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Tomoki Inoue

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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Indivisible Commodities and an Equivalence Theorem on the Strong Core*

Tomoki Inoue†‡

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†Institute of Mathematical Economics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany; tomoki.inoue@gmail.com.

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Abstract

We consider a pure exchange economy with finitely many indivisible commodities that are available only in integer quantities. We prove that in such an economy with a sufficiently large number of agents, but finitely many agents, the strong core coincides with the set of cost-minimized Walras allocations. Because of the indivisibility, the preference maximization does not imply the cost minimization. A cost-minimized Walras equilibrium is a state where, under some price vector, all agents satisfy both the preference maximization and the cost minimization.
Keywords: Indivisible commodities, Strong core, Cost-minimized Walras equilibrium, Core equivalence.
JEL Classification: C71, D51.

1 Introduction

The core is an institution-free concept, but it is known that in an economy with perfectly divisible commodities, the core contains all Walras allocations and any core allocation can be approximately decentralized by prices, as the number of agents becomes large. The purpose of this paper is to prove that, if every commodity is indivisible, then the core coincides with the set of Walras allocations with a certain property, even though the number of economic agents is finite. In particular, any core allocation in a large finite economy can be exactly decentralized by prices.

The perfect divisibility of commodities is usually assumed in economic theory for the convenience of the analysis. In this paper, we assume that every commodity can be consumed only in integer quantities. Agents can consume multiple types of commodities and can consume multiple units of each commodity. Thus, the commodity space is given by the products of the set of integers. Our argument shows that the inherent properties of the set of integers such as countability, discreteness, and additivity are helpful, rather than obstructive, to prove the core equivalence.

In our economy, because of the discreteness of the commodity space, agents' preference relations are necessarily locally satiated. Thus, the preference maximization does not imply the cost minimization. On the other hand, since any consumption vector cannot have local cheaper consumption vector under any prices, the cost minimization does not imply the preference maximization. Thus, approximate equilibrium in an economy with perfectly divisible commodities such as pseudo-equilibrium or quasi-equilibrium is not an approximate concept in our economy. Therefore, a cost-minimized Walras equilibrium which satisfies not only the preference maximization but also the cost minimization is a stronger concept than a Walras equilibrium. In our economy, the set of cost-minimized Walras allocations coincides with the core.

In the literature, several notions of improvement by a coalition are used to define cores. In an economy with indivisible commodities, the size of the core depends heavily on which notion of improvement is adopted. Accordingly, a clear distinction among the several competing notions of cores should be made. The core we focus on is defined by the weak improvement. The weak improvement requires that some members in a coalition to be better off and other members not be worse off by redistribution of their endowments. This is the same core that Debreu and Scarf [5] analyzed. We refer to this notion of the core as the *strong core*.

Debreu and Scarf [5] considered a sequence of replica economies with convex consumption sets. Two agents who have the same preference relation and the same endowment vector are said to be of the same type. An economy where in each type there are n times as many agents as the original economy is called the n-fold replica economy. If agents' preference relations are strongly convex, then agents of the same type are allocated the same consumption bundle by a strong core allocation or by a Walras allocation in any replica economy. (We refer to this property of strong core allocations as the *strong* equal treatment property.) Thus, by choosing a representative agent from each type, we can regard strong core allocations and Walras allocations for any replica economy as having the same dimensions as allocations for the original economy. Therefore, we can compare the size of strong cores or the size of Walras allocations of economies with a different number of replications. Although the size of Walras allocations is constant, the sequence of strong cores is shrinking as the number of replications increases, because possible coalitions increase. Under the assumptions of strong convexity and local nonsatiation of preference relations, Debreu and Scarf [5] proved that the limit of the decreasing sequence of strong cores coincides with the set of Walras allocations. By the local nonsatiation, Walras equilibria are cost-minimized Walras equilibria and, therefore, Debreu and Scarf's result implies that the limit of the sequence of strong cores coincides with the set of cost-minimized Walras allocations. Our main theorem gives the same core equivalence in a finite economy with indivisible commodities.

Anderson [1] used the notion of a core defined by strong improvement that requires all members in a coalition to be better off by redistribution of their endowments. We refer to this as the *weak core*. Without any assumptions, Walras allocations are always in the weak core. Anderson [1] considered a more general sequence of economies than the sequence of replica economies. In Anderson's model, all agents may belong to different types, and agents' preference relations need not be convex. Anderson [1] proved that under the assumption of monotonicity of preference relations (a stronger assumption than local nonsatiation), if the number of agents whose endowment vectors are in a given bounded set increases, then some measure of the non-Walras degree of weak core allocations tends to zero. Therefore, in a large finite economy, weak core allocations can be approximately decentralized by prices.

Both in Debreu and Scarf's [5] theorem and in Anderson's [1] theorem, the assumptions of the local nonsatiation of preference relations and the convexity of consumption sets play essential roles. Strictly speaking, Anderson's [1] measure of the non-Walras degree represents the distance between weak core allocation and quasi-equilibrium. In an economy with convex consumption sets, if agents' endowment vectors lie in the interior of their consumption sets, then so-called "minimum wealth condition" is met and, hence, quasi-equilibrium is Walras equilibrium. In our economy, in contrast, as mentioned earlier, quasi-equilibrium is not necessarily close to Walras equilibrium; therefore, even if we can show that Anderson's measure tends to zero, we cannot say that the strong core is close to the set of Walras allocations. Accordingly, we consider a more restrictive economy than that investigated by Anderson [1], but our economy is still more general than replica economy investigated by Debreu and Scarf [5]. We can prove that, if the number of agents' types is finite and if each type has many agents, then the strong core coincides with the set of cost-minimized Walras allocations. It should be noted that our economy does not have properties essential in theorems by Debreu and Scarf [5] and by Anderson [1] and, in contrast to Debreu and Scarf's [5] theorem, our theorem is not a "limit theorem."

In our economy, because of indivisibility, the strong core and the set of cost-minimized Walras allocations can be empty. Thus, the equivalence may be vacuous. To guarantee the nonemptiness of the strong core or the existence of a cost-minimized Walras equilibrium, some combinatorial conditions are needed. First, the type set is important. There exists an economy which does not have a cost-minimized Walras equilibrium regardless of the number of agents of each type. Example 3 illustrates this fact. Second, the relative ratio of the number of agents of each type to the size of economy is important. In Example 2, we give an economy with two types s and t. If the number of agents of type s is smaller than the number of agents of type t, a cost-minimized Walras equilibrium always exists. On the other hand, if the number of agents of type s is larger than the number of agents of type t, then a cost-minimized Walras equilibrium does not exist unless these two numbers are in a certain ratio.

In the process of the proof of our core equivalence theorem, the weak equal treatment property of strong core allocations is shown. That is, consumption bundles allocated by a strong core allocation to agents of the same type have the same utility level with respect to the common preference relation. As discussed earlier, in an economy with convex consumption sets and strongly convex preference relations, strong core allocations have the strong equal treatment property: all members of the same type receive the same consumption bundle. This property depends heavily on the strong convexity of preference relations. In addition, as Green [7] pointed out, this property is inherent to replica economies. Green [7] proved that for almost all economies where the greatest common divisor of the numbers of agents of each type is one, there exists a strong core allocation that does not even have the weak equal treatment property. It should be emphasized that the method of proof of our equal treatment property is very different from that provided by Debreu and Scarf [5]. Moreover, our equal treatment property holds even if the greatest common divisor of the numbers of agents of each type is one.

In other work, Inoue [11] obtained the same core equivalence in an atomless economy under weaker assumptions on agents' preference relations. In our theorem, we assume that any two commodities are substitutable, whereas in Inoue's [11] theorem, lexicographic preference relations are permitted. In our theorem, we give a bound of the size of economies above which the core equivalence holds. To clarify such bound, we need stronger assumptions and lengthier argument than Inoue's [11] proof in which Lyapunov's convexity theorem can be applied.

Inoue [9] obtained another type of core equivalence in an atomless economy. He introduced a core defined by improvement as an intermediate notion between the weak and the strong improvement. Accordingly, such a core is also an intermediate concept between the strong and the weak core. We denote this as the *core*. Inoue [9] proved that the core coincides with the set of exactly feasible Walras allocations. Our theorem and Inoue's [11] theorem imply that large finite economy and atomless economy produce the same equivalence on the strong core. On the core, in contrast, the equivalence holds only in atomless economies. Actually, Inoue [12] gave examples of the sequence of replica economies such that every economy has a core allocation which is not a Walras allocation;

therefore, the set of exactly feasible Walras allocations is strictly smaller than the core in any replica economy and only in the limit, both sets coincide.

Shapley and Scarf [15] analyzed yet another type of indivisible commodity market. Their economic model has finitely many agents, and each agent has only one indivisible commodity (e.g., a house). Commodities are also differentiated; therefore, the number of agents is equal to the number of commodities. In their model, it is assumed that every agent prefers his commodity to nothing. Hence, any individually rational feasible allocation can be represented by a permutation of the initial allocation. By using David Gale's top-trading-cycle method, Shapley and Scarf proved that Walras equilibria always exist, even though the strong core can be empty. Subsequently, Roth and Postlewaite [14] proved that, if agents' preference relations do not admit indifference among consumptions of one unit of one commodity, then the strong core coincides with the set of Walras allocations (which are also cost-minimized Walras allocations by the no-indifference assumption) and these sets consist of only one allocation; therefore, the same core equivalence holds as ours and also it is not a vacuous equivalence. Wako [17] proved that, even if agents' preference relations have indifference, the strong core coincides with the set of cost-minimized Walras allocations, although these sets are possibly empty. This is an improvement of his previous work (Wako [16]), where he proved that any strong core allocation is a Walras allocation. These results in the Shapley-Scarf model can be easily extended to replica economies so that our assumption that the number of agents of each type is large can be satisfied. Roth and Postlewaite's [14] and Wako's [17] proofs depend heavily on the model specification, so theirs are very different from the proof in this paper. In addition, their core equivalence holds in any replica economy, whereas our core equivalence holds only in a sufficiently large economy which is not necessarily a replica economy.

The paper itself is organized as follows. In Section 2, we present our model and the main theorem. Section 3 provides an outline of the proof of the theorem. Section 4 gives a formal proof. Purely technical results used in the proof of the main theorem are relegated to the Appendix.

2 Model and Main Theorem

We begin with some notation. Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ be the sets of real numbers, rational numbers, and integers, respectively. For $x = (x^{(1)}, \dots, x^{(m)})$ and $y = (y^{(1)}, \dots, y^{(m)})$ in $\mathbb{R}^m$ $(m \geq 2)$, we write $x \geq y$ if $x^{(j)} \geq y^{(j)}$ for all $j \in \{1, \dots, m\}$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x^{(j)} > y^{(j)}$ for all $j \in \{1, \dots, m\}$. The symbol 0 denotes the origin in $\mathbb{R}^m$, as well as the real number zero. Let χ_i be the ith unit vector, i.e., $\chi_i^{(i)} = 1$ and $\chi_i^{(j)} = 0$ if $j \neq i$. $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x \geq 0\}$; $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m \mid x \gg 0\}$. $\mathbb{Q}^m_+$ and $\mathbb{Z}^m_+$ are defined in a similar way. $\mathbb{Z}_{++}$ is the set of natural numbers. The inner product $\sum_{j=1}^m x^{(j)} y^{(j)}$ of x and y in $\mathbb{R}^m$ is denoted by $x \cdot y$. The cardinality of a finite set A is denoted by $\#A$.

We consider a pure exchange economy with L indivisible commodities, where L is a natural number with $L \geq 2$.[1] Every commodity in our economy is available in integer quantities; therefore, the commodity space is given by $\mathbb{Z}^L$. For simplicity, we assume that all agents have the same consumption set $\mathbb{Z}^L_+$. An agent a is characterized by his preference relation $\precsim_a$ on $\mathbb{Z}^L_+$ and his endowment vector $e(a) \in \mathbb{Z}^L_+$. Any preference relation $\precsim$ is a binary relation on $\mathbb{Z}^L_+$ which is required to be reflexive, transitive, complete, and weakly monotone.[2] Let $\mathcal{P}$ be the set of all preference relations on $\mathbb{Z}^L_+$. Given a preference relation $\precsim$, we define binary relations $\succ$ and $\sim$ as follows: $x \succ y$ if and only if not $(x \precsim y)$; $x \sim y$ if and only if $x \precsim y$ and $y \precsim x$. We sometimes write $x \succsim y$ for $y \precsim x$ and write $x \prec y$ for $y \succ x$.

The space of agents' characteristics is then $\mathcal{P} \times \mathbb{Z}^L_+$. A mapping $\mathcal{E}$ of a finite set A of agents into $\mathcal{P} \times \mathbb{Z}^L_+$, $\mathcal{E}(a) = (\precsim_a, e(a))$ for all $a \in A$, is an *economy* if $\sum_{a \in A} e(a) \gg 0$. Given an economy $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}^L_+$, an *allocation* for $\mathcal{E}$ is a mapping of A into $\mathbb{Z}^L_+$. An allocation $f : A \to \mathbb{Z}^L_+$ for $\mathcal{E}$ is *exactly feasible* if the equality $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$ holds. A *coalition* is a nonempty subset of A.

The core we focus on is the strong core defined by the weak improvement. The precise definition is as follows:

[1]For an economy with only one commodity, we can easily show that, if every agent's preference relation is strongly monotone, then the strong core coincides with the set of cost-minimized Walras allocations, and these sets contain only endowment allocation.

[2]A preference relation $\precsim$ is *weakly monotone* if, for all x and y in $\mathbb{Z}^L_+$, $x \leq y$ implies that $x \precsim y$.

