Working Papers

Institute of Mathematical Economics

406

September 2008

On the foundations of Lévy finance: Equilibrium for a single-agent financial market with jumps

Frederik Herzberg

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?406
ISSN: 0931-6558

On the foundations of Lévy finance: Equilibrium for a single-agent financial market with jumps

Frederik S. Herzberg

This work was funded through a research grant of the German Academic Exchange Service (*Deutscher Akademischer Austauschdienst*, DAAD). I am deeply grateful to Professor Robert M. Anderson for numerous discussions, many helpful comments and a great number of fruitful ideas.

Key words: Financial equilibrium; asset pricing; representative agent models; Lévy processes; nonstandard analysis.

Journal of Economic Literature Classification. G13; D52.

2000 Mathematics Subject Classification. Primary 91B24, 91B28; Secondary 03H05, 60G51.

Department of Mathematics, University of California, Berkeley, CA 94720-3840, United States of America. E-mail: herzberg@math.berkeley.edu.

Current address: Institut für Mathematische Wirtschaftsforschung, Universität Bielefeld, Universitätsstraße 25, D-33615 Bielefeld, Germany. E-mail: fherzberg@uni-bielefeld.de.

Abstract. For a continuous-time financial market with a single agent, we establish equilibrium pricing formulae under the assumption that the dividends follow an exponential Lévy process. The agent is allowed to consume a lump at the terminal date; before, only flow consumption is allowed. The agent's utility function is assumed to be additive, defined via strictly increasing, strictly concave smooth felicity functions which are bounded below (thus, many CRRA and CARA utility functions are included). For technical reasons we require that only pathwise continuous trading strategies are permitted in the demand set.

The resulting equilibrium prices depend on the agent's risk-aversion through the felicity functions. It turns out that the these prices will be the (stochastic) exponential of a Lévy process essentially only if this process is geometric Brownian motion.

1. Introduction

This article addresses the equilibrium foundations of Lévy finance by studying a continuous-time financial market with a single ("representative") agent who trades assets whose terminal dividends are given by the exponential of a Lévy process, evaluated at the terminal date.

Lévy processes where introduced into financial modelling as a means of parametrically generalizing the original Black-Scholes [**16**] model to account for jumps in stock prices. This field of research originated two decades ago with papers by Madan and Seneta [**55, 56, 57**] on financial applications of the variance-gamma process, although Merton [**59**] was the first to model logarithmic stock prices through a Lévy process other than Brownian motion (by adding a Poisson process). Since then, there has been a fast-growing number of notable contributions to this field, e.g. by Eberlein and Keller [**33**], Barndorff-Nielsen [**13**], Chan [**21**] and — even with some empirical validation — by Carr, Geman, Madan and Yor [**19, 20**], to mention but a few. For more references, consult the volume edited by Barndorff-Nielsen, Mikosch and Resnick [**14**] as well as Applebaum's survey article on Lévy processes [**10**] or textbooks such as Boyarchenko and Levendorskiĭ [**17**], Schoutens [**67**] and Applebaum [**11**]. Examples of more recent research on asset pricing based on exponential Lévy process models (including interest rate models) are papers by Eberlein, Kluge and Papapantoleon [**34**], Filipović and Tappe [**36**], Almendral and Osterlee [**4, 5**], and Herzberg [**42**].

Existence proofs and explicit equilibrium asset pricing formulae for certain continuous-time financial markets have previously been established by Cox, Ingersoll and Ross [**22**], Bick [**15**], Duffie and Skiadas [**30**], He and Leland [**38**], and Raimondo [**64**]. The methodological choices of these authors are quite diverse: For example, Cox, Ingersoll and Ross [**22**] extensively use Itô calculus, Duffie and Skiadas [**30**] employ Gateaux derivatives, whilst Raimondo [**64**] works with a hyperfinite discretization in the sense of Robinson's [**65**] nonstandard analysis. However, except for Duffie and Skiadas [**30**], all of this research is limited to the case of asset-price processes with continuous paths, whereas Duffie and Skiadas [**30**] do not discuss concrete models for the dividend processes.

The present paper proves the existence of an equilibrium and provides explicit equilibrium asset pricing formulae for a single-agent, continuous-time financial market, under the assumption that dividends are only paid at the terminal date, when they are given by the exponential realization of some Lévy process at that date. In addition, we only allow path-continuous predictable processes as admissible trading strategies.

It turns out that even in this simple model, no asset price process will ever end up being an exponential Lévy process or the stochastic exponential of a Lévy process, the only exception being the very special case when the asset price process becomes geometric Brownian motion. Instead, asset prices will generically be more complicated stochastic integrals with respect to Lévy processes. Hence, an economically sound generalization of the Black-Scholes model needs to assume that asset price processes are given by general stochastic integrals with respect to Lévy processes, not just exponential Lévy processes or stochastic exponentials of Lévy processes.

Our pricing formulae show that generically the increments of the logarithmic asset prices are not stationary. This is not surprising: Intuitively, it is clear that discontinuous behavior in the economy, in our case in the guise of dividends which are analytic functions of jump processes, should mean that risk premia of financial instruments increase with the maturity date.

In order to prove both the existence of equilibrium and the pricing formulae, we extend the hyperfinite discretization technique of Raimondo [**64**] in two directions: First, rather than imposing a short-sale constraint and applying Radner's theorem [**63**], we follow Anderson and Raimondo [**9**] by invoking the Duffie-Shafer theorem [**31, 32**] to obtain an equilibrium in the discrete hyperfinite economy. Secondly, we utilize the recently developed nonstandard theory of Lévy processes — which was devised in a seminal paper by Lindstrøm [**51**] and subsequently expanded by Albeverio and Herzberg [**2**], Lindstrøm [**52**], Herzberg and Lindstrøm [**43**] as well as Herzberg [**41**] — in order to construct and analyze the appropriate hyperfinite discretization. In addition to Lindstrøm's [**51**] account, there are also other nonstandard approaches to Lévy processes, due to Albeverio and Herzberg [**3**] as well as Ng [**62**], but Lindstrøm's [**51**] treatment is the most useful for the purposes of the present paper.

Our pricing formulae could perhaps also be derived through an application of Duffie and Skiadas's [**30**] results. Rather than verifying the various technical assumptions of that paper in the situation of the present article, we have chosen to provide a proof from scratch. For, unlike the approach of Duffie and Skiadas [**30**], the methodology of this paper can be — and, for the case of Brownian information, already has been — applied to study both equilibrium derivative pricing and multi-agent financial markets (cf. Anderson and Raimondo [**8, 9**]).

Nonstandard analysis has a long history of successful application in both equilibrium theory (e.g. Brown and Robinson [**18**] and Keisler [**47**]) and asset pricing (e.g. Cutland, Kopp and Willinger [**23, 24, 25, 26**], Cutland, Kopp, Willinger and Wyman [**27**], Khan and Sun [**48, 49**], and Ng [**61**]). An excellent introduction to nonstandard methods in economics is Anderson's *Handbook of Mathematical Economics* article [**7**]. Another classical exposition of nonstandard analysis, with emphasis on applications in stochastic analysis, is Albeverio, Fenstad, Lindstrøm and Høegh-Krohn [**1**]. To be sure, the results of this paper are "standard" in the sense that they can (and will) be formulated without any reference to nonstandard analysis. Notions from nonstandard analysis will only appear in the proof section.

The application of nonstandard methodology to stochastic analysis with jump processes entails some technical limitations which force us to allow only admissible trading strategies with continuous paths in the demand set of our model. For, the interplay between standard stochastic processes and their liftings (nonstandard analogues) becomes much more delicate as soon as the underlying filtration is generated by a process with discontinuous paths — even if these processes have *càdlàg*[1] paths. In particular, there is no canonical notion of lifting in that situation: The lifting notions which are based on the pathwise *càdlàg* property (such as Hoover and Perkins' [**44**] *SDJ* liftings, or liftings which are well-behaved in Lindstrøm's

[1] *Càdlàg* is the French acronym for 'right-continuous with left limits'.

[**50**] sense) are incompatible with the lifting notions needed for stochastic integration with respect to square-integrable martingales (viz. 2-liftings, cf. Lindstrøm [**50**]). This implies that there exists as yet no lifting notion which would match, given some asset price process and a general *càdlàg* admissible trading strategy, the difference between the current portfolio value (a pathwise scalar product) and the gains from trading (a stochastic integral) associated with that trading strategy. For this reason, we need to restrict the demand set to allow only for admissible trading strategies with continuous paths. We conjecture that in our model, every *càdlàg* — and maybe even more general — admissible trading strategy can be approximated, in terms of the utility of the consumption plan financed by that trading strategy, by a path-continuous admissible trading strategy. Then, the restriction to path-continuous strategies does not lead to a different notion of equilibrium.

The results of this paper motivate several related questions which we hope to address in future research:

(1) Implications for equilibrium derivative pricing: The valuation of derivatives in Lévy market models by means of martingale measures is conceptually troublesome since there are infinitely many martingale measures and hence generically infinitely many derivative prices. In practice, this is overcome by choosing the martingale measure which maximizes some utility functional; however, there is no clear economic justification for preferring one martingale measure over the other. By contrast, equilibrium pricing only provides a unique derivative price for each equilibrium.
(2) Numerical aspects of equilibrium asset pricing: As suggested by Anderson and Raimondo [**8**], one can probably expand the existing nonstandard literature on continuity corrections for discrete-time stochastic option-pricing models (e.g. Cutland, Kopp and Willinger [**23, 24, 25, 26**]) to obtain discrete approximations of the equilibrium pricing formulae studied both in this paper and in Anderson and Raimondo's article [**8**].
(3) Vindication of the restriction to pathwise continuous trading strategies: We expect that the notion of equilibrium in this paper ultimately does not depend on the assumption of pathwise continuous admissible trading strategies (as was discussed above), but we have to leave this to future research.
(4) Extension to multi-agent models: Just as Raimondo's paper [**64**] was the first in a series of papers on the equilibrium foundations of continuous-time finance with Brownian stochasticity which eventually culminated in Anderson and Raimondo's [**9**] existence proof for a multi-agent financial market with Brownian information, we hope that the methodology in this paper can be extended to analyze multi-agent models for financial markets with Lévy stochasticity.

2. Model

We modify the model of Raimondo [**64**] through replacing the d-dimensional Brownian motion β by a general d-dimensional exponentially integrable Lévy process. For technical reasons that were outlined in the introduction, we can only admit pathwise continuous trading strategies into the demand set.

Hence, the model of our economy as follows. Herein and in all of this paper, the Lebesgue measure on $[0,T]$ will be denoted by λ.

(1) The economy has a single agent.

(2) Stochasticity in the market is modeled as follows: Take a vector $\gamma_x \in \mathbb{R}^d$, a symmetric nonnegative-definite matrix $C_x \in \mathbb{R}^{d\times d}$, and a Lévy measure[2] ν_x on $\mathbb{R}^d$ such that $y \mapsto \sum_{k=1}^d \mathrm{e}^{y_k}$ (the sum of exponential components) is ν_x-integrable, and let $\left(\Omega, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space, which will be further specified later on,[3] such that there exists a Lévy process $x : \Omega \times [0,T] \to \mathbb{R}^d$ on $(\Omega, \mathcal{G}_T, \mathbb{P})$ which is adapted to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ and such that the drift, covariance matrix and Lévy (jump intensity) measure of x are γ_x, C_x, ν_x, respectively.[4] Every $\mathbb{P}\otimes\lambda$-measurable map $y : \Omega \times [0,T] \to \mathbb{R}^m$ (for some $m \in \mathbb{N}$) will be referred to as a *stochastic process*. The exponential integrability assumption on ν_x ensures that $\exp(x(\cdot,t)) \in L^1(\mathbb{P})$ for all $t \in [0,T]$.

