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Characterizing Core Stability with Fuzzy Games*†

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Abstract

This paper investigates core stability of cooperative, TU games via a fuzzy extension of the totally balanced cover of a TU game. The stability of the core of the fuzzy extension of a game, the concave extension, is shown to reflect the core stability of the original game and vice versa. Stability of the core is then shown to be equivalent to the existence of an equilibrium of a certain correspondence.

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1 Introduction

The core of a cooperative game is called stable if it coincides with the stable set in the sense of [22]. The problem of core stability is an important problem in cooperative game theory for numerous reasons. A characterization of core stability is desirable because it provides one with an existence theorem for von Neumann-Morgenstern stable sets for a certain class of games as well as insights into the core, which itself is a central paradigm of cooperative game theory. This is not to mention that the stable core is also a very convincing solution concept. There are numerous papers providing conditions for core stability in cooperative, TU games, however, a complete characterization of core stability via the coalition function of a cooperative, TU game is still lacking. For a recent paper providing an overview of previous results as well as new results on core stability, see [20].

The difficulty of characterizing core stability (via the coalition function or some other criteria that are simple to verify) has led researchers to consider new ways of analyzing the problem of core stability, namely via fuzzy games. The fuzzy game paradigm utilized in this paper was introduced by [1]. (For other approaches to defining and analyzing fuzzy "extensions" of cooperative TU games, see, e.g., [3] and [17].) The analysis, presented in this paper, continues the line of work of [6] and [5]. In [6], the authors discuss properties related to core stability for fuzzy games in

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general. In [5], the concavification of a cooperative game is introduced and characterizations for numerous properties intimately related to core stability of cooperative, TU games are provided. Other papers also relevant to the analysis presented here are [10] and [11]. The goal of this paper is to use the tools of [5] to derive new necessary and sufficient conditions for core stability of cooperative, TU games. It is shown that core stability of a cooperative, TU game is equivalent to the existence of an equilibrium of a certain correspondence.

This paper is structured as follows. In Section 1.1 the relevant cooperative, TU game theory definitions and notation are introduced. Section 1.2 is devoted to presenting all the necessary fuzzy game concepts and definitions as well as an example to demonstrate an important property relating to core stability that had been neglected by the literature. In Section 1.3, the concavification of the coalition function is introduced and Theorem 1.27 states that the fuzzy game defined by the concavification has a stable core if and only if the original cooperative, TU game has a stable core. In Section 1.4 important properties of the concavification and its superdifferential are investigated that then form the basis of the final section. In the last section, Section 1.5, Theorem 1.46 provides new necessary and sufficient conditions for core stability. It is also shown in this section that a certain important correspondence closely resembling the correspondence in Theorem 1.46 satisfies many nice properties.

1.1 Preliminaries - TU Games

In this section relevant definitions and concepts for the entire paper are provided. The definitions provided here may be found in [22] unless specified otherwise.

A *game* here is a pair, (N, v), where $N \subseteq \mathbb{N}$ is a finite, nonempty subset with numbers representing players. For the sake of simplifying the notation it will be assumed that $|N| = n$ (here and in the following, if D is a finite set, then $|D|$ denotes the cardinality of D). In addition, nonempty subsets of N will be referred to as coalitions. v is the *coalition function*, $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, which intuitively describes the worth of a coalition. For $S \subseteq N$ denote by $\mathbb{R}^S$ the set of all real functions on S. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. A *payoff* to the players is generated by a vector x, $x \in \mathbb{R}^N$. To simplify the notation, one often introduces the following convention for a vector $x \in \mathbb{R}^N$ and a set $S \subseteq N$: $x(S) := \sum_{i \in S} x_i$, where each x_i stands for the i^{th} component of the vector x ($x(\emptyset) = 0$). Let x_S denote the restriction of x to S, i.e. $x_S := (x_i)_{i \in S}$. In addition, if $x, y \in \mathbb{R}^S$, then write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$ and $x \gg y$ if $x_i > y_i$ for all $i \in S$. For a game (N, v) the **set of imputations** $I(N, v)$ is defined as, $I(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x_i \geq v(\{i\}) \ \forall\ i \in N\}$ and the **core**, $\mathcal{C}(N, v)$, is given by $\mathcal{C}(N, v) := \{x \in I(N, v) \mid x(S) \geq v(S)\ \forall S \subseteq N\}$. Let (N, v) be a game. $\eta \in \mathbb{R}^N$ is said to *dominate* $\zeta \in \mathbb{R}^N$ via the coalition D if η satisfies $\eta(D) \leq v(D)$ as well as $\eta_D \gg \zeta_D$. In the case that η dominates ζ via the coalition D one writes $\eta\ \mathrm{dom}_D\ \zeta$ and one writes $\eta\ \mathrm{dom}\ \zeta$ in case there is a coalition D such that $\eta\ \mathrm{dom}_D\ \zeta$.

Definition 1.1. Let (N, v) be a game. (N, v) **has a stable core** if for all $x \in I(N, v) \backslash \mathcal{C}(N, v)$ there exists $y \in \mathcal{C}(N, v)$ such that $y \,\mathrm{dom}\, x$.

In order to state the Bondareva-Shapley Theorem (see [8] and [19]), which provides necessary and sufficient conditions for the existence of the core, the following definitions are necessary. For

$T \subseteq N$, denote by $\chi^T \in \mathbb{R}^N$ the *characteristic vector* of T, defined by

$$\chi_i^T = \begin{cases} 1 & \text{, if } i \in T, \\ 0 & \text{, if } i \in N \setminus T. \end{cases}$$

A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* (over N) if positive numbers $\delta^S, S \in \mathcal{B}$, exist such that $\sum_{S\in\mathcal{B}} \delta^S \chi^S = \chi^N$. The collection $(\delta^S)_{S\in\mathcal{B}}$ is called a system of *balancing weights* for $\mathcal{B}$. The *totally balanced cover* of (N, v), $(N, \bar{v})$, is given by

$$\bar{v}(S) = \max \left\{ \sum_{T\in\mathcal{B}} \delta^T v(T) \;\middle|\; \begin{array}{l} \mathcal{B} \text{ is a balanced collection over } S \text{ and} \\ (\delta^T)_{T\in\mathcal{B}} \text{ is system of balancing weights for } \mathcal{B} \end{array} \right\} \forall S \subseteq N. \quad (1.1)$$

The *Bondareva-Shapley* Theorem states that a game (N, v) has a nonempty core if and only if $\bar{v}(N) = v(N)$. A game will be called *balanced* if it has a nonempty core.

1.2 Fuzzy Games

In this section all necessary definitions and concepts relating to fuzzy games will be provided. Unless stated otherwise, all definitions stated here can be found in [6].

