automatically satisfied as a consequence of the consumer's credit taking behavior. It is therefore clear that as long as consumers' predictions for future non-capital income and future bond returns are correct or at least sufficiently precise, bankruptcy is excluded by the behavior of consumers themselves. Clearly, if expectations fail to be correct and actual realizations deviate too much from their predicted values, a potential problem of bankruptcy comes into play (which may still be avoided if dividend payments and/or asset prices are sufficiently large).

4 Demand behavior with CRRA utility

The previous section has established the existence of solution to the consumer's multi-period decision problem and of a continuous demand functions under quite general assumptions on the consumer's preferences. To derive additional properties of these demand function this section assumes a specific functional form of the utility function u in Assumption 2 which exhibits constant absolute risk aversion (CRRA).[4] To this end we make the following assumption.

Assumption 5
The utility function u in Assumption 2 is of the form

$$u(c;\gamma) = \begin{cases} \frac{1}{\gamma} c^\gamma & \gamma \neq 0 \\ \ln c & \gamma = 0 \end{cases} \tag{22}$$

Consider the recursive solution to the consumer's decision problem studied in Section 3. We first treat the case where $\gamma \neq 0$ in (22). The following proposition shows that in this case the value functions V_n defined by (12) possess a particularly convenient form.

Proposition 2
Let Assumptions 1– 5 be satisfied and suppose $\gamma \neq 0$ in (22). Define the values $\hat{E}_n$, $n = 1, \ldots, N$ as in (3). Then the following holds true.

(i) For $n = 1, \ldots, N$ and $w_n \geq -\hat{E}_n$ the value functions V_n defined by (12) satisfy

$$V_n(w_n, s_1^n) \;=\; \beta^n\, u\Big([w_n + \hat{E}_n] v_n(s_1^n); \gamma\Big)$$

where $v_N \equiv 1$ and the mappings $v_n : \mathbb{S}^n \longrightarrow \mathbb{R}_{++}$ are defined recursively as

$$\begin{cases} v_n(s_1^n) \;=\; \Big(1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\Big)^{\frac{1-\gamma}{\gamma}} \\ v_n^\star(s_1^n) \;=\; \gamma \max\limits_{\theta \geq 0, \theta^\top p_n \leq 1} \Big\{ \beta \int_{\mathbb{S}} u\Big([\hat{R}_n(1 - \theta^\top p_n) + \theta \oplus s] v_{n+1}(s_1^n, s); \gamma\Big) Q_{n+1}(s_1^n, ds) \Big\}. \end{cases}$$

(ii) The solution functions defined in equation (16) take the form

$$\begin{aligned} c_n^\star(w_n, s_1^n) &= \bar{c}_n(s_1^n)(w_n + \hat{E}_n) \\ x_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n(s_1^n))(w_n + \hat{E}_n)\theta_n(s_1^n) \\ y_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n(s_1^n))(w_n + \hat{E}_n)(1 - p_n^\top \theta_n(s_1^n)) - \hat{E}_n \end{aligned}$$

for each $n = 1, \ldots, N$ where $\bar{c}_n(s_1^n) := \big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\big]^{-1}$ and

$$\theta_n(s_1^n) \;:=\; \operatorname*{argmax}_{\theta \geq 0, \theta^\top p_n \leq 1} \Big\{ \beta \int_{\mathbb{S}} u\Big([\hat{R}_n(1 - \theta^\top p_n) + \theta \oplus s] v_{n+1}(s_1^n, s); \gamma\Big) Q_{n+1}(s_1^n, ds) \Big\}.$$

[4] Most of the subsequent results have been derived in a slightly different setting by Hakansson (1969). All proofs of the following results are placed in Appendix A.

Noting from (3) that $\hat{\epsilon}_0 = \hat{e}_1 + \hat{E}_1$ It follows from Proposition 2 and equation (19) that an optimal decision for $t = 0$ may be obtained from the following optimization problem:

$$\max_{(c,z)\in\mathbb{C}\times\mathbb{Z}} \left\{ u(c;\gamma) + \beta \int_{\mathbb{S}} u\Big([Ry + \hat{\epsilon}_0 + x \oplus s]v_1(s);\gamma\Big)\nu_1(ds) \Big| y = w - c - x^\top p \geq -\hat{\epsilon}_0/R \right\}. \tag{23}$$

One also observes from their recursive definition in Proposition 2 that the functions v_n, $n = 1, \ldots, N$ are exclusively determined by the consumer's expectations $\hat{R}$ for future bond returns and the distribution ν for asset prices and dividends. Given these results the following theorem characterizes the demand functions derived from (23). Given the previous derivations the proof is straightforward by applying Lemma 6 given in Appendix A.

Theorem 3
Given the planning horizon $N \geq 1$ let Assumptions 1 – 5 be satisfied and suppose $\gamma \neq 0$. Let expectations $\hat{e} \in \mathbb{R}^N_+$, $\hat{R} \in \mathbb{R}^{N-1}_{++}$ and $\nu \in \text{Prob}(\mathbb{S}^N)$ be given. Furthermore, define $\hat{\epsilon}_0 \geq 0$ as in (3) and let the functions v_n, $n = 1, \ldots, N$ be defined as in Proposition 2. Then for all $(R, p) \gg 0$ and $w > -\hat{\epsilon}_0/R$ the demand functions in (20) take the form

$$\begin{aligned}
\varphi_c(R, p, w; \hat{e}, \hat{R}, \nu) &= \bar{c}(R, p; \hat{R}, \nu)(w + \hat{\epsilon}_0/R) \\
\varphi_x(R, p, w; \hat{e}, \hat{R}, \nu) &= (1 - \bar{c}(R, p; \hat{R}, \nu))(w + \hat{\epsilon}_0/R)\,\theta(R, p; \hat{R}, \nu) \\
\varphi_y(R, p, w; \nu_q, \hat{e}, \hat{R}) &= (1 - \bar{c}(R, p; \hat{R}, \nu))(w + \hat{\epsilon}_0/R)(1 - p^\top\theta(R, p; \hat{R}, \nu)) - \hat{\epsilon}_0/R
\end{aligned} \tag{24}$$

where $\bar{c}(R, p; \hat{R}, \nu) := \left[1 + v^\star(R, p; \hat{R}, \nu)^{\frac{1}{1-\gamma}}\right]^{-1}$ and

$$\begin{aligned}
\theta(R, p; \hat{R}, \nu) &:= \arg\max_{\theta\geq 0, \theta^\top p \leq 1} \left\{ \beta \int_{\mathbb{S}} u\Big([R(1 - \theta^\top p) + \theta \oplus s]v_1(s);\gamma\Big)\nu_1(ds) \right\} \\
v^\star(R, p; \hat{R}, \nu) &:= \gamma \max_{\theta\geq 0, \theta^\top p \leq 1} \left\{ \beta \int_{\mathbb{S}} u\Big([R(1 - \theta^\top p) + \theta \oplus s]v_1(s);\gamma\Big)\nu_1(ds) \right\}.
\end{aligned}$$

One observes that consumption at time $t = 0$ is given by a fraction $\bar{c}(R, p; \hat{R}, \nu)$ of the sum of current wealth w and the discounted expected non-capital income stream $\hat{\epsilon}_0/R$. This sum will be called *lifetime income*. Such a consumption behavior strongly supports the so-called permanent income hypothesis (see, e.g., Romer 1996). Clearly, only the quantity w is directly available to the consumer while the quantity $\hat{\epsilon}_0/R$ has to be borrowed by issuing bonds. This is the reason for the appearance of the term $-\hat{\epsilon}_0/R$ in the bond demand function. The optimal investment in shares is determined by the solution $\theta(R, p; \hat{R}, \nu)$ which is exclusively determined by current and expected financial prices.

Consider now the case with logarithmic utility where $\gamma = 0$ in (22). Since in this case the function u is not defined at zero Assumption 2 must be relaxed to hold only on the

interior of the consumption set $\mathbb{C}$. However, Lemma 1 implies that the wealth process $(W_n(Z, s_1^n))_{n=1}^N$ induced by any potentially optimal strategy $(C, Z) \in \mathscr{B}(R, p, w)$ has to satisfy $W_n(Z, s_1^n) > -\hat{E}_n\ \nu-$ a.s. Otherwise one easily shows that $c_n(s_1^n) = 0$ with positive probability which implies $\mathbb{E}_\nu\left[U_0(C, \cdot)\right] = -\infty < V_0(R, p, w)$. It follows from this observation that the decision problem (9) remains well-defined even in the log case.
To alleviate the subsequent notation we define $\beta_n := 1 + \beta + \ldots + \beta^{N-n}$ for each $n = 0, 1, \ldots, N$. The following result establishes the form of the value functions for the log-utility case.

Proposition 3
Let Assumptions 1– 5 be satisfied and let $\gamma = 0$ in (22). Then the following holds true:

(i) For $n = 1, \ldots, N$ and $w_n > -\hat{E}_n$ the value functions V_n defined by (12) satisfy

$$V_n(w_n, s_1^n) = \beta_n \ln\left(w_n + \hat{E}_n\right) + v_n(s_1^n)$$

where the functions $v_n : \mathbb{S}^n \longrightarrow \mathbb{R}$ are defined recursively as $v_N \equiv 0$ and

$$\begin{aligned} v_n(s_1^n) \quad = \quad & \max_{\theta \geq 0, p^\top p \leq 1} \left\{ (\beta_n - 1) \int_{\mathbb{S}} \ln\Big(\hat{R}_n(1 - \theta^\top p_n) + \theta \oplus s\Big) Q_{n+1}(s_1^n, ds) \Big| \theta^\top p_n \leq 1 \right\} \\ & + \beta \int_{\mathbb{S}} v_{n+1}(s_1^n, s) Q_{n+1}(s_1^n, ds) + (\beta_n - 1)\ln(\beta_n - 1) - \beta_n \ln(\beta_n). \end{aligned}$$

