Working Papers

Institute of Mathematical Economics

423

November 2009

Convexity and Complementarity in Network Formation: Implications for the Structure of Pairwise Stable Networks

Tim Hellmann

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?423
ISSN: 0931-6558

Convexity and Complementarity in Network Formation: Implications for the Structure of Pairwise Stable Networks

Tim Hellmann*

November 19, 2009

Abstract

This paper studies the properties of convexity (concavity) and strategic complements (substitutes) in network formation and the implications for the structure of pairwise stable networks. First, different definitions of convexity (concavity) in own links from the literature are put into the context of diminishing marginal utility of own links. Second, it is shown that there always exists a pairwise stable network as long as the utility function of each player satisfies convexity in own links and strategic complements. For network societies with a profile of utility functions satisfying concavity in own links and strategic complements, a local uniqueness property of pairwise stable networks is derived. The results do neither require any specification on the utility function nor any other additional assumptions such as homogeneity.

Keywords: Networks, Network Formation, Game Theory, Supermodularity, Increasing Differences, Stability, Existence, Uniqueness

JEL-Classification: D85, C72, L14

*Institute of Mathematical Economics, Bielefeld University, postbox 100131, D-33501 Bielefeld, Germany. Email: thellmann@wiwi.uni-bielefeld.de, phone: +49 521 106 2574, fax: +49 521 106 2997.

1 Introduction

In the seminal paper of Jackson and Wolinsky (1996) the concept of pairwise stability for the formation of undirected networks is introduced. Since then, pairwise stability has been the most commonly used notion of stability in the vast growing literature of network formation. Although pairwise stable networks have been analyzed widely in different models of network formation, not a lot is understood yet with respect to the general structure of pairwise stable networks. While imposing a specific functional form of utility over networks leads to specific results in terms of pairwise stable networks, the question remains which properties stable networks generally have. Even with respect to the existence of pairwise stable networks not a lot can be found in the literature. There are two studies which derive sufficient conditions for existence or uniqueness of pairwise stable networks. Jackson and Watts (2002b) directly address the question of existence. They show that the existence of a function similar to a network potential function is sufficient for ruling out cycles and thus guaranteeing the existence of pairwise stable networks. A different objective can be found in Chakrabarti and Gilles (2007), in which they analyze network potentials. However, they show by a corollary of a result by Jackson and Watts (2002b) that for network societies having an ordinal network potential function there always exists a pairwise stable network. Both conditions in Jackson and Watts (2002b) and Chakrabarti and Gilles (2007) are strong, for instance in the case of Chakrabarti and Gilles (2007), a link between two players needs to be either beneficial to both or to none.[1] In most models of network formation this condition is not satisfied. The assumption needed in Jackson and Watts (2002b) is similar. Hence, both results require strong assumptions just to prove existence of pairwise stable networks. Note that both papers do not present any examples of models from the literature of network formation which satisfy their assumptions.

In this paper, I also approach the question of existence conditions for pairwise stable networks. In contrast to previous work, I aim at analyzing the structure of pairwise stable networks by neither imposing restrictive assumptions such as the existence of an ordinal potential nor specifying the utility function. Instead, I impose only very natural conditions on the profile of utility functions such as convexity (concavity) in own links and strategic complements (substitutes). The former assumption captures that players have increasing (decreasing) marginal returns from own links, while the latter implies increasing (decreasing) marginal utility from other players' links. The notion of convexity (concavity) respectively strategic complements (substitutes) is not new and has been used in some models of network formation.[2] In Bloch and Jackson (2007), and Calvó-Armengol and Ilkiliç (2009) convexity (concavity) in own links is defined with respect to marginal utilities of link deletion. Instead, Goyal and Joshi (2006a) define convexity (concavity) with respect to link addition. However, in their paper the utility function does not take into account the whole network structure, but merely focuses on one particular network statistic, the degree distribution. I generalize their definition of convexity (concavity) and show an equivalence result for all collected notions of convexity (concavity).

[1] Definition 7 is required for the result.

[2] See for instance Bloch and Jackson (2007), Goyal and Joshi (2006a), and Calvó-Armengol and Ilkiliç (2009).

The main result of this paper is that in a network society with a profile of utility functions which is convex in own links and satisfies the strategic complements property there always exists a pairwise stable network. This result only requires these very natural assumptions, and there are several models in the literature of networks formation, in which these are satisfied. Besides the conditions of convexity and strategic complements, the result does not rely on any specification or homogeneity of the utility function. Since there are several examples in the literature, where the conditions of the theorem are satisfied and the conditions have natural interpretations of non-diminishing marginal utility, the existence result seems to be more appealing than those which require the existence of a potential-like function. Furthermore, the implications of concavity and strategic substitutes in network formation are studied. It is generally not possible to establish a corresponding existence result as I show by a counterexample. However, concavity and strategic substitutes also have strong implications for the general structure of pairwise stable networks. In generic cases, pairwise stable networks are unique for the range of networks that can be reached by either only adding links or by only deleting links. Generic cases can be ruled out by either imposing a no-indifference property or simply requiring strict inequalities in the definition of pairwise stability. Again, the result is of very general nature, since neither a specification of the utility function nor a homogeneous profile of utility functions is required. The assumptions of concavity and substitutability are very intuitive: they resemble non-increasing marginal utility from additional own respectively other players' links. Several model can be found in the literature of network formation which satisfy both conditions.

The driving force in these results is that both the effects of additional own and other players' links on marginal utility are positive respectively negative. Therefore, we are not able to establish results of the same generality for the other two combinations, i.e. convexity and strategic substitutes respectively concavity and complements.

Most closely related to my approach is the work of Goyal and Joshi (2006a). They also use different combinations of the four conditions convexity and concavity in own links, as well as, strategic complements and strategic substitutes to obtain existence and uniqueness results for utility functions that have a particular structure: in Goyal and Joshi's model, each player's utility function only depends on the degree distribution. Moreover, Goyal and Joshi (2006a) provide qualitative results in terms of special architectures of pairwise stable networks, which are driven by the particular structure of their utility function. My approach is more general, since no specification of the utility function is assumed. Even in this very general framework, I am able to show the effects of the above conditions, which have strong implications for existence and uniqueness of pairwise stable networks.

Another study which discusses the properties of convexity and concavity in own links is the paper by Calvó-Armengol and Ilkiliç (2009). They use the property to show the relations between pairwise stability and pairwise Nash stability. Pairwise Nash stability is defined as the intersection between the set of pairwise stable networks and the set of networks, which are supported by a Nash equilibrium of the link announcement game introduced by Myerson (1991).[3]

[3]See also Bloch and Jackson (2006) for a definition and a study of relation between different stability concepts.

The paper is organized as follows: First, the formal model is presented and several notions of convexity and concavity are discussed, which can be found in the literature. I clarify the relation of the ones from the literature and the notion of non-diminishing (non-increasing) marginal utility from own links and present an equivalence result of all these definitions. Second, the result of Jackson and Watts (2002b) and other techniques to exclude the existence of closed improvement cylces are presented and thereby showing existence of pairwise stable networks. As an implication, I show in Section 4 that the assumptions of convexity and strategic complements are sufficient to exclude the existence of closed improvement cylces, which implies by the result of Jackson and Watts (2002b) the existence of pairwise stable networks. In Section 5, the effects of assuming concavity and strategic substitutes are analyzed. The main result in this section is that the these conditions imply a local uniqueness property in generic cases. The final section concludes. The proofs of all results can be found in the appendix.

2 The Model

Throughout the paper that set of nodes or vertices is assumed to be finite and given by $N = \{1, ..., n\}$. I will refer to nodes of the network as individuals or players. I focus here on undirected networks, the set of all possible edges of the graph is defined as the set of all unordered pairs of players of size 2, $g^N := \{K \subset N : |K| = 2\}$. A network is a collection or a set of edges or links, giving g^N the interpretation of the complete network, since it contains all possible links. The set of all undirected networks can hence be defined as $G := \{g : g \subseteq g^N\}$. Given a network $g \in G$, players i and j are directly connected in g, if the corresponding edge is contained in g, that is $\{i, j\} \in g$. For short notation, I denote a link also as $(ji =)ij := \{i, j\}$. Individuals have a preference ordering over the set of networks. For each player, this preference ordering can be presented by a utility function $u_i : G \to \mathbb{R}$, with the usual assumptions on the preference ordering.[4] By $u = \prod_{i \in N} u_i$, I denote the profile of utility functions. Given the set of all players N, the set of all possible networks G and the profile of utility functions u, we say that the triple $\mathbb{G} = (N, G, u)$ defines a *network society*. In a network $g \in G$, the set of *neighbors* of player $i \in N$ is given by $N_i(g) := \{j \in N : ij \in g\}$. Similarly, $L_i(g) := \{ij \in g : j \in N\}$ denotes the set of player i's *links* in g. I denote the set of links obtained by deleting player i and all of his links by $L_{-i}(g) := \{jk \in g : jk \notin L_i(g)\}$. Obviously it holds that $g = L_i(g) \cup L_{-i}(g)$ for all $g \in G$.

When self-interested players form links, one may ask which networks evolve and persist. While best-responses may also lead to cycles, we want to look for networks that are not altered by self-interested players. As an analog to equilibrium in non-cooperative Game Theory these networks are referred to as stable. There are two distinct approaches how stable networks are defined in the literature. One looks at the link announcement game defined in Myerson (1991) and uses well-known equilibrium concepts of non-cooperative Game Theory. The second approach defines desired properties of stability directly on the set of networks. I introduce here only the well-known concept of pairwise stability introduced by Jackson and Wolinsky (1996).[5]

[4] In particular, completeness and transitivity.

[5] A game theoretic foundation and a comparison of the several definitions of stability can be found in

Pairwise Stability. A network g is pairwise stable (PS) if no link will be cut by a single player, and no two players want to form a link:

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \backslash ij)$ and $u_j(g) \geq u_j(g \backslash ij)$ and

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

In words, (i) implies that all links in a pairwise stable network must be beneficial to the two involved players and (ii) says, that there are no additional links (links not contained in g) which are beneficial to both players. This definition reflects the behavior of self interested players who are in control of their links: two players will form a link if it is beneficial to both, while any single player will reject a link that is not beneficial. Pairwise stability is a basic notion that can be refined in multiple ways (e.g. unilateral stability, Buskens and Van de Rijt, 2005; strong stability, Dutta and Mutuswami, 1997; or bilateral stability, Goyal and Vega-Redondo, 2007). Since stability depends on the network society $\mathbb{G} = (N, G, u)$, I denote the set of stable networks, in this case the set of all pairwise stable networks as $PS(\mathbb{G})$.

