Universität Bielefeld

Fakultät für Mathematik

Dissertation zur Erlangung des Doktorgrades der Mathematik (Dr. math.)

A combinatorial classification of thick subcategories of derived and stable categories

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Oktober 2011

Gedruckt auf alterungsbeständigem Papier nach ISO 9706.

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Datum der mündlichen Prüfung: 19. Dezember 2011

Abstract. In this thesis, we investigate thick subcategories for stable categories of certain Frobenius categories and for derived categories of hereditary abelian categories. Both types of categories arise in the representation theory of finite-dimensional algebras. There is a close relation between such stable categories and the derived categories of hereditary abelian categories. We show that this relation is well-behaved concerning thick subcategories.

Then, we give a classification of the thick subcategories of $\mathcal{D}^b(\text{mod}(A))$ where A is a finite-dimensional hereditary algebra of finite or tame representation type.

This enables us to classify the thick subcategories of algebraic triangulated categories with finitely many indecomposable objects and the thick subcategories of the stable module categories of an important class of self-injective algebras of tame representation type.

The classification is of a combinatorial nature and we emphasise combinatorial aspects such as counting and the lattice structure.

Zusammenfassung. In dieser Doktorarbeit untersuchen wir die dicken Unterkategorien für stabile Kategorien von gewissen Frobenius Kategorien und für derivierte Kategorien von erblichen abelschen Kategorien. Beide Arten von Kategorien kommen in der Darstellungstheorie von endlich-dimensionalen Algebren vor. Es gibt einen engen Zusammenhang zwischen solchen stabilen Kategorien und den derivierten Kategorien von erblichen abelschen Kategorien. Wir zeigen, dass diese Beziehung verträglich ist mit den dicken Unterkategorien.

Dann klassifizieren wir die dicken Unterkategorien von $\mathcal{D}^b(\text{mod}(A))$, wobei A eine endlich-dimensionale erbliche Algebra vom endlichen oder vom zahmen Darstellungstyp ist.

Dies versetzt uns in die Lage, die dicken Unterkategorien von algebraischen triangulierten Kategorien mit endlich vielen unzerlegbaren Objekten zu klassifizieren. Ebenso klassifizieren wir die dicken Unterkategorien einer wichtigen Klasse von selbst-injektiven Algebren vom zahmen Darstellungstyp.

Die Klassifikation ist kombinatorischer Natur und wir legen besonderen Wert auf kombinatorische Aspekte wie Zählen und die Verbandsstruktur.

Acknowledgements. First and foremost I wish to express my thanks to my supervisor, Professor Henning Krause. I thank him for his support, for helpful discussions and for coming up with the interesting topic of this thesis.

I have created important parts of the thesis during my research stay in Paris with Professor Bernhard Keller in spring 2010. I sincerely thank him for giving me this opportunity, and for his very profitable assistance during this time.

I would also like to thank the members and the visitors of the Paderborn Representation Theory Group and the Bielefeld Representation Theory Group for helpful discussions. I am particularly grateful to Professor Hideto Asashiba, Estanislao Herscovich, Dirk Kussin, Adam-Christiaan van Roosmalen, Greg Stevenson, Jan Stovicek and Dieter Vossieck. Moreover, I thank Philipp Lampe and Greg Stevenson for proofreading parts of the thesis.

I would like to thank Martin Rubey for helpful references.

Finally, I am indebted to the IRTG 'Geometry and Analysis of Symmetries' in Paderborn for the financial support in form of a scholarship. I am also grateful to the CRC 701 'Spectral Structures and Topological Methods in Mathematics' in Bielefeld for the financial support during the final period of my PhD studies.

Contents

CHAPTER 1

Introduction

A full subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called *thick* if it is a triangulated subcategory of $\mathcal{T}$ which is in addition closed under taking direct summands.

The study of thick subcategories is a way to extract a mathematical structure from the triangulated category. More precisely, the set of all thick subcategories of a triangulated category forms a lattice. This yields an invariant of the category which can help to understand the category itself.

The structure can be used to gain information about the objects of the ambient triangulated category. For example one can make a statement about the vanishing of morphisms between two given objects of the category. (We will present this as an application of our classification at the end of this thesis.)

The classification problem was approached in various mathematical fields as in stable homotopy theory, in commutative algebra and in the representation theory of finite groups. The first work was done by Hopkins and Smith concerning the p-local finite stable homotopy category [**48**]. Hopkins [**47**] and Neeman [**66**] studied the concept for the category of perfect complexes $\mathcal{D}^{\mathrm{per}}(R)$ of a noetherian ring R. They showed that the thick subcategories of $\mathcal{D}^{\mathrm{per}}(R)$ correspond to specialisation closed subsets of the prime ideal spectrum of R. Benson, Carlson and Rickard [**14**] classified the thick subcategories of the stable module category of the group algebra kG for a p-group G in terms of closed subvarietes of the maximal ideal spectrum of the group cohomology ring.

The existing classifications which have been mentioned all depend on some tensor structure. In this thesis, we consider triangulated categories where such additional structure is not available. Thus, we need to develop techniques which are completely different from the existing ones. An alternative approach towards triangulated categories without tensor structure can be found in the recent thesis of Stevenson (see [**82**] and [**83**]); he exploits the external action of another tensor triangulated category.

1. The main results

The triangulated categories of our interest are the stable categories of certain Frobenius categories and the derived categories of abelian hereditary categories. Both types of categories arise in the representation theory of finite-dimensional algebras. Actually, these two kinds of triangulated categories and their thick subcategories are strongly related.

Namely, we consider algebraic triangulated categories, that is, triangulated k-categories $\mathcal{T}$ which are triangle equivalent to the stable category of a Frobenius category. If $\mathcal{T}$ has only finitely many indecomposable objects and if it is connected and standard and if the field k is algebraically closed, then by a Theorem of Amiot [**1**], $\mathcal{T}$ is triangle equivalent to an orbit category $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle\Phi\rangle$ where $\vec{\Delta}$ is a quiver of Dynkin type and Φ is an admissible automorphism on $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$. An example of such a category is the stable module category $\underline{\mathrm{mod}}(\Lambda)$ of a self-injective algebra Λ of finite representation type.

If the self-injective algebra Λ is not of finite, but of tame representation type, then there is no comparable classification for $\underline{\mathrm{mod}}(\Lambda)$ so far. But important examples for these algebras are algebras which are isomorphic to orbit algebras $\hat{A}/G$ where A is a tame hereditary algebra and G is an admissible group of automorphisms on $\hat{A}$. The stable module category of $\hat{A}$ in turn is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}(A))$. This is due to Happel [**46**].

Hence, in both cases the stable category is related to the derived category of the module category of a hereditary algebra. Thus, the first step in the classification is a comparison of the thick subcategories of the stable category to those of the derived category.

We can formulate this comparison in a more general setting (not only for finite or tame representation type). Namely, we consider orbit categories $\mathcal{D}^b(\mathrm{mod}(A))/\langle\Phi\rangle$ where A is any hereditary algebra and Φ is an automorphism on $\mathcal{D}^b(\mathrm{mod}(A))$. Now for an arbitrary automorphism Φ it is not clear whether the orbit category is triangulated. Keller proves in [**56**] that it is triangulated making certain assumptions on Φ and we formulate our theorem in this setting.

THEOREM 1.1. *Let A be a hereditary k-algebra. Let $\Phi\colon \mathcal{D}^b(\mathrm{mod}(A)) \to \mathcal{D}^b(\mathrm{mod}(A))$ be an automorphism such that $\mathcal{D}^b(\mathrm{mod}(A))/\langle\Phi\rangle$ is triangulated. Then, there is a bijective correspondence between*

- *the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))/\langle\Phi\rangle$, and*
- *the set of thick $\langle\Phi\rangle$-invariant subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$.*

For self-injective algebras of the form $\hat{A}/G$ which are not necessarily triangle equivalent to such an orbit category, this comparison comes along as follows. We need to assume here that the associated push-down functor $F_\lambda\colon \mathrm{mod}(\hat{A}) \to \mathrm{mod}(\hat{A}/G)$ is dense and that the automorphism group G' on $\mathcal{D}^b(\mathrm{mod}(A))$ induced by G (via the equivalence $\underline{\mathrm{mod}}(\hat{A}) \cong \mathcal{D}^b(\mathrm{mod}(A))$) fulfils the assumptions (of Keller's Theorem) such that the orbit category $\mathcal{D}^b(\mathrm{mod}(A))/G'$ is triangulated.

THEOREM 1.2. *Let A be a hereditary algebra and let G be a cyclic admissible group of automorphisms on its repetitive category $\hat{A}$. Then, there is a bijective correspondence between*

- *the set of thick subcategories of $\underline{\mathrm{mod}}(\hat{A}/G)$, and*
- *the set of G'-invariant thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$.*

Being aware of this, the next step in the classification of thick subcategories of a stable category is a classification of the thick subcategories of the derived category of a hereditary algebra. Also this is an interesting task on its own, of course.

There is already a classification of the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ which are generated by an exceptional sequence. We call this set $\mathrm{Th}_{\mathrm{exc}}(A)$. This classification is due to Ingalls and Thomas [**53**] if A is the path algebra of a quiver of Dynkin or extended Dynkin type and it is generalised by Igusa, Schiffler and Thomas [**52**] to arbitrary path algebras over an algebraically closed field. The statement also holds without the assumption that the field is algebraically closed (see Krause [**59**]). The classification of $\mathrm{Th}_{\mathrm{exc}}(A)$ is in terms of the poset of noncrossing partitions $\mathrm{NC}(W,c)$ of the Weyl group W associated to A with $c \in W$ being the Coxeter transformation.

As soon as A is not of finite representation type, there are thick subcategories which are not generated by an exceptional sequence. We complete the classification in case that A is a hereditary k-algebra of tame representation type and k is an arbitrary field. Parts of this are based on Dichev's classification [**34**] for tame path algebras over algebraically closed fields.

For a tame hereditary algebra it is known (see for instance [**75**]) that the subcategory of regular modules decomposes into uniserial categories $\coprod_{j\in J}\mathcal{H}_j \times \coprod_{i=1}^{s}\mathcal{U}_{n_i}$ of which at most three factors $\mathcal{U}_{n_1},\dots,\mathcal{U}_{n_s}$ are of rank > 1. The classification depends on this decomposition. Namely, we introduce the set of noncrossing arcs $\mathrm{NA}(n)$ on a circle with n points. Fixing the above notation for the decomposition of the regular part we get the following classification.

THEOREM 1.3. *Let A be a tame hereditary k-algebra. The set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ is given by the union of sets*

$$\mathrm{Th}_{\mathrm{exc}}(A) \cup \mathrm{Th}_{\mathrm{reg}}(A)$$

where $\mathrm{Th}_{\mathrm{exc}}(A)$ corresponds bijectively to $\mathrm{NC}(W,c)$, $\mathrm{Th}_{\mathrm{reg}}(A)$ corresponds bijectively to a set of tuples

$$\{(p, x_1, \dots, x_s) \mid p \in 2^J, x_i \in \mathrm{NA}(n_i)\}.$$

Moreover, the intersection of $\mathrm{Th}_{\mathrm{exc}}(A)$ and $\mathrm{Th}_{\mathrm{reg}}(A)$ equals the set of such tuples with an additional assumption on the elements of $\mathrm{NA}(n)$.

Now we can combine the Theorems 1.1 or 1.2 with the last Theorem 1.3 in order to get a classification for thick subcategories of algebraic triangulated categories. First to the algebraic triangulated categories $\mathcal{T}$ with finitely many indecomposable objects. As said these are always of the form $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle\Phi\rangle$. From this we define the type of $\mathcal{T}$ to be (Δ, r, t) where r and t depend on Φ. Moreover, there is a list of all possible automorphisms Φ due to Xiao and Zhu [**85**].

THEOREM 1.4. *Let $\mathcal{T}$ be a triangulated category with finitely many indecomposable objects which is connected, algebraic and standard of type (Δ, r, t) excluding the cases $(D_n, r, 2)$ for n even and $(D_4, r, 3)$. Put $s = \gcd(p, h_\Delta)$ where p is a natural number depending on the type of $\mathcal{T}$ and h_Δ is the Coxeter number of Δ. Let W_Δ be the associated Weyl group and let $c \in W_\Delta$ be the Coxeter transformation. Then, there is a bijective correspondence between*

- *the set of thick subcategories of $\mathcal{T}$, and*

- *the set* $(\mathrm{NC}(W_\Delta, c))^{c^s}$ *of noncrossing partitions which are invariant under* s*-fold conjugation by* c.

We have to exclude the two mentioned types in this theorem since in these cases it is not possible to transfer the action of Φ to the noncrossing partitions. Therefore, we classify the thick subcategories for the type $(D_4, r, 3)$ by hand. The missing type $(D_n, r, 2)$ for an even n is covered by an alternative formulation of the above theorem for Dynkin types A und D. Namely, for these types there is an alternative description of $\mathrm{NC}(W_\Delta, c)$ motivated by the the poset of intersection subspaces of the hyperplanes of the root system of W. For $\Delta = A_n$ this is the original poset of noncrossing partitions invented by Kreweras [**61**]. Altogether, this gives a complete classification of thick subcategories for these algebraic triangulated categories of finite type.

An important class of self-injective algebras of tame representation type are the r-fold trivial extension algebras ($r \in \mathbb{N}$) of tame hereditary algebras A, i.e. orbit algebras of the form $\hat{A}/\langle \nu_{\hat{A}}^r \rangle$ where $\nu_{\hat{A}}$ is the Nakayama automorphism. In this case it is possible to combine Theorem 1.2 with Theorem 1.3 to get the following classification. Again we fix the notation for the decomposition of the regular part of $\mathrm{mod}(A)$ described above.

THEOREM 1.5. *Let A be a tame hereditary algebra. The set of thick subcategories of* $\underline{\mathrm{mod}}(\hat{A}/\langle \nu_{\hat{A}}^r \rangle)$ *corresponds bijectively to*

$$\mathrm{Th}_{\mathrm{exc}}(A)^{\mathrm{inv}} \cup \mathrm{Th}_{\mathrm{reg}}(A)^{\mathrm{inv}}.$$

Here $\mathrm{Th}_{\mathrm{exc}}(A)^{\mathrm{inv}}$ *corresponds bijectively to*

$$(\mathrm{NC}(W, c))^{c^r} = \{w \in \mathrm{NC}(W, c) \mid c^r w c^{-r} = w\}$$

and $\mathrm{Th}_{\mathrm{reg}}(A)^{\mathrm{inv}}$ *corresponds bijectively to*

$$\{(p, x_1, \ldots, x_s) \mid p \in 2^J, x_i \in (\mathrm{NA}(n_i))^r\}$$

where $(\mathrm{NA}(n_i))^r$ *denotes the set of noncrossing arcs on a circle which are invariant under rotation by* $r\frac{2\pi}{n_i}$ *of the circle.*

Again we also describe the intersection of the two sets.

In general, the whole machinery is applicable to a variety of triangulated categories. Explicitly, we also execute it for $\underline{\mathrm{mod}}(\Lambda_q)$ where Λ_q is the tame self-injective algebra $k\langle X, Y\rangle/(X^2, Y^2, XY - qYX)$ for $q \in k^*$, and for the cluster category $\mathcal{C}_{\vec{\Delta}}$ where Δ is of Dynkin or extended Dynkin type.

In all the cases we emphasise combinatorial aspects and questions such as the poset or the lattice structure and the number of thick subcategories.

2. Outline

The thesis is organised as follows. In Chapter 2 we present the category theoretical background of the thesis. We introduce the different notions of categories and relate them to each other. After this we are able to introduce orbit categories and we do this in Chapter 3. This chapter is also already concerned with the thick subcategories of orbit categories in general, that is, Theorem 1.1 is discussed here.

In the sequel the general structure is the following. First we present and discuss the necessary background on the relevant categories and then we classify the thick subcategories of those categories.

According to this, Chapter 4 is about hereditary algebras A and their module categories $\text{mod}(A)$, Chapter 5 introduces noncrossing partitions (which are needed for the upcoming classification and related to hereditary algebras via roots and Weyl groups) and finally, in Chapter 6 we present the classification of thick subcategories of $\mathcal{D}^b(\text{mod}(A))$.

Analogously, we proceed concerning algebraic triangulated categories. In the Chapters 7 and 8 we present everything which is relevant for these categories, and the Chapters 9 and 10 contain the classification of their thick subcategories.

The final three chapters are devoted to related topics which are interesting in the whole context.

In Chapter 11 we briefly present other classification approaches for thick subcategories (as mentioned in the very beginning of this chapter) and compare them to the approach of this thesis.

Chapter 12 is called Hom-Vanishing and contains an application (already mentioned above) of our classification. Namely, we show how the classification provides information about the question whether or not there are non-zero morphism between two objects of the triangulated category.

At last, in Chapter 13 we give the classification of the thick subcategories for the cluster category.

3. Basic assumptions and conventions

Throughout, k denotes a field. We say explicitly if we need k to be algebraically closed.

Every category in this thesis is k-linear, small, i.e. its objects form a set, and Hom-finite which means that the morphism spaces are finite-dimensional.

