Uniqueness of solutions to Fokker-Planck equations related to singular SPDE driven by Lévy and cylindrical Wiener noise

Dissertation zur Erlangung des Doktorgrades
an der Fakultät für Mathematik
der Universität Bielefeld

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im September 2011

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Für die im Rahmen des Promotionsverfahrens auf Papier eingereichten Exemplare gilt:

Gedruckt auf alterungsbeständigem Papier gemäß DIN-ISO 9706.

Contents

1. Introduction

Stochastic Partial Differential Equations (SPDE)

The main subject of this thesis are semilinear stochastic partial differential equations (SPDE) in a separable real Hilbert space H driven by a stochastic process Y, which is either an H-valued Lévy process or the sum of a Lévy and a cylindrical Wiener process. A prototypical formulation of such an equation is

$$\begin{cases} \mathrm{d}X(t) = \Big[AX(t) + F\big(t, X(t)\big)\Big]\,\mathrm{d}t + \mathrm{d}Y(t) \\ \quad X(s) = x \in H\,, \quad 0 \le s \le t \le T, \end{cases} \tag{SPDE}$$

where A is a self-adjoint linear operator in H (more particularly, the infinitesimal generator of a C_0-semigroup of operators denoted by e^{tA}), F a possibly singular and/or multivalued map, Y a centered Lévy process (or the sum of such a Lévy process with a cylindrical Wiener process) and T a finite positive real number (see Chapter 2 below for a precise exposition of the framework, and the following sections of this Introduction for an explanation of some of the basic concepts underlying this thesis).

SPDE have been a very active topic of research in Stochastic Analysis for a number of decades. Several mathematical approaches have been established to obtain pathwise solutions to different classes of such equations. By calling a solution "pathwise", we mean that it specifies the development of an individual solution path (or 'trajectory') in space and time. See for example the textbooks [PR07] for an introduction to the so-called "variational" approach, and [DPZ92], [PZ07] for introductions to the so-called semigroup (or "mild") approach to pathwise solutions for SPDE. (In particular, the last named reference also includes a short overview of the history of the topic of SPDE with Lévy noise; all three of them contain fairly exhaustive lists of references.)

Fokker-Planck Equations related to SPDE

It turns out, however, that the regularity requirements on the coefficients, which are needed to show existence and uniqueness of pathwise solutions (with current methods), are necessarily restrictive. In cases outside these restrictions, the aim is to at least determine their distributions.

To better understand this approach, let us first assume that the coefficients A and F in

(SPDE) are sufficiently regular to allow the identification of a unique pathwise solution $X(t,s,x)$, which has the Markov property. We use this to define the family of transition evolution operators $(P_{s,t})_{0\le s\le t\le T}$ on the Banach space $\mathcal{B}_b(H)$ of bounded, measurable functions $H \to \mathbb{R}$ by

$$P_{s,t}\varphi(x) := \mathbb{E}\Big[\varphi\big(X(t,s,x)\big)\Big] \quad \text{for all } \varphi \in \mathcal{B}_b(H)$$

(which fulfills the Chapman-Kolmogorov equation $P_{r,t} \circ P_{s,r} = P_{s,t}$ for all $0 \le s \le r \le t \le T$), and define a family $(\eta_t)_{t\ge 0}$ in $\mathcal{M}_1(H)$, the space of probability measures on H, by

$$\eta_t(\mathrm{d}x) := (P_{s,t})^* \zeta(\mathrm{d}x)\ , \quad t \ge s,$$

for an initial condition ζ in $\mathcal{M}_1(H)$. As usual, by P^* we denote the adjoint of an operator P. If we set $\zeta := \delta_x$ (the Dirac measure on H with mass in the starting point x), this family η of measures describes the evolution of the distribution of the solution X of (SPDE) over time; we see that, by definition,

$$\int_H \varphi(x)\ \eta_t(\mathrm{d}x) = \int_H P_{s,t}\varphi(x)\ \zeta(\mathrm{d}x) \quad \text{for all } \varphi \in \mathcal{B}_b(H).$$

Now, denote the Kolmogorov operator for (SPDE) by L, and its restriction to some suitable test function space $\mathcal{W}_{T,A}$ by L_0, specified as

$$L_0\psi(t,\,\cdot\,) = D_t\psi(t,\,\cdot\,) + \big\langle D\psi(t,\,\cdot\,)\,,\, F(t,\,\cdot\,)\big\rangle + U\psi(t,\,\cdot\,) \quad \text{for all } \psi \in \mathcal{W}_{T,A}.$$

Here U denotes the Ornstein-Uhlenbeck operator related to (SPDE) in the case $F = 0$. It takes the form

$$U\psi(t,x) = \int_H \big[i\langle A\xi, x\rangle - \lambda(\xi)\big] \cdot e^{i\langle \xi,x\rangle}\ \mathcal{F}^{-1}\big(\psi(t,\,\cdot\,)\big)(\mathrm{d}\xi)$$

for all $\psi \in \mathcal{W}_{T,A}$ (cf. [LR02]), where λ denotes a negative-definite function related to the Lévy process Y (the so-called characteristic exponent, or 'symbol' of Y), and $\mathcal{F}^{-1}$ the inverse Fourier transform. Then, some computations based on Itô's formula (cf. Lemma 4.1.4 below) establish the fact, that our family η of distributions solves the Fokker-Planck equation

$$\int_H \psi(t,x)\ \eta_t(\mathrm{d}x) = \int_H \psi(s,x)\ \zeta(\mathrm{d}x) + \int_s^t \int_H L_0\psi(r,x)\ \eta_r(\mathrm{d}x)\ \mathrm{d}r \qquad \text{(FPE)}$$
$$\text{for all } \psi \in \mathcal{W}_{T,A} \text{ and almost all } t \in [s,T],$$

assuming that the integrals in (FPE) exist.

At the heart of the approach followed in this thesis lies the realization, that it is possible to identify (by approximation) the Kolmogorov operator L even for equations of type (SPDE) with singular coefficients, for which there exists no pathwise solution. In this case, finding a family (η_t) that solves (FPE), and thus finding the distribution of the solution to (SPDE), has proven to be an interesting target.

Like much of the research in Stochastic Analysis, the study of existence and uniqueness of solutions to Fokker-Planck equations related to SPDE first started in finite dimensions; see e.g. [BDPR04], [BDPRS07], [BDPR08], [LBL08], [RZ10] and [Fig08] (where results for FPE-type equations are used to derive so-called martingale ("weak") solutions for the initial stochastic equation – an interesting aspect of this approach, which we are, however, not going to extend upon in this thesis), and the references therein. (See also related fundamental work for transport equations in [DL89].) In more recent years, Fokker-Planck equations related to SPDE in infinite dimensional spaces have received more attention; see e.g. [AF09], [BDPRS09], [BDPR09], [BDPR10], [BDPR11] and the references therein. However, while it seems impossible to check the hundreds of papers that are referring to the papers cited above in detail, to the best of our knowledge all of the current and past research seems to have focused exclusively on the case of SPDE perturbed by Wiener noise. Finally, let us mention recent work by S. Shaposhnikov (currently on the way to publication), where examples for Fokker-Planck equations are identified, for which the solutions are not unique.

Before we proceed to an overview of the scope and structure of this thesis, let us explain some concepts and terms, which are underlying this thesis. Keep in mind that, given that a thorough treatment of any of these concepts easily fills chapters in a textbook, our explanations have to remain a little rough.

Lévy Processes

Let us start with some observations and facts concerning Lévy processes in Hilbert space, before we introduce 'our' process. As mentioned above, most of the published results in the theory of SPDE are concerned with the case of equations 'driven' by a time-continuous Lévy process (i.e., a Brownian motion or Wiener process). However, recently (at least since [CM87]) the analysis of SPDE with possibly non-continuous Lévy noise has received increasingly more attention. By itself, the theory of Lévy processes is almost as old (see e.g. the classical reference [Lév34], or [App04] for an introduction to the topic including a historical overview) as the more well-known theory of Wiener processes or Brownian motions (see e.g. the classical, more than a hundred years old references [Bac00] and [Ein05], and also [JS95] for a historical overview with a focus on Wiener's role).

We start with the definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a an abstract probability space, H a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$, and $\big(Y(t)\big)_{t\geq 0} = \big(Y(t,\omega)\big)_{t\geq 0}$ an H-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$.[1] Y is called a *Lévy process*, if the following four conditions are fulfilled:

- Y has *independent increments*; that is, the increment $Y(t) - Y(s)$ is stochastically independent of the behavior of Y before time s for any $0 \leq s \leq t < \infty$
- Y has *stationary increments*; that is, for any $s \in (0, \infty)$ the distribution of the increment $Y(t+s) - Y(t)$ is independent of the choice of $t \in [0, \infty)$
- $Y(0) = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$
- Y is *stochastically continuous*; that is, for any $\varepsilon > 0$ and any $t \geq 0$ we have that

$$\lim_{h\to 0} \mathbb{P}\Big[\big|Y(t+h) - Y(t)\big| > \varepsilon\Big] = 0 \quad \text{(independent of sign } h\text{).}$$

Note that, in the last condition, the $\mathbb{P}$-zero set of paths with jumps at time t may depend on t. A Lévy process Y, for which the map $t \mapsto Y(t, \omega)$ is continuous for all ω, is a *Wiener process* (cf. e.g. [PZ07, Sect. 4.4]). By a standard result (cf. e.g. [FR00, Thm. 5.1], [PZ07, Thm. 4.3]), we may assume that the individual trajectories $t \mapsto Y(t)$ of a Lévy process Y are *cadlag* (continuous to the right, left limits exist). A Lévy process with the property, that $\mathbb{E}\big[Y(t)\big] = 0$ for all $t \geq 0$, is called *centered*.

The core result for the whole theory of Lévy processes, and an important asset for any analysis involving such processes, is the following fact, due to Lévy and Khintchine (see e.g. [Lin86, Thm. 5.7.3], [Par67, Thm. VI.4.10] for the Hilbert space case): The characteristic function of a Lévy process at time $t \geq 0$ can be written as

$$\mathbb{E}\big[e^{i\langle \xi, Y(t)\rangle}\big] = e^{-t\lambda(\xi)} \quad \text{for any } \xi \in H,$$

where the so-called *characteristic exponent* (or, *Lévy symbol*) $\lambda : H \to \mathbb{C}$ is a negative definite function with $\lambda(0) = 0$, which fulfills a continuity condition (the so-called Sazonov continuity[2]) and can be represented as

$$\lambda(\xi) = -i\langle \xi, b\rangle + \frac{1}{2} \cdot \langle \xi, Q\xi\rangle - \int_H \exp\big[i\langle \xi, x\rangle\big] - 1 - \frac{i\langle \xi, x\rangle}{1 + |x|^2}\, M(\mathrm{d}x)\ . \tag{LKD}$$

Here, b is an element of H, $Q \in L(H)$ is a nonnegative, symmetric trace-class operator, and M is a *Lévy measure* on H; that is, a Borel measure satisfying

$$M(\{0\}) = 0 \quad \text{and} \quad \int_H \big(1 \wedge |x|^2\big)\, M(\mathrm{d}x) < \infty\ .$$

[1] We generally assume, that the filtration $(\mathcal{F}_t)$ fulfills "the usual conditions" as defined in [PZ07, Section 3.1]: It is right-continuous, and every $\mathcal{F}_t$ contains all $\mathbb{P}$-zero sets in $\mathcal{F}$.

[2] The *Sazonov topology* on H is the coarsest topology, such that the seminorm $x \mapsto \|Sx\|$ is continuous for all Hilbert-Schmidt operators S on H.

The Lévy measure M characterizes the distribution of the jumps of the Lévy process Y; note, that the continuous diffusion part and the jump part of a Lévy process are mutually stochastically independent (cf. e.g. [PZ07, Thm. 4.23]). As shown in [Par67, Chap. VI, Thms. 2.4 and 4.8], the function

$$\xi \mapsto \int_H \exp\left[i\langle\xi, x\rangle\right] - 1 - \frac{i\langle\xi, x\rangle}{1+|x|^2}\, M(\mathrm{d}x)$$

is Sazonov continuous for every Lévy measure M on H (i.e., the Sazonov continuity of this term can be shown directly). It is actually the characteristic function of a probability measure $e_G(M)$ on H. In [Lin86, Prop. 5.4.7] this probability measure $e_G(M)$ is called the *generalized exponent* of the Lévy measure M. The representation of λ (and thus, the Lévy process $\big(Y(t)\big)_{t\geq 0}$ and its one dimensional time marginals $(\gamma_t)_{t\geq 0}$) through the triple $[b, Q, M]$ is unique. It is usually referred to as the *Lévy-Khintchine decomposition*. Among other things, it trivially implies that $\lambda(-\xi) = \overline{\lambda(\xi)}$ (where $\bar{c}$ denotes the complex conjugation of $c \in \mathbb{C}$). The result holds in both directions: any function, that can be described in this way by such a triple, is a characteristic function of a Lévy process (see e.g. [Par67, Chap. VI, Thm. 4.10]).

Another classical fact from the theory of Lévy processes (see e.g. [PZ07, Sect. 4.1]) is, that the distributions $(\gamma_t)_{t\geq 0}$ of a Lévy process are *infinitely divisible*.[3] The Lévy-Khintchine decomposition of any infinitely divisible probability measure γ_1 on H with characteristic triple $[b, Q, M]$ is uniquely given by the following convolution of probability measures:

$$\gamma_1 = e_G(M) * N_Q * \delta_b\ ,$$

where δ_x denotes the Dirac measure with mass in $x \in H$, N_Q the centered Gaussian measure with covariance operator Q and $e_G(M)$ the generalized exponent of the Lévy measure M, all of which are again infinitely divisible probability measures on H (see e.g. [Lin86, Sect. 5.7] for details).

Note that a function $\lambda : H \to \mathbb{C}$ is called *negative definite*, if for any $n \in \mathbb{N}$ and all n-tupels $(\xi_1, \ldots, \xi_n) \in (H)^n$, the $n \times n$-matrix $\big(\lambda(\xi_i) + \overline{\lambda(\xi_j)} - \lambda(\xi_i - \xi_j)\big)_{ij}$ is positive Hermitian; that is, if for any $(c_1, \ldots, c_n) \in \mathbb{C}^n$

$$\sum_{i,j=1}^{n} \big(\lambda(\xi_i) + \overline{\lambda(\xi_j)} - \lambda(\xi_i - \xi_j)\big) \cdot c_i \bar{c}_j \geq 0\ .$$

By a theorem of Schoenberg (see e.g. [BF75, Thm. II.7.8 and Cor. II.8.4]), this is equivalent to $\xi \mapsto e^{-t\lambda(\xi)}$ being positive definite. For an introduction into the theory of nega-

[3] A probability measure μ on H is called *infinitely divisible*, if for any $n \in \mathbb{N}$ there exists a probability measure μ_n on H, such that its n-fold convolution fulfills $(\mu_n)^{*n} = \mu$. Equivalently, the Fourier transforms fulfill $\hat{\mu}(\xi) = \big(\hat{\mu}_n(\xi)\big)^n$ for all $\xi \in H$.

tive definite functions, we refer to [BF75, Chap. II, Sect. 7 and 8]

Textbooks about Lévy processes, their distributions and related analytical and probabilistic theory include the quite recent [App09], which in its introduction offers an overview of other existing textbooks about the theory of Lévy processes. Specifically, the analysis of SPDE driven by Lévy processes in infinite dimensional spaces is the topic of [PZ07], which offers an exhaustive list of references to research publications in the area. Detailed introductions to the theory of infinitely divisible measures in infinite dimensional spaces are given in [Par67] and [Lin86]. The stochastic integration in infinite dimensional spaces with respect to Lévy processes has been studied e.g. in [AR05], [Sto05], [App06] (see also the references therein). Below, we also use recent results from [MPR10], which establishes (among other results) existence and uniqueness of solutions to SPDE with multiplicative Lévy noise in infinite dimensional spaces.

Stochastic Processes in this Thesis

To include in our framework also cylindrical Wiener processes, we consider our equation (SPDE) to be perturbed by a process Y, which is the sum of a Lévy process J with characteristic triplet $[0, 0, M]$ and a (possibly cylindrical) Wiener process W. The characteristic function of this new process $Y = J + \sqrt{Q}W$ (which possibly does not take values in the Hilbert space H itself, but only in a larger space) again has a representation of type (LKD), only that in our case the covariance operator Q representing the continuous diffusion part is not necessarily of trace-class. (Obviously, if Q turns out to be of trace-class, then Y is again an H-valued Lévy process as described before.) For details on cylindrical Wiener processes and stochastic analysis with such processes, we refer to the textbooks [DPZ92] and [PR07].

Let us point out, that the results presented in Chapters 3 and 4, and in Section 5.1 (that is, the case of an m-dissipative nonlinear drift part F), hold true in the case of (SPDE) driven by the general noise introduced above in this section (including, of course, H-valued Lévy noise). The results in Section 5.2 (the case of a merely measurable nonlinear drift part F) explicitly require, that $Q^{-1} \in L(H)$, i.e. we need a Wiener noise, which is essentially equally strong in all directions of the underlying orthonormal basis. This excludes the case of an H-valued Lévy process.

Finally, we would like to mention, that a different modification of the concept of Lévy processes towards a 'cylindrical Lévy process' has been studied in [AR10].

Ornstein-Uhlenbeck Processes and Generalized Mehler Semigroups

Consider the case of (SPDE) with $F \equiv 0$. The pathwise solution to such a linear equation is called an *Ornstein-Uhlenbeck process with jumps* (referring to the fundamental work of

Ornstein and Uhlenbeck, see e.g. [OU30]), or simply Ornstein-Uhlenbeck process. In this case, the transition evolution operator for $\varphi \in \mathcal{B}_b(H)$ takes the form

$$S_t\varphi(x) := \mathbb{E}\Big[\varphi\big(X(t,0,x)\big)\Big] = \int_H \varphi(e^{tA}x + y)\ \mu_t(\mathrm{d}y)\ ,$$

where μ_t is the distribution of the *stochastic convolution*

$$Y_A(t) := \int_0^t e^{sA}\ \mathrm{d}Y(s)\ , \quad t \geq 0\ .$$

As shown in [Sto05, Sect. 4.1] (see also [CM87], [App06], [PZ07] and the references therein), Y_A is well-defined for any C_0-semigroup $(e^{tA})_{t\geq 0}$ and cadlag in time (given a generalized contraction property to be fulfilled by (e^{tA}); cf. [PZ07, Sect. 9.4]). Its Fourier transform can be explicitly computed as

$$\hat{\mu}_t(\xi) = \exp\Big[-\int_0^t \lambda(e^{sA}\xi)\ \mathrm{d}s\Big] \quad \text{for all } t \geq 0, \xi \in H$$

(see e.g. [Knä08, Lem. 4.2]), where λ is the characteristic exponent of Y introduced above. As in [FR00, Sect. 2.1], we observe that

$$\begin{aligned}
\lambda_t &:= \int_0^t \lambda(e^{sA}\xi)\ \mathrm{d}s \\
&= -\int_0^t i\langle \xi, e^{sA}b\rangle\ \mathrm{d}s + \frac{1}{2}\int_0^t \langle e^{sA}\xi\,,\, Qe^{sA}\xi\rangle\ \mathrm{d}s \\
&\qquad - \int_0^t \int_H \exp\big[i\langle \xi, e^{sA}x\rangle\big] - 1 - \frac{i\langle \xi, e^{sA}x\rangle}{1+|x|^2}\ M(\mathrm{d}x)\ \mathrm{d}s \\
&= -i\langle \xi, b_t\rangle + \frac{1}{2}\cdot\langle \xi, Q_t\xi\rangle - \int_H \exp\big[i\langle \xi, x\rangle\big] - 1 - \frac{i\langle \xi, x\rangle}{1+|x|^2}\ M_t(\mathrm{d}x)\ ,
\end{aligned} \tag{LKD$_t$}$$

where we denote

$$\begin{aligned}
Q_t &:= \int_0^t e^{sA}Qe^{sA}\ \mathrm{d}s \\
b_t &:= \int_0^t e^{sA}b\ \mathrm{d}s + \int_0^t \int_H e^{sA}x \cdot \left(\frac{1}{1+|e^{sA}x|^2} - \frac{1}{1+|x|^2}\right) M(\mathrm{d}x)\ \mathrm{d}s\ ,
\end{aligned}$$

and define the measures M_t, $t > 0$, by

$$M_t(B) := \int_0^t M\big((e^{sA})^{-1}B \setminus \{0\}\big)\ \mathrm{d}s \quad \text{for all } B \in \mathcal{B}(H).$$

We emphasize that (LKD) for λ implies (LKD$_t$) for λ_t without assuming Q to be trace-class. If Q is trace-class, then by the theory of trace-class (or, nuclear) operators, the Q_t,

$t > 0$, are known to be again symmetric, trace-class and nonnegative (see e.g. [PR07, Rem. B.0.6]). We stress, however, that it is enough to assume each Q_t to be trace-class so that $\exp[-\lambda_t]$ is the Fourier transform of a measure on H. Below we shall always work under this weaker assumption (see condition (H.l2) below). Furthermore, we observe that

$$\int_H \left(1 \wedge |x|^2\right) \, M_t(\mathrm{d}x) = \int_0^t \int_H \left(1 \wedge |e^{sA}x|^2\right) \, M(\mathrm{d}x) \, \mathrm{d}s < \infty \, .$$

Consequently, μ_t is again infinitely divisible, since we have a Lévy-Khintchine decomposition for its characteristic function with the characterizing triplet $[b_t, Q_t, M_t]$.

Another crucial observation is, that (S_t) is indeed a semigroup of operators on $\mathcal{B}_b(H)$: We have $S_t\big(S_s\varphi\big) = S_{s+t}\varphi$ and $S_0\varphi = \varphi$. According to [BRS96, Prop 2.2], this is equivalent to the fact, that the family (μ_t) forms an $(e^{tA})_{t\geq 0}$-*convolution semigroup of measures*:

$$\mu_{t+s} = \big(\mu_t \circ (e^{sA})^{-1}\big) * \mu_s \quad \text{for all } 0 \leq s \leq s+t \leq T \, ,$$

where $\mu \circ B^{-1}$ denotes the image measure of a measure μ under the linear operator B, and $*$ denotes convolution of measures.

Thus, we obtain that transition semigroups for Ornstein-Uhlenbeck processes are generalized Mehler semigroups: By definition (cf. [BRS96, Prop. 2.2 and Def. 2.4]), a family $(p_t)_{t\geq 0}$ of operators on $\mathcal{B}_b(H)$ defined as

$$p_t f(x) := \int_H f(T_t x + y) \, \nu_t(\mathrm{d}y) \, , \quad x \in H, \ t \geq 0, \ f \in \mathcal{B}_b(H),$$

is a semigroup, called *generalized Mehler semigroup*, if (T_t) is a C_0-semigroup of operators on H and (ν_t) is a (T_t)-convolution semigroup of probability measures on H.

The concept of generalized Mehler semigroups has been introduced in [BRS96]. In recent years, they have been a topic of intense study, see e.g. [FR00], [DPT01], [SS01], [LR02], [RW03], [LR04], [App07] and the references therein. In Chapter 3 of this thesis, we generalize results on generalized Mehler semigroups to the case of explicitly time-dependent test functions. These results are then used in the subsequent chapters.

Operator Semigroups and Dissipativity

A family[4] of continuous linear operators $(T_t)_{t\geq 0}$ on a Banach space $\mathbb{B}$ is called a C_0-*semigroup* (or, *strongly continuous semigroup*) of operators, if $T_0 = I$ is the identity opera-

[4] Most of this paragraph is collected, in a very condensed form, from [Paz83, Chap. 1]. A classical reference for the role of operator semigroups in the theory of SPDE is [DPZ92].

tor on $\mathbb{B}$, if $T_{t+s} = T_t \circ T_s$ for any $s, t \geq 0$ and if

$$\lim_{t \searrow 0} T_t \varphi = \varphi \quad \text{for all } \varphi \in \mathbb{B}.$$

For such semigroups, the *infinitesimal generator* $\big(G, D(G)\big)$ is defined by

$$D(G) := \left\{ \varphi \in \mathbb{B} \,\middle|\, \lim_{t \searrow 0} \frac{T_t \varphi - \varphi}{t} \quad \text{exists} \right\}$$

$$G\varphi \quad := \lim_{t \searrow 0} \frac{T_t \varphi - \varphi}{t} \quad \text{for all } \varphi \in D(G).$$

If G is a bounded operator on $\mathbb{B}$, then we can write the semigroup generated by G as $(e^{tG})_{t \geq 0}$. Since the generator of a C_0-semigroup – and, likewise, the C_0-semigroup generated by an operator G – is unique, this notation is often extended even to cases, where G is not bounded. In this thesis, we stick to this, even though it is formally an abuse of notation. Important facts include that the infinitesimal generator of a C_0-semigroup of operators on $\mathbb{B}$ is densely defined in $\mathbb{B}$, and that it is a closed linear operator.

A linear[5] operator $\big(G, D(G)\big)$ on a Banach space $\mathbb{B}$ is called *dissipative*, if for any $\varphi \in \mathbb{B}$ there exists a φ^* in the dual space $\mathbb{B}^*$, such that ${}_{\mathbb{B}^*}\langle \varphi^*, \varphi \rangle_{\mathbb{B}} = \|\varphi^*\|^2_{\mathbb{B}^*} = \|\varphi\|^2_{\mathbb{B}}$ and ${}_{\mathbb{B}^*}\langle \varphi^*, G\varphi \rangle_{\mathbb{B}} \leq 0$. This is equivalent to $\|\alpha\varphi - G\varphi\|_{\mathbb{B}} \geq \alpha \cdot \|\varphi\|_{\mathbb{B}}$ for all $\varphi \in D(G)$ and $\alpha > 0$. It turns out, that every dissipative operator is closable. A core result about dissipativity of linear operators is the following theorem (see e.g. [Paz83, Chap. I, Thm. 4.3], [Ebe99, Chap. 1, App. A], or the original reference [LP61]):

Theorem ("Lumer-Phillips Theorem")**.** *Let G be a densely defined linear operator on $\mathbb{B}$. Then, G is the generator of a C_0-semigroup of contractions on $\mathbb{B}$ if and only if G is* m-dissipative *(i.e., G is dissipative and* $\operatorname{Range}\big(\alpha I - G, D(G)\big) = \mathbb{B}$ *for some (equivalently, for all)* $\alpha > 0$*).*

This theorem makes it obvious, why the question of dissipativity of Kolmogorov operators is considered interesting: As soon as we know, that such an operator is m-dissipative, we gain 'for free' the insight, that it generates a (uniquely determined) C_0-semigroup of contractions.

The definition of dissipativity can be extended to nonlinear maps; we refer to Appendix B for more.

Finally, the concept of operator semigroups and their generator can be generalized to the case of not necessarily strongly continuous semigroups. This has been pursued in the theory of so-called weakly continuous semigroups, or *π-semigroups* (see e.g. [Cer94], [Cer95], [CG95], [Pri99], [Man06], [Man08a] and the overview in Appendix A of this thesis). A linear operator $\big(K, D(K)\big)$ is defined to be the generator of a π-semigroup

[5] Most of this paragraph is collected, in a very condensed form, from [Paz83, Chap. 1] and [Ebe99, Chap. 1, App. A].

$(P_t)_{t\geq 0}$ of operators on a space of bounded and continuous functions as follows:

$$\varphi \in D(K) \quad \text{and} \quad K\varphi = f$$
$$\Leftrightarrow \quad \begin{cases} \lim\limits_{h\to 0} \dfrac{P_h\varphi(x) - \varphi(x)}{h} = f(x) \quad \text{for all } x \in H \\ \sup\limits_{\substack{h\in(0,T],\\ x\in H}} \dfrac{\left|P_h\varphi(x) - \varphi(x)\right|}{h} < \infty \,. \end{cases}$$

In this thesis we show, that the extensions of L_0 and $U + D_t$, respectively, are generators in this sense of the semigroups of transition evolution operators related to (SPDE) if F is sufficiently smooth and if $F = 0$, respectively. However, note that in this situation (generators of π- instead of C_0-semigroups) we have to prove the m-dissipativity separately.

Scope and Structure of this Thesis

The main parts of this thesis are the following:

Chapter 3: We generalize results from the literature on generalized Mehler semigroups (in particular, [LR02]) to the case of test functions with explicit time-dependence.

Chapter 4: We generalize results from [BDPR09] about Kolmogorov operators for SPDE with regular coefficients driven by Wiener noise to our framework (in particular, the case of noise with jumps).

Chapter 5: We show, that the research presented in [BDPR09] and [BDPR11], establishing uniqueness of the solution to the Fokker-Planck equation related to (SPDE) in the case, that the nonlinearity F is m-dissipative or merely measurable, can be generalized from the situation of an SPDE driven by Wiener noise to the situation of an SPDE driven by noise with jumps.

In addition, we can generalize and thus reinforce the observation, that in the case of an equation of type (SPDE), the m-dissipativity of the coefficients A and F implies the m-dissipativity of the Kolmogorov operator in the following sense: Denote for $p \geq 1$ the closure of L_0 in the space $L^p\big([0,T] \times H; \eta\big)$ by L_p. Then L_p is m-dissipative in $L^p\big([0,T] \times H; \eta\big)$ for every η in a large class of measures (see Subsection 2.1.3 below), which includes all solutions to (FPE) for any initial probability measure ζ with finite p-th moments.

It turns out, that our chosen test function space, an explicitly time-dependent extension of the space used in the existing literature on generalized Mehler semigroups, allows us to optimize some crucial estimates obtained in [BDPR09]. Consequently, some of the technical conditions imposed in the Wiener noise case (in particular, integrability conditions; cf. Remarks 2.2.4, 2.2.8, 2.2.12 and 2.2.13 below) can be relaxed. Essentially, this is due to the fact that the mapping $(s,t) \mapsto S_t\psi(s,x)$ enjoys better continuity properties for a ψ from our test function space $\mathcal{W}_{T,A}$ (see Subsection 2.1.2 below for the definition, and Lemma 3.4.2 for the continuity result) compared to the test function space $\mathcal{E}_A$ of exponential functions used e.g. in [BDPR09] and [BDPR11].

A description of the technical framework, our assumptions and main results, and differences compared to existing work, are given in **Chapter 2**.

An example is included in **Chapter 6**, and in the **appendices** we offer short introductory overviews of basic definitions and results concerning π-semigroups and the Yosida approximation of dissipative maps.

Finally, let us note that it seems realistic to hope, that the existence results for solutions to Fokker-Planck equations, as obtained in [BDPR10], can also be generalized to the case of (SPDE) driven by noise with jumps. This will be a topic of future research.

Acknowledgements

This thesis is a result of the author's PhD studies as an associated doctoral researcher within the International Research Training Group "Stochastics and Real World Models". Financial support of the DFG through grant no. IRTG 1132 for participation in a number of meetings and workshops is gratefully acknowledged. During the first months of the PhD project, the author was partly supported by the Collaborative Research Centre "Spectral Structures and Topological Methods in Mathematics". The financial support of the DFG through grant no. CRC 701 is also gratefully acknowledged.

The author would like to thank his PhD supervisor, Professor Michael Röckner, for his support and a number of very helpful discussions.[6]

[6] This Acknowledgement might be considered very short by some, in particular in comparison to statements by other PhD candidates (see also [Hoe11]). Please keep in mind, though, that this thesis, while written in English, is still written by a Westphalian. That is, by one of those people, from whose mouth "this is edible" may well be the highest and, in particular, the most wordy praise you might ever hear for really great food.

2. Framework and main results

2.1. Framework and notation

As laid out in the Introduction, we consider the equation

$$\begin{cases} \mathrm{d}X(t) = \Big[AX(t) + F\big(t, X(t)\big)\Big]\,\mathrm{d}t + \mathrm{d}Y(t) \\ \quad X(s) = x \in H\,, \quad 0 \le s \le t \le T, \end{cases} \tag{(SPDE)}$$

where the self-adjoint operator A is the infinitesimal generator of a C_0-semigroup of operators on H denoted by (e^{tA}), which we assume to be quasi-contractive; $F : D(F) \subset H \to H$ a possibly singular and/or multivalued map (see concrete hypotheses below); and T a finite positive real number. Heuristically, we think of Y as the sum of a centered Lévy process J in H with characteristic triplet $[0, 0, M]$ and a (possibly cylindrical) Wiener process $\sqrt{Q}W$. But we rarely use these notions below. Essentially, we only need the characteristic exponents λ of Y and λ_t of μ_t, but we assume Sazonov continuity only for λ_t (cf. Hypothesis (H.l2) below).

2.1.1. Spaces of functions and measures

Let H be a separable real Hilbert space, identified via the Riesz isomorphism with its own dual space H^* (i.e., the space of continuous linear functionals $H \to \mathbb{R}$). We denote the inner product in H by $\langle\,\cdot\,,\,\cdot\,\rangle$ and the norm by $|\cdot|$.

As usual, $L(H)$ denotes the space of bounded linear operators on H, and $\|\cdot\|_{L(H)}$ the corresponding canonical operator norm.

By $\mathcal{B}(H)$, we denote the Borel σ-algebra on H (that is, the σ-algebra generated by all open sets). Since H is separable, $\mathcal{B}(H)$ is generated by H^*.