(ii) The solution functions defined in equation (16) take the form

$$\begin{aligned} c_n^\star(w_n, s_1^n) &= \bar{c}_n(w_n + \hat{E}_n) \\ x_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n)(w_n + \hat{E}_n)\theta_n(s_1^n) \\ y_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n)(w_n + \hat{E}_n)(1 - p_n^\top \theta_n(s_1^n)) - \hat{E}_n \end{aligned}$$

for each $n = 1, \ldots, N$ where $\bar{c}_n := \left[1 + \beta_n\right]^{-1}$ and

$$\theta_n(s_1^n) \quad := \quad \underset{\theta \geq 0, \theta^\top p_n \leq 1}{\operatorname{argmax}} \left\{ \int_{\mathbb{S}} \ln\Big(\hat{R}_n(1 - \theta^\top p_n) + \theta \oplus s\Big) Q_{n+1}(s_1^n, ds) \right\}.$$

Utilizing Proposition 3 and the fact that equation (3) implies $\hat{e}_0 = \hat{e}_1 + \hat{E}_1$, one obtains from (19) the following one-stage decision problem for $t = 0$ defined for all $(R, p) \gg 0$ and $w > -\hat{e}_0/R$:

$$\max_{(c,z) \in \mathbb{C} \times \mathbb{Z}} \left\{ \ln(c) + (\beta_0 - 1) \int_{\mathbb{S}} \ln\Big(W(z, s, \hat{e}_0, R)\Big) \nu_1(ds) \ \Big| c + y + x^\top p = w, y \geq -\hat{e}_0/R \right\}. \quad (25)$$

Here $\beta_0 := (1 + \beta + \ldots + \beta^N)$ and the additive constant $v_0 := \int_{\mathbb{S}} v_1(s)\nu_1(ds)$ has been omitted since it does not affect the solution to the problem. Utilizing Lemma 6 the optimal consumption and investment decision for $t = 0$ can be determined from (25). The resulting form of the demand functions is stated in the following theorem. Given the previous derivations the proof is again straightforward by applying Lemma 6 given in Appendix A.

Theorem 4
Let Assumptions 1 – 5 be satisfied and suppose $\gamma = 0$ in (22). Let the consumer's planning horizon $N \geq 1$ and expectations $\hat{e} \in \mathbb{R}^N_+$, $\hat{R} \in \mathbb{R}^{N-1}_{++}$ and $\nu \in \mathrm{Prob}(\mathbb{S}^N)$ be given. Furthermore, define $\hat{\epsilon}_0 \geq 0$ as in (3) and the marginal distribution ν_1 as in Lemma 4. Then for all $(R,p) \gg 0$ and $w > -\hat{\epsilon}_0/R$ determined by (1) the consumer's demand behavior can be described by the functions

$$\begin{aligned}
\varphi_c(R,p,w;\nu_1,\hat{e},\hat{R}) &= \bar{c}(w+\hat{\epsilon}_0/R) \\
\varphi_x(R,p,w;\nu_1,\hat{e},\hat{R}) &= (1-\bar{c})(w+\hat{\epsilon}_0/R)\,\theta(R,p;\nu_1) \qquad (26)\\
\varphi_y(R,p,w;\nu_1,\hat{e},\hat{R}) &= (1-\bar{c})(w+\hat{\epsilon}_0/R)(1-p^\top\theta(R,p;\nu_1)) - \hat{\epsilon}_0/R
\end{aligned}$$

where $\bar{c} := \left[1+\beta+\ldots+\beta^N\right]^{-1}$ *and*

$$\theta(R,p;\nu_1) := \arg\max_{\theta\geq 0,\theta^\top p\leq 1}\left\{\int_{\bar{\mathbb{S}}_1}\ln\Big(R(1-\theta^\top p)+\theta\oplus s\Big)\nu_1(ds)\right\}. \qquad (27)$$

Compared to the previous case where $\gamma \neq 0$ the number $\bar{c}$ defining the marginal propensity to consumption (out of lifetime income) depends exclusively on the subjective discount factor β and the consumer's remaining lifetime N. This implies that the optimal consumption decision is independent of any expectations for future asset prices and dividends. The remaining lifetime income $(1-\bar{c})(w+\hat{\epsilon}_0/R)$ is invested into the safe asset $m=0$ and shares $m=1,\ldots,M$. In this regard, the amount invested in shares is determined by the solution θ to (27). The structure of this problem corresponds exactly to the portfolio decision problem solved by a consumer with a one-period planning horizon who is endowed with one unit of wealth and who is neither allowed to take short sales nor credit. In particular, no expectations for asset prices and dividends which lie further than one period ahead enter the problem. This property is called complete myopia or myopic investment behavior and is well-known to hold with logarithmic utility, see, e.g., Ingersoll (1987), or Hakansson (1970). Note though that expectations for future bond returns and non-capital income strongly influence the decision through the term $\hat{\epsilon}_0$.
We note from the integral in (27) that only the sum of next period's prices and dividends $s_1 = (p_1, d_1)$ enter the problem defining the cum-dividend price $q := p_1 + d_1$ of the following period. This sum being a measurable function of the random variable s_1 permits us to define an induced measure ν_q for the random variable q corresponding to the image measure induced by ν_1. Clearly, since by (ii) of Assumption 5 and Lemma 9, ν_1 is supported on the compact set $\bar{\mathbb{S}}_1 \subset \mathbb{S}$, the support of ν_q will be a compact subset $\bar{\mathbb{Q}} \subset \mathbb{R}^M_{++}$. Exploiting the change-of-variable formula the function in (27) can equivalently be defined as

$$\hat{\theta}(R,p;\nu_q) := \arg\max_{\theta\in\mathbb{X}}\left\{\int_{\bar{\mathbb{Q}}}\ln(R+\theta^\top(q-Rp))\;\nu_q(d\,q)\;\Big|\;\theta^\top p\leq 1\;\right\}.$$

We close this section with the following corollary which specializes Theorem 4 to the case with a single risky asset where $M=1$. The proof follows immediately.

Corollary 3
Under the hypotheses of Theorem 4 suppose $M = 1$. Then the demand functions (26) take the form

$$\begin{aligned}
\varphi_c(R, p, w; \nu_q, \hat{e}, \hat{R}) &= \bar{c}(w + \hat{\epsilon}_0/R) \\
\varphi_x(R, p, w; \nu_q, \hat{e}, \hat{R}) &= (1 - \bar{c})(w + \hat{\epsilon}_0/R)\, \tilde{\theta}(Rp; \nu_q)/p \\
\varphi_y(R, p, w; \nu_q, \hat{e}, \hat{R}) &= (1 - \bar{c})(w + \hat{\epsilon}_0/R)(1 - \tilde{\theta}(Rp; \nu_q)) - \hat{\epsilon}_0/R
\end{aligned}$$

where $\bar{c}$ is defined as above and the share of (lifetime) income invested in shares is determined by the continuous map $\tilde{\theta}(\cdot, \nu_q) : \mathbb{R}_{++} \longrightarrow [0, 1]$

$$\tilde{\theta}(\pi; \nu_q) := \arg \max_{\tilde{\theta} \in [0,1]} \left\{ \int_{\bar{\mathbb{Q}}} \ln(\pi + \tilde{\theta}(q - \pi))\ \nu_q(d\, q) \right\}.$$

5 Conclusions

This paper offers a general mathematical formulation of a multi-period consumption-investment problem in the presence of uncertainty due to stochastic investment possibilities. Based on the notion of a consumption-investment strategy a multi-period expected utility maximization problem was formulated under general assumptions on preferences and expectations. The existence and properties of a solution to the problem were studied under general assumptions as well as for the special case with CRRA preferences permitting closed form solutions. The formulation of a demand function parameterized in current prices, wealth and expectations offers a convenient possibility to utilize the results of this paper in macroeconomic and financial models with uncertainty and consumers facing a multi-period planning horizon. A first application may be found in Hillebrand (2007) and Böhm & Hillebrand (2007) who study the role of pension systems in a stochastic OLG model with multi-period lived consumers. The proposed structure of the decision problem should also offer a generally applicable framework for most decision problems in portfolio theory and stochastic macroeconomics. Modifications of the proposed setup to include short selling and/or the maximization of expected utility of terminal wealth are straightforward.

A Mathematical Proofs

A.1 Proof of Lemma 2

It suffices to show that there exists an upper bound $\bar{U}$ such that $\mathbb{E}_\nu[U_0(C,\cdot)] \le \bar{U}$ for all $(C,Z) \in \mathscr{B}(R,p,w)$. Let $(C,Z) \in \mathscr{B}(R,p,w)$ be arbitrary. It suffices to show that $U_0(C,s_1^N) < \bar{U}$ for all $s_1^N \in \bar{\mathbb{S}}$. Since $\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \ldots, \times \bar{\mathbb{S}}_N$ with each $\bar{\mathbb{S}}_n$ being compact there exist upper and lower bounds $\underline{s}_n \in \mathbb{S}$ and $\bar{s}_n \in \mathbb{S}$ such that $\underline{s}_n \le s_n \le \bar{s}_n$ ν - a.s. for each $n = 1, \ldots, N$. Setting e.g. $\underline{s} := \min\{\underline{s}_n | n = 1, \ldots, N\}$ and $\bar{s} := \max\{\underline{s}_n | n = 1, \ldots, N\}$ these bound can be chosen independently of n. Moreover, letting $\underline{q} := \underline{p} + \underline{d}$, $\bar{q} := \bar{p} + \bar{d}$ and $q_n := p_n + d_n$ for each $n = 1, \ldots, N$ it follows that $0 \ll \underline{p} \le p_n \le \bar{p}$ ν - a.s. and $0 \ll \underline{q} \le \bar{q}_n \le \bar{q}$ ν - a.s. for each $n = 1, \ldots, N$. Define the wealth process $\{W_n(Z, s_1^n)\}_{n=1}^N$ associated with strategy $(C,Z) \in \mathscr{B}(R,p,w)$ as in (2). By virtue of Definition 2 and Lemma 1 we have $c_0 \le w + \hat{\epsilon}_0/R$ and $c_n(s_1^n) \le W_n(Z,s_1^n) + \hat{E}_n$ for each $s_1^n \in \mathbb{S}^n$, $n = 1, \ldots, N$. It is therefore sufficient to prove that for each $n = 1, \ldots, N$ there exists $\bar{W}_n \in \mathbb{R}$ such that

$$W_n(Z, s_1^n) \le \bar{W}_n \quad \nu \text{ - a.s.} \tag{28}$$

We prove (28) for $n = 1$ and then apply an induction argument. To this end, note that any investment decision (y_0, x_0) made at stage $n = 0$ satisfies the budget constraint $c_0 + y_0 + x_0^\top p = w$. Utilizing Lemma 1 one obtains bounds on this investment as $-\hat{\epsilon}_0/R \le y_0 \le w =: \bar{y}_0$ and $x_0^{(m)} \le (w + \hat{\epsilon}_0/R)/p^{(m)} =: \bar{x}_0^{(m)}$ for each $m = 1, \ldots, M$. Setting $\bar{x}_0 := (\bar{x}_0^{(1)}, \ldots, \bar{x}_0^{(M)})^\top$ gives

$$W_1(Z, s_1) = \hat{e}_1 + R\, y_0 + x_0^\top q_1 \le \hat{e}_1 + R\, \bar{y}_0 + \bar{x}_0^\top \bar{q}_1 =: \bar{W}_1 \quad \nu \text{ - a.s.}$$

