On stochastic completeness of weighted graphs

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A Dissertation Submitted for the Degree of Doctor
at
the Department of Mathematics Bielefeld University

4 June 2011

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Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
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4 June 2011

Gedruckt auf alterungsbeständigem Papier nach DIN–ISO 9706

Abstract

In this thesis we are concerned with the long time behavior of continuous time random walks on infinite graphs. The following three related problems are considered.

1. Stochastic completeness of the random walk. We characterize the stochastic completeness of the random walk in terms of function-theoretic and geometric properties of the underlying graph.

2. Uniqueness of the Cauchy problem for the discrete heat equation in certain function classes. We provide a uniqueness class on an arbitrary graph in terms of the growth of the L^2-norm of solutions and show its sharpness. An application of this results to bounded solutions yields a criterion for stochastic completeness in terms of the volume growth with respect to a so-called adapted distance. In special cases, this leads to a volume growth criterion with respect to the graph distance as well.

3. Escape rate of the random walk. We provide upper rate functions for stochastically complete random walks in terms of the volume growth function.

Acknowledgment It is a pleasure to thank the many people who helped make this thesis possible.

First of all, I would like to express my sincere gratitude to my supervisor, Dr. Alexander Grigor'yan. He continuously supported me in various ways with his enthusiasm, knowledge, inspiration and encouragement.

During the whole procedure of writing this thesis, I benefitted from inspiring conversations with many people. Many thanks to Radek Wojciechowski, Matthias Keller, Daniel Lenz and Jun Masamune who have close interest with me and generously shared their understanding of the subject. I learned a large part of mathematics that I know from my fellow students in Bielefeld: Ante Mimica, Shunxiang Ouyang, Wei Liu, Zhe Han and Zhiwei Li. They patiently answered many (sometimes naive) questions of mine. I also had happy time discussing with Jiaxin Hu, Józef Dodziuk, and Elton Hsu. I wish to thank them in addition.

Mrs Epp and the SFB web-team helped me a lot when I met with problems in daily life. I especially appreciate their work.

I am indebted to my best friends, Yang Liu, Yi Li, Jiahua Fan and Tian Zhang for their emotional support which helped me get through the most difficult times.

The largest achievement being abroad for me is meeting a charming lady, Feng Ji who later became my wife. She provides me a loving environment and always has confidence in me. Lastly but most importantly, I wish to deeply thank my parents far away in China. They supported me throughout and taught me the philosophy of hard work and persistence. This thesis is dedicated to them.

Bielefeld, June 4, 2011 Xueping Huang

Contents

Chapter 0

Introduction

0.1 General overview

In this thesis we are concerned with long time behavior of continuous time random walks (Markov chains) on infinite graphs. We are interested in the following three related problems.

(1) Stochastic completeness of the random walk.

The random walk is stochastically complete if it has infinite lifetime with probability 1. Our results about stochastic completeness are of two kinds:

(a) characterizations of stochastic completeness using certain function-theoretic properties of the graph (the weak Omori-Yau maximum principle and the Khas'minskii criterion for graphs).

(b) relation of the stochastic completeness to the geometric properties of the underlying graph, such as bounds of degree and volume growth.

(2) Uniqueness class for the Cauchy problem for the heat equation.

Analogous to the classical Cauchy problem for the heat equation, one can define a similar problem on a graph and ask in what class of functions is the solution unique. For example, uniqueness in the class of bounded functions is equivalent to the stochastic completeness. We obtain the uniqueness class on graphs in terms of the growth of certain integrals of functions. Unlike the classical uniqueness class of Tichonov, that consists of functions bounded by $e^{c|x|^2}$, the uniqueness class on a simplest graph $\mathbb{Z}$ consists of functions bounded by $e^{\varepsilon|x|\ln|x|}$, and the class is sharp.

(3) Escape rate for random walks on a graph, that is, how far away can the random walk move in a given time t. This question only makes sense on stochastically

complete graphs. We prove upper bounds on the escape rate in terms of the volume growth of the graph.

0.2 Setup

We briefly outline the settings of this thesis. A more detailed account of the framework is provided in Chapter 1 following the work of Keller and Lenz [33].

A weighted graph is a triple (V, w, μ) where V is a countably infinite vertex set and $w(x, y)$ and $\mu(x)$ are nonnegative weight functions on $V \times V$ and V respectively such that

(1) $\mu(x) > 0$ for all $x \in V$;

(2) $w(x, x) = 0$ for all $x \in V$;

(3) $w(x, y) = w(y, x)$ for all $x, y \in V$;

(4) $\sum_{y \in V} w(x, y) < +\infty$ for all $x \in V$.

We can view μ as a measure on V and construct the function spaces $l^p(V, \mu)$ in the usual way. The function w defines an edge set E by

$$x \sim y \Leftrightarrow w(x, y) > 0, \quad \text{and} \quad E = \{(x, y) \in V \times V : x \sim y\},$$

that equips V with an undirected, simple (i.e. without loops and multiedges), infinite graph structure. *Throughout the thesis, all graphs will be assumed to be of this type.* We call x and y neighbors if $x \sim y$ holds. When the underlying graph (V, E) is connected, there is a natural graph distance ρ on V, namely, the length of the shortest path between every two points.

An analogue of the classical Laplacian on Euclidean spaces or more generally on Riemannian manifolds, the so-called formal Laplacian Δ ([33]) on a weighted graph (V, w, μ) can be constructed as

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(f(x) - f(y)) \tag{0.2.1}$$

where f is any real valued function on V such that (0.2.1) makes sense. For example, let (V, E) be a locally finite and connected graph. It is natural to consider the weight function w with $w(x, y) = 1$ for $x \sim y$, and $w(x, y) = 0$ otherwise. Then there are two natural choices of μ (and consequently, Δ) on V.

(1) The normalized (or combinatorial) Laplacian case: $\mu(x) = \deg(x)$, where $\deg(x) = \#\{y \in V : y \sim x\}$ is the number of neighbors of x. And

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y\in V, y\sim x} (f(x) - f(y)).$$

(2) The so-called physical Laplacian case (named by Weber [53]): μ is the counting measure (i.e. $\mu(x) \equiv 1$), and hence

$$\Delta f(x) = \sum_{y\in V, y\sim x} (f(x) - f(y)).$$

Let Δ_0 be the restriction of Δ to the space $C_c(V)$ of finitely supported functions on V and let L be the Friedrichs extension of Δ_0 on $l^2(V, \mu)$. The (minimal) heat semigroup $\{P_t\}_{t\geq 0}$ on (V, w, μ) is constructed as

$$P_t = \exp(-tL),$$

and can be extended from $l^2(V, \mu)$ to $l^\infty(V)$, the space of bounded functions on V.

A weighted graph (V, w, μ) (or the formal Laplacian Δ) is called stochastically complete if

$$\text{for all } \ t > 0, \quad P_t 1 = 1.$$

One reason why the stochastic completeness problem is interesting is that it is related to a number of other equivalent properties and can be investigated from different points of view. In the analytic aspect, an equivalent property is that for some/any $T > 0$, the Cauchy problem of the heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0, \\ \lim\limits_{t\to 0+} u(x,t) = 0, \end{cases} \tag{0.2.2}$$

has only zero solution in the class of bounded functions on $V \times (0, T]$.

From the probabilistic point of view, the heat semigroup P_t serves as the transition semigroup of a minimal, reversible, continuous time Markov chain $\{X_t\}_{t\geq 0}$ on V which can be naturally constructed from the weight functions w and μ. The stochastic completeness of (V, w, μ) is equivalent to the non-explosion of $\{X_t\}_{t\geq 0}$, that is, for all $(x, t) \in V \times (0, \infty)$,

$$\mathbb{P}_x (X_t \in V) = 1.$$

Let d be a distance function on V, such that all d-balls are finite. If the weighted graph (V, w, μ) is stochastically complete then one can ask whether there exists a function $R(t)$ on $[0, \infty)$ such that for some $x_0 \in V$,

$$\mathbb{P}_{x_0}\{d(X_t, x_0) \leq R(t) \text{ for all sufficiently large t}\} = 1.$$

Such a function $R(t)$ is called an upper rate function and can be considered as a quantitative way of understanding the stochastic completeness.

Now return to the two examples of weights on a locally finite and connected graph (V, E). The normalized Laplacian, is not interesting for the point of view of stochastic completeness. Due to the boundedness of Δ, the corresponding random walk is always stochastically complete in this case.

In the pioneering work of Dodziuk [8] and Dodziuk, Matthai [9], they first considered the question of stochastic completeness of the physical Laplacian which turned out particularly interesting. Inspired by the work of Dodziuk and Matthai, Weber [53] and Wojciechowski [54] independently studied the stochastic completeness of the physical Laplacian in depth. Following their terminology, we will call a graph (V, E) stochastically complete if so is the physical Laplacian on it. Keller and Lenz [33, 34] first considered the stochastic completeness problem for general weighted graphs and set up a convenient framework based on the theory of Dirichlet forms.

0.3 Main results

For simplicity, we state all results for the physical Laplacian on graphs, whereas in the main body we prove them in a more general form for weighted graphs. In this section, Δ denotes the physical Laplacian and μ denotes the counting measure.

We first study the weak Omori-Yau maximum principle for graphs which provide a convenient equivalent condition for stochastic completeness.

Theorem 0.1 (=Theorem 2.1.2)**.** *A graph (V, E) is said to satisfy the weak Omori-Yau maximum principle if for every nonnegative bounded function f on V with and for every $\alpha > 0$,*

$$\sup_{\Omega_\alpha} \Delta f \geq 0,$$

where

$$\Omega_\alpha = \{x \in V : f(x) > \sup_V f - \alpha\}.$$

Then a locally finite and connected graph (V, E) is stochastically complete if and only if it satisfies the weak Omori-Yau maximum principle.

Remark 0.3.1. Pigola, Rigoli, and Setti [42, 43] first studied the weak Omori-Yau maximum principle on manifolds, based on earlier work by Omori [41] and Yau [57].

As a consequence, we obtain a simple proof of the Khas'minskii criterion for graphs. For the classical Khas'minskii criterion on manifolds, see Khas'minskii [36].

Theorem 0.2 (=Theorem 2.3.1). *Let (V, E) be a locally finite and connected graph. If there exists a nonnegative function γ on V such that*

$$\gamma(x) \to +\infty$$

as x leaves every finite set and

$$\Delta\gamma(x) + \lambda\gamma(x) \geq 0$$

outside a finite set, then (V, E) is stochastically complete.

An application of the Khas'minskii criterion for graphs is the following theorem.

Theorem 0.3 (=Theorem 2.5.4). *Let (V, E) be a locally finite and connected graph. Let $f \in C^1([0, +\infty))$ be some positive, increasing function such that*

$$\int_0^{+\infty} \frac{dr}{f(r)} = +\infty.$$

If for some fixed $x_0 \in V$,

$$\Delta\rho(x_0, x) \geq -f(\rho(x_0, x))$$

outside a finite set, then (V, E) is stochastically complete.

Remark 0.3.2. Previously, a similar result was proved by Weber [53] for $f = \text{const}$. Both results are analogous to the curvature type criteria for stochastic completeness of manifolds by Yau [58] and later Ichihara [30], Varapoulos [52] and Hsu [27].

To obtain criteria for stochastic completeness in terms of the volume growth function, we need introduce the notion of adapted distances. Our key observation is that the volume growth with respect to the graph distance is not an adequate quantity for the stochastic completeness problem. An essential feature of the geodesic distance on a Riemannian manifold is that

$$|\nabla d(x_0, x)| \leq 1 \tag{0.3.3}$$

which is important in constructing cut-off functions with controlled energy density.

A natural analogue of (0.3.3) for a distance d on a graph (V, E) is that

$$\sum_{y\in V, y\sim x} d^2(x,y) \leq 1 \tag{0.3.4}$$

for all $x \in V$. In the physical Laplacian case, a short calculation shows that generally the graph distance ρ does not satisfy (0.3.4). This observation naturally leads us to adopt a notion of adapted distances on weighted graphs to the stochastic completeness problem. The following definition is first introduced by Frank, Lenz and Wingert [15] in the more general setting of nonlocal Dirichlet forms when studying spectral properties (they use the phrase "intrinsic metric" instead).

Definition 0.4 (=Definition 1.6.2)**.** Let (V, E) be a locally finite and connected graph. We call a distance d on V adapted if (0.3.4) holds for all $x \in V$ and $d(x, y) \leq 1$ whenever $x \sim y$.

Such type of distances always exist on a connected graph. Let (V, E) be a locally finite and connected graph. Define a function $\sigma(x, y)$ for all pairs of neighbors $x \sim y$ by

$$\sigma(x,y) = \min\left\{\frac{1}{\sqrt{\deg(x)}}, \frac{1}{\sqrt{\deg(y)}}, 1\right\}. \tag{0.3.5}$$

It naturally induces a distance d on X as follows: for all pairs of distinct points x, y,

$$d(x,y) := \inf\{\sum_{i=0}^{n-1}\sigma(x_i, x_{i+1}) : x_0 = x, x_n = y, \forall 0 \leq i \leq n-1, x_i \sim x_{i+1}\}. \tag{0.3.6}$$

Our main result is the following.

Theorem 0.5 (=Theorem 4.1)**.** *Let (V, E) be a locally finite and connected graph. Let d be an adapted distance on (V, E). Assume that for some point $x_0 \in V$, for some constants $C > 0$ and $0 < c < \frac{1}{2}$, the volume of balls $\mu(B_d(x_0, r))$ satisfies*

$$\mu(B_d(x_0,r)) \leq C\exp\left(cr\ln r\right), \tag{0.3.7}$$

for all $r > 0$ large enough. Then (V, E) is stochastically complete.

Note that for a geodesically complete Riemannian manifold M, if

$$\int^{\infty} \frac{rdr}{\ln \mathrm{vol}(B\left(x_0, r\right))} = \infty \tag{0.3.8}$$

for some $x_0 \in M$, then M is stochastically complete. This sharp volume growth criterion is due to Grigor'yan [19]. The borderline of stochastic completeness and incompleteness for manifolds lies around e^{cr^2} type volume growth in contrast to (0.3.7). We do not know whether Theorem 0.5 is sharp or not for graphs.

It is desirable to have criteria for stochastic completeness in terms of the volume function relative to the graph distance ρ. Wojciechowski [56] first showed that for each $\varepsilon > 0$, there are stochastically incomplete graphs called anti-trees with $cr^{3+\varepsilon}$ type volume growth with respect to the graph distance. Our result here is as follows.

Theorem 0.6 (=Theorem 4.2.2)**.** *Let (V, E) be a locally finite and connected graph. If for some point $x_0 \in V$, and some constant $c > 0$,*

$$\mu(B_\rho(x_0, r)) \leq cr^3 \tag{0.3.9}$$

for all $r \in \mathbb{N}_+$, then (V, E) is stochastically complete.

Interestingly, the theorem above is proven as a corollary of Theorem 0.5. Unlike the manifold case, the borderline between stochastic completeness and incompleteness goes on cubic volume line rather than quadratic exponential one. This is also part of our motivation to consider adapted distances.

As it was already mentioned above, the stochastic completeness is equivalent to the Cauchy problem in the class of bounded functions. In fact, we obtain our Theorem 0.5 as a consequence of a more general result about uniqueness class.

Theorem 0.7 (=Theorem 3.1)**.** *Let (V, E) be a locally finite and connected graph. Let d be an adapted distance on V such that all d-balls are finite. Let $u(x, t)$ be a solution to the Cauchy problem (0.2.2) on $V \times [0, T]$ for some $T > 0$. If there are an increasing sequence of positive numbers $\{R_n\}_{n\in\mathbb{N}}$ with*

$$\lim_{n\to\infty} R_n = +\infty,$$

and two constants $C > 0$, $0 < c < \frac{1}{2}$ such that for some $x_0 \in V$,

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x, t)\mu(x)dt \leq C \exp\left(cR_n \ln R_n\right), \tag{0.3.10}$$

then $u(x, t) \equiv 0$ on $V \times [0, T]$.

The proof uses the approach of Grigor'yan [19] via the integrated maximum principle for solutions of the heat equation. However, due to the discreteness of the Laplacian, a direct application of the method of [19] does not work. The key point of the proof is a new integrated maximum principle specific to the graph setting, that is stated in Lemma 3.1.1.

The uniqueness class given by (0.3.10) is sharp up to the constant c. We will show in Section 3.3 that on the simplest graph $\mathbb{Z}$ there are nonzero solutions $u(x,t)$ to the Cauchy problem (0.2.2) with $\exp(c_1 R \ln R)$ type growth for some $c_1 > 0$. For this counterexample, we use the approach of Tichonov [51] to construct a solution $u(n,t)$ of the heat equation on $\mathbb{Z}$ with $u(n,0) \equiv 0$ in the form

$$u(n,t) = \begin{cases} g(t), & n = 0, \\ g(t) + \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1), & n \geq 1, \\ u(-n-1,t), & n \leq -1, \end{cases} \tag{0.3.11}$$

where

$$g(t) = \exp\left(-\frac{1}{t^2}\right).$$

However, due to the discreteness of the setting, the resulting solution is entirely different from that of Tichonov and grows at a much slower rate $\exp(c_1 R \ln R)$.

Remark 0.8. It is a classical problem to find the uniqueness classes in the setting of heat equation on Euclidean spaces, see for example the work of Tichonov [51] and Täcklind [50]. Our uniqueness class (0.3.10) in the integrated form is more in the spirit of the work of Oleinik and Radkevich [40], Gushchin [26] in the Euclidean case, and Grigor'yan [19] in the manifold case.

In the probabilistic approach, we study the upper rate function of the escape rate for the continuous time random walk corresponding to the physical Laplacian.

Theorem 0.9. *Let (V,E) be a locally finite and connected graph with an adapted distance d. Let $\{X_t\}_{t\geq 0}$ be the continuous time Markov chain associated with the physical Laplacian.*

(1) If for some constant $0 < c < \frac{1}{2}$ and for all $r \geq 2$,

$$\mu(B_d(x_0,r)) \leq \exp(cr\ln r),$$

Then for any $a > \frac{1}{1-2c}$, there is some constant $C > 0$ such that

$$R(t) = Ct^a \ln t$$

is an upper rate function for $\{X_t\}_{t\geq 0}$.

(2) If for some constant $M > 0$ such that

$$\mu(B_d(x_0,r)) \leq \exp Mr$$

for all $r \geq 2$*, then there exists some constant* $C > 0$ *such that the inverse function* $\psi^{-1}(t)$ *of*

$$\psi(R) = C \int_8^R \frac{rdr}{f(r) + \ln\ln(r)} \tag{0.3.12}$$

is an upper rate function for $\{X_t\}_{t\geq 0}$.

In the case that the graph has at most exponential type volume growth, our result (0.3.12) coincides with the recent results on manifolds by Hsu and Qin [28]. Previously, similar results are obtained by Grigor'yan and Hsu [24] for Cartan-Hardamad manifolds. However for $\exp(cr \ln r)$ type volume growth, our result is different.

It remains unclear whether this is a technical difference or essential. In the view of the sharpness of Theorem 0.7 about uniqueness class, one could expect that for some $c > 0$, $\exp(cr \ln r)$ is a borderline case for stochastic completeness and the escape rate in this case is sharp. However, we still do not have evidence for that. Alternatively, it could happen that our method for proving stochastic completeness is not sharp enough and the volume growth criterion (0.3.8) still holds if we consider adapted distances. By calculations of concrete examples, we are more inclined to the latter possibility.

0.4 Structure of the thesis

We now briefly review the contents of the main chapters not explicitly mentioned above. In the first chapter we survey the foundations for our work: the analytical framework of Keller and Lenz [33] and the construction of the minimal continuous time Markov chain. Theorem 1.5.1 gathers various equivalent conditions for stochastic incompleteness. A more detailed comparison between adapted distances and the graph distance is given via examples. The next chapter is devoted to an alternative approach to the geometric criteria for stochastic completeness (stochastic incompleteness) of Wojciechowski [54, 55, 56] and Weber [53]. We develop the weak Omori-Yau maximum principle and prove the Khas'minskii criterion for general weighted graphs. Combining them, we are able to give a unified approach to many know geometric criteria together with simpler proofs. Stability of stochastic incompleteness of weighted graphs is discussed in Section 2.4. Chapter 3 deals with the uniqueness class problem. An extension of Theorem 0.7 is proved with the help of an important technical tool, Lemma 3.1.1. We also show that Theorem 0.7 is close to be sharp by examples. In Chapter 4, we prove Theorem 0.5 and Theorem

0.6 for a class of weighted graphs similar to physical Laplacian case. The probabilistic point of view is taken up in the last chapter. We give more explicit upper rate functions for different types of volume growth functions in Theorem 5.3.

Note that the historical notes here are by no means complete. In particular, we should point out that there are extensive literature on the stochastic completeness problem in the general context of continuous time Markov chain. See for example the work of Feller [12, 13], Reuter [45] and Chung [3]. For the stochastic completeness problem on manifolds, we refer the survey paper of Grigor'yan [22] for a comprehensive historical account. We will provide more references at the beginning of each chapter as well.

Chapter 1

Foundations

This chapter is expository and nothing is claimed to be original. Most of the materials here are taken from the pioneering work [33], [34], [53], and [54] with some modifications to fit the need of this thesis. For the general theory of Dirichlet forms and the corresponding Hunt processes, we refer to the classical monograph [17]. We will also make use of results from Davies' book [4]. It is worth clarifying that many results here are presented not in their possibly more general original form in order to fit our work later on.

We first briefly summary the framework of weighted graphs set up by Keller and Lenz [33]. We refer the reader to [33] and [34] for most proofs. However, for the sake of completeness, we will include some facts such as the parabolic minimum principle Theorem 1.4.15 which do not directly appear in their paper. These proofs here are based on the ideas of Weber [53] and Keller and Lenz [33] and are *not* claimed to be new. The main theorem of this chapter is Theorem 1.5.1 which gives a big list of equivalent conditions for stochastic incompleteness. It allows us to transfer a question about heat semigroups to questions about the uniqueness of solutions to elliptic and parabolic (partial) difference equations. The conditions there are *not* new. In the setting of weighted graphs, most of them are due to Weber [53], Wojciechowski [54], Keller and Lenz [33, 34]. However, we added two more conditions ((7) and (7')) in analogue with the smooth setting. These two conditions are important for the study of uniqueness class in Chapter 3. See [22] for the equivalent conditions for stochastic incompleteness on manifolds. After the thesis was written up, we noticed that Keller, Lenz and Wojciechowski [35] just fixed the gap in the literature and covered the parabolic minimum principle and the conditions (7) and (7') in Theorem 1.5.1.

Then for preparation of the study of the uniqueness of solutions of parabolic (partial) difference equations, we introduce the notion of adapted distances and compare them with the usual graph distance. Adapted distances naturally leads to cut-off functions that fit the classical Caccioppoli type estimates which will be our key technique in Chapter 3. This notion is first introduced by Frank, Lenz and Wingert [15]. Inspired by the integrability conditions for a Lévy measure, the work of Masamune and Uemura [37] implicitly contains the same notion. Folz [14] also came up with similar ideas with the goal to obtain heat kernel estimates on weighted graphs. In the case of strongly local Dirichlet forms on distance spaces, the corresponding notion of intrinsic distances is classical and has been applied to stochastic completeness problems. See for example [49]. The first work introducing different distances on graphs seems to be Davies [6]. Based on communications with Grigor'yan and Wojciechowski, we found the idea of applying the adapted distances to the stochastic completeness problem of weighted graphs. A different notion of weighted distance has also been introduced by Colin de Verdière, Torki-Hamza and Truc [7] in the context of essential self-adjointness.

Besides the analytical aspect, we describe the probabilistic side as well, that is, the minimal right continuous Markov chain corresponding to a weighted graph. This topic is classical and there are many good monographs: Chung [3], Freedman [16], Norris [39], and Stroock [48], just to name a few. So we only briefly survey the results that we need without proof. This part will only be used in Chapter 5.

1.1 Weighted graphs

Throughout the paper, V will be a countably infinite set with the discrete topology and the associated trivial Borel $\sigma-$algebra. We denote the space of compactly supported (i.e. finitely supported) functions on V by $C_c(V)$.

Let $\mu(x) : V \to (0, \infty)$ be a positive function on V. It can also be viewed as a fully supported (Radon) measure on V. For the measure space (V, μ), we naturally associates the function spaces $l^p(V, \mu)$ for $p \in [1, \infty)$:

$$l^p(V, \mu) = \{f : V \to \mathbb{R} | \sum_{x \in V} |f(x)|^p \mu(x) < \infty\}.$$

For $p = \infty$, $l^p(V, \mu)$ is just the space of bounded functions on V and is in fact independent of μ. So it is proper to denote it by $l^\infty(V)$ for simplicity. We will use the notation

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x) \mu(x)$$

for $f \in l^p(V, \mu)$ and $g \in l^q(V, \mu)$ where $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$ (when $p = 1$, we adopt the convention that $q = \infty$). Note that obviously $C_c(V) \subseteq l^p(V, \mu)$ for all

$p \in [1, \infty]$.

To make V a weighted graph, we need another function $w : V \times V \to [0, \infty)$ such that the following holds:

(1) $w(x,x) = 0$ for all $x \in V$;

(2) $w(x,y) = w(y,x)$ for all $x, y \in V$;

(3) $\sum_{y \in V} w(x,y) < +\infty$ for all $x \in V$.

