Working Papers

Institute of Mathematical Economics

401

September 2008

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http://www.wiwi.uni-bielefeld.de/˜imw/Papers/showpaper.php?401
ISSN: 0931-6558

A utility representation theorem with weaker continuity condition

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September 4, 2008

Abstract

We prove that a preference relation which is continuous on every straight line has a utility representation if its domain is a convex subset of a finite dimensional vector space. Our condition on the domain of a preference relation is stronger than Eilenberg (1941) and Debreu (1959, 1964), but our condition on the continuity of a preference relation is strictly weaker than theirs.

JEL classification: C60; D11

Keywords: Linear continuity; Utility representation

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[†]I would like to express my gratitude to Takuya Masuzawa and Shin-ichi Suda for helpful discussions. I also would like to thank Chiaki Hara, Atsushi Kajii, Tadashi Sekiguchi, and Walter Trockel for their valuable comments. A part of the results of this paper was presented at the Third Asian Workshop on General Equilibrium Theory (GETA2006) held at Academia Sinica. I acknowledge a Grant-in-Aid for Research Fellow of the Japan Society for the Promotion of Science (JSPS) and financial support from the German Research Foundation (DFG).

1 Introduction

In their expected utility representation, Herstein and Milnor (1953) used a weaker notion of continuity of a preference relation than the usual continuity. It requires that preference relation is continuous in the parameter space. When we regard the operation of convex combination as the mixture operation of Herstein and Milnor (1953), a preference relation continuous in their sense is continuous on any straight line in the domain of the preference relation. We refer to this notion of continuity as the linear continuity. By assuming the independence axiom, Herstein and Milnor (1953) proved that every linearly continuous preference relation has an expected utility representation. As far as only utility representation is concerned, the independence axiom is dispensable. We prove that every linearly continuous preference relation on a convex subset of a finite dimensional vector space has a utility representation (Theorem 1).

Eilenberg (1941) (see also Debreu (1959, 1964)) proved that every continuous preference relation on a separable connected topological space has a continuous utility representation. Since any convex subset of a finite dimensional vector space is separable and connected with respect to the Euclidean topology, our condition on the domain of preference relation is stronger than Eilenberg's theorem. On the other hand, as was shown by Young and Young (1910), the linear continuity is strictly weaker than the usual continuity, so our condition on the continuity of preference relation is weaker than Eilenberg's theorem.

If a linearly continuous preference relation is not continuous, its utility representation cannot be continuous. Thus, there may not exist a maximal element for a linearly continuous preference relation in a compact set. This fact limits the application of our theorem, but Inoue (2008) proved that if a linearly continuous preference relation is convex or weakly monotone, it is upper semi-continuous and, therefore, it has a maximal element in a compact set.

The main step in the proof of our utility representation theorem is to show that every linearly continuous preference relation is countably bounded (Proposition 1), i.e., there exists a countable set of vectors such that any vector is preordered between some vectors in the countable set. Since the linear continuity is equivalent to the usual continuity on any straight line, from Eilenberg's theorem, for any two vectors, there exists a utility function on the segment connecting those vectors. The countable boundedness of preference relation enables us to extend this utility function on the segment to the whole space by repeated application of Eilenberg's theorem.

If the set of discontinuity points of a linearly continuous preference relation is small enough, we can obtain its utility representation by direct application of Eilenberg's theorem. As an example, assume that a linearly continuous preference relation on X has only one discontinuity point x. Since $X \setminus \{x\}$ is still separable and connected, we can apply Eilenberg's theorem to $X \setminus \{x\}$ and obtain a utility function u on $X \setminus \{x\}$. For any vector (except x) on a straight line in X which passes through x, its utility has already been defined. Thus, we can define $u(x)$ by the limit of the utilities of vectors on the straight line. Then, we obtain a utility representation on the whole domain X. Since Young and Young's (1910) example tells us that there exists a linearly continuous preference relation whose discontinuity points make an uncountable dense subset of the domain, it is not clear whether the above procedure is valid for any linearly continuous preference relation. We prove that for any linearly continuous preference relation, the set of its discontinuity points is small enough to apply Eilenberg's theorem to the set of continuity points and small enough to define the utilities of discontinuity points properly (Propositions 2-4). It should be emphasized that this result does not mean that our utility representation theorem is dispensable, because we rely on the utility representation when we show the smallness of the set of discontinuity points.

The linear continuity of preference relation is defined by using the one-dimensional Euclidean topology and, therefore, it is defined free from the topology of the domain of preference relation. In the case of finite dimensional vector space, any Hausdorff linear topology is equivalent to the Euclidean topology. Therefore, the linear continuity of preference relation is of special interest when the domain of preference relation is infinite dimensional (see the introduction of Herstein and Milnor (1953) and the notes of Chapter

4 of Debreu (1959)). However, our utility representation theorem cannot be extended to a nonseparable infinite dimensional topological vector space, because from Estévez Toranzo and Hervés Beloso (1995), it follows that any nonseparable infinite dimensional topological vector space has a continuous preference relation which cannot have a utility representation.

This paper is organized as follows. In Section 2, we give the definition of linear continuity and give an example of linearly continuous preference relation which is not continuous. In Section 3, we prove the utility representation theorem of linearly continuous preference relation. In Section 4, we discuss the relationship between our utility representation theorem and Eilenberg's theorem.

2 Linearly continuous preference relations

Let X be a nonempty convex subset of the L-dimensional vector space $\mathbb{R}^L$ which is equipped with the Euclidean topology.[1] A preference relation $\succsim$ on X is a reflexive, transitive, and complete binary relation on X. Given a preference relation $\succsim$, we define binary relations $\succ$ and $\sim$ on X as follows: $x \succ y$ if and only if not $y \succsim x$; $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. A *utility function representing a preference relation* $\succsim$ or a *utility representation of* $\succsim$ is a real-valued function u on X such that $x \succsim y$ if and only if $u(x) \geq u(y)$.

In their expected utility representation, Herstein and Milnor (1953) used a weaker continuity than the usual continuity. It requires that a preference relation is continuous in the parameter space. As seen in Remark 1 below, this continuity geometrically means that a preference relation is continuous on any straight line. We refer to this continuity as linear continuity.[2] For $x, y \in X$, let $I(x, y) = \{t \in \mathbb{R} \,|\, (1-t)x + ty \in X\}$. Since X is

[1] As we see below, linear continuity is defined independently from the topology on X. We equip X with the Euclidean topology, because in Section 4 we discuss the relationship between Eilenberg's theorem on a Euclidean space and our utility representation theorem. Note that any Hausdorff linear topology on a finite dimensional vector space is equivalent to the Euclidean topology.

[2] In decision theory, this continuity is called *mixture continuity*. This term is suitable when X is a set

convex, $I(x, y)$ is an interval which contains $[0, 1]$.

Definition 1 A preference relation $\succsim$ on X is *linearly continuous* if for every $x, y, z \in X$, the sets $\{t \in I(x, y) \,|\, (1-t)x + ty \succsim z\}$ and $\{t \in I(x, y) \,|\, z \succsim (1-t)x + ty\}$ are closed in $I(x, y)$ with respect to the Euclidean topology on $\mathbb{R}$.

Remark 1 For $x, y \in X$, let $X(x, y)$ be the straight line in X which passes through x and y, i.e., $X(x, y) = \{(1-t)x + ty \,|\, t \in \mathbb{R}\} \cap X = \{(1-t)x + ty \,|\, t \in I(x, y)\}$. A preference relation $\succsim$ on X is linearly continuous if and only if for every $x, y, z \in X$, the sets $\{w \in X(x, y) \,|\, w \succsim z\}$ and $\{w \in X(x, y) \,|\, z \succsim w\}$ are closed in $X(x, y)$.

Remark 2 If a preference relation $\succsim$ on X is continuous, i.e., for every $x \in X$, the sets $\{y \in X \,|\, y \succsim x\}$ and $\{y \in X \,|\, x \succsim y\}$ are closed in X, then $\succsim$ is linearly continuous.

The inverse of Remark 2 is not true. Actually, the binary relation generated from the function of Young and Young's (1910) example is linearly continuous but is not continuous. We also will give a simple example later, but first we give the definition of linear continuity of a real-valued function and second we state the relationship of (linear) continuities between a preference relation and its utility representation.

Definition 2 A real-valued function $u : X \to \mathbb{R}$ is *linearly continuous* if for every $\alpha \in \mathbb{R}$ and every $x, y \in X$, the sets $\{t \in I(x, y) \,|\, u((1-t)x + ty) \geq \alpha\}$ and $\{t \in I(x, y) \,|\, \alpha \geq u((1-t)x + ty)\}$ are closed in $I(x, y)$.