(3) There are J risky assets $A_1, \ldots, A_J$ (wherein $0 \le J \le d$), one risk-free asset A_0, and one consumption good C. Trading occurs over times in the interval $[0,T]$. The vector of *cum-dividendis* price processes of the securities will be denoted by p_A, whereas p_C will be the price process of the consumption good.

(4) We shall identify the securities with their dividend processes, which are given by

$$\forall \omega \in \Omega \quad \forall t \in [0,T) \qquad A_j(\omega,t) = 0, \qquad A_j(\omega,T) = \exp(x_j(\omega,T))$$

for all $j \in \{1,\ldots,J\}$, and

$$\forall \omega \in \Omega \quad \forall t \in [0,T) \qquad A_0(\omega,t) = 0, \qquad A_0(\omega,T) = 1$$

(5) The agent's endowment process is given by

$$\forall \omega \in \Omega \quad \forall t < T \qquad e(\omega,t) = 1, \qquad e(\omega,T) = \rho(x(\omega,T))$$

for some continuous function $\rho : \mathbb{R}^d \to \mathbb{R}$ which satisfies

$$\exists r \in \mathbb{R}_{\ge 0} \quad \forall \bar{x} \in \mathbb{R}^d \qquad 0 \le \rho(\bar{x}) \le r + e^{r\,|\bar{x}|}.$$

Hence, there is a flow endowment throughout the time interval as well as a lump endowment at the end of the time horizon.

(6) A *(cum-dividendis) securities price process* is a $\mathbb{R}_{\ge 0}^{J+1}$-valued square-integrable stochastic process which happens to be a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale p_A. A *consumption price process* is a $\mathbb{R}_{\ge 0}$-valued stochastic process p_C.

(7) An *admissible trading strategy*, given a securities price process p_A is a $\mathbb{R}^{J+1}$-valued predictable[5] stochastic process z such that
   - for all $j \in \{0,\ldots,J\}$, z_j is square-integrable with respect to the Doléans measure of p_{A_j},[6] and
   - $z_0(\cdot,0) = 0$, $z_j(\cdot,0) = 1$ for all $j \in \{1,\ldots,J\}$.

(8) A *consumption plan* is a $\mathbb{R}_{\ge 0}$-valued stochastic process c.

---

[2] A measure ν on $\mathbb{R}^d$ is called a *Lévy measure* if and only if both $\nu\{0\} = 0$ and $\int_{\mathbb{R}^d} 1 \wedge x^2 \ \nu(\mathrm{d}x) < +\infty$.

[3] Our approach is to provide a "strong solution" in the sense of stochastic analysis to the pricing problem, thus assuming that we are already working on a rich probability space.

[4] In Lindstrøm's [**51**] terminology, (γ_x, C_x, ν_x) is the *generating triplet* of x.

[5] The σ-algebra of *predictable sets* is the smallest σ-algebra which contains all sets of the form $\{0\} \times G_0$ and $(s,t] \times G$ for all $s < t$ and all $G_0 \in \mathcal{G}_0$, $G \in \mathcal{G}_s$.

[6] The *Doléans measure* of a square-integrable martingale N is defined on the σ-algebra of predictable sets as the measure ν_N such that both

$$\nu_N[(s,t] \times G] = \mathbb{E}\left[|N(\cdot,t) - N(\cdot,s)|^2 \chi_G\right]$$

and

$$\nu_N[\{0\} \times G_0] = 0$$

for all $s < t$ and all $G_0 \in \mathcal{G}_0$, $G \in \mathcal{G}_s$. (Cf. Doléans [**28**] and Métiviér [**60**].)

(9) The agent's *utility* U is a function of the consumption plan c, defined via two felicity functions φ_1, φ_2, with flow consumption throughout the time interval as well as a lump consumption at the end of the time-horizon:

$$U : c \mapsto \mathbb{E}\left[\int_0^T \varphi_1\left(c(\cdot,t)\right)\mathrm{d}t + \varphi_2\left(c(\cdot,T)\right)\right].$$

$\varphi_1, \varphi_2 : \mathbb{R}_{>0} \to \mathbb{R}$ are assumed to be twice continuously differentiable, strictly increasing, strictly concave and bounded from below. Furthermore, we assume that

$$\forall c \in (0,1] \qquad \varphi_2'(c) \leq \frac{\gamma}{c^r} \tag{1}$$

holds for some $\gamma, r \in \mathbb{R}$.[7]

(10) The *budget set* for price processes p_A, p_C is the set of all real-valued stochastic processes c for which there exists some admissible trading strategy z such that c satisfies the *(intertemporal) budget constraint* generated by z:

$$p_A(\cdot,t)\cdot z(\cdot,t) \;=\; \mathbf{1}\cdot p_A(0) + \int_0^t z\mathrm{d}p_A + \int_0^t p_C(\cdot,s)\left(e(\cdot,s)-c(\cdot,s)\right)\mathrm{d}s$$

$\mathbb{P}$-almost surely for every $t < T$ and

$$\begin{aligned} p_A(\cdot,T)\cdot z(\cdot,T) \;=\; & \mathbf{1}\cdot p_A(0) + \int_0^T z\mathrm{d}p_A + \int_0^T p_C(\cdot,s)\left(e(\cdot,s)-c(\cdot,s)\right)\mathrm{d}s \\ & + \left(z(\cdot,T)\cdot A(\cdot,T) + e(\cdot,T) - c(\cdot,T)\right)p_C(\cdot,T) \end{aligned}$$

$\mathbb{P}$-almost surely. Herein, we use the notation $\mathbf{1} := \underbrace{(1,\ldots,1)}_{J \text{ times}}^\top$

(11) The *continuous-strategy demand set* for price processes p_A, p_C consists of all those pairs (z, c) of an admissible trading strategy z and a consumption plan c such that
- z has continuous paths $\mathbb{P}$-almost surely,
- c satisfies the budget constraint generated by z, and
- $U(c)$ is the maximum of U on the budget set for p_A, p_C.

(12) A *continuous-strategy (securities-market) equilibrium* for the economy (e, A, φ) is a quadruple (p_A, p_C, z, c) such that
- (z, c) is an element of the continuous-strategy demand set for p_A, p_C, and
- the securities and goods markets clear:

$$\forall t \in [0,T] \qquad z_0(\cdot,t) = 0, \quad \forall j \in \{1,\ldots,J\} \quad z_j(\cdot,t) = 1$$

$$\forall t \in [0,T) \quad c(\cdot,t) = 1, \qquad c(\cdot,T) = e(\cdot,T) + \sum_{j=1}^J A_j(\cdot,T)$$

everywhere on Ω.

We put $x_0 := 0$, so that x can also be viewed as a $\mathbb{R}^{d+1}$-valued Lévy process.

[7] A sufficient condition for this estimate, given the other assumptions on φ_2, is that φ_2 exhibits *bounded scaled risk aversion* on $(0,1]$ in the sense that $\sup_{c\in(0,1]} -c^q \frac{\varphi_2''(c)}{\varphi_2'(c)} < +\infty$ for some $q \geq 1$. In particular, this condition is satisfied if φ_2 exhibits bounded relative risk aversion on $(0,1]$ — for instance, if φ_2 is of constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA). See Lemma A.1 in Appendix A.

3. Main result

Our result is the existence of an equilibrium for the stochastic continuous-time economy described in Section 2 and an explicit pricing formula in terms of the Lévy process x. We will use the abbreviation $R:\mathbb{R}^d\to\mathbb{R}$, $y\mapsto\rho(y)+\sum_{k=1}^{J}\mathrm{e}^{y_k}$.

THEOREM 3.1. *There exists a continuous-strategy securities-market equilibrium* (p_A,p_C,z,c) *for the economy* (e,A,φ). *One has* $p_C(\cdot,t)=\varphi'(1)$ *for all* $t<T$ *as well as* $p_C(\cdot,T)=\varphi_2'(R(x(\cdot,T)))$ *with probability* 1. *Furthermore, with probability* 1, *for all* $t\in[0,T]$ *and* $j\in\{0,\dots,J\}$,

$$p_{A_j}(\cdot,t)=\mathbb{E}\left[\varphi_2'\left(R\left(x(\cdot,T)\right)\right)\exp\left(x_j(\cdot,T)\right)\middle|\,\mathcal{G}_t\right],$$

hence

$$p_{A_j}(\omega,t)=\exp\left(x_j(\omega,t)\right)\int\varphi_2'\left(R\left(x(\omega,t)+z\right)\right)\mathrm{e}^{z_j}\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)$$

for $\mathbb{P}$*-almost all* $\omega\in\Omega$.

4. Proof

Our proof is based on nonstandard analysis, more precisely: on a hyperfinite, i.e. formally finite, discretization of the continuous-time economy. This proof technique was introduced into the equilibrium theory of financial markets by Anderson and Raimondo [**64, 8, 9**]. Our argument consists essentially of three parts: First, we give a rigorous description of the hyperfinite economy, wherein Lindstrøm's [**51**] theory of hyperfinite Lévy processes will be put to use. Then, secondly, we establish, by means of the Duffie-Shafer [**31, 32**] theorems, the existence of equilibrium in a perturbed hyperfinite discretization of the continuous-time economy (Theorem 4.1) as well as a characterization of the equilibrium asset prices (Theorem 4.3). The last step will be the proof that the standard part of the hyperfinite economy is indeed an equilibrium of the standard economy (Theorem 4.4). Our main result, Theorem 3.1, is just the combination of Theorem 4.1, Theorem 4.3 and Theorem 4.4.

4.1. The hyperfinite economy. Our hyperfinite discretization of the continuous-time economy is a generalization of the hyperfinite economy of Raimondo [**64**], obtained through replacing Anderson's [**6**] random walk $\hat{\beta}$ by a hyperfinite Lévy process X (in the sense of Lindstrøm [**51**]) whose right standard part will be the Lévy process x.

Lindstrøm [**51**] has shown that for for all triples (γ_x,C_x,ν_x) consisting of a vector in $\mathbb{R}^d$, a symmetric nonnegative-definite matrix in $\mathbb{R}^{d\times d}$, and a Lévy measure on $\mathbb{R}^d$, respectively, there exists a hyperfinite Lévy process X whose right standard part has generating triplet (γ_x,C_x,ν_x). Hence, we may assume that x is the right standard part of some hyperfinite Lévy process $X:\Omega\times\mathbb{T}\to{}^*\mathbb{R}^d$ defined on some hyperfinite probability space (Ω,P) (where Ω is a hyperfinite set and P is an internal finitely-additive probability measure on the internal power-set of Ω), wherein, $\mathbb{T}$ is the hyperfinite time-line

$$\mathbb{T}:=\left\{\frac{n}{N}T\ :\ n\in{}^*\mathbb{N}_0,\quad n\le N\right\}$$

(for some $N\in{}^*\mathbb{N}\setminus\mathbb{N}$), and we shall write $\Delta t:=\frac{1}{N}$ for the — infinitesimal — spacing of $\mathbb{T}$.[8] If we denote by I the increment set of X, then (Ω,P) may be chosen

[8] An internal map $X:\Omega\times\mathbb{T}\to{}^*\mathbb{R}^d$ is a *hyperfinite Lévy process* if and only if

(1) $X(\cdot,0)=0$.
(2) For all $t_0<\cdots<t_n\in\mathbb{T}$, the internal random variables $X(\cdot,t_1-t_0),\dots,X(\cdot,t_n-t_{n-1})$ are *-independent under P.
(3) For all $a\in{}^*\mathbb{R}^d$ and $t\in\mathbb{T}$, $P\{X(\cdot,t+\Delta t)-X(\cdot,t)=a\}=P\{X(\cdot,\Delta t)=a\}$.

such that both

$$\Omega = I^{\mathbb{T}\setminus\{T\}} = \{\omega : \mathbb{T} \setminus \{T\} \to I \ : \ \omega \text{ internal}\}$$

and

$$\forall t \in \mathbb{T} \quad \forall \omega \in \Omega \qquad X(\omega, t) = \sum_{u \in \mathbb{T} \cap [0,t)} \omega(u).$$

