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Frederik Herzberg

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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LINEAR HYPERFINITE LÉVY INTEGRALS

Abstract. This article shows that the nonstandard approach to stochastic integration with respect to (C^2 functions of) Lévy processes is consistent with the classical theory of pathwise stochastic integration with respect to (C^2 functions of) jump-diffusions with finite-variation jump part.

It is proven that internal stochastic integrals with respect to hyperfinite Lévy processes possess right standard parts, and that these standard parts coincide with the classical pathwise stochastic integrals, provided the integrator's jump part is of finite variation. If the integrator's Lévy measure is bounded from below, one can obtain a similar result for stochastic integrals with respect to C^2 functions of Lévy processes.

As a by-product, this yields a short, direct nonstandard proof of the generalized Itô formula for stochastic differentials of smooth functions of Lévy processes.

1. Introduction

Stochastic analysis with Lévy-process integrators has received much attention in the past decade, for at least two independent reasons. First, there is the remarkable elegance and methodological richness of the theory of Lévy processes, due to celebrated representation results via infinitesimal generators of space-translation invariant semigroups or Fourier transforms of infinitely divisible distributions (Lévy-Khintchine formulae). The second reason lies in the demand of mathematical finance for an analytic framework to employ jump diffusions in financial modelling (cf. e.g. Barndorff-Nielsen, Mikosch and Resnick [9], Cont and Tankov [12] or Schoutens [28]). There are now numerous expository works on Lévy processes in general (e.g. Bertoin [11] or Sato [27]) and on its relationship with stochastic analysis in particular (cf. Applebaum [7]). See also Applebaum [6] for a survey article.

Recently, some authors have studied Lévy processes by means of Robinsonian nonstandard analysis. Most notable therein is Lindstrøm's theory of *hyperfinite Lévy processes* [21] which has inspired some other papers in this area (e.g. Lindstrøm [22], Albeverio and Herzberg [3], as well as Albeverio, Fan and Herzberg [1]; different approaches to Lévy processes from the vantage point of nonstandard analysis are Albeverio and Herzberg [4] as well as Ng [26]). This approach provides a rigorous framework to treat Lévy processes as if they were random walks; in particular, it entails a canonical definition of the (internal) stochastic integral with a Lévy process as integrator, viz. as a hyperfinite—i.e. formally finite—Riemann-Stieltjes sum.

The classical route to a pathwise definition of the stochastic integral with respect to a Lévy process with finite-variation jump part addresses the diffusion part and the jump part separately with different methods. Whilst the Itô theory is employed for the integral with respect to the diffusion part, an ordinary pathwise Riemann-Stieltjes integral (or, equivalently, integration with respect to a signed measure) consitutes the integral with respect to the jump part (cf. Millar [25]).

The present paper establishes a link between this classical pathwise approach to Lévy stochastic integrals and the aforementioned nonstandard methodology.

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First, we will show that internal stochastic integrals with respect to Lipschitz functions of hyperfinite Lévy processes Z admit a right standard part (Lemma 3.1). Then, given a *generating triplet* of a real-valued Lévy process with finite-variation jump part (i.e. a triple consisting of the drift coefficient, the diffusion coefficient, and the Lévy measure ν, which is assumed to satisfy $\int_{-1}^{1} |x| \ \nu(\mathrm{d}x) < +\infty$), we shall construct its *Lindstrøm lifting* Z as a slight refinement of Lindstrøm's representation theorem [21]. This Z is a particularly simple hyperfinite Lévy process which admits an internal jump-diffusion decomposition, where the internal jump part J can be written as a difference of two increasing hyperfinite Lévy processes. This entails an explicit jump-diffusion decomposition for the standard part ${}^\circ Z$ of Z as well.

The standard part of the internal stochastic integral with respect to J will be shown to coincide pathwise with the jump part of the classical pathwise stochastic integral with respect to ${}^\circ J$ (a consequence of Theorem 5.1). The diffusion part of the internal stochastic integral equals, as was shown as early as Anderson's [5] seminal paper, a path-continuous modification of the Itô integral with respect to the standard diffusion part. Combining the results for the drift and diffusion part, we obtain the right standard part of the internal stochastic integral of Z to be the the classical pathwise stochastic integral with respect to ${}^\circ Z$.

Furthermore, under the assumption that the Lévy measure ν is concentrated on a set that is bounded from below, we will consider the internal integral with respect to twice continuously differentiable functions of Lindstrøm liftings Z. We will prove (in Theorem 6.1) that its standard part equals the stochastic integral with respect to the function of ${}^\circ Z$, when defined via the generalized Itô formula for Lévy integrals (cf. Applebaum [7]). As a by-product of this result, we obtain a short nonstandard proof of this generalized Itô formula (Theorem 6.2).

Hence, the use of Lindstrøm lifings of Lévy processes allows for an intuitive pathwise definition of the stochastic integral for Lévy processes as integrators.

A different route to the characterization of internal stochastic integrals, even with respect to general hyperfinite Lévy processes (rather than reduced liftings), based on SL^2-martingales, has been proposed by Lindstrøm [21, 22]. He proved first that hyperfinite Lévy processes with finite increments can be decomposed into an internal drift part and a hyperfinite martingale part [21, Corollary 2.5] and that hyperfinite Lévy processes have finite increment except for a set of arbitrarily small positive probability. Later, Lindstrøm [22] applies the SL^2-martingale theory of stochastic integration (cf. Lindstrøm [18, 19, 20], Hoover and Perkins [15, 16] and Albeverio *et al.* [2]) to the martingale part. This reflects the methodological choice of important expositions on Lévy stochastic calculus, such as Applebaum's [7], which also base their definition of Lévy stochastic integrals on L^2-martingale theory, since this does not require further restrictions on the Lévy measure. Our approach is on the one hand more restrictive, but on the other hand much more intuitive than SL^2-martingale analysis. Our proofs do not utilize the internal drift-martingale decomposition [22, Corollary 1.7], but they depend on a certain lifting theorem (Theorem 4.5) which assumes that $\int_{-1}^{1} |x| \ \nu(\mathrm{d}x) < +\infty$. Of course, the connection between nonstandard and classical pathwise stochastic integrals is an interesting question in its own right.

The use of nonstandard methods is often dubbed "non-constructive", because it relies on the ultrafilter existence theorem (which is a consequence of the Axiom of Choice, albeit not equivalent to it, cf. Banaschewski [8]). Notwithstanding this, recent research has shown that there do exist definable nonstandard models of the reals and even definable fully-fledged nonstandard universes, cf. Kanovei and Shelah [17] as well as Herzberg [14]. (Herein, "definable" means definable over ZFC, i.e.

Zermelo-Fraenkel set theory with the Axiom of Choice.) The nonstandard world is hence much more accessible than popular opinion assumed only five years ago.

The present paper is organized as follows. Section 2 reviews hyperfinite Lévy processes. In Section 3, we define internal stochastic integrals with respect to Lipschitz continuous functions of hyperfinite Lévy processes and prove that these internal integrals (when viewed as internal stochastic processes) admit a right standard part. Section 4 reviews the Lévy-Khintchine formula and proves the existence of Lindstrøm liftings whenever $\int_{-1}^{1}|x|\ \nu(\mathrm{d}x)<+\infty$. In Section 5, we show that the standard part of stochastic integrals whose integrator is a Lindstrøm lifting coincides with a pathwise definition of the stochastic integral for Lévy processes with a finite-variation jump part. Finally, Section 6 is devoted to stochastic integrals with respect to smooth functions of hyperfinite Lévy processes and to the generalized Itô formula for Lévy processes with finite-variation jump part.

For all of this paper, we fix some hyperfinite probability space (Ω,P). We define a time line by $\mathbb{T}:=\{n\,\Delta t\ :\ n\leq N!\}$, wherein $N\in{}^*\mathbb{N}\setminus\mathbb{N}$ and $\Delta t:=\frac{T}{N!}$ for some $T\in\mathbb{Q}_{>0}$. It follows that $[0,T]\cap\mathbb{Q}\subset\mathbb{T}$.

This induces a standard probability space $\mathsf{L}(\Omega):=\left(\Omega,\sigma\left(2^{\Omega}\right),\mathsf{L}\left(P\right)\right)$, wherein 2^{Ω} denotes the internal algebra of internal subsets of Ω, $\sigma\left(2^{\Omega}\right)$ denotes the smallest σ-algebra containing 2^{Ω}, and $\mathsf{L}(P)$, the *Loeb probability measure* associated with P, is the Carathéodory measure completion of the finitely-additive measure $A\mapsto{}^{\circ}P(A)$ (cf. Loeb [23]).

2. Review of hyperfinite Lévy processes

Let $d\in\mathbb{N}$. Consider an ${}^*\mathbb{R}^d$-valued internal map $X:\Omega\times\mathbb{T}$. For any such map X, we define the *infinitesimal increment operator* Δ by

$$\forall t\in\mathbb{T}\setminus\{T\}\qquad \Delta X_t:=X_{t+\Delta t}-X_t.$$

Next we reproduce Lindstrøm's definition of a hyperfinite Lévy process [21, Definitions 1.1, 1.3]:

2.1. Definition *Let $d\in\mathbb{N}$ and let (Ω,P) be a hyperfinite probability space. An internal map $X:\Omega\times\mathbb{T}\to{}^*\mathbb{R}^d$ is called a* hyperfinite random walk *if and only if there exists a hyperfinite set $A\subset{}^*\mathbb{R}^d$ and a hyperfinite set $\{p_a\}_{a\in A}\subset{}^*\mathbb{R}_{\geq 0}$ such that $\sum_{a\in A}p_a=1$ and X satisfies all of the following properties:*

- $X_0=0$.
- *The internal random variables $\Delta X_0,\ldots,\Delta X_{T-\Delta t}$ are *-independent under P.*
- *For all $t\in\mathbb{T}\setminus\{T\}$, $P\{\Delta X_t=a\}=p_a$.*

A a hyperfinite random walk X is called hyperfinite Lévy process *if* $\mathsf{L}(P)\left[\bigcap_{t\in\mathbb{T}}\{X_t\ \textit{finite}\}\right]=1$.

The two most well-known examples of hyperfinite Lévy processes are Anderson's [5] random walk and Loeb's internal Poisson process [23]. The *reduced lifting* of any given Lévy process, constructed by Albeverio and Herzberg [3], is a particularly simple hyperfinite Lévy process.