Moreover, all considered categories are supposed to be idempotent complete, i.e. each idempotent in the category splits. In particular, this implies that the categories are Krull-Remak-Schmidt.

Concerning (valued) diagrams we make the following convention. If $\vec{\Delta}$ is an oriented (valued) diagram, we denote by Δ its underlying diagram. If the respective object does not depend on the orientation, we will just write Δ.

CHAPTER 2

Preliminaries

Throughout this thesis, the subject of study will always be some category and in particular its subcategories. There will appear different concepts and kinds of categories and this chapter intends to distinguish and connect the several notions.

1. Additive and abelian categories

In principle all our categories $\mathcal{A}$ are *additive*, i.e. every finite family of objects has a product, each set of morphisms $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group and the composition of morphisms is bilinear.

DEFINITION 2.1. An additive category $\mathcal{A}$ is said to be *abelian* if every morphism has a kernel and a cokernel and if every monomorphism is the kernel and every epimorphism is the cokernel of some morphism.

As said in the introduction, all considered categories are idempotent complete and hence Krull-Remak-Schmidt.

DEFINITION 2.2. An additive category is called *Krull-Remak-Schmidt category* if every object decomposes into a finite direct sum of objects having local endomorphism ring.

This property is very important, in particular for the classification of subcategories. It turns out that for the sort of subcategories we are considering, it suffices then to name the indecomposable objects of the subcategory. Actually, a decomposition into indecomposable direct summands is unique.

THEOREM 2.3 (Krull-Remak-Schmidt). *Let $X = X_1^{a_1} \oplus \ldots \oplus X_r^{a_r}$ be a decomposition of an object in a Krull-Remak-Schmidt category such that the X_i are pairwise non-isomorphic indecomposable objects with $a_i \geq 1$. If $X = Y_1^{b_1} \oplus \ldots \oplus Y_s^{b_s}$ is another decomposition of this form, then $r = s$ and, up to reordering, $X_i \cong Y_i$ and $a_i = b_i$.* □

Moreover, all appearing abelian categories $\mathcal{A}$ will be *finite length categories*, i.e. each object X in $\mathcal{A}$ has a finite *composition series*

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X$$

with *composition factors* X_i/X_{i-1} being simple objects. Actually, for an abelian category the finite length property implies that the category is Krull-Remak-Schmidt. Also, for a finite length object, being indecomposable and having a local endomorphism ring are equivalent.

The property of a category being abelian yields an additional structure on it given by *exact sequences* which are sequences of objects in the category and morphisms between them such that the image of one morphism equals the kernel of the following morphism. In particular, there are the *short exact sequences*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where f is a monomorphism, g is an epimorphism and $\mathrm{Im}(f) = \mathrm{Ker}(g)$. That is, A is essentially a subobject of B and C is isomorphic to the corresponding quotient. Also, short exact sequences define *extensions* of objects in the abelian category. Moreover, these short exact sequences make abelian categories into *exact categories* in the sense of Quillen [**67**]. An *exact category* is an additive category together with a class of sequences $A \to B \to C$ fulfilling certain axioms.

Definition 2.4. A full subcategory $\mathcal{W}$ of an abelian category $\mathcal{A}$ is called a *wide subcategory* (after Hovey [**49**]) if it is abelian and closed under extensions. For a full subcategory $\mathcal{W}$ being *abelian* means that if there is a morphism $f\colon A \to B$ in $\mathcal{W}$, then $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{W}$.

This definition of a wide subcategory makes sure that the category itself is an abelian category whereas the kernel and the cokernel are the same as in the ambient abelian category.

Note that a wide subcategory is automatically closed under direct summands. Moreover, for a wide subcategory the so-called 2-*out-of*-3-*property* holds, that is whenever two objects of a short exact sequence are contained in $\mathcal{W}$, then so is the third. For arbitrary abelian categories it is not true that the 2-out-of-3-property implies wideness. But it is true if we assume the following property.

Definition 2.5. An abelian category $\mathcal{H}$ is called *hereditary* if

$$\mathrm{Ext}^n_{\mathcal{H}}(A, B) = 0$$

for all $n \geq 2$ and all $A, B \in \mathcal{H}$.

In case of an hereditary category $\mathcal{H}$, Dichev [**34**, Theorem 3.3.1] points out that a subcategory $\mathcal{W}$ is wide if and only if it is closed under direct summands and if it fulfils the 2-out-of-3-property. Thus, one has to look at short exact sequences when it comes to the classification of wide subcategories.

Example 2.6. Let A be an algebra over a field. Then, the category $\mathrm{Mod}(A)$ of all left (or right) modules over A is an abelian category. The same holds for the category $\mathrm{mod}(A)$ of all finite-dimensional modules over A.

This leads us to a comment on notation and convention. If not stated otherwise, we will always consider finite-dimensional left modules and denote the category by $\mathrm{mod}(A)$.

This is our main example of an abelian category. Precisely, we are going to consider algebras whose module category is equivalent to the category of representations of certain diagrams. These categories are also hereditary. More on that topic in Chapter 4.

2. Triangulated categories

There are important categories which do not admit an exact structure as described above. Therefore, one tries to endow them with another comparable structure, in this case a triangulated structure. An example of such a category is the derived category of an abelian category which we will define further down.

We take the formulations and notations for the definition of derived and triangulated categories from [**58**]. Originally, these concepts were invented by Verdier in [**84**].

Definition 2.7. Let $\mathcal{T}$ be an additive category with an equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (α, β, γ) of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X.$$

A morphism between two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple

$$(\phi_1, \phi_2, \phi_3)$$

of morphisms in $\mathcal{T}$ making the following diagram commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X'
\end{array}$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a class of distinguished triangles (called *exact triangles*) satisfying the following axioms.

(TR1) A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is exact. Each map α fits into an exact triangle (α, β, γ).

(TR2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.

(TR3) Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of maps ϕ_1 and ϕ_2 satisfying $\phi_2 \circ \alpha = \alpha' \circ \phi_1$ can be completed to a morphism

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X'
\end{array}$$

of triangles.

(TR4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\
X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
 & & W & = & W & \xrightarrow{\beta_3} & \Sigma Y \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\
 & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & &
\end{array}$$

Example 2.8. Let A be a finite-dimensional self-injective algebra, i.e. projective and injective A-modules coincide. Let $\underline{\mathrm{mod}}(A)$ be the stable module category of A which is given as follows. The objects of $\underline{\mathrm{mod}}(A)$ are the same as in $\mathrm{mod}(A)$ and morphisms are by definition

$$\underline{\mathrm{Hom}}_A(X,Y) := \mathrm{Hom}_A(X,Y)/\{X \to P \to Y \mid P \text{ projective}\}.$$

Then, $\underline{\mathrm{mod}}(A)$ is triangulated. The equivalence Σ is given by

$$X \mapsto \mathrm{Coker}(X \mapsto E(X))$$

where $E(X)$ denotes the injective hull of X. Exact triangles in $\underline{\mathrm{mod}}(A)$ arise from short exact sequences in $\mathrm{mod}(A)$. Namely, given a short exact sequence $0 \to X \to Y \to Z \to 0$, the fact that $E(X)$ is injective and the universal property of the cokernel yield morphisms in $\mathrm{mod}(A)$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \sim} & & \\ 0 & \longrightarrow & X & \longrightarrow & E(X) & \longrightarrow & \Sigma X & \longrightarrow & 0 \end{array}$$

commutative. This defines an exact triangle $X \to Y \to Z \to \Sigma X$. Now one can check the axioms (TR1) to (TR4) for $\underline{\mathrm{mod}}(A)$. This turns out to be possible since the algebra is self-injective.

More generally, this holds for arbitrary Frobenius categories. In the context of an exact category with a set E of exact sequences we may introduce the notion of projective and injective objects (for example Happel calls this *E-projective* or *E-injective* in [**46**]) such that it coincides with the notion of projectivity or injectivity if the category is abelian. Hence, it makes sense to define the following for an exact category.

Definition 2.9. An exact category is called *Frobenius* if it has enough projective and injective objects and if moreover, projective and injective objects coincide.

A triangulated category is called *algebraic* if it is triangle equivalent to the stable category of a Frobenius category.

Theorem 2.10 (Happel [**46**]). *Let $\mathcal{F}$ be a Frobenius category. Then, its stable category $\underline{\mathcal{F}}$ is triangulated.* □

The idea of the triangulated structure of $\underline{\mathcal{F}}$ is as described above for the stable module category of a self-injective algebra.

By the way these are our main examples for a triangulated category. More on algebraic triangulated categories in Chapter 7 and on self-injective algebras in Chapter 8.

The lattice of thick subcategories. The *correspondent* of a wide subcategory of an abelian category in the triangulated world is the following. (For hereditary abelian categories it will literally turn out to be a correspondent).

Definition 2.11. A full subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called *thick* if the following conditions hold.

- Let $X \to Y \to Z \to \Sigma X$ be an exact triangle. If two of $\{X, Y, Z\}$ are contained in $\mathcal{S}$, then so is the third.
- $X \in \mathcal{S}$ implies $\Sigma^n X \in \mathcal{S}$ for all $n \in \mathbb{Z}$.
- $\mathcal{S}$ is closed under direct summands.

This definition ensures that a thick subcategory is a triangulated category itself. Note that we also have a 2-out-of-3-property here.

The following concept will help us to describe thick subcategories

DEFINITION 2.12. We say that a thick subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is *generated* by a set of objects E in $\mathcal{T}$ if $\mathcal{S} = \text{Thick}(E)$ where $\text{Thick}(E)$ denotes the smallest thick subcategory of $\mathcal{T}$ containing the objects in E.

The set $T(\mathcal{T})$ of all thick subcategories of a triangulated category has an interesting combinatorial property.

DEFINITION 2.13. A *lattice* is a partially ordered set in which any two elements have a unique supremum called the *join* $\vee$ and an infimum called the *meet* $\wedge$.

The set $T(\mathcal{T})$ forms a lattice. The partial order is given by inclusion, the join of two thick subcategories $\mathcal{R}$ and $\mathcal{S}$ is its intersection which is thick again. The meet of $\mathcal{R}$ and $\mathcal{S}$ is $\text{Thick}(\mathcal{S} \cup \mathcal{R})$.

Moreover, in certain situations there is a *lattice complement*.

DEFINITION 2.14. Let $\mathcal{S}$ be a thick subcategory of a triangulated category $\mathcal{T}$. Then we call

$$\mathcal{S}^{\perp} = \{Y \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) = 0 \ \forall \ X \in \mathcal{S}\}$$

and

$$^{\perp}\mathcal{S} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) = 0 \ \forall \ Y \in \mathcal{S}\}$$

the *right* respectively *left perpendicular categories* with respect to $\mathcal{S}$.

One can check that the perpendicular categories are thick subcategories again.

Now we need the assumption that our triangulated category is *locally finite*, that is for each indecomposable object $X \in \mathcal{T}$ there are at most finitely many isomorphism classes of indecomposable objects $Y \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}(X, Y) \neq 0$.

PROPOSITION 2.15 (Krause [**59**], Proposition 4.4). *Let $\mathcal{T}$ be a locally finite triangulated category. Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. Then,*

$$\mathcal{S} \vee \mathcal{S}^{\perp} = \mathcal{T} \text{ and } \mathcal{S} \wedge \mathcal{S}^{\perp} = 0.$$

□

3. Auslander-Reiten theory

One of the most important tools used in this thesis is Auslander-Reiten theory. Namely, it is essential for the classification of the categories we work with. Moreover, we are going to use it all the time in order to picture categories and highlight subcategories within these categories.

Originally, Auslander-Reiten theory was defined for abelian categories in [**9**]. Happel introduced Auslander-Reiten triangles for triangulated categories in [**45**]. In [**59**] and [**64**] the theory is explained in the setting of a Krull-Remak-Schmidt category which covers both concepts. We keep with this general setting without going too much into details, i.e. there will be no proofs. Hence, throughout this section, let $\mathcal{C}$ be a Krull-Remak-Schmidt category.

The idea of Auslander-Reiten theory is to illustrate the category by its smallest building blocks. For objects these are the indecomposable objects. To get something appropriate for morphisms we need some preparation.

Definition 2.16. The *radical* of $\mathcal{C}$ is by definition the collection of subgroups

$$\operatorname{Rad}_{\mathcal{C}}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{C}}(X,Y)$$

for each pair X, Y of objects in $\mathcal{C}$ where

$$\operatorname{Rad}_{\mathcal{C}}(X,Y) = \{\phi \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \mid \operatorname{id}_X - \phi'\phi \text{ is invertible } \forall\, \phi' \colon Y \to X\}.$$

Definition 2.17. A morphism ϕ is called *irreducible* if ϕ is neither a section nor a retraction, and if $\phi = \phi''\phi'$ is a factorisation, then ϕ' is a section or ϕ'' is a retraction.

For each $n > 1$ and each pair X, Y define recursively $\operatorname{Rad}^n_{\mathcal{C}}(X,Y)$ to be the set of morphisms ϕ that admit a factorisation $\phi = \phi''\phi'$ with $\phi' \in \operatorname{Rad}_{\mathcal{C}}(X,Z)$ and $\phi'' \in \operatorname{Rad}^{n-1}_{\mathcal{C}}(Z,Y)$ for some object Z. Then we set

$$\operatorname{Irr}_{\mathcal{C}}(X,Y) = \operatorname{Rad}_{\mathcal{C}}(X,Y)/\operatorname{Rad}^2_{\mathcal{C}}(X,Y).$$

This is a bimodule over the division rings $\Delta(X)$ and $\Delta(Y)$ where

$$\Delta(X) = \operatorname{End}_{\mathcal{C}}(X)/\operatorname{Rad}_{\mathcal{C}}(X,X).$$

Note that a morphism $X \to Y$ is irreducible if and only if it belongs to

$$\operatorname{Rad}_{\mathcal{C}}(X,Y) \setminus \operatorname{Rad}^2_{\mathcal{C}}(X,Y).$$

Definition 2.18. A morphism $\phi \colon X \to Y$ is called *right almost split* if ϕ is not a retraction and if every morphism $X' \to Y$ that is not a retraction factors through ϕ. The morphism ϕ is *right minimal* if every endomorphism $\alpha \colon X \to X$ with $\phi\alpha = \phi$ is invertible. Dually, we define *left almost split* and *left minimal*.

Definition 2.19 (Liu [**64**]). A sequence of morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathcal{C}$ is called *Auslander-Reiten sequence* if

- α is minimal left almost split and a weak kernel of β,
- β is minimal right almost split and a weak cokernel of α, and
- $Y \neq 0$.

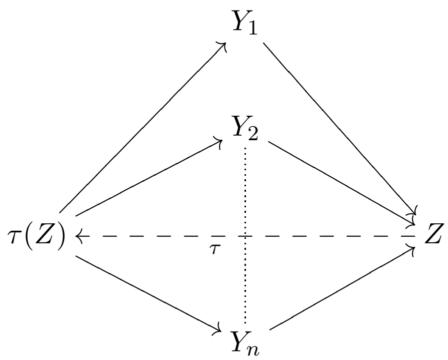
Theorem 2.20 (Liu [**64**]). *Let $\mathcal{C}$ be a Krull-Remak-Schmidt category and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be an Auslander-Reiten sequence.*

- *Up to isomorphism, the sequence is the unique Auslander-Reiten sequence starting with X and ending with Z.*
- *Each irreducible morphism $f_1 \colon X \to Y_1$ or $g_1 \colon Y_1 \to Z$ fits into an Auslander-Reiten sequence*

$$X \xrightarrow{(f_1 \;\; f_2)} Y_1 \amalg Y_2 \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} Z.$$

□

This theorem justifies to define the *Auslander-Reiten translation* $\tau(Z)$ of an object Z in $\mathcal{C}$ to be the unique object X whenever there is an Auslander-Reiten sequence ending in Z. Analogously, $\tau^{-1}(X) = Z$. Moreover, it makes sense to define the *Auslander-Reiten quiver* $\Gamma_{\mathcal{C}}$ of $\mathcal{C}$ in the following manner. The vertex set consists of isomorphism classes of indecomposable objects of $\mathcal{C}$. For vertices X, Y there is a unique arrow $X \to Y$ with valuation $(\dim_{\Delta(X)} \operatorname{Irr}(X,Y), \dim_{\Delta(Y)} \operatorname{Irr}(X,Y))$ if and only if $\operatorname{Irr}(X,Y) \neq 0$. Since the irreducible morphism fit into Auslander-Reiten sequences, the Auslander-Reiten quiver consists of *meshes* like



representing an Auslander-Reiten sequence $\tau(Z) \to Y_1 \amalg \ldots \amalg Y_n \to Z$.

Throughout this thesis, we will mostly not distinguish between the vertices and arrows of the Auslander-Reiten quiver and the indecomposable objects and morphisms they stand for.