The Banach space of all bounded, $\mathcal{B}(H)$-measurable functions $H \to \mathbb{R}$ is denoted by $\mathcal{B}_b(H)$, with the norm

$$\|\varphi\|_0 := \sup_{x\in H}\big|\varphi(x)\big| \quad \text{for all } \varphi \in \mathcal{B}_b(H)\,.$$

By $\mathcal{C}_u(H)$ we denote the closed subspace of $\mathcal{B}_b(H)$ of all functions $H \to \mathbb{R}$, which are uniformly continuous; the space of merely continuous elements of $\mathcal{B}_b(H)$ is denoted by $\mathcal{C}_b(H)$. The space $\mathcal{C}_{u,k}(H)$, $k \in \mathbb{N}$, contains all functions $\varphi : H \to \mathbb{R}$, such that the mapping $x \mapsto \frac{\varphi(x)}{1+|x|^k}$ is in $\mathcal{C}_u(H)$. We use the norm

$$\|\varphi\|_{u,k} := \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^k} \quad \text{for } k \in \mathbb{N},\ \varphi \in \mathcal{C}_{u,k}(H).$$

For any $k \in \mathbb{N}$, the space $\mathcal{C}_u^k(H)$, is made up of all functions in $\mathcal{C}_u(H)$ with continuous and bounded derivatives of order ℓ for any $\ell \leq k$.

By $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$, we denote the space of all functions $\varphi : [0,T] \times H \to \mathbb{R}$, such that $x \mapsto \varphi(t,x)$ is in $\mathcal{C}_u(H)$ for any $t \in [0,T]$, and $t \mapsto \varphi(t,\,\cdot\,)$ is continuous with respect to the sup-norm on $\mathcal{C}_u(H)$. On $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$, we define the norm

$$\|\varphi\|_{0,T} := \sup_{t\in[0,T]} \big\|\varphi(t,\,\cdot\,)\big\|_0 \,.$$

Furthermore, $\mathcal{C}\big([0,T];\mathcal{C}_{u,k}(H)\big)$, $k \in \mathbb{N}$, is the space of all functions $\varphi : [0,T] \times H \to \mathbb{R}$, such that the mapping $(t,x) \mapsto \frac{\varphi(t,x)}{1+|x|^k}$ is in $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$. Here, we use the norm

$$\|\varphi\|_{u,k,T} := \sup_{t\in[0,T]} \big\|\varphi(t,\,\cdot\,)\big\|_{u,k} \quad \text{for } k \in \mathbb{N},\ \varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,k}(H)\big).$$

Remark 2.1.1. *(i) Let $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$. Then the family $\big\{\varphi(t,\,\cdot\,) \;\big|\; t \in [0,T]\big\} \subset \mathcal{C}_u(H)$ is equi-uniformly continuous: For each $\varepsilon > 0$ there exists a $\delta_0 > 0$, such that*

$$|x-y| < \delta_0 \quad \textit{implies} \quad \sup_{t\in[0,T]} \big|\varphi(t,x) - \varphi(t,y)\big| < \varepsilon \,.$$

Indeed, given $\varepsilon > 0$, by definition of $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ there exists $\delta > 0$, such that

$$|t-s| < \delta \quad \textit{implies} \quad \big\|\varphi(t,\,\cdot\,) - \varphi(s,\,\cdot\,)\big\|_0 < \frac{\varepsilon}{3} \,.$$

On the other hand, let $r_1, \dots, r_N \in [0,T]$, such that

$$[0,T] \subset \bigcup_{i=1}^N (r_i - \delta\,,\, r_i + \delta) \,.$$

Again by definition of $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ there exists a δ_0, such that

$$|x-y| < \delta_0 \quad \textit{implies} \quad \big|\varphi(r_i,x) - \varphi(r_i,y)\big| < \frac{\varepsilon}{3} \textit{ for all } 1 \leq i \leq N.$$

Now, let $|x-y| < \delta_0$ and $t \in [0,T]$. Then, there is an $i \in \{1,\dots,N\}$, such that

$t \in (r_i - \delta\,,\, r_i + \delta)$. Hence,

$$\begin{aligned}
&\big|\varphi(t,x) - \varphi(t,y)\big| \\
&\quad \le \big|\varphi(t,x) - \varphi(r_i,x)\big| + \big|\varphi(r_i,x) - \varphi(r_i,y)\big| + \big|\varphi(r_i,y) - \varphi(t,y)\big| \\
&\quad < 2 \cdot \big\|\varphi(t,\,\cdot\,) - \varphi(r_i,\,\cdot\,)\big\|_0 + \frac{\varepsilon}{3} \\
&\quad < \varepsilon\,.
\end{aligned}$$

(ii) We have

$$\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big) = \mathcal{C}_u\big([0,T]\times H\big)\,.$$

Indeed, if $\varphi \in \mathcal{C}_u\big([0,T]\times H\big)$, then by definition for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\big|(t,x) - (s,y)\big| < \delta$ implies $\big|\varphi(t,x) - \varphi(s,y)\big| < \varepsilon$. Now, uniform continuity with respect to the space coordinate for fixed time is immediate. For continuity in time with respect to the supremum norm over H, we need to establish that for each $\varepsilon > 0$ there is a $\delta > 0$, such that $|t-s| < \delta$ implies $\big\|\varphi(t,\,\cdot\,) - \varphi(s,\,\cdot\,)\big\|_0 = \sup_{x\in H}\big|\varphi(t,x) - \varphi(s,x)\big| < \varepsilon$. This is, however, again immediate by definition for any $\varphi \in \mathcal{C}_u\big([0,T]\times H\big)$.

Conversely, if $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ and $\varepsilon > 0$, then there exists a $\delta > 0$, such that

$$|t-s| < \delta \quad \textit{implies} \quad \big\|\varphi(t,\,\cdot\,) - \varphi(s,\,\cdot\,)\big\|_0 < \frac{\varepsilon}{2}\,,$$

and (by (i))

$$|x-y| < \delta \quad \textit{implies} \quad \sup_{r\in[0,T]}\big|\varphi(r,x) - \varphi(r,y)\big| < \frac{\varepsilon}{2}\,.$$

Thus, if $\big|(t,x) - (s,y)\big| < \delta$, then

$$\begin{aligned}
&\big|\varphi(t,x) - \varphi(s,y)\big| \le \big|\varphi(t,x) - \varphi(t,y)\big| + \big|\varphi(t,y) - \varphi(s,y)\big| \\
&\quad \le \sup_{r\in[0,T]}\big|\varphi(r,x) - \varphi(r,y)\big| + \big\|\varphi(t,\,\cdot\,) - \varphi(s,\,\cdot\,)\big\|_0 < \varepsilon\,.
\end{aligned}$$

The Schwartz function space $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ is the space of all functions $\varphi : \mathbb{R}^d \to \mathbb{C}$, which are differentiable infinitely often and which fulfill

$$\|\varphi\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d}\big|x^\alpha D^\beta \varphi(x)\big| < \infty$$

for all d-tuples α, β of nonnegative integers. See e.g. [RS80, Sect. 5.3] or [BB03, Sect. 12.1] for more details about $\mathcal{S}$, the elements of which are also called functions of rapid decrease, or Schwartz test functions.

The (Frechet) derivative of a function with respect to space is denoted by D; the derivative with respect to time is denoted by D_t.

Spaces of measures are generally denoted by $\mathcal{M}$. The space of probability measures on H is denoted by $\mathcal{M}_1(H)$ and the space of complex-valued measures on H with bounded total variation by $\mathcal{M}_b^{\mathbb{C}}(H)$. The Fourier transform of a measure μ is denoted by $\mathcal{F}\mu = \hat{\mu}$, and the inverse Fourier transform by $\mathcal{F}^{-1}$.

2.1.2. The test function space $\mathcal{W}_{T,A}$

In contrast to the Wiener noise case, we need a different test function space, which can be considered as a space of (linear combinations of) Fourier transforms of certain measures. The time-independent test function space $\mathcal{W}_A$ is used in the literature on generalized Mehler semigroups (see e.g. [BRS96], [LR02]). However, since in our case the coefficients in (SPDE) depend explicitly on time, we need to introduce a time-dependent version, which we denote by $\mathcal{W}_{T,A}$.

We consider the following spaces of functions:

$\mathcal{W}_{A,\mathbb{C}}$ — Functions $\varphi : H \to \mathbb{C}$, such that there exists an $m \in \mathbb{N}$ with

$$\varphi(x) = f_m\big(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle\big) \quad \text{for all } x \in H,$$

where $f_m \in \mathcal{S}(\mathbb{R}^m;\mathbb{C})$ and $\{\xi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis (ONB) of H, with each ξ_i being an eigenvector of A (see Hypothesis (H.l1) on page 21 below).

$\mathcal{W}_A$ — Real-valued elements of $\mathcal{W}_{A,\mathbb{C}}$.

We note that for any fixed $m \in \mathbb{N}$, all $\varphi \in \mathcal{W}_A$, which correspond to Schwartz functions $f_m = f_\varphi$ from the space $\mathcal{S}(\mathbb{R}^m;\mathbb{R})$ and the same subset $\{\xi_1, \ldots, \xi_m\}$ of the ONB $\{\xi_i\}_{i\in\mathbb{N}}$ of H, form a vector space ('limited' vector space property).

Remark 2.1.2. *Observe that we can write any test function $\varphi \in \mathcal{W}_A$ as*

$$\varphi(x) = f_m(P_m x)$$

for all $x \in H$ (where m is uniquely determined by φ, and P_n, $n \in \mathbb{N}$, is the orthogonal projection of H onto $\operatorname{span}(\{\xi_1, \ldots, \xi_n\})$ *($\equiv \mathbb{R}^n$), defined by*

$$P_n x := \sum_{j=1}^{n} \langle x, \xi_j\rangle \xi_j \quad \textit{for all } x \in H).$$

$\mathcal{W}_{T,A}$ — The linear span of all functions $\psi : [0,T] \times H \to \mathbb{R}$, such that there is an $m \in \mathbb{N}$ with

$$\psi(t,x) = \phi(t) \cdot f_m\big(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle\big) \quad \forall\, (t,x) \in [0,T] \times H, \tag{2.1.1}$$

where $f_m \in \mathcal{S}(\mathbb{R}^m;\mathbb{R})$, the ONB $\{\xi_i\}_{i\in\mathbb{N}}$ is as above and $\phi \in \mathcal{C}^2([0,T])$, with the additional requirement that $\phi(T) = 0$.

The following remark about the structure of $\psi \in \mathcal{W}_{T,A}$ as (linear combinations of) Fourier transforms of measures in $\mathcal{M}_b^{\mathbb{C}}(H)$ is similar to the time-independent case described in [LR02, Rem. 1.1].

Remark 2.1.3. *Choose any $\psi \in \mathcal{W}_{T,A}$ of the form* (2.1.1) *and fix it. By definition of $\mathcal{W}_{T,A}$, there are an $m \in \mathbb{N}$ and an orthonormal set $\{\xi_1, \ldots, \xi_m\} \subset H$ of A-eigenvectors, such that*

$$\psi(t,x) = \phi(t) \cdot f_m\big(\langle \xi_1, x\rangle\,,\ \ldots\ ,\ \langle \xi_m, x\rangle\big)\ .$$

Now, denote the inverse Fourier transform of f_m by

$$g_m : \mathbb{R}^m \to \mathbb{C}\ .$$

Note that $g_m \in \mathcal{S}(\mathbb{R}^m;\mathbb{C})$, see e.g. [RS80, Ch. IX]*, and that g_m is uniquely determined by the requirement that*

$$f_m(y) = \int_{\mathbb{R}^m} e^{i(r,y)_{\mathbb{R}^m}}\ g_m(r)\mathrm{d}r \quad \textit{for all } y \in \mathbb{R}^m.$$

Furthermore, recall that the test functions in $\mathcal{W}_{T,A}$ are real-valued: $f_m(y)$ is real-valued if and only if $g_m(-r) = \overline{g_m(r)}$ for all $m \in \mathbb{N}$ and $r \in \mathbb{R}^m$. (As usual, we denote by $\overline{g}$ the complex conjugation of g.)

We set

$$\nu_m(\mathrm{d}r) := g_m(r)\mathrm{d}r\ .$$

Observe that both ν_m is in $\mathcal{M}_b^{\mathbb{C}}(\mathbb{R}^m)$ for each $m \in \mathbb{N}$. Now consider the embedding

$$\begin{array}{cccc} \Pi_m & : & \mathbb{R}^m & \to & H \\ & & (r_1, \ldots, r_m) & \mapsto & \sum_{j=1}^{m} r_j \xi_j \end{array}$$

with ξ_i as above, and define[1]

$$\nu_t := \phi(t)\,\nu_m \circ \Pi_m^{-1} \quad (\in \mathcal{M}_b^{\mathbb{C}}(H)). \tag{2.1.2}$$

[1] Note that the starting point for this construction is one element $\psi \in \mathcal{W}_{T,A}$. The number m is (uniquely) characteristic for ψ. In particular, ν_t is related to ψ and thus (implicitly) to this characteristic number m. Consequently, there is no need to have (2.1.2) "consistent for different choices of $m \in \mathbb{N}$".

Similar to [BLR99, Lem. 1.3], *we see that for all* $t \in [0, T]$ *and* $x \in H$,

$$\begin{aligned}\mathcal{F}(\nu_t)(x) &= \int_H e^{i\langle y,x\rangle}\ \nu_t(\mathrm{d}y) = \phi(t) \cdot \int_{\mathbb{R}^m} e^{i\langle \Pi_m(r),x\rangle}\ \nu_m(\mathrm{d}r)\\ &= \phi(t) \cdot \int_{\mathbb{R}^m} \exp\Big[i \sum_{j=1}^{m} r_j \cdot \langle \xi_j, x\rangle\Big]\ g_m(r)\mathrm{d}r = \phi(t) \cdot f_m\big(\langle \xi_1, x\rangle, \dots, \langle \xi_m, x\rangle\big)\\ &= \psi(t, x)\ .\end{aligned}$$

We need the following fact: Any $\varphi \in \mathcal{C}\big([0, T]; \mathcal{C}_u(H)\big)$ can be approximated pointwise in space for each $t \in [0, T)$ by some sequence $\{\psi_n\} \subset \mathcal{W}_{T,A}$. The presentation below is based on the approach in [DP04a, Prop. 1.2], but due to the time–dependence and the different nature of the test functions, the technical approach is changed here. Note that, since by definition the test functions take the value 0 at time T, the approximation can not include the end point of the time interval $[0, T]$.

Lemma 2.1.4. *For all* $\varphi \in \mathcal{C}\big([0, T]; \mathcal{C}_u(H)\big)$*, there exists a three-index sequence* $\{\psi_{n_1,n_2,n_3}\} \subset \mathcal{W}_{T,A}$*, such that for all* $t \in [0, T)$

(i) $\|\psi_{n_1,n_2,n_3}\|_{0,T} \le \|\varphi\|_{0,T} + 1$ *for any* $n_1, n_2, n_3 \in \mathbb{N}$.

(ii) $\lim\limits_{n_1\to\infty} \lim\limits_{n_2\to\infty} \lim\limits_{n_3\to\infty} \psi_{n_1,n_2,n_3}(t, x) = \varphi(t, x)$ *for all* $x \in H$.

Proof of Lemma 2.1.4. As observed in Remark 2.1.1, φ is an element of $\mathcal{C}_u\big([0, T] \times H\big)$.

Assume at first that $H = \mathbb{R}^d$. For any $n_1 \in \mathbb{N}$ (assume without loss of generality that $n_1 > \frac{2}{T}$), we can find (by multiplication with an appropriate, smooth 'bump function') a function $\varphi_{n_1} \in \mathcal{C}_u\big([0, T] \times \mathbb{R}^d\big)$, such that

(i) φ_{n_1} is supported on $\big[0, T - \frac{1}{n_1}\big] \times \big[-n_1 - \frac{1}{2}, n_1 + \frac{1}{2}\big]^d$

(ii) $\varphi_{n_1}(t, x) = \varphi(t, x)$ for all $(t, x) \in \big[0, T - \frac{2}{n_1}\big] \times \big[-n_1 + \frac{1}{2}, n_1 - \frac{1}{2}\big]^d$

(iii) $\big|\varphi_{n_1}(t, x)\big| \le \big|\varphi(t, x)\big|$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Of course, we find that $\varphi_{n_1}(t, x) \xrightarrow{n_1\to\infty} \varphi(t, x)$ for all $(t, x) \in [0, T) \times \mathbb{R}^d$.

Observe, that $\varphi_{n_1} \in \mathcal{C}_\infty\big(\mathrm{dom}(n_1)\big)$, where we set $\mathrm{dom}(n_1) := [0, T) \times (-n_1 - 1, n_1 + 1)^d$, and that $\mathcal{W}_{T,A,n_1} := \big\{\psi \in \mathcal{W}_{T,A}\ \big|\ \mathrm{supp}\,\psi \subset \mathrm{dom}(n_1)\big\}$ forms a sub-algebra of $\mathcal{C}_\infty\big(\mathrm{dom}(n_1)\big)$ (see also Remark 2.1.5 below), which separates the points of $\mathrm{dom}(n_1)$ and contains for each pair $(t, x) \in \mathrm{dom}(n_1)$ an element $\tilde{\psi}$, such that $\tilde{\psi}(t, x) \neq 0$. Thus, we can use the version of the Stone-Weierstraß theorem for locally compact spaces, as presented e.g. in [Sim63, §7.38], to obtain that $\mathcal{W}_{T,A,n_1}$ is dense in $\mathcal{C}_\infty\big(\mathrm{dom}(n_1)\big)$ with respect to uniform convergence. Hence, for each $n_1 \in \mathbb{N}$ there exists a sequence $(\psi_{n_1,n_2})_{n_2\in\mathbb{N}} \subset \mathcal{W}_{T,A,n_1} \subset \mathcal{W}_{T,A}$, such that $\psi_{n_1,n_2} \xrightarrow{n_2\to\infty} \varphi_{n_1}$ converges uniformly on

$\mathrm{dom}(n_1)$. By taking away its first N_{n_1} elements, if necessary, we may assume without loss of generality, that the approximating sequence fulfills

$$\big|\psi_{n_1,n_2}(t,x)\big| \le \big|\varphi_{n_1}(t,x)\big| + 1 \le \big|\varphi(t,x)\big| + 1 \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}.$$

Now, let H be infinite dimensional. Choose any $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ and consider for each $n_3 \in \mathbb{N}$ and $t \in [0,T]$ the mapping $\varphi(t, P_{n_3}\,\cdot\,)$. By the first part of the proof, for each $n_3 \in \mathbb{N}$ there is a double-index sequence $\{\psi_{n_1,n_2,n_3}\}_{n_1\in\mathbb{N},n_2\in\mathbb{N}} \subset \mathcal{W}_{T,A}$, such that for all $t \in [0,T)$

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty} \psi_{n_1,n_2,n_3}(t,P_{n_3}x) = \varphi(t,P_{n_3}x) \quad \text{for all } x \in H$$
$$\|\psi_{n_1,n_2,n_3}\|_{0,T} \le \|\varphi\|_{0,T} + 1\ .$$

Thus we obtain that for all $t \in [0,T)$

$$\lim_{n_3\to\infty}\lim_{n_1\to\infty}\lim_{n_2\to\infty} \psi_{n_1,n_2,n_3}(t,P_{n_3}x) = \varphi(t,x) \quad \text{for all } x \in H.$$ □

Remark 2.1.5. *$\mathcal{W}_{T,A}$ is a sub-algebra of $\mathcal{C}_u\big([0,T]\times H\big)$.*

The only non-obvious observation necessary is, that for any $\psi_1, \psi_2 \in \mathcal{W}_{T,A}$ of the form (2.1.1)

$$\begin{aligned}\psi_1(t,x)&\cdot\psi_2(t,x)\\ &= \phi_1(t)\cdot f_{1,m_1}\big(\langle\xi_1,x\rangle,\dots,\langle\xi_{m_1},x\rangle\big)\cdot\phi_2(t)\cdot f_{2,m_2}\big(\langle\xi_1,x\rangle,\dots,\langle\xi_{m_2},x\rangle\big)\\ &= (\phi_1\cdot\phi_2)(t)\cdot f_{m_1\vee m_2}\big(\langle\xi_1,x\rangle,\dots,\langle\xi_{m_1\vee m_2},x\rangle\big)\ ,\end{aligned}$$

where $f_{m_1\vee m_2} \in \mathcal{S}(\mathbb{R}^{m_1\vee m_2};\mathbb{R})$, since the product of a Schwartz function on $\mathbb{R}^{m_1\vee m_2}$ and a bounded $\mathcal{C}^\infty$-function on $\mathbb{R}^{m_1\vee m_2}$ is again a Schwartz function on $\mathbb{R}^{m_1\vee m_2}$.

Corollary 2.1.6. *For any $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,k}(H)\big)$, $k \in \mathbb{N}$, there exists a triple-index sequence $\{\psi_{n_1,n_2,n_3}\} \subset \mathcal{W}_{T,A}$, such that for all $(t,x) \in [0,T)\times H$*

(i) $\big|\psi_{n_1,n_2,n_3}(t,x)\big| \le \big(\|\varphi\|_{u,k,T}+1\big)\cdot\big(1+|x|^k\big)$ *for all* $n_1,n_2,n_3 \in \mathbb{N}$

(ii) $\lim\limits_{n_1\to\infty}\lim\limits_{n_2\to\infty}\lim\limits_{n_3\to\infty} \psi_{n_1,n_2,n_3}(t,x) = \varphi(t,x).$

Proof. Let $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,k}(H)\big)$. By a similar localization and approximation procedure as in the proof of the Lemma above (note that the localization of φ to $\mathrm{dom}(n_1)$ for any $n_1 \in \mathbb{N}$ is again in $\mathcal{C}_\infty\big(\mathrm{dom}(n_1)\big)$), we find a triple-index sequence $\{\psi_{n_1,n_2,n_3}\} \subset \mathcal{W}_{T,A}$, which approximates φ pointwise as claimed. Since the supremum norm of φ is not necessarily finite, we use the fact that $\varphi(t,x) = \frac{\varphi(t,x)}{1+|x|^k}\cdot\big(1+|x|^k\big)$, to obtain the claimed upper bound. □

2.1.3. Spaces of probability kernels η on H

Let $s \in [0,T]$ and $\zeta \in \mathcal{M}_1(H)$, and recall the formulation of the Fokker-Planck equation:

$$\int_H \psi(t,x)\ \eta_t(\mathrm{d}x) = \int_H \psi(s,x)\ \zeta(\mathrm{d}x) + \int_s^t \int_H L_0\psi(r,x)\ \eta_r(\mathrm{d}x)\ \mathrm{d}r \qquad \text{((FPE))}$$
$$\text{for all } \psi \in \mathcal{W}_{T,A} \text{ and almost all } t \in [s,T].$$

We use the following families of probability kernels:

$\mathcal{K}_s^0$ — positive Borel measures on $[s,T] \times H$, such that
$\eta(\mathrm{d}t, \mathrm{d}x) = \eta_t(\mathrm{d}x)\mathrm{d}t$,
where $\eta_t \in \mathcal{M}_1(H)$ for all $t \in [s,T]$
and $t \mapsto \eta_t(B)$ is measurable on $[s,T]$ for all $B \in \mathcal{B}(H)$

$\mathcal{K}_{s,\zeta}^0$ — elements η of $\mathcal{K}_s^0$, which fulfill (FPE) with initial condition $\zeta \in \mathcal{M}_1(H)$

$\mathcal{K}_{s,\leq\beta}^0$ — elements η of $\mathcal{K}_s^0$, such that there exists a $\beta \geq 0$ with

$$\int_{[s,T]\times H} L_0\psi(r,x)\ \eta(\mathrm{d}r,\mathrm{d}x) \leq \beta \cdot \int_{[s,T]\times H} \psi(r,x)\ \eta(\mathrm{d}r,\mathrm{d}x) \qquad (2.1.3)$$
$$\text{for all } \psi \in \mathcal{W}_{T,A} \text{ with } \psi \geq 0.$$

Remark 2.1.7. *Obviously, $\mathcal{K}_{s,\zeta}^0 \subset \mathcal{K}_{s,\leq\beta}^0$ (see e.g.* (2.2.3)*).*

Note, that we have to make sure in the different frameworks under consideration in this thesis (by appropriate assumptions on the kernels and F), that the integrals in (FPE) *exist; notably, that*

$$\int_{[s,T]\times H} \left| L_0\psi(r,x) \right|\ \eta(\mathrm{d}r,\mathrm{d}x) < \infty\ .$$

2.2. Hypotheses and main results

In this section, we give an overview of our framework and hypotheses, and the main results obtained under each set of hypotheses.

2.2.1. The linear case

As announced in the Introduction, in this part of the thesis we extend existing results about the generalized Mehler semigroup related to (SPDE) in the case $F \equiv 0$ to the case of explicitly time-dependent test functions. Proofs of the results are included in Chapter 3 below.

We need the following hypotheses for the linear case. Additional remarks concerning each of these follow below.

(H.l1) H has an orthonormal basis $\{\xi_i\}_{i\in\mathbb{N}}$ of eigenvectors of A, and A is self-adjoint and such that $\langle Ax, x\rangle \leq \omega \cdot |x|^2$ for some $\omega \geq 0$ and all $x \in D(A)$.

(H.l2) The function $\lambda : H \to \mathbb{C}$ is negative definite and of the form (LKD), but we assume the trace-class property only for Q_t and not necessarily for Q. The Lévy measure M in the decomposition (LKD) has finite q-th moments for a $q > 2$.[2]

For any $n \in \mathbb{N}$ and $F_n := \operatorname{span}\{\xi_1, \ldots, \xi_n\}$, the restriction $\lambda|_{F_n}$ is in $\mathcal{C}^\infty(F_n)$. Furthermore, $\ker Q_t = \{0\}$ for all $t \geq 0$ (which is the case, for example, if $\ker Q = \{0\}$).

(H.l3) $e^{tA}(H) \subset Q_t^{1/2}(H)$ for all $t > 0$.

Furthermore, for each $t \in (0, T]$ there is a $\Lambda_t \in L(H)$, such that $Q_t^{1/2}\Lambda_t = e^{tA}$ and

$$\int_0^T \|\Lambda_t\|_{L(H)}\, \mathrm{d}t < \infty\,.$$

Remark 2.2.1. *Hypothesis (H.l1) is crucial for the construction of our test function space $\mathcal{W}_{T,A}$. In addition, we note the following:*

(i) The hypothesis implies, that $\big(A, D(A)\big)$ is the generator of a quasi-contractive C_0-semigroup $(e^{tA})_{t\geq 0}$ on H; i.e., there is an $\omega \geq 0$, such that

$$\|e^{tA}\|_{L(H)} \leq e^{t\omega}$$

for all $t \geq 0$.

(ii) We denote the eigenvalues of $A = A^*$ (which due to the self-adjointness are real numbers) by α_i, $i \in \mathbb{N}$:

$$A\xi_i = \alpha_i \xi_i \quad \text{for all } i \in \mathbb{N}.$$

Remark 2.2.2. *Note that the assumptions on Q_t are standard assumptions for the existence of the stochastic convolution from the Wiener noise case (which of course is a special case of our situation; cf. e.g. the classical textbook* [DPZ92] *or the lecture notes* [Hai09]*). Some further observations concerning Hypothesis (H.l2):*

(i) As explained in the Introduction, the function

$$\xi \mapsto \lambda_t(\xi) = \int_0^t \lambda\big(e^{sA}\xi\big)\, \mathrm{d}s\,, \quad t \in [0, T],$$

[2]See pages 4 and 7 of the Introduction for details on the decompositions of λ and λ_t.

is of the form (LKD$_t$)*. It is Sazonov-continuous by the trace-class property of Q_t, together with the observation, that for any Lévy measure M on H the function*

$$\xi \mapsto \int_H \exp\big[i\langle \xi, x\rangle\big] - 1 - \frac{i\langle \xi, x\rangle}{1+|x|^2}\, M(\mathrm{d}x)$$

is Sazonov continuous on H. In fact, for each Lévy measure M on H this latter function is the characteristic function of the generalized exponent $e_G(M)$ of M, which itself is an infinitely divisible probability measure on H (cf. e.g. [Par67, Chap. VI, Thms. 2.4 and 4.8]*). Note that λ_t inherits from λ the property of being negative definite.*

(ii) By Sazonov continuity and negative definiteness of λ_t for all $t \in [0, T]$, there exists a family $(\mu_t)_{t\in[0,T]}$ of infinitely divisible probability measures, such that for all $t \in [0, T]$

$$\mathcal{F}(\mu_t)(\xi) = \hat{\mu}_t(\xi) = e^{-\lambda_t(\xi)} \quad \textit{for all } \xi \in H \tag{2.2.1}$$

and $\mu_t = e_G(M_t) * N_{Q_t} * \delta_{b_t}$, as explained in the Introduction.

Also, recall that the family $(\mu_t)_{t\geq 0}$ is an e^{tA}-convolution semigroup of probability measures on H:

$$\mu_{t+s} = \mu_t * \big(\mu_s \circ (e^{tA})^{-1}\big)$$

or, equivalently,

$$\hat{\mu}_{t+s}(\xi) = \hat{\mu}_t(\xi) \cdot \hat{\mu}_s(e^{tA}\xi) \quad \textit{for all } \xi \in H.$$

(iii) Note that trivially

$$\int_H |x|^2\, M(\mathrm{d}x) = \int_{\{|x|\leq 1\}} |x|^2\, M(\mathrm{d}x) + \int_{\{|x|>1\}} |x|^2\, M(\mathrm{d}x)\ .$$

The first summand is finite by virtue of M being a Lévy measure. The second summand is (up to a constant) smaller than the q-th moment of M for any $q > 2$. Consequently, we see immediately, that M has finite second moments. By [Lin83, Rem. 2 on p. 81] *this implies, that μ_t also has finite second moments.*

(iv) The smoothness condition on finitely based restrictions of λ is needed in particular to achieve that $S_t(\mathcal{W}_A) \subset \mathcal{W}_A$ (cf. Remark 3.2.2), which in turn is crucial for the approximation results obtained in Section 3.4, which are needed for the proof of all main results of this thesis.

See [LR02, Sect. 3] *for a possible approach to the situation without this restriction (however, only in the time-independent case).*

(v) *Infinite differentiability of finitely-based restrictions of λ holds for example, if the Lévy measure M in the Lévy-Khintchine decomposition of λ fulfills*

$$M(\mathrm{d}x) = \mathbb{I}_{\{a|\varepsilon \le \|a\| \le 1/\varepsilon\}}(x) \cdot M(\mathrm{d}x)$$

for some $\varepsilon > 0$ (cf. [LR02, Prop. 3.3]*).*

An example for a negative definite, Sazonov continuous function $\lambda : H \to \mathbb{R}$, which is C^∞ on H (not only on finitely based restrictions) is

$$\lambda(\xi) = \frac{m \cdot \|C\xi\|^2}{m + \|C\xi\|^2}\,, \quad m > 0,$$

where $C : H \to H$ is assumed to be symmetric, positive definite and of trace-class (cf. [LR02, Rem. 4.2]*).*

Remark 2.2.3. *Concerning Hypothesis (H.l3) let us first recall, that for any nonnegative operator B, we can uniquely identify another nonnegative operator C, such that $C^2 = B$. C is usually denoted as $B^{1/2}$ (see e.g.* [RS80, Thm. VI.9]*). If an operator $B \in L(H)$ is not injective, B^{-1} denotes the pseudo-inverse (see e.g.* [PR07, App. C]*):*

$$B^{-1} := \left(B|_{\ker(B)^\perp}\right)^{-1} \quad : \quad B\big(\ker(B)^\perp\big) = B(H) \quad \to \quad \ker(B)^\perp\,.$$

Hypothesis (H.l3) is needed for the proof of the integration by parts formula in Lemma 3.1.1. The latter in turn is required to establish, that the generalized Mehler semigroup (S_t) has the strong Feller property (see Lemma 3.1.2), on which our proof of the approximation result presented in Theorem 1 below relies.

For any $t \ge 0$, define the generalized Mehler semigroup (S_t) by

$$S_t\varphi(x) := \int_H \varphi(e^{tA}x + y)\; \mu_t(\mathrm{d}y)\,, \quad x \in H,\; \varphi \in \mathcal{B}_b(H).$$

The infinitesimal generator U of the semigroup (S_t) of operators, restricted to the test function space $\mathcal{W}_{T,A}$, has been identified in [FR00, Rem. 4.4] and [LR02, Thm. 1.1] as

$$U\psi(t,x) = \int_H \big(i\langle A\xi, x\rangle - \lambda(\xi)\big) \cdot e^{i\langle \xi, x\rangle}\; \nu_t(\mathrm{d}\xi)$$

for all $\psi \in \mathcal{W}_{T,A}$, $t \in [0,T]$, $x \in H$, where $\hat{\nu}_t(x) = \psi(t,x)$. Define the operator

$$V_0 := D_t + U\,, \quad D(V_0) := \mathcal{W}_{T,A}\,,$$

and consider the space-time homogenization $(S^T_\tau)_{\tau \ge 0}$ of (S_t), defined for elements φ of a particular subspace of $\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$ as

$$(S_\tau^T\varphi)(t,x) := \begin{cases} S_\tau\varphi(t+\tau,\,\cdot\,)(x) & \text{if } t+\tau\le T \\ 0 & \text{else.}\end{cases}$$

Then the generator $\big(V,D(V)\big)$ of $(S_\tau^T)_{\tau\ge 0}$ on $D(V)\subset\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$ in the sense of π-semigroups (similar to [Pri99]; see also Appendix A of this thesis) is an extension of $\big(V_0,D(V_0)\big)$, and we obtain the following approximation result (cf. Corollary 3.4.4), which is used throughout the rest of this thesis:

Theorem 1. *Let $u\in D(V)$ and let η be a finite nonnegative Borel measure on $[0,T]\times H$. Assume that Hypotheses (H.l1)–(H.l3) hold.*

Then, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}\subset\mathcal{W}_{T,A}$ and an $n_0\in\mathbb{N}$, such that for a finite $C>0$

$$\big|\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| \le C\cdot\big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big)$$

for all $(t,x)\in[0,T]\times H$, $n\ge n_0$, and

$$\psi_n\to u\ ,\quad \langle D\psi_n,h\rangle\to\langle Du,h\rangle\ ,\quad V_0\psi_n\to Vu$$

converge in measure η as $n\to\infty$ for any $h\in H$.

