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Coalition Formation in the Airport Problem

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Abstract

We have studied the incentives of forming coalitions in the Airport Problem. It has shown that in this class of games, if coalitions form freely, the Shapley value does not lead to the formation of grand or coalitions with many players. Just a coalition with a few number of players forms to act as the producer and other players would be the consumers of the product. We have found the two member coalition which forms and we have checked its stability.

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1 Introduction

One of the most important problems in cooperative game theory is that which coalitions will form. Coalition formation today has a rich literature in game theory. Examples of coalition formation include professors who would like to choose a university to join among different ones, soccer players who want to change their teams, persons who would like to join communities and clubs, countries which calculate the benefits and costs of joining different international societies such as OPEC, WTO, NAFTA, EU, NATO, etc.

Besides this, the question of how to allocate the worth of a coalition to different players participating in it arises. There are different solution concepts for this question in the cooperative game theory literature. One of the most important of these allocation rules is the Shapley value, suggested by Shapley in 1953. This solution concept has several nice properties such as it is the unique solution of a cooperative game that satisfies efficiency, additivity, symmetry, and dummy player property. This solution concept can be suggested for each finite TU cooperative game. Shapley suggests the value $\Phi_i(v)$ to player i in a cooperative game (N, v) as:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

This is a weighted average of all possible marginal worths for player i. As it has many desirable properties, the Shapley value has many applications in cooperative game theory. Here, we consider this allocation rule as a currently accepted rule for the allocation of the value of a cooperative game among its players. One should notice that we can consider a cooperative game from two different points of view. In one, there is a government which allocates the worth of a coalition to its members and the players should accept their assigned payoffs or leave the game. In the other point of view, we assume that coalitions form freely and there is no government. So, players who form a coalition should decide themselves how to allocate the worth of the coalition among themselves. In this paper we take the second point of view which is more usual in real world situations. That is, some firms (players) in order to increase their profits, form a coalition and allocate its worth among themselves.

Forming a coalition and playing a cooperative game has some gains for all the players participating in the game, but in real world forming a coalition usually takes a long time and has some difficulties. One of these difficulties is that although all the player are aware of the benefits of forming a coalition, they usually do not agree on how to distribute the benefits of the cooperation among themselves immediately. They usually put a lot of time and effort on bargaining to agree on how to do it. This process gets more complicated and difficult to settle down as the number of players increases. Although there are some solution concepts for the problem of how to distribute the benefits of a cooperative game among players such as Shapley value and Nucleolus, these concepts just can settle down by a governor and it is not done in a free cooperative game in which there is no government and so the bargaining power or other kinds of players' power determine the distribution structure.

Another reason which describes why coalitions with more than a specified number of members in some games do not form is that it is more profitable for some players to form a small coalition -which is also easier to form rather than forming a large one- and then charge other players for using the product of their cooperation.

Now we take a look at some types of coalition formation models used in the literature.

2 Different types of coalition formation

2.1 Open Membership

Under this rule of coalition formation, in a game membership in a coalition is open to all players who are willing to abide the rule of that coalition. This type of coalition formation is designed to model an environment in which players are free to form any coalition as long as no player would like to join a coalition.

2.2 Infinite Horizon Coalition Unanimity Game

In this rule of coalition formation, a coalition forms if and only if all potential members agree to form that coalition. The process is as follows: First player 1 makes a proposal for a coalition e.g. $\{1, 4, 6, 8\}$. Then the player with the smallest label in the proposed coalition -here player 4- accepts or rejects the proposal. If player 4 accepts, then it is the time for player 6 to accept or reject the proposal, and the process goes on until we reach the player with the maximum index on the coalition proposed by player 1. If any of the potential members rejects the proposal of player 1, then the proposal is thrown out and the player who first rejects the proposal starts over by proposing another coalition. If instead, all members of the proposal made by player 1 accept the proposal, then they form the coalition. The remaining players continue the coalition formation game, starting by the player with the smallest index making a proposal to the rest of players. In this rule of coalition formation once a coalition forms, it can not break apart, admit new members, or merge with other coalition, regardless how to act the rest of players. If all potential coalitions got rejected, the game finishes and all players will act alone.

2.3 Equilibrium Binding Agreements

Ray and Vohra (94) suggest the Equilibrium Binding Agreements (EBA) as a rule of coalition formation as follows:

A coalition structure $C = \{S_1, ..., S_m\}$ is stable if and only if there do not exist (1) a sub coalition of coalition $\hat{S}_i$ in C and (2) a more refined coalition structure C' which can be induced by deviation by these $\hat{S}_i$ leading perpetrators, such that these $\hat{S}_i$ leading perpetrators are better off under C' than under C.

In EBA theory a strategy vector for the grand coalition is blocked if a leading perpetrator $S \subset N$ can induce another strategy under a finer coalition structure which would make itself better off. This new coalition structure should be stable such that coalitions should not break apart due to some further deviators. In addition, if the deviation of S involves the changing in other coalitions as well, all newly formed coalitions must be better off than before the deviation of S.