By way of induction, suppose that for some $n \in \{1, \ldots, N-1\}$ there exists an upper bound $\bar{W}_n$ such that $W_n(Z, s_1^n) \le \bar{W}_n$ ν - a.s. Lemma 1 and the induction hypothesis imply that the investment plans $y_n(\cdot)$ and $x_n(\cdot)$ at stage n satisfy $-\hat{E}_n \le y_n(s_1^n) \le W_n(Z,s_1^n) \le \bar{W}_n =: \bar{y}_n$ ν - a.s. and $x_n^{(m)}(s_1^n) \le (W_n(Z,s_1^n) + \hat{E}_n)/p_n^{(m)} \le (\bar{W}_n + \hat{E}_n)/\underline{p}^{(m)} =: \bar{x}_n^{(m)}$ ν - a.s. for each $m = 1, \ldots, M$. Setting $\bar{x}_n := (\bar{x}_n^{(1)}, \ldots, \bar{x}_n^{(M)})^\top$ gives

$$W_{n+1}(Z, s_1^{n+1}) = \hat{e}_{n+1} + \hat{R}_n\, y_n(s_1^n) + x_n(s_1^n)^\top q_{n+1} \le \hat{e}_{n+1} + \hat{R}_n\, \bar{y}_n + \bar{x}_n^\top \bar{q} =: \bar{W}_{n+1} \quad \nu \text{ - a.s.}$$

which proves the claim (28). Equation (28) and Lemma 1 imply that

$$c_n(s_1^n) \le \bar{c}_n := \bar{W}_n + \hat{E}_n \quad \nu \text{ - a.s.}$$

for each $n = 1, \ldots, N$ and, exploiting the monotonicity of u

$$U_0(C, s_1^N) = u(c_0) + \sum_{n=1}^N \beta^n u(c_n(s_1^n)) \le \bar{U} := \sum_{n=0}^N \beta^n u(\bar{c}_n) \quad \nu \text{ - a.s.}$$

The last equation implies that $\mathbb{E}_\nu[U_0(C,\cdot)] \le \bar{U}$ for all $(C,Z) \in \mathscr{B}(R,p,w)$ and therefore $V_0(R,p,w) \le \bar{U} < \infty$. ■

A.2 Proof of Lemma 4

For each $n = 1, \ldots, N$ let $\pi_n : \mathbb{S}^N \longrightarrow \mathbb{S}$ denote the nth projection mapping defined as $\pi_n(s_1, \ldots, s_n, \ldots, s_N) = s_n$ and by $\pi_{1,n} : \mathbb{S}^N \longrightarrow \mathbb{S}^n$, $\pi_{1,n}(s_1, \ldots, s_n, \ldots, s_N) = (s_1, \ldots, s_n)$ the projection of $\mathbb{S}^N$ onto its first n components. In the sequel we will utilize the following factorization lemma the proof of which can be found in Gänssler & Stute (1977, p.198, Satz 5.3.21) and Arnold (1998, p. 23, Satz 1.4.3).

Lemma 5
Let $(\Omega_1, \mathcal{A}_1)$ be a measurable space and $(\Omega_2, \mathcal{B}(\Omega_2))$ be a Polish space equipped with its Borelian σ-Algebra generated by the open subsets of Ω_2. Then for each probability measure $\nu_{1,2}$ on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{B}(\Omega_2))$ there exists a transition probability $Q_2 : \Omega_1 \times \mathcal{B}(\Omega_2) \longrightarrow [0, 1]$ and a marginal probability $\nu_1 : \mathcal{A}_1 \longrightarrow [0, 1]$ such that one has the factorization

$$\nu_{1,2}(A) = \int_{\Omega_1} \int_{\Omega_2} 1_A(\omega_1, \omega_2) Q_2(\omega_1, d\omega_2) \nu_1(d\omega_1). \tag{29}$$

for each measurable $A \in \mathcal{A}_1 \otimes \mathcal{B}(\Omega_2)$. The measure ν_1 on $(\Omega_1, \mathcal{A}_1)$ is defined by the projection mapping $\pi_1 : \Omega_1 \times \Omega_2 \longrightarrow \Omega_1$, $\pi_1(\omega_1, \omega_2) = \omega_1$ such that $\nu_1 = \pi_1\nu := \nu \circ \pi_1^{-1}$. The factorization in (29) is $\nu_{1,2}$ - a.s. unique.

To apply Lemma 5 we first show that the space $\mathbb{S} = \mathbb{R}^M_{++} \times \mathbb{R}^M_{+}$ is Polish. Following Bauer (1992, p.179, Beispiele 1-4), the Euclidean space $\mathbb{R}^M$ is Polish, hence $\mathbb{R}^M_{++}$ and $\mathbb{R}^M_{+}$ being open and closed subspaces of a Polish space are Polish and hence also their product. This shows that the space $\mathbb{S}$ satisfies the requirements of Lemma 5.
The desired factorization (11) of the measure ν in Assumption 1 is now achieved by a repeated application of Lemma 5. In a first step, using the notation of Lemma 5, set $\Omega_1 = \mathbb{S}^{N-1}$, $\Omega_2 = \mathbb{S}$ and $\nu_{1,2} = \nu$ to obtain the factorization

$$\nu(B) \;=\; \int_{\mathbb{S}^{N-1}} \int_{\mathbb{S}} 1_B(s_1^{N-1}, s_N) Q_N(s_1^{N-1}, ds_N) \nu_{N-1}(ds_1^{N-1})$$

for each measurable $B \in \mathcal{B}(\mathbb{S}^N)$ with a transition probability $Q_N : \mathbb{S}^{N-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0, 1]$ and a marginal probability $\nu_{N-1} = \pi_{1,N-1}\nu := \nu \circ \pi_{1,N-1}^{-1}$.
In a second step, set $\Omega_1 = \mathbb{S}^{N-2}$, $\Omega_2 = \mathbb{S}$ and $\nu_{1,2} = \nu_{N-1}$ to obtain a factorization of ν_{N-1} into transition probability $Q_{N-1} : \mathbb{S}^{N-2} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0, 1]$ and marginal probability $\nu_{N-2} = \pi_{1,N-2}\nu_{N-1} := \nu_{N-1} \circ \pi_{1,N-2}^{-1}$. Continuing in this fashion one obtains a sequence of transition probabilities Q_n, $n = N, \ldots, 2$ and marginal probabilities ν_n, $n = N-1, \ldots, 1$. At each stage $n > 1$ the measure ν_n is factorized into a transition probability $Q_n : \mathbb{S}^{n-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0, 1]$ and a marginal probability $\nu_{n-1} = \pi_{1,n-1}\nu_n := \nu_n \circ \pi_{1,n-1}^{-1}$. In the final step one obtains a factorization of the measure ν_2 into transition probability $Q_2 : \mathbb{S} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0, 1]$ and marginal probability ν_1 which satisfies $\nu_1 = \pi_1\nu := \nu \circ \pi_1^{-1}$ completing the proof of Lemma 4. ■

A.3 Proof of Theorem 1

(i) Utilizing the properties of the value function V_1 stated in Proposition 1 and the non-redundancy of the measure ν_1 stated in Assumption 4 (ii) the proof is straightforward by following steps $1-3$ in the proof of Proposition 1.

(ii) For ease of notation we shall adopt the following conventions. For any strategy $(C, Z) \in \mathscr{B}(R, p, w)$ we set $c_n(s_1^n) := c_0$, $z_n(s_1^n) := z_0$ and $W(z_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) := w$ if $n = 0$ as well as $\hat{R}_0 := R$ as before. Recall from Definition 2 that the terminal plan of *any* feasible strategy satisfies $c_N(s_1^N) = W(z_{N-1}(s_1^{N-1}), s_N, \hat{e}_N, \hat{R}_{N-1})$ and $z_N(s_1^N) = 0$ $\forall s_1^N \in \mathbb{S}^N$. Given the optimal decision $(c_0^\star, z_0^\star)$ from (i) and the functions $(c_n^\star, z_n^\star)(\cdot)$ defined in Proposition 1 (iii) let the strategy $(C^\star, Z^\star)$ be defined as in (ii) of Theorem 1 and set $c_n^\star(w_n, s_1^n) := c_0^\star$ and $z_n^\star(w_n, s_1^n) := z_0^\star$ if $n = 0$. Finally, recall from the definition (12) of the value functions V_n, $n = 1, \ldots, N$, that for each $\hat{w} \geq -\hat{E}_n$ and $s_1^n \in \mathbb{S}^n$

$$V_n(\hat{w}, s_1^n) = u(c_n^\star(\hat{w}, s_1^n)) + \beta \int_{\mathbb{S}} V_{n+1}(W(z_n^\star(\hat{w}, s_1^n), s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds). \quad (30)$$

Let $(\hat{C}, \hat{Z}) = (\hat{c}_0, \hat{c}_1(\cdot), \ldots, \hat{c}_N(\cdot), \hat{z}_0, \hat{z}_1(\cdot), \ldots, \hat{z}_N(\cdot)) \in \mathscr{B}(R, p, w)$ be an arbitrary strategy. The claim will follow if we show that $\mathbb{E}_\nu[U_0(C^\star, \cdot)] \geq \mathbb{E}_\nu\big[U_0(\hat{C}, \cdot)\big]$. If $N = 1$, this task is trivial since any strategy reduces to a decision for $t = 0$ and the inequality is therefore implied by (i). Hence the remainder assumes that $N \geq 2$. The idea of the proof is to construct a list of induced strategies $(\hat{C}^{(n)}, \hat{Z}^{(n)})$, $n = 1, \ldots, N-1$ obtained by successively replacing the plans $(\hat{c}_n, \hat{z}_n)(\cdot)$ in $(\hat{C}, \hat{Z})$ with the potentially optimal plans for stage n defined by the functions $(c_n^\star, z_n^\star)(\cdot)$ from Proposition 1 (iii) and to show that

$$\mathbb{E}_\nu[U_0(C^\star, \cdot)] \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(1)}, \cdot)\right] \geq \ldots \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(N-1)}, \cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C}, \cdot)\right]. \quad (31)$$