Definition 1. Let $f : A \to \mathbb{Z}^L_+$ be an allocation for an economy $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}^L_+$. A coalition S can *weakly improve upon* f if there exists a mapping $g : S \to \mathbb{Z}^L_+$ such that

$$\sum_{a\in S} g(a) = \sum_{a\in S} e(a),$$
$$g(a) \succ_a f(a) \quad \text{for some } a \in S, \text{ and}$$
$$g(a) \succsim_a f(a) \quad \text{for all } a \in S.$$

The set of all exactly feasible allocations for $\mathcal{E}$ that cannot be weakly improved upon by any coalition is called the *strong core* of $\mathcal{E}$ and is denoted by $C_S(\mathcal{E})$.

Obviously, every strong core allocation is then Pareto-efficient. If there exist only two agents in an economy, strong core allocation is equal to individually rational Pareto-efficient allocation; therefore, there always exists a strong core allocation. If the size of an economy is larger than two agents, then, in contrast, the strong core can be empty. In Example 3 below, we give an economy with the empty strong core.

In our economy, the strong core is completely characterized by cost-minimized Walras equilibria whose definition is as follows:

Definition 2. Let $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}^L_+$ be an economy. A pair (p, f) of a price vector $p \in \mathbb{Q}^L_+$ and an exactly feasible allocation $f : A \to \mathbb{Z}^L_+$ is called a *cost-minimized Walras equilibrium* for $\mathcal{E}$ if

(i) for all $a \in A$, $p \cdot f(a) \le p \cdot e(a)$;

(ii) for all $a \in A$, if $x \in \mathbb{Z}^L_+$ and $x \succ_a f(a)$, then $p \cdot x > p \cdot e(a)$; and

(iii) for all $a \in A$, if $x \in \mathbb{Z}^L_+$ and $x \succsim_a f(a)$, then $p \cdot x \ge p \cdot e(a)$.

An exactly feasible allocation $f : A \to \mathbb{Z}^L_+$ is called a *cost-minimized Walras allocation* for $\mathcal{E}$ if there exists a price vector $p \in \mathbb{Q}^L_+$ such that (p, f) is a cost-minimized Walras equilibrium for $\mathcal{E}$. The set of all cost-minimized Walras allocations for $\mathcal{E}$ is denoted by $W_{CM}(\mathcal{E})$.

From the exact feasibility of f and condition (i), it follows that $p \cdot f(a) = p \cdot e(a)$ for all $a \in A$.[3] Putting together this with condition (ii), we obtain the preference maximization:

[3]This follows also from the exact feasibility of f and condition (iii), or from conditions (i) and (iii).

for all $a \in A$, $f(a) \succsim_a x$ holds for all $x \in \{y \in \mathbb{Z}^L_+ \,|\, p \cdot y \leq p \cdot f(a)\}$. Also, putting together with condition (iii), we obtain the cost minimization: for all $a \in A$, $p \cdot f(a) \leq p \cdot x$ holds for all $x \in \{y \in \mathbb{Z}^L_+ \,|\, y \succsim_a f(a)\}$.

When a pair (p, f) of a price vector $p \in \mathbb{Q}^L_+$ and an exactly feasible allocation f satisfies conditions (i) and (ii), it is called a *Walras equilibrium.* We denote the set of all Walras allocations for $\mathcal{E}$ by $W(\mathcal{E})$. Note that, because of indivisibility, even a Walras equilibrium may not exist.[4]

In an economy with perfectly divisible commodities, if agents' preference relations are locally nonsatiated, then the preference maximization implies the cost minimization; therefore, Walras equilibrium is cost-minimized Walras equilibrium. On the other hand, if agents' consumption sets are convex, preference relations are continuous, and the minimum wealth condition is met, then the cost minimization implies the preference maximization (see Debreu [4, Theorem (1), Section 9, Chapter 4]); therefore, if (p, f) satisfies conditions (i) and (iii), then it also satisfies condition (ii). In our economy, in contrast, since the commodity space is discrete, one of the preference maximization and the cost minimization does not imply the other. Thus, there can exists a Walras equilibrium that is not a cost-minimized Walras equilibrium. Actually, in Examples 1 and 3 below, we give an economy with a Walras equilibrium that is not a cost-minimized Walras equilibrium.

If (p, f) is a cost-minimized Walras equilibrium, then for all $\alpha \in \mathbb{Q}_{++}$, $(\alpha\, p, f)$ is also a cost-minimized Walras equilibrium. Thus, for every cost-minimized Walras allocation, there exists an associated *integral* equilibrium price vector. It should be noted that we can restrict the space of price vectors to $\mathbb{Q}^L_+$ without loss of generality. In fact, if a pair of a vector $p \in \mathbb{R}^L_+ \setminus \mathbb{Q}^L_+$ and an exactly feasible allocation f satisfies conditions (i)-(iii) of Definition 2, then there exists a price vector $p^* \in \mathbb{Q}^L_+$ such that (p^*, f) is a cost-minimized Walras equilibrium.[5]

By an argument similar to the proof of the first welfare theorem, we can show that, for all economy $\mathcal{E}$, cost-minimized Walras allocations for $\mathcal{E}$ are strong core allocations for

[4]Henry [8] gave an example of an economy with one indivisible commodity and two divisible commodities such that a Walras equilibrium does not exist. For economies where every commodity is indivisible, Shapley and Scarf [15, Section 8] gave an example of the nonexistence of a Walras equilibrium.

[5]This fact is clear from the last part of the proof of our main theorem.

$\mathcal{E}$, i.e., $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$.

In our main theorem, we place restrictions on preference relations. For all $k \in \mathbb{Z}$ with $k \geq 2$,[6] we define a subset $\mathcal{P}_k$ of $\mathcal{P}$ as follows: $\precsim \in \mathcal{P}_k$ if and only if (i) $\precsim \in \mathcal{P}$ and (ii) for all $h, i \in \{1, \ldots, L\}$ with $h \neq i$ and all $x \in \mathbb{Z}^L_+$, if $x^{(i)} \geq 1$, then $x + k\,\chi_h - \chi_i \succ x$.[7] From (i) and (ii), it follows that, for all $h \in \{1, \ldots, L\}$ and all $x \in \mathbb{Z}^L_+$, $x + k\,\chi_h \succ x$ holds.[8] Condition (ii) means that agents whose preference relations are in $\mathcal{P}_k$ are willing to give up one unit of a commodity in exchange for k units of another commodity. Therefore, preference relations in $\mathcal{P}_k$ have uniformly positive marginal rates of substitution. In particular, the lexicographic ordering is excluded.

In our main theorem, we consider an economy where there exist many agents who have the same preference relation and the same endowment vector. To make this more precise, we introduce some notation. Let $k \in \mathbb{Z}$ with $k \geq 2$ and let $T \subseteq \mathcal{P}_k \times \mathbb{Z}^L_+$ be a nonempty finite set. The set T is a type set of agents. For all $t \in T$, we write $t = (\precsim_t, e_t)$. Given an economy $\mathcal{E} : A \to T$ and a type $t \in T$, denote the set of agents of type t by A_t, i.e., $A_t = \mathcal{E}^{-1}(\{t\}) = \{a \in A \,|\, (\precsim_a, e(a)) = t\}$.

We first give $r \in \mathbb{R}$ with $r \geq 1$, $k \in \mathbb{Z}$ with $k \geq 2$, and $T \subseteq \mathcal{P}_k \times \mathbb{Z}^L_+$. Then, we consider an economy $\mathcal{E} : A \to T$ such that (1) $\#A$ is sufficiently large and (2) $\#A_t/\#A \geq 1/r$ for all $t \in T$. Thus, the number $1/r$ represents a lower bound of the ratio of agents of each type t to the whole economy. Conditions (1) and (2) guarantee that there exist many agents whose types are t, i.e., $\#A_t$ is large enough. If number r and economy $\mathcal{E} : A \to T$ satisfy that $r = \#T$ and $\#A_t/\#A \geq 1/r$ for all $t \in T$, this economy is the $\#A/r$-fold replica economy. Hence, our theorem covers more general economies than replica economies.

We can now state our main result.

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[6]The reason why k is assumed to be greater than 1 is that $\mathcal{P}_1$, which is defined similar to $\mathcal{P}_k$ with $k \geq 2$, is empty. Indeed, if $\precsim \in \mathcal{P}_1$, then $\chi_1 = \chi_2 + \chi_1 - \chi_2 \succ \chi_2$ and $\chi_2 = \chi_1 + \chi_2 - \chi_1 \succ \chi_1$, contradicting the irreflexivity of $\succ$. This fact was pointed out by Akiyoshi Shioura.

[7]Condition (ii) is related to the equi-monotonicity of preference relations in an economy with perfectly divisible commodities. Let $\mathcal{Q}$ be the space of continuous and strongly monotone preference relations on the consumption set $\mathbb{R}^L_+$. It can be shown that for every finite subset $\mathcal{Q}'$ of $\mathcal{Q}$ and every compact subset K of $\mathbb{R}^L_{++}$, there exists a positive number δ such that, for all $\precsim \in \mathcal{Q}'$, all $h, i \in \{1, \ldots, L\}$, and all $x \in K$, $x + \chi_h - \delta\,\chi_i \succ x$ holds.

[8]This fact can be shown as follows. Let $i \neq h$. Then, $x + k\,\chi_h = (x + \chi_i) + k\,\chi_h - \chi_i \succ x + \chi_i \succsim x$.

Theorem. *For all* $r \in \mathbb{R}$ *with* $r \geq 1$*, all* $k \in \mathbb{Z}$ *with* $k \geq 2$*, and all* $T \subseteq \mathcal{P}_k \times \mathbb{Z}^L_+$ *with* $\#T \leq r$ *and* $\sum_{t \in T} e_t \gg 0$*, there exists an* $N \in \mathbb{Z}_{++}$ *such that if* $\mathcal{E} : A \to T$*,* $\#A \geq N$*, and* $\#A_t/\#A \geq 1/r$ *for all* $t \in T$*, then*

(1) *all strong core allocations for* $\mathcal{E}$ *have the weak equal treatment property, i.e., for all* $f \in C_S(\mathcal{E})$*, all* $t \in T$*, and all* $a, b \in A_t$*,* $f(a) \sim_a f(b)$ *holds; and*

(2) *the strong core of* $\mathcal{E}$ *coincides with the set of cost-minimized Walras allocations for* $\mathcal{E}$*, i.e.,* $C_S(\mathcal{E}) = W_{CM}(\mathcal{E})$*.*

The number N depends on r, k, L, and $M = \max\{\|e\|_\infty \mid (\precsim, e) \in T\}$.[9] The proof of the theorem is given in Section 4. From the proof, it is clear that the size of economies which satisfy the weak equal treatment property of strong core allocations is smaller than the size of economies which also satisfy the equivalence between the strong core and the set of cost-minimized Walras allocations.

Since the inclusion $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$ holds for any economy $\mathcal{E}$, our theorem says the converse holds if the size of economy is sufficiently large. A small economy $\mathcal{E}_1$ can have a strong core allocation f_1 that is not a cost-minimized Walras allocation. In any replica economy $\mathcal{E}_n$ of $\mathcal{E}_1$, the replica allocation of f_1 is not a cost-minimized Walras allocation for $\mathcal{E}_n$. From our theorem, if the number n of replications is sufficiently large, then the replica allocation of f_1 is not a strong core allocation for replica economy $\mathcal{E}_n$. The next example illustrates this fact.

Example 1. Let $L = 2$ and $T = \{s, t\}$. Each type's endowment vector is given by $e_s = (3, 1)$ and $e_t = (2, 2)$. Each type's preference relation is represented by the following utility functions (see Figures 1 and 2):

$$u_s(x^{(1)}, x^{(2)}) = \begin{cases} 2x^{(1)} + x^{(2)} & \text{if} \quad x^{(1)} \leq 1, \\ \frac{1}{2}(x^{(1)} + 2x^{(2)} + 3) & \text{if} \quad x^{(1)} \geq 2, \end{cases} \quad \text{and}$$

$$u_t(x^{(1)}, x^{(2)}) = x^{(1)} + x^{(2)}.$$

Clearly, both $\precsim_s$ and $\precsim_t$ are in $\mathcal{P}_3$.

For all $n \geq 1$, let $A_{n,s} = \{(s, 1), \ldots, (s, n)\}$, $A_{n,t} = \{(t, 1), \ldots, (t, n)\}$, and $A_n = A_{n,s} \cup A_{n,t}$. For all $n \geq 1$, define economy $\mathcal{E}_n : A_n \to T$ by $\mathcal{E}_n(s, i) = s$ and $\mathcal{E}_n(t, i) = t$ for

[9] For $x \in \mathbb{R}^L$, $\|x\|_\infty = \max\{|x^{(j)}| \mid j = 1, \ldots, L\}$.