Through its right standard part, every hyperfinite Lévy process X gives rise to an ordinary $\mathbb{R}^d$-valued stochastic process on the probability space $\left(\Omega,\sigma\left(2^{\Omega}\right),\mathsf{L}\left(P\right)\right)$ (cf. Lindstrøm [21, Theorem 6.6]). Let us briefly recall how right standard parts are defined (cf. Albeverio *et al.* [2, Definitions 4.2.9, 4.2.11], Lindstrøm [21, Definitions 6.1, 6.2]):

2.2. Definition *Consider an internal function $F:\mathbb{T}\to{}^*\mathbb{R}$. Let $r\in[0,T]$ and $\alpha\in\mathbb{R}$. α is the S-*right limit *(the S-*left limit*, respectively) of F at r if and only if* *for all* $\varepsilon \in \mathbb{R}_{>0}$ *there exists some* $\delta \in \mathbb{R}_{>0}$ *such that for all* $u \in {}^*(r, r+\delta) \cap \mathbb{T}$ *with* $u \not\simeq r$ *(for all* $u \in {}^*(r-\delta, r) \cap \mathbb{T}$ *with* $u \not\simeq r$*, respectively), one has* $|F(u) - \alpha| < \varepsilon$*. In this case, we denote* α *by* $S\text{-}\lim_{s\downarrow r} F(s)$ *(by* $S\text{-}\lim_{s\uparrow r} F(s)$*, respectively).*

F *is said to have* S-one-sided limits *if and only if it has an* S*-right limit and an* S*-left limit at all* $r \in [0,T]$*.*

If F *has* S-one-sided limits*, then the function* ${}^\circ F : t \mapsto S\text{-}\lim_{s\downarrow t} F(s)$ *will be called the* right standard part *of* F*.*

Finally, let $W : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ *be an internal stochastic process on an internal probability space* (Ω, P) *and assume that for* $\mathsf{L}(P)$*-almost all* ω*, the path* $W(\omega) : t \mapsto W_t(\omega)$ *has* S*-one-sided limits. Then the stochastic process* ${}^\circ W : (\omega, t) \mapsto S\text{-}\lim_{s\downarrow t} W_s(\omega)$ *(which is well-defined for* $\mathsf{L}(P)$*-almost all* ω*) will be called the* right standard part *of* W*.*

2.3. Remark *Suppose* F *has* S*-one-sided limits. For all* $r \in [0,T)$*, there exists some* $t \in {}^*(r,T] \cap \mathbb{T}$ *such that* $F(t) \simeq {}^\circ F(r)$*.*

Proof. Let $r \in [0,T)$. The remark is a consequence of “overspill”: For all $n \in \mathbb{N}$, the internal formula

$$\exists m \geq n \quad \exists t \in {}^*\left(r, r + \frac{1}{m}\right) \qquad |F(t) - F(r)| < \frac{1}{n}$$

is true. Therefore, it must be true also for some $n \in {}^*\mathbb{N} \setminus \mathbb{N}$. □

Since $\mathbb{T}$ was chosen such that $[0,T] \cap \mathbb{Q} \subset \mathbb{T}$, the definition of a right standard part and the density of $\mathbb{Q}$ in $\mathbb{R}$ immediately yield:

2.4. Remark *Suppose* F *has* S*-one-sided limits. The limit* $\lim_{\mathbb{Q}\ni s\downarrow t} {}^\circ(F(s))$ *exists and equals* ${}^\circ F(t)$ *for all* $t \in [0,T) \cap \mathbb{Q}$*.*

As noted above, Lindstrøm [21, Theorem 6.6] showed that for $\mathsf{L}(P)$-almost all ω, the path $X(\omega) : t \mapsto X_t(\omega)$ has S-one-sided limits. Hence, the right standard part ${}^\circ X$ exists. Moreover, due to Lindstrøm [21, Theorem 6.6], it is an $\mathbb{R}^d$-valued *Lévy process* on the Loeb probability space $\left(\Omega, \sigma\left(2^\Omega\right), \mathsf{L}\left(P\right)\right)$:

2.5. Definition *A stochastic process* $x : \Gamma \times [0,T] \to \mathbb{R}^d$ *on some probability space* $(\Gamma, \mathcal{C}, Q)$ *is called* Lévy process *if and only if it has all of the following properties:*

- $x_0 = 0$
- *For* $n \in \mathbb{N}$ *and* $0 \leq t_0 \leq \cdots \leq t_n \leq T$*, the random variables* $x_{t_1} - x_{t_0}, \ldots, x_{t_n} - x_{t_{n-1}}$ *are independent under* Q*.*
- *For all* $s \leq t \leq T$*,* $x_t - x_s$ *has the same distribution as* x_{t-s}
- *For* Q*-almost all* $\omega \in \Gamma$*, the* sample path $x(\omega) : t \mapsto x_t(\omega)$ *is right-continuous with left limits (*càdlàg*).*

In other words, a Lévy process is a stochastic process, starting in zero, with stationary and independent increments, almost all of whose paths are right-continuous with left limits.

3. Stochastic integration with respect to hyperfinite Lévy processes

Let $m \in \mathbb{N}$. For every pair of internal processes $W, Y : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^m$, one can define the *hyperfinite stochastic integral* as an internal Riemann-Stieltjes sum via

$$\forall \omega \in \Omega \quad \forall t \in \mathbb{T} \setminus \{T\} \qquad \int_0^t Y(\omega)\, \mathrm{d}W(\omega) := \sum_{u<t} Y_u(\omega) \Delta W_u(\omega). \tag{1}$$

In this section, we will assume that X is a ${}^*\mathbb{R}^d$-valued hyperfinite Lévy process, and that W depends on X through $W = f(X)$. We will impose more assumptions on f and Y, and therefore we review some terminology here.

First, we call $f : {}^*\mathbb{R}^d \to {}^*\mathbb{R}^m$ *S-continuous* if and only if

- f is internal,
- for all finite $x, y \in {}^*\mathbb{R}^d$ with $x \simeq y$, one has $f(x) \simeq f(y)$, and
- $f(x)$ is finite for all finite $x \in {}^*\mathbb{R}^d$.

(Some authors drop the last requirement; we use the definition employed by Lindstrøm [22, discussion preceding Definition 3.1] here.) For instance, the *-image of a standard continuous function $f : \mathbb{R}^d \to \mathbb{R}^m$ is S-continuous.

If $f : {}^*\mathbb{R}^d \to {}^*\mathbb{R}^m$ is S-continuous, then for all finite $a \in {}^*\mathbb{R}^d$ overspill yields

$$\forall \varepsilon \in \mathbb{R}_{>0} \quad \exists \delta \in \mathbb{R}_{>0} \quad \forall x \in {}^*\mathbb{R}^d \qquad (|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)\,. \tag{2}$$

Later on, we will require f to be even S-Lipschitz continuous.

This definition can be generalized by replacing ${}^*\mathbb{R}^d$ by some S-dense subset of a *-interval, for instance by $\mathbb{T}$. Hence, an internal map $F : \mathbb{T} \to {}^*\mathbb{R}^m$ is S-continuous if and only if $F(t) \simeq F(u)$ for all $u \simeq t \in \mathbb{T}$, and $F(t)$ is finite for all $t \in \mathbb{T}$.

We shall assume that the internal stochastic process Y (the integrand) is S-continuous in the sense that for almost all $\omega \in \Omega$, the path $Y(\omega) : \mathbb{T} \to \Omega$ is S-continuous . Hence, almost all paths of Y are bounded by a positive real.

The first result gives a criterion for $\int Y \,\mathrm{d}f(X)$ to have a standard part and hence to be meaningful as a stochastic process in the standard sense.

3.1. Lemma *Consider an S-continuous ${}^*\mathbb{R}^m$-valued internal process Y and an S-continuous $f : {}^*\mathbb{R}^d \to {}^*\mathbb{R}^m$. The internal process $\left(\int_0^t Y \,\mathrm{d}f(X)\right)_{t\in\mathbb{T}}$ has S-one sided limits. Thus, it has a right standard part, denoted $\overset{\circ}{\int} Y \,\mathrm{d}f(X)$.*

Proof. Consider any $r \in [0, T]$. Choose some ω such that the internal path $X(\omega)$ has S-one sided limits (the set of such ω has probability 1 by Lindstrøm [21, Proposition 6.3]). By the definition of an S-right limit (cf. Lindstrøm [21, Definitions 6.1-6.2]), there exists for all $\varepsilon' \in \mathbb{R}_{>0}$ some $\delta \in \mathbb{R}_{>0}$ such that for all $u, v \in \mathbb{T}$ with $u, v \not\simeq r$ and $u, v \in (r, r+\delta)$, one has $|X_u(\omega) - X_v(\omega)| < \varepsilon'$. Let us now consider some $\varepsilon \in \mathbb{R}_{>0}$. If $\varepsilon' \in \mathbb{R}_{>0}$ has been chosen small enough, the S-continuity of f (see Formula (2)) yields that $|f(X_u(\omega)) - f(X_v(\omega))| < \varepsilon$ and hence

$$\left|\int_0^u Y \,\mathrm{d}f(X) - \int_0^v Y \,\mathrm{d}f(X)\right| < \varepsilon \max_{t\in\mathbb{T}} |Y_t(\omega)| \tag{3}$$

for all $u, v \in \mathbb{T}$ with $u, v \not\simeq r$ and $u, v \in (r, r+\delta)$. However, $\max_{t\in\mathbb{T}} |Y_t(\omega)|$ is finite. (For, the path $Y(\omega) : \mathbb{T} \to {}^*\mathbb{R}^m$ is S-continuous and therefore S-bounded on $\mathbb{T}$.) Therefore, Estimate (3) already shows that the internal path $t \mapsto \int_0^t Y(\omega) \,\mathrm{d}f(X(\omega))$ has an S-right limit for $\mathsf{L}(P)$-a.e. $\omega \in \Omega$. Analogously, one can prove that the internal path $t \mapsto \int_0^t Y(\omega) \,\mathrm{d}f(X(\omega))$ has an S-left limit for $\mathsf{L}(P)$-a.e. $\omega \in \Omega$. □

For the following Lemma, we shall impose additional assumptions:

- $m = 1$.
- The integrand Y is S-bounded, i.e. there exists some $M_Y \in \mathbb{R}_{>0}$ (referred to as the *S-bound* of Y) such that $\mathsf{L}(P)\left[\bigcap_{t\in\mathbb{T}} \{|Y_t(\omega)| \le M_Y\}\right] = 1$.
- $f : {}^*\mathbb{R}^m_{\ge 0} \to {}^*\mathbb{R}$ is S-Lipschitz continuous, i.e. f is internal and there exists some $C_f \in \mathbb{R}_{>0}$ (referred to as *Lipschitz constant* of f), such that for all finite $x, y \in {}^*\mathbb{R}^d$, one has $|f(x) - f(y)| \le C_f|x-y|$.
- $f(X)$ is increasing, i.e. $P\{f(X_u) \le f(X_t)\} = 1$ for all $u \le t \in \mathbb{T}$.

For example, $f(X)$ will be increasing if $d = 1$ and $f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ is increasing and $P\{\Delta X_0 \ge 0\} = 1$ (or $A \subseteq {}^*\mathbb{R}_{\ge 0}$).

3.2. Lemma *Suppose Y is an S-bounded S-continuous ${}^*\mathbb{R}$-valued internal stochastic process. Suppose that $f : {}^*\mathbb{R}^d \to {}^*\mathbb{R}$ is S-Lipschitz continuous and that $f(X)$ is increasing. Then*

(1) *For all* $\varepsilon \in \mathbb{R}_{>0}$ *there exists some* $\delta \in \mathbb{R}_{>0}$ *such that for all* $u, v \in \mathbb{T}$ *with* $|u - v| < \delta$ *one has*

$$P\left\{\left|\int_0^v Y \, \mathrm{d}f(X) - \int_0^u Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \leq \varepsilon.$$

(2) *For all* $t \in \mathbb{T}$, *one has* ${}^\circ\left(\int_0^t Y \, \mathrm{d}f(X)\right) = {}^\circ\!\int_0^{{}^\circ t} Y \, \mathrm{d}f(X)$ *with* $\mathsf{L}(P)$*-probability* 1.

(3) *One has*

$$\mathsf{L}(P)\left\{\forall t \in [0,T] \cap \mathbb{Q} \quad \forall s \in \mathbb{T} \quad \left(s \simeq t \Rightarrow \int_0^s Y \, \mathrm{d}f(X) \simeq {}^\circ\!\int_0^t Y \, \mathrm{d}f(X)\right)\right\} = 1.$$

This Lemma generalizes a finding by Lindstrøm [21, Lemma 6.4] who proved a similar result for the special case where $\int Y \, \mathrm{d}f(X)$ is a hyperfinite Lévy process (i.e. for $m = d$, $f = \mathrm{id}$ and $Y = 1$).