Following the above conventions, define for each stage $n = 0, 1, \ldots, N$ the induced strategy $(\hat{C}^{(n)}, \hat{Z}^{(n)}) = (\hat{c}_0^{(n)}, \hat{c}_1^{(n)}(\cdot), \ldots, \hat{c}_N^{(n)}(\cdot), \hat{z}_0^{(n)}, \hat{z}_1^{(n)}(\cdot), \ldots, \hat{z}_N^{(n)}(\cdot))$ as follows:

$$\begin{array}{lcll} \hat{c}_m^{(n)}(s_1^m) & := & \hat{c}_m(s_1^m), & m = 0, 1, \ldots, n-1 \\ \hat{z}_m^{(n)}(s_1^m) & := & \hat{z}_m(s_1^m), & m = 0, 1, \ldots, n-1 \\ \hat{c}_m^{(n)}(s_1^m) & := & c_m^\star(W(\hat{z}_{m-1}^{(n)}(s_1^{m-1}), s_m, \hat{e}_m, \hat{R}_{m-1}), s_1^m) & m = n, \ldots, N \\ \hat{z}_m^{(n)}(s_1^m) & := & z_m^\star(W(\hat{z}_{m-1}^{(n)}(s_1^{m-1}), s_m, \hat{e}_m, \hat{R}_{m-1}), s_1^m) & m = n, \ldots, N. \end{array} \quad (32)$$

Observe that each strategy $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ is feasible and coincides with the original strategy $(\hat{C}, \hat{Z})$ up to period stage $n-1$ (the same is obviously true for any strategy $(\hat{C}^{(m)}, \hat{Z}^{(m)})$ with $m > n$). In particular, the strategies induce the same wealth process until stage n and hence yield the same random variable $W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$ defining wealth at stage n. From stage n onwards the plans of the strategy $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ are defined by the functions $((c_m^\star, z_m^\star)(\cdot))_{m=n}^N$. Observe, however, that this does *not* imply that $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ coincides with $(\hat{C}^\star, \hat{Z}^\star)$ from stage n onwards because in general $W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) \neq W(z_{n-1}^\star(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$.

Since the above conventions imply that $(\hat{C}^{(N)}, \hat{Z}^{(N)}) = (\hat{C}, \hat{Z})$ and $(\hat{C}^{(0)}, \hat{Z}^{(0)}) = (\hat{C}^\star, \hat{Z}^\star)$, the claim (31) will follow if we show that $\mathbb{E}_\nu\left[U_0(\hat{C}^{(n)}, \cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(n+1)}, \cdot)\right]$ for each $n = 0, \ldots, N-1$. Since by definition (32) the strategies $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ and $(\hat{C}^{(n+1)}, \hat{Z}^{(n+1)})$ coincide until stage $n-1$, it suffices to show that for each $n = 0, \ldots, N-1$

$$\mathbb{E}_\nu\left[\sum_{m=n}^{N} \beta^{m-n} u(\hat{c}_m^{(n)}(\cdot))\right] \geq \mathbb{E}_\nu\left[\sum_{m=n}^{N} \beta^{m-n} u(\hat{c}_m^{(n+1)}(\cdot))\right]. \tag{33}$$

Case 1: Assume that $n > 0$. Let $s_1^n \in \mathbb{S}^n$ be arbitrary but fixed and set $\hat{w}_n := W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) \geq -\hat{E}_n$. Recall from (32) that the strategy $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ is defined by the functions $((c_m^\star, z_m^\star)(\cdot))_{m=n}^N$ from stage n onwards. Using the conditional distributions Q_m, $m = n+1, \ldots, N$ from Lemma 4 and equation (30) this implies that

$$\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n}^{N} \beta^{m-n} u(\hat{c}_m^{(n)}(s_1^m)) Q_N(s_1^{N-1}, ds_N) \cdots Q_{n+1}(s_1^n, ds_{n+1}) = V_n(\hat{w}_n, s_1^n). \tag{34}$$

Likewise, for any fixed $s_1^{n+1} \in \mathbb{S}^{n+1}$ and $\hat{w}_{n+1} := W(\hat{z}_n(s_1^n), s_{n+1}, \hat{e}_{n+1}, \hat{R}_n) \geq -\hat{E}_{n+1}$ the plans of strategy $(\hat{C}^{(n+1)}, \hat{Z}^{(n+1)})$ satisfy

$$\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n+1}^{N} \beta^{m-n} u(\hat{c}_m^{(n+1)}(s_1^m)) Q_N(s_1^{N-1}, ds_N) \ldots Q_{n+2}(s_1^{n+1}, ds_{n+2}) = \beta\, V_{n+1}(\hat{w}_{n+1}, s_1^{n+1}). \tag{35}$$

Combining (34) and (35) and recalling from (32) that $\hat{c}_n^{(n+1)}(s_1^n) = \hat{c}_n(s_1^n)$ and $\hat{z}_n^{(n+1)}(s_1^n) = \hat{z}_n(s_1^n)$ one has for each fixed $s_1^n \in \mathbb{S}^n$

$$\begin{aligned} &\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n}^{N} \beta^{m-n} \left(u(\hat{c}_m^{(n)}(s_1^m)) - u(\hat{c}_m^{(n+1)}(s_1^m))\right) Q_N(s_1^{N-1}, ds_N) \ldots Q_{n+1}(s_1^n, ds_{n+1}) \\ = \;& V_n(\hat{w}_n, s_1^n) - u(\hat{c}_n(s_1^n)) - \beta \int_{\mathbb{S}} V_{n+1}(W(\hat{z}_n(s_1^n), s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s)) Q_{n+1}(s_1^n, ds) \geq 0, \end{aligned} \tag{36}$$

where as before $\hat{w}_n = W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$. Clearly, (36) holds with equality if and only if $\hat{c}_n(s_1^n) = c_n^\star(\hat{w}_n, s_1^n)$ and $\hat{z}_n(s_1^n) = z_n^\star(\hat{w}_n, s_1^n)$. Using the notation introduced in the proof of Lemma 4 let $\nu_n := \pi_{1,n}\nu$ denote the joint marginal distribution of the random variables $s_1, \ldots, s_n$. The inequality in (36) being true for all $s_1^n \in \mathbb{S}^n$ implies that it is preserved under integration with respect to ν_n.[5] Hence, integrating both sides

[5] Alternatively, one could integrate (36) successively over the conditional distributions $Q_n, \ldots, Q_2$ and the marginal distribution ν_1 defined in Lemma 4.

of (36) with respect to ν_n and exploiting the factorization Lemma 4 gives

$$
\begin{aligned}
& \mathbb{E}_\nu\left[\sum_{m=n}^{N}\beta^{m-n}(u(\hat{c}_m^{(n)}(\cdot))\right] - \mathbb{E}_\nu\left[\sum_{m=n}^{N}\beta^{m-n}u(\hat{c}_m^{(n+1)}(\cdot))\right] \\
= & \int_{\mathbb{S}^n}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=n}^{N}\beta^{m-n}u(\hat{c}_m^{(n)}(s_1^m))Q_N(s_1^{N-1},ds_N)\cdots Q_{n+1}(s_1^n,ds_{n+1})\nu_n(ds_1^n) \\
- & \int_{\mathbb{S}^n}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=n}^{N}\beta^{m-n}u(\hat{c}_m^{(n+1)}(s_1^m))Q_N(s_1^{N-1},ds_N)\ldots Q_{n+1}(s_1^n,ds_{n+1})\nu_n(ds_1^n) \\
= & \int_{\mathbb{S}^n}V_n(W(\hat{z}_{n-1}(s_1^{n-1}),s_n,\hat{e}_n,\hat{R}_{n-1}),s_1^n)\nu_n(ds_1^n) - \int_{\mathbb{S}^n}u(\hat{c}_n(s_1^n))\nu_n(ds_1^n) \\
- & \beta\int_{\mathbb{S}}V_{n+1}(W(\hat{z}_n(s_1^n),s,\hat{e}_{n+1},\hat{R}_n),s_1^n,s))Q_{n+1}(s_1^n,ds)\nu_n(ds_1^n) \geq 0.
\end{aligned}
$$

This proves (33) for the case $n > 0$.

Case 2: Assume that $n = 0$. In this case, one has $(\hat{C}^{(0)}, \hat{Z}^{(0)}) = (\hat{C}^\star, \hat{Z}^\star)$ and the strategies may only differ with respect to the decisions $(c_0^\star, z_0^\star)$ and $(\hat{c}_0^{(1)}, \hat{z}_0^{(1)}) = (\hat{c}_0, \hat{z}_0)$. Using an analogous reasoning as in the previous case one has from (30) and (32) for each fixed $s_1 \in \mathbb{S}$

$$
\begin{aligned}
\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=1}^{N}\beta^m u(\hat{c}_m^{(0)}(s_1^m))Q_N(s_1^{N-1},ds_N)\ldots Q_2(s_1,ds_2) &= \beta V_1(W(z_0^\star,s_1,\hat{e}_1,R),s_1) \\
\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=1}^{N}\beta^m u(\hat{c}_m^{(1)}(s_1^m))Q_N(s_1^{N-1},ds_N)\ldots Q_2(s_1,ds_2) &= \beta V_1(W(\hat{z}_0,s_1,\hat{e}_1,R),s_1).
\end{aligned}
$$

Hence one obtains, utilizing the factorization Lemma 4

$$
\begin{aligned}
& \mathbb{E}_\nu\left[\sum_{m=0}^{N}\beta^m u(\hat{c}_m^{(0)}(\cdot))\right] - \mathbb{E}_\nu\left[\sum_{m=0}^{N}\beta^m u(\hat{c}_m^{(1)}(\cdot))\right] \\
= & \int_{\mathbb{S}}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=0}^{N}\beta^m(u(\hat{c}_m^{(0)}(s_1^m))Q_N(s_1^{N-1},ds_N)\ldots Q_2(s_1,ds_2)\nu_1(ds_1) \\
- & \int_{\mathbb{S}}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=0}^{N}\beta^m u(\hat{c}_m^{(1)}(s_1^m))Q_N(s_1^{N-1},ds_N)\ldots Q_2(s_1,ds_2)\nu_1(ds_1) \\
= & u(c_0^\star) + \beta\int_{\mathbb{S}}V_1(W(z_0^\star,s,\hat{e}_1,R),s_1)\nu_1(ds) - u(\hat{c}_0) - \beta\int_{\mathbb{S}}V_1(W(\hat{z}_0,s,\hat{e}_1,R),s_1)\nu_1(ds) \\
\geq & 0,
\end{aligned}
$$

where equality holds if and only if $(\hat{c}_0, \hat{z}_0) = (c_0^\star, z_0^\star)$.