2.4 Two Stage Model

This model of coalition formation is used in different types by Myerson (78), Shenoy (79), Hart and Kurz (83), Aumann and Myerson (88), Bloch (95, 96), Ray and Vohra (94, 95) and Yi and Shin (95). The model considers the formation of multiple coalitions and examines the equilibrium number and sizes of coalitions. In this model, in the first stage, players form coalitions and in the second stage, coalitions engage in a noncooperative game, given the coalition structure determined from the first stage. Under some assumptions, the equilibrium is unique for any coalition structure. Although different types of this model share the common objective of analyzing the equilibrium, each of these types adopts a different notion of stability of a coalition structure. Bloch (95, 96) examines an infinite horizon coalition unanimity game in which a coalition forms if and only if all members agree to form the coalition. Ray and Vohra (94) consider the Equilibrium Binding Agreements rule under which coalitions are allowed only to break up into smaller subcoalitions. Yi and Shin (95) investigate the Open Membership game in which members can join an existing coalition without the agreement of the members of the coalition which he is now a member of. The main difference between these rules of coalition formation is the question that if an existing coalition can break apart, admit new members, or merge with other coalitions or not? It is obvious that different rules of coalition formation may lead to different coalition structures.

2.5 Proposal Model

Perry and Reny (94, Econometrica) suggested this process of coalition formation. First we have a definition:

Definition 2.1 *A* **proposal** *is a pair* (x, S) *in which* $S \subseteq N$ *and* x *is a vector where for all* $i \in S$, x_i *is the payoff assigned to player* i *and* $\sum_{i \in S} x_i = v(S)$.

The method of coalition formation which is proposed by Perry and Reny is as follows: The game starts at $t = 0$ at which a player in N can choose to make a proposal or to be quiet. Time is continuous. At every positive time t, any player can choose to make a proposal, accept the current proposal (x, S) if he is a member of S, be quiet, or leave and consume. A proposal becomes binding, when it is accepted by all members of the relevant coalition. Once a player accepts a proposal, he must remain quiet i.e. he can neither make a new proposal nor leave to consume, until that proposal becomes binding, or it is replaced by a new proposal. Once a proposal (x, S) gets binding, we say that the coalition S has formed. After this formation, any player i in S can leave and consume x_i which he gets from his cooperation in coalition S or look for a better proposal.

Now, we take a look at the coalition formation in the games that the formation of a coalition has some externalities for the players who do not participate in the coalition.

First we have a definition.

Definition 2.2 *A* **Partition Function** *is a function which assigns a value to each coalition in a coalition structure as a function of the entire coalition structure.*

This definition means that the value of a coalition does not just depend on the members of the coalition but also on the coalition structure in which that coalition is a member. Yi (97) shows that under a set of conditions on the partition function in the coalition formation in which the formation of coalitions produce negative externalities to nonmembers, the grand coalition is an equilibrium outcome under the Open Membership rule, but typically not under the Coalition Unanimity rule nor the Equilibrium Binding Agreements rule. In addition he shows that the Coalition Unanimity rule and the Equilibrium Binding Agreements rule do better than Open Membership rule, but the grand coalition is typically not a stable outcome under these two rules either. He also proves that in the coalition formation with positive externalities -those coalition which their formation have positive externalities for non members- unlike the case of negative externalities, Open Membership does not support the grand coalition as an equilibrium outcome.

3 Airport Problem

Suppose n types of airplanes would like to build a runway for landing on. The types of airplanes are different and so the characteristics of bands which they need and the cost of construction of bands are different. We assume that it costs C_i to player type i to build the band alone and if two airplanes have the same cost, we consider them as the same type. Without loss of generality we can assume $C_1 < C_2 < ... < C_n$. One of the assumptions in the Airport Problem is that if coalition $\{(i_1, i_2, ..., i_k)\}, i_j \in \{1, 2, ..., n\}$ such that $i_1 < i_2 < ... < i_k$ forms and builds the band, all players of type $j, j \leq i_k$ can use the band built by this coalition but no player $i > i_k$ can use it.

The standard cooperative game in this problem is that if $N = \{1, 2, ..., n\}$ for each $S \subseteq N$ the cost for coalition S to build the band is:

$$c(S) = Max\{C_j \mid j \in S\}$$

in which $c(S)$ is the characteristic cost function. Obviously,

$$c(N) = C_n$$

The saving game related to this cost structure is $G = (N, s)$ in which:

$$s(S) = \sum_{j \in S} C_j - c(S)$$

Definition 3.1 *A Cooperative game is a pair* $G = (N, v)$ *in which* $N = \{1, 2, ..., n\}$ *is the finite set of players and* $v : 2^N \to \Re$ *is the characteristic function such that* $v(\phi) = 0$.