The triple (V, w, μ) is called a weighted graph in this thesis. The function w naturally induces a symmetric relation $E \subseteq V \times V$ on V, that is,

$$(x,y) \in E \Leftrightarrow w(x,y) > 0.$$

We call such a pair of x, y neighbors and denote it by $x \sim y$. Viewing V as the vertex set and putting single edges between neighbors, this gives (V, w, μ) an underlying graph structure (V, E). We often need consider subsets of V and the following definition is useful.

Definition 1.1.1. Let (V, w, μ) be a weighted graph and U is a subset of V. We define the (outer) boundary ∂U of U as

$$\partial U = \{x \in U^c : \exists y \in U, x \sim y\}.$$

And the closure $\bar{U}$ of U is defined to be

$$\bar{U} = U \cup \partial U.$$

The weighted graph (V, w, μ) is called locally finite if every vertex in V has only finite many neighbors. We call a pair of points $x \neq y$ in V connected if there is a chain of points $\{x_0, \cdots, x_n\}$ in V such that

$$x_0 = x, x_n = y, x_k \sim x_{k+1} \text{ for all } 0 \leq k \leq n-1.$$

With the convention that every point is connected with itself, this induces an e-quivalence relation on V. We call a weighted graph connected if for all pairs of points in V are connected. For the general case, we can naturally define connected components. Note that the notion of connected components also makes sense for a subset U of V by directly restricting w to $U \times U$ and viewing U as a subgraph. We will discuss the notion of subgraphs in more details later.

Remark 1.1.2. The underlying graph of a weighted graph is undirected, loop-less and without multiedges but not necessarily locally finite or connected. Such graphs are often called simple graphs. Another way to construct a weighted graph is to start from such a graph and to put weights on the vertex set and the edge set. See Example 1.1.5 below.

Remark 1.1.3. In [33], the framework of Keller and Lenz is in fact more general. They allow to include a potential term and introduce a notion of stochastic incompleteness *at infinity*. In view of the Dirichlet subgraphs introduced by them, this is in fact a more natural setting. Nevertheless, in this thesis, we adopt the more "classical" setting and focus on the stochastic completeness problem without a potential term.

In analogue with the degree function on locally finite graphs (without weights), we have a notion of weighted degree:

$$\mathrm{Deg}(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y).$$

Its meaning is clearer from the probabilistic point of view to be introduced in Section 1.7.

The main object of our study is the so called formal Laplacian Δ introduced by Keller and Lenz [33]:

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(f(x) - f(y))$$

where f is a function on V in the domain $\mathcal{D}$ of Δ:

$$\mathcal{D} = \{g : V \to \mathbb{R} | \sum_{y \in V} w(x, y)|g(y)| < \infty \ \text{ for all } \ x \in V\}.$$

The operator Δ can be viewed as a discrete version of the Laplace-Beltrami operator on Riemannian manifolds.

Remark 1.1.4. It is easy to see that $l^\infty(V) \subseteq \mathcal{D}$. When the underlying graph of (V, w, μ) is locally finite, $\mathcal{D}$ is just the space of real valued functions on V.

Example 1.1.5. Let (V, E) be an infinite, locally finite simple graph with V the vertex set and E the edge set viewed as a symmetric subset of $V \times V$. The weight function $w(x, y)$ on $V \times V$ is supported on E and satisfies that $w(x, y) = 1$ if $(x, y) \in E$. The vertex weight function $\mu(x)$ is simply defined to be identically 1. In this case the formal Laplacian is

$$\Delta f(x) = \sum_{y, y \sim x} (f(x) - f(y)). \tag{1.1.1}$$

This is the so-called physical Laplacian studied first by Dodziuk [8] and Dodziuk, Matthai [9], and then independently by Weber [53] and Wojciechowski [54]. See also the work of Keller, Lenz and Wojciechowski [35] for recent developments. The weighted degree function in this case is $\mathrm{Deg}(x) = \deg(x)$ where $\deg(x)$ is the normally defined degree function of a graph, that is, the number of neighbors of vertices. The physical Laplacian case offers a large family of weighted graphs whose stochastic completeness problem is interesting. Most of our results are aimed (though not restricted) to understand this family.

As the physical Laplacian is the best understood case, it is useful to generalize it to a more general family of weighted graphs that share most of the good properties. Direct restrictions are locally finiteness and connectedness. Further more, we introduce two assumptions on the weights.

Assumption 1.1.6. The weights on vertices of the weighted graph (V, w, μ) have a positive lower bound, namely

$$C_\mu = \inf_{x \in V} \mu(x) > 0.$$

Assumption 1.1.7. The weights on vertices and edges of the weighted graph (V, w, μ) satisfies the following relation:

$$w(x, y) \leq C_w \mu(x)\mu(y) \quad \text{for all } x, y \in X, \tag{1.1.2}$$

for some constant $C_w > 0$.

Example 1.1.8. Let (V, E) and $w(x, y)$ as in Example 1.1.5. This time we choose the vertex weight to be $\mu(x) = \deg(x)$. The formal Laplacian now becomes

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y, y \sim x} (f(x) - f(y)).$$

It is easy to see that the weighted degree function is $\mathrm{Deg}(x) \equiv 1$. This is the most common setting in the analytical study of random walks. However, from the point of view of stochastic completeness, it is not an interesting case as such weighted graphs are always stochastically complete. See [8], [9] for example.

1.2 Dirichlet forms, semigroups and resolvents

There is a natural Markovian symmetric quadratic form Q_c on $C_c(V)$ for a weighted graph (V, w, μ) defined as:

$$Q_c(u,v) = \frac{1}{2}\sum_{x\in V}\sum_{y\in V} w(x,y)\,(u(x)-u(y))\,(v(x)-v(y))\,.$$

Such a quadratic form is determined by its diagonal value. Consider its maximal extension Q_{max} on $l^2(V,\mu)$ with diagonal

$$Q_{max}(u) = Q_{max}(u,u) = \frac{1}{2}\sum_{x\in V}\sum_{y\in V} w(x,y)\,(u(x)-u(y))^2\,,$$

where the value ∞ is allowed. Viewed as a extended real valued function on $l^2(V,\mu)$, $Q_{max}(u)$ is lower semicontinuous in u by Fatou's Lemma. By Theorem 1.2.1. in [4], Q_{max} is a closed form and hence Q_c is closable. We denote the closure of Q_c by $\mathcal{E}$ with domain $\mathcal{F}$. By standard results (Theorem 3.1.1. in [17]), the pair $(\mathcal{E},\mathcal{F})$ is a Dirichlet form. By construction, it is in fact a regular Dirichlet form.

The general machinery of Dirichlet forms provides us several (families of) operators corresponding to $(\mathcal{E},\mathcal{F})$. First, there exists a unique selfadjoint operator L on $l^2(V,\mu)$ such that $\mathcal{F}$ is the domain of $L^{1/2}$ and

$$\mathcal{E}(f,f) = \langle L^{1/2}f, L^{1/2}f\rangle$$

for $f \in \mathcal{F}$. The operator L defined as above is the Friedrichs extension of Δ_0 which is the restriction of the formal Laplacian Δ to $C_c(V)$. The operator L then generates a strongly continuous semigroup

$$\{P_t = \exp(-tL), t > 0\},$$

and a strongly continuous resolvent

$$\{G_\alpha = (\alpha + L)^{-1}, \alpha > 0\}$$

on $l^2(V,\mu)$. One significant connection between the semigroup and the resolvent is

$$G_\alpha u = \int_0^\infty e^{-t\alpha} P_t u dt \tag{1.2.3}$$

for any $u \in l^2(V,\mu)$. Again by standard theory ([17]), P_t and G_α have the positivity preserving property, that is, they map nonnegative functions to nonnegative functions.

To introduce the notion of stochastic completeness, we need extend P_t to $l^\infty(V)$. This is done by first taking monotone approximations in $l^2(V,\mu)$ for nonnegative functions in $l^\infty(V)$ and then extend by linearity. Positivity preserving is essential in this extension. For details, we refer to p.49 in [17]. In fact, as in [4], P_t (and the corresponding resolvent G_α) can be extended to $l^p(V,\mu)$ for all $p \in [1,\infty]$. These semigroups and resolvents are strongly continuous for $p \in [1,\infty)$ and weak $*$ continuous for $p = \infty$. They are consistent on their common domains. So when there is no risk of confusion, we will denote all of them by P_t and G_α respectively. They are selfadjoint ([4]) in the sense that

$$P_t^{(q)} = \left(P_t^{(p)}\right)^*, G_\alpha^{(q)} = \left(G_\alpha^{(p)}\right)^*$$

for all $p \in [1,\infty)$ and $\frac{1}{p}+\frac{1}{q} = 1$. For any $p \in [1,\infty)$, as the semigroup $P_t^{(p)}$ is strongly continuous, it has a corresponding generator $L^{(p)}$ with a dense domain $\mathcal{D}(L^{(p)})$ in $l^p(V,\mu)$. The $p = \infty$ case is subtler, as the generator is defined through the resolvent

$$(\alpha + L^{(\infty)})^{-1} = \left((\alpha + L^{(1)})^{-1}\right)^*,$$

with domain $G_\alpha^{(\infty)}(l^\infty(V))$ which is not generally dense in $l^\infty(V)$. It is worth pointing out that the relation (1.2.3) remains true for $G_\alpha^{(\infty)}$ and $P_t^{(\infty)}$ on $l^\infty(V)$.

Remark 1.2.1. Note that the Dirac function

$$\delta_x(y) = \frac{1}{\mu(x)}\chi_{\{x\}}(y) \in C_c(V),$$

where χ_U is the characteristic function of a subset U of V. The semigroup P_t then has a natural kernel

$$p(t,x,y) = (P_t\delta_x)(y),$$

since

$$\langle P_t\delta_x, f\rangle = \langle \delta_x, P_t f\rangle = P_t f(x)$$

for all $f \in l^2(V,\mu), x \in V$. Through this kernel, the semigroup and the resolvent can be extended on a general class of functions including all nonnegative functions. Note also that $p(t,x,y) = p(t,y,x)$ since P_t is a bounded symmetric operator. See [33] for more details.

An immediate consequence of the weak $*$ continuity of $P_t^{(\infty)}$ is that $P_t v(x)$ is continuous in t on $(0,\infty)$ for all $x \in V$ when $v \in l^\infty(V)$. In the meanwhile,

$$\lim_{t\to 0}\left(P_t 1\right)(x) = \lim_{t\to 0}\langle \delta_x, P_t 1\rangle = \lim_{t\to 0}\langle P_t\delta_x, 1\rangle = \langle \delta_x, 1\rangle = 1,$$

for all $x \in V$ as P_t is strongly continuous on $l^1(V, \mu)$. Now we can give the definition of stochastic completeness:

Definition 1.2.2. A weighted graph (V, w, μ) is said to be stochastically complete if and only if the corresponding semigroup P_t satisfies that

$$P_t 1 = 1$$

for all $t > 0$. Otherwise (V, w, μ) is called stochastically incomplete.

1.3 Minimum principles

In this section, we introduce the elliptic minimum principle of Keller and Lenz [33] and develop a parabolic version of minimum principle. The parabolic minimum principle is certainly classical in the PDE theory. See the book [44] for example. In the setting of physical Laplacian on locally finite graphs, it is first proven independently by Weber [53] and Wojciechowski [54] in a slightly different form (maximum principle). It is already known to Keller and Lenz in the more general setting of weighted graphs though it is not stated explicitly in their paper. Here we will present a proof of the parabolic minimum principle for the sake of completeness.

Theorem 1.3.1. *(Elliptic Minimum Principle) Let (V, w, μ) be a weighted graph. Let $U \subseteq V$ be a given subset with connected components $\{U_i\}_{i \in I}$. Assume a function f on V satisfies:*

(1) $(\Delta + \alpha) f \geq 0$ on U for some $\alpha > 0$;

(2) $f|_{U_i}$ attains its minimum on each connected component U_i of U;

(3) $f \geq 0$ for all $x \in U^c$.

Then, $u \equiv 0$ or $u > 0$ on each connected component of U. In particular $u \geq 0$.

Theorem 1.3.2. *(Parabolic Minimum Principle) Let (V, w, μ) be a weighted graph. Let $U \subseteq V$ be a given finite subset and $T > 0$. Assume that a function u on $V \times [0, T]$ satisfies:*

(1) $u(x, t)$ is continuous and differentiable in t on $[0, T]$ for all $x \in U$;

(2) as a function of $x \in V$, $u(x, t) \in \mathcal{D}$ for all $t \in [0, T]$;

(3) $\frac{\partial}{\partial t} u + \Delta u \geq 0$ on $U \times [0, T]$;

(4) $u \geq 0$ *on* $U^c \times [0, T]$*, and* $u(x, 0) \geq 0$ *for all* $x \in U$*.*

Then $u \geq 0$*.*

Proof. As $U \times [0, T]$ is compact, $u|_{U\times[0,T]}$ attains its minimum at some point (x_0, t_0). If $u < 0$ at some point in $U \times [0, T]$, then

$$u(x_0, t_0) < 0.$$

Note that $u \geq 0$ on $U^c \times [0, T]$. Together with the fact that $u(x_0, t_0)$ is the minimum on $U \times [0, T]$, we have

$$\begin{aligned}
\Delta u(x_0, t_0) &= \frac{1}{\mu(x_0)} \sum_{y \in V} w(x_0, y) \left(u(x_0, t_0) - u(y, t_0)\right) \\
&= \frac{1}{\mu(x_0)} \sum_{y \in U} w(x_0, y) \left(u(x_0, t_0) - u(y, t_0)\right) \\
&+ \frac{1}{\mu(x_0)} \sum_{y \in U^c} w(x_0, y) \left(u(x_0, t_0) - u(y, t_0)\right) \\
&\leq 0.
\end{aligned}$$

We first consider the case that $\frac{\partial}{\partial t} u + \Delta u > 0$ on $U \times [0, T]$. Since $u(x, 0) \geq 0$ for all $x \in U$, we see that $t_0 \in (0, T]$ and as a consequence

$$\frac{\partial}{\partial t} u(x_0, t_0) \leq 0.$$

Hence we have

$$\Delta u(x_0, t_0) > -\frac{\partial}{\partial t} u(x_0, t_0) \geq 0.$$

A contradiction.

For the general case, consider the function

$$v_\varepsilon(x, t) = u(x, t) + \varepsilon t$$

where $\varepsilon > 0$. We have

$$\frac{\partial}{\partial t} v_\varepsilon(x, t) + \Delta v_\varepsilon(x, t) = \frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) + \varepsilon > 0.$$

The assumptions (1), (2) and (3) for u also hold for v_ε as

$$v_\varepsilon(x, 0) = u(x, 0), v_\varepsilon(x, t) \geq u(x, t).$$

From the previous argument, we see that

$$u(x,t) + \varepsilon T \geq v_\varepsilon(x,t) \geq 0.$$

As $\varepsilon > 0$ is arbitrary, the assertion follows. □

1.4 Dirichlet subgraphs

To understand the finer properties of the semigroup P_t such as minimality of $P_t 1$ and the explicit form of the generators, we need make use of the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ and develop the approximation of P_t by its restrictions on subgraphs.

For $U \subseteq V$, taking the weights to be $w|_{U\times U}, \mu|_U$, we obtain a naive definition of subgraph. The stability of stochastic incompleteness under the operation of taking subgraphs will be discussed in Chapter 2.

In this section, we mainly use a notion of Dirichlet subgraphs introduced by Keller and Lenz [33]. Denote $\mu|_U$ by μ_U. Let $i_U : l^2(U, \mu_U) \to l^2(V, \mu)$ be the canonical embedding, that is, extension by zero outside U. Let $p_U : l^2(V, \mu) \to l^2(U, \mu_U)$ be the canonical projection, that is, the adjoint of i_U. Then we have a selfadjoint operator $L_U^{(D)}$ on $l^2(U, \mu_U)$ by

$$L_U^{(D)} = p_U L i_U.$$

The operator naturally induces a Dirichlet form on $l^2(U, \mu_U)$ by

$$Q_U^{(D)}(u) = \mathcal{E}(i_U u).$$

We will denote the corresponding semigroup on $l^2(U, \mu_U)$ by $P_t^{U,(D)}$ and the resolvent by $G_\alpha^{U,(D)}$. Afterwards, for simplicity, we will omit the superscript (D) which hints that the operator $L_U^{(D)}$ is obtained by restriction to a subgraph with the Dirichlet boundary condition.

For the case that U is finite, we can easily see by direct calculation that

$$p_U L i_U = L_U = p_U \Delta i_U.$$

Recall that L is a selfadjoint operator on $l^2(V, \mu)$ while Δ is the formal Laplacian on its domain $\mathcal{D}$.

The elliptic and parabolic minimum principles introduced in the previous section imply the monotonicity structure of G_α^U and P_t^U with respect to U. More explicitly, we state the following two theorems which are due to Keller and Lenz [33] in the setting of weighted graphs.

Theorem 1.4.1. *(Elliptic domain Monotonicity) Let (V, w, μ) be a weighted graph. Let $K_1 \subseteq K_2 \subseteq V$ be given with K_1 and K_2 finite. Then, for all $f \in l^2(V, \mu)$ with $f \geq 0$,*

$$i_{K_1} G_\alpha^{K_1} p_{K_1} f \leq i_{K_2} G_\alpha^{K_2} p_{K_2} f$$

pointwise.

Theorem 1.4.1 is proven through the elliptic minimum principle Theorem 1.3.1. It essentially use that fact that

$$L_U = p_U \Delta i_U$$

when U is finite. Similarly by the parabolic minimum principle, we have a parabolic domain monotonicity theorem.

Theorem 1.4.2. *(Parabolic domain Monotonicity) Let (V, w, μ) be a weighted graph. Let $K_1 \subseteq K_2 \subseteq V$ be given with K_1 and K_2 finite. Then, for all $f \in l^2(V, \mu)$ with $f \geq 0$,*

$$i_{K_1} P_t^{K_1} p_{K_1} f \leq i_{K_2} P_t^{K_2} p_{K_2} f$$

pointwise.

Proof. Let

$$u_i = i_{K_i} P_t^{K_i} p_{K_i} f$$

for $i = 1, 2$. By the general theory of strongly continuous semigroups on Hilbert spaces, we have

$$\frac{\partial}{\partial t} \left(P_t^{K_i} p_{K_i} f \right) + L_{K_i} \left(P_t^{K_i} p_{K_i} f \right) = 0.$$

Note that here "$\frac{\partial}{\partial t}$" is the strong derivative on $l^2(K_i, \mu_{K_i})$. However, since the Dirac function $\delta_x(y) \in l^2(V, \mu)$, the above equation also holds in the pointwise sense.

Consider $L_{K_i} = p_{K_i} \Delta i_{K_i}$ and

$$p_{K_i} i_{K_i} = Id$$

on $l^2(K_i, \mu_{K_i})$, it follows that

$$p_{K_i}\left(\frac{\partial}{\partial t} u_i + \Delta u_i\right) = p_{K_i} \frac{\partial}{\partial t} \left(i_{K_i} P_t^{K_i} p_{K_i} f \right) + p_{K_i} \Delta i_{K_i} \left(P_t^{K_i} p_{K_i} f \right) = 0. \tag{1.4.4}$$

Now let $v = u_2 - u_1$. Obviously $v \geq 0$ on $K_1^c \times [0,T]$. It is also clear that

$$v(x,0) = i_{K_2} p_{K_2} f(x) - i_{K_1} p_{K_1} f(x) \geq 0.$$

By (1.4.4), for any fixed $T > 0$, we have that

$$\frac{\partial}{\partial t} v(x,t) + \Delta v(x,t) = 0 \tag{1.4.5}$$

on $K_1 \times (0,T]$. Since v is a bounded function and is continuous on $K_1 \times [0,T]$, as observed in [33], by the differential mean value theorem, (1.4.5) extends to $K_1 \times [0,T]$. (See also Remark 1.4.8.) Then $v \geq 0$ on $V \times [0,T]$ by the parabolic minimum principle Theorem 1.3.2. As $T > 0$ is arbitrarily chosen, the assertion holds. □

Remark 1.4.3. For general Dirichlet forms, domain monotonicity has already been shown in the works [46], [47] before [33].

The following theorem of Keller and Lenz [33] makes serious use of the regularity of the Dirichlet form.

Theorem 1.4.4. *Let (V, w, μ) be a weighted graph and $(\mathcal{E}, \mathcal{F})$ be the associated regular Dirichlet form. Let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of V with $V = \cup_{n\in\mathbb{N}} K_n$. Then for any $f \in C_c(V)$,*

$$\lim_{n\to\infty} i_{K_n} G_\alpha^{K_n} p_{K_n} f = G_\alpha f$$

in $l^2(V,\mu)$. The corresponding results also holds for the semigroups $P_t^{K_n}$.

In view of the domain monotonicity Theorem 1.4.1 and Theorem 1.4.2, the approximations in Theorem 1.4.4 are in fact monotone both for the resolvents and the semigroups. So for each nonnegative function $g \in l^p(V,\mu)$ for some $p \in [0,\infty]$, we can find a sequence of nonnegative functions $g_n \in C_c(V)$ monotonically increasing to g. By the construction of P_t (or G_α), the sequence $P_t g_n$ (or $G_\alpha g_n$) increases to $P_t g$ (or $G_\alpha g$). Fix a sequence of increasing finite sets $K_n \subseteq V$. Then by Theorem 1.4.2 and Theorem 1.4.4, the sequence of functions $i_{K_m} P_t^{K_m} p_{K_m} g_n$ converges monotonically to $P_t g_n$. And by Theorem 1.4.1 and Theorem 1.4.4, the sequence of functions $i_{K_m} G_\alpha^{K_m} p_{K_m} g_n$ converges monotonically to $G_\alpha g_n$. These monotone approximations allow Keller and Lenz [33] to establish finer properties of the semigroups and resolvents. Above all, they show that the generators $L^{(p)}$ with $p \in [1,\infty]$ are restrictions of Δ on their domain.

Theorem 1.4.5. *Let (V, w, μ) be a weighted graph. Let $p \in [1,\infty]$. For any $g \in$*

$l^p(V,\mu)$, $G_\alpha g$ is in the domain $\mathcal{D}$ of Δ and

$$(\Delta+\alpha)\, G_\alpha g = g.$$

As a consequence, $L^{(p)} f = \Delta f$ for any $f \in \mathcal{D}(L^{(p)})$.

Remark 1.4.6. We would like to remind that confusions might appear if we do not distinguish an operator and its certain restrictions. Take $g \in l^\infty(V)$ for example. As stated above,

$$(\Delta+\alpha)\, G_\alpha g = g,$$

since $G_\alpha g \in \mathcal{D}$ and $L^{(\infty)}$ is a restriction of Δ on its domain. However,

$$G_\alpha\, (\Delta+\alpha)\, g = g$$

may fail as g is not necessarily in the domain of $L^{(\infty)}$.

Together with the construction of P_t on $l^\infty(V)$, an immediate consequence of this explicit form of generators is the differentiability of $P_t f$ for $f \in l^\infty(V)$.

Definition 1.4.7. Let (V,w,μ) be a weighted graph. Let $u(x,t)$ be a function on $V\times[0,\infty)$ (or on $V\times[0,T]$ for some $T>0$). Then $u(x,t)$ is said to be a solution to the Cauchy problem of the heat equation with initial condition $f(x)$ on $V\times[0,\infty)$ (or on $V\times[0,T]$) if it satisfies the following conditions:

(1) as a function of $x\in V$, $u(x,t)\in\mathcal{D}$ for all $t\in[0,\infty)$ (or for all $t\in[0,T]$);

(2) as a function of $t\in[0,\infty)$ (or of $t\in[0,T]$), $u(x,t)$ is differentiable for all $x\in V$;

(3) $u(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t)+\Delta u(x,t)=0, \\ u(x,0)=f(x), \end{cases} \tag{1.4.6}$$

on $V\times[0,\infty)$ (or on $V\times[0,T]$ respectively).

Remark 1.4.8. As observed in [33], when $u(x,t)$ is assumed further to be bounded, it is enough to have that $u(x,t)$ is differentiable in t on $(0,\infty)$ and continuous in t on $[0,\infty)$, and satisfies (1.4.6) on $V\times(0,\infty)$ with $u(x,0)=f(x)$. Note that

$$\lim_{t\to0+}\frac{\partial}{\partial t}u(x,t) = -\lim_{t\to0+}\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)(u(x,t)-u(y,t)) = -\Delta f(x)$$

by Lebesgue's dominated convergence theorem. Then by the differential mean value theorem, $\frac{\partial}{\partial t}u(x,0)$ exists and satisfies that

$$\frac{\partial}{\partial t}u(x,0) + \Delta u(x,0) = 0.$$

Roughly speaking, (1.4.6) automatically extends from $(0,\infty)$ to $[0,\infty)$. In the case that the underlying graph is locally finite, this automatic extension even holds without assuming the boundedness of $u(x,t)$. The same argument applies to bounded time intervals $[0,T]$.

Theorem 1.4.9. *Let (V,w,μ) be a weighted graph. Let P_t on $l^\infty(V)$ be defined as before. For any $f \in l^\infty(V)$, the function $u(x,t) = P_t f(x)$ is differentiable in t on $[0,\infty)$ for each $x \in V$ and is a solution to the Cauchy problem of the heat equation with initial condition $f(x)$.*

Remark 1.4.10. In the setting of physical Laplacians on locally finite graphs, this result is shown by Weber [53] and Wojciechowski [54] independently. The generalization to weighted graphs is due to Keller and Lenz [33].