Proof of Lemma 3.2. In order to prove the first assertion, let $\varepsilon \in \mathbb{R}_{>0}$ be given. Note that

$$\begin{aligned} & P\left\{\left|\int_0^v Y \, \mathrm{d}f(X) - \int_0^u Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \\ = \; & P\left\{\left|\int_u^v ((Y \vee 0) + (Y \wedge 0)) \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \\ = \; & P\left\{\left|\int_u^v (Y \vee 0) \, \mathrm{d}f(X)\right| + \left|\int_u^v (Y \wedge 0) \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \\ \leq \; & P\left\{\left|\int_u^v (Y \vee 0) \, \mathrm{d}f(X)\right| \geq \frac{\varepsilon}{2}\right\} + P\left\{\left|\int_u^v (Y \wedge 0) \, \mathrm{d}f(X)\right| \geq \frac{\varepsilon}{2}\right\}. \end{aligned}$$

Therefore, we only need to prove the first assertion for nonnegative Y.

Furthermore, we may assume that X has finite increments, since there exists some hyperfinite Lévy process $\bar{X}$ with finite increments such that $P\left[\bigcup_{t \in \mathbb{T}} \{X_t \neq \bar{X}_t\}\right] \leq \frac{\varepsilon}{2}$ (cf. Lindstrøm [21, Proposition 3.4]). But for hyperfinite Lévy processes with finite increments, both $\mu_X := \frac{1}{\Delta t} E[\Delta X_0]$ and $\sigma_X := \frac{1}{\Delta t} E\left[|\Delta X_0|^2\right]$ are finite (cf. Lindstrøm [21, Corollary 2.4]). Furthermore,

$$\forall t \in \mathbb{T} \qquad E\left[|X_t|^2\right] = \sigma_X^2 t + |\mu_X|^2 t (t - \Delta t) \tag{4}$$

(cf. Lindstrøm [21, Lemma 1.2]). On the other hand, when we apply Chebyshev's inequality and exploit that Y is nonnegative and that $\Delta f(X)_u \geq 0$ for all $u \in \mathbb{T}$ with probability 1, we obtain

$$\begin{aligned} & P\left\{\left|\int_0^v Y \, \mathrm{d}f(X) - \int_0^u Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} = P\left\{\left|\sum_{u \leq t < v} Y_t \Delta f(X)_t\right| \geq \varepsilon\right\} \\ \leq \; & \varepsilon^{-2} E\left[\left|\sum_{u \leq t < v} Y_t \Delta f(X)_t\right|^2\right] \leq \varepsilon^{-2} E\left[\left|M_Y \sum_{u \leq t < v} \Delta f(X)_t\right|^2\right] \\ \leq \; & \varepsilon^{-2} M_Y^2 E\left[\left|f(X_v) - f(X_u)\right|^2\right] \end{aligned}$$

(assuming without loss of generality $u < v$), wherein M_Y denotes the S-bound of Y. Denoting the Lipschitz constant of f by C_f, we get

$$\begin{aligned} & P\left\{\left|\int_0^v Y \, \mathrm{d}f(X) - \int_0^u Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \\ \leq\ & \varepsilon^{-2} M_Y^2 C_f^2 E\left[|X_v - X_u|^2\right] = \varepsilon^{-2} M_Y^2 C_f^2 E\left[|X_{v-u}|^2\right]. \end{aligned}$$

By Equation (4), we arrive at

$$\begin{aligned} & P\left\{\left|\int_0^v Y \, \mathrm{d}f(X) - \int_0^u Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \\ \leq\ & \varepsilon^{-2} M_Y^2 C_f^2 \left(\sigma_X^2 (v-u) + |\mu_X|^2 (v-u)(v-u-\Delta t)\right) \\ \leq\ & \varepsilon^{-2} M_Y^2 C_f^2 \left(\sigma_X^2 \delta + |\mu_X|^2 \delta^2\right) \longrightarrow 0 \text{ as } \delta \downarrow 0 \end{aligned}$$

Therefore, by choosing δ sufficiently small, we can ensure that $P\left\{\left|\int_0^u Y \, \mathrm{d}f(X) - \int_0^v Y \, \mathrm{d}f(X)\right| \geq \varepsilon\right\} \leq \frac{\varepsilon}{2}$.

The second assertion follows from the uniqueness of limits in probability and the definition of S-right limits: If $t \in \mathbb{T}$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$ is such that $t < u_n$ for all $n \in \mathbb{N}$ and ${}^\circ u_n \downarrow {}^\circ t$ as $n \to \infty$, then ${}^\circ\left(\int_0^{u_n} Y \, \mathrm{d}f(X)\right)$ converges to ${}^\circ\left(\int_0^t Y \, \mathrm{d}f(X)\right)$ in $\mathsf{L}(P)$-probability by the first assertion of the Lemma. On the other hand, $\overset{\circ}{\int} Y \, \mathrm{d}f(X)$ being an S-right limit pathwise and hence pathwise right-continuous, one has ${}^\circ\left(\int_0^{u_n} Y \, \mathrm{d}f(X)\right) \longrightarrow \overset{\circ}{\int}_0^t Y \, \mathrm{d}f(X)$ as $n \to \infty$ $\mathsf{L}(P)$-almost surely and hence also in $\mathsf{L}(P)$-probability. Therefore, $\overset{\circ}{\int}_0^t Y \, \mathrm{d}f(X) = {}^\circ\left(\int_0^t Y \, \mathrm{d}f(X)\right)$ with $\mathsf{L}(P)$-probability 1.

The last statement in the Lemma is an immediate consequence of the second assertion. □

In particular, when we put $m = d$ and $f = \mathrm{id}$ in Lemmas 3.1 and 3.2, we obtain the following results:

3.3. Lemma *Consider an S-continuous ${}^*\mathbb{R}^d$-valued internal process Y. The internal process $\left(\int_0^t Y \, \mathrm{d}X\right)_{t\in\mathbb{T}}$ has S-one sided limits. Thus, it has a right standard part, denoted $\overset{\circ}{\int} Y \, \mathrm{d}X$.*

3.4. Lemma *Let Y be an S-bounded S-continuous ${}^*\mathbb{R}$-valued internal process, and assume that X is an ${}^*\mathbb{R}$-valued increasing hyperfinite Lévy process (i.e. $A \subseteq {}^*\mathbb{R}_{\geq 0}$).*

(1) *Let $\varepsilon \in \mathbb{R}_{>0}$. There exists some $\delta \in \mathbb{R}_{>0}$ such that for all $u, v \in \mathbb{T}$ satisfying $|u - v| < \delta$ one has*

$$P\left\{\left|\int_0^v Y \, \mathrm{d}X - \int_0^u Y \, \mathrm{d}X\right| \geq \varepsilon\right\} \leq \varepsilon.$$

(2) *For all $t \in \mathbb{T}$, one has ${}^\circ\left(\int_0^t Y \, \mathrm{d}X\right) = \overset{\circ}{\int}_0^{{}^\circ t} Y \, \mathrm{d}X$ with $\mathsf{L}(P)$-probability 1.*

(3) *One has*

$$\mathsf{L}(P)\left\{\forall t \in [0,T] \cap \mathbb{Q} \quad \forall s \in \mathbb{T} \qquad \left(s \simeq t \Rightarrow \int_0^s Y \, \mathrm{d}X = \overset{\circ}{\int}_0^t Y \, \mathrm{d}X\right)\right\} = 1.$$

In Section 4, increasing hyperfinite Lévy processes will play an important role.

4. The Lévy-Khintchine formula and Lindstrøm liftings

The Lévy-Khintchine formula says that for all one-dimensional Lévy processes z there exist two real numbers $\sigma > 0$ and γ as well as some Borel measure ν on $\mathbb{R}$ with $\nu\{0\} = 0$ and $\int \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < +\infty$ such that the Fourier transform of z_1 is given by

(5)
$$\forall u \in \mathbb{R} \quad E\left[\exp\left(\mathrm{i}uz_1\right)\right] = \exp\left(\mathrm{i}\gamma u - \frac{\sigma^2 u^2}{2} + \int \left(\exp\left(\mathrm{i}ux\right) - 1 - \mathrm{i}ux\chi_{(-1,1)}\right) \nu(\mathrm{d}x)\right).$$

Given ν, the parameters γ, σ, ν are uniquely determined. Any Borel measure ν on $\mathbb{R}$ with $\nu\{0\} = 0$ and $\int \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < +\infty$ is called *Lévy measure.*

Conversely, given such γ, σ, ν, there exists a Lévy process z satisfying Equation (5), and if some Lévy process z' also satisfies (5), then z and z' have the same finite-dimensional distributions.

Thus, the Lévy-Khintchine formula yields a one-to-one correspondence, which motivates the following definition:

4.1. Definition *A triple (γ, σ, ν), consisting of a real γ, a positive real σ and a Lévy measure ν is called the* generating triplet *of some real-valued Lévy process z if and only if the Lévy-Khintchine formula* (5) *holds. In this case, we also say that the process z* corresponds *to the generating triplet (γ, σ, ν).*

Given a generating triplet (γ, σ, ν), let z be a corresponding Lévy process. Let us assume that $\int_{-1}^{+1} |x| \; \nu(\mathrm{d}x) < +\infty$. In this case, after a change of γ, the Lévy-Khintchine formula can be simplified to

$$\forall u \in \mathbb{R} \quad E\left[\exp\left(\mathrm{i}uz_1\right)\right] = \exp\left(\mathrm{i}\gamma u - \frac{\sigma^2 u^2}{2} + \int \left(\exp\left(\mathrm{i}ux\right) - 1\right) \nu(\mathrm{d}x)\right).$$

Moreover, if $\int_{-1}^{+1} |x| \; \nu(\mathrm{d}x) < +\infty$, the Lévy-Itô decomposition (cf. e.g. Applebaum [7, Theorem 2.4.16]) yields the existence of a Lévy process j, as well as a normalized Wiener process b such that

$$\forall t \in [0,T] \qquad z_t = \sigma b_t + \gamma t + j_t \quad \text{almost surely} \tag{6}$$

and

$$\forall u \in \mathbb{R} \quad E\left[\exp\left(\mathrm{i}uj_1\right)\right] = \exp\left(\int \left(\exp\left(\mathrm{i}ux\right) - 1\right) \nu(\mathrm{d}x)\right). \tag{7}$$

Furthermore, this j, called the *jump part* of z, has then finite variation (cf. Bertoin [11, p. 15] or Sato [27, Theorem 21.9(i)]). Conversely, if $\int_{-1}^{+1} |x| \; \nu(\mathrm{d}x) = +\infty$, then z does not have finite variation (cf. Sato [27, Theorem 21.9(ii)])

In general, we shall refer to any Lévy process j satisfying Equation (7) for some Lévy measure with $\int_{-1}^{+1} |x| \; \nu(\mathrm{d}x) < +\infty$ as a *pure-jump finite-variation Lévy process* with Lévy measure ν.

Lindstrøm has shown that for any given generating triplet (γ, σ, ν), there exists some hyperfinite Lévy process whose standard part corresponds to that triplet.

We shall now slightly refine this result. Herein, we need a couple of definitions.

4.2. Definition *By an* Andersonian random walk *on the internal probability space (Ω, P), we mean a hyperfinite random walk B with increment set $\left\{-\sqrt{\Delta t}, \sqrt{\Delta t}\right\}$ and transition probabilities $p_{\sqrt{\Delta t}} = p_{-\sqrt{\Delta t}} = \frac{1}{2}$.*

As Anderson [5] showed, any such Andersonian random walk is a normalized Wiener process.

4.3. Definition *A hyperfinite random walk is called* increasing *if and only if its increment set A is a subset of* ${}^*\mathbb{R}_{\geq 0}$.