(iii) This is an immediate consequence of the last statement. ■

A.4 Proof of Proposition 2

The proof uses the following lemma which is proved first.

Lemma 6
Let $\hat{\nu}$ be a probability measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ supported on the compact set $\bar{\mathbb{S}} \subset \mathbb{S}$ which induces non-redundant assets in the sense of Definition 4. Given values $\hat{\epsilon} \geq 0$, $\hat{R} > 0$, $w > -\hat{\epsilon}/\hat{R}$, $p \gg 0$ and $\hat{\beta} > 0$ l et $g : \mathbb{S} \longrightarrow \mathbb{R}_{++}$ be some continuous function and define $u(\cdot;\gamma)$ as in Assumption 5. Then the solution to the problem

$$\max_{(c,y,x)\in\mathbb{C}\times\mathbb{Y}\times\mathbb{X}} \left\{ u(c;\gamma) + \hat{\beta} \int_{\mathbb{S}} u\Big([\hat{R}y + \hat{\epsilon} + x \oplus s]g(s);\gamma\Big)\hat{\nu}(ds) \Big| c + y + x^\top p = w, y \geq -\hat{\epsilon}/\hat{R} \right\} \tag{37}$$

is uniquely defined and takes the form

$$\begin{aligned} c^\star &= \bar{c}\big(w + \hat{\epsilon}/\hat{R}\big), \\ x^\star &= (1-\bar{c})\big(w + \hat{\epsilon}/\hat{R}\big)\theta^\star \\ y^\star &= (1-\bar{c})\big(w + \hat{\epsilon}/\hat{R}\big)(1 - p^\top\theta^\star) - \hat{\epsilon}/\hat{R}. \end{aligned} \tag{38}$$

where $\bar{c} = \big[1 + v^{\star\frac{1}{1-\gamma}}\big]^{-1}$ and

$$\theta^\star := \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\mathbb{S}} u\big([\hat{R}(1-p^\top\theta) + \theta\oplus s)]g(s);\gamma\big)\hat{\nu}(ds) \Big| \theta^\top p \leq 1 \right\}. \tag{39}$$

$$v^\star := \begin{cases} \hat{\beta} & \gamma = 0 \\ \gamma \max_{\theta\in\mathbb{X}} \left\{ \hat{\beta} \int_{\mathbb{S}} u\big([\hat{R}(1-p^\top\theta) + \theta\oplus s)]g(s);\gamma\big)\hat{\nu}(ds) \Big| \theta^\top p \leq 1 \right\} & \gamma \neq 0 \end{cases} \tag{40}$$

Proof: We first show that the value $\theta^\star$ in (39) is well-defined. For each $\theta \in B(p) := \{\theta \in \mathbb{R}^M_+ | p^\top\theta \leq 1\}$, define the function

$$U(\theta;\hat{R},p,\hat{\nu}) := \int_{\bar{\mathbb{S}}} u\big([\hat{R}(1-p^\top\theta) + \theta\oplus s)]g(s);\gamma\big)\hat{\nu}(ds). \tag{41}$$

Applying Lemma 8 the function $U(\cdot;\hat{R},p,\hat{\nu})$ being defined on the compact set $B(p)$ is continuous implying the existence of a solution to (39). Uniqueness will follow if we show that $U(\cdot;\hat{R},p,\nu)$ is strictly concave. For this purpose, let $\theta',\theta'' \in B(p)$ with $\theta' \neq \theta''$. Set $z' := (1-p^\top\theta',\theta') \in \mathbb{Z}$ and $z'' := (1-p^\top\theta'',\theta'') \in \mathbb{Z}$ and observe that the non-redundancy property of the measure $\hat{\nu}$ implies that the measurable set $A(z',z'',\hat{R}) := \{s \in \mathbb{S} | W(z',s,0,\hat{R}) \neq W(z'',s,0,\hat{R})\}$ has positive measure, i.e., $\hat{\nu}(A(z',z'',\hat{R})) > 0$. Employing a similar argument as in step 2 in the proof of Proposition 1 and exploiting the linearity of the function W it is now straightforward to show that $U(\lambda\theta' + (1-\lambda)\theta'';\hat{R},p,\hat{\nu}) > \lambda U(\theta';\hat{R},p,\hat{\nu}) + (1-\lambda)U(\theta'';\hat{R},p,\hat{\nu})$ for all $\lambda \in]0,1[$ proving the strict concavity of the function $U(\cdot;\hat{R},p,\hat{\nu})$. This together with the convexity of the set $B(p)$

ensures the uniqueness of the solution $\theta^\star$ to (39). It follows that the value $v^\star$ defined in (40) is indeed well-defined and satisfies $v^\star > 0$ due to the properties of the function $u(\cdot;\gamma)$. We claim that the value $c^\star$ defined above satisfies

$$c^\star = \arg\max_{c\in\mathbb{C}}\Big\{u(c;\gamma) + v^\star u(w+\hat{\epsilon}/\hat{R}-c;\gamma)\,\Big|\, c \le w+\hat{\epsilon}/\hat{R}\Big\}. \tag{42}$$

A routine calculation shows that the solution to (42) is of the form $c^\star = \bar{c}\big(w+\hat{\epsilon}/\hat{R}\big)$ with $\bar{c}$ being defined as above. Now let the values $x^\star = \big(w+\hat{\epsilon}/\hat{R}-c^\star\big)\theta^\star$ and $y^\star$ be defined as above. We show that the triple $(c^\star, y^\star, x^\star)$ is the unique maximizer to (37). Note first that $0 < c^\star < w+\hat{\epsilon}/\hat{R}$ and $y^\star = w - c^\star - p^\top x^\star \ge -\hat{\epsilon}/\hat{R}$. Hence the constraints in (37) are satisfied. Define the value

$$V^\star := u(c^\star;\gamma) + \hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([\hat{R}y^\star + \hat{\epsilon} + x^\star \oplus s]g(s);\gamma\Big)\hat{\nu}(ds). \tag{43}$$

Let $(\tilde{c},\tilde{y},\tilde{x}) \ne (c^\star, y^\star, x^\star)$ be another triple satisfying $\tilde{c} + \tilde{x}^\top p + \tilde{y} = w$ and $\tilde{y} \ge -\hat{\epsilon}/\hat{R}$. Defining

$$\tilde{V} := u(\tilde{c};\gamma) + \hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([\hat{R}\tilde{y} + \hat{\epsilon} + \tilde{x} \oplus s]g(s);\gamma\Big)\hat{\nu}(ds) \tag{44}$$

we show that $V^\star > \tilde{V}$. The remainder treats the cases $\gamma \ne 0$ and $\gamma = 0$ separately.
Case 1: $\gamma \ne 0$. The definitions of $x^\star$, $\theta^\star$ and $v^\star$ and equation (43) imply the relationship

$$\begin{aligned} V^\star &= u(c^\star;\gamma) + \hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([w+\hat{\epsilon}/\hat{R}-c^\star][\hat{R}(1-p^\top\theta^\star) + \theta^\star \oplus s)]g(s);\gamma\Big)\hat{\nu}(ds) \\ &= u(c^\star;\gamma) + u\Big(w+\hat{\epsilon}/\hat{R}-c^\star;\gamma\Big)\gamma\hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([\hat{R}(1-p^\top\theta^\star) + \theta^\star \oplus s)]g(s);\gamma\Big)\hat{\nu}(ds) \\ &= u(c^\star;\gamma) + u\Big(w+\hat{\epsilon}/\hat{R}-c^\star;\gamma\Big)v^\star. \end{aligned} \tag{45}$$

Suppose $\tilde{c} = w + \hat{\epsilon}/\hat{R}$. Then the budget conditions imply $\tilde{y} = -\hat{\epsilon}/\hat{R}$ and $\tilde{x} = 0$. Equations (42), (44) and (45) together with $c^\star \ne \tilde{c}$ then yield

$$\begin{aligned} \tilde{V} = u(\tilde{c};\gamma) &= u(\tilde{c};\gamma) + u(w+\hat{\epsilon}/\hat{R}-\tilde{c};\gamma)\,v^\star \\ &< u(c^\star;\gamma) + u(w+\hat{\epsilon}/\hat{R}-c^\star;\gamma)\,v^\star = V^\star. \end{aligned}$$

Suppose $\tilde{c} < w + \hat{\epsilon}/\hat{R}$. Defining $\tilde{\theta} := [w+\hat{\epsilon}/\hat{R}-\tilde{c}]^{-1}\tilde{x}$ and observing that $\tilde{\theta} \in B(p)$ gives

$$\begin{aligned} \tilde{V} &= u(\tilde{c};\gamma) + \hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([w+\hat{\epsilon}/\hat{R}-\tilde{c}][\hat{R}(1-p^\top\tilde{\theta}) + \tilde{\theta} \oplus s)]g(s);\gamma\Big)\hat{\nu}(ds) \quad (46) \\ &= u(\tilde{c};\gamma) + u(w+\hat{\epsilon}/\hat{R}-\tilde{c};\gamma)\tilde{v} \end{aligned}$$

where

$$\tilde{v} := \gamma\hat{\beta}\int_{\bar{\mathbb{S}}} u\Big([\hat{R}(1-p^\top\tilde{\theta}) + \tilde{\theta} \oplus s)]g(s);\gamma\Big)\hat{\nu}(ds). \tag{47}$$

Note that our assumption $(\tilde{c}, \tilde{y}, \tilde{x}) \neq (c^\star, y^\star, x^\star)$ implies $(\tilde{c}, \tilde{\theta}) \neq (c^\star, \theta^\star)$. Suppose first that $\gamma > 0$. Then equation (40) gives $v^\star \geq \tilde{v}$ (with strict inequality if $\tilde{\theta} \neq \theta^\star$). Equations (45) and (46) then imply

$$\begin{aligned} \tilde{V} &\leq u(\tilde{c}; \gamma) + u(w + \hat{\epsilon}/\hat{R} - \tilde{c}; \gamma) v^\star \\ &\leq u(c^\star; \gamma) + u(w + \hat{\epsilon}/\hat{R} - c^\star; \gamma) v^\star = V^\star \end{aligned} \tag{48}$$

with at least one inequality being strict. Second, suppose $\gamma < 0$. Then equation (40) gives $v^\star \leq \tilde{v}$ (with strict inequality if $\tilde{\theta} \neq \theta^\star$). Since utility is negative-valued one has $u(c; \gamma) v^\star \geq u(c; \gamma) \tilde{v}$ for each $c \geq 0$. Repeating the previous argument shows that the inequality (48) also holds in case $\gamma < 0$. Hence $\tilde{V} < V^\star$.