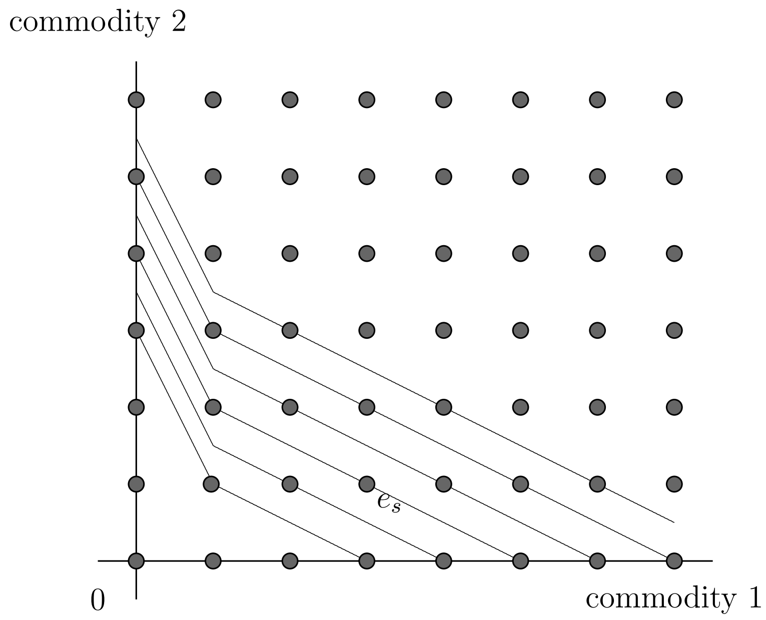

Figure 1: Endowment vector and indifference curves of agents of type s

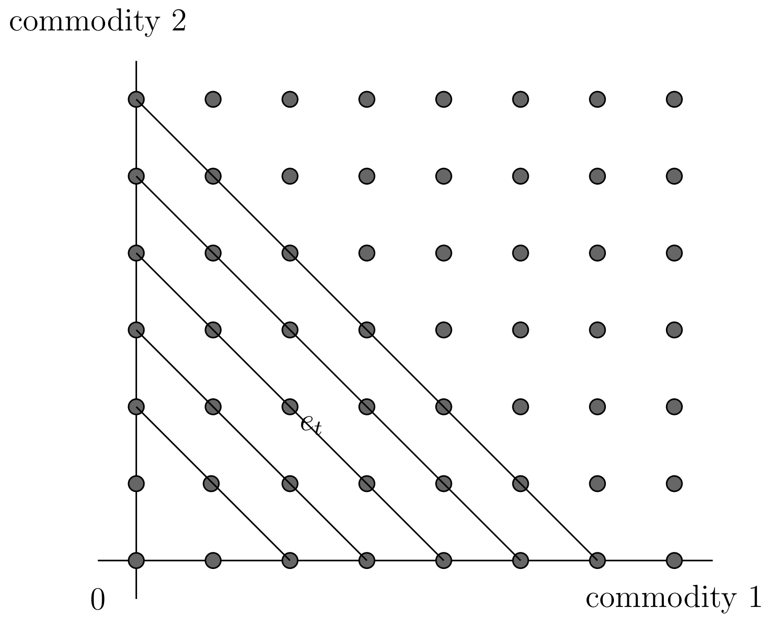

Figure 2: Endowment vector and indifference curves of agents of type t

all $i \in \{1, \ldots, n\}$. Thus, economy $\mathcal{E}_n$ is the n-fold replica economy of $\mathcal{E}_1$, and economy $\mathcal{E}_n$ consists of n agents of type s and n agents of type t. An allocation $f_1 : A_1 \to \mathbb{Z}^2_+$ for $\mathcal{E}_1$ is defined by $f_1(s,1) = (1,2)$ and $f_1(t,1) = (4,1)$. One could check that $f_1 \in C_S(\mathcal{E}_1)$ and $f_1 \notin W_{CM}(\mathcal{E}_1)$.

For all $n \geq 2$, let $f_n : A_n \to \mathbb{Z}^2_+$ be a replica allocation of f_1; that is, $f_n(s,i) = f_1(s,1)$ and $f_n(t,i) = f_1(t,1)$ for all $i \in \{1, \ldots, n\}$. For all $n \geq 2$, f_n is not a cost-minimized Walras allocation for $\mathcal{E}_n$ for the same reason that $f_1 \notin W_{CM}(\mathcal{E}_1)$. Although f_1 is a strong core allocation for $\mathcal{E}_1$, for all $n \geq 2$, f_n is not a strong core allocation for $\mathcal{E}_n$ as we show in the following.

Let $n \geq 2$. Consider $S = \{(s,1),(s,2),(t,1)\}$. Define $g : S \to \mathbb{Z}^2_+$ by

$$\begin{aligned} g(s,i) &= f_1(s,1) = (1,2) \quad \text{for } i = 1,2, \text{ and} \\ g(t,1) &= (6,0). \end{aligned}$$

Then,

$$\sum_{(r,i)\in S} g(r,i) = \sum_{(r,i)\in S} e_n(r,i) \quad \text{and} \quad g(t,1) \succ_t f_1(t,1),$$

where $e_n : A_n \to \mathbb{Z}^2_+$ is the endowment allocation for $\mathcal{E}_n$. Therefore, $f_n \notin C_S(\mathcal{E}_n)$ for all $n \geq 2$.

From our theorem, it follows that there exists an $n_0 \in \mathbb{Z}_{++}$ such that, for all $n \geq n_0$, $C_S(\mathcal{E}_n) = W_{CM}(\mathcal{E}_n)$ holds. In this example, we can choose $n_0 = 2$. It can be shown that, for all $n \geq 2$,

$$C_S(\mathcal{E}_n) = \left\{ f : A_n \to \mathbb{Z}^2_+ \;\middle|\; \begin{array}{l} f(s,i) = (1,3) \quad \text{and} \quad f(t,i) = (4,0) \\ \text{for all } i \in \{1, \ldots, n\} \end{array} \right\}.$$

Thus, for all $n \geq 2$, every allocation in $C_S(\mathcal{E}_n)$ is a cost-minimized Walras allocation under the price vector $(1,1)$. Hence, for all $n \geq 2$, $\emptyset \neq C_S(\mathcal{E}_n) = W_{CM}(\mathcal{E}_n)$.

Note that, for all $n \geq 1$, endowment allocation e_n is a Walras allocation under the price vector $p = (1, p^{(2)})$ with $1 < p^{(2)} < 3/2$, but e_n is not a cost-minimized Walras allocation. Summing up these facts, for all $n \geq 2$, $\emptyset \neq C_S(\mathcal{E}_n) = W_{CM}(\mathcal{E}_n) \subsetneq W(\mathcal{E}_n)$ holds.

The next example illustrates that combinatorial condition on the relative ratios $\#A_t/\#A$ $(t \in T)$ is needed for the existence of cost-minimized Walras equilibrium.

Example 2. Consider the type set T form Example 1. In Example 1, we considered economies where the number of agents of type s is equal to the number of agents of type t. Here, we consider economies where these two numbers are different. By an argument similar to Example 1, we can show that, if economy $\mathcal{E} : A \to T$ satisfies that $0 < \#A_s/\#A \leq \#A_t/\#A$, then any feasible allocation $f : A \to \mathbb{Z}^2_+$ such that

$$\begin{aligned} f(a) &= (1,3) \quad \text{for all } a \in A_s, \text{ and} \\ \|f(a)\|_1 &= 4 \quad \text{for all } a \in A_t \end{aligned}$$

is a cost-minimized Walras allocation for $\mathcal{E}$.[10]

We consider economy $\mathcal{E} : A \to T$ with $\#A_s/\#A > \#A_t/\#A > 0$. If $\#A_s/\#A = 2/3$ and $\#A_t/\#A = 1/3$, then allocation $f : A \to \mathbb{Z}^2_+$ defined by

$$f(a) = \begin{cases} (1,2) & \text{if } a \in A_s, \\ (6,0) & \text{if } a \in A_t, \end{cases}$$

is exactly feasible. In addition, f is a cost-minimized Walras allocation under $p = (1,2)$.

If economy $\mathcal{E} : A \to T$ satisfies that $\#A_s/\#A > \#A_t/\#A > 0$ and $\#A_s/\#A \neq 2/3$, then $\mathcal{E}$ does not have a cost-minimized Walras equilibrium as shown in the following.

- Under price vector $p = (1, p^{(2)})$ with $p^{(2)} < 1$, commodity 2 is in excess demand.
- Under price vector $p = (1, p^{(2)})$ with $p^{(2)} > 2$, commodity 1 is in excess demand.
- Under price vector $p = (1, p^{(2)})$ with $1 < p^{(2)} < 2$, agents of type s do not have consumption vectors that satisfy both the preference maximization and the cost minimization.
- Under price vector $p = (1, p^{(2)})$ with $p^{(2)} = 1$ or $p^{(2)} = 2$, preference-maximized allocations are not feasible.

Summing up these results, we have

- If economy $\mathcal{E} : A \to T$ satisfies that $0 < \#A_s/\#A \leq \#A_t/\#A$, then $W_{CM}(\mathcal{E}) \neq \emptyset$.

[10] For $x \in \mathbb{R}^L$, $\|x\|_1 = \sum_{i=1}^{L} |x^{(i)}|$.

- If economy $\mathcal{E} : A \to T$ satisfies that $\#A_s/\#A = 2/3$ and $\#A_t/\#A = 1/3$, then $W_{CM}(\mathcal{E}) \neq \emptyset$.
- If economy $\mathcal{E} : A \to T$ satisfies that $\#A_s/\#A > \#A_t/\#A > 0$ and $\#A_s/\#A \neq 2/3$, then $W_{CM}(\mathcal{E}) = \emptyset$.

Hence, in order to guarantee the existence of cost-minimized Walras equilibrium, the relative ratio of the number of agents of each type to the size of economy is essential. From our theorem, this relative ratio is essential also for the nonemptiness of the strong core.

Not only the relative ratios $\#A_t/\#A$ $(t \in T)$ but also the type set T is essential for the existence of a cost-minimized Walras equilibrium. The next example illustrates this fact.

Example 3. Let $L = 2$ and $T = \{t\}$. Type t's endowment vector e_t is given by $(1, 2)$. The preference relation $\precsim_t$ of agents of type t is represented by a utility function $u_t : \mathbb{Z}^2_+ \to \mathbb{R}$ defined by

$$u_t(x^{(1)}, x^{(2)}) = \begin{cases} 3.5 & \text{if} \quad (x^{(1)}, x^{(2)}) = (3, 0), \\ x^{(1)} + x^{(2)} & \text{otherwise.} \end{cases}$$

(See Figure 3.) Then, $\precsim_t \in \mathcal{P}_2$. Although the indifference curves drawn in the figure are not convex, this preference relation is discretely convex in the sense that, for all $x \in \mathbb{Z}^2_+$, we have $\operatorname{co}\left(\{y \in \mathbb{Z}^2_+ \mid y \succsim_t x\}\right) \cap \mathbb{Z}^2 = \{y \in \mathbb{Z}^2_+ \mid y \succsim_t x\}$, where $\operatorname{co}(C)$ denotes the convex hull of set C.[11] For all $n \geq 1$, let $A_n = \{(t, 1), \ldots, (t, n)\}$. Define $\mathcal{E}_n : A_n \to T$ by $\mathcal{E}_n(t, i) = t$ for all $i \in \{1, \ldots, n\}$. Let $e_n : A_n \to \mathbb{Z}^2_+$ be the endowment allocation for $\mathcal{E}_n$.

In every economy $\mathcal{E}_n$ with $n \geq 1$, if price vector $p = (1, p^{(2)})$ satisfies that $p^{(2)} \leq 1/2$, then commodity 2 is in excess demand; if $p = (1, p^{(2)})$ satisfies that $p^{(2)} \geq 1$, then commodity 1 is in excess demand. Under price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, a pair (p, e_n) is a Walras equilibrium, but is not a cost-minimized Walras equilibrium. Since any other allocation than e_n cannot be a Walras allocation under $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, we have $\emptyset = W_{CM}(\mathcal{E}_n) \subsetneq W(\mathcal{E}_n) = \{e_n\}$ for all $n \geq 1$.

[11] The discrete convexity of preference relation is related to the nonemptiness of the weak core. See Inoue [10].

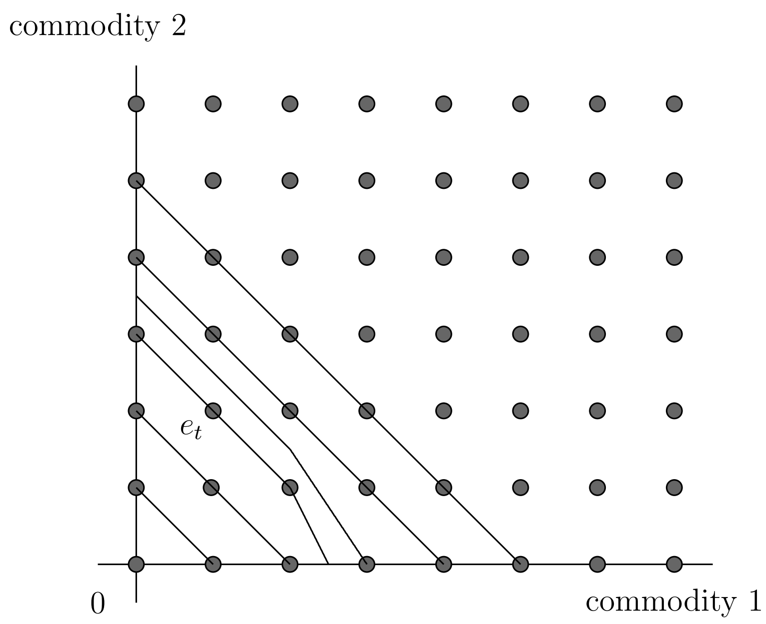

Figure 3: Endowment vector and indifference curves of agents of type t

From our theorem, it follows that $C_S(\mathcal{E}_n) = \emptyset = W_{CM}(\mathcal{E}_n)$ for n large enough. Indeed, this equivalence is met for all $n \geq 4$; one can show that $C_S(\mathcal{E}_n) = \emptyset$ for all $n \geq 4$. On the other hand, if $2 \leq n \leq 3$, there exists a strong core allocation that is not a cost-minimized Walras allocation. Actually, endowment allocation e_2 for $\mathcal{E}_2$ is a strong core allocation, but as mentioned above, e_2 is not a cost-minimized Walras allocation. Also, an allocation $g : \{(t,1),(t,2),(t,3)\} \to \mathbb{Z}^2_+$ for $\mathcal{E}_3$ defined by

$$g(t,1) = (3,0), \quad \text{and}$$
$$g(t,i) = (0,3) \quad \text{for all } i \in \{2,3\}$$

is a strong core allocation but is not a Walras allocation for $\mathcal{E}_3$.

3 Outline of the Proof

We give an outline of the proof here and give a formal proof in the next section. Let $r \in \mathbb{R}$ with $r \geq 1$, $k \in \mathbb{Z}$ with $k \geq 2$, and $T \subseteq \mathcal{P}_k \times \mathbb{Z}^L_+$ with $\#T \leq r$ and $\sum_{t \in T} e_t \gg 0$. Let $\mathcal{E} : A \to T$ be an economy with $\#A_t/\#A \geq 1/r$ for all $t \in T$. Later we assume that $\#A$ is sufficiently large, but at this point A may be an arbitrary finite set of agents.