In particular, we can see that the function $u = 1 - P_t 1$ solves the Cauchy problem of the heat equation with zero initial condition:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + \Delta u(x,t) = 0, \\ u(x,0) = 0. \end{cases} \tag{1.4.7}$$

A direct calculation shows the following:

Lemma 1.4.11. *Let u be a bounded solution to the Cauchy problem of the heat equation with zero initial condition (1.4.7). Then for any $\alpha > 0$, the function*

$$g = \int_0^\infty e^{-t\alpha} u dt$$

satisfies that

$$(\Delta + \alpha) g = 0. \tag{1.4.8}$$

Remark 1.4.12. Let $f \in l^\infty(V)$. If the Cauchy problem of the heat equation with zero initial condition (1.4.7) has a nonzero bounded solution, we can not expect $P_t f$ to be a unique bounded solution to the Cauchy problem of the heat equation with initial condition f. There is also bounded solutions to the equation

$$(\Delta + \alpha) g = f$$

other than $G_\alpha f$.

However, if $f \in l^\infty(V)$ is nonnegative, $P_t f$ and $G_\alpha f$ have the minimal property.

Theorem 1.4.13. *Let (V, w, μ) be a weighted graph. Let $f \in l^\infty(V)$ be nonnegative. Then $G_\alpha f$ is the smallest nonnegative function such that*

$$(\Delta + \alpha) g \geq f.$$

Remark 1.4.14. This theorem is taken from [33]. In fact, the $f \in l^\infty(V)$ condition can be neglected. For details we refer to [33].

Theorem 1.4.15. *Let (V, w, μ) be a weighted graph. Let $f \in l^\infty(V)$ be nonnegative. For any $T > 0$, $P_t f$ is the smallest nonnegative solution to the Cauchy problem of the heat equation with initial condition f (1.4.6) on $V \times [0, T]$.*

Proof. Let $u(x, t)$ be another nonnegative solution. As described before, let $f_n \in C_c(V)$ be a sequence of nonnegative functions monotonically increasing to f and $K_n \subseteq V$ be a sequence of finite sets monotonically increasing to V. Define a double sequence of functions

$$g_{m,n} = i_{K_m} P_t^{K_m} p_{K_m} f_n.$$

Then for each n, $g_{m,n}$ increasingly converges to $P_t f_n$. And $P_t f_n$ increasingly converges to $P_t f$. The function $g_{m,n}$ satisfies that

$$\begin{aligned}
\frac{\partial}{\partial t} g_{m,n} &= \frac{\partial}{\partial t} i_{K_m} P_t^{K_m} p_{K_m} f_n \\
&= -i_{K_m} L^{K_m} P_t^{K_m} p_{K_m} f_n \\
&= -i_{K_m} p_{K_m} \Delta i_{K_m} P_t^{K_m} p_{K_m} f_n \\
&= -i_{K_m} p_{K_m} \Delta g_{m,n}.
\end{aligned}$$

And the initial value of $g_{m,n}$ is

$$g_{m,n}(x, 0) = i_{K_m} p_{K_m} f_n(x).$$

Consider $v_{m,n} = u - g_{m,n}$. We see that

$$\begin{aligned}
p_{K_m} \frac{\partial}{\partial t} v_{m,n} + p_{K_m} \Delta v_{m,n} &= -p_{K_m} \frac{\partial}{\partial t} g_{m,n} - p_{K_m} \Delta g_{m,n} \\
&= p_{K_m} i_{K_m} p_{K_m} \Delta g_{m,n} - p_{K_m} \Delta g_{m,n} \\
&= 0,
\end{aligned}$$

as $p_{K_m} i_{K_m} = Id$. Hence
$$\frac{\partial}{\partial t} v_{m,n} + \Delta v_{m,n} = 0$$
on $K_m \times [0,T]$. It is clear that $v_{m,n} \geq 0$ on $K_m^c \times [0,T]$ and

$$v_{m,n}(x,0) = u(x,0) - g_{m,n}(x,0) = f(x) - i_{K_m} p_{K_m} f_n \geq 0.$$

By the parabolic minimum principle Theorem 1.3.2, we have $v_{m,n} \geq 0$, that is,

$$u(x,t) \geq g_{m,n}(x,t)$$

on $V \times [0,T]$. It follows that

$$u(x,t) \geq P_t f_n(x).$$

Finally we can conclude that
$$u(x,t) \geq P_t f(x)$$
on $V \times [0,T]$. □

1.5 The equivalence theorem

After all these preparations, we can obtain the following big list of equivalent conditions for stochastic incompleteness. It is the starting point of our further investigation in this thesis.

Theorem 1.5.1. *Let (V, w, μ) be a weighted graph. The following statements are equivalent:*

(1) The weighted graph (V, w, μ) is stochastically incomplete. In other words, there is some $t > 0$ and some $x \in V$ such that $P_t 1(x) < 1$.

(2) The function $\int_0^\infty e^{-t\alpha}(1 - P_t 1)dt$ is nonzero for any $\alpha > 0$.

(2') The function $\int_0^\infty e^{-t\alpha}(1 - P_t 1)dt$ is nonzero for some $\alpha > 0$.

(3) For any $\alpha > 0$, there is a nonzero, nonnegative bounded function $g(x)$ on V such that
$$(\Delta + \alpha)\, g = 0.$$

(3') For some $\alpha > 0$, there is a nonzero, nonnegative bounded function $g(x)$ on V such that
$$(\Delta + \alpha)\, g = 0.$$

(4) For any $\alpha > 0$, there is a nonzero, bounded function $g(x)$ on V such that

$$(\Delta + \alpha)\, g = 0.$$

(4') For some $\alpha > 0$, there is a nonzero, bounded function $g(x)$ on V such that

$$(\Delta + \alpha)\, g = 0.$$

(5) For any $\alpha > 0$, there is a nonzero, nonnegative bounded function $g(x)$ on V such that

$$(\Delta + \alpha)\, g \leq 0.$$

(5') For some $\alpha > 0$, there is a nonzero, nonnegative bounded function $g(x)$ on V such that

$$(\Delta + \alpha)\, g \leq 0.$$

(6) There exists a nonzero, nonnegative bounded function $u(x,t)$ on $V \times [0,\infty)$ such that u solves the Cauchy problem of the heat equation with zero initial condition (1.4.7).

(6') There exists a nonzero, bounded function $u(x,t)$ on $V \times [0,\infty)$ such that u solves the Cauchy problem of the heat equation with zero initial condition (1.4.7).

(7) For any $T > 0$, there exists a nonzero, bounded function $u(x,t)$ on $V \times [0,T]$ such that u solves the Cauchy problem of the heat equation with zero initial condition (1.4.7).

(7') For some $T > 0$, there exists a nonzero, bounded function $u(x,t)$ on $V \times [0,T]$ such that u solves the Cauchy problem of the heat equation with zero initial condition (1.4.7).

Proof. $(1) \Rightarrow (2)$: This follows from the fact that $P_t1(x)$ is continuous in t for all $x \in V$.

$(2) \Rightarrow (2'), (3) \Rightarrow (3'), (4) \Rightarrow (4'), (5) \Rightarrow (5'), (7) \Rightarrow (7')$: Obvious.

$(2) \Rightarrow (3), (2') \Rightarrow (3')$: The $1 - P_t1$ is a nonnegative bounded solution to the Cauchy problem of the heat equation with zero initial condition (1.4.7). So by Lemma 1.4.11, the function $\int_0^\infty e^{-t\alpha}(1 - P_t1)dt$ is a nonnegative bounded solution to

$$(\Delta + \alpha)\, g = 0.$$

And by assumption, it is nonzero.

$(3) \Rightarrow (4), (3') \Rightarrow (4'), (6) \Rightarrow (6'),$: Obvious.

$(4) \Rightarrow (5), (4') \Rightarrow (5')$: Let f be a nonzero bounded function on V and satisfies

$$(\Delta + \alpha) f = 0.$$

Divide f into positive and negative parts as $f = f_+ - f_-$. If $f_+ = 0$, then $-f$ is a nonzero, nonnegative bounded solution to

$$(\Delta + \alpha) g = 0.$$

Otherwise, f_+ is a nonzero, nonnegative bounded function. For $x \in V$, if $f_+(x) = 0$, then it is clear that $f_+(x) - f_+(y) \le 0$ for any $y \in V$. Otherwise $f_+(x) > 0$, then for all $y \in V$,

$$f_+(x) - f_+(y) = f(x) - f_+(y) \le f(x) - f(y).$$

Hence f_+ satisfies that for all $x \in V$,

$$(\Delta + \alpha) f_+(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(f_+(x) - f_+(y)) + \alpha f_+(x) \le 0.$$

$(5) \Rightarrow (2), (5') \Rightarrow (2')$: Let g be a nonzero, nonnegative bounded function on V such that

$$(\Delta + \alpha) g \le 0.$$

Without loss of generality, we can assume that $g \le 1$. Then we have that $g' = 1 - g$ is nonnegative bounded and satisfies

$$(\Delta + \alpha) g' \ge \alpha.$$

Let $f = \int_0^\infty \alpha e^{-t\alpha}(1 - P_t 1)dt$. Then $0 \le f \le 1$ and

$$f' = 1 - f = \int_0^\infty \alpha e^{-t\alpha} P_t 1 dt = G_\alpha(\alpha 1).$$

By Theorem 1.4.13, we have $f' \le g'$ and hence $g \le f$. Since g is nonzero and nonnegative, f is nonzero.

$(4') \Rightarrow (7)$: Consider the functions $u(x, t) = e^{\alpha t} g(x)$ and $v(x, t) = P_t g(x)$ where g is as in $(4')$. Then it is easy to see that they are both bounded solutions to the Cauchy problem of the heat equation with initial condition g on $V \times [0, T]$ for any $T > 0$. On the other hand, for any $t > 0$,

$$\sup_{x \in V} |v(x, t)| \le \sup_{x \in V} |g(x)| < e^{\alpha t} \sup_{x \in V} |g(x)| = \sup_{x \in V} |u(x, t)|,$$

as g is nonzero and bounded. So the function $w(x,t) = u(x,t) - v(x,t)$ is a nonzero bounded solution to the Cauchy problem of the heat equation with zero initial condition on $V \times [0,T]$ for any $T > 0$.

$(7') \Rightarrow (1)$: Let $u(x,t)$ be a nonzero bounded solution to the Cauchy problem of the heat equation with zero initial condition on $V \times [0,T]$. Without loss of generality, we can assume that

$$\sup_{x\in V, t\in[0,T]} u(x,t) > 0, \quad \sup_{x\in V, t\in[0,T]} |u(x,t)| < 1.$$

So the function $v = 1 - u$ is a positive solution to the Cauchy problem of the heat equation with initial condition 1 and

$$\inf_{x\in V, t\in[0,T]} v < 1.$$

By Theorem 1.4.15, we have

$$P_t 1 \le v.$$

Thus for some $(x,t) \in V \times [0,T]$, $P_t1(x) < 1$.

$(1) \Rightarrow (6)$: As we already showed before, $1 - P_t 1$ is a nonzero, nonnegative bounded solution to the Cauchy problem of the heat equation with zero initial condition on $V \times [0,\infty)$.

$(6') \Rightarrow (7)$: Obvious. □

Remark 1.5.2. This list of equivalent conditions for stochastic incompleteness is classical in the smooth setting. The proof given here is a combination of those in [33] and [22].

Remark 1.5.3. Theorem 1.5.1 translates the stochastic incompleteness problem to the uniqueness problem of certain linear (partial) difference equations. In Chapter 2, we will make use of the equivalent conditions related to the equation

$$(\Delta + \alpha)\, g = 0,$$

and the inequality

$$(\Delta + \alpha)\, g \le 0.$$

In Chapter 4, the conditions related to the Cauchy problem of the heat equation with zero initial condition will be used extensively.

1.6 The graph distance and adapted distances

The main technical tool we adopt in the study of the Cauchy problem of the heat equation with zero initial condition is a priori estimates. More explicitly, we use a discrete analogue of the classical Caccioppoli type estimate. To briefly explain the idea, let f be a function in the domain $\mathcal{D}$ of the formal Laplacian Δ on a weighted graph (V, w, μ). Let η be a finitely supported function on V as a "cut off" function.

Lemma 1.6.1.

$$-\sum_{x\in V}\Delta f(x)\cdot f(x)\eta^2(x)\mu(x) \le \frac{1}{2}\sum_{x\in V} f^2(x)\sum_{y\in V} w(x,y)\left(\eta(x)-\eta(y)\right)^2.$$

We omit the proof here as it is essentially contained in the proof of Lemma 3.1.1. Intuitively, the left hand side of the above inequality corresponds to the "local mass"

$$\sum_{x\in V} f^2(x)\eta^2(x)\mu(x)$$

if we assume further $(\Delta+\alpha)f = 0$ for some $\alpha > 0$, or the rate of change of it if f also depends on time t and satisfies

$$(\frac{\partial}{\partial t}+\Delta)f = 0.$$

The right hand side will be a multiple of the "local mass" as well if we have

$$\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)\left(\eta(x)-\eta(y)\right)^2 \le C \tag{1.6.9}$$

for some $C > 0$. This leads to a quantitative comparison of the "local mass" of f at different space(-time) regions. In practice, η is often constructed as a tent function of the form

$$\eta(x) = C(R - d(x, x_0))_+$$

with respect to a distance d on V. Our task in this section is to compare the distances on a weighted graph to determine those that fulfill our need.

For a connected weighted graph, we can introduce a natural graph distance with respect to its underlying graph structure. Let x and y be two distinct points in V. We call a sequence of points $x_0, \cdots, x_n$ a chain connecting x and y if $x_0 = x, x_n = y$ and $x_i \sim x_{i+1}$ for all $i = 0, 1, ..., n-1$. The number n is called the length of this chain. A natural graph distance ρ can be defined on X as the minimal length of chains connecting two distinct points. It is easy to see that the graph distance is finer than the discrete metric. However, the graph distance can not distinguish different weighted graphs with the same underlying graph structure.

We will make use of the graph distance in the study of the physical Laplacian in Chapter 2. So we introduce some notations for preparation. We fix a point $x_0 \in V$ as a reference point of the graph and define

$$r(x) = \rho(x, x_0).$$

A key feature of the graph distance is that if $x \sim y$, then

$$|r(x) - r(y)| \leq 1.$$

We use further the following notations for a locally finite connected weighted graph.

$$S_R = \{y \in V : r(y) = R\},$$

$$B_R = \cup_{n=0}^{R} S_n = \{y \in V : r(y) \leq R\},$$

$$m_{\pm}(x) = \#\{y : y \sim x, r(y) = r(x) \pm 1\},$$

$$K_{\pm}(r) = \max_{x \in S_r} m_{\pm}(x),$$

and

$$k_{\pm}(r) = \min_{x \in S_r} m_{\pm}(x),$$

that have clear geometric meanings.

Tent functions constructed from the graph distance generally do not satisfy (1.6.9) as it does not see the quantitative information of weights. So in order to use Caccioppoli type estimates, it is necessary to introduce a new family of distances that are sensitive to weights.

Definition 1.6.2. We call a distance d on a connected weighted graph (V, w, μ) adapted if

(1)

$$\frac{1}{\mu(x)} \sum_{y \in V} w(x, y) d^2(x, y) \leq 1 \qquad (1.6.10)$$

for every $x \in V$,

(2) $d(x, y) \leq 1$ whenever $w(x, y) > 0$.

Remark 1.6.3. Note that the quantity

$$\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)\left(\eta(x)-\eta(y)\right)^2$$

can be viewed as a discrete analogue of $|\nabla\eta|^2(x)$. So (1.6.10) is an analogue of the fact that

$$|\nabla d|^2 \leq 1$$

where d is the geodesic distance on a Riemannian manifold. Tent functions with respect to an adapted distance automatically satisfy (1.6.9) by the triangulated inequality.

In the physical Laplacian case, the graph distance ρ on a locally finite and connected graph is generally not an adapted distance as

$$\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)(r(x)-r(y))^2 = m_+(x)+m_-(x),$$
$$\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)\rho^2(x,y) = \deg(x).$$

However the following construction shows that such a distance always exists on a connected weighted graph.

Definition 1.6.4. Define a function $\sigma(x,y)$ for all for all pairs of neighbors $x \sim y$ by

$$\sigma(x,y) = \min\left\{\frac{1}{\sqrt{\mathrm{Deg}(x)}}, \frac{1}{\sqrt{\mathrm{Deg}(y)}}, 1\right\}. \tag{1.6.11}$$

It naturally induces a distance d on X as follows: for all pairs of distinct points x, y,
(1.6.12)

$$d(x,y) := \inf\{\sum_{i=0}^{n-1}\sigma(x_i,x_{i+1}) : x_0, x_1, \cdots, x_n \text{ is a chain connecting } x \text{ and } y\}.$$

Remark 1.6.5. It is easy to see by definition that

$$d(x,y) \leq \sigma(x,y) \leq 1$$

if $x \sim y$. A direct consequence is that for any $x, y \in V$,

$$d(x,y) \leq \rho(x,y).$$

So the volume growth with respect to an adapted distance is generally larger than that with respect to the graph distance. When the weighted degree function $\mathrm{Deg}(x)$ is bounded from above by some constant $C > 1$, we have that

$$\frac{1}{\sqrt{C}}\rho(x, y) \leq d(x, y) \leq \rho(x, y).$$

In this case, the adapted distance and the graph distance have similar properties.

For a locally finite and connected weighted graph (V, w, μ), the closed balls in the graph distance are compact (finite). This is a nice topological property and the tent functions are finitely supported. However, it is not necessarily true for an adapted distance on (V, w, μ). So we propose the following assumption that is frequently adopted in the Chapters 3, 4 and 5.

Assumption 1.6.6. There exists an adapted distance d on (V, w, μ) such that the d-balls $B_d(x, r)$ are finite sets for any $x \in V, r > 0$.

Remark 1.6.7. Suppose we are looking for a sufficient condition in terms of volume growth with respect to an adapted distance d. Then necessarily the balls in d have finite measure. Suppose that the weighted graphs are not too far from the physical Laplacian case in the sense that they satisfy Assumption 1.1.6. That is, the weights $\mu(x)$ on vertices have a positive lower bound. It follows that Assumption 1.6.6 is automatically fulfilled. So Assumption 1.6.6 is not so restricted as it looks at first.

Remark 1.6.8. Note that since we always assume that weighted graphs are infinite, a weighted graph that satisfies Assumption 1.6.6 is necessarily of infinite radius in the adapted distance there.

To provide some intuition for the adapted distances, we introduce the special families of weakly symmetric graphs that allow explicit calculations in many cases. They are also important in the later chapters. The stochastic completeness problem of them was first systematically studied by Wojciechowski [56].

Definition 1.6.9. Let (V, E) be a locally finite and connected graph. Fix a point $x_0 \in V$ as a reference point. The graph V is called weakly symmetric (with respect to x_0) if it satisfies

$$m_+(x) = g_+(\rho(x, x_0)), m_-(x) = g_-(\rho(x, x_0))$$

with functions $g_+(r), g_-(r) : \mathbb{N} \to \mathbb{N}$.

Example 1.6.10 (Model Trees)**.** Let (V, E) be an infinite, locally finite tree, that is, an infinite undirected, connected graph such that any two vertices are connected by exactly one simple chain (a chain with no duplicate vertices). Assume further that (V, E) is symmetric with respect to a reference point x_0. In other words, $m_-(x_0) = 0$, $m_-(x) = 1$ for $x \neq x_0$ and $m_+(x) = f(\rho(x, x_0))$ where $f : \mathbb{N} \to \mathbb{N}_+$ is a integer valued function.

We consider the physical Laplacian on (V, E) and the adapted distance d constructed in Definition 1.6.4. Recall the notation $r(x) = \rho(x, x_0)$. The we have $\mathrm{Deg}(x_0) = f(0)$ and $\mathrm{Deg}(x) = 1 + f(r(x))$ for $x \neq x_0$. Let $f(n) = [(n+1)^s]$ where $s > 0$ and $[c]$ is the integer part of c . Then for $x \in S_n$ and $y \in S_{n+1}$ such that $x \sim y$, we have

$$\sigma(x, y) \asymp \frac{1}{(n+1)^{s/2}},$$

where " $\asymp$ " means that the two sides are bounded by each other up to positive constants. So for $x \in V$ such that $r(x) = n > 1$, we have

$$d(x, x_0) \asymp \begin{cases} (n+1)^{1-s/2},, & \text{if } 0 < s < 2, \quad (1.6.13) \\ \ln(n+2), & \text{if } s = 2. \quad (1.6.13') \end{cases}$$

If $s > 2$, we see that (V, E) is bounded in d. Assumption 1.6.6 is fulfilled only when $0 < s \leq 2$. So the relation between adapted distances and the graph distance can be subtle. Unboundedness and locally finiteness may fail in adapted distances. Similar calculations can also be done for the following example constructed by Wojciechowski [56].

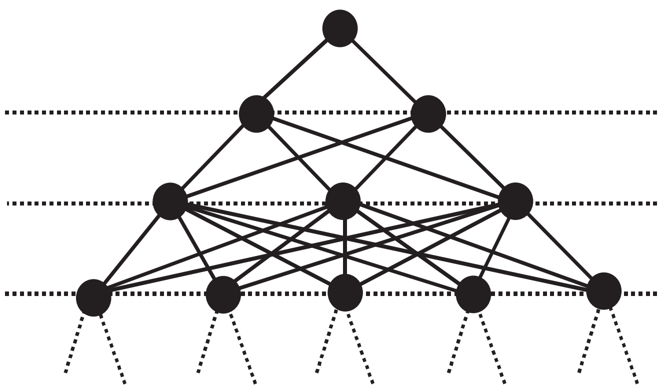

Figure 1: Anti-tree of Wojciechowski

Example 1.6.11 (Anti-trees). Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint, finite, nonempty sets with $S_0 = \{x_0\}$. Denote $\#S_n$ by $S(n)$. Let

$$V = \bigcup_{n\in\mathbb{N}} S_n$$

and

$$E = \{(x, y) \in V \times V | \exists n \in \mathbb{N}, x \in S_n, y \in S_{n\pm 1}\}.$$

In other words, we connect every vertex in S_r to every vertex in S_{r+1} to get a graph (V, E) that is symmetric with respect to x_0. The resulting graph (V, E) is infinite, locally finite, connected and simple.

1.7 Continuous time Markov chains

The materials in this section are standard and we include them just for the sake of completeness. We mainly follow the notations and constructions in [39]. In particular, we only consider the minimal right-continuous Markov chains which is closely related to the semigroup P_t we constructed in Section 1.2. For the much more subtle non-minimal chains, see [3] and [16]. To the author's knowledge, most of the results are due to Feller [11, 12], Doob [10], Chung [3] etc.

Let (V, w, μ) be a weighted graph. Define

$$q_{xy} = \frac{w(x, y)}{\mu(x)}$$

for $x \neq y$ and

$$q_{xx} = -\mathrm{Deg}(x).$$

The matrix $Q = (q_{xy})_{V\times V}$ satisfies that

(1) $0 \leq -q_{xx} < \infty$ for all x;

(2) $q_{xy} \geq 0$ for all $x \neq y$;

(3) $\sum_{y\in V} q_{xy} = 0$ for all x.

This kind of matrices are called Q-matrices in [39]. There is a natural jump matrix $\Pi = (\pi_{xy})_{V\times V}$ associated with Q as:

$$\pi_{x,y} = \begin{cases} \dfrac{q_{xy}}{|q_{xx}|}, & \text{if } x \neq y, \text{ and } q_{xx} \neq 0, \\ 0, & \text{if } x \neq y, \text{ and } |q_{xx}| = 0; \end{cases}$$

$$\pi_{x,x} = \begin{cases} 0, & \text{if } q_{xx} \neq 0, \\ 1, & \text{if } q_{xx} = 0. \end{cases}$$

Following [39], we can construct a minimal right-continuous Markov chain $\{X_t\}_{t\geq 0}$ corresponding to the Q-matrix. First we adjoin a cemetery point ∂ to V and denote the set $V \cup \partial$ by V_∂. From the jump matrix Π we can construct a discrete time Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ on V. Let $T_1, T_2, \cdots$ be a sequence of independent exponential random variables of parameter 1 that are independent of $\{Y_n\}_{n\in\mathbb{N}}$. Set $S_n = T_n/q(Y_{n-1})$ and $J_n = S_1 + \cdots + S_n$ with the convention $J_0 = 0$. Define the (first) explosion time ζ by

$$\zeta = \sup_n J_n,$$

which is the first time that X_t jumps out of V. Then the continuous time Markov chain $\{X_t\}_{t\geq 0}$ on V_∂ is defined as

$$X_t = \begin{cases} Y_n, & \text{if } \ J_n \leq t < J_{n+1} \text{ for some } n \in \mathbb{N}, \\ \partial, & \text{if } \ t \geq \zeta. \end{cases}$$

For all $n \in \mathbb{N}_+$, conditioning on $Y_0 = x_0, \cdots, Y_{n-1} = x_{n-1}$, $S_1, \cdots, S_n$ are independent exponential random variables with parameters $q(x_0), \cdots, q(x_{n-1})$ respectively. The process $\{X_t\}_{t\geq 0}$ has the (time homogenous) Markov property in the sense that for all $n \in \mathbb{N}$, all sequences of time $0 \leq t_0 \leq \cdots \leq t_{n+1}$, and all sequences of points $x_0, \cdots, x_{n+1}$ in V_∂,

$$\mathbb{P}(X_{t_{n+1}} = x_{n+1} | X_{t_0} = x_0, \cdots, X_{t_n} = x_n) = \mathbb{P}(X_{t_{n+1}-t_n} = x_{n+1} | X_0 = x_n).$$

Remark 1.7.1. (a) In [39], it is shown that the Markov chain $\{X_t\}_{t\geq 0}$ has the strong Markov property.