4.4. Definition *Consider a generating triplet (γ, σ, ν). An ${}^*\mathbb{R}$-valued hyperfinite Lévy process is called a* Lindstrøm lifting *based on (γ, σ, ν) if and only if*

- *${}^\circ Z$ corresponds to that triplet and*
- *there are two increasing hyperfinite Lévy processes J^+ and J^- and an Andersonian random walk B such that*

$$\forall t \in \mathbb{T} \qquad Z_t = \gamma t + \sigma B_t + J_t^+ - J_t^- \tag{8}$$

A Lindstrøm lifting is called pure *if and only if ${}^\circ J^+$ and ${}^\circ J^-$ are pure-jump finite-variation Lévy processes.*

In the definition of a Lindstrøm lifting, J^+ and J^- are increasing and finite for almost all paths (as they are hyperfinite Lévy processes). Therefore, their standard parts are always finite-variation Lévy processes.

4.5. Theorem *Consider a generating triplet (γ, σ, ν) and assume $\int_{-1}^{+1} |x| \ \nu(\mathrm{d}x) < +\infty$. Then there exists a pure Lindstrøm lifting based on (γ, σ, ν).*

Proof of Theorem 4.5. Lindstrøm [21, Theorem 9.1] has established the existence of some hyperfinite Lévy processes Z and J as well as an Andersonian random walk B such that

$$\forall t \in \mathbb{T} \qquad Z_t = \gamma t + \sigma B_t + J_t,$$

and such that ${}^\circ Z$ corresponds to (γ, σ, ν) and $j := {}^\circ J$ has Lévy measure ν.

We next define

$$\forall t \in \mathbb{T} \qquad J_t^+ := \sum_{\substack{s<t \\ \Delta J_s \geq 0}} \Delta J_s$$

and

$$\forall t \in \mathbb{T} \qquad J_t^- := - \sum_{\substack{s<t \\ \Delta J_s \leq 0}} \Delta J_s.$$

Then, J^+ and J^- are hyperfinite random walks, and obviously they are increasing.

We shall now prove that J^+ and J^- are hyperfinite Lévy proceses, too. Herein, we shall utilize Lindstrøm's characterization of hyperfinite Lévy processes [21, Theorem 4.3]. Let us, for this sake, denote the set of increments of J by A and its set of transition probabilities by $\{p_a\}_{a \in A}$. Let us put $A^+ := A \cap {}^*\mathbb{R}_{\geq 0}$ and $A^- := A \cap {}^*\mathbb{R}_{\leq 0}$ as the sets of increments for J^+ and J^-, respectively. The corresponding sets of transition probabilities for J^+ and J^- are given by

$$\forall a \in A^+ \setminus \{0\} \qquad p_a^+ := p_a, \quad p_0^+ := 1 - \sum_{a' \in A^+ \setminus \{0\}} p_{a'}^+$$

and

$$\forall a \in A^- \setminus \{0\} \qquad p_a^- := p_a, \quad p_0^- := 1 - \sum_{a' \in A^+ \setminus \{0\}} p_{a'}^-,$$

respectively. Conditions (ii) and (iii) are obviously satisfied by the pairs $A^+, \{p_a^+\}_{a \in A^+}$ and $A^-, \{p_a^-\}_{a \in A^-}$ since they are satisfied by the pair $A, \{p_a\}_{a \in A}$ (as J is a hyperfinite Lévy process). In order to check Condition (i) of Lindstrøm's characterization of hyperfinite Lévy processes [21, Theorem 4.3], it is enough to prove that $\frac{1}{\Delta t} \sum_{|a| \leq k} |a| \, p_a$ is finite for all finite k. This can be seen as follows.

First, recall from the proof of Lindstrøm's representation result [21, Proof of Theorem 9.1] how A and $\{p_a\}_{a \in A}$ were constructed. Partition the set $B_N := \left\{ x \in {}^*\mathbb{R} \ : \ \frac{1}{N} \leq |x| \leq N \right\}$ by means of a lattice of infinitesimal spacing, and choose simultaneously and internally one element from each partition class. We may assume that this element has been chosen minimally in norm. The resulting set is A. Denote for any $a \in A$ its partition class by $[a]$ and define $p_a = {}^*\nu([a])\,\Delta t$.

Since ν is by assumption a Lévy measure satisfying $\int_{-1}^{+1} |x| \; \nu(\mathrm{d}x) < +\infty$, we have $\int_{-k}^{k} |x| \; \nu(\mathrm{d}x) < +\infty$ and therefore the finiteness of $\int_{-k}^{k} |x| \; {}^*\nu(\mathrm{d}x)$ for all finite k. This implies that

$$\frac{1}{\Delta t} \sum_{\substack{a \in A \\ |a| \le k}} |a|\, p_a = \sum_{\substack{a \in A \\ |a| \le k}} |a| \, {}^*\nu([a]) \le \int_{-k}^{k} |x| {}^*\nu(\mathrm{d}x)$$

(where we exploit that a is minimal in norm in $[a]$), wherein the right-hand side is finite. Hence, $\frac{1}{\Delta t} \sum_{\substack{a \in A \\ |a| \le k}} |a|\, p_a$ is finite for all finite k, and therefore, Condition (i) follows even for the pairs $A^+, \{p_a^+\}_{a \in A^+}$ and $A^-, \{p_a^-\}_{a \in A^-}$ of increments and transition probabilities for J^+ and J^-. Thus, J^+ and J^- are indeed hyperfinite Lévy processes.

Finally, we have to prove that $j^+ := {}^\circ J^+$ and $j^- := {}^\circ J^-$ are pure-jump finite-variation processes.

From the hyperfinite Lévy-Khintchine formula (cf. Lindstrøm [21, Theorem 8.1]), we can derive the following approximate identity for $E\left[\exp\left(\mathrm{i} y J_1^+\right)\right]$ for all finite $y \in {}^*\mathbb{R}$:

$$E\left[\exp\left(\mathrm{i} y J_1^+\right)\right] = \exp\left(\mathrm{i} \int_{\{a \,:\, |a| > \eta\}} \left(e^{\mathrm{i} y a} - 1\right) \; \hat{\nu}^+(\mathrm{d}a)\right)$$

wherein

$$\hat{\nu}^+(B) := \frac{1}{\Delta t} \sum_{\substack{a \in B \\ a > 0}} p_a$$

for all internal $B \subseteq {}^*\mathbb{R}$. Using basic Loeb measure theory, this leads to

$$\forall u \in \mathbb{R} \qquad E\left[\exp\left(\mathrm{i} u j_1^+\right)\right] = \exp\left(\mathrm{i} \int \left(e^{\mathrm{i} u x} - 1\right) \; \nu_{j^+}(\mathrm{d}x)\right), \tag{9}$$

wherein

$$\nu_{j^+}(C) := \lim_{\varepsilon \downarrow 0} \mathsf{L}\left(\hat{\nu}^+\right)\left(\mathsf{st}^{-1}\left\{x \in C \;:\; |x| \ge \varepsilon\right\}\right)$$

for all Borel-measurable $C \subseteq \mathbb{R}$. Now, if we define $\hat{\nu} : B \mapsto \frac{1}{\Delta t} \sum_{a \in B} p_a$, we have $\hat{\nu}^+(B) \le \hat{\nu}(B)$ for all internal $B \subseteq {}^*\mathbb{R}$ and therefore,

$$\nu_{j^+}(C) \le \nu_j(C) := \lim_{\varepsilon \downarrow 0} \mathsf{L}\left(\hat{\nu}\right)\left(\mathsf{st}^{-1}\left\{x \in C \;:\; |x| \ge \varepsilon\right\}\right) \tag{10}$$

for all Borel-measurable $C \subseteq \mathbb{R}$. However, a comparison between the hyperfinite Lévy-Khintchine formula (cf. Lindstrøm [21, Theorem 8.1]) and the standard Lévy-Khintchine formula shows (using basic Loeb measure theory) that ν_j must be the Lévy measure of ${}^\circ J$, which is just ν. Thus, we have proven that $\nu_{j^+}(C) \le \nu(C)$ for all Borel-measurable $C \subseteq \mathbb{R}$ and conclude that $\int_{-1}^{+1} |x| \nu_{j^+}(\mathrm{d}x) < +\infty$. In light of Equation (9), we obtain that j^+ is indeed a pure-jump finite-variation process.

Symmetrically, one can prove that j^- is a pure-jump finite-variation process, too. $\square$

Any pure Lindstrøm lifting entails an explicit decomposition of $z := {}^\circ Z$ as

$$\forall t \in [0, T] \qquad z_t = \gamma t + \sigma b_t + j_t^+ - j_t^-, \tag{11}$$

wherein $b := {}^\circ B$, $j^+ := {}^\circ J^-$ and $j^- := {}^\circ J^-$. This is in accordance with Equation (6), since the j therein is a finite-variation process and hence can be written as the difference of two increasing processes. These increasing processes can be chosen as Lévy processes: Just compare Equations (11) and (6), and note that $j^+ := {}^\circ J^-$

and $j^- := {}^\circ J^-$ are Lévy processes. It follows that they are bounded. Furthermore, all their paths are right-continuous with left limits—see our definition of a Lévy process. (In fact, the existence of a càdlàg modification follows already from the continuity of the semigroup of finite-dimensional distributions and hence it is a property exhibited by all Feller processes, cf. e.g. Sato [27].)

4.6. Remark *A different Lindstrøm lifting based on (γ, σ, ν) and hence an alternative proof of Theorem 4.5 (which then leads to a decomposition in the form of Equation* (11)*) can be obtained as follows. For sufficiently small Δt, Albeverio and Herzberg* [3] *(building on previous work by Lindstrøm* [21]*) proved the existence of a hyperfinite Lévy process Z whose right standard part corresponds to (γ, σ, ν) and which can be written the sum of two *-independent hyperfinite Lévy processes, one being a multiple σB of an Andersonian random walk with some hyperreal drift γ, and the other one being a superposition J of hyperfinitely many Loeb Poisson processes.*

In other words,

$$\forall t \in \mathbb{T} \qquad Z_t = \sigma B_t + \gamma t + J_t, \tag{12}$$

wherein B and J are independent and J is the internal superposition of hyperfinitely many internal Poisson processes. (That is, the distribution of ΔJ is the convolution of $M \in {}^*\mathbb{N}$ independent random variables I_n, wherein for each $n < M$, I_n is distributed according to $(1 - \lambda_n)\,\delta_0 + \lambda_n \delta_{x_n}$, where the x_n are pairwise distinct elements of ${}^*\mathbb{R} \setminus \{0\}$ and $\{\lambda_n \,:\, n < M\} \subset {}^*\mathbb{R}_{>0}$.) Such a hyperfinite Lévy process Z is called a reduced lifting *of its right standard part $z := {}^\circ Z$.*

J can be written as the difference of two independent hyperfinite Lévy processes $J = J^+ - J^-$, such that both J^+ and J^- are increasing: In order to define J^+, we let the internal distribution of ΔJ^+ under P be given by the convolution of all internal random variables I_n such that $x_n > 0$, and in order to define J^-, we let the internal distribution of ΔJ^- under P be given by the convolution of all internal random variables $-I_n$ for which $x_n < 0$. Since $\Delta J_t = \Delta J_t^+ - \Delta J_t^-$ for all $t \in \mathbb{T}$, one obviously has $J = J^+ - J^-$, and for each $\omega \in \Omega$, the paths $J^+_\cdot(\omega) : t \mapsto J_t^+(\omega)$ and $J^-_\cdot(\omega) : t \mapsto J_t^-(\omega)$ are increasing. In order to verify that J^+ and J^- are indeed hyperfinite Lévy processes (and not merely hyperfinite random walks), we can proceed as in the proof of Theorem 4.5, by combining the assumption $\int_{-1}^{+1} |x|\ \nu(\mathrm{d}x) < +\infty$ with Lindstrøm's characterization of hyperfinite Lévy processes [21, Theorem 4.3].