One can also define an internal filtration $(\mathcal{F}_u)_{u\in\mathbb{T}}$ on Ω: For all $u \in \mathbb{T}$, define an equivalence relation $\sim_u$ on Ω by

$$\forall \omega, \omega' \in \Omega \qquad \omega \sim_u \omega' :\Leftrightarrow \forall v < u \quad \omega(v) = \omega'(v),$$

and define $\mathcal{F}_u$ to be the algebra of all internal, $\sim_u$-respecting sets. In other words:

$$\mathcal{F}_u := \left\{ C \in 2^\Omega \ : \ \forall \omega, \omega' \in \Omega \left(\omega \sim_u \omega' \Rightarrow (\omega \in C \Leftrightarrow \omega' \in C)\right)\right\}.$$

The probability space of the standard model, $\left(\Omega, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}\right)$, can now be specified as follows:

- Ω is the hyperfinite set $\Omega = I^{\mathbb{T}\setminus\{T\}}$, the first component of the domain of X.
- $\mathbb{P}$ is the Loeb measure[9] generated by P, in symbols: $\mathbb{P} = \mathsf{L}(P)$.
- For all $t \in [0,T]$, the σ-algebra $\mathcal{G}_t$ is the $\mathsf{L}(P)$-completion of the algebra

$$\left\{ C \in 2^\Omega \ : \ \forall u \simeq t \quad \forall \omega, \omega' \in \Omega \left((\forall v < u \quad \omega(v) = \omega'(v)) \Rightarrow (\omega \in C \Leftrightarrow \omega' \in C)\right)\right\}$$

  (equivalently, $\mathcal{G}_t$ is the $\mathsf{L}(P)$-completion of the algebra $\bigcap_{u\simeq t} \mathcal{F}_u$).
- x is the right standard part of X.

Here is the model of the (discrete) hyperfinite economy:

(1) The economy has a single agent.
(2) Randomness is given by the hyperfinite probability space (Ω, P). An *internal (stochastic) process* is an internal map $Y : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^m$ (for $m \in \mathbb{N}$). An internal process is called *nonanticipating* if and only if it is adapted with respect to the internal filtration $\mathcal{F} = (\mathcal{F}_u)_{u\in\mathbb{T}}$ (i.e. if $Y(\cdot, u)$ is $\mathcal{F}_u$-measurable for all $u \in \mathbb{T}$).[10]
(3) There are $J+1$ securities.
(4) The internal dividend processes of the securities are $\hat{A}_0, \ldots, \hat{A}_J$, where $\hat{A}_0$ is risk-free.
(5) The agent in the hyperfinite economy has an endowment process $\hat{e} : \Omega \times \mathbb{T} \to {}^*\mathbb{R}_{\geq 0}$.
(6) A *(cum-dividendis) securities price process* for the hyperfinite economy is an internal process $\hat{p}_A : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^{J+1}_{\geq 0}$. A *consumption price process* for the hyperfinite economy is an internal process $\hat{p}_C : \Omega \times \mathbb{T} \to {}^*\mathbb{R}_{\geq 0}$.
(7) An *admissible trading strategy* for the hyperfinite economy is an internal process $\hat{z} : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^{J+1}$ such that
    - $\hat{z}(\cdot, t)$ is $\mathcal{F}_{t-\Delta t}$-measurable for all $t \in \mathbb{T} \setminus \{0\}$, and
    - $\hat{z}_0(\cdot, 0) = 0$, $\hat{z}_j(\cdot, 0) = 1$ for all $j \in \{1, \ldots, J\}$.[11]
(8) A *consumption plan* for the hyperfinite economy is an internal process $\hat{c} : \Omega \times \mathbb{T} \to {}^*\mathbb{R}_{\geq 0}$.

The internal set $a \in {}^*\mathbb{R}^d \ : \ P\{X(\cdot, \Delta t) = a\} > 0$ is called the *increment set* of X.

[9] The Loeb measure $\mathsf{L}(\mu)$ generated by an internal finitely-additive finite measure μ on some internal algebra $\mathcal{A}$ is the Carathéodory extension of the finitely additive measure $A \mapsto {}^\circ\mu(A)$. (Cf. Loeb [**53**] and Albeverio, Fenstad, Høegh-Krohn and Lindstrøm [**1**].)

[10] An equivalent definition would be to demand that for all $\omega, \omega' \in \Omega$ and $t \in \mathbb{T}$, if $\omega(s) = \omega'(s)$ for all $s < t$, then $Y(\omega, t) = Y(\omega', t)$.

[11] Anderson and Raimondo [**9**] employ a slightly different terminology. They would call the process $(\hat{z}(\cdot, t - \Delta t))_{t\in\mathbb{T}}$, rather than $\hat{z}$, an (admissible) trading strategy.

(9) The agent's utility in the hyperfinite economy is an internal function of the consumption plan $\hat{c}$, defined by

$$\hat{U} : \hat{c} \mapsto E\left[\left(\sum_{u<T} {}^*\varphi_1\left(\hat{c}(\cdot,u)\right)\Delta t\right) + {}^*\varphi_2\left(\hat{c}(\cdot,T)\right)\right],$$

wherein ${}^*\varphi_1, {}^*\varphi_2$ are the *-images of the functions $\varphi_1, \varphi_2 : \mathbb{R}_{>0} \to \mathbb{R}$ that defined the utility function in the standard economy (which were assumed to be twice continuously differentiable, strictly increasing, concave and bounded from below).

(10) The *budget set* for price processes $\hat{p}_A, \hat{p}_C$ is the set of all consumption plans $\hat{c}$ for which there exists some admissible trading strategy $\hat{z}$ such that $\hat{c}$ satisfies the *(intertemporal) budget constraint* generated by $\hat{z}$:

$$\begin{aligned} & \hat{p}_A(\cdot,u)\cdot\hat{z}(\cdot,u) \\ = \; & \mathbf{1}\cdot\hat{p}_A(0) + \sum_{v<u}\hat{z}(\cdot,v)\Delta\hat{p}_A(\cdot,v) + \sum_{v<u}\hat{p}_C(\cdot,v)\left(\hat{e}(\cdot,v)-\hat{c}(\cdot,v)\right)\Delta t \end{aligned}$$

for every $u \in \mathbb{T}$ and

$$\begin{aligned} & \hat{p}_A(\cdot,T)\cdot\hat{z}(\cdot,T) \\ = \; & \mathbf{1}\cdot\hat{p}_A(0) + \sum_{v<T}\hat{z}(\cdot,v)\Delta\hat{p}_A(\cdot,v) + \sum_{v<T}\hat{p}_C(\cdot,v)\left(\hat{e}(\cdot,v)-\hat{c}(\cdot,v)\right)\Delta t \\ & + \left(\hat{z}(\cdot,T)\cdot\hat{A}(\cdot,T) + \hat{e}(\cdot,T) - \hat{c}(\cdot,T)\right)\hat{p}_C(\cdot,T) \end{aligned}$$

(11) The *demand set* for price processes $\hat{p}_A, \hat{p}_C$ consists of all those pairs $(\hat{z}, \hat{c})$ where $\hat{z}$ is an admissible trading strategy and $\hat{c}$ is a consumption plan such that
- $\hat{c}$ satisfies the budget constraint generated by $\hat{z}$, and
- $\hat{U}(\hat{c})$ is the maximum of $\hat{U}$ on the budget set for $\hat{p}_A, \hat{p}_C$.

(12) A *(securities-market) equilibrium* for the hyperfinite economy $\left(\hat{e}, \hat{A}, {}^*\varphi\right)$ is a quadruple $(\hat{p}_A, \hat{p}_C, \hat{z}, \hat{c})$ such that
- $(\hat{z}, \hat{c})$ lies in the demand set for $\hat{p}_A, \hat{p}_C$,
- and the securities and goods markets clear, i.e.

$$\begin{aligned} \forall u \in \mathbb{T} \quad & \hat{z}_0(\cdot,u) = 0, & \forall j \in \{1,\ldots,J\} \quad & \hat{z}_j(\cdot,u) = 1 \\ \forall u \in \mathbb{T}\setminus\{T\} \quad & \hat{c}(\cdot,u) = 1, & \hat{c}(\cdot,T) = \hat{e}(\cdot,T) + & \sum_{j=1}^{J}\hat{A}_j(\cdot,T) \end{aligned}$$

everywhere on Ω.

Again, we will put $X_0 = 0$, so that X can also be regarded as a ${}^*\mathbb{R}^{d+1}$-valued hyperfinite Lévy process.

It is worth noting that for every S-continuous trading strategy in the hyperfinite economy, the difference between the internal current portfolio value and the internal gains process is itself an S-continuous process:

LEMMA 4.1. *Suppose $\hat{y}$ is S-continuous, and, $\mathsf{L}(P)$-almost surely, $\hat{p}_A(\cdot,u)$ is finite for all $u \in \mathbb{T}$. Then $\hat{y}\hat{p}_A - \int \hat{y}\mathrm{d}\hat{p}_A$ is S-continuous.*

PROOF. For all $u, v \in \mathbb{T}$,

$$\begin{aligned}
& \hat{y}(\cdot,v)\hat{p}_A(\cdot,v) - \int_0^v \hat{y}\ \mathrm{d}\hat{p}_A - \left(\hat{y}(\cdot,u)\hat{p}_A(\cdot,u) - \int_0^u \hat{y}\ \mathrm{d}\hat{p}_A\right) \\
= \ & \hat{y}(\cdot,v)\hat{p}_A(\cdot,v) - \hat{y}(\cdot,u)\hat{p}_A(\cdot,u) - \int_u^v \hat{y}\ \mathrm{d}\hat{p}_A \\
= \ & \hat{y}(\cdot,v)\hat{p}_A(\cdot,v) - \hat{y}(\cdot,u)\hat{p}_A(\cdot,u) - \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
& + \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) - \int_u^v \hat{y}\ \mathrm{d}\hat{p}_A \\
= \ & \left(\hat{y}(\cdot,v) - \hat{y}(\cdot,u)\right)\hat{p}_A(\cdot,u) + \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) - \int_u^v \hat{y}\ \mathrm{d}\hat{p}_A.
\end{aligned}$$

whilst $\hat{p}_A(\cdot,u)$ is finite $\mathsf{L}(P)$-almost surely.

Now the first addend on the right-hand side of this equation is zero if $u \simeq v$, due to the S-continuity of $\hat{y}$ and because $\hat{p}_A(\cdot,u)$ is finite $\mathsf{L}(P)$-almost surely. The second part can be estimated as follows:

$$\begin{aligned}
& \left(\hat{y}(\cdot,v) - \max_{t\in[u,v]} \hat{y}(\cdot,t) \vee \min_{t\in[u,v]} \hat{y}(\cdot,t)\right)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
= \ & \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
& - \left(\max_{t\in[u,v]} \hat{y}(\cdot,t) \vee \left(-\min_{t\in[u,v]} \hat{y}(\cdot,t)\right)\right)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
\leq \ & \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) - \int_u^v \hat{y}\ \mathrm{d}\hat{p}_A \\
\leq \ & \hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
& - \left(\min_{t\in[u,v]} \hat{y}(\cdot,t) \wedge \left(-\max_{t\in[u,v]} \hat{y}(\cdot,t)\right)\right)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) \\
= \ & \left(\hat{y}(\cdot,v) - \min_{t\in[u,v]} \hat{y}(\cdot,t) \wedge \left(-\max_{t\in[u,v]} \hat{y}(\cdot,t)\right)\right)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right),
\end{aligned}$$

wherein $\min, \max, \wedge, \vee$ are taken componentwise. But if $u \simeq v$, then $\hat{y}(\cdot,v) \simeq \min_{t\in[u,v]} \hat{y}(\cdot,t) \simeq \max_{t\in[u,v]} \hat{y}(\cdot,t)$ whilst $\hat{p}_A(\cdot,v), \hat{p}_A(\cdot,u)$ are finite, therefore both the very left-hand side and the very right-hand side of this chain of inequalities are infinitesimal. For this reason, $\hat{y}(\cdot,v)\left(\hat{p}_A(\cdot,v) - \hat{p}_A(\cdot,u)\right) - \int_u^v \hat{y}\ \mathrm{d}\hat{p}_A$ is infinitesimal whenever $u \simeq v$. □

4.2. The hyperfinite equilibrium. We shall now establish the existence of a securities-market equilibrium for some hyperfinite economy that is an infinitesimal perturbation of the hyperfinite economy whose primitives correspond to those of the standard economy.