For the following the subsequent notation of link addition and link deletion proof networks adapted from Chakrabarti and Gilles (2007) is useful for the results:

Link Addition Proof Networks. A network g is *link addition proof* if no two players want to form a link: $\quad \forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.
Link Deletion Proof Networks. A network g is *link deletion proof* if no link will be cut by a single player: $\quad \forall ij \in g, \quad u_i(g) \geq u_i(g \backslash ij)$.

In a link addition proof network no link will be added and in a link deletion proof network no links will be deleted by self interested (myopic) players. Both conditions simply coincide with the two conditions of pairwise stability. Let us denote the set of link addition proof networks by $G^a(\mathbb{G}) := \{g \in G \mid \forall ij \in g^N \backslash g : mu_i(g \cup ij, ij) > 0 \implies mu_j(g \cup ij, ij) < 0\}$, and the set of link deletion proof networks by $G^d(\mathbb{G}) := \{g \in G \mid \forall ij \in g : mu_i(g, ij) \geq 0\}$. Trivially, a network which is link addition proof and link deletion proof is pairwise stable, $G^a(\mathbb{G}) \cap G^d(\mathbb{G}) = PS(\mathbb{G})$. Furthermore, the empty network is deletion proof, $g^\emptyset \in G^d$, since there exists no link which can be deleted in the empty network, and analogously the complete network is link addition proof, $g^N \in G^a$.

2.1 Concavity and Convexity in Network Formation

Consider a network society $\mathbb{G} = (N, G, u)$ as defined above. The decision to form or to sever a link typically depends on players' marginal utility from a given link. If the marginal utility from a given link positive, the player has an incentive to form that link, since it provides him with additional positive payoff. Similar considerations hold, when we consider marginal utilities of sets of links. Depending on a given network g, let us denote player i's marginal utility of a set of links *currently* in network g as $mu_i(g, l) = u_i(g) - u_i(g \setminus l)$, s.t. $l \subseteq g$. In words, the marginal utility of a set of links is the difference

Bloch and Jackson (2006).

of utilities from a given network g containing the set of links l and the network that is obtained by deleting the set of links l. Similarly we can denote player i's marginal utility of a set of *new* links by $mu_i(g \cup l, l) = u_i(g \cup l) - u_i(g)$ for $l \subseteq g^N \setminus g$.

A common assumption on utility functions in economic theory is convexity or concavity, representing increasing respectively diminishing marginal utility. Convexity and concavity, however, are defined for functions on an interval in the real numbers. In the model of bilateral links, the decision variables (the set of own links) are discrete. Thus, it does not really make sense to speak about the curvature or derivative of the utility function. We may, however, think about diminishing or increasing marginal utility of a given link with respect to the set inclusion ordering "$\subseteq$". In the literature on network formation several definitions of convexity with respect to own links can be found. For instance, Bloch and Jackson (2007) define the following:

Definition 1 (Bloch and Jackson (2007)). *The utility function u_i of player i is convex (concave) in own* current *links, if $\forall g \in G$ and $\forall l_i \subseteq L_i(g)$ the following holds:*

$$mu_i(g, l_i) \geq (\leq) \sum_{ij \in l_i} mu_i(g, ij).$$

Bloch and Jackson (2007) define convexity in own links with respect to the set of own links currently contained in a network g. Marginal payoffs from a given set of links already contained in g should be at least as high as the sum of the marginal payoffs from separate links. By defining the property on the set of links already in network g, the property of convexity is defined with respect to link deletion: the marginal utility of deleting the set l is as least as high as deleting each link contained in l separately from g and summing over the marginals. Since only deletion is considered, convexity in own current links will turn out to be equivalent to concavity in link addition as we will see later. Note that Definition 1 is also given in Calvó-Armengol and Ilkiliç (2009) labeled as supermodularity in own links.[6]

The natural dual to Definition 1 is to consider link addition instead of link deletion. Calvó-Armengol and Ilkiliç (2009) define such a property and call it strong submodularity.[7] Adapting the definition of Calvó-Armengol and Ilkiliç (2009) to our framework, we define *convexity (concavity) in own new links*, by simply requiring the property to hold for all links that can be potentially added instead of requiring it for all links that are already (currently) contained in g.

Definition 2. *The utility function u_i of player i is convex (concave) in own* new *links, if $\forall g \in G$ and $\forall l_i \subseteq L_i(g)$ the following holds:*

$$mu_i(g \cup l_i, l_i) \geq (\leq) \sum_{ik \in l_i} mu_i(g \cup ik, ik).$$

[6]Calvó-Armengol and Ilkiliç (2009) consider convexity weighted with a factor α. However, taking $\alpha = 1$ in their definition of α-supermodularity gives us Definition 1.

[7]Calvó-Armengol and Ilkiliç (2009) again introduce a weight β in their definition, which is omitted here and also allow for simultaneous link deletion. Note that the use of super- and submodularity is justified, as can be shown by considering the partial ordering $\subseteq$ on the set of own links $L_i(g)$.

Both notions of convexity (concavity) are defined on different sets and indeed have different interpretations.[8] While the first only considers links that are currently in network g, the latter considers only potential outside links. By looking more closely at the definitions, convexity (concavity) in own current links expresses something similar than concavity (convexity) in own new links: Marginal utility of a given link seems to be non-increasing (non-decreasing) when adding links. However it is not so clear whether both definitions are actually equivalent: If so, why would we need different definitions for the same property? A third definition can be found in Goyal and Joshi (2006a). In Goyal and Joshi's model, however, the utility function depends only on the degree distribution. Hence in their paper, convexity in own links is defined by increasing (decreasing) marginal utility in the *number* of own links. Instead of comparing numbers, we adopt their definition by defining convexity with respect to the set inclusion ordering "$\subseteq$".

Definition 3. *A utility function u_i of player i is convex (concave) in own links, if $\forall g \in G$, $\forall l_i \subseteq L_i(g^N \setminus g)$, and $\forall ij \notin g \cup l_i$:*

$$mu_i(g \cup ij, ij) \leq (\geq) mu_i(g \cup l_i \cup ij, ij).$$

This definition exactly represents the intuition of non-diminishing (non-increasing) marginal utility of a given link from own own links: by adding some own links, the marginal utility of a given link does not decrease (increase). Equivalently to Definition 3, we may say that a utility function u_i is convex in own links, if for any two networks g, g', which only differ in the links of player i and $g \subseteq g'$, the marginal utility of adding any link $ij \in g^N \setminus g$ is larger in the g' than in g, i.e. $mu_i(g \cup ij, ij) \leq (\geq) mu_i(g' \cup ij, ij)$. Writing the property this way exactly captures what we mean by non-decreasing with respect to the set inclusion ordering: the marginal utility of a given link in a network g is not larger than the marginal utility in a g', which includes g, i.e. $g \subseteq g'$. I consider this as the most natural definition of convexity in network formation, since it captures non-diminishing marginal utility property of utility functions.

Definitions 1, 2, and 3 are all giving a formalization of convexity in network formation. While the first is defined on the set of links contained in a network, the other two are defined on the set of links that can potentially be added. Thus, the definitions point into different directions, i.e. link deletion and link addition. Let us try to organize the three notions of convexity. Reversing convexity and concavity in Definition 1, the following result shows that all definitions are equivalent:

Proposition 1. *Let $u_i : G \to \mathbb{R}$ the utility function of player i. Then the following statements are equivalent:*

(1) u_i is convex (concave) in own links.

(2) u_i is convex (concave) in own new links.

(3) u_i is concave (convex) in own current links.

The proof can be found in the appendix. The result shows that in fact all three definitions of convexity are equivalent. Although it may seem odds, let me point out again that

[8]Subsequently, I show that concavity in own new links is equivalent to convexity in own current links.

concavity in own current links is equivalent to convexity in own links. The reason is simply that the definition of concavity in own current links is misleading, since it is defined on the links already contained in a network. Proposition 1 shows that the definitions are substitutable, which is used in some of the proofs. Furthermore, the introduction of Definition 3 helps understand convexity in network formation by thinking about diminishing marginal utility with respect to own links. Since all three notions are equivalent, I will thus only refer to convexity in own links, or short convexity, according to Definition 3.

Up to now, we have defined and discussed the effects of changing own links on the marginal utility of a given link. Marginal utilities, however, may also be affected by the links of other players. Again, the marginal utility of a given link can differ significantly, when other players change links. If the effect of additional links of other players on marginal utility of a given link is non-negative, then Goyal and Joshi (2006a) speak about strategic complementarity in network formation. This label is quite natural, since it corresponds to the definition of complementary goods: here, the goods are the links. However, in contrast to microeconomic theory or industrial organization the domain of our utility function is discrete. Thus, we cannot assume differentiability of the utility function. In order to find a reasonable definition of strategic complementary, we have to consider the set inclusion ordering $\subseteq$. However, we cannot turn to Goyal and Joshi (2006a) and take their definition of strategic complements, since in their work the domain of the utility function is not the network itself, but the number of links. This assumption, however, is itself quite restrictive: Two networks which have the same number of own and other players's links imply the same utility. Thus they define strategic complements respectively substitutes as increasing marginal utility of a given link in the number of other players links. Rather, I adapt their definition to our more general class of utility functions by defining it with respect to set inclusion ordering:

Definition 4. *A utility function u_i of player i satisfies the strategic complements (substitutes) property, if for all $g \in G$ and any set of links $l_{-i} \subseteq L_{-i}(g^N \setminus g)$ it holds that*

$$mu_i(g, ij) \leq (\geq) mu_i(g \cup l_{-i}, ij). \tag{1}$$

In words, if the utility function satisfies the strategic complements (substitutes) property and other players add links such that player i is not involved, then the marginal utility of a given link does not decrease (increase).

Although both notions of convexity (concavity) and complementarity (substitutability) may seem restrictive, since both have to hold for the whole set of networks G, we find many examples in the literature of network formation, which satisfy the properties. I present some of them subsequently.

3 The Existence of Pairwise Stable Networks

The main focus of this paper is to elaborate on the effects of imposing convexity (concavity) respectively strategic complements (substitutes) in network formation on the structure of pairwise stable networks. Particularly, we are interested in the implications for the existence of these, since existence results in the literature usually require restrictive assumptions such as the property of network potentials. Rather than potential functions, I will use here the notions of convexity (concavity) and complementarity (substitutability) and study their implications for the structure of pairwise stable networks.