Remark 2.2.4. *This result has been shown in the Wiener noise case in* [BDPR09, Cor. A.3] *(which in turn generalizes* [DPT01, Sect. 2]*). However, due to the different family of test functions used in these references, the upper bound achieved there grows proportional to $\big(1+|x|^2\big)$ in space. This is essentially due to the fact, that the results equivalent to Lemma 3.4.2 in these references contain continuity of the map $(s,t)\mapsto S_t\psi(s,\,\cdot\,)$ in the topology of $\mathcal{C}_{u,2}(H)$ only.*

2.2.2. Regular nonlinearity F

In this part of the thesis, we show m-dissipativity of L on $L^p\big([0,T]\times H;\eta\big)$ for each $\eta\in\mathcal{K}^0_{0,\le\beta}$ fulfilling certain integrability conditions, and existence of a solution η to (FPE) for the case of (SPDE) with a nonlinear drift part F, which fulfills quite strong regularity conditions (see Hypothesis (H.c1) below). Proofs are included in Chapter 4.

For any $s\in[0,T]$ and $p\in[0,\infty)$, we define

$$\mathcal{K}_s^p := \left\{\eta\in\mathcal{K}_s^0\ \middle|\ \int_{[s,T]\times H}|x|^p\,\eta(\mathrm{d}t,\mathrm{d}x)<\infty\right\}.$$

Furthermore, we set for any $s \in [0,T]$, $p \in [0,\infty)$ and $\beta \geq 0$

$$\mathcal{K}^p_{s,\zeta} := \mathcal{K}^p_s \cap \mathcal{K}^0_{s,\zeta}$$
$$\mathcal{K}^p_{s,\leq\beta} := \left\{ \eta \in \mathcal{K}^0_{s,\leq\beta} \;\middle|\; \int_{[s,T]\times H} |x|^p + |x|^p \cdot \big|F(t,x)\big|^p \;\eta(\mathrm{d}t,\mathrm{d}x) < \infty \right\}.$$

In addition to Hypotheses (H.l1)–(H.l3), we assume the following.

(H.c1) Both $F : [0,T] \times H \to H$ and $DF(t,\cdot\,) : H \to L(H)$ (the latter for any $t \in [0,T]$) are continuous.

Further, there is a $K > 0$, such that

$$\big|F(t,x) - F(t,y)\big| \leq K \cdot |x - y| \quad \text{for all } x,y \in H,\, t \in [0,T].$$

Remark 2.2.5. *(H.c1) implies, that $L_0\psi$ is η-integrable (i.e., the integrals in* (FPE) *exist) for any $\eta \in \mathcal{K}^1_s$. It also implies, that $\mathcal{K}^{2p}_{s,\zeta} \subset \mathcal{K}^p_{s,\leq\beta}$.*

Let us recall the formulation of the Kolmogorov operator on the test function space $\mathcal{W}_{T,A}$: We have

$$L_0\psi(t,\cdot\,) = D_t\psi(t,\cdot\,) + \big\langle D\psi(t,\cdot\,)\,,\,F(t,\cdot\,)\big\rangle + U\psi(t,\cdot\,)\,.$$

The following remark transfers [BDPR09, Rem. 1.1] into our framework.

Remark 2.2.6. *Recall the formulation of the Fokker-Planck equation: Let $\zeta \in \mathcal{M}_1(H)$ and $s \in [0,T]$ and consider a solution η to*

$$\int_H \psi(t,x)\;\eta_t(\mathrm{d}x) = \int_H \psi(s,x)\;\zeta(\mathrm{d}x) + \int_s^t \int_H L_0\psi(r,x)\;\eta_r(\mathrm{d}x)\;\mathrm{d}r \qquad \text{((FPE))}$$
$$\text{for all } \psi \in \mathcal{W}_{T,A} \text{ and almost all } t \in [s,T].$$

We note the following:

(i) Independent of the (non)regularity of F, any $\eta \in \mathcal{K}^0_{0,\zeta}$ fulfills for all $\psi \in \mathcal{W}_{T,A}$

$$\int_0^T \int_H L_0\psi(r,x)\;\eta_r(\mathrm{d}x)\;\mathrm{d}r = -\int_H \psi(0,x)\;\zeta(\mathrm{d}x) \qquad (2.2.2)$$

and, consequently,

$$\int_0^T \int_H L_0\psi(r,x)\;\eta_r(\mathrm{d}x)\;\mathrm{d}r \leq 0 \quad \text{for all } \psi \in \mathcal{W}_{T,A} \text{ with } \psi \geq 0. \qquad (2.2.3)$$

Note that equivalence of (FPE) *(for almost all $t \in [0,T]$) and* (2.2.2) *(as e.g. in* [BDPR08, Lemma 2.1]*) does not seem to hold in our framework.*

(ii) If $\psi \in \mathcal{W}_{T,A}$, then $\psi^2 \in \mathcal{W}_{T,A}$ (cf. Remark 2.1.5), and the square field operator Γ takes the form

$$\begin{aligned}\Gamma(\psi,\psi)(t,x) &:= L_0\psi^2(t,x) - 2\psi(t,x)\cdot L_0\psi(t,x) \\ &= \Big\langle D\psi(t,x)\,,\, Q\big(D\psi(t,x)\big)\Big\rangle + \int_H \big(\psi(t,x)-\psi(t,x+y)\big)^2\, M(\mathrm{d}y)\,,\end{aligned} \tag{2.2.4}$$

where Q and M are from (LKD). *We note that both summands on the right hand side are nonnegative.*

Proof. We only show (2.2.4); the rest is obvious. Using [LR04, Prop. 4.1] for the Ornstein-Uhlenbeck component U, we see that

$$\begin{aligned}L_0\psi^2(t,x) &= D_t\big(\psi^2(t,x)\big) + \big\langle D\big(\psi^2(t,x)\big)\,,\, F(t,x)\big\rangle + U\big(\psi^2(t,x)\big) \\ &= 2\psi(t,x)\cdot D_t\psi(t,x) + \big\langle 2\psi(t,x)\cdot D\psi(t,x)\,,\, F(t,x)\big\rangle \\ &\quad + 2\psi(t,x)\cdot U\psi(t,x) + \Big\langle D\psi(t,x)\,,\, Q\big(D\psi(t,x)\big)\Big\rangle \\ &\quad + \int_H \big(\psi(t,x)-\psi(t,x+y)\big)^2\, M(\mathrm{d}y)\,,\end{aligned}$$

which proves the claim. □

From (2.2.2) and (2.2.4) we get that for all $\psi \in \mathcal{W}_{T,A}$, since $\mathcal{W}_{T,A}$ is an algebra,

$$\begin{aligned}&\int_0^T \int_H \psi(t,x)\cdot L_0\psi(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t \\ &= \frac{1}{2}\int_0^T\int_H L_0\psi^2(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t - \frac{1}{2}\int_0^T\int_H \Gamma(\psi,\psi)(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t \\ &= -\frac{1}{2}\int_H \psi^2(0,x)\ \eta_0(\mathrm{d}x) - \frac{1}{2}\int_0^T\int_H \Gamma(\psi,\psi)(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t\,,\end{aligned}$$

and even if (η_t) satisfies only (2.2.3) instead of (2.2.2), we still have that

$$\begin{aligned}&\int_0^T \int_H \psi(t,x)\cdot L_0\psi(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t \le -\frac{1}{2}\int_0^T\int_H \Gamma(\psi,\psi)(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t \\ &\le 0 \quad \text{for all } \psi \in \mathcal{W}_{T,A} \text{ with } \psi \ge 0.\end{aligned}$$

Remark 2.2.7. *Hypothesis (H.c1) implies m-dissipativity of $x \mapsto F(t,x) - Kx$ for all $t \in [0,T]$.*

In this framework, we define the transition operators related to the solution of (SPDE) (in the genuinely semilinear case) by

$$P_{s,t}\varphi(x) := \mathbb{E}\big[\varphi\big(X(t,s,x)\big)\big]\,,\quad 0 \le s \le t \le T\,,\ \varphi \in \mathcal{C}_{u,1}(H).$$

We show, that the generator on $\mathcal{W}_{T,A}$ of the space-time homogenization P_τ^T of $P_{s,t}$ is identical to L_0, and we establish the extension of L_0 and its m-dissipativity.

More precisely, we obtain the following results (cf. Proposition 4.3.1 and Proposition 4.4.3):

Theorem 2. *Let Hypotheses (H.l1)–(H.l3) and (H.c1) hold. Let $s \in [0,T]$ and $\zeta \in \mathcal{M}_1(H)$, such that $\int_H |x| \, \zeta(\mathrm{d}x) < \infty$. Define the family $(\eta_t)_{s\leq t\leq T} \subset \mathcal{M}_1(H)$ by setting $\eta_t := P_{s,t}^*\zeta$. Then, $\eta \in \mathcal{K}_{s,\zeta}^1$. Furthermore, if $\int_H |x|^2 \, \zeta(\mathrm{d}x) < \infty$, then $\eta \in \mathcal{K}_{s,\zeta}^2$.*

Choose any $p \in [1,\infty)$ and let $\eta \in \mathcal{K}_{s,\leq\beta}^p$. Then, the closure L_p of L_0 in $L^p\big([s,T] \times H;\eta\big)$ is m-dissipative in $L^p\big([s,T] \times H;\eta\big)$ and thus generates a C_0-semigroup on this L^p-space, which is Markov.

In particular, if $\int_H |x|^2 \, \zeta(\mathrm{d}x) < \infty$, then the closure L_1 of L_0 in $L^1\big([s,T] \times H; (P_{s,t}^*\zeta)_{t\geq s}\big)$ is m-dissipative in this L^1-space and thus generates a Markovian C_0-semigroup on the L^1-space. It follows from the subsequent study of the possibly singular, dissipative case, that this solution $\eta = (P_{s,t}^*\zeta)_{t\geq s}$ is actually unique.

Remark 2.2.8. *A similar result for the case of* (SPDE) *driven by Wiener noise has been obtained in* [BDPR09, Thm. 2.8]. *As mentioned in the Introduction, our choice of $\mathcal{W}_{T,A}$ as test function space allows for relaxed integrability conditions; in the above named reference, the condition*

$$\int_{[0,T]\times H} |x|^{2p} + |x|^{2p} \cdot \big|F(t,x)\big|^p \, \eta(\mathrm{d}t,\mathrm{d}x) < \infty$$

is required for the m-dissipativity result. Accordingly, in [BDPR09] *the m-dissipativity of the Kolmogorov operator L with respect to the solution η of* (FPE) *requires, that the initial condition ζ has finite third moments.*

2.2.3. m-dissipative nonlinearity F

In this part of the thesis, we still consider the problem (SPDE) and the related Fokker-Planck equation as introduced above. However, we now allow F to be less regular than before: we only assume, that F is m-dissipative (cf. Definition 2.2.9 below and Appendix B). For a large family of kernels η and some $p \in [1,\infty)$, we show m-dissipativity in $L^p\big([0,T] \times H;\eta\big)$ for the closure L_p of L_0 in this L^p-space. We also show, that solutions to (FPE) are unique under certain integrability conditions. Proofs are included in Section 5.1.

Definition 2.2.9. Let $\big(F(t,\,\cdot\,)\big)_{t\in[0,T]}$ be a family of mappings $F(t,\,\cdot\,) : D\big(F(t,\,\cdot\,)\big) \subset H \to 2^H$, where $D\big(F(t,\,\cdot\,)\big)$ is a Borel set in H for each $t \in [0,T]$.

This family is called *m-quasi-dissipative*, if the following conditions are fulfilled:

- There is a $K \geq 0$ independent of t, such that for any $t \in [0, T]$

$$\langle u - v, x - y \rangle \leq K \cdot |x - y|^2 \qquad \text{for all } x, y \in D\big(F(t, \cdot\,)\big),\ u \in F(t, x),\ v \in F(t, y).$$

- For $t \in [0, T]$ and any $\alpha > K$,

$$\operatorname{Range}\big(\alpha I - F(t, \cdot\,)\big) = \bigcup_{x \in D(F(t, \cdot\,))} \big(\alpha x - F(t, x)\big) = H \,.$$

If an m-dissipative family F fulfills the first of the two conditions even for $K = 0$, it is called *m-dissipative*.

Remark 2.2.10. *Let F be an m-dissipative mapping as defined above. Note the following:*

(i) For any $x \in D\big(F(t, \cdot\,)\big)$, the set $F(t, x)$ is closed, non-empty and convex (see e.g. [Bar76, Prop. 3.5(iv), Chap. II]*, or Appendix B).*

(ii) For any $x \in D\big(F(t, \cdot\,)\big)$, we set

$$F_0(t, x) := y_0(t) \,,$$

where $y_0(t) \in F(t, x)$ is chosen such that $\big|y_0(t)\big| = \min_{y \in F(t,x)} |y|$.

As a consequence of the Yosida approximation of F (see Chapter 5 or Appendix B), we gain that the function $x \mapsto F_0(t, x)$ is Borel-measurable for each $t \in [0, T]$.

(iii) As stated in [BDPR09, Rem. 3.1(i)]*, the results below (in this part of the thesis) extend to the case of m-quasi-dissipative F: All proofs remain valid for F replaced by $\tilde{F} := F + F_2$, where F_2 is a C^∞- and Lipschitz-continuous map; in particular, this holds for $F_2 := K \cdot I$.*

For this part of the thesis, we require that in addition to Hypotheses (H.l1)–(H.l3) the following holds true:

(H.d1) $\big(F(t, \cdot\,)\big)_{t \in [0,T]}$ is a family of m-dissipative mappings in H, such that for all $t \in [0, T]$ we have $0 \in D\big(F(t, \cdot\,)\big)$ and $F_0(t, 0) = 0$.

(As a rule, we set $\big|F_0(t, x)\big| = +\infty$ if $(t, x) \notin D(F)$.)

(H.d2) For some $p \in [1, \infty)$, the set $\mathcal{K}^{p,\mathrm{diss}}_{0,\leq\beta}$ (to be defined below) is not empty.

We define for $p \in [1, \infty)$ and $s \in [0, T]$

$$\mathcal{K}^{p,\mathrm{diss}}_s := \left\{ \eta \in \mathcal{K}^0_s \;\middle|\; \int_{[s,T] \times H} |x|^p + \big|F_0(t, x)\big|^p + |x|^p \cdot \big|F_0(t, x)\big|^p \, \eta(\mathrm{d}t, \mathrm{d}x) < \infty \right\} .$$

We furthermore set

$$\mathcal{K}^{p,\mathrm{diss}}_{s,\zeta} := \mathcal{K}^0_{s,\zeta} \cap \mathcal{K}^{p,\mathrm{diss}}_s \quad \text{and} \quad \mathcal{K}^{p,\mathrm{diss}}_{s,\leq\beta} := \mathcal{K}^0_{s,\leq\beta} \cap \mathcal{K}^{p,\mathrm{diss}}_s .$$

Remark 2.2.11. *Let us note the following observations concerning the hypotheses:*

(i) By (H.d1), any $\eta \in \mathcal{K}^{p,\mathrm{diss}}_0$, $p \geq 1$, must have the property that $\eta\big([0,T] \times H \setminus D(F)\big) = 0$.

(ii) Recall that for any $\psi \in \mathcal{W}_{T,A}$, we have $L_0\psi = V_0\psi + \langle D\psi, F\rangle$. By Remark 3.3.1 below, there exists an $M_{V,\psi} \in (0,\infty)$, such that $V_0\psi \leq M_{V,\psi}$. Since $|D\psi|$ is bounded for every $\psi \in \mathcal{W}_{T,A}$, we have that $\langle D\psi, F\rangle \leq \|D\psi\|_{0,T} \cdot |F|$ pointwise on $[0,T] \times H$.

Thus, $L_0\psi \in L^p\big([0,T] \times H; \eta\big)$ for p as in Hypothesis (H.d2), $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{0,\leq\beta}$ and $\psi \in \mathcal{W}_{T,A}$.

In this framework, we obtain the following result (see Proposition 5.1.2):

Theorem 3. *Assume that Hypotheses (H.l1)–(H.l3) and (H.d1)–(H.d2) hold, and let $p \in [1,\infty)$ be as in Hypothesis (H.d2).*

Then, for each $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{s,\leq\beta}$ the closure L_p of L_0 in $L^p\big([s,T] \times H; \eta\big)$ is m-dissipative in $L^p\big([s,T] \times H; \eta\big)$. It generates a Markov semigroup, and the resolvent set $\varrho(L_p)$ is equal to $\mathbb{R}$.

In the Wiener-noise case, there are existence results for such measures η solving the Fokker-Planck equation (FPE) (or an equivalent formulation). See e.g. [BDPR09, Rem. 3.5] for references.

We obtain the following uniqueness result (cf. Proposition 5.1.3):

Theorem 4. *Let $\zeta \in \mathcal{M}_1(H)$ and $s \in [0,T]$. Given Hyptheses (H.l1)–(H.l3) and (H.d1), the set $\mathcal{K}^{1,\mathrm{diss}}_{s,\zeta}$ contains at most one element.*

Remark 2.2.12. *The two theorems above generalize Theorems 3.3 and 3.6 in* [BDPR09] *from the case of Wiener noise to that of Lévy noise with jumps (and the sum of such noise with a cylindrical Wiener process). Given the work in the parts of this thesis preceding Chapter 5, the remaining steps to prove Theorems 3 and 4, as included in Section 5.1 of this thesis, are essentially similar to those in* [BDPR09]*; they are, however, included for the convenience of the reader.*

As before, our choice of the test function space allows us to achieve uniqueness with relaxed integrability conditions on η: In [BDPR09]*, the uniqueness result requires solutions η to* (FPE) *to fulfill the condition*

$$\int_{[s,T]} \int_H |x|^2 + \big|F_0(t,x)\big| + |x|^2 \cdot \big|F_0(t,x)\big| \; \eta_t(\mathrm{d}x) \; \mathrm{d}t < \infty ,$$

which is more restrictive than our condition.

2.2.4. Measurable nonlinearity F

In this part of the thesis, which generalizes the Wiener noise case results presented in [BDPR11, Section 4] to our framework, we show uniqueness for the solution of the Fokker-Planck equation related to (SPDE) in the case of a merely measurable nonlinearity F. Proofs are included in Section 5.2.

In addition to Hypotheses (H.l1)–(H.l3), we require the following throughout this part of the thesis:

(H.m1) $F : D(F) \to H$ is a measurable map, where $D(F) \in \mathcal{B}\big([0,T] \times H\big)$.

(As a rule, we set $\big|F(t,x)\big| = +\infty$ if $(t,x) \notin D(F)$.)

(H.m2) $Q^{-1} \in L(H)$.

It turns out, that we have to 'pay' for the relaxed requirements on F by having to assume a stricter integrability condition on the family of possible solution measures η, and by restricting ourselves to (SPDE) driven by noise, which has a ("full") cylindrical Wiener noise part. Apart from this, the main idea of the proof (to show and then use a dense range condition for L) is similar to the proof of uniqueness in the m-dissipative case treated before. (Even though, the method to establish this dense range condition is different.)

Let $s \in [0,T]$ and $\zeta \in \mathcal{M}_1(H)$. Set

$$\mathcal{K}_{s,\zeta}^{\mathrm{meas}} := \left\{ \eta \in \mathcal{K}_{s,\zeta}^0 \;\middle|\; \int_{[s,T]\times H} |x|^2 + \big|F(t,x)\big|^2 + |x|^2 \cdot \big|F(t,x)\big|^2 \, \eta(\mathrm{d}t, \mathrm{d}x) < \infty \right\}.$$

We observe that $\mathcal{K}_{s,\zeta}^{\mathrm{meas}}$ is a convex subset of $\mathcal{K}_{s,\zeta}^0$.

We obtain the following result (cf. Proposition 5.2.6 and the notes preceding the proposition):

Theorem 5. *Assume that Hypotheses (H.l1)–(H.l3) and (H.m1)–(H.m2) hold.*
Then, $\mathcal{K}_{s,\zeta}^{\mathrm{meas}}$ contains at most one element.

Remark 2.2.13. *This result generalizes* [BDPR11, Theorem 4.1] *from the cylindrical Wiener noise case to the case of* (SPDE) *driven by the sum of Lévy noise with jumps and a cylindrical Wiener process. It uses a gradient estimate for the square-field operator Γ introduced in Remark 2.2.6(ii) above. Given this estimate and the preparations in Chapters 3 and 4, we can use ideas from* [BDPR11] *to prove Theorem 5.*

As before, we obtain relaxed moment conditions for η; in [BDPR11], *the uniqueness of the solution to* (FPE) *is shown only in*

$$\left\{ \eta \in \mathcal{K}_{s,\zeta}^0 \;\middle|\; \int_s^T \int_H |x|^4 + \big|F(t,x)\big|^2 + |x|^4 \cdot \big|F(t,x)\big|^2 \, \eta_t(\mathrm{d}x)\, \mathrm{d}t < \infty \right\}.$$

The differences in the moment conditions are caused by our approach pursued in the first chapters of this thesis and the estimates obtained there.

Remark 2.2.14. *As pointed out in the Introduction, it seems reasonable to assume, that the existence results for solutions to Fokker-Planck equations, as obtained in* [BDPR10]*, can also be generalized to the case of noise with jumps (more precisely, Lévy noise plus cylindrical Wiener noise). This will be a topic of future research.*

3. The linear case

As indicated in the Introduction, this chapter generalizes the appendix of [BDPR09] to our situation, adapting and extending methods and results from the literature on generalized Mehler semigroups (particularly, [LR02] and [LR04]).

Throughout this chapter, we assume the Hypotheses (H.l1)–(H.l3) to hold.

3.1. The generalized Mehler semigroup (S_t)

We start with an integration by parts formula. Before we formulate the result, let us recall the following (see e.g. [DP04a, p.11f]): The range of $Q_t^{1/2}$ is a proper subset of H:

$$Q_t^{1/2}(H) \subsetneq H \quad \text{(actually: } N_{Q_t}\big(Q_t^{1/2}(H)\big) = 0\text{).}$$

Thus, the *white noise function* W is not defined on all of H:

$$\begin{aligned} W : Q_t^{1/2}(H) &\to L^2(H, N_{Q_t}) \\ f \quad &\mapsto \langle \,\cdot\, , Q_t^{-1/2} f \rangle =: W_f \, . \end{aligned}$$

However, from the following computation (for $f \in Q_t^{1/2}(H)$; using [DP04a, (1.17)]),

$$\int_H \langle x, Q_t^{-1/2} f \rangle^2 \, N_{Q_t}(\mathrm{d}x) = \big\langle Q_t Q_t^{-1/2} f \, , \, Q_t^{-1/2} f \big\rangle = |f|^2 \, , \tag{3.1.1}$$

we gather that W is an isometric isomorphism:

$$\|W_f\|^2_{L^2(H, N_{Q_t})} = |f|^2 \, .$$

Since $Q_t^{1/2}(H)$ is dense in H (due to (H.l2); particularly, $\ker Q_t = \{0\}$), the white noise function can thus be extended uniquely to a mapping $H \to L^2(H, N_{Q_t})$, which is still denoted by W. Even for this extension, the notation

$$W_f(\,\cdot\,) =: \langle \,\cdot\, , Q_t^{-1/2} f \rangle \, , \quad f \in H,$$

is used (though it formally is an abuse of notation).

Lemma 3.1.1. *For $\varphi \in \mathcal{C}_{u,1}(H)$, $h \in H$, $t \in (0,T]$, we have for all $x \in H$, that*

$$\begin{aligned}
&\big\langle DS_t\varphi(\,\cdot\,)(x)\,,\,h\big\rangle \\
&= \int_H \int_H \Big(\int_H \varphi(e^{tA}x + y_1 + y_2 + y_3) \cdot \langle \underbrace{Q_t^{-1/2} e^{tA}}_{=:\Lambda_t} h\,,\, Q_t^{-1/2} y_1 \rangle \; N_{Q_t}(\mathrm{d}y_1) \Big) \\
&\qquad\qquad e_G(M_t)(\mathrm{d}y_2)\; \delta_{b_t}(\mathrm{d}y_3)\;.
\end{aligned}$$

Let us repeat, that the extension of the white noise function (to use "$\langle\,\cdot\,, Q_t^{-1/2} y_1\rangle$" for arbitrary $y_1 \in H$) requires that $\ker Q_t = \{0\}$ (cf. (H.l2)). The following proof adapts a classical argument from the Wiener noise case to our more general framework; see e.g. [DP04a, Cor. 1.6 and Prop. 1.7] or [DPZ02, Thm. 6.2.2] for the Wiener noise case.

Proof. Recall the Lévy-Khintchine decomposition

$$\mu_t = N_{Q_t} * e_G(M_t) * \delta_{b_t}\;.$$

Using the definition of (S_t) and this decomposition, we observe that

$$\begin{aligned}
\big\langle DS_t\varphi(\,\cdot\,)(x)\,,\,h\big\rangle &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \Big(\int_H \varphi(e^{tA}x + y + \varepsilon\cdot e^{tA}h) - \varphi(e^{tA}x + y)\; \mu_t(\mathrm{d}y) \Big) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_H \int_H \Big(\int_H \varphi(e^{tA}x + y_1 + y_2 + y_3 + \varepsilon\cdot e^{tA}h) - \varphi(e^{tA}x + y_1 + y_2 + y_3) \\
&\qquad\qquad N_{Q_t}(\mathrm{d}y_1) \Big)\; e_G(M_t)(\mathrm{d}y_2)\; \delta_{b_t}(\mathrm{d}y_3) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_H \int_H \Big(\int_H \varphi(e^{tA}x + y_1 + y_2 + y_3) \cdot \big(e^{\langle \Lambda_t(\varepsilon h), Q_t^{-1/2} y_1\rangle - \frac{1}{2}|\Lambda_t(\varepsilon h)|^2} - 1 \big) \\
&\qquad\qquad N_{Q_t}(\mathrm{d}y_1) \Big)\; e_G(M_t)(\mathrm{d}y_2)\; \delta_{b_t}(\mathrm{d}y_3)\;,
\end{aligned}$$

where we used the Cameron-Martin formula in the last step (cf. e.g. [DP04a, Thm. 1.4]). Let us identify an N_{Q_t}-integrable upper bound (independent of ε) for the following term: By the intermediate value theorem, there is for any $\varepsilon \in (0,1]$ an $\varepsilon_0 \in (0,\varepsilon)$, such that

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \cdot \Big(\exp\big[\langle \varepsilon\cdot\Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle - \frac{\varepsilon^2}{2} \cdot |\Lambda_t h|^2 \big] - 1 \Big) \right| \\
&= \big| \langle \Lambda_t h\,,\, Q_t^{-1/2} y_1\rangle - \varepsilon_0 |\Lambda_t h|^2 \big| \cdot \exp\big[\langle \varepsilon_0 \cdot \Lambda_t h\,,\, Q_t^{-1/2} y_1\rangle - \frac{\varepsilon_0^2}{2} \cdot |\Lambda_t h|^2 \big] \\
&\le \Big(\underbrace{\big|\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1\rangle\big|}_{\le \exp[|\langle \Lambda_t h, Q_t^{-1/2} y_1\rangle|]} + |\Lambda_t h|^2 \Big) \cdot \exp\Big[\big| \langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle \big| \Big]
\end{aligned}$$

$$\leq \left(1 + |\Lambda_t h|^2\right) \cdot \exp\Big[2 \cdot \big|\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle\big|\Big]\,,$$

where in the last estimate we use the fact, that the argument of the exponential function is positive. Since

$$\begin{aligned}
&\int_H \exp\Big[2 \cdot \big|\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle\big|\Big]\; N_{Q_t}(\mathrm{d}y_1) \\
&\quad \leq \int_H \exp\big[-2\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle\big] + \exp\big[2\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle\big]\; N_{Q_t}(\mathrm{d}y_1) = 2e^{2|\Lambda_t h|^2}
\end{aligned}$$

since $\langle \Lambda_t h\,,\, Q_t^{-1/2} y_1 \rangle = W_{\Lambda_t h}(y_1)$ fulfills $N_{Q_t} \circ W^2_{\Lambda_t h} \sim N\big(0, |\Lambda_t h|^2\big)$ (cf. e.g. [DP06, Prop. 1.15]), we may use Lebesgue's dominated convergence theorem to obtain, that

$$\begin{aligned}
&\big\langle D S_t \varphi(\,\cdot\,)(x)\,,\, h \big\rangle \\
&\quad = \int_H \int_H \int_H \varphi(e^{tA}x + y_1 + y_2 + y_3) \cdot \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(e^{\varepsilon\langle \Lambda_t h, Q_t^{-1/2} y_1 \rangle - \frac{\varepsilon^2}{2}|\Lambda_t h|^2} - 1\big) \\
&\qquad\qquad N_{Q_t}(\mathrm{d}y_1)\; e_G(M_t)(\mathrm{d}y_2)\; \delta_{b_t}(\mathrm{d}y_3)\,,
\end{aligned}$$

which proves the claim. □

The following result and its proof are quite similar to the Wiener-case (see [DPZ02, Prop. 11.2.5] and [Cer95]). Only the integration by parts formula used in the proof is formulated differently (see Lemma 3.1.1 above), but the estimates remain essentially the same.

Lemma 3.1.2. *We have the following Feller properties for (S_t):*

(i) For $\varphi \in \mathcal{C}_u(H)$ and all $(t, x) \in [0, T] \times H$,

$$\big|S_t\varphi(x)\big| = \left|\int_H \varphi(e^{tA}x + y)\; \mu_t(\mathrm{d}y)\right| \leq \|\varphi\|_0\,.$$

(ii) For all $\varphi \in \mathcal{C}_{u,k}(H)$, $k \in \mathbb{N}$ and $t \in [0, T]$, we have

$$\|S_t\varphi\|_{u,k} \leq Ce^{\omega t} \cdot \|\varphi\|_{u,k} \cdot \int_H 1 + |y|^k\; \mu_t(\mathrm{d}y)\,,$$

where $C \in (0, \infty)$ is independent of t, x and φ.

(iii) For $\varphi \in \mathcal{C}_u(H)$ and all $(t, x) \in (0, T] \times H$,

$$\big|DS_t\varphi(x)\big| \leq \|\Lambda_t\|_{L(H)} \cdot \|\varphi\|_0\,.$$

(iv) For all $(t,x) \in (0,T] \times H$ and $\varphi \in \mathcal{C}_{u,1}(H)$, we have

$$\frac{\big|DS_t\varphi(x)\big|}{1+|x|} \leq \tilde{C}e^{t\omega} \cdot \|\Lambda_t\|_{L(H)} \cdot \left(\int_H 1+|y|^2 \; \mu_t(\mathrm{d}y)\right)^{1/2} \cdot \|\varphi\|_{u,1} \, ,$$

where $\tilde{C} \in (0,\infty)$ is independent of t, x and φ.

We note that a version of (iv) can also be shown for $\varphi \in \mathcal{C}_{u,k}(H)$ for $k > 1$, and that our assumptions imply the existence of finite first and second moments for μ_t (i.e., in our framework the upper estimate in (ii) is finite for $k \in \{1,2\}$).

Proof. There is nothing to prove for (i).

For (ii), let $x \in H$, $t \in [0,T]$, $k \in \mathbb{N}$ and $\varphi \in \mathcal{C}_{u,k}(H)$. Then, there exists a $C \in (0,\infty)$ independent of t, x and φ, such that

$$\begin{aligned}
\frac{\big|S_t\varphi(x)\big|}{1+|x|^k} &= \left|\int_H \frac{\varphi(e^{tA}x+y)}{1+|e^{tA}x+y|^k} \cdot \frac{1+|e^{tA}x+y|^k}{1+|x|^k} \; \mu_t(\mathrm{d}y)\right| \\
&\leq \|\varphi\|_{u,k} \cdot \int_H \frac{1+|e^{tA}x+y|^k}{1+|x|^k} \; \mu_t(\mathrm{d}y) \\
&\leq Ce^{\omega t} \cdot \|\varphi\|_{u,k} \cdot \int_H \underbrace{\frac{1+|x|^k+|y|^k}{1+|x|^k}}_{\leq 1+|y|^k} \; \mu_t(\mathrm{d}y) \, ,
\end{aligned}$$

which proves the claim.