The size of this perturbation is determined by two infinitesimal constants $\psi(\Delta t), \chi(\Delta t)$. Using, as before, the abbreviation R for the function $R : \mathbb{R}^d \to \mathbb{R}$, $y \mapsto \rho(y) + \sum_{k=1}^J \mathrm{e}^{y_k}$ and its *-image, we define these constants as

$$\psi(\Delta t) = \frac{\Delta t}{\max_{\omega\in\Omega} \max_{\xi\in[0,1]} \left|{}^*\varphi_2''\left(R\left(X(\omega,T)\right) + \xi\right)\right| \max_{\omega\in\Omega} \max_k \exp\left(X_k(\omega,T)\right)}$$

and

$$\chi(\Delta t) = \frac{\Delta t}{\max_{\omega\in\Omega} \max_{\xi\in[0,1]} \left|{}^*\varphi_2''\left(R\left(X(\omega,T)\right) + \xi\right)\right|}.$$

THEOREM 4.1. *The hyperfinite economy* $\left(\hat{e}, \hat{A}, {}^*\varphi\right)$ *has a securities-market equilibrium* $(\hat{p}_A, \hat{p}_C, \hat{z}, \hat{c})$ *for some* $\hat{e}, \hat{A}$ *which satisfy*

$$\forall t \in \mathbb{T}\setminus\{T\} \qquad 0 \leq \hat{e}(\cdot,t) - 1 \leq \psi(\Delta t)$$
$$0 \leq \hat{e}(\cdot,T) - {}^*\rho(X(\cdot,T)) \leq \psi(\Delta t)$$

as well as for all $j \in \{0,\dots,J\}$,

$$\forall t \in \mathbb{T}\setminus\{T\} \qquad 0 \leq \hat{A}_j(\cdot,t) \leq \chi(\Delta t)$$
$$\exp(X_j(\cdot,T)) \leq \hat{A}_j(\cdot,T) \leq \exp(X_j(\cdot,T)) + \chi(\Delta t).$$

For $\mathsf{L}(P)$*-almost all* ω*, one has*

$$\forall t \in \mathbb{T}\setminus\{T\} \quad \hat{p}_C(\omega,t) = {}^*\varphi_1'(1), \qquad \hat{p}_C(\omega,T) \simeq {}^*\varphi_2'(R(X(\omega,T))),$$

as well as for all $j \in \{0,\dots,J\}$,

$$\text{(2)} \quad \forall t \in \mathbb{T} \quad \hat{p}_{A_j}(\omega,t) \simeq \exp(X_j(\omega,t)) \int {}^*\varphi_2'(R(X(\omega,t)+y))\,\mathrm{e}^{y_j} P_{X(\cdot,T-t)}(\mathrm{d}y).$$

PROOF. The existence of the equilibrium for a perturbed economy follows from the Duffie-Shafer theorem [**31, 32**] transferred to the nonstandard universe.

Next, we prove the formula (2) for $\hat{p}_A$. The first-order conditions (cf. e.g. Magill and Quinzii [**58**, §22, pp. 230-231]) imply that for all $j \in \{0,\dots,J\}$ and $t \in \mathbb{T}$,

$$\text{(3)} \qquad \hat{p}_{A_j}(\cdot,t) = E\left[{}^*\varphi_2'\left(\hat{e}(\cdot,T) + \sum_{k=1}^{J}\hat{A}_k(\cdot,T)\right)\hat{A}_j(\cdot,T)\middle|\mathcal{F}_t\right].$$

Let us combine this equation with the estimates on $\hat{e}$ and $\hat{A}$ which the Theorem assumes. Then there exist random variables ζ, η such that

$$\hat{p}_{A_j}(\cdot,t) \quad = \quad E\left[{}^*\varphi_2'(R(X(\cdot,T))+\zeta)(\exp(X_j(\cdot,T))+\eta)\middle|\mathcal{F}_t\right]$$

and

$$0 \leq \hat{e}(\cdot,T) + \sum_{k=1}^{J}\hat{A}_k(\cdot,T) - {}^*\rho(X(\cdot,T)) - \sum_{k=1}^{J}\exp(X_k(\cdot,T)) = \zeta \leq \psi(\Delta t) + J\chi(\Delta t)$$

as well as

$$0 \leq \hat{A}_j(\cdot,T) - \exp(X_j(\cdot,T)) = \eta \leq \chi(\Delta t).$$

Combining these bounds on ζ with the intermediate-value theorem and the choice of $\psi(\Delta t), \chi(\Delta t)$, we obtain

$$\begin{aligned}
& \left|{}^*\varphi_2'(R(X(\cdot,T))+\zeta) - {}^*\varphi_2'(R(X(\cdot,T)))\right| \\
\leq \;& \zeta \max_{\xi\in[0,\zeta]}\left|{}^*\varphi_2''(R(X(\cdot,T))+\xi)\right| \\
\leq \;& (\psi(\Delta t)+J\chi(\Delta t)) \max_{\xi\in[0,\psi(\Delta t)+J\chi(\Delta t)]}\left|{}^*\varphi_2''(R(X(\cdot,T))+\xi)\right| \\
\leq \;& (\psi(\Delta t)+J\chi(\Delta t)) \max_{\omega\in\Omega}\max_{\xi\in[0,\psi(\Delta t)+J\chi(\Delta t)]}\left|{}^*\varphi_2''(R(X(\omega,T))+\xi)\right| \simeq 0
\end{aligned}$$

everywhere on Ω. Therefore

$$\text{(4)} \qquad \forall\omega\in\Omega \qquad {}^*\varphi_2'(R(X(\omega,T))+\zeta) \simeq {}^*\varphi_2'(R(X(\omega,T)))$$

Since also $|\eta| \leq \chi(\Delta t) \simeq 0$ and $X(\cdot,T)$ is finite $\mathsf{L}(P)$-almost surely, we conclude that

$${}^*\varphi_2'(R(X(\cdot,T))+\zeta)(\exp(X_j(\cdot,T))+\eta) \simeq {}^*\varphi_2'(R(X(\cdot,T)))\exp(X_j(\cdot,T))$$

holds $\mathsf{L}(P)$-almost surely. (In fact, our choice of $\psi(\Delta t), \chi(\Delta t)$ even makes sure that this approximate identity holds everywhere on Ω.) This implies, by Loeb integration theory, that

$$\begin{aligned} \hat{p}_{A_j}(\cdot,t) &= E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)+\zeta\right)\left(\exp\left(X_j(\cdot,T)\right)+\eta\right)\middle|\,\mathcal{F}_t\right] \\ &\simeq E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right] \end{aligned} \tag{5}$$

$\mathsf{L}(P)$-almost surely. Since X is a hyperfinite Lévy process, we may finally deduce

$$\hat{p}_{A_j}(\cdot,t) \simeq \int {}^*\varphi_2'\left(R\left(X(\cdot,t)+y\right)\right)\exp\left(X_j(\cdot,t)+y_j\right)P_{X(\cdot,T-t)}(\mathrm{d}y)$$

$\mathsf{L}(P)$-almost surely, which is the same as the approximate identity (2).

As the last step, we prove the formulae for $\hat{p}_C$. At the terminal date, the first-order conditions yield $\hat{p}_C(\cdot,T) = {}^*\varphi_2'\left(\hat{c}(\cdot,T)\right)$. Combining this with market clearing and Equation (4), we obtain

$$\hat{p}_C(\cdot,T) = {}^*\varphi_2'\left(\hat{c}(\cdot,T)\right) = {}^*\varphi_2'\left(\hat{e}(\cdot,T)+\sum_{k=1}^{J}\hat{A}_k(\cdot,T)\right) \simeq {}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right).$$

Finally, the individual consumption equals the social consumption and hence $\hat{c}(\cdot,t)=1$ for all $t\in\mathbb{T}\setminus\{T\}$, which leads, via the first-order conditions, to

$$\forall t\in\mathbb{T}\setminus\{T\} \quad \hat{p}_C(\cdot,t) = {}^*\varphi_1'\left(\hat{c}(\cdot,t)\right) = {}^*\varphi_1'(1).$$

□

THEOREM 4.2. *$\hat{p}_A$ has a right standard part and is nonanticipating as well as S-square integrable. Its right standard part ${}^\circ\hat{p}_A$ is a square-integrable martingale.*

PROOF. Due to Equation (5) (which follows from the first-order conditions), $\hat{p}_A$ is a martingale and nonanticipating.

Next, fix $j\in\{0,\ldots,J\}$. φ_2' is nonnegative and R is the *-image of the continuous function $\rho+\sum_{k=1}^{J}\exp\left((\cdot)_k\right)$, whence the internal processes $(\omega,t)\mapsto E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\wedge n\middle|\,\mathcal{F}_t\right](\omega)$, $n\in\mathbb{N}$ are adapted functions of X (in the terminology of Fajardo and Keisler [**35**]) and, moreover, bounded internal martingales. Hence, $(\omega,t)\mapsto E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right](\omega)$ is the monotone limit of adapted functions which are martingales. Also, X is a right lifting of $x={}^\circ X$.

Therefore, the model theory of stochastic processes teaches, by means of the Adapted Lifting Theorem (cf. Fajardo and Keisler [**35**]) that the internal process $(\omega,t)\mapsto E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right](\omega)$ has a right standard part (for all j) which is a martingale.

In addition, if we combine the growth estimate (1) on φ_2' with the assumption that φ_2' is decreasing, we obtain S-square integrability of $(\omega,t)\mapsto E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right](\omega)$ through an estimate on the standard part of ${}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)$.

This implies that the right standard part of the internal process $(\omega,t)\mapsto E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right](\omega)$ is a square-integrable martingale.

In light of Equation (5) we finally conclude that $\hat{p}_{A_j}:(\omega,t)\mapsto\hat{p}_{A_j}(\omega,t)\simeq E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right](\omega)$ has a right standard part which is a square-integrable martingale. □

LEMMA 4.2. *For all $t\in\mathbb{T}$ and all finite $a\in{}^*\mathbb{R}^J$, ${}^*\varphi_2'\left(R\left(a+X(\cdot,T-t)\right)\right)$ is $SL^1(P)$.*

PROOF. Note that R grows exponentially and ${}^*\varphi_2'(c)\le\frac{\gamma}{c^r}$ holds for all $c\in(0,1]$ by estimate (1). Therefore, ${}^*\varphi_2'\circ R$ exhibits exponential growth. On the other hand $\exp(X_j(\cdot,T-t))$ is S-integrable with respect to P for all $j\in\{0,\dots,J\}$. Therefore, ${}^*\varphi_2'(R(a+X(\cdot,T-t)))$ must be S-integrable with respect to P, too. □

LEMMA 4.3. *For all* $t\in\mathbb{T}$, $\mathsf{L}\left(P_{X(\cdot,t)}\right)\circ\mathsf{st}^{-1}=\mathsf{L}(P)_{x(\cdot,{}^\circ t)}$

PROOF. Consider a Borel measurable $B\subseteq\mathbb{R}^J$. Lindstrøm [**51**, Lemma 6.4] has shown that ${}^\circ(X(\omega,t))=x(\omega,{}^\circ t)$ for $\mathsf{L}(P)$-almost every $\omega\in\Omega$. Therefore, $\mathsf{L}(P)\{x(\omega,{}^\circ t)\in B\}=\mathsf{L}(P)\{{}^\circ(X(\cdot,t))\in B\}$ and hence $\mathsf{L}(P)\{x(\omega,{}^\circ t)\in B\}=\mathsf{L}(P)\left\{X(\cdot,t)\in\mathsf{st}^{-1}[B]\right\}=\mathsf{L}\left(P_{X(\cdot,t)}\right)\circ\mathsf{st}^{-1}[B]$. □

LEMMA 4.4. *For all* $j\in\{0,\dots,J\}$ *and all* $t\in\mathbb{T}$,

$$ {}^\circ\left(\hat p_{A_j}(\omega,t)\right)=\exp\left(x_j(\omega,{}^\circ t)\right)\int\varphi_2'\left(R\left(x(\omega,{}^\circ t)+z\right)\right)\mathrm{e}^{z_j}\,\mathsf{L}(P)_{x(\cdot,T-{}^\circ t)}(\mathrm{d}z) \tag{6}$$

for $\mathsf{L}(P)$*-almost all* ω.