5. Stochastic integration with respect to Lindstrøm liftings

Consider a bounded adapted (path-)continuous real-valued process y and a standard real-valued Lévy process with decomposition as in Equation (11) for two increasing càdlàg processes j^+, j^-. (In light of the Lévy-Itô decomposition, it suffices that the Lévy measure ν of z satisfies $\int_{-1}^{1} |x|\ \nu(\mathrm{d}x) < +\infty$, cf. Bertoin [11, p. 15].)

The classical pathwise definition of the stochastic integral (cf. e.g. Millar [25]) puts

$$\forall t \in [0, T] \qquad \int_0^t y\ \mathrm{d}z := \gamma t + \sigma \int_0^t y\ \mathrm{d}b + \int_0^t y\ \mathrm{d}j^+ - \int_0^t y\ \mathrm{d}j^-, \tag{13}$$

wherein $\int y\ \mathrm{d}b$ is the Itô integral of y with respect to b, and for the following, we will always assume that $\int y\ \mathrm{d}b$ has been chosen as a path-continuous modification thereof. The integrals $\int y\ \mathrm{d}j^+$ and $\int y\ \mathrm{d}j^-$ can be defined pathwise, because for $\mathsf{L}(P)$-almost all $\omega \in \Omega$, the paths $t \mapsto j_t^+(\omega)$ and $t \mapsto j_t^-(\omega)$ are increasing, bounded and right-continuous with left limits (see the discussion of Equation (11) above) and thus may be viewed as measures. Alternatively, one can define the

Riemann-Stieltjes integral with respect to the paths of j^+, j^- or j, because all of these paths have finite variation almost surely.

To be more specific, consider any such path i (either $= j^+(\omega) : t \mapsto j_t^+(\omega)$ or $= j^-(\omega) : t \mapsto j_t^-(\omega)$ for some $\omega \in \Omega$) and note that this i induces a Borel measure on $[0,T]$ via

$$\forall s \in (0,T] \qquad i\left([0,s]\right) := i(s), \quad i\left(\{s\}\right) := i(s) - \lim_{u \uparrow s} i(u), \quad i\left(\{0\}\right) = 0.$$

Now, i being a finite Borel measure on $[0,T]$, the integral $\int \cdot \mathrm{d}i$ is well-defined for all bounded y. In this way, the integral with respect to j^+ and j^- can be defined pathwise almost surely as some Lebesgue integral. Of course, the integral difference $\int y \ \mathrm{d}j^+ - \int y \ \mathrm{d}j^-$ then coincides with the pathwise Riemann-Stieltjes integral of y with respect to the finite-variation process $j = j^+ - j^-$.

So far, we have reviewed the definition of the classical pathwise stochastic integral with respect to z. In light of Theorem 4.5, there is a process which has the same finite-dimensional distributions as z and furthermore is the standard part of a Lindstrøm lifting Z. We will from now on assume that $z := {}^\circ Z$, and that y is an adapted, bounded, (path-)continuous process on $\mathsf{L}(\Omega)$. Furthermore, the decomposition of Z in Equation (8) will again be written as

$$\forall t \in \mathbb{T} \qquad Z_t = \gamma t + \sigma B_t + J_t^+ - J_t^-,$$

which also yields a decomposition of z (as in Equation (11)):

$$\forall t \in [0,T] \qquad z_t = \gamma t + \sigma b_t + j_t^+ - j_t^-,$$

wherein $b := {}^\circ B$, $j^+ := {}^\circ J^-$ and $j^- := {}^\circ J^-$.

Now we introduce the following important convention:

The stochastic integral with respect to z will always be understood as in Equation (13) *with $b := {}^\circ B$, $j^+ := {}^\circ J^-$ and $j^- := {}^\circ J^-$.*

The process y allows for an S-bounded, pathwise S-continuous lifting Y, thus being also an SL^2-lifting in the sense of Albeverio *et al.* [2] (cf. also Lindstrøm [20], Hoover and Perkins [15, 16] or Stroyan and Bayod [29]). With this choice of Y, the right standard part of $\int Y_t \ \mathrm{d}Z_t$ exists due to Lemma 3.3. In view of the decomposition of Z, we have

$$\int Y_u \ \mathrm{d}Z_u = \sigma \int Y_u \ \mathrm{d}B_u + \gamma \int Y_u \ \mathrm{d}u + \int Y_u \ \mathrm{d}J_u^+ - \int Y_u \ \mathrm{d}J_u^-. \tag{14}$$

Recalling Anderson's [5] treatment of stochastic integrals with respect to B, we know that the standard part of $\sigma \int Y_t \ \mathrm{d}B_t + \gamma \int Y_t \mathrm{d}t$ exists and equals $\sigma \int y_t \ \mathrm{d}b_t + \gamma \int y_t \mathrm{d}t$ $\mathsf{L}(P)$-almost surely, wherein ${}^\circ B$ is the (path-continuous) standard part of the Andersonian random walk B. Therefore, in order to show that the right standard part ${}^\circ\!\int Y \ \mathrm{d}Z$ of $\int Y \ \mathrm{d}Z$ equals the classical pathwise stochastic integral of $\int y_t \ \mathrm{d}z_t$, we need to show that the right standard parts of the internal stochastic integrals of Y with respect to the hyperfinite Lévy processes J^+ and J^- (whose existence also follows from Lemma 3.3) equal the classical stochastic integrals of y with respect to j^+ and j^-.

The following theorem accomplishes just that.

5.1. Theorem *Let J^+ be an increasing hyperfinite Lévy process with right standard part $j^+ = {}^\circ J^+$, and let Y be an S-bounded S-continuous internal process with right standard part y. For all $t \in [0,T]$,*

$$\int_0^t y_s \ \mathrm{d}j_s^+ = {}^\circ\!\int_0^t Y_s \ \mathrm{d}J_s^+ \qquad \mathsf{L}(P)\textit{-almost surely.}$$

Proof. Recall from Lemma 3.4 (for $Y := 1$) that for all $t \in [0,T]$ and all $s \in \mathbb{T}$ with $s \simeq t$, one has $J_s^+ = {}^\circ J_t^+$ with $\mathsf{L}(P)$-probability 1. (For this special case, one can also refer to Lindstrøm [21, Lemma 6.4].) Hence,

$$\forall t \in [0,T] \cap \mathbb{Q} \quad \forall s \in \mathbb{T} \qquad \left(s \simeq t \Rightarrow J_s^+ = {}^\circ J_t^+\right)$$

holds with $\mathsf{L}(P)$-probability 1. Let this event be denoted Ω_0, and consider the event Ω_1 of all ω such that that the path $j^+(\omega)$ is bounded and càdlàg. This also has $\mathsf{L}(P)$-probability 1. Finally, consider the event Ω_2, consisting of all ω such that the internal path $t \mapsto Y_t(\omega)$ has a right standard part. Again, by Lindstrøm [21, Proposition 6.3], this event has $\mathsf{L}(P)$-probability 1.

Hence $\mathsf{L}(P)\left[\Omega_0 \cap \Omega_1 \cap \Omega_2\right] = 1$. Let us fix some $\omega \in \Omega_0 \cap \Omega_1 \cap \Omega_2$, and put $K = J^+(\omega)$ as well as $k = j^+(\omega)$.

K and k can be interpreted as measures: We have already remarked that since k is bounded, càdlàg and increasing, k induces a Borel measure, abusing notation also called k, defined by

$$\forall s \in (0,T] \qquad k\left([0,s]\right) = k(s), \quad k\left(\{s\}\right) = k(s) - \lim_{u \uparrow s} k(u), \quad k\left(\{0\}\right) = 0. \tag{15}$$

Similarly, the internal, S-bounded and increasing path $K = J^+(\omega)$ induces an internal measure on the hyperfinite power-set $2^{\mathbb{T}}$ via

$$\forall t \in \mathbb{T} \qquad K\left({}^*[0,t] \cap \mathbb{T}\right) = K\left(t\right) \tag{16}$$

(in particular, Equation (16) holds for $t \in [0,T] \cap \mathbb{Q}$). Below, we will show that the composition of the corresponding Loeb measure $\mathsf{L}(K)$ with the inverse standard-part operator, equals the measure k defined in Equation (15): $\mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot) \cap \mathbb{T}\right) = k$.

Next, observe that it the Theorem is established as soon as we have shown that $\int_0^t y_s(\omega)\, \mathrm{d}k(s) = {}^\circ\!\!\int_0^t Y_s(\omega)\, \mathrm{d}J_s^+(\omega)$ holds at least for all *rational* $t \in [0,T]$: Since the path $k : t \mapsto j_t^+(\omega)$ is càdlàg, so must be integrals of bounded continuous functions with respect to the measure k defined in Equation (15). In particular, the function $t \mapsto \int_0^t y_s(\omega)\, \mathrm{d}k_s$ (which equals $t \mapsto \int_0^t y_s(\omega)\, \mathrm{d}j_s^+(\omega)$) will be càdlàg. However, as a pathwise right standard part, the function $t \mapsto {}^\circ\!\!\int_0^t Y(\omega)\, \mathrm{d}J^+(\omega)$ also is càdlàg whereever it is defined (viz. $\mathsf{L}(P)$-almost surely because of Lemma 3.3). Thus, both sides of the equation $\int_0^t y_s(\omega)\, \mathrm{d}k(s) = {}^\circ\!\!\int_0^t Y_s(\omega)\, \mathrm{d}J_s^+(\omega)$ are càdlàg, whence it is sufficient to prove it for all $t \in [0,T] \cap \mathbb{Q}$. (The identity will then follow for all $t \in [0,T]$.)

Next, note that

$$\forall t \in [0,T] \cap \mathbb{Q} \qquad {}^\circ\!\!\int_0^t Y_s(\omega)\, \mathrm{d}J_s^+(\omega) = {}^\circ\left(\int_0^t Y_u(\omega)\, \mathrm{d}J_u^+(\omega)\right) \text{ for } \mathsf{L}(P)\text{-a.e. } \omega \in \Omega$$

due to Lemma 3.4. On the other hand,

$$\begin{aligned}
\int_0^t Y_u(\omega)\ \mathrm{d}J_u^+(\omega) &= \sum_{u<t} Y_u(\omega)\ \left(J_{u+\Delta t}^+(\omega) - J_u^+(\omega)\right) = \sum_{u<t} Y_u(\omega)\ K\left(\{u+\Delta t\}\right) \\
&= \sum_{0<u\le t} Y_{u-\Delta t}(\omega)\ K\left(\{u\}\right) = \int_{^*(0,t]\cap\mathbb{T}} Y_{u-\Delta t}(\omega)\ K\left(\mathrm{d}u\right) \\
&\simeq \int_{^*(0,t]\cap\mathbb{T}} {}^\circ\left(Y_{u-\Delta t}(\omega)\right)\ \mathsf{L}(K)\left(\mathrm{d}u\right) \\
&= \int_{^*(0,t]\cap\mathbb{T}} {}^\circ\left(Y_u(\omega)\right)\ \mathsf{L}(K)\left(\mathrm{d}u\right) = \int_{^*[0,t]\cap\mathbb{T}} {}^\circ\left(Y_u(\omega)\right)\ \mathsf{L}(K)\left(\mathrm{d}u\right) \\
&= \int_{[0,t]} {}^\circ\left(Y(\omega)\right)(s)\ \mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right)(\mathrm{d}s) \\
&= \int_{[0,t]} y_s(\omega)\ \mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right)(\mathrm{d}s)\,, \qquad (17)
\end{aligned}$$

wherein we have used the S-continuity of the internal path $u \mapsto Y_u(\omega)$ (which ensures that its standard part is constant on each monad $\mathsf{st}^{-1}\{s\}$) as well as the fact that $K\{0\} = 0$ and therefore $\mathsf{L}(K)\{0\} = 0$.