To show (iii), let $(t,x) \in (0,T] \times H$ and $\varphi \in \mathcal{C}_u(H)$. Using Lemma 3.1.1 together with Hölder's and Jensen's inequality (and the extension of the white noise function introduced above), for any $h \in H$,

$$\begin{aligned}
&\big|\langle DS_t\varphi(x)\,,\, h\rangle\big|^2 \\
&= \Bigg|\int_H\int_H\left(\int_H \varphi(e^{tA}x+y_1+y_2+y_3) \cdot \langle \Lambda_t h\,,\, Q_t^{-1/2}y_1\rangle \; N_{Q_t}(\mathrm{d}y_1)\right) \\
&\qquad\qquad e_G(M_t)(\mathrm{d}y_2) \; \delta_{b_t}(\mathrm{d}y_3)\Bigg|^2 \\
&\leq \int_H\int_H\Bigg(\int_H \big|\varphi(e^{tA}x+y_1+y_2+y_3)\big|^2 \; N_{Q_t}(\mathrm{d}y_1) \\
&\qquad \cdot \underbrace{\int_H \big|\langle \Lambda_t h\,,\, Q_t^{-1/2}y_1\rangle\big|^2 \; N_{Q_t}(\mathrm{d}y_1)\Bigg)}_{\overset{(3.1.1)}{\leq} |\Lambda_t h|^2} \; e_G(M_t)(\mathrm{d}y_2) \; \delta_{b_t}(\mathrm{d}y_3) \\
&\leq |\Lambda_t h|^2 \cdot \int_H \big|\varphi(e^{tA}x+y)\big|^2 \; \mu_t(\mathrm{d}y) \, ,
\end{aligned}$$

which implies

$$\left|DS_t\varphi(x)\right|^2 \le \|\Lambda_t\|^2_{L(H)} \cdot \int_H \left|\varphi(e^{tA}x+y)\right|^2 \mu_t(\mathrm{d}y) \le \|\Lambda_t\|^2_{L(H)} \cdot \|\varphi\|_0^2 \,.$$

For (iv), as above we obtain, that for any $h \in H$, $x \in H$, $\varphi \in \mathcal{C}_{u,1}(H)$ and $t \in (0,T]$

$$\left|\langle DS_t\varphi(x)\,,\, h\rangle\right|^2 \le |\Lambda_t h|^2 \cdot \int_H \left|\varphi(e^{tA}x+y)\right|^2 \mu_t(\mathrm{d}y) \,.$$

Consequently (similar to the argument for (ii)), there exists a constant $\tilde{C} \in (0,\infty)$ independent of φ or x, such that

$$\begin{aligned}\frac{\left|DS_t\varphi(x)\right|^2}{\left(1+|x|\right)^2} &\le \|\Lambda_t\|^2 \cdot \|\varphi\|^2_{u,1} \cdot \int_H \frac{\left(1+|e^{tA}x+y|\right)^2}{\left(1+|x|\right)^2}\,\mu_t(\mathrm{d}y)\\ &\le \tilde{C}^2 e^{2\omega t} \cdot \|\Lambda_t\|^2 \cdot \|\varphi\|^2_{u,1} \cdot \int_H 1+|y|^2\,\mu_t(\mathrm{d}y) \,. \qquad \square\end{aligned}$$

3.2. The infinitesimal generator U of (S_t)

As pointed out in [LR02, p. 300] (see also Remark 3.3.4 below), we have that for all $\varphi \in \mathcal{F}\big(\mathcal{M}_b^{\mathbb{C}}(H)\big)$,

$$S_t\varphi(x) \xrightarrow{t\to 0} \varphi(x) \quad \text{for all } x \in H$$

(recall that $\mathcal{W}_A \subset \mathcal{F}\big(\mathcal{M}_b^{\mathbb{C}}(H)\big)$). The fact, that this convergence is only pointwise in H (and not with respect to the supremum norm in the function space), takes (S_t) out of direct reach of the theory of C_0-semigroups.

Let us repeat some of the main results of [LR02]. We introduce the time-dependent versions later.

Fact 3.2.1. *Define the linear operator U by*

$$U\psi(x) := \int_H \big(i\langle A\xi, x\rangle - \lambda(\xi)\big) \cdot e^{i\langle\xi,x\rangle}\,\mathcal{F}^{-1}(\psi)(\mathrm{d}\xi)\,, \quad \psi \in \mathcal{W}_A. \tag{3.2.1}$$

Then, the following holds:

(i) The operator U maps $\mathcal{W}_A(H)$ into $\mathcal{C}_b(H)$ (actually even into $\mathcal{W}_A(H)$; see the note preceding Remark 3.3.1 below).

(ii) For all $\psi \in \mathcal{W}_A$ and $x \in H$, we have $S_t\psi(x) - \psi(x) = \int_0^t S_sU\psi(x)\,\mathrm{d}s$.

(iii) $S_t(\mathcal{W}_A) \subset \mathcal{W}_A$.

(The precise references for the three items are, in order of appearance, [LR02, Thm. 1.1(i), Thm. 1.1(ii), Thm. 1.3(i)]. Note that, in contrast to our setting, λ is assumed to be Sazonov-continuous throughout [LR02]. We show the analogous results for our time-dependent case without this assumption.)

Remark 3.2.2. *We note the following:*

(i) The proof to Fact 3.2.1(iii) (cf. [LR02, Thm. 1.3(i)]*) can be generalized to the time-dependent case immediately. See Remark 3.3.3 below. (Note in particular, that for ψ in $\mathcal{W}_{T,A}$ or in $\mathcal{W}_A$ corresponding to an $f_m \in \mathcal{S}(\mathbb{R}^m;\mathbb{R})$, $S_t\psi$ always corresponds to a function $\tilde{f}_m$ from the same space $\mathcal{S}(\mathbb{R}^m;\mathbb{R})$ – in other words, both ψ and $S_t\psi$ are cylinder functions depending on the same A-eigenspaces.)*

(ii) In [LR04, Prop. 3.5]*, a more explicit formulation of U has been established, which we need in the proof of Lemma 4.1.4:*

Let $\psi \in \mathcal{W}_{T,A}(H)$ of the form (2.1.1) *and $x \in H$. Then, recalling that*

$$\psi(t,x) = \int_H e^{i\langle \xi, x\rangle}\, \nu_t(\mathrm{d}\xi)$$

(cf. Remark 2.1.3) and using the Lévy-Khintchine decomposition of λ, as explained in the Introduction,[1] we have that

$$\begin{aligned} U\psi(t,\,\cdot\,)(x) &= \int_H \big(i\langle A\xi, x\rangle - \lambda(\xi)\big)\cdot e^{i\langle \xi,x\rangle}\, \nu_t(\mathrm{d}\xi) \\ &= \big\langle AD\psi(t,x)\,,\,x\big\rangle + \big\langle D\psi(t,x)\,,\,b\big\rangle - \frac{1}{2}\int_H \langle \xi, Q\xi\rangle \cdot e^{i\langle \xi,x\rangle}\, \nu_t(\mathrm{d}\xi) \\ &\quad + \int_H \psi(t,x+y) - \psi(t,x) - \frac{\big\langle D\psi(t,x)\,,\,y\big\rangle}{1+|y|^2}\, M(\mathrm{d}y)\,. \end{aligned}$$

3.3. The generalized Mehler semigroup in $\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$

Let

$$V_0\psi(t,\,\cdot\,) := D_t\psi(t,\,\cdot\,) + U\psi(t,\,\cdot\,)\,, \quad \psi \in \mathcal{W}_{T,A}.$$

The following remark extends Fact 3.2.1(i) (cf. [LR02, Thm. 1.1(i)]); the proof is adapted from [LR02], using the structure of the test functions in $\mathcal{W}_{T,A}$ as explained in Remark 2.1.3. The observation, that U maps the test function space into itself, has not been made in [LR02], even though it follows from the proof presented there (given Hypothesis (H.l2); note, however, that the analogon of this hypothesis has been made only throughout parts of [LR02]).

[1] Recall again, that in our framework Q is not assumed to be trace-class, and thus λ is not Sazonov-continuous.

Remark 3.3.1. *For $\psi \in \mathcal{W}_{T,A}$, we have $U\psi \in \mathcal{W}_{T,A}$ and $V_0\psi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$.*

Proof. Using the formulation of U in (3.2.1), we see that for $\psi \in \mathcal{W}_{T,A}$ (assume without loss of generality that ψ is of the form (2.1.1)) and any $t \in [0,T]$, $x \in H$,

$$\begin{aligned} V_0\psi(t,\cdot)(x) &= (U + D_t)(\psi)(t,\cdot)(x) \\ &= \int_H \big(i\langle A\xi, x\rangle - \lambda(\xi)\big)\cdot e^{i\langle \xi, x\rangle}\,\nu_t(\mathrm{d}\xi) \\ &\quad + D_t\Big(\phi(t)\cdot f_m\big(\langle \xi_1,\cdot\rangle,\dots,\langle \xi_m,\cdot\rangle\big)\Big)(x)\,, \end{aligned} \tag{3.3.1}$$

where ν_t is determined by $\mathcal{F}(\nu_t)(\,\cdot\,) = \psi(t,\cdot)$.

We start with the second summand.

$$D_t\Big(\phi(t)\cdot f_m\big(\langle \xi_1,x\rangle,\dots,\langle \xi_m,x\rangle\big)\Big) = \phi'(t)\cdot f_m\big(\langle \xi_1,x\rangle,\dots,\langle \xi_m,x\rangle\big)\,,$$

and by construction of $\mathcal{W}_{T,A}$, this term is in $\mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$.

For the first summand, using (3.3.1) and the structure of ν_t, we obtain that

$$\begin{aligned} &U\psi(t,\cdot)(x) \\ &= \phi(t)\cdot\int_{\mathbb{R}^m}\Big(i\Big\langle A\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\,x\Big\rangle - \lambda\Big(\sum_{j=1}^m r_j\xi_j\Big)\Big)\cdot\exp\Big[i\Big\langle\sum_{k=1}^m r_k\xi_k\,,\,x\Big\rangle\Big]\,g_m(r)\mathrm{d}r \\ &= \phi(t)\cdot\Big(\sum_{j=1}^m A_j(x) + B(x)\Big)\,, \end{aligned} \tag{3.3.2}$$

where

$$\begin{aligned} A_j(x) &:= i\langle \alpha_j\xi_j, x\rangle\cdot\underbrace{\int_{\mathbb{R}^m} r_j\cdot\exp\Big[i\Big\langle\sum_{k=1}^m r_k\xi_k\,,\,x\Big\rangle\Big]\,g_m(r)\mathrm{d}r}_{=:a_j(x)} \\ B(x) &:= -\int_{\mathbb{R}^m}\lambda\Big(\sum_{j=1}^m r_j\xi_j\Big)\cdot\exp\Big[i\Big\langle\sum_{k=1}^m r_k\xi_k\,,\,x\Big\rangle\Big]\,g_m(r)\mathrm{d}r\,. \end{aligned}$$

We first note, that in the second line of (3.3.2) all factors of the integrand fulfill the identity $f(-r) = \overline{f(r)}$ (for the Lévy symbol λ, this fact has been observed in the Introduction). By symmetry of the Lebesgue measure, this implies that $U\psi(t,\cdot)(x)$ is real-valued for all $(t,x) \in [0,T]\times H$. It remains to show the regularity properties.

As a Fourier transform of a Schwartz function, the restriction $a_j(x)|_{\mathrm{span}\{\xi_1,\dots,\xi_n\}}$ is again a Schwartz function in space. Considering that

$$0 \le \Big|\lambda\Big(\sum_{k=1}^m r_k\xi_k\Big)\Big| \le C_\xi\cdot\Big(1 + \sum_{k=1}^m r_k^2\Big)$$

(cf. [LR02, Lem. 3.2], [BF75, Cor. 7.16]), we can use the fact that by the smoothness property of λ on finite-dimensional subspaces of H (Hypothesis (H.l2)),

$$r \mapsto \lambda\Big(\sum_{j=1}^{m} r_j \xi_j\Big) \cdot g_m(r)$$

is an element of $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$. Thus, similar to the a_j the term B only depends on the m-dimensional subspace $\operatorname{span}\big(\{\xi_1, \dots, \xi_m\}\big)$, and the respective restriction is an element of $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$. Regularity with respect to t is obvious by the right hand side of (3.3.2), as ϕ remains untouched by U. $\square$

To extend the operator V_0 to a larger domain, we consider for each $\alpha \in \mathbb{R}$

$$R_\alpha^V f(t, \cdot)(x) := \int_t^T e^{-\alpha(s-t)} \cdot S_{s-t} f(s, \cdot)(x) \, \mathrm{d}s \,, \quad f \in \mathcal{C}\big([0,T]; \mathcal{C}_{u,1}(H)\big).$$

By Lemma 3.1.2, $R_\alpha^V f \in \mathcal{C}\big([0,T]; \mathcal{C}_{u,1}(H)\big)$ for any $f \in \mathcal{C}\big([0,T]; \mathcal{C}_{u,1}(H)\big)$.

Remark 3.3.2. *R_α^V fulfills the resolvent identity*

$$R_\alpha^V - R_\kappa^V = (\kappa - \alpha) \cdot R_\kappa^V R_\alpha^V \quad \textit{for all } \kappa, \alpha \in \mathbb{R}.$$

Proof. On the one hand,

$$\begin{aligned}
(R_\alpha^V - R_\kappa^V) f(t,x) &= R_\alpha^V f(t,x) - R_\kappa^V f(t,x) \\
&= \int_t^T \big(e^{-\alpha(s-t)} - e^{-\kappa(s-t)}\big) \cdot S_{s-t} f(s, \cdot)(x) \, \mathrm{d}s \,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\kappa - \alpha) \cdot R_\kappa^V R_\alpha^V f(t,x) &= (\kappa - \alpha) \cdot R_\kappa^V \bigg(\int_t^T e^{-\alpha(s-t)} \cdot S_{s-t} f(s, \cdot)(x) \, \mathrm{d}s\bigg) \\
&= (\kappa - \alpha) \cdot \int_t^T e^{-\kappa(r-t)} \cdot S_{r-t} \bigg(\int_r^T e^{-\alpha(s-r)} \cdot S_{s-r} f(s, \cdot)(x) \, \mathrm{d}s\bigg) \, \mathrm{d}r
\end{aligned}$$

and, using $S_{r-t} S_{s-r} = S_{s-t}$ and $e^{-\alpha(s-r)} = e^{-\alpha(s-t)} \cdot e^{\alpha(r-t)}$,

$$\begin{aligned}
&= (\kappa - \alpha) \cdot \int_t^T e^{-(\kappa-\alpha)(r-t)} \cdot \bigg(\int_r^T e^{-\alpha(s-t)} \cdot S_{s-t} f(s, \cdot)(x) \, \mathrm{d}s\bigg) \, \mathrm{d}r \\
&= \bigg[-e^{-(\kappa-\alpha)(r-t)} \cdot \int_r^T e^{-\alpha(s-t)} \cdot S_{s-t} f(s, \cdot)(x) \, \mathrm{d}s\bigg]_{r=t}^{r=T} \\
&\quad - \int_t^T \big(-e^{-(\kappa-\alpha)(r-t)}\big) \cdot \big(-e^{-\alpha(r-t)}\big) \cdot S_{r-t} f(r, \cdot)(x) \, \mathrm{d}r
\end{aligned}$$

$$= \int_t^T e^{-\alpha(s-t)} \cdot S_{s-t} f(s,\,\cdot\,)(x)\;\mathrm{d}s - \int_t^T e^{-\kappa(r-t)} \cdot S_{r-t} f(r,\,\cdot\,)(x)\;\mathrm{d}r\,,$$

which proves the claim. □

As a consequence of Remark 3.3.2, we obtain that the range $R_\alpha^V\big(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))\big)$ does not depend on the choice of α. We also observe that for any $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$

$$\begin{aligned}
\alpha \cdot R_\alpha^V \varphi(t,x) &= \alpha \cdot \int_t^T e^{-\alpha(s-t)} \cdot S_{s-t}\varphi(s,\,\cdot\,)(x)\;\mathrm{d}s \\
&= \alpha \cdot \int_0^{T-t} e^{-\alpha s} \cdot S_s \varphi(s+t,\,\cdot\,)(x)\;\mathrm{d}s = \int_0^{\alpha(T-t)} e^{-s} \cdot S_{s/\alpha}\varphi\Big(\frac{s}{\alpha}+t,\,\cdot\,\Big)(x)\;\mathrm{d}s \\
&\xrightarrow{\alpha\to\infty} \varphi(t,x)\;.
\end{aligned}$$

Thus, R_α^V is injective and continuous for each α, with $D(R_\alpha^V) := \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$. Consequently, for each α the inverse operator $(R_\alpha^V)^{-1}$ exists and is a closed linear operator on $R_\alpha^V\big(D(R_\alpha^V)\big)$. Which implies that

$$V := \alpha I - (R_\alpha^V)^{-1}$$

is a closed linear operator defined on

$$D(V) := R_\alpha^V\big(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))\big) \quad \big(\subset \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)\big)\;.$$

(Note again, as explained above, that this definition is independent of the choice of α.)

The family $(S_\tau^T)_{\tau\geq 0}$ of operators given by the space-time homogenization

$$(S_\tau^T\varphi)(t,x) := \begin{cases} S_\tau\varphi(t+\tau,\,\cdot\,)(x) & \text{if } \tau+t\leq T \\ 0 & \text{else.}\end{cases} \tag{3.3.3}$$

in the space

$$\mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big) := \Big\{\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)\;\Big|\;\varphi(T,x) = 0 \text{ for all } x\in H\Big\}$$

forms a semigroup, since by (3.3.3) and the semigroup property of $(S_t)_{t\geq 0}$ for any $\tau,\varrho \geq 0$ and $\varphi \in \mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big)$ we have that

$$\big(S_\tau^T(S_\varrho^T\varphi)\big)(t,x) = \begin{cases} \big(S_\tau^T S_\varrho\varphi\big)(t+\varrho,x) & \text{if } t+\varrho\leq T \\ 0 & \text{if } t+\varrho > T\end{cases} \tag{3.3.4}$$

$$= \begin{cases} S_\tau S_\varrho\varphi(t+\varrho+\tau,\,\cdot\,)(x) & \text{if } t+\varrho+\tau\leq T \\ 0 & \text{if } t+\varrho\leq T \text{ and } t+\varrho+\tau > T \\ 0 & \text{if } t+\varrho > T\end{cases}$$

$$= \begin{cases} S_{\tau+\varrho}\varphi(t+\varrho+\tau,\,\cdot\,)(x) & \text{if } t+\varrho+\tau \le T \\ 0 & \text{else} \end{cases}$$
$$= (S^T_{\tau+\varrho}\varphi)(t,x)\;.$$

Observe furthermore, that for any $\varphi \in \mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big)$ we have

$$\int_0^\infty e^{-\alpha r}\cdot (S^T_r\varphi)(t,x)\;\mathrm{d}r = \int_0^{T-t} e^{-\alpha r}\cdot S_r\varphi(t+r,\,\cdot\,)(x)\;\mathrm{d}r$$
$$= \int_t^T e^{-\alpha(r-t)}\cdot S_{r-t}\varphi(r,\,\cdot\,)(x)\;\mathrm{d}r = R^V_\alpha\varphi(t,\,\cdot\,)(x)$$

The following remark generalizes [LR02, Thm. 1.3(i)] to the time-dependent case. See also Remark 3.2.2(i).

Remark 3.3.3. *For each $\tau \in [0,T]$, we have both $S^T_\tau(\mathcal{W}_{T,A}) \subset \mathcal{W}_{T,A}$ and $S_\tau(\mathcal{W}_{T,A}) \subset \mathcal{W}_{T,A}$.*

The proof is a slightly more detailed (and time-dependent) version of that in [LR02]. Recall, that the Fourier transform of μ_t has the form

$$\hat{\mu}_t(\xi) = \exp\Big[-\int_0^t \lambda(e^{sA}\xi)\;\mathrm{d}s\Big] \quad \text{for all } t\ge 0,\ \xi\in H.$$

In the following proof we denote by $B_{\tau,m}$ the diagonal $(m\times m)$-matrix

$$B_{\tau,m} = \begin{pmatrix} e^{\tau\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{\tau\alpha_m} \end{pmatrix}, \quad \tau\in[0,T],\ m\in\mathbb{N},$$

where $\{\alpha_i\}_{i\in\mathbb{N}}$ are the eigenvalues of A corresponding to the eigenvectors $\{\xi_i\}_{i\in\mathbb{N}}$.

Proof of Remark 3.3.3. Let $\psi \in \mathcal{W}_{T,A}$ (of the form (2.1.1)) and $(s,\tau)\in[0,T]\times[0,T]$. Assume w.l.o.g. that $s+\tau\le T$. Then,

$$(S^T_\tau\psi)(s,x) = \int_H \psi(s+\tau,\,e^{\tau A}x+y)\;\mu_\tau(\mathrm{d}y)$$
$$= \underbrace{\phi(s+\tau)}_{=:\phi_\tau(s)}\int_H\int_{\mathbb{R}^m}\exp\Big[i\big\langle \Pi_m(r)\,,\,e^{\tau A}x+y\big\rangle\Big]\;\nu_m(\mathrm{d}r)\;\mu_\tau(\mathrm{d}y)\;.$$

Since the absolute value of the integrand is bounded by 1, $\nu_m\in\mathcal{M}^{\mathbb{C}}_b(\mathbb{R}^m)$ and $\mu_\tau\in\mathcal{M}_1(H)$, we can apply Fubini's theorem to obtain that

$$(S^T_\tau\psi)(s,x)$$
$$= \phi_\tau(s)\int_{\mathbb{R}^m}\exp\Big[i\big\langle\Pi_m(r)\,,\,e^{\tau A}x\big\rangle\Big]\cdot\Big(\int_H\exp\big[i\langle\Pi_m(r),y\rangle\big]\;\mu_\tau(\mathrm{d}y)\Big)\;g_m(r)\mathrm{d}r$$

$$= \phi_\tau(s) \int_{\mathbb{R}^m} \underbrace{\exp\Big[i\sum_{j=1}^m r_j \cdot \langle e^{\tau\alpha_j}\xi_j, x\rangle\Big]}_{=e^{i(B_\tau r\,,(\langle\xi_1,x\rangle,\dots,\langle\xi_m,x\rangle))_{\mathbb{R}^m}}} \cdot \underbrace{\exp\Big[-\int_0^\tau \lambda\Big(\sum_{j=1}^m r_j \cdot e^{u\alpha_j}\xi_j\Big)\,\mathrm{d}u\Big]}_{=\hat{\mu}(\Pi_m(r))\ \ (\leq 1)} g_m(r)\mathrm{d}r$$

$$= \phi_\tau(s) \int_{\mathbb{R}^m} \exp\big[i(B_{\tau,m}r, P_m x)_{\mathbb{R}^m}\big] \cdot \hat{\mu}_\tau\big(\Pi_m(B_{\tau,m}^{-1}B_{\tau,m}r)\big)\, g_m(B_{\tau,m}^{-1}B_{\tau,m}r) \cdot \frac{\det B_{\tau,m}}{\det B_{\tau,m}}\,\mathrm{d}r$$

(now apply the transformation theorem)

$$= \phi_\tau(s) \int_{\mathbb{R}^m} \exp\big[i(r, P_m x)_{\mathbb{R}^m}\big] \cdot \underbrace{\hat{\mu}_\tau\big(\Pi_m(B_{\tau,m}^{-1}r)\big)\, g_m(B_{\tau,m}^{-1}r) \cdot \frac{1}{\det B_{\tau,m}}}_{=:\tilde{g}_m(r)}\,\mathrm{d}r\,,$$

where ϕ_τ is again in $\mathcal{C}^2\big([0,T]\big)$, and fulfills $\phi_\tau(T) = 0$. Due to the regularity properties of $\lambda|_{F_m}$, we have that $\tilde{g}_m(r) \in \mathcal{S}(\mathbb{R}^m;\mathbb{C})$. The fact that $S_\tau^T\psi$ is real-valued follows by construction.

A similar argument shows that $S_\tau(\mathcal{W}_{T,A}) \subset \mathcal{W}_{T,A}$ as well. □

Remark 3.3.4. *The semigroup $(S_\tau^T)_{\tau\geq 0}$ actually is a π-semigroup on $\mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big)$. We check the conditions specified in Appendix A:*

(i) Fix any $\varphi \in \mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big)$ and $(t,x) \in [0,T]\times H$. The continuity of $\tau \mapsto (S_\tau^T\varphi)(t,x)$ is trivial for any $\tau > T-t$.

For $\tau < T-t$, we adapt the proof of [Man06, Prop. 4.6(iii)] *to our framework: Let $X(t,x)$ denote, only for the purpose of this proof, the solution at time t of* (SPDE) *in the linear case (i.e., $F = 0$), starting at time 0 in the point $x \in H$. Recall, that (S_t) is the transition semigroup of this process. Below, considering $S_{\tau+h}^T$ for $h > 0$, we always assume without loss of generality, that $\tau + h < T - t$.*

$$\begin{aligned}
&(S_{\tau+h}^T\varphi)(t,x) - (S_\tau^T\varphi)(t,x)\\
&\quad = S_{\tau+h}\varphi(t+\tau+h,\,\cdot\,)(x) - S_\tau\varphi(t+\tau,\,\cdot\,)(x)\\
&\quad = \mathbb{E}\Big[\varphi\big(t+\tau+h\,,\,X(\tau+h,x)\big) - \varphi\big(t+\tau\,,\,X(\tau,x)\big)\Big]\\
&\quad = \mathbb{E}\Bigg[\bigg(\frac{\varphi\big(t+\tau+h\,,\,X(\tau+h,x)\big)}{1+\big|X(\tau+h,x)\big|} - \frac{\varphi\big(t+\tau\,,\,X(\tau,x)\big)}{1+\big|X(\tau,x)\big|}\bigg)\\
&\qquad\qquad \cdot \Big(1+\big|X(\tau+h,x)\big|\Big)\Bigg]\\
&\qquad + \mathbb{E}\Bigg[\frac{\varphi\big(t+\tau\,,\,X(\tau,x)\big)}{1+\big|X(\tau,x)\big|} \cdot \Big(\big|X(\tau+h,x)\big| - \big|X(\tau,x)\big|\Big)\Bigg]\,.
\end{aligned}$$

By uniform continuity of the function $(t,x) \mapsto \frac{\varphi(t,x)}{1+|x|}$, *mean-square continuity of* $X(t,x)$ *in time (cf. e.g.* [Sto05, Thm. 4.1.7]*), Lipschitz-continuity of* $X(t,x)$ *with respect to the initial condition* x *and the fact that* μ_t *(i.e.,* X*) has finite moments, the first summand converges to zero as* $h \to 0$. *The second summand is bounded by* $\|\varphi\|_{u,1,T} \cdot \mathbb{E}\big[|X(\tau+h,x)| - |X(\tau,x)|\big]$, *which again converges to zero as* $h \to 0$ *by mean-square continuity of* $X(t,x)$ *in time.*

In the point $\tau = T - t$, *the observations*

$$\lim_{h\searrow 0}\big|\underbrace{(S^T_{\tau+h}\varphi)(t,x)}_{\substack{=0,\\ \text{by def. of } (S^T_\tau)}} - \underbrace{(S^T_\tau\varphi)(t,x)}_{\substack{=0,\\ \text{since } \varphi\in\mathcal{C}_T}}\big| = 0$$

and

$$\begin{aligned}&\lim_{h\nearrow 0}\big|(S^T_{\tau+h}\varphi)(t,x) - \underbrace{(S^T_\tau\varphi)(t,x)}_{=0}\big|\\ &= \lim_{h\nearrow 0}\left|\int_H \varphi\big(T+h,\, e^{(T-t+h)A}x + y\big)\ \mu_{T-t+h}(\mathrm{d}y)\right|\end{aligned}$$

(which converges to zero again by dominated convergence) imply continuity.

(ii) This condition is again fulfilled by dominated pointwise convergence.

(iii) This condition is fulfilled by construction, with $M = 1$ *and* $\omega = 0$.

To show that V, as the extension of V_0, generates the semigroup $(S^T_\tau)_{\tau\geq 0}$ in the sense of π-semigroups (arguing as in [Pri99]), we use the following criteria:

$$\begin{aligned}&u \in D(V) \quad \text{and} \quad Vu = f \\ \Leftrightarrow \quad &\begin{cases}\lim\limits_{h\to 0}\dfrac{(S^T_h u)(t,x) - u(t,x)}{h} = f(t,x) \quad \text{for all } (t,x)\in[0,T]\times H \\ \sup\limits_{\substack{h\in(0,T],\\ (t,x)\in[0,T]\times H}} \dfrac{\big(1+|x|\big)^{-1}}{h}\cdot\big|(S^T_h u)(t,x) - u(t,x)\big| < \infty\,.\end{cases}\end{aligned} \tag{3.3.5}$$

For the first condition, we generalize [LR02, Thm. 1.1(ii)] (see also Fact 3.2.1(ii) above) to our situation (in particular: time-dependent test functions, and the generator V (respectively, V_0) consisting of time and space derivative).

Lemma 3.3.5. *For any $\psi \in \mathcal{W}_{T,A}$, $x \in H$, $t \in [0,T)$ and $h \in [0,T)$ we have that*

$$(S_h^T\psi)(t,x) - \psi(t,x) = \int_0^h (S_s^T V_0\psi)(t,x)\ \mathrm{d}s\ .$$

Apart from some adjustments due to the time-dependence, the proof is quite similar to that in [LR02, pp. 303–305]. Conveniently it turns out, that the additional terms caused by the time-dependence of $\psi \in \mathcal{W}_{T,A}$ and of $V_0 = U + D_t$ cancel each other out.

Proof. Assume without loss of generality, that $\psi \in \mathcal{W}_{T,A}$ is of the form (2.1.1). By Remark 3.3.1, we know that $V_0\psi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ for all $\psi \in \mathcal{W}_{T,A}$. Thus, $S_s^T V_0\psi$ is well-defined, and by the definitions of S_s^T and V_0 we see that

$$(S_s^T V_0\psi)(t,x) = \begin{cases} S_s V_0\psi(t+s,\,\cdot\,)(x) & \text{if } s \le T-t \\ 0 & \text{if } s+t > T \end{cases}$$

(from here on, assume first that the first case holds)

$$\begin{aligned} &= \int_H V_0\psi(t+s, e^{sA}x+y)\ \mu_s(\mathrm{d}y) \\ &= \int_H U\psi(t+s, e^{sA}x+y)\ \mu_s(\mathrm{d}y) + \int_H (D_t\psi)(t+s, e^{sA}x+y)\ \mu_s(\mathrm{d}y)\ . \end{aligned} \tag{3.3.6}$$

We start by considering the summands separately.

$$\begin{aligned} &\int_H U\psi(t+s, e^{sA}x+y)\ \mu_s(\mathrm{d}y) \\ &\overset{(3.3.1)}{=} \int_H \int_H \big(i\langle A\xi, e^{sA}x+y\rangle - \lambda(\xi)\big) \cdot e^{i\langle \xi, e^{sA}x+y\rangle}\ \nu_{t+s}(\mathrm{d}\xi)\ \mu_s(\mathrm{d}y) \\ &= \underbrace{\int_H \int_H i\langle A\xi, y\rangle \cdot e^{i\langle \xi, e^{sA}x+y\rangle}\ \nu_{t+s}(\mathrm{d}\xi)\ \mu_s(\mathrm{d}y)}_{=:B_1(s,t,x)} \\ &\quad + \underbrace{\int_H \big(i\langle A\xi, e^{sA}x\rangle - \lambda(\xi)\big) \cdot e^{i\langle \xi, e^{sA}x\rangle} \cdot \left(\int_H e^{i\langle \xi, y\rangle}\ \mu_s(\mathrm{d}y)\right) \nu_{t+s}(\mathrm{d}\xi)}_{=:B_2(s,t,x)} \end{aligned}$$

(the use of Fubini's theorem in the last step is justified by the boundedness argument in the end of the proof of Remark 3.3.1 above). *[Note: The following derivation of the first equation in* (3.3.9) *is very similar to* [LR02]. *Readers familiar with this reference might want to skip the following two pages, which are included for the convenience of all other readers.]* Now, by the structure of $\psi \in \mathcal{W}_{T,A}$, there exists an $m \in \mathbb{N}$, such that

$$\begin{aligned} &B_1(s,t,x) \\ &\overset{(2.1.2)}{=} \phi(s+t) \cdot \int_H \int_{\mathbb{R}^m} i\langle \underbrace{A\Big(\sum_{j=1}^m r_j\xi_j\Big)}_{=\sum_{j=1}^m \alpha_j r_j \xi_j}, y\rangle \cdot \exp\Big[i\langle \underbrace{\Big(\sum_{j=1}^m r_j\xi_j\Big)}_{=\Pi_m(r)}, e^{sA}x + y\rangle\Big]\, g_m(r)\mathrm{d}r \\ &\qquad \mu_s(\mathrm{d}y) \\ &= \phi(s+t) \cdot \sum_{j=1}^m \alpha_j \int_H \int_{\mathbb{R}^m} i\langle \xi_j, y\rangle \cdot \underbrace{r_j \cdot \exp\big[i\langle \Pi_m(r), e^{sA}x + y\rangle\big] \cdot g_m(r)}_{=e^{i\langle \Pi_m(r),y\rangle}\cdot h_{j,m,s}(r)}\, \mathrm{d}r \\ &\qquad \mu_s(\mathrm{d}y)\ , \end{aligned} \tag{3.3.7}$$

where $h_{j,m,s}(r) := e^{i\langle \Pi_m(r), e^{sA}x\rangle} \cdot g_m(r) \cdot r_j$ and $\alpha_1, \ldots, \alpha_m$ are the eigenvalues related to the A-eigenvectors $\xi_1, \ldots, \xi_m$. Note that for any choice of x, the mapping $r \mapsto h_{j,m,s}(r)$ is an element of $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$, and that by construction

$$\big|e^{i\langle \Pi_m(r),y\rangle} \cdot h_{j,m,s}(r)\big| = |h_{j,m,s}(r)| = |r_j \cdot g_m(r)| \xrightarrow{|r|\to\infty} 0\ . \tag{3.3.8}$$

Considering the integrand on the right hand side of (3.3.7) and using that

$$i\langle \xi_j, y\rangle \cdot e^{i\langle \Pi_m(r),y\rangle} \cdot h_{j,m,s}(r) = \frac{\partial}{\partial r_j}\big(e^{i\langle \Pi_m(r),y\rangle}\big) \cdot h_{j,m,s}(x)\ ,$$

together with (3.3.8), by iterated integration by parts we obtain that

$$\begin{aligned} B_1(s,t,x) &= \phi(s+t) \cdot \sum_{j=1}^m \alpha_j \int_H \left(\int_{\mathbb{R}^m} \frac{\partial}{\partial r_j}\big(e^{i\langle \Pi_m(r),y\rangle}\big) \cdot h_{j,m,s}(r)\ \mathrm{d}r \right) \mu_s(\mathrm{d}y) \\ &= \phi(s+t) \cdot \sum_{j=1}^m \alpha_j \int_H \left(- \int_{\mathbb{R}^m} e^{i\langle \Pi_m(r),y\rangle} \cdot \frac{\partial}{\partial r_j}\big(h_{j,m,s}(r)\big)\ \mathrm{d}r \right) \mu_s(\mathrm{d}y) \\ &= -\phi(s+t) \cdot \sum_{j=1}^m \alpha_j \int_{\mathbb{R}^m} \frac{\partial}{\partial r_j}\big(h_{j,m,s}(r)\big) \cdot \underbrace{\int_H e^{i\langle \Pi_m(r),y\rangle}\ \mu_s(\mathrm{d}y)}_{=\hat{\mu}_s(\Pi_m(r))}\ \mathrm{d}r \\ &= \phi(s+t) \cdot \sum_{j=1}^m \alpha_j \int_{\mathbb{R}^m} h_{j,m,s}(r) \cdot \frac{\partial}{\partial r_j}\Big(\hat{\mu}_s\big(\Pi_m(r)\big)\Big)\ \mathrm{d}r\ . \end{aligned}$$

(Note that in all steps above we use the fact that $h_{j,m,s}$ is in $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$, so there are no non-zero boundary terms when integrating by parts.)