PROOF. Fix $j\in\{0,\dots,J\}$ and $t\in\mathbb{T}$. In the following calculations, the first equation is just Equation (2), the second equation follows from Lemma 4.2 and Loeb integration theory, and the last equation is due to Lemma 4.3:

$$\begin{aligned}
\hat p_{A_j}(\omega,t) &\simeq \exp\left(X_j(\omega,t)\right)\int{}^*\varphi_2'\left(R\left(X(\omega,t)+y\right)\right)\mathrm{e}^{y_j}P_{X(\cdot,T-t)}(\mathrm{d}y)\\
&\simeq \exp\left(X_j(\omega,t)\right)\int{}^\circ\left({}^*\varphi_2'\left(R\left(X(\omega,t)+y\right)\right)\mathrm{e}^{y_j}\right)\mathsf{L}\left(P_{X(\cdot,T-t)}\right)(\mathrm{d}y)\\
&= \exp\left(X_j(\omega,t)\right)\int{}^\circ\left({}^*\varphi_2'\left(R\left(X(\omega,t)+{}^\circ y\right)\right)\mathrm{e}^{y_j}\right)\mathsf{L}\left(P_{X(\cdot,T-t)}\right)(\mathrm{d}y)\\
&= \exp\left(X_j(\omega,t)\right)\int{}^\circ\left({}^*\varphi_2'\left(R\left(X(\omega,t)+z\right)\right)\mathrm{e}^{z_j}\right)\mathsf{L}\left(P_{X(\cdot,T-t)}\right)\circ\mathsf{st}^{-1}(\mathrm{d}z)\\
&= \exp\left(X_j(\omega,t)\right)\int{}^\circ\left({}^*\varphi_2'\left(R\left(X(\omega,t)+z\right)\right)\mathrm{e}^{z_j}\right)\mathsf{L}(P)_{x(\cdot,T-{}^\circ t)}(\mathrm{d}z)
\end{aligned}$$

for all $\omega\in\Omega$. However, the function $\xi\mapsto{}^*\varphi_2'\left(R(\xi+z)\right)\mathrm{e}^{\xi_j}$ is S-continuous (for all $z\in\mathbb{R}^J$) and so is exp, and additionally we have by Lindstrøm [**51**, Lemma 6.4] that $X(\omega,t)\simeq x(\omega,{}^\circ t)$ for $\mathsf{L}(P)$-almost every $\omega\in\Omega$. This finally yields

$$\hat p_{A_j}(\omega,t)\;\simeq\;\exp\left(x_j(\omega,{}^\circ t)\right)\int\varphi_2'\left(R\left(x(\omega,{}^\circ t)+z\right)\right)\mathrm{e}^{z_j}\,\mathsf{L}(P)_{x(\cdot,T-{}^\circ t)}(\mathrm{d}z)$$

for $\mathsf{L}(P)$-almost every $\omega\in\Omega$. □

4.3. Equilibrium for the standard economy. The model theory of stochastic processes can again be used to derive, from the pricing formulae in Theorem 4.1, an explicit formula for the right standard part ${}^\circ\hat p_A$ of $\hat p_A$.

THEOREM 4.3. *With Loeb probability* 1, *one has for all* $t\in[0,T]$ *and* $j\in\{0,\dots,J\}$,

$$ {}^\circ\hat p_{A_j}(\cdot,t)=\mathbb{E}\left[\varphi_2'\left(R\left(x(\cdot,T)\right)\right)\exp\left(x_j(\cdot,T)\right)\middle|\,\mathcal{G}_t\right], \tag{7}$$

hence for $\mathsf{L}(P)$*-almost all* ω,

$$ {}^\circ\hat p_{A_j}(\omega,t)=\exp\left(x_j(\omega,t)\right)\int\varphi_2'\left(R\left(x(\omega,t)+z\right)\right)\mathrm{e}^{z_j}\,\mathsf{L}(P)_{x(\cdot,T-t)}(\mathrm{d}z) \tag{8}$$

for all $t\in[0,T]$ *and* $j\in\{0,\dots,J\}$.

Proof. Equation (5) tells us that with $\mathsf{L}(P)$-probability 1, one has for all $t \in \mathbb{T}$,

$$\hat{p}_A(\cdot,t) \simeq E\left[{}^*\varphi_2'\left(R\left(X(\cdot,T)\right)\right)\exp\left(X_j(\cdot,T)\right)\middle|\,\mathcal{F}_t\right].$$

Combining this with the Adapted Lifting Theorem (cf. Fajardo and Keisler [**35**]) completes the proof for the conditional-expectation representation of ${}^\circ\hat{p}_A$ in Equation (7). Equation (8) follows immediately from Equation (7) by the properties of Lévy processes:

$$\begin{aligned} & \mathbb{E}\left[\varphi_2'\left(R\left(x(\cdot,T)\right)\right)\exp\left(x_j(\cdot,T)\right)\middle|\,\mathcal{G}_t\right] \\ = & \int \varphi_2'\left(R\left(x\left(\omega,t\right)+z\right)\right)\exp\left(x_j\left(\omega,t\right)+z_j\right)\mathsf{L}(P)_{x(\cdot,T-t)}(\mathrm{d}z) \\ = & \exp\left(x_j(\omega,t)\right)\int \varphi_2'\left(R\left(x\left(\omega,t\right)+z\right)\right)\mathrm{e}^{z_j}\mathsf{L}(P)_{x(\cdot,T-t)}(\mathrm{d}z). \end{aligned}$$

□

The combination of Theorem 4.3 with Lemma 4.4 gives another characterization of ${}^\circ\hat{p}_A$ (which, as we shall see, is the price process of the standard economy) through $\hat{p}_A$:

Corollary 4.1. *For all $j \in \{0,\ldots,J\}$ and all $t \in \mathbb{T}$, one has*

$${}^\circ\hat{p}_{A_j}\left(\cdot,{}^\circ t\right) = {}^\circ\left(\hat{p}_{A_j}(\omega,t)\right)$$

for $\mathsf{L}(P)$-almost all $\omega \in \Omega$.

Finally, we prove that the standard part of the hyperfinite securities-market equilibrium is indeed a (standard) securities-market equilibrium for the (standard) continuous-time economy. Herein, by standard part we mean, as before, the — pathwise — right standard part, and it should be noted that this is the canonical notion of standard part when studying stochastic processes whose paths are right-continuous functions with left limits: For, the right standard part of a function $[0,T] \to \mathbb{R}^d$ which is right-continuous with left limits coincides with the topological standard part of that function, both in the Skorokhod J_1 topology (cf. Hoover and Perkins [**44**]) and in the Kolmogorov metric (cf. Stroyan and Bayod [**69**]).

Theorem 4.4. $(p_A, p_C, z, c) := ({}^\circ\hat{p}_A, {}^\circ\hat{p}_C, {}^\circ\hat{z}, {}^\circ\hat{c})$ *is a continuous-strategy (securities-market) equilibrium for the standard continuous-time economy (e, A, φ, ρ, x).*

In the following, we shall denote by $\tilde{A} := (A_1,\ldots,A_J)$ the vector of risky assets.

Proof of Theorem 4.4. The internal market clearing condition ensures that $c(\cdot,u) = 1$ for all $u \in \mathbb{T}\setminus\{T\}$, therefore, we also get

$$\forall t \in [0,T] \qquad c(\cdot,t) = 1.$$

Moreover, by the choice of the equilibrium endowment and dividend processes $\hat{e}, \hat{A}$ in Theorem 4.1 and the fact that $X(\cdot,T) \simeq x(\cdot,T)$ (due to Lindstrøm [**51**, Lemma 6.4]), we know that $\hat{e}(\cdot,T) \simeq \rho\left(x\left(\cdot,T\right)\right)$ and $\hat{A}_j(\cdot,T) \simeq \exp\left(x_j(\cdot,T)\right)$, hence by market clearing

$$c(\cdot,T) \simeq \hat{c}(\cdot,T) = \rho\left(x\left(\cdot,T\right)\right) + \sum_{k=1}^{J}\exp\left(x_k(\cdot,T)\right).$$

On the other hand, by Loeb integration theory, we have $U(c) \simeq \hat{U}\left(\hat{c}\right)$, so

$$\hat{U}\left(\hat{c}\right) \;\simeq\; U(c) = T\varphi_1(1) + \mathbb{E}\left[\varphi_2\left(\rho\left(x\left(\cdot,T\right)\right) + \sum_{k=1}^{J}\exp\left(x_k(\cdot,T)\right)\right)\right]$$

which is $> -\infty$ since x is exp-integrable.

Furthermore, we have

$$^\circ\int_0^t \hat{z}\ \mathrm{d}\hat{p}_A = \mathbf{1}\cdot {}^\circ\hat{p}_A(\cdot,t) - \mathbf{1}\cdot {}^\circ\hat{p}_A(\cdot,0) = \mathbf{1}\cdot p_A(\cdot,t) - \mathbf{1}\cdot p_A(\cdot,0) = \int_0^t z\ \mathrm{d}p_A$$

for all $t \in [0,T]$ on a set of Loeb probability 1 (viz. where $\hat{p}_A$ has a right standard part). Hence,

$$\int_0^t z\ \mathrm{d}p_A - p_A(\cdot,t)z(\cdot,t) = {}^\circ\int_0^t \hat{z}\ \mathrm{d}\hat{p}_A - {}^\circ\hat{p}_A(\cdot,t)^\circ\hat{z}(\cdot,t)$$

for all $t \in [0,T]$ on a set with Loeb probability 1. Also, by Loeb integration theory, we have

$$\forall t \in \mathbb{T} \quad \int_0^{^\circ t} p_C(\cdot,s)\left(e(\cdot,s) - c(\cdot,s)\right)\ \mathrm{d}s = \sum_{u<t} \hat{p}_C(\cdot,u)\left(\hat{e}(\cdot,u) - \hat{c}(\cdot,u)\right)\ \Delta t$$

on a set of Loeb probability 1.

Now, since $\hat{c}$ satisfies the internal budget constraint generated by $\hat{z}$, taking the right standard part of the internal budget equation proves that c must satisfy the budget constraint generated by z. Also, note that the securities and consumption goods markets clear for the economy (p_A, p_C, z, c).

Therefore, if (p_A, p_C, z, c) was not an equilibrium then c could not maximize U over the budget set. Hence, there would be an admissible trading strategy y with continuous paths and a consumption plan c' in the budget set generated by y such that $U(c') > U(c)$.

Since c' is in the domain of U, we must have $c' \in L^1(\lambda \otimes \mathsf{L}(P))$; furthermore, c' is continuous and adapted. Therefore, c' has a 1-lifting $\bar{c}'$ (by Anderson [**6**, Lemma 31]). For all $\varepsilon \in {}^*\mathbb{R}_{>0}$, let $\bar{c}'_\varepsilon := \max\{\varepsilon, \bar{c}'\}$.