For the existence of Auslander-Reiten sequences we need to get more concrete now. Thus, for the rest of this section consider the abelian category $\operatorname{mod}(A)$ of a finite-dimensional algebra A over a field k. We keep with [**7**] for the theory in this context. Here, we can define the Auslander-Reiten translate τ of a module M concretely. Namely, take a minimal projective resolution

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

of M. Apply the functor $\operatorname{Hom}_A(-, A)$ to this sequence and define $\operatorname{Tr}(M) := \operatorname{Coker}(\operatorname{Hom}_A(p_1, A))$. This gives a module over the opposite algebra of A. Hence, applying the standard duality $D = \operatorname{Hom}_k(-, k)$ on this brings us back to $\operatorname{mod}(A)$. Then, define the Auslander-Reiten translation to be

$$\tau = D \operatorname{Tr} \text{ and } \tau^{-1} = \operatorname{Tr} D.$$

This is the appropriate definition for τ which is compatible with the above general definition of it as the unique object an Auslander-Reiten sequence starts in. Indeed, in this context we can make a statement on the existence of Auslander-Reiten sequences.

THEOREM 2.21. *For any indecomposable non-projective A-module M there exists an Auslander-Reiten sequence $0 \to \tau M \to E \to M \to 0$ in $\operatorname{mod}(A)$.*

For any indecomposable non-injective A-module N there exists an Auslander-Reiten sequence $0 \to N \to F \to \tau^{-1} N \to 0$. □

One can prove this using the following Auslander-Reiten formulas which are anyway very useful.

THEOREM 2.22. *Let M and N be indecomposable A-modules. Then, there exist isomorphisms*

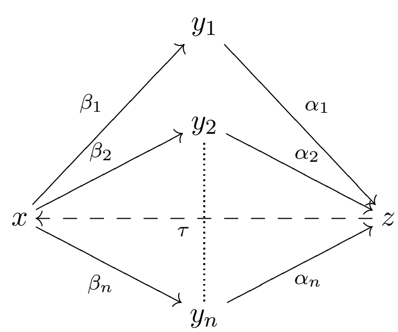
$$\mathrm{Ext}^1_A(M,N) \cong D\,\underline{\mathrm{Hom}}_A(\tau^{-1}N, M) \cong D\,\overline{\mathrm{Hom}}_A(N, \tau M).$$

Here $\overline{\mathrm{Hom}}$ is the analogue of $\underline{\mathrm{Hom}}$ defined in Section 2 in the sense that one factors out morphisms which factor through an injective module (instead of projective).

Note that in case the algebra is hereditary, the above formulas appear as (see [**2**, Corollary 5.1.2])

$$\mathrm{Ext}^1_A(M,N) \cong D\,\mathrm{Hom}_A(\tau^{-1}N, M) \cong D\,\mathrm{Hom}_A(N, \tau M).$$

As said Auslander-Reiten theory of a category is an important tool to encode information about it. This works out to a certain extend. Ideally, the Auslander-Reiten quiver tells us everything about the category as in the following case. A category $\mathcal{C}$ is said to be *standard* if $\mathrm{ind}(\mathcal{C})$ is equivalent to the *mesh category* of its Auslander-Reiten quiver. The mesh category $k(\Gamma)$ of an Auslander-Reiten quiver Γ is defined as follows. The *path category* of Γ is by definition the category whose objects are the vertices of Γ and given two vertices x,y, the k-space of morphisms from x to y is given by the k-vectorspace with basis the set of all paths from x to y. The composition of morphisms is induced from the usual composition of paths. The *mesh ideal* of a path category is the ideal generated by the elements $\sum_{i=1}^n \alpha_i\beta_i$ for all meshes



in Γ. Finally the *mesh category* $k(\Gamma)$ is the quotient category of the path category of Γ by the mesh ideal.

4. Derived categories

Following up the previous sections, we now start with an abelian category and define its derived category which turns out to be triangulated. The definition of a derived category is not particularly catchy, thus we will recall the definition briefly and focus on important notions, properties and examples we will need in this thesis.

Again, originally the concept was invented by Verdier [**84**]. Our reference here is [**58**].

4.1. The formal definition. Let $\mathcal{A}$ be an abelian category. A *complex* in $\mathcal{A}$ is a sequence of maps

$$X\colon \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. A map $\phi\colon X \to Y$ between complexes consists of maps $\phi^n\colon X^n \to Y^n$ with $d_Y^n \circ \phi^n = \phi^{n+1} \circ d_X^n$. Then, denote by $\mathcal{C}(\mathcal{A})$ the category of complexes.

For a complex X denote by ΣX or $X[1]$ the shifted complex with $(\Sigma X)^n = X^{n+1}$ and $d_{\Sigma X}^n = -d_X^{n+1}$.

A map $\phi\colon X \to Y$ is *null-homotopic* if there are maps $\rho^n\colon X^n \to Y^{n-1}$ such that $\phi^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$. The null-homotopic maps form an ideal in $\mathcal{C}(\mathcal{A})$ and the *homotopy category* $\mathcal{K}(\mathcal{A})$ is the quotient of $\mathcal{C}(\mathcal{A})$ with respect to this ideal.

Let X be a complex. For each $n \in \mathbb{Z}$ the *cohomology* is defined as

$$H^n X = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

A map $\phi\colon X \to Y$ between complexes induces a map $H^n\phi\colon H^n X \to H^n Y$ in each degree, and ϕ is called a *quasi-isomorphism* if $H^n\phi$ is an isomorphism in each degree.

Finally, the *derived category* $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathcal{K}(\mathcal{A})$ by formally inverting all quasi-isomorphisms, that is

$$\mathcal{D}(\mathcal{A}) = \mathcal{K}(\mathcal{A})[S^{-1}]$$

the localisation of $\mathcal{K}(\mathcal{A})$ with respect to the class S of all quasi-isomorphisms.

Mostly in this thesis we will be concerned with the so-called *bounded derived category* $\mathcal{D}^b(\mathcal{A})$ which is the derived category as constructed above but this time we only consider bounded complexes, i.e. complexes in

$$\mathcal{C}^b(\mathcal{A}) = \{X \in \mathcal{C}(\mathcal{A}) \mid X^n = 0 \text{ for } |n| \gg 0\}.$$

4.2. The derived category of a hereditary category. If the category $\mathcal{A}$ is hereditary, there is an explicit description of the objects and the morphisms of $\mathcal{D}(\mathcal{A})$. Namely, we have the following for objects caused by the vanishing of $\operatorname{Ext}^2(-,-)$.

LEMMA 2.23. *Let $\mathcal{A}$ be an abelian hereditary category and let X be a complex in $\mathcal{D}(\mathcal{A})$. Then, X is isomorphic to the complex*

$$\cdots \to H^{n-1}X \xrightarrow{0} H^n X \xrightarrow{0} H^{n+1}X \to \cdots .$$

□

Hence, indecomposable objects of $\mathcal{D}(\mathcal{A})$ are given by complexes

$$\cdots \to 0 \to A \to 0 \to \cdots$$

concentrated in degree n where A is indecomposable in $\mathcal{A}$. For an object A in $\mathcal{A}$ we denote its associated complex concentrated in degree 0 also by A. Note that there is a canonical embedding $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ sending an object $A \in \mathcal{A}$ to this particular complex.

The description of morphisms in case of $\mathcal{A}$ hereditary goes back to the following observation.

LEMMA 2.24. *Let $\mathcal{A}$ be an abelian category. Let A, B objects in $\mathcal{A}$ and $n \in \mathbb{Z}$. There is a canonical isomorphism*

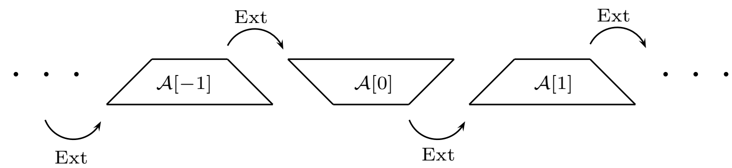
$$\operatorname{Ext}^n_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A, \Sigma^n B).$$

□

Since this is zero for $n \geq 2$, morphisms in $\mathcal{D}(\mathcal{A})$ reduce to morphisms and extensions in $\mathcal{A}$. Precisely,

$$\mathcal{D}(\mathcal{A}) = \coprod_{n \in \mathbb{Z}} \Sigma^n \mathcal{A}$$

with non-zero maps $\Sigma^i \mathcal{A} \to \Sigma^j \mathcal{A}$ only if $j - i \in \{0, 1\}$. This structure is visualised in the following figure.



In particular, this picture appears if $\mathcal{A}$ is equal to $\operatorname{mod}(k\vec{\Delta})$ where Δ is a Dynkin diagram or an extended Dynkin diagram and $k\vec{\Delta}$ is its associated path algebra (for background on representations of diagrams see Chapter 4). Namely, in this case the Auslander-Reiten quiver of $\mathcal{D}^b(\mathcal{A})$ has a particular shape.

Definition 2.25. Let $\vec{\Delta}$ be a quiver. The *repetition* of $\vec{\Delta}$ [**70**] is the translation quiver $\mathbb{Z}\Delta = \mathbb{Z}\vec{\Delta}$ defined as follows. The vertices of $\mathbb{Z}\Delta$ are the pairs (i, x) with $i \in \mathbb{Z}$ and $x \in \vec{\Delta}_0$. To each arrow $\alpha \colon x \to y$ in $\vec{\Delta}$ and each $i \in \mathbb{Z}$ there is an arrow $(i, \alpha) \colon (i, x) \to (i, y)$ and an arrow $\sigma(i, \alpha) \colon (i-1, y) \to (i, x)$. The *translation* τ is defined on $(\mathbb{Z}\Delta)_0$ via $\tau(i, x) = (i-1, x)$. The repetitive quiver does not depend on the orientation of $\vec{\Delta}$.

Given a translation quiver $\mathbb{Z}\Delta$ with translation τ, we define $\mathbb{Z}\Delta/\langle \tau^r \rangle$ for $r \in \mathbb{N}$ to be the quiver which is obtained from $\mathbb{Z}\Delta$ by identifying any vertex x of $\mathbb{Z}\Delta$ with $\tau^r(x)$ and proceeding compatibly with the arrows.

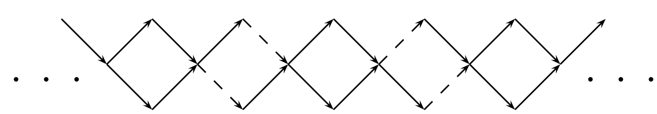
Proposition 2.26 (Happel [**46**], Section 5.6). *Let $A = k\vec{\Delta}$ be a finite-dimensional hereditary k-algebra.*

(1) *If Δ is a Dynkin diagram, then the Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod}(k\vec{\Delta}))$ is of the form $\mathbb{Z}\Delta$.*
(2) *If Δ is an extended Dynkin diagram, then the connected components of the Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod}(k\vec{\Delta}))$ are of the form $\mathbb{Z}\Delta$ and $\mathbb{Z}A_\infty/\langle \tau^r \rangle$ for some $r \in \mathbb{N}$.*

□

Here A_∞ denotes the infinite Dynkin diagram of type A which is infinite in one direction. That is, $\mathbb{Z}A_\infty/\langle \tau^r \rangle$ is nothing else than a stable tube of rank r.

Example 2.27. Let $\vec{A}_3$ be the quiver $1 \to 2 \to 3$. Then the Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod}(k\vec{A}_3))$ is of this form.



Dotted arrows stand for morphisms in the derived category coming from extensions in the module category.

In fact, this encodes all the information about $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$ as we deduce from the following statement.

Proposition 2.28 (Happel [**46**], Proposition 5.6). *Let Δ be a Dynkin diagram. Then,*

$$\mathrm{ind}(\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta})))$$

is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. □

As the translation quiver suggests, the Auslander-Reiten translation τ on the module category may be extended to the derived category. For the module category of a finite-dimensional k-algebra A of finite global dimension we can do this explicitly. Namely, by understanding each object of $\mathcal{D}^b(\mathrm{mod}(A))$ as its projective resolution $P^\bullet$ we identify $\mathcal{D}^b(\mathrm{mod}(A))$ with $\mathcal{K}^b(\mathrm{proj}(A))$ where $\mathrm{proj}(A)$ denotes the full subcategory of $\mathrm{mod}(A)$ consisting of projective modules. Then,

$$\tau(P^\bullet) = \Sigma^{-1}\nu_A(P^\bullet)$$

does the job (see Happel [**46**, Section 4.9]). Here $\nu_A = D\,\mathrm{Hom}_A(-, A)$ is the Nakayama functor which sends projective to injective modules and vice versa. Note that for a non-projective A-module understood as an object of the derived category this definition of τ coincides with the definition of τ for modules. Moreover, an indecomposable projective module is mapped to the Σ^{-1}-shift of the corresponding indecomposable injective module. Actually, τ gives an auto-equivalence on $\mathcal{D}^b(\mathrm{mod}(A))$.

4.3. The triangulated structure. As announced $\mathcal{D}(\mathcal{A})$ is a triangulated category. To be correct we remark that one has to define the triangulated structure firstly for $\mathcal{K}(\mathcal{A})$ and note then that it carries over to $\mathcal{D}(\mathcal{A})$. Anyhow, we formulate it directly for $\mathcal{D}(\mathcal{A})$. The equivalence $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is given by the shift $\Sigma = [1]$ defined above. Concerning triangles we state exemplarily how we extend a morphism $\alpha\colon X \to Y$ of complexes to a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X.$$

In fact, the exact triangles are just the triangles which are isomorphic to triangles of this particular shape. Namely, let Z be the complex with $Z^n = X^{n+1} \amalg Y^n$ and $d_Z^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ \alpha^{n+1} & d_Y^n \end{pmatrix}$. Then, the triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is degreewise defined by

$$X^n \xrightarrow{\alpha^n} Y^n \xrightarrow{\binom{0}{\mathrm{id}}} X^{n+1} \amalg Y^n \xrightarrow{(-\mathrm{id}\;\;0)} X^{n+1}.$$

To verify the axioms (TR1) to (TR4) see [**84**, II.1.3.2].

The same construction turns the bounded derived category $\mathcal{D}^b(\mathcal{A})$ into a triangulated category.

Finally, it is important to emphasise that the triangulated structure of $\mathcal{D}(\mathcal{A})$ is again related to the exact structure of $\mathcal{A}$. Namely, the canonical embedding $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ sends every short exact sequence

$$E\colon 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

in $\mathcal{A}$ to an exact triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ in $\mathcal{D}(\mathcal{A})$ where γ is the morphism induced by E under $\mathrm{Ext}^1_{\mathcal{A}}(C, A) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(C, \Sigma A)$. By the way, this also induces a *long exact sequence in cohomology*

$$\ldots \to H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to H^{n+1}(B) \to \ldots$$

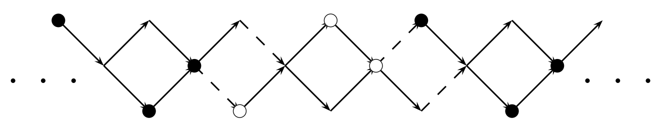
in $\mathcal{A}$.

For this reason we can compare thick subcategories of the triangulated category $\mathcal{D}^b(\mathcal{A})$ to wide subcategories of the abelian category $\mathcal{A}$.

THEOREM 2.29 (Brüning [**27**]). *Let $\mathcal{A}$ be a hereditary abelian category. The assignments $\mathcal{S} \mapsto \{H^0C \mid C \in \mathcal{S}\}$ and $\mathcal{W} \mapsto \{C \in \mathcal{D}^b(\mathcal{A}) \mid H^nC \in \mathcal{W}\ \forall n \in \mathbb{Z}\}$ induce a bijective correspondence between*

- *the class of thick subcategories of $\mathcal{D}^b(\mathcal{A})$, and*
- *the class of wide subcategories of $\mathcal{A}$.* □

EXAMPLE 2.30. Let $\vec{A}_3$ be as above. The following figure pictures a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$ (black and white dots indicating the indecomposable objects of the subcategory) together with its corresponding wide subcategory of $\mathrm{mod}(k\vec{A}_3)$ (only white dots).



CHAPTER 3

Orbit categories

Many of the categories we are interested in are equivalent to orbit categories of triangulated categories. Therefore, we discuss them in this chapter, in particular the question under which circumstances they are triangulated and if they are, their triangulated structure. The general reference for all this is [**56**].

If there is a triangulated structure on the orbit category, it makes sense to ask for thick subcategories. Hence, at the end of this chapter we state one of the main theorems of this thesis which explains the thick subcategories of the orbit category in terms of the original triangulated category. This is essential for the later classification.

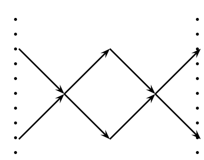
Definition 3.1. Let $\mathcal{T}$ be an additive category, let $F\colon \mathcal{T} \to \mathcal{T}$ be an automorphism and let $\langle F \rangle$ be the group of automorphisms generated by F. The *orbit category* $\mathcal{T}/F = \mathcal{T}/\langle F \rangle$ has the same objects as $\mathcal{T}$ and the morphisms $X \to Y$ are in bijection to

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^n Y).$$

The composition of morphisms is defined in a natural way (compare [**31**]).

To illustrate this definition we consider an example already known from above.