(b) From the above construction, we see that $\frac{1}{\mathrm{Deg}(x)}$ gives the expected holding time of the process at $x \in V$. This is a probabilistic interpretation of the weighted degree function.

(c) It is also direct to see that at the time when the process $\{X_t\}_{t\geq 0}$ leaves a point $x \in V$, it can only jump to the neighbors of x.

The following definition of explosion of a Markov chain is taken from [39].

Definition 1.7.2. The Markov chain $\{X_t\}_{t\geq 0}$ is called explosive if for some $x \in V$,

$$\mathbb{P}_x(\zeta < \infty) > 0.$$

Otherwise $\{X_t\}_{t\geq 0}$ is called nonexplosive.

The quantity

$$\tilde{p}(t, x, y) = \mathbb{P}_x(X_t = y) = \mathbb{P}(X_t = y | X_0 = x)$$

is called the transition probability of $\{X_t\}_{t\geq 0}$. For any $t \geq 0$, define the matrix $\mathcal{P}_t$

by $\mathcal{P}_t = (\tilde{p}(t,x,y))_{V\times V}$. The family $\{\mathcal{P}_t\}_{t\geq 0}$ is in fact a matrix semigroup as

$$\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s, \mathcal{P}_0 = Id,$$

for all $s \geq 0, t \geq 0$. From the construction of $\{X_t\}_{t\geq 0}$, we see that $\{X_t\}_{t\geq 0}$ is explosive if and only if for some $x \in V$ and some $t > 0$,

$$(\mathcal{P}_t 1)(x) = \sum_{y\in V} \tilde{p}(t,x,y) = \mathbb{P}_x(X_t \in V) < 1.$$

The matrix semigroup $\mathcal{P}_t$ can be also viewed as an operator semigroup as

$$\mathcal{P}_t f(x) = \sum_{y\in V} \tilde{p}(t,x,y)f(y) = \sum_{y\in V} f(y)\mathbb{P}_x(X_t = y) = \mathbb{E}_x\left(f(X_t)\right)$$

for $f \in l^2(V,\mu)$.

The following properties of $\{X_t\}_{t\geq 0}$ and $\mathcal{P}_t$ are showed in Theorem 2.8.3 and Theorem 2.8.4 of [39].

Proposition 1.7.3. *Let (V, w, μ) be a weighted graph and $\{X_t\}_{t\geq 0}$ be the corresponding minimal right-continuous Markov chain constructed as before. Then the semigroup $\mathcal{P}_t = (\tilde{p}(t,x,y))_{V\times V}$ is the minimal nonnegative solution to the backward equation*

$$\frac{\partial}{\partial t}\mathcal{P}_t = Q\mathcal{P}_t, \mathcal{P}_0 = Id.$$

In other words, for all $x, y \in V, t \geq 0$,

$$\frac{\partial}{\partial t}\tilde{p}(t,x,y) = \sum_{z\in V} q_{xz}\tilde{p}(t,z,y) = \frac{1}{\mu(x)}\sum_{z\in V} w(x,z)(\tilde{p}(t,z,y) - \tilde{p}(t,x,y)),$$

and

$$\tilde{p}(0,x,y) = \chi_x(y).$$

Proposition 1.7.3 allows us to relate $\mathcal{P}_t$ to the semigroup P_t we constructed before. Recall Theorem 1.4.15, for each $y \in V$, the function

$$p(t,x,y)\mu(y) = p(t,y,x)\mu(y) = (P_t\delta_y)(x)\mu(y)$$

is the minimal nonnegative solution to

$$\begin{cases} \dfrac{\partial}{\partial t}p(t,x,y)\mu(y) = \dfrac{1}{\mu(x)}\displaystyle\sum_{z\in V} w(x,z)(p(t,z,y)\mu(y) - p(t,x,y)\mu(y)), \\ p(0,x,y)\mu(y) = \mu(y)\delta_x(y) = \chi_x(y). \end{cases}$$

From the minimality of both $\tilde{p}(t,x,y)$ and $p(t,x,y)$, we have the relation

$$\tilde{p}(t,x,y) = p(t,x,y)\mu(y).$$

Thus

$$\mathcal{P}_t f(x) = \sum_{y\in V} \tilde{p}(t,x,y)f(y) = \sum_{y\in V} p(t,x,y)f(y)\mu(y) = P_t f(x).$$

The semigroup $\mathcal{P}_t$ coincides with P_t that we constructed using Dirichlet form theory and thus the notions of explosion and stochastic incompleteness just coincide.

Proposition 1.7.4. *A weighted graph (V,w,μ) is stochastically incomplete if and only if the corresponding minimal right continuous Markov chain $\{X_t\}_{t\geq 0}$ is explosive.*

Now we consider the restriction of the process $\{X_t\}_{t\geq 0}$ to a finite set U of V. Let τ_U be the first exit time of U, that is

$$\tau_U = \inf\{t \geq 0 : X_t \in U^c\}.$$

We can define a semigroup $\mathcal{P}_t^U$ on $l^2(U,\mu_U)$ as (see (4.1.2) in [17] for example)

$$\mathcal{P}_t^U f(x) = \mathbb{E}_x\left(f(X_t)1_{\{t<\tau_U\}}\right).$$

Since U is finite, $\{\mathcal{P}_t^U\}_{t\geq 0}$ is a semigroup of finite matrices. The corresponding Markov chain $\{X_t^U\}_{t\geq 0}$ can be viewed as constructed from $\{X_t\}_{t\geq 0}$ in the way that at the first time when X_t runs out of U, we send it to the cemetery point ∂ and it never gets back. It is a standard calculation from the construction of $\{X_t\}_{t\geq 0}$ and the definition of τ_U to obtain that $\mathcal{P}_t^U$ satisfies

$$\frac{\partial}{\partial t}\mathcal{P}_t^U = Q^U\mathcal{P}_t^U, \mathcal{P}_0^U = Id_U,$$

where the matrix $Q^U = (q_{xy}^U)_{U\times U}$ is

$$q_{xy}^U = \begin{cases} \dfrac{w(x,y)}{\mu(x)}, & x\neq y,\, x\in U, y\in U \\ -\dfrac{1}{\mu(x)}\displaystyle\sum_{z\in V} w(x,z), & x = y\in U. \end{cases}$$

Remark 1.7.5. Note that as U is finite, $\mathcal{P}_t^U$ is simply $\exp\left(tQ^U\right)$.

The following proposition will be used in Chapter 5.

Proposition 1.7.6. *Let (V, w, μ) be a locally finite weighted graph with the formal Laplacian Δ. Let W and U be two finite subsets of V such that $\bar{U} \subseteq W$. Define a function u on $W \times [0, \infty)$to be*

$$u(x, t) = \mathbb{P}_x(\tau_U \le t).$$

Then $u(x, 0) \equiv 0$ and $u(x, t)$ satisfies

$$\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0,$$

for all $x \in U$ and $t \ge 0$.

Proof. By the definition of $\mathcal{P}_t^W$, we have that

$$u(x, t) = \mathbb{P}_x(\tau_W \le t) = 1 - \mathbb{E}_x\left(1_{\{\tau_W > t\}}\right) = 1 - \mathcal{P}_t^W 1_W(x).$$

It is clear that $u(x, 0) \equiv 0$ on W. Moreover, viewing 1_W as a column vector, for all $x \in W$,

$$\frac{\partial}{\partial t} u(x, t) = -\frac{\partial}{\partial t} \mathcal{P}_t^W 1_W(x) = -Q^W \mathcal{P}_t^W 1_W(x).$$

Since $\bar{U} \subseteq W$, for all $x \in U$, we have that $y \in W$ if $w(x, y) > 0$. Hence for all $x \in U$,

$$\begin{aligned} Q^W \mathcal{P}_t^W 1_W(x) &= \sum_{y \in W} \frac{w(x, y)}{\mu(x)} \mathcal{P}_t^W 1_W(y) - \sum_{y \in V} \frac{w(x, y)}{\mu(x)} \mathcal{P}_t^W 1_W(x) \\ &= \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(\mathcal{P}_t^W 1_W(y) - \mathcal{P}_t^W 1_W(x)) \\ &= \Delta u(x, t). \end{aligned}$$

The equation

$$\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0$$

then holds for all $x \in U$ and $t \ge 0$. □

Chapter 2

The weak Omori-Yau maximum principle

In this chapter we first prove that the stochastic completeness of a weighted graph is equivalent to a discrete analogue of the weak Omori-Yau maximum principle. The latter notion was introduced by Pigola, Rigoli, and Setti [42], [43], where they proved the aforementioned equivalence in the setting of manifolds and gave many applications. For the original form of Omori-Yau maximum principle, see [41], [57].

Then we apply the weak Omori-Yau maximum principle to obtain an analogue of the Khas'minskii criterion [36] for weighted graphs. The proof here is inspired by the one in [43]. Together with the weak Omori-Yau maximum principle, the Khas'minskii criterion gives most known criteria for stochastic completeness or stochastic incompleteness as well as new criteria. Stability of stochastic incompleteness is also investigated through the weak Omori-Yau maximum principle. Most of the contents of this chapter are taken from the author's paper [29] with possible modifications.

Recently, Keller, Lenz and Wojciechowski [35] obtained another proof of the Khas'minskii criterion for weighted graphs in a slightly different form.

2.1 Equivalence to stochastic completeness

From now on, we will denote the supremum of a function f on V by f^*.

Definition 2.1.1. A weighted graph (V, w, μ) is said to satisfy the weak Omori-Yau maximum principle if for every nonnegative function f on V with $f^* = \sup_V f < +\infty$ and for every $\alpha > 0$,

$$\sup_{\Omega_\alpha} \Delta f \geq 0,$$

where

$$\Omega_\alpha = \{x \in V : f(x) > f^* - \alpha\}.$$

Theorem 2.1.2. *For any weighted graph (V, w, μ), the weak Omori-Yau maximum principle is equivalent to stochastic completeness.*

Proof. Assume that the weak Omori-Yau maximum principle holds but the graph is stochastically incomplete. By Theorem 1.5.1, there exists a bounded, non-negative, nonconstant solution f of the equation $\Delta f + \lambda f = 0$ for some $\lambda > 0$. Choosing $\alpha = \frac{f^*}{2} > 0$, we have

$$\sup_{\Omega_\alpha} \Delta f = \sup_{\Omega_\alpha} (-\lambda f) \leq -\lambda \frac{f^*}{2} < 0$$

which is a contradiction.

Conversely, if (V, w, μ) is stochastically complete but the weak Omori-Yau maximum principle does not hold, there exists a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ and $c > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f < -2c.$$

Define

$$f_\alpha = (f + \alpha - f^*)_+,$$

which is obviously nonconstant, nonnegative and bounded. Setting $\lambda = \frac{c}{\alpha}$, we claim that

$$\Delta f_\alpha + \lambda f_\alpha \leq 0,$$

which implies stochastic incompleteness by Theorem 1.5.1 and leads to a contradiction.

For $x \in \Omega_\alpha^c$, $f_\alpha(x) = 0$, so the claim is trivially true.

For $x \in \Omega_\alpha$, we have

$$\lambda f_\alpha(x) \leq \lambda \alpha = c,$$

and

$$f_\alpha(x) - f_\alpha(y) = f(x) - f^* + \alpha - f_\alpha(y) \leq f(x) - f(y).$$

Hence

$$\begin{aligned}\Delta f_\alpha(x) + \lambda f_\alpha(x) &= \frac{1}{\mu(x)} \sum_y w(x,y)(f_\alpha(x) - f_\alpha(y)) + \lambda f_\alpha(x) \\ &\le \frac{1}{\mu(x)} \sum_y w(x,y)(f(x) - f(y)) + c \\ &= \Delta f(x) + c \le -c.\end{aligned}$$

□

2.2 A key lemma

The following lemma describes some elementary properties of a function f that violates the weak Omori-Yau maximum principle.

Lemma 2.2.1. *Suppose that a weighted graph (V, w, μ) is stochastically incomplete. Let f be a nonnegative function on V such that $f^* < +\infty$ and for some $\alpha > 0$ and $c > 0$,*

$$\sup_{\Omega_\alpha} \Delta f \le -c.$$

Let $\alpha' = \min\{\alpha, c\}$. Then the following is true.

(1) f cannot attain its supremum f^* on V, and in particular, is nonconstant;

(2) $\sup_{\Omega_{\alpha'}} \Delta f \le -\alpha'$;

(3) for every $n \ge 1$, and every $x \in \Omega_{\frac{\alpha'}{n}}$,

$$\mathrm{Deg}(x) = \frac{1}{\mu(x)} \sum_y w(x,y) \ge n.$$

In other words,

$$\Omega_{\frac{\alpha'}{n}} \subseteq \{x \in V : \mathrm{Deg}(x) \ge n\}.$$

Proof. (1) Suppose that there exists $x_0 \in V$ such that $f(x_0) = f^*$. In particular, $x_0 \in \Omega_\alpha$. We have that

$$\frac{1}{\mu(x_0)} \sum_{y \in V} w(x_0, y)(f(y) - f(x_0)) = -\Delta f(x_0) \ge c > 0.$$

Thus there exists $y \in V$ such that $f(y) > f(x_0)$, a contradiction.

(2) Since $\alpha' \le \alpha$, we have $\Omega_{\alpha'} \subseteq \Omega_\alpha$. So

$$\sup_{\Omega_{\alpha'}} \Delta f \le \sup_{\Omega_\alpha} \Delta f < -c \le -\alpha'.$$

(3) For $x \in \Omega_{\frac{\alpha'}{n}}$, set

$$l = \frac{1}{\mu(x)} \sum_{y: f(y) > f(x)} w(x,y),$$

we have

$$\alpha' \le -\Delta f(x) \le \frac{1}{\mu(x)} \sum_{y: f(y)>f(x)} w(x,y)(f(y)-f(x)) \le \frac{l\alpha'}{n}.$$

Therefore $l \ge n$ and, in particular, $\mathrm{Deg}(x) \ge n$ for all $x \in \Omega_{\frac{\alpha'}{n}}$. □

Note that stochastic incompleteness is a global property while the weighted degree function is a local quantity. We can define a "global weighted degree function" in an iterative way.

Lemma 2.2.2. *Fix a non-decreasing sequence $\Theta = \{a_k\}_{k\ge 0}$ of nonnegative real numbers. We use the convention that*

$$\sum_{y, y \in \emptyset} w(x,y) = 0.$$

For $x \in V$ and $k \in \mathbb{N}$, define

$$\mathrm{Deg}_{\Theta,0}(x) = \mathrm{Deg}(x),$$

and

$$\mathrm{Deg}_{\Theta,k+1}(x) = \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{\Theta,k}(y) \ge a_k} w(x,y).$$

Then for any $x \in V$, $\{\mathrm{Deg}_{\Theta,k}(x)\}_{k\ge0}$ forms a non-increasing, nonnegative sequence. In particular,

$$\mathrm{Deg}_{\Theta,\infty}(x) = \lim_{k\to\infty} \mathrm{Deg}_{\Theta,k}(x)$$

exists for all $x \in V$.

Proof. The sequence $\{\mathrm{Deg}_{\Theta,k}(x)\}_{k\ge0}$ obviously has nonnegative entries. We only need to prove that for any $k \ge 0$,

$$\mathrm{Deg}_{\Theta,k+1}(x) \le \mathrm{Deg}_{\Theta,k}(x).$$

For $k = 0$, we have

$$\mathrm{Deg}_{\Theta,1}(x) = \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}(y) \geq a_0} w(x,y) \leq \frac{1}{\mu(x)} \sum_{y} w(x,y) = \mathrm{Deg}_{\Theta,0}(x).$$

Assume that the assertion holds for $k = n-1 \geq 0$, that is

$$\mathrm{Deg}_{\Theta,n}(x) \leq \mathrm{Deg}_{\Theta,n-1}(x).$$

Since $a_n \geq a_{n-1}$, we see that for $k = n$,

$$\begin{aligned}
\mathrm{Deg}_{\Theta,n}(x) &= \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{\Theta,n-1}(y) \geq a_{n-1}} w(x,y) \\
&\geq \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{\Theta,n-1}(y) \geq a_n} w(x,y) \\
&\geq \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{\Theta,n}(y) \geq a_n} w(x,y) \\
&= \mathrm{Deg}_{\Theta,n+1}(x).
\end{aligned}$$

The assertion follows by induction. □

Definition 2.2.3. We call $\mathrm{Deg}_{\Theta,\infty}(x)$ the global weighted degree of x with respect to the sequence Θ. For the special case when $a_k \equiv n \geq 0$, we denote $\mathrm{Deg}_{\Theta,\infty}(x)$ by $\mathrm{Deg}_{n,\infty}(x)$ and call it the global weighted degree of x with parameter n.

Remark 2.2.4. Note that unlike the weighted degree $\mathrm{Deg}(x)$, the global weighted degree of x contains information of points that may not be neighbors of x.

The definition of global weighted degree is a good one in the sense that it is "stable" and there is no need to define something like $\mathrm{Deg}_{n,\infty+1}(x)$.

Lemma 2.2.5. *Let* $\mathrm{Deg}_{n,\infty}(x)$ *be the global weighted degree function of some weighted graph* (V, w, μ) *for some* $n \geq 0$*. Then the following holds for all* $x \in V$*:*

$$\frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{n,\infty}(x) \geq n} w(x,y) = \mathrm{Deg}_{n,\infty}(x).$$

Proof. Fix some $x \in V$. We abuse the notation and view $w(x, \cdot)$ as a finite measure on V as

$$\frac{1}{\mu(x)} \sum_{y} w(x,y) < \infty.$$

Note that $\{\mathrm{Deg}_{n,k}(x)\}_{k=1}^{\infty}$ decreases to $\mathrm{Deg}_{n,\infty}(x)$. We have that

$$\mathrm{Deg}_{n,\infty}(x) \geq n \Leftrightarrow \forall k \in \mathbb{N}, \mathrm{Deg}_{n,k}(x) \geq n.$$

Denote by A_k the set

$$\{y \in V : \mathrm{Deg}_{n,k}(x) \geq n\}.$$

Then as a non-increasing sequence of sets

$$\lim_{k\to\infty} A_k = \{y \in V : \mathrm{Deg}_{n,\infty}(x) \geq n\}.$$

We obtain by the finiteness of $w(x, \cdot)$,

$$\begin{aligned}
&\frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{n,\infty}(x) \geq n} w(x,y) \\
=&\frac{1}{\mu(x)} w(x, \lim_{k\to\infty} A_k) \\
=&\frac{1}{\mu(x)} \lim_{k\to\infty} w(x, A_k) \\
=&\lim_{k\to\infty} \mathrm{Deg}_{n,k}(x) \\
=&\mathrm{Deg}_{n,\infty}(x).
\end{aligned}$$

□

Now we state the monotonicity of $\mathrm{Deg}_{n,\infty}(x)$ in n.

Lemma 2.2.6. *For $m > n \geq 0$, $k \in \mathbb{N}$, the following holds for any $x \in V$,*

$$\mathrm{Deg}_{n,k}(x) \geq \mathrm{Deg}_{m,k}(x).$$

In particular, for any $x \in V$,

$$\mathrm{Deg}_{n,\infty}(x) \geq \mathrm{Deg}_{m,\infty}(x).$$

Proof. The first assertion can be proven by an induction procedure similar to the proof of Lemma 2.2.2. The $k = 0$ case is obvious as

$$\mathrm{Deg}_{n,0}(x) = \mathrm{Deg}(x) = \mathrm{Deg}_{m,0}(x).$$

Assume that for all $x \in V$,

$$\mathrm{Deg}_{n,k}(x) \geq \mathrm{Deg}_{m,k}(x).$$

Then we have

$$\begin{aligned}\mathrm{Deg}_{n,k+1}(x) &= \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{n,k}(y) \geq n} w(x,y) \\ &\geq \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{m,k}(y) \geq n} w(x,y) \\ &\geq \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{m,k}(y) \geq m} w(x,y) \\ &= \mathrm{Deg}_{m,k+1}(x).\end{aligned}$$

This completes the induction. The last assertion follows by taking the limit $k \to \infty$ in

$$\mathrm{Deg}_{n,k}(x) \geq \mathrm{Deg}_{m,k}(x).$$

□

The notion of the global weighted degree function allows us to improve Lemma 2.2.1 as follows.

Theorem 2.2.7. *Suppose that a weighted graph (V, w, μ) is stochastically incomplete. Let f be a nonnegative function on V such that $f^* < +\infty$ and for some $\alpha > 0$,*

$$\sup_{\Omega_\alpha} \Delta f \leq -\alpha.$$

Then for any $n \geq 1$,

$$\Omega_{\frac{\alpha}{n}} \subseteq \{x \in V : \mathrm{Deg}_{n,\infty}(x) \geq n\}.$$

And for any $m > n \geq 1$,

$$\Omega_{\frac{\alpha}{m}} \subseteq \{x \in V : \mathrm{Deg}_{n,\infty}(x) \geq m\}.$$

As a consequence, (V, w, μ) has unbounded global weighted degree for any parameter $n \geq 1$.

Proof. In the proof of part (3) of Lemma 2.2.1, we already showed that for $x \in \Omega_{\frac{\alpha}{n}}$,

$$l = \frac{1}{\mu(x)} \sum_{y: f(y) > f(x)} w(x,y) \geq n.$$

We claim that for all $x \in \Omega_{\frac{\alpha}{n}}$, $k \in \mathbb{N}$,

$$n \leq l \leq \mathrm{Deg}_{n,k}(x).$$

Assuming the claim, we see that for any $x \in \Omega_{\frac{\alpha}{n}}$,

$$n \leq l \leq \mathrm{Deg}_{n,\infty}(x).$$

Hence

$$\Omega_{\frac{\alpha}{n}} \subseteq \{x \in V : \mathrm{Deg}_{n,\infty}(x) \geq n\}.$$

Now we complete the proof of the claim. For all $x \in \Omega_{\frac{\alpha}{n}}$,

$$\mathrm{Deg}_{n,0}(x) = \frac{1}{\mu(x)} \sum_{y} w(x,y) \geq \frac{1}{\mu(x)} \sum_{y: f(y) > f(x)} w(x,y) = l.$$

Assume that the claim is true for k. In other words, for all $x \in \Omega_{\frac{\alpha}{n}}$,

$$\mathrm{Deg}_{n,k}(x) \geq l \geq n.$$

Note that if $f(y) > f(x)$ for $x \in \Omega_{\frac{\alpha}{n}}$, y is necessarily in $\Omega_{\frac{\alpha}{n}}$ and consequently,

$$\mathrm{Deg}_{n,k}(y) \geq l \geq n.$$

Thus we have

$$\mathrm{Deg}_{n,k+1}(x) = \frac{1}{\mu(x)} \sum_{y, \mathrm{Deg}_{n,k}(y) \geq n} w(x,y) \geq \frac{1}{\mu(x)} \sum_{y: f(y) > f(x)} w(x,y) = l$$

for any $x \in \Omega_{\frac{\alpha}{n}}$. The claim follows by induction.

By Lemma 2.2.6, we see that for $m > n \geq 1$,

$$\Omega_{\frac{\alpha}{m}} \subseteq \{x \in V : \mathrm{Deg}_{m,\infty}(x) \geq m\} \subseteq \{x \in V : \mathrm{Deg}_{n,\infty}(x) \geq m\}.$$

The set $\Omega_{\frac{\alpha}{m}}$ is nonempty for any $m > n$, so that the function $\mathrm{Deg}_{n,\infty}(x)$ is necessarily unbounded for any $n \geq 1$. □

An immediate consequence of the properties of the global weighted degree function is the following result.

Proposition 2.2.8. *Let (V, w, μ) be a weighted graph. For some $n \geq 1$, let $\mathrm{Deg}_{n,\infty}$ be the global weighted degree function as before. Suppose that for any sequence of points $\{x_k\}_{k=1}^{\infty}$ with $x_k \sim x_{k+1}$ for any $k \in \mathbb{N}$,*

$$\sum_{k=1}^{\infty} \frac{1}{\mathrm{Deg}_{n,\infty}(x_k)} = \infty.$$

Then (V, w, μ) *is stochastically complete.*

Proof. Assume the contrary is true. Then there exists a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f \leq -\alpha.$$

It follows as in the proof of part (1) of Lemma 2.2.1 that for any point $x \in \Omega_\alpha$, there is some y with $y \sim x$ such that $f(y) > f(x)$. Take some $x_1 \in \Omega_{\frac{\alpha}{n}}$. Suppose that x_k has been chosen. Choose some $x_{k+1} \sim x_k$ such that

$$f(x_{k+1}) \geq \sup_{y, y \sim x_k} f(y) - \frac{1}{2^k},$$

and

$$f(x_{k+1}) > f(x_k).$$

So $x_k \in \Omega_{\frac{\alpha}{n}}$ for any $k \in \mathbb{N}$.