Recalling (2.2.1),

$$\hat{\mu}_s\big(\Pi_m(r)\big) = \exp\left[-\int_0^s \lambda\big(e^{\theta A}\Pi_m(r)\big)\,\mathrm{d}\theta\right],$$

we derive

$$\begin{aligned} B_1(s,t,x) = \phi(s+t)\cdot\sum_{j=1}^m \alpha_j \int_{\mathbb{R}^m} &\big(e^{i\langle \Pi_m(r), e^{sA}x\rangle}\cdot g_m(r)\cdot r_j\big) \\ &\cdot \exp\left[-\int_0^s \lambda\big(e^{\theta A}\Pi_m(r)\big)\,\mathrm{d}\theta\right] \\ &\cdot \left(-\int_0^s D\lambda\big(e^{\theta A}\Pi_m(r)\big)\cdot e^{\theta A}\xi_j\,\mathrm{d}\theta\right)\mathrm{d}r\,. \end{aligned}$$

Applying the fundamental theorem of calculus (which applies due to the continuous differentiability of $\lambda|_{F_n}$) to $u\mapsto \lambda\big(e^{uA}\Pi_m(r)\big)$, we see that

$$\begin{aligned} \lambda\big(e^{sA}\Pi_m(r)\big) - \lambda\big(\Pi_m(r)\big) &= \int_0^s D\lambda\big(e^{\theta A}\Pi_m(r)\big)\cdot\big(e^{\theta A}A\Pi_m(r)\big)\,\mathrm{d}\theta \\ &= \sum_{j=1}^m \alpha_j r_j \int_0^s D\lambda\big(e^{\theta A}\Pi_m(r)\big)\cdot\big(e^{\theta A}\xi_j\big)\,\mathrm{d}\theta\,, \end{aligned}$$

and consequently

$$\begin{aligned} B_1(s,t,x) &= \phi(s+t)\cdot\int_{\mathbb{R}^m}\exp\left[i\langle \Pi_m(r), e^{sA}x\rangle - \int_0^s \lambda\big(e^{\theta A}\Pi_m(r)\big)\,\mathrm{d}\theta\right] \\ &\qquad\cdot\big(\lambda(\Pi_m(r)) - \lambda(e^{sA}\Pi_m(r))\big)\,g_m(r)\mathrm{d}r \\ &= \int_H \exp\big[i\langle\xi, e^{sA}x\rangle\big]\cdot\underbrace{\exp\left[-\int_0^s\lambda(e^{\theta A}\xi)\,\mathrm{d}\theta\right]}_{=\hat{\mu}_s(\xi)}\cdot\big(\lambda(\xi)-\lambda(e^{sA}\xi)\big)\,\nu_{s+t}(\mathrm{d}\xi)\,. \end{aligned}$$

We can trivially rewrite B_2 as follows:

$$B_2(s,t,x) = \int_H \big(i\langle e^{sA}A\xi, x\rangle - \lambda(\xi)\big)\cdot e^{i\langle e^{sA}\xi,x\rangle}\cdot\hat{\mu}_s(\xi)\,\nu_{t+s}(\mathrm{d}\xi)\,.$$

Thus,

$$\begin{aligned} &\int_0^h B_1(s,t,x) + B_2(s,t,x)\,\mathrm{d}s \qquad (3.3.9) \\ &= \int_0^h\int_H\Big(\big(\lambda(\xi)-\lambda(e^{sA}\xi)\big) + \big(i\langle e^{sA}A\xi,x\rangle - \lambda(\xi)\big)\Big) \\ &\qquad\cdot\exp\big[i\langle\xi, e^{sA}x\rangle\big]\cdot\hat{\mu}_s(\xi)\,\nu_{s+t}(\mathrm{d}\xi)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_0^h \int_H \big(i\langle e^{sA}A\xi, x\rangle - \lambda(e^{sA}\xi)\big) \\
&\qquad\qquad \cdot \exp\Big[i\langle e^{sA}\xi, x\rangle - \int_0^s \lambda(e^{\theta A}\xi)\ \mathrm{d}\theta\Big]\ \nu_{t+s}(\mathrm{d}\xi)\ \mathrm{d}s \\
&= \int_0^h \int_{\mathbb{R}^m} \frac{\mathrm{d}}{\mathrm{d}s}\bigg(\exp\Big[i\langle e^{sA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\, x\rangle - \int_0^s \lambda(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big))\ \mathrm{d}\theta\Big]\bigg) \\
&\qquad\qquad \cdot \phi(s+t)\cdot g_m(r)\mathrm{d}r\ \mathrm{d}s\,.
\end{aligned}
$$

Now we apply Fubini's Theorem to the last line above. Using integration by parts on the new "inner" integral with respect to time, we see that for each fixed $r \in \mathbb{R}^m$

$$
\begin{aligned}
&\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}\bigg(\exp\Big[i\langle e^{sA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\, x\rangle - \int_0^s \lambda(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big))\ \mathrm{d}\theta\Big]\bigg)\cdot\phi(s+t)\cdot g_m(r)\ \mathrm{d}s \\
&= \bigg[\exp\Big[i\langle e^{sA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\, x\rangle - \int_0^s \lambda(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big))\ \mathrm{d}\theta\Big]\cdot\phi(s+t)\cdot g_m(r)\bigg]_{s=0}^{s=h} \\
&\quad - \int_0^h \exp\Big[i\langle e^{sA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\, x\rangle - \int_0^s \lambda(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big))\ \mathrm{d}\theta\Big] \\
&\qquad\quad \cdot\bigg(\frac{\mathrm{d}}{\mathrm{d}s}\phi(t+s)\bigg)\cdot g_m(r)\ \mathrm{d}s\,.
\end{aligned}
\tag{3.3.10}
$$

Now we return to the time derivative summand in (3.3.6):

$$
\begin{aligned}
&\int_H (D_t\psi)(t+s, e^{sA}x+y)\ \mu_s(\mathrm{d}y) \\
&= \int_H D_t\Big(\phi(t+s)\cdot f_m\big(\langle\xi_1, e^{sA}x+y\rangle, \ldots, \langle\xi_m, e^{sA}x+y\rangle\big)\Big)\ \mu_s(\mathrm{d}y) \\
&= \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\phi(t+s)\bigg)\cdot\int_H\int_{\mathbb{R}^m}\exp\big[i\Big(\sum_{j=1}^m r_j\langle\xi_j, e^{sA}x+y\rangle\Big)\big]\ g_m(r)\mathrm{d}r\ \mu_s(\mathrm{d}y) \\
&= \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\phi(t+s)\bigg)\cdot\int_H\int_{\mathbb{R}^m}\exp\big[i\Big(\sum_{j=1}^m r_j\langle\xi_j, y\rangle\Big)\big]\cdot\exp\big[i\Big(\sum_{j=1}^m r_j\langle\xi_j, e^{sA}x\rangle\Big)\big] \\
&\qquad\qquad g_m(r)\mathrm{d}r\ \mu_s(\mathrm{d}y) \\
&= \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\phi(t+s)\bigg)\cdot\int_{\mathbb{R}^m}\underbrace{\int_H\exp\big[i\langle\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\, y\rangle\big]\ \mu_s(\mathrm{d}y)}_{=\hat{\mu}_s(\sum r_j\xi_j)}\cdot\exp\big[i\Big(\sum_{j=1}^m r_j\langle\xi_j, e^{sA}x\rangle\Big)\big] \\
&\qquad\qquad g_m(r)\mathrm{d}r
\end{aligned}
$$

$$= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\phi(t+s)\Big)\cdot \int_{\mathbb{R}^m} \exp\Big[-\int_0^s \lambda\big(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big)\big)\,\mathrm{d}\theta + i\Big(\sum_{j=1}^m r_j\langle \xi_j, e^{sA}x\rangle\Big)\Big]$$
$$g_m(r)\mathrm{d}r\ .$$

Finally, using (3.3.9), (3.3.10) and the equation above we obtain:

$$\int_0^h (S_s^T V_0\psi)(t,x)\,\mathrm{d}s \tag{3.3.11}$$
$$= \int_0^h B_1(s,t,x)+B_2(s,t,x)\,\mathrm{d}s + \int_0^h\int_H (D_t\psi)(t+s,e^{sA}x+y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$
$$= \int_{\mathbb{R}^m}\Bigg[\exp\Big[i\langle e^{sA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\,x\rangle - \int_0^s \lambda\big(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big)\big)\,\mathrm{d}\theta\Big]$$
$$\cdot\,\phi(s+t)\cdot g_m(r)\Bigg]_{s=0}^{s=h}\mathrm{d}r$$
$$= \phi(h+t)\cdot\int_{\mathbb{R}^m}\exp\Big[i\langle e^{hA}\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\,x\rangle - \int_0^h \lambda\big(e^{\theta A}\Big(\sum_{j=1}^m r_j\xi_j\Big)\big)\,\mathrm{d}\theta\Big]\,g_m(r)\mathrm{d}r$$
$$-\,\phi(t)\cdot\int_{\mathbb{R}^m}\exp\big[i\langle\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\,x\rangle\big]\,g_m(r)\mathrm{d}r\ .$$

Considering the two summands separately, we see that

$$\phi(t)\cdot\int_{\mathbb{R}^m}\exp\big[i\langle\Big(\sum_{j=1}^m r_j\xi_j\Big)\,,\,x\rangle\big]\,g_m(r)\mathrm{d}r = \psi(t,x)$$

and, using the different re-formulations of $\hat{\mu}_h(\xi)$,

$$\int_H \exp\Big[i\langle e^{hA}\xi,x\rangle - \int_0^h \lambda(e^{\theta A}\xi)\,\mathrm{d}\theta\Big]\,\nu_{h+t}(\mathrm{d}\xi)$$
$$= \int_H \underbrace{\int_H \exp\big[i\langle \xi, e^{hA}x+y\rangle\big]\,\nu_{h+t}(\xi)}_{=\psi(t+h,e^{hA}x+y)}\,\mu_h(\mathrm{d}y)$$
$$= S_h\psi(t+h,\,\cdot\,)(x)\ .$$

Recalling that

$$(S_h^T\psi)(t,x)-\psi(t,x) = S_h\psi(t+h,\,\cdot\,)(x)-\psi(t,x)\ ,$$

we conclude the proof … almost.

In the beginning of the proof we limited ourselves to the case that $s \le T-t$. So now, let $s+t>T$. Then, by definition of the family $(S_\tau^T)_{\tau\geq 0}$,

$$(S_s^T\psi)(t,x)-\psi(t,x)=-\psi(t,x)\ ,$$

whereas on the other hand, using the definitions and the result proved above for the case $s+t\leq T$

$$\begin{aligned}\int_0^s (S_r^T V_0\psi)(t,x)\ \mathrm{d}r &= \int_0^{T-t}(S_r^T V_0\psi)(t,x)\ \mathrm{d}r + 0\\ &= (S_{T-t}^T\psi)(t,x)-\psi(t,x) = S_{T-t}\psi(T,x)-\psi(t,x) = -\psi(t,x)\ ,\end{aligned}$$

which proves the assertion. □

For the second criterium in (3.3.5), we show the following, stronger result (which actually implies that $(S_h^T)_{h\geq 0}$, restricted to the test function space $\mathcal{W}_{T,A}$, is a C_0-semigroup):

Lemma 3.3.6. *For all $\psi\in\mathcal{W}_{T,A}$,*

$$\sup_{\substack{h\in(0,T],\\ (t,x)\in[0,T]\times H}} \frac{\left|(S_h^T\psi)(t,x)-\psi(t,x)\right|}{h} < \infty\ .$$

Proof. If $t=T$, then by definition of $\mathcal{W}_{T,A}$ we have that $(S_h^T\psi)(t,x)=\psi(t,x)=0$, and the claim is fulfilled. From here on, let $t<T$.

As before, assume first that $t+h\leq T$. Choose $\psi\in\mathcal{W}_{T,A}$ and assume without loss of generality, that $\psi(t,x)=\phi(t)\cdot f_m(P_m x)$. If $\psi=0$, the assertion is trivially fulfilled; assume that $\|\psi\|_{0,T}>0$. Use Lemma 3.3.5 to see that

$$(S_h^T\psi)(t,x)-\psi(t,x)=\int_0^h (S_s^T V_0\psi)(t,x)\ \mathrm{d}s \leq h\cdot\sup_{s\in[0,h]}\left|(S_s^T V_0\psi)(t,x)\right|\ .$$

By definition of $(S_t^T)_{t\geq 0}$ (and recalling that $t+h\leq T$, i.e. $s+t\leq T$ for $s\in[0,h]$),

$$(S_h^T\psi)(t,x)-\psi(t,x)\leq h\cdot\sup_{s\in[0,h]}\left|S_s V_0\psi(t+s,\,\cdot\,)(x)\right|\ .$$

Recall that using Remark 3.3.1, there is a $\tilde{\psi}\in\mathcal{W}_{T,A}$, such that

$$V_0\psi(t+s,\,\cdot\,)(x)=\phi'(t+s)\cdot f_m(P_m x)+\tilde{\psi}(t+s,x)\ .$$

Now we can use Lemma 3.1.2(i) to obtain, that

$$\sup_{x\in H}\left|(S_h^T\psi)(t,x)-\psi(t,x)\right|\leq h\cdot\underbrace{\sup_{(t,x)\in[0,T]\times H}\left|\phi'(t)\cdot f_m(P_m x)+\tilde{\psi}(t,x)\right|}_{=:C}\ ,\qquad (3.3.12)$$

where $C \in (0, \infty)$ is independent of h, s, t and x.

Consider the case $t + h > T$. There exists an $\varepsilon > 0$, such that $h > \varepsilon$, and by definition of $(S_h^T)_{h \geq 0}$,

$$(S_h^T \psi)(t, x) - \psi(t, x) = -\psi(t, x) \, ,$$

which proves the claim, since $h > \varepsilon$ and $\mathcal{W}_{T,A} \subset \mathcal{C}\big([0, T]; \mathcal{C}_u(H)\big)$. □

3.4. A core for V

The following result and proof is adapted from [BDPR09, Prop. A.2], which in turn generalizes [DPT01, Prop. 2.5]. Note that we are working on a different space of test functions, which changes some of the technical arguments (in particular, the continuity argument; cf. Lemma 3.4.2 and (3.4.8)). One advantage of these changes is, that the upper bounds (3.4.1) and (3.4.11) depend only linearly (and not quadratically) on $|x|$.

Proposition 3.4.1. *Let $u \in D(V)$, $\varepsilon > 0$, and η a finite nonnegative Borel measure on $[0, T] \times H$. Then there exist a sequence $(\psi_n) \subset \mathcal{W}_{T,A}$, a constant $c \in (0, \infty)$ and an $n_0 \in \mathbb{N}$, such that*

$$\big|\psi_n(t, x)\big| + \big|V_0 \psi_n(t, x)\big| \leq (cT + 1) \cdot \big(\|Vu\|_{u,1,T} + 1\big) \cdot \big(1 + |x|\big) \tag{3.4.1}$$

for all $(t, x) \in [0, T] \times H$ and $n \geq n_0$, and

$$\psi_n \xrightarrow{n \to \infty} u \quad \text{and} \quad V\psi_n = V_0 \psi_n \xrightarrow{n \to \infty} Vu$$

converge in measure η on $[0, T) \times H$.

Proof. Replacing η by $\frac{1}{1+|x|} \cdot \eta$ we may assume, that $\int_H 1 + |x| \; \eta(\mathrm{d}t, \mathrm{d}x) < \infty$.

Let $f \in \mathcal{C}\big([0, T]; \mathcal{C}_{u,1}(H)\big)$ and $u = -R_0^V f = V^{-1} f$, i.e., for all $(t, x) \in [0, T] \times H$,

$$u(t, x) = -\int_t^T S_{s-t} f(s, \,\cdot\,)(x) \; \mathrm{d}s = -(T - t) \cdot \int_0^1 S_{(T-t)r} f\big((T - t)r + t, \,\cdot\,\big)(x) \; \mathrm{d}r \, .$$

Note that, by definition of V, all $u \in D(V)$ are of this form.

By Corollary 2.1.6, we can identify a triple-index sequence $(\psi_{n_1,n_2,n_3})_{n_1,n_2,n_3 \in \mathbb{N}} \subset \mathcal{W}_{T,A}$, such that for all $(t, x) \in [0, T) \times H$

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \psi_{n_1,n_2,n_3}(t, x) = f(t, x) \quad \text{and} \tag{3.4.2}$$
$$\big|\psi_{n_1,n_2,n_3}(t, x)\big| \leq \big(\|f\|_{u,1,T} + 1\big) \cdot \big(1 + |x|\big) = \big(\|Vu\|_{u,1,T} + 1\big) \cdot \big(1 + |x|\big)$$
$$\text{for all } n_1, n_2, n_3.$$

To simplify notation, we denote the triple-index n_1, n_2, n_3 by $\bar{n}$ and the triple-limit $\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty}$ by $\lim_{\bar{n} \rightrightarrows \infty}$ for the rest of this proof; that is, (3.4.2) now reads as follows:

$$\lim_{\bar n \rightrightarrows \infty} \psi_{\bar n}(t,x) = f(t,x) \quad \text{and} \tag{(3.4.2)}$$
$$\big|\psi_{\bar n}(t,x)\big| \le \big(\|f\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) = \big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \ \ \forall\, \bar n.$$

Now we set, for each $\bar n$ and all $(t,x) \in [0,T]\times H$,

$$\begin{aligned} u_{\bar n}(t,x) &:= V^{-1}\psi_{\bar n}(t,x) = -\int_t^T S_{s-t}\psi_{\bar n}(s,\,\cdot\,)(x)\,\mathrm{d}s \\ &= -(T-t)\cdot\int_0^1 S_{(T-t)r}\psi_{\bar n}\big((T-t)r+t,\,\cdot\,\big)(x)\,\mathrm{d}r\,, \end{aligned}$$

which means that $Vu_{\bar n} = \psi_{\bar n}$. From Lemma 3.1.2 and (3.4.2), we conclude that there exists a $c \in (0,\infty)$, such that for all $(t,x) \in [0,T)\times H$

$$\lim_{\bar n \rightrightarrows \infty} u_{\bar n}(t,x) = u(t,x) \quad \text{and} \tag{3.4.3}$$
$$\begin{aligned} \big|u_{\bar n}(t,x)\big| &\le (T-t)\cdot \sup_{s\in[t,T]}\big(S_{s-t}\psi_{\bar n}(s,\,\cdot\,)(x)\big) \\ &\le c(T-t)\cdot\big(\|f\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \le cT\cdot\big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \quad \forall\, \bar n. \end{aligned}$$

Furthermore, $Vu_{\bar n}(t,x) = VV^{-1}\psi_{\bar n}(t,x) = \psi_{\bar n}$, hence by (3.4.2)

$$\lim_{\bar n \rightrightarrows \infty} Vu_{\bar n}(t,x) = Vu(t,x) = f(t,x) \quad \text{and} \tag{3.4.4}$$
$$\begin{aligned} &\big|Vu_{\bar n}(t,x)\big| = \big|\psi_{\bar n}(t,x)\big| \\ &\quad \le \big(\|f\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) = \big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \ \ \forall\, \bar n. \end{aligned}$$

Next, we construct sequences of elements of $\mathcal{W}_{T,A}$, which approximate the $u_{\bar n}$ (which in turn are elements of $R^V_\alpha(\mathcal{W}_{T,A}) \subset \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$). We set

$$\begin{aligned} &\Sigma := \big\{\text{partitions } \sigma_N = \{t_0,\dots,t_N\} \text{ of } [0,1] \;\big|\; 0 = t_0 < t_1 < \cdots < t_N = 1\big\} \\ &|\sigma_N| := \max_{i=1,\dots,N} |t_i - t_{i-1}|\,. \end{aligned}$$

For any given $\sigma_N = \{t_0, t_1, \dots, t_N\} \in \Sigma$, triple-index $\bar n$ and $(t,x) \in [0,T]\times H$ we set

$$u_{\bar n,\sigma_N}(t,x) := -(T-t)\cdot\sum_{k=1}^N S_{(T-t)t_k}\psi_{\bar n}\big((T-t)t_k+t,\,\cdot\,\big)(x)\cdot(t_k - t_{k-1})\,. \tag{3.4.5}$$

As pointed out in Remark 3.2.2(ii), $S_t(\mathcal{W}_{T,A}) \subset \mathcal{W}_{T,A}$, and $S_t\psi$ depends on the same A-eigenspaces as ψ. Thus, the sum in the definition of $u_{\bar n,\sigma_N}$ is still in $\mathcal{W}_{T,A}$. Consider furthermore

$$V_0 u_{\bar{n},\sigma_N}(t,x) := -(T-t)\cdot \sum_{k=1}^{N} S_{(T-t)t_k} V_0 \psi_{\bar{n}}\big((T-t)t_k + t,\,\cdot\,\big)(x)\cdot (t_k - t_{k-1})\ .$$

(3.4.2), (3.4.3) and (3.4.5) together imply that for all $(t,x)\in[0,T)\times H$

$$\lim_{\bar{n}\rightrightarrows\infty}\lim_{|\sigma_N|\to 0} u_{\bar{n},\sigma_N}(t,x) = u(t,x) \quad \text{and} \tag{3.4.6}$$

$$\big|u_{\bar{n},\sigma_N}(t,x)\big| \le cT\cdot\big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \quad \text{for all } \bar{n} \text{ and any } N \text{ large enough}$$

and similarly

$$\lim_{\bar{n}\rightrightarrows\infty}\lim_{|\sigma_N|\to 0} V_0 u_{\bar{n},\sigma_N}(t,x) = Vu(t,x)\ . \tag{3.4.7}$$

By Lemma 3.4.2 below, the mapping $(t,s)\mapsto S_s\psi(t,\,\cdot\,)(x)$ is continuous in the topology of $\mathcal{C}_u(H)$ for any $\psi\in\mathcal{W}_{T,A}$. Consequently,

$$u_{\bar{n},\sigma_N}(t,x) \xrightarrow{|\sigma_N|\to 0} \underbrace{-(T-t)\cdot\int_0^1 S_{(T-t)r}\psi_{\bar{n}}\big((T-t)r+t,\,\cdot\,\big)(x)\,\mathrm{d}r}_{=u_{\bar{n}}(t,x)} \tag{3.4.8}$$

converges in the topology of $\mathcal{C}_u(H)$ for each $t\in[0,T]$. Thus, there is a $\delta>0$, such that if $|\sigma_N|<\delta$, then for all $\bar{n},t,x$ and all N big enough,

$$\left|Vu_{\bar{n}}(t,x) - \Big(-(T-t)\cdot\sum_{k=1}^{N} S_{(T-t)t_k}V_0\psi_{\bar{n}}\big((T-t)t_k+t,\,\cdot\,\big)(x)\cdot(t_k-t_{k-1})\Big)\right| \le 1$$

which is equivalent to

$$\underbrace{\big|Vu_{\bar{n},\sigma_N}(t,x)\big|}_{=V_0 u_{\bar{n},\sigma_N}(t,x)} \le \big|Vu_{\bar{n}}(t,x)\big| + 1 \quad \text{for all } \bar{n},t,x \text{ and all } N \text{ big enough.} \tag{3.4.9}$$

Now, let $\sigma_N\in\Sigma$ be chosen as $\sigma_N = \{0, 1/2^N, 2/2^N, \dots, 1\}$. Clearly, $|\sigma_N| \xrightarrow{N\to\infty} 0$. For $\bar{n}$ fixed, $u_{\bar{n},\sigma_N}(t,x) \xrightarrow{N\to\infty} u_{\bar{n}}(t,x)$. By (3.4.3) and (3.4.9), the pointwise convergences in (3.4.6) and (3.4.7) imply $L^1(\eta)$-convergence on $[0,T)\times H$ in both cases through the dominated convergence theorem of Lebesgue. Finally, we choose a sequence of elements ψ_n from the net $u_{\bar{n},\sigma_N}$, which preserves the convergences of ψ_n and $V_0\psi_n$ to u and Vu, respectively, in $L^1(\eta)$ and thus in measure η. Without loss of generality, this sequence can be chosen such, that for an n_0 big enough

$$\big|V_0\psi_n(t,x)\big| \le \big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big) \quad \text{for all } t,x \text{ and all } n\ge n_0$$

(using (3.4.9) and (3.4.4)). □

Lemma 3.4.2. *The mapping*

$$\begin{aligned} [0,T]\times[0,T] &\to \mathcal{C}_u(H) \\ (t,s) &\mapsto S_s\psi(t,\,\cdot\,) \end{aligned}$$

is continuous in the topology of $\mathcal{C}_u(H)$ for all $\psi \in \mathcal{W}_{T,A}$.

Proof. Fix $(s,t) \in [0,T]\times[0,T]$, a test function $\psi \in \mathcal{W}_{T,A}$ and a sequence $\big((s_n,t_n)\big)_{n\in\mathbb{N}}$ converging to (s,t) as $n\to\infty$. Assume without loss of generality, that $\psi(t,x)$ is of the form $\phi(t)\cdot f_m(P_m x)$. We show that

$$\lim_{n\to\infty}\sup_{x\in H}\big|S_{s_n}\psi(t_n,\,\cdot\,)(x) - S_s\psi(t,\,\cdot\,)(x)\big| = 0\,.$$

Observe that

$$\begin{aligned} &\big\|S_{s_n}\psi(t_n,\,\cdot\,) - S_s\psi(t,\,\cdot\,)\big\|_0 \\ &\quad\le \big\|S_{s_n}\psi(t_n,\,\cdot\,) - S_{s_n}\psi(t,\,\cdot\,)\big\|_0 + \big\|S_{s_n}\psi(t,\,\cdot\,) - S_s\psi(t,\,\cdot\,)\big\|_0\,, \end{aligned} \tag{3.4.10}$$

and consider the two summands on the right hand side separately.

Start with the first one:

$$\begin{aligned} &\big\|S_{s_n}\psi(t_n,\,\cdot\,) - S_{s_n}\psi(t,\,\cdot\,)\big\|_0 \\ &\quad\le \big|\phi(t_n)-\phi(t)\big|\cdot\underbrace{\Big\|\int_H f_m\big(P_m(e^{s_nA}\,\cdot\,+y)\big)\;\mu_{s_n}(\mathrm{d}y)\Big\|_0}_{\le\sup_{z\in H} f_m(P_m z)<\infty} \xrightarrow{n\to\infty} 0\,, \end{aligned}$$

since $\phi \in \mathcal{C}^2\big([0,T]\big)$.

For the second summand on the right hand side of (3.4.10), by the semigroup property of (S_t) it is sufficient to consider the case $s = 0$ (i.e., $s_n \xrightarrow{n\to\infty} 0$):

$$\begin{aligned} &\big\|S_{s_n}\psi(t,\,\cdot\,) - S_0\psi(t,\,\cdot\,)\big\|_0 \\ &\quad\le \big\|S_{s_n}\psi(t,\,\cdot\,) - S_{s_n}\psi(t+s_n,\,\cdot\,)\big\|_0 + \big\|S^T_{s_n}\psi(t,\,\cdot\,) - \psi(t,\,\cdot\,)\big\|_0\,. \end{aligned}$$

Here, the first summand on the right hand side converges to 0 as $n\to\infty$ by the same argument as above, and for the second summand we obtain convergence to 0 using Lemma 3.3.6 (resp., equation (3.3.12) in its proof). □

Lemma 3.4.3. *If $u \in D(V)$, then it is differentiable in space for all $t \in [0,T]$, and*

$$Du(t,x) = -\int_t^T DS_{s-t}Vu(s,\,\cdot\,)(x)\;\mathrm{d}s\,.$$

Proof. Let $u \in D(V) = R^V_\alpha\big(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))\big)$. Then there is an $f \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$ with $f = Vu$, and we can write u as

$$u(t,x) = -\int_t^T S_{s-t}f(s,\,\cdot\,)(x)\;\mathrm{d}s\;.$$

Recall that, by Lemma 3.1.2, there is a $c \in (0,\infty)$ independent of t and x, such that

$$\big|DS_\theta f(t,x)\big| \le c\|\Lambda_\theta\|_{L(H)} \cdot \left(\int_H 1 + |y|^2\,\mu_\theta(\mathrm{d}y)\right)^{1/2} \cdot \big\|f(t,\,\cdot\,)\big\|_{u,1} \cdot \big(1+|x|\big)$$

for any $\theta > 0$, $x \in H$ and $f \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$. Thus, integration and differentiation may be exchanged by (H.l3). □

Corollary 3.4.4. *Let $u \in D(V)$ and η a finite nonnegative Borel measure on $[0,T]\times H$.*

Then, there exists a sequence $(\psi_n) \subset \mathcal{W}_{T,A}$, such that for a $c \in (0,\infty)$ and an $n_0 \in \mathbb{N}$ large enough (similar to Proposition 3.4.1 above), we have

$$\begin{aligned}&\big|\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| \\ &\quad\le \left(cT + 1 + \int_0^T \|\Lambda_s\|_{L(H)}\;\mathrm{d}s\right)\cdot\big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big)\end{aligned} \tag{3.4.11}$$

for all $(t,x) \in [0,T]\times H$ and $n \ge n_0$, and $\psi_n \to u$, $\langle D\psi_n, h\rangle \to \langle Du, h\rangle$, $V_0\psi_n \to Vu$ converge in measure η as $n\to\infty$ for any $h \in H$.

Observe that $\int_0^T \|\Lambda_s\|_{L(H)}\,\mathrm{d}s < \infty$ by Hypothesis (H.l3).

Proof. Let (ψ_n) be the approximating sequence constructed in Proposition 3.4.1 above. By Lemma 3.1.2, Lemma 3.4.3 and Proposition 3.4.1, we have

$$D\psi_n(t,x) \le \int_0^T \|\Lambda_s\|_{L(H)}\;\mathrm{d}s\cdot\big(\|Vu\|_{u,1,T}+1\big)\cdot\big(1+|x|\big)$$

for all $(t,x) \in [0,T]\times H$ and any $n \in \mathbb{N}$ big enough. Thus, the claimed upper bound is valid in light of the proposition.

It remains to show the convergence of $\langle D\psi_n, h\rangle \to \langle Du, h\rangle$ as stated in the claim. We use the convergence result in the proposition, that $V_0\psi_n \to Vu$ converges in measure η on $[0,T)\times H$. Applying the integration by parts formula in Lemma 3.1.1 together with Lemma 3.1.2 and Lemma 3.4.3, we see that for each $h \in H$, $(t,x) \in [0,T)\times H$,

$$\begin{aligned}&\lim_{n\to\infty}\big\langle D\psi_n(t,x)\,,\,h\big\rangle = \left\langle -\lim_{n\to\infty}\int_t^T DS_{s-t}V_0\psi_n(s,x)\;\mathrm{d}s\,,\,h\right\rangle \\ &\quad= \left\langle -\int_t^T DS_{s-t}Vu(s,x)\;\mathrm{d}s\,,\,h\right\rangle = \big\langle Du(t,x)\,,\,h\big\rangle\;.\end{aligned}$$ □

4. Regular nonlinearity F

As announced in the Introduction, in this chapter we generalize results for (SPDE) with a regular nonlinearity F from [BDPR09, Sect. 2] to the case of (SPDE) driven by Lévy noise, or Lévy noise plus a cylindrical Brownian motion. We show the existence of a solution to (FPE) (uniqueness follows from the uniqueness result in the next chapter) and the m-dissipativity of the related Kolmogorov operator L.

Throughout this chapter we assume, that Hypotheses (H.l1)–(H.l3) and (H.c1) hold.

4.1. The transition evolution operators $P_{s,t}$

We use the following fact (e.g. from [MPR10, Thm. 2.4], where actually even the multiplicative case is covered; see also [MR10, Thm. 12 and Rem. 13]):

Fact 4.1.1. *Given Hypotheses (H.l1)–(H.l3) and (H.c1),* (SPDE) *has for any* $s \geq 0$ *a mild solution* $\big(X(t,s,x)\big)_{s\leq t\leq T}$ *with cadlag sample paths, given by*

$$X(t,s,x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A}F\big(r, X(r,s,x)\big)\, \mathrm{d}r + \underbrace{\int_s^t e^{(t-r)A}\, \mathrm{d}Y(r)}_{=Y_A(t-s)}$$

for all $0 \leq s \leq t \leq T$*. The map* $x \mapsto X(t,s,x)$ *is Lipschitz continuous, and the solution has the Markov property.*

Lemma 4.1.2. *Let* $q > 2$ *as in Hypothesis (H.l2). Then, we have for all* $0 \leq s \leq t \leq T$, $\tilde{q} \in [2,q]$ *and* $x \in H$*, that*

$$\mathbb{E}\Big[\big|X(t,s,x)\big|^{\tilde{q}}\Big] \leq C \cdot \big(1 + |x|^{\tilde{q}}\big)$$

for a $C \in (0,\infty)$ *independent of* s, t *and* x*.*

(Actually, the estimate can be specified quite explicitly; see (4.1.1) *below.)*

The proof is essentially the same as in the Wiener case (cf. [BDPR09, Lem. 2.2]; we nevertheless include (a slightly extended version of) the proof and others, which share this similarity, for the convenience of the reader).