Example 3.2. Let $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$ with $\vec{A}_3$ as above. Let F be $\tau^2\colon \mathcal{D}^b(\mathrm{mod}(k\vec{A}_3)) \to \mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$. The definition of morphisms in the orbit category makes objects lying in the same orbit under F isomorphic. Therefore, we identify each object X in $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$ with $\tau^2(X)$. Thus, $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))/\tau^2$ would look like



where we understand vertices on the dotted lines as identified. By the way, this is the translation quiver $\mathbb{Z}A_3/\langle\tau^2\rangle$ where τ denotes also the translation on the repetition $\mathbb{Z}A_3$.

Now let $\mathcal{T}$ be triangulated. Is $\mathcal{T}/F$ then triangulated, too? This is not at all obvious and not true in all cases. We would like to endow the orbit category with a triangulated structure such that the projection $\pi\colon \mathcal{T} \to \mathcal{T}/F$

is a *triangle functor*, that is a functor preserving exact triangles. Here is the problem with the triangulated structure of $\mathcal{T}/F$. Let $u\colon X \to Y$ be a morphism in $\mathcal{T}/F$. This is usually a sum of morphisms $u_1, \ldots, u_N$ in $\mathcal{T}$ with $u_i\colon X \to F^{n_i}Y$. We would like to extend this to a triangle in the orbit in such way that it is compatible with how we would do it in $\mathcal{T}$. But u does not just *lift* to a morphism in $\mathcal{T}$ and hence it is not obvious how we can use the triangulated structure of $\mathcal{T}$ to define one in the orbit category.

In his article on triangulated orbit categories [**56**] Keller introduces a framework, assuming some conditions on $\mathcal{T}$ and F, in which we get that $\mathcal{T}/F$ is triangulated. In fact, all the categories we study do fulfil these conditions.

Next to those, a prominent example of an orbit category which is triangulated is the cluster category

$$\mathcal{C}_{\vec{\Delta}} = \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\tau^{-1} \circ \Sigma$$

associated to a quiver $\vec{\Delta}$ without oriented cycles. We will discuss this in Chapter 13.

THEOREM 3.3 (Keller [**56**]). *Let $\mathcal{H}$ be a connected hereditary abelian category admitting a tilting object. Let $\mathcal{T} = \mathcal{D}^b(\mathcal{H})$, let $F\colon \mathcal{T} \to \mathcal{T}$ be an automorphism, and let $\Sigma\colon \mathcal{T} \to \mathcal{T}$ be the shift. Assume that the following hypotheses hold:*

- *For each indecomposable U of $\mathcal{H}$ there are only finitely many integers i such that F^iU lies in $\mathcal{H}$.*
- *There is an integer $N \geq 0$ such that the $\langle F\rangle$-orbit of each indecomposable of $\mathcal{T}$ contains an object $\Sigma^n U$ for some $0 \leq n \leq N$ and some indecomposable object U of $\mathcal{H}$.*

Then, the orbit category $\mathcal{T}/F$ admits a triangulated structure such that the projection $\mathcal{T} \to \mathcal{T}/F$ is triangulated.

We recall the idea of Keller's proof since we will need some aspects of it further down.

PROOF. The idea is that we embed the orbit category into a bigger category which is triangulated. Then, we extend a morphism in the orbit category to a triangle in the ambient category and show that the extension also lies in the orbit category.

The ambient triangulated category is given by the derived category of a differential graded (dg) category. For the basics about dg categories we refer to [**57**].

Fix a tilting object T of $\mathcal{H}$ and let $A = \mathrm{End}(T)$ be the endomorphism algebra. Then $\mathcal{D}^b(\mathrm{mod}(A))$ is triangle equivalent to $\mathcal{D}^b(\mathcal{H})$. Let $\mathcal{A} = \mathcal{C}^b_{dg}(\mathrm{proj}(A))$ be the dg category of bounded complexes of finitely generated projective A-modules. We assume that F is a standard equivalence, i.e. it is isomorphic to the derived tensor product by a complex of A-A-bimodules and this defines a dg functor F as well. Let $\mathcal{B}$ be the dg orbit category of $\mathcal{A}$ with respect to F. This yields

$$\mathcal{D}(\mathrm{Mod}(A)) \cong \mathcal{D}\mathcal{A} \quad \text{and} \quad H^0\mathcal{B} \cong \mathcal{D}^b(\mathrm{mod}(A))/F.$$

Now let $\mathcal{M}$ be the triangulated subcategory of $\mathcal{DB}$ generated by the representable functors $\mathcal{B}(-,X) =: \hat{X}$ for $X \in \mathcal{B}$. Embed the orbit category

$$\mathcal{D}^b(\mathrm{mod}(A))/F \cong H^0\mathcal{B} \hookrightarrow \mathcal{M} \subseteq \mathcal{DB},\ X \mapsto \hat{X}.$$

Consider the projection $\pi\colon \mathcal{A} \to \mathcal{B}$, the restriction

$$\pi_*\colon \mathcal{DB} \to \mathcal{D}(\mathrm{Mod}(A))$$

along π and the left adjoint $\pi^*\colon \mathcal{D}(\mathrm{Mod}(A)) \to \mathcal{DB}$ to π_*.

Then, π^* restricts to the canonical projection

$$\mathcal{D}^b(\mathrm{mod}(A)) \to \mathcal{D}^b(\mathrm{mod}(A))/F.$$

Hence, a morphism in $\mathcal{D}^b(\mathrm{mod}(A))/F$ is of the form $\pi^*\hat{X} \to \pi^*\hat{Y}$ where $X, Y \in \mathcal{A}$ and $\hat{X} = \mathcal{A}(-,X)$. Extend this to a triangle in $\mathcal{M}$ and apply π_* to this triangle. We get a triangle

$$\pi_*\pi^*(\hat{X}) \to \pi_*\pi^*(\hat{Y}) \to \pi_*(E) \to \Sigma\pi_*\pi^*(\hat{X})$$

in $\mathcal{D}(\mathrm{Mod}(A))$. For the elements of the triangle we have

$$\begin{aligned} \pi_*\pi^*(\hat{X}) &\cong \pi_*(\mathcal{B}(-,\pi(X))) = \mathcal{B}(\pi(-),\pi(X)) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \mathcal{A}(F^n-,X) = \bigoplus_{n\in\mathbb{Z}} F^n(\hat{X}). \end{aligned}$$

By the first assumption of the theorem these objects lie in $\mathcal{D}(\mathrm{mod}(A))$ and so does $\pi_*(E)$. Using this finiteness property together with the second assumption of the theorem, we can show that $\pi_*(E)$ is a sum of finitely many $\langle F\rangle$-orbits of shifted indecomposables $Z_1,\dots,Z_m$ of $\mathcal{H}$. The adjoint of the inclusion $Z := \bigoplus_{i=1}^m Z_i \hookrightarrow \pi_*(E)$ yields an isomorphism $\pi^*(Z) \cong E$ where $Z \in \mathcal{D}^b(\mathrm{mod}(A))$. □

Given such a triangulated orbit category of the above form, it is evident that the thick subcategories of the orbit category $\mathcal{T}/F$ are somehow related to the thick subcategories of $\mathcal{T}$ and since the projection $\pi\colon \mathcal{T} \to \mathcal{T}/F$ is a triangle functor and also crucial for the definition of the orbit category, it is a good guess to use this functor for a correspondence. Clearly, if $\mathcal{S}$ is a thick subcategory of $\mathcal{T}/F$, then the preimage $\pi^{-1}(\mathcal{S})$ is immediately seen to be thick in $\mathcal{T}$ since π is a triangle functor. What we can also say about $\pi^{-1}(\mathcal{S})$ is that it is invariant under $\langle F\rangle$. This shows that not all thick subcategories of $\mathcal{T}$ appear.

But, given a thick subcategory $\mathcal{S}$ of $\mathcal{T}$ which is invariant under $\langle F\rangle$, it is not that easy to show that $\pi(\mathcal{S})$ is thick in $\mathcal{T}/F$. Here the same problems show up as discussed above for the triangulated structure. Nevertheless, it is possible and stated and proven in the sequel.

THEOREM 3.4. *Let A be a hereditary k-algebra. Let $\Phi\colon \mathcal{D}^b(\mathrm{mod}(A)) \to \mathcal{D}^b(\mathrm{mod}(A))$ be an automorphism such that the hypotheses of Keller's Theorem hold. Then, the canonical projection $\pi\colon \mathcal{D}^b(\mathrm{mod}(A)) \to \mathcal{D}^b(\mathrm{mod}(A))/\Phi$ induces a bijective correspondence between*

- *the set of thick $\langle\Phi\rangle$-invariant subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$, and*
- *the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))/\Phi$.*

Proof. Let $\mathcal{S}$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))/\Phi$. Then, as seen in the preparation of this theorem, $\pi^{-1}(\mathcal{S})$ is thick and $\langle\Phi\rangle$-invariant.

Now let $\mathcal{S}$ be a thick $\langle\Phi\rangle$-invariant subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$. We want to show that $\pi(\mathcal{S})$ is thick in the orbit category. As seen above it is not obvious how to lift a given triangle

$$\pi(X) \to \pi(Y) \to \pi(Z) \to \Sigma\pi(X)$$

with $\pi(X), \pi(Y) \in \pi(\mathcal{S})$ in $\mathcal{D}^b(\mathrm{mod}(A))/\Phi$ to a triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{D}^b(\mathrm{mod}(A))$. But as in the proof of Keller's Theorem we may apply π_* and obtain a triangle

$$\bigoplus_{n\in\mathbb{Z}} \Phi^n(X) \to \bigoplus_{n\in\mathbb{Z}} \Phi^n(Y) \to \bigoplus_{n\in\mathbb{Z}} \Phi^n(Z) \to \Sigma \bigoplus_{n\in\mathbb{Z}} \Phi^n(X)$$

in $\mathcal{D}(\mathrm{mod}(A))$. This yields a long exact sequence

$$\begin{aligned} \ldots \to H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X)) \to H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Y)) \to H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Z)) \\ \to H^{p+1}(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X)) \to H^{p+1}(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Y)) \to \ldots \end{aligned}$$

in $\mathrm{mod}(A)$. Consider the terms of this sequence.

$$\begin{aligned} H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X)) &= H^0(\Sigma^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X))) = H^0(\bigoplus_{n\in\mathbb{Z}} \Phi^n(\Sigma^p X)) \\ &= H^0(\bigoplus_{|n|<r} \Phi^n(\Sigma^p X)) \end{aligned}$$

for some $r \in \mathbb{N}$ large enough. We find this r because of the first assumption of Keller's Theorem. Since $\mathcal{S}$ is thick and $\langle\Phi\rangle$-invariant and $X \in \mathcal{S}$, $\bigoplus_{|n|<r} \Phi^n(\Sigma^p X)$ lies in $\mathcal{S}$ for each $p \in \mathbb{Z}$ and therefore, $H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X))$ lies in $H^0(\mathcal{S})$ for each $p \in \mathbb{Z}$.

By Brüning's Theorem 2.29, $H^0(\mathcal{S})$ is a wide subcategory of $\mathrm{mod}(A)$. Hence, the cokernel of

$$H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X)) \to H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Y))$$

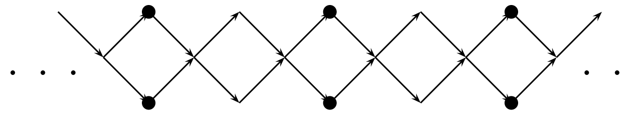
as well as the kernel of

$$H^{p+1}(\bigoplus_{n\in\mathbb{Z}} \Phi^n(X)) \to H^{p+1}(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Y))$$

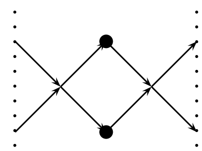
are contained in $H^0(\mathcal{S})$ for each $p \in \mathbb{Z}$. Moreover, $H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n(Z))$ is an extension of these two objects and therefore, it is also an object of $H^0(\mathcal{S})$ for each $p \in \mathbb{Z}$.

Z is a direct summand of $\bigoplus_{n\in\mathbb{Z}} \Phi^n Z$ and hence, $H^p(Z)$ is a direct summand of $H^p(\bigoplus_{n\in\mathbb{Z}} \Phi^n Z)$ for each p. Since $H^0(\mathcal{S})$ is closed under direct summands, we have that $H^p(Z) \in H^0(\mathcal{S})\ \forall p$. Applying Brüning's correspondence again, we conclude that $Z \in \mathcal{S}$ and thus $\pi(Z) \in \pi(\mathcal{S})$. □

Example 3.5. Consider again $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$ and $F = \tau^2$. Here is an example of a $\langle\tau^2\rangle$-invariant thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))$.



It corresponds to its projection into the orbit category $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_3))/\langle\tau^2\rangle$.



CHAPTER 4

Hereditary algebras

An algebra A is called *hereditary* if all submodules of projective A-modules are projective again. Equivalently, $\mathrm{mod}(A)$ is an hereditary category.

Later we want to classify the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ for a tame hereditary algebra A. This chapter collects all the needed properties of $\mathrm{mod}(A)$.

The main reference for the following definitions and facts is the preliminary chapter of [**75**].

For the time being, let A be a finite-dimensional hereditary algebra over an arbitrary field k. Let $1_A = e_1 + \ldots + e_s$ be a decomposition of the identity into pairwise orthogonal idempotents. We assume that A is basic, i.e. $Ae_i \not\cong Ae_j$ for all $i \neq j$. Denote the indecomposable projective modules Ae_i by P_i. Hence, $A \cong \oplus_{i=1}^{s} P_i$. Note that this means that we also have s corresponding simple modules denoted by $S_1, \ldots, S_s$ and injective modules denoted by $I_1, \ldots, I_s$.

1. Diagrams, dimension vectors and quadratic forms

We assign to A an oriented diagram in the following manner. Decompose $A/\operatorname{rad}(A)$ into a product $\prod_{i=1}^{s} F_i$ of division rings F_i. Concretely, $F_i \cong \mathrm{End}(P_i)$. This is possible since A is basic. Consider $\operatorname{rad}(A)/\operatorname{rad}^2(A)$ as $A/\operatorname{rad}(A)$-$A/\operatorname{rad}(A)$-bimodule and decompose it as the direct sum of F_i-F_j-bimodules ${}_iM_j$. The oriented diagram of A is given by s vertices and an arrow from a vertex i to a vertex j provided ${}_iM_j \neq 0$. Moreover, we add to such an arrow a valuation $(\dim_{F_i}({}_iM_j), \dim_{F_j}({}_iM_j))$. Denote the associated valued diagram (forgetting the orientation of the edges) by $(\Gamma_A, d_A) = (\Gamma, d)$.

This defines a bilinear form associated to A. Denote $f_i = \dim_k(F_i)$ and $d_{ij} = \dim_{F_i}({}_iM_j)$. Define a bilinear form $B = B_A$ on $\mathbb{R}^s$ as follows. For $x, y \in \mathbb{R}^s$,

$$B(x,y) = \sum_i f_i x_i y_i - \frac{1}{2} \sum_{i,j} d_{ij} f_i x_i y_j.$$

This also gives a quadratic form on $\mathbb{R}^s$ defined as

$$q(x) = B(x,x)$$

for $x \in \mathbb{R}^s$.

Both forms are of great importance for the representation theory of A. The *Grothendieck group* $K_0(A)$ of an algebra A makes A-modules approachable for q and B. Namely, let G be the free abelian group generated by the isomorphism classes $[X]$ for $X \in \mathrm{mod}(A)$. Then, the Grothendieck group is the factor group G/H where H is generated by the elements $[X]-[Y]+[Z]$ for all short exact sequences $0 \to X \to Y \to Z \to 0$. Now we have a bijection

$$\underline{\dim}\colon K_0(A) \xrightarrow{\sim} \mathbb{Z}^s, [M] \mapsto (m_1, \dots, m_s)$$

where m_j is the multiplicity of the simple module S_j as a composition factor of M.

We call $\underline{\dim}([M]) = \underline{\dim}(M)$ the *dimension vector* or the *dimension type* of M. There is another way to express $\underline{\dim}(M)$ and we will use this later in this thesis, namely

$$(\underline{\dim}(M))_i = \dim_{\mathrm{End}(P_i)} \mathrm{Hom}(P_i, M)$$

for all $i \in \{1, \dots, s\}$. See for instance [**12**, Lemma 1.7.6].

It is very convenient that in this homological context and if A is hereditary, the bilinear form B simplifies to

$$B(\underline{\dim}(M), \underline{\dim}(N)) = \dim_k \mathrm{Hom}_A(M,N) - \dim_k \mathrm{Ext}^1_A(M,N)$$

for A-modules M, N.

Having this connection in mind, we can state how the quadratic form q determines the representation type of A.