For a nonnegative vector $Q \in \mathbb{R}^N$, let $F(Q)$ be the box $F(Q) = \{c \in \mathbb{R}^N | 0 \leq c \leq Q\}$. The point Q represents the grand coalition and every $c \in F(Q)$ represents a fuzzy coalition. The *support* of a fuzzy coalition $c \in F(Q)$ is the set $supp(c) := \{i \in N \mid c_i > 0\}$. In addition, $|c|$ stands for the l^1 norm of c, that is $|c| = \sum_{i=1}^n |c_i|$ and for two vectors $x, y \in \mathbb{R}^N$, $x \cdot y = \sum_{i=1}^n x_i y_i$.

Definition 1.2. A **fuzzy game** is a pair (Q, v) so that
i) $Q \in \mathbb{R}_+^N$ and
ii) $v : F(Q) \to \mathbb{R}$ is bounded and satisfies $v(0) = 0$.

Definition 1.3. Let (Q, v) be a fuzzy game. The **set of imputations**, $I(Q, v)$, is defined as

$$I(Q, v) = \{x \in \mathbb{R}^N \mid x \cdot Q = v(Q),\ x_i Q_i \geq v(Q_i \chi^{\{i\}})\ \forall\, i \in N\}.$$

Definition 1.4. Let (Q, v) be a fuzzy game. The **core**, denoted by $\mathcal{C}(Q, v)$, is defined as

$$\mathcal{C}(Q, v) = \{x \in \mathbb{R}^N \mid x \cdot Q = v(Q),\ x \cdot c \geq v(c)\ \forall\, c \in F(Q)\}.$$

In accordance with the terminology for cooperative, TU games, if the core of a fuzzy game is not empty, then the game will be called *balanced*. (see [4] for details).

Definition 1.5. Let (Q, v) be a fuzzy game and $0 \neq c \in F(Q)$. Then $x \in \mathbb{R}^N$ is said to **dominate** $y \in \mathbb{R}^N$ via the fuzzy coalition c, $x \text{ dom}_c\ y$, if $x \cdot c \leq v(c)$ and $x_i > y_i$ for every $i \in supp(c)$. x is said to dominate y, $x \text{ dom } y$, if there exists a $0 \neq c \in F(Q)$ such that $x \text{ dom}_c\ y$.

Definition 1.6. Let (Q, v) be a fuzzy game. The fuzzy game (Q, v) has a **stable core** if, for every imputation $y \notin \mathcal{C}(Q, v)$, there exists an $x \in \mathcal{C}(Q, v)$ such that $x \text{ dom } y$.

Definition 1.7. Let (Q, v) be a fuzzy game. A fuzzy coalition $c \in F(Q)$ is **exact** if there exists $x \in \mathcal{C}(Q, v)$ such that $x \cdot c = v(c)$.

Definition 1.8. Let (Q, v) be a fuzzy game. (Q, v) has a **large core** if for every $y \in \mathbb{R}^N$ that satisfies $y \cdot c \geq v(c)$ for every $c \in F(Q)$, there exists $x \in \mathcal{C}(Q, v)$ such that $x \leq y$.

In [6] a number of relationships between the above concepts are demonstrated. The authors, however, did not consider the relationship between core stability and largeness. Here it will be shown that for fuzzy games, in contrast to cooperative, TU games, largeness of the core does not imply, in general, that a fuzzy game has a stable core, even for balanced fuzzy games (to see that core stability does not imply largeness of the core one can consider the obvious fuzzy game arising from Example one in [21]). In addition, there exist games which have a large core that is empty.

Example 1.9. Let (Q, v) be a fuzzy game defined as follows. $Q = (1, 1)$ and for $x = (x_1, x_2) \in F(Q)$

$$v(x) = \begin{cases} 2(x_1 + x_2)^2, & \text{if } 0 \leq x_1 + x_2 < 1, \\ (x_1 + x_2)^2, & \text{if } 1 \leq x_1 + x_2 \leq 2. \end{cases}$$

Then it follows that $I(Q, v) = \{\alpha(1, 3) + (1 - \alpha)(3, 1) \mid 0 \leq \alpha \leq 1\}$. Also for $\zeta \in \mathbb{R}^2$,

$$\zeta \cdot x \geq v(x) \; \forall \, x \in F(Q) \iff \zeta_i \geq 2, \; i = 1, 2.$$

This will now be proven. To prove the if direction, let $\zeta \in \mathbb{R}^2$ and let $\zeta_i \geq 2, \; i = 1, 2$. Then for all $x \in F(Q)$ it follows that $\zeta \cdot x \geq 2(x_1 + x_2) \geq v(x)$. So, to prove the other direction let $x^1 = (x_1, 0)$ and $x^2 = (0, x_2)$ for $0 < x_i < 1$, $i = 1, 2$. Then $\zeta \cdot x^i = \zeta_i x_i \geq 2x_i^2$ for $i = 1, 2$ implies that $\zeta_i \geq 2x_i$ for all $0 < x_i < 1$ and $i = 1, 2$, hence $\zeta_i \geq 2$ for $i = 1, 2$. Therefore, $\mathcal{C}(Q, v) = \{(2, 2)\}$ and the game (Q, v) has a large core. However, the game (Q, v) does not have a stable core, as not a single imputation $y \neq (2, 2)$ can be dominated by $(2, 2)$. To demonstrate this, choose $\eta \in I(Q, v) \backslash \mathcal{C}(Q, v)$ and assume, without loss of generality, that $\eta_1 < 2$. If $x \in F(Q)$ were to exist such that $(2, 2) \, \mathrm{dom}_x \, \eta$, then it would follow that $supp(x) = \{1\}$. However, if $x = (1, 0)$, then it follows that $2 = (2, 2)(1, 0) > v((1, 0)) = 1$ and if $x = (x_1, 0)$ with $0 < x_1 < 1$, then $(2, 2)(x_1, 0) = 2x_1 > 2x_1^2 = v(x)$.

Example 1.10. By letting $Q = 1$ and $v(x) = x + 1$ for all $x \in [0, 1)$ and $v(1) = 1$, it follows that the fuzzy game (Q, v) has a large core that is empty.

1.3 The Concavification of the Coalition Function

In [5], the authors introduced the concavification of the coalition function. In this section, it will be shown that the fuzzy game defined by the concavification preserves core stability of the original cooperative, TU game.

To define the concavification, let $\mathbb{H} := \{x \in \mathbb{R}^N \mid 0 \leq x_i \leq 1 \; \forall \, i \in N\}$.