We must now prove that the right-hand side of this last Equation (17) equals $\int_0^t y_s(\omega)\mathrm{d}k(s)$. In order to accomplish this, we will show that $\mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right) = k$ (viewing K and k as measures).

Using the identity $\mathsf{st}^{-1}\left([0,s]\right)\cap\mathbb{T} = \bigcap_{\mathbb{Q}\ni t>s} {}^*[0,t]\cap\mathbb{T}$ and the choice of $\omega\in\Omega_0$, we obtain

$$\begin{aligned}
\mathsf{L}(K)\left(\mathsf{st}^{-1}[0,s]\cap\mathbb{T}\right) &= \mathsf{L}(K)\left(\bigcap_{\mathbb{Q}\ni t>s} {}^*[0,t]\cap\mathbb{T}\right) = \lim_{\mathbb{Q}\ni t\downarrow s} \mathsf{L}(K)\left({}^*[0,t]\cap\mathbb{T}\right) \\
&= \lim_{\mathbb{Q}\ni t\downarrow s} {}^\circ\left(K(t)\right) = \lim_{\mathbb{Q}\ni t\downarrow s} k(t) = k(s) = k\left([0,s]\right).
\end{aligned}$$

for all $s \in [0,T]\cap\mathbb{Q}$. In a similar fashion, the identity $\mathsf{st}^{-1}\{s\}\cap\mathbb{T} = \bigcap_{\varepsilon\in\mathbb{Q}_{>0}} {}^*(s-\varepsilon,s+\varepsilon]\cap\mathbb{T}$ enables us to derive that

$$\begin{aligned}
\mathsf{L}(K)\left(\mathsf{st}^{-1}\{s\}\cap\mathbb{T}\right) &= \mathsf{L}(K)\left(\bigcap_{\varepsilon\in\mathbb{Q}_{>0}} {}^*(s-\varepsilon,s+\varepsilon]\cap\mathbb{T}\right) \\
&= \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} \mathsf{L}(K)\left({}^*(s-\varepsilon,s+\varepsilon]\cap\mathbb{T}\right) \\
&= \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} {}^\circ\left(K(s+\varepsilon)\right) - \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} {}^\circ\left(K(s-\varepsilon)\right) \\
&= \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} k(s+\varepsilon) - \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} k(s-\varepsilon) = k(s) - \lim_{\mathbb{Q}\ni t\uparrow s} k(t) = k\{s\}
\end{aligned}$$

for all $s\in\mathbb{Q}\cap(0,T]$ and $\omega\in\Omega_0$

Therefore, we obtain both $\mathsf{L}(K)\left(\mathsf{st}^{-1}[0,s]\cap\mathbb{T}\right) = k\left([0,s]\right)$ and $\mathsf{L}(K)\left(\mathsf{st}^{-1}\{s\}\cap\mathbb{T}\right) = k\{s\}$ for all $s\in[0,T]\cap\mathbb{Q}$. However, both $\mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right)$ and k are finite Borel measures on $[0,T]$ and therefore regular (both from the inside and from the outside, cf. e.g. Bauer [10, Lemma 26.2]). So, $\mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right) = k$.

This readily yields

$$\int_{[0,t]} y_s(\omega)\ \mathsf{L}(K)\left(\mathsf{st}^{-1}(\cdot)\cap\mathbb{T}\right)(\mathrm{d}s) = \int_{[0,t]} y_s(\omega)\ k\left(\mathrm{d}s\right). \qquad (18)$$

Therefore, by Equation (17) and the definition of k, we finally obtain

$$\int_0^t Y_u(\omega)\ \mathrm{d}J_u^+(\omega) \simeq \int_{[0,t]} y_s(\omega)\ k\left(\mathrm{d}s\right) = \int_{[0,t]} y_s(\omega)\ \mathrm{d}j_s^+(\omega) \text{ for } \mathsf{L}(P)\text{-a.e. } \omega\in\Omega.$$

□

Based on Theorem 5.1, we deduce the following:

5.2. Theorem *Let Z be a Lindstrøm lifting with right standard part z, and let Y be S-bounded and S-continuous with right standard part y. For all $t \in [0,T]$,*

$$\int_0^t y_s \ \mathrm{d}z_s = {}^\circ\!\int_0^t Y_s \ \mathrm{d}Z_s \qquad \mathsf{L}(P)\text{-}almost\ surely.$$

Proof. Anderson [5] has proven that the standard part of $\int Y_s \ \mathrm{d}B_s$ exists and equals $\int y_s \ \mathrm{d}b_s$ (recall that $b := {}^\circ B$). Inserting this, together with Theorem 5.1 (applied to both J^+ and J^- *en lieu* of J^+), into Equation (1) yields

$$\begin{aligned} {}^\circ\!\int_0^t Y_s \ \mathrm{d}Z_s &= \gamma t + \sigma\, {}^\circ\!\int_0^t Y_s \ \mathrm{d}B_s + {}^\circ\!\int_0^t Y_s \ \mathrm{d}J_s^+ - {}^\circ\!\int_0^t Y_s \ \mathrm{d}J_s^- \\ &= \gamma t + \sigma \int_0^t y_s \ \mathrm{d}b_s + \int_0^t y_s \ \mathrm{d}j_s^+ - \int_0^t y_s \ \mathrm{d}j_s^- = \int_0^t y_s \ \mathrm{d}z_s. \end{aligned}$$

□

For $\mathbb{R}^d$-valued Lévy processes z as integrators and $\mathbb{R}^d$-valued bounded adapted and continuous integrands, we can now simply note that the components $z^{(1)}, \ldots, z^{(d)}$ of z are Lévy processes, too, and thus define

$$\int y \ \mathrm{d}z = \sum_{i=1}^d \int y^{(i)} \ \mathrm{d}z^{(i)}.$$

If $Z^{(1)}, \ldots, Z^{(d)}$ are Lindstrøm liftings with standard parts $z^{(1)}, \ldots, z^{(d)}$, respectively, then

$$\forall t \in [0,T] \quad \forall i \in \{1,\ldots,d\} \qquad \int_0^t y_s^{(i)} \ \mathrm{d}z_s^{(i)} = {}^\circ\!\int_0^t Y_s^{(i)} \ \mathrm{d}Z_s^{(i)} \quad \mathsf{L}(P)\text{-almost surely}$$

by our previous result about one-dimensional Lévy stochastic integrals (Theorem 5.2, applied for each $i \in \{1,\ldots,d\}$). Defining $\int Y \ \mathrm{d}Z = \sum_{i=1}^d \int Y^{(i)} \ \mathrm{d}Z^{(i)}$, we finally obtain

$$\forall t \in [0,T] \qquad \int_0^t y_s \ \mathrm{d}z_s = {}^\circ\!\int_0^t Y_s \ \mathrm{d}Z_s \qquad \mathsf{L}(P)\text{-almost surely.}$$

6. The Itô formula

In this section, we shall establish a link between the right standard parts of internal Riemann-Stieltjes sums with respect to smooth functions of hyperfinite Lévy processes (as in Equation (1)) and standard stochastic integrals with respect to functions of (standard) Lévy processes, wherein the stochastic differential of a smooth function of a (standard) Lévy process is given by the generalized Itô formula for (standard) Lévy processes (cf. e.g. Applebaum [7, Theorem 4.4.10]). *En passant,* we obtain a short, direct nonstandard proof of this generalized Itô formula.

We use the abbreviation $z_{t-} = \lim_{s\uparrow t} z_s$ for all $t \in (0,T]$, with the convention $z_{0-} := z_0$. Also, we will call a subset $B \subset \mathbb{R}$ *bounded from below* if and only if there exists some $\eta \in \mathbb{R}_{>0}$ such that $B \subseteq \mathbb{R} \setminus [-\eta, \eta]$.

For all $d \in \mathbb{N}$, for all ${}^*\mathbb{R}^d$-valued hyperfinite Lévy processes Z and any $\eta \in {}^*\mathbb{R}_{>0}$, we shall denote by $Z^{\leq\eta}$ the hyperfinite Lévy process given by

$$\forall t \in \mathbb{T} \quad \forall \omega \in \Omega \qquad Z_t^{\leq\eta}(\omega) = \sum_{\substack{s<t \\ |\Delta Z_s(\omega)|\leq\eta}} \Delta Z_s(\omega),$$

and by $Z^{>\eta}$ the hyperfinite Lévy process given by

$$\forall t \in \mathbb{T} \quad \forall \omega \in \Omega \qquad Z_t^{>\eta}(\omega) = \sum_{\substack{s<t \\ |\Delta Z_s(\omega)|>\eta}} \Delta Z_s(\omega).$$

The results of this section continue to depend on the existence of Lindstrøm liftings as established in Theorem 4.5. In addition, we shall impose an even stronger assumptions on the generating triplet under consideration, by requiring its Lévy measure ν to be concentrated on a bounded-below set. Since anyway $\int(1 \wedge |x|^2)\nu(\mathrm{d}x) < +\infty$ (by virtue of the regularity properties of Lévy measures), this already implies that $\int_{-1}^{1} |x| \ \nu(\mathrm{d}x) < +\infty$ whence Theorem 4.5 may be applied.

6.1. Theorem *Consider a generating triplet (γ, σ, ν), and assume that ν is concentrated on a set that is bounded from below. There exists a Lindstrøm lifting $Z = \left(\gamma t + \sigma B_t + J_t^+ - J_t^-\right)_{t\in\mathbb{T}}$ based on (γ, σ, ν) such that $P\left[\left\{0 < \Delta J_0^+ < \eta\right\} \cup \left\{0 < \Delta J_0^- < \eta\right\}\right] = 0$. For any such Z, for all twice continuously differentiable $f : \mathbb{R} \to \mathbb{R}$, for all S-continuous S-bounded adapted processes Y with right standard part y and for all $t \in [0, T]$, we have*

$$\begin{aligned} {}^{\circ}\!\!\int_0^t Y_u \ \mathrm{d}f(Z_u) &= \int_0^t y_s f'(z_{s-}) \, \mathrm{d}z_s + \int_0^t y_s f''(z_{s-}) \frac{\sigma^2}{2} \mathrm{d}s \\ &\quad - \sum_{s\in[0,t]} y_s f'(z_{s-})(z_s - z_{s-}) + \sum_{s\in[0,t]} y_s \left(f(z_s) - f(z_{s-})\right) \end{aligned}$$

$\mathsf{L}(P)$*-almost surely.*

Note that, in view of the (standard) generalized Itô formula, the right-hand side of the equation in Theorem 6.1 is commonly defined as $\int_0^t y_s \ \mathrm{d}f(z_s)$.

Proof. By assumption, there exists some $\eta \in \mathbb{R}_{>0}$ such that ν is concentrated on $\mathbb{R}\backslash[-\eta, \eta]$. As we have remarked already, combining this concentration of ν with the regularity properties of ν as a Lévy measure (in particular $\int(1 \wedge |x|^2)\nu(\mathrm{d}x) < +\infty$) yields that ν has finite mass. Therefore, $\int_{-1}^{1} |x| \ \nu(\mathrm{d}x) < +\infty$, and we are entitled to apply Theorem 4.5.

