Working Papers

Institute of Mathematical Economics

415

March 2009

# Irreversible Investment in Oligopoly

Jan-Henrik Steg

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?415
ISSN: 0931-6558

# Irreversible Investment in Oligopoly

Jan-Henrik Steg
Institute of Mathematical Economics, IMW
Bielefeld University
jhsteg@wiwi.uni-bielefeld.de

**Abstract**

We offer a new perspective on games of irreversible investment under uncertainty in continuous time. The basis is a particular approach to solve the involved stochastic optimal control problems which allows to establish existence and uniqueness of an oligopolistic open loop equilibrium in a very general framework without reliance on any Markovian property. It simultaneously induces quite natural economic interpretation and predictions by its characterization of optimal strategies through first order conditions. The construction of equilibrium policies is then enabled by a stochastic representation theorem. A stepwise specification of the general model leads to further economic conclusions. We obtain explicit solutions for Lévy processes.

*JEL subject classification:* C73, D43, D92
*Keywords:* Irreversible Investment, Stochastic Game, Oligopoly, Real Options, Equilibrium

# 1 Introduction

The purpose of this work is to provide a new perspective on irreversible investment equilibria, which admits to derive very general results and interesting economic interpretations at the same time. Games of irreversible investment belong to the intersection of real options theory and game theory, which is becoming more and more important. It is widely acknowledged that a competitive environment may have a considerable impact on the valuation of real options, since when exercise strategies of opponents influence the value of the underlying asset, optimal policies cannot be determined in isolation, as in classical models of real options, see [8]. In an irreversible investment context, preemptive concerns reduce the classical option value of waiting, which requires that investment is undertaken only when the net present value exceeds some strictly positive threshold. If one introduces perfect competition, this option value is completely eliminated as intuition suggests. This case has been analysed in a very general framework by Baldursson and Karatzas [2]. We will be mainly interested in the intermediate setting of a finite number of players who may irreversibly invest in the same industry, as often and in amounts as small as they like. In typical instances, the determination of optimal investment policies will involve singular control problems. One is the model by Grenadier [9], who assumes that an inverse demand function determining spot revenue is influenced by a diffusion and investment costs are purely proportional. Optimization is then performed with the help of hypothetical myopic investors, which are also used by Baldursson and Karatzas. In the oligopolistic case, Back and Paulsen [1] thoroughly conduct this approach in the presence of a diffusion. Furthermore, they discuss the nature of equilibrium, it is based on open loop strategies. This means that investment happens conditional on the revelation of uncertainty but does not react to deviations by opponents. Our different approach does not rely on artificial myopic investors, neither do we need any Markovian assumption. In fact, it allows us to construct the unique oligopolistic open loop equilibrium at the same level of generality as the perfectly competitive equilibrium by Baldursson and Karatzas. A particular benefit is that optimal strategies are characterized by first order conditions, which give important economic insights and predictions by themselves. They have already been used by Bertola [7] in the context of a single price-taking firm. Since investment is irreversible, the stream of marginal revenue from any moment onwards must optimally never be worth more than current investment cost. Moreover, whenever investment happens, the agent optimally has to be indifferent towards a further infinitesimal unit. After deriving first order conditions along these lines for equilibrium, we give a constructive existence proof using a stochastic representation theorem dedicatively discussed by Bank and El Karoui [3]. In fact, we turn the inequality which the first order conditions form most of the time into an equality. This representation problem has a solution by the theorem alluded to. The stochastic process identified thereby allows to construct our equilbrium strategies. This approach has been introduced by Bank and Riedel [5] in the context of optimal intertemporal consumption, but has a much broader realm. To present it and illustrate its usefulness with respect to economic interpretation, we apply it to the case of perfect competition in Section 2. The results of Baldursson and Karatzas are reproduced with its help in a much more direct way. Specifically, the conditions for aggregate investment to support a perfectly competitive equilbrium are equivalent to first order conditions of the above type for a social planner's investment problem. We construct the social planner's optimal policy through direct application of the stochastic representation theorem and it delivers immediately the equilibrium exercise times. In Section 3 we consider the case of oligopoly, to derive an existence and uniqueness result of equilibrium in open loop strategies in a conceivably general framework. The only restrictions with respect to the underlying stochastics are to ensure measurability and integrability. Regarding the control variables, we will assume concavity of the profit flows in own capital and that opponent capital is not a too strong strategic complement if it tends to be. Given homogeneous firms, the equilibrium is symmetric. In Section 4, the general model will be stepwise specified to obtain more economic predictions. Cournot-type spot competition will be shown to induce that in equilibrium with heterogeneous initial capital levels, the smallest firm(s) necessarily will catch up before any other invests, which pushes symmetry. Also we will illustrate the diminishment of the option value of waiting under increasing competition. In the limit, when the number of competitors tends to infinity, a perfectly competitive equilibrium as in Section 2 is attained, with investment at the zero net present value threshold. Finally, we will derive explicit solutions when inverse demand is of constant elasticity and is multiplied by an exponential Lévy-process, so in a more general case than the original Grenadier model [9].

# 2 Perfect competition

Consider first a perfectly competitive setting, where an individual firm's action does not influence the revenue opportunities of any other firm in the industry. There is a non-atomic continuum $[0, \infty)$ of homogeneous investors, all owning a perpetual option to enter the market. Exercising such an option starts a noncallable stochastic profit stream. To model the underlying uncertainty, let $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. An entering strategy is then a stopping time $\tau$ with respect to the given filtration, that is the decision whether to exercise the option at any point in time $t$ has to be based on the information reflected by the $\sigma$-algebra $\mathscr{F}_t$. Although a single investor is negligibly small so that his entry does not increase the level of capital in the industry, the entering firms collectively generate an aggregate investment process. We identify the current capital stock, denoted $Q_t$, by the measure of firms having entered so far. Since exit is not allowed and depreciation abstracted from for expositional simplicity, the process $(Q_t)_{t\geq 0}$ will be nondecreasing, and an individual firm will take it as given. Formally, any such process belongs to $\mathscr{A}$, the class of feasible aggregate investment processes.

$$\mathscr{A} \triangleq \{Q \text{ adapted, nondecreasing, left-continuous, with } Q_0 = 0 \ \mathbf{P}\text{-a.s.}\}$$

The paths are considered to be left-continuous, so that new capital will become working *after* the information triggering investment has been learned. The capital stock clearly is assumed to influence any active firm's instantaneous profit, which is thus modeled as a random field $\pi(\omega, t, q) : \Omega \times [0, \infty) \times [0, \infty) \to \mathbb{R}$, where dependence on time $t$ incorporates possible discounting, and $q$ is some capital stock. Here, we chose an infinite horizon, but note that one might as well consider a finite horizon together with some scrap value function as terminal payoff, conditional on having entered the market before. Finally, we introduce the cost, at which any of the options can be exercised. It may be random as well and is thus formulated as a stochastic process $k$.