Case 2: $\gamma = 0$. Note that the boundary cases $\tilde{c} = w + \hat{\epsilon}/\hat{R}$ and $\tilde{c} = 0$ both give $\tilde{V} = -\infty$ such that trivially $V^\star > \tilde{V}$ in this case. Suppose therefore that $0 < \tilde{c} < w + \hat{\epsilon}/\hat{R}$ and define $\tilde{\theta}$ as before. In this case equations (43) and (45) may be written as

$$\begin{aligned} V^\star &= u(c^\star; \gamma) + \hat{\beta}\, u\big(w + \hat{\epsilon}/\hat{R} - c^\star; \gamma\big) + \hat{\beta} \int_{\bar{\mathbb{S}}} u\big([\hat{R}(1 - p^\top \theta^\star) + \theta^\star \oplus s)]g(s); \gamma\big) \hat{\nu}(ds) \\ \tilde{V} &= u(\tilde{c}; \gamma) + \hat{\beta}\, u\big(w + \hat{\epsilon}/\hat{R} - \tilde{c}; \gamma\big) + \hat{\beta} \int_{\bar{\mathbb{S}}} u\big([\hat{R}(1 - p^\top \tilde{\theta}) + \tilde{\theta} \oplus s)]g(s); \gamma\big) \hat{\nu}(ds). \end{aligned}$$

Recalling from (40) that $v^\star = \hat{\beta}$ one observes from equations (39) and (41) that

$$\begin{aligned} u(\tilde{c}; \gamma) + \hat{\beta}\, u\big(w + \hat{\epsilon}/\hat{R} - \tilde{c}; \gamma\big) &\leq u(c^\star; \gamma) + \hat{\beta}\, u\big(w + \hat{\epsilon}/\hat{R} - c^\star; \gamma\big) \\ \int_{\bar{\mathbb{S}}} u\big([\hat{R}(1 - p^\top \tilde{\theta}) + \tilde{\theta} \oplus s)]g(s); \gamma\big) \hat{\nu}(ds) &\leq \int_{\bar{\mathbb{S}}} u\big([\hat{R}(1 - p^\top \theta^\star) + \theta^\star \oplus s)]g(s); \gamma\big) \hat{\nu}(ds). \end{aligned}$$

with at least one of the inequalities being strict. This shows that $V^\star > \tilde{V}$ also in this case. Since $(\tilde{c}, \tilde{y}, \tilde{x})$ was arbitrary, the triple $(c^\star, y^\star, x^\star)$ is indeed the unique maximizer of (37). $\square$

Proof of Proposition 2

Since $v_N \equiv 1$ and $\hat{E}_N = 0$ the claim in (i) is obviously satisfied for $n = N$. We therefore apply an induction argument by supposing there exists $n \in \{1, \ldots, N-1\}$ and a continuous function $v_{n+1} : \mathbb{S}^{n+1} \longrightarrow \mathbb{R}_{++}$ such that V_{n+1} is of the form $V_{n+1}(w_{n+1} s_1^{n+1}) = u\big([w_{n+1} + \hat{E}_{n+1}] v_{n+1}(s_1^{n+1}); \gamma\big)$. It will be shown that this implies the functional forms of V_n given in (i) and the solution functions defined in (ii). Let $s_1^n \in \mathbb{S}^n$, $w_n > -\hat{E}_n$ be arbitrary but fixed. Noting from (3) that $\hat{e}_{n+1} + \hat{E}_{n+1} = \hat{R}_n \hat{E}_n$ it follows from (12) and the induction hypothesis that the value function V_n satisfies

$$V_n(w_n, s_1^n) = \max_{(c,z) \in \mathbb{C} \times \mathbb{Z}} \Big\{ u(c; \gamma) + \beta \int_{\mathbb{S}_{n+1}} u\Big([\hat{R}_n(y + \hat{E}_n) + x \oplus s] v_{n+1}(s_1^n, s); \gamma\Big) Q_{n+1}(s_1^n, ds) \;\Big|\; y = w_n - c - x^\top p_n \geq -\hat{E}_n \Big\} \tag{49}$$

Utilizing Lemma 6 (setting $\hat{\nu} = Q_{n+1}(s_1^n, \cdot)$, $\bar{\mathbb{S}} = \bar{\mathbb{S}}_{n+1}$, $\hat{R} = \hat{R}_n$, $\hat{\epsilon} = \hat{E}_n \hat{R}_n$, $w = w_n$, $p = p_n$, $\hat{\beta} = \beta$ and $g = v_{n+1}(s_1^n, \cdot)$ the solution to the maximization problem in (52) takes the form

$$\begin{aligned} c_n^\star(w_n, s_1^n) &= \bar{c}(s_1^n)\big(w_n + \hat{E}_n\big) \\ x_n^\star(w_n, s_1^n) &= (1 - \bar{c}(s_1^n))\big(w_n + \hat{E}_n\big)\theta_n(s_1^n) \\ y_n^\star(w_n, s_1^n) &= (1 - \bar{c}(s_1^n))\big(w_n + \hat{E}_n\big)(1 - p_n^\top \theta_n(s_1^n)) - \hat{E}_n \end{aligned} \tag{50}$$

where $\bar{c}_n(s_1^n) := \big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\big]^{-1}$ and

$$\begin{aligned} v_n^\star(s_1^n) &:= \gamma \max_{\theta \geq 0, p_n^\top \theta \leq 1} \left\{ \beta \int_{\mathbb{S}} u\big([\hat{R}_n(1 - p_n^\top \theta) + \theta \oplus s)] v_{n+1}(s_1^n, s); \gamma\big) Q_{n+1}(s_1^n, ds) \right\} \\ \theta_n(s_1^n) &:= \operatorname*{arg\,max}_{\theta \geq 0, p_n^\top \theta \leq 1} \left\{ \int_{\mathbb{S}} u\big([\hat{R}_n(1 - p_n^\top \theta) + \theta \oplus s)] v_{n+1}(s_1^n, s); \gamma\big) Q_{n+1}(s_1^n, ds) \right\}. \end{aligned}$$

This proves the second claim (ii) of the proposition. Substituting the solutions in (50) into (49) and omitting the arguments (w_n, s_1^n) for notational convenience gives

$$\begin{aligned} V_n(w_n, s_1^n) &= u(c_n^\star; \gamma) + \beta \int_{\bar{\mathbb{S}}_{n+1}} u\Big([\hat{R}_n(y_n^\star + \hat{E}_n) + x_n^\star \oplus s] v_{n+1}(s_1^n, s); \gamma\Big) Q_{n+1}(s_1^n, ds) \\ &= u\big(\bar{c}_n(s_1^n)(w_n + \hat{E}_n); \gamma\big) + u\big((1 - \bar{c}_n(s_1^n))(w_n + \hat{E}_n); \gamma\big) \\ &\quad \gamma\beta \int_{\mathbb{S}} u\big([\hat{R}_n(1 - p_n^\top \theta_n(s_1^n)) + \theta_n(s_1^n) \oplus s)] v_{n+1}(s_1^n, s); \gamma\big) Q_{n+1}(s_1^n, ds) \\ &= u\big(\bar{c}_n(s_1^n)(w_n + \hat{E}_n); \gamma\big) + u\big((1 - \bar{c}_n(s_1^n))(w_n + \hat{E}_n); \gamma\big) v_n^\star(s_1^n) \\ &= u\big(w_n + \hat{E}_n; \gamma\big) \Big(\bar{c}_n(s_1^n)^\gamma + v_n^\star(s_1^n)(1 - \bar{c}_n(s_1^n))^\gamma\Big) \end{aligned}$$

Recalling that $\bar{c}_n(s_1^n) = \big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\big]^{-1}$ the second term may be written as

$$\begin{aligned} \bar{c}_n(s_1^n)^\gamma + v_n^\star(s_1^n)(1 - \bar{c}_n(s_1^n))^\gamma &= \frac{1 + v_n^\star(s_1^n) v_n^\star(s_1^n)^{\frac{\gamma}{1-\gamma}}}{\Big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\Big]^\gamma} \\ &= \Big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\Big]^{1-\gamma} \end{aligned}$$

Setting $v_n(s_1^n) := \big[1 + v_n^\star(s_1^n)^{\frac{1}{1-\gamma}}\big]^{-1}$ then gives the claim (i) for the function V_n. The fact that each v_n is continuous is an immediate consequence of Proposition 1. ■