Similar to the proofs of part (3) of Lemma 2.2.1 and of Theorem 2.2.7, we have

$$\begin{aligned}
\alpha &\leq -\Delta f(x_k) \\
&\leq \frac{1}{\mu(x_k)} \sum_{y: f(y) > f(x_k)} w(x_k, y)(f(y) - f(x_k)) \\
&\leq (f(x_{k+1}) + \frac{1}{2^k} - f(x_k)) \frac{1}{\mu(x_k)} \sum_{y: f(y) > f(x_k)} w(x_k, y) \\
&\leq (f(x_{k+1}) + \frac{1}{2^k} - f(x_k)) \mathrm{Deg}_{n,\infty}(x_k).
\end{aligned}$$

Note that

$$f(x_1) + \sum_{k=1}^{\infty} (f(x_{k+1}) - f(x_k)) \leq f^* < \infty.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{\mathrm{Deg}_{n,\infty}(x_k)} \leq \sum_{k=1}^{\infty} \frac{1}{\alpha} (f(x_{k+1}) + \frac{1}{2^k} - f(x_k)) < \infty.$$

A contradiction. □

Remark 2.2.9. With the global weighted degree function replaced with the weighted degree function, this result is already known to Wojciechowski, Keller and Lenz through analytical methods. From the probabilistic point of view, this is an easy standard result, see [39] for example.

2.3 Khas'minskii criterion

Now we are ready to prove the following analogue of Khas'minskii criterion for stochastic completeness.

Theorem 2.3.1. *Assume that for some $n_0 \geq 1$, the global weighted degree function* $\mathrm{Deg}_{n_0,\infty}(x)$ *is unbounded for the weighted graph (V, w, μ). If there exists a nonnegative function $\gamma \in \mathcal{D}$ on V such that*

$$\gamma(x) \to +\infty \quad \textit{as} \quad \mathrm{Deg}_{n_0,\infty}(x) \to +\infty \tag{2.3.1}$$

and

$$\Delta\gamma(x) + \lambda\gamma(x) \geq 0 \tag{2.3.2}$$

outside a set A of bounded global weighted degree $\mathrm{Deg}_{n_0,\infty}$ *for some $\lambda > 0$, then (V, w, μ) is stochastically complete.*

Proof. We only need to prove that (V, w, μ) satisfies the weak Omori-Yau maximum principle. If not, there exists a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f \leq -\alpha.$$

Let

$$\sup\{\mathrm{Deg}_{n_0,\infty}(x) : x \in A\} = M < +\infty.$$

By Theorem 2.2.7, changing α if necessary, we can assume that $\mathrm{Deg}_{n_0,\infty} \geq M + n_0$ for all $x \in \Omega_\alpha$. It is clear that $\Omega_\alpha \cap A = \emptyset$.

Let

$$u = f - c\gamma,$$

where the parameter $c > 0$ will be chosen later.

Since $f^* < +\infty$ and

$$\gamma(x) \to +\infty \quad \text{as} \quad \mathrm{Deg}_{n_0,\infty}(x) \to +\infty,$$

there exists $N(c) > M + n_0$ such that

$$\sup_{\{x \in V : \mathrm{Deg}_{n_0,\infty}(x) < N(c)\}} u(x) = u^* := \sup_V u(x) < +\infty.$$

Let $0 < \eta < \min(\frac{\alpha}{2}, \frac{\alpha}{2\lambda})$. We can choose $\bar{x}$ such that

$$f(\bar{x}) > f^* - \frac{\eta}{2}.$$

Choose $c = c(\eta, \bar{x}) > 0$ small enough to ensure that $c\gamma(\bar{x}) < \frac{\eta}{2}$.

For $n \in \mathbb{N}_+$, we can choose x_n with $\mathrm{Deg}_{n0,\infty}(x_n) < N(c)$ such that

$$u(x_n) + \frac{1}{n} > u^* \geq u(\bar{x}) = f(\bar{x}) - c\gamma(\bar{x}) > f^* - \frac{\eta}{2} - \frac{\eta}{2}.$$

We have

$$f(x_n) + \frac{1}{n} \geq f(x_n) - c\gamma(x_n) + \frac{1}{n} > f(\bar{x}) - c\gamma(\bar{x}) > f^* - \eta,$$

and

$$c\gamma(x_n) < f(x_n) - f^* + \eta + \frac{1}{n} < \eta + \frac{1}{n}.$$

So for every index $n > \frac{2}{\eta}$, note that by definition $\eta < \frac{\alpha}{2}$ and $\eta < \frac{\alpha}{2\lambda}$,

$$f(x_n) \geq u(x_n) > f^* - \eta - \frac{1}{n} > f^* - \frac{3}{2}\eta > f^* - \alpha,$$

$$c\lambda\gamma(x_n) < \frac{3}{2}\lambda\eta < \frac{3}{4}\alpha.$$

In particular, for every index $n > \frac{2}{\eta}$, $x_n \in \Omega_\alpha$ and $x \notin A$. It follows that for all $n > \frac{2}{\eta}$,

$$\Delta\gamma(x_n) + \lambda\gamma(x_n) \geq 0,$$

and

$$\Delta f(x_n) \leq -\alpha.$$

Then

$$\begin{aligned} \Delta(f - c\gamma)(x_n) &= \Delta f(x_n) - c\Delta\gamma(x_n) \\ &\leq -\alpha + c\lambda\gamma(x_n) < -\alpha/4. \end{aligned} \tag{2.3.3}$$

On the other hand, if $u(y) > u(x_n)$ where $n > \frac{2}{\eta}$, we have

$$f(y) \geq u(y) > u(x_n) > f^* - \alpha.$$

So the following inclusions of sets hold for all $n > \frac{2}{\eta}$,

$$\{y \in V : u(y) > u(x_n)\}$$

$$
\begin{aligned}
&\subseteq \{y \in V : f(y) > f^* - \alpha\} \\
&\subseteq \{y \in V : \mathrm{Deg}_{n_0,\infty}(y) \geq M + n_0\} \\
&\subseteq \{y \in V : \mathrm{Deg}_{n_0,\infty}(y) \geq n_0\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Delta(f - c\gamma)(x_n) &= \Delta u(x_n) \\
&= \frac{1}{\mu(x_n)} \sum_y w(x_n, y)(u(x_n) - u(y)) \\
&\geq \frac{1}{\mu(x_n)} \sum_{y, u(y) > u(x_n)} w(x_n, y)(u(x_n) - u^*) \\
&\geq \frac{1}{\mu(x_n)} \sum_{y, f(y) > f^* - \alpha} w(x_n, y)(-\frac{1}{n}) \\
&\geq \frac{1}{\mu(x_n)} \sum_{y, \mathrm{Deg}_{n_0,\infty}(y) \geq n_0} w(x_n, y)(-\frac{1}{n}) \\
&= -\frac{\mathrm{Deg}_{n_0,\infty}(x_n)}{n} > -\frac{N(c)}{n}.
\end{aligned}
$$

Choosing sufficiently large n, we obtain a contradiction to (2.3.3). □

Remark 2.3.2. Note that unlike in the case of manifolds we do not require that the exceptional set A is compact.

Remark 2.3.3. We conjecture that the converse to Theorem 2.3.1 is true. Namely, if a weighted graph (V, w, μ) is stochastically complete, then there should exist a function $\gamma(x) \in \mathcal{D}$ on V satisfying the conditions (2.3.1), (2.3.2). This is motivated by Nakai's result [38] that the converse to Khas'minskii criterion [36] for parabolic Riemannian manifolds is true.

A convenient version of Khas'minskii criterion on manifolds is given in [43] . We give the discrete analogue here.

Theorem 2.3.4. *Let (V, w, μ) be a weighted graph. If there exists a nonnegative function $\sigma \in \mathcal{D}$ on V with*

$$
\sigma(x) \to +\infty \quad \text{as} \quad \mathrm{Deg}_{n,\infty}(x) \to +\infty
$$

satisfying:

$$
\Delta\sigma(x) + f(\sigma(x)) \geq 0
$$

outside a set A of bounded global weighted degree $\mathrm{Deg}_{n,\infty}$ *for some positive, increasing* *function* $f \in C^1([0,+\infty))$ *with*

$$\int_0^{+\infty} \frac{dr}{f(r)} = +\infty,$$

then (V,w,μ) *is stochastically complete.*

Proof. Let

$$\phi(r) = \exp\Big(\int_0^r \frac{ds}{f(s)+s}\Big),$$

we have $\phi(r) \to +\infty$ as $r \to +\infty$ (cf. Lemma 2.3.5 below).

The function $\phi(r)$ is increasing and concave since:

(1) $\phi'(r) = \frac{\phi(r)}{f(r)+r} > 0$;

(2) $\phi''(r) = -\frac{\phi(r)f'(r)}{(f(r)+r)^2} \leq 0$.

Therefore for $r, s \geq 0$ we have

$$\phi(r) - \phi(s) \geq \phi'(r)(r-s).$$

Thus

$$\begin{aligned} \Delta\phi(\sigma(x)) &= \frac{1}{\mu(x)} \sum_{y \in V} w(x,y)(\phi(\sigma(x)) - \phi(\sigma(y))) \\ &\geq \phi'(\sigma(x)) \frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(\sigma(x)-\sigma(y)) \\ &= \phi'(\sigma(x))\Delta\sigma(x), \end{aligned} \tag{2.3.4}$$

which also shows that $\phi(\sigma(x)) \in \mathcal{D}$. Now, consider $\gamma(x) = \phi(\sigma(x))$, then

$$\gamma(x) \to +\infty \quad \text{as} \quad \mathrm{Deg}_{n,\infty}(x) \to +\infty.$$

On the complement of A we have

$$\begin{aligned} \Delta\gamma(x) + \gamma(x) &= \Delta\phi(\sigma(x)) + \phi(\sigma(x)) \\ &\geq \phi'(\sigma(x))\Delta\sigma(x) + \phi(\sigma(x)) \\ &= \phi'(\sigma(x))\Big(\Delta\sigma(x) + \frac{\phi(\sigma(x))}{\phi'(\sigma(x))}\Big) \\ &= \phi'(\sigma(x))(\Delta\sigma(x) + f(\sigma(x)) + \sigma(x)) \\ &\geq \phi'(\sigma(x))(\Delta\sigma(x) + f(\sigma(x)) \geq 0 \end{aligned} \tag{2.3.5}$$

Theorem 2.3.1 applied to $\gamma(x)$ with $\lambda = 1$ implies stochastic completeness. □

In the previous proof, we have made use of the following elementary fact.

Lemma 2.3.5. *Let $f \in C^1([0,+\infty))$ be a positive, increasing function. Assume further that*

$$\int_0^{+\infty} \frac{dr}{f(r)} = +\infty.$$

Then

$$\int_0^{+\infty} \frac{dr}{f(r)+r} = +\infty.$$

For the sake of completeness, we give a proof here.

Proof. Note that the integral is only improper at $+\infty$ since f is positive and increasing on $[0,+\infty)$. Assume that

$$\int_0^{+\infty} \frac{dr}{f(r)+r} < +\infty.$$

For all $x > 0$, we have

$$0 < \frac{x}{2} \cdot \frac{1}{f(x)+x} \leq \int_{\frac{x}{2}}^{x} \frac{dr}{f(r)+r} \leq \int_{\frac{x}{2}}^{+\infty} \frac{dr}{f(r)+r}.$$

The third integral necessarily goes to 0 as x approaches $+\infty$. Thus there exists $r_0 > 0$ such that for any $r > r_0$,

$$\frac{r}{f(r)+r} \leq \frac{1}{2}.$$

It follows that $f(r) \geq r$ for all $r > r_0$. But then

$$\int_{r_0}^{+\infty} \frac{dr}{f(r)+r} \geq \int_{r_0}^{+\infty} \frac{dr}{2f(r)} = +\infty.$$

A contradiction. □

2.4 Stability of stochastic incompleteness

In this section we show that after certain surgeries, a stochastically incomplete graph will remain stochastically incomplete. The weak Omori-Yau maximum principle allows us to pass from the stability of existence of certain functions to the stability of stochastic incompleteness. Roughly speaking, Theorem 2.2.7 implies that a perturbation of bounded global weighted degree does not affect the stochastic incompleteness. This intuition is made explicit by the following theorems.

Theorem 2.4.1. *Let (V, w, μ) be a weighted graph and $W \subseteq V$. Let the subgraph $(W, w|_{W\times W}, \mu|_W)$ be stochastically incomplete. If one of the following two conditions holds, (V, w, μ) is also stochastically incomplete.*

(1) For some $m > n \geq 1$, $\sup\{\mathrm{Deg}^W_{n,\infty}(x) : x \in W, \exists y \in V\backslash W, w(x,y) > 0\} < m$;

(2) There exists $n \geq 1$, such that $\forall x \in W$,

$$\frac{1}{\mu(x)} \sum_{y\in V\backslash W} w(x,y) < n.$$

Proof. (1) Since W is stochastically incomplete there exists a nonnegative function f on W and $\alpha > 0$ such that

$$\sup_{\Omega^W_\alpha} \Delta^W f \leq -\alpha.$$

Here

$$\Omega^W_\alpha = \{x \in W : f(x) > f^* - \alpha\},$$

and

$$\Delta^W f(x) = \frac{1}{\mu(x)} \sum_{y\in W} w(x,y)(f(x) - f(y))$$

for $x \in W$.

Define a function u on V by

$$u(x) = \begin{cases} (f(x) + \dfrac{\alpha}{m} - f^*)_+, & x \in W, \quad (2.4.6)\\ 0, & x \in V\backslash W. \quad (2.4.6') \end{cases}$$

We see that $u^* = \frac{\alpha}{m}$ and

$$\Omega^V_{\frac{\alpha}{m}} = \{x \in V : u(x) > 0\} = \{x \in W : f(x) > f^* - \frac{\alpha}{m}\} \subseteq \{x \in W : \mathrm{Deg}^W_{n,\infty}(x) \geq m\}$$

by Theorem 2.2.7.

Thus for any $x \in \Omega^V_{\frac{\alpha}{m}}$, $y \in V\backslash W$, we have $w(x,y) = 0$. Hence for every $x \in \Omega^V_{\frac{\alpha}{m}}$

$$
\begin{aligned}
\Delta^V u(x) &= \frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(u(x) - u(y)) \\
&= \frac{1}{\mu(x)} \sum_{y\in W} w(x,y)(u(x) - u(y)) \\
&\le \frac{1}{\mu(x)} \sum_{y\in W} w(x,y)(f(x) - f(y)) \\
&= \Delta^W f(x) \le -\alpha.
\end{aligned} \tag{2.4.7}
$$

The stochastic incompleteness of (V, w, μ) then follows from Theorem 2.1.2.

(2) As in (1), there is a nonnegative function f on W and $\alpha > 0$ such that

$$\sup_{\Omega^W_\alpha} \Delta^W f \le -\alpha$$

since W is stochastically incomplete by assumption.

Define a function u on V by

$$
u(x) = \begin{cases} (f(x) + \frac{\alpha}{2n} - f^*)_+, & x \in W, \quad (2.4.8) \\ 0, & x \in V\backslash W. \quad (2.4.8') \end{cases}
$$

We see that $u^* = \frac{\alpha}{2n}$ and

$$\Omega^V_{\frac{\alpha}{2n}} = \{x \in V : u(x) > 0\} = \{x \in W : f(x) > f^* - \frac{\alpha}{2n}\}.$$

So for any $x \in \Omega^V_{\frac{\alpha}{2n}}$,

(2.4.9)

$$
\begin{aligned}
\Delta^V u(x) &= \frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(u(x) - u(y)) \\
&= \frac{1}{\mu(x)} \sum_{y\in W} w(x,y)(u(x) - u(y)) + \frac{1}{\mu(x)} \sum_{y\in V\backslash W} w(x,y)(u(x) - u(y)) \\
&\le \frac{1}{\mu(x)} \sum_{y\in W} w(x,y)(f(x) - f(y)) + \frac{1}{\mu(x)} \sum_{y\in V\backslash W} w(x,y)\frac{\alpha}{2n} \\
&\le \Delta^W f(x) + \frac{\alpha}{2} < -\frac{\alpha}{2}.
\end{aligned}
$$

The stochastic incompleteness of (V, w, μ) then follows from Theorem 2.1.2. □

Remark 2.4.2. Part (2) of Theorem 2.4.1 was first proved by Keller and Lenz [33]. Our proof here is more elementary.

In Theorem 2.4.1 we derive stochastic incompleteness of graphs from that of subgraphs. The weak Omori-Yau maximum principle allows also to obtain implications in the opposite direction, as in the next statement.

Theorem 2.4.3. *Let (V, w, μ) be a stochastically incomplete weighted graph and $m > n \geq 1$. The subgraph*

$$W = \{x \in V : \mathrm{Deg}^V_{n,\infty}(x) \geq m\}$$

with weights $(W, w|_{W\times W}, \mu|_W)$ is stochastically incomplete as well.

Proof. There exists a nonnegative function f on V and $\alpha > 0$ such that

$$\sup_{\Omega^V_\alpha} \Delta^V f \leq -\alpha.$$

We will show that $f|_W$ is a function violating the weak Omori-Yau maximum principle.

From Lemma 2.2.1 and Theorem 2.2.7, we see that

$$\sup_W f = \sup_V f,$$

and

$$\Omega^W_{\frac{\alpha}{m}} = \Omega^V_{\frac{\alpha}{m}}.$$

We claim that for any $x \in \Omega^W_{\frac{\alpha}{m}}$,

$$\Delta^W f(x) \leq \Delta^V f(x) \leq -\alpha.$$

In fact, for $x \in \Omega^W_{\frac{\alpha}{m}}, y \in V\backslash W$, we claim that $f(y) \leq f(x)$. If not

$$f(y) > f(x) > f^* - \frac{\alpha}{m},$$

so that $y \in \Omega^W_{\frac{\alpha}{m}} \subseteq W$, a contradiction.

Then for any $x \in \Omega^W_{\frac{\alpha}{m}}$, we obtain

$$\begin{aligned}
-\alpha \geq \Delta^V f(x) &= \frac{1}{\mu(x)} \sum_{y \in V} w(x,y)(f(x) - f(y)) \\
&= \frac{1}{\mu(x)} \sum_{y \in W} w(x,y)(f(x) - f(y)) \\
&+ \frac{1}{\mu(x)} \sum_{y \in V \setminus W} w(x,y)(f(x) - f(y)) \\
&\geq \frac{1}{\mu(x)} \sum_{y \in W} w(x,y)(f(x) - f(y)) = \Delta^W f(x).
\end{aligned}$$

The stochastic incompleteness of (V, w, μ) then follows from Theorem 2.1.2. □

Remark 2.4.4. The results in this section remains true if we replace the global weighted degree $\mathrm{Deg}_{n,\infty}$ by weighted degree Deg briefly because

$$\mathrm{Deg}(x) \geq \mathrm{Deg}_{n,\infty}(x)$$

for all x and $n \geq 0$. This version of these results are proven in [29].

2.5 Applications to the physical Laplacian

In this section, we apply the weak Omori-Yau maximum principle and Khas'minskii criterion to the physical Laplacian on an (un-weighted) graph. We assume that (V, E) is an undirected, locally finite, connected infinite graph without loops and multi-edges where V is the set of vertices and E is the set of edges. Recall that the physical Laplacian on (V, E) corresponds to the weights that $w(x, y) \in \{0, 1\}$, $\mu(x) \equiv 1$ and $w(x, y) = 1 \Leftrightarrow (x, y) \in E$. Throughout this section, we use (V, E) to denote this weighted graph.

Let ρ be the graph distance on (V, E) as before and fix a point $x_0 \in V$ as a reference point. We will frequently use the notations introduced in Section 1.6.

Recall that the weighted degree function

$$\mathrm{Deg}(x) = \sum_{y \in V} w(x,y) = \#\{y \in V : y \sim x\},$$

is exactly the number of neighbors of x in V, i.e. $\deg(x)$. And the formal Laplacian in this case is

$$\Delta f(x) = \sum_{y, y\sim x} (f(x) - f(y)). \tag{2.5.10}$$

Here f can now be an arbitrary function on V because of the locally finiteness. For example,

$$\Delta r(x) = m_-(x) - m_+(x). \tag{2.5.11}$$

The machinery of the weak Omori-Yau maximum principle and Khas'minskii criterion can be applied in two ways.

(1) Choose a series $\sum_{n=0}^{\infty} a_n$ with nonnegative terms, and define the function

$$f(x) = \sum_{n=0}^{r(x)} a_n$$

which then can be used in the weak Omori-Yau maximum principle and Khas'minskii criterion. Choosing the series appropriately we obtain sufficient conditions for stochastic completeness and incompleteness.

(2) Alternatively, one can determine "natural" values of a_n by solving certain difference equations or inequalities.

Before going into details we would like to point out that for a locally finite graph of unbounded degree, $\deg(x) \to +\infty$ implies $r(x) \to +\infty$. Thus Theorem 2.3.1 can be restated in a weaker form:

Theorem 2.5.1. *Let (V, E) be a locally finite and connected graph. Assume that the degree function* $\deg(x)$ *is unbounded. If there exists a nonnegative function γ on V with*

$$\gamma(x) \to +\infty \quad \textit{as} \quad r(x) \to +\infty$$

satisfying

$$\Delta\gamma(x) + \lambda\gamma(x) \geq 0 \quad \textit{outside a finite set } A$$

for some $\lambda > 0$, then (V, E) is stochastically complete.

Remark 2.5.2. Keller, Lenz and Wojciechowski [35] also obtained independently a similar form of Khas'minskii criterion for general weighted graphs using a different method.

2.5.1 Criteria for stochastic completeness

In what follows, $\sum_{n=0}^{\infty} a_n$ is a series with nonnegative terms.

Theorem 2.5.3. *Let (V,E) be a locally finite and connected graph. If $\sum_{n=0}^{\infty} a_n = +\infty$ and for some $\lambda > 0$, the following inequality*

$$m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} \leq \lambda \sum_{n=0}^{r(x)} a_n$$

holds outside a finite set, then V is stochastically complete.

Proof. Let $\gamma(x) = \sum_0^{r(x)} a_n$, then

$$\Delta\gamma(x) + \lambda\gamma(x) = m_-(x)a_{r(x)} - m_+(x)a_{r(x)+1} + \lambda \sum_0^{r(x)} a_n \geq 0$$

outside a finite set and $\gamma(x) \to +\infty$ as $r(x) \to +\infty$. By Theorem 2.5.1, (V,E) is stochastically complete. □

Theorem 2.5.3 already gives some nontrivial results through some obvious choices of a_n. One natural choice is $a_n \equiv 1$. Then a sufficient condition for stochastic completeness is

$$m_+(x) - m_-(x) \leq \lambda r(x)$$

outside a finite set for some $\lambda > 0$. This improves the curvature type criterion of Weber [53] where the sufficient condition is

$$m_+(x) - m_-(x) \leq C$$

for some constant $C > 0$.

One can improve this result by choosing divergent series with smaller terms. We do this via Theorem 2.3.4.

Theorem 2.5.4. *Let (V,E) be a locally finite and connected graph. If for some positive, increasing function $f \in C^1([0,+\infty))$ with $\int_0^{+\infty} \frac{dr}{f(r)} = +\infty$,*

$$m_+(x) - m_-(x) \leq f(r(x))$$

outside a finite set, then (V,E) is stochastically complete.

Proof. We only need to take $\sigma(x) = r(x)$ in Theorem 2.3.4. □

Remark 2.5.5. Can we conclude that (V, E) is stochastically incomplete if

$$m_+(x) - m_-(x) \geq f(r(x)),$$

where $f(r) > 0$ and $\sum_{r=0}^{\infty} \frac{1}{f(r)} < +\infty$? This may be a useful complement to Theorem 2.5.4.

The following result was first obtained by Wojciechowski [56]. We give a shorter proof, based on Theorem 2.5.1.

Theorem 2.5.6. *Let (V, E) be a locally finite and connected graph. If $\sum_{r=0}^{\infty} \frac{1}{K_+(r)} = +\infty$, then (V, E) is stochastically complete.*

Proof. Let

$$\gamma(x) = \sum_{r=0}^{r(x)-1} \frac{1}{K_+(r)}$$

for $r(x) > 0$, and $\gamma(x_0) = 0$. We then have that

$$\gamma(x) \to +\infty \quad \text{as} \quad r(x) \to +\infty,$$

and outside a finite set

$$\Delta\gamma(x) + \gamma(x) = m_-(x)\frac{1}{K_+(r(x)-1)} - m_+(x)\frac{1}{K_+(r(x))} + \gamma(x) \geq \gamma(x) - 1 \geq 0.$$

The assertion follows from Theorem 2.5.1. □

2.5.2 Criteria for stochastic incompleteness

Similarly, using test series to define functions that violate the weak Omori-Yau maximum principle, we obtain a curvature type criterion for stochastic incompleteness:

Theorem 2.5.7. *Let (V, E) be a locally finite and connected graph. If $\sum_{l=0}^{\infty} a_l < +\infty, a_l \geq 0$ and for some $n \in \mathbb{N}, c > 0$, the inequality*

$$m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} \geq c$$

holds for $r(x) > n$, then (V, E) is stochastically incomplete.

Proof. Let

$$f(x) = \sum_{l=0}^{r(x)} a_l.$$

Then

$$f^* = \sum_{r=0}^{\infty} a_r < +\infty.$$

Let $\alpha = \sum_{l=n+1}^{\infty} a_l$. Then $f(x) > f^* - \alpha$ implies $r(x) > n$. So in this case,

$$-\Delta f(x) = m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} \geq c.$$

By Theorem 2.1.2, (V, E) is stochastically incomplete. □

Theorem 2.1.2 can also be used to derive the following result about stochastic incompleteness obtained by Wojciechowski [55].

Theorem 2.5.8. *Let (V, E) be a locally finite and connected graph. If*

$$\sum_{r=1}^{\infty} \max_{x \in S_r} \frac{m_-(x)}{m_+(x)} < +\infty, \tag{2.5.12}$$

then V is stochastically incomplete.