Note that even if u is a utility representation of a continuous preference relation $\succsim$, u may not be continuous. For example, the usual ordering $\geq$ on $\mathbb{R}$ is continuous and any decreasing function on $\mathbb{R}$ is a utility function representing $\geq$, but decreasing function may be discontinuous at some points. Therefore, the continuity of a preference relation and the continuity of its utility representation are not equivalent. The following remark

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of lotteries and we interpret $(1-t)x + ty$ $(0 \leq t \leq 1,\ x, y \in X)$ as the mixed lottery of lotteries x and y with the respective probabilities $1-t$ and t.

gives the relationship of (linear) continuities between a preference relation and its utility representation.

Given a real-valued function $u : X \to \mathbb{R}$, a preference relation $\succsim^u$ on X is defined by $x \succsim^u y$ if and only if $u(x) \geq u(y)$. It is clear that u is a utility representation of $\succsim^u$.

Remark 3 (1) If $u : X \to \mathbb{R}$ is linearly continuous (resp. continuous at $x \in X$), then $\succsim^u$ is linearly continuous (resp. continuous at x).[3]

(2) Let $\succsim$ be a linearly continuous preference relation (resp. a preference relation continuous at $x \in X$) and let u be its utility representation. If $u(X)$ is an interval, u is linearly continuous (resp. continuous at x).

Now, we are ready to give an example which illustrates that the linear continuity is strictly weaker than the continuity.

Example 1 A real-valued function u on $\mathbb{R}^2$ is defined by

$$u(x, y) = \begin{cases} \dfrac{2x^2y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Clearly, u is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$. At $(0, 0)$, u is not continuous, because $u(0, 0) = 0$ and $u(x, x^2) = 1$ for any $x \neq 0$. Since u is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$, it is continuous on any straight line which does not pass through $(0, 0)$. In addition, it can be easily shown that u is continuous on any straight line passing through $(0, 0)$. Thus, u is linearly continuous.

We now prove that $u(\mathbb{R}^2)$ is an interval. Since u is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and $\mathbb{R}^2 \setminus \{(0, 0)\}$ is connected, $u(\mathbb{R}^2 \setminus \{(0, 0)\})$ is an interval. From $u(0, 0) = 0 \in u(\mathbb{R}^2 \setminus \{(0, 0)\})$, it follows that $u(\mathbb{R}^2) = u(\mathbb{R}^2 \setminus \{(0, 0)\})$. Therefore, $u(\mathbb{R}^2)$ is an interval.

From Remark 3, $\succsim^u$ is linearly continuous but is not continuous at $(0, 0)$.

[3] A preference relation $\succsim$ on X is *continuous at* $x \in X$ if for every $w, y \in X$ with $w \succ x \succ y$, there exists an open subset U of $\mathbb{R}^L$ such that $w \succ x' \succ y$ for every $x' \in X \cap U$.

Because of the lack of continuity, a linearly continuous preference relation may not have a maximal element in a compact set (Inoue, 2008, Example 2). If a linearly continuous preference relation $\succsim$ is convex or weakly monotone, however, it recovers the upper semi-continuity and, therefore, it has a maximal element in a compact set (Inoue, 2008, Theorems 1 and 3).

Following Young and Young (1910), we can construct a linearly continuous preference relation whose discontinuity points make a dense subset of $\mathbb{R}^2$.

Example 2 Let $\mathbb{Q}^2 = \{(a_1, b_1), (a_2, b_2), \ldots\}$, where $\mathbb{Q}$ is the set of rational numbers. For every natural number n, define $u_n : \mathbb{R}^2 \to \mathbb{R}$ by $u_n(x, y) = u(x - a_n, y - b_n)$, where u is the function in Example 1. The function $U(x, y) = \sum_{n=1}^{\infty} 2^{-n} u_n(x, y)$ is well-defined, because $\max_{(x,y)\in\mathbb{R}^2} |u(x, y)| = 1$. Since u is linearly continuous and u is discontinuous only at $(0, 0)$, the function U is linearly continuous and is not continuous at any $(a, b) \in \mathbb{Q}^2$. Since $U(\mathbb{R}^2)$ is an interval, from Remark 3, $\succsim^U$ is linearly continuous and is not continuous at any $(a, b) \in \mathbb{Q}^2$.

Young and Young (1910) constructed a linearly continuous function such that the set of its discontinuity points is an uncountable dense subset of $\mathbb{R}^2$. In Section 4, we will discuss the size of the set of discontinuity points of a linearly continuous preference relation.

3 Representation by a utility function

We prove that the linear continuity is sufficient for the utility representation.

Theorem 1 *Let X be a nonempty convex subset of $\mathbb{R}^L$. If a preference relation $\succsim$ on X is linearly continuous, then there exists a real-valued function $u : X \to \mathbb{R}$ such that (i) $a \succsim b$ if and only if $u(a) \geq u(b)$, (ii) $u(X)$ is an interval, and (iii) u is linearly continuous.*

Before giving a proof, we compare this theorem with related works in the literature. Eilenberg (1941) (see also Debreu (1959, 1964)) proved that every continuous preference relation on a separable connected topological space can be represented by a continuous utility function. Monteiro (1987) proved that a continuous preference relation $\succsim$ on a path connected topological space X has a continuous utility representation if and only if it is *countably bounded*, i.e, there exists a countable subset Y of X such that for every $x \in X$, there exist $y, z \in Y$ with $y \succsim x \succsim z$. The domain of a preference relation in Eilenberg's theorem and in Monteiro's theorem may not be a vector space and even if it is a vector space, it may not be finite dimensional. Thus, our condition on the domain of a preference relation is stronger than their conditions. (Recall that a convex subset of $\mathbb{R}^L$ is separable, connected, and path connected with respect to the Euclidean topology.) On the other hand, as we saw in the previous section, the linear continuity is strictly weaker than the usual continuity.

In our finite dimensional topological vector space framework, as we show in the next proposition, the countably boundedness follows from the linear continuity of a preference relation, although it is a necessary and sufficient condition for continuous utility representation in Monteiro's framework.

Proposition 1 *Let X be a nonempty convex subset of $\mathbb{R}^L$. If a preference relation $\succsim$ on X is linearly continuous, then there exists a countable subset Y of X such that for every $x \in X$, there exist $y, z \in Y$ with $y \succsim x \succsim z$.*

Proof of Proposition 1. We only prove that there exists an upward countable subset Y of X such that for every $x \in X$, there exists a $y \in Y$ with $y \succsim x$. By a similar manner, we can prove the existence of a downward countable subset of X. We prove by induction on the dimension of the affine hull $\operatorname{aff}(X)$ of X. Let $k = \dim \operatorname{aff}(X)$. Note that $k \leq L$. Under an appropriate affine transformation, $\operatorname{aff}(X)$ can be identified with $\mathbb{R}^k$.[4] Thus, we may assume that X is a subset of $\mathbb{R}^k$. When $k = 0$, the proposition is clear. When $k = 1$, X is an interval. Therefore, X can be represented as a countable union of closed intervals, say, $X = \bigcup_{n=1}^{\infty} [a_n, b_n]$. Since $\succsim$ is linearly continuous, there exists a maximal element for

[4]Note that an affine transformation maps straight lines to straight lines. Thus, the linear continuity of a preference relation is not affected by an affine transformation.

$\succsim$ on every $[a_n, b_n]$. Namely, for every n, there exists a $y_n \in [a_n, b_n]$ such that for every $x \in [a_n, b_n]$, $y_n \succsim x$. Let $Y = \{y_1, y_2, \ldots\}$. Then, Y satisfies the required property.

Suppose that the proposition is true for $k \leq l$ but not true for $k = l + 1$. Then, we have:

(a) for every countable subset Y of X, there exist x_1 and x_2 in X such that for every $y \in Y$, $x_1 \succ x_2 \succ y$.

Let $\mathrm{pr}_1 : \mathbb{R}^{l+1} \to \mathbb{R}$ be the projection into the first coordinate, i.e., $\mathrm{pr}_1(x^{(1)}, \ldots, x^{(l+1)}) = x^{(1)}$. Since $\dim \mathrm{aff}(X) = l + 1$, $\mathrm{pr}_1(X)$ is a nondegenerate interval. Therefore, $\mathrm{pr}_1(X) \cap \mathbb{Q}$ is a countably infinite set, where $\mathbb{Q}$ is the set of rational numbers. Hence, we may write $\mathrm{pr}_1(X) \cap \mathbb{Q} = \{q_1, q_2, \ldots\}$. Since for every n, the set $X \cap (\{q_n\} \times \mathbb{R}^l)$ is a convex set with at most dimension l, by the induction hypothesis, we have:

(b) for every n, there exists a countable subset Y_n of $X \cap (\{q_n\} \times \mathbb{R}^l)$ such that for every $x \in X \cap (\{q_n\} \times \mathbb{R}^l)$, there exists a $y \in Y_n$ with $y \succsim x$.