Definition 1.11. Let (N, v) be a game. The **concavification of** v, $\hat{v}$, is defined as the minimum of all concave, positively homogeneous functions $f : \mathbb{H} \to \mathbb{R}$ such that $f(\chi^S) \geq v(S)$ for every coalition S.

In [5], the authors prove the following.

Proposition 1.12. Let (N, v) be a game and $\hat{v}$ the concavification of v. For every $q \in \mathbb{H}$,

$$\hat{v}(q) = \max\{\sum_{S\subseteq N} \alpha_S v(S) \mid \sum_{S\subseteq N} \alpha_S \chi^S = q, \alpha_S \geq 0\}.$$

Let (N, v) be a game. Note that the concavification of v, $\hat{v}$, coincides with the totally balanced cover of v, $\bar{v}$, on all χ^S, that is, for a coalition S, $\hat{v}(\chi^S) = \bar{v}(S)$. For the next definition, note that $F(\chi^N) = \mathbb{H}$.

Definition 1.13. Let (N, v) be a game and $\hat{v}$ the concavification of v. The fuzzy game $(\chi^N, \hat{v})$ is called **the concave extension of** (N, v).

Remark 1.14. The reader should note that the concave extension does not "extend" the coalition function v over $\mathbb{H}$, however, the totally balanced cover of v over $\mathbb{H}$. To avoid clumsy formulations, however, the fuzzy game $(\chi^N, \hat{v})$ will still be referred to as the concave extension of (N, v).

For a game (N, v), the concavification of the coalition function v is piecewise linear, positively homogeneous and totally balanced (see [5] and [23]). Here it will also be proven to be continuous on $\mathbb{H}$ (in [5] the authors mention that the concavification is continuous but give no proof of this claim). To this end, four definitions and a proposition, which will also be relevant for later sections, will be given here. These definitions can be found in, e.g., [9]. Throughout, let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$.

Definition 1.15. A **correspondence** is a map $\varphi : X \to 2^Y$. One writes $\varphi : X \rightsquigarrow Y$.

Definition 1.16. A correspondence $\varphi : X \rightsquigarrow Y$ is **lower hemi-continuous** (l.h.c.) at $x \in X$ if for every $y \in \varphi(x)$ and all sequences $\{x^t\}_{t\in\mathbb{N}}$ in X with $x^t \to \bar{x}$, there exists a sequence $\{y^t\}_{t\in\mathbb{N}}$ with $y^t \to \bar{y}$ so that $y^t \in \varphi(x^t)$ for all t. φ is l.h.c. if it is l.h.c. at all $x \in X$.

Definition 1.17. A correspondence $\varphi : X \rightsquigarrow Y$ is **upper hemi-continuous** (u.h.c.) at x if, for every neighborhood $V \supseteq \varphi(x)$, there exists a neighborhood U of x such that $\varphi(y) \subseteq V$ for all $y \in U \cap X$. φ is u.h.c. if it is u.h.c. at all $x \in X$.

To simplify a number of proofs in this paper, the following proposition will be useful (see, e.g., [9]).

Proposition 1.18. Let $\varphi : X \rightsquigarrow Y$ be a correspondence and let Y be compact. Then $\varphi : X \rightsquigarrow Y$ is **upper hemi-continuous** (u.h.c.) at $\bar{x} \in X$, if for all sequences $\{x^t\}_{t\in\mathbb{N}}$ in X with $x^t \to \bar{x}$ and all sequences $\{y^t\}_{t\in\mathbb{N}}$ with $y^t \to \bar{y}$ and $y^t \in \varphi(x^t)$ for all t, it follows that $\bar{y} \in \varphi(\bar{x})$.

Definition 1.19. A correspondence $\varphi : X \rightsquigarrow Y$ is **continuous** at $x \in X$ if φ is both u.h.c. and l.h.c. at x. φ is continuous if it is continuous at all $x \in X$.

It will now be proven that the concavification of a coalition function is continuous. To do so, it will be shown that the following correspondence is continuous, an interesting result in its own right. Let $\Phi : \mathbb{H} \to \mathbb{R}^{2^N\setminus\{\emptyset\}}$ be defined by,

$$\Phi(x) = \{(\alpha_S)_{S\in 2^N\setminus\{\emptyset\}} \mid \alpha_S \geq 0, \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S \chi^S = x\}. \tag{1.2}$$

Remark 1.20. In the proof the following simple fact will be used. For a finite set I and $\alpha_i \in \mathbb{R}_+$, $i \in I$, and $\gamma \in \mathbb{R}_+$, if $\gamma \geq \max_{i\in I} \alpha_i$, then $\sum_{i\in I}(\alpha_i - \gamma)_+ = 0$. If there exists $i^* \in I$ with $\alpha_{i^*} \geq \gamma$, then $\sum_{i\in I}(\alpha_i - \gamma)_+ \leq \sum_{i\in I\setminus i^*} \alpha_i + \alpha_{i^*} - \gamma = \sum_{i\in I} \alpha_i - \gamma$.

Remark 1.21. In addition, one notices immediately that the correspondence Φ is from $\mathbb{H}$ to a compact subset W of $\mathbb{R}^{2^N\setminus\{\emptyset\}}$, whereby

$$W := \{\eta \in \mathbb{R}^{2^N\setminus\{\emptyset\}} \mid \eta = (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}}, 0 \leq \alpha_S \leq 1 \; \forall \; S \in 2^N\setminus\{\emptyset\}\}.$$

Lemma 1.22. Φ is a continuous correspondence.

Proof:
1$^{\text{st}}$STEP :
It will first be proven that Φ is a l.h.c. correspondence. Take $\bar{x} \in \mathbb{H}$ and $\eta = (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}} \in \Phi(\bar{x})$. For $x \in \mathbb{H}$, let $\delta^x = \max_{i\in N}(\bar{x}_i - x_i)$ and define for all coalitions S, $|S| \geq 2$, $\alpha_S^x = (\alpha_S - \delta^x)_+$. Then, by Remark 1.20, it follows, for all $i \in N$, that one can choose $\alpha_{\{i\}}^x \geq 0$, so that $\sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S^x \chi^S = x$. Let $x^t \to \bar{x}$ and let $\eta^t := (\alpha_S^{x^t})_{S\in 2^N\setminus\{\emptyset\}}$, with the $\alpha_{\{i\}}^{x^t}$ chosen so that $\sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S^{x^t} \chi^S = x^t$. Then it follows immediately that $\eta^t \to \eta$ and $\eta^t \in \Phi(x^t)$ for all $t \in \mathbb{N}$.
2$^{\text{nd}}$STEP :
To prove that Φ is an u.h.c. correspondence, let $\bar{x} \in \mathbb{H}$, let $x^t \to \bar{x}$ and let $\eta^t := (\alpha_S^t)_{S\in 2^N\setminus\{\emptyset\}} \to \eta := (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}}$ with $\eta^t \in \Phi(x^t)$ for all $t \in \mathbb{N}$. Then for all $\epsilon > 0$ one can choose $M \in \mathbb{N}$ such that for all $k > M$ it follows that $|\eta^k - \eta| < \frac{\epsilon}{2|N|}$ and $|\bar{x} - x^k| < \frac{\epsilon}{2}$. Hence, for all $\epsilon > 0$ there exists a $k \in \mathbb{N}$, such that

$$\begin{aligned}|\bar{x} - \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S \chi^S| &= |\bar{x} - \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S^k \chi^S + \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S^k \chi^S - \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S \chi^S| \\ &\leq |\bar{x} - x^k| + |\sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S^k - \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S||S| \leq \frac{\epsilon}{2} + |\eta^k - \eta||N| < \epsilon.\end{aligned}$$