By virtue of Theorem 4.5, we can find some Z (the Lindstrøm lifting) whose right standard part corresponds to (γ, σ, ν) and such that $Z = \left(\gamma t + \sigma B_t + J_t^+ - J_t^-\right)_{t\in\mathbb{T}}$, wherein J^+ and J^- are increasing hyperfinite Lévy processes with

$$\forall u \in \mathbb{T} \qquad \Delta J_u^+, \Delta J_u^- \in {}^*\mathbb{R}_{\geq\eta} \cup \{0\}$$

for all $u \in \mathbb{T}$ (since the increment set A of J is derived from ${}^*\nu$, which is concentrated on ${}^*\mathbb{R} \setminus {}^*[-\eta, \eta]$).

We define an increasing ${}^*\mathbb{N}_0$-sequence $\{\tau_n\}_{n\in{}^*\mathbb{N}_0}$ of internal stopping times $\tau_n : \Omega \to \mathbb{T}$ by means of the following recursion on ${}^*\mathbb{N}_0$:

$$\begin{aligned} & \tau_0 && := && 0 \\ \forall n \in {}^*\mathbb{N} \qquad & \tau_n && := && \min\left\{u \in \mathbb{T} \ : \ u > \tau_{n-1}, \quad \Delta J_{u-\Delta t}^+ \vee \Delta J_{u-\Delta t}^- \geq \eta\right\} \wedge T \\ & && = && \min\left\{u \in \mathbb{T} \ : \ u > \tau_{n-1}, \quad \left|\Delta J_{u-\Delta t}^+\right| \vee \left|\Delta J_{u-\Delta t}^-\right| \neq 0\right\} \wedge T. \end{aligned}$$

(Herein, we adopt the convention $\min\emptyset = {}^*\infty$.)

Let us choose some $\omega \in \Omega$ such that Z, J^+ and J^- have a right standard part; since Z, J^+ and J^- are hyperfinite Lévy processes, the set of such ω has $\mathsf{L}(P)$-probability 1 (cf. Lindstrøm [21, Proposition 6.3] and see Lemma 3.3). It follows that already for some finite N, one has $\tau_N(\omega) = T$. Since $Z_u = \sigma B_u + \gamma u + J_u^+ - J_u^-$ for all $u \in \mathbb{T}$, we obtain that

$$(19) \qquad \forall u \in \mathbb{T} \setminus \{\tau_1(\omega) - \Delta t, \ldots, \tau_N(\omega) - \Delta t\} \qquad \Delta Z_u(\omega) = \sigma \, \Delta B_u(\omega) + \gamma \, \Delta t,$$

so

(20)
$$\forall n \le N \quad \forall u \in [\tau_n(\omega), \tau_{n+1}(\omega)) \cap \mathbb{T} \qquad Z_u(\omega) = \sigma B_u(\omega) + \gamma u + J^+_{\tau_n(\omega)} - J^-_{\tau_n(\omega)}.$$

Since $B(\omega) : u \mapsto B_u(\omega)$ is S-continuous by choice of ω, we have that $Z(\omega)$ is S-continuous on $[\tau_n(\omega), \tau_{n+1}(\omega)) \cap \mathbb{T}$ for all $n < N$. Furthermore, since $z(\omega)$ is the right standard part of $Z(\omega)$, we must have $z_t(\omega) \neq z_{t-}(\omega)$ if and only if there exists some $u \in \mathsf{st}^{-1}\{t\} \cap \mathbb{T}$ such that $\Delta J^+_u(\omega) > 0$ or $\Delta J^-_u(\omega) > 0$ (which is equivalent to $\Delta J^+_u(\omega) \ge \eta$ or $\Delta J^-_u(\omega) \ge \eta$). Hence, $z_t(\omega) \neq z_{t-}(\omega)$ if and only if ${}^\circ\tau_n(\omega) = t$ for some $n < N$. However, due to the finiteness of N, the set $\mathbb{T}(t) := \mathsf{st}^{-1}\{t\} \cap \{\tau_n(\omega)\}_{n<N}$ is finite and hence internal for all $t \in [0, T]$, and non-empty only for finitely many $t_1, \dots, t_m$.

Let us now fix some $t \in [0, T] \cap \mathbb{Q}$. Using the notation

$$\forall i \in \{1, \dots, m\} \qquad u_{2i-1} := \min \mathbb{T}(t_i) \wedge t, \quad u_{2i} := \max \mathbb{T}(t_i) \wedge t,$$

combined with $u_0 := t_0 := 0$ and $u_{2m+1} := t_{m+1} := t$, we first observe that Equation (20) implies, from now on oppressing the argument ω,

(21)
$$\forall i \in \{0, \dots, m\} \quad \forall u \in (u_{2i}, u_{2i+1}) \qquad Z_u = \sigma B_u + \gamma u + J^+_{u_{2_i}} - J^-_{u_{2_i}}.$$

Therefore, the nonstandard version of Itô's formula (cf. Albeverio *et al.* [2, Proposition 4.4.13]) yields that for all $i \in \{0, \dots, m\}$,

$$\begin{aligned}
&\sum_{u \in (u_{2i}, u_{2i+1})} Y_{u-\Delta t} \Delta f(Z_{u-\Delta t}) \\
\simeq &\sum_{u \in (u_{2i}, u_{2i+1})} Y_{u-\Delta t} f'(Z_{u-\Delta t}) (\sigma\, \Delta B_{u-\Delta t} + \gamma\, \Delta t) \\
&+ \sum_{u \in (u_{2i}, u_{2i+1})} Y_{u-\Delta t} f''(Z_{u-\Delta t}) \frac{\sigma^2}{2} \Delta t.
\end{aligned}$$

On the other hand, by the properties of a right standard part (see Remark 2.3),

(22)
$$\forall i \in \{1, \dots, m\} \qquad Z_{u_{2i}} \simeq z_{t_i}, \quad \forall u \in [u_{2i}, u_{2i+1}) \quad Z_u \simeq z_{{}^\circ u-}.$$

Therefore, since Y and Z are S-continuous on (u_{2i}, u_{2i+1}), the nonstandard Itô formula (in combination with lifting theorems about the stochastic integral, cf. e.g. Albeverio *et al.* [2, Theorem 4.4.17]) actually yields

$$\begin{aligned}
&\sum_{u \in (u_{2i}, u_{2i+1})} Y_{u-\Delta t} \Delta f(Z_{u-\Delta t}) \\
\simeq &\int_{t_i}^{t_{i+1}} y_s f'(z_{s-}) (\sigma \mathrm{d}b_s + \gamma\, \mathrm{d}s) + \int_{t_i}^{t_{i+1}} y_s f''(z_{s-}) \frac{\sigma^2}{2} \mathrm{d}s \\
= &\int_{t_i}^{t_{i+1}} y_s f'(z_{s-}) \mathrm{d}z_s + \int_{t_i}^{t_{i+1}} y_s f''(z_{s-}) \frac{\sigma^2}{2} \mathrm{d}s - \sum_{s \in (t_i, t_{i+1}]} y_s f'(z_{s-}) (z_s - z_{s-})
\end{aligned}$$

Note that for $i \in \{1, \dots, m\}$, the S-continuity of Y yields

(23)
$$\sum_{u \in [u_{2i-1}, u_{2i}]} Y_{u-\Delta t} \Delta f(Z_{u-\Delta t}) \simeq y_{t_i} \left(f(Z_{u_{2i}}) - f(Z_{u_{2i-1}-\Delta t}) \right)$$

(To see this, consider a hyperfinite U and $\phi,\psi : U \to {}^*\mathbb{R}$, wherein $\mathsf{st}\circ\phi$ is constant. Then

$$\begin{aligned}
\sum_u \phi(u)\psi(u) &= \underbrace{\sum_{\substack{u\\ \psi(u)>0}} \phi(u)\psi(u)}_{\min\phi\sum_{\phi(u)>0}\psi(u)\leq\cdot\leq\max\phi\sum_{\phi(u)>0}\psi(u)} \\
&\quad + \underbrace{\sum_{\substack{u\\ \psi(u)<0}} \phi(u)\psi(u)}_{\max\phi\sum_{\phi(u)<0}\psi(u)\leq\cdot\leq\min\phi\sum_{\phi(u)<0}\psi(u)} \\
&\simeq \mathsf{st}\circ\phi \sum_{\substack{u\\ \psi(u)>0}} \psi(u) + \mathsf{st}\circ\phi \sum_{\substack{u\\ \psi(u)<0}} \psi(u) = \mathsf{st}\circ\phi \sum_u \psi(u).
\end{aligned}$$

Applying this result to $U := [u_{2i-1}, u_{2i}]$, $\phi : u \mapsto Y_{u-\Delta t}$ and $\psi : u \mapsto \Delta f\left(Z_{u-\Delta t}\right)$ leads, via

$$\sum_{u\in[u_{2i-1},u_{2i}]} \Delta f\left(Z_{u-\Delta t}\right) = f\left(Z_{u_{2i}}\right) - f\left(Z_{u_{2i-1}-\Delta t}\right),$$

to Equation (23).)

However, by Equation (22), $Z_{u_{2i-1}-\Delta t} \simeq z_{t_i-}$ whilst $Z_{u_{2i}} \simeq z_{t_i}$. Exploiting the continuity of f, Equation (23) can therefore be written as

$$\sum_{u\in[u_{2i-1},u_{2i}]} Y_{u-\Delta t}\Delta f\left(Z_{u-\Delta t}\right) \simeq y_{t_i}\left(f\left(z_{t_i}\right) - f\left(z_{t_i-}\right)\right).$$

We now calculate as follows: For every $t \in [0,T]\cap\mathbb{Q}$, one has

$$\begin{aligned}
&\int_0^t Y \,\mathrm{d}f(Z) = \sum_{0\leq v<t} Y_v \Delta f\left(Z_v\right) = \sum_{0<u\leq t} Y_{u-\Delta t}\Delta f\left(Z_{u-\Delta t}\right) \\
= &\sum_{i=0}^m \sum_{u\in(u_{2i},u_{2i+1})} Y_{u-\Delta t}\Delta f\left(Z_{u-\Delta t}\right) + \sum_{i=1}^m \sum_{u\in[u_{2i-1},u_{2i}]} Y_{u-\Delta t}\Delta f\left(Z_{u-\Delta t}\right) \\
\simeq &\sum_{i=0}^m \int_{t_i}^{t_{i+1}} y_s f'\left(z_{s-}\right)\mathrm{d}z_s + \int_{t_i}^{t_{i+1}} y_s f''\left(z_{s-}\right)\frac{\sigma^2}{2}\mathrm{d}s \\
&- \sum_{s\in(t_i,t_{i+1}]} y_s f'\left(z_{s-}\right)\left(z_s - z_{s-}\right) \\
&+ \sum_{i=1}^m y_{t_i}\left(f\left(z_{t_i}\right) - f\left(z_{t_i-}\right)\right) \\
= &\int_0^t y_s f'\left(z_{s-}\right)\mathrm{d}z_s + \int_0^t y_s f''\left(z_{s-}\right)\frac{\sigma^2}{2}\mathrm{d}s - \sum_{s\in[0,t]} y_s f'\left(z_{s-}\right)\left(z_s - z_{s-}\right) \\
&+ \sum_{s\in[0,t]} y_s\left(f\left(z_s\right) - f\left(z_{s-}\right)\right)
\end{aligned}$$

Note that the right-hand side is right-continuous in t. This implies ${}^\circ\left(\int_0^t Y \,\mathrm{d}f(Z)\right) = \lim_{\mathbb{Q}\ni s\downarrow t} {}^\circ\left(\int_0^s Y \,\mathrm{d}f(Z)\right)$ for all $t \in [0,T)\cap\mathbb{Q}$, and therefore ${}^\circ\left(\int_0^t Y \,\mathrm{d}f(Z)\right) = {}^\circ\!\int_0^t Y \,\mathrm{d}f(Z)$ for all $t \in [0,T)\cap\mathbb{Q}$ by Remark 2.4.