Definition 3.2 *A cooperative game* $G = (N, v)$ *is called* **superadditive** *if:*

$$\forall K, L \subseteq N; K \cap L = \phi \Longrightarrow v(K \cup L) \geq v(K) + v(L)$$

Definition 3.3 *A cooperative game* $G = (N, v)$ *is* **convex** *if:*

$$\forall K, L \subseteq N; v(K \cup L) + v(K \cap L) \geq v(K) + v(L)$$

Definition 3.4 *A cost game* $G = (N, c)$*is called* **subadditive** *if:*

$$\forall K, L \subseteq N; c(K \cup L) \leq c(K) + c(L)$$

In the Airport Problem if the cost game is subadditive, the saving game would be superadditive. This game is also convex.

Littlechild and Owen(1973) have shown that if the grand coalition forms, the Shapley value assigns the cost y_i to player i by:

$$y_i = y_{i-1} + \frac{C_i - C_{i-1}}{\sum_{k=i}^{n} n_k}$$

in which n_k is the number of players of type k and $y_0 = C_0 = 0$. For simplicity we assume $n_i = 1$ for all $i \in \{1, 2, ..., n\}$. So, the Shapley value allocate the cost :

$$y_i = y_{i-1} + \frac{C_i - C_{i-1}}{n - i + 1}$$

to player i for using the service which is produced by the grand coalition. In this paper we would like to explore if this allocation rule could lead to a stable grand coalition or if it is more profitable (or with less cost for using the band) for some players to leave the grand coalition and form another coalition and behave in a different way.

4 Coalition Formation in the Airport Problem

To form a coalition, players should bargain to achieve an agreement on how to distribute the value of the coalition among themselves. We assume that at any time, each player chooses the best choice possible for him. In this problem, the best choice is the one offering him the service which he needs by the lowest cost. If coalition $S = \{(i_1, i_2, ..., i_k)\}, i_j \in \{1, 2, ..., n\} \forall j = 1, 2, ..., k$ in which $i_1 < i_2 < ... < i_k$ forms, all players $l < i_k$ can use the service produced by this coalition. This coalition can potentially charge other players C_{i_k-1}. By potential charge we mean that if the coalition S is the only coalition which has formed, it can suggest the use of the band built by it to all players type $l < i_k$. The reason that they can charge others C_{i_k-1} is that the best thing that players $\{1, 2, ..., i_k\} \backslash S$ can do is that to form the coalition $\{1, 2, ..., i_k\} \backslash S$ and build the band themselves and the cost of building the band for this coalition would be C_{i_k-1}.

The process of forming a coalition is as follows: Any player $i \in N$ can propose a coalition S if $i \in S$. A coalition will form if all its members agree on forming it and if so, they would write a contract and then they would build the band. If player i at the same time is a member of several proposed coalitions, he would accept the one which suggest him using the band with the lowest cost. Obviously he would not like to be a member of two coalition as he should pay additional cost which is not necessary as one band would be sufficient for him. Due to the characteristic of Airport Problem, if $n \notin S$, in the case that coalition S forms and build the band, by sure another band would be built for the use of player n. So, if $n \notin S$ and S forms, it would not be the only producer of the band. So, coalition S can not charge other players who do not cooperate with S the amount C_{i_k-1} and the charging amount would diminish because if one of the two producer charges the amount a for a type of airplane, the other would suggest $a-\varepsilon$ to the same type as the marginal cost of using the band is zero, so the previous one would suggest $a - 2\varepsilon$ and so on. As a result, in order that coalition S forms and it can charge other players $j < i_k, j \notin S$, we should have $n \in S$. Therefore, player n is the one who decides to cooperate with whom and to form which coalition.

We claim that in the Airport Problem, if coalitions form freely as in open Membership method, the Shapley value allocation rule does not lead to form the grand coalition. To prove this claim, we consider a society in which the Shapley value is accepted as the rule of distributing the values of coalitions among their members or allocating the cost of building a product among those who use it. We show that in this society grand coalition does not form.

4.1 Norms of the society for cooperation

We assume that the society of the players follows two norms in their cooperation:

1- If two players form a coalition, they would allocate the cost of producing the service among themselves by the Shapley value cost allocation rule or proportional to their cost of producing the service when acting alone, i.e. if player i and j such that $i < j$ form a coalition, player i would pay $\frac{C_i}{C_i+C_j}.C_j$ and player j would pay $\frac{C_j}{C_i+C_j}.C_j$ to build the band with the cost of C_j

2- After producing the service by a coalition, each member of it would get proportional of the cost that he has paid to produce the service from what the coalition may earn. For example, if player i has paid 30 percent of the cost of producing the service by the coalition, he would get 30 percent of the amount that the coalition get as the charging fee from other players who would like to use their service and pay the charging fee.

5 Examples:

5.1 Example 1.

$C_i = i; i \in \{1, 2, ..., n\}$.