In this section, I briefly introduce here the techniques developed in Jackson and Watts (2002b) to show existence of pairwise stable networks. Some of the existence results in Jackson and Watts (2002b) are shown by improving paths. An improving path is a sequence of networks such that each two consecutive networks in the sequence only differ in one link and the addition (or deletion) of that link is improving for both (one of the two) involved players. I adapt the formal definition from Jackson and Watts (2002b):

Improving Paths. An *improving path* from network g to network g' is a finite sequence of networks $(g_1, ..., g_K)$ such that $g_k \in G$ for all $k = 1, ..., K$, $g_1 = g$, $g_K = g'$, and for all $k = 1, ..., K-1$ it holds that either

$g_{k+1} = g_k \setminus ij$ and $u_i(g_k \setminus ij) > u_i(g_k)$, or

$g_{k+1} = g_k \cup ij$ and $u_i(g_k \cup ij) > u_i(g_k)$ and $u_j(g_k \cup ij) \geq u_j(g_k)$.

Thus, given a network g_k the next element in an improving path g_{k+1} is formed either by one player beneficially cutting a link or by two players creating a link, which is beneficial to both, reflecting again the idea that two players need to agree about forming a link, but one player can delete any link by himself. [9] We can trivially observe that a network g is pairwise stable if and only if there is no improvement path leaving g.

Given the notion of improving paths, Jackson and Watts (2002b) define an (improvement) cycle C as an improving path $(g_1, ..., g_K)$ such that $g_1 = g_K$. Thus, in an improvement cycle, players myopically add and cut links but finally arrive at the same network. Furthermore, Jackson and Watts (2002b) speak about a *closed cycle*, if for all networks $g \in C$ there does not exists an improving path leading to a network $g' \notin C$. In a dynamic framework, where players can only add or sever one link at a time and play a myopic best response, then closed cycles and pairwise stable networks would constitute recurrent classes: Once a closed cycle is reached, no player will add or cut links that lead to a network outside the closed cycle.[10] Therefore, closed cycles represent something similar than pairwise stable networks: If we assume myopic players, who can only alter one link at a time and always play a best response, then once a stable network or a closed cycle is reached, it will never be abandoned. In a sense, closed cycles are not less stable than pairwise stable networks.

With these definitions, Jackson and Watts (2002b) get the following important result stated as a lemma, which I adapt to my framework:

Lemma 1 (Jackson and Watts, 2002b)**.** *For any network society $\mathbb{G}$, there exists at least one pairwise stable network or a closed cycle of networks.*

[9] Implicitly it is assumed here that players are myopic: when adding or severing a link they do not take into account the final network in the sequence, but only see the myopic improvement. If we assume farsighted behavior, then individuals do not compare two consecutive elements of the sequence, but rather the current network to the resulting network. For a study on farsighted behavior, see Page (2004).

[10] See Jackson and Watts (2002a) for such a setup and the observation that pairwise stable networks and closed cycles are the only recurrent classes.

In Jackson and Watts (2002b) and in Chakrabarti and Gilles (2007) this lemma is used to show existence of a pairwise stable network. Both papers show in a similar fashion that a network society with a utility function, which has an ordinal potential, implies non-existence of cycles, and hence the existence of pairwise stable networks by Lemma 1. I will use the same technique to show the existence result in Theorem 1.

Lemma 1 implies that one way to proof existence of pairwise stable networks is to show non-existence of *closed* cycles. With the property of ordinal potentials however, it is shown in both works of Jackson and Watts (2002b) and Chakrabarti and Gilles (2007) that no cycles exist. However, to show existence of pairwise stable networks, we do not need to rule out cycles, just *closed* cycles. The following lemma shows conditions under which closed cycles fail not exist:

Lemma 2. *Suppose an improvement cycle C either that does not contain*

- *a link addition proof network, i.e* $\forall g \in C \implies g \notin G^a$*, or*
- *a link deletion proof network, i.e* $\forall g \in C \implies g \notin G^d$*.*

Then C cannot be a closed cycle.

The proof is straightforward and presented in the appendix. If a cycle does not contain a link addition proof network, then it cannot be closed, since there always exists an improving path to a link addition proof network. This is trivial, since we can always add links in a non-link addition proof network. Thus there exists an improving path to either the complete network (which is link addition proof) or another link addition proof network. But we assumed that a link addition proof network is not part of the improvement cycle. Since we have constructed an improvement path leading out of the cycle, the cycle cannot be closed. The second part is shown analogously.

The lemma is helpful for proving the main result. In the proof of Theorem 1, I show that no addition proof network can be part of any improvement cycle, ruling out the existence of closed cycles and thus implying the existence of pairwise stable networks.

4 Convexity and Strategic Complements

In this part, I show the implications for the existence of pairwise stable networks, if a profile of utility functions satisfies the assumptions of convexity and strategic complements. Recall, that convexity in own links means that the marginal utility from a given link is non-decreasing when adding other own links. In other words, the returns from own links are non-decreasing with respect to the set inclusion ordering. The assumption of strategic complements refers to the effects of additional links of other players on marginal utility. The incentive (marginal utility) to form a given link is non-decreasing when other players add links. The effects of both adding own and other players links are non-negative for the incentive to form a given link. The main result for a network society such that the profile of utility functions satisfies convexity in own links and the strategic complements property is that there always exists a pairwise stable network. The intuition behind this result, presented in Theorem 1, is the following: if a network g is link deletion proof, then any improvement from g can only involve the addition of links. In an improving path, a successor of a link deletion proof network g is again link deletion proof since marginal utilities of all current (and new) links have not decreased, if convexity and strategic complements are satisfied. Continuing in this manner a pairwise stable networks has to be reached eventually. In other words, no cycle can contain a deletion proof network since otherwise it cannot be a cycle. The following result summarizes this intuition.

Lemma 3. *Let $\mathbb{G}$ be a network society and suppose that u satisfies the strategic complements property and convexity in own links. Then:*

(1) No link addition proof network $g \in G^a$ can be part of an improvement cycle.

(2) No link deletion proof network $g \in G^d$ can be part of an improvement cycle.

The proof is presented in the appendix. Convexity and strategic complements imply that adding links to a deletion proof network does not decrease the marginal utility of a given link since the effects of own links and the effects of other links are non-negative on marginal utility. Thus, once a link deletion proof network is reached, any improving path emanating from it only involves link addition. Hence, an improvement cycle cannot contain a link deletion proof network since otherwise it cannot be a cycle (since no links will ever be deleted). Analogous considerations hold for link addition proofness. The following theorem summarizes the results obtained in Lemma 1-3.

Theorem 1. *Suppose a profile of utility functions $u = (u_1, ..., u_n)$ of a network society $\mathbb{G}$ satisfies the strategic complements property and convexity in own links. Then:*

(1) There does not exist a closed improvement cycle.

(2) There exists a pairwise stable network.

The theorem is an immediate implication of Lemma 2 and Lemma 3. The proof of Theorem 1 is thus omitted since by Lemma 3 no addition proof network can be part of any improvement cycle in a network society $\mathbb{G}$ with a profile of utility functions which satisfies convexity and strategic complements. Then, Lemma 2 implies that there does not exist a closed cycle, implying part (1) of Theorem 1. By the result of Jackson and Watts (2002b) (see Lemma 1) we get thus existence of pairwise stable networks. This result shows that convexity and strategic complements are indeed sufficient for the existence of pairwise stable networks. Particularly appealing is generality of the result, and Theorem 1 is therefore in the spirit of the existence results of Jackson and Watts (2002b) and Chakrabarti and Gilles (2007). However, the assumptions imposed here seem more intuitive and less restrictive, as they simply reflect non-diminishing marginal utility. Furthermore, they are easy to check and instead of the papers above, I can easily find models in the literature which satisfy the assumptions of convexity and strategic complements. Among them is the model of “Provision of a Pure Public” by Goyal and Joshi (2006a), presented subsequently.

By simply requiring the properties of convexity and strategic complements, we thus arrive at a general result: there always exists a pairwise stable network. In some models of network formation these properties result from the setup in the model and I will subsequently present an example from the literature which satisfies the assumption. In general, however, a more natural assumption on the utility function is concavity, i.e. diminishing marginal utility together with substitutability. These are models, where additional (own and other) links decrease the incentive to form (the marginal utility of) a given link, i.e. links are substitutable. These assumptions will be discussed in the next subsection. Let us look at a network formation model taken from Goyal and Joshi (2006a).

Example 1 (Goyal and Joshi (2006a), Provision of a Pure Public Good)**.** *In this model there are n players choosing an output level x_i (second stage) to produce a public good which is valuable for everybody $\tilde{\pi}_i(x) = \sum_{i \in N} x_i$. Players can collaborate (first stage) and share their knowledge about production of the public good, which reduces the marginal costs of producing the output, but is costly with $c > 0$. The marginal costs of producing the public good is given by $f_i(x_i, g) = \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2$, for all $i \in N$, where d_i represents player i's degree, i.e. $d_i(g) = |L_i(g)|$.*

Given $d_i(g)$, player i's maximization problem at the second stage is thus $max_{x_i \in \mathbb{R}_+} x_i + \sum_{j \in N \setminus i} x_j - \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2$. This implies optimal output of $x_i^*(g) = (d_i(g)+1)^2$. Hence, in equilibrium every player chooses optimal output $x_i^*(g)$ for all $i \in N$, yielding the utility function

$$u_i^{PG}(g) = \frac{1}{2}(d_i(g)+1)^2 + \sum_{j \in N \setminus \{i\}} (d_j(g)+1)^2 - cd_i(g),$$

where the first term is the difference of own (equilibrium-) output and production costs, the second term is the (equilibrium-) output of all other agents, and the last term is the costs of collaboration. Marginal utility of a given link ij satisfies

$$mu_i^{PG}(g \cup ij, ij) = 9/2 + d_i(g) + 2d_j(g) - c.$$

Thus, marginal utility of a given link is increasing in both d_i and d_j, implying convexity and strategic complements.

The reason that the "Provision of a Pure Public" model satisfies convexity and strategic complements is primely due to the structure of marginal cost of producing the output: an additional link lowers marginal costs quadratically, hence increasing optimal output quadratically. Since the utility function is linear in own and other player's public good output, we get convexity in of own links and strategic complements.

5 Concavity and Strategic Substitutes

In many models of network formation the effects of own and other players' links on marginal utility are just the other way around: marginal utility is decreasing in own links and links are substitutes rather than complements. This is also more intuitive if we think about markets and goods. A common assumption in economic theory is diminishing marginal utility. The more an individual consumes the less valuable is an additional consumption good. This is also natural if we think about network formation: we find many models and I present some of them in this Chapter, where concavity in own links is satisfied instead of convexity. Here concavity and substitutability, again, have a common interpretation: the effects of additional own and other players' links on marginal utility of any given link are *non-positive*, in other words, links are substitutable. This is especially true in models where connectivity to other players matters, i.e. models where the utility function is decreasing in distances to other players, such as the connections model.[11]

In the case of network society such that convexity and strategic complements are satisfied, there always exists a pairwise stable network. This is not true anymore in the case of concavity and substitutability. Consider the following example, which is kept as simple as possible to show that even though both substitutability and convexity are satisfied, there does not exist a pairwise stable network.