Remark 4.1.3. *As shown in* [MPR10, Prop. 3.3] *(see also* [MR10, Lem. 4]*), the existence of* *finite q-th moments for M implies, that*

$$\underbrace{\mathbb{E}\Big[\sup_{s\in[0,T]}\big|Y_A(s)\big|^q\Big]}_{=:M_{T,A,q}} < \infty\ .$$

By virtue of Remark 2.2.2(iii), this result immediately extends to all $\tilde{q} \in [2, q]$.

Proof of Lemma 4.1.2. Set $Z(t) := X(t,s,x) - Y_A(t-s)$. By construction, Z satisfies the following equation in the mild sense:

$$\begin{cases} \mathrm{d}Z(t) = \Big[AZ(t) + F\big(t, Z(t) + Y_A(t-s)\big)\Big]\,\mathrm{d}t \\ Z(s) = x\ , \quad t \geq s. \end{cases}$$

We set $C_1 := \sup_{t\in[0,T]}\big|F(t,0)\big| \quad (<\infty)$.

Similarly e.g. to [DP04a, Sect. 3.1], we define the Yosida approximation $(A_k)_{k\in\mathbb{N}}$ of A by $A_k := kA(k-A)^{-1}$ for each $k \in \mathbb{N}$ and consider

$$\begin{cases} \mathrm{d}Z_k(t) = \Big[A_k Z_k(t) + F\big(t, Z_k(t) + Y_A(t-s)\big)\Big]\,\mathrm{d}t \\ Z_k(s) = x\ , \quad t \geq s. \end{cases}$$

As seen e.g. in [DPZ92, Thm. A.2], we have that $\|e^{tA_k}\|_{L(H)} \leq e^{\omega_k t}$ for $k > \omega$, where $\omega_k := \frac{\omega k}{k-\omega}$. In particular, $\omega_k \leq \omega + 1$ for each k large enough ($k \geq \omega^2 + \omega$). Since the A_k are bounded, we may consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|Z_k(t)\big|^{\tilde{q}} = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} Z_k(t)\ ,\ \tilde{q}\cdot\big|Z_k(t)\big|^{\tilde{q}-2}\cdot Z_k(t)\right\rangle,$$

and thus obtain (using Hypotheses (H.l1) and (H.c1)) that

$$\begin{aligned} \frac{1}{\tilde{q}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\big|Z_k(t)\big|^{\tilde{q}} &\leq \omega_k\big|Z_k(t)\big|^{\tilde{q}} + \Big\langle F\big(t, Y_A(t-s)\big)\ ,\ Z_k(t)\Big\rangle\cdot\big|Z_k(t)\big|^{\tilde{q}-2} \\ &\quad + \Big\langle F\big(t, Z_k(t) + Y_A(t-s)\big) - F\big(t, Y_A(t-s)\big)\ ,\ Z_k(t)\Big\rangle\cdot\big|Z_k(t)\big|^{\tilde{q}-2} \\ &\leq \omega_k\big|Z_k(t)\big|^{\tilde{q}} + \Big|F\big(t, Y_A(t-s)\big)\Big|\cdot\big|Z_k(t)\big|^{\tilde{q}-1} + K\cdot\big|Z_k(t)\big|^{\tilde{q}} \\ &\leq \underbrace{\left(\frac{\omega_k + K}{2} + \frac{\tilde{q}-1}{\tilde{q}}\right)}_{\substack{\leq \frac{K+\omega+1}{2}+1 =: \frac{\omega_1}{\tilde{q}} \\ \text{for } k \text{ large enough}}}\cdot\big|Z_k(t)\big|^{\tilde{q}} + \frac{1}{\tilde{q}}\cdot\Big|F\big(t, Y_A(t-s)\big)\Big|^{\tilde{q}}\ . \end{aligned}$$

where we used Young's inequality in the last step.[1] Note that ω_1 is finite, strictly positive for K large enough and independent of k. Considering this inequality in integral form and applying Gronwall's lemma (see e.g. [Wie06, Thm. A.0.1]), we obtain that

$$\begin{aligned}\left|Z_k(t)\right|^{\tilde{q}} &\leq \left|Z_k(s)\right|^{\tilde{q}} \cdot \exp\left[\int_s^t \omega_1 \,\mathrm{d}r\right] + \int_s^t \left|F(r, Y_A(r-s))\right|^{\tilde{q}} \cdot \exp\left[\int_r^t \omega_1 \,\mathrm{d}u\right] \mathrm{d}r \\ &= \exp\big[(t-s)\omega_1\big] \cdot |x|^{\tilde{q}} + \int_s^t \Big|F\big(r, Y_A(r-s)\big)\Big|^{\tilde{q}} \cdot \exp\big[(t-r)\omega_1\big] \,\mathrm{d}r\,,\end{aligned}$$

which gives us an upper bound for $\left|Z_k(t)\right|^{\tilde{q}}$ independent of $k \in \mathbb{N}$, and thus also for $\left|Z(t)\right|^{\tilde{q}} = \lim_{k\to\infty}\left|Z_k(t)\right|^{\tilde{q}}$. In fact, for each choice of s, t, x and k we have

$$\begin{aligned}&Z_k(t) - Z(t) \\ &\quad = e^{tA_k}x - e^{tA}x \\ &\qquad + \int_s^t e^{(t-r)A_k} F\big(r, Z_k(r) + Y_A(r-s)\big) - e^{(t-r)A} F\big(r, Z(r) + Y_A(r-s)\big) \,\mathrm{d}r\,.\end{aligned}$$

Since for any Yosida approximation we have $\lim_{k\to\infty} e^{tA_k}x = e^{tA}x$ for all $x \in H$ uniformly in $t \in [0, T]$, we obtain by the Lipschitz property of F in space, that

$$\begin{aligned}&\int_s^t \Big|e^{(t-r)A_k} F\big(r, Z_k(r) + Y_A(r-s)\big) - e^{(t-r)A} F\big(r, Z(r) + Y_A(r-s)\big)\Big| \,\mathrm{d}r \\ &\quad \leq \int_s^t \Big|e^{(t-r)A_k} F\big(r, Z_k(r) + Y_A(r-s)\big) - e^{(t-r)A_k} F\big(r, Z(r) + Y_A(r-s)\big)\Big| \\ &\qquad\qquad + \Big|e^{(t-r)A_k} F\big(r, Z(r) + Y_A(r-s)\big) - e^{(t-r)A} F\big(r, Z(r) + Y_A(r-s)\big)\Big| \,\mathrm{d}r \\ &\quad \leq \int_s^t \Big|e^{T(\omega+1)} \cdot K \cdot \big|Z(r) - Z_k(r)\big|\Big| \\ &\qquad\qquad + \underbrace{\Big|\big(e^{(t-r)A_k} - e^{(t-r)A}\big)\, F\big(r, Z(r) + Y_A(r-s)\big)\Big|}_{\xrightarrow{k\to\infty} 0} \,\mathrm{d}r\,.\end{aligned}$$

With Gronwall's lemma, we finally see that that $Z_k \xrightarrow{k\to\infty} Z$.

By definition of Z, we get that there is a finite $C_2 \in (0, \infty)$, independent of t, s and x, such that

$$\left|X(t, s, x)\right|^{\tilde{q}} \leq C_2 \cdot \left|Z(t)\right|^{\tilde{q}} + C_2 \cdot \left|Y_A(t-s)\right|^{\tilde{q}}\,,$$

[1] Recall: Let $a, b, t, s, \varepsilon > 0$ and assume $\frac{1}{s} + \frac{1}{t} = 1$. Then, $ab \leq \frac{1}{s}(\varepsilon a)^s + \frac{1}{t}(b\varepsilon^{-1})^t$.
In our case, we choose $s = \tilde{q}$, $t = \frac{\tilde{q}}{\tilde{q}-1}$ and $\varepsilon = 1$.

and by the Lipschitz property of F we see that

$$\left|F(t,x)\right| \leq \left|F(t,0)\right| + \left|F(t,x) - F(t,0)\right| \leq C_1 + K \cdot |x|$$

for all $(t,x) \in [0,T] \times H$, and thus, finally,

$$\begin{aligned}
&\mathbb{E}\left[\left|X(t,s,x)\right|^{\tilde{q}}\right] && (4.1.1)\\
&\leq C_2 \cdot \left(e^{(t-s)\omega_1} \cdot |x|^{\tilde{q}} + \int_s^t \left|C_1 + K \cdot M_{T,A,\tilde{q}}\right|^{\tilde{q}} \cdot e^{(t-r)\omega_1}\, \mathrm{d}r\right) + C_2 \cdot M_{T,A,\tilde{q}}\\
&\leq C \cdot \left(1 + |x|^{\tilde{q}}\right)
\end{aligned}$$

for a C depending on $T, K, \omega, \tilde{q}$ and F – but not on s, t or x. □

Define the transition evolution operator related to (SPDE) by

$$P_{s,t}\varphi(x) := \mathbb{E}\left[\varphi\big(X(t,s,x)\big)\right], \quad 0 \leq s \leq t \leq T,\ \varphi \in \mathcal{C}_u(H).$$

Since F is Lipschitz, we can use Lemma 4.1.2 to show that there exist constants $C_q \in (0,\infty)$ independent of s, t and x, such that for $\tilde{q} \in [2,q]$, where $q > 2$ as in (H.l2), we have

$$P_{s,t}|\cdot|^{\tilde{q}}(x) \leq C_q \cdot \left(1 + |x|^{\tilde{q}}\right) \quad \text{for all } x \in H,\ 0 \leq s \leq t \leq T. \tag{4.1.2}$$

By the Lipschitz property of both F (with respect to space) and X (with respect to the initial condition), we also have that there exists a $C \in (0,\infty)$ independent of s, t and x, such that

$$P_{s,t}|\cdot|(x) \leq C \cdot \left(1 + |x|\right) \quad \text{for all } x \in H,\ 0 \leq s \leq t \leq T, \tag{4.1.3}$$

Due to the Markov property of the solution, the family $(P_{s,t})_{0 \leq s \leq t \leq T}$ fulfills the Chapman-Kolmogorov equation: $P_{s,t} = P_{r,t} \circ P_{s,r}$ for any $0 \leq s \leq r \leq t \leq T$.

Lemma 4.1.4. *For any $0 \leq s \leq t \leq T$, we have that $P_{s,t}\big(\mathcal{C}_u(H)\big) \subset \mathcal{C}_u(H)$. Furthermore, observe that for all $\psi \in \mathcal{W}_{T,A}$ we have*

$$P_{s,t}\psi(t,x) = \psi(s,x) + \int_s^t P_{s,r}L_0\psi(r,x)\, \mathrm{d}r \quad \textit{for any } 0 \leq s \leq t \leq T \textit{ and } x \in H.$$

The first part of the result extends to $\mathcal{C}_{u,1}(H)$ and $\mathcal{C}_{u,2}(H)$, similar to Lemma 3.1.2. We will use some additional notation: For a cadlag trajectory $t \mapsto Y(t)$, we denote

$$Y(t-) := \lim_{s \nearrow t} Y(s), \quad \text{and} \quad \Delta Y_t := Y(t) - Y(t-).$$

The quadratic variation is denoted by $[Y,Y]_t$ and its continuous part by $[Y,Y]^c_t$. Hence,

$$[Y,Y]_t = [Y,Y]^c_t + \sum_{0\le s\le t} (\Delta Y_s)^2$$

(assuming that $Y(0-) = 0$).

Proof. Let $\varphi \in \mathcal{C}_u$. It suffices to show the uniform continuity of $x \mapsto P_{s,t}\varphi(x)$ for $\varphi \in \mathcal{C}^1_u(H)$ (since $\mathcal{C}^1_u \subset \mathcal{C}_u$ is dense; see [LL86]). For such φ, we see that for any $0 \le s \le t \le T$ and all $x, y \in H$,

$$\big|P_{s,t}\varphi(x) - P_{s,t}\varphi(y)\big| = \Big|\mathbb{E}\Big[\varphi\big(X(t,s,x)\big) - \varphi\big(X(t,s,y)\big)\Big]\Big| \le M_{s,t}\cdot |x-y| \ ,$$

where $M_{s,t} \in (0,\infty)$ by the Lipschitz continuity of both φ and the map $x \mapsto X(t,s,x)$ (cf. Fact 4.1.1 above). Furthermore, for any $0 \le s \le t \le T$, any $\varphi \in \mathcal{C}_u(H)$ and all $x \in H$ we have that

$$\big|P_{s,t}\varphi(x)\big| = \Big|\mathbb{E}\Big[\varphi\big(X(t,s,x)\big)\Big]\Big| \le \|\varphi\|_0 \ ,$$

which concludes the proof of the first claim.

To show the second part of the assertion, let $\psi \in \mathcal{W}_{T,A}$ and assume without loss of generality, that ψ is of the form (2.1.1). As established in Remark 2.1.3, there exists an $m \in \mathbb{N}$, such that for any $(t,x) \in [0,T] \times H$,

$$\psi(t,x) = \phi(t) \cdot f_m\big(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle\big) \ .$$

Fix any $x \in H$, $s \in [0,T]$ and denote for all $t \in [s,T]$

$$\xi^X(t) := \Big(\big\langle \xi_1, X(t,s,x)\big\rangle, \ldots, \big\langle \xi_m, X(t,s,x)\big\rangle\Big) = P_m X(t,s,x) \ .$$

Recall that e.g. by [PZ07, Sect. 9.3] we know that the mild solution to (SPDE) is equivalent to the (analytically) weak solution: from Fact 4.1.1 we thus conclude that for any element ξ_i of the ONB $\{\xi_i\}_{i\in\mathbb{N}}$ of H,

$$\begin{aligned}
\xi^X_i(t) &= \big\langle \xi_i, X(t,s,x)\big\rangle \\
&= \langle \xi_i, x\rangle + \int_s^t \underbrace{\big\langle A\xi_i\,,\, X(u,s,x)\big\rangle}_{=\langle \alpha_i\xi_i, \xi^X_i \xi_i\rangle_H} \,\mathrm{d}u + \int_s^t \Big\langle \xi_i\,,\, F\big(u, X(u,s,x)\big)\Big\rangle \,\mathrm{d}u \\
&\quad + \big\langle \xi_i\,,\, Y(t-s)\big\rangle \ .
\end{aligned}$$

(Note that in the Hilbert space rigging $H_1' \subset H \subset H_1$, where H_1 is the space in which the cylindrical diffusion part of Y takes values, H_1' can be chosen such, that $\{\xi_i\}_{i\in\mathbb{N}} \subset H_1'$.) As explained in the Introduction, the last term in the sum above can be decomposed as follows:

$$\big\langle \xi_i, Y(t-s) \big\rangle = \big\langle \xi_i, J(t-s) \big\rangle + \underbrace{\big\langle \xi_i, \sqrt{Q} W(t-s) \big\rangle}_{=: W^Q_{\xi_i}(t-s)} ,$$

where J is an H-valued Lévy process with characteristic triplet $[0, 0, M]$ and W is a cylindrical Wiener process; note, that for the latter we have (cf. e.g. [DPZ92, Prop. 4.11])

$$\mathbb{E}\big[W^Q_h(t-s) W^Q_{\tilde h}(t-s)\big] = (t-s) \cdot \langle Qh, \tilde h \rangle \quad \text{for all } h, \tilde h \in H_1', \, 0 \le s \le t \le T.$$

Also, observe that $\langle \xi_i, J \rangle$ is again an ($\mathbb{R}$-valued) Lévy process (cf. e.g. [PZ07, Sect. 4.8]).

On the other hand, by Itô's formula (see e.g. [Pro05, Section II.7]), we have

$$\begin{aligned}
&\psi\big(t, X(t,s,x)\big) = \phi(t) \cdot f_m\big(\xi^X(t)\big) \\
&= \phi(s) \cdot f_m\big(\xi^X(s)\big) + \int_s^t \Big\langle \phi(u) \cdot D f_m\big(\xi^X(u-)\big) \, , \, \mathrm{d}\xi^X(u) \Big\rangle \\
&\quad + \int_s^t D_t \phi(u) \cdot f_m\big(\xi^X(u-)\big) \, \mathrm{d}u \\
&\quad + \frac{1}{2} \int_s^t \phi(u) \cdot D^2 f_m\big(\xi^X(u-)\big) \, \mathrm{d}\big[\xi^X(\,\cdot\,), \xi^X(\,\cdot\,)\big]^c_u \\
&\quad + \sum_{s \le u \le t} \Bigg[\phi(u) \cdot \Big(f_m\big(\xi^X(u)\big) - f_m\big(\xi^X(u-)\big) \Big) \\
&\qquad\qquad - \phi(u) \cdot \Big\langle D f_m\big(\xi^X(u-)\big) \, , \, \xi^X(u) - \xi^X(u-) \Big\rangle \Bigg] .
\end{aligned}$$

Let us take a closer look at the 4th summand on the right hand side. Recall that

$$f_d(y) = \int_{\mathbb{R}^d} e^{i(r,y)_{\mathbb{R}^d}} \, g_d(r) \mathrm{d}r \quad \text{for any } y \in \mathbb{R}^d$$

(cf. Remark 2.1.3). Consequently, for $1 \le j, k \le d$,

$$\frac{\partial}{\partial y_j} \frac{\partial}{\partial y_k} f_d(y) = - \int_{\mathbb{R}^d} e^{i(r,y)_{\mathbb{R}^d}} \cdot r_j r_k \, g_d(r) \mathrm{d}r \, .$$

We apply this to our 4th summand identified above, to obtain that

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{1}{2} \sum_{1 \le i,j \le m} \int_s^t \phi(u) \cdot \frac{\partial^2}{\partial x_i \partial x_j} f_m\big(\xi^X(u-)\big) \, \mathrm{d}\big[\xi^X_i(\,\cdot\,), \xi^X_j(\,\cdot\,)\big]^c_u \Bigg] \\
&= \mathbb{E}\Bigg[-\frac{1}{2} \sum_{j=1}^m \int_s^t \phi(u) \cdot \bigg(\int_{\mathbb{R}^m} e^{i(r, \xi^X(u-))_{\mathbb{R}^m}} \cdot r_j^2 \, g_m(r) \mathrm{d}r \bigg) \cdot \langle \xi_j, Q \xi_j \rangle \, \mathrm{d}u \Bigg]
\end{aligned}$$

$$= \mathbb{E}\Big[-\frac{1}{2}\sum_{j=1}^{m}\int_s^t \phi(u)\cdot\int_{\mathbb{R}^m} e^{i(r,\xi^X(u-))_{\mathbb{R}^m}}\cdot\langle r_j\xi_j, Qr_j\xi_j\rangle\; g_m(r)\mathrm{d}r\;\mathrm{d}u\Big]$$
$$= \mathbb{E}\Big[-\frac{1}{2}\int_s^t\int_H e^{i\langle y,X(u-,s,x)\rangle}\cdot\langle y,Qy\rangle\;\nu_u(\mathrm{d}y)\;\mathrm{d}u\Big]\,.$$

Using [Pro05, Thm.s I.36 and I.38] for the jump term, we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\psi\big(t,X(t,s,x)\big)\Big]-\psi(s,x)\\
&\quad= \mathbb{E}\Big[\int_s^t \big\langle AD\psi\big(u,X(u-,s,x)\big)\,,\,X(u,s,x)\big\rangle\\
&\qquad\qquad + \Big\langle D\psi\big(u,X(u-,s,x)\big)\,,\,F\big(u,X(u,s,x)\big)\Big\rangle\;\mathrm{d}u\Big]\\
&\qquad+\mathbb{E}\Big[\int_s^t D_t\psi\big(u,X(u-,s,x)\big)\;\mathrm{d}u\Big]\\
&\qquad-\frac{1}{2}\cdot\mathbb{E}\Big[\int_s^t\int_H e^{i\langle y,X(u-,s,x)\rangle}\cdot\langle y,Qy\rangle\;\nu_u(\mathrm{d}y)\;\mathrm{d}u\Big]\\
&\qquad+\mathbb{E}\Big[\int_s^t\int_H \psi\big(u,X(u-,s,x)+y\big)-\psi\big(u,X(u-,s,x)\big)\\
&\qquad\qquad\qquad-\frac{\big\langle D\psi\big(u,X(u-,s,x)\big)\,,\,y\big\rangle}{1+|y|^2}\;M(\mathrm{d}y)\;\mathrm{d}u\Big]\\
&\quad=\mathbb{E}\Big[\int_s^t L_0\psi\big(u,X(u,s,x)\big)\;\mathrm{d}u\Big]\,,
\end{aligned}$$

where we used Remark 3.2.2(ii) in the last step.

Now let us recall that $L_0\psi = V_0\psi + \langle D\psi, F\rangle$, and that by Remark 3.3.1 we have $V_0\psi \in \mathcal{C}\big([0,T];\mathcal{C}_u(H)\big)$ for all $\psi\in\mathcal{W}_{T,A}$. By the regularity assumptions on $\psi\in\mathcal{W}_{T,A}$ and the sublinearity of F, the second summand $\langle D\psi, F\rangle$ is also sublinear in space. Thus, the function $(t,x)\mapsto\big\langle D\psi(t,x)\,,\,F(t,x)\big\rangle$ is integrable in time over $[0,T]$ for each fixed x, and we may apply Fubini's theorem to obtain that

$$\begin{aligned}
&\int_s^t P_{s,u}L_0\psi(u,\,\cdot\,)\;\mathrm{d}u = \int_s^t\mathbb{E}\Big[L_0\psi\big(u,X(u,s,\,\cdot\,)\big)\Big]\;\mathrm{d}u\\
&\quad=\mathbb{E}\Big[\int_s^t L_0\psi\big(u,X(u,s,\,\cdot\,)\big)\;\mathrm{d}u\Big]\,.
\end{aligned}$$
□

4.2. Extension of the generator L_0 to $\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$

Let $\alpha \in \mathbb{R}$ and $(s,x) \in [0,T] \times H$. Define

$$R^L_\alpha \varphi(s,x) := \int_s^T e^{-\alpha(r-s)} \cdot P_{s,r}\varphi(r,\,\cdot\,)(x)\;\mathrm{d}r\;, \quad \varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big).$$

Remark 4.2.1. *R^L_α satisfies the resolvent equation*

$$R^L_\alpha - R^L_{\alpha'} = (\alpha' - \alpha) \cdot R^L_{\alpha'} R^L_\alpha \quad \textit{for all } \alpha, \alpha' \in \mathbb{R}.$$

This result (which is proven similar to Remark 3.3.2) in turn implies that the range $R^L_\alpha\big(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))\big)$ is independent of α (cf. [MR92, Prop. 1.5]). Note in addition, that

$$\begin{aligned}
\alpha \cdot R^L_\alpha \varphi(s,x) &= \alpha \cdot \int_0^{T-s} e^{-\alpha r} \cdot P_{s,s+r}\varphi(s+r,\,\cdot\,)(x)\;\mathrm{d}r \\
&= \int_0^{\alpha(T-s)} e^{-r} \cdot P_{s,s+\frac{r}{\alpha}}\varphi\Big(s + \frac{r}{\alpha},\,\cdot\,\Big)(x)\;\mathrm{d}r \\
&\xrightarrow{\alpha\to\infty} \varphi(s,x) \quad \text{for all } \varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big),\ \alpha \in \mathbb{R},\ (s,x) \in [0,T] \times H\;.
\end{aligned}$$

Consequently, we have that R^L_α is injective and $D(R^L_\alpha)$ is $\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$. We conclude that $(R^L_\alpha)^{-1}$ exists and is closed on $R^L_\alpha\big(D(R^L_\alpha)\big)$. Thus, $L := \alpha I - (R^L_\alpha)^{-1}$ is also closed as a densely defined operator on $\mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)$. It is independent of α, and

$$R^L_\alpha = (\alpha I - L)^{-1} \quad \text{and} \quad D(L) = R^L_\alpha\big(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))\big) \quad \text{for all } \alpha \in \mathbb{R}.$$

The space-time homogenization P^T_τ of $P_{s,t}$ in the space

$$\mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big) := \Big\{\varphi \in \mathcal{C}\big([0,T];\mathcal{C}_{u,1}(H)\big)\;\Big|\;\varphi(T,x) = 0 \text{ for all } x \in H\Big\}$$

given by

$$\begin{aligned}
&(P^T_\tau\varphi)(t,x) \\
&\quad := \begin{cases} P_{t,t+\tau}\varphi(t+\tau,\,\cdot\,)(x) = \mathbb{E}\big[\varphi\big(t+\tau, X(t+\tau,t,x)\big)\big] & \text{for } t+\tau \le T \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

is a semigroup by the same argument as in (3.3.4) above; similar to $(S^T_\tau)_{\tau\ge 0}$ before, it can be shown that $(P^T_\tau)_{\tau\ge0}$ is again a π-semigroup on $\mathcal{C}_T\big([0,T];\mathcal{C}_{u,1}(H)\big)$. By construction, $(P^T_\tau)_{\tau\ge0}$ is generated by L in the sense of π-semigroups. This means in particular that, similar to [BDPR09, (2.10)], we can adapt [Pri99] to have the following criterium:

$$
\begin{aligned}
&u \in D(L) \quad \text{and} \quad Lu = \varphi \qquad\qquad (4.2.1)\\
\Leftrightarrow \quad &\begin{cases} \lim\limits_{h\to 0} \dfrac{1}{h}\big((P_h^T u)(t,x) - u(t,x)\big) = \varphi(t,x) \quad \text{for all } (t,x) \in [0,T]\times H \\ \sup\limits_{\substack{h\in(0,1],\\ (t,x)\in[0,T]\times H}} \dfrac{\big(1+|x|\big)^{-1}}{h} \cdot \big|(P_h^T u)(t,x) - u(t,x)\big| < \infty \,. \end{cases}
\end{aligned}
$$

To establish that L extends L_0, we need to show that $\mathcal{W}_{T,A} \subset D(L)$ and that $L\psi = L_0\psi$ for all $\psi \in \mathcal{W}_{T,A}$. However, these facts both follow immediately from (4.2.1) together with Lemma 4.1.4 and an argument similar to the proof of Lemma 3.3.6: In particular, for any $\psi \in \mathcal{W}_{T,A}$, $t < T$ and $t+h \leq T$, we have that there exists a $C \in (0,\infty)$ such that

$$
\begin{aligned}
(P_h^T\psi)(t,x) - \psi(t,x) &= P_{t,t+h}\psi(t+h,\,\cdot\,)(x) - \psi(t,x) = \int_t^{t+h} P_{t,r}L_0\psi(r,x)\,\mathrm{d}r\\
&\leq h\cdot C\big(1+|x|\big)\,,
\end{aligned}
$$

where we use the definition of L_0 and the estimate (4.1.3).

4.3. Existence of a solution to the Fokker-Planck equation

We note that for any $\zeta \in \mathcal{M}_1(H)$ and $0 \leq s \leq t \leq T$ we have $P_{s,t}^*\zeta \in \mathcal{M}_1(H)$, where for any $t \in [s,T]$ we define

$$
\int_H \varphi(x)\ (P_{s,t}^*\zeta)(\mathrm{d}x) := \int_H P_{s,t}\varphi(x)\ \zeta(\mathrm{d}x) \quad \text{for all } \varphi \in \mathcal{B}_b(H).
$$

Proposition 4.3.1. *Let $\zeta \in \mathcal{M}_1(H)$ such that $\int_H |x|\ \zeta(\mathrm{d}x) < \infty$, and $s \in [0,T]$. Define the family $(\eta_t)_{s\leq t\leq T} \subset \mathcal{M}_1(H)$ by setting $\eta_t := P_{s,t}^*\zeta$. Then, η_t is a solution to* (FPE) *for all $t \in [s,T]$. Furthermore, for $q > 2$ as in (H.l2), there is a $C_q \in (0,\infty)$, such that for all $t \in [s,T]$ the family $(\eta_t)_{t\geq s}$ fulfills the estimate*

$$
\int_H |x|^{\tilde q}\ \eta_t(\mathrm{d}x) \leq C_q \cdot \left(1 + \int_H |x|^{\tilde q}\ \zeta(\mathrm{d}x)\right) \quad \textit{for all } \tilde q \in [2,q]; \tag{4.3.1}
$$

in particular, (4.1.3) *implies that* $\int_H |x|\ \eta_t(\mathrm{d}x) < \infty$ *for all $t \in [s,T]$.*

Proof. Let $\psi \in \mathcal{W}_{T,A}(H)$. By definition of η_t, for any $t \in [s,T]$,

$$
\int_H \psi(t,x)\ \eta_t(\mathrm{d}x) = \int_H P_{s,t}\psi(t,\,\cdot\,)(x)\ \zeta(\mathrm{d}x)\,.
$$

Now, Lemma 4.1.4 implies that η_t solves (FPE), and (4.3.1) follows from (4.1.2). □

4.4. m-dissipativity of L

Lemma 4.4.1. *Let $f \in \mathcal{C}\big([0,T];\mathcal{C}_u^1(H)\big)$ and $\alpha \in \mathbb{R}$. Set $u := (\alpha I - L)^{-1} f$. Then,*

(i) $Du \in \mathcal{C}\big([0,T];\mathcal{C}_b(H;H)\big)$

(ii) $u \in D(V)$, *and*

$$\alpha u - Vu - \langle Du, F\rangle = f \ .$$

In particular, $Lu = Vu + \langle Du, F\rangle$.

Using the results in [MPR10] about differentiability of $X(t,s,x)$ with respect to the initial condition, the proof of the lemma remains the same as in the Wiener noise case. The same holds for Corollary 4.4.2 below (its proof relies on the approximation result for the linear case obtained in Corollary 3.4.4).

Proof. By construction, $u \in D(L)$ $(= R_\alpha^L(\mathcal{C}([0,T];\mathcal{C}_{u,1}(H))))$. For all $(t,x) \in [0,T] \times H$, we have

$$u(t,x) = R_\alpha^L f(t,x) = \int_t^T e^{-\alpha(r-t)} \cdot P_{t,r} f(r,\,\cdot\,)(x)\ \mathrm{d}r \ . \tag{4.4.1}$$

(i) By definition of $P_{t,r}$ and the smoothness conditions on F, using [MPR10, Thm. 2.7] we have that for any $0 \le t \le r \le T$,

$$DP_{t,r} f(r,\,\cdot\,)(x) = \mathbb{E}\Big[DX(r,t,x)^*\, Df\big(r, X(r,t,x)\big)\Big]$$

is bounded and continuous in space. Thus, $Du \in \mathcal{C}\big([0,T];\mathcal{C}_b(H;H)\big)$.

(ii) Fix $t \in [0,T]$ and $h > 0$, such that $t + h \le T$. Set

$$Z(t+h,t,x) := e^{hA}x + \int_t^{t+h} e^{(t+h-r)A}\ \mathrm{d}Y(r) \ .$$

Then,

$$X(t+h,t,x) = Z(t+h,t,x) + \underbrace{\int_t^{t+h} e^{(t+h-s)A} F\big(s, X(s,t,x)\big)\ \mathrm{d}s}_{=:g(t+h,t,x)} \ .$$

(As usual, we omit the ω-dependence of the processes in the notation. Note that, of course, g depends on $\omega \in \Omega$.) Observe, that $\lim_{h\to 0} g(t+h,t,x) = 0$ by the right-continuity of the cadlag path $s \mapsto X(s,t,x)$ and the regularity properties of F.