Let $Y = \bigcup_{n=1}^{\infty} Y_n$. Then, Y is countable and, therefore, from (a), it follows that:

(c) there exist x_1^* and x_2^* in X such that for every $y \in Y$, $x_1^* \succ x_2^* \succ y$.

Since $\mathrm{pr}_1(X)$ is nondegenerate, there exists a $w \in X$ such that $w^{(1)} \neq x_1^{*(1)}$. From the linear continuity of $\succsim$, we have:

(d) there exists a $t_0 \in [0, 1[$ such that for every $t \in [t_0, 1]$, $(1 - t)w + tx_1^* \succ x_2^*$.

Since $t_0 < 1$ and $w^{(1)} \neq x_1^{*(1)}$, there exists a $t^* \in [t_0, 1]$ such that $(1 - t^*)w^{(1)} + t^* x_1^{*(1)} \in \mathbb{Q}$. Therefore, for some n^*, $(1 - t^*)w^{(1)} + t^* x_1^{*(1)} = q_{n^*}$. By (b), there exists a $y^* \in Y$ such that $y^* \succsim (1 - t^*)w + t^* x_1^*$. On the other hand, from (c) and (d), it follows that $(1 - t^*)w + t^* x_1^* \succ x_2^* \succ y^*$, which is a contradiction. This completes the proof of Proposition 1. ■

Once we know that a preference relation is countably bounded, we can prove Theorem 1 by applying Eilenberg's (1941) theorem repeatedly. The formal proof is as follows:

Proof of Theorem 1. From Proposition 1, there exist two countable sets $\{y_1, y_2, \ldots\}$ and $\{z_1, z_2, \ldots\}$ with $\cdots \succsim y_2 \succsim y_1 \succsim z_1 \succsim z_2 \succsim \cdots$ such that for every $x \in X$, there exists n with $y_n \succsim x \succsim z_n$. Since $[y_1, z_1] = \{(1-t)y_1 + tz_1 \,|\, 0 \leq t \leq 1\}$ is separable and connected, and $\succsim$ is continuous on $[y_1, z_1]$, from Eilenberg's (1941) theorem (see also Debreu (1959, 1964)), there exists a continuous utility function u on $[y_1, z_1]$. Note that $u([y_1, z_1])$ is a bounded interval, because u is continuous on $[y_1, z_1]$ and $[y_1, z_1]$ is connected and compact. Let y_1' (resp. z_1') be a maximal (resp. minimal) element on $[y_1, z_1]$. Since for every $x \in X$ with $y_1' \succsim x \succsim z_1'$, there exists a $w_x \in [y_1, z_1]$ with $x \sim w_x$, we can extend u to the set $\{x \in X \,|\, y_1' \succsim x \succsim z_1'\}$ by defining $u(x) = u(w_x)$. If $y_n \succ y_1'$ for some n, u has not been defined on a subinterval $[y_n, v[= \{(1-t)y_n + tv \,|\, 0 \leq t < 1\}$ of $[y_n, y_1']$ such that $u(v)$ has already been defined. Note that $v \sim y_1'$. Again, from Eilenberg's theorem, there exists a continuous function u_n on $[y_n, v]$ with $u_n(v) = u(v) = u(y_1')$. Note that $u_n([y_n, v])$ is a bounded closed interval. Let y_n' be a maximal element on $[y_n, v]$. Since for every $x \in X$ with $y_n' \succsim x \succsim y_1'$, there exists a $w_x \in [y_n, v]$ with $x \sim w_x$, we can define $u(x) = u_n(w_x)$. Thus, we have extended u to the set $\{x \in X \,|\, y_n' \succsim x \succsim z_1'\}$. Note that, by construction of u, $u(\{x \in X \,|\, y_n' \succsim x \succsim z_1'\})$ is a bounded closed interval. By repeating this argument, we can extend u to the whole space X, because $\succsim$ is countably bounded. By construction, u is a representation of $\succsim$ and $u(X)$ is an interval. Therefore, from Remark 3, u is linearly continuous. This completes the proof of Theorem 1. ■

4 Relationship with Eilenberg's (1941) theorem

We discuss the relationship between our utility representation theorem (Theorem 1) and Eilenberg's (1941) theorem. Let X be a nonempty convex subset of $\mathbb{R}^L$ and let $\succsim$ be a linearly continuous preference relation on X as in Theorem 1. Also, let $D = \{x \in X \,|\, \succsim$ is not continuous at $x\}$. Suppose that D is small enough in the following two senses. First, $X \setminus D$ is connected. Second, every discontinuity point is linearly accessible from the set $X \setminus D$ of continuity points, i.e., for every $x \in D$, there exists a $y \in X$ and a sequence $(x_n)_n$ in $(X \setminus D) \cap [x, y]$ such that $x_n \to x$. Then, from the connectedness of

$X \setminus D$, we can apply Eilenberg's theorem to $X \setminus D$ and obtain a continuous utility function u on $X \setminus D$. Since u is continuous on $X \setminus D$ and $X \setminus D$ is connected, $u(X \setminus D)$ is an interval. By using the fact that every discontinuity point is linearly accessible from the set of continuity points, we can extend the function u to the whole space X with preserving that the extended function u is a utility representation of $\succsim$ and $u(X)$ is an interval. Therefore, from Remark 3, the utility function is linearly continuous. Hence, if the set D of discontinuity points is small enough, from Eilenberg's theorem, we can obtain the utility representation of a linearly continuous preference relation.

In Proposition 2, we prove that every discontinuity point is linearly accessible from the set of continuity points. In Proposition 3, we prove that the set $X \setminus D$ of continuity points is connected. Finally, in Proposition 4, with the help of Propositions 2 and 3 and Eilenberg's theorem, we prove that every linearly continuous preference relation can be represented by a linearly continuous utility function.

It should be emphasized that Proposition 4 does not mean our utility representation theorem (Theorem 1) is dispensable, because we essentially rely on the utility representation theorem when we show the smallness of the set of discontinuity points. Actually, in the proof of Lemma 2 below, we use our utility representation theorem in order to apply Kershner's theorem. Kershner (1943) characterized the set of discontinuity points of a unicontinuous function which is weaker than a linearly continuous function. In the proof of Kershner's theorem, the following facts are used: the set of discontinuity points of any real-valued function is a F_σ-set; a continuous function on a compact set is uniformly continuous. Our utility representation theorem enables us to use Kershner's theorem.

We give the precise statements of Kershner's theorem (Kershner, 1943, Theorem 6) and Kuratowski-Ulam theorem (Kuratowski and Ulam, 1932; Kuratowski, 1966, pp. 246-247; Oxtoby, 1971, Theorem 15.1) which play the important roles in the following. Let $k \geq 2$. A real-valued function u on a rectangle $\prod_{j=1}^{k}[a^{(j)}, b^{(j)}]$ is *unicontinuous* if for every $i \in \{1, \dots, k\}$ and every $(\bar{x}^{(1)}, \dots, \bar{x}^{(i-1)}, \bar{x}^{(i+1)}, \dots, \bar{x}^{(k)}) \in \prod_{j \neq i}[a^{(j)}, b^{(j)}]$, the function $[a^{(i)}, b^{(i)}] \ni x^{(i)} \mapsto u(\bar{x}^{(1)}, \dots, \bar{x}^{(i-1)}, x^{(i)}, \bar{x}^{(i+1)}, \dots, \bar{x}^{(k)}) \in \mathbb{R}$ is continuous. Note that a linearly continuous function on a rectangle is unicontinuous. For every $i \in \{1, \dots, k\}$, define $\mathrm{pr}^k_{-i}: \mathbb{R}^k \to \mathbb{R}^{k-1}$ by $\mathrm{pr}^k_{-i}(x^{(1)}, \ldots, x^{(k)}) = (x^{(1)}, \ldots, x^{(i-1)}, x^{(i+1)}, \ldots, x^{(k)})$.