Example 2. *Let $\mathbb{G} = (N, G, u)$ such that $N = \{1, 2, 3\}$. Suppose that $mu_i(ij, ij) > 0$ for all $i, j \in N$, meaning that any player wants to form a link to any other player in the empty network. Furthermore, in the complete network any player wants to delete links: $mu_i(g^N, ij) < 0$ for all $i, j \in N$, implying also that no player wants to form an additional link, when two links are already in the network. In the case of two links, let the marginals of* current *links satisfy:*

network	*player 1*	*player 2*	*player 3*
$g = \{12, 13\}$	$mu_1(\{12, 13\}, 13) > 0$	$mu_2(\{12, 13\}, 12) < 0$	$mu_3(\{12, 13\}, 13) > 0$
$g = \{12, 23\}$	$mu_1(\{12, 23\}, 12) > 0$	$mu_2(\{12, 23\}, 12) > 0$	$mu_3(\{12, 23\}, 23) < 0$
$g = \{13, 23\}$	$mu_1(\{13, 23\}, 13) < 0$	$mu_2(\{12, 23\}, 23) > 0$	$mu_3(\{12, 23\}, 23) > 0$

Suppose now, that the utility functions satisfies the above conditions on the marginals. With these assumptions only, it is easy to see that there does not exist a pairwise stable network. This is illustrated Figure 1.

The networks presented in Figure 1 form a closed cycle. Hence, none of those can be pairwise stable. Note that the only networks not shown in Figure 1 are the empty and the complete network which are trivially on an improving path to the closed cycle C, and therefore not stable, since $mu_i(ij, ij) > 0$ and $mu_i(g^N, ij) < 0$ for all $i, j \in N$.

I show now, that one can easily construct a utility function satisfying concavity and strategic substitutes and the above assumptions on the marginals. Consider, for instance, the following profile of utility functions, such that $u_i(g^\emptyset) = 0$ and $u_i(g^N) = -1$ for all $i \in N$, $u_i(ij) = 2$, and $u_k(ij) = 1$ forall $k, i, j \in N$ such that $k \neq i, j$. Furthermore, for $|g| = 2$ it holds that $u_j(ij, jk) = 3$ for all $i, j, k \in N$ and $u_2(12, 13) = 0$ and $u_3(12, 13) = 2$, $u_1(12, 23) = 2$ and $u_3(12, 23) = 0$, and $u_1(13, 23) = 0$ and $u_2(13, 23) = 2$. It is easy to check that this particular utility function satisfies the above assumption on the marginals as well as concavity and strategic substitutes. Consider, for instance player 1*. Calculating the marginal utilities for the above specified utility function gives:*

$$mu_1(\{12\}, 12) > mu_1(\{12, 13\}, 12), \qquad mu_1(\{12, 23\}, 12) > mu_1(g^N, 12),$$
$$mu_1(\{13\}, 13) > mu_1(\{12, 13\}, 13), \qquad mu_1(\{13, 23\}, 13) > mu_1(g^N, 13),$$

[11] Calvó-Armengol and Ilkiliç (2009) show that the homogeneous connections model satisfies concavity in own links. A more general proof is given in Büchel and Hellmann (2009) for the heterogeneous connections model.

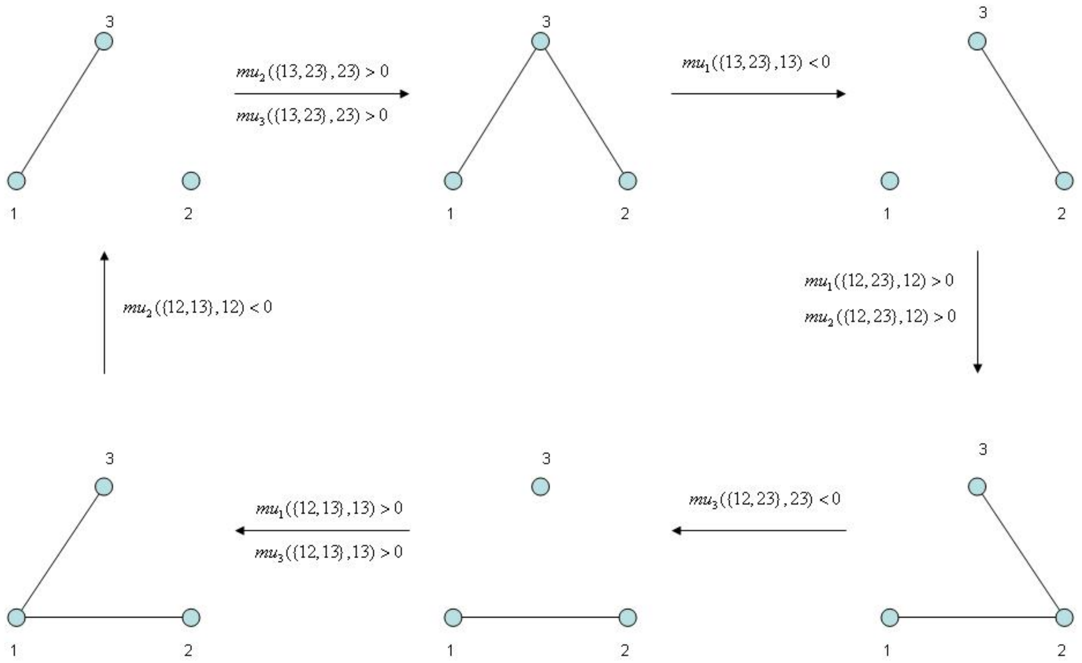

Figure 1: The closed cycle C of networks of Example 2

implying that concavity is satisfied for player 1. Furthermore, since

$$mu_1(\{12\},12) > mu_1(\{12,23\},12), \quad mu_1(\{12,13\},12) > mu_1(g^N,12)$$
$$mu_1(\{13\},13) > mu_1(\{13,23\},13), \quad mu_1(\{12,13\},13) > mu_1(g^N,13),$$

the strategic substitutes property is satisfied. Moreover, it is easy to see that the marginal utilities satisfy the conditions above. Analogous considerations hold for the other two players. Hence, there does not exist a pairwise stable network, although the profile of utility functions satisfies concavity and substitutes.

Generally, Example 2 shows that existence of a pairwise stable network is not guaranteed in a network society with a profile of utility functions satisfying concavity and strategic substitutes. In other words, these conditions are not sufficient for the existence of pairwise stable networks. Lemma 1 states that non-existence of a pairwise stable network implies existence of a closed cycle. This is of course true in Example 2. However, one should not be misled and suppose that convexity and strategic substitutes are sufficient for the existence of a closed cycle. It is easy to see that existence of a closed cycle is generally not guaranteed: consider, for instance, a network society with a profile of utility functions such that own and other players' links do not have an effect on marginal utilities, i.e. $mu_i(g,ij) = mu_i(g',ij)$ for all $g, g' \in G$ and for all $ij \in g^N$. Here, both concavity and convexity together with strategic substitutes and strategic complements are satisfied. Thus, by Theorem 1 there does not exist a closed cycle, although strategic substitutes and concavity are satisfied. This will also be true for generic cases since altering the marginals a little bit will not necessarily lead to different results. Thus, we can neither guarantee existence of pairwise stable networks nor existence of closed cycles, when the utility function only satisfies concavity and strategic substitutes.

The existence results for network societies with a profile of utility functions satisfying concavity and strategic substitutes are negative. However, the assumptions put some structure on the incentives of players, which may thus be helpful deriving some general properties of stable networks and improving paths in cases where concavity and strategic substitutes are satisfied. By Lemma 2, we got that any deletion (addition) proof network can only have a *successor* in an improving path that is again deletion (addition) proof, if convexity ans strategic complements are satisfied, which was the main idea to show the existence of pairwise stable networks. However, the effects of own and other players' links are just the other way around if we have concavity and strategic substitutes. Thus, for a deletion proof network in an improving path involving only link addition the *predecessor* needs also be deletion proof. This is expressed in the following result.

Lemma 4. *Let there be a network society $\mathbb{G}$ such that u satisfies the strategic substitutes property and concavity in own links. Then the following holds:*

(1) If a network $g \in G$ is link deletion proof, then all networks $g' \subseteq g$ are link deletion proof, $g' \in G^d$.

(2) If a network $g \in G$ is link addition proof, then all networks $g' \supseteq g$ are addition proof, $g' \in G^a$.

The proof is straightforward and presented in the appendix of this chapter. However, the assertion is strong. Any subnetwork of a deletion proof network is deletion proof and any supernetwork of an addition-proof network is addition proof. Thus, the improving path result in Lemma 2 is just reversed in the case of concavity and strategic substitutes: a link addition proof network that is reached in an improving path by deleting a link has a *predecessor* which is addition proof. In Lemma 2, where convexity and strategic complements are satisfied each link addition proof network (if it is not pairwise stable) has a *successor* in an improving path that is link addition proof.

Lemma 4 has some trivial implications: since any supernetwork (subnetwork) of a link addition (deletion) proof network is link addition (deletion) proof, all networks are link addition (deletion) proof if the empty (complete) network is link addition (deletion) proof. Of course in this case the empty (complete) network is pairwise stable. Hence, there cannot exist any cycle since all other networks are link addition (deletion) proof in that case.

Furthermore, note that a pairwise stable network g is both link deletion proof and link addition proof. Thus, by Lemma 4 any supernetwork of pairwise stable network g needs to be addition proof and any subnetwork of g needs to be deletion proof. Therefore, Lemma 4 suggests that in generic cases there may be no super- or subnetworks of a pairwise stable network g which are also pairwise stable. The intuition is the following: suppose there exists another pairwise stable network g' and suppose $g \subsetneq g'$. Since any subnetwork of g' (since g' is pairwise stable) is link deletion proof and any supernetwork of g (since g is pairwise stable) is link addition proof, this immediately implies that all $\tilde{g}$ such that $g \subseteq \tilde{g} \subseteq g'$ are also pairwise stable. To exclude this non-generic case consider the following definition adapted from Jackson and Watts (2002b).[12]

Definition 5 (Jackson and Wolinsky, 1996)**.** *The utility function u_i of player i exhibits no indifference if for all $g \in G$ and for any link $ij \in L_i(g^N \setminus g)$ the following holds: $u_i(g) \neq u_i(g \cup ij)$.*

It is clear that in case of no indifference and concavity and strategic substitutes the above described case cannot occur since otherwise any network in between both networks is stable and thus some players are indifferent between pairwise stable networks, which is excluded by Definition 5. The following result summarizes this intuition.