Whence it follows that $\eta \in \Phi(\bar{x})$ **q.e.d.**

To demonstrate that the concavification is continuous one other property of the correspondence Φ needs to be demonstrated.

Definition 1.23. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ and let $\varphi : X \rightsquigarrow Y$ be a correspondence. φ is **compact valued** if $\varphi(x)$ is compact for all $x \in X$.

Lemma 1.24. The correspondence Φ is compact valued.

Proof: Let $x \in \mathbb{H}$. It follows from Remark 1.21 that $\Phi(x)$ is bounded. To demonstrate that $\Phi(x)$ is closed, let $\{\eta^t\}_{t\in\mathbb{N}}$, $\eta^t := (\alpha_S^t)_{S\in 2^N\setminus\{\emptyset\}}$, be a sequence such that $\eta^t \in \Phi(x)$, for all t, and $\eta^t \to \eta$. Let $\eta = (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}}$ and let $\bar{x} = \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S \chi^S$. Then for all $\epsilon > 0$ it follows that there exists an $M \in \mathbb{N}$ such that for all $k > M$, $k \in \mathbb{N}$,

$$\epsilon > |\eta^k - \eta||N| \geq \sum_{S\in 2^N\setminus\{\emptyset\}} |\alpha_S^k - \alpha_S||S| = \sum_{S\in 2^N\setminus\{\emptyset\}} |\alpha_S^k \chi^S - \alpha_S \chi^S| = |x - \bar{x}|.$$

Hence, one has that $\eta \in \Phi(x)$. **q.e.d.**

An application of the following theorem, first proven in [7], will then be used to show that, for a game (N, v), the concavification $\hat{v}$ is continuous.

Theorem 1.25. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ and let $\varphi : X \rightsquigarrow Y$ be a compact valued correspondence. Let $f : Y \to \mathbb{R}$ be continuous. Define $\mu : X \rightsquigarrow Y$ by

$$\mu(x) = \{y \in \varphi(x) \mid y \text{ maximizes } f \text{ on } \varphi(x)\}$$

and $F : X \to \mathbb{R}$ by $F(x) = f(y)$ for $y \in \mu(x)$. If φ is continuous at x, then F is continuous at x.

Corollary 1.26. Let (N, v) be a game and $\hat{v}$ the concavification of v. Then $\hat{v}$ is continuous.

Proof: For $\eta := (\alpha_S)_{S \in 2^N \setminus \{\emptyset\}} \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, define the function $f : \mathbb{R}^{2^N \setminus \{\emptyset\}} \to \mathbb{R}$ by $f(\eta) = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S \chi^S$. As Φ defined by Equation 1.2 is a compact valued, continuous correspondence and f is clearly continuous, it follows from the previous theorem that $\hat{v}$ is continuous. **q.e.d.**

The main reason for studying the concave extension is because of a property that it shares with the totally balanced cover of a cooperative game. That is, a balanced game (N, v) has a stable core if and only if its concave extension $(\chi^N, \hat{v})$ has a stable core. This result will now be proven. Note that $I(\chi^N, \hat{v}) = I(N, v)$ for balanced games.

Theorem 1.27. Let (N, v) be a balanced game and let $(\chi^N, \hat{v})$ be the concave extension of (N, v). Then (N, v) has a stable core if and only if $(\chi^N, \hat{v})$ has a stable core.

Proof:
1$^{\text{st}}$STEP :
For the only if implication, let (N, v) have a stable core. If $x \in I(\chi^N, \hat{v}) \setminus \mathcal{C}(\chi^N, \hat{v})$, then there exists $0 \neq q \in \mathbb{H}$ such that $x \cdot q < \hat{v}(q)$. This implies, however, that there exists a coalition S such that $x(S) < v(S)$. As the game (N, v) has a stable core and $x \in I(N, v)$, there exists a $y \in \mathcal{C}(N, v)$ and a coalition R satisfying $v(R) = \hat{v}(\chi^R)$[1] and $y \operatorname{dom}_R x$. Now $y(P) \geq v(P)$ for all $P \subseteq N$ and hence, $y \cdot q \geq \hat{v}(q)$ for all $q \in \mathbb{H}$. Therefore, $y \in \mathcal{C}(\chi^N, \hat{v})$ and $(\chi^N, \hat{v})$ has a stable core.
2$^{\text{nd}}$STEP :
It will now be proven that if the game $(\chi^N, \hat{v})$ has a stable core, then (N, v) has a stable core. So let $x \in I(N, v) \setminus \mathcal{C}(N, v)$. Then there exists a coalition S such that $x(S) < v(S)$. Hence, $x \cdot \chi^S < \hat{v}(\chi^S)$ and therefore $x \notin \mathcal{C}(\chi^N, \hat{v})$ and $x \in I(\chi^N, \hat{v})$. As the game $(\chi^N, \hat{v})$ has a stable core, there exists $y \in \mathcal{C}(\chi^N, \hat{v})$ and a non zero $q \in \mathbb{H}$ such that $y \operatorname{dom}_q x$. Then there exists a coalition S such that $y(S) = v(S)$ (see Proposition 3.11 in [21]) and $S \subseteq \mathit{supp}(q)$. Now $y_i > x_i$ for all $i \in S$ and therefore, $y \operatorname{dom}_S x$ and one also has that $y \in \mathcal{C}(N, v)$. Hence, (N, v) has a stable core. **q.e.d.**

Remark 1.28. One should note that the fuzzy game defined by the Choquet extension of a TU game (see, e.g., [12]) also satisfies the previous theorem (the proof is even simpler than the one just given). The Choquet extension, however, is concave if and only if the coalition function v is convex (see [16]). As concavity plays an important role for the rest of this paper, this justifies the current author's negligence of the coming analysis for the Choquet extension of a TU game.