We consider coalitions with two members. If player n forms a coalition with player $n-1$, they can potentially charge other players $n-2$ units. As we assumed, they will distribute the benefit of $n-2$ proportional to the amount that they have paid. Here we assume they allocate the cost of building the band between themselves by the Shapley value. Due to simplified Shapley value formula player k should pay Φ_k to produce the service as:

$\Phi_n^{\{n,n-1\}} = 1 + \frac{n-1}{2} = \frac{n+1}{2}$

$\Phi_{n-1}^{\{n,n-1\}} = \frac{n-1}{2}$

Potential revenue of charging other players would be:

$R(n)^{\{n,n-1\}} = \frac{(n+1)/2}{n}.(n-2)$

$R(n-1)^{\{n,n-1\}} = \frac{(n-1)/2}{n}.(n-2)$

Net cost of using the service will be:

$NC(n)^{\{n,n-1\}} = \frac{n+1}{2} - \frac{n+1}{2}.\frac{n-2}{n} = \frac{n+1}{2}.(1 - \frac{n-2}{n}) = \frac{n+1}{n}$

$NC(n-1)^{\{n,n-1\}} = \frac{n-1}{2}.(1 - \frac{n-2}{n}) = \frac{n-1}{n}$

If player n forms a coalition with player $k, k \in \{1, 2, ..., n-2\}$ we will have:

$\Phi_n^{\{n,k\}} = n - k/2$

$\Phi_k^{\{n,k\}} = k/2$

Potential revenue for each of the two players will be:

$R(n)^{\{n,k\}} = \frac{n-k/2}{n}.(n-1)$

$R(k)^{\{n,k\}} = \frac{k/2}{n}.(n-1)$

So, the net cost of using the service for player n and k would be:

$NC(n)^{\{n,k\}} = (n - \frac{k}{2}).(1 - \frac{n-1}{n}) = (n - \frac{k}{2}).\frac{1}{n}$

$NC(k)^{\{n,k\}} = \frac{k}{2}.(1 - \frac{n-1}{n}) = \frac{k}{2n}$

Player n minimizes his net cost over all players k. It is obvious that the net cost for player n is minimized when k gets the largest possible amount which is here $n-2$. Thus, the minimum net cost which player n should pay would be:

$$(n - \frac{n-2}{2}).\frac{1}{n} = \frac{n+2}{2n}$$

To find the partner who minimizes the net cost of using the service, player n compares $\frac{n+2}{2n}$ and $\frac{n+1}{n}$ and decides to form the coalition with player $n-2$ as his net cost would be $\frac{n+2}{2n}$ which is absolutely less than $\frac{n+1}{n}$. Thus coalition $\{n, n-2\}$ forms.

One may think of that it may diminish the cost for player n to produce the service alone and then charge others. In this case he should pay the cost of one unit which is more than $\frac{n+2}{2n}$ for all $n > 2$. As n increases player n has more incentive to cooperate with player $n-2$ as his net cost of using the band decreases more.
Now we have the definition of stability:

Definition 5.1 *We call a coalition* **stable** *if the cost for all its members are minimized when they join that coalition rather than forming any other coalition.*

Note that this definition which we use in our model is different from the definition of stability in 2.3.

Not considering coalition $\{n, n-2\}$, the coalition that minimizes the cost for player $n-2$ is $\{n-2, n-4\}$ in which his cost would be $\frac{n-2+2}{2(n-2)}$. Comparing this cost with $\frac{n-2}{2n}$ we observe that $\frac{n}{2(n-2)} > \frac{n-2}{2n)}$ for all amounts of n . We should also consider the coalition $\{n-2, n-1\}$. In this coalition we have:

$NC(n-2)^{\{n-1,n-2\}} = \frac{n-2}{2}(1 - \frac{n-3}{n-1})$ (I)

While in coalition $\{n, n-2\}$ we have:

$NC(n-2)^{\{n,n-2\}} = \frac{n-2}{2}(1 - \frac{n-1}{n})$ (II)

For stability we should have (II)<(I). We see that:

$1 - \frac{n-1}{n} < 1 - \frac{n-3}{n-1} \Leftrightarrow \frac{1}{n} < \frac{2}{n-1} \Leftrightarrow n - 1 < 2n$

which always occurs.

As a result, the coalition $\{n, n-2\}$ is stable as it minimizes the cost for both players n and $n-2$.

If player n wants to form the grand coalition with all other players which minimizes his cost in the Shapley value cost allocation rule, he should pay the cost:

$$\sum_{i=1}^{n} \frac{1}{n} = \frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{2} + 1$$

which is much more than $\frac{n+2}{2n}$. For example if we let $n = 5$ the cost will be 0.7 for player n while in the Shapley value allocation rule it would be:

$$\frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1 = \frac{137}{60}$$

This amount is even more than 2 while in other way it would be 0.7. For the case $n = 15$, the net cost that player n should pay in coalition $\{n, n-2\}$ is 0.57 while in the grand coalition he should pay 3.32. As n increases player n has more incentive to form coalition $\{n, n-2\}$ rather than the grand coalition for two reasons: The first reason is that it gets much more difficult and it takes more time so that the grand coalition forms and the second one is that the cost decreases noticeably in the case of forming the coalition $\{n, n-2\}$ rather than the grand coalition.