Thus, we have established that

$$\begin{aligned} {}^{\circ}\!\!\int_0^t Y\,\mathrm{d}f(Z) &= \int_0^t y_s f'(z_{s-})\,\mathrm{d}z_s + \int_0^t y_s f''(z_{s-})\frac{\sigma^2}{2}\mathrm{d}s \\ &\quad - \sum_{s\in[0,t]} y_s f'(z_{s-})(z_s - z_{s-}) + \sum_{s\in[0,t]} y_s (f(z_s) - f(z_{s-})) \end{aligned}$$

holds for all $t \in [0,T) \cap \mathbb{Q}$. Since both sides of this equation are right-continuous with left limits, the equation even holds for all $t \in [0,T]$. □

6.2. Theorem *Consider a generating triplet (γ,σ,ν), and assume that ν is concentrated on a set that is bounded from below. Let z be a Lévy processes that corresponds to (γ,σ,ν). Then we have for all twice continuously differentiable f and for all $t \in [0,T]$,*

$$\begin{aligned} f(z_t) - f(z_0) &= \int_0^t f'(z_{s-})\ \mathrm{d}z_s + \frac{\sigma^2}{2}\int_0^t f''(z_{s-})\ \mathrm{d}s \\ &\quad + \sum_{s<t} \left(f(z_s) - f(z_{s-}) - (z_s - z_{s-}) f'(z_{s-})\right) \end{aligned} \tag{24}$$

$\mathsf{L}(P)$*-almost surely.*

Proof. There exists a Lindstrøm lifting $Z = \left(\gamma t + \sigma B_t + J_t^+ - J_t^-\right)_{t\in\mathbb{T}}$ based on (γ,σ,ν) such that $P\left[\left\{0 < \Delta J_0^+ < \eta\right\} \cup \left\{0 < \Delta J_0^- < \eta\right\}\right] = 0$. Let z be its right standard part (which corresponds to (γ,σ,ν)).

First, since f is continuous, the right standard part of $f(Z)$ is $f(z)$, and hence the right standard part of $f(Z) - f(Z_0)$ is $f(z) - f(z_0)$. Since $\int_0^t \mathrm{d}f(Z_t) = f(Z_t) - f(Z_0)$ for all $t \in \mathbb{T}$, this implies that

$$\forall t \in [0,T] \qquad {}^{\circ}\!\!\int_0^t \mathrm{d}f(Z) = f(z_t) - f(z_0).$$

Now we can already apply Theorem 6.1 (with $Y := 1$) to deduce Equation (24).

However, in order to make the proof more self-contained, we shall give, in addition, a direct derivation of Equation (24) under the more restrictive assumption of a thrice continuously differentiable f with compact support. Let $t_1,\ldots,t_m$ and $u_0,\ldots,u_{2m+1}$ as in the proof of Theorem 6.1. Oppressing the argument ω, we recall from Equation (21) that

$$\forall i \in \{0,\ldots,m\} \quad \forall u \in (u_{2i},u_{2i+1}) \qquad Z_u = \sigma B_u + \gamma u + J_{u_{2i}}^+ - J_{u_{2i}}^-.$$

Therefore, for all $i \in \{0,\ldots,m\}$,

$$\begin{aligned} \forall u \in (u_{2i},u_{2i+1}) \quad \Delta f(Z_{u-\Delta t}) &= f'(Z_{u-\Delta t})(\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t) \\ &\quad + f''(Z_{u-\Delta t})\frac{1}{2}\left|\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t\right|^2 \\ &\quad + \frac{1}{6} f'''(\xi)(\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t)^3 \end{aligned}$$

for some ξ. Note that $(\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t)^3$ is of order $\Delta t^{3/2}$ (since $|\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t|$ is of order $\sqrt{\Delta t}$), and that $|\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t|^2 = \sigma^2\Delta t +$ terms of order $\Delta t^{3/2}$. Since $(u_{2i},u_{2i+1}) \cap \mathbb{T}$ has cardinality $\leq \frac{T}{\Delta t}$, we conclude that

$$\begin{aligned} \sum_{u\in(u_{2i},u_{2i+1})} \Delta f(Z_{u-\Delta t}) &\simeq \sum_{u\in(u_{2i},u_{2i+1})} f'(Z_{u-\Delta t})(\sigma\,\Delta B_{u-\Delta t} + \gamma\,\Delta t) \\ &\quad + \sum_{u\in(u_{2i},u_{2i+1})} f''(Z_{u-\Delta t})\frac{\sigma^2}{2}\Delta t \end{aligned} \tag{25}$$

for all $i \in \{0, \ldots, m\}$.

Recall from Equation (22) that

$$\forall i \in \{1, \ldots, m\} \qquad Z_{u_{2i}} \simeq z_{t_i}, \quad \forall u \in [u_{2i}, u_{2i+1}) \quad Z_u \simeq z_{{}^\circ u-}.$$

Inserting this into Equation (25) yields, in combination with lifting theorems for stochastic integrals (cf. Albeverio *et al.* [2, Theorem 4.4.17]):

$$\begin{aligned}
&\sum_{u \in (u_{2i}, u_{2i+1})} \Delta f\left(Z_{u-\Delta t}\right) \\
\simeq\ & \int_{t_i}^{t_{i+1}} f'\left(z_{s-}\right)\left(\sigma \mathrm{d} b_s + \gamma\ \mathrm{d} s\right) + \int_{t_i}^{t_{i+1}} f''\left(z_{s-}\right) \frac{\sigma^2}{2} \mathrm{d} s \\
=\ & \int_{t_i}^{t_{i+1}} f'\left(z_{s-}\right) \mathrm{d} z_s + \int_{t_i}^{t_{i+1}} f''\left(z_{s-}\right) \frac{\sigma^2}{2} \mathrm{d} s - \sum_{s \in (t_i, t_{i+1}]} f'\left(z_{s-}\right)\left(z_s - z_{s-}\right).
\end{aligned}$$

On the other hand, by Equation (22), $Z_{u_{2i-1}-\Delta t} \simeq z_{t_i-}$ and $Z_{u_{2i}} \simeq z_{t_i}$, whence the continuity of f ensures that

$$\sum_{u \in [u_{2i-1}, u_{2i}]} \Delta f\left(Z_{u-\Delta t}\right) = f\left(Z_{u_{2i}}\right) - f\left(Z_{u_{2i-1}-\Delta t}\right) \simeq f\left(z_{t_i}\right) - f\left(z_{t_i-}\right).$$

Combining our equations for $\sum_{u \in (u_{2i}, u_{2i+1})} \Delta f\left(Z_{u-\Delta t}\right)$ and for $\sum_{u \in [u_{2i-1}, u_{2i}]} \Delta f\left(Z_{u-\Delta t}\right)$, we obtain

$$\begin{aligned}
& f\left(Z_t\right) - f\left(Z_0\right) = \sum_{u \leq t} \Delta f\left(Z_{u-\Delta t}\right) \\
=\ & \sum_{i=0}^{m} \sum_{u \in (u_{2i}, u_{2i+1})} \Delta f\left(Z_{u-\Delta t}\right) + \sum_{i=1}^{m} \sum_{u \in [u_{2i-1}, u_{2i})} \Delta f\left(Z_{u-\Delta t}\right) \\
\simeq\ & \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f'\left(z_{s-}\right) \mathrm{d} z_s + \int_{t_i}^{t_{i+1}} f''\left(z_{s-}\right) \frac{\sigma^2}{2} \mathrm{d} s - \sum_{s \in (t_i, t_{i+1}]} f'\left(z_{s-}\right)\left(z_s - z_{s-}\right) \\
& + \sum_{i=1}^{m} y_{t_i}\left(f\left(z_{t_i}\right) - f\left(z_{t_i-}\right)\right) \\
=\ & \int_0^t f'\left(z_{s-}\right) \mathrm{d} z_s + \int_0^t f''\left(z_{s-}\right) \frac{\sigma^2}{2} \mathrm{d} s - \sum_{s \in [0,t]} f'\left(z_{s-}\right)\left(z_s - z_{s-}\right) \\
& + \sum_{s \in [0,t]} y_s\left(f\left(z_s\right) - f\left(z_{s-}\right)\right)
\end{aligned}$$

for all $t \in [0, T] \cap \mathbb{Q}$. On the other hand, Lemma 3.4 (applied to $Y := 1$) yields that for all $t \in [0, T] \cap \mathbb{Q}$, one has ${}^\circ\left(Z_t\right) = z_t$ with $\mathsf{L}(P)$-probability 1. (For this special case, one can also refer to Lindstrøm [21, Lemma 6.4].) Since f is continuous, we may deduce

$$\text{(26)} \quad \forall t \in [0, T] \cap \mathbb{Q} \qquad f\left(Z_t\right) - f\left(Z_0\right) \simeq f\left(z_t\right) - f\left(z_0\right) \qquad \mathsf{L}(P)\text{-almost surely.}$$

Combining this Equation (26) with our previous calculations in this proof, one arrives at Equation (24) for all $t \in [0, T] \cap \mathbb{Q}$. As both sides of the equation are right-continuous with left limits, the equation follows for arbitrary $t \in [0, T]$.

Finally, recall that hyperfinite adapted probability spaces are universal in the model-theoretic sense, based on the language of adapted probability logic (cf. e.g. Fajardo and Keisler [13]). Therefore, Equation (24) does not only hold when z is the standard part of Z, but for every Lévy process z corresponding to the generating triplet (γ, σ, ν). □

6.3. Remark *Nonstandard methods can be used to prove a generalization of the Itô formula even for local L^2-martingales (cf. Lindstrøm* [20, pp. 327-330, in particular Theorem 15]*, which corresponds to a slightly earlier result by Métiviér* [24]*), based on a corresponding formula for internal SL^2-martingales (cf. Lindstrøm* [19, Theorem 22]*). An alternative nonstandard proof of Theorem 6.2 could therefore be based on the SL^2-martingale theory and an internal drift-martingale decomposition (cf. Lindstrøm* [21, Corollary 2.5]*). Our proof, however, makes no use whatsoever of either of these results, but instead utilizes our refinement (Theorem 4.5) of Lindstrøm's representation theorem* [21, Theorem 9.1] *and is therefore technically more accessible.*

7. Conclusion

For any generating triplet of a Lévy process with finite-variation jump part, there is a particularly simple hyperfinite Lévy process, whose internal jump part can be decomposed into two increasing hyperfinite Lévy processes (Theorem 4.5). Hyperfinite stochastic integration with respect to this hyperfinite Lévy process is consistent with classical pathwise stochastic integration with respect to its standard part (Theorem 5.2). If the Lévy measure is even concentrated on a set that is bounded from below, we can show that stochastic integration with respect to smooth functions of this hyperfinite Lévy process is consistent with classical pathwise stochastic integration based on the generalized Itô formula (Theorem 6.1). In particular, this reasoning leads to a short, direct nonstandard proof of the generalized Itô formula for Lévy processes with Lévy measures that are concentrated on bounded-below sets (Theorem 6.2).

Hence, the theory of hyperfinite Lévy processes leads to a simple pathwise definition of the stochastic integral with respect to functions of Lévy processes with finite-variation jump part. What is more, the càdlàg property of the paths of these stochastic integrals follows without further argument from the existence of a right standard part (Lemma 3.1).

By the model-theoretic universality and saturation of hyperfinite adapted probability spaces (cf. e.g. Fajardo and Keisler [13]), most probabilistic results about the standard parts of hyperfinite Lévy processes can be generalized to arbitrary Lévy processes.

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