1.4 The Superdifferential of $\hat{v}$

Let (N, v) be a game and let $(\chi^N, \hat{v})$ be the concave extension of (N, v). In this section properties of the superdifferential of $\hat{v}$, $\partial \hat{v}$, will be investigated. In the last section of this paper, this analysis

[1] As demonstrated in [14], domination can always be achieved via a coalition S with $v(S) = \bar{v}(S)$.

will be used to characterize when the concave extension $(\mathbb{R}^N_+, \hat{v})$ over $\mathbb{R}^N_+$ of a game (N, v) (see below) has a stable core.

Remark 1.29. Let (N, v) be a game. To continue the analysis, the domain of the concavification $\hat{v}$ of v will be extended over $\mathbb{R}^N_+$ via its homogeneity. By a certain abuse of notation, the extension of $\hat{v}$ over $\mathbb{R}^N_+$ will also be denoted by $\hat{v}$. So, let $q \in \mathbb{R}^N_+$, $q \neq 0$. Then $\hat{v}(q) := q(N)\hat{v}(\frac{q}{q(N)})$, where the $\hat{v}$ on the left hand side is defined on $\mathbb{R}^N_+$ and the $\hat{v}$ on the right hand side is the concavification defined on $\mathbb{H}$. To make it clear that the extension over $\mathbb{R}^N_+$ is being considered the notation $(\mathbb{R}^N_+, \hat{v})$ will be used. One can then consider the pair $(\mathbb{R}^N_+, \hat{v})$ as an extended type of fuzzy game, which will be called **the concave extension over** $\mathbb{R}^N_+$ of a game (N, v). For the concave extension over $\mathbb{R}^N_+$ of a game (N, v), the core and the set of imputations will be defined as follows, $\mathcal{C}(\mathbb{R}^N_+, \hat{v}) := \mathcal{C}(\chi^N, \hat{v})$ and $I(\mathbb{R}^N_+, \hat{v}) := I(\chi^N, \hat{v})$. Notice that for $x \in \mathcal{C}(\mathbb{R}^N_+, \hat{v})$, for all $q \in \mathbb{R}^N_+$, $q \neq 0$, one has $x \cdot q = q(N)x \cdot \frac{q}{q(N)} \geq q(N)\hat{v}(\frac{q}{q(N)}) = \hat{v}(q)$.

Remark 1.30. By applying the definitions of domination and a stable core for fuzzy games to the concave extension $(\mathbb{R}^N_+, \hat{v})$ over $\mathbb{R}^N_+$ of a game (N, v), one notices, because of the homogeneity of $\hat{v}$, that the concave extension $(\mathbb{R}^N_+, \hat{v})$ over $\mathbb{R}^N_+$ has a stable core if and only if the game $(\chi^N, \hat{v})$ has a stable core.

The notation in the next definition follows that of, e.g., [18].

Definition 1.31. A vector $x \in \mathbb{R}^N$ is a **supergradient** of a concave function $f : \mathbb{R}^N_+ \to \mathbb{R}$ at a point $q \in \mathbb{R}^N_+$ if

$$f(z) \leq f(q) + x \cdot (z - q) \; \forall \, z \in \mathbb{R}^N_+. \tag{1.3}$$

The set of all supergradients of f at q is called the **superdifferential** of f at q and is denoted by $\partial f(q)$.

Proposition 1.32. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then

$$\partial \hat{v}(q) = \{x \in \mathbb{R}^N \mid x \cdot q = \hat{v}(q), \; x \cdot p \geq \hat{v}(p) \; \forall p \in \mathbb{R}^N_+\}.$$

Proof: Let $x \in \partial \hat{v}(q)$. To prove that $x \cdot p \geq \hat{v}(p)$ for all $p \in \mathbb{R}^N_+$, take $z = p + q$ in Equation (1.3). Then from the homogeneity and concavity of the concavification, it follows that

$$\hat{v}(p) + \hat{v}(q) \leq \hat{v}(p + q) \leq \hat{v}(q) + x \cdot (p + q - q)$$

and hence, $x \cdot p \geq \hat{v}(p)$. To prove that $x \cdot q = \hat{v}(q)$, take $z = \frac{q}{2}$ in Equation (1.3) and one has

$$\hat{v}(\frac{q}{2}) \leq \hat{v}(q) + x \cdot (\frac{q}{2} - q).$$

From this, and the homogeneity of $\hat{v}$, it follows that $x \cdot q \leq \hat{v}(q)$, but as $x \cdot p \geq \hat{v}(p)$ for all $p \in \mathbb{R}^N_+$, one has $x \cdot q = \hat{v}(q)$. To show the other implication, let $y \in \{x \in \mathbb{R}^N \mid x \cdot p \geq \hat{v}(p) \; \forall p \in \mathbb{R}^N_+, \; x \cdot q = \hat{v}(q)\}$. Then it follows from $y \cdot q = \hat{v}(q)$ and $y \cdot z \geq \hat{v}(z)$ for all $z \in \mathbb{R}^N_+$ that $y \cdot z - y \cdot q \geq \hat{v}(z) - \hat{v}(q)$. Hence, $y \cdot (z - q) + \hat{v}(q) \geq \hat{v}(z)$. **q.e.d.**

The following result, for super additive games, appears in a slightly different setting, e.g., in the first edition of [2], p. 213.

Corollary 1.33. Let (N, v) be a balanced game and let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. Then $\partial \hat{v}(\chi^N) = \mathcal{C}(N, v)$.

For later results it will also be necessary to demonstrate that for all $q \in \mathbb{R}^N_+$, $\partial\hat{v}(q) \neq \emptyset$. To do so the following proposition is necessary, which can be found in [18] (*int* stands for the interior of a set).

Proposition 1.34. Let $f : \mathbb{R}^N_+ \to \mathbb{R}$ be a closed[2], concave function and let $q \in int\mathbb{R}^N_+$. Then $\partial f(q)$ is a nonempty, compact set.

Lemma 1.35. Let (N, v) be a game and $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. Then for all $q \in \mathbb{R}^N_+$, $\partial\hat{v}(q) \neq \emptyset$.