In order to give our model some more structure and to guarantee that a solution exists, let the profit function satisfy the following assumption.

**Assumption 1.**

i. For any $(\omega, t) \in \Omega \times [0, \infty)$, the mapping $q \mapsto \pi(\omega, t, q)$ is continuously decreasing from $\pi(\omega, t, 0) = +\infty$ to $\pi(\omega, t, +\infty) \leq 0$.

ii. For $q \in \mathbb{R}_+$ fixed, $(\omega, t) \mapsto \pi(\omega, t, q)$ is progressively measurable and $\mathbf{P} \otimes dt$-integrable.

Furthermore, we assign the following properties to the investment cost process $k$.

**Assumption 2.** The nonnegative process k is a right-continuous supermartingale with $k_0 < \infty$ and $k_\infty = 0$.

*Remark* 2.1. Assumption 2 is satisfied in the common case where the investment cost is constant but discounted at a nonnegative optional or deterministic rate. The supermartingale requirement however can be dropped, but then we would have to let instantaneous profit really become negative for large capital stocks. $k$ may then be any optional process with $k_\infty = 0$ and a finite supremum over all stopping times of $\mathbf{E}k_\tau$, and which is lower-semicontinuous in expectation.

## 2.1 Characterization of equilibrium

Since equilibrium as usual will require optimal individual behavior, we are now formalizing the optimal stopping problem of an individual firm. It faces a particular aggregate investment process $Q \in \mathscr{A}$, which results from the entering decisions of all other firms.

The firm then has to choose a rule of when to enter the market if at all, foresight not being possible, thus in form of a stopping time $\tau$ taking values in $[0,\infty]$. Let $\mathscr{T}$ denote the set of all such stopping times, being the strategy space of each individual firm. All firms evaluate a strategy $\tau \in \mathscr{T}$ by the implied expected payoff

$$j(\tau|Q) \triangleq \mathbf{E}\left[\int_\tau^\infty \pi(t,Q_t)\,dt - k_\tau\right] \quad (Q \in \mathscr{A}). \tag{2.1}$$

We call the supremum of expected payoff taken over all stopping times *option value*, depending of course on the given $Q$. Before further inquiring these generic individual optimization problems, remember that we are eventually looking for an equilibrium. So let us first argue that the requirement of optimal behavior on behalf of *all* firms, incumbents and persistent potential entrants, limits the observational variety of equilibrium outcomes. In fact, because staying outside the market gives zero profit and in our model there is always a positive measure of option holders not having exercised yet, further exercise at any stopping time cannot yield positive expected payoff in equilibrium. On the other hand, a positive measure of firms enter at any time of increase of aggregate investment. Further exercise at these times cannot yield negative expected payoff in equilibrium, because then due to the continuity assured in Assumption 1, some of the entering firms would better stay out. These considerations shall suffice to characterize a perfectly competitive equilibrium in exercise strategies by the resulting aggregate investment process.

**Definition 2.2.** $Q^* \in \mathscr{A}$ is a *perfectly competitive equilibrium* investment process if $\sup_{\tau\in\mathscr{T}} j(\tau|Q^*)$ — the option value given $Q^*$ — is zero, and exercising is optimal whenever $Q^*$ increases, i.e. at all stopping times $\tau^*(x) \triangleq \inf\{t \geq 0 | Q^*_t > x\}$, $x \in \mathbb{R}_+$.

Note that at the times when equilibrium investment increases all option holders are indifferent whether to exercise immediately or to keep waiting, possibly forever. Thus we conclude that there is an equilibrium in individual strategies where just enough firms enter at any such time to support the aggregate equilibrium investment. The reasoning up to this point will be formalized stronger when we determine the — as we will see unique — equilibrium investment process in the following.

For this purpose, let us now introduce a fictitious *social planner*, like it is common practice in finding perfectly competitive equilibria. Consider that this authority can control how many firms enter at each moment, but still without foresight. Its objective is to pursue an efficient irreversible investment process in the sense of maximizing the aggregate expected profit, net of investment cost. If the firm level profit flow $\pi$ is inverse demand minus variable production cost, the planner is benevolent in the classical meaning that consumer surplus shall be maximal while taking account of all incurred costs. Formally, this leads to the irreversible investment problem of maximizing

$$J(Q) \triangleq \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t)\, dt - \int_0^\infty k_t\, dQ_t\right] \tag{2.2}$$

over all $Q \in \mathscr{A}$, where the random field $\Pi : \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ relates to $\pi$ by

$$\Pi(\omega, t, q) = \int_0^q \pi(\omega, t, y)\, dy. \tag{2.3}$$

Consequently, it inherits the measurability assertion of Assumption 1 and is concave in capital with continuous partial derivative $\Pi_q \triangleq \partial\Pi/\partial q$. Furthermore, by the integrability assumption, the negative part of $\Pi$ is also $\mathbf{P} \otimes dt$-integrable for fixed $q \in \mathbb{R}_+$.

By (2.3), we know a lower bound on achievable revenue, but to have a meaningful stochastic control problem, we impose the additional

**Assumption 3.** The process $(\omega, t) \mapsto \sup_{q\in\mathbb{R}_+} \Pi(\omega, t, q)^+$ is $\mathbf{P}\otimes dt$-integrable.

In combination with Assumption 2, the value of the problem is finite and it suffices to consider *admissible* controls with bounded expected cost. Since the problem is of the monotone follower type with concave objective functional $J$, we solve it by the approach developped by Bank and Riedel [5] in the context of intertemporal utility maximization. Thus, the solution is characterized by a first order condition, and then an optimal control policy will be constructed with the help of a stochastic representation theorem. This methodology has proven to be useful for a variety of (so far single agent) control problems, see [4]. For our purpose, it is very illustrative, because the relation between the social planner's control problem and equilibrium determination becomes immediate in the first order condition for an optimal control policy. The latter is based on the following gradient, which has also been used by Bertola [7] for a more specific single-agent problem. Let $\nabla J(Q)$ denote for any $Q \in \mathscr{A}$ the unique optional process such that

$$\nabla J(Q)_\tau = \mathbf{E}\left[\int_\tau^\infty \Pi_q(t, Q_t)\, dt \,\middle|\, \mathscr{F}_\tau\right] - k_\tau \quad \text{for all stopping times } \tau \in \mathscr{T}. \tag{2.4}$$

Heuristically, it describes the marginal profit from *irreversible* investment at any stopping time, see the discussion below. The first order condition in terms of this gradient in fact coincides with our argued definition of an equilibrium investment process as formalized in the following proposition.