Theorem 2. *Let $\mathbb{G}$ be a network society and suppose u satisfies the strategic substitutes property, concavity in own links, and exhibits no indifference. Then:*

(1) If g is pairwise stable, then for all $g' \subset g$ and for all $g' \supset g$ it holds that $g' \notin [PS(\mathbb{G})]$.

(2) If g^N is pairwise stable, then there exists no other pairwise stable network.

(3) If $g^{\emptyset}$ is pairwise stable, then there exists no other pairwise stable network.

As usual the proof can be found in the appendix of this chapter. Note that in the proof the no-indifference property is only needed locally, i.e. only needed for the pairwise stable networks. We could similarly put a slightly stronger assumption on the pairwise stable networks, requiring $mu_k(g \cup ij, ij) < 0$ for at least one $k \in \{i, j\}$ instead of $mu_i(g \cup ij, ij) > 0 \implies mu_j(g \cup ij, ij) < 0$ as property (ii) in the definition of pairwise stability. With that notion we would have the same statement with weaker requirements since if u satisfies the no-indifference property, then for any pairwise stable network we have $mu_k(g \cup ij, ij) < 0$ for at least one $k \in \{i, j\}$. Thus, the no-indifference property (or weaker: the adjusted pairwise stability concept) rules out the non-generic case, where players are indifferent between a pairwise stable network and an adjacent network.

Hence, rather than existence of pairwise stability we get a uniqueness result in case of concavity and strategic substitutes together with no indifference: a pairwise stable network g (if it exists) is "locally" unique, there exists no other pairwise stable network which contains g or is contained in g. In other words, there exists no other pairwise stable network which can be attained by only adding respectively only deleting any set of links from g. Since G together with the set inclusion ordering $\subseteq$ is a partially ordered set, local uniqueness can also be interpreted the following way: if a network $g \in G$ is pairwise stable, then for any network $g' \in G$ such that g and g' are *ordered* by the bilateral relation $\subseteq$ it holds that g' is pairwise stable if and only if $g' = g$. Furthermore, if the complete or the empty network is stable, then it is the only stable network.

As mentioned in the beginning there are several models of network formation in the literature which satisfy the assumptions of concavity and strategic substitutes. Consider the following model, taken from Goyal and Joshi (2006a).

[12]At first glance, the definition in Jackson and Watts (2002b) seems different from Definition 5. However, both are equivalent as one can easily verify.

Example 3 (Goyal and Joshi (2006a), Friendships Networks)**.** *In the friendship model, there are n individuals who derive utility from social interaction. Individuals can form friendships and their utility is increasing in the number of friends and the time each individual is able to spend with his friends. Each player has a fixed amount of time available and allocates it equally among his friends. One representation by a utility function capturing the above described setting is given by:*

$$u_i(g) = \sqrt{d_i(g)} + \sum_{j \in N_i(g)} \frac{1}{d_j(g)},$$

resembling the above assumptions. Here, again, $d_i(g) := |L_i(g)|$ is the number of player i's links, also called degree. The marginal utility of a given link can be calculated:

$$mu_i(g \cup ik, ik) = \sqrt{d_i(g) + 1} - \sqrt{d_i(g)} + \frac{1}{d_k(g)}.$$

Therefore, $mu_i(g \cup ik, ik)$ is decreasing in own degree and thus in own links and decreasing in player k's degree and thus non-increasing in other players' links. Hence, concavity and strategic substitutes are satisfied.

Other examples, satisfying the properties of concavity and strategic substitutes are the Free-Trade-Agreements-Model by Goyal and Joshi (2006b) and the Patent Races Model by Goyal and Joshi (2006a). Therefore, as a concluding remark of the discussion of networks societies with a profile of utility functions satisfying concavity and strategic substitutes, we see that these properties are satisfied in many models in the literature and hence can be seen as very natural and intuitive properties.

6 Other Properties Sufficient for the Existence of Pairwise Stable Networks

I have analyzed, so far, the effects of convexity and strategic complements respectively concavity and substitutes on the structure of pairwise stable networks. The other two combinations, i.e. convexity and strategic substitutes respectively concavity and strategic complements, can also be found in some models in the literature, and depending on the interpretation can also be very intuitive. However in these combinations, the effects of own and other players' links on marginal utility point into different directions. This makes it difficult to find results concerning existence or to say something meaningful about the general structure of pairwise stable networks. Hence the analysis of these combinations is omitted here. Rather, I present here some properties that also guarantee existence of pairwise stable networks.

First, imagine that other players' linking behavior has no effect on the marginals. This is the case when the utility function satisfies both strategic substitutes and strategic complements. Hence, own incentives to form links are not effected by other links. However, with just these two properties we cannot exclude cycles or guarantee existence of pairwise stable networks since the name or the labels of the players could potentially matter. If these labels do not matter, the only characteristic of a network that influences the utility is the number of own links, in other words the own degree. Let us first define formally the property that we just described in words that labels of players do not matter for the incentives.

Definition 6. *The utility function u_i of player i in a network society $\mathbb{G}$ satisfies anonymity in marginal utilities if for all $j, k \in N$ and for all g such that $N_j(g) = N_k(g)$, it holds that*

$$mu_i(g \cup ij, ij) = mu_i(g \cup ik, ik). \tag{2}$$

In this definition the marginal utility to connect to two players j and k, who have equal network positions, i.e. two players which are symmetric, in a given network g, is equal. Two players j and k have equal network position if the permutation $\pi : N \to N$ such that $\pi(i) = i$ for all $i \neq \{j, k\}$ and $\pi(j) = k$, does not change the network, hence $g = g_\pi$. It is easy to see that two players j and k have equal network positions if and only if $N_j(g) = N_k(g)$. Hence, the definition exactly captures that links to equals have the same marginal utility. Together with the strategic complements and substitutes property, we get existence of pairwise stable networks.

Proposition 2. *Let $\mathbb{G}$ be a network society and suppose that u satisfies both the strategic complements and substitutes property and suppose anonymity is satisfied. Then:*

(1) For any $i \in N$ such that $g_{-i} = g'_{-i}$ and $d_i(g) = d_i(g')$ it holds that $u_i(g) = u_i(g')$.

(2) There does not exist a closed cycle of networks.

(3) There exists a pairwise stable network.

A profile of utility function satisfying both the strategic complements and substitutes property implies that links of other players have no effect on *marginal utility* of a given link. Note that this does not imply non-existence of externalities,[13] which captures the effects on *absolute* utility. Together with anonymity in marginal utilities the assumptions of strategic complements and substitutes implies existence of pairwise stable networks. The intuition behind this result is that other players' links do not matter for the decision of a single player. Since labels of players do not matter, no player cares of whom to connect to and thus only optimizes the number of links. A similar result is shown by Büchel (2009). Büchel shows that if the utility function only depends on own degree, such that $u_i(g) = \tilde{u}_i(d_i(g))$, then there always exist a pairwise stable network.

I discussed the (strong) assumption of *pairwise sign compatibility (PSC)* by Chakrabarti and Gilles (2007) briefly in the beginning. It requires that any link is either beneficial to both or to none of the involved players. This assumption especially implies that no disagreement about the formation of a given link between two myopic players is possible.

Definition 7 (Chakrabarti and Gilles, 2007)**.** *A profile of utility functions u satisfies pairwise sign compatibility (PSC) if for all $g \in G$ and for all links $ij \in g$ it holds that:* $sgn\left(mu_i(g, ij)\right) = sgn\left(mu_j(g, ij)\right).$

[13]See for instance Büchel and Hellmann (2009) for a definition of externalities in network formation and the implications for the tension between stability and efficiency

In the existence result of Chakrabarti and Gilles (2007) pairwise sign compatibility among other properties is required to show the existence result. Pairwise sign compatibility is also implicitly assumed in the existence result of Jackson and Watts (2002b). Thus, we could imagine that together with our properties on the utility function, we are able to establish an even stronger result.

Proposition 3. *Let $\mathbb{G}$ be a network society and suppose that u satisfies PSC, anonymity and the strategic complements property. Then either the empty network or the complete network is pairwise stable. Furthermore, if no indifference is satisfied, then one of those is uniquely pairwise stable.*

With these strong conditions of PSC, anonymity and the strategic complements, we get also strong results: the empty network and the complete network are the only candidates for stability and at least one of them is pairwise stable. No indifference additionally ensures that exactly one of them is stable. Although the assumptions in this result are strong, let me emphasize that the utility function is not specified. The results only depend on the general assumptions. Let us take a look at one more property taken from the literature. Besides convexity in *own* links that has been analyzed extensively in this chapter, Bloch and Jackson (2007) define also a stronger property, namely convexity *all* links.

Definition 8 (Bloch and Jackson, 2007)**.** *The utility function u_i is convex (concave) in all links if $\forall i \in N$, $\forall g \in G$, $\forall l \in g^N \setminus g$, and $\forall jk \notin g \cup l$:*

$$mu_i(g \cup jk, jk) \leq (\geq) mu_i(g \cup l \cup jk, jk).$$

The convexity property in this definition trivially includes convexity in *own* links according to Definition 3. Moreover it includes the effects of other players links in a non-negative fashion. Thus the following result is obvious.

Lemma 5. *Let $\mathbb{G}$ be a network society and suppose that u satisfies convexity (concavity) in all links. Then,*

(1) u is convex (concave) in own links.

(2) u satisfies the strategic complements (substitutes) property.

Since the definition of convexity in *all* links implies convexity in *own* links and the strategic complements property, we can immediately conclude that a network society with a profile of utility functions satisfying convexity in all links possesses a pairwise stable network by Theorem 1. In fact, convexity in *all* links is stronger than convexity in *own* links and complementarity, as one can easily check.

This concluding section collects some definitions taken from the literature that are sufficient for existence. However, most of the assumptions are very strong. Hence, this part is kept very brief and can be seen as an extension to the analysis of the more natural properties that only include the effects of own and other players' links on marginal utilities.

7 Conclusion

In this paper, I have studied conditions which are sufficient for the existence of pairwise stable networks. I have focused on definitions which seem quite natural and are widely used in economics and network formation: convexity and concavity, describing the effects of own links on marginal utility, and complementarity and substitutability representing the effects of other players links on marginal utility. In the case of convexity and concavity several definitions can be found in the literature. All of them are equivalent, and I put them into relation of non-diminishing and non-increasing marginal utilities with respect to the set inclusion ordering.