Kershner's theorem *Let u be a unicontinuous function on $\prod_{j=1}^k [a^{(j)}, b^{(j)}]$ with $k \geq 2$ and $a^{(j)} < b^{(j)}$ for every $j \in \{1, \ldots, k\}$. Let $D = \{x \in \prod_{j=1}^k [a^{(j)}, b^{(j)}] \mid u$ is not continuous at $x\}$. Then, for every $i \in \{1, \ldots, k\}$, $\mathrm{pr}^k_{-i}(D)$ is of the first category in $\prod_{j \neq i} [a^{(j)}, b^{(j)}]$.*

Kuratowski-Ulam theorem *Let T_1 and T_2 be topological spaces such that T_2 has a countable base. If $D \subset T_1 \times T_2$ is of the first category in $T_1 \times T_2$, there exists a $P \subset T_1$ of the first category in T_1 such that for every $x \in T_1 \setminus P$, the set D_x is of the first category in T_2, where D_x is the x-section of D, i.e., $D_x = \{y \in T_2 \mid (x, y) \in D\}$.*

The first lemma is a variation of Kuratowski-Ulam theorem.

Lemma 1 *Let $k \geq 2$ and $D \subset \mathbb{R}^k$ be of the first category in $\mathbb{R}^k$. Let $x_0, y_0 \in \mathbb{R}^k$ with $x_0 \neq y_0$. Let $\{v_1, \ldots, v_{k-1}\}$ be an orthonormal basis of $(\mathrm{span}\{x_0 - y_0\})^\perp$. Then, for every $\varepsilon > 0$, there exists a $P \subset]-\varepsilon, \varepsilon[$ of the first category in $\mathbb{R}$ such that for every $t \in]-\varepsilon, \varepsilon[\setminus P$ the set $D \cap \mathrm{co}(\{x_0\} \cup Y^\varepsilon_{tv_1})$ is of the first category in $\mathrm{aff}(\{x_0\} \cup Y^\varepsilon_{tv_1})$, where $Y^\varepsilon_{tv_1} = \{y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i \mid s_i \in]-\varepsilon, \varepsilon[, \, i = 2, \ldots, k-1\}$.*

In Figure 1, the set $\mathrm{co}(\{x_0\} \cup Y^\varepsilon_{tv_1})$ with $k = 3$ is drawn. Note that $\mathrm{aff}(\{x_0\} \cup Y^\varepsilon_{tv_1}) = \mathrm{span}\{y_0 + tv_1 - x_0, v_2, \ldots, v_{k-1}\} + \{x_0\}$.

Proof of Lemma 1. Since a countable union of sets of the first category in $\mathbb{R}$ is of the first category in $\mathbb{R}$, it suffices to prove that if D is nowhere dense in $\mathbb{R}^k$, then for every $\varepsilon > 0$, there exists a $P \subset [-\varepsilon, \varepsilon[$ of the first category in $\mathbb{R}$ such that for every $t \in]-\varepsilon, \varepsilon[\setminus P$, the set $D \cap \mathrm{co}(\{x_0\} \cup Y^\varepsilon_{tv_1})$ is nowhere dense in $\mathrm{aff}(\{x_0\} \cup Y^\varepsilon_{tv_1})$. Let D be nowhere dense in $\mathbb{R}^k$. Since the closure of D is also nowhere dense in $\mathbb{R}^k$, we may assume that D is closed. Let $G = \mathbb{R}^k \setminus D$. Then, G is an open dense subset of $\mathbb{R}^k$. Let $\{V_n \mid n \in \mathbb{N}\}$ be a countable base for $]-\varepsilon, \varepsilon[^{k-2} \times]0, 1[$ such that $V_n \neq \emptyset$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$G_n = \{t \in]-\varepsilon, \varepsilon[\quad | \quad \text{there exists } (s_2, \ldots, s_{k-1}, \alpha) \in V_n \text{ with}$$

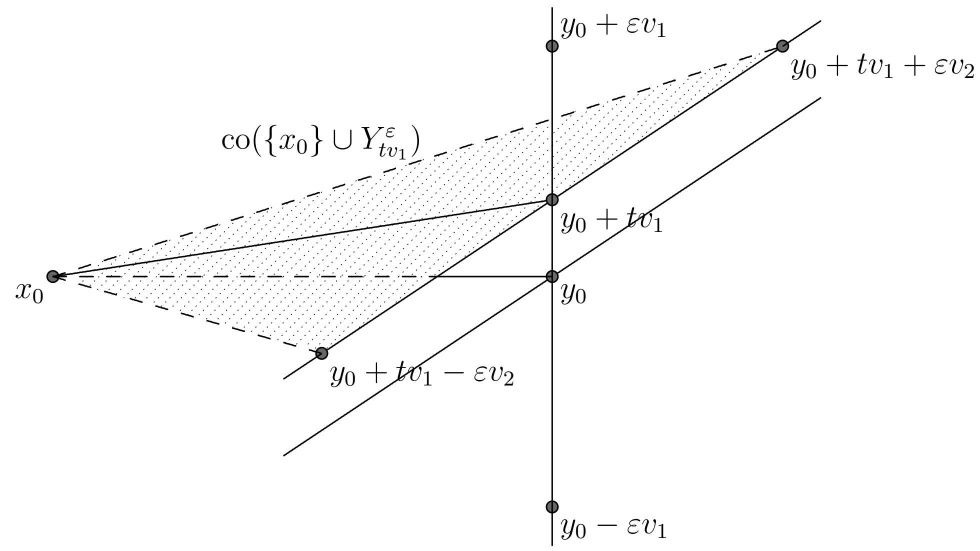

Figure 1: $\mathrm{co}(\{x_0\} \cup Y^{\varepsilon}_{tv_1})$ when $k = 3$

$$\alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i) \in G\}.$$

Claim 1 For every $n \in \mathbb{N}$, G_n is open.

Proof of Claim 1. Let $n \in \mathbb{N}$ and $t \in G_n$. Then, there exists $(s_2, \ldots, s_{k-1}, \alpha) \in V_n$ with $\alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i) \in G$. Define $g :]-\varepsilon, \varepsilon[\to \mathbb{R}^k$ by $g(\hat{t}) = \alpha x_0 + (1-\alpha)(y_0 + \hat{t}v_1 + \sum_{i=2}^{k-1} s_i v_i)$. Since g is continuous, $g^{-1}(G)$ is open and, therefore, from $t \in g^{-1}(G) \subset G_n$, it follows that G_n is open. ■

Define $h :]-\varepsilon, \varepsilon[^{k-1} \times]0,1[\to \mathbb{R}^k$ by $h(t, s_2, \ldots, s_{k-1}, \alpha) = \alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i)$.

Claim 2 h is an open mapping.

Proof of Claim 2. Note that the family of all sets such that $]\underline{t}, \overline{t}[\times \prod_{i=2}^{k-1}]\underline{s}_i, \overline{s}_i[\times]\underline{\alpha}, \overline{\alpha}[\subset]-\varepsilon, \varepsilon[^{k-1} \times]0,1[$ forms a base for $]-\varepsilon, \varepsilon[^{k-1} \times]0,1[$. Note further that $h(]\underline{t}, \overline{t}[\times \prod_{i=2}^{k-1}]\underline{s}_i, \overline{s}_i[\times]\underline{\alpha}, \overline{\alpha}[) = \mathrm{int}(\mathrm{co}\{\alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i) \,|\, \alpha \in \{\underline{\alpha}, \overline{\alpha}\}, t \in \{\underline{t}, \overline{t}\}, s_i \in \{\underline{s}_i, \overline{s}_i\}, i = 2, \ldots, k-1\})$. Thus, h is an open mapping. ■

Claim 3 For every $n \in \mathbb{N}$, G_n is dense in $]-\varepsilon, \varepsilon[$.