**Proposition 2.3.** *If Assumptions 1,2, and 3 are satisfied, a control policy $Q^* \in \mathscr{A}$ maximizes the social planner's objective* (2.2) *if it is a perfectly competitive equilibrium investment process according to Definition 2.2, because then*

$$\nabla J(Q^*) \leq 0 \quad \textit{and} \quad \int_0^\infty \nabla J(Q^*)_s \, dQ^*_s = 0 \quad \mathbf{P}\text{-}a.s. \tag{2.5}$$

*Proof.* We first show the claimed optimality, so let $Q^* \in \mathscr{A}$ satisfy (2.5). The equality therein implies that $J(Q^*) \geq 0$ and consequently that the expected investment cost of $Q^*$ is finite, so the control is admissible. To see this, use the process $Q^0 \equiv 0$ in the estimation below. But generally, consider some arbitrary $Q \in \mathscr{A}$ with $J(Q) > -\infty$. Since $\Pi$ is concave in $q$ by its definition (2.3) and Assumption 1, we can estimate

$$\begin{aligned} J(Q) - J(Q^*) &= \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t) - \Pi(t, Q^*_t)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\ &\leq \mathbf{E}\left[\int_0^\infty \Pi_q(t, Q^*_t)\big(Q_t - Q^*_t\big)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\ &= \mathbf{E}\left[\int_0^\infty \Pi_q(t, Q^*_t)\Big(\int_0^t d(Q_s - Q^*_s)\Big)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\ &= \mathbf{E}\left[\int_0^\infty \int_s^\infty \Pi_q(t, Q^*_t)\, dt \, d(Q_s - Q^*_s) - \int_0^\infty k_s \, d(Q_s - Q^*_s)\right] \\ &= \mathbf{E}\left[\int_0^\infty \nabla J(Q^*)_s \, d(Q_s - Q^*_s)\right]. \end{aligned}$$

In the second last line, we use Fubini's theorem to change the order of integration. By the first order condition (2.5), the last expression above is nonpositive. So we conclude $J(Q) \leq J(Q^*)$.

Now we show that if $Q^* \in \mathscr{A}$ is a perfectly competitive equilibrium investment process according to Definition 2.2, it satisfies (2.5). Remember $\Pi_q = \pi$ and an individual firm's objective (2.1). So, the Definition 2.2 of an equilibrium investment process translates into (i) $\mathbf{E}[\nabla J(Q^*)_\tau] \leq 0$ for all stopping times $\tau \in \mathscr{T}$, which implies the inequality in (2.5), and (ii) $\mathbf{E}[\nabla J(Q^*)_{\tau^*(x)}] = 0$ for all $x \in \mathbb{R}_+$ and $\tau^*(x)$ as in Definition 2.2. To deduce the required equality, note that $\tau^*$ is the right-continuous inverse of the monotone $Q^*$ (see also (2.8) below). This permits to use the change-of-variable formula

$$\int_0^\infty \nabla J(Q^*)_s \, dQ^*_s = \int_0^\infty \nabla J(Q^*)_{\tau^*(x)} \, dx \quad \mathbf{P}\text{-a.s.},$$

cf. [2]. The integrand on the right-hand side is zero **P**-a.s. by the equilibrium property, which completes the proof. □

The intuition conveyed by the first order condition reveals the connection between the optimal control problem and equilibrium determination. Our social planner may consider to increase his investment at any stopping time. This incremental investment may be arbitrarily small, so let us think of it as infinitesimal. Then, since investment is irreversible, the reward is a flow of marginal profit from that moment onwards. Given some investment plan already worked out, the profitability calculation for such an additional bit is thus the same as that of an individual firm owning the option to enter a market with the same assumed capital expansion. Optimality of investment of course requires that expected payoff cannot be increased by even an infinitesimal additional investment, which corresponds to no individual firm strictly preferring to enter in equilibrium. On the other hand, there must not be regret of having invested even infinitesimally too much, like a firm's of having entered.

Note that if there exists an optimal control policy, it will be unique due to strict concavity of the planner's objective functional $J(Q)$. So, solving his investment problem is really equivalent to finding a perfectly competitive equilibrium of our initial game. However, the above characterization of an optimal control is not more constructive yet, since it takes the form of an inequality most of the time. To overcome this difficulty, we now make use of the mentioned stochastic representation theorem.

## 2.2 Construction of equilibrium investment

The heart of our chosen approach to optimal stochastic control is the formulation of a stochastic representation problem, turning the first order condition into an equality. Its solution provides a quite direct way to construct the optimal control policy. Our assumptions made so far will suffice to guarantee its existence. The representation problem is to find the (unique) optional process $l$ satisfying

$$\mathbf{E}\left[\int_\tau^\infty \pi(t, \sup_{\tau \le u < t} l_u) \, dt \middle| \mathscr{F}_\tau \right] - k_\tau = 0 \quad \text{for all stopping times } \tau \in \mathscr{T}. \tag{2.6}$$

In comparison to the first order condition (2.5), we replaced $Q^*$ by the running supremum (reset at $\tau$) of the process $l$ to be determined, while enforcing equality to hold $\mathbf{P}$-a.s. Bank and El Karoui [3] discuss this representation problem in detail and we can use their central result [3, Theorem 3] to assure the existence of a solution $l$ to (2.6) under our Assumptions 1,2, and 3. For practical purposes, the representation problem will typically be solved numerically by backward induction, backed by the theoretical foundation of existence and uniqueness. But under some quite common specifications of $\pi$ and $k$, one can derive closed-form solutions. We will discuss these in the oligopoly case below, the limit of which turning out to be the present perfectly competitive equilibrium.

Once we derived the seemingly abstract process $l$, we obtain the social planner's optimal control policy — resp. the perfectly competitive equilibrium investment process — as follows.