5.2 Example 2.

$C_i = i^2, i \in \{1, 2, ..., n\}$.

If coalition $\{n, n-1\}$ forms, by the Shapley value cost allocation rule we have:

$\Phi_n^{\{n,n-1\}} = n^2 - \frac{(n-1)^2}{2}$

$\Phi_{n-1}^{\{n,n-1\}} = \frac{(n-1)^2}{2}$

and so the net cost which player n should pay for using the service would be:

$NC(n)^{\{n,n-1\}} = (n^2 - \frac{(n-1)^2}{2})(1 - \frac{(n-2)^2}{n^2})$ (III)

If player n forms a coalition with player $k, k \in \{1, 2, ..., n-2\}$, his net cost would be:

$NC(n)^{\{n,k\}} = (n^2 - \frac{k^2}{2})(1 - \frac{(n-1)^2}{n^2})$ (IV)

As they potentially can charge other players $(n-1)^2$ units.

Player n minimized the net cost in (IV) over all k and $k = n-2$ is the answer of this minimization. If he forms the coalition with player $n-2$ his net cost would be:

$NC(n)^{\{n,n-2\}} = (n^2 - \frac{(n-2)^2}{2})(1 - \frac{(n-1)^2}{n^2})$ (V)

Comparing this amount with the net cost of using the service in the case of formation the coalition with player $n-1$ we observe that always (V)<(III).

Proof. $(n^2 - \frac{(n-2)^2}{2})(1 - \frac{(n-1)^2}{n^2}) < (n^2 - \frac{(n-1)^2}{2})(1 - \frac{(n-2)^2}{n^2}) \Leftrightarrow$

$(n^2 - (n-2)^2).(n^2 - (n-1)^2) < (2n^2 - (n-1)^2).(n^2 - (n-2)^2) \Leftrightarrow$

$(2n^2 - n^2 - 4 + 4n).(n^2 - n^2 - 1 + 2n) < (2n^2 - n^2 - 1 + 2n).(n^2 - n^2 - 4n + 4n) \Leftrightarrow$

$(n^2 - 4n + 4n).(2n - 1) < (n^2 + 2n - 1).(4n - 4) \Leftrightarrow$

$\frac{3}{2} < n.$
which is always true.

So $\{n, n-2\}$ is the best coalition for player n to form as it minimizes his cost of using the service among all possible coalitions with two players.

Stability:
By a simple calculation like example 1, one could check that $\{n, n-2\}$ is stable.

6 Theorems

Theorem 6.1 *In the Airport Problem, if the Shapley value is the rule of allocating the worth of a coalition among its members, the grand coalition would not form. The only two member stable coalition which forms would be*$\{n, n-2\}$.

Proof. By the characteristic of the cost structure and the assumption that if j is the maximum type of airplanes cooperating in coalition S, the band built by this coalition is usable for all aircrafts of type $i \leq j$, if $n \notin S$ the band built by coalition S can not support all types of airplanes especially the type n, so another band for the use of player n must be built. If two bands are built, as the marginal cost of using the band is zero, the charge fee that the two coalitions which have built the bands can get from other players not cooperating with these coalitions would be much less than when there is only one band. Because if one of them charges the amount a for a type of airplane, the other would suggest $a-\varepsilon$ to the same type as the marginal cost of using the band is zero, so the previous one would suggest $a-2\varepsilon$ and so on. This process continues until the charge fee of those types of planes outside the coalitions decreases to a small amount that it is better for the members of the coalition to join the grand coalition. In this case, the cost which each player of coalition S should pay is much higher than the charge fee suggested by the coalition in which player n is a member of, thus S would not form. As a result, a coalition S will form only if $n \in S$. As all the players are aware of this fact, they just cooperate with coalition S if $n \in S$. Now we explore which type of planes is the best one for player n to cooperate with. For the case that the coalition has two members, the best type to cooperate with is the one which minimizes the net cost of using the band for player n. The net cost of player n is the cost which he pays for the construction of the band minus his share of the revenue of the coalition. Here the revenue of the coalition is the amount which other types of aircrafts pays to the coalition to use the band built by it.