Proof of Claim 3. Let $n \in \mathbb{N}$. Let U be a nonempty open subset of $[-\varepsilon, \varepsilon[$. Then, $U \times V_n$ is a nonempty subset of $]-\varepsilon, \varepsilon[^{k-1} \times]0,1[$. From Claim 2, h is an open mapping and, therefore, $h(U \times V_n)$ is nonempty and open. Since G is dense in $\mathbb{R}^k$, we have $h(U \times V_n) \cap G \neq \emptyset$. Thus, there exist $t \in U$ and $(s_2, \ldots, s_{k-1}, \alpha) \in V_n$ such that $\alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i) \in G$. Hence, $t \in G_n$ and, therefore, $G_n \cap U \neq \emptyset$. Thus, G_n is dense in $]-\varepsilon, \varepsilon[$. ■

Let $P =]-\varepsilon, \varepsilon[\setminus \bigcap_{i=1}^{\infty} G_n$. By Claims 1 and 3, for every $n \in \mathbb{N}$, G_n is open dense in $]-\varepsilon, \varepsilon[$ and, therefore, P is of the first category in $\mathbb{R}$. Let $t \in \bigcap_{n=1}^{\infty} G_n =]-\varepsilon, \varepsilon[\setminus P$. Define $h_t :]-\varepsilon, \varepsilon[^{k-2} \times]0,1[\to \mathbb{R}^k$ by $h_t(s_2, \ldots, s_{k-1}, \alpha) = \alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i)$. Note that $h_t(\prod_{i=2}^{k-1}]\underline{s}_i, \overline{s}_i[\times]\underline{\alpha}, \overline{\alpha}[) = \mathrm{ri}(\mathrm{co}\{\alpha x_0 + (1-\alpha)(y_0 + tv_1 + \sum_{i=2}^{k-1} s_i v_i) \mid \alpha \in \{\underline{\alpha}, \overline{\alpha}\}, s_i \in \{\underline{s}_i, \overline{s}_i\}, i = 2, \ldots, k-1\})$, where $\mathrm{ri}(C)$ stands for the relative interior of the set C. Thus, if we restrict the range of h_t to $\mathrm{aff}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}) = \mathrm{span}\{y_0 + tv_1 - x_0, v_2, \ldots, v_{k-1}\} + \{x_0\}$, h_t is an open mapping. Note also that the family of all sets $h_t(\prod_{i=2}^{k-1}]\underline{s}_i, \overline{s}_i[\times]\underline{\alpha}, \overline{\alpha}[)$ with $\prod_{i=2}^{k-1}]\underline{s}_i, \overline{s}_i[\times]\underline{\alpha}, \overline{\alpha}[\subset]-\varepsilon, \varepsilon[^{k-2} \times]0,1[$ forms a base for $\mathrm{ri}(\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}))$. Since $\{V_n \mid n \in \mathbb{N}\}$ is a base for $]-\varepsilon, \varepsilon[^{k-2} \times]0,1[$, $\{h_t(V_n) \mid n \in \mathbb{N}\}$ is a base for $\mathrm{ri}(\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}))$. Since $t \in \bigcap_{n=1}^{\infty} G_n$, for every $n \in \mathbb{N}$, there exists $(s_2, \ldots, s_{k-1}, \alpha) \in V_n$ with $h_t(s_2, \ldots, s_{k-1}, \alpha) \in G$. Therefore, $h_t(V_n) \cap G \neq \emptyset$ for every $n \in \mathbb{N}$. This means that $G \cap \mathrm{ri}(\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}))$ is dense in $\mathrm{ri}(\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}))$. Thus, $D \cap \mathrm{ri}(\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon}))$ is nowhere dense in $\mathrm{aff}(\{x_0\} \cup Y_{tv_1}^{\varepsilon})$. Since the relative boundary of $\mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon})$ is nowhere dense in $\mathrm{aff}(\{x_0\} \cup Y_{tv_1}^{\varepsilon})$, $D \cap \mathrm{co}(\{x_0\} \cup Y_{tv_1}^{\varepsilon})$ is nowhere dense in $\mathrm{aff}(\{x_0\} \cup Y_{tv_1}^{\varepsilon})$. This completes the proof of Lemma 1. ■

Corollary 1 *Let $k \geq 2$. Let $D \subset \mathbb{R}^k$ be of the first category in $\mathbb{R}^k$. Let $x_0, y_0 \in \mathbb{R}^k$ with $x_0 \neq y_0$. Then, for every $\varepsilon > 0$, there exists a $y' \in \mathbb{R}^k$ such that $y' \neq x_0$, $\|y' - y_0\| < \varepsilon$, and $D \cap [x_0, y']$ is of the first category in $[x_0, y']$, where $\|\cdot\|$ is the Euclidean norm.*

Proof of Corollary 1. Let $\{v_1, \ldots, v_{k-1}\}$ be an orthonormal basis of $(\mathrm{span}\{x_0 - y_0\})^{\perp}$. Let $\varepsilon > 0$. By Lemma 1, there exists a $t_1 \in]-\varepsilon/(k-1), \varepsilon/(k-1)[$ such that $D \cap \mathrm{co}(\{x_0\} \cup$

$Y^{\varepsilon/(k-1)}_{t_1v_1}$) is of the first category in $\mathrm{aff}(\{x_0\}\cup Y^{\varepsilon/(k-1)}_{t_1v_1}) = \mathrm{span}\{y_0+t_1v_1-x_0, v_2, \dots, v_{k-1}\}+\{x_0\}$. If $k \geq 3$, we can apply Lemma 1 again to $D \cap \mathrm{co}(\{x_0\} \cup Y^{\varepsilon/(k-1)}_{t_1v_1})$ and, therefore, there exists a $t_2 \in]-\varepsilon/(k-1), \varepsilon/(k-1)[$ such that $D \cap \mathrm{co}(\{x_0\} \cup Y^{\varepsilon/(k-1)}_{t_1v_1,t_2v_2})$ is of the first category in $\mathrm{aff}(\{x_0\} \cup Y^{\varepsilon/(k-1)}_{t_1v_1,t_2v_2})$, where $Y^{\varepsilon/(k-1)}_{t_1v_1,t_2v_2} = \{y_0 + t_1v_1 + t_2v_2 + \sum_{i=3}^{k-1} s_iv_i \mid s_i \in]-\varepsilon/(k-1), \varepsilon/(k-1)[, i = 3, \dots, k-1\}$. Note that $\mathrm{aff}(\{x_0\} \cup Y^{\varepsilon/(k-1)}_{t_1v_1,t_2v_2}) = \mathrm{span}\{y_0 + t_1v_1 + t_2v_2 - x_0, v_3, \dots, v_{k-1}\} + \{x_0\}$. Hence, by applying Lemma 1 repeatedly, there exist $t_1, \dots, t_{k-1} \in]-\varepsilon/(k-1), \varepsilon/(k-1)[$ such that $D \cap \mathrm{co}(\{x_0\} \cup Y^{\varepsilon/(k-1)}_{t_1v_1,\dots,t_{k-1}v_{k-1}})$ is of the first category in $\mathrm{aff}(\{x_0\}\cup Y^{\varepsilon/(k-1)}_{t_1v_1,\dots,t_{k-1}v_{k-1}})$, where $Y^{\varepsilon/(k-1)}_{t_1v_1,\dots,t_{k-1}v_{k-1}} = \{y_0+\sum_{i=1}^{k-1} t_iv_i\}$. Thus, $D \cap [x_0, y_0 + \sum_{i=1}^{k-1} t_iv_i]$ is of the first category in $[x_0, y_0 + \sum_{i=1}^{k-1} t_iv_i]$. By the definition of $\{v_1, \dots, v_{k-1}\}$, we have $x_0 \neq y_0 + \sum_{i=1}^{k-1} t_iv_i$. In addition,

$$\left\| y_0 + \sum_{i=1}^{k-1} t_iv_i - y_0 \right\| = \left\| \sum_{i=1}^{k-1} t_iv_i \right\| \leq \sum_{i=1}^{k-1} |t_i|\|v_i\| = \sum_{i=1}^{k-1} |t_i| < (k-1)\frac{\varepsilon}{k-1} = \varepsilon.$$

This completes the proof of Corollary 1. ■

Lemma 2 *Let $L \geq 2$. Let X be a nonempty convex subset of $\mathbb{R}^L$ with $\mathrm{aff}(X) = \mathbb{R}^L$ and $\succsim$ be a linearly continuous preference relation on X. Let $D = \{x \in X \mid \succsim$ is not continuous at $x\}$. Then,*

(1) for every rectangle $A = \prod_{j=1}^{L}[a^{(j)}, b^{(j)}] \subset X$ with $a^{(j)} < b^{(j)}$ for every $j \in \{1, \dots, L\}$, and every $i \in \{1, \dots, L\}$, the set $\mathrm{pr}^L_{-i}(D \cap A)$ is of the first category in $\prod_{j\neq i}[a^{(j)}, b^{(j)}]$,

(2) D is of the first category in $\mathrm{aff}(X)$.

Proof of Lemma 2. By Theorem 1, there exists a utility representation u of $\succsim$ such that $u : X \to \mathbb{R}$ is linearly continuous and $u(X)$ is an interval. Thus, from Remark 3, it follows that $D = \{x \in X \mid u$ is not continuous at $x\}$. Since u is unicontinuous on A, by Kershner's theorem, we obtain (1).