**Theorem 2.4.** *Under Assumptions 1,2, and 3, the unique maximizer for the social planner's objective functional* (2.2) *is given by*

$$Q_t^* \triangleq \big( \sup_{0 \le u < t} l_u \big)^+ \quad (t \ge 0), \tag{2.7}$$

*where $l$ is the optional process solving the representation problem* (2.6)*.*

*Proof.* The process $Q^*$ defined by (2.7) clearly belongs to $\mathscr{A}$. Thus we only need to show that it satisfies the first order condition (2.5). Indeed, for any stopping time $\tau \in \mathscr{T}$ we have by this definition of $Q^*$ and since $\pi$ is decreasing in $q$

$$\begin{aligned} \nabla J(Q^*)_\tau &= \mathbf{E}\left[ \int_\tau^\infty \pi(t, Q_t^*)\, dt \middle| \mathscr{F}_\tau \right] - k_\tau \\ &\le \mathbf{E}\left[ \int_\tau^\infty \pi(t, \sup_{\tau \le u < t} l_u)\, dt \middle| \mathscr{F}_\tau \right] - k_\tau, \end{aligned}$$

where the last expression is zero exactly by representation (2.6). So, by arbitrariness of $\tau$, $\nabla J(Q^*)$ is nonpositive. To check the equality in (2.5) holds true $\mathbf{P}$-a.s., note that if $dQ_s^* > 0$, we may ignore for any $t > s$ the earlier history of $l$, then $Q_t^* = \sup_{0 \le u < t} l_u = \sup_{s \le u < t} l_u > 0$ for all $t > s$, implying $\nabla J(Q^*)_s = 0$. □

Combining our results up to this point, we completely described the equilibrium investment process we were looking for. If an individual firm expects aggregate investment to follow the social planner's preferred control, it is optimal for the firm to exercise its entry option at any time of increase of this control. Importantly, optimal entry timing merely yields zero expected net profit, implying that we may expect consistency of individual with aggregate behavior. By this requirement, we have somewhat circumvented solving the individual firms' optimal stopping problems. Nevertheless, they remain the basic components of equilibrium, so they shall be presented properly, too.

## 2.3 Optimal entry times

A further motivation to formally analyze the individual stopping time problems is to illustrate the familiar connection between (singular) optimal stochastic control and optimal stopping. In this context, the value of our pursued approach will become quite clear, namely that it provides a more direct way of solving for the equilibrium than the earlier proposed way via artificial myopic agents.

First, consider the optimal stopping problem of a firm rationally expecting the investment process identified by Theorem 2.4 to prevail. Recall the objective defined in (2.1). It is maximized by the stopping times used in Definition 2.2 of an equilibrium investment process.

**Corollary 2.5.** *Given the conditions of Theorem 2.4 and the process $Q^*$ identified therein, $j(\tau|Q^*)$ is maximized by any stopping time $\tau^*(x) = \inf\{t \geq 0 | Q^*_t > x\}$, $x \in \mathbb{R}_+$. Then, the option value is $j(\tau^*(x)|Q^*) = 0$.*

*Proof.* Use the definition of $Q^*$ and the representation (2.6) of $k_\tau$ to obtain for any $\tau \in \mathscr{T}$

$$\begin{aligned} j(\tau|Q^*) &= \mathbf{E}\left[\int_\tau^\infty \pi(t, Q^*_t)\, dt - k_\tau\right] \\ &= \mathbf{E}\left[\int_\tau^\infty \pi(t, \sup_{0\leq u<t} l_u)\, dt - \int_\tau^\infty \pi(t, \sup_{\tau\leq u<t} l_u)\, dt\right] \end{aligned}$$

As $\pi$ is decreasing in $q$, the last expectation is nonpositive. Now, fix an $x \in \mathbb{R}_+$ and the corresponding $\tau^*(x) \in \mathscr{T}$. Then, for any $t > \tau^*(x)$ by the definition of $Q^*$, $\sup_{0\leq u<t} l_u = \sup_{\tau^*(x)\leq u<t} l_u$ and the two integrands cancel. Thus, $\tau^*(x)$ is optimal. □

This further perspective formally completes our perfectly competitive equilibrium. From the process $l$ solving representation problem (2.6), we have obtained aggregate investment $Q^*$, and the record-setting times of $l$ also yield the optimal entry times. Let us now compare this procedure to the approach involving myopic agents, which are basically just a construct to aid interpretation when the singular control problem is solved via Snell envelopes. These hypothetical agents solve similar stopping problems as rational ones, they only assume that aggregate capital remains fixed forever at some level, say $x \geq 0$. In all other respects, they have the same knowledge as the rational agents. Formally, the myopic agents evaluate any stopping time $\tau \in \mathscr{T}$ they may choose by

$$j^m(\tau|x) \triangleq \mathbf{E}\left[\int_\tau^\infty \pi(t,x)\,dt - k_\tau\right].$$

By the argument used in Corollary 2.5 it is easy to see that $\tau^*(x)$ is optimal for a myopic firm facing the specific capital level $x$. Thus, it of course also turns out here that the set of optimal stopping times for all myopic agents is the same as that of the rational ones derived before.

The relation between the equilibrium investment process and the myopic optimal stopping times can also be expressed in the reverse way,

$$Q_t^* = \sup\{x \in [0,\infty) : \tau^*(x) < t\} \quad t \in [0,\infty). \tag{2.8}$$

This is actually how Baldursson and Karatzas [2] determine the equilibrium investment process. To do so, first the myopic stopping time for every possible capital level has to be known, with amounts to calculating a continuum of Snell envelopes. In contrast, our approach directly solves for the investment process and delivers the stopping times as an immediate consequence, without the necessity to consider myopic agents.

# 3 Oligopoly

Now that we have demonstrated the concepts we will keep working with, we move on to study an oligopolistic industry, i.e. in which individual firms *can* influence the state of the industry by their investment decisions. In fact, we will derive an oligopolistic equilibrium at the same level of generality as for the perfectly competitive case before. Note that in particular we have made extremely little assumptions regarding the underlying uncertainty, for instance we have never relied on any Markov property. The most specific restriction of our model has been the monotone dependence of instantaneous profit on aggregate capital. Familiar Cournot-type instances, which actually belong to this class, will be discussed in the subsequent section. But first, in this section, we show that our approach to optimal control can handle a very general model of oligopolistic irreversible investment.