In Lemma 1, we prove that strong core allocations are uniformly bounded; there exists a $\xi \in \mathbb{Z}_+$ such that $\|f(a)\|_\infty \leq \xi$ for all $f \in C_S(\mathcal{E})$ and all $a \in A$. It should be emphasized that the upper bound ξ depends only on exogenous variables r, k, L, and $M = \max\{\|e\|_\infty \,|\, (\precsim, e) \in T\}$, and it does not depend on the size $\#A$ of economy. Since T is a finite set, agents' net trade vectors are also uniformly bounded; $\|f(a) - e(a)\|_\infty \leq \xi$ for all $f \in C_S(\mathcal{E})$ and all $a \in A$. (From the definition of ξ which is made precise in the formal proof, we can take the same upper bound as the one of strong core allocations.)

Let $X_{L,\xi} = \{x \in \mathbb{Z}^L \,|\, \|x\|_\infty \leq \xi\}$. Then, for all $f \in C_S(\mathcal{E})$ and all $a \in A$, $f(a) - e(a) \in X_{L,\xi}$. Since $X_{L,\xi}$ is a finite set and every strong core allocation f is exactly feasible, i.e., $\sum_{a \in A}(f(a) - e(a)) = 0$, we can expect that, if $\#A$ is sufficiently large, then there exists a nonempty subset B of A such that f is exactly feasible within B, i.e., $\sum_{a \in B}(f(a) - e(a)) = 0$. This expectation is true as proved in Lemma 3 in the Appendix. We can take a subset B whose cardinality is bounded by $\mu(L, \xi)$. Literally, this upper bound depends only on dimension L and the upper bound ξ of vectors $f(a) - e(a)$ $(a \in A)$.

After these preparations, we can show that all strong core allocations have the weak equal treatment property; if $\#A > r\mu(L, \xi)$, then, for all $f \in C_S(\mathcal{E})$, all $t \in T$, and all $a, b \in A_t$, $f(a) \sim_t f(b)$ holds (Lemma 2). This is proved by a contradiction argument.

- Strong core allocation f is exactly feasible within B and within $A \backslash B$, i.e., $\sum_{a \in B}(f(a) - e(a)) = 0$ and $\sum_{a \in A \backslash B}(f(a) - e(a)) = 0$.
- If $B \cap A_t \neq \emptyset$ for some $t \in T$, then $(A \setminus B) \cap A_t \neq \emptyset$. (This follows from the assumption $\#A > r\mu(L, \xi)$.)

These facts play essential roles to construct a coalition which can weakly improve upon f.

Assume that $\#A$ is sufficiently large. (The required size of economy is made precise in the formal proof.) Let $f \in C_S(\mathcal{E})$. For every $t \in T$, let

$$\psi_t = \{x \in \mathbb{Z}^L_+ \,|\, x \succsim_t f(a)\} - \{e_t\},$$

where $a \in A_t$. By the weak equal treatment property, ψ_t is well-defined. A cost-minimized Walras equilibrium price vector can be obtained in two steps. First, we find a price vector p_0 under which strong core allocation f satisfies the cost minimization. Second, we move

p_0 slightly and find a price vector $\bar{p}$ under which f satisfies both the cost minimization and the preference maximization. In both steps, we use a well-known separation theorem for convex sets. To obtain price vector p_0, it suffices to prove that

$$0 \notin \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t \in T} \psi_t\right)\right),$$

where $\mathrm{int}(C)$ denotes the interior of set C. This is proved by a contradiction argument. Thus, we suppose that $0 \in \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t \in t} \psi_t\right)\right)$ and we find a coalition which can weakly improve upon f. From $0 \in \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t \in t} \psi_t\right)\right)$, 0 can be represented as a convex combination of elements of $\bigcup_{t \in T} \psi_t$. Roughly speaking, the denomination of coefficients of the convex combination represents the size of improving coalition. Thus, we have to find a convex combination whose coefficients are *rational* and their *denominators are bounded*. To obtain such convex combination, we need a mathematical lemma (Lemma 6 in the Appendix). The size $\#A$ of economy $\mathcal{E}$ must be larger than these denominators.

Assume that we could prove that $0 \notin \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t \in t} \psi_t\right)\right)$. Then, by the separation theorem for convex sets, there exists a $p_0 \in \mathbb{R}^L \setminus \{0\}$ such that $p_0 \cdot z \geq 0$ for all $z \in \bigcup_{t \in T} \psi_t$. Under price vector p_0, f satisfies the cost minimization, but there may exist an $x \in \mathbb{Z}^L_+$ such that $x \succ_a f(a)$ and $p_0 \cdot (x - e(a)) = 0$ for some $a \in A$. In such a case, we put $H_0 = \{z \in \mathbb{R}^L \,|\, p_0 \cdot z = 0\}$ and, by the same argument as above, we prove that $0 \notin \mathrm{ri}\left(\mathrm{co}\left(\bigcup_{t \in T} \psi_t \cap H_0\right)\right)$, where $\mathrm{ri}(C)$ denotes the relative interior of set C. Again, by the separation theorem for convex sets, there exists a $p_1 \in \mathrm{span}\left(\bigcup_{t \in T} \psi_t \cap H_0\right) \setminus \{0\}$ such that $p_1 \cdot z \geq 0$ for all $z \in \bigcup_{t \in T} \psi_t \cap H_0$. If we take sufficiently small $\varepsilon_1 > 0$, then, for all $z \in \bigcup_{t \in T} \psi_t$ with $p_0 \cdot z > 0$, $(p_0 + \varepsilon_1 p_1) \cdot z > 0$ holds and, for all $z \in \bigcup_{t \in T} \psi_t \cap H_0$, $(p_0 + \varepsilon_1 p_1) \cdot z \geq 0$ holds. Namely, if $x \in \mathbb{Z}^L_+$ is outside the budget set under p_0, then x is still outside the new budget set under $p_0 + \varepsilon_1 p_1$. This is possible because agents' consumption set $\mathbb{Z}^L_+$ is discrete. When there exists an $x \in \mathbb{Z}^L_+$ such that $x \succ_a f(a)$ and $(p_0 + \varepsilon_1 p_1) \cdot (x - e(a)) = 0$ for some $a \in A$, by repeating the same argument, we can obtain a price vector $\bar{p}$ such that, for all $a \in A$,

- if $x \in \mathbb{Z}^L_+$ and $x \succ_a f(a)$, then $\bar{p} \cdot (x - e(a)) > 0$, and
- if $x \in \mathbb{Z}^L_+$ and $x \succsim_a f(a)$, then $\bar{p} \cdot (x - e(a)) \geq 0$.

(Note that the dimension of span $\left(\bigcup_{t\in T}\psi_t \cap H_0\right)$ is one less than the dimension of span $\left(\bigcup_{t\in T}\psi_t\right)$. Since the dimension decreases by one at each step, we can obtain $\bar{p}$ in finite steps.) The pair $(\bar{p}, f)$ satisfies the conditions of cost-minimized Walras equilibrium except that $\bar{p} \in \mathbb{Q}^L_+$. Finally, by applying Inoue's [9] separation theorem (Lemma 7 in the Appendix), we obtain an integral price vector p^* under which f is a cost-minimized Walras allocation.

4 Proof of Theorem

Let $r \in \mathbb{R}$ with $r \geq 1$, $k \in \mathbb{Z}$ with $k \geq 2$, and $T \subseteq \mathcal{P}_k \times \mathbb{Z}^L_+$ with $\#T \leq r$ and $\sum_{t\in T} e_t \gg 0$. Let

$$M = \max\{\|e\|_\infty \mid (\precsim, e) \in T\} \quad \text{and}$$
$$\xi = \max\{rM^2L^2(ML+1),\ (kL+1)ML\}.$$

Note that the number ξ depends only on exogenous variables. We will prove that strong core allocations are uniformly bounded regardless of the size of economy. In the case of perfectly divisible commodities, Bewley [3, Theorem 1] first proved that strong core allocations are uniformly bounded. His proof uses a contradiction argument, so the bound of strong core allocations is not clear. On the other hand, Mas-Colell's [13, Lemma 7.4.10] proof clarifies the bound. Our proof is based on Mas-Colell's proof.

Lemma 1. *For every finite set A of agents, if economy $\mathcal{E} : A \to T$ satisfies that $\#A_t/\#A \geq 1/r$ for all $t \in T$, then $\|f(a)\|_\infty \leq \xi$ for all $f \in C_S(\mathcal{E})$ and all $a \in A$.*

Proof. Let A be the set of agents and $\mathcal{E} : A \to T$ be an economy such that $\#A_t/\#A \geq 1/r$ for all $t \in T$. Let $f \in C_S(\mathcal{E})$. By a simple calculation, it follows that, if $\#A \leq rML^2(ML+1)$, then $\|f(a)\|_\infty \leq \xi$ for all $a \in A$. Therefore, in the remainder of the proof, we assume that $\#A > rML^2(ML+1)$. Let $J = \{j \in \{1, \dots, L\} \mid f^{(j)}(a) \geq ML+1 \text{ for some } a \in A\}$ and $J' = \{j \in \{1, \dots, L\} \mid f^{(j)}(a) \geq (kL+1)ML \text{ for some } a \in A\}$. Then, $J' \subseteq J$. If $J = \emptyset$, the proof has been completed. Thus, the set J is supposed to be nonempty. For $j \in J$, we choose $a_j \in \operatorname{argmax}\{f^{(j)}(a) \mid a \in A\}$. Note that $a_i = a_j$ may hold for some distinct indices i and j. Let $B = \{a_j \mid j \in J\}$. Then, $\#B \leq \#J \leq L$. The excess demand of the coalition B is denoted by $y = \sum_{a\in B}(f(a) - e(a)) \in \mathbb{Z}^L$. Let $J'' = \{j \in \{1, \dots, L\} \mid y^{(j)} \leq -1\}$. By a simple calculation, $y^{(j)} \geq 1$ for all $j \in J$. Thus, $J \cap J'' = \emptyset$. We also have:

(1) $y^{(j)} \geq kML^2$ for all $j \in J'$.

In addition, if $J' = \emptyset$, then $\|f(a)\|_\infty \leq \xi$ for all $a \in A$. Therefore, the proof is completed if we can prove that $J' = \emptyset$.

Claim 1. $J' = \emptyset$.

Proof. Suppose, to the contrary, that $J' \neq \emptyset$. Let $C = A \setminus B$. Then, $\#C = \#A - \#B \geq \#A - L$. Since $\#A > rML^2(ML+1) > L$, we have $C \neq \emptyset$. Define $\tilde{y} \in \mathbb{Z}^L$ by

$$\tilde{y} = y - \sum_{j \in J''} y^{(j)} \chi_j.$$

Clearly, $\tilde{y} \geq 0$. Moreover, from (1), it follows that, for all $j \in J'$, $\tilde{y}^{(j)} = y^{(j)} \geq kML^2$.

Subclaim 1.1. $J'' \neq \emptyset$.

Proof. Suppose, to the contrary, that $J'' = \emptyset$. We have

$$\begin{aligned} kML^2 \textstyle\sum_{j \in J'} \chi_j \leq \tilde{y} &= y \\ &= \textstyle\sum_{a \in B}(f(a) - e(a)) \\ &= -\textstyle\sum_{a \in C}(f(a) - e(a)) \\ &= \textstyle\sum_{a \in C}(e(a) - f(a)). \end{aligned}$$

The third equality follows from the exact feasibility of strong core allocation f. Since $C \neq \emptyset$, we can choose an agent a^* of C. Define a mapping $g : C \to \mathbb{Z}^L_+$ by

$$g(a) = \begin{cases} f(a^*) + \sum_{c \in C}(e(c) - f(c)) & \text{if} \quad a = a^*, \\ f(a) & \text{if} \quad a \in C \setminus \{a^*\}. \end{cases}$$

Because $g(a^*) \geq f(a^*) + k\chi_j$ for all $j \in J'$ and $\precsim_{a^*} \in \mathcal{P}_k$, we have $g(a^*) \succ_{a^*} f(a^*)$. We also have

$$\sum_{a \in C} g(a) = \sum_{a \in C} f(a) + \sum_{c \in C}(e(c) - f(c)) = \sum_{a \in C} e(a).$$

This contradicts that $f \in C_S(\mathcal{E})$. Thus, we have established the proof of Subclaim 1.1. □

For all $j \in J''$, let $C_j = \{a \in C \mid f^{(j)}(a) \geq 1\}$.

Subclaim 1.2. $\#C_j > ML^2$ for all $j \in J''$.