We now prove (2). Note that $\mathrm{int}(X)$ can be represented as a countable union of rectangles, say, $\mathrm{int}(X) = \bigcup_{n=1}^{\infty} \prod_{j=1}^{L}[a_n^{(j)}, b_n^{(j)}]$ with $a_n^{(j)} < b_n^{(j)}$ for every $j \in \{1, \dots, L\}$ and every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, since $D \cap \prod_{j=1}^{L}[a_n^{(j)}, b_n^{(j)}] \subset [a_n^{(1)}, b_n^{(1)}] \times \mathrm{pr}^L_{-1}(D \cap \prod_{j=1}^{L}[a_n^{(j)}, b_n^{(j)}])$, from (1), it follows that $D \cap \prod_{j=1}^{L}[a_n^{(j)}, b_n^{(j)}]$ is of the first category in $\mathbb{R}^L$.

Therefore, $D \cap \mathrm{int}(X) = \bigcup_{n=1}^{\infty}(D \cap \prod_{j=1}^{L}[a_n^{(j)}, b_n^{(j)}])$ is of the first category in $\mathbb{R}^L$. Since X is convex, the boundary $\mathrm{bd}(X)$ of X is nowhere dense in $\mathbb{R}^L$. Since $D \subset (D \cap \mathrm{int}(X)) \cup \mathrm{bd}(X)$, D is of the first category in $\mathbb{R}^L$. ■

Proposition 2 *Let X be a nonempty convex subset of $\mathbb{R}^L$ and $\succsim$ be a linearly continuous preference relation on X. Let $D = \{x \in X \mid \succsim \text{ is not continuous at } x\}$. Then, for every $x, y \in X$ and every $\varepsilon > 0$, there exists a $y' \in X$ such that $y' \neq x$, $\|y' - y\| < \varepsilon$, and $D \cap [x, y']$ is of the first category in $[x, y']$. In particular, every $x \in D$ is linearly accessible from $X \setminus D$, i.e., for every $x \in D$, there exists a $y \in X$ and a sequence $(x_n)_n$ in $(X \setminus D) \cap [x, y]$ such that $x_n \to x$.*

Proof of Proposition 2. Let $k = \dim \mathrm{aff}(X)$. If $k \leq 1$, then $D = \emptyset$ and, therefore, the statement is clear. Let $k \geq 2$. Under an appropriate affine transformation, $\mathrm{aff}(X)$ can be identified with $\mathbb{R}^k$. Thus, we may assume that X is a subset of $\mathbb{R}^k$. Let $x, y \in X$ and $\varepsilon > 0$. Then, there exists a $z \in \mathrm{int}(X)$ with $z \neq x$ and $\|z - y\| < \varepsilon/2$. Since $z \in \mathrm{int}(X)$, there exists an $\varepsilon_1 \in]0, \varepsilon]$ with $\{w \in \mathbb{R}^k \mid \|w - z\| < \varepsilon_1/2\} \subset X$. By Lemma 2, D is of the first category in $\mathbb{R}^k$ and, therefore, by Corollary 1, there exists a $y' \in \mathbb{R}^k$ such that $y' \neq x$, $\|y' - z\| < \varepsilon_1/2$, and $D \cap [x, y']$ is of the first category in $[x, y']$. From $\|y' - z\| < \varepsilon_1/2$, it follows that $y' \in X$. Since $\|y' - y\| \leq \|y' - z\| + \|z - y\| < \varepsilon_1/2 + \varepsilon/2 \leq \varepsilon$, we obtained the required result. ■

Lemma 3 *Let $k \geq 2$ and $A = \prod_{j=1}^{k}[a^{(j)}, b^{(j)}]$ with $a^{(j)} < b^{(j)}$ for every $j \in \{1, \ldots, k\}$. Let $D \subset A$. If for every $i \in \{1, \ldots, k\}$, the set $\mathrm{pr}_{-i}^{k}(D)$ is of the first category in $\prod_{j \neq i}[a^{(j)}, b^{(j)}]$, then $A \setminus D$ is connected.*

Proof of Lemma 3. We prove by induction on k. Let $k = 2$. Suppose, to the contrary, that $A \setminus D$ is not connected. Then, there exist open subsets U and V of $\mathbb{R}^2$ such that $U \cap (A \setminus D) \neq \emptyset$, $V \cap (A \setminus D) \neq \emptyset$, $U \cap V \cap (A \setminus D) = \emptyset$, and $U \cup V \supset A \setminus D$.

Claim 4 There exist $x^* \in U \cap A$, $y^* \in V \cap A$, and $i \in \{1, 2\}$ such that $x^{*(i)} = y^{*(i)} \notin \mathrm{pr}_i(D)$, where $\mathrm{pr}_i(z^{(1)}, z^{(2)}) = z^{(i)}$.

Proof of Claim 4. By assumption, $\text{pr}_1(D) = \text{pr}^2_{-2}(D)$ is of the first category in $[a^{(1)}, b^{(1)}]$. Thus, $\text{pr}_1(D) \times [a^{(2)}, b^{(2)}]$ is of the first category in $A = [a^{(1)}, b^{(2)}] \times [a^{(2)}, b^{(2)}]$. Since $U \subset \mathbb{R}^2$ is open and $U \cap A \neq \emptyset$, we have $\text{int}(U \cap A) \neq \emptyset$. By Baire's category theorem, $U \cap A$ is of the second category in A and, therefore, we can pick $\bar{x} \in (U \cap A) \setminus (\text{pr}_1(D) \times [a^{(2)}, b^{(2)}])$. Similarly, we can pick $\hat{y} \in (V \cap A) \setminus ([a^{(1)}, b^{(1)}] \times \text{pr}_2(D))$. We consider two distinct cases.

Case 1. $V \cap (\{\bar{x}^{(1)}\} \times [a^{(2)}, b^{(2)}]) \neq \emptyset$.

In this case, we can pick $\bar{y} \in V \cap (\{\bar{x}^{(1)}\} \times [a^{(2)}, b^{(2)}])$. Then, $\bar{y} \in V \cap A$, $\bar{x} \in U \cap A$, and $\bar{y}^{(1)} = \bar{x}^{(1)} \notin \text{pr}_1(D)$. Therefore, we obtained the desired property in Case 1.

Case 2. $V \cap (\{\bar{x}^{(1)}\} \times [a^{(2)}, b^{(2)}]) = \emptyset$.

From $\bar{x}^{(1)} \notin \text{pr}_1(D)$, it follows that $\{\bar{x}^{(1)}\} \times [a^{(2)}, b^{(2)}] \subset A \setminus D \subset U \cup V$. Thus, we have $\{\bar{x}^{(1)}\} \times [a^{(2)}, b^{(2)}] \subset U$. Let $\hat{x} = (\bar{x}^{(1)}, \hat{y}^{(2)})$. Then, $\hat{x} \in U \cap A$, $\hat{y} \in V \cap A$, and $\hat{x}^{(2)} = \hat{y}^{(2)} \notin \text{pr}_2(D)$. Therefore, we obtained the desired property in Case 2. This completes the proof of Claim 4. $\blacksquare$

Let x^* and y^* be those of Claim 4 with $x^{*(1)} = y^{*(1)} \notin \text{pr}_1(D)$. Then, $\{x^{*(1)}\} \times [a^{(2)}, b^{(2)}] \subset A \setminus D$. We have $x^* \in U \cap (\{x^{*(1)}\} \times [a^{(2)}, b^{(2)}])$, $y^* \in V \cap (\{x^{*(1)}\} \times [a^{(2)}, b^{(2)}])$, $U \cap V \cap (\{x^{*(1)}\} \times [a^{(2)}, b^{(2)}]) \subset U \cap V \cap (A \setminus D) = \emptyset$, and $U \cup V \supset A \setminus D \supset \{x^{*(1)}\} \times [a^{(2)}, b^{(2)}]$, but this contradicts that $\{x^{*(1)}\} \times [a^{(2)}, b^{(2)}]$ is connected. Thus, $A \setminus D$ is connected if $k = 2$.

Assume now that the lemma is true for $k = l - 1$ but not for $k = l$. Then, there exist open subsets U and V of $\mathbb{R}^l$ such that $U \cap (A \setminus D) \neq \emptyset$, $V \cap (A \setminus D) \neq \emptyset$, $U \cap V \cap (A \setminus D) = \emptyset$, and $U \cup V \supset A \setminus D$.

Claim 5 For every $x^{(1)} \in [a^{(1)}, b^{(1)}]$, either $U \cap (\{x^{(1)}\} \times \prod_{j=2}^{l} [a^{(j)}, b^{(j)}]) \neq \emptyset$ or $V \cap (\{x^{(1)}\} \times \prod_{j=2}^{l} [a^{(2)}, b^{(j)}]) \neq \emptyset$ holds.