First we consider the coalition $\{n, n-1\}$:

The Shapley value allocates the cost of:

$\Phi(n)^{\{n,n-1\}} = C_n - \frac{C_{n-1}}{2}$

$\Phi(n-1)^{\{n,n-1\}} = \frac{C_{n-1}}{2}$

Potential charge of other players is C_{n-2} because $c(\{1, 2, ..., n\} \backslash \{n, n-1\}) = C_{n-2}$. So the net cost for each of the players will be:

$NC(n)^{\{n,n-1\}} = (C_n - \frac{C_{n-1}}{2}).(1 - \frac{C_{n-2}}{C_n})(I)$

$NC(n-1)^{\{n,n-1\}} = (\frac{C_{n-1}}{2}).(1 - \frac{C_{n-2}}{C_n})$

If the coalition $\{n, k\}, k \in \{1, 2, ..., n-2\}$ forms, we would have:

$\Phi(n)^{\{n,k\}} = C_n - \frac{C_k}{2}$

$\Phi(n)^{\{n,k\}} = \frac{C_k}{2}$

in this case, potential charge of other players would be C_{n-1} as $c(\{1, 2, ..., n\} \backslash \{n, n-2\}) = C_{n-1}$, so the net cost for each of the will be:

$NC(n)^{\{n,k\}} = (C_n - \frac{C_k}{2}).(1 - \frac{C_{n-1}}{C_n})$

$NC(k)^{\{n,k\}} = \frac{C_k}{2}.(1 - \frac{C_{n-1}}{C_n})$

Player n minimizes his cost. So C_k should be maximized, thus $k = n-2$ is the best player for player n to form the two member coalition with among all players in the set $\{1, 2, ..., n-2\}$.

$Min_k NC(n)^{\{n,k\}} = (C_n - \frac{C_{n-2}}{2}).(1 - \frac{C_{n-1}}{C_n})(II)$

Min$\{(I), (II)\} = (II)$ as:

$(C_n - \frac{C_{n-2}}{2}).(1 - \frac{C_{n-1}}{C_n}) < (C_n - \frac{C_{n-1}}{2}).(1 - \frac{C_{n-2}}{C_n}) \Leftrightarrow$

$(2C_n - C_{n-2})(C_n - C_{n-1}) < (2C_n - C_{n-1})(C_n - C_{n-1}) \Leftrightarrow$

$2C_n^2 - 2C_nC_{n-1} - C_nC_{n-2} + C_{n-1}C_{n-2} < 2C_n^2 - 2C_nC_{n-2} - C_nC_{n-1} + C_{n-1}C_{n-2} \Leftrightarrow$

$C_n(C_{n-2} - C_{n-1}) < 0.$

The last inequality always holds as due to our assumption $C_{n-2} < C_{n-1}$. More is the difference of C_{n-1} and C_{n-2} , more incentive has player n to form the coalition with $n-2$ rather than $n-1$.

Stability:

Now we calculate the net cost for player $n-2$ in the coalition $\{n, n-2\}$:

$NC(n-2)^{\{n,n-2\}} = (\frac{C_{n-2}}{2}).(1 - \frac{C_{n-1}}{C_n})$

This amount is the minimum net cost of using the band for player $n-2$ because if he does not accept this choice the best thing that he can do is to form the coalition $\{1, 2, ..., n-2, n-1\}$ in which he should pay the cost of:

$y_{n-2}^{\{1,2,...,n\}} = \frac{C_1}{n-1} + \frac{C_2 - C_1}{n-2} + ... + C_{n-2} - C_{n-3}$

which is absolutely higher than his net cost in the case of cooperating with player n. Therefore the coalition $\{n, n-2\}$ is stable as it minimizes the cost of using the band for both the players n and $n-2$ among all possible cases of cooperation or acting individually.

If player n wants to form the grand coalition and accept the Shapley value allocation rule to allocate the cost of building the band among the players, the cost which he should pay would be:

$NC(n)^{\{1,2,...,n\}} = \frac{C_1}{n} + \frac{C_2 - C_1}{n-1} + ... + C_n - C_{n-1}.$

which is much higher than $NC(n)^{\{n,n-2\}}$.

If coalition $\{(i_1, i_2, ..., i_k)\}, i_j \in \{1, 2, ..., n\}, i_1 < i_2 < ... < i_k$ forms, the allocated cost to player i_j by the Shapley value would be:

$NC(i_j)^{\{(i_1,i_2,...,i_k)\}} = \frac{C_{i_1}}{k} + \frac{C_{i_2}-C_{i_1}}{k-1} + ... + C_{i_j} - C_{i_{j-1}}.(III)$

From the above formula it is clear that even participating in coalitions with many players if the rule of allocating the cost is the Shapley value is not optimum for player n as his net cost in coalition $\{n, n-2\}$ is much less than (III).

Definition 6.1 Proportionality rule: *In a coalition each player of the coalition gets as much as the cost that he has paid proportional to the total cost of the coalition -to produce the service on which the members of the coalition cooperate- from the revenue of the cooperation.*

Theorem 6.2 *In the Airport Problem, if proportionality is the rule of allocating the worth of a coalition among its members, grand coalition would not form. If the condition:*

$$(C_n - C_{n-1}).(C_1 + ... + C_{n-3} + C_{n-1}) < C_{n-1}.(C_{n-2} + C_n)$$

holds, The only two member stable coalition which forms would be $\{n, n-2\}$.

Proof.