Proof: By Proposition 1.34, $\partial\hat{v}(r) \neq \emptyset$ for all $r \in int\mathbb{R}^N_+$. Let $q \in \mathbb{R}^N_+ \backslash int\mathbb{R}^N_+$. Because $\hat{v}$ is piece-wise linear, it follows that there exists a $\delta > 0$ so that for all $t \in \mathbb{R}^N_+$ with $|t - q| < \delta$ one has, for all $0 \leq \alpha \leq 1$,

$$\alpha\hat{v}(t) + (1 - \alpha)\hat{v}(q) = \hat{v}(\alpha t + (1 - \alpha)q).$$

Let $r \in int\mathbb{R}^N_+$, $|r - q| < \delta$ and define $p = \frac{1}{2}r + \frac{1}{2}q \in int\mathbb{R}^N_+$. Let $y \in \partial\hat{v}(p)$. Then one has, from the linearity of y and $\hat{v}$, $\frac{1}{2}y \cdot r + \frac{1}{2}y \cdot q = \frac{1}{2}\hat{v}(r) + \frac{1}{2}\hat{v}(q)$, however, $y \cdot r \geq \hat{v}(r)$ and $y \cdot q \geq \hat{v}(q)$, hence, one can conclude that $y \in \partial\hat{v}(q)$. **q.e.d.**

Another well-known, simple result (cf. [18]), which will be useful later on, is the following.

Proposition 1.36. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then $\partial\hat{v}(q)$ is a convex set.

To prove some more results about $\partial\hat{v}$, which will be used in the last section of this paper, a characterization of $\hat{v}$ will now be given. In the following, let (N, v) be a game and

$$U(N, v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S),\ \forall\ S \subseteq N\}.$$

Proposition 1.37. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then

$$\hat{v}(q) = \min_{x \in U(N,v)} \{x \cdot q\}. \tag{1.4}$$

Proof: Let $q \in \mathbb{R}^N_+$. By Lemma 1.35, for all $q \in \mathbb{R}^N_+$, $\partial\hat{v}(q) \neq \emptyset$. However, all $y \in \partial\hat{v}(q)$ are such that $y \cdot q = \hat{v}(q)$ and $y \cdot p \geq \hat{v}(p)$ otherwise. Hence, all such y are elements of $U(N, v)$ and satisfy $\hat{v}(q) = y \cdot q = \min_{x \in U(N,v)}\{x \cdot q\}$. **q.e.d.**

Let $extC$ stand for the extreme points of a set $C \subseteq \mathbb{R}^N_+$. As is well-known, for each $q \in \mathbb{R}^N_+$, the minimum in Equation (1.4) is attained by an extreme point of the set $U(N, v)$. Note also that the number of extreme points of $U(N, v)$ is finite (see [13]). Let $H \in extU(N, v)$ and let

$$\hat{x} := \sum_{\{S \subseteq N \,:\, v(S) = H(S)\}} \chi^S. \tag{1.5}$$

Then, from Equation (1.4),

$$\hat{v}(\hat{x}) = H \cdot \hat{x} < G \cdot \hat{x}\ \forall\ G \in extU(N, v) \backslash \{H\} \tag{1.6}$$

Via the continuity of $\hat{v}$, this allows one, for $q \in \mathbb{R}^N_+$, to rewrite $\hat{v}$ in the following form.

$$\hat{v}(q) = \min_{H \in extU(N,v)} \{H \cdot q\}. \tag{1.7}$$

[2] A concave function $f : \mathbb{R}^N_+ \to \mathbb{R}$ is closed if the set $\{(p, a) \in \mathbb{R}^N_+ \times \mathbb{R} \mid a \leq f(p)\}$ is closed.

Using the previous results, a proposition concerning the structure of the set $\partial\hat{v}(q)$ for $q \in \mathbb{R}^N_+$ will be proven, which will be useful later on. Before that can be done, some well-known results, which will also be relevant for later analysis, will be presented. In the following, let $convH(C)$ stand for the convex hull of a set C (the following proposition and definition can be found in, e.g., [18]).

Proposition 1.38. Let $C \subseteq \mathbb{R}^N$ be a compact, convex set. Then $C = convH(ext\, C)$.

Definition 1.39. A hyperplane $\mathcal{H}$ to a set $C \subseteq \mathbb{R}^N$ is **supporting** if C is contained in one of the closed half spaces defined by $\mathcal{H}$ and also $\mathcal{H} \cap C \neq \emptyset$.

As the set $extU(N, v)$ is finite, it follows that for each extreme point, x, of $U(N, v)$ there is a supporting hyperplane through x which contains no other point of $U(N, v)$ (i.e. extreme points are also exposed, see [15]). To simplify the statement of the next result, let

$$\mathcal{J}(q) := \{H \in extU(N, v) \mid H \cdot q = \hat{v}(q)\}. \tag{1.8}$$

Proposition 1.40. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in int\mathbb{R}^N_+$. Then $\partial\hat{v}(q) = convH(\mathcal{J}(q))$.

Proof: Let $q \in int\mathbb{R}^N_+$. First of all, it is clear from the definition of $\hat{v}$, in Equation (1.7), that $\partial\hat{v}(q) \supseteq \mathcal{J}(q)$. By Proposition 1.34 and Proposition 1.36, it follows that $\partial\hat{v}(q)$ is a nonempty, compact, convex set. By Proposition 1.38, it follows that this set can be written as the convex hull of its extreme points. Assume now, per absurdum, that there exists an extreme point $\eta \in ext\,\partial\hat{v}(q)$ such that $\eta \neq H$ for all $H \in \mathcal{J}(q)$. Because η is an extreme (and hence exposed) point of $U(N, v)$ there exists a vector $y \in \mathbb{R}^N$ (which defines a supporting hyperplane intersecting only η in $\partial\hat{v}(q)$) so that $\eta \cdot y > H \cdot y$ for all $H \in \mathcal{J}(q)$. In addition, as $\hat{v}$ is continuous and all $H \in extU(N, v)\backslash\mathcal{J}(q)$ are also continuous (considered as functions of $q \in \mathbb{R}^N_+$) and also $\hat{v}(q) < \min_{H \in extU(N,v)\backslash\mathcal{J}(q)}\{H \cdot q\}$, it follows that there exists a $\delta > 0$ so that for $p \in N_\delta(q) := \{z \in \mathbb{R}^N_+ \mid |z - q| < \delta\}$ one has $\hat{v}(p) < \min_{H \in extU(N,v)\backslash\mathcal{J}(q)}\{H \cdot p\}$. Let $\epsilon > 0$ so that $q - \epsilon y \in N_\delta(q)$. Because $\eta \cdot q = H \cdot q$ for all $H \in \mathcal{J}(q)$, it follows that

$$\eta \cdot (q - \epsilon y) < \min_{H \in \mathcal{J}(q)}\{H \cdot (q - \epsilon y)\} = \hat{v}(q - \epsilon y).$$

A contradiction, because $\eta \in \partial\hat{v}(q)$. **q.e.d.**

For the general case, see [18][3].