Proof of Claim 5. Suppose, to the contrary, that there exists an $x^{(1)} \in [a^{(1)}, b^{(1)}]$ such that $U \cap (\{x^{(1)}\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) = \emptyset$ and $V \cap (\{x^{(1)}\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) = \emptyset$. Since $U \cup V \supset A \setminus D$, we have $\{x^{(1)}\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}] \subset D$. Then, $\mathrm{pr}_{-1}^{l}(D) = \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]$. This contradicts that $\mathrm{pr}_{-1}^{l}(D)$ is of the first category in $\prod_{j=2}^{l}[a^{(j)}, b^{(j)}]$. This completes the proof of Claim 5. ■

By Claim 5, we may assume that $V \cap (\{b^{(1)}\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. Let $\alpha = \sup\{t \in [a^{(1)}, b^{(1)}] \,|\, U \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset\}$. Since $U \cap A \neq \emptyset$, the right-hand set defining α is nonempty.

Claim 6 There exists a nondegenerate subinterval I of $[a^{(1)}, b^{(1)}]$ such that for every $t \in I$, $U \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$ and $V \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$.

Proof of Claim 6. We consider two distinct cases.

Case 1. $U \cap (\{\alpha\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$.

In this case, since U is open, we have $\alpha = b^{(1)}$. Since both U and V are open, there exists an $\varepsilon > 0$ such that for every $t \in]b^{(1)} - \varepsilon, b^{(1)}]$, $U \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$ and $V \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. Thus, Claim 6 holds in Case 1.

Case 2. $U \cap (\{\alpha\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) = \emptyset$.

By Claim 5, we have $V \cap (\{\alpha\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. Since V is open, there exists an $\varepsilon_1 > 0$ such that for every $t \in]\alpha - \varepsilon_1, \alpha]$, $V \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. From the definition of α, there exists a $t^* \in]\alpha - \varepsilon_1, \alpha] \cap [a^{(1)}, b^{(1)}]$ such that $U \cap (\{t^*\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. Since U is open, there exists an $\varepsilon_2 > 0$ such that for every $t \in]t^* - \varepsilon_2, t^* + \varepsilon_2[$, $U \cap (\{t\} \times \prod_{j=2}^{l}[a^{(j)}, b^{(j)}]) \neq \emptyset$. The interval $]\alpha - \varepsilon_1, \alpha] \cap]t^* - \varepsilon_2, t^* + \varepsilon_2[\cap [a^{(1)}, b^{(1)}]$ is nondegenerate and has the desired property. Thus, Claim 6 holds in Case 2. This completes the proof of Claim 6. ■

By assumption, for every $i \in \{2, \ldots, l\}$, $\mathrm{pr}_{-i}^{l}(D)$ is of the first category in $\prod_{j \neq i}[a^{(j)}, b^{(j)}]$.

Thus, by Kuratowski-Ulam theorem, for every $i \in \{2, \ldots, l\}$, there exists a $P_i \subset [a^{(1)}, b^{(1)}]$ of the first category in $[a^{(1)}, b^{(1)}]$ such that for every $x^{(1)} \in [a^{(1)}, b^{(1)}] \setminus P_i$, $(\mathrm{pr}^l_{-i}(D))_{x^{(1)}} = \mathrm{pr}^{l-1}_{-i}(D_{x^{(1)}})$ is of the first category in $\prod_{2 \le j \le l : j \ne i}[a^{(i)}, b^{(i)}]$, where $E_{x^{(1)}}$ is the $x^{(1)}$-section of set E. Since $\bigcup_{i=2}^l P_i$ is of the first category in $[a^{(1)}, b^{(1)}]$, we can pick $x^{*(1)} \in I \setminus \bigcup_{i=2}^l P_i$, where I is the nondegenerate interval obtained in Claim 6. From $x^{*(1)} \notin \bigcup_{i=2}^l P_i$, it follows that for every $i \in \{2, \ldots, l\}$, the set $\mathrm{pr}^{l-1}_{-i}(D_{x^{*(1)}})$ is of the first category in $\prod_{2 \le j \le l : j \ne i}[a^{(j)}, b^{(j)}]$. By the induction hypothesis, $\prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}$ is connected.

Since $x^{*(1)} \in I$, we have $U \cap (\{x^{*(1)}\} \times \prod_{j=2}^l [a^{(j)}, b^{(j)}]) \ne \emptyset$ and $V \cap (\{x^{*(1)}\} \times \prod_{j=2}^l [a^{(j)}, b^{(j)}]) \ne \emptyset$. This implies that $U_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}] \ne \emptyset$ and $V_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}] \ne \emptyset$. Since U and V are open subsets of $\mathbb{R}^l$, $U_{x^{*(1)}}$ and $V_{x^{*(1)}}$ are open subsets of $\mathbb{R}^{l-1}$ and, therefore, we have $\mathrm{int}(U_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}]) \ne \emptyset$ and $\mathrm{int}(V_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}]) \ne \emptyset$. Hence, by Baire's category theorem, $U_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}]$ and $V_{x^{*(1)}} \cap \prod_{j=2}^l [a^{(j)}, b^{(j)}]$ are of the second category in $\prod_{j=2}^l [a^{(j)}, b^{(j)}]$. From $x^{*(1)} \notin P_2$, it follows that $\mathrm{pr}^{l-1}_{-2}(D_{x^{*(1)}})$ is of the first category in $\prod_{j=3}^l [a^{(j)}, b^{(j)}]$. Since $D_{x^{*(1)}} \subset [a^{(2)}, b^{(2)}] \times \mathrm{pr}^{l-1}_{-2}(D_{x^{*(1)}})$, the set $D_{x^{*(1)}}$ is of the first category in $\prod_{j=2}^l [a^{(j)}, b^{(j)}]$. Therefore, $U_{x^{*(1)}} \cap (\prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}) \ne \emptyset$ and $V_{x^{*(1)}} \cap (\prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}) \ne \emptyset$. In addition, $U_{x^{*(1)}} \cap V_{x^{*(1)}} \cap (\prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}) = \emptyset$ and $U_{x^{*(1)}} \cup V_{x^{*(1)}} \supset \prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}$. This contradicts that $\prod_{j=2}^l [a^{(j)}, b^{(j)}] \setminus D_{x^{*(1)}}$ is connected. Thus, $A \setminus D$ is connected when $k = l$. This completes the proof of Lemma 3. ■

Proposition 3 *Let X be a nonempty convex subset of $\mathbb{R}^L$ and $\succsim$ be a linearly continuous preference relation on X. Let $D = \{x \in X \mid \succsim$ is not continuous at $x\}$. Then, $X \setminus D$ is connected.*

Proof of Proposition 3. Let $k = \dim \mathrm{aff}(X)$. If $k \le 1$, then $D = \emptyset$ and, therefore, $X \setminus D$ is connected. Let $k \ge 2$. Under an appropriate affine transformation, $\mathrm{aff}(X)$ can be identified with $\mathbb{R}^k$. Thus, we may assume that X is a subset of $\mathbb{R}^k$. Suppose, to the contrary, that $X \setminus D$ is not connected. Then, there exist open subsets U and V of $\mathbb{R}^k$ such that $U \cap (X \setminus D) \ne \emptyset$, $V \cap (X \setminus D) \ne \emptyset$, $U \cap V \cap (X \setminus D) = \emptyset$, and $U \cup V \supset X \setminus D$.

Claim 7 There exists a rectangle $A = \prod_{j=1}^k [a^{(j)}, b^{(j)}]$ such that $a^{(j)} < b^{(j)}$ for every

$j \in \{1, \dots, k\}$, $A \subset X$, $U \cap A \neq \emptyset$, and $V \cap A \neq \emptyset$.