THEOREM 4.1. *Let A be a finite-dimensional hereditary algebra, let (Γ, d) be the associated valued diagram and let $q\colon \mathbb{R}^s \to \mathbb{R}$ be the associated quadratic form. Then,*

1. *A is of finite representation type $\Leftrightarrow$ q is positive definite $\Leftrightarrow$ (Γ, d) is a Dynkin diagram,*
2. *A is of tame representation type $\Leftrightarrow$ q is positive semidefinite but not definite $\Leftrightarrow$ (Γ, d) is an extended Dynkin diagram.*

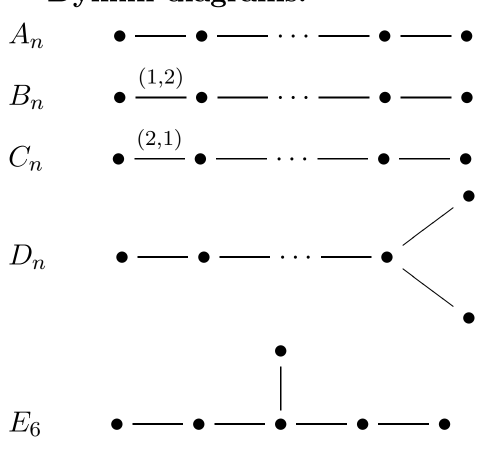
PROOF. The relation between the quadratic form and the diagram in both cases can be derived from Theorems 1 and 4 of [**22**, Chapter VI].

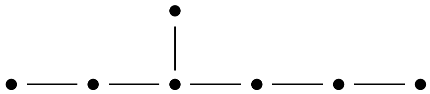
The other equivalence is for finite representation type well-known as Gabriel's Theorem. One finds a proof in [**10**, Theorem 3.6].

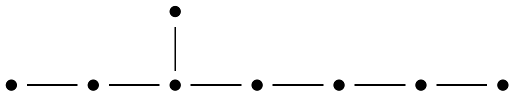
Concerning tame representation type, we may leave the property of having a semidefinite but not definite quadratic form as a definition of tame type as Ringel does in [**75**]. □

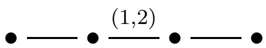
Here is a complete list of Dynkin and extended Dynkin diagrams.

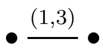
Dynkin diagrams.



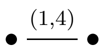
E_7 

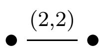
E_8 

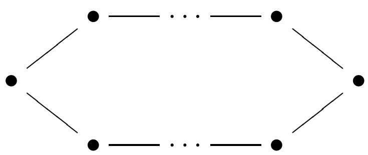
F_4 

G_2 

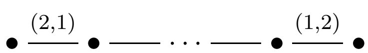
Extended Dynkin diagrams.

$\tilde{A}_{11}$ 

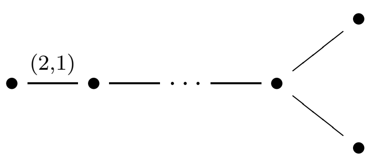
$\tilde{A}_{11}$ 

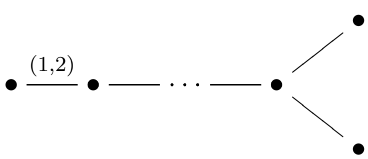
$\tilde{A}_n$ 

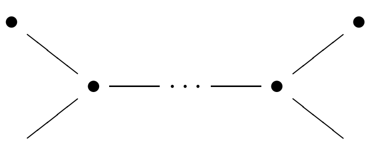
$\tilde{B}_n$ 

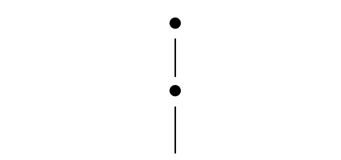
$\tilde{C}_n$ 

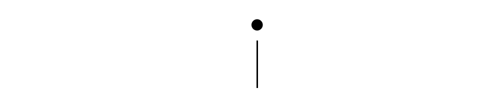
$\tilde{BC}_n$ 

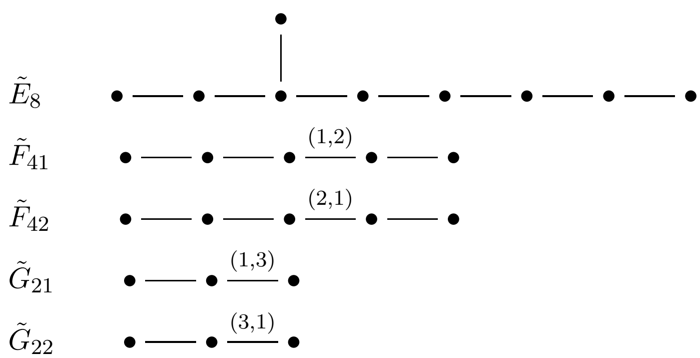
$\tilde{BD}_n$ 

$\tilde{CD}_n$ 

$\tilde{D}_n$ 

$\tilde{E}_6$ 

$\tilde{E}_7$ 



REMARK 4.2. If we assume the field k to be perfect, then Gabriel points out in [**42**, Section 8] that mod(A) is Morita equivalent to the category of finite-dimensional representations of the valued graph (Γ, d). We will talk about these representations further down.

Moreover, it is well-known that if the field is algebraically closed, then we may associate an actual quiver to the algebra instead of a valued oriented graph. In fact, each finite-dimensional hereditary basic algebra over an algebraically closed field is isomorphic to the path algebra of a finite acyclic quiver. For algebras of finite or tame representation type these quivers have underlying diagrams A_n, D_n, E_6, E_7, E_8 or $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, respectively.

2. The Weyl group and roots

In the cases described in the remark above the indecomposable modules can by completely described by their dimension type. This is done by the root system of the corresponding diagram. The definition of this needs some preparation. Note that all the definitions apply to an arbitrary hereditary algebra using its associated diagram. This is going to be important for the introduction of noncrossing partitions.

Let (Γ, d) be a valued graph and $B(-,-)$ its associated bilinear form. For every vertex $i \in \Gamma$ we define a *simple reflection* $s_i \colon \mathbb{R}^s \to \mathbb{R}^s$ by

$$s_i(x) = x - 2\frac{B(x, e_i)}{B(e_i, e_i)} e_i$$

where e_i is the vector with ith entry 1 and 0 elsewhere.

DEFINITION 4.3. The *Weyl group* $W = W_\Gamma$ associated to Γ is the group of linear transformations of $\mathbb{R}^s$ generated by the simple reflections s_i, $i \in \Gamma$.

An important element of the Weyl group is the following. A product of all simple reflections of W is called a *Coxeter element* of W. Two different Coxeter elements (coming from a different order of the simple reflections) are conjugate to each other (see [**4**, Lemma 2.6.2]). If we fix the following regulation for the order, the Coxeter element is uniquely determined. Namely, we call a numbering of the vertex set of $\vec{\Gamma}$ *admissible* if the following rule holds. If there is an arrow $j \to i$, then $j > i$. We may reorder our vertex set $\{1, \dots, s\}$ to an admissible one if necessary. Indeed, this is always possible if there are no oriented cycles. The Coxeter elements coming from two different admissible numberings are equal. See for instance [**7**, VII.4].

DEFINITION 4.4. Let $\{1, \ldots, s\}$ be an admissible numbering of $\vec{\Gamma}$. Then, we call

$$c = s_1 \cdots s_s$$

the *Coxeter transformation.*

This has an important homological meaning. As in [**36**] one could define the Auslander-Reiten translation τ via a product of *Coxeter functors* according to an admissible numbering. This explains the idea of the fact that

$$\underline{\dim}(\tau(M)) = c\,\underline{\dim}(M) \text{ and } \underline{\dim}(\tau^{-1}(N)) = c^{-1}(\underline{\dim}(N))$$

for all $M \in \mathrm{mod}(A)$ not projective and $N \in \mathrm{mod}(A)$ not injective. For the indecomposable projectives one gets $c\,\underline{\dim}(P_i) = -\underline{\dim}(I_i)$. This fits together with the picture that $\tau(P_i) = \Sigma^{-1}(I_i)$ in the derived category.

DEFINITION 4.5. A vector $x \in \mathbb{R}^s$ is called *positive* if $x_i \geq 0$ for all i. It is called *sincere* if all its components are non-zero.

A vector $x \in \mathbb{R}^s$ is called a *real root* if it is of the form $w(e_i)$ for some $w \in W$ and some $i \in \Gamma$.

An important class of indecomposable modules whose dimension vector is a real root is given by the exceptional ones.

THEOREM 4.6 (Ringel [**76**], Corollary 2). *Let X be an indecomposable A-module with $\mathrm{Ext}^1(X, X) = 0$, then $\underline{\dim}(X)$ is a real root.* □

THEOREM 4.7 (Ringel [**75**]). *Let q be a quadratic form associated to a tame hereditary algebra. Then, q vanishes precisely on a one-dimensional subspace of $\mathbb{R}^s$ generated by a sincere vector h.* □

DEFINITION 4.8. We call an indecomposable module X *homogeneous* if $\underline{\dim}(X)$ is a multiple of h.

In [**75**] one finds the precise values for h but this is not relevant for us. It is only important that h is sincere.

If the module category of the algebra is equivalent to the category of representations of its associated graph, then we can even make the following statement. For the moment denote the category of finite-dimensional representations of the valued graph $\Gamma = (\Gamma, d)$ by $\mathcal{L}(\Gamma)$. For the precise definition of this see Section 4 below in this chapter.

THEOREM 4.9 (Dlab, Ringel [**36**]). *Let A be an algebra such that* $\mathrm{mod}(A)$ *is equivalent to $\mathcal{L}(\Gamma)$ for a valued graph Γ.*

If Γ is Dynkin, then $\underline{\dim}$ induces a bijective correspondence between the positive real roots of Γ and the indecomposable A-modules.

If Γ is extended Dynkin, then, $\underline{\dim}$ induces a bijective correspondence between the positive real roots of Γ and the indecomposable non-homogeneous A-modules. □

REMARK 4.10. In the context of the theorem we also have that a positive vector v is a real root if and only if $q(v) = 1$ which is maybe the more famous definition of a real root. See for instance [**55**, Lemma 2.1].

3. Preprojective, regular and preinjective modules

The structure of the module category of a hereditary algebra A is very well understood in terms of its Auslander-Reiten quiver. This holds in particular if A is of finite or tame representation type. Recall from Chapter 2.3 the basics about Auslander-Reiten theory.

DEFINITION 4.11. An indecomposable module X of finite length is called *preprojective* if $X \cong \tau^{-n}(P)$ for some $n \in \mathbb{N}_0$ and some indecomposable projective P. Analogously, X is called *preinjective* if $X \cong \tau^{n}(I)$ for some $n \in \mathbb{N}_0$ and some indecomposable injective module I.

Finally, X is called *regular* if it is neither preprojective nor preinjective. We call an arbitrary module regular if it does not have preprojective or preinjective direct summands.

In any case, the preprojective and the preinjective modules are classified by a countable series of modules $\tau^{-n}(P_i)$ and $\tau^{n}(I_i)$ where $n \in \mathbb{N}_0$ and $i \in \{1, \dots, s\}$.

If the algebra is of finite representation type, then there are no regular modules and the preprojective and the preinjective modules coincide.

If the algebra is of tame representation type, then one knows the structure of its regular part completely.

THEOREM 4.12 (Ringel [**75**]). *Let A be a tame hereditary algebra. The regular A-modules of finite length form an exact abelian extension-closed subcategory $\mathcal{R}$ of* $\operatorname{mod}(A)$. □

Since $\mathcal{R}$ is abelian, we can consider simple objects and composition series within $\mathcal{R}$. We call a module *simple regular* if it is regular and simple in $\mathcal{R}$. Analogously, we define the *regular socle* and the *regular length* of an object in $\mathcal{R}$.

THEOREM 4.13 (Ringel [**75**]). *Let T be the set of orbits of simple regular modules under the action of τ. All these orbits are finite and all but at most three of them are one element sets.*

The regular part $\mathcal{R}$ decomposes as the direct sum of uniserial categories $\mathcal{R}_t$ where t runs through T. □

We call the smallest natural number r such that $\tau^r(R) \cong R$ for all $R \in \mathcal{R}_i$ the *rank* of the factor $\mathcal{R}_i$. For later considerations we specify the notation of the above decomposition to

$$\mathcal{R} = \coprod_{j \in J} \mathcal{H}_j \times \coprod_{i=1}^{s} \mathcal{U}_{n_i}.$$

Here $\mathcal{H}_j$ are the factors of rank 1 and $\mathcal{U}_{n_i}$ are the s (at most three) factors of rank n_i greater than 1. Hence, the index set J arises from T. In general, J is not easy to describe. But if the field k is algebraically closed, then it can be identified with the points of the projective line $P_1(k)$ over k.

Since $\mathcal{R}$ is uniserial, its indecomposable modules are up to isomorphism uniquely determined by its regular socle S and its regular length n. Denote such a module by $S_{[n]}$. Analogously, $S^{[n]}$ denotes the module with regular top S and regular length n. With this notation in hand, we can state some facts about the structure of $\mathcal{R}$. We do this in the general setting of a uniserial category.

PROPOSITION 4.14 (Chen-Krause [**30**]). *Let $\mathcal{U}$ be a uniserial category. For a morphism $\phi\colon A \to B$ between indecomposable objects, the following are equivalent:*

1. *The morphism ϕ is irreducible.*
2. *There exists a simple object S and an integer n such that ϕ is, up to isomorphism, of the form $S_{[n]} \rightarrowtail S_{[n+1]}$ or $S^{[n+1]} \twoheadrightarrow S^{[n]}$.* □

PROPOSITION 4.15 (Chen-Krause [**30**]). *Let $\mathcal{U}$ be a connected* Hom*-finite k-linear uniserial length category admitting a Serre-functor τ. Suppose that $\mathcal{U}$ has $m < \infty$ simple objects. Then, the* Ext*-quiver of $\mathcal{U}$ is of type $\tilde{A}_m$ (with cyclic orientation).* □

This means that we may enumerate the simple objects of a connected component of our category by $R_1, \dots, R_m$ where $\tau(R_i) = R_{i+1}$ for $1 \leq i < m$ and $\tau(R_m) = R_1$, and for these simples we have

$$\dim_{\mathrm{End}(R_i)} \mathrm{Ext}^1(R_i, R_{i+1}) = 1 \text{ for } 1 \leq i < m,\ \dim_{\mathrm{End}(R_m)} \mathrm{Ext}^1(R_m, R_1) = 1,$$

and there are no non-trivial extensions between simple objects in all other constellations.

Together with the above properties this shows that the Auslander-Reiten quiver of a connected component of $\mathcal{R}$ with m simple objects is a tube $\mathbb{Z}A_\infty/\langle\tau^m\rangle$.

Finally, we should think about the position of the elements in the regular part $\mathcal{R}$ within the root system. Let $S_{[j]}$ be an indecomposable element in a uniserial connected component of $\mathcal{R}$ of rank n. Then, again by [**75**] this is homogeneous if and only if j is an integer multiple of the rank n.

Above we already established the name $\mathcal{R}$ for the regular part of the category $\mathrm{mod}(A)$. In the same manner, we denote by $\mathcal{P}$ the preprojective and by $\mathcal{Q}$ the preinjective part. To complete the picture, the next proposition explains the vanishing or not vanishing of morphisms and extensions between the different parts.

PROPOSITION 4.16. *Let A be a tame hereditary algebra with $\mathcal{P}$, $\mathcal{R}$, $\mathcal{Q}$ the preprojective, regular and preinjective part of* $\mathrm{mod}(A)$*. Then,*

$$\mathrm{Hom}(\mathcal{Q}, \mathcal{P}) = 0 = \mathrm{Ext}^1(\mathcal{P}, \mathcal{Q}),$$
$$\mathrm{Hom}(\mathcal{R}, \mathcal{P}) = 0 = \mathrm{Ext}^1(\mathcal{P}, \mathcal{R}),$$
$$\mathrm{Hom}(\mathcal{Q}, \mathcal{R}) = 0 = \mathrm{Ext}^1(\mathcal{R}, \mathcal{Q}).$$

PROOF. We only prove the first assertion, the rest is dual. The proof is taken from [**32**, Section 7] for path algebras but as we will see now there are no difficulties to adapt it. Let $X \in \mathcal{Q}$ and $Y \in \mathcal{P}$ indecomposable. Since X is not projective we have $X \cong \tau^{-i}\tau^i(X)$ for $i \geq 0$. Thus,

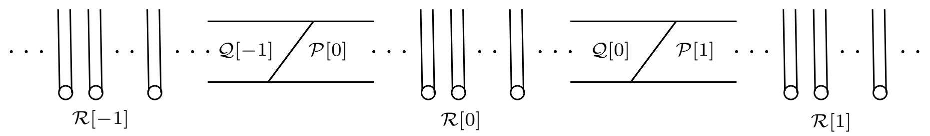
$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\tau^{-i}\tau^i(X), Y) \cong \mathrm{Hom}(\tau^i(X), \tau^i(Y))$$

for a suitable $i \geq 0$ such that $\tau^i(Y)$ is projective. If $\phi\colon \tau^i(X) \to \tau^i(Y)$ was a non-zero morphism, then $\mathrm{Im}(\phi)$ would be non-zero projective since the category is hereditary, but then $\mathrm{Im}(\phi)$ would be a direct summand of $\tau^i(X)$.

This would imply $\tau^i(X) \cong \mathrm{Im}(\phi)$ since $\tau^i(X)$ is indecomposable. This is a contradiction since $\tau^i(X)$ is not projective. Hence, $\mathrm{Hom}(X, Y) = 0$.