Remark 1.41. Note that one can show more than the result in the previous proposition. Let $q \in int\mathbb{R}^N_+$. Then, from Equations (1.5) and (1.6), it follows that $ext\partial\hat{v}(q) = \mathcal{J}(q)$.

1.5 The Inverse Domination Correspondence

In this section the question of core stability will be considered from the perspective of nonlinear analysis. In addition, a closely related correspondence will be investigated and important properties of this correspondence will be proven. The focus of this section is on the following correspondence.

[3] That is, $\partial\hat{v}(q) = K(q) + convH(\mathcal{J}(q))$, where $K(q)$ is the normal cone at q.

Definition 1.42. Let (N, v) be a game. For an imputation x, let

$$\operatorname{dom} x := \{y \in I(N, v) \mid y \operatorname{dom} x\}.$$

In particular, one can say that the game (N, v) has a stable core if and only if for all $x \in I(N, v) \backslash \mathcal{C}(N, v)$, it follows that $\operatorname{dom} x \cap \mathcal{C}(N, v) \neq \emptyset$, i.e., if and only if $0 \in \operatorname{dom} x - \mathcal{C}(N, v)$. As it stands, this result is not very useful. In the following, however, it shall be rewritten in a form which is amenable to the techniques of nonlinear analysis. To do so, the extension of $\operatorname{dom} x$ over $\mathbb{R}_+^N$, for the concave extension $(\mathbb{R}_+^N, \hat{v})$ over $\mathbb{R}_+^N$ of a game (N, v), will be considered. The extended definition of $\operatorname{dom} x$ is clear (one can now dominate via $q \in \mathbb{R}_+^N$ and not just via the χ^S, $S \subseteq N$). To begin the analysis some lemmata are required.

Lemma 1.43. Let (N, v) be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$ and let $q \in \mathbb{R}_+^N$ be an exact coalition. Then for all $\lambda > 0$ it follows that

$$\hat{v}(q + \lambda\chi^N) = \hat{v}(q) + \lambda\hat{v}(\chi^N).$$

Proof: Let $q \in \mathbb{R}_+^N$ be an exact coalition. Then there exists an $x \in \mathcal{C}(\mathbb{R}_+^N, \hat{v})$ such that $x \cdot q = \hat{v}(q)$. Let $\lambda > 0$. Then, by homogeneity and concavity of $\hat{v}$,

$$\hat{v}(q + \lambda\chi^N) \leq x \cdot (q + \lambda\chi^N) = x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N) \leq \hat{v}(q + \lambda\chi^N)$$

and the result follows. **q.e.d.**

Using this, one can now prove the following result.

Lemma 1.44. Let (N, v) be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$, let $q \in \mathbb{R}_+^N$ be an exact coalition and let $\lambda > 0$. Then

$$\partial\hat{v}(q + \lambda\chi^N) = \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N).$$

Proof: Let $q \in \mathbb{R}_+^N$ be an exact coalition and let $\lambda > 0$. If $x \in \partial\hat{v}(q + \lambda\chi^N)$, then $x \cdot (q + \lambda\chi^N) = \hat{v}(q + \lambda\chi^N)$, whence one has that $x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ from the exactness of q and the previous lemma. However, $x \cdot q \geq \hat{v}(q)$ and $x \cdot \chi^N \geq \hat{v}(\chi^N)$, which imply that $x \cdot q = \hat{v}(q)$ and $x \cdot \chi^N = \hat{v}(\chi^N)$ (and $x \cdot p \geq \hat{v}(p)$ otherwise, as $x \in \partial\hat{v}(q + \lambda\chi^N)$). It then follows that $x \in \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N)$. The other implication is clear, as if $x \cdot q = \hat{v}(q)$ and $x \cdot \chi^N = \hat{v}(\chi^N)$ (and $x \cdot p \geq \hat{v}(p)$ otherwise), then $x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ and, from the previous result, it follows that, because q is an exact coalition, $\hat{v}(q + \lambda\chi^N) = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ and hence, $x \cdot (q + \lambda\chi^N) = \hat{v}(q + \lambda\chi^N)$. Whence $x \in \partial\hat{v}(q + \lambda\chi^N)$. **q.e.d.**

Utilizing this result, one can now formulate necessary and sufficient conditions for core stability of the the concave extension $(\mathbb{R}_+^N, \hat{v})$ over $\mathbb{R}_+^N$ of a game (N, v). Before that is done, some more notation needs to be introduced. For two sets $C_1, C_2 \subseteq \mathbb{R}_+^N$, let $C_1 - C_2 := \{x - y \mid x \in C_1, y \in C_2\}$. In addition, define the following set for a game (N, v) and its concave extension $(\mathbb{R}_+^N, \hat{v})$ over $\mathbb{R}_+^N$.

Definition 1.45. Let (N, v) be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$, let $x \in \mathbb{R}^N$ and $q \in \mathbb{R}_+^N$.

$$D(q, x) := \{y \in I(\mathbb{R}_+^N, \hat{v}) \mid y \cdot q = \hat{v}(q), y_i \geq x_i \;\forall\, i \in \mathit{supp}(q)\}.$$

The desired result is as follows (for a set C, riC stands for the relative interior of C).

Theorem 1.46. Let (N, v) be a game and let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. Then $(\mathbb{R}^N_+, \hat{v})$ has a stable core if and only if for all $x \in I(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ there exists a non-zero, exact $q \in \mathbb{R}^N_+$ and $\lambda > 0$ so that

$$0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N).$$

Proof: First of all, if the game $(\mathbb{R}^N_+, \hat{v})$ has a stable core, then for all imputations $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ there exists a non-zero $q \in \mathbb{R}^N_+$ and $y \in \mathcal{C}(\mathbb{R}^N_+, \hat{v}) \cap \mathrm{dom}_q\, x$. Hence, q is exact, $y \in riD(q, x)$ and $y \in \partial\hat{v}(q)$. Therefore, for $\lambda > 0$, $y \in \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N) = \partial\hat{v}(q + \lambda\chi^N)$, by Lemma 1.44, and hence, $0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N)$. To prove the other implication, assume that $x \in I(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{C}(\mathbb{R}^N_+, \hat{v})$. Then there exists a non zero, exact $q \in \mathbb{R}^N_+$ and $\lambda > 0$ such that $0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N)$. That is, there exists a $z \in \partial\hat{v}(q + \lambda\chi^N)$ such that $z\, \mathrm{dom}_q\, x$, from which the concave extension $(\mathbb{R}^N_+, \hat{v})$ over $\mathbb{R}^N_+$ has a stable core, as by Lemma 1.44, $z \in \mathcal{C}(\mathbb{R}^N_+, \hat{v})$. **q.e.d.**

The reason for the interest in such a result is based on the properties of the following correspondence, called the inverse domination correspondence. Let (N, v) be a game and $(\mathbb{R}^N_+, \hat{v})$ its concave extension over $\mathbb{R}^N_+$, let $x \in I(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ and let $\lambda > 0$ be fixed. Define for $q \in \mathbb{R}^N_+$ the correspondence $F_x : \mathbb{R}^N_+ \rightsquigarrow \mathbb{R}^N$ by

$$F_x(q) := D(q, x) - \partial\hat{v}(q + \lambda\chi^N). \tag{1.9}$$

Note that the choice of λ in the definition is irrelevant.