Proof. Let $j \in J''$. Since $J \cap J'' = \emptyset$, it follows that $j \notin J$. Thus, $f^{(j)}(a) \leq ML$ for all $a \in A$. Since $C \setminus C_j = \{a \in C \,|\, f^{(j)}(a) < 1\} = \{a \in C \,|\, f^{(j)}(a) = 0\}$, we have

$$\begin{aligned}\sum_{a \in A} f^{(j)}(a) &\leq \{\#A - (\#C - \#C_j)\}ML \\ &= (\#C_j)ML + (\#A - \#C)ML \\ &\leq (\#C_j)ML + ML^2.\end{aligned}$$

On the other hand, since $\sum_{t \in T} e_t \gg 0$, there exists a type $t_j \in T$ such that $e^{(j)}_{t_j} \geq 1$. Thus,

$$\frac{\#A}{r} \leq \#A_{t_j} \leq \sum_{a \in A_{t_j}} e^{(j)}(a) \leq \sum_{a \in A} e^{(j)}(a).$$

Because strong core allocation f is exactly feasible, we have

$$\frac{\#A}{r} \leq \sum_{a \in A} e^{(j)}(a) = \sum_{a \in A} f^{(j)}(a) \leq (\#C_j)ML + ML^2.$$

Thus,

$$\#C_j \geq \frac{\#A}{rML} - L > \frac{rML^2(ML+1)}{rML} - L = ML^2.$$

This completes the proof of Subclaim 1.2. □

For all $j \in J''$, we have

$$\begin{aligned}-1 \;\geq\; y^{(j)} &= \textstyle\sum_{a \in B}(f^{(j)}(a) - e^{(j)}(a)) \\ &\geq -\textstyle\sum_{a \in B} e^{(j)}(a) \\ &\geq -M(\#B) \\ &\geq -ML.\end{aligned}$$

Therefore, by Subclaim 1.2, there exists $\{G_j \,|\, j \in J''\}$ such that

$$\begin{aligned}&G_j \subseteq C_j && \text{for all } j \in J'', \\ &\#G_j = -y^{(j)} && \text{for all } j \in J'', \text{ and} \\ &G_j \cap G_\ell = \emptyset && \text{if } j \neq \ell.\end{aligned}$$

Since $J' \neq \emptyset$, by (1), there exists an $h \in \{1, \ldots, L\}$ such that $\tilde{y}^{(h)} = y^{(h)} \geq kML^2$. Define a mapping $\hat{g} : C \to \mathbb{Z}^L$ by

$$\hat{g}(a) = \begin{cases} f(a) + k\chi_h - \chi_j & \text{if} \quad a \in G_j \ (j \in J''), \\ f(a) & \text{if} \quad a \in C \setminus \bigcup_{j \in J''} G_j. \end{cases}$$

For all $a \in G_j$, $f^{(j)}(a) \geq 1$ holds since $G_j \subseteq C_j$. Thus, $\hat{g}(a) \in \mathbb{Z}^L_+$ for all $a \in C$. Because $\precsim_a \in \mathcal{P}_k$ for all $a \in C$, we have $\hat{g}(a) \succ_a f(a)$ for all $a \in \bigcup_{j \in J''} G_j$. We also have

$$\begin{aligned}
\sum_{a \in C} \hat{g}(a) &= \sum_{a \in C} f(a) + k \sum_{j \in J''} (\# G_j)\chi_h - \sum_{j \in J''} (\# G_j)\chi_j \\
&= \sum_{a \in C} f(a) - k \left(\sum_{j \in J''} y^{(j)} \right) \chi_h + \sum_{j \in J''} y^{(j)} \chi_j \\
&\leq \sum_{a \in C} f(a) + kML^2 \chi_h + \sum_{j \in J''} y^{(j)} \chi_j \\
&\leq \sum_{a \in C} f(a) + \tilde{y} + \sum_{j \in J''} y^{(j)} \chi_j \\
&= \sum_{a \in C} f(a) + y \\
&= \sum_{a \in C} f(a) + \sum_{a \in B} (f(a) - e(a)) \\
&= \sum_{a \in A} f(a) - \sum_{a \in B} e(a) \\
&= \sum_{a \in A} e(a) - \sum_{a \in B} e(a) \\
&= \sum_{a \in C} e(a).
\end{aligned}$$

Although $\hat{g}$ may not be exactly feasible within coalition C, coalition C can weakly improve upon f because agents' preference relations are weakly monotone. This contradicts that $f \in C_S(\mathcal{E})$. This completes the proof of Claim 1. □

Thus, we have established the proof of Lemma 1. □

Next, we prove that in an economy with a large number of agents, every strong core allocation has the weak equal treatment property. Before giving the precise statement, we introduce some notation. Let $X_{L,\xi} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq \xi\}$. A set $\mathcal{X}_{L,\xi}$ is defined as follows. A mapping $\alpha : X_{L,\xi} \to \mathbb{Z}_+$ belongs to $\mathcal{X}_{L,\xi}$ if and only if $\sum_{x \in X_{L,\xi}} \alpha(x) \geq 1$, $\sum_{x \in X_{L,\xi}} \alpha(x)x = 0$, and there exists no mapping $\beta : X_{L,\xi} \to \mathbb{Z}_+$ such that

$$\begin{aligned}
&\sum_{x \in X_{L,\xi}} \beta(x) \geq 1, \quad \sum_{x \in X_{L,\xi}} \beta(x)x = 0,\\
&\beta(x) \leq \alpha(x) \quad \text{for all } x \in X_{L,\xi}, \text{ and}\\
&\beta(y) < \alpha(y) \quad \text{for some } y \in X_{L,\xi}.
\end{aligned}$$

Let

$$\mu(L,\xi) = \sup \left\{ \sum_{x \in X_{L,\xi}} \alpha(x) \,\middle|\, \alpha \in \mathcal{X}_{L,\xi} \right\}.$$

The important fact is that $\mu(L,\xi)$ is finite and, therefore, it is a natural number. This fact is shown in Lemma 3 in the Appendix.

The first statement of the theorem follows from the following lemma.

Lemma 2. *Let A be the set of agents such that $\#A > r\mu(L,\xi)$. If economy $\mathcal{E} : A \to T$ satisfies that $\#A_t/\#A \geq 1/r$ for all $t \in T$, then all strong core allocations for $\mathcal{E}$ have the weak equal treatment property, i.e., for all $f \in C_S(\mathcal{E})$, all $t \in T$, and all $a, b \in A_t$, we have $f(a) \sim_t f(b)$.*

Proof. Suppose, to the contrary, that there exists an allocation $f \in C_S(\mathcal{E})$, a type $t \in T$, and two agents $a, b \in A_t$ such that $f(a) \succ_t f(b)$. Without loss of generality, we can assume that $f(a) \succsim_t f(c) \succsim_t f(b)$ for all $c \in A_t$. By Lemma 1, $\|f(c)\|_\infty \leq \xi$ for all $c \in A$. Since both $f(c)$ and $e(c)$ are nonnegative vectors, we have $\|f(c) - e(c)\|_\infty \leq \max\{\xi,\ M\} = \xi$ for all $c \in A$. Thus, $f(c) - e(c) \in X_{L,\xi}$ for all $c \in A$. Define $\alpha : X_{L,\xi} \to \mathbb{Z}_+$ by, for all $x \in X_{L,\xi}$,

$$\alpha(x) = \#\{c \in A \,|\, f(c) - e(c) = x\}.$$

Note that

$$\begin{aligned}
&\sum_{x \in X_{L,\xi}} \alpha(x)x = \sum_{c \in A} (f(c) - e(c)) = 0 \quad \text{and}\\
&\sum_{x \in X_{L,\xi}} \alpha(x) = \#A > r\mu(L,\xi).
\end{aligned}$$

Thus, by the definition of $\mu(L,\xi)$, there exists a natural number $\ell > r$ and $\beta_j \in \mathcal{X}_{L,\xi}$ ($j = 1,\dots,\ell$) such that for all $x \in X_{L,\xi}$, $\alpha(x) = \sum_{j=1}^{\ell} \beta_j(x)$. Therefore, there exists a partition $\{B_1,\dots,B_\ell\}$ of A such that for all $j \in \{1,\dots,\ell\}$ and all $x \in X_{L,\xi}$,

$$\beta_j(x) = \#\{c \in B_j \mid f(c) - e(c) = x\}.$$

Without loss of generality, we can assume $b \in B_1$. Since $\beta_1 \in \mathcal{X}_{L,\xi}$, we have

$$\begin{aligned}&\sum_{c\in B_1}(f(c)-e(c)) = \sum_{x\in X_{L,\xi}} \beta_1(x)x = 0 \quad \text{and}\\ &\#B_1 = \sum_{x\in X_{L,\xi}} \beta_1(x) \le \mu(L,\xi).\end{aligned}$$

Note that the set $A_t \setminus B_1$ is nonempty, because $\#A_t \ge \#A/r > \mu(L,\xi) \ge \#B_1$.

Claim 2. $f(c) \sim_t f(b)$ for all $c \in A_t \setminus B_1$.

Proof. Suppose, to the contrary, that $f(c^*) \succ_t f(b)$ for some $c^* \in A_t \setminus B_1$. We consider a coalition $C_1 = (A \setminus (B_1 \cup \{c^*\})) \cup \{b\}$. Define $g_1 : C_1 \to \mathbb{Z}_+^L$ by

$$g_1(c) = \begin{cases} f(c^*) & \text{if} \quad c = b,\\ f(c) & \text{if} \quad c \in C_1 \setminus \{b\}.\end{cases}$$

Since $\sum_{c\in A\setminus B_1}(f(c) - e(c)) = 0$ and agents b and c^* have the same type, we have $\sum_{c\in C_1}(g_1(c) - e(c)) = 0$. In addition, we have $g_1(b) = f(c^*) \succ_t f(b)$. This contradicts that $f \in C_S(\mathcal{E})$. This completes the proof of Claim 2. □

From Claim 2, it follows that $a \in B_1$. Since $A_t \setminus B_1$ is nonempty, we can pick $c' \in A_t \setminus B_1$. We consider a coalition $C_2 = (B_1 \setminus \{a\}) \cup \{c'\}$. Define $g_2 : C_2 \to \mathbb{Z}_+^L$ by

$$g_2(c) = \begin{cases} f(a) & \text{if} \quad c = c',\\ f(c) & \text{if} \quad c \in C_2 \setminus \{c'\}.\end{cases}$$

Since $\sum_{c\in B_1}(f(c)-e(c)) = 0$ and agents a and c' have the same type, we have $\sum_{c\in C_2}(g_2(c)-e(c)) = 0$. In addition, from Claim 2, it follows that $g_2(c') = f(a) \succ_t f(b) \sim_t f(c')$. This contradicts that $f \in C_S(\mathcal{E})$. This completes the proof of Lemma 2. □

We next prove the second statement of the theorem. Let

$$\rho = \xi + k(L-1)\xi \quad \text{and}$$
$$q = \max\{\mu(L,\xi),\ 2^{L-1}L^{L/2}\rho^{L+1}(1+\rho)\}.$$

Note that both numbers ρ and q depend only on exogenous variables. Let A be the set of agents such that $\#A > rq$ and let $\mathcal{E} : A \to T$ be an economy such that $\#A_t/\#A \geq 1/r$ for all $t \in T$. We prove that $C_S(\mathcal{E}) \subseteq W_{CM}(\mathcal{E})$. (Recall that the opposite inclusion $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$ is always satisfied.) Let $f \in C_S(\mathcal{E})$.

Claim 3. $\rho\,\chi_i \succ_a f(a)$ for all $i \in \{1, \dots, L\}$ and all $a \in A$.

Proof. Let $i \in \{1, \dots, L\}$ and $a \in A$. We consider two distinct cases.

Case 1. $f^{(j)}(a) = 0$ for all $j \neq i$.

Since $\precsim_a \in \mathcal{P}_k$, we have

$$f(a) \prec_a f(a) + k\,\chi_i = \left(f^{(i)}(a) + k\right)\chi_i.$$

By Lemma 1, we have

$$f^{(i)}(a) + k \leq \xi + k < \rho.$$

Since $\precsim_a$ is weakly monotone, we have $f(a) \prec_a \rho\,\chi_i$.

Case 2. $f^{(j)}(a) \geq 1$ for some $j \neq i$.

Since $\precsim_a \in \mathcal{P}_k$, we have

$$f(a) \prec_a f(a) - \sum_{j\neq i} f^{(j)}(a)\chi_j + k\sum_{j\neq i} f^{(j)}(a)\chi_i = \left(f^{(i)}(a) + k\sum_{j\neq i} f^{(j)}(a)\right)\chi_i.$$

By Lemma 1, we have

$$f^{(i)}(a) + k\sum_{j\neq i} f^{(j)}(a) \leq \xi + k(L-1)\xi = \rho.$$

Since $\precsim_a$ is weakly monotone, we have $f(a) \prec_a \rho\,\chi_i$. This completes the proof of Claim 3. □

For every $t \in T$, let

$$\varphi_t = \{z \in \mathbb{Z}^L \mid z + e_t \in \mathbb{Z}^L_+ \quad \text{and} \quad z + e_t \succ_t f(a)\} = \{x \in \mathbb{Z}^L_+ \mid x \succ_t f(a)\} - \{e_t\}$$

and

$$\psi_t = \{z \in \mathbb{Z}^L \mid z + e_t \in \mathbb{Z}^L_+ \quad \text{and} \quad z + e_t \succsim_t f(a)\} = \{x \in \mathbb{Z}^L_+ \mid x \succsim_t f(a)\} - \{e_t\},$$

where $a \in A_t$. By Lemma 2, since $\#A > r\mu(L, \xi)$, φ_t and ψ_t are both well-defined for all $t \in T$. For all $t \in T$, define the set φ'_t of minimal elements of φ_t as follows: $z \in \varphi'_t$ if and only if $z \in \varphi_t$ and there exists no $y \in \varphi_t$ with $y < z$. The set ψ'_t of minimal elements of ψ_t is defined similarly. Since every φ_t and every ψ_t is bounded from below, by Gordan's lemma (Lemma 4 in the Appendix), φ'_t and ψ'_t are nonempty and finite, and satisfies that $\varphi_t \subseteq \varphi'_t + \mathbb{Z}^L_+$ and $\psi_t \subseteq \psi'_t + \mathbb{Z}^L_+$. Since agents' preference relations are weakly monotone, we have $\varphi_t = \varphi'_t + \mathbb{Z}^L_+$ and $\psi_t = \psi'_t + \mathbb{Z}^L_+$ for all $t \in T$.

Let $X_{L,\rho} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq \rho\}$.

Claim 4. $\varphi'_t \subseteq X_{L,\rho}$ and $\psi'_t \subseteq X_{L,\rho}$ for all $t \in T$.