Also $\mathrm{Ext}^1(Y, X) \cong D\,\mathrm{Hom}(\tau^-(X), Y) = 0$. $\square$

Altogether this leads to the following picture of $\mathcal{D}^b(\mathrm{mod}(A))$.



The order of the parts in this picture comes from the existence of non-zero morphisms between the parts. The picture is supposed to suggest that the only non-zero morphisms are from left to right.

Also the picture gives an idea of what we mean by a regular object of $\mathcal{D}^b(\mathrm{mod}(A))$ from now on.

4. Representations of valued graphs and quivers

As mentioned above, in certain situations all modules of a hereditary algebra correspond to representations of valued graphs or quivers. Here is briefly the definition of the category of representations. A reference for this is [**36**].

Definition 4.17. A *valued graph* (Γ, d) is a finite set of vertices together with non-negative integers d_{ij} for all pairs $i, j \in \Gamma$ such that $d_{ii} = 0$ and subject to the condition that there exist non-zero natural numbers f_i satisfying $d_{ij} f_j = d_{ji} f_i \ \forall i, j \in \Gamma$. We call pairs i, j with $d_{ij} \neq 0$ *edges*.

An *orientation* Ω of a valued graph (Γ, d) is given by prescribing, for each edge i, j an order (indicated by an arrow $i \to j$).

A *k-modulation* $\mathcal{M}$ of a valued graph (Γ, d) is a set of division rings F_i, $i \in \Gamma$, together with an F_i-F_j-bimodule ${}_iM_j$ and an F_j-F_i-bimodule ${}_jM_i$ for all edges i, j of (Γ, d) such that

(1) each F_i contains k in its center and $[F_i : k] < \infty$ for all i,
(2) k operates centrally on each ${}_iM_j$,
(3) there are F_j-F_i-bimodule isomorphisms
$$ {}_jM_i \cong \mathrm{Hom}_{F_i}({}_iM_j, F_i) \cong \mathrm{Hom}_{F_j}({}_iM_j, F_j), $$
(4) $\dim_{F_j}({}_iM_j) = d_{ij}$.

A *k-species* $(\mathcal{M}, \Omega)$ of a valued graph (Γ, d) is a k-modulation $\mathcal{M}$ of (Γ, d) together with an admissible orientation.

Definition 4.18. A *representation* $X = (X_i, {}_j\varphi_i)$ of a k-species $(\mathcal{M}, \Omega)$ of (Γ, d) is a set of finite-dimensional right F_i-spaces X_i, $i \in \Gamma$, together with F_j-linear mappings
$$ {}_j\varphi_i \colon X_i \otimes_{F_i} {}_iM_j \to X_j $$
for all oriented edges $i \to j$.

A *morphism* $\alpha \colon X \to X'$, $X = (X_i, {}_j\varphi_i)$, $X' = (X'_i, {}_j\varphi'_i)$, of representations is defined as a set $\alpha = (\alpha_i)$ of F_i-linear mappings $\alpha_i \colon X_i \to X'_i$, $i \in \Gamma$, satisfying
$$ {}_j\varphi'_i(\alpha_i \otimes 1) = \alpha_j({}_j\varphi_i) $$
for all $i \to j$.

The representations of a k-species $(\mathcal{M}, \Omega)$ form an abelian category which we denote by $\mathcal{L}(\mathcal{M}, \Omega) =: \mathcal{L}$. In $\mathcal{L}$ each object has finite length and admits a decomposition into indecomposable objects.

Clearly, if we have an algebra A such that $\mathrm{mod}(A) \cong \mathcal{L}(\mathcal{M}, \Omega)$ for some oriented graph and some modulation, then the graph is the one we get out of the algebra as described in Section 1 at the head of this chapter.

REMARK 4.19. A quiver and its representations is a particular case of the definition above. A quiver is an oriented graph without the valuation and a representation of a quiver is a tuple of finite-dimensional vector spaces corresponding to the vertices and linear maps between these vector spaces corresponding to arrows of the quiver. The category of finite-dimensional representations of a quiver is equivalent to $\mathrm{mod}(A)$ where A is the path algebra of the quiver.

CHAPTER 5

Noncrossing partitions

The purpose of this chapter is to introduce the relevant background about noncrossing partitions and their combinatorics. Noncrossing partitions are the tool to classify thick subcategories in this thesis.

We start with an arbitrary *Coxeter group* W, i.e. a group with presentation

$$\langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$, $m_{ij} \geq 2$ for $i \neq j$ and $m_{ij} = \infty$ if there is no such relation. We call the generating set $S := \{s_1, \dots, s_n\}$ the set of *simple reflections*. Next, let $R := \{wsw^{-1} \mid w \in W, s \in S\}$ be the set of *reflections*. For $w \in W$ we define the *absolute length* $l(w)$ to be the shortest expression of w as a product of elements in R. This differs from the usual notion of length in W which is the shortest expression of W as a product of elements in S. With respect to the absolute length we define a partial order $\leq$ on W via the rule

$$v \leq w \Leftrightarrow l(v) + l(v^{-1}w) = l(w)$$

for $v, w \in W$. We call this order the *absolute order*.

A *Coxeter element* c is a product of the simple reflections in some order. The *Coxeter number* h is the order of an Coxeter element. Since different Coxeter elements are conjugate to each other, the Coxeter number does not depend on the chosen Coxeter element.

The following definition is due to Brady/Watt [**24**] and Bessis [**16**].

Definition 5.1. For a Coxeter group W and a Coxeter element c the set of *noncrossing partitions* with respect to W and c is defined as

$$\mathrm{NC}(W, c) := \{w \in W \mid \mathrm{id} \leq w \leq c\}$$

In the context of this thesis, W is given by a Weyl group W_Γ associated with a graph Γ which is in turn associated with a finite-dimensional k-algebra. We discussed this in Chapter 4. Recall that here the simple reflections are given by $s_i(x) = x - 2\frac{B(x,e_i)}{B(e_i,e_i)}e_i$ for $x \in \mathbb{R}^s$ according to the vertices i of Γ. For such a reflection group we can name the reflections in R explicitly. Namely, there is a bijective correspondence between the positive real roots and the reflections. The Bijection sends a positive real root $\beta = w(e_i)$ to s_β where

$$s_\beta(x) = x - 2\frac{B(x, \beta)}{B(\beta, \beta)}\beta$$

for $x \in \mathbb{R}^s$. Of course, for simple roots this yields $s_{e_i} = s_i$. The correspondence works out since by [**51**, Section 1.2] for a non-zero vector $\alpha \in \mathbb{R}^s$ and $w \in W$ we have

$$s_{w(\alpha)} = w s_\alpha w^{-1}$$

and hence $s_\beta = s_{w(e_i)} = w s_i w^{-1} \in R$.

For $\Gamma = A_{s-1}$ together with a certain orientation the above definition of noncrossing partitions coincides with the original definition of noncrossing partitions of the set $\{1, \dots, s\}$ due to Kreweras [**61**] which also motivates the name. In fact, this is a very useful intuitive description of the elements in $\mathrm{NC}(W_\Gamma, c)$. Next to A_n, one has something similar for Γ of type B_n and D_n. The next three sections are devoted to these cases.

1. The A_n-type

In order to motivate this description, we need to slightly change the context to actual *real reflection groups*. This is the original geometric approach to reflection groups motivated by the classification of semisimple Lie algebras.

For the definitions concerning reflection groups we keep with [**51**].

Let Φ be a finite generating set in the Euclidean space $\mathbb{R}^n$ with inner product $(x, y) = \sum_{i=1}^n x_i y_i$ for $x = (x_i), y = (y_i) \in \mathbb{R}^n$. For $\alpha \in \Phi$ define

$$s_\alpha(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$$

for $x \in \mathbb{R}^n$. This is the reflection in the hyperplane

$$H_\alpha := \{x \in \mathbb{R}^n \mid (x, \alpha) = 0\}$$

orthogonal to α. Then, Φ is called a *root system* if

- $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$, and
- $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,
- $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Then, $W = W(\Phi)$ is the subgroup of automorphisms of $\mathbb{R}^n$ generated by the reflections s_α for $\alpha \in \Phi$.

The connection from reflection groups to Coxeter groups is the following. To a Coxeter group $W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1\rangle$ one defines a Coxeter diagram with vertices corresponding to the simple reflections and edges depending on the numbers m_{ij} in a certain way (see [**51**, Chapter 2.1]). Then, the finite irreducible Coxeter groups are precisely those with Coxeter diagram of *Coxeter-Dynkin type*. These are precisely the Dynkin diagrams listed above plus the exceptional diagrams

H_3 $\quad \bullet \text{---} \bullet \overset{5}{\text{---}} \bullet,$

H_4 $\quad \bullet \text{---} \bullet \text{---} \bullet \overset{5}{\text{---}} \bullet,$

$I_2(m)$ $\quad \bullet \overset{m}{\text{---}} \bullet.$

To a Coxeter diagram Γ in turn, one can define a root system Φ such that $W(\Phi)$ is isomorphic to the original Coxeter group W. Also, this is isomorphic to the Weyl group W_Γ associated with a Dynkin diagram Γ. Note that in this geometrical context here the roots and reflections are not necessarily precisely the same as defined in Chapter 4, but the two concepts correspond to each other.

For a Coxeter group of type A_n this means concretely the following. A reference for this is [**51**, Chapter 2.10]. Let V be the subspace of $\mathbb{R}^{n+1}$ for which the coordinates sum to zero. Let Φ be the set of vectors of squared length 2. Then, Φ consists of the vectors

$$e_i - e_j \text{ for } 1 \leq i \neq j \leq n+1$$

and Φ forms a root system. For a root $\alpha = e_i - e_j \in \Phi$ the reflection s_α is the reflection in the hyperplane $H_\alpha = \{x \in \mathbb{R}^{n+1} \mid x_i = x_j\}$ or shortly $H_\alpha = \{x_i = x_j\}$. For two roots α and β, the product $s_\alpha s_\beta$ is a rotation fixing the intersection $H_\alpha \cap H_\beta$.

Let $\Pi^A(n+1)$ be the poset of intersection subspaces of the hyperplanes of the root system Φ. By the particular shape of the hyperplanes we can view these intersection subspaces as partitions w of the set $\{1, \ldots, n+1\}$ into disjoint blocks. For example for $n+1 = 6$ the intersection $\{x_1 = x_4\} \cap \{x_4 = x_5\} \cap \{x_2 = x_6\} = \{x_1 = x_4 = x_5\} \cap \{x_2 = x_6\}$ corresponds to the partition $\{1,4,5\} \cup\!\!\!\cdot\, \{2,6\} \cup\!\!\!\cdot\, \{3\}$.

Up to now, everything we defined in this section was supposed to give an understanding for the following connection. The following is actually relevant for later calculations.

We place the numbers $1, 2, \ldots, n+1$ clockwise around a circle in order ($n+1$ adjacent to 1) and we draw a chord of the circle between i and j if they are in the same block of w and no other elements strictly between them when going clockwise from i to j around the circle are also in this block. Then, $\mathrm{NC}^A(n+1)$ consists of the elements of $\Pi^A(n+1)$ in which all these chords may be drawn without crossing each other.

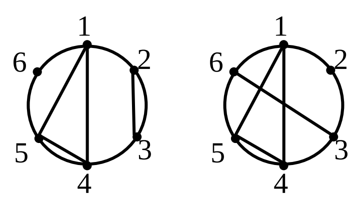


FIGURE 1. A noncrossing and a crossing partition of $\{1, \ldots, 6\}$.

From now on we fix the following numbering of A_n

$$1 \text{ ——— } 2 \text{ ——— } \cdots \text{ ——— } n-1 \text{ ——— } n$$

and according to this a Coxeter element $c = s_1 \cdots s_n$.

Let $W = W_{A_n}$ be the corresponding Weyl group. Identify the simple reflection $s_i \in W$ with the transposition $(i, i+1)$ for $1 \leq i \leq n$ in the symmetric group $\mathcal{S}_{n+1}$. This yields an isomorphism $W_{A_n} \cong \mathcal{S}_{n+1}$. One can easily check the relations for W_{A_n}.

THEOREM 5.2 (Brady [**23**], Chapter 3). *There is a bijection*

$$f\colon\ \mathrm{NC}(W_{A_n}, c) \xrightarrow{\sim} \mathrm{NC}^A(n+1).$$

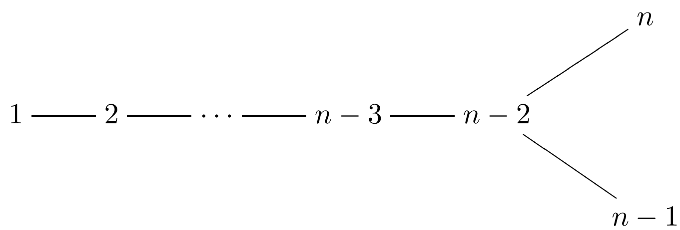
Write $w \in \mathrm{NC}(W_{A_n}, c) \subset \mathcal{S}_{n+1}$ as a product of disjoint cycles. Then, $f(w)$ is the disjoint union of blocks formed by these cycles. □

EXAMPLE 5.3. Let $n = 5$ and consider $v = (2,3)(1,4)(4,5) \in \mathcal{S}_6$. Then, one can check that $l(v) + l(v^{-1}c) = 3 + 3 = 6 = l(c)$ and hence, $v \in \mathrm{NC}(W_{A_6}, c)$. This is sent to $f(v) = \{2,3\} \cup\!\!\!\cdot\, \{1,4,5\} \cup\!\!\!\cdot\, \{6\}$ which corresponds to the noncrossing partition in the figure above.

2. The D_n-type

Thanks to Reiner and Athanasiadis [**8**], there is a analogue description for the noncrossing partitions associated with a graph of type D_n.

In this section we fix the following numbering for D_n.



Let $\Pi^D(n)$ the poset of intersection subspaces of the hyperplanes of the root system of type D_n, i.e. the integer vectors in $V = \mathbb{R}^n$ of length $\sqrt{2}$. Choose as simple roots $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_{n-1} + e_n$. Hence, the root system Φ consists of vectors $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$. Then, the hyperplanes orthogonal to these roots are of the form

$$\{x_i = \pm x_j \mid 1 \leq i < j \leq n\}.$$

Thus, we can consider the elements of $\Pi^D(n)$ as partitions w of the set $[\pm n] = \{1, 2, \ldots, n, -1, -2, \ldots, -n,\}$ into blocks such that

- if B is a block, then also $-B$ is a block,
- there is at most one zero block, i.e. a block containing both i and $-i$,
- the zero block, if present, does not consist of a single pair $\{i, -i\}$.

We call those elements *D_n-partitions*.

Now draw the numbers

$$1, \ldots, n-1, -1, \ldots, -(n-1)$$

clockwise around a circle and label its centroid by n and $-n$. Given a D_n-partition w and a block B of w, let $\operatorname{con}(B)$ denote the convex hull of the set of points labeled with the elements of B. Two distinct blocks B and B' of w are said to *cross* if $\operatorname{con}(B)$ and $\operatorname{con}(B')$ do not coincide and one of them contains a point of the other in its relative interior.

The poset $\mathrm{NC}^D(n)$ is defined as the D_n-partitions w with the property that no two blocks of w cross.

In order to make the visualisation of the D_n-partitions well-defined, we additionally label the convex hull of a non-zero block containing n or $-n$ by $+$ or $-$, respectively.

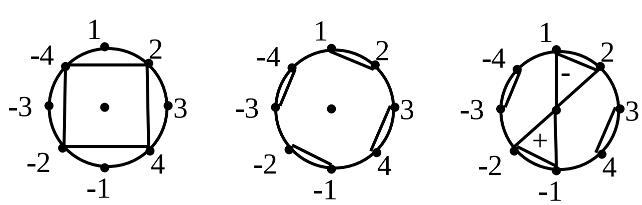


FIGURE 2. Examples for noncrossing D_5-partitions.

The Weyl group associated with D_n looks as follows. Denote by $\mathcal{S}_{2n}$ the symmetric group on $[\pm n]$. For $i \neq -j$ we write $((i,j)) = (i,j)(-i,-j)$. By identifying the simple reflections s_i in W_{D_n} with $((i,i+1))$ for $i < n$ and s_n with $((-(n-1),n))$, we get that the Weyl group W_{D_n} is isomorphic to the subgroup of $\mathcal{S}_{2n}$ generated by the reflections $((i,j))$ for $i \neq -j$.

For a cycle $z = (i_1,\ldots,i_k)$ in $\mathcal{S}_{2n}$ denote by $\bar{z}$ the cycle $(-i_1,\ldots,-i_k)$. We call $z\bar{z}$ a *paired cycle* if z and $\bar{z}$ are disjoint whereas a cycle $z = \bar{z} = (i_1,\ldots,i_k,-i_1,\ldots,-i_k)$ is called a *balanced cycle*.

One can show that each element of W_{D_n} is a product of disjoint paired and balanced cycles.

Fix a Coxeter element $c = s_1 \cdots s_n$.