So, consider now that instead of a continuum there are $n$ homogeneous firms with the option to repeatedly invest in the same underlying industry, at any time and in amounts as small as they like. As before, anticipation as well as capital retrieval is impossible, so formally the strategy spaces coincide with that of the social planner above, $\mathscr{A}$. Let $Q^i \in \mathscr{A}$ denote the strategy chosen by firm $i$, $i = 1 \dots n$. Again, we assume that the instantaneous profit of each firm depends not only on its own activity but also on aggregate capital in the industry, on $Q \triangleq \sum_{j=1\dots n} Q^j \in \mathscr{A}$. Reflecting the point of view of firm $i$, this dependency will be equivalently modeled by accounting for its opponent capital $Q^{-i} \triangleq Q - Q^i$. Given a combination of strategies from $\mathscr{A}^n$, firm $i$ then receives the payoff

$$J^i(Q^i|Q^{-i}) \triangleq \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t^i, Q_t^{-i})\, dt - \int_0^\infty k_t\, dQ_t^i\right], \tag{3.1}$$

where we redefine the random field $\Pi : \Omega \times [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ to include opponent capital. Consequently, we have to make new assumptions to clarify the properties we require.

**Assumption 4.**

i. For any $(\omega, t) \in \Omega \times [0, \infty)$, the mapping $(q^i, q^{-i}) \mapsto \Pi(\omega, t, q^i, q^{-i})$ is continuously differentiable. For $q^{-i} \in \mathbb{R}_+$ fixed, the partial derivative $\Pi_{q^i} \triangleq \partial\Pi/\partial q^i$ decreases in $q^i$.

ii. For $(q^i, q^{-i}) \in \mathbb{R}_+^2$ fixed, $(\omega, t) \mapsto \Pi(\omega, t, q^i, q^{-i})$ is progressively measurable and $\mathbf{P} \otimes dt$-integrable.

iii. For any $Q \in \mathscr{A}$, $\Pi(\omega, t, 0, Q_t) = 0$ $\mathbf{P}$-a.s.

iv. The process $(\omega, t) \mapsto \sup_{(q^i, q^{-i}) \in \mathbb{R}_+^2} \Pi(\omega, t, q^i, q^{-i})^+$ is $\mathbf{P} \otimes dt$-integrable.

This is a quite natural extension of Assumptions 1 and 3 to account for opponent capital as a further parameter. Note that revenue is still concave in own capital. Hence, the integrability assumption on $\Pi$ also applies to marginal revenue $\Pi_{q^i}$. We assume again that there is no revenue as long as no capital has been invested. Finally, we already added a condition to encounter finite optimization problems. These restrictions are sufficient to characterize best replies, but to construct the equilibrium, we will need a further condition.

**Assumption 5.** $\Pi_{q^i}$ decreases in $q^i$ along the ray $q^{-i} = (n-1)q^i$ from $+\infty$ to a nonpositive value.

*Remark* 3.1. Assumption 5 concerns the relative influences of own and opponent capital on marginal revenue and also appears in the literature on Cournot competition [10, Sec. 4.2]. Formulated in terms of second derivatives, it is in subscript notation

$$\Pi_{q^i q^i} + (n-1) \cdot \Pi_{q^i q^{-i}} \leq 0. \tag{3.2}$$

It is among the weakest known requirements to guarantee uniqueness of equilibrium in the static Cournot game with payoff $\Pi$, where we would neglect investment cost. A sufficient condition for this property is that for $q^i \in \mathbb{R}_+$ fixed, $\Pi_{q^i}$ does not increase in $q^{-i}$, which would also imply existence of the static game's equilibrium.

Regarding the investment cost process $k$, Assumption 2 shall remain valid. Thus, since revenue opportunities are again limited by assumption, we only consider admissible strategies with finite expected cost.

That each firm views aggregate opponent investment as a given adapted process has a clear implication for the type of equilibrium we will derive, since it restricts interaction. This is not because we do not include all individual capital levels in each profit stream, which would mainly complicate things just by notation. It is because firms cannot condition their investment during the run of the game on deviating capital levels. The firms choose investment plans contingent only on the revelation of information by nature, once at the beginning of the game. After having committed to these plans, there is no more strategic interaction. Thus we have to classify the available strategies as *open loop strategies*, and to add this connotation to our equilibrium concept.

**Definition 3.2.** $(Q^{*_1}, \ldots, Q^{*_n}) \in \mathscr{A}^n$ is an *open loop investment equilibrium* if $Q^{*_i}$ maximizes $J^i(Q^{*_i}|Q^{*_{-i}})$ over $\mathscr{A}$ for all $i = 1 \ldots n$, where $Q^{*_{-i}} = \sum_{j=1\ldots n, j\neq i} Q^{*_j}$.

In this setting, the optimal irreversible investment problem of each firm $i$, which faces a given process $Q^{-i} \in \mathscr{A}$ of opponent investment, is structurally the same as that of the social planner above, since then $\Pi(\omega, t, q^i, Q_t^{-i}(\omega))$ with Assumptions 4 and 5 satisfies Assumptions 1 and 3 (where of course $\Pi_{q^i} = \pi$). Thus, we can solve firm $i$'s problem using the results already derived. Let us state without further proof the first order condition which characterizes optimal investment for each firm for the sake of completeness and easier reference. It is now formulated in terms of the gradient $\nabla J^i(Q^i|Q^{-i})$, which is for any $(Q^i, Q^{-i}) \in \mathscr{A}^2$ the unique optional process satisfying $\nabla J^i(Q^i|Q^{-i})_\tau = \mathbf{E}[\int_\tau^\infty \Pi_{q^i}(t, Q_t^i, Q_t^{-i})\, dt | \mathscr{F}_\tau] - k_\tau$ for all stopping times $\tau \in \mathscr{T}$.

**Proposition 3.3.** *If Assumptions 2 and 4 are satisfied, a control policy $Q^{*i} \in \mathscr{A}$ maximizes firm i's objective* (3.1) *for a given process $Q^{-i} \in \mathscr{A}$ if*

$$\nabla J^i(Q^{*i}|Q^{-i}) \leq 0 \quad \textit{and} \quad \int_0^\infty \nabla J^i(Q^{*i}|Q^{-i})_s \, dQ^{*i}_s = 0 \quad \mathbf{P}\text{-}a.s. \tag{3.3}$$

As before, if $Q^{*i}$ satisfies the equality in (3.3), it yields nonnegative payoff and is an admissible strategy. Instead of deriving the best response of each firm to every possible opponent investment process and then searching for a fixed point in function space, we will directly identify the unique symmetric equilibrium using the ideas presented in the previous section. Indeed, considering the first order condition (3.3) for optimal individual behavior in combination with hypothesized symmetry leads us to formulate a similar stochastic representation problem as above, the solution to which will again let us quite directly identify equilibrium investment. For this aim, suppose that we want to find an optimal investment process for firm $i$, by the illustrated approach of first turning its first order condition into an equality with the help of an auxiliary process $L$. Furthermore suppose that the investment of all opponents *happens to coincide* with that of firm $i$. Then, the representation problem becomes to find an optional process $L$ that satisfies

$$\mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, \sup_{\tau \leq u < t} L_u, (n-1) \cdot \sup_{\tau \leq u < t} L_u) \, dt \middle| \mathscr{F}_\tau \right] - k_\tau = 0 \quad \text{for all } \tau \in \mathscr{T}. \tag{3.4}$$

Our Assumptions 4 and 5, together with Assumption 2 concerning $k$, warrant that we can still apply the result by Bank and El Karoui [3, Theorem 3] to infer existence of a unique solution $L$. It allows us to directly construct the symmetric oligopolistic equilibrium as follows.