In the main result of this chapter it is shown that the properties of convexity and complementarity are sufficient for the existence of pairwise stable networks. Past studies needed strong and restrictive assumptions to derive sufficient conditions for the existence of pairwise stable networks and were not able to find models in the literature satisfying the assumptions. The properties of convexity and complementarity, however, can be found in some models of which I presented one example here. An even more intuitive assumption on the utility function is concavity and substitutability representing non-increasing marginal utility. These conditions are, however, not sufficient for existence of pairwise stable networks. Instead, conditional on existence, pairwise stable networks are locally unique: in generic cases, there exists no other network that can be reached by link addition or link deletion which is pairwise stable. Again, many network formation models from the literature can be found satisfying the assumptions of concavity and substitutability.

A particular feature of this study is the generality of the analysis. The utility function is not specified, it is only restricted to natural settings which are not strong as many models in the literature share them. The contribution of this paper to the network formation literature is three-fold. First, the notion of convexity and concavity is clarified and definitions in the literature are organized. Second, I am able to establish an existence result only depending on very natural settings compared to past work. Third, the results elaborate on the general structure of pairwise stable networks. Some of them may help characterize pairwise stable networks in different models of network formation, using e.g. the uniqueness result.

APPENDIX

Proof of Proposition 1.

(1)⇒ (2) Suppose that for a player $i \in N$, u_i is convex (concave) in own links. I show that then u_i is also convex in own new links, i.e. $mu_i(g \cup l_i, l_i) \geq \sum_{ij \in l_i} mu_i(g \cup ij, ij)$ for every $g \in G$, and for any set of own *new* links $l_i \subseteq L_i(g^N \backslash g)$. Let $g \in G$ and $l_i \subseteq L_i(g^N \backslash g)$. Since any network is a set of single links, l_i can also be written as $l_i = \{ij_1, ..., ij_m\}$. By definition of marginal utility we get $mu_i(g \cup l_i, l_i) = u_i(g \cup l_i) - u_i(g)$. We can add zeros and rearrange the summation to get:

$$\begin{aligned} mu_i(g \cup l_i, l_i) &= u_i(g \cup l_i) - u_i(g) \\ &= u_i(g \cup l_i) - u_i(g \cup l_i \setminus ij_1) + u_i(g \cup l_i \setminus ij_1) \\ &\quad - u_i(g \cup l_i \setminus (ij_1 \cup ij_2)) + u_i(g \cup l_i \setminus (ij_1 \cup ij_2)) - ... \\ &\quad - u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{m-1} ij_k\}) + u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{m-1} ij_k\}) - u_i(g) \\ &= \sum_{x=1}^{m} \left(u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{x-1} ij_k\}) - u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{x} ij_k\}) \right) \end{aligned} \tag{3}$$

We can now apply convexity in own links by leaving out the links $l_i \setminus \{\bigcup_{k=1}^{x} ij_k\}$ in every summand and get:

$$\begin{aligned} &\sum_{x=1}^{m} \left(u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{x-1} ij_k\}) - u_i(g \cup l_i \setminus \{\bigcup_{k=1}^{x} ij_k\}) \right) \\ \geq &\sum_{x=1}^{m} (u_i(g \cup ij_x) - u_i(g)) \\ = &\sum_{ij \in l_i} (u_i(g \cup ij) - u_i(g)) = \sum_{ij \in l_i} mu_i(g \cup ij, ij), \end{aligned}$$

implying convexity in own *new* links, since l_i and g, where chosen arbitrarily.

(1)⇒(3) This step can be shown analogously to step 1. Suppose that for a player $i \in N$, u_i is convex in own links. I show that then u_i is also concave in own current links, i.e. $mu_i(g, l_i) \leq \sum_{ij \in l_i} mu_i(g, ij)$ for every $g \in G$, and $l_i \subseteq L_i(g)$.

Let $g \in G$ and $l_i \subseteq L_i(g)$. We can write l_i as a list of its links, $l_i = \{ij_1, ..., ij_m\}$. By definition of marginal utility we get $mu_i(g, l_i) = u_i(g) - u_i(g \setminus l_i)$. Similar to step 1, I add zeros and rearrange the summation to get:

$$\begin{aligned} mu_i(g, l_i) &= u_i(g) - u_i(g \setminus l_i) \\ &= u_i(g) - u_i(g \setminus ij_1) + u_i(g \setminus ij_1) - u_i(g \setminus (ij_1 \cup ij_2)) + u_i(g \setminus (ij_1 \cup ij_2)) \\ &\quad - ... + ... - u_i(g \setminus \{\bigcup_{k=1}^{m-1} ij_k\}) + u_i(g \setminus \{\bigcup_{k=1}^{m-1} ij_k\}) - u_i(g \setminus l_i) \\ &= \sum_{x=1}^{m} \left(u_i(g \setminus \{\bigcup_{k=1}^{x-1} ij_k\}) - u_i(g \setminus \{(\bigcup_{k=1}^{x}) ij_k\}) \right) \end{aligned}$$

By convexity in own links, adding a set of links increases the marginal utility of a given link. Thus, adding $\{\bigcup_{k=1}^{x-1} ij_k\}$ to the network $g \setminus \{\bigcup_{k=1}^{x-1} ij_k\}$ in every summand yields higher marginals:

$$
\begin{aligned}
& \sum_{x=1}^{m} \left(u_i(g \setminus \{\bigcup_{k=1}^{x-1} ij_k\}) - u_i(g \setminus \{\bigcup_{k=1}^{x}\} ij_k\}) \right) \\
\leq & \sum_{x=1}^{m} (u_i(g) - u_i(g \setminus ij_x)) \\
= & \sum_{ij \in l_i} (u_i(g) - u_i(g \setminus ij)) = \sum_{ij \in l_i} mu_i(g, ij),
\end{aligned}
$$

implying concavity in own *current* links, since $g \in G$ and $l_i \subseteq L_i(g)$ where chosen arbitrarily.

(2)⇒(1) Now, suppose u_i is convex in own new links. By Definition 2 it holds for all $g \in G$ and for any set of own links $l_i \subseteq L_i(g^N \setminus g)$ that:

$$
mu_i(g \cup l_i, l_i) \geq \sum_{ij \in l_i} mu_i(g \cup ij, ij) \tag{4}
$$

Applying (4) to a set of own links $\bar{l}_i \subseteq L_i(g^N \setminus g)$ of size two, e.g. $\bar{l}_i = \{ik, il\}$ for some links $ik, il \in L_i(g^N \setminus g)$, implies:

$$
mu_i(g \cup ik \cup il, ik \cup il) \geq mu_i(g \cup ik, ik) + mu_i(g \cup il, il), \tag{5}
$$

for all $g \in G$, and for any two links $ik, il \in L_i(g^N \setminus g)$. By equivalently rearranging equation (5) we get:

$$
\begin{aligned}
\Leftrightarrow \quad & u_i(g \cup ik \cup il) - u_i(g) && \geq u_i(g \cup ik) - u_i(g) + u_i(g \cup il) - u_i(g) \\
\Leftrightarrow \quad & u_i(g \cup ik \cup il) - u_i(g \cup il) && \geq u_i(g \cup ik) - u_i(g) \\
\Leftrightarrow \quad & mu_i(g \cup il \cup ik, ik) && \geq mu_i(g \cup ik, ik) \qquad (6)
\end{aligned}
$$

This holds again for all $g \in G$, and for any two links $ik, il \in L_i(g^N \setminus g)$. I complete the proof of this step by showing that this implies convexity in own links. Let there be a network $\tilde{g} \in N$, a link $ij \notin \tilde{g}$ and set of own links $\{ij_1, ..., ij_m\} = \tilde{l}_i \subseteq L_i(g^N \setminus (\tilde{g} \cup ij))$. Define a sequence of networks $(g_0, g_1, ..., g_m)$ such that $g_0 = \tilde{g}$ and $g_k = \tilde{g} \cup (\bigcup_{l=1}^{k} ij_l)$ for all $k = 1, ..., m$. Applying inequality (6) to every g_k in the sequence, such that $k = 0, 1, ..., m-1$, we get:

$$
mu_i(g_k \cup ij, ij) \overset{(6)}{\leq} mu_i(g_k \cup ij_{k+1} \cup ij, ij) = mu_i(g_{k+1} \cup ij, ij), \tag{7}
$$

for all $k = 0, 1, ..., m-1$. Since the inequality holds for every two consecutive elements in the sequence, it holds especially for the first and last element ($g_0 = \tilde{g}$ and $g_m = \tilde{g} \cup \tilde{l}_i$) in the sequence:

$$
(7) \Rightarrow \quad mu_i(\tilde{g}, ij) \leq mu_i(g_m \cup ij, ij) = mu_i(\tilde{g} \cup \tilde{l}_i \cup ij, ij)
$$

which, again holds for all networks $\tilde{g} \in G$, for all links $ij \notin \tilde{g}$ and $\tilde{l}_i \subseteq L_i(g^N \setminus (\tilde{g} \cup ij))$, implying convexity in own links.

(3)⇒ (1) We show this step similarly to step 3. By definition of concavity in own current links we get

$$mu_i(g, l_i) \leq \sum_{ij \in l_i} mu_i(g, ij), \tag{8}$$

for all $i \in N$, for all $g \in G$ and for any set of own current links $l_i \subseteq L_i(g)$. Letting $l_i = \{ik, il\}$ for any two links $ij, ik \in L_i(g)$, (8) implies:

$$mu_i(g, ik \cup il) \leq mu_i(g, ik) + mu_i(g, il). \tag{9}$$

Rearranging (9) gives:

$$\begin{aligned}
\Leftrightarrow \quad u_i(g) - u_i(g \setminus \{ik, il\}) &\leq u_i(g) - u_i(g \setminus ik) + u_i(g) - u_i(g \setminus il) \\
\Leftrightarrow \quad u_i(g \setminus il) - u_i(g \setminus \{ik, il\}) &\leq u_i(g) - u_i(g \setminus ik) \\
\Leftrightarrow \quad u_i(g' \cup ik) - u_i(g') &\leq u_i(g' \cup ik \cup il) - u_i(g' \cup il)) \\
\Leftrightarrow \quad mu_i(g' \cup ik, ik) &\leq mu_i(g' \cup il \cup ik, ik), \qquad (10)
\end{aligned}$$

with $g' := g \setminus \{il, ik\}$. Equation (10) holds for all $i \in N$, for all $g' \in G$, and for any two links $ik, il \in L_i(g^N \setminus g')$, and thus is equivalent to (6), completing the proof, since (6) implies convexity in own links, as shown in step 3.

To show the analogous equivalences for concavity in own links one can simply invert all "$\leq$"-signs. □

Proof of Lemma 2.