THEOREM 5.4 (Athanasiadis/Reiner [**8**]). *There is a bijection*

$$f\colon\ \mathrm{NC}(W_{D_n},c) \to \mathrm{NC}^D(n).$$

For $w \in \mathrm{NC}(W_{D_n},c)$, $f(w)$ is the partition of $[\pm n]$

- *whose nonzero blocks are formed by the paired cycles of w and*
- *whose zero block is the union of the elements of all balanced cycles of w if such exist.*

The inverse $g\colon\ \mathrm{NC}^D(n) \to \mathrm{NC}(W_{D_n},c)$ maps a partition w to the product of disjoint cycles $g(w)$

- *whose paired cycles are formed by the nonzero blocks of w, each ordered with respect to the order $-1,-2,\ldots,-n,1,2,\ldots,n$ and*
- *whose balanced cycles are $(n,-n)$ and the cycle formed by the entries of the zero block of w other than n and $-n$ ordered as above, if the zero block exists.* □

3. The B_n-type

Again as in [**51**, Chapter 2.10] we choose $V = \mathbb{R}^n$ and as a root system Φ the set of all vectors of squared length 1 or 2. Then, Φ consists of the vectors $\pm e_i$ for $1 \leq i \leq n$ and $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$. Thus, the hyperplanes orthogonal to these roots are of the form $\{x_i = 0\}$ and $\{x_i = \pm x_j\}$. Hence, $\Pi^B(n)$ is defined as $\Pi^D(n)$ except that the zero block might consist of a single pair $\{i,-i\}$.

Then, we draw the numbers $1,\ldots,n,-1,\ldots,-n$ clockwise around a circle and define $\mathrm{NC}^B(n)$ to be the set of partitions in $\Pi^B(n)$ which do not cross when visualised in this circle. This description goes back to Reiner [**69**].

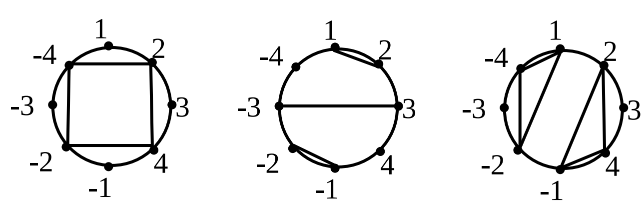


FIGURE 3. Examples for noncrossing B_4-partition

As the picture suggests, $\mathrm{NC}^B(n)$ is the same as the subset of $\mathrm{NC}^A(2n)$ consisting of those A_{2n}-partitions which are invariant under rotation by π. More generally, the following holds.

Proposition 5.5 (Reiner [**69**], Proposition 1). *Let $n, s \geq 2$ be integers. The subset of elements of* $\mathrm{NC}^A(sn)$ *invariant under rotation by* $\frac{2\pi}{s}$ *is isomorphic to* $\mathrm{NC}^B(n)$. □

It is also this relationship to the A_{2n}-case which gives a bijection between $\mathrm{NC}^B(n)$ and $\mathrm{NC}(W_{B_n}, c)$ for a suitable choice of the numbering of B_n and c. This is done in [**19**].

4. The lattice structure and the Kreweras complement

Recall that a lattice is a partially ordered set in which any two elements have a unique supremum and a unique infimum.

The set of noncrossing partitions is a partially ordered set by means of the absolute order on W. Is it also a lattice? This is quite clear for the classical cases A, B and D where one can use the alternative descriptions of $\mathrm{NC}(W, c)$ described above. For $\mathrm{NC}^A(n)$ Kreweras showed in [**61**] the lattice property. Here the meet of two partitions is the intersection and the join is the *noncrossing closure*, i.e. we join blocks of the two partitions which would otherwise cross.

For arbitrary Coxeter groups the problem was surprisingly hard to solve. Finally it was done by Brady and Watt in the finite case.

Theorem 5.6 (Brady/Watt [**23**], Theorem 7.8). *Let W be a finite real reflection group and let c be a Coxeter element. Then,* $\mathrm{NC}(W, c)$ *is a lattice.* □

An important construction in this context is the following

Definition 5.7. Let W be a Coxeter group and let c be a Coxeter element. For an element $w \in W$ the element $K^c(w) := w^{-1}c$ is called the *Kreweras complement*.

This defines an order-reversing map from the poset $\mathrm{NC}(W, c)$ into itself.
Moreover, the Kreweras complement is actually a complement.

Theorem 5.8 (Armstrong [**4**], Theorem 2.6.14). *Let W be a finite Coxeter group and let c be a Coxeter element. Let $w \in \mathrm{NC}(W, c)$. Then,*

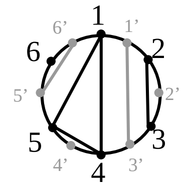
$$w \vee K^c(w) = c \text{ and } w \wedge K^c(w) = \mathrm{id}\,.$$

That is, $K^c(w)$ is the lattice complement of w in $\mathrm{NC}(W, c)$. □

In the lattice $\mathrm{NC}^A(n)$ there is an explicit construction of the Kreweras complement which is compatible with the above general definition. In fact, this is the original Kreweras complement due to Kreweras. Let $w \in \mathrm{NC}^A(n)$. Draw primed numbers $1', \ldots, n'$ clockwise around a circle so that the primed numbers interlace the unprimed numbers $1, \ldots, n$. Then, $K(w)$ is the unique maximal partition of $\{1', \ldots, n'\}$ such that $w \cup K(w)$ is a noncrossing partition of $\{1, 1', 2, 2', \ldots, n, n'\}$. Forgetting the primes, we obtain an element $K(w) \in \mathrm{NC}^A(n)$.

For example the Kreweras complement of $\{1, 4, 5\} \cup \{2, 3\} \cup \{6\}$ in $\mathrm{NC}^A(6)$ is equal to $\{1, 3\} \cup \{2\} \cup \{4\} \cup \{5, 6\}$.
This is confirmed by a quick calculation in $\mathcal{S}_6$, namely

$$((1, 4, 5)(2, 3))^{-1}(1, 2, 3, 4, 5, 6) = (1, 3)(5, 6).$$



5. Counting noncrossing partitions and the cyclic sieving phenomenon

The number of elements in $\mathrm{NC}^A(n)$ is a very famous number, the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

This is generalisable to the number of elements in $\mathrm{NC}(W, c)$ for W a finite irreducible Coxeter group. For type B_n there are $\mathrm{Cat}(B_n) := \binom{2n}{n}$ noncrossing partitions (see [**69**, Proposition 6]) and for type D_n there are $\mathrm{Cat}(D_n) := \binom{2n}{n} - \binom{2n-2}{n-1}$ noncrossing partitions (see [**8**, Theorem 1.2]).

But we cannot only count the number of all elements but also the number of certain invariant elements in $\mathrm{NC}(W, c)$. On the one hand, this is currently of great interest in combinatorics, and on the other hand, it will turn out to be relevant for the classification of thick subcategories further down in this thesis.

Independently from the whole topic of noncrossing partitions, Reiner, Stanton and White [**68**] defined the following phenomenon and identified it in many examples.

Definition 5.9. Let X be a finite set, let G be a finite cyclic group acting on X and let $f(q) \in \mathbb{N}[q]$ be a polynomial. For $g \in G$ denote by $X^g := \{x \in X \mid gx = x\}$ the fixed point set. The triple $(X, G, f(q))$ is said to exhibit the *cyclic sieving phenomenon* if for all $g \in G$

$$\#X^g = f(z_{o(g)})$$

where $o(g)$ denotes the order of g and $z_d = e^{\frac{2\pi i}{d}}$ denotes the dth root of unity.

We have such a cyclic sieving phenomenon for noncrossing partitions. In order to define the polynomial in this context we need to go a bit further back in the theory of reflection groups. Out of the classification of finite irreducible complex reflection groups W on $\mathbb{C}^n$ by Shephard and Todd [**78**], there arise certain invariants $d_1 \leq \ldots \leq d_n$ called the *degrees* of W. The exact definition of the degrees is not relevant for this thesis and can be found in [**78**]. Any real reflection groups can be regarded as a complex reflection group and hence in particular, our finite irreducible Coxeter groups are complex reflection groups and we can associate to them the following degrees. This list is taken from [**77**].

Group	degrees
A_n	$2, 3, 4, \ldots, n+1$
B_n, C_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
$I_2(m)$	$2, m$

Now let W be a finite irreducible complex reflection group with degrees $d_1, \ldots, d_n$. We state the theorem in this generality but mostly think of W as of a finite irreducible Coxeter group. Define the *W-q-Catalan number* to be

$$\mathrm{Cat}(W, q) := \prod_{i=1}^{n} \frac{[h + d_i]_q}{[d_i]_q}$$

where $[n]_q := 1 + q + \ldots + q^{n-1}$ and $h := d_n$. Note that h is the Coxeter number for the finite Coxeter groups.

Let c be a Coxeter element. This acts on $\mathrm{NC}(W, c)$ by conjugation.

THEOREM 5.10 (Bessis/Reiner [**17**], Theorem 1.1). *The triple*

$$(\mathrm{NC}(W, c), \langle c \rangle, \mathrm{Cat}(W, q))$$

exhibits the cyclic sieving phenomenon. □

In order to keep this in mind, we should translate again what this actually means, namely for $1 \le t \le h$

$$\#(\mathrm{NC}(W, c))^{c^t} = \mathrm{Cat}(W, z_{o(c^t)})$$

where

$$(\mathrm{NC}(W, c))^{c^t} = \{w \in W \mid c^t w c^{-t} = w\}.$$

Of course for $t = h$ these are all elements in $\mathrm{NC}(W, c)$ and its number is given by $\mathrm{Cat}(W, 1)$. Moreover, as one expects, $\mathrm{Cat}(W_{A_n}, 1) = C_n$.

Next, we figure out what conjugation by the Coxeter element means for the classical cases $\mathrm{NC}^A(n)$ and $\mathrm{NC}^D(n)$. Recall from Theorem 5.2 the isomorphism $f\colon \mathrm{NC}(W_{A_{n-1}}, c) \to \mathrm{NC}^A(n+1)$. As in the context of this theorem we assume that A_n has the same numbering and we consider the same Coxeter element $c = s_1 \cdots s_n$ as there.

PROPOSITION 5.11. *Let $w \in \mathrm{NC}(W_{A_n}, c)$ and let $t \in \mathbb{N}$. Then, $f(c^t w c^{-t})$ is the partition $f(w)$ rotated by $t\frac{2\pi}{n+1}$.*

PROOF. The Coxeter element is of the form

$$c = s_1 \cdots s_n = (1, 2) \cdots (n, n+1) = (1, 2, 3, \ldots, n+1).$$

We can write an element of $\mathrm{NC}(W_{A_n}, c) \subset \mathcal{S}_{n+1}$ as a product of disjoint cycles. It is sufficient to observe the conjugation by the Coxeter element of one of these cycles,

$$\begin{aligned}(1,2,\dots,n+1)&(p_1,p_2,\dots,p_k)(n+1,n,\dots,2,1)\\ &=([p_1+1],\dots,[p_k+1]),\end{aligned}$$

where $[p_j+1]=(p_j+1)\bmod(n+1)$ if $p_j+1\neq n+1$ and $[p_j+1]=n+1$ otherwise, and we see that this is just the clockwise rotation. □

Remark 5.12. T. Araya [**3**] also observes this phenomenon concerning rotation of 'non-crossing spanning trees' which are in correspondence with exceptional sequences of $\mathcal{D}^b(\mathrm{mod}(k\vec{A}_n))$.

In the D_n-case, simple rotation is not enough. Therefore, we introduce the following action on $\mathrm{NC}^D(n)$.

Definition 5.13. Let $w\in\mathrm{NC}^D(n)$. Denote by $\rho\colon\ \mathrm{NC}^D(n)\to\mathrm{NC}^D(n)$ the rotation of w by $\frac{\pi}{n-1}$ and denote by $\sigma\colon\ \mathrm{NC}^D(n)\to\mathrm{NC}^D(n)$ the following operation: If $w\in\mathrm{NC}^D(n)$ contains a non-zero block containing n or $-n$, the visualisation of this block is labeled by a sign $+$ or $-$. Then, σ changes this sign. On blocks not containing n or $-n$ or on zero-blocks σ acts like the identity.

Now let $f\colon\ \mathrm{NC}(W_{D_n},c)\to\mathrm{NC}^D(n)$ be as in Theorem 5.4 where D_n is numbered as there and where $c=s_1\cdots s_n$.

Lemma 5.14. *Let* $w\in\mathrm{NC}(W_{D_n},c)$ *and let* $f(w)$ *be the corresponding element in* $\mathrm{NC}^D(n)$. *Then,*

$$f(cwc^{-1})=(\sigma\rho)(f(w)).$$

Proof. The Coxeter element is given by

$$c=s_1\cdots s_n=(1,2,\dots,n-1,-1,\dots,-(n-1))(n,-n).$$

Let $w\in\mathrm{NC}(W_{\vec{D}_n},c)$. Then, $f(w)$ is a disjoint union of blocks corresponding to a disjoint product w of cycles in $\mathcal{S}_{2n}$. Thus, it is sufficient to consider the blocks separately.

If n is not contained in the block, conjugation by the Coxeter element of the corresponding cycle yields as in the A_n-case a rotation by $\frac{2\pi}{2(n-1)}=\frac{\pi}{n-1}$.

If n is contained in a non-zero block, this block corresponds to one part $(i_1,\dots,i_k)$ of a paired cycle with $i_k=n$ and one can easily check that

$$c(i_1,\dots,i_k)c^{-1}$$

is given by the 'rotated cycle' in which additionally n is replaced by $-n$.

If we have a zero-block, this corresponds to the balanced cycles $(n,-n)$ and the cycle formed by the entries of the zero block other than n and $-n$. The latter conjugated by the Coxeter element is again given by the 'rotated cycle' and

$$c(n,-n)c^{-1}=(n,-n).$$

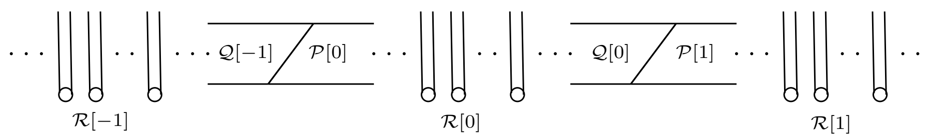
Hence, we get the 'rotated zero-block'. □

CHAPTER 6

Thick subcategories for tame hereditary algebras

In this chapter we classify the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ where A is a hereditary algebra over a field k of finite or of tame representation type.

Recall from Chapters 4 and 2.4 the shape of $\mathcal{D}^b(\mathrm{mod}(A))$.



Moreover, recall from Chapter 2 the two notions of subcategories in this context, namely the *wide subcategories* of $\mathrm{mod}(A)$ and the *thick subcategories* of $\mathcal{D}^b(\mathrm{mod}(A))$. By Theorem 2.29 they correspond to each other and therefore, we are allowed to jump between the two concepts if this is convenient.

Our classification is divided into two parts. First we present the classification of the thick subcategories generated by an exceptional sequence.

DEFINITION 6.1. An A-module X is called *exceptional* if X is indecomposable and $\mathrm{Ext}^1(X, X) = 0$. A sequence $(X_1, X_2, \dots, X_r)$ of A-modules is called *exceptional* if each X_i is exceptional and

$$\mathrm{Hom}_A(X_j, X_i) = 0 = \mathrm{Ext}^1_A(X_j, X_i) \quad \text{for all} \quad i < j.$$

Such a sequence is called *complete* if r equals the number of simple A-modules.

Thick subcategories generated by exceptional sequences are classified via noncrossing partitions. We denote the set of these thick subcategories by $\mathrm{Th}_{\mathrm{exc}}(A)$.

But not each thick subcategory is of this form. For example, let A be the Kronecker algebra. It is well-known that its regular part $\mathcal{R}$ decomposes into a direct sum of uniserial categories of rank one. By Theorem 4.12, $\mathcal{R}$ is a wide subcategory of $\mathrm{mod}(A)$ for each tame hereditary algebra and hence, it induces a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$. But none of the objects in $\mathcal{R}$ is exceptional and thus, it is certainly not generated by an exceptional sequence.

Hence, in a second step, we classify the thick subcategories of the regular part $\mathcal{R}$.

Finally, we study how the two classes fit together and state the classification theorem about all thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$.

1. Thick subcategories and noncrossing partitions

First of all, a remark on the history of this classification theorem. The first formulation is due to Ingalls and Thomas [**53**] in the context where A is a path algebra of a Dynkin or an extended Dynkin quiver. A generalisation of this is given by Igusa, Schiffler and Thomas in [**52**] for path algebras of arbitrary quivers over an algebraically closed field using braid group actions. The statement also holds without the assumption that the field is algebraically closed (see Krause [**59**]).

THEOREM 6.2 (Ingalls-Thomas, Igusa-Schiffler-Thomas, Krause). *Let A be a hereditary k-algebra. Let $\vec{\Gamma}$ be the associated valued oriented graph, let $W = W_\Gamma$ be the associated Weyl group and let $\{1, \dots, n\}$ be an admissible numbering of $\vec{\Gamma}$. Let c be the Coxeter transformation $c = s_1 \cdots s_n$. There is an order preserving bijective correspondence between*

- *the set $\mathrm{Th}_{\mathrm{exc}}(A)$ of thick subcategories $\mathcal{S}$ of $\mathcal{D}^b(\mathrm{mod}(A))$ generated by an exceptional sequence in $\mathrm{mod}(A)$, and*
- *the set $NC(W, c)$ of noncrossing partions.*

PROOF. For a detailed proof see [**59**]. □

We state some explanations concerning the above theorem.