**Theorem 3.4.** *Under Assumptions 2, 4 and 5, the unique symmetric open loop investment equilibrium is given by*

$$Q^{*i}_t \triangleq \big(\sup_{0 \leq u < t} L_u\big)^+ \quad (t \geq 0) \tag{3.5}$$

*for all $i = 1 \dots n$, where $L$ is the optional process solving the representation problem* (3.4).

*Proof.* The process $Q^{*i}$ defined as above clearly belongs to $\mathscr{A}$. We only need to show that it satisfies the first order condition (3.3) if all opponents behave identically to firm $i$. Indeed, for any stopping time $\tau \in \mathscr{T}$ we have due to the monotonicity by Assumption 5 and the definition of $Q^{*i}$

$$\begin{aligned}\nabla J^i(Q^{*i}|Q^{*-i})_\tau &= \mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, Q_t^{*i}, Q_t^{*-i})\,dt\,\middle|\,\mathscr{F}_\tau\right] - k_\tau \\ &\leq \mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, \sup_{\tau\leq u<t} L_u, (n-1)\cdot \sup_{\tau\leq u<t} L_u)\,dt\,\middle|\,\mathscr{F}_\tau\right] - k_\tau,\end{aligned}$$

where the last expression is zero exactly by representation (3.4). To check that the equality in (3.3) holds true $\mathbf{P}$-a.s., consider $dQ_s^{*i} > 0$. Then, $Q_t^{*i} = \sup_{0\leq u<t} L_u = \sup_{s\leq u<t} L_u > 0$ for all $t > s$, implying the required equality. □

Thus, to find the symmetric open loop equilibrium for an oligopoly, we only have to solve the backward equation (3.4), given any specification of our model primitives. While this constructive existence and uniqueness result is appealing for its generality, we are of course also interested in some concise economic predictions. These however have to await some stepwise specialization of the competitive setting, which we will conduct in the next section. For certain familiar cases, we will even obtain closed form solutions. Yet, we will answer an important question from the economic point of view while we are still in the general framework, because it arises from an even further generalization.

## 3.1 Asymmetric capital levels

Namely, we now allow for some heterogeneity by considering that the firms may have individual levels of capital already installed at the beginning of the above game. This situation is not only important at the start, but it also mimics possible intermediate stages of the game and thus has some predictive power. Before we adapt all concerned notions and results, let us analyze the situation, to see how much we have to revise.

For what we want to show, it is necessary to state a little more precisely the relative influences of own and opponent capital on marginal revenue. In fact, assume that for a fixed level of aggregate capital, marginal revenue is the smaller, the greater own installed capital is. This assumption is for instance very naturally satisfied for Cournot-type competition, since it follows from inverse demand, resp. price, being decreasing in aggregate supply.

**Assumption 6.** For any $(\omega, t, q) \in \Omega \times [0,\infty) \times [0,\infty)$, $\Pi_{q^i}(\omega, t, q^i, q - q^i)$ decreases in $q^i, 0 \leq q^i \leq q$.

Then, as long as the levels of installed capital are not all the same, only the smallest firm(s) will invest. Since this result will easily generalize, consider for the sake of the argument the case of two firms.

**Proposition 3.5.** *Set $n = 2$. Assume that firm $i$, $i = 1, 2$, has capital $\underline{Q}^i$ installed before the investment game starts, with $\underline{Q}^1 > \underline{Q}^2$. Then, if Assumptions 2, 4, 5 and 6 are satisfied, in an open loop equilibrium, $dQ_s^{*1} = 0$ as long as $\underline{Q}^1 > \underline{Q}^2 + \int_0^t dQ_s^{*2}$.*

*Proof.* Interpret the equilibrium investment processes as including the respective initial capital, i.e. $Q_0^{*i} = \underline{Q}^i$, $\mathbf{P}$-a.s. Suppose firm 1 invests at time $\tau_1$, before firm 2 invests for the first time at $\tau_2$. Then the first order conditions (3.3) at $\tau_1$ become

$$\mathbf{E}\left[\int_{\tau_1}^{\tau_2} \Pi_{q^i}(t, Q_t^{*1}, Q_0^{*2})\, dt \middle| \mathscr{F}_{\tau_1}\right] + \mathbf{E}\left[\int_{\tau_2}^{\infty} \Pi_{q^i}(t, Q_t^{*1}, Q_t^{*2})\, dt \middle| \mathscr{F}_{\tau_1}\right] - k_{\tau_1} = 0 \tag{3.6}$$

for firm 1 and

$$\mathbf{E}\left[\int_{\tau_1}^{\tau_2} \Pi_{q^i}(t, Q_0^{*2}, Q_t^{*1})\, dt \middle| \mathscr{F}_{\tau_1}\right] + \mathbf{E}\left[\int_{\tau_2}^{\infty} \Pi_{q^i}(t, Q_t^{*2}, Q_t^{*1})\, dt \middle| \mathscr{F}_{\tau_1}\right] - k_{\tau_1} \leq 0 \tag{3.7}$$

for firm 2.