(1) Let C be an improvement cycle and suppose for all $g \in C$ it holds that $g \notin G^a$, i.e. no link addition proof network is part of the improvement cycle C. Trivially for all networks $g \notin G^a$ there exists an improvement path leading to a link addition proof network: Take $g \notin G^a$, then there exists by definition a link $ij \in g^N \setminus g$ such that $mu_i(g \cup ij, ij) > 0$ and $mu_j(g \cup ij, ij) \geq 0$. Thus the link can be added as an improvement in the sense of improvement paths. If the new network is addition proof, then we are done, otherwise there exists another link which can be added. Hence, we can construct an improvement path that only involves the addition of links. Because the number of links is finite, this process leads either to a link addition proof network or eventually to the complete network, which is trivially addition proof. Thus, C cannot be a closed cycle, since it does not contain an addition proof network, implying that there exists an improving path to an addition proof network, which is not contained in C.

(2) This part is completely analogous, since if no network in an improvement cycle is deletion proof then there exists an improvement path which only involves link deletion, leading eventually to an deletion proof network, e.g. the empty network. □

Proof of Lemma 3.

(1) Let $g \in G^a$, and suppose the contrary of the proposition is true, i.e. suppose there exists an improvement path from g to itself, labeled by $(g_1, ..., g_m)$, such that

$g_1 = g_m = g$. For any improvement path emanating from g, links can only be deleted by assumption, since $g \in G^a$. Since $(g_1, ..., g_m)$ is a cycle, there have to be links, which are added along the improvement path. Let g_k be the first network in the improvement path, which is reached by adding a link from network g_{k-1}. Label the first link that is added along the sequence as ij. This link ij is either not contained in g (i.e. is not part of the first network in the sequence) or is deleted along the sequence. For $h < k$, define g_h as the network in sequence such that is reached by deleting the link ij from network g_{h-1} (set $h = 1$ if the link ij is not contained in g), i.e. $g_h = g_{h-1} - ij$ or let $h = 1$ if $ij \notin g$. Since we assume that $(g_1, ..., g_m)$ is an improving path, we get $0 > mu_i(g_h \cup ij, ij)$ for at least one of the involved players, if $h \geq 2$. Furthermore, since g is assumed to be addition proof, we get either $0 \geq mu_i(g_h \cup ij, ij)$ for both players or $0 > mu_i(g_h \cup ij, ij)$ for at least one of the involved players if $h = 1$. Let $l := g_h \setminus g_{k-1}$, $l_i := L_i(g_h \setminus g_{k-1})$ and $l_{-i} := l \setminus l_i$. We defined g_k as the first network in the sequence that is reached by adding a link to the predecessor. Hence, up to g_{k-1} links have only been deleted along the sequence. Since $h < k$, we get thus $g_{k-1} \cup l_i \cup l_{-i} = g_h$. Now, since u satisfies convexity in own links we get:

$$mu_i(g_{k-1} \cup ij, ij) \leq mu_i(g_{k-1} \cup l_i \cup ij, ij). \tag{11}$$

Similarly by strategic complements,

$$mu_i(g_{k-1} \cup l_i \cup ij, ij) \leq mu_i(g_{k-1} \cup l_i \cup l_{-i} \cup ij, ij). \tag{12}$$

Notice that $g_{k-1} \cup l_i \cup l_{-i} = g_h$ implying by (11) and (12): $mu_i(g_{k-1} \cup ij, ij) \leq mu_i(g_h \cup ij, ij)$. But for $mu_i(g_h \cup ij, ij)$ it holds that $mu_i(g_h \cup ij, ij) < 0$ for one player or $mu_i(g_h \cup ij, ij) \leq 0$ for both involved players. Hence the link is not added along the improvement path, contradicting the supposition that there exists an improvement cycle, where $g \in G^a$ is part of.

(2) Similarly to (1), suppose that $g \in G^d$ and the contrary of the proposition is true. Take such an improvement path from g to itself, labeled by $(g_1, ..., g_m)$ such that $g_1 = g_m = g$. For any improvement from g, links can only be added by assumption. Let g_k be the first network in the sequence that is reached by deleting a link, say $g_k = g_{k-1} - ij$ and let i be the player, such that $mu_i(g_{k-1}, ij) < 0$. This link has either been added along the improvement path or has initially been part of g. Let $h < k$ be such that $g_h = g_{h-1} + ij$ or let $h = 1$ if $ij \in g$. By definition of g and improving paths, we get $0 \leq mu_i(g_h, ij)$. Since g_{k-1} is a reached by only adding links from g_h, let $l = g_{k-1} \setminus g_h$ be the set of links that are added. Let $l_i = l \cap L_i(g^N)$ the subset of those links, where player i is involved and $l_{-i} = l \setminus l_i$ the set of links, where player i is not involved. By strategic complements and convexity, it holds that: $mu_i(g_h, ij) \leq mu_i(g_h \cup l_i, ij) \leq mu_i(g_h \cup l_i \cup l_{-i}, ij) = mu_i(g_{k-1}, ij)$, contradicting the supposition that there exists an improvement cycle, where $g \in G^d$ is part of. □

Proof of Lemma 4.

(1) Let $g \in G^d$ and consider a network $g' \subseteq g$. Since g is link deletion proof, we have for all links $ij \in g : mu_i(g, ij) \geq 0$. Let $l := g \setminus g'$, $l_i := l \cap L_i(g^N)$ and $l_{-i} := l \setminus l_i$. Then we get by strategic substitutes and concavity for all $ij \in G$:

$$mu_i(g', ij) \geq mu_i(g' \cup l_i, ij) \geq mu_i(g' \cup l_i \cup l_{-i}, ij) = mu_i(g, ij) \geq 0.$$

(2) The proof is analogous, take $g \in G^a$, $g' \supseteq g$. As above by substitutes and concavity we get for all $ij \notin g'$:

$$mu_i(g' \cup ij, ij) \leq mu_i(g \cup ij, ij).$$

By g being addition-proof, it follows that $g' \in G^a$. □

Proof of Theorem 2.

(1) Suppose that the profile of utility functions satisfies concavity in own links and the strategic substitutes property and exhibits no indifference. Suppose the contrary of the proposition is true and there exist two networks $\tilde{g}, \hat{g} \in G$ such that $\tilde{g} \subset \hat{g}$ and $\tilde{g}, \hat{g} \in PS(\mathbb{G})$. Let $\tilde{g} \subseteq g \subseteq \hat{g}$. By Lemma 4, it holds that $g \in G^a(\mathbb{G})$, since $g \supseteq \tilde{g}$ and $\tilde{g} \in PS(\mathbb{G})$. Furthermore, by Lemma 4, it holds that $g \in G^d(\mathbb{G})$, since $g \subseteq \hat{g}$ and $\hat{g} \in PS(\mathbb{G})$. Thus, $g \in G^a(\mathbb{G}) \cap G^d(\mathbb{G}) = PS(\mathbb{G})$. Since $\tilde{g} \subsetneq \hat{g}$, there exists at least one link $ij \in g^N$ such that $\tilde{g} \cup ij \subseteq \hat{g}$, implying that $\tilde{g} \cup ij$ is pairwise stable. Particularly it holds that $mu_i(g \cup ij, ij) \geq 0$ and $mu_j(g \cup ij, ij) \geq 0$. However, since $\tilde{g}$ is also pairwise stable we have $mu_i(g \cup ij, ij) > 0 \Rightarrow mu_j(g \cup ij, ij) < 0$. Thus, $mu_i(g \cup ij, ij) = mu_j(g \cup ij, ij) = 0$, contradicting that u exhibits no indifference.

(2), (3) Both statements follow directly from (1), since every network is a superset of the empty network and every network is a subset of the complete network. □

Proof of Proposition 2. Let u satisfy both the strategic complements and substitutes property and anonymity. We show the proposition in 4 steps:

Claim 1: Under the above assumptions we get that $mu_i(g, ij) = mu_i(g_i, ij)$ for all $i \in N$, and for any network $g \in G$ such that $ij \in g$, where g_i is the ego-network of player i.

This is immediately implied since the profile of utility functions satisfies both the strategic complements and substitutes property, that is it holds for all $i \in N$, for all $g \in G$, for all $l_{-i} \in L_{-i}(g^N \setminus g)$ and for all $ij \in g$ that

$$mu_i(g, ij) = mu_i(g \cup l_{-i}, ij). \tag{13}$$

Thus since $g \setminus g_i \subseteq L_{-i}(g^N)$ the first claim follows by equation (13). In other words, the marginal utility of any link ij in two networks g, g' is the same as long as both networks imply the same ego-network $g_i = g'_i$ for player i.

Claim 2: Under the above assumptions, it holds that $mu_i(g, ij) = mu_i(g, ik)$ for all players $i \in N$, for all networks $g \in G$ and for all links $ij, ik \in g$.

We have to show that in a given network g any two links of the network have the same marginal utility. To proof this claim, consider the neighbors of players j and k and let $l^1 := \{kx \notin g | x \in N_j(g)\}$ be the set of links that k needs to add and $l^2 := \{jy \notin g_j | y \in N_k(g)\}$ be the set of links j needs to add to make the set of neighbors of both players equal. Let $\tilde{g} = g \cup l^1 \cup l^2$. Note that $l^1, l^2 \in L_{-i}(g^N)$, since $ij, ik \in g$, and for the sets of neighbors it holds that $N_j(\tilde{g}) = N_k(\tilde{g})$. Hence,

$$mu_i(g, ij) \overset{(13)}{=} mu_i(g \cup l^1 \cup l^2, ij) \overset{(2)}{=} mu_i(g \cup l^1 \cup l^2, ik) \overset{(13)}{=} mu_i(g, ik) \qquad (14)$$

Equivalently, we also get $mu_i(g \cup ij, ij) = mu_i(g \cup ik, ik)$, for all players $i \in N$, for all networks $g \in G$ and for all links $ij, ik \notin g$.

Claim 3: Under the above assumptions, it holds that $u_i(g) = u_i(g')$ for all players $i \in N$, for all networks $g, g' \in G$ such that $g_{-i} = g'_{-i}$ and $d_i(g) = d_i(g')$.

Let there be two networks $g, g' \in G$ such that $g_{-i} = g'_{-i}$ and $d_i(g) = d_i(g')$. Let $l^1 := g \setminus g'$ and $l^2 := g' \setminus g$. Note that $l^1, l^2 \in L_i(g)$, since $g_{-i} = g'_{-i}$ and $|l^1| = |l^2|$, since $d_i(g) = d_i(g')$. We prove the claim by induction over $|l^1| = |l^2| = k$.