Proof. We only prove that $\varphi'_t \subseteq X_{L,\rho}$. The inclusion $\psi'_t \subseteq X_{L,\rho}$ can be proved similarly. Suppose, to the contrary, that $\varphi'_t \nsubseteq X_{L,\rho}$ for some $t \in T$. Since $\varphi'_t \subseteq \mathbb{Z}^L_+ - \{e_t\} \subseteq \mathbb{Z}^L_+ - \{(M, \dots, M)\}$ and $M < \rho$, we have, for all $z \in \varphi'_t$ and all $h \in \{1, \dots, L\}$, $z^{(h)} > -\rho$. Therefore, from $\varphi'_t \nsubseteq X_{L,\rho}$, there exists a $z \in \varphi'_t$ and an $h \in \{1, \dots, L\}$ such that $z^{(h)} > \rho$. Since $\rho\,\chi_h \succ_t f(a)$ for all $a \in A_t$ by Claim 3, we have $\rho\,\chi_h - e_t \in \varphi_t$. For coordinate h, we have

$$z^{(h)} > \rho \geq \rho\,\chi_h^{(h)} - e_t^{(h)}.$$

Since $z \geq -e_t$, for coordinate i with $i \neq h$, we have

$$z^{(i)} \geq -e_t^{(i)} = \rho\,\chi_h^{(i)} - e_t^{(i)}.$$

Thus, $z > \rho\,\chi_h - e_t$ and $\rho\,\chi_h - e_t \in \varphi_t$. This contradicts that $z \in \varphi'_t$. We have established the proof of Claim 4. □

By Claim 4, for all $t \in T$, we have

$$\begin{aligned} \psi_t &= \psi_t' + \mathbb{Z}_+^L \\ &= (\psi_t' \cap X_{L,\rho}) + \mathbb{Z}_+^L \\ &\subseteq (\psi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L \\ &\subseteq \psi_t + \mathbb{Z}_+^L \\ &= \psi_t. \end{aligned}$$

Thus, $\psi_t = (\psi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L$ for all $t \in T$. Therefore, $\bigcup_{t \in T} \psi_t = \left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho}\right) + \mathbb{Z}_+^L$.

Claim 5. $0 \notin \operatorname{int}\left(\operatorname{co}\left(\bigcup_{t \in T} \psi_t\right)\right)$ if and only if $0 \notin \operatorname{int}\left(\operatorname{co}\left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho}\right)\right)$, where $\operatorname{int}(C)$ and $\operatorname{co}(C)$ denote the interior and the convex hull of set C, respectively.

Proof. It suffices to prove the sufficiency. Assume that $0 \notin \operatorname{int}\left(\operatorname{co}\left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho}\right)\right)$. By the separation theorem for convex sets, there exists a $p \in \mathbb{R}^L \setminus \{0\}$ such that, for all $z \in \operatorname{co}\left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho}\right)$, $p \cdot z \geq 0$.

We prove that $p \geq 0$. Suppose, to the contrary, that $p^{(h)} < 0$ for some $h \in \{1, \ldots, L\}$. Since, by Claim 3, $\rho\, \chi_h \succ_t f(a)$ for all $a \in A_t$ and all $t \in T$, and since $e_t \geq 0$ and $\precsim_t$ is weakly monotone, we have

$$\rho\, \chi_h + e_t \succsim_t \rho\, \chi_h \succ_t f(a).$$

Thus, $\rho\, \chi_h \in \varphi_t \cap X_{L,\rho} \subseteq \psi_t \cap X_{L,\rho}$. Then, by the consequence of the separation theorem for convex sets,

$$0 \leq p \cdot (\rho\, \chi_h) = \rho\, p^{(h)},$$

a contradiction. We have then $p \geq 0$.

Since $\bigcup_{t \in T} \psi_t = \left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho}\right) + \mathbb{Z}_+^L$, we have $p \cdot z \geq 0$ for all $z \in \operatorname{co}\left(\bigcup_{t \in T} \psi_t\right)$. Hence, $0 \notin \operatorname{int}\left(\operatorname{co}\left(\bigcup_{t \in T} \psi_t\right)\right)$. This completes the proof of Claim 5. □

We next find a price vector $\bar{p}$ under which strong core allocation f satisfies both the cost minimization and the preference maximization. To obtain such price vector $\bar{p}$, we first find a price vector p_0 under which f satisfies the cost minimization. Then, we move p_0 in an appropriate direction slightly and make the resulting price vector satisfy the desired properties. The following claim will be used not only when we find the first price vector p_0 but also when we move p_0 in an appropriate direction.

Claim 6. Let H be a linear subspace of $\mathbb{R}^L$. If $\bigcup_{t\in T}\varphi_t \cap X_{L,\rho} \cap H \neq \emptyset$, then $0 \notin \mathrm{ri}\left(\mathrm{co}\left(\bigcup_{t\in T}\psi_t \cap X_{L,\rho} \cap H\right)\right)$, where ri$(C)$ denotes the relative interior of set C.

Proof. Suppose, to the contrary, that $0 \in \mathrm{ri}\left(\mathrm{co}\left(\bigcup_{t\in T}\psi_t \cap X_{L,\rho} \cap H\right)\right)$. Let $z_0 \in \bigcup_{t\in T}\varphi_t \cap X_{L,\rho} \cap H$. Then, there exists a $t_0 \in T$ with $z_0 \in \varphi_{t_0}$. Since every $\succ_t$ is irreflexive, $0 \notin \bigcup_{t\in T}\varphi_t$ and, therefore, $z_0 \neq 0$. Let $\bar{s} = \min\{s \in \mathbb{R} \,|\, s\, z_0 \in \mathrm{co}\left(\bigcup_{t\in T}\psi_t \cap X_{L,\rho} \cap H\right)\}$. Then, by Lemma 6, $|\bar{s}| \leq \rho$ and there exist $q_0 \in \mathbb{Z}_{++}$ with $q_0 \leq 2^{L-1}L^{L/2}\rho^{L+1}$, $\{x_{t,j} \,|\, j = 1, \dots, m_t\} \subseteq \psi_t \cap X_{L,\rho} \cap H$ $(t \in T)$, and $(\alpha_t^{(1)}, \dots, \alpha_t^{(m_t)}) \in \mathbb{Q}_+^{m_t}$ $(t \in T)$ such that

$$\begin{aligned}
&\sum_{t\in T}\sum_{j=1}^{m_t}\alpha_t^{(j)} = 1,\\
&q_0\,\alpha_t^{(j)} \in \mathbb{Z}_+ \quad \text{for all } j \in \{1,\dots,m_t\} \text{ and all } t \in T,\\
&q_0\,\bar{s} \in -\mathbb{Z}_{++}, \text{ and}\\
&\sum_{t\in T}\sum_{j=1}^{m_t}\alpha_t^{(j)}\,x_{t,j} = \bar{s}\,z_0.
\end{aligned}$$

Since $\#A > rq = r\max\{\mu(L,\xi),\ 2^{L-1}L^{L/2}\rho^{L+1}(1+\rho)\}$, we have, for all $t \in T$, $\#A_t \geq \#A/r > 2^{L-1}L^{L/2}\rho^{L+1}(1+\rho)$. Thus, for all $t \in T \setminus \{t_0\}$, there exists a mutually disjoint family $\{C_{t,j} \,|\, j = 1, \dots, m_t\}$ of subsets of A_t such that

$$\#C_{t,j} = q_0\,\alpha_t^{(j)} \quad \text{for all } j \in \{1,\dots,m_t\}.$$

Since $q_0 + q_0\,\bar{s} \leq 2^{L-1}L^{L/2}\rho^{L+1}(1+\rho)$, there exists a mutually disjoint family $\{C_{t_0,j} \,|\, j = 1, \dots, m_{t_0}\} \cup \{D\}$ of subsets of A_{t_0} such that

$$\begin{aligned}
&\#C_{t_0,j} = q_0\,\alpha_{t_0}^{(j)} \quad \text{for all } j \in \{1,\dots,m_{t_0}\}, \text{ and}\\
&\#D = q_0\,|\bar{s}|.
\end{aligned}$$

Let $S = \bigcup_{t\in T}\bigcup_{j=1}^{m_t} C_{t,j} \cup D$. Define $g : S \to \mathbb{Z}_+^L$ by

$$g(a) = \begin{cases} x_{t,j} + e_t & \text{if } a \in C_{t,j}\ (j = 1,\dots,m_t;\ t \in T),\\ z_0 + e_{t_0} & \text{if } a \in D. \end{cases}$$

Then,

$$\begin{aligned}\sum_{a\in S} g(a) &= \sum_{t\in T}\sum_{j=1}^{m_t} (\#C_{t,j})\, x_{t,j} + (\#D)\, z_0 + \sum_{a\in S} e(a)\\ &= q_0\left(\sum_{t\in T}\sum_{j=1}^{m_t} \alpha_t^{(j)}\, x_{t,j} + |\bar{s}|\, z_0\right) + \sum_{a\in S} e(a)\\ &= \sum_{a\in S} e(a).\end{aligned}$$

Therefore, g is exactly feasible within coalition S. For $a \in C_{t,j}$ ($j = 1, \dots, m_t$; $t \in T$), since $x_{t,j} \in \psi_t$, we have $g(a) = x_{t,j} + e_t \succsim_t f(a)$. For $a \in D$, since $z_0 \in \varphi_{t_0}$, we have $g(a) = z_{t_0} + e_{t_0} \succ_{t_0} f(a)$. This contradicts that $f \in C_S(\mathcal{E})$. Hence, $0 \notin \mathrm{ri}\left(\mathrm{co}\left(\bigcup_{t\in T} \psi_t \cap X_{L,\rho} \cap H\right)\right)$. This completes the proof of Claim 6. □

By Claim 4, $\bigcup_{t\in T} \varphi_t \cap X_{L,\rho} \neq \emptyset$ and then, by Claim 6, $0 \notin \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t\in T} \psi_t \cap X_{L,\rho}\right)\right)$. Thus, by Claim 5, we have $0 \notin \mathrm{int}\left(\mathrm{co}\left(\bigcup_{t\in T} \psi_t\right)\right)$. By the separation theorem for convex sets, there exists a $p_0 \in \mathbb{R}^L \setminus \{0\}$ such that $p_0 \cdot z \geq 0$ for all $z \in \mathrm{co}\left(\bigcup_{t\in T} \psi_t\right)$. From the weak monotonicity of preference relations, $\mathrm{co}\left(\bigcup_{t\in T} \psi_t\right) = \mathrm{co}\left(\bigcup_{t\in T} \psi_t\right) + \mathbb{R}^L_+$. Hence, we have $p_0 \geq 0$.

Claim 7. $p_0 \cdot (f(a) - e(a)) = 0$ for all $a \in A$.

Proof. Since $f(a) - e(a) \in \bigcup_{t\in T} \psi_t$ for all $a \in A$, we have $p_0 \cdot (f(a) - e(a)) \geq 0$. By the exact feasibility of f, we have $p_0 \cdot (f(a) - e(a)) = 0$ for all $a \in A$. □

Claim 8. $p_0 \gg 0$.

Proof. Suppose, to the contrary, that there exists an $h \in \{1, \dots, L\}$ with $p_0^{(h)} = 0$. Since $p_0 \in \mathbb{R}^L_+ \setminus \{0\}$, there exists a $j \in \{1, \dots, L\}$ with $p_0^{(j)} > 0$. Since $\sum_{a\in A} f^{(j)}(a) = \sum_{a\in A} e^{(j)}(a) > 0$, there exists an agent $a \in A$ with $f^{(j)}(a) \geq 1$. From $\precsim_a \in \mathcal{P}_k$, it follows that $f(a) \prec_a f(a) - \chi_j + k\,\chi_h$. Thus,

$$f(a) - \chi_j + k\,\chi_h - e(a) \in \bigcup_{t\in T} \varphi_t \subseteq \bigcup_{t\in T} \psi_t.$$

By the consequence of the separation theorem and by Claim 7, we have

$$0 \leq p_0 \cdot (f(a) - \chi_j + k\,\chi_h - e(a)) = -p_0^{(j)} + k\,p_0^{(h)} = -p_0^{(j)},$$

which is a contradiction. Thus, $p_0 \gg 0$. □

Claim 9. There exists a $\bar{p} \in \mathbb{R}^L_{++}$ such that

(1) $\bar{p} \cdot z > 0$ for all $z \in \bigcup_{t \in T} \varphi_t$, and

(2) $\bar{p} \cdot z \geq 0$ for all $z \in \bigcup_{t \in T} \psi_t$.

Proof. Let $H_0 = \{z \in \mathbb{R}^L \,|\, p_0 \cdot z = 0\}$. If $\bigcup_{t \in T} \varphi_t \cap H_0 = \emptyset$, then p_0 satisfies the desired properties. Assume that $\bigcup_{t \in T} \varphi_t \cap H_0 \neq \emptyset$.

Subclaim 9.1. $\bigcup_{t \in T} \psi_t \cap H_0 \subseteq X_{L,\rho}$.