Since over the interval $\tau_1 \leq t < \tau_2$ firm 1 is larger, $Q_t^{*1} > Q_t^{*2} = Q_0^{*2}$ by hypothesis, Assumption 6 implies that the conditional expectation of marginal revenue over this interval is greater for firm 2. Furthermore since firm 2 optimally invests at $\tau_2$, its second conditional expectation equals $\mathbf{E}[k_{\tau_2}|\mathscr{F}_{\tau_1}]$. Also due to the first order condition, the second conditional expectation of firm 1 cannot exceed $\mathbf{E}[k_{\tau_2}|\mathscr{F}_{\tau_1}]$. Summing up, the left hand side of (3.7) is greater than that of (3.6), clearly a contradiction to optimality. We conclude that firm 1 will not invest in equilibrium as long as it has more capital installed than firm 2. □

The result allows us to extend the game in the following way. Let there be a vector $(\underline{Q}^1, \ldots, \underline{Q}^n) \in \mathbb{R}_+^n$. Then the strategy space for firm $i$, $i = 1, \ldots, n$ is

$$\mathscr{A}^i \triangleq \{Q^i \text{ adapted, nondecreasing, left-continuous, with } Q_0^i = \underline{Q}^i \ \mathbf{P}\text{-a.s.}\}.$$

Proposition 3.5 now tells us that it is not difficult to adjust Theorem 3.4 for the construction of an open loop equilibrium, since we know that the smallest firms will catch up before any other invests. Once all firms are equally sized, they act identically as suggested by the theorem.

# 4 Cournot competition

For a further analysis from an economic perspective, we will now specify instantaneous revenue some more. The first step is to consider spot Cournot competition. This will be modelled by an inverse demand function with aggregate supply set equal to installed capital. Uncertainty is reflected in the spot price, say it depends on some stochastic process. Formally, let revenue in the following be given by

$$\Pi(\omega, t, q^i, q^{-i}) = e^{-rt} P(X_t(\omega), q^i + q^{-i}) q^i,$$

where the stochastic process $X : \Omega \times [0, \infty) \to \mathbb{R}$ captures randomness and $P : \mathbb{R} \times [0, \infty) \to [0, \infty)$ shall be continuous and have the usual property that for given $x \in \mathbb{R}$, the mapping $q \mapsto P(x, q)$ is decreasing in $q$. For ease of notation, assume that the positive discount factor $r$ is fixed, and in the same spirit set the spot price of capital equal to one, so $k_t = e^{-rt}$, which satisfies Assumption 2. However, Assumptions 4 and 5 impose some restrictions on the choice of $P$ and $X$. Since we will focus on the dependence on $q$, assume regarding the randomness simply that $X$ is sufficiently well behaved. Concerning capital, $P$ is required to be continuously differentiable in $q$, so denote the partial derivative by $P_q$, which is negative by our specification. Then, the monotonicity assumption implies that $P$ must not be too convex, if at all. For a symmetric $n$-firm equilibrium, we need in terms of the second partial derivative

$$(n + 1)P_q + qP_{qq} < 0 \quad (q \in \mathbb{R}_+), \tag{4.1}$$

which is equivalent to (3.2) in combination with symmetry.

This specification already enables us to draw some conclusions, the first of which has already been mentioned. Namely, revenue defined as above satisfies Assumption 6. Observe that, for given aggregate capital $q \in \mathbb{R}_+$,

$$\Pi_{q^i}(\omega, t, q^i, q - q^i) = P(X_t(\omega), q) + q^i P_q(X_t(\omega), q)$$

is indeed decreasing in $q^i$, since we specified $P_q$ to be negative. Thus, Cournot-type competition implies by Proposition 3.5 the catching-up property in any open loop equilibrium with heterogeneous starting levels. This result, that firms will eventually be of equal size, given sufficient incentive to invest, is actually reflected in the related game with perfectly reversible investment. Here, the optimal capital level equates marginal revenue and the user cost of capital, $r$ in our current setting. So the optimal reversible capital level is a function of the realisation of the process $X$, say $R^{*i}(x)$. Formally, in equilibrium, $P(x, R^*(x)) + R^{*i}(x)P_q(x, R^*(x)) = r$ for all firms $i = 1 \ldots n$, where again $R^*(x) = \sum_{i=1 \ldots n} R^{*i}(x)$. Consequently, all firms must choose the same reversible equilibrium output, $R^{*i}(x) = \frac{1}{n} R^*(x)$.

The current level of specification also lends itself to illustrate the value of waiting to invest and that it diminishes with increasing competition. Let us take a look at the first order condition (3.3) in a symmetric equilibrium of the current setting. For any $\tau \in \mathscr{T}$ it takes the form

$$\mathbf{E}\left[\int_\tau^\infty e^{-rt}\Big(P(X_t, Q_t^*) + \frac{Q_t^*}{n} P_q(X_t, Q_t^*)\Big)\, dt \,\Big|\, \mathscr{F}_\tau\right] - e^{-r\tau} \leq 0, \tag{4.2}$$

where we neglect to indicate that equilibrium investment $Q^*$ varies with $n$. If we increase competition, i.e. the number of firms, the partial derivative $P_q$ looses weight until we arrive in the limit at the first order condition for a perfectly competitive equilibrium,

$$\mathbf{E}\left[\int_\tau^\infty e^{-rt} P(X_t, Q_t^*)\, dt \,\Big|\, \mathscr{F}_\tau\right] - e^{-r\tau} \leq 0. \tag{4.3}$$

Here, because equality holds at any time of investment, the expected revenue generated by the last infinitesimal unit of capital minus its cost is zero, so it has zero net present value. If we consider now a time of investment in the oligopoly equilibrium, when (4.2) is binding, we conclude that the capital level there must be lower than in the perfectly competitive equilbrium at the same time, since $P_q$ is negative and otherwise (4.3) would be violated. Thus, investment in oligopoly is slower than under perfect competition, and only happens when it has a strictly positive net present value. But with an increasing number of firms this value of the option to wait diminishes until the zero net present value investment rule is finally reached.