Definition 1.47. Let $X \subseteq \mathbb{R}^n_+$ and $Y \subseteq \mathbb{R}^m_+$. Let $\varphi : X \rightsquigarrow Y$ be a correspondence. φ is **convex valued** if $\varphi(x)$ is convex for all $x \in X$.

Let (N, v) be a game, $(\mathbb{R}^N_+, \hat{v})$ its concave extension over $\mathbb{R}^N_+$ and let $x \in \mathbb{R}^N$. As is clear, $D(\cdot, x)$ is a correspondence taking compact and convex values. Note also that for $q \in int\mathbb{R}^N_+$, by Proposition 1.40, $\partial\hat{v}$ is also a nonempty, compact and convex valued correspondence. Another important property shared by both correspondences, $\partial\hat{v}$ and $D(\cdot, x)$, is that they are u.h.c. correspondences.

Proposition 1.48. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in int\mathbb{R}^N_+$. Then $\partial\hat{v}(q)$ is u.h.c. at q.

Proof: Let $q \in int\mathbb{R}^N_+$ and let $\mathcal{J}(q)$ be defined as in Equation (1.8). As $\hat{v}$ is continuous and all $H \in extU(N, v) \backslash \mathcal{J}(q)$ are also continuous (considered as functions of $q \in \mathbb{R}^N_+$) and also $\hat{v}(q) < \min_{H \in extU(N,v) \backslash \mathcal{J}(q)} \{H \cdot q\}$, it follows that there exists a $\delta > 0$ so that for $p \in N_\delta(q) := \{z \in \mathbb{R}^N_+ \mid |z - q| < \delta\}$ one has $\hat{v}(p) < \min_{H \in extU(N,v) \backslash \mathcal{J}(q)} \{H \cdot p\}$. Hence, one has for all $p \in N_\delta(q) \cap int\mathbb{R}^N_+$ (by Proposition 1.40) that $\partial\hat{v}(p) \subseteq \partial\hat{v}(q)$. **q.e.d.**

For the general case, see [2].

Proposition 1.49. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $x \in \mathbb{R}^N$. Then $D(\cdot, x)$ is a u.h.c. correspondence.

Proof: Note that the range of $D(\cdot, x)$ is contained in $I(\mathbb{R}^N_+, \hat{v})$, a compact set. Let $q \in \mathbb{R}^N_+$. If $x \cdot q > \hat{v}(q)$, then $D(q, x) = \emptyset$ and the result clearly follows. Hence, let $x \cdot q \leq \hat{v}(q)$ and let $\{q^t\}_{t \in \mathbb{N}}$ be a sequence in $\mathbb{R}^N_+$ such that $q^t \to q$ and $\{y^t\}_{t \in \mathbb{N}}$ be a sequence such that $y^t \to y$ and $y^t \in D(q^t, x)$ for all t. Hence, one clearly has $y_i \geq x_i$ for all $i \in supp(q)$ and also that

$y \in I(\mathbb{R}^N_+, \hat{v})$ (because there exists $\bar{t} \in \mathbb{N}$ such that for all $t > \bar{t}$, $y^t_i \geq x^t_i$, for all $i \in supp(q)$, and $y^t \in I(\mathbb{R}^N_+, \hat{v})$). Because $\hat{v}$ is a continuous function, it follows that for $q^t \to q$ one has $\hat{v}(q^t) \to \hat{v}(q)$ and because $y^t \to y$ and $y^t q^t = \hat{v}(q^t)$ for all $t \in \mathbb{N}$, one can conclude that $yq = \hat{v}(q)$.
q.e.d.

Finally, it also important to note that for two u.h.c. correspondences φ_1 and φ_2 that one also has the following result (cf. [9]).

Proposition 1.50. Let $\varphi_1 : X \rightsquigarrow Y$ and $\varphi_2 : X \rightsquigarrow Y$ be u.h.c. correspondences. Then $\varphi_1 - \varphi_2$ is also u.h.c.

Proof: Let O be a neighborhood of $\varphi_1(x) - \varphi_2(x)$ and let O_1 be a neighborhood of $\varphi_1(x)$ and O_2 a neighborhood of $\varphi_2(x)$ such that $O_1 - O_2 \subseteq O$. Then there exist neighborhoods U_1 and U_2 of x so that $\varphi_1(x^1) \subseteq O_1$ for all $x^1 \in U_1$ and $\varphi_2(x^2) \subseteq O_2$ for all $x^2 \in U_2$. Choose a neighborhood, $U \subseteq U_1, U_2$, of x and it follows that $\varphi_1(z) - \varphi_2(z) \subseteq O$ for all $z \in U$. **q.e.d.**

Corollary 1.51. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. Let $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ be an imputation, let $\lambda > 0$ and let $F_x : \mathbb{R}^N_+ \rightsquigarrow \mathbb{R}^N$ be defined as in Equation (1.9). Then F_x is a nonempty, u.h.c., compact and convex valued correspondence.

Proof: As $\partial\hat{v}(q)$ is a u.h.c. correspondence for all $q \in int\mathbb{R}^N_+$ and for two compact (convex) sets $C_1, C_2 \in \mathbb{R}^N_+$, $C_1 - C_2$ is also compact (convex), the corollary follows from the definition of F_x.
q.e.d.

Remark 1.52. Let (N, v) be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $\lambda > 0$. Note that if for all $x \in I(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ there exists an exact $q \in \mathbb{R}^{\mathbb{N}}_+$ such that $0 \in riF_x(q) = riD(q, x) - ri\partial\hat{v}(q + \lambda\chi^N) \subseteq riD(q, x) - \partial\hat{v}(q + \lambda\chi^N)$, then it follows that $(\mathbb{R}^N_+, \hat{v})$ has a stable core. For a characterization of exactness, see [23] (note that the statements in both [23] and [5] characterizing when a coalition is exact are actually incorrect. In addition to the conditions stated in both papers, one also requires, for a coalition S, that $\hat{v}(\chi^S) = v(S)$ for the claimed results to be correct).

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