Proof. Denote $\max_{x \in S_r} \frac{m_-(x)}{m_+(x)}$ by $\eta(r)$. Let

$$f(x) = \sum_{r=1}^{r(x)-1} \eta(r)$$

for $r(x) \geq 2$, and $f(x) = 0$ elsewhere. Then

$$f^* = \sup f(x) = \sum_{r=1}^{\infty} \eta(r) < +\infty.$$

Choose $r_0 > 2$ sufficiently large so that

$$0 < \alpha = \sum_{r=r_0-1}^{\infty} \eta(r) < \frac{1}{2}.$$

Then

$$\begin{aligned}\Omega_\alpha &= \{x \in V : f(x) > f^* - \alpha\} \\ &= \{x \in V : f(x) > \sum_{r=1}^{r_0-2} \eta(r)\} \\ &= \{x \in V : \sum_{r=1}^{r(x)-1} \eta(r) > \sum_{r=1}^{r_0-2} \eta(r)\} \\ &= \{x \in V : r(x) > r_0 - 1\} \\ &= B^c_{r_0-1}.\end{aligned}$$

But for $x \in B^c_{r_0-1}$, we have $r(x) - 1 \geq r_0 - 1$ and thus

$$\eta(r(x) - 1) < \alpha < \frac{1}{2}.$$

Hence we obtain that for $x \in \Omega_\alpha$,

$$\begin{aligned}\Delta f(x) &= m_-(x)\eta(r(x) - 1) - m_+(x)\eta(r(x)) \\ &\leq \frac{1}{2}m_-(x) - m_-(x) \\ &\leq -\frac{1}{2}m_-(x) \leq -\frac{1}{2}.\end{aligned}$$

By Theorem 2.1.2 (V, E) is stochastically incomplete. □

Remark 2.5.9. Theorem 2.5.8 first appeared in a slightly weaker form as Theorem 3.4 in [55]. There stochastic incompleteness is established under the condition

$$\sum_{r=1}^{\infty} \frac{K_-(r)}{k_+(r)} < +\infty$$

instead of (2.5.12).

2.5.3 The weakly symmetric graphs

Recall the Definition 1.6.9 of weakly symmetric graphs and the notations there. For such graphs, Wojciechowski [56] proved the following criterion. Here we present a proof based on the weak Omori-Yau maximum principle.

Theorem 2.5.10. *A weakly symmetric graph* (V, E) *is stochastically complete if* *and only if*

$$\sum_{r=0}^{\infty} \frac{V(r)}{g_+(r)S(r)} = +\infty$$

where $S(r) = \#S_r$ *and* $V(r) = \#B_r$*.*

Proof. Since

$$m_+(x) = g_+(r(x)), m_-(x) = g_-(r(x)),$$

we see that

$$g_-(r)S(r) = g_+(r-1)S(r-1).$$

Let

$$\gamma(x) = \sum_{r=0}^{r(x)-1} \frac{V(r)}{g_+(r)S(r)}$$

for $r(x) > 0$, and $\gamma(x_0) = 0$. We have

$$\begin{aligned} \Delta\gamma(x) &= g_-(r(x))\frac{V(r(x)-1)}{g_+(r(x)-1)S(r(x)-1)} - g_+(r(x))\frac{V(r(x))}{g_+(r(x))S(r(x))} \\ &= \frac{V(r(x)-1)}{S(r(x))} - \frac{V(r(x))}{S(r(x))} = -1 \end{aligned} \tag{2.5.13}$$

for $r(x) \geq 1$.

If $\gamma(x) \to +\infty$ as $r(x) \to +\infty$, then

$$\Delta\gamma(x) + \gamma(x) = \gamma(x) - 1 \geq 0$$

outside a finite set. The stochastic completeness then follows from Theorem 2.5.1.

For the other implication suppose that $\gamma^* = \sup \gamma(x) < +\infty$. Letting $\alpha = \gamma^*$, we see that on $\Omega_\alpha = B_0^c$, by the calculation (2.5.13) above,

$$\Delta\gamma(x) = -1.$$

The stochastic incompleteness then follows from Theorem 2.1.2. □

Remark 2.5.11. Theorem 2.5.10 is analogous to the classical stochastic completeness criterion for model manifolds proven by Grigor'yan [20], Ichihara [30], Khas'minskii [36] (see Proposition 3.2 in [22]). As pointed out by Wojciechowski [56], it is interesting to notice that for a weakly symmetric graph, the edges between points on the same sphere play no role in stochastic completeness. See also [35] for further studies of weakly symmetric graphs.

Now we apply Theorem 2.5.10 to the Examples 1.6.10 and 1.6.11. The criteria here are first due to Wojciechowski [54], [56].

Example 2.5.12 (Model Trees)**.** Let (V, E) and f be as in Example 1.6.10. Then (V, E) is stochastically complete if and only if

$$\sum_{r=0}^{\infty} \frac{1}{f(r)} = +\infty. \tag{2.5.14}$$

Proof. We have that $g_+(r) = f(r)$ for $r \in \mathbb{N}$ and

$$S(r) = \prod_{n=0}^{r-1} f(n) = f(r-1)S(r-1),$$

for $r \in \mathbb{N}_+$. By Theorem 2.5.10, the stochastic completeness of (V, E) is equivalent to that

$$\sum_{r=1}^{\infty} \frac{V(r)}{f(r)S(r)} = +\infty.$$

First assume that there exists some $N \in \mathbb{N}_+$ such that for all $n > N$, $f(n) \geq 2$. Then for $r \geq N+1$,

$$S(r) \leq V(r) = \sum_{n=0}^{r} S(n) \leq \sum_{n=N+1}^{r} \frac{S(r)}{2^{r-n}} + V(N) \leq 2S(r) + C,$$

where $C = V(N)$. And

$$\sum_{r=N+1}^{\infty} \frac{1}{f(r)} \leq \sum_{r=N+1}^{\infty} \frac{V(r)}{f(r)S(r)} \leq 2 \sum_{r=N+1}^{\infty} \frac{1}{f(r)} + C \sum_{r=N+1}^{\infty} \frac{1}{f(r)S(r)}.$$

Note that in this case

$$\sum_{r=0}^{\infty} \frac{1}{f(r)S(r)} = \sum_{r=1}^{\infty} \frac{1}{S(r)} \leq \sum_{r=1}^{\infty} \frac{1}{2^r} < \infty.$$

Hence the stochastic completeness of (V, E) is equivalent to that

$$\sum_{r=0}^{\infty} \frac{1}{f(r)} = +\infty.$$

For the rest case, that is, there is a strictly increasing sequence of $\{n_k \in \mathbb{N}\}_{k \in \mathbb{N}}$ such that $f(n_k) = 1$. In this case, it is easy to see that (2.5.15) holds. On the other hand, by Proposition 2.2.8, (V, E) is stochastically complete. This completes the proof. □

Example 2.5.13 (Anti-trees)**.** Let (V, E) and $S(n)$ be as in Example 1.6.11. Then by Theorem 2.5.10, the stochastic completeness of (V, E) is equivalent to that

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} S(n)}{S(r)S(r+1)} = +\infty. \tag{2.5.15}$$

Remark 2.5.14. Taking $S(r) = [(r+1)^{2+\varepsilon}]$ with some $\varepsilon > 0$, we see that (V, E) is stochastically incomplete whereas

$$V(r) \leq Cr^{3+\varepsilon}$$

for some $C > 0$. This interesting fact, due to Wojciechowski [56], is significantly different from the volume growth criteria in the classical case. It is then interesting to ask what is the smallest possible volume growth with respect to the graph distance for stochastically incomplete graphs. It is natural to conjecture that for the physical Laplacian on graphs, the condition

$$\mu(B_r) \leq Cr^3, C > 0, \tag{2.5.16}$$

implies stochastic completeness. This is first proven by Grigor'yan, Huang and Masamune [25]. One goal of the next chapters is to offer two proofs through different approachs.

Note that for geodesically complete Riemannian manifolds, the almost sharp condition ([5], [19], [27], [32])

$$\mu(B_r) \leq \exp Cr^2, C > 0, \tag{2.5.17}$$

with respect to the geodesic distance implies stochastic completeness. The big d-ifference between (2.5.16) and (2.5.17) indicates that the graph distance should be replaced by some distances which are more suitable to give volume growth criteria for stochastic completeness.

Remark 2.5.15. This construction of Wojciechowski [56], at the same time gives a counterexample to the converse to Theorem 2.5.8. The graph (V, E) with $S(r) = (r+1)^3$ satisfies

$$\sum_{r=1}^{\infty} \max_{x \in S_r} \frac{m_-(x)}{m_+(x)} = \sum_{r=1}^{\infty} \frac{(r-1)^3}{(r+1)^3} = +\infty,$$

but is stochastically incomplete.

Remark 2.5.16. On the other hand, there exist stochastically complete graphs with arbitrarily large volume growth. For example, take a set of vertices $\{0, 1, 2, \cdots, n \cdots\}$ with edges $n \sim n+1$. For each vertex n, we associate a distinct finite set V_n and add extra edges between n and points in V_n. The resulting graph (V, E) is then a tree whose volume growth can be chosen to be arbitrarily large. One can observe that every vertex in V has at most two neighbors which have degree larger than 1. It is then of bounded global weighted degree with parameter 1 and hence is stochastically complete by Theorem 2.2.7. The stochastic completeness of V can be shown via Theorem 2.4.3 as well.

For a weakly symmetric graph (V, E), the quantity $g_+(r)S(r)$ can be interpreted as $\#\partial B_r$ where ∂B_r is the boundary of B_r as in Definition 1.1.1. Then the stochastic completeness of a weakly symmetric graph (V, E) is equivalent to

$$\sum_{r=0}^{\infty} \frac{V(r)}{\#\partial B_r} = +\infty.$$

In an earlier version of [35], Keller and Wojciechowski proposed the following conjecture for a general (V, E).

Conjecture 2.5.17. *Let (V, E) be a locally finite and connected graph. If for some fixed point $x_0 \in V$ as a reference point,*

$$\sum_{r=0}^{\infty} \frac{\#B_r}{\#\partial B_r} = +\infty, \tag{2.5.18}$$

then (V, E) is stochastically complete.

This is an analogue of a conjecture for the stochastic completeness of manifold proposed by Grigor'yan in [22]. However, recently Bär and Bessa [2] constructed a counterexample to Grigor'yan's conjecture. Their idea can also be applied to the physical Laplacian as follows.

Take a stochastically complete tree (V_1, E_1) with a reference point x_1, for example, a binary tree. Then (V_1, E_1) has exponential volume growth with respect to the graph distance. Choose a stochastically incomplete graph with only polynomial volume growth, for example, an anti-tree (V_2, E_2) as in Example 1.6.11 with $S(r) = (r+1)^3$. Denote the reference point by x_2. Now we make a single extra edge between x_1 and x_2 resulting in a new graph (V, E). Since the gluing happens at only one point at (V_2, E_2), the graph (V, E) is stochastically incomplete by Theorem 2.4.1. However, for any fixed point $x_0 \in V$ as a reference point, the quantities $\#B_r$ and $\#\partial B_r$ are always of the order 2^n. So we know that V satisfies (2.5.18) while it is stochastically incomplete. This example is simpler than the example of [2] in the manifold case, thanks to special features of the discrete setting.

Chapter 3

Uniqueness class

As shown in Theorem 1.5.1, stochastic completeness of a weighted graph (V, w, μ) is closely connected to the Cauchy problem of the heat equation with zero initial condition (1.4.7) where Δ is the formal Laplacian associated to (V, w, μ). The main result of this chapter is the following uniqueness class for the Cauchy problem of the heat equation with zero initial condition.

Theorem 3.1. *Let (V, w, μ) be a weighted graph such that its underlying graph is locally finite and connected. We also assume that d is an adapted distance on V such that Assumption 1.6.6 holds. Let $u(x,t)$ be a solution to the Cauchy problem of the heat equation with zero initial condition (1.4.7) on $V \times [0,T]$ for some $T > 0$. If there are an increasing sequence of positive numbers $\{R_n\}_{n\in\mathbb{N}}$ with*

$$\lim_{n\to\infty} R_n = +\infty,$$

and two constants $C > 0$, $1 > \epsilon > 0$ such that for some $x_0 \in V$,

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x,t)\mu(x)dt \leq C \exp\left(\frac{1}{2}(1-\epsilon)R_n \ln R_n\right), \tag{3.0.1}$$

then $u(x,t) \equiv 0$ on $V \times [0,T]$.

This uniqueness class is given in an integrated form in the spirit of the classical results of Oleinik and Radkevich [40], Gushchin [26] and Grigor'yan [19, 22]. Tichonov [51] is among the first to obtain

$$|u(x,t)| \leq \exp\left(c|x|^2\right)$$

type uniqueness class the Cauchy problem of heat equation on Euclidean spaces.

Täcklind [50] fist obtain the sharp form

$$|u(x,t)| \leq \exp f(|x|)$$

where f is a positive increasing function on $[0,\infty)$ such that

$$\int^{\infty} \frac{rdr}{f(r)} = \infty.$$

We first establish a discrete integrated maximum principle, Lemma 3.1.1 inspired by Grigor'yan [21] and use it as a key tool to prove Theorem 3.1. The integrated maximum principle on manifolds dates back to Aronson [1]. The discreteness of the formal Laplacian causes serious difficulties and the resulting integrated maximum principle is significantly different from the manifold case. Then we show by an easy example that Theorem 3.1 is almost sharp. The example is based on the classical example of Tichonov but has a much different feature.

3.1 Integrated maximum principle

Let (V, w, μ) be a weighted graph such that the underlying graph is locally finite and connected. Recall that for a subset A of V, the closure $\bar{A}$ of A is

$$\bar{A} = A \cup \partial A = \{x : x \in A; \text{ or } x \in A^c, \text{ and } \exists y \in A, \text{ s.t. } x \sim y\}.$$

The following lemma can be viewed as a discrete version of Grigor'yan's "integrated maximum principle" [21].

Lemma 3.1.1. *Let (V, w, μ) be a weighted graph that is locally finite and connected. Let $\Omega \subseteq V$ be a subset of V. Fix some $T > 0$. Let $u(x,t)$ be a function on $\bar{\Omega} \times (0,T]$ that is differentiable in t on $(0,T]$ and has pointwise zero right limit at time 0. Assume further that $u(x,t)$ solves the heat equation*

$$\frac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0, \tag{3.1.2}$$

on $\Omega \times (0,T]$. Take two auxiliary functions $\eta(x)$ on V and $\xi(x,t)$ on $V \times [0,T]$ such that

(1) the function $\eta(x) \geq 0$ is finitely supported and $\mathrm{supp}\eta \subseteq \Omega$;

(2) $\xi(x,t)$ is continuously differentiable in t on $[0,T]$ for each $x \in V$;

(3) *the inequality* $(\eta^2(x) - \eta^2(y))(e^{\xi(x,t)} - e^{\xi(y,t)}) \geq 0$ *holds for all* $x \sim y$ *and* $t \in [0, T]$*;*

(4) *the inequality* $\mu(x)\frac{\partial}{\partial t}\xi(x,t) + \frac{1}{2}\sum_{y\in V} w(x,y)(1 - e^{\xi(y,t)-\xi(x,t)})^2 \leq 0$ *holds for any* $x \in V$ *and* $t \in [0, T]$.

Then for any $\tau \in (0, T]$*, we have the following estimate:*

(3.1.3)

$$\sum_{x\in\Omega} u^2(x,\tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x) \leq 2\int_0^\tau \sum_{x\in\bar{\Omega}}\sum_{y\in\bar{\Omega}} w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)}dt.$$

Remark 3.1.2. In the same spirit of generalizing Grigor'yan's work to the graph setting, Folz [14] develops a different version of "integrated maximum principle" independently of us.

Before giving a proof of Lemma 3.1.1, we present an elementary fact that we will use frequently.

Lemma 3.1.3. *Let* $\{a_i\}_{i=1}^\infty$ *and* $\{b_i\}_{i=1}^\infty$ *be two sequences of real numbers such that*

$$\sum_{i=1}^\infty a_i^2 < \infty, \sum_{i=1}^\infty b_i^2 < \infty.$$

The inequality

$$\sum_{i=1}^\infty a_i b_i \leq \frac{\delta}{2}\sum_{i=1}^\infty a_i^2 + \frac{1}{2\delta}\sum_{i=1}^\infty b_i^2 \tag{3.1.4}$$

holds for all $\delta > 0$.

Proof. By the Cauchy-Schwarz inequality we have that

$$\sum_{i=1}^\infty a_i b_i \leq \left(\delta\sum_{i=1}^\infty a_i^2\right)^{\frac{1}{2}}\left(\frac{1}{\delta}\sum_{i=1}^\infty b_i^2\right)^{\frac{1}{2}}.$$

Then the desired inequality follows by the AM-GM inequality. □

Proof of Lemma 3.1.1. Note that by an argument similar to that of Remark 1.4.8, $u(x,t)$ can be extended by 0 to satisfy the heat equation (3.1.2) on $\Omega \times [0, T]$. We multiply the heat equation in (3.1.2) by $u(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)$ and sum over $x \in \Omega$:

(3.1.5) $$\sum_{x\in\Omega}\frac{\partial}{\partial t}u(x,t)\cdot u(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)$$

$$+\sum_{x\in\Omega}\sum_{y\in\bar{\Omega}} w(x,y)(u(x,t)-u(y,t))\cdot u(x,t)\eta^2(x)e^{\xi(x,t)} = 0.$$

Note that since $\eta(x)$ is finitely supported, the sums in (3.1.5) are of finite type. Furthermore, since supp$\eta \subseteq \Omega$, if we make a sum over $x \in \Omega$ of some multiple of $\eta^2(x)$, it is equivalent to do it over $\bar{\Omega}$. By symmetry of $w(x,y)$, we have

$$\begin{aligned}(3.1.6)\quad &\sum_{x\in\bar{\Omega}} \frac{\partial}{\partial t}u^2(x,t)\cdot\eta^2(x)e^{\xi(x,t)}\mu(x)\\ &+\sum_{x\in\bar{\Omega}}\sum_{y\in\bar{\Omega}} w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)}-u(y,t)\eta^2(y)e^{\xi(y,t)}) = 0.\end{aligned}$$

In this proof, further through, the sums without specification of range will be understood to be over $\bar{\Omega}$.

Using the fact that

$$\frac{\partial}{\partial t}u^2(x,t)\cdot e^{\xi(x,t)} = \frac{\partial}{\partial t}(u^2(x,t)e^{\xi(x,t)}) - u^2(x,t)\cdot e^{\xi(x,t)}\frac{\partial}{\partial t}\xi(x,t),$$

and the finiteness of the sums, we obtain

$$\begin{aligned}(3.1.7)\quad &\frac{\partial}{\partial t}(\sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)) = \sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\frac{\partial}{\partial t}\xi(x,t)\\ &-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)}-u(y,t)\eta^2(y)e^{\xi(y,t)}).\end{aligned}$$

We split the sum in the last line:

$$(3.1.8)\quad -\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)}-u(y,t)\eta^2(y)e^{\xi(y,t)})$$

$$(3.1.9)\quad = -\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)}$$

$$(3.1.10)\quad -\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)}$$

$$(3.1.11)\quad -\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)\eta^2(y)(e^{\xi(x,t)}-e^{\xi(y,t)}),$$

which is a discrete analogue to the Leibniz rule. Then we apply Lemma 3.1.3 to (3.1.10) and (3.1.11) to cancel the term in (3.1.9).

First, for any $\delta_1 > 0$,

$$
\begin{aligned}
&-\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)} \\
&\le \frac{\delta_1}{2}\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta(x)+\eta(y))^2 e^{\xi(x,t)} \\
&+ \frac{1}{2\delta_1}\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(y,t)e^{\xi(x,t)}.
\end{aligned}
$$

Applying the elementary fact

$$
(\eta(x)+\eta(y))^2 \le 2(\eta^2(x)+\eta^2(y)),
$$

we have that by the symmetry of $w(x,y)$,

$$
\begin{aligned}
&-\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)} \\
&\le \delta_1 \sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)+\eta^2(y))e^{\xi(x,t)} \\
&+ \frac{1}{2\delta_1}\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(y,t)e^{\xi(x,t)} \\
&= \frac{\delta_1}{2}\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)+\eta^2(y))(e^{\xi(x,t)}+e^{\xi(y,t)}) \\
&+ \frac{1}{2\delta_1}\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(y,t)e^{\xi(x,t)}.
\end{aligned}
$$

Using the condition (3) of Lemma 3.1.1, it follows that

$$
\begin{aligned}
&-\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)} \\
&\le \delta_1 \sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)e^{\xi(x,t)}+\eta^2(y)e^{\xi(y,t)}) \\
&+ \frac{1}{2\delta_1}\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(y,t)e^{\xi(x,t)} \\
&= 2\delta_1 \sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)} \\
&+ \frac{1}{2\delta_1}\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)}.
\end{aligned}
\tag{3.1.12}
$$

Similarly, for any $\delta_2 > 0$,

$$
\begin{aligned}
&-\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)\eta^2(y)(e^{\xi(x,t)}-e^{\xi(y,t)}) \\
&= -\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))u(x,t)\eta^2(x)(e^{\xi(x,t)}-e^{\xi(y,t)}) \\
&\le \frac{\delta_2}{2}\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)} \\
&+ \frac{1}{2\delta_2}\sum_x \sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2 u^2(x,t)\eta^2(x)e^{\xi(x,t)}.
\end{aligned}
$$

Choose $\delta_1 = 1/4$ and $\delta_2 = 1$ and apply the above estimates of (3.1.10) and (3.1.11) to (3.1.8). It follows that

$$
\begin{aligned}
&-\sum_x \sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)}-u(y,t)\eta^2(y)e^{\xi(y,t)}) \\
&\quad\le 2\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)} \\
&\quad+ \frac{1}{2}\sum_x \sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2 u^2(x,t)\eta^2(x)e^{\xi(x,t)}.
\end{aligned}
$$

And hence by (3.1.7),

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)) \\
&\le \sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\frac{\partial}{\partial t}\xi(x,t) \\
&+ 2\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)} \\
&+ \frac{1}{2}\sum_x \sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2 u^2(x,t)\eta^2(x)e^{\xi(x,t)} \\
&\le 2\sum_x \sum_y w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)},
\end{aligned}
$$

where in the last inequality we used the condition (4) on the auxiliary function. Integrate the above inequality with respect to t on $[0,\tau]$, we get the desired inequality as $u(x,0) \equiv 0$. □

Remark 3.1.4. The idea behind Lemma 3.1.1 is the Caccioppoli type estimate Lemma 1.6.1. In practical applications, $\eta(x)$ is often chosen to be some kind of "cut off" function. As explained before, the estimate Lemma 3.1.1 together with the "cut off" property of $\eta(x)$, allow us to quantitatively compare the "local mass"

$$\sum_x u^2(x,t)\mu(x)$$

at different space-time regions.

3.2 Uniqueness class

Now we can prove the main Theorem 3.1 in this chapter by specifying the auxiliary functions in Lemma 3.1.1. In the following, (V, w, μ) will be a weighted graph that is locally finite and connected, and d is an adapted distance on V to make Assumption 1.6.6 hold.

Lemma 3.2.1. *Fix a point $x_0 \in V$. Define*

$$\xi(x,t) = -\frac{1}{2}\alpha^2 e^{2\alpha} t - \alpha\left(d(x,x_0) - \delta R\right)_+ .$$

where $\alpha > 0$, $R > 0$ and $0 < \delta < \frac{1}{2}$ are to be chosen later. The condition (2) of Lemma 3.1.1 clearly holds for $\xi(x,t)$. And $\xi(x,t)$ also satisfies the condition (4) in Lemma 3.1.1, that is, the following inequality holds for any $x \in V$ and $t \in [0,T]$:

$$\mu(x)\frac{\partial}{\partial t}\xi(x,t) + \frac{1}{2}\sum_{y\in V} w(x,y)(1 - e^{\xi(y,t)-\xi(x,t)})^2 \le 0. \tag{3.2.13}$$

Proof. Note the following elementary inequality for $a \in \mathbb{R}$:

$$(e^a - 1)^2 \le e^{2|a|}a^2.$$

Hence

$$\begin{aligned}\frac{1}{2}\sum_{y\in V} w(x,y)(1 - e^{\xi(y,t)-\xi(x,t)})^2 &\le \frac{1}{2}\sum_{y\in V} w(x,y)(\xi(y,t) - \xi(x,t))^2 e^{2|\xi(y,t)-\xi(x,t)|}\\ &\le \frac{1}{2}\sum_{y\in V} w(x,y)\alpha^2 d^2(x,y) e^{2\alpha d(x,y)}\\ &\le \frac{1}{2}\alpha^2\mu(x)e^{2\alpha},\end{aligned}$$

where in the last inequality we used the facts that

$$w(x,y) > 0 \Leftrightarrow x \sim y$$

and that $d(x,y) \leq 1$ when $x \sim y$. Then the inequality (3.2.13) follows easily. □

Remark 3.2.2. Note that $e^{\xi(x,t)}$ is decaying quickly in space. This will help cancel the growth of

$$\int_0^T \sum_{x \in B_d(x_0,R_n)} u^2(x,t)\mu(x)dt.$$

Lemma 3.2.3. *Fix a point $x_0 \in V$. Define*

$$\eta(x) = \min\{\frac{(R-1-d(x,x_0))_+}{\delta R}, 1\},$$

where $R > 0$ and $0 < \delta < \frac{1}{2}$ to be chosen are the same as in Lemma 3.2.1. Then $\eta(x)$ satisfies the following inequality for any $x \in V$:

$$\sum_{y \in V} w(x,y)(\eta(x) - \eta(y))^2 \leq \frac{1}{\delta^2 R^2}\mu(x)\chi_{\{(1-\delta)R-2 \leq d(x,x_0) \leq R\}}, \tag{3.2.14}$$

where χ is the characteristic function of a set.