Setting, as before,

$$S_t u(x) := \int_H u(e^{tA}x + y)\ \mu_t(\mathrm{d}y) = \mathbb{E}\big[u\big(Z(t,0,x)\big)\big]\ ,$$

and S_h^T as defined in (3.3.3), we have for any $h \in (0, T-t]$, that

$$\begin{aligned}
(S_h^T u)(t,x) &= S_h u(t+h,\,\cdot\,)(x) = \mathbb{E}\Big[u\big(t+h\,,\,Z(h,0,x)\big)\Big] \\
&= \mathbb{E}\Big[u\big(t+h\,,\,Z(t+h,t,x)\big)\Big] = \mathbb{E}\Big[u\big(t+h\,,\,X(t+h,t,x) - g(t+h,t,x)\big)\Big] \\
&= \underbrace{\mathbb{E}\Big[u\big(t+h\,,\,X(t+h,t,x)\big)\Big]}_{=(P_h^T u)(t,x)} \\
&\quad - \int_0^1 \mathbb{E}\Big[\Big\langle Du\big(t+h\,,\,X(t+h,t,x) - (1-\xi)\cdot g(t+h,t,x)\big)\,, \\
&\hspace{18em} g(t+h,t,x)\Big\rangle\Big]\ \mathrm{d}\xi\ .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{h}\big((S_h^T u)(t,x) - u(t,x)\big) \\
&\quad= \frac{1}{h}\big((P_h^T u)(t,x) - u(t,x)\big) \\
&\qquad - \frac{1}{h}\int_0^1 \mathbb{E}\Big[\Big\langle Du\big(t+h\,,\,X(t+h,t,x) - (1-\xi)\cdot g(t+h,t,x)\big)\,, \\
&\hspace{18em} g(t+h,t,x)\Big\rangle\Big]\ \mathrm{d}\xi\ .
\end{aligned} \tag{4.4.2}$$

Now, since $u \in D(L)$ by construction, we obtain using (4.2.1), that

$$\lim_{h\to 0}\frac{1}{h}\big((P_h^T u)(t,x) - u(t,x)\big) = Lu(t,x)\ .$$

For the second summand on the right hand side of (4.4.2), we observe that

$$\begin{aligned}
\lim_{h\to 0}\frac{g(t+h,t,x)}{h} &= \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} e^{(t+h-s)A}F\big(s, X(s,t,x)\big)\ \mathrm{d}s \\
&= \lim_{h\to 0} e^{(t+h)A}\left(\frac{1}{h}\cdot\int_t^{t+h} e^{-sA}F\big(s, X(t,s,x)\big)\ \mathrm{d}s\right) \\
&= e^{tA}e^{-tA}F(t,x)\ ,
\end{aligned}$$

using the right-continuity of $s \mapsto X(s,t,x)$ and the regularity of F. Also,

$$\lim_{h\to 0} Du\big(t+h\,,\,X(t+h,t,x) - (1-\xi)\cdot g(t+h,t,x)\big) = Du(t,x)\ ,$$

again using the right-continuity of g and X in time, and the continuity of Du both in time and space. Thus, for all $(t,x) \in [0,T] \times H$,

$$\lim_{h\to 0} \frac{1}{h}\big((S_h^T u)(t,x) - u(t,x)\big) = Lu(t,x) - \big\langle Du(t,x)\,,\,F(t,x)\big\rangle\;.$$

By (3.3.5), it remains to show the following inequality to obtain that $u \in D(V)$ and $Vu = Lu - \langle Du, F\rangle$:

$$\sup_{\substack{h\in(0,1],\\(t,x)\in[0,T]\times H}} \frac{\big(1+|x|\big)^{-1}}{h} \cdot \big((S_h^T u)(t,x) - u(t,x)\big) < \infty\;. \tag{4.4.3}$$

Note that by (4.1.3) and the Lipschitz continuity of both $x \mapsto F(s,x)$ for any $s \in [0,T]$ and $x \mapsto X(t,s,x)$ for any $0 \le s \le t \le T$, there is a $C_{F,X} \in (0,\infty)$, such that $\mathbb{E}\big[F\big(s, X(s,t,x)\big)\big] \le C_{F,X} \cdot \big(1+|x|\big)$ for any $0 \le s \le t \le T$ and all $x \in H$. $C_{F,X}$ is independent of s, t and x. Since furthermore, by (H.l1), we have that $\|e^{rA}\|_{L(H)} \le 1 \vee e^{\omega T}$ for any $r \in [0,T]$, we obtain that for any $0 \le t \le T$, $h \in (0, T-t]$, $x \in H$,

$$\begin{aligned} g(t+h,t,x) &= \int_t^{t+h} e^{(t+h-s)A} F\big(s, X(s,t,x)\big)\;\mathrm{d}s \\ &\le h \cdot (1 \vee e^{\omega T}) \cdot C_{F,X} \cdot \big(1+|x|\big)\;, \end{aligned}$$

and thus

$$\begin{aligned} &\frac{1}{h}\bigg|\int_0^1 \mathbb{E}\Big[\Big\langle Du\big(t+h\,,\,X(t+h,t,x) - (1-\xi)g(t+h,t,x)\big)\;, \\ &\qquad\qquad\qquad\qquad\qquad\qquad g(t+h,t,x)\Big\rangle\Big]\;\mathrm{d}\xi\bigg| \\ &\quad\le \|Du\|_0 \cdot C_{F,X}(1 \vee e^{\omega T}) \cdot \big(1+|x|\big) = c \cdot \big(1+|x|\big) \end{aligned}$$

for a $c \in (0,\infty)$. Together with (4.2.1) and (4.4.2), this proves (4.4.3). □

Corollary 4.4.2. *Let $f \in \mathcal{C}\big([0,T];\mathcal{C}_u^1(H)\big)$ and $\alpha \in \mathbb{R}$. Set $u := (\alpha I - L)^{-1} f$. Then, for any bounded Borel measure η on $[0,T] \times H$, there exists a sequence $(\psi_n) \subset \mathcal{W}_{T,A}$, such that for all $h \in H$*

$$\psi_n \to u\;, \quad \langle D\psi_n, h\rangle \to \langle Du, h\rangle\;, \quad V_0\psi_n \to Vu$$

converge in measure η as $n \to \infty$, and thus also $L_0\psi_n \to Lu$.

Furthermore, for some $C_1 \in (0,\infty)$,

$$\big|\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| \le C_1 \cdot \big(1+|x|\big)$$

for all $(t,x) \in [0,T] \times H$ *(uniformly for all* $n \in \mathbb{N}$*).*

Proof. By Lemma 4.4.1, u is in $D(V)$. Thus, by Corollary 3.4.4 we can find a sequence $(\psi_n) \subset \mathcal{W}_{T,A}$ as claimed. The convergence of $L_0\psi_n \to Lu$ follows from Lemma 4.4.1(ii). □

Proposition 4.4.3. *Let Hypotheses (H.l1)–(H.l3) and (H.c1) hold, let* $p \in [1,\infty)$ *and* $\eta \in \mathcal{K}^p_{0,\le\beta}$.

Then, $\big(L_0 - \frac{\beta}{p}\big)$ *is dissipative in the space* $L^p\big([0,T] \times H;\eta\big)$*. Consequently,* $\big(L_0 - \frac{\beta}{p}\big)$ *is closable. Its closure* $\big(L_p - \frac{\beta}{p}\big)$ *is m-dissipative in* $L^p\big([0,T] \times H;\eta\big)$*. Thus,* L_p *generates a* C_0*-semigroup* $(e^{\tau L_p})_{\tau\ge 0}$ *on* $L^p\big([0,T] \times H;\eta\big)$*; this semigroup is Markov.*

Apart from changes in the estimates caused by our choice of $\mathcal{W}_{T,A}$ as test function space, and the optimized estimates we established before, the following proof is essentially the same as that of [BDPR09, Thm. 2.8].

Proof. By the dissipativity criterium in [Ebe99, App. A, p. 31], (2.1.3) implies that the operator $\big(L_0 - \frac{\beta}{p}, \mathcal{W}_{T,A}\big)$ is dissipative in $L^p\big([0,T] \times H;\eta\big)$ for $p \in [1,\infty)$.

We show below, that for any $\alpha \in \mathbb{R}$ the closure of the range of $(\alpha - L_0)$ includes $\mathcal{C}\big([0,T];\mathcal{C}^1_u(H)\big)$, which is dense in $L^p\big([0,T] \times H;\eta\big)$. The remaining part of the assertion follows from the proof of Proposition 5.1.2.

Let $f \in \mathcal{C}\big([0,T];\mathcal{C}^1_u(H)\big)$, $\alpha \in \mathbb{R}$ and $u = (\alpha - L)^{-1}f$. By Lemma 4.4.1, we know that u is in $D(V)$ and differentiable in space, and it fulfills

$$\alpha u - Vu - \langle Du, F\rangle = f \; .$$

By Corollary 4.4.2, there is a sequence $(\psi_n) \subset \mathcal{W}_{T,A}$, such that for some $c_1 \in (0,\infty)$ and all $n \in \mathbb{N}$, $(t,x) \in [0,T] \times H$,

$$\big|\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| \le c_1 \cdot \big(1 + |x|\big) \; ,$$

and for all $h \in H$

$$\psi_n \to u\,, \quad V_0\psi_n \to Vu\,, \quad \langle D\psi_n, h\rangle \to \langle Du, h\rangle$$

converge in measure η. Now, define

$$f_n := \alpha\psi_n \underbrace{- V_0\psi_n - \langle D\psi_n, F\rangle}_{=-L_0\psi_n} \; .$$

Then we have $f_n \to f$ in measure η, and there exists a constant $c_2 \in (0,\infty)$, such that

$$\big|f_n(t,x)\big| \le c_2 \cdot \Big(1 + |x| + |x| \cdot \big|F(t,x)\big|\Big) \quad \text{for all } (t,x) \in [0,T] \times H.$$

Thus, by the dominated convergence theorem (using the fact that $\eta \in \mathcal{K}^p_{0,\leq\beta}$), $f_n \to f$ converges in $L^p\big([0,T] \times H; \eta\big)$. $\square$

5. Uniqueness results for the singular case

5.1. m-dissipative nonlinearity F

In this section, we show the m-dissipativity of L and the uniqueness of the solution to (FPE), as claimed in Chapter 2, for the case of (SPDE) with a merely m-dissipative nonlinear drift part. Throughout this section, we assume that Hypotheses (H.l1)–(H.l3) and (H.d1)–(H.d2) hold.

Let us note, that the technical steps described in this section are essentially similar to those in Section 3 of [BDPR09]. The major differences between our approach and [BDPR09], which lead to changes in some of the estimates below, arise in the two preceding chapters concerning the Ornstein-Uhlenbeck case and the case of (SPDE) with regular coefficients. The application of the results of the two preceding chapters in the approximation approach below is basically the same as in [BDPR09] and included only for the convenience of the reader.

5.1.1. Regular approximations of F

To be able to use the results from the previous chapter in this setting, consider the Yosida approximation of F, given by

$$F_a(t,x) := F\big(J_a(t,x)\big) = \frac{1}{a}\big(J_a(t,x) - x\big) \quad \text{and} \quad J_a(t,x) := \big(I - aF(t,\,\cdot\,)\big)^{-1}(x)$$

for all $x \in H$, $t \in [0,T]$ and $a > 0$.

Facts from the classical theory (see e.g. [DPZ92, App. D.3], [Bar76, §II.3.1] or Appendix B below) include that $\lim_{a\to 0} F_a(t,x) = F_0(t,x) = F(t,x)$ and $\big|F_a(t,x)\big| \le F_0(t,x)$ for all $x \in D\big(F(t,\,\cdot\,)\big)$ and any $t \in [0,T]$. Furthermore, $F_a(t,\,\cdot\,)$ is Lipschitz continuous with constant $\frac{2}{a}$. Note that in our case the condition $F_0(t,0) = 0$ implies that $F_a(t,0) = 0$. Since, however, F_a is not in general differentiable, we take one further step.

Let $B : D(B) \subset H \to H$ be a self-adjoint, negative definite operator, such that B^{-1} is of trace class. Set

$$F_{a,q}(t,x) := \int_H e^{qB} F_a(t, e^{qB}x + y)\; N_{\frac{1}{2}B^{-1}(e^{2qB}-1)}(\mathrm{d}y) \quad \text{for all } a,q > 0,\ t \in \mathbb{R},\ x \in H$$

(see also [DPR02, p. 266] for a similar approximation strategy).

Again we collect some facts from the literature. $F_{a,q}(t,\,\cdot\,)$ is dissipative and of class $\mathcal{C}^\infty$.

For any choice of t and a, we have pointwise convergence $F_{a,q}(t,\,\cdot\,) \xrightarrow{q\to 0} F_a(t,\,\cdot\,)$. Also, $F_{a,q}(t,\,\cdot\,)$ is Lipschitz continuous with Lipschitz constant $\frac{2}{a}$ and thus fulfills Hypothesis (H.c1), which allows us to use results from the previous chapter to treat (SPDE) with the m-dissipative F replaced by the approximation $F_{a,q}$.

Finally, for $x = 0$ we get the estimate

$$\left|F_{a,q}(t,0)\right| \leq \int_H \left|F_a(t,y)\right| \, N_{\frac{1}{2}B^{-1}(e^{2qB}-1)}(\mathrm{d}y) \leq \frac{2}{a} \underbrace{\int_H |y| \, N_{\frac{1}{2}B^{-1}(e^{2qB}-1)}(\mathrm{d}y)}_{=:C_q \xrightarrow{q\to 0} 0} .$$

5.1.2. m-dissipativity of L

We still consider the framework of Hypotheses (H.l1)–(H.l3) and (H.d1)–(H.d2). Let $p \in [1,\infty)$ and $s \in [0,T]$. Similar to the regular case (cf. the first paragraph in the proof of Proposition 4.4.3, which applies here as well), we observe the following.

Fact 5.1.1. *From the definition of $\mathcal{K}^{p,\mathrm{diss}}_{s,\leq\beta}$ we conclude, that $\big(L_0 - \frac{\beta}{p}, \mathcal{W}_{T,A}\big)$ is dissipative and thus closable in $L^p\big([s,T] \times H;\eta\big)$ for all $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{s,\leq\beta}$, where $p \in [1,\infty)$ is as in Hypothesis (H.d2).*

The closure is denoted by $\big(L_p - \frac{\beta}{p}, D(L_p)\big)$.

Our next aim is to show the m-dissipativity of L_p. Therefore, we consider the (approximating) equation

$$\alpha u_{a,q} - V u_{a,q} - \langle D u_{a,q}, F_{a,q}\rangle = \varphi \;, \quad a,q > 0, \tag{5.1.1}$$

where $\alpha > 0$ and $\varphi \in \mathcal{C}\big([0,T];\mathcal{C}^1_u(H)\big)$.

As observed in Lemma 4.4.1 (see also (4.4.1)), (5.1.1) is uniquely solved by

$$u_{a,q}(t,x) = \int_t^T e^{-\alpha(r-t)} \cdot \mathbb{E}\big[\varphi\big(r, X_{a,q}(r,t,x)\big)\big] \, \mathrm{d}r \;, \quad t \in \mathbb{R},\; x \in H.$$

Here, $X_{a,q}$ is the mild solution to

$$\begin{cases} \mathrm{d}X_{a,q}(s,t,x) = \Big[A X_{a,q}(s,t,x) + F_{a,q}\big(t, X_{a,q}(s,t,x)\big)\Big] \, \mathrm{d}s + \mathrm{d}Y(t) \\ \quad X_{a,q}(t,t,x) = x \in H \;; \quad s \geq t. \end{cases}$$

Hence, we are in the situation of Chapter 4, with $X_{a,q}$ replacing X and $u_{a,q}$ replacing u.

To be able to use Lebesgue's theorem below, we need to find an upper bound for $D u_{a,q}$, which is independent of a and q. We first observe, that by [MPR10, Thm. 2.7] we have for any $h \in H$, that

$$\big\langle Du_{a,q}(t,x), h \big\rangle = \int_t^T e^{-\alpha(s-t)} \cdot \mathbb{E}\Big[\big\langle D\varphi\big(s, X_{a,q}(s,t,x)\big) \,,\, \theta^h_{a,q}(s,t,x) \big\rangle \Big] \,\mathrm{d}s \,, \tag{5.1.2}$$

where

$$\theta^h_{a,q}(s,t,x) := DX_{a,q}(s,t,x)\, h$$

is the mild solution to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\theta^h_{a,q}(s,t,x) = A\theta^h_{a,q}(s,t,x) + DF_{a,q}\big(s, X_{a,q}(s,t,x)\big)\theta^h_{a,q}(s,t,x) \\ \quad \theta^h_{a,q}(t,t,x) = h \,, \quad s \geq t. \end{cases}$$

We thus get

$$\begin{aligned} &\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}s} \big|\theta^h_{a,q}(s,t,x)\big|^2 \\ &\quad = \big\langle A\theta^h_{a,q}(s,t,x) \,,\, \theta^h_{a,q}(s,t,x) \big\rangle + \Big\langle DF_{a,q}\big(s, X_{a,q}(s,t,x)\big)\, \theta^h_{a,q}(s,t,x) \,,\, \theta^h_{a,q}(s,t,x) \Big\rangle \\ &\quad \leq \omega \cdot \big|\theta^h_{a,q}(s,t,x)\big|^2 \,, \end{aligned}$$

where we used (H.l1) and the dissipativity of $F_{a,q}$ in the last step. Considering this as an integral equation and applying Gronwall's inequality, we see that

$$\big|\theta^h_{a,q}(s,t,x)\big|^2 \leq \big|\theta^h_{a,q}(t,t,x)\big|^2 \cdot \exp\big[(s-t)\cdot 2\omega\big] \leq |h|^2 \cdot \exp\big[(s-t)\cdot 2\omega\big] \,.$$

Note that these computations are not formally rigorous, since $\theta^h_{a,q}(s,t,x)$ is not necessarily in $D(A)$; however, this can be remedied by using the Yosida approximation of A similar to the proof of Lemma 4.1.2. Thus, using (5.1.2) and assuming that $\alpha > \omega$, we obtain that

$$\big|Du_{a,q}(t,x)\big| \leq \frac{1}{\alpha - \omega} \cdot \sup_{\substack{t\in[0,T],\\ x\in H}} \big|D\varphi(t,x)\big| \quad \text{for all } (t,x) \in [0,T] \times H. \tag{5.1.3}$$

Proposition 5.1.2. *Let $p \in [1,\infty)$ be as in Hypothesis (H.d2), $s \in [0,T]$ and $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{s,\leq\beta}$. Given Hypotheses (H.l1)–(H.l3) and (H.d1)–(H.d2), $\big(L_p - \frac{\beta}{p}\big)$ is m-dissipative in $L^p\big([s,T] \times H;\eta\big)$. Thus, L_p generates a C_0-semigroup $(e^{\tau L_p})_{\tau\geq 0}$ on $L^p\big([s,T]\times H;\eta\big)$. Furthermore, this semigroup is Markov, i.e. positivity preserving and $e^{\tau L_p}1 = 1$ for all $\tau \geq 0$. Finally, the resolvent set $\varrho(L_p)$ of L_p is $\mathbb{R}$.*

Proof. Let $\varphi \in \mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$ and let $u_{a,q}$ be the solution to (5.1.1). Assume that $\alpha > \omega$.

Claim 1.

$$\lim_{a\to 0}\lim_{q\to 0}\big\langle Du_{a,q}(t,x)\,,\,F_{a,q}(t,x)-F_0(t,x)\big\rangle = 0 \quad \text{in } L^p\big([s,T]\times H;\eta\big).$$

Using (5.1.3), we see that there is an $M_u \in (0,\infty)$, such that

$$\begin{aligned} I_{a,q} &:= \int_s^T \int_H \big|\big\langle Du_{a,q}(t,x)\,,\,F_{a,q}(t,x)-F_0(t,x)\big\rangle\big|^p \,\mathrm{d}\eta_t(x)\,\mathrm{d}t \\ &\le M_u^p \cdot \int_s^T \int_H \big|F_{a,q}(t,x)-F_0(t,x)\big|^p \,\mathrm{d}\eta_t(x)\,\mathrm{d}t \end{aligned}$$

Recall that by construction for any fixed $a>0$, $F_{a,q}(t,\,\cdot\,)$ is Lipschitz continuous with Lipschitz constant $\frac{2}{a}$; since $F_{a,q}(t,0) \le \frac{2}{a}\cdot C_q$ and $C_q \to 0$ as $q\to 0$, we can find a constant $C_a \in (0,\infty)$, such that

$$\big|F_{a,q}(t,x)\big| \le C_a \cdot \big(1+|x|\big) \quad \text{for all } x,t \text{ and all } q \text{ small enough.}$$

So, since $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{s,\le\beta}$ and using dominated convergence,

$$\limsup_{q\to 0} I_{a,q} \le M_u^p \cdot \int_s^T \int_H \big|F_a(t,x)-F_0(t,x)\big|^p \,\mathrm{d}\eta_t(x)\,\mathrm{d}t\,.$$

We have by construction, that $|F_a - F_0| \le 2|F_0|$. Thus, we can repeat the dominated convergence argument for $a\to 0$, to conclude the proof of Claim 1.

Claim 2. $u_{a,q}$ is in $D(L_p)$, and

$$\alpha u_{a,q} - L_p u_{a,q} = \varphi + \langle Du_{a,q}\,,\,F_{a,q}-F_0\rangle\,.$$

Apply Lemma 4.4.1 and Corollary 4.4.2 to the situation of (5.1.1), to see that $u_{a,q}$ is in $D(V)$, differentiable in space, and that there are $(\psi_n) \subset \mathcal{W}_{T,A}$, such that for all $h\in H$

$$\psi_n \to u_{a,q}\,,\quad \langle D\psi_n, h\rangle \to \langle Du_{a,q}, h\rangle\,,\quad V_0\psi_n \to Vu_{a,q} \quad \text{in } \eta\text{-measure as } n\to\infty,$$

and there exists a $c_1 \in (0,\infty)$, such that

$$\big|\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| \le c_1\cdot\big(1+|x|\big) \quad \text{for all } (t,x)\in[s,T]\times H.$$

In particular, $(L_0\psi_n)_{n\in\mathbb{N}}$ converges in η-measure as $n\to\infty$, and since $\eta \in \mathcal{K}^{p,\mathrm{diss}}_{s,\le\beta}$, the sequence $\big(|L_0\psi_n|\big)_{n\in\mathbb{N}}$ is uniformly bounded by an element of $L^p\big([s,T]\times H;\eta\big)$. Thus, $(L_0\psi_n)$ converges in $L^p\big([s,T]\times H;\eta\big)$, and consequently, we see that $L_p u_{a,q} = Vu_{a,q} + \langle Du_{a,q}, F_0\rangle$, which settles Claim 2 together with (5.1.1).

Claim 3. We have $\varphi \in \mathrm{Range}(\alpha - L_p)$.

This follows from Claims 1 and 2.

Since $\mathcal{C}\big([s,T];\mathcal{C}_u^1(H)\big)$ is dense in $L^p\big([s,T]\times H;\eta\big)$, the first assertion of the Proposition (m-dissipativity) follows. The second one is a consequence of the first by the Lumer-Phillips Theorem. The third assertion follows from [Ebe99, Lemma 1.9] and the fact that $L_p 1 = 0$.

It remains to prove the fourth and last assertion, that $\varrho(L_p) = \mathbb{R}$. Choose any $\delta \in \mathbb{R}$. Let $\alpha > 0$, such that $\alpha + \delta > \frac{\beta}{p}$, and $\varphi \in L^p\big([s,T]\times H;\eta\big)$. Then, by the m-dissipativity of L_p, there is a $v \in D(L_p)$, such that

$$(\alpha + \delta - L_p)v = e_\alpha \varphi \ ,$$

where $e_\alpha\varphi(t,x) := e^{\alpha t}\varphi(t,x)$ for $(t,x) \in [s,T]\times H$. Set $u := e_{-\alpha}v$. A similar approximation argument as executed above for $u_{a,q}$ shows that $u \in D(L_p)$; furthermore, $(\delta - L_p)u = e_{-\alpha}(\alpha+\delta-L_p)v = \varphi$. Thus, $\big(\delta - L_p, D(L_p)\big)$ is surjective. By injectivity of $\big(\delta + \alpha - L_p, D(L_p)\big)$, we know that also $\big(\delta - L_p, D(L_p)\big)$ is injective. Thus, $\delta \in \varrho(L_p)$, since $\big(\delta - L_p, D(L_p)\big)$ is closed. □

5.1.3. Uniqueness of the solution to the Fokker-Planck equation

Proposition 5.1.3. *Let $\zeta \in \mathcal{M}_1(H)$ and $s \in [0,T]$. Given Hypotheses (H.l1)–(H.l3) and (H.d1), we have that $\mathcal{K}^{1,\mathrm{diss}}_{s,\zeta}$ contains at most one element.*

Proof. Let $\eta^{(1)}, \eta^{(2)} \in \mathcal{K}^{1,\mathrm{diss}}_{s,\zeta}$ and set $\mu := \frac{1}{2}\eta^{(1)} + \frac{1}{2}\eta^{(2)}$. Then $\mu \in \mathcal{K}^{1,\mathrm{diss}}_{s,\zeta}$, and $\eta^{(i)} = \sigma_i\mu$ for some measurable functions $\sigma_i : [s,T]\times H \to [0,2]$.

By (2.2.2), we obtain

$$\int_{[s,T]\times H} L_0\psi \,\mathrm{d}\eta^{(1)} = \int_{[s,T]\times H} L_0\psi \,\mathrm{d}\eta^{(2)} \quad \text{for all } \psi \in \mathcal{W}_{T,A},$$

in other words,

$$\int_{[s,T]\times H} L_0\psi \,(\sigma_1 - \sigma_2)\mathrm{d}\mu = 0 \quad \text{for all } \psi \in \mathcal{W}_{T,A}.$$

However, by Proposition 5.1.2 (which applies, since $\mathcal{K}^{1,\mathrm{diss}}_{s,\zeta} \subset \mathcal{K}^{1,\mathrm{diss}}_{s,\leq\beta}$), the range of $\big(L_0, \mathcal{W}_{T,A}\big)$ is dense in $L^1\big([s,T]\times H;\mu\big)$. Also, $(\sigma_1 - \sigma_2)$ is bounded by definition. Thus, we see that $\sigma_1 = \sigma_2$. □

5.2. Measurable nonlinearity F

In this section, we prove the uniqueness of the solution to (FPE) in the case of (SPDE) with a merely measurable nonlinear drift part F. Throughout this section, we assume that conditions (H.l1)–(H.l3) and (H.m1)–(H.m2) hold.

A main ingredient to the proof of this uniqueness result is a gradient estimate using the square-field operator Γ introduced in Remark 2.2.6(ii) above. It is this estimate, which requires us to assume that $Q^{-1} \in L(H)$. Using this, we can adapt ideas from [BDPR11, Sect. 4] (uniqueness in the cylindrical Wiener noise case), to obtain uniqueness of the solution to (FPE) in the case of Lévy plus cylindrical Wiener noise. Similar to Section 5.1, our proof uses an approximation procedure (albeit a different one) and rests on the results obtained in Chapters 3 and 4. Again, differences between our results in those Chapters above and results in the literature on the Wiener noise case enable us to obtain relaxed moment conditions compared to [BDPR11] (cf. Remark 2.2.13).

5.2.1. The dense range condition

The central problem of this subsection is to establish, that the dense range condition

$$L_0\big(D(L_0)\big) \quad \text{is dense in} \quad L^1\big([s,T] \times H; \eta\big) \tag{5.2.1}$$

is fulfilled for any η in a convex set $\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$ of measures to be defined below, where as before we choose $s \in [0,T]$.

Recall that L_0 is the restriction of L to the test function space $\mathcal{W}_{T,A}(H)$; in other words, $D(L_0) = \mathcal{W}_{T,A}$.

Definition 5.2.1. For $\alpha \geq 0$, we set

$$\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha} := \left\{ \eta \in \mathcal{K}^0_{s,\leq 2\alpha} \;\middle|\; \int_{[s,T]\times H} |x|^2 + \big|F(t,x)\big|^2 + |x|^2 \cdot \big|F(t,x)\big|^2 \, \eta(\mathrm{d}t,\mathrm{d}x) < \infty \right\}.$$

We observe that $\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$ is a convex subset of $\mathcal{K}^0_{s,\leq 2\alpha}$.

Lemma 5.2.2. *Let $\alpha \geq 0$, $s \in [0,T]$ and $\eta \in \mathcal{K}^0_{s,\leq 2\alpha}$. Then, for all $\psi \in \mathcal{W}_{T,A}$,*

$$\begin{aligned} &\int_{[s,T]\times H} \psi(t,x)\, L_0\psi(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x) \\ &\quad \leq \alpha \int_{[s,T]\times H} \psi^2(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x) - \frac{1}{2} \int_{[s,T]\times H} \Gamma(\psi,\psi)(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x)\,, \end{aligned} \tag{5.2.2}$$

where, as in Remark 2.2.6(ii), Γ is the square field operator.

In particular, $\big(L_0, \mathcal{W}_{T,A}\big)$ is quasi-dissipative, hence closable in $L^2\big([0,T] \times H, \eta\big)$ for any $\eta \in \mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$; we denote the closure by $\big(L_2, D(L_2)\big)$.

Proof. By Remarks 2.1.5 and 2.2.6(ii), for any $\psi \in \mathcal{W}_{T,A}$, we have $\psi^2 \in \mathcal{W}_{T,A}$ and

$$\begin{aligned} &\int_{[s,T]\times H} \psi(t,x) \cdot L_0\psi(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x) \\ &\quad = \frac{1}{2} \int_{[s,T]\times H} L_0\psi^2(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x) - \frac{1}{2} \int_{[s,T]\times H} \Gamma(\psi,\psi)(t,x)\, \eta(\mathrm{d}t,\mathrm{d}x)\,, \end{aligned}$$

which proves the claim using (2.1.3). $\square$

Lemma 5.2.3. *Let $\alpha > 0$, $s \in [0,T]$, $\eta \in \mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$ and $f \in \mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$. Let $F_c : [0,T] \times H \to H$ fulfill Hypothesis (H.c1) and assume that it fulfills the integrability condition in the definition of $\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$ above. Then, by Lemma 4.4.1, there is a $u_c \in D(V)$ with*

$$\alpha u_c - V u_c - \big\langle D u_c \,,\, F_c \big\rangle = f \;.$$

By the m-dissipativity of L in the regular case, we obtain that $\|u_c\|_{0,T} \leq \frac{1}{\alpha} \cdot \|f\|_{0,T}$. In this situation, the following assertions hold:

(i) $u_c \in D(L_2)$ and

$$\alpha u_c - L_2 u_c = f + \big\langle D u_c \,,\, F_c - F \big\rangle \quad \text{in } L^2\big([s,T] \times H; \eta\big) \;. \tag{5.2.3}$$

(ii) Assuming $\|Q^{-1}\| < \infty$, we have

$$\begin{aligned} &\int_{[s,T]\times H} \big|D u_c(t,x)\big|^2 \,\eta(\mathrm{d}t,\mathrm{d}x) \\ &\quad \leq 4\|Q^{-1}\| \cdot \frac{1}{\alpha^2} \cdot \|f\|^2_{0,T} \\ &\qquad \cdot \left(\alpha \cdot (T-s) + \|Q^{-1}\| \int_{[s,T]\times H} \big|F_c(t,x) - F(t,x)\big|^2 \,\eta(\mathrm{d}t,\mathrm{d}x) \right) \;. \end{aligned}$$

Note that, since Q has a finite inverse and η is chosen from $\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$, the right hand side is finite for any $f \in \mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$.

Proof. By Corollary 3.4.4, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{T,A}$, such that for a constant $C_3 \in (0,\infty)$, depending on T, $\|V u_c\|_{0,T}$ and $(\Lambda_t)_{t\in[0,T]}$, we have

$$\big|\psi_n(t,x)\big| + \big|D\psi_n(t,x)\big| + \big|V_0\psi_n(t,x)\big| \leq C_3 \cdot \big(1 + |x|\big)$$

for all $(t,x) \in [0,T] \times H$, $n \in \mathbb{N}$, and

$$\psi_n \to u_c \;, \quad \langle D\psi_n, h\rangle \to \langle D u_c, h\rangle \;, \quad V_0\psi_n \to V u_c$$

converge for any $h \in H$ in measure η as $n \to \infty$.

Thus,

$$L_0\psi_n = V_0\psi_n + \big\langle D\psi_n \,,\, F\big\rangle \xrightarrow{n\to\infty} V u_c + \big\langle D u_c \,,\, F\big\rangle$$

converges in measure η and $|L_0\psi_n(t,\cdot)(x)| \leq C_3 \cdot \big(1 + |x|\big) \cdot \big(1 + |F(t,x)|\big)$. By the choice of $\eta \in \mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$, we can apply Lebesgue's dominated convergence theorem to observe, that

$$L_0\psi_n \xrightarrow{n\to\infty} Vu_c + \langle Du_c\,,\, F\rangle \quad \text{converges in } L^2\big([s,T]\times H;\eta\big)$$

and, consequently, $u_c \in D(L_2)$.

To complete the proof of (i), we simply recall that, by construction of L_2 and by assumption, we have

$$-Vu_c = -L_2u_c + \langle Du_c\,,\, F\rangle \quad \text{and} \quad \alpha u_c - Vu_c = f + \langle Du_c\,,\, F_c\rangle\,,$$

which implies (5.2.3).

Let us consider (ii). Note that, since $\eta \in \mathcal{K}^{\text{meas}}_{s,\le 2\alpha}$, we can obtain from a similar approximation as above, that the estimate (5.2.2) holds for $u_c \in D(L_2)$. This implies, that

$$\begin{aligned}
&\frac{1}{2}\int_{[s,T]\times H}\Gamma(u_c,u_c)(t,x)\;\eta(\mathrm{d}t,\mathrm{d}x) \\
&\quad\le \alpha\int_{[s,T]\times H}u_c^2(t,x)\;\eta(\mathrm{d}t,\mathrm{d}x) - \int_{[s,T]\times H}u_c(t,x)\cdot L_2u_c(t,x)\;\eta(\mathrm{d}t,\mathrm{d}x)\,.
\end{aligned} \tag{5.2.4}$$

On the other hand, if we multiply (5.2.3) by u_c, we have

$$\alpha u_c^2 - u_c\cdot L_2u_c = u_c\cdot f + u_c\cdot\langle Du_c\,,\, F_c - F\rangle \quad \text{in } L^2\big([s,T]\times H;\eta\big). \tag{5.2.5}$$

By (5.2.4), (5.2.5) and the definition of the square field operator Γ we conclude that

$$\begin{aligned}
&\frac{1}{2}\int_{[s,T]\times H}\big|Q^{1/2}\big(Du_c(t,x)\big)\big|^2\;\eta(\mathrm{d}t,\mathrm{d}x) \\
&\qquad+\underbrace{\int_{[s,T]\times H}\int_H\big(u_c(t,x)-u_c(t,x+y)\big)^2\;M(\mathrm{d}y)\;\eta(\mathrm{d}t,\mathrm{d}x)}_{\ge 0} \\
&\quad\le \int_{[s,T]\times H}\big|u_c(t,x)\big|\cdot\big|f(t,x)\big|\;\eta(\mathrm{d}t,\mathrm{d}x) \\
&\qquad+\int_{[s,T]\times H}\big|u_c(t,x)\big|\cdot\big|Du_c(t,x)\big|\cdot\big|F_c(t,x)-F(t,x)\big|\;\eta(\mathrm{d}t,\mathrm{d}x) \\
&\quad\le (T-s)\cdot\frac{1}{\alpha}\cdot\|f\|_{0,T}^2 \\
&\qquad+\int_{[s,T]\times H}\frac{1}{2}\cdot\frac{1}{2\|Q^{-1}\|}\cdot\big|Du_c(t,x)\big|^2 \\
&\qquad\qquad+\frac{1}{2}\cdot 2\|Q^{-1}\|\cdot\frac{1}{\alpha^2}\cdot\|f\|_{0,T}^2\cdot\big|F_c(t,x)-F(t,x)\big|^2\;\eta(\mathrm{d}t,\mathrm{d}x)\,,
\end{aligned}$$

where we used Young's inequality in the last step. This implies the assertion of the Lemma. $\square$

Proposition 5.2.4. *Let $\alpha > 0$, $s \in [0,T]$ and $\eta \in \mathcal{K}^{\text{meas}}_{s,\leq 2\alpha}$. Then, the dense range condition* (5.2.1) *is fulfilled.*

Proof. For any measurable map $F : D(F) \subset [0,T] \times H \to H$ and $\eta \in \mathcal{K}^{\text{meas}}_{s,\leq 2\alpha}$, there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of functions, which fulfill Hypothesis (H.c1) and

$$\lim_{n\to\infty} \int_{[s,T]\times H} |F_n - F|^2 \, \mathrm{d}\eta = 0 \, . \tag{5.2.6}$$

Let $f \in \mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$. Then, by Lemma 4.4.1, there exists for any $n \in \mathbb{N}$ a function $u_n \in D(V)$, such that

$$\alpha u_n - V u_n - \big\langle Du_n \, , \, F_n \big\rangle = f \, .$$

By m-dissipativity of L in the regular case, we have $\|u\|_{0,T} \leq \frac{1}{\alpha} \cdot \|f\|_{0,T}$. Thus, by Lemma 5.2.3(i), for any $n \in \mathbb{N}$

$$\alpha u_n - L_2 u_n = f + \big\langle Du_n \, , \, F_n - F \big\rangle \quad (\text{in } L^2), \tag{5.2.7}$$

and from Lemma 5.2.3(ii) we obtain that

$$\sup_{n\in\mathbb{N}} \int_{[s,T]\times H} \big|Du_n(t,x)\big|^2 \, \eta(\mathrm{d}t, \mathrm{d}x) < \infty \, . \tag{5.2.8}$$

Together, (5.2.6)–(5.2.8) imply (as $n \to \infty$), that f is in the closure of $(\alpha - L_0)\big(\mathcal{W}_{T,A}\big)$ in $L^1\big([s,T] \times H; \eta\big)$.