Proof of Claim 7. Let $x \in U \cap (X \setminus D)$ and $y \in V \cap (X \setminus D)$. Since V is open, by Proposition 2, there exists a $y' \in V \cap \mathrm{int}(X)$ such that $y' \neq x$ and $D \cap [x, y']$ is of the first category in $[x, y']$. Since $y' \in \mathrm{int}(X)$ and X is convex, we have $]x, y'] \subset \mathrm{int}(X)$. Define $t^* = \sup\{t \in [0,1] \,|\, (1-t)x + ty' \in U\}$. Since $x \in U$ and U is open, $t^* > 0$ follows. Let $z = (1-t^*)x + t^*y'$. Then, $z \in \mathrm{int}(X)$ and, therefore, there exists an $r > 0$ such that $A := \prod_{j=1}^{k}[z^{(j)} - r, z^{(j)} + r] \subset X$. By the definition of t^*, we have $U \cap A \neq \emptyset$. We prove that $V \cap A \neq \emptyset$. Suppose, to the contrary, that $V \cap A = \emptyset$. Since $y' \in V$, we have $t^* < 1$. Again, by the definition of t^*, $\{(1-t)x + ty' \,|\, t \in]t^*, 1]\} \cap U = \emptyset$. Since $V \cap A = \emptyset$, we have $\{(1-t)x + ty' \,|\, t \in]t^*, 1]\} \cap A \cap (U \cup V) = \emptyset$. From $U \cup V \supset X \setminus D$, it follows that $\{(1-t)x + ty' \,|\, t \in]t^*, 1]\} \cap A \subset D$. Since $\{(1-t)x + ty' \,|\, t \in]t^*, 1]\} \cap A$ is a nondegenerate segment in $[x, y']$, this contradicts that $D \cap [x, y']$ is of the first category in $[x, y']$. Then, we have proved that $V \cap A \neq \emptyset$. This completes the proof of Claim 7. ■

By Lemma 2, for every $i \in \{1, \dots, k\}$, the set $\mathrm{pr}^k_{-i}(D \cap A)$ is of the first category in $\prod_{j \neq i}[a^{(j)}, b^{(j)}]$. Thus, by Lemma 3, $A \setminus D$ is connected. Since U and V are open, we have $\mathrm{int}(U \cap A) \neq \emptyset$ and $\mathrm{int}(V \cap A) \neq \emptyset$, and, therefore, from Baire's category theorem, both sets $U \cap A$ and $V \cap A$ are of the second category in A. By Lemma 2, D is of the first category in $\mathbb{R}^k$. Hence, $U \cap (A \setminus D) \neq \emptyset$ and $V \cap (A \setminus D) \neq \emptyset$. In addition, we have $U \cap V \cap (A \setminus D) \subset U \cap V \cap (X \setminus D) = \emptyset$ and $U \cup V \supset X \setminus D \supset A \setminus D$. This contradicts that $A \setminus D$ is connected. Thus, $X \setminus D$ is connected. ■

Proposition 4 *Let X be a nonempty convex subset of $\mathbb{R}^L$ and $\succsim$ be a linearly continuous preference relation on X. Then, by using Propositions 2 and 3, from Eilenberg's (1941) theorem (see also Debreu (1959, 1964)), it can be shown that there exists a real-valued function $u : X \to \mathbb{R}$ such that (i) $a \succsim b$ if and only if $u(a) \geq u(b)$, (ii) $u(X)$ is an interval, and (iii) u is linearly continuous.*

Proof of Proposition 4. Let $D = \{x \in X \,|\, \succsim \text{ is not continuous at } x\}$. Then, from Proposition 3, $X \setminus D$ is connected. Since $X \setminus D$ is also separable, from Eilenberg's (1941) theorem (see also Debreu (1959, 1964)), there exists a continuous function u on $X \setminus D$ such that $a \succsim b$ if and only if $u(a) \geq u(b)$. Since u is continuous on $X \setminus D$ and $X \setminus D$ is connected, $u(X \setminus D)$ is an interval. We may assume that $u(X \setminus D)$ is bounded.[5]

Let $x \in D$. If there exist a and b in $X \setminus D$ with $a \succsim x \succsim b$, from the connectedness of $X \setminus D$ and the continuity of $\succsim$ on $X \setminus D$, there exists a $w_x \in X \setminus D$ with $w_x \sim x$. In this case, $u(x)$ is defined by $u(x) = u(w_x)$. Let $Y = \{x \in X \,|\, \text{there exists a } w_x \in X \setminus D \text{ with } w_x \sim x\}$. Then, u has been defined on Y. By construction, $u(Y)$ is a bounded interval, and for every $a, b \in Y$, $a \succsim b$ if and only if $u(a) \geq u(b)$.

By the definition of Y, for every $x \in X \setminus Y = D \setminus Y$, either (i) $x \succ a$ for every $a \in X \setminus D$, or (ii) $a \succ x$ for every $a \in X \setminus D$. For $x \in D \setminus Y$ with $x \succ a$ for every $a \in X \setminus D$, $u(x)$ is defined by $u(x) = \sup_{a \in X \setminus D} u(a)$ and for $x \in D \setminus Y$ with $a \succ x$ for every $a \in X \setminus D$, $u(x)$ is defined by $u(x) = \inf_{a \in X \setminus D} u(a)$. Then, u has been defined in the whole space X. Note that $u(X)$ is an interval. It remains to prove that u is a utility representation of $\succsim$. Before proving that, we prove the following claim.

Claim 8 Let $x \in D \setminus Y$.

(1) If $x \succ y$ (resp. $y \succ x$) for some $y \in X$, then there exists an $a \in X \setminus D$ with $a \succ y$ (resp. $y \succ a$).

(2) If $x \succ y$ (resp. $y \succ x$) for some $y \in Y$, then there exists an $a \in X \setminus D$ with $x \succ a \succ y$ (resp. $y \succ a \succ x$).

(3) If $x \succ y$ for some $y \in D \setminus Y$, then $x \succ a \succ y$ for every $a \in X \setminus D$.

Proof of Claim 8.

(1) We only prove the case where $x \succ y$ for some $y \in X$. By Proposition 2, there exists a $z \in X$ and a sequence $(x_n)_n$ in $(X \setminus D) \cap [x, z]$ such that $x_n \to x$. Since $\succsim$ is

[5]Since $\mathrm{Tan}^{-1} : \mathbb{R} \to]-\pi/2, \pi/2[$ is strictly monotone, continuous, and bounded, $\mathrm{Tan}^{-1} \circ u$ is also a continuous utility function on $X \setminus D$ such that $\mathrm{Tan}^{-1} \circ u(X \setminus D)$ is a bounded interval.

linearly continuous, the set $\{t \in I(x,z) \mid (1-t)x + tz \succ y\}$ is open in $I(x,z)$ and, therefore, $x_n \succ y$ for sufficiently large n. Since $x_n \in X \setminus D$ for every n, we have proved (1).

(2) We only prove the case where $x \succ y$ for some $y \in Y$. By (1), there exists an $a \in X \setminus D$ with $a \succ y$. Since $y \in Y$, there exists a $w_y \in X \setminus D$ with $x \succ w_y \sim y$. Since $x \in D \setminus Y$, this implies that $x \succ b$ for every $b \in X \setminus D$. Therefore, $x \succ a \succ y$.

(3) Let $x, y \in D \setminus Y$ with $x \succ y$. By (1), there exist $a, b \in X \setminus D$ with $a \succ y$ and $x \succ b$. Since $x, y \in D \setminus Y$, this implies that $x \succ a \succ y$ for every $a \in X \setminus D$.

This completes the proof of Claim 8. ■

Now we are ready to prove that u is a utility representation of $\succsim$. It suffices to prove that (a) $x \sim y$ implies $u(x) = u(y)$, and (b) $x \succ y$ implies $u(x) > u(y)$. Let x and y in X with $x \sim y$. Note that in this case, either $x, y \in Y$ or $x, y \in D \setminus Y$ holds. If $x, y \in Y$, it is clear that $u(x) = u(y)$. If $x, y \in D \setminus Y$ and if $x \sim y \succ a$ (resp. $a \succ x \sim y$) for every $a \in X \setminus D$, by definition $u(x) = \sup_{a \in X \setminus D} u(a) = u(y)$ (resp. $u(x) = \inf_{a \in X \setminus D} u(a) = u(y)$). Therefore, $x \sim y$ implies $u(x) = u(y)$.

Let x and y in X with $x \succ y$. If $x, y \in Y$, then it is clear that $u(x) > u(y)$. If $x \in D \setminus Y$ and $y \in Y$, then from Claim 8(2), there exists a $b \in X \setminus D$ such that $x \succ b \succ y$. Thus, $u(x) = \sup_{a \in X \setminus D} u(a) \geq u(b) > u(y)$. By a similar manner, we can prove the case where $x \in Y$ and $y \in D \setminus Y$. If $x, y \in D \setminus Y$, from Claim 8(3), $x \succ a \succ y$ for every $a \in X \setminus D$. From Claim 8(2), there exist $a, b \in X \setminus D$ such that $x \succ b \succ a \succ y$. Therefore, $u(x) = \sup_{z \in X \setminus D} u(z) \geq u(b) > u(a) \geq \inf_{z \in X \setminus D} u(z) = u(y)$. Therefore, $x \succ y$ implies $u(x) > u(y)$. Hence, u is a utility representation of $\succsim$.

Since $u(X)$ is an interval, from Remark 3, u is linearly continuous. ■

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