Proof. Without loss of generality, we can assume that $R > 0$ is large enough. For $x \in V$ such that

$$d(x,x_0) < (1-\delta)R - 2,$$

we see that if $y \sim x$,

$$d(y,x_0) \leq d(x,x_0) + d(x,y) \leq (1-\delta)R - 1.$$

Hence $\eta(x) = \eta(y) = 1$. Similarly, for x such that

$$d(x,x_0) > R,$$

if $y \sim x$,

$$d(y,x_0) \geq d(x,x_0) - d(x,y) \geq R - 1.$$

And then $\eta(x) = \eta(y) = 0$.

Finally, for x such that

$$(1-\delta)R - 2 \leq d(x,x_0) \leq R,$$

we have

$$\sum_{y \in V} w(x,y)(\eta(x) - \eta(y))^2 \leq \frac{1}{\delta^2 R^2} \sum_{y \in V} w(x,y)d^2(x,y) \leq \frac{1}{\delta^2 R^2}\mu(x).$$

□

Proof of Theorem 3.1. Recall the notations in Theorem 3.1, Lemma 3.2.1 and Lemma 3.2.3. Let x_0 be the fixed point there. Let $0 < \delta < \frac{1}{2}$ be small enough such that

$$(1 - \frac{\epsilon}{4})(1 - 2\delta) \geq 1 - \frac{\epsilon}{2}. \tag{3.2.15}$$

Define $\xi(x, t)$ as in Lemma 3.2.1 with $R > \frac{2}{1-2\delta}$ and $\alpha > 0$ to be chosen later. Define $\eta(x)$ as in Lemma 3.2.3.

It is easy to check the conditions (1), (2) and (3) in Lemma 3.1.1 and the condition (3) is proven in Lemma 3.2.1. So we can apply Lemma 3.1.1 in the special case of $\Omega = V$ to assert that for any $\tau \in (0, T]$,

$$\sum_{x \in V} u^2(x, \tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x) \leq 2\int_0^\tau \sum_{x \in V}\sum_{y \in V} w(x, y)(\eta(x) - \eta(y))^2 u^2(x, t)e^{\xi(y,t)}dt. \tag{3.2.16}$$

Note that since $R > \frac{2}{1-2\delta}$, we have $\eta(x) = 1$ when $d(x, x_0) \leq \delta R$. For the left side of (3.2.16), we have

$$\sum_{x \in B_d(x_0,\delta R)} u^2(x, \tau)\mu(x)e^{-\frac{1}{2}\tau\alpha^2 e^{2\alpha}} \leq \sum_{x \in V} u^2(x, \tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x).$$

For the right side of (3.2.16), the following estimate holds

$$\begin{aligned}
&2\int_0^\tau \sum_{x \in V}\sum_{y \in V} w(x, y)(\eta(x) - \eta(y))^2 u^2(x, t)e^{\xi(y,t)}dt \\
\leq &2e^\alpha \int_0^\tau \sum_{x \in V}\sum_{y \in V} w(x, y)(\eta(x) - \eta(y))^2 u^2(x, t)e^{\xi(x,t)}dt \\
= &2e^\alpha \int_0^\tau \sum_{x \in V} u^2(x, t)e^{\xi(x,t)} \sum_{y \in V} w(x, y)(\eta(x) - \eta(y))^2 dt \\
\leq &\frac{2}{\delta^2 R^2} e^\alpha \int_0^\tau \sum_{x \in V} \mu(x)\chi_{\{(1-\delta)R-2 \leq d(x,x_0) \leq R\}} u^2(x, t)e^{-\alpha(d(x,x_0)-\delta R)_+} dt \\
\leq &\frac{2}{\delta^2 R^2} e^{3\alpha-(1-2\delta)\alpha R} \int_0^\tau \sum_{x \in B_d(x_0,R)} \mu(x)u^2(x, t)dt.
\end{aligned}$$

Put them together, we have

$$\sum_{x \in B_d(x_0,\delta R)} u^2(x, \tau)\mu(x)e^{-\frac{1}{2}\tau\alpha^2 e^{2\alpha}} \leq \frac{2}{\delta^2 R^2} e^{3\alpha-(1-2\delta)\alpha R} \int_0^\tau \sum_{x \in B_d(x_0,R)} \mu(x)u^2(x, t)dt. \tag{3.2.17}$$

Let R_n be as in Theorem 3.1. As in (3.0.1),

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x,t)\mu(x)dt \le C\exp\left(\frac{1}{2}(1-\epsilon)R_n\ln R_n\right).$$

For n big enough such that $R_n > \max\{\frac{2}{1-2\delta},1\}$, choose $\alpha = \frac{1}{2}(1-\frac{\epsilon}{4})\ln R_n$ and let $R = R_n$ in (3.2.17). We have

$$\begin{aligned}
&\sum_{x\in B_d(x_0,\delta R_n)} u^2(x,\tau)\mu(x)\\
&\le \frac{2}{\delta^2 R_n^2} e^{\frac{1}{2}\tau\alpha^2 e^{2\alpha}+3\alpha-(1-2\delta)\alpha R}\int_0^\tau \sum_{x\in B_d(x_0,R_n)} \mu(x)u^2(x,t)dt\\
&\le \frac{2C}{\delta^2 R_n^2}\exp\left(\frac{1}{2}\tau\alpha^2 e^{2\alpha}+3\alpha-(1-2\delta)\alpha R+\frac{1}{2}(1-\epsilon)R_n\ln R_n\right)\\
&\le \frac{2C}{\delta^2 R_n^2}\exp\{\frac{1}{8}\tau(1-\frac{\epsilon}{4})^2(\ln R_n)^2 R_n^{1-\frac{\epsilon}{4}}+\frac{3}{2}(1-\frac{\epsilon}{4})\ln R_n\\
&+\frac{1}{2}(1-\epsilon)R_n\ln R_n-\frac{1}{2}(1-2\delta)(1-\frac{\epsilon}{4})R_n\ln R_n\}.
\end{aligned} \tag{3.2.18}$$

Apply (3.2.15), we arrive at

$$\begin{aligned}
&\sum_{x\in B_d(x_0,\delta R_n)} u^2(x,\tau)\mu(x)\\
&\le \frac{2}{\delta^2 R_n^2}\exp\left(\frac{1}{8}\tau(1-\frac{\epsilon}{4})^2(\ln R_n)^2 R_n^{1-\frac{\epsilon}{4}}+\frac{3}{2}(1-\frac{\epsilon}{4})\ln R_n-\frac{\epsilon}{4}R_n\ln R_n\right).
\end{aligned} \tag{3.2.19}$$

Let n approaches to $+\infty$. Since R_n increases to $+\infty$, we can see that the right side of (3.2.19) tends to zero while the left side is nonnegative and nondecreasing. Hence

$$u(x,\tau)\equiv 0$$

for all $x\in V$. Note that τ is arbitrarily chosen in $(0,T]$, the theorem follows. □

3.3 A sharpness example

The example here is a discrete analogue of the classical construction of Tichonov [51]. See also the textbook of John [31]. However, the discrete case turns out to have a different behavior.

We equip $\mathbb{Z}$ with a graph structure such that for $m, n \in \mathbb{Z}$,

$$m \sim n \Leftrightarrow |m - n| = 1.$$

Let the weights $\mu(n) \equiv 1$ and $w(m, n) \in \{0, 1\}$. The heat equation takes a simple form in this case:

$$\frac{\partial}{\partial t} u(n, t) + 2u(n, t) - u(n - 1, t) - u(n + 1, t) = 0. \tag{3.3.20}$$

As before, we have a natural graph distance ρ on $\mathbb{Z}$ which is just given by $\rho(m, n) = |m - n|$. By choosing $\sigma(n, n + 1) = \frac{\sqrt{2}}{2}$ in Definition 1.6.4, we have that $d = \frac{\sqrt{2}}{2}\rho$ is an adapted distance on $(\mathbb{Z}, w, \mu)$. It is also direct to see that Assumption 1.6.6 holds.

Let $g(t)$ be a smooth function on $\mathbb{R}$ such that all orders of derivatives of it goes to 0 at $t = 0$. For example, we can take

$$g(t) = \begin{cases} \exp(-t^{-\beta}), & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{3.3.21}$$

where $\beta > 1$ is a constant.

For $0 < T < +\infty$, we define a function $u(n, t)$ on $\mathbb{Z} \times [0, T]$ as follows:

$$u(n, t) = \begin{cases} g(t), & n = 0, \\ g(t) + \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!}(n + k) \cdots (n + 1)n \cdots (n - k + 1), & n \geq 1, \\ u(-n - 1, t), & n \leq -1. \end{cases} \tag{3.3.22}$$

In the above definition, the function $(n+k) \cdots (n+1)n \cdots (n-k+1)$ plays the role of the power x^{2k} in the continuous setting. However, the main difference is that

$$(n + k) \cdots (n + 1)n \cdots (n - k + 1)$$

vanishes for all $k > |n|$. Hence the sums in (3.3.22) are in fact all of finite type. It is an elementary calculation to check that $u(n, t)$ solves the heat equation. By the property of $g(t)$, we see that $u(n, t)$ also satisfies the zero initial condition.

We choose $g(t)$ as in (3.3.21). The following estimate is taken from [31] (p.172): for all $0 < \theta < 1$ small enough,

$$|g^{(k)}(t)| < \frac{k!}{(\theta t)^k} \exp(-\frac{1}{2} t^{-\beta}), \tag{3.3.23}$$

holds for all $k \in \mathbb{N}$. Hence for $n \geq 1$, $t > 0$, we have
(3.3.24)

$$|u(n,t)| \leq \exp(-t^{-\beta}) + \sum_{k=1}^{n} \frac{k!}{(2k)!(\theta t)^k}(n+k)\cdots(n+1)n\cdots(n-k+1)\exp(-\frac{1}{2}t^{-\beta}).$$

We are going to make an estimate of $|u(n,t)|$ independent of t. The following estimate is elementary.

Lemma 3.3.1. *Let $k > 0, \theta > 0$. Then for $t > 0$,*

$$\frac{1}{(\theta t)^k}\exp(-\frac{1}{2}t^{-\beta}) \leq \left(\frac{2k}{e\beta\theta^\beta}\right)^{\frac{k}{\beta}}.$$

Proof. Note that

$$\frac{1}{t^k}\exp(-\frac{1}{2}t^{-\beta}) = \exp(-k\ln t - \frac{1}{2}t^{-\beta}).$$

Consider

$$(-k\ln t - \frac{1}{2}t^{-\beta})' = -\frac{k}{t} + \frac{\beta}{2}t^{-\beta-1}\begin{cases} > 0, & 0 < t < \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}, \\ = 0, & t = \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}, \\ < 0, & t > \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}. \end{cases}$$

Then we see that $\frac{1}{t^k}\exp(-\frac{1}{2}t^{-\beta})$ attains its maximum at $\left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}$. The assertion follows. □

So now we have that

$$|u(n,t)| \leq 1 + \sum_{k=1}^{n} \frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^\beta}\right)^{\frac{k}{\beta}}. \tag{3.3.26}$$

Roughly speaking, the following lemma shows that the last term in the estimate (3.3.26) of $|u(n,t)|$ is the dominating one.

Lemma 3.3.2. *Fix some $n \geq 2$. Let $1 \leq k \leq n-1$. Denote*

$$\frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^\beta}\right)^{\frac{k}{\beta}}$$

by a_k. Then for $0 < \theta < 1$ small enough, we have that for any $1 \leq k \leq n-1$,

$$a_k \leq a_{k+1}.$$

Proof. By direct calculations,

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\frac{(k+1)!}{(2k+2)!}(n+k+1)\cdots(n+1)n\cdots(n-k)\left(\frac{2k+2}{e\beta\theta^\beta}\right)^{\frac{k+1}{\beta}}}{\frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^\beta}\right)^{\frac{k}{\beta}}}\\ &= \frac{(k+1)(n+k+1)(n-k)}{(2k+2)(2k+1)}\times(1+\frac{1}{k})^{\frac{k}{\beta}}\left(\frac{2}{e\beta\theta^\beta}\right)^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}}\\ &\geq \frac{(k+1)(2k+2)}{(2k+2)(2k+1)}\left(\frac{2}{e\beta\theta^\beta}\right)^{\frac{1}{\beta}}\\ &\geq \frac{1}{2\theta}\left(\frac{2}{e\beta}\right)^{\frac{1}{\beta}}.\end{aligned}$$

Hence for

$$0<\theta<\frac{1}{2}\left(\frac{2}{e\beta}\right)^{\frac{1}{\beta}},$$

we have

$$\frac{a_{k+1}}{a_k}\geq 1.$$

□

By Lemma 3.3.1 and Lemma 3.3.2, we have for $\theta > 0$ small enough, for all $n \geq 1, t > 0$,

$$\begin{aligned}|u(n,t)| &\leq 1+na_n\\ &= 1+\frac{n!}{(2n)!}(2n)!\left(\frac{2n}{e\beta\theta^\beta}\right)^{\frac{n}{\beta}}\\ &= 1+n!\left(\frac{2n}{e\beta\theta^\beta}\right)^{\frac{n}{\beta}}.\end{aligned}$$

For $\theta > 0$ small enough, it is direct to see that the sequence

$$1+n!\left(\frac{2n}{e\beta\theta^\beta}\right)^{\frac{n}{\beta}}$$

is nondecreasing.

Choose $x_0 = 0$, and a sequence $R_m = \frac{\sqrt{2}}{2}m$. For any $T \in (0, +\infty)$, we have

$$\int_0^T \sum_{x\in B_d(x_0,R_m)} u^2(x,t)\mu(x)dt \leq \int_0^T \sum_{n=-m}^{m} u^2(n,t)dt$$

$$
\begin{aligned}
&\le 2\int_0^T \sum_{n=0}^{m} u^2(n,t)dt \\
&\le 2\int_0^T (1+m)\left(1+m!\left(\frac{2m}{e\beta\theta^\beta}\right)^{\frac{m}{\beta}}\right)^2 dt \\
&\le 2T(1+m)\left(1+m!\left(\frac{2m}{e\beta\theta^\beta}\right)^{\frac{m}{\beta}}\right)^2 .
\end{aligned}
$$

Apply Stirling's approximation to the last term in the above inequality, we have that for all big enough m,

$$
\int_0^T \sum_{x\in B_d(x_0,R_m)} u^2(x,t)\mu(x)dt \le CT\exp\left(2\sqrt{2}(1+\frac{1}{\beta})(1+\epsilon)R_m\ln R_m\right), \tag{3.3.27}
$$

where $\epsilon > 0$ is a constant and $C > 1$ is a large enough constant depending on ϵ.

Remark 3.3.3. Note that the gap between the estimate for a nonzero solution (3.3.27) and the uniqueness class bound (3.0.1) only lies in the constant in the exponent. This is very different from the known uniqueness class in the smooth setting where the gap appears as different classes of functions in the exponent.

Chapter 4

Stochastic completeness and volume growth

Applying the integrated uniqueness class (3.0.1) to bounded solutions of the Cauchy problem of the heat equation with zero initial condition, we will get sufficient conditions for stochastic completeness in terms of volume growth with respect to adapted distances.

Theorem 4.1. *Let* (V, w, μ) *be a locally finite and connected weighted graph that satisfies Assumption 1.1.6. Let* d *be an adapted distance on* (V, w, μ)*. Assume that for some point* $x_0 \in V$*, the volume of balls* $\mu(B_d(x_0, r))$ *satisfies*

$$\mu(B_d(x_0, r)) \leq C \exp(cr \ln r), \tag{4.0.1}$$

for some constants $C > 0$*, and* $0 < c < \frac{1}{2}$ *and for all* $r > 0$ *large enough. Then* (V, w, μ) *is stochastically complete.*

Interestingly, for a certain class of weighted graphs including the physical Laplacian case, Theorem 4.1 implies a volume growth criterion in terms of the graph distance that is a good complement to the anti-tree example of Wojciechowski. The first volume growth type criterion for Riemannian manifolds is given by Gaffney [18]. Sharper criteria are exploited in depth by many authors, Grigor'yan [19], Davies [5], Hsu [27], Karp and Li [32], just to name a few.

By concrete calculations, we are more inclined to believe that Theorem 4.1 is not sharp. If this is the case, it will be a significant difference from the discrete setting to the manifold setting where the integrated uniqueness class given by Grigor'yan [19, 22] gives the sharp volume growth criteria for stochastic completeness.

4.1 Volume growth criteria in the adapted distances

Having got the uniqueness class criterion in the previous chapter, we can apply it to the stochastic completeness problem for a class of graphs.

Proof of Theorem 4.1. Let $u(x,t)$ be a bounded solution to the Cauchy problem (1.4.7) with zero initial condition on $V \times [0,T]$ for some $T > 0$. By Theorem 1.5.1, it suffices to show that

$$u(x,t) \equiv 0,$$

on $V \times [0,T]$. Without loss of generality, we can assume that

$$|u(x,t)| \leq 1.$$

As already mentioned in Remark 1.6.7, d is in fact an adapted distance such that Assumption 1.6.6 is fulfilled. Note that (4.0.1) implies that the volume of balls $\mu(B_d(x_0,r))$ of (V,w,μ) are finite for every $r > 0$. Together with Assumption 1.1.6 that

$$C_\mu = \inf_{x \in V} \mu(x) > 0,$$

it follows that the balls $\mu(B_d(x_0,r))$ are finite.

Then by (4.0.1), for any $T > 0$, and for all $r > 0$ large enough,

$$\int_0^T \sum_{x \in B_d(x_0,r)} u^2(x,t)\mu(x)dt \leq T\mu(B_d(x_0,r)) \leq CT \exp\left(cr \ln r\right).$$

Since $0 < c < \frac{1}{2}$, we have that $u(x,t) \equiv 0$ on $V \times [0,T]$ by Theorem 3.1. □

Remark 4.1.1. In [25], together with Grigor'yan and Masamune, we prove a criterion of stochastic completeness similar to Theorem 4.1 through a different approach originally due to Davies [5] in the smooth setting. The criterion in [25] applies to a large class of jump processes on general locally compact metric spaces.

4.2 Volume growth criteria in the graph distance

The volume growth criterion of stochastic completeness in adapted distances helps us understand the corresponding criterion in terms of the graph metric. The following is a direct consequence of Theorem 4.1.

Corollary 4.2.1. *Let (V, w, μ) be a locally finite and connected weighted graph that satisfies Assumption 1.1.6. Let d be the adapted distance on (V, w, μ) defined as in Definition 1.6.4. If for some point $x_0 \in V$,*

$$\mu(B_d(x_0, r)) \leq \exp Cr$$

for some constant $C > 0$ and all $r > 0$ large enough, then (V, w, μ) is stochastically complete.

The following theorem is a slight generalization of Theorem 1.4 in [25] and can be viewed as a partial complement to Wojciechowski's example of stochastically incomplete graphs with polynomial volume growth with respect to the graph distance [56]. See also Example 4.3.2 and Remark 2.5.14.

Theorem 4.2.2. *Let (V, w, μ) be a locally finite and connected weighted graph that satisfies Assumptions 1.1.6 and 1.1.7. Let ρ be the graph distance on (V, w, μ) as before. If for some point $x_0 \in V$, and some constant $c > 0$,*

$$\mu(B_\rho(x_0, r)) \leq cr^3 \tag{4.2.2}$$

for all $r \in \mathbb{N}_+$, then (V, w, μ) is stochastically complete.

Proof. For any non-negative integer r set

$$S_\rho(r) = \{x \in X : \rho(x, x_0) = r\}.$$

Set

$$V(x_0, n) = \mu(B_\rho(x_0, n)) = \sum_{r=0}^{n} \mu(S_\rho(r)).$$

Put $\varepsilon = \frac{1}{5}$ and $\alpha = 200c$ where c is the constant in (4.2.2). It follows from (4.2.2) that, for any $n \geq 1$,

$$\#\{r \in [n-1, 2n+1] : \mu(S_\rho(r)) > \alpha n^2\} \leq \frac{c(2n+1)^3}{\alpha n^2} \leq \varepsilon n.$$

It follows that

$$\#\{r \in [n+1, 2n] : \max_{i=-2,-1,0,1} \mu(S_\rho(r+i)) > \alpha n^2\} \leq 4\varepsilon n,$$

and hence,

$$\#\{r \in [n+1, 2n] : \max_{i=-2,-1,0,1} \mu(S_\rho(r+i)) \leq \alpha n^2\} \geq (1-4\varepsilon)n. \tag{4.2.3}$$

Recall Assumption 1.1.7 that

$$w(x,y) \le C_w \mu(x)\mu(y)$$

for some constant $C_w > 0$. For any point $x \in S_\rho(r)$ we have

$$\text{(4.2.4)} \qquad \operatorname{Deg}(x) = \frac{1}{\mu(x)} \sum_{y, y\sim x} w(x,y)$$

$$\text{(4.2.5)} \qquad \le C_w \sum_{y,y\sim x} \mu(y)$$

$$\text{(4.2.6)} \qquad \le C_w \{\mu(S_\rho(r-1)) + \mu(S_\rho(r)) + \mu(S_\rho(r+1))\}.$$

So it follows from (4.2.3) that

$$\text{(4.2.7)} \qquad \#\{r \in [n+1, 2n] : \max_{x\in S_\rho(r-1)\cup S_\rho(r)} \operatorname{Deg} x \le 3C_w \alpha n^2\} \ge (1-4\varepsilon)n.$$

Let d be the adapted distance as in Definition 1.6.4. It follows that, for any

$$n \ge \frac{1}{\sqrt{3\alpha C_w}},$$

and any $r \in [n+1, 2n]$ as in (4.2.7) such that

$$\text{(4.2.8)} \qquad \max_{x\in S_\rho(r-1)\cup S_\rho(r)} \operatorname{Deg} x \le 3C_w \alpha n^2,$$

every pair of $x \sim y$ with $x \in S_\rho(r-1)$, $y \in S_\rho(r)$ satisfies

$$\text{(4.2.9)} \qquad \sigma(x,y) \ge \frac{1}{\sqrt{3\alpha C_w} n}.$$

For any chain connecting a vertex $x \in S_\rho(n)$ with a vertex $y \in S_\rho(2n)$ and for any $r \in [n+1, 2n]$ there is an edge $x_r \sim y_r$ from this chain such $x_r \in S_\rho(r-1)$ and $y_r \in S_\rho(r)$.

The length L of this chain is bounded below by $\sum_{r=n+1}^{2n} \sigma(x_r, y_r)$. Restricting the summation to those r that satisfy (4.2.8) and noticing that for any such r, $\sigma(x_r, y_r) \ge \frac{1}{\sqrt{3\alpha C_w} n}$, we obtain

$$L \ge \frac{1}{\sqrt{3\alpha C_w} n} (1-4\varepsilon)\, n = \frac{1-4\varepsilon}{\sqrt{3\alpha C_w}} =: \delta.$$

Now we can estimate $d\left(x_0, x\right)$ for any vertex $x \notin B_\rho\left(x_0, R\right)$, where

$$R > 4 \vee \frac{4}{\sqrt{3\alpha C_w}}.$$

Choose a positive integer k so that

$$2^k \leq R < 2^{k+1}.$$

Any chain connecting x_0 and x contains a subsequence $\{x_i\}_{i=1}^k$ of vertices such that $x_i \in S_\rho\left(x_0, 2^i\right)$. By the previous argument, the length of the chain between x_{i-1} and x_i is bounded from below by a constant δ, for any $i = k_0, k_0+1, ..., k-1$ where

$$k_0 = \{[\log_2 \frac{1}{\sqrt{3\alpha C_w}}] \vee 0\} + 1.$$

It follows that the length of the whole chain is bounded below by $\delta(k - k_0)$, whence

$$d\left(x_0, x\right) \geq \delta(k - k_0) \geq \delta\left(\log_2 R - k_0 - 1\right).$$

Setting

$$r = \delta\left(\log_2 R - k_0 - 1\right),$$

we obtain

$$B_d(x_0, r) \subseteq B_\rho(x_0, R),$$

whence for all $r > 0$ large enough,

$$\mu\left(B_d(x_0, r)\right) \leq \mu\left(B_\rho(x_0, R)\right) \leq cR^3 \leq C \exp\left(br\right),$$

for some constants C and b. Hence, the volume growth with respect to the adapted distance d is at most exponential, and we conclude by Corollary 4.2.1 that (V, w, μ) is stochastically complete. □

4.3 Examples

Although the uniqueness class in Theorem 3.1 is close to be sharp, the volume growth criteria of stochastic completeness derived from it, Theorem 4.1, seems to be not sharp. This is indicated by calculations of the volume growth with respect to adapted distances of some natural examples whose stochastic completeness is easy to be determined.

In the following two examples of graphs, we consider the physical Laplacians and the adapted distance d as defined in Definition 1.6.4 on them.