A.5 Proof of Proposition 3

Since the claim in (i) is again immediately satisfied for $n = N$ we apply the same induction argument as in the proof of Proposition 2 supposing there exists $n \in \{1, \ldots, N-1\}$ such that the claim (i) holds for V_{n+1}. Let $s_1^n \in \mathbb{S}^n$ and $w_n > -\hat{E}_n$ be arbitrary but fixed. Using the recursive definition (12) yields the value function V_n under the induction hypothesis as

$$V_n(w_n, s_1^n) = \tilde{V}_n(w_n, s_1^n) + \beta \int_{\mathbb{S}} v_{n+1}(s_1^n, s) Q_{n+1}(s_1^n, ds) \tag{51}$$

where, noting that $\beta\,\beta_{n+1} = (\beta_n - 1)$ and that $\hat{e}_{n+1} + \hat{E}_{n+1} = \hat{R}_n \hat{E}_n$ by virtue of (3)

$$\begin{aligned}\tilde{V}_n(w_n, s_1^n) \;:=\; \max_{(c,y,x)\in\mathbb{C}\times\mathbb{Y}\times\mathbb{X}} \Big\{ &\ln(c) + (\beta_n - 1) \int_{\bar{\mathbb{S}}_{n+1}} \ln\big(\hat{R}_n(y + \hat{E}_n) + x \oplus s\big) Q_{n+1}(s_1^n, ds) \\ &\Big| y = w_n - c - x^\top p_n \geq -\hat{E}_n \Big\}. \end{aligned} \tag{52}$$

Utilizing Lemma 6 (setting $\hat{\nu} = Q_{n+1}(s_1^n, \cdot)$, $\bar{\mathbb{S}} = \bar{\mathbb{S}}_{n+1}$, $\hat{R} = \hat{R}_n$, $\hat{\epsilon} = \hat{E}_n \hat{R}_n$, $w = w_n$, $p = p_n$, $\hat{\beta} = \beta_n - 1$ and $g = v_{n+1}(s_1^n, \cdot)$ the solution to the maximization problem in (52) takes the form

$$\begin{aligned} c_n^\star(w_n, s_1^n) &= \bar{c}_n\big(w_n + \hat{E}_n\big) \\ x_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n)\big(w_n + \hat{E}_n\big)\theta_n^\star \\ y_n^\star(w_n, s_1^n) &= (1 - \bar{c}_n)\big(w_n + \hat{E}_n\big)(1 - p_n^\top \theta_n^\star) - \hat{E}_n \end{aligned} \tag{53}$$

where $\bar{c}_n := \frac{1}{\beta_n}$ and

$$\theta_n(s_1^n) := \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\bar{\mathbb{S}}_{n+1}} \ln\Big(\hat{R}_n(1 - \theta^\top p_n) + \theta \oplus s\Big) Q_{n+1}(s_1^n, ds) \Big| \theta^\top p_n \leq 1 \right\}.$$

This proves the claim in (ii). Substituting the solution (53) into (52) and omitting the arguments (w_n, s_1^n) for convenience yields

$$\begin{aligned} \tilde{V}_n(w_n, s_1^n) &= \ln(c_n^\star) + (\beta_n - 1) \int_{\mathbb{S}} \ln\big(\hat{R}_n(y_n^\star + \hat{E}_n) + x_n^\star \oplus s\big) Q_{n+1}(s_1^n, ds) \\ &= \ln(c_n^\star) + (\beta_n - 1) \ln\big(w_n + \hat{E}_n - c_n^\star\big) \\ &\quad + \int_{\bar{\mathbb{S}}_{n+1}} \ln\Big(\hat{R}_n(1 - \theta_n(s_1^n)^\top p_n) + \theta_n(s_1^n) \oplus s\Big) Q_{n+1}(s_1^n, ds) \\ &= \ln\big(w_n + \hat{E}_n\big) + (\beta_n - 1) \ln\big(\beta_n - 1\big) - \beta_n \ln \beta_n \\ &\quad + \int_{\bar{\mathbb{S}}_{n+1}} \ln\Big(\hat{R}_n(1 - \theta_n(s_1^n)^\top p_n) + \theta_n(s_1^n) \oplus s\Big) Q_{n+1}(s_1^n, ds). \end{aligned} \tag{54}$$

Combining (51) and (54) gives the claim (i). The fact that each v_n is continuous follows again from Proposition 1. ∎

B Technical Results

The following part collects some technical devices that are used in this paper. Although most of the results are standard, a proof is included here for ease of reference.

Lemma 7
Let $(\Omega, \mathcal{B}, \nu)$ *be a probability space and* $f, g : \Omega \longrightarrow \mathbb{R}$ *be measurable functions which are* ν*-integrable and satisfy the conditions* $f(\omega) \geq g(\omega)$ *for all* $\omega \in \Omega$ *and* $\nu(\{\omega \in \Omega | f(\omega) > g(\omega)\}) > 0$. *Then*

$$\int_{\Omega} f(\omega)\nu(d\omega) > \int_{\Omega} g(\omega)\nu(d\omega).$$

Proof: Consider the function $h := f - g$ which is measurable and ν-integrable. By definition $h(\omega) \geq 0$ for all $\omega \in \Omega$ and $\nu(\{\omega \in \Omega | h(\omega) > 0\}) > 0$. By the monotonicity of integrals one has $\int_{\Omega} h(\omega)\nu(d\omega) \geq 0$. Assume by way of contradiction that $\int_{\Omega} h(\omega)\nu(d\omega) = 0$. Following Bauer (1992, Satz 13.2, p.81), this is equivalent to $\nu(\{\omega \in \Omega | h(\omega) > 0\}) = 0$, which is a contradiction. Hence $\int_{\Omega} h(\omega)\nu(d\omega) > 0$ and, by the linearity of integrals, $\int_{\Omega} f(\omega)\nu(d\omega) > \int_{\Omega} g(\omega)\nu(d\omega)$. ■

Lemma 8
Let Ω *be a topological space and* $(\Omega, \mathcal{B}(\Omega), \nu)$ *be a probability space with* ν *being supported on the compact subset* $\bar{\Omega} \in \mathcal{B}(\Omega)$, *i.e.,* $\nu(\bar{\Omega}) = 1$. *Let* $\Theta \subset \mathbb{R}^m_{++}$ *be a compact set and* $h : \Theta \times \Omega \longrightarrow \mathbb{R}$ *be a continuous function. Then* h *is* ν*-integrable and the map* $H : \Theta \longrightarrow \mathbb{R}$ *defined as*

$$H(\theta) := \int_{\Omega} h(\theta, \omega)\nu(d\omega)$$

is continuous.

Proof: We show that the requirements $(a) - (c)$ of Lemma 16.1 in Bauer (1992, p. 101) are satisfied.
(a) Since ν is supported on the compact set $\bar{\Omega}$, the function H can be written as

$$H(\theta) = \int_{\bar{\Omega}} h(\theta, \omega)\nu(d\omega).$$

For each fixed $\theta \in \Theta$ the continuous mapping $h(\theta, \cdot) : \bar{\Omega} \longrightarrow \mathbb{R}$ is Borel-measurable and bounded from above by the value $\bar{h}(\theta) := \max\{h(\theta, \omega) | \omega \in \bar{\Omega}\}$ and from below by $\underline{h}(\theta) := \min\{h(\theta, \omega) | \omega \in \bar{\Omega}\}$. Note that both bounds are well-defined due to the compactness of $\bar{\Omega}$. This implies that the map $h(\theta, \cdot)$ is ν-integrable for all $\theta \in \Theta$.
(b) By assumption, for each fixed $\omega \in \Omega$ the map $h(\cdot, \omega) : \Theta \longrightarrow \mathbb{R}$ is continuous and hence continuous at each point $\theta_0 \in \Theta$.
(c) Exploiting compactness of the set $\Theta \times \bar{\Omega}$ by virtue of Tychonoff's theorem (cf. Lipschutz 1965, p. 171, Theorem 12.9), the value $\bar{h} := \max\{|h(\theta, \omega)| : (\theta, \omega) \in \Theta \times \bar{\Omega}\}$ is well-defined and satisfies $|h(\theta, \omega)| \leq \bar{h}$ for all $(\theta, \omega) \in \Theta \times \bar{\Omega}$.
Thus we see that the map h fulfills the requirements (a)-(c) in Bauer (1992, Lemma 16.1, p. 101), which implies that the map $H(\cdot)$ is continuous. ■

Lemma 9
Suppose that the measure ν defined in Assumption 1 is supported on the measurable subset $\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \ldots \times \bar{\mathbb{S}}_N \in \mathcal{B}(\mathbb{S}^N)$. Then each conditional distribution $Q_n(s_1^{n-1}, \cdot)$, $n = 2, \ldots, N$ is supported on $\bar{\mathbb{S}}_n$ while the marginal distribution ν_1 is supported on $\bar{\mathbb{S}}_1$.

Proof: For each measurable set $B \in \mathcal{B}(\mathbb{S}^N)$ we have $B = (B \cap \bar{\mathbb{S}}) \uplus (B \cap \bar{\mathbb{S}}^c)$ where $\uplus$ denotes the union of disjoint sets and $\bar{\mathbb{S}}^c$ is the complement of $\bar{\mathbb{S}}$. Since ν is supported on $\bar{S}$ and $B \cap \bar{\mathbb{S}}^c \subset \bar{\mathbb{S}}^c$ we have $\nu(B \cap \bar{\mathbb{S}}^c) = 0$ and therefore

$$\nu(B) = \nu(B \cap \bar{\mathbb{S}}) + \nu(B \cap \bar{\mathbb{S}}^c) = \nu(B \cap \bar{\mathbb{S}}) \quad \forall B \in \mathcal{B}(\mathbb{S}^N). \tag{55}$$

Noting that $\mathbf{1}_{B\cap\bar{\mathbb{S}}}(s_1^N) = \mathbf{1}_B(s_1^N) \cdot \mathbf{1}_{\bar{\mathbb{S}}}(s_1^N)$ and $\mathbf{1}_{\bar{\mathbb{S}}}(s_1^N) = \prod_{n=1}^N \mathbf{1}_{\bar{\mathbb{S}}_n}(s_n)$ one obtains the factorization (11) as

$$\begin{aligned}
\nu(B) &= \nu(B \cap \bar{\mathbb{S}}) \qquad (56)\\
&= \int_{\mathbb{S}}\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \mathbf{1}_B(s_1^N)\, \mathbf{1}_{\bar{\mathbb{S}}_N}(s_N) Q_N(s_1^{N-1}, ds_N) \cdots \mathbf{1}_{\bar{\mathbb{S}}_2}(s_2) Q_2(s_1, ds_2) \mathbf{1}_{\bar{\mathbb{S}}_1}(s_1) \nu_1(ds_1)\\
&= \int_{\bar{\mathbb{S}}_1}\int_{\bar{\mathbb{S}}_2} \cdots \int_{\bar{\mathbb{S}}_N} \mathbf{1}_B(s_1^N)\, Q_N(s_1^{N-1}, ds_N) \cdots Q_2(s_1, ds_2) \nu_1(ds_1).
\end{aligned}$$