$k = 1$: In this case only the link l^1 (singleton set) needs to be cut and only the link l^2 needs to be added in order to move from g to g'. Thus by Claim 2 it follows:

$$\begin{aligned} & u_i(g) - u_i(g') \\ &= u_i(g) - u_i(g \setminus l_1) + u_i(g \setminus l_1) - u_i((g \cup l_2 \setminus l_1)) + u_i((g \cup l_2 \setminus l_1)) - u_i(g') \\ &= \underbrace{mu_i(g, l_1) - mu_i((g \cup l_2) \setminus l_1, l_2)}_{\overset{(14)}{=} 0} + \underbrace{u_i(g') - u_i(g')}_{=0} = 0 \end{aligned}$$

$k \rightarrow k+1$: Suppose the claim holds for all $g, g' \in G$ such that the assumptions of the claim are satisfied and that $|g \setminus g'| = |g' \setminus g| = k$. We show that it then also holds for $\tilde{g}$ and $\bar{g}$ such that the assumptions of claim 3 are satisfied and $|\tilde{g} \setminus \bar{g}| = |\bar{g}' \setminus \tilde{g}| = k+1$. Denote $l^1 := \tilde{g} \setminus \bar{g}$ and $l^2 := \bar{g} \setminus \tilde{g}$, and label the links in $l^1 = \{ij_1, ..., ij_k, ij_{k+1}\}$ and the links in $l^2 = \{im_1, ..., im_k, ij_{k+1}\}$ and define the following network $\hat{g} := (g \cup \{im_1, ..., im_k\}) \setminus \{ij_1, ..., ij_k\}$. Since the proposition holds for k, we get $u_i(\tilde{g}) = u_i(\hat{g})$, since $|\tilde{g} \setminus \bar{g}| = k$. We get thus the following :

$$\begin{aligned} u_i(\tilde{g}) - u_i(\bar{g}) &= u_i(\hat{g}) - u_i(\bar{g}) \\ &= u_i(\hat{g}) - u_i(\hat{g} \setminus ij_{k+1}) + u_i(\hat{g} \setminus ij_{k+1}) \\ &\quad - u_i((\hat{g} \cup im_{k+1}) \setminus ij_{k+1}) + u_i((\hat{g} \cup im_{k+1}) \setminus ij_{k+1}) - u_i(\bar{g}') \\ &= \underbrace{mu_i(\hat{g}, ij_{k+1}) - mu_i((\hat{g} \cup im_{k+1}) \setminus ij_{k+1}, im_{k+1})}_{\overset{(14)}{=} 0} + u_i(\bar{g}') - u_i(\bar{g}') \\ &= 0. \end{aligned}$$

Hence, $u_i(g) = u_i(g')$ for all players $i \in N$, for all networks $g, g' \in G$ such that $g_{-i} = g'_{-i}$ and $d_i(g) = d_i(g')$.

Claim 4: Under the assumptions above, then for any deletion proof $g \in G^d$ it holds that it is either pairwise stable or there exists an improving path to a pairwise stable network.

Let $g \in G^d$ be deletion-proof. Suppose that g is not stable (otherwise there is nothing to show). Thus, there exists players (at least two), who want to add links with other players. Construct a sequence of networks $(g_k)_{k=0,\dots,K}$ such that $g_0 = g$ and for all $k = 1, ..., K$ it holds that $g_k := g_{k-1} + ij$ for some link $ij \notin g_{k-1}$, with $mu_i(g_{k-1} \cup ij, ij) \geq 0$ and $mu_j(g_{k-1} \cup ij, ij) > 0$ and let g_K be the network where no such link ij can be found. Clearly K must be finite, since there are only a finite number of links. Also by construction g_k defeats g_{k-1} for all $k = 1, ..., K$. We show that in g_K no player wants to delete a link, i.e. the marginal utility of any link in g_K is not negative. Take a link $ij \in g_K$. We show that neither player i nor j wants to cut the link. Suppose first that player i added a link in the improving path $(g_k)_{k=0,\dots,K}$. Let g_H be the network along the improving path where player i added his last link, such that $H := max\,\{k \in \{1...K\} | g_k = g_{k-1} \cup im, m \in N\}$ and let $im := g_H \setminus g_{H-1}$ the last link that i added. Thus, $g_K \setminus g_H \subseteq L_{-i}(g^N)$, and hence:

$$mu_i(g_K, ij) \overset{(13)}{=} mu_i(g_H, ij) \overset{(14)}{=} mu_i(g_H, im) \geq 0.$$

Since im is added along the improving path, we get $mu_i(g_H, im) \geq 0$. If player i did not add a link in the improving path, then $g_K \setminus g \subseteq L_{-i}(g^N)$ and

$$mu_i(g_K, ij) \overset{(13)}{=} mu_i(g, ij) \geq 0,$$

since g is by assumption deletion-proof. This shows the claim. Furthermore, since for any deletion-proof g, there exists an improving path to a pairwise stable network, g cannot be part of a closed improvement cycle. By Lemma 2, there cannot exist a closed improvement cycle and hence there has to exist a pairwise stable network. □

Proof of Proposition 3.

Suppose the empty network is not stable (otherwise there is nothing to show). Then there exists at least one player $i \in N$ such that $mu_i(g^{\emptyset} \cup ij, ij) > 0$ for some $ij \in L_i(g^N)$. Notice that for all $ik \in L_i(g^N)$ this implies by anonymity :

$$mu_i(g^{\emptyset} \cup ik, ik) > 0,$$

since $g_k = \emptyset$ for all $k \in N$, and hence we get by PSC:

$$mu_k(g^{\emptyset} \cup ik, ik) > 0.$$

Notice that for all $k, m \in N$ this implies again by anonymity :

$$mu_k(g^{\emptyset} \cup km, km) > 0. \tag{15}$$

Claim 1: Under the assumptions above, if the empty network is not stable, then it holds for all $i \in N$, for all $l_i \subset L_i(g^N)$ and for all $ij \in L_i(g^N \setminus l_i)$ that: $mu_i(l_i \cup ij, ij) > 0$.

We show the claim by induction over $|l_i|$. Let $|l_i| = 0$, then 15 immediately implies the claim.

Now suppose the claim holds for all $l_i \in L_i(g^N)$ such that $|l_i| = k$. Let $\tilde{l}_i \in L_i(g^N)$ such that $|\tilde{l}_i| = k+1$ and leave out some link $im \in \tilde{l}_i$. Then for all $ij \in L_i(g^N \setminus (\tilde{l}_i))$ we get by strategic complements $mu_j((\tilde{l}_i \setminus im) \cup ij, ij) \leq mu_j(\tilde{l}_i \cup ij, ij)$, since $im \notin L_j(g^N)$. Because we have $|\tilde{l}_i \setminus im| = k$, we get by assumption $0 < mu_j((\tilde{l}_i \setminus im) \cup ij, ij)$, and hence $0 < mu_j(\tilde{l}_i \cup ij, ij)$, implying $0 < mu_i(\tilde{l}_i \cup ij, ij)$ by PSC, which shows the claim.

Claim 2: Under the assumptions above, if the empty network is not stable, then the complete network is stable.

Assume the contrary, that the complete network and the empty network are not pairwise stable. Since the complete network is not pairwise stable, there exists a player $i \in N$ and a link $ij \in L_i(g^N)$ such that $mu_i(g^N, ij) < 0$. Hence, for this link we get by strategic complements $mu_i(L_i(g^N), ij) \leq mu_i(g^N, ij) < 0$, since $L_{-i}(g^N) = g^N \setminus L_i(g^N)$. This, however, contradicts claim 1, since taking $l_i := L_i(g^N) \setminus ij$, we get $mu_i(l_i \cup ij, ij) = mu_i(L_i(g^N), ij) < 0$. Hence the complete network has to be pairwise stable, if the empty network is not pairwise stable. □

Proof of Lemma 5. Both steps are trivial. Convexity in all links implies for all players $i \in N$, for all networks $g \in G$, for any set of links $l \in g^N \setminus g$, and for any link $jk \notin g \cup l$ that $mu_i(g \cup jk, jk) \leq mu_i(g \cup l \cup jk, jk)$. By simply restricting attention to the a set of own links $l \in L_i(g^N \setminus g)$ and the marginal utility of a link $jk \in L_i(g^N \setminus (g \cup l_i))$, convexity in own links immediately follows. Strategic complements captures the effect of other links on marginal utility. Letting $l \in L_{-i}(g^N \setminus g)$ and comparing marginals of a link $jk \in L_i(g^N \setminus (g \cup l_i))$ in the definition of convexity in all links implies the strategic complements property. □

References

Bloch, F., and Jackson, M. (2006): Equilibrium Definitions in Network Formation Games. International Journal of Game Theory 34, 305-318.

Bloch, F., and Jackson, M. (2007): The Formation of Networks with Transfers among Players. Journal of Economic Theory 127(1), 83-110.

Büchel, B. (2009): Preference-based Network Formation. Doctoral Thesis, Bielefeld University.

Büchel, B., and Hellmann, T. (2009): Under-connected and Over-connected Networks. mimeo.

Buskens, V., and Van de Rijt, A. (2005): A Nash Equilibrium Refinement for Myerson's Network Formation Game. Utrecht University, Mimeo.

Calvó-Armengol, A., and Ilkiliç, R. (2007): Pairwise-Stability and Nash Equilibria in Network Formation. International Journal of Game Theory 38, 5179.

Chakrabarti, S., and Gilles, R.P. (2007): Network Potentials. Review of Economic Design 11, 13-52.

Dutta, B., and Mutuswami, S. (1997): Stable Networks. Journal of Economic Theory 76, 322344.

Goyal, S., and Joshi, S. (2006a): Unequal Connections. International Journal of Game Theory 34, 319-349.

Goyal, S., and Joshi, S. (2006b): Bilateralism and Free-trade, International Economic Review 47(3), 749-778.

Goyal, S., and Vega-Redondo, F. (2007): Structural Holes in Social Networks. Journal of Economic Theory 137(1), 460-492.

Jackson, M., and Watts, A. (2002a): The Evolution of Social and Economic Networks. Journal of Economic Theory 106, 265-295.

Jackson, M., and Watts, A. (2002b): The Existence of Pairwise Stable Networks. Seoul Journal of Econonomics 14(3), 299-321.

Jackson, M., and Wolinsky, A. (1996): A Strategic Model of Social and Economic Networks. Journal of Economic Theory 71, 44-74.

Myerson, R.B. (1991): Game Theory: Analysis of Conflict. Cambridge University Press.

Page, F.(2004): Farsighted Stability in Network Formation. In: Demange, G., and Wooders, M. (eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press.