Example 4.3.1 (Model Trees)**.** Let (V, E) and f be as in Example 1.6.10. Take $f(n) = [(n+1)^s]$ for some $s > 0$. By Example 2.5.12, (V, E) is stochastically complete if and only if $0 < s \leq 1$. As is already shown, for $x \in V$ such that $r(x) = n > 1$,

$$d(x, x_0) \asymp (n+1)^{1-s/2}.$$

And for $n > 1$,

$$\mu(B_\rho(x_0, n)) \asymp S(n) \asymp (n!)^s.$$

By Stirling's formula, there exist constants $0 < c_1 < c_2, 0 < C_1 < C_2$ such that

$$C_1 \exp(c_1 n \ln n) \leq \mu(B_\rho(x_0, n)) \leq C_2 \exp(c_2 n \ln n),$$

for all n large enough. Hence, for some constants $0 < c_3 < c_4, 0 < C_3 < C_4$ for all $r > 0$ large enough,

$$C_3 \exp\left(c_3 r^{\frac{2}{2-s}} \ln r\right) \leq \mu(B_d(x_0, r)) \leq C_4 \exp\left(c_4 r^{\frac{2}{2-s}} \ln r\right).$$

As $s > 0$, we see that $\frac{2}{2-s} > 1$ and this type volume growth is far beyond the scope of Theorem 4.1.

Example 4.3.2 (Anti-trees)**.** Let (V, E) and $S(n)$ be as in Example 1.6.11. First we take $S(n) = [(n+1)^s]$ for some $s > 0$. By Example 4.3.2, (V, E) is stochastically complete if and only if $0 < s \leq 2$. Similar to the previous example, for $x \in V$ such that $r(x) = n > 1$,

$$d(x, x_0) \asymp (n+1)^{1-s/2},$$

if $0 < s < 2$ and

$$d(x, x_0) \asymp \ln(n+1)$$

if $s = 2$. For $n > 1$, we have

$$\mu(B_\rho(x_0, n)) \asymp n^{1+s}.$$

So when $0 < s < 2$, for all $r > 0$ large enough,

$$C_1 r^{\frac{2+2s}{2-s}} \leq \mu(B_d(x_0, r)) \leq C_2 r^{\frac{2+2s}{2-s}}$$

for some constants $0 < C_1 < C_2$. And similarly for the case $s = 2$,

$$C_1 \exp c_1 r \leq \mu(B_d(x_0, r)) \leq C_2 \exp c_2 r$$

for some constants $0 < c_1 < c_2, 0 < C_1 < C_2$ and for all $r > 0$ large enough.

So far everything seems within the scope of Theorem 4.1. But we can make subtler choices of $S(n)$ such as $S(n) = [(n+1)^2 \ln^s(n+e)]$ with $n > 0$. Again by Example 4.3.2, (V, E) is stochastically complete if and only if $0 < s \le 1$.

To calculate the adapted distance d, note that for $x \in S_n$ and $y \in S_{n+1}$ such that $x \sim y$, we have for all n large enough

$$\sigma(x, y) \asymp \frac{1}{(n+1)\ln^{s/2}(n+e)}.$$

A direct calculation shows that for $x \in V$ such that $r(x) = n > 1$,

$$d(x, x_0) \asymp \ln^{1-s/2}(n).$$

For $n > 1$, we have

$$\mu(B_\rho(x_0, n)) \asymp (n+1)^3 \ln^s(n+e).$$

Hence, there exist constants $0 < c_1 < c_2, 0 < C_1 < C_2$ such that

$$C_1 \exp\left(c_1 r^{\frac{2}{2-s}}\right) \le \mu(B_d(x_0, r)) \le C_2 \exp\left(c_2 r^{\frac{2}{2-s}}\right),$$

for all $r > 0$ large enough. Theorem 4.1 again fails to detect the borderline of volume growth.

As shown by the previous examples and in analogue with the classical result of Grigor'yan [22] on manifolds, we are more inclined to think that the borderline of volume growth between stochastic completeness and incompleteness should be at

$$\int_1^\infty \frac{r dr}{\ln \mu(B_d(x_0, r))} = \infty, \tag{4.3.10}$$

type. More precisely, let d be an adapted distance on (V, w, μ) that satisfies Assumption 1.1.6. If for some reference point $x_0 \in V$, the volume of balls $\mu(B_d(x_0, r))$ satisfies (4.3.10), then (V, w, μ) is conjectured to be stochastically complete.

The conjectural volume growth criteria (4.3.10) has exactly the same form as Grigor'yan's on manifolds [19]. However, in contrast to the manifold case, one can not expect to prove this as a consequence of some uniqueness class type result. So some special features of stochastic completeness (or stochastic incompleteness) should be taken into count and definitely new ideas may emerge.

Chapter 5

Escape rate

In this chapter, we assume that (V, w, μ) is a locally finite and connected weighted graph such that Assumption 1.1.6 holds. Let $\{X_t\}_{t\geq 0}$ be the corresponding minimal right continuous Markov chain as constructed in Section 1.7. We are interested in an upper bound on how "fast" such a Markov chain can go.

Definition 5.1. Let (V, w, μ) be a locally finite and connected weighted graph such that Assumption 1.1.6 holds. Let d an adapted distance on (V, w, μ). Fix a reference point $x_0 \in V$. A function $R(t)$ is called an upper rate function (with respect to d) for the minimal process $\{X_t\}_{t\geq 0}$ (or equivalently, for (V, w, μ)) if

$$\mathbb{P}_{x_0}\{d(X_t, x_0) \leq R(t) \text{ for all sufficiently large t}\} = 1.$$

Remark 5.2. Let $R(t)$ be an upper rate function for (V, w, μ) with respect to the adapted distance d as in the above definition. Assume further that Assumption 1.6.6 holds. Then from the definition we see that the existence of an upper rate function implies stochastic completeness of (V, w, μ).

Theorem 5.3. *Under the settings in Definition 5.1, we assume that for all* $r \geq 2$,

$$\ln \mu(B(x_0, r)) \leq f(r), \tag{5.0.1}$$

where f *is a positive, increasing continuous function on* $[0, \infty)$. *For technical reasons, we consider two special classes of functions* f.

(1) *There is some constant* $M > 0$ *such that*

$$\frac{f(r)}{r} \leq M \tag{5.0.2}$$

for all $r \geq 2$. *In this case, there exists some constant* $C > 0$ *such that the* *inverse function* $\psi^{-1}(t)$ *of*

$$\psi(R) = C\int_8^R \frac{rdr}{f(r)+\ln\ln(r)} \tag{5.0.3}$$

is an upper rate function for $\{X_t\}_{t\geq 0}$.

(2) The function $\frac{f(r)}{r}$ *is increasing for* $r \geq 2$ *and*

$$\int_1^\infty \frac{rdr}{f(r)\exp\left(C_0\frac{f(r)}{r}\right)} = \infty \tag{5.0.4}$$

for some constant $C_0 > 2$. *Then there is some constant* $C > 0$, *such that the inverse function* $\psi^{-1}(t)$ *of*

$$\psi(R) = C\int_1^R \frac{rdr}{f(r)\exp\left(C_0\frac{f(r)}{r}\right)} \tag{5.0.5}$$

is an upper rate function for $\{X_t\}_{t\geq 0}$.

Remark 5.4. (a) Note that by Remark 1.6.7, Assumption 1.6.6 is automatically fulfilled since we assume Assumption 1.1.6 and (5.0.1).

(b) The upper rate function as the inverse function of ψ in (5.0.3) seems to be sharp since it coincides with the results in the manifold case [24, 28]. It would be interesting to see whether it is true even without the restriction (5.0.2).

We apply Theorem 5.3 to calculate some upper rate functions.

Example 5.5. (1) Take

$$f(r) = c\ln r.$$

This corresponds to that the volume is bounded from above by some power function of distance. In this case, the $\ln\ln r$ term will be small compared to $f(r)$. So we have

$$R(t) = C\sqrt{t\ln t}$$

is an upper rate function for some constant $C > 0$.

(2) Take

$$f(r) = cr^\alpha$$

for some $0 < \alpha \leq 1$. Then

$$R(t) = Ct^{\frac{1}{2-\alpha}}$$

is an upper rate function for some constant $C > 0$.

(3) Take
$$f(r) = cr \ln r.$$
Here the constant $c > 0$ becomes important. If $0 < c < \frac{1}{2}$, then for any $1/c > C_0 > 2$, there exists some constant $C > 0$ (depending on C_0) such that
$$R(t) = Ct^{\frac{1}{1-cC_0}} \ln t$$
is an upper rate function. If $c \geq 1/2$, (5.0.4) cannot be true. So we are not able to judge the existence of an upper rate function.

Remark 5.6. The last example above gives yet another proof of Theorem 4.1.

The proof of Theorem 5.3 consists of two parts. In the first part, using probabilistic arguments, one reduces the question to estimates of certain solutions to the heat equation. The probabilistic argument based on the Borel-Cantelli lemma is standard (see, for example, [24], [28]). The required estimates are obtained then by analytic methods. We apply the integrated maximum principle Lemma 3.1.1 in a similar way to Chapter 3.

5.1 Main strategy

We generally follow the strategy in [23], and [24] for upper rate functions of the Brownian motions on Riemannian manifolds. Recall that $B_d(x, r)$ denotes a closed ball centered at x with radius r in d. Let $\{R_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of positive numbers to be chosen later such that $\lim_{n\to\infty} R_n = \infty$. Denote the balls $B_d(x_0, R_n)$ by B_n and define a sequence of stopping times τ_n by
$$\tau_n = \tau_{B_n}.$$
Suppose for a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ we have that
$$\sum_{1}^{\infty} \mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) < \infty.$$
Then by Borel-Cantelli lemma, it follows that $\mathbb{P}_{x_0}$ almost surely
$$\tau_n - \tau_{n-1} > c_n$$
for all n large enough. Let $T_n = \sum_1^n c_k$. With $\mathbb{P}_{x_0}$ probability 1, we have that $\tau_n > T_n - T_0$ for all n large enough, where T_0 is some random number. Suppose

that we can find a strictly increasing homeomorphism $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$T_{n-1} - \psi(R_n) \to +\infty \tag{5.1.6}$$

as $n \to \infty$. Then with $\mathbb{P}_{x_0}$ probability 1, for n large enough and for t such that

$$\psi(R_{n-1}) < t \le \psi(R_n) < T_{n-1} - T_0,$$

we have that

$$d(X_t) \le R_{n-1} \le \psi^{-1}(t). \tag{5.1.7}$$

Notice that $\lim_{n\to\infty} \psi(R_n) = \infty$, so (5.1.7) holds for all large enough t, $\mathbb{P}_{x_0}$ almost surely. In other words, $\psi^{-1}(t)$ is an upper rate function for $\{X_t\}_{t\ge 0}$.

5.2 Exit time estimate

With the main strategy in hand, the key technical problem is to estimate the quantity

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \le c_n).$$

By the strong Markov property of the minimal Markov chain $\{X_t\}_{t\ge 0}$, we have

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \le c_n) = \mathbb{E}_{x_0}(\mathbb{P}_{X_{\tau_{n-1}}}\{\tau_n \le c_n\}). \tag{5.2.8}$$

By the construction of $\{X_t\}_{t\ge 0}$, since (V, w, μ) is locally finite, we know that

$$X_{\tau_{n-1}} \in \partial B_{n-1}, X_{\tau_n} \in \partial B_n.$$

Define

$$r_n = R_n - R_{n-1} - 1$$

and assume $r_n > 2$ for $n \ge 1$. Hence X_t must run out of a ball $B_d(X_{\tau_{n-1}}, r_n)$ before it leaves B_n. So it follows that

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \le c_n) \le \sup_{z\in\partial B_{n-1}} \mathbb{P}_z\{\tau_{B_d(z,r_n)} \le c_n\}.$$

For a fixed $z \in \partial B_{n-1}$, define

$$u(x,t) = \mathbb{P}_x(\tau_{B_d(z,r_n)} \le t)$$

as a function on $B_d(z, r_n) \times (0, \infty)$. It is direct to see that $0 \leq u(x,t) \leq 1$. Note that $\bar{B}_d(z, r_n - 1) \subseteq B_d(z, r_n)$. By Proposition 1.7.6, the function

$$u(x,t) = 1 - \mathcal{P}_t^{B_d(z,r_n)} 1,$$

is a solution to the heat equation

$$\frac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0,$$

on $B_d(z, r_n - 1) \times [0, \infty)$ with $u(x, 0) \equiv 0$.

Fix some n, we can then apply Lemma 3.1.1 to $u(x,t)$. Choose the auxiliary functions to be

$$\xi(x,t) = -2\alpha_n^2 e^{4\alpha_n} t - 2\alpha_n d(x,z), \quad \text{and} \quad \eta(x) = \frac{(e^{\alpha_n(r_n-1)} - e^{\alpha_n d(x,z)})_+}{e^{\alpha_n(r_n-1)} - 1},$$

with $\alpha_n > 0$ to be determined later. Concerning $\eta(x)$, we have the estimate

$$\begin{aligned} & \sum_{y \in V} w(x,y)(\eta(x) - \eta(y))^2 \\ \leq & \frac{1}{(e^{\alpha_n(r_n-1)} - 1)^2} \sum_{y \in V} w(x,y)(e^{\alpha_n d(x,z)} - e^{\alpha_n d(y,z)})^2 \\ \leq & \frac{\alpha_n^2 e^{2\alpha_n d(x,z) + 2\alpha_n}}{(e^{\alpha_n(r_n-1)} - 1)^2} \sum_{y \in V} w(x,y) d^2(x,y) \\ \leq & \frac{\alpha_n^2 e^{2\alpha_n d(x,z) + 2\alpha_n}}{(e^{\alpha_n(r_n-1)} - 1)^2} \mu(x). \end{aligned} \tag{5.2.9}$$

It is direct to check that they fulfill the conditions in Lemma 3.1.1, so we have that for any $\tau > 0$

$$\begin{aligned} & \sum_{x \in B_d(z, r_n - 1)} u^2(x,\tau)\eta^2(x) e^{\xi(x,\tau)} \mu(x) \\ \leq & 2\int_0^\tau \sum_{x \in \bar{B}_d(z, r_n - 1)} \sum_{y \in \bar{B}_d(z, r_n - 1)} w(x,y)(\eta(x) - \eta(y))^2 u^2(y,t) e^{\xi(x,t)} dt \\ \leq & 2\int_0^\tau \sum_{x \in \bar{B}_d(z, r_n - 1)} \sum_{y \in \bar{B}_d(z, r_n - 1)} w(x,y)(\eta(x) - \eta(y))^2 e^{\xi(x,t)} dt, \end{aligned} \tag{5.2.10}$$

where we used the symmetry of $w(x,y)$ and the fact that $0 \leq u \leq 1$. Plug in the explicit forms of the auxiliary functions, we have

$$u^2(z,\tau)\mu(z) e^{-2\alpha_n^2 e^{4\alpha_n} \tau} \tag{5.2.11}$$

$$\leq 2\int_0^\tau \sum_{x\in B_d(z,r_n)} \sum_{y\in B_d(z,r_n)} w(x,y)(\eta(x)-\eta(y))^2 e^{\xi(x,t)}dt \tag{5.2.12}$$

$$\leq 2\int_0^\tau \sum_{x\in B_d(z,r_n)} \frac{\alpha_n^2 e^{2\alpha_n d(x,z)+2\alpha_n}}{(e^{\alpha_n(r_n-1)}-1)^2} \times e^{-2\alpha_n d(x,z)-2\alpha_n^2 e^{4\alpha_n t}}\mu(x)dt \tag{5.2.13}$$

$$= \frac{e^{-2\alpha_n}(1-e^{-2\alpha_n^2 e^{4\alpha_n}\tau})}{(e^{\alpha_n(r_n-1)}-1)^2}\mu(B_d(z,r_n)). \tag{5.2.14}$$

Recall that we assume that for any $x \in V$,

$$\mu(x) \geq C_\mu > 0.$$

It follows that

$$u^2(z,\tau) \leq \frac{1}{C_\mu}\frac{e^{-2\alpha_n}(e^{2\alpha_n^2 e^{4\alpha_n}\tau}-1)}{(e^{\alpha_n(r_n-1)}-1)^2}\mu(B_d(z,r_n)) \tag{5.2.15}$$

Let $\tau = c_n$, we have

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) \leq \sup_{z\in\partial B_{n-1}} u(z,c_n) \leq \sqrt{\frac{\mu(B_n)}{C_\mu}}\frac{e^{\alpha_n^2 e^{4\alpha_n}c_n}}{e^{\alpha_n r_n}-e^{\alpha_n}}.$$

Note an elementary fact that if $r_n > 2$, then

$$\frac{1}{2}\alpha_n r_n < \alpha_n(r_n-1)$$

and hence

$$\frac{1}{e^{\alpha_n r_n}-e^{\alpha_n}} \leq \frac{\alpha_n(r_n-1)+1}{\alpha_n(r_n-1)}e^{-\alpha_n r_n} \leq (1+\frac{2}{\alpha_n r_n})e^{-\alpha_n r_n}.$$

Together with (5.0.1), we have

$$\mathbb{P}_{x_0}(\tau_n-\tau_{n-1}\leq c_n) \leq \frac{1}{\sqrt{C_\mu}}(1+\frac{2}{\alpha_n r_n})\exp\{\alpha_n^2 e^{4\alpha_n}c_n + \frac{1}{2}f(R_n) - \alpha_n r_n\}. \tag{5.2.16}$$

Now we prove Theorem 5.3 by specifying R_n, c_n and α_n. We give separate proofs for the two classes of functions f.

5.3 Upper rate function for case (1)

Proof of Theorem 5.3, case (1). We choose $R_n = 2^{n+3}$ for $n \geq 0$ and then

$$r_n = R_n - R_{n-1} - 1 = 2^{n+2} - 1 \geq \frac{1}{4} R_n \geq 4$$

for $n \geq 1$. Let

$$\alpha_n = \frac{2f(R_n) + 2\ln\ln R_n}{r_n}.$$

Then we have by condition (5.0.2)

$$\alpha_n \leq \frac{8f(R_n)}{R_n} + \frac{2\ln(n+3)}{2^{n+2}-1} \leq 8M + 1,$$

and

$$1 + \frac{2}{\alpha_n r_n} = 1 + \frac{2}{2f(R_n) + 2\ln\ln R_n} \leq 1 + \frac{1}{f(0)}.$$

Write

$$C_1 = \frac{1}{8e^{32M+4}}$$

for short. Choose

$$c_n = C_1 \frac{r_n^2}{f(R_n) + \ln\ln R_n}.$$

The estimate (5.2.16) gives

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) \leq \frac{1 + \frac{1}{f(0)}}{\sqrt{C_\mu}} \exp\{\frac{1}{2}(f(R_n) + \ln\ln R_n) + \frac{1}{2} f(R_n) - 2(f(R_n) + \ln\ln R_n)\}$$

$$\leq \frac{1 + \frac{1}{f(0)}}{\sqrt{C_\mu}} \exp\{-\frac{3}{2}\ln\ln R_n\}. \tag{5.3.17}$$

It easily follows that

$$\sum_{n=1}^{\infty} \mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) < \infty.$$

We can now determine a function ψ such that (5.1.6) holds. Consider

$$\begin{aligned} T_n &= \sum_{m=1}^{n} c_m \\ &\geq \frac{C_1}{16} \sum_{m=1}^{n} \frac{R_m^2}{f(R_m) + \ln\ln R_m} \\ &= \frac{C_1}{32} \sum_{m=1}^{n} \frac{R_{m+1}(R_{m+1} - R_m)}{f(R_m) + \ln\ln R_m} \end{aligned}$$

$$\geq \frac{C_1}{32}\int_{R_1}^{R_{n+1}} \frac{rdr}{f(r)+\ln\ln r}.$$

Set

$$\psi(R) = \frac{C_1}{64}\int_{8}^{R} \frac{rdr}{f(r)+\ln\ln r}.$$

By condition (5.0.2) we can see that

$$T_n - \psi(R_{n+1}) \to \infty \text{ as } n \to \infty.$$

Thus $\psi^{-1}(t)$ is the desired upper rate function. □

5.4 Upper rate function for case (2)

Proof of Theorem 5.3, case (2). Let

$$\varepsilon = (\frac{C_0}{2} - 1) \wedge 1 > 0$$

and

$$\frac{8}{\varepsilon} + 1 \geq k_0 = [\frac{8}{\varepsilon}] + 1 \geq \frac{8}{\varepsilon} \geq 8.$$

We choose $R_n = k_0^n$ such that for all $n \geq 1$,

$$r_n = k_0^n - k_0^{n-1} - 1 \geq (1 - \frac{2}{k_0})R_n \geq (1 - \frac{\varepsilon}{4})R_n > 2.$$

Choose

$$\alpha_n = \frac{(\frac{1}{2} + \frac{\varepsilon}{4})f(R_n)}{r_n}.$$

We have that

$$1 + \frac{2}{\alpha_n r_n} = 1 + \frac{2}{(\frac{1}{2} + \frac{\varepsilon}{4})f(R_n)} < 1 + \frac{4}{f(0)}.$$

Then choose

$$c_n = \frac{\varepsilon f(R_n)}{8\alpha_n^2 e^{4\alpha_n}}.$$

The estimate (5.2.16) gives

$$\mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) \leq \frac{1 + \frac{4}{f(0)}}{\sqrt{C_\mu}} \exp\{-\frac{\varepsilon}{8} f(R_n)\}. \tag{5.4.18}$$

By the assumed monotonicity of $\frac{f(r)}{r}$ for $r \geq 1$, we obtain that

$$\begin{aligned}\sum_1^\infty \mathbb{P}_{x_0}(\tau_n - \tau_{n-1} \leq c_n) &\leq \frac{1+\frac{4}{f(0)}}{\sqrt{C_\mu}} \sum_1^\infty \exp\{-\frac{\varepsilon}{8} f(R_n)\} \\ &\leq \frac{1+\frac{4}{f(0)}}{\sqrt{C_\mu}} \sum_1^\infty \exp\{-\frac{\varepsilon}{8} f(1) R_n\} < \infty.\end{aligned}$$

We can now determine a function ψ such that (5.1.6) holds. Recall that

$$r_n = (1 - \frac{1}{k_0} - \frac{1}{k_0^n}) R_n \geq (1 - \frac{\varepsilon}{4}) R_n.$$

Note the elementary fact that

$$\frac{1+\frac{\varepsilon}{2}}{1-\frac{\varepsilon}{4}} \leq 1 + \varepsilon,$$

where $0 < \varepsilon \leq 1$. Then we can estimate c_n as

$$\begin{aligned}c_n = \frac{\varepsilon f(R_n)}{8\alpha_n^2 e^{4\alpha_n}} &= \frac{\varepsilon}{8(\frac{1}{2}+\frac{\varepsilon}{4})^2} \frac{r_n^2}{f(R_n)\exp\{2(1+\frac{\varepsilon}{2})\frac{f(R_n)}{r_n}\}} \\ &\geq \frac{\varepsilon(1-\frac{\varepsilon}{4})^2}{8(\frac{1}{2}+\frac{\varepsilon}{4})^2} \frac{R_n^2}{f(R_n)\exp\{2(1+\varepsilon)\frac{f(R_n)}{R_n}\}}.\end{aligned}$$

Write

$$C = \frac{(1-\frac{\varepsilon}{4})^2 \varepsilon}{16(\frac{1}{2}+\frac{\varepsilon}{4})^2 \frac{8}{\varepsilon}(\frac{8}{\varepsilon}+1)}$$

for short. Hence

$$\begin{aligned}T_n &= \sum_{m=1}^n c_m \\ &\geq \frac{\varepsilon(1-\frac{\varepsilon}{4})^2}{8(\frac{1}{2}+\frac{\varepsilon}{4})^2} \sum_{m=1}^n \frac{R_m^2}{f(R_m)\exp\{2(1+\varepsilon)\frac{f(R_m)}{R_m}\}} \\ &= \frac{(1-\frac{\varepsilon}{4})^2 \varepsilon}{8(\frac{1}{2}+\frac{\varepsilon}{4})^2 k_0(k_0-1)} \sum_{m=1}^n \frac{R_{m+1}(R_{m+1}-R_m)}{f(R_m)\exp\{2(1+\varepsilon)\frac{f(R_m)}{R_m}\}} \\ &\geq 2C \int_{R_1}^{R_{n+1}} \frac{r\,dr}{f(r)\exp\{C_0 \frac{f(r)}{r}\}},\end{aligned}$$

where in the lase inequality we have applied the assumption that $\frac{f(r)}{r}$ is increasing for $r \geq 1$. Set

$$\psi(R) = C \int_1^R \frac{r dr}{f(r) \exp\{C_0 \frac{f(r)}{r}\}}.$$

By condition (5.0.4) we can see that

$$T_n - \psi(R_{n+1}) \to \infty \quad \text{as} \quad n \to \infty.$$

Thus $\psi^{-1}(t)$ is the desired upper rate function. $\square$

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Notations

$\mathbb{N}$	the set of nonnegative integer numbers $\{0, 1, 2, 3, \dots\}$.
$\mathbb{N}_+$	the set of positive integer numbers $\{1, 2, 3, \dots\}$.
$\mathbb{Z}$	the set of integer numbers.
$\#A$	the cardinality of the set A.
(V, w, μ)	a weighted graph, a triple where V is a countably infinite set, w is a nonnegative function on $V \times V$, and μ is a nonnegative function on V.
(V, E)	a graph, V is a countably infinite set and E is a symmetric subset of $V \times V$.
$\{X_t\}_{t \geq 0}$	a reversible minimal continuous time Markov chain.
∂A	the (outer) boundary of a subset A of a graph (V, E).
$\bar{A}$	the closure of a subset A of a graph (V, E).
Δ	the formal Laplacian on a weighted graph.
P_t	the minimal heat semigroup on a weighted graph.