Like the first part of the proof of theorem 6.1, we should have $n \in S$ so that coalition S forms and could charge other players who are not a member of S. Now we would find the best partner for player n to form a two member coalition. If coalition $\{n, n-1\}$ forms, the cost which each of the two players should pay to build the runway would be:

$y(n)^{\{n,n-1\}} = \frac{C_n}{C_{n-1}+C_n}.C_n$

$y(n-1)^{\{n,n-1\}} = \frac{C_{n-1}}{C_{n-1}+C_n}.C_n$

This coalition potentially can charge other players C_{n-2}. As they get from the charged fee their share proportional to what they have paid to build the band -We have assumed this is a norm in the society of players- so, the net cost of using the band for each of them would be:

$$NC(n)^{\{n,n-1\}} = \frac{C_n}{C_{n-1}+C_n}.C_n - \frac{C_n}{C_{n-1}+C_n}.C_{n-2} = \frac{C_n}{C_{n-1}+C_n}[C_n - C_{n-2}](I)$$

$$NC(n-1)^{\{n,n-1\}} = \frac{C_{n-1}}{C_{n-1}+C_n}[C_n - C_{n-2}]$$

If coalition $\{n,k\}, k \in \{1,2,...,n-2\}$ forms, the cost allocated to each of the players would be:

$$y(n)^{\{n,k\}} = \frac{C_n}{C_k+C_n}.C_n$$

$$y(k)^{\{n,k\}} = \frac{C_k}{C_k+C_n}.C_n$$

and the net cost of each player of using the service would be:

$$NC(n)^{\{n,k\}} = \frac{C_n}{C_k+C_n}.[C_n - C_{n-1}]$$

$$NC(k)^{\{n,k\}} = \frac{C_k}{C_k+C_n}.[C_n - C_{n-1}]$$

The minimum of the net cost for player n among all players $k, k \in \{1,2,...,n-2\}$ takes place when C_k is maximized. As we have $C_1 < C_2 < ... < C_n$, this minimum takes place at $k = n-2$. So we have:

$$Min_k NC(n)^{\{n,k\}} = Min\frac{C_n}{C_k+C_n}.[C_n - C_{n-1}] = \frac{C_n}{C_{n-2}+C_n}.[C_n - C_{n-1}](II)$$

$$Min\{(I),(II)\} = Min\{\frac{C_n}{C_{n-1}+C_n}.[C_n - C_{n-2}], \frac{C_n}{C_{n-2}+C_n}.[C_n - C_{n-1}]\}$$

We claim that always $(II) < (I)$.

Proof.

$\frac{C_n}{C_{n-1}+C_n}.[C_n - C_{n-2}] > \frac{C_n}{C_{n-2}+C_n}.[C_n - C_{n-1}] \Leftrightarrow \frac{C_n - C_{n-2}}{C_{n-1}+C_n} > \frac{C_n - C_{n-1}}{C_{n-2}+C_n} \Leftrightarrow$

$C_n^2 - C_{n-2}^2 > C_n^2 - C_{n-1}^2 \Leftrightarrow C_{n-2}^2 < C_{n-1}^2 \Leftrightarrow C_{n-2} < C_{n-1}.$

due to the assumption of $C_1 < C_2 < ... < C_n$, the last inequality always hold. So the best partner for player n would be $n-2$ to form a two member coalition to produce the band and then charge the other users as the coalition $\{n, n-2\}$ minimized the net cost of using the band for player n among all the two member coalitions containing player n.

On the other hand the cost of using the band for player n in the grand coalition would be:

$NC(n)^{\{1,2,...,n\}} = \frac{C_n}{C_1+C_2+...+C_n}.C_n(III)$

we claim that always $(II) < (III)$.

Proof.
$\frac{C_n}{C_{n-2}+C_n}.[C_n - C_{n-1}] < \frac{C_n}{C_1+C_2+...+C_n}.C_n \Leftrightarrow$

$[C_1 + C_2 + ... + C_n].(C_n - C_{n-1}) < C_n.(C_{n-2} + C_n) \Leftrightarrow$

$C_n.[C_n - C_{n-2}] + C_n.[C_1 + ... + C_{n-3} + C_{n-1}] - C_{n-1}.[C_1 + ... + C_n] <$

$C_n.(C_{n-2} + C_n) \Leftrightarrow C_n.(\sum_{i=1}^n C_i - C_{n-2} - C_n) < C_{n-1}.\sum_{i=1}^n C_i$

$-C_n.(C_{n-2} + C_n) < (C_{n-1} - C_n).\sum_{i=1}^n C_i \Leftrightarrow$

$(C_n - C_{n-1}).(C_1 + ... + C_{n-3} + C_{n-1} + C_{n-2} + C_n) < C_n.(C_{n-2} + C_n) \Leftrightarrow$

$(C_n - C_{n-1}).(C_1 + ... + C_{n-3} + C_{n-1}) + (C_n - C_{n-1}).(C_{n-2} + C_n) <\Leftrightarrow$

$C_n.(C_{n-2}+C_n) \Leftrightarrow (C_n - C_{n-1}).(C_1+...+C_{n-3}+C_{n-1}) < C_{n-1}.(C_{n-2}+C_n).(IV)$

The inequality (IV) holds if $(C_n - C_{n-1})$ is not too large like the case of Airport Problem in real situations. If $(C_n - C_{n-1})$ be so large such that this inequality does not hold, player n would not need to cooperate with any player and he would build the runway alone. The inequality (IV) also shows that if the rule of cooperation be proportionality, when player n would cooperate and when he would act alone.