Proof. Let $z \in \bigcup_{t \in T} \psi_t \cap H_0$. Then, $p_0 \cdot z = 0$. Since $p_0 \gg 0$ by Claim 8 and $p_0 \cdot y \geq 0$ for all $y \in \text{co}\left(\bigcup_{t \in T} \psi_t\right)$, we have, by Claim 4, $z \in \bigcup_{t \in T} \psi'_t \subseteq X_{L,\rho}$. □

By Subclaim 9.1, we have

$$\emptyset \neq \bigcup_{t \in T} \varphi_t \cap H_0 \subseteq \bigcup_{t \in T} \psi_t \cap H_0 \subseteq X_{L,\rho}.$$

Thus, $\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \cap H_0 \neq \emptyset$ and, by Claim 6, we have

$$0 \notin \text{ri}\left(\text{co}\left(\bigcup_{t \in T} \psi_t \cap X_{L,\rho} \cap H_0\right)\right) = \text{ri}\left(\text{co}\left(\bigcup_{t \in T} \psi_t \cap H_0\right)\right).$$

By the separation theorem for convex sets, there exists a $p_1 \in \text{span}\left(\bigcup_{t \in T} \psi_t \cap H_0\right) \setminus \{0\}$ such that $p_1 \cdot z \geq 0$ for all $z \in \bigcup_{t \in T} \psi_t \cap H_0$. Let

$$E_0 = \bigcup_{t \in T} \psi_t \setminus H_0 = \left\{ z \in \bigcup_{t \in T} \psi_t \,\middle|\, p_0 \cdot z > 0 \right\}.$$

Let E'_0 be the set of minimal elements of E_0, i.e., $x \in E'_0$ if and only if $x \in E_0$ and there exists no $y \in E_0$ with $y < x$. Since E_0 is nonempty and bounded from below, by Gordan's lemma (Lemma 4 in the Appendix), E_0' is a nonempty finite subset of E_0 and satisfies that $E_0 \subseteq E_0' + \mathbb{Z}_+^L$. Since $p_0 \gg 0$, we have

$$0 < \min\{p_0 \cdot z \,|\, z \in E_0'\} = \inf\{p_0 \cdot z \,|\, z \in E_0\}.$$

Since the mapping $p \mapsto \min\{p \cdot z \,|\, z \in E_0'\}$ is continuous, there exists an open neighborhood U_0 of p_0 such that, for all $p \in U_0$, $\min\{p \cdot z \,|\, z \in E_0'\} > 0$. Since $p_0 \gg 0$, by taking a sufficiently small $\varepsilon_1 > 0$, we have $p_0 + \varepsilon_1 p_1 \gg 0$ and $p_0 + \varepsilon_1 p_1 \in U_0$. Then,

$$0 < \min\{(p_0 + \varepsilon_1 p_1) \cdot z \,|\, z \in E_0'\} = \inf\{(p_0 + \varepsilon_1 p_1) \cdot z \,|\, z \in E_0\}.$$

Summing up, we have obtained that

(a) $p_0 + \varepsilon_1 p_1 \in \mathbb{R}_{++}^L$;

(b) $(p_0 + \varepsilon_1 p_1) \cdot z > 0 \quad$ for all $z \in \bigcup_{t \in T} \psi_t \setminus H_0$; and

(c) $(p_0 + \varepsilon_1 p_1) \cdot z \geq 0 \quad$ for all $z \in \bigcup_{t \in T} \psi_t \cap H_0$.

Let $H_1 = \{z \in \mathbb{R}^L \,|\, (p_0 + \varepsilon_1 p_1) \cdot z = 0\}$. If $\bigcup_{t \in T} \varphi_t \cap H_0 \cap H_1 = \emptyset$, then $p_0 + \varepsilon_1 p_1$ satisfies the desired properties. Assume that $\bigcup_{t \in T} \varphi_t \cap H_0 \cap H_1 \neq \emptyset$. Then, by the same argument as above, there exists a $p_2 \in \operatorname{span}\left(\bigcup_{t \in T} \psi_t \cap H_0 \cap H_1\right) \setminus \{0\}$ and an $\varepsilon_2 > 0$ such that

(a′) $p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2 \in \mathbb{R}_{++}^L$;

(b′) $(p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2) \cdot z > 0 \quad$ for all $z \in \bigcup_{t \in T} \psi_t \setminus (H_0 \cup H_1)$; and

(c′) $(p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2) \cdot z \geq 0 \quad$ for all $z \in \bigcup_{t \in T} \psi_t \cap H_0 \cap H_1$.

Let $H_2 = \{z \in \mathbb{R}^L \,|\, (p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2) \cdot z = 0\}$. If $\bigcup_{t \in T} \varphi_t \cap H_0 \cap H_1 \cap H_2 = \emptyset$, then $p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2$ satisfies the desired properties. If $\bigcup_{t \in T} \varphi_t \cap H_0 \cap H_1 \cap H_2 \neq \emptyset$, then, by repeating the same argument, say, m times ($m \leq L$), we could obtain that

$$\bigcup_{t \in T} \varphi_t \cap H_0 \cap H_1 \cap \cdots \cap H_{m-1} = \emptyset,$$

because $0 \notin \bigcup_{t \in T} \varphi_t$ and

$$\dim\left(H_0 \cap H_1 \cap H_2\right) < \dim\left(H_0 \cap H_1\right) < \dim H_0 = L - 1.$$

The vector $p_0 + \sum_{i=1}^{m-1} \varepsilon_i p_i$ satisfies the desired properties. This completes the proof of Claim 9. □

Let $\bar{p}$ be the vector obtained in Claim 9. Let

$$F = \left\{ z \in \bigcup_{t \in T} \psi_t \,\middle|\, \bar{p} \cdot z > 0 \right\} \quad \text{and} \quad V = \left\{ z \in \bigcup_{t \in T} \psi_t \,\middle|\, \bar{p} \cdot z = 0 \right\}.$$

Since $f(a) - e(a) \in \bigcup_{t \in T} \psi_t$ for all $a \in A$, we have, by Claim 9, $\bar{p} \cdot (f(a) - e(a)) \geq 0$ for all $a \in A$. By the exact feasibility of f, we have

$$\bar{p} \cdot (f(a) - e(a)) = 0 \quad \text{for all } a \in A$$

and, therefore, $f(a) - e(a) \in V$ for all $a \in A$. Thus, by Claim 9, the pair $(\bar{p}, f)$ satisfies conditions (i)-(iii) of the definition of cost-minimized Walras equilibrium, but $\bar{p}$ may not be a rational vector. Finally, we find an integral price vector p^* under which f is a cost-minimized Walras allocation. Note that $\mathrm{co}(F) \cap V = \emptyset$. Since $\bar{p} \gg 0$, we have $\mathrm{co}(F + \mathbb{Z}^L_+) \cap V = \emptyset$ and, then, $\mathrm{co}(F) \cap (V - \mathbb{R}^L_+) = \emptyset$. By Inoue's [9] separation theorem (Lemma 6 in the Appendix), there exists a $p^* \in V^\perp \cap \mathbb{Z}^L_+$ and an $\varepsilon > 0$ such that $p^* \cdot z \geq \varepsilon$ for all $z \in F$. We prove that (p^*, f) is a cost-minimized Walras equilibrium. Since $p^* \in V^\perp$ and $f(a) - e(a) \in V$ for all $a \in A$, we have

$$p^* \cdot (f(a) - e(a)) = 0 \quad \text{for all } a \in A.$$

Let $a \in A$ and $x \in \mathbb{Z}^L_+$ with $x \succ_a f(a)$. Then, $x - e(a) \in \bigcup_{t \in T} \varphi_t \subseteq F$. Thus, $p^* \cdot (x - e(a)) \geq \varepsilon > 0$.

Let $a \in A$ and $x \in \mathbb{Z}^L_+$ with $x \succsim_a f(a)$. Then, $x - e(a) \in \bigcup_{t \in T} \psi_t \subseteq F \cup V$. Thus, $p^* \cdot (x - e(a)) \geq 0$. Therefore, (p^*, f) is a cost-minimized Walras equilibrium. We have established that $C_S(\mathcal{E}) \subseteq W_{CM}(\mathcal{E})$.

Appendix

The following results are used in the proof of the theorem.

Lemma 3. *Let L and N be natural numbers. Let $X_{L,N} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq N\}$. Define a set $\mathcal{X}_{L,N}$ as follows: A mapping $\alpha : X_{L,N} \to \mathbb{Z}_+$ belongs to $\mathcal{X}_{L,N}$ if and only if $\sum_{x \in X_{L,N}} \alpha(x) \geq 1$, $\sum_{x \in X_{L,N}} \alpha(x)x = 0$, and there exists no mapping $\beta : X_{L,N} \to \mathbb{Z}_+$ such* *that* $\sum_{x \in X_{L,N}} \beta(x) \geq 1$, $\sum_{x \in X_{L,N}} \beta(x)x = 0$, $\beta(x) \leq \alpha(x)$ *for all* $x \in X_{L,N}$, *and* $\beta(y) < \alpha(y)$ *for some* $y \in X_{L,N}$. *Then, the number*

$$\mu(L, N) := \sup \left\{ \sum_{x \in X_{L,N}} \alpha(x) \,\middle|\, \alpha \in \mathcal{X}_{L,N} \right\}$$

is finite. Hence, $\mu(L, N)$ *can be achieved by some* $\alpha \in \mathcal{X}_{L,N}$. *In particular,*

$$\begin{aligned} &\mu(1, 1) = 2, \\ &\mu(1, N) \leq N^2(N + 1)/2 \quad \textit{for all } N \geq 2, \textit{and} \\ &\mu(L + 1, N) \leq \mu(L, N)\mu(1, N(N + 1)\mu(L, N)) \quad \textit{for all } L, N \in \mathbb{Z}_{++}. \end{aligned}$$

Proof. We prove the lemma by induction on L. Let $L = 1$. Clearly, $\mu(1, 1) = 2$. We consider the case where $N \geq 2$.

Claim 10. $\mu(1, N) \leq N^2(N + 1)/2$ for all $N \geq 2$.

Proof. Suppose, to the contrary, that there exists a mapping $\alpha \in \mathcal{X}_{1,N}$ such that $\sum_{x \in X_{1,N}} \alpha(x) > N^2(N + 1)/2$. Because $\sum_{x \in X_{1,N}} \alpha(x) > 2$, by the definition of $\mathcal{X}_{1,L}$, $\alpha(0) = 0$ and there exists no $\ell \in X_{1,N} \setminus \{0\}$ such that $\alpha(\ell) \geq 1$ and $\alpha(-\ell) \geq 1$. Thus, $\#\{x \in X_{1,N} \,|\, \alpha(x) \geq 1\} \leq N$. Since $\sum_{x \in X_{1,N}} \alpha(x) > N^2(N+1)/2$, there exists an integer $m \in X_{1,N} \setminus \{0\}$ such that

$$\alpha(m) > N(N + 1)/2.$$

We may assume that $m \geq 1$.

Subclaim 10.1. If $\ell \in \mathbb{Z}$ and $1 \leq \ell \leq N$, then $\alpha(-\ell) < m$.

Proof. Suppose, to the contrary, that there exists an $\ell \in \mathbb{Z}$ such that $1 \leq \ell \leq N$ and $\alpha(-\ell) \geq m$. Note that $\alpha(m) > N(N+1)/2 > N \geq \ell$. We define a mapping $\beta : X_{1,N} \to \mathbb{Z}_+$ by

$$\beta(x) = \begin{cases} m & \text{if} \quad x = -\ell, \\ \ell & \text{if} \quad x = m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\sum_{x \in X_{1,N}} \beta(x) \geq 1,$$
$$\beta(x) \leq \alpha(x) \quad \text{for all } x \in X_{1,N}, \text{ and}$$
$$\beta(m) < \alpha(m).$$

Moreover, we have $\sum_{x \in X_{1,N}} \beta(x)x = m(-\ell) + \ell m = 0$. This contradicts that $\alpha \in \mathcal{X}_{1,N}$. Therefore, we have established the proof of Subclaim 10.1. $\square$

Since

$$0 = \sum_{x \in X_{1,N}} \alpha(x)x = \sum_{\ell=1}^{N} \alpha(\ell)\ell + \sum_{\ell=1}^{N} \alpha(-\ell)(-\ell),$$

we have $\sum_{\ell=1}^{N} \alpha(-\ell)\ell = \sum_{\ell=1}^{N} \alpha(\ell)\ell$. On the other hand, from Subclaim 10.1, it follows that

$$\begin{aligned}\sum_{\ell=1}^{N} \alpha(-\ell)\ell &< m\sum_{\ell=1}^{N} \ell \\ &= mN(N+1)/2 \\ &< \alpha(m)m \\ &\leq \sum_{\ell=1}^{N} \alpha(\ell)\ell,\end{aligned}$$

which is a contradiction. This completes the proof of Claim 10. $\square$

Let $K \in \mathbb{Z}_{++}$. Assume that for all $N \in \mathbb{Z}_{++}$ and all $L \in \mathbb{Z}_{++}$ with $L \leq K$, $\mu(L, N)$ is finite. We now prove that $\mu(K+1, N)$ is finite for all $N \in \mathbb{Z}_{++}$.

Claim 11. $\mu(K+1, N) \leq \mu(K, N)\mu(1, N(N+1)\mu(K, N))$ for all $N \in \mathbb{Z}_{++}$.