Now, since f is arbitrarily chosen from $\mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$, and since $\mathcal{C}\big([s,T];\mathcal{C}^1_u(H)\big)$ is dense in $L^1\big([s,T] \times H; \eta\big)$, the dense range condition (5.2.1) is shown. □

Remark 5.2.5. *Of course, any m-dissipative operator fulfills, by definition of m-dissipativity, a dense range condition of the form* (5.2.1). *Thus, the conceptual approach taken in this section to establish uniqueness of solutions to* (FPE) *is not too different from that pursued in Section 5.1 above. One difference is that, while in this section the moment conditions on η are stricter, the stronger assumptions on F in Section 5.1 allow us to reduce the moment conditions on η (compare the definition of $\mathcal{K}^{1,\text{diss}}_{s,\zeta}$ on page 29 for the case of dissipative F with the moment condition in the definition of $\mathcal{K}^{\text{meas}}_{s,\zeta}$ for the case of measurable F on page 30). The main reason for this is, that the Yosida-approximation of F used in Section 5.1 requires its dissipativity property. The approximation approach in the present section instead requires $Q^{-1} \in L(H)$, to ensure an L^2-upper bound for Du_c.*

5.2.2. Uniqueness of the solution to the Fokker-Planck equation

We return to our initial aim, to show the uniqueness of the solution to the Fokker-Planck equation related to the equation (SPDE) in the case of a merely measurable nonlinearity

F.

Since $\mathcal{K}^{\mathrm{meas}}_{s,\leq 2\alpha}$ is by definition a superset of $\mathcal{K}^{\mathrm{meas}}_{s,\zeta}$ (by virtue of $\mathcal{K}^{0}_{s,\leq 2\alpha}$ being a superset of $\mathcal{K}^{0}_{s,\zeta}$; see Remark 2.2.6(i) and (2.2.2)), we know by the preceding section that any $\eta \in \mathcal{K}^{\mathrm{meas}}_{s,\zeta}$ fulfills the dense range condition (5.2.1). As a consequence, we get the uniqueness of the solution to the Fokker-Planck equation from the following result:

Proposition 5.2.6. *Let $\tilde{\mathcal{K}} \subset \mathcal{K}^{0}_{s,\zeta}$ be a convex subset, such that the dense range condition* (5.2.1) *is fulfilled for any $\eta \in \tilde{\mathcal{K}}$.*
Then, $\tilde{\mathcal{K}}$ contains at most one element.

The proof of this claim is the same as that of Proposition 5.1.3 above.

6. Example

One classical example for semilinear equations of type (SPDE) are reaction-diffusion equations, where the linear part describes the diffusion of substances e.g. in a fluid, and the nonlinear part describes the space-time development of a reaction (e.g. chemical or biological) between these substances.

In this chapter, we explain the application of our results to such a situation. Let us note two things. First, it is not proven at this moment, whether (and in which way) existing existence results for solutions to (FPE) from the Wiener noise case can be adapted to our framework. We thus have to assume for this chapter, that existence results similar to those from the Wiener noise case exist within our framework, which is of course not entirely satisfying. Second, what follows below is naturally not the first description of how to apply abstract results for Fokker-Planck equations characterizing the solutions of SPDE to reaction-diffusion problems. Our presentation below is essentially the same as that in [BDPR11, Sect. 6].

Let $H := L^2\big((0,1)\big)$ (the L^2-space with respect to the Lebesgue measure on $(0,1) \subset \mathbb{R}$) with norm $|\cdot|_H := |\cdot|_{L^2((0,1))}$. The separability of this space is a classical fact from measure and integration theory.

Define the linear operator $A : D(A) \subset H \to H$ by $Ax(r) := \frac{\mathrm{d}^2}{\mathrm{d}r^2}x(r)$ and $D(A) := H^2\big((0,1)\big) \cap H_0^1\big((0,1)\big)$. This very simple Laplace operator fulfills Hypothesis (H.l1).

The nonlinear drift part $F : D(F) \to H$ is defined, for a given $m \in \mathbb{N}$, on $D(F) := [0,T] \times L^{2m}\big((0,1)\big)$ by

$$F(t,x)(r) := f\big(r,t,x(r)\big) + h\big(r,t,x(r)\big) \quad \text{for all } r \in (0,1),\, (t,x) \in D(F),$$

where we assume that $f, h : (0,1) \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions, which fulfill the following conditions:

(0) For any fixed $r \in (0,1)$, the functions $f(r,\cdot,\cdot)$ and $h(r,\cdot,\cdot)$ are continuous on $[0,T] \times \mathbb{R}$.

(f1) *(polynomial growth).* There exist an odd integer $m \in \mathbb{N}$ and a nonnegative $c_1 \in L^2\big([0,T]\big)$, such that

$$\big|f(r,t,z)\big| \le c_1(t) \cdot \big(1 + |z|^m\big) \quad \text{for all } t \in [0,T],\, z \in \mathbb{R},\, r \in (0,1).$$

(f2) *(quasi-dissipativity).* There exists a nonnegative $c_2 \in L^1([0,T])$, such that

$$\big[f(r,t,z_1) - f(r,t,z_2)\big] \cdot (z_1 - z_2) \le c_2(t) \cdot |z_1 - z_2|^2 \qquad \text{for all } t \in [0,T];\, z_1, z_2 \in \mathbb{R};\, r \in (0,1).$$

(h1) *(linear growth).* There exists a nonnegative $c_3 \in L^2([0,T])$, such that

$$\big|h(r,t,z)\big| \le c_3(t) \cdot \big(1 + |z|\big) \quad \text{for all } t \in [0,T], z \in \mathbb{R}, r \in (0,1).$$

For such a nonlinear drift part, it is not yet established, whether there exists a pathwise solution to equations of type (SPDE) (for either Wiener or more general noise). However, F as described above fits into the framework of Section 5.2, fulfilling (H.m1).

Assume, that Hypotheses (H.l2)–(H.l3) and (H.m2) are fulfilled. Define, for $N \in \mathbb{N}$,

$$V_N(t,x) := \begin{cases} \underbrace{2\big[c_1(t) + c_3(t) + 1\big]}_{=:c_4(t)} \cdot \big(1 + |x|^N_{L^{2N}((0,1))}\big) & \\ & \text{if } (t,x) \in [0,T] \times L^{2N}\big((0,1)\big) \\ +\infty & \text{else.} \end{cases}$$

Observe that (f1) and (h1) imply

$$\big|F(t,x)\big| \le V_m(t,x) < \infty\,. \tag{6.0.1}$$

Assume that from here on $N \ge m$.

Assumption 6.0.7. *For any $\zeta \in \mathcal{M}_1(H)$ with $\int_H |x|^{2N}_{L^{2N}((0,1))}\,\zeta(\mathrm{d}x) < \infty$, there exists a solution $\eta \in \mathcal{K}^1_s$ to* (FPE)*, such that $t \mapsto \eta_t(A)$ is measurable on $[s,T]$ for all $A \in \mathcal{B}(H)$.*

This solution has the following additional properties:

$$\begin{aligned} &\sup_{t\in[s,T]} \int_H |x|^2_{L^2((0,1))}\,\eta_t(\mathrm{d}x) < \infty \\ &t \mapsto \int_H \psi(t,x)\,\eta_t(\mathrm{d}x) \quad \textit{is continuous for all } \psi \in \mathcal{W}_{T,A} \\ &\int_s^T \int_H V_N^2(r,x) + \big|(-A)^\delta x\big|^2_{L^2((0,1))}\,\eta_r(\mathrm{d}x)\,\mathrm{d}r \\ &\quad \le C_5 \cdot \int_s^T \int_H V_N^2(r,x)\,\zeta(\mathrm{d}x)\,\mathrm{d}r < \infty \quad \textit{for a } C_5 \in (0,\infty) \textit{ and any } \delta \in \big(\tfrac{1}{4},\tfrac{1}{2}\big). \end{aligned} \tag{6.0.2}$$

(Note, that any Dirac measure δ_x with $x \in L^{2N}\big((0,1)\big)$ fulfills the requirements for ζ indicated above.)

Remark 6.0.8. *This 'Assumption' is proven to be valid for the Wiener noise case in* [BDPR10, Sect. 4]. *The generalization of these existence results to the case of Lévy noise plus cylindrical Wiener noise will be a topic of future research.*

Choose now $N := m+2$ and observe that by construction of V_N and (6.0.1) (note that $c_4(t) \geq 1$ for all t by definition),

$$\begin{aligned}
\int_s^T \int_H \left|F(t,x)\right|^2 \eta_t(\mathrm{d}x)\ \mathrm{d}t &\leq \int_s^T \int_H V_m^2(t,x)\ \eta_t(\mathrm{d}x)\ \mathrm{d}t \\
&\leq \int_s^T \int_H \left[c_4(t) \cdot \left(1 + |x|^m_{L^{2m}((0,1))}\right)\right]^2 \eta_t(\mathrm{d}x)\ \mathrm{d}t \\
&\leq \int_s^T c_4^2(t) \int_H 1 + |x|^{2m}_{L^{2m}((0,1))}\ \eta_t(\mathrm{d}x)\ \mathrm{d}t\ ,
\end{aligned}$$

which implies together with the properties of our assumedly existing solution η (particularly, (6.0.2)), that η is in $\mathcal{K}^{\text{meas}}_{s,\zeta}$ (as defined on page 30) if the initial condition ζ fulfills

$$\int_H |x|^{2(m+2)}_{L^{2(m+2)}((0,1))}\ \zeta(\mathrm{d}x) < \infty\ .$$

Consequently, by Theorem 5, η is the unique solution to (FPE) with coefficients and initial data as specified above.

A. π-semigroups

Below, we give a short introductory overview of concepts and results from the theory of π-semigroups. The theory as presented in [Pri99] has been developed on the foundations of the theory of weakly continuous semigroups (see e.g. [Cer94], [Cer95] and [CG95]). More recent related works include [Man06]. The following overview is based on [Pri99]. While we limit ourselves to the case of bounded functions in the following overview, note that the extension of the theory of weakly continuous semigroups to the case of functions with polynomial growth has been treated e.g. in [Man06] and [Cer95]. We let E be a separable metric space and define $\mathcal{C}_u(E)$ as before.

A family $(T_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{C}_u(E)$ is called a *π-semigroup*, if the following criteria are fulfilled:

(o) $(T_t)_{t\geq 0}$ is a semigroup of operators; that is, $T_0 = I$ and $T_{t+s} = T_t \circ T_s$ for $s, t \geq 0$.

(i) For any $f \in \mathcal{C}_u(E)$, $x \in E$, the function $t \mapsto T_t f(x)$, defined for $t \in [0, \infty)$, is continuous.

(ii) For any bounded sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}_u(E)$, such that f_n converges pointwise to $f \in \mathcal{C}_u(E)$ as $n \to \infty$ (we denote this as $f_n \xrightarrow{\pi} f$), we have that $T_t f_n \xrightarrow{\pi} T_t f$ for all $t \in [0, \infty)$.

(iii) There exist $M \geq 1$ and $\omega \geq 0$, such that for all $t \in [0, \infty)$, the operator norm of T_t is bounded from above as follows:

$$\|T_t\|_{L(\mathcal{C}_u(E))} \leq M \cdot e^{\omega t} \ .$$

The *type ω of a π-semigroup* (T_t) is defined as

$$\omega := \inf\Big\{\alpha \geq 0 \ \Big| \ \text{there exists an } M_\alpha \geq 1, \text{ such that} \\ \|T_t\|_{L(\mathcal{C}_u(E))} \leq M_\alpha \cdot e^{\alpha t} \text{ for all } t \geq 0\Big\} \ .$$

The *generator G of a π-semigroup* (T_t) is defined as follows:

$$\begin{cases} D(G) := \Big\{ f \in \mathcal{C}_u(E) \;\Big|\; \text{there exist } g \in \mathcal{C}_u(E) \text{ and } \delta > 0, \\ \qquad\qquad\qquad\qquad \text{such that } \sup_{h\in(0,\delta]} \|\Delta_h f\|_0 < \infty \\ \qquad\qquad\qquad\qquad \text{and } \lim_{h\searrow 0} \Delta_h f(x) = g(x) \text{ for all } x \in E \Big\} \\ Gf(x) := \lim_{h\searrow 0} \Delta_h f(x) \quad \text{for all } f \in D(g),\, x \in E, \end{cases}$$

where we denote $\Delta_h := \frac{1}{h}(T_h - I)$.

Note that, in contrast to the infinitesimal generator of a C_0-semigroup, the generator of a π-semigroup is not necessarily densely defined (with respect to the supremum norm on $\mathcal{C}_u(E)$) or m-dissipative. On the other hand, if (T_t) is even a C_0-semigroup, then the 'weak' generator as defined above and the 'strong' generator as defined in the Introduction coincide (see e.g. [Pri99, Thm. 3.7]).

Two further definitions: A linear operator $L : D(L) \subset \mathcal{C}_u(E) \to \mathcal{C}_u(E)$ is called *π-closed*, if

$$f_n \xrightarrow{\pi} f \quad \text{and} \quad Lf_n \xrightarrow{\pi} g$$

imply that

$$f \in D(L) \quad \text{and} \quad Lf = g\,.$$

A subset $V \subset \mathcal{C}_u(E)$ is called *π-dense in* $\mathcal{C}_u(E)$, if for any $f \in \mathcal{C}_u(E)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset V$, such that $f_n \xrightarrow{\pi} f$.

Some basic properties of generators of π-semigroups (cf. [Pri99, Prop. 3.2 and 3.4]): If G is the generator of a π-semigroup (T_t) of type ω on $\mathcal{C}_u(E)$, then we have the following for any $f \in D(G)$:

(i) $T_t f \in D(G)$ and $GT_t f = T_t G f$ hold for all $t \geq 0$.

(ii) For any $x \in E$, the map $t \mapsto T_t f(x)$, $t \in [0,\infty)$, is continuously differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t f(x) = T_t G f(x) \quad \text{for all } t \geq 0.$$

(iii) $D(G)$ is π-dense in $\mathcal{C}_u(E)$.

(iv) G is a π-closed operator on $\mathcal{C}_u(E)$.

(v) The family of operators $(R^G_\alpha)_{\alpha>\omega}$ defined as

$$R^G_\alpha f(x) := \int_0^\infty e^{-\alpha u} \cdot T_u f(x)\, \mathrm{d}u \quad \text{for all } f \in \mathcal{C}_u(E),\, x \in E,\, \alpha > \omega,$$

is a family of bounded operators, which is identical to the resolvent of G and which fulfills the following estimate (cf. [Pri99, Prop. 3.6]):

$$\|R_\alpha^G f\|_0 \leq \frac{M}{\alpha - \omega} \cdot \|f\|_0 \quad \text{for all } f \in \mathcal{C}_u(E), \alpha > \omega.$$

B. m-dissipative maps and their Yosida approximation

The following introductory overview is based on [Bar76, §II.3.1]. As before, let H be a separable real Hilbert space, and 2^H its power set.

A map $F : D(F) \subset H \to 2^H$ is called *dissipative*, if for any $x_1, x_2 \in D(F)$ we have

$$\langle y_1 - y_2\,,\, x_1 - x_2 \rangle \leq 0 \quad \text{for any } y_1 \in F(x_1), y_2 \in F(x_2).$$

A dissipative map $F : D(F) \subset H \to 2^H$ is called *m-dissipative*, if for any $\alpha > 0$ we have

$$\text{Range}(\alpha I - F) = H\,.$$

Note that in a Hilbert space, a map F is m-dissipative if and only if it is *maximal dissipative*, that is, F has no proper dissipative extension. (In the case of dissipative maps on general Banach spaces, as covered in [Bar76], the two notions do not in general coincide.)

We call a map $F : D(F) \subset H \to 2^H$ *quasi-dissipative*, if there is a $K > 0$, such that for any $x_1, x_2 \in D(F)$ we have

$$\langle x_1 - x_2\,,\, y_1 - y_2 \rangle \leq K \cdot |x_1 - x_2|^2 \quad \text{for all } y_1 \in F(x_1), y_2 \in F(x_2).$$

Since this is equivalent to $\big\langle x_1 - x_2\,,\, (y_1 - Kx_1) - (y_2 - Kx_2) \big\rangle \leq 0$, we observe that for a quasi-dissipative map F as defined above, the map $(F - KI)$ is dissipative. Finally, a quasi-dissipative map is called *m-quasi-dissipative*, if for any $\alpha > K$

$$\text{Range}(\alpha I - F) = H\,.$$

Because the quasi-dissipative case can be reduced to the dissipative case as indicated above, we only consider (m-)dissipative maps below.

Some more notation: For any closed set $S \subset H$, we define

$$\lfloor S \rfloor := \inf\big\{ |x| \;\big|\; x \in S \big\}\,,$$

and for $x \in D(F)$ se set

$$F_0(x) := \big\{ y \in F(x) \;\big|\; |y| = \lfloor F(x) \rfloor \big\}\,.$$

This way, $\left|F_0(x)\right|$ is well-defined right away. We will see later, that $F(x)$ is not only closed but also convex, which will make $F_0(x)$ well-defined as well.

The following geometric observation holds for any $x, y \in H$:

$$
\begin{aligned}
&|x| \le |x - \beta y| \quad \text{for all } \beta > 0 \\
&\quad \Leftrightarrow \quad \langle x, y\rangle \le 0 \, .
\end{aligned}
$$

This implies the following characterization of dissipative maps:

(o) A map $F : D(F) \subset H \to 2^H$ is dissipative if and only if for any $\beta > 0$ and $x_1, x_2 \in D(F)$ we have

$$
|x_1 - x_2| \le \left|(x_1 - \beta y_1) - (x_2 - \beta y_2)\right| \quad \text{for all } y_1 \in F(x_1), y_2 \in F(x_2).
$$

As a consequence, we observe that for any dissipative map $F : D(F) \subset H \to 2^H$ and every $\beta > 0$, the map $(I - \beta F)^{-1}$ is well-defined and non-expansive on $\mathrm{Range}(I - \beta F)$. We define for $x \in \mathrm{Range}(I - \beta F)$ the maps

$$
\begin{aligned}
J_\beta(x) &:= (I - \beta F)^{-1}(x) \\
F_\beta(x) &:= \frac{1}{\beta} \cdot \left(J_\beta(x) - x\right) \, .
\end{aligned}
$$

Due to the similarity of this construction to Yosida's approximation of linear operators, F_β is also called Yosida approximation of F.

From now on, we let $F : D(F) \subset H \to 2^H$ be an m-dissipative map, and $\beta > 0$. Then, the maps J_β and F_β have the following properties (proofs are included at the end of this appendix):

(i) $\left|J_\beta(x) - J_\beta(y)\right| \le |x - y| \quad$ for all $x, y \in H$.

(ii) $F_\beta(x) \in F\big(J_\beta(x)\big) \quad$ for all $x \in H$.

(iii) $F_\beta : H \to H$ is dissipative and Lipschitz-continuous with Lipschitz constant $\frac{2}{\beta}$.

(iv) $\left|F_\beta(x)\right| \le \left|F_0(x)\right| \quad$ for any $x \in D(F)$.

(v) $\lim_{\beta \to 0} J_\beta(x) = x \quad$ for any $x \in D(F)$.

For some further results used in Section 5.1, we need the following additional definition (denoting by "$\rightharpoonup$" the weak convergence in H): We call a map $F : D(F) \subset H \to 2^H$ *demiclosed*, if for any pair of sequences $(x_n)_{n\in\mathbb{N}} \subset D(F)$ and $(y_n)_{n\in\mathbb{N}} \subset H$ with $y_n \in F(x_n)$, $n \in \mathbb{N}$, such that $x_n \to x_0$ and $y_n \rightharpoonup y_0$ as $n \to \infty$, we have that $x_0 \in D(F)$ and $y_0 \in F(x_0)$.

Let $F : D(F) \subset H \to 2^H$ again be m-dissipative. We see the following:

(vi) F is demiclosed.

(vii) Let $(x_\beta)_{\beta\geq 0} \subset H$ be such that for $\beta \to 0$ we have $x_\beta \to x_0$ and $\{F_\beta(x_\beta)\}_{\beta>0}$ is a bounded set. Then, $x_0 \in D(F)$. Furthermore, there is a sequence $(\beta_n)_{n\in\mathbb{N}}$, such that $\beta_n \to 0$ and $F_{\beta_n}(x_{\beta_n}) \rightharpoonup y_0$ as $n \to \infty$, and $y_0 \in F(x_0)$.

(viii) For any $x \in H$, the function $\beta \mapsto |F_\beta(x)|$ is monotone nonincreasing for $\beta \in (0,\infty)$, and

$$\lim_{\beta\to 0}|F_\beta(x)| = |F_0(x)| \ .$$

(ix) For any $x \in D(F)$, the set $F(x) \subset H$ is closed and convex.

(x) $\lim_{\beta\to 0} F_\beta(x) = F_0(x) \quad$ for all $x \in D(F)$.

Proofs. (i) is obvious by the definition of J_β (and the argument preceding the definition).
(ii): For any $x \in H$, $\beta > 0$, we have

$$F_\beta(x) \in \frac{1}{\beta} \cdot \Big(J_\beta(x) - \big((I-\beta F)\circ J_\beta\big)(x)\Big) = \big(F \circ J_\beta\big)(x) \ .$$

(iii): By definition of F_β, for any $\beta > 0$, $x_1, x_2 \in H$,

$$\big|F_\beta(x_1) - F_\beta(x_2)\big| = \frac{1}{\beta} \cdot \big|J_\beta(x_1) - x_1 - J_\beta(x_2) + x_2\big| \leq \frac{2}{\beta} \cdot |x_1 - x_2| \ ,$$

where we used (i) in the last estimate.
Dissipativity follows from the observation, that for each $x_1, x_2 \in H$, $\beta > 0$,

$$\begin{aligned}&\big\langle x_1 - x_2\, ,\, J_\beta(x_1) - x_1 - J_\beta(x_2) + x_2\big\rangle \\ &\quad = -\underbrace{\big\langle x_1 - x_2\, ,\, x_1 - x_2\big\rangle}_{=|x_1-x_2|^2} + \underbrace{\big\langle x_1 - x_2\, ,\, J_\beta(x_1) - J_\beta(x_2)\big\rangle}_{\leq |x_1-x_2|\cdot|x_1-x_2| \text{ with (i)}} \leq 0 \ .\end{aligned}$$

(iv): Let $x \in D(F)$, $\beta > 0$. We have by definition of F_β and (i), that

$$\big|F_\beta(x)\big| = \frac{1}{\beta} \cdot \Big|J_\beta(x) - \big(J_\beta \circ (I - \beta F)\big)(x)\Big| \leq \frac{1}{\beta}|\beta y|$$

for each $y \in F(x)$.
(v): For any $x \in D(F)$, $\beta > 0$, we have using (iv):

$$\big|J_\beta(x) - x\big| = \beta \cdot \big|F_\beta(x)\big| \leq \beta \cdot \big|F_0(x)\big| \xrightarrow{\beta\to 0} 0 \ .$$

(vi): Let $(x_n)_{n\in\mathbb{N}} \subset D(F)$ and $y_n \in F(x_n)$, $n \in \mathbb{N}$, such that $x_n \to x_0$ and $y_n \rightharpoonup y_0$ as $n \to \infty$. Then, for all $n \in \mathbb{N}$,

$$\langle y_n - y\,,\, x_n - x\rangle \le 0 \quad \text{for each pair } x \in D(F), y \in f(x),$$

and consequently

$$\langle y_0 - y\,,\, x_0 - x\rangle \le 0 \quad \text{for each pair } x \in D(F), y \in f(x).$$

This, however, implies that $x_0 \in D(F)$ and $y_0 \in F(x_0)$ by the maximality of F. (Otherwise, there would be an extension $\tilde{F}$ of F with $D(\tilde{F})$ containing the disjoint union $D(F) \,\dot\cup\, \{x_0\}$, which would contradict the m-dissipativity of F.)

(vii): Let $(x_\beta)_{\beta\ge 0} \subset H$ as described in the assertion. By boundedness of $\{F_\beta(x_\beta)\}$, there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$, such that for $n \to \infty$ we have $\beta_n \to 0$ and $\big(F_{\beta_n}(x_{\beta_n})\big)_{n\in\mathbb{N}}$ converges weakly, say, to $y_0 \in H$:

$$F_{\beta_n}(x_{\beta_n}) \rightharpoonup y_0 \quad \text{as } n \to \infty.$$

Using that $F_{\beta_n}(x_{\beta_n}) = F\big(J_{\beta_n}(x_{\beta_n})\big)$ and $J_{\beta_n}(x_{\beta_n}) \xrightarrow{n\to\infty} x_0$, by (vi) we have $x_0 \in D(F)$ and $y_0 \in F(x_0)$.

(viii): Let $x \in H$. Recall that for any $\beta > 0$, we have $J_\beta(x) \in D(F)$ and $F_\beta(x) \in F\big(J_\beta(x)\big)$. Also, by definition of F_β, we have that $x = J_\beta(x) - \beta F_\beta(x)$ for all $\beta > 0$ and any $x \in H$. Using the dissipativity property of F, we obtain for any $\beta > \delta > 0$ and $x \in H$, that

$$\begin{aligned}
&\big\langle J_\beta(x) - J_\delta(x)\,,\, F_\beta(x) - F_\delta(x)\big\rangle \le 0\\
\Leftrightarrow\quad &\big\langle \beta F_\beta(x) - \delta F_\delta(x)\,,\, F_\beta(x) - F_\delta(x)\big\rangle \le 0\\
\Leftrightarrow\quad &\big\langle \delta F_\beta(x) - \delta F_\delta(x)\,,\, \beta F_\beta(x) - \delta F_\delta(x)\big\rangle \le 0\\
\Leftrightarrow\quad &\big\langle \beta F_\beta(x) - \delta F_\delta(x)\,,\, \beta F_\beta(x) - \delta F_\delta(x)\big\rangle \le \big\langle (\beta-\delta)\cdot F_\beta(x)\,,\, \beta F_\beta(x) - \delta F_\delta(x)\big\rangle\\
\Rightarrow\quad &\big|\beta F_\beta(x) - \delta F_\delta(x)\big|^2 \le |\beta - \delta| \cdot \big|F_\beta(x)\big| \cdot \big|\beta F_\beta(x) - \delta F_\delta(x)\big|\\
\Rightarrow\quad &\big|\beta F_\beta(x) - \delta F_\delta(x)\big| \le |\beta - \delta| \cdot \big|F_\beta(x)\big|\,.
\end{aligned}$$

Thus, if $\beta > \delta$, then $\big|F_\delta(x)\big| \ge \big|F_\beta(x)\big|$ for any $x \in H$, which shows the monotonicity.

Since the monotone function $\beta \to \big|F_\beta(x)\big|$ is bounded for $\beta > 0$ and $x \in H$, it must converge: Let $x \in H$ fixed and set

$$r := \lim_{\beta\to 0}\big|F_\beta(x)\big|\,.$$

By (iv), we have $r \le \big|F_0(x)\big|$. On the other hand, let $(\beta_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence, such that $\beta_n \to 0$ from above and $F_{\beta_n}(x) \rightharpoonup y$, both as $n \to \infty$. Because $|y| \le r$ and (by (vii)) $y = |F_0(x)|$, we see that $|F_0(x)| \leq r$, which concludes the proof of the claimed convergence.

(ix): Since F is maximal dissipative, $F(x_0)$ for any $x_0 \in D(F)$ is given by

$$F(x_0) = \{y_0 \in H \mid \langle y - y_0 \, , \, x - x_0 \rangle \leq 0 \text{ for all pairs } x \in D(F), y \in F(x)\} \ .$$

The convexity of this set is easily seen: If $\{y_i\}_{i=1,2}$ fulfill

$$\langle y - y_i \, , \, x - x_0 \rangle \leq 0 \quad \text{for a pair } (x, y) \in H \times H,$$

then also $\langle y - y_r \, , \, x - x_0 \rangle \leq 0$ holds for any $r \in (0, 1)$ and $y_r := \frac{1}{r} y_1 + \frac{r-1}{r} y_2$.

(x): Let $x \in D(F)$. By (iv), we have that $\{|F_\beta(x)|\}_{\beta > 0}$ is bounded by $|F_0(x)|$. Thus, by (vii), we can find a sequence $\{\beta_n\}_{n \in \mathbb{N}}$, $\beta_n \xrightarrow{n \to \infty} 0$, such that $F_{\beta_n}(x)$ converges weakly to an element of H, say, y_0:

$$F_{\beta_n}(x) \rightharpoonup y_0 \quad \text{as } n \to \infty,$$

and $y_0 \in F(x)$. And with (viii),

$$\lim_{n \to \infty} |F_{\beta_n}(x)| = |y_0| = |F_0(x)| \ .$$

Together with [Bar76, Prop. I.1.4], the claimed convergence follows. □

C. Zusammenfassung

According to the graduation rules of the Department of Mathematics at Bielefeld University, a one-page summary in German language has to be submitted along with the PhD thesis. This summary is included on the following page.

Zusammenfassung

der Dissertation

Uniqueness of solutions to Fokker-Planck equations related to singular SPDE driven by Lévy and cylindrical Wiener noise

von Sven Wiesinger

Thema dieser Dissertation ist die Eindeutigkeit von Lösungen stochastischer partieller Differentialgleichungen (SPDG). Die Existenz und Eindeutigkeit von Lösungen solcher Gleichungen ist seit mehreren Jahrzehnten ein Kernthema der Forschung in der Wahrscheinlichkeitstheorie. In der Regel versteht man dabei unter dem Begriff „Lösung“ einen Prozess, der den Pfad des durch die SPDG beschriebenen Systems abhängig von Zeit, Anfangs- oder Randbedingungen und Zufallseinfluss beschreibt.

Wie schon in der Theorie der nicht stochastischen (deterministischen) partiellen Differentialgleichungen existiert auch in der Theorie der SPDG das Problem, dass zu vielen, durchaus anwendungsrelevanten Gleichungen der Nachweis der Existenz und Eindeutigkeit von „pfadweisen“ Lösungen mit gegenwärtigen mathematischen Methoden nicht möglich ist. In einigen dieser Fälle hat sich der Ansatz bewährt, anstelle der SPDG die daraus abgeleitete Fokker-Planck-Gleichung zu untersuchen, deren Lösung zwar nicht den Lösungspfad der SPDG angibt, aber immerhin die zeitliche Entwicklung der Lösungsverteilungen. Dieser Ansatz, der in den vergangenen Jahren von mehreren Gruppen von Autoren für unendlichdimensionale Zustandsräume verallgemeinert wurde, steht im Mittelpunkt der vorliegenden Dissertation. Während in der Vergangenheit vorwiegend SPDG untersucht wurden, deren stochastischer Teil durch einen unendlichdimensionalen Wiener-Prozess gegeben ist, wird in der vorliegenden Arbeit die Verallgemeinerung einiger aktueller Ergebnisse zur Eindeutigkeit von Lösungen von Fokker-Planck-Gleichungen für den Fall von SPDG gezeigt, in deren stochastischem Teil ein unendlichdimensionaler Lévy-Prozess mit Sprüngen (bzw. die Summe eines solchen Prozesses mit einem zylindrischen Wiener-Prozess) steht.

Grundlage für diese neuen Ergebnisse zur Eindeutigkeit der Lösung von Fokker-Planck-Gleichungen sind Resultate zur Verallgemeinerung der Theorie sogenannter Mehler-Halbgruppen von Operatoren auf Funktionenräumen, die am Anfang dieser Dissertation entwickelt werden. Die Übergangshalbgruppen linearer SPDG, auch Ornstein-Uhlenbeck-Halbgruppen genannt, gehören in die Familie der (verallgemeinerten) Mehler-Halbgruppen. Ihre Theorie wurde vor einigen Jahren für den Fall unendlichdimensionaler linearer SPDG mit Lévy-Rauschen verallgemeinert. Im ersten Teil dieser Dissertation werden Ergebnisse zur Konstruktion des infinitesimalen Generators solcher Mehler-Halbgruppen erweitert für den Fall explizit zeitabhängiger Testfunktionen. Dadurch wird im Weiteren die Untersuchung von SPDG mit explizit zeitabhängigem Drift („nicht autonomer Fall“) möglich, und somit der Beweis der oben beschriebenen Ergebnisse zur Eindeutigkeit der Lösung von Fokker-Planck-Gleichungen.

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Index of notation

Below we list notation, which is used systematically throughout the thesis. Page numbers refer to definitions and, in some cases, to hypotheses specifying particular assumptions.