7 Conclusion

In this paper we have considered the Airport Problem and we have shown that if we have n types of airplanes such that $C_1 < C_2 < ... < C_n$, by the assumption that each player can build the band usable for himself or those who have a smaller cost of band construction than his, it is more profitable for some players like player n to form another coalition than the grand coalition and then charge the other players for using the product of their cooperation. Furthermore, in the case that the coalitions with two players form, we have found the coalition which would form independent of the cost structure which can be linear, concave or convex would be $\{n, n-2\}$.
In addition the optimum charge fees for the total players who are not participating in the coalition are determined such that they prefer the choice of paying the charged fee rather than to form another coalition. We have shown that if coalition $\{n, n-2\}$ forms and build the runway and then charges other players such as we described in the paper, the cost of using the band would be much less than the case that players $n, n-2$ cooperate with the grand coalition and obey the Shapley value for the distribution the worth of the coalition or the cost of building the band. As a result, some players like player n will not participate in the grand or even large coalitions with many players because they can have more profit by not cooperating with them. Therefore, the grand coalition or coalitions with many players will lead to less profit for the some players if the rule of distribution the value of coalition is the Shapley value or the proportional rule. Hence, the Shapley value rule of distribution the gain of cooperation does not lead to form the grand coalition in the Airport Problem.

References

1- Bloch, F. 1996. "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division", Games and Economic Behavior, Vol. 14. 90-123.

2- Bloch, F. 1995. "Endogenous Structures of Association in Ologopolies", The RAND Journal of Economics, Vol. 26, No. 3. 537-556.

3- Bogomolnaia, A. and Jackson, M. O. 2002. "The Stability of Hedonic Structures", Games and Economic Behavior, Vol. 38. 201-230.

4- Brink, R. and Laan, G. 2005. "A Class of Consistent Share Functions for Games in Coalition Structure", Games and Economic Behavior, Vol. 51. 193-212

5- Driessen, T. 1988. "Cooperative Games, Solutions and Applications". Boston: Kluwer Academic Publishers.

6- Dubey, P. 1982. "The Shapley Value as Aircraft Landing Fees-Revisited", Management Science. Vol. 28. No. 8. Aug. PP. 869-874.

7- Friedman, J. W. 1989. "Game Theory with Application to Economics". Oxford: Oxford University Press.

8- Hart, S. and Kurz, M. 1983. "Endogenous Formation of Coalitions", Econometrica. Vol. 51. No. 4. 1047-1064.

9- Huang, C. Y. and Sjostrom, T. 2003. "Consistent Solutions for Cooperative Games with Externalities", Games and Economic Behavior, Vol. 43. 196-213.

10- Littlechild, S. C. and G. Owen. 1973. "A Simple Expression for Shapley Value in a Special Case", Management Science. Theory Series. Vol. 20. No. 3. Nov. 370-372.

11- Littlechild, S. C. and G. F. Thompson. 1977. "Aircraft Landing Fees: A Game Theory Approach", The Bell Journal of Economics. Vol. 8. No. 1. Spring. 186-204.

12- Neumann, V. and O. Morgenstern. 1953. "Theory of Games and Economic Behavior". Princeton: Princeton University press, 3rd ed.

13- Peleg, B. and P. Sudholter. 2003. "Introduction to the Theory of Cooperative Games". Boston: Kluwer Academic Publishers.

14- Perry, M. and Reny, P. J. 1994."A Noncooperative View of Coalition Formation and the Core", Econometrica, Vol. 62, No. 4. 795-817.

15- Potters, J. and Sudholter. 1999. "Airport problems and Consistent Allocation Rules", Mathematical Social Science. Vol. 1. No. 38. 83-102.

16- Ray, D. and Vohra, R. 1997. "Equilibrium Binding Agreements", Journal of Economic Theory, Vol. 73. Article No. ET962236. 30-78.

17- Ray, D. and Vohra, R. 1999. "A Theory of Endogenous Coalition Structures", Games and Economic Behavior, Vol. 26. 286-336.

18- Shapley, L. S. 1953. "The Value of an N-Person Game: Contribution to the Theory of Games". Princeton: Princeton University Press.

19- Tijs, S. H. and T. S. Driessen. 1986. "Game Theory and Cost Allocation Problems", Management Science. Vol. 32. No. 8. Aug. 1015-1028.

20- Yi, S. S. 1997. "Stable Coalition Structures with Externalities", Games and Economic Behavior, Vol. 20. 201-237.

21- Zhou, L. 1994. "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation", Games and Economic Behavior, Vol. 6. 512-526.