By finding estimates on $\varphi_1(\bar{c}'_\varepsilon(\omega,t))$ and $\varphi_2(\bar{c}'_\varepsilon(\omega,T))$ for all $t \in [0,T]$ and almost all ω, one can prove, via Lebesgue's Dominated Convergence Theorem that ${}^\circ\hat{U}(\bar{c}'_\varepsilon) \longrightarrow U(c')$ as $\varepsilon \to 0$ in $\mathbb{R}_{>0}$. Hence, there exists some $\varepsilon' \simeq 0$ such that ${}^\circ\hat{U}(\bar{c}'_\varepsilon) = U(c')$. Define $\hat{c}' := \bar{c}'_{\varepsilon'}$. It is clear that then $\hat{c}'$ also is a 1-lifting of c'. Hence

$$(9)\quad \forall t \in \mathbb{T} \quad \int_0^{^\circ t} p_C(\cdot,s)\left(e(\cdot,s) - c'(\cdot,s)\right)\ \mathrm{d}s \simeq \sum_{u<t} \hat{p}_C(\cdot,u)\left(\hat{e}(\cdot,u) - \hat{c}'(\cdot,u)\right)\ \Delta t$$

on a set of Loeb probability 1.

Based on y, we next choose some process $\hat{y}$ such that $\hat{c}'$ violates the budget constraint generated by that process $\hat{y}$ merely infinitesimally (at all times $t \in \mathbb{T}$).

Since y is continuous by assumption, it must have an S-continuous lifting. Let $\hat{y}$ be such a lifting. In order to prove that $\hat{y}$ finances $\hat{c}'$ up to an infinitesimal at all internal times $t \in \mathbb{T}$, first note that $y(\cdot, {}^\circ u) = {}^\circ(\hat{y}(\cdot,u))$ for all $u \in \mathbb{T}$ (as $\hat{y}$ is S-continuous). Second, since $\hat{p}_A$ is a right-continuous martingale and y is square-integrable with respect to the Doléans measure of p_A, the S-continuous lifting $\hat{y}$ must even be a 2-lifting of y with respect to p_A. Thus, the SL^2 theory of stochastic integration (cf. Lindstrøm [**50**, Theorem 17]) yields ${}^\circ\int_0^t \hat{y}\ \mathrm{d}\hat{p}_A = \int_0^t y\ \mathrm{d}p_A$ for all $t \in [0,T]$. Hence, the right standard part of $\hat{p}_A\hat{y} - \int \hat{y}\mathrm{d}\hat{p}_A$ is $p_A y - \int y\ \mathrm{d}p_A$. On the other hand, by Lemma 4.1, the internal process $\hat{p}_A\hat{y} - \int \hat{y}\mathrm{d}\hat{p}_A$ must be S-continuous.

These deliberations yield that with Loeb probability 1,

$$\forall t \in \mathbb{T} \quad \hat{p}_A(\cdot,t)\hat{y}(\cdot,t) - \int_0^t \hat{y}\mathrm{d}\hat{p}_A \simeq p_A(\cdot, {}^\circ t)\, y(\cdot, {}^\circ t) - \int_0^{^\circ t} y\ \mathrm{d}p_A.$$

Combining this result with Equation (9) and the assumption that c' satisfies the budget constraint generated by y, we get that indeed with Loeb probability 1, the consumption plan $\hat{c}'$ violates the budget constraint generated by $\hat{y}$ at all times at most by an infinitesimal.

Since the time line $\mathbb{T}$ is hyperfinite, the maximal amount over time by which this violation occurs is infinitesimal, too. Put formally, if we define f by

$$f(\omega) \quad := \quad \max_{t\in\mathbb{T}} \begin{pmatrix} \hat{p}_A(\omega,t)\hat{y}(\omega,t) - \int_0^t \hat{y}(\omega,u)\mathrm{d}\hat{p}_A(\omega,u) - \mathbf{1}\cdot\hat{p}_A(0) \\ -\int_0^t \hat{p}_C(\omega,u)\left(\hat{e}(\omega,u)-\hat{c}'(\omega,u)\right)\,\mathrm{d}u \end{pmatrix}$$

for every $\omega\in\Omega$, then $f\simeq 0$ with Loeb probability 1.

Consider next the function g defined by

$$\forall\omega\in\Omega \quad g(\omega) := \frac{1}{1+\sum_{u\in\mathbb{T}}\hat{e}(\omega,u)} \min_{j\in\{0,\ldots,J\}} \frac{\min_{t\in\mathbb{T}}\left(\hat{p}_{A_j}(\omega,t)\wedge\hat{p}_C(\omega,t)\right)}{\max_{t\in\mathbb{T}}\left(\hat{p}_{A_j}(\omega,t)\vee\hat{p}_C(\omega,t)\right\}}.$$

By Equation (2), the second factor in this equation is non-infinitesimal with Loeb probability 1, and the first factor is Loeb almost surely non-infinitesimal as X is a hyperfinite Lévy process (and therefore is almost surely finite). Hence, g is non-infinitesimal with Loeb probability 1.

Thus, there exists some infinitesimal $\delta>0$ such that both $|f|<\delta$ and $g>\sqrt{\delta}$ with Loeb probability 1. (For, the set $\{\delta\in{}^*\mathbb{R}_{>0} \;:\; P\{|f|\geq\delta\}\leq\delta\}$ is internally defined, hence internal, and contains all positive reals, hence it must contain a positive infinitesimal hyperreal δ as well. But on the other hand, ${}^\circ P\left[\left\{g\leq\sqrt{\delta}\right\}\right]\leq \mathsf{L}(P)\{g\simeq 0\}=0$. Therefore, the internal set $\Omega'=\{|f|<\delta\}\cap\left\{g>\sqrt{\delta}\right\}$ satisfies $\mathsf{L}(P)[\Omega']=1$.)

In order to produce a contradiction, we shall now construct a consumption plan $\hat{\hat{c}}$ and an admissible trading strategy $\hat{\hat{y}}$ that finances it such that $\hat{U}\left(\hat{\hat{c}}\right)>\hat{U}\left(\hat{c}\right)$. $\hat{\hat{y}}$ will be the following modification of $\hat{y}$: We put $\hat{\hat{y}}_j=\hat{y}_j$ for every $j\geq 1$, and for all $\omega\in\Omega$, we set $\hat{\hat{c}}(\omega,t)=0$ and invest the resulting savings on consumption into the bond (formally, $\hat{\hat{y}}_0$ will be recursively defined via the budget constraint), until the first $t=t_1(\omega)$ such that $\hat{p}_{A_0}\left(\hat{\hat{y}}_0-\hat{y}_0\right)(\omega,t)\geq\sqrt{\delta}$.

For all $\omega\in\Omega$ and $t>t_1(\omega)$, we put $\hat{\hat{c}}(\omega,t)=\hat{c}'(\omega,t)$ and $\hat{\hat{y}}_0(\omega,t)=\hat{y}_0(\omega,t)$, and the vector $\hat{\hat{y}}_{\tilde{A}}(\omega,t)$ is set at whatever value is then needed to finance the consumption.

Define $\hat{\hat{c}}$ and $\hat{\hat{y}}$ in this way for all t, given ω, starting from $t_1(\omega)$ until the first time $t=t_2(\omega)$ for which these formulae would yield $\hat{p}_A\cdot\hat{\hat{y}}(\omega,t)<\hat{p}_A\cdot\hat{y}(\omega,t)$.

For all $t\geq t_2(\omega)$, put $\hat{\hat{y}}(\omega,t)=\hat{y}(\omega,t)$ and let $\hat{\hat{c}}$ be chosen, recursively in t, in such a way that the budget constraint will be met, i.e. the consumption $\hat{\hat{c}}$ will be reduced such that for all $t\geq t_2(\omega)$,

$$\hat{p}_A(\omega,t)\cdot\hat{y}(\omega,t)-\mathbf{1}\cdot\hat{p}_A(0)=\sum_{u<t}\hat{p}_C\left(\hat{e}-\hat{\hat{c}}\right)(\omega,u)\Delta t+\hat{\hat{y}}\cdot\Delta\hat{p}_A(\omega,u).$$

For all (ω, t) with $t < t_2(\omega)$, the choice of $\hat{\hat{c}}$ yields, via the budget constraint for $\hat{\hat{c}}$ and $\hat{\hat{y}}$, the following equations:

$$\begin{aligned}
\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) &= -\hat{p}_A \cdot \hat{y}(\omega, t) + \mathbf{1} \cdot \hat{p}_A(0) \\
&\quad + \sum_{t_1(\omega) \le u < t} \hat{p}_C \left(\hat{e} - \hat{\hat{c}}\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&\quad + \sum_{u < t_1(\omega)} \hat{p}_C \left(\hat{e} - \hat{\hat{c}}\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&= -\hat{p}_A \cdot \hat{y}(\omega, t) + \mathbf{1} \cdot \hat{p}_A(0) \\
&\quad + \sum_{t_1(\omega) \le u < t} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&\quad + \sum_{u < t_1(\omega)} \hat{p}_C \hat{e}(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u)
\end{aligned}$$

By the choice of $t_2(\omega)$, this means that for all (ω, t) with $t_1(\omega) < t < t_2(\omega)$, one has

$$\begin{aligned}
\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) &\ge -\hat{p}_A \cdot \hat{\hat{y}}(\omega, t) + \mathbf{1} \cdot \hat{p}_A(0) \\
&\quad + \sum_{t_1(\omega) \le u < t} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&\quad + \sum_{u < t_1(\omega)} \hat{p}_C \hat{e}(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&\ge -\hat{p}_A \cdot \hat{\hat{y}}(\omega, t) + \mathbf{1} \cdot \hat{p}_A(0) \\
&\quad + \sum_{t_1(\omega) \le u < t} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u) \\
&\quad + \sum_{u < t_1(\omega)} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{\hat{y}} \cdot \Delta \hat{p}_A(\omega, u).
\end{aligned}$$

Finally, the construction of $\hat{\hat{y}}, \hat{\hat{c}}$ ensures that $\hat{c}'$ never violates the budget constraint generated by $\hat{\hat{y}}$ as much as the budget constraint generated by $\hat{y}$. Therefore, we can find, for all (ω, t) with $t_1(\omega) < t < t_2(\omega)$, the following lower bound for $\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t)$:

$$\begin{aligned}
\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) &\ge -\hat{p}_A \cdot \hat{y}(\omega, t) + \mathbf{1} \cdot \hat{p}_A(0) \\
&\quad + \sum_{t_1(\omega) \le u < t} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{y} \cdot \Delta \hat{p}_A(\omega, u) \\
&\quad + \sum_{u < t_1(\omega)} \hat{p}_C \left(\hat{e} - \hat{c}'\right)(\omega, u)\Delta t + \hat{y} \cdot \Delta \hat{p}_A(\omega, u) \\
&\ge -f(\omega) \qquad (10)
\end{aligned}$$

Since t_1 and t_2 are internal stopping times, $\hat{\hat{y}}$ is an admissible internal trading strategy. Note that $t_1(\omega) \simeq 0$ for almost all ω due to Lemma 4.4. Moreover, one can prove that $t_2(\omega) > T$ for almost all ω. (For, if $\omega \in \Omega'$ and $t \ge t_1(\omega)$, then inequality (10) yields first of all $-\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) \le f(\omega) < \delta \le g(\omega)\sqrt{\delta} \le g(\omega)\hat{p}_{A_0} \cdot \left(\hat{\hat{y}}_{A_0} - \hat{y}_{A_0}\right)(\omega, t_1(\omega)) = g(\omega)\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t_1(\omega))$. However, if in addition $\left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) < 0$, this would mean that

$$\begin{aligned}
&-\frac{\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t_1(\omega))}{\hat{p}_A \cdot \left(\hat{\hat{y}} - \hat{y}\right)(\omega, t)} \\
&> \left(1 + \sum_{u \in \mathbb{T}} \hat{e}(\omega, u)\right) \max_{j \in \{0, \ldots, J\}} \frac{\max_{t \in \mathbb{T}} \left(\hat{p}_{A_j}(\omega, t) \vee \hat{p}_C(\omega, t)\right)}{\min_{t \in \mathbb{T}} \left(\hat{p}_{A_j}(\omega, t) \wedge \hat{p}_C(\omega, t)\right)}.
\end{aligned}$$

Hence, the negative relative portfolio loss generated by the trading strategy $\left(\hat{\hat{y}} - \hat{y}\right)$ would be more than the maximal relative gain from investing while reinvesting all endowments. This is a contradiction. Therefore, $\left(\hat{\hat{y}} - \hat{y}\right)(\omega, t) \not\lesssim 0$ for all $\omega \in \Omega'$, $t > t_1(\omega)$. Since $\mathsf{L}(P)[\Omega'] = 1$, we arrive at $t_2 > T$ with $\mathsf{L}(P)$-probability 1.)