This equality being true for all $B \in \mathcal{B}(\mathbb{S}^N)$ requires that for each $n = 2, \ldots, N$ and all $s_1^{n-1} \in \mathbb{S}^{n-1}$ the set $\bar{\mathbb{S}}_n$ must have full measure, i.e. $\bar{Q}_n(s_1^{n-1}, \bar{\mathbb{S}}_n) = 1$. Hence the support of $\bar{Q}_n(s_1^{n-1}, \cdot)$ must be a subset of $\bar{\mathbb{S}}_n$. The fact that the support of ν_1 is the set $\bar{\mathbb{S}}_1$ follows immediately from the fact that $\nu_1(\cdot) = \nu \circ \pi_1^{-1}(\cdot)$ is induced by the projection π_1 (cf. proof of Lemma 4). ■

Lemma 10
Given $\hat{E} \geq 0$, the correspondence $\mathbb{B} : [-\hat{E}, \infty[\times \mathbb{R}^M_{++} \rightrightarrows \mathbb{C} \times \mathbb{Y} \times \mathbb{X}$,

$$\mathbb{B}(w, p) := \left\{ (c, y, x) \in \mathbb{C} \times \mathbb{Y} \times \mathbb{X} \middle| c + y + x^\top p = w,\ y \geq -\hat{E} \right\}$$

is (upper- and lower-hemi-) continuous.

Proof: Noting that $\mathbb{B}$ is non-empty- and compact-valued, we may apply the definitions given in Stokey & Lucas (1994, p. 56) to show that $\mathbb{B}$ is lower- and upper-hemi-continuous at each point $(w_0, p_0) \in [-\hat{E}, \infty[\times\mathbb{R}^M_{++}$.
L.h.c. Let the point $(w_0, p_0) \in [-\hat{E}_n, \infty[\times\mathbb{R}^M_{++}$ be arbitrary and let $(w_n, p_n)_{n\geq 1}$ be a sequence taking values in $[-\hat{E}_n, \infty[\times\mathbb{R}^M_{++}$ which satisfies $\lim_{n\to\infty}(w_n, p_n) = (w_0, p_0)$. Let (c_0, y_0, x_0) be an arbitrary point in $\mathbb{B}(w_0, p_0)$. We show that there exists a sequence $(c_n, y_n, x_n)_{n\geq 1}$ which satisfies $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(c_n, y_n, x_n) = (c_0, y_0, x_0)$. Assume first that $w_0 > -\hat{E}$ and define for each $n \geq 1$

$$c_n := \frac{w_n + \hat{E}}{w_0 + \hat{E}} c_0, \quad x_n^{(m)} := \frac{w_n + \hat{E}}{w_0 + \hat{E}} \frac{p_0^{(m)}}{p_n^{(m)}} x_0^{(m)},\ m = 1, \ldots, M \quad \text{and } y_n := w_n - c_n - x_n^\top p_n.$$

Then we have $\lim_{n\to\infty} c_n = c_0 \geq 0$, $\lim_{n\to\infty} x_n^{(m)} = x_0^{(m)} \geq 0$ for each $m = 1, \ldots, M$ and $\lim_{n\to\infty} y_n = y_0 \geq -\hat{E}$. By construction, $c_n + y_n + x_n^\top p_n = w_n$ for all n and

$$y_n = w_n - \frac{w_n + \hat{E}}{w_0 + \hat{E}}(c_0 + p_0^\top x_0) \geq w_n - \frac{w_n + \hat{E}}{w_0 + \hat{E}}(w_0 + \hat{E}) = -\hat{E}$$

which shows that indeed $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n. In the second case where $w_0 = -\hat{E}$ one observes from the definition of $\mathbb{B}$ that the set $B(w_0, p_0)$ is a singleton and the point (c_0, y_0, x_0) must satisfy $c_0 = 0$, $y_0 = -\hat{E}$ and $x_0 = 0$. In this case, define $c_n = 0$, $y_n = w_n$ and $x_n = 0$ for each $n \geq 1$ to see that $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(c_n, y_n, x_n) = (0, -\hat{E}, 0) = (c_0, y_0, x_0)$. This proves that $\mathbb{B}$ is indeed l.h.c. on its domain of definition.

U.h.c. Let $(w_n, p_n)_{n\geq 1}$ and $(c_n, y_n, x_n)_{n\geq 1}$ be arbitrary sequences taking values in $[-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$ and $\mathbb{C} \times \mathbb{Y} \times \mathbb{X}$ which satisfy $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(w_n, p_n) = (w_0, p_0) \in [-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$. We show that there exists a convergent subsequence $(c_{n_k}, y_{n_k}, x_{n_k})_{k\geq 1}$ of $(c_n, y_n, x_n)_{n\geq 1}$ which satisfies $\lim_{k\to\infty}(c_{n_k}, y_{n_k}, x_{n_k}) = (c_0, y_0, x_0)$ where $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. Note first that $(w_n, p_n)_{n\geq 1}$ being a convergent sequence implies that it must be bounded. In particular there exist values $\bar{w} \geq -\hat{E}$ and $\underline{p}^{(m)} > 0$, $m = 1, \ldots, M$ such that $w_n \leq \bar{w}$ and $\underline{p}^{(m)} \leq p_n^{(m)}$ for all $n \geq 1$. Since $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n this implies $0 \leq c_n \leq \bar{w} + \hat{E}$, $-\hat{E} \leq y_n \leq \bar{w}$ and $0 \leq x_n^{(m)} \leq (\bar{w} + \hat{E})/\underline{p}^{(m)}$, $m = 1, \ldots, M$. Hence the sequence $(c_n, y_n, x_n)_{n\geq 1}$ is bounded implying the existence of a convergent subsequence $(c_{n_k}, y_{n_k}, x_{n_k})_{k\geq 1}$. Let $(c_0, y_0, x_0) := \lim_{k\to\infty}(c_{n_k}, y_{n_k}, x_{n_k})$. It therefore remains to show that $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. Since $c_{n_k} \geq 0$, $y_{n_k} \geq -\hat{E}$ and $x_{n_k} \geq 0$, $m = 1, \ldots, M$ for all $k \geq 1$ one has $c_0 \geq 0$, $y_0 \geq -\hat{E}$ and $x_0^{(m)} \geq 0$, $m = 1, \ldots, M$. Furthermore, since $(w_n, p_n)_{n\geq 1}$ converges to (w_0, p_0), so does the subsequence $(w_{n_k}, p_{n_k})_{k\geq 1}$ (cf. Lipschutz 1965, p.65). Therefore, since $\lim_{k\to\infty}(w_{n_k}, p_{n_k}) = (w_0, p_0)$ and for each $k \geq 1$ one has $c_{n_k} + y_{n_k} + x_{n_k}^\top p_{n_k} - w_{n_k} = 0$ this yields $\lim_{k\to\infty}(c_{n_k} + y_{n_k} + x_{n_k}^\top p_{n_k} - w_{n_k}) = c_0 + y_0 + x_0^\top p_0 - w_0 = 0$ and therefore $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. ■

References

Arnold, L. (1998): *Random Dynamical Systems.* Springer-Verlag, Berlin a.o.

Bauer, H. (1992): *Maß– und Integrationstheorie.* Walter de Gruyter Verlag, Berlin a.o., 2ndAuflage.

Bellman, R. (1957): *Dynamic Programming.* Princeton University Press.

Böhm, V. & M. Hillebrand (2007): "On the Inefficiency of Pay-As-You-Go Pension Systems in Stochastic Economies with Assets", Discussion Paper 565, Department of Economics, Bielefeld University.

DUFFIE, D. (1992): *Dynamic Asset Pricing Theory.* Princton University Press, Princton (NJ) a.o.

FAMA, E. (1970): "Multiperiod Consumption-Investment Decisions", *American Economic Review*, 60, 163–174.

——— (1976): "Multiperiod Consumption-Investment Decisions - A Correction", *American Economic Review*, 66, 723–724.

GÄNSSLER, P. & W. STUTE (1977): *Wahrscheinlichkeitstheorie.* Springer Verlag, Berlin a.o.

GRANDMONT, J.-M. (1982): "Temporary General Equilibrium Theory", in *Handbook of Mathematical Economics*, ed. by K. Arrow & M. Intrilligator. North-Holland Publishing Company, Amsterdam a.o.

GRANDMONT, J.-M. & W. HILDEBRAND (1974): "Stochastic Processes of Temorary Equilibria", *Journal of Mathematical Economics*, S. 247–277.

HAKANSSON, N. (1969): "Optimal Investment and Consumption Strategies under Risk, an Uncertain Lifetime and Insurance", *International Economic Review*, 10, 443–466.

HAKANSSON, N. H. (1970): "Optimal Investment and Consumption Strategies under Risk for a Class of Utility Functions", *Econometrica*, 38, 587–607.

HILLEBRAND, M. (2007): "The Role of Pension Systems and Demographic Change for Asset Prices and Capital Formation", Discussion Paper 560, Department of Economics, Bielefeld University.

HILLEBRAND, M. & J. WENZELBURGER (2006): "On the Dynamics of Asset Prices and Portfolios in a Multiperiod CAPM", *Chaos, Solitons and Fractals*, 29(3), 578–594.

INGERSOLL, J. E. (1987): *Theory of Financial Decision Making.* Rowman & Littlefield, Totowa (NJ).

LIPSCHUTZ, S. (1965): *General Topology.* Schaum Publishing Company.

PHELPS, E. S. (1965): "The Accumulation of Risky Capital: A Sequential Utility Analysis", *Econometrica*, 30(4), 729–743.

PLISKA, S. (1997): *Introduction to Mathematical Finance.* Blackwell Publishers Inc, Massachusetts.

ROMER, D. (1996): *Advanced Macroeconomics.* McGraw–Hill, New York.

STOKEY, N. L. & R. E. LUCAS (1994): *Recursive Methods in Economic Dynamics.* Harvard University Press, Cambridge.