## 4.1 Explicit solutions

Now we take a further step in specifying the model, to demonstrate the derivation of explicit solutions. In the above, let uncertainty influence inverse demand as a factor. To ensure that it does not become negative, let $X$ be an exponential process. Formally, we set

$$P(x, q) = x \cdot p(q) \quad \text{and} \quad X_t = e^{Y_t},$$

with a decreasing function $p : [0, \infty) \to [0, \infty)$ and a Lévy-process $(Y_t)_{t \geq 0}$ without negative jumps. Precisely, let inverse demand be of constant elasticity, which means

$$p(q) = q^{-\frac{1}{\alpha}},$$

where $\alpha$ is positive to ensure that price decreases in quantity. This is basically the model considered by Grenadier [9], but we allow for more general stochastic processes than geometric brownian motion. Now, the single structural restriction (4.1) we have to make is equivalent to $\alpha > \frac{1}{n}$, required in

[9], too. The integrability requirements of Assumption 4 depend of course on a concrete process $Y$, assume they are satisfied. Given these conditions, let us solve the representation problem (3.4) to construct the unique open loop equilibrium. Observe first that in the current setting marginal instantaneous revenue is given by

$$\Pi_{q^i}(t, q^i, q^{-i}) = e^{-rt} X_t (q^i + q^{-i})^{-\frac{1}{\alpha}} \Big(1 - \frac{1}{\alpha} \frac{q^i}{q^i + q^{-i}}\Big),$$

where in a hypothesized symmetric situation $q^i + q^{-i} = n \cdot q^i$. We now guess that for a given $n$, the solution $L$ to representation problem (3.4) takes the form

$$L_t = \frac{1}{n} \kappa^\alpha X_t^\alpha \quad (t \geq 0), \tag{4.4}$$

with some constant parameter $\kappa$. Consequently, investment in equilibrium given by $Q_t^{*i} = \sup_{0 \leq u < t} L_u, i = 1 \dots n$, will occur whenever the factor $X$ sets a new record. Such a policy is what one would intuitively expect for Markovian processes positively influencing revenue. It seems that aggregate investment is independent of the number of firms, but $\kappa$ will actually depend on $n$. Plugging the hypothesized process $L$ into (3.4) with marginal revenue as above yields for any $\tau \in \mathscr{T}$

$$\mathbf{E}\Big[\int_\tau^\infty e^{-rt} X_t \Big(n \sup_{\tau \leq u < t} \frac{1}{n} \kappa^\alpha X_u^\alpha\Big)^{-\frac{1}{\alpha}} \Big(1 - \frac{1}{\alpha n}\Big)\, dt \Big| \mathscr{F}_\tau\Big] - e^{-r\tau} = 0,$$

which we can simplify, since $\alpha$ is positive, to

$$\mathbf{E}\Big[\int_\tau^\infty e^{-rt} \kappa^{-1} \frac{X_t}{\sup_{\tau \leq u < t} X_u} \Big(1 - \frac{1}{\alpha n}\Big)\, dt \Big| \mathscr{F}_\tau\Big] - e^{-r\tau} = 0.$$

This is in terms of the Lévy process $Y$ equivalent to

$$\mathbf{E}\Big[\int_\tau^\infty e^{-rt} e^{\inf_{\tau \leq u < t} Y_t - Y_u}\, dt \Big| \mathscr{F}_\tau\Big] = e^{-r\tau} \kappa \Big(\frac{\alpha n}{\alpha n - 1}\Big).$$

Now we can make use of the fact that the increments $Y_t - Y_\tau$ and $Y_u - Y_\tau$ given $\mathscr{F}_\tau$ have the same distribution as $Y_{t-\tau}$ and $Y_{u-\tau}$ under $\mathscr{F}_0$ to make a shift in the time variable and find

$$\mathbf{E}\Big[\int_0^\infty e^{-rt} e^{\inf_{0 \leq u < t} Y_t - Y_u}\, dt\Big] = \kappa \Big(\frac{\alpha n}{\alpha n - 1}\Big),$$

which is independent of the stopping time $\tau$. So this last equation completely determines the parameter $\kappa$ for which (3.4) in our current specification is satisfied at any $\tau \in \mathscr{T}$. Further, note that $\inf_{0\le u<t} Y_t - Y_u$ has the same distribution as $\inf_{0\le u<t} Y_u = -\sup_{0\le u<t} -Y_u$ so that

$$\kappa\Big(\frac{\alpha n}{\alpha n-1}\Big) = \mathbf{E}\Big[\int_0^\infty e^{-rt} e^{-\sup_{0\le u<t} -Y_u}\,dt\Big] = \frac{1}{r}\mathbf{E}\Big[e^{-\sup_{0\le u<\tau(r)} -Y_u}\Big],$$

where $\tau(r)$ is an independent exponentially distributed time with parameter $r$. Now see [6, ch. VII] that the running supremum of a Lévy process without positive jumps, $-Y$, stopped at an independent exponential time is itself exponentially distributed with rate $\Phi^{-Y}(r)$ and thus

$$\kappa\Big(\frac{\alpha n}{\alpha n-1}\Big) = \frac{\Phi^{-Y}(r)}{r\big(1+\Phi^{-Y}(r)\big)}, \tag{4.5}$$

where $\Phi^{-Y}(r)$ is the Laplace exponent of $-Y$ at $r$. Since the right hand side is constant, $\kappa$ is increasing in $n$, and so is aggregate investment $Q_t^* = \sup_{0\le u<t} n\cdot L_u$ with $L$ as in (4.4). In fact, if we denote the right hand side of (4.5) by $\kappa_\infty$, to which $\kappa$ converges as the number of firms grows to infinity, then $L^c \triangleq \kappa_\infty^\alpha X^\alpha$ drives investment in a perfectly competitive equilibrium, with zero net present value as discussed above.

To check that we obtain the same results as Grenadier [9] and Back and Paulsen [1], note that if $Y_t = \mu t + \sigma B_t$ for standard Brownian motion $B$ and constants $\mu$ and $\sigma$, the Laplace exponent becomes

$$\Phi^{-Y}(r) = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

# References


[1] K. Back and D. Paulsen. Open loop equilibria and perfect competition in option exercise games. *Review of Financial Studies*, 2009. forthcoming.

[2] F.M. Baldursson and I. Karatzas. Irreversible investment and industry equilibrium. *Finance Stochast.*, 1:69–89, 1997.

[3] P. Bank and N. El Karoui. A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.*, 32:1030–1067, 2004.

[4] P. Bank and H. Föllmer. American options, multi-armed bandits, and optimal consumption plans: A unifying view. In *Paris-Princeton Lectures on Mathematical Finance*, volume 1814 of *Lecture Notes in Math.*, pages 1–42. Springer-Verlag, Berlin, 2002.

[5] P. Bank and F. Riedel. Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab.*, 11:750–788, 2001.

[6] J. Bertoin. *Lévy Processes.* Cambridge University Press, Cambridge, 1996.

[7] G. Bertola. Irreversible investment. *Research in Economics*, 52:3–37, 1998.

[8] A. Dixit and R. Pindyck. *Investment under uncertainty.* Princeton University Press, Princeton, N.J., 1994.

[9] S. Grenadier. Option exercise games: An application to the equilibrium investment strategies of firms. *Review of Financial Studies*, 15(3):691–721, 2002.

[10] X. Vives. *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, Cambridge, Mass., 1999.