This shows that $\hat{\hat{c}}(\omega, t) = \hat{c}'(\omega, t)$ for almost every (ω, t). Since φ_1 and φ_2 are bounded below, ${}^\circ\hat{U}\left(\hat{\hat{c}}\right) = {}^\circ\hat{U}\left(\hat{c}'\right) = U\left(c'\right) > U(c) = {}^\circ\hat{U}\left(\hat{c}\right)$, which contradicts the choice of $(\hat{p}_A, \hat{p}_C, \hat{z}, \hat{c})$ as an internal securities-market equilibrium of the hyperfinite economy. □

5. Discussion

Theorem 3.1 has interesting consequences for the foundations of Lévy finance. It implies that even for a model as simple as ours, the resulting asset-price process will never be an exponential Lévy process (i.e. the composition of exp and some Lévy process) or the stochastic exponential (cf. Doléans-Dade [**29**], see also Applebaum [**11**]) of a Lévy process. This is the thrust of the following Remark 5.1, which generalizes the findings of Raimondo [**64**, Remark 1, p. 273] to the case of general exponential Lévy (not just geometric Brownian) dividends:

Remark 5.1. *Fix $j \in \{1, \ldots, J\}$. From Theorem 3.1 one can derive that the discounted equilibrium price of asset j, viz. the ratio*

$$\frac{p_{A_j}}{p_{A_0}} = \frac{{}^\circ\hat{p}_{A_j}}{{}^\circ\hat{p}_{A_0}} : (\omega, t) \mapsto \exp\left(x_j(\omega, t)\right) \frac{\int \varphi_2'\left(R\left(x\left(\omega, t\right) + z\right)\right) \mathrm{e}^{z_j} \mathbb{P}_{x(\cdot, T-t)}(\mathrm{d}z)}{\int \varphi_2'\left(R\left(x\left(\omega, t\right) + z\right)\right) \mathbb{P}_{x(\cdot, T-t)}(\mathrm{d}z)}$$

will generically be neither an exponential Lévy process nor the stochastic exponential of a Lévy process. Essentially, the only exception is the special arrangement of primitives of the economy where all of the following propositions are true:

1. $\varphi_2 : c \mapsto \gamma c^\alpha$ *for some* $\gamma > 0$, $\alpha \in (0, 1)$
2. $\rho = 0$, *i.e.* $R : z \mapsto \sum_{k=1}^J \mathrm{e}^{z_j}$,
3. $d = 1$.
4. x *is a constant multiple of one-dimensional Brownian motion.*

If these conditions hold, then $\frac{p_{A_j}}{p_{A_0}}$ will be the exponential martingale corresponding to x (geometric Brownian motion with drift equal to the negative halved square of the diffusion coefficient).

If one of these four conditions fails, then $\frac{p_{A_j}}{p_{A_0}}$ will be neither an exponential Lévy process nor the stochastic exponential of a Lévy process, except perhaps under some knife-edge circumstances which make the correction factor cancel out for all $t \in [0, T]$.

The second and third condition in Remark 5.1 can be summarized as $R = \exp : \mathbb{R} \to \mathbb{R}_{>0}$. The first condition implies that the felicity function φ_2 exhibits Constant Relative Risk Aversion. The second condition means that there is no endowment at the terminal date.

In this sense, the pricing formulae in Theorem 3.1 relate the shape of the representative agent's utility function and the dynamics of the asset price process for models with general log-Lévy dividends, and thus substantially generalize the results of He and Leland [**38**] and Raimondo [**64**].

Proof of Remark 5.1. Note that Raimondo [**64**, Remark 1, p. 273] has already argued, albeit merely in the case of geometric Brownian dividends, that only if the first three conditions are satisfied, there is any hope to simplify $\frac{p_{A_j}}{p_{A_0}}$. Exactly the same reasoning, however, applies to our setting. Hence, we may assume that the first three conditions are satisfied.

Under this assumption, $\frac{p_{A_1}}{p_{A_0}}$ can be written as the product of an exponential Lévy process and a deterministic function of the time argument:

$$\begin{aligned}\frac{p_{A_1}}{p_{A_0}}:(\omega,t)\mapsto \quad & \exp\left(x_1(\omega,t)\right)\frac{\int\varphi_2'\left(R\left(x\left(\omega,t\right)+z\right)\right)\mathrm{e}^{z_1}\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}{\int\varphi_2'\left(R\left(x\left(\omega,t\right)+z\right)\right)\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}\\ &=\exp\left(x(\omega,t)\right)\frac{\int\gamma\alpha\exp\left(\left(\alpha-1\right)\left(x\left(\omega,t\right)+z\right)\right)\mathrm{e}^{z}\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}{\int\gamma\alpha\exp\left(\left(\alpha-1\right)\left(x\left(\omega,t\right)+z\right)\right)\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}\\ &=\exp\left(x(\omega,t)\right)\frac{\int\mathrm{e}^{\alpha z}\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}{\int\mathrm{e}^{(\alpha-1)z}\mathbb{P}_{x(\cdot,T-t)}(\mathrm{d}z)}\\ &=\exp\left(x(\omega,t)\right)\frac{\mathbb{E}\left[\exp\left(\alpha x(\cdot,T-t)\right]\right.}{\mathbb{E}\left[\exp\left((\alpha-1)x(\cdot,T-t)\right]\right.}.\end{aligned}$$

Therefore, $\frac{p_{A_1}}{p_{A_0}}$ can only be an exponential Lévy process or the stochastic exponential of a Lévy process if the second — deterministic — factor $\kappa(t):=\frac{\mathbb{E}[\exp(\alpha x(\cdot,T-t)]}{\mathbb{E}[\exp((\alpha-1)x(\cdot,T-t)]}$ is the exponential of an affine function of the time argument t.

However, the factor $\kappa(t)$ can only be simplified further if x is a stable process. But even for rotationally stable Lévy processes (of Hurst index H, say), we have $\mathbb{E}\left[\exp\left(\delta x(\cdot,T-t)\right]=\mathbb{E}\left[\exp\left(\delta\left(T-t\right)^H x(\cdot,1)\right)\right]\right.$ (for every $\delta>0$), which — in light of the Lévy-Khintchine formula (cf. Applebaum [**11**] or Sato [**66**]) — will only be the exponential of an affine function of t if $H=1/2$, i.e. if x is a constant multiple of Brownian motion. Hence $\frac{p_{A_1}}{p_{A_0}}$ will essentially only be an exponential Lévy process or the stochastic exponential of a Lévy process if — in addition to the first three conditions — the Lévy process x is just a constant multiple of Brownian motion. □

In the introduction, it was already mentioned that nonstandard analysis has been used fruitfully in both equilibrium theory and mathematical finance for several decades. This success in economic applications notwithstanding, popular opinion used to view, until less than five years ago, nonstandard analysis as an intrinsically non-constructive tool, due to its heavy dependence on a non-principal ultrafilter. For, although the ultrafilter existence theorem does not imply the Axiom of Choice (cf. Halpern and Lévy [**37**], see also Banaschewski [**12**]), it does entail the existence of non-Lebesgue measurable sets (cf. e.g. Luxemburg [**54**]) and therefore is independent of Zermelo-Fraenkel set theory without the Axiom of Choice (cf. Solovay [**68**]). However, recent research in mathematical logic has finally established the existence of *definable* nonstandard universes, under the assumption of Zermelo-Fraenkel set theory plus Axiom of Choice (cf. Kanovei and Shelah [**46**], Kanovei and Reeken [**45**], Herzberg [**39, 40**]).

Appendix A. Bounded scaled risk aversion

LEMMA A.1. *Suppose $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is twice continuously differentiable, strictly increasing, strictly concave, bounded from below and* $\sup_{c\in(0,1]} -c^q \frac{\varphi''(c)}{\varphi'(c)} < +\infty$ *for some $q \geq 1$. Then there exist $\gamma, r \in \mathbb{R}$ such that*

$$\forall c \in (0,1] \qquad \varphi'(c) \leq \frac{\gamma}{c^r}$$

PROOF BY CONTRAPOSITION. Suppose that $\{\varphi'(c)\, c^r \ : \ c \in (0,1]\}$ is unbounded for all $r \in \mathbb{R}$. Define, for all $c_0 > 0$, the maximum of 1 and the maximal relative risk aversion on $[c_0, 1]$ by

$$r(c_0) := \max_{c\in[c_0,1]} -c\frac{\varphi''(c)}{\varphi'(c)} \vee 1.$$

Fix now $c_0 > 0$ and consider the function $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined by

$$\forall c > 0 \qquad \psi(c) := \varphi'(c) c^{r(c_0)}.$$

Since its derivative is given by

$$\forall c > 0 \qquad \psi'(c) = c^{r(c_0)-1}\left(r(c_0)\,\varphi'(c) + c\varphi''(c)\right),$$

one has $\psi'(c) \geq 0$ if and only if $r(c_0) \geq -c\frac{\varphi''(c)}{\varphi'(c)}$. The latter estimate holds for all $c \in [c_0, 1]$, therefore ψ is increasing on $[c_0, 1]$. Hence

$$\forall c \in [c_0,1] \qquad \varphi'(c_0)\, c_0^{r(c_0)} \leq \varphi'(c) c^{r(c_0)} \leq \varphi'(1). \tag{11}$$

Note that r is decreasing. Furthermore, we may assume that $r(c) \uparrow \infty$ as $c \downarrow 0$, for otherwise we would get

$$\forall c \in (0,1] \qquad \varphi'(c) c^{\sup r} \leq \varphi'(c) c^{r(c)} \leq \varphi'(1),$$

whence estimate (1) would already be established.