Proof. Suppose, to the contrary, that $\mu(K+1, N) > \mu(K, N)\mu(1, N(N+1)\mu(K, N))$ for some $N \in \mathbb{Z}_{++}$. Then, there exists a mapping $\alpha \in \mathcal{X}_{K+1,N}$ such that

$$\sum_{x \in X_{K+1,N}} \alpha(x) > \mu(K, N)\mu(1, N(N+1)\mu(K, N)).$$

We define a mapping $\beta : X_{K,N} \to \mathbb{Z}_+$ by $\beta(y) = \sum_{\ell \in X_{1,N}} \alpha(\ell, y)$ for all $y \in X_{K,N}$. From $\alpha \in \mathcal{X}_{K+1,N}$, it follows that

$$\sum_{y \in X_{K,N}} \beta(y)y = \sum_{y \in X_{K,N}} \sum_{\ell \in X_{1,N}} \alpha(\ell, y)y = 0.$$

We also have

$$\begin{aligned} \sum_{y \in X_{K,N}} \beta(y) &= \sum_{y \in X_{K,N}} \sum_{\ell \in X_{1,N}} \alpha(\ell, y) \\ &= \sum_{x \in X_{K+1,N}} \alpha(x) \\ &> \mu(K,N)\mu(1, N(N+1)\mu(K,N)). \end{aligned}$$

Thus, there exists a natural number $k > \mu(1, N(N+1)\mu(K,N))$ and mappings $\tilde{\gamma}_j \in \mathcal{X}_{K,N}$ ($j = 1, \ldots, k$) such that, for all $y \in X_{K,N}$, $\sum_{j=1}^{k} \tilde{\gamma}_j(y) = \beta(y)$. Since $\sum_{j=1}^{k} \tilde{\gamma}_j(y) = \sum_{\ell \in X_{1,N}} \alpha(\ell, y)$ for all $y \in X_{K,N}$, there exist mappings $\gamma_j : X_{K+1,N} \to \mathbb{Z}_+$ ($j = 1, \ldots, k$) such that

$$\begin{aligned} &\sum_{\ell \in X_{1,N}} \gamma_j(\ell, y) = \tilde{\gamma}_j(y) \quad \text{for all } y \in X_{K,N} \text{ and all } j \in \{1, \ldots, k\} \text{ and} \\ &\sum_{j=1}^{k} \gamma_j(\ell, y) = \alpha(\ell, y) \quad \text{for all } (\ell, y) \in X_{K+1,N}. \end{aligned}$$

For every $j \in \{1, \ldots, k\}$, let

$$\delta_j = \sum_{\ell \in X_{1,N}} \sum_{y \in X_{K,N}} \gamma_j(\ell, y)\ell \in \mathbb{Z}.$$

Since $\alpha \in \mathcal{X}_{K+1,N}$, we have

$$\sum_{j=1}^{k} \delta_j = \sum_{\ell \in X_{1,N}} \sum_{y \in X_{K,N}} \sum_{j=1}^{k} \gamma_j(\ell, y)\ell = \sum_{\ell \in X_{1,N}} \sum_{y \in X_{K,N}} \alpha(\ell, y)\ell = 0.$$

Since $\tilde{\gamma}_j \in \mathcal{X}_{K,N}$, it follows that for all $\ell \in X_{1,N}$,

$$\sum_{y \in X_{K,N}} \gamma_j(\ell, y) \leq \sum_{y \in X_{K,N}} \tilde{\gamma}_j(y) \leq \mu(K,N).$$

Therefore, for all $j \in \{1, \ldots, k\}$,

$$|\delta_j| \leq \sum_{\ell \in X_{1,N}} |\ell|\mu(K,N) = N(N+1)\mu(K,N).$$

Thus, $\delta_j \in X_{1,N(N+1)\mu(K,N)}$ for all $j \in \{1, \dots, k\}$. Since $\sum_{j=1}^{k} \delta_j = 0$ and $k > \mu(1, N(N+1)\mu(K,N))$, there exists a subset J of $\{1, \dots, k\}$ such that $\emptyset \neq J \subsetneq \{1, \dots, k\}$ and $\sum_{j \in J} \delta_j = 0$. Define a mapping $\zeta : X_{K+1,N} \to \mathbb{Z}_+$ by, for all $x \in X_{K+1,N}$,

$$\zeta(x) = \sum_{j \in J} \gamma_j(x).$$

We have

$$\begin{aligned}\sum_{x \in X_{K+1,N}} \zeta(x) x^{(1)} &= \sum_{\ell \in X_{1,N}} \sum_{y \in X_{K,N}} \sum_{j \in J} \gamma_j(\ell, y)\ell \\ &= \sum_{j \in J} \delta_j \\ &= 0.\end{aligned}$$

Since $\tilde{\gamma}_j \in \mathcal{X}_{K,N}$ for all j, we have

$$\begin{aligned}\sum_{(\ell,y) \in X_{K+1,N}} \zeta(\ell, y) y &= \sum_{y \in X_{K,N}} \sum_{\ell \in X_{1,N}} \zeta(\ell, y) y \\ &= \sum_{j \in J} \sum_{y \in X_{K,N}} \sum_{\ell \in X_{1,N}} \gamma_j(\ell, y) y \\ &= \sum_{j \in J} \sum_{y \in X_{K,N}} \tilde{\gamma}_j(y) y \\ &= 0.\end{aligned}$$

Thus, $\sum_{x \in X_{K+1,N}} \zeta(x) x = 0$. Since $J \neq \emptyset$, we have $\sum_{x \in X_{K+1,N}} \zeta(x) \geq 1$. It is obvious that for all $x \in X_{K+1,N}$,

$$\zeta(x) = \sum_{j \in J} \gamma_j(x) \leq \sum_{j=1}^{k} \gamma_j(x) = \alpha(x).$$

Since $J \subsetneq \{1, \dots, k\}$ and $\sum_{x \in X_{K+1,N}} \gamma_j(x) \geq 1$ for all $j \in \{1, \dots, k\}$, there exists an element x^* of $X_{K+1,N}$ such that

$$\zeta(x^*) = \sum_{j \in J} \gamma_j(x^*) < \sum_{j=1}^{k} \gamma_j(x^*) = \alpha(x^*).$$

This contradicts that $\alpha \in \mathcal{X}_{K+1,N}$. This completes the proof of Claim 11. □

Hence, we have established the proof of Lemma 3. □

Lemma 4 (Gordan's lemma). *Let E be a nonempty subset of $\mathbb{Z}^L$. Define a subset E' of E as follows: $x \in E'$ if and only if $x \in E$ and there exists no $y \in E$ with $y < x$. If E is bounded from below, then E' is a nonempty finite set and satisfies that $E \subseteq E' + \mathbb{Z}^L_+$.*

Proof. See, e.g., Inoue [9, Lemma 5.1]. □

Lemma 5 (Hadamard's inequality). *If $B = (b_1, \dots, b_\ell)$ is an $\ell \times \ell$ matrix of real numbers, then*

$$|\det B| \leq \prod_{j=1}^{\ell} \|b_j\|,$$

where $\det B$ *is the determinant of matrix B and $\|\cdot\|$ is the Euclidean norm.*

Proof. See Dunford and Schwartz [6, pp.1018-1019]. □

Lemma 6. *Let $E \subseteq X_{L,N}$, $z^* \in E \setminus \{0\}$, and $0 \in \mathrm{ri}\,(\mathrm{co}(E))$, where $\mathrm{ri}(C)$ denotes the relative interior of set C. Let $m = \dim \mathrm{span}(E)$, $\bar{s} = \min\{s \in \mathbb{R} \mid s\, z^* \in \mathrm{co}(E)\}$, and $q = 2^{m-1} m^{m/2} N^{m+1}$. Then, $1 \leq m \leq L$, $|\bar{s}| \leq N$, and there exist $q_0 \in \mathbb{Z}_{++}$ with $q_0 \leq q$, $\{x_1, \dots, x_m\} \subseteq E$, and $(\alpha^{(1)}, \dots, \alpha^{(m)}) \in \mathbb{Q}^m_+$ such that*

$$\begin{aligned}
&\sum_{j=1}^{m} \alpha^{(j)} = 1, \\
&q_0\, \alpha^{(j)} \in \mathbb{Z}_+ \text{ for all } j \in \{1, \dots, m\}, \\
&q_0\, \bar{s} \in -\mathbb{Z}_{++}, \text{ and} \\
&\sum_{j=1}^{m} \alpha^{(j)} x_j = \bar{s}\, z^*.
\end{aligned}$$

Proof. From $E \subseteq X_{L,N}$ and $z^* \in E \setminus \{0\}$, it follows that $1 \leq m \leq L$. Since E is a nonempty finite set, $\mathrm{co}(E)$ is compact. Since $z^* \neq 0$, $\bar{s}$ is well-defined and, from $0 \in \mathrm{ri}\,(\mathrm{co}(E))$, $\bar{s} < 0$. In addition, from $\bar{s}\, z^* \in \mathrm{co}(E) \subseteq \mathrm{co}(X_{L,N})$ and $z^* \neq 0$, it follows that $|\bar{s}| \leq N/\|z\|_\infty \leq N$.

Since $\bar{s}\, z^*$ lies on the relative boundary of polytope $\mathrm{co}(E)$, there exists a $(m-1)$-dimensional facet F of $\mathrm{co}(E)$ such that $\bar{s}\, z^* \in F$. Hence, there exist affinely independent vectors $\{x_1, \dots, x_m\} \subseteq E$ such that $\bar{s}\, z^* \in \mathrm{co}\{x_1, \dots, x_m\} \subseteq F$.

Claim 12. $\{x_1 - x_m, \ldots, x_{m-1} - x_m, z^*\}$ is linearly independent.

Proof. Suppose, to the contrary, that $\{x_1 - x_m, \ldots, x_{m-1} - x_m, z^*\}$ is linearly dependent. Since $\{x_1 - x_m, \ldots, x_{m-1} - x_m\}$ is linearly independent (because $\{x_1, \ldots, x_m\}$ is affinely independent), there exists a $(\beta^{(1)}, \ldots, \beta^{(m-1)}) \in \mathbb{R}^{m-1} \setminus \{0\}$ such that

$$z^* = \sum_{j=1}^{m-1} \beta^{(j)}(x_j - x_m).$$

Let $p \in \mathrm{span}(E) \setminus \{0\}$ be a normal vector to facet F such that

$$p \cdot x \geq p \cdot (\bar{s}\, z^*) \quad \text{for all } x \in \mathrm{co}(E).$$

Then, for all $j \in \{1, \ldots, m-1\}$, $p \cdot (x_j - x_m) = 0$ and, therefore, we have

$$p \cdot z^* = \sum_{j=1}^{m-1} \beta^{(j)}\, p \cdot (x_j - x_m) = 0.$$

Hence, $p \cdot x \geq p \cdot (\bar{s}\, z^*) = 0$ for all $x \in \mathrm{co}(E)$. This contradicts that $0 \in \mathrm{ri}\,(\mathrm{co}(E))$. We have established the proof of Claim 12. □

We may assume that $\{x_1 - x_m, \ldots, x_{m-1} - x_m, z^*, \chi_{m+1}, \ldots, \chi_L\}$ is a basis of $\mathbb{R}^L$. For every $j \in \{1, \ldots, L\}$, let $\hat{x}_j = (x_j^{(1)}, \ldots, x_j^{(m)})^T \in \mathbb{Z}^m$, where the symbol T is the transposition operator of vectors. Also, let $\hat{z}^* = (z^{*(1)}, \ldots, z^{*(m)})^T \in \mathbb{Z}^m$. Since $\bar{s}\, z^* \in \mathrm{co}\{x_1, \ldots, x_m\}$, there exists $(\alpha^{(1)}, \ldots, \alpha^{(m)}) \in \mathbb{R}^m_+$ such that

$$\sum_{j=1}^{m} \alpha^{(j)} = 1 \quad \text{and}$$
$$\bar{s}\, z^* = \sum_{j=1}^{m} \alpha^{(j)}\, x_j = \sum_{j=1}^{m-1} \alpha^{(j)}\, (x_j - x_m) + x_m.$$

Let $B = (\hat{x}_1 - \hat{x}_m, \ldots, \hat{x}_{m-1} - \hat{x}_m, \hat{z}^*)$. Then, B is a nonsingular $m \times m$ matrix. Since

$$B \begin{pmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(m-1)} \\ -\bar{s} \end{pmatrix} = -\hat{x}_m,$$

by Cramer's rule, we have

$$\alpha^{(j)} = \frac{\det B_j}{\det B} \quad \text{for all } j \in \{1, \dots, m-1\},$$

where

$$B_j = (\hat{x}_1 - \hat{x}_m, \dots, \hat{x}_{j-1} - \hat{x}_m, -\hat{x}_m, \hat{x}_{j+1} - \hat{x}_m, \dots, \hat{x}_{m-1} - \hat{x}_m, \hat{z}^*).$$

Since all elements in matrices B and B_j $(j = 1, \dots, m-1)$ are integral, we have

$$\begin{aligned} &|\det B| \in \mathbb{Z}_{++} \quad \text{and} \\ &|\det B|\, \alpha^{(j)} = |\det B_j| \in \mathbb{Z}_+ \quad \text{for all } j \in \{1, \dots, m-1\}. \end{aligned}$$

Hence,

$$|\det B|\, \alpha^{(m)} = |\det B| - \sum_{j=1}^{m-1} |\det B|\, \alpha^{(j)} \in \mathbb{Z}_+.$$

In addition, by Hadamard's inequality (Lemma 5), we have

$$|\det B| \le \prod_{j=1}^{m-1} \|\hat{x}_j - \hat{x}_m\| \times \|\hat{z}^*\| \le m^{m/2}(2N)^{m-1}N = 2^{m-1}m^{m/2}N^m.$$

Although $|\det B|\, \bar{s}$ may not be integral, from $|\det B|\, \bar{s}\, z^* = \sum_{j=1}^{m} |\det B|\, \alpha^{(j)}\, x_j \in \mathbb{Z}^L$, it follows that $|\det B|\, \bar{s}\, \|z^*\|_\infty \in \mathbb{Z}$. Let $q_0 = |\det B|\, \|z^*\|_\infty \in \mathbb{Z}_{++}$. Since $z^* \in X_{L,N}$, we have

$$q_0 \le 2^{m-1}m^{m/2}N^m \cdot N = q.$$

Summing up, we have

$$\begin{aligned} &q_0 \in \mathbb{Z}_{++} \quad \text{with} \quad q_0 \le q, \\ &\{x_1, \dots, x_m\} \subseteq E, \\ &(\alpha^{(1)}, \dots, \alpha^{(m)}) \in \mathbb{Q}_+^m, \\ &\sum_{j=1}^{m} \alpha^{(j)} = 1, \\ &q_0\, \alpha^{(j)} \in \mathbb{Z}_+ \quad \text{for all } j \in \{1, \dots, m\}, \\ &q_0\, \bar{s} \in -\mathbb{Z}_{++}, \quad \text{and} \\ &\sum_{j=1}^{m} \alpha^{(j)}\, x_j = \bar{s}\, z^*. \end{aligned}$$

This completes the proof of Lemma 6. □

Lemma 7 (Inoue's [9] separation theorem). *Let F be a nonempty subset of $\mathbb{Z}^L$ and let V be a linear subspace of $\mathbb{R}^L$ spanned by some elements of $\mathbb{Z}^L$. If F is bounded from below and if $\mathrm{co}(F) \cap (V - \mathbb{R}^L_+) = \emptyset$, then there exists a $p \in V^\perp \cap \mathbb{Z}^L_+$ and an $\varepsilon > 0$ such that*

$$p \cdot z \geq \varepsilon \quad \text{for all } z \in \mathrm{co}(F),$$

where $V^\perp$ is the orthogonal complement of V.

Proof. See Inoue [9, Theorem 5.2]. □

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