Now, by estimate (11), we get $\varphi'(c) \geq \varphi'(c_0)\left(\frac{c_0}{c}\right)^{r(c_0)}$ for all $c \in [c_0, 1]$, hence for all sufficiently small c_0 (such that $r(c_0) > 1$), we obtain

$$\begin{aligned}
\varphi(c_0) - \varphi(1) &= -\int_{c_0}^{1} \varphi'(c)\,\mathrm{d}c \\
&\leq -\int_{c_0}^{1} \varphi'(c_0)\left(\frac{c_0}{c}\right)^{r(c_0)} \mathrm{d}c = -\varphi'(c_0)\, c_0^{r(c_0)} \left.\frac{c^{1-r(c_0)}}{1-r(c_0)}\right|_{c=c_0}^{c=1} \\
&= \frac{\varphi'(c_0)\, c^{r(c_0)}}{r(c_0)-1} - \frac{\varphi'(c_0)\, c_0}{r(c_0)-1} \leq \frac{\varphi'(1)}{r(c_0)-1} - \frac{\varphi'(c_0)\, c_0^q}{c_0^{q-1} r(c_0) - c_0^{q-1}}
\end{aligned}$$

But since $q \geq 1$, we may calculate

$$c_0^{q-1} r(c_0) - c_0^{q-1} \leq c_0^{q-1} r(c_0) = \max_{c\in[c_0,1]} -c_0^{q-1} c\frac{\varphi''(c)}{\varphi'(c)} \leq \sup_{c\in(0,1]} -c^q \frac{\varphi''(c)}{\varphi'(c)},$$

so

$$\varphi(c_0) - \varphi(1) \quad \leq \quad \frac{\varphi'(1)}{r(c_0)-1} - \frac{\varphi'(c_0)\, c_0^q}{\sup_{c\in(0,1]} -c^q \frac{\varphi''(c)}{\varphi'(c)}}.$$

Note that $\frac{\varphi'(1)}{r(c_0)-1} \longrightarrow 0$ as $c_0 \downarrow 0$, whilst $\{\varphi'(c)\, c^q \ : \ c \in (0,1]\}$ is assumed to be unbounded and $\sup_{c\in(0,1]} -c^q \frac{\varphi''(c)}{\varphi'(c)}$ is finite. Therefore, the last estimate shows that the set $\{\varphi(c_0) - \varphi(1) \ : \ c_0 \in (0,1]\}$ is not bounded from below. Hence, the function φ cannot be bounded from below either, a contradiction. □

References

[1] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, and T. Lindstrøm. *Nonstandard methods in stochastic analysis and mathematical physics*. Pure and Applied Mathematics. 122. Orlando, FL: Academic Press, 1986.

[2] S. Albeverio and F.S. Herzberg. A combinatorial infinitesimal representation of Lévy processes and an application to incomplete markets. *Stochastics*, 78(5):301–325, 2006.

[3] S. Albeverio and F.S. Herzberg. Lifting Lévy processes to hyperfinite random walks. *Bulletin des Sciences Mathématiques*, 130(8):697–706, 2006.

[4] A. Almendral and C. Oosterlee. Accurate evaluation of European and American options under the CGMY process. *SIAM Journal on Scientific Computing*, 29(1):93–117, 2007.

[5] A. Almendral and C. Oosterlee. On American options under the variance gamma process. *Applied Mathematical Finance*, 14(2):131–152, 2007.

[6] R.M. Anderson. A non-standard representation for Brownian motion and Itô integration. *Israel Journal of Mathematics*, 25(1-2):15–46, 1976.

[7] R.M. Anderson. Non-standard analysis with applications to economics. In W. Hildenbrand and H. Sonnenschein, editors, *Handbook of mathematical economics. Volume IV*, pages 2145–2208. Handbooks in Economics. Amsterdam: North-Holland, 1991.

[8] R.M. Anderson and R.C. Raimondo. Market clearing and derivative pricing. *Economic Theory*, 25:21–34, 2005.

[9] R.M. Anderson and R.C. Raimondo. Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets. *Econometrica*, forthcoming, 2008.

[10] D. Applebaum. Lévy processes—from finance to probability and quantum groups. *Notices of the American Mathematical Society*, 51(11):1336–1347, 2004.

[11] D. Applebaum. *Lévy processes and stochastic calculus*. Cambridge Studies in Advanced Mathematics. 93. Cambridge, UK: Cambridge University Press, 2004.

[12] B. Banaschewski. The power of the ultrafilter theorem. *Journal of the London Mathematical Society. Second Series*, 27(2):193–202, 1983.

[13] O.E. Barndorff-Nielsen. Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, 24(1):1–13, 1997.

[14] O.E. Barndorff-Nielsen, T. Mikosch, and S.I. Resnick, editors. *Lévy processes: Theory and applications*. Boston, MA: Birkhäuser, 2001.

[15] A. Bick. On viable diffusion price processes of the market portfolio. *Journal of Finance*, 45(2):673–689, 1990.

[16] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[17] S.I. Boyarchenko and S.Z. Levendorskiĭ. *Non-Gaussian Merton-Black-Scholes theory*. Singapore: World Scientific, 2002.

[18] D.J. Brown and A. Robinson. Nonstandard exchange economies. *Econometrica*, 43:41–55, 1975.

[19] P. Carr, H. Geman, D.B. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75(2):305–333, 2002.

[20] P. Carr, H. Geman, D.B. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13(3):345–382, 2003.

[21] T. Chan. Pricing contingent claims on stocks driven by Lévy processes. *Annals of Applied Probability*, 9(2):504–528, 1999.

[22] J.C. Cox, J.E. Ingersoll, and S.A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384, 1985.

[23] N.J. Cutland, P.E. Kopp, and W. Willinger. A nonstandard approach to option pricing. *Mathematical Finance*, 1(4):1–38, 1991.

[24] N.J. Cutland, P.E. Kopp, and W. Willinger. A nonstandard treatment of options driven by Poisson processes. *Stochastics and Stochastics Reports*, 42(2):115–133, 1993.

[25] N.J. Cutland, P.E. Kopp, and W. Willinger. From discrete to continuous financial models: New convergence results for option pricing. *Mathematical Finance*, 3(2):101–123, 1993.

[26] N.J. Cutland, P.E. Kopp, and W. Willinger. From discrete to continuous stochastic calculus. *Stochastics and Stochastics Reports*, 52(3-4):173–192, 1995.

[27] N.J. Cutland, P.E. Kopp, W. Willinger, and M.C. Wyman. Convergence of Snell envelopes and critical prices in the American put. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of derivative securities*, pages 126–140. Publications of the Newton Institute. 15. Cambridge, UK: Cambridge University Press, 1997.

[28] C. Doléans. Existence du processus croissant naturel associé à un potentiel de la classe (D). *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 9(4):309–314, 1968.

[29] C. Doléans-Dade. Quelques applications de la formule de changement de variables pour les semi-martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 16(3):181–194, 1970.

[30] C. Duffie, D. Skiadas. Continuous-time security pricing. *Journal of Mathematical Economics*, 23:107–131, 1994.

[31] D. Duffie and W. Shafer. Equilibrium in incomplete markets. I. *Journal of Mathematical Economics*, 14:285–300, 1985.

[32] D. Duffie and W. Shafer. Equilibrium in incomplete markets. II. *Journal of Mathematical Economics*, 15:199–216, 1986.

[33] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[34] E. Eberlein, W. Kluge, and A. Papapantoleon. Symmetries in Lévy term structure models. *International Journal of Theoretical and Applied Finance*, 9(6):967–986, 2006.

[35] S. Fajardo and H.J. Keisler. *Model theory of stochastic processes.* Lecture Notes in Logic. 14. Urbana, IL: Association for Symbolic Logic. Natick, MA: A.K. Peters, 2002.

[36] D. Filipović and S. Tappe. Existence of Lévy term structure models. *Finance and Stochastics*, 12(1):83–115, 2008.

[37] J.D. Halpern and A. Levy. The Boolean prime ideal theorem does not imply the axiom of choice. In *Axiomatic Set Theory. Proceedings of Symposia in Pure Mathematics. XIII. Part 1*, pages 83–134. Providence, RI: American Mathematical Society, 1971.

[38] H. He and H. Leland. An intertemporal general equilibrium model of asset prices. *Review of Financial Studies*, 6(3):593–617, 1993.

[39] F.S. Herzberg. A definable nonstandard enlargement. *Mathematical Logic Quarterly*, 54(2):167–175, 2008.

[40] F.S. Herzberg. Addendum to “A definable nonstandard enlargement”. *Mathematical Logic Quarterly*, to appear, 2008.

[41] F.S. Herzberg. Linear hyperfinite Lévy integrals. *Tbilisi Mathematical Journal*, submitted, 2008.

[42] F.S. Herzberg. Perpetual Bermudan continuity corrections and a multi-dimensional Wiener-Hopf type result. *Stochastic Analysis and Applications*, to appear, 2008.

[43] F.S. Herzberg and T. Lindstrøm. Erratum and addendum to “Hyperfinite Lévy processes” (*Stochastics and Stochastics Reports*, 76(6):517–548, 2004). *Stochastics*, submitted, 2008.

[44] D.N. Hoover and E. Perkins. Nonstandard construction of the stochastic integral and applications to stochastic differential equations. I. *Transactions of the American Mathematical Society*, 275:1–29, 1983.

[45] V. Kanovei and M. Reeken. *Nonstandard analysis, axiomatically.* Springer Monographs in Mathematics. Berlin: Springer, 2004.

[46] V. Kanovei and S. Shelah. A definable nonstandard model of the reals. *Journal of Symbolic Logic*, 69(1):159–164, 2004.

[47] H.J. Keisler. Getting to a competitive equilibrium. *Econometrica*, 64(1):29–49, 1996.

[48] M.A. Khan and Y. Sun. The capital asset-pricing model and arbitrage pricing theory: A unification. *Proceedings of the National Academy of Sciences of the United States of America*, 94(8):4229–4232, 1997.

[49] M.A. Khan and Y. Sun. Asymptotic arbitrage and the APT with or without measure-theoretic structures. *Journal of Economic Theory*, 101(1):222–251, 2001.

[50] T. Lindstrøm. Hyperfinite stochastic integration. II: Comparison with the standard theory. *Mathematica Scandinavica*, 46(2):293–314, 1980.

[51] T. Lindstrøm. Hyperfinite Lévy processes. *Stochastics and Stochastics Reports*, 76(6):517–548, 2004.

[52] T. Lindstrøm. Nonlinear stochastic integrals for hyperfinite Lévy processes. *Logic and Analysis*, 1(2):91–129, 2008.

[53] P.A. Loeb. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Transactions of the American Mathematical Society*, 211:113–122, 1975.

[54] W.A.J. Luxemburg. What is nonstandard analysis? *American Mathematical Monthly*, 80 (Supplement)(1):38–67, 1973.

[55] D.B. Madan and E. Seneta. Chebyshev polynomial approximations and characteristic function estimation. *Journal of the Royal Statistical Society. Series B. Methodological*, 49(2):163–169, 1987.

[56] D.B. Madan and E. Seneta. Chebyshev polynomial approximations for characteristic function estimation: some theoretical supplements. *Journal of the Royal Statistical Society. Series B. Methodological*, 51(2):281–285, 1989.

[57] D.B. Madan and E. Seneta. The variance gamma (V.G.) model for share market returns. *Journal of Business*, 63(4):511–524, October 1990.

[58] M. Magill and M. Quinzii. *Theory of incomplete markets.* Cambridge, MA: Massachusetts Institute of Technology Press, 1996.

[59] R.C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1-2):125–144, 1976.

[60] M. Métiviér. *Reelle und vektorwertige Quasimartingale und die Theorie der Stochastischen Integration.* Lecture Notes in Mathematics. 607. Heidelberg: Springer, 1977.

[61] S.-A. Ng. *Hypermodels in mathematical finance: Modelling via infinitesimal analysis.* Singapore: World Scientific, 2003.

[62] S.-A. Ng. A nonstandard Lévy-Khintchine formula and Lévy processes. *Acta Mathematica Sinica*, 24(2):241–252, 2008.

[63] R. Radner. Existence of equilibrium of plans, prices and price expectiations in a sequence of markets. *Econometrica*, 40(2):289–303, 1972.

[64] R.C. Raimondo. Market clearing, utility functions, and security prices. *Economic Theory*, 25:265–285, 2005.

[65] A. Robinson. *Non-standard analysis.* Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland, 1966.

[66] K.-I. Sato. *Lévy processes and infintely divisible distributions.* Cambridge, UK: Cambridge University Press, 1999.

[67] W. Schoutens. *Lévy processes in finance: Pricing financial derivatices.* Wiley Series in Probability and Statistics. Chichester, UK: John Wiley & Sons, 2003.

[68] R.M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92(1):1–56, 1970.

[69] K.D. Stroyan and J.M. Bayod. *Foundations of infinitesimal stochastic analysis.* Studies in Logic and the Foundations of Mathematics. 119. Amsterdam: North-Holland, 1986.