REMARK 6.3. (1) Note that originally, the formulation of Ingalls and Thomas was a bijection between the noncrossing partitions and the wide subcategories of $\mathrm{mod}(A)$ but as mentioned these correspond to the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$. However, this explains why we speak of exceptional sequences in $\mathrm{mod}(A)$ and also this explains how an exceptional sequence in $\mathrm{mod}(A)$ generates a thick subcategory in $\mathcal{D}^b(\mathrm{mod}(A))$.

(2) The correspondence of the theorem is given as follows. We denote it by cox. Let $\mathcal{S}$ be a thick subcategory generated by an exceptional sequence $E = (E_1, \dots, E_r)$. Then

$$\mathrm{cox}(\mathcal{S}) := s_{E_1} \cdots s_{E_r}$$

where $s_{E_i} = s_{\underline{\dim}(E_i)}$. Recall from Chapter 5 that there is a bijection between reflections s_β and positive real roots β. Since E_i is exceptional, $\underline{\dim}(E_i)$ is a positive real root by Theorem 4.6 and hence it makes sense to define cox like this.

(3) Note that if $\{1, \dots, n\}$ is an admissible numbering, then the sequence of simple modules $(S_1, \dots, S_n)$ in this very order is an exceptional sequence. Thus, the whole category $\mathcal{D}^b(\mathrm{mod}(A))$ which is generated by this sequence corresponds to the Coxeter transformation c.

REMARK 6.4. If the algebra is of finite representation type, then one can easily see that each thick subcategory is generated by an exceptional sequence. Hence, for the finite type the classification is done at this point.

Observe that the theorem implies that for most of the Weyl groups which are not finite the poset of noncrossing partitions does not form a lattice with respect to the absolute order since $\mathrm{Th}_{\mathrm{exc}}(A)$ does not and since cox preserves the order of the posets. For instance, consider a tame hereditary algebra whose regular part admits a tube of rank 2 with regular simple modules R_1 and R_2. These are exceptional modules and hence $\text{Thick}(R_1)$,$\text{Thick}(R_2) \in \text{Th}_{\text{exc}}(A)$. But $\text{Thick}(R_1) \vee \text{Thick}(R_2) = \text{Thick}(R_1, R_2)$ is the whole tube and we cannot find an exceptional sequence in it.

On the other hand, for finite Weyl groups it confirms the lattice property for $\text{NC}(W, c)$ since here $\text{Th}_{\text{exc}}(A)$ coincides with the set of all thick subcategories and this is a lattice.

An important construction in this context is the Kreweras complement introduced in Chapter 5. The next proposition is concerned with the corresponding operation on the level of thick subcategories.

Proposition 6.5. *Let A be a hereditary k-algebra, let* cox*, c, W be as in Theorem 6.2. Let $\mathcal{S}$ be a thick subcategory which is generated by an exceptional sequence. Then,*

$$K^c(\text{cox}(\mathcal{S})) = \text{cox}({}^{\perp}\mathcal{S}).$$

□

This is exactly what one would expect since both the Kreweras complement and the perpendicular category are lattice complements as soon as the respective poset forms a lattice.

The proof of the above fact can be deduced from the definition of the Kreweras complement and the following important observation.

Proposition 6.6 (Krause [**59**], Proposition 6.6). *Let A be a hereditary k-algebra. Let $\mathcal{S}$ be a thick subcategory of $\mathcal{D}^b(\text{mod}(A))$ which is generated by an exceptional sequence $(E_1, \ldots, E_r)$ in* $\text{mod}(A)$*. Then, one can complete this to a complete exceptional sequence $(E_1, \ldots, E_n)$ such that ${}^{\perp}\mathcal{S} = \text{Thick}(E_{r+1}, \ldots, E_n)$.* □

Braid group actions. The key to the proof of Theorem 6.2 is the introduction of braid group actions on sequences of reflections and on exceptional sequences. The *braid group on n strands* has the following presentation

$$B_n := \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_r\sigma_s = \sigma_s\sigma_r \rangle$$

where $1 \leq i \leq n-2$ and $|r - s| \geq 2$. On the set of isomorphism classes of complete exceptional sequences of $\text{mod}(A)$ this acts via

$$\sigma_i((X_1, \ldots, X_n)) = (X_1, \ldots, X_{i-1}, L, X_i, X_{i+2}, \ldots, X_n)$$

for all $1 \leq i \leq n-1$ where L is the unique module making the new sequence exceptional.

On the set of sequences of reflections $(x_1, \ldots, x_n)$ such that $c = x_1 \cdots x_n$, B_n acts like this:

$$\sigma_i((x_1, \ldots, x_n)) = (x_1, \ldots, x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, x_{i+2}, \ldots, x_n)$$

for all i.

Theorem 6.7 (Crawley-Boevey [**33**], Ringel [**76**], Igusa-Schiffler [**52**]). *Let A be a connected hereditary k-algebra with simple modules $S_1, \ldots, S_n$ forming an exceptional sequence $(S_1, \ldots, S_n)$. Let W be the associated Weyl group and fix a Coxeter element $c = s_1 \cdots s_n$. Then, the braid group B_n on n strings acts transitively on*

- *the isomorphism classes of complete exceptional sequences*
$$(X_1, \dots, X_n)$$
in $\mathrm{mod}(A)$*, and*
- *the sequences* $(x_1, \dots, x_n)$ *of reflections such that* $c = x_1 \cdots x_n$*.*

Moreover, $\sigma(X_1, \dots, X_n) = (Y_1, \dots, Y_n)$ *implies*
$$\sigma(s_{X_1}, \dots, s_{X_n}) = (s_{Y_1}, \dots, s_{Y_n})$$
for all $\sigma \in B_n$*.* □

This means that we get all possible factorisations of the Coxeter element into reflections by starting with $c = s_1 \cdots s_n$ and applying the braid group action. In particular, it gives us all noncrossing partitions as prefixes of these factorisations.

Moreover, it shows how to compute the inverse of the bijection cox. Namely, given $w \in \mathrm{NC}(W, c)$. Write this as a product $x_1 \cdots x_r$ of reflections. Since $w \in \mathrm{NC}(W, c)$, we can extend this to a product of n reflections $x_1 \cdots x_r x_{r+1} \cdots x_n$ factorising c. By the theorem, there is $\sigma \in B_n$ such that $\sigma((s_1, \dots, s_n)) = (x_1, \dots, x_n)$. Apply the same σ to the exceptional sequence $(S_1, \dots, S_n)$ and denote the resulting exceptional sequence by $(X_1, \dots, X_n)$. Setting $\mathcal{S} := \mathrm{Thick}(X_1, \dots, X_r)$ we have
$$\mathrm{cox}(\mathcal{S}) = s_{X_1} \cdots s_{X_r} = x_1 \cdots x_r = w.$$

2. Thick subcategories of the regular part

Let $\mathcal{R}$ be the subcategory of regular modules of $\mathrm{mod}(A)$. Recall from Chapter 4.3 that this is of the form
$$\mathcal{R} = \coprod_{j \in J} \mathcal{H}_j \times \coprod_{i=1}^{s} \mathcal{U}_{n_i}$$
where all the notations are as described there.

By Theorem 4.12 the regular part $\mathcal{R}$ is a wide subcategory of $\mathrm{mod}(A)$ and hence it is an abelian category. Therefore, we may consider wide subcategories of $\mathcal{R}$ and these are in turn wide subcategories of $\mathrm{mod}(A)$.

We call an object in $\mathcal{D}^b(\mathrm{mod}(A))$ *regular* if it is isomorphic to some shift of a regular object in $\mathrm{mod}(A)$. By Theorem 2.29 and again 4.12, the regular objects of $\mathcal{D}^b(\mathrm{mod}(A))$ form a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$. We denote the set of thick subcategories of this thick subcategory by $\mathrm{Th}_{\mathrm{reg}}(A)$. Equivalently, this is the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ consisting only of regular objects. Moreover, it corresponds to the set of wide subcategories of $\mathcal{R}$.

The aim is to understand $\mathrm{Th}_{\mathrm{reg}}(A)$. In order to do so, we classify the wide subcategories of $\mathcal{R}$. We start with the wide subcategories of the uniserial direct factors of $\mathcal{R}$ separately.

All the following is loosely based on Dichev's classification [**34**] of wide subcategories of the category of nilpotent representations of $\tilde{A}_n$.

Definition 6.8. An indecomposable object X in an abelian category is called a *brick* if its endomorphism ring $\mathrm{End}(X)$ is a division ring. Two bricks X and X' are called *orthogonal* if $\mathrm{Hom}(X, X') = 0 = \mathrm{Hom}(X', X)$.

Definition 6.9. Place the numbers $1, \ldots, n$ clockwise around a circle. An *arc* on the circle is a pair $a = (i, j)$, $i, j \in \{1, \ldots, n\}$. Corresponding to an arc (i, j) we draw an arc clockwise around the circle from i to j. For an arc $a = (i, j)$ we denote by $s(a) = i$ its start and by $e(a) = j$ its end point. Note that (i, j) is different from (j, i). If $i = j$ we make the convention that (i, i) corresponds to an arc which circles the circle once.

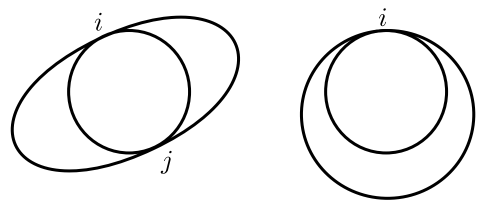


Figure 1. Arcs (i, j), (j, i) and (i, i).

We say that two arcs *cross* if the corresponding arcs on the circle intersect whereas we make the following convention concerning the coincidence in one point. Two arcs a and b are not regarded as crossing if $s(a) = e(b)$ or vice versa (and if they do not cross anywhere else), but they are crossing if $s(a) = s(b)$ or $e(a) = e(b)$.

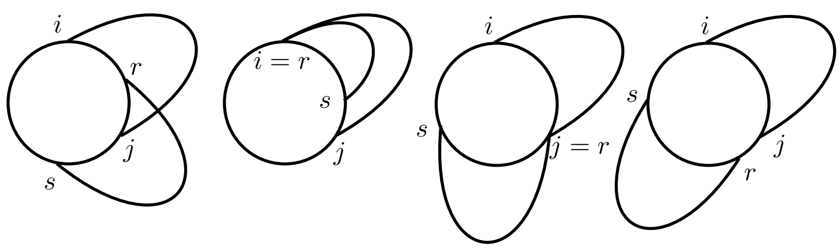


Figure 2. The arcs (i, j), (r, s) in the figure cross, cross, do not cross, do not cross (from left to right).

Then, the set of *noncrossing arcs on a circle with n points* is by definition the set $\mathrm{NA}(n)$ of families $(a_1, \ldots, a_r)$ of arcs which pairwise do not cross.

By definition the empty family consisting of no arcs at all is an element of $\mathrm{NA}(n)$.

Proposition 6.10. *Let $\mathcal{U}_n$ be a connected uniserial length category with n simple objects $R_1, \ldots, R_n$. There is a bijective correspondence between*

1. *the set of wide subcategories of $\mathcal{U}_n$,*
2. *the set of families of pairwise orthogonal bricks in $\mathcal{U}_n$, and*
3. *the set $\mathrm{NA}(n)$ of noncrossing arcs on a circle with n points.*

Proof. First note that the subcategory $\{0\}$ corresponds to an empty family of bricks which corresponds to the empty family of noncrossing arcs. Now we can assume that all objects are non-zero or non-empty, respectively.

We begin with the correspondence of the first two sets. Let $\mathcal{C}$ be a wide subcategory of $\mathcal{U}_n =: \mathcal{U}$. Let $(E_1, \dots, E_r)$ be a complete family of simple objects in $\mathcal{C}$. We claim that this gives us a family of pairwise orthogonal bricks. Since $\mathcal{C}$ is closed under kernels and cokernels, we can argue as in Schur's lemma: If $f\colon S \to S'$ is a non-zero morphism between two simple objects in $\mathcal{C}$, then $\mathrm{Ker}(f), \mathrm{Coker}(f), \mathrm{Im}(f)$ belong to $\mathcal{C}$. If f is not an isomorphism, then either the kernel is a proper non-zero subobject of S or the image is a proper non-zero subobject of S'. But then, S or S' would not be simple, a contradiction. Hence, the simple objects are pairwise orthogonal. The argument above can also be used if $S = S'$, i.e. if S is simple, there are no non-invertible endomorphisms of S and hence, $\mathrm{End}(S)$ is a division ring.

Vice versa, if $(E_1, \dots, E_r)$ is a family of pairwise orthogonal bricks, we assign to it the wide subcategory $\mathrm{Thick}(E_1, \dots, E_r)$, i.e. the smallest wide subcategory containing $\{E_1, \dots, E_r\}$. (We use here the same notation as for thick subcategories to avoid a new name again and since it corresponds anyway.)

In order to show that these two assignments give a bijective correspondence, it remains to show that

- if $(E_1, \dots, E_r)$ is a family of pairwise orthogonal bricks, then the bricks $E_1, \dots, E_r$ are the simple objects in $\mathrm{Thick}(E_1, \dots, E_r)$, and
- if $E_1, \dots, E_r$ are the simple objects of a thick subcategory $\mathcal{C}$, then $\mathcal{C} = \mathrm{Thick}(E_1, \dots, E_r)$.

Let $E = (E_1, \dots, E_r)$ be a family of pairwise orthogonal bricks. Let $\mathcal{E}$ be the full subcategory of $\mathcal{U}_n$ consisting of objects which have a composition series with composition factors isomorphic to one of $E_1, \dots, E_r$. In [**80**, Chapter X] we find the proof that $\mathcal{E}$ is an exact abelian extension-closed subcategory of $\mathcal{U}$. (It is shown in the setting that the ambient category is a module category, but there are no properties needed which are not given here.) Therefore, since $\mathrm{Thick}(E_1, \dots, E_r)$ is the smallest exact abelian extension-closed subcategory of $\mathcal{U}$ containing $E_1, \dots, E_r$, $\mathrm{Thick}(E_1, \dots, E_r) \subseteq \mathcal{E}$. The other inclusion is immediate by induction on the composition length of an object in $\mathcal{E}$. Hence, $\mathrm{Thick}(E_1, \dots, E_r) = \mathcal{E}$ and clearly, $E_1, \dots, E_r$ are the simple objects in $\mathcal{E}$ and therefore in $\mathrm{Thick}(E_1, \dots, E_r)$.

Next, if $E_1, \dots, E_r$ are the simple objects in a thick subcategory $\mathcal{C}$, then clearly, $\mathcal{C}$ is the same as the full subcategory of objects in $\mathcal{U}$ which admit a composition series with composition factors isomorphic to one of the E_i. As seen above this is equal to $\mathrm{Thick}(E_1, \dots, E_r)$.

Finally, we show the correspondence between the second and the third set. It is given as follows. Let $R_1, \dots, R_n$ be the simple objects in $\mathcal{U}_n$ numbered as indicated in the sequel of Proposition 4.15. Recall that the indecomposable objects in $\mathcal{U}_n$ are uniquely determined by their socle and their length. For convenience denote here by $R_i^r := R_i^{[r]}$ the indecomposable object with length r and socle R_i (the lower index should not be confused with the length as in the notation using the top to classify the object). Let c denote the permutation $c = (1, 2, \dots, n)$ of the set $\{1, \dots, n\}$. Now let $E = R_i^r$ be a brick in $\mathcal{U}_n$. Then, we assign to E the arc $(i, c^r(i))$. This actually gives an arc since by [**36**, Theorem 3.5] the length of E is $\leq n$ since E is a brick. In this manner we assign to a family of orthogonal bricks

$(E_1, \dots, E_s)$ a family of arcs $(a_1, \dots, a_s)$. Vice versa, if we have an arc (i, j), there is a number $1 \le r \le n$ with $j = c^r(i)$, and we assign to (i, j) the unique brick R_i^r and proceed analogously with a family of noncrossing arcs. By definition the composition of these two assignments equals the identity. It remains to show that actually two arcs $(i, c^r(i))$ and $(j, c^p(j))$ cross if and only if $\mathrm{Hom}(R_i^r, R_j^p) \neq 0$ or $\mathrm{Hom}(R_j^p, R_i^r) \neq 0$.

Suppose that $(i, c^r(i))$ and $(j, c^p(j))$ cross. Without loss of generality we may assume the following situation: the start point j of the second arc *lies between* i and $c^r(i)$ and the end point $c^p(j)$ *lies beyond* $c^r(i)$ going clockwise around the circle. In other words, there is a number $0 \le q < r$ with $j = c^q(i)$. Hence, $(j, c^r(i)) = (j, c^{r-q}(j))$ with $r - q > 0$. Moreover, $p \ge r - q$ since $c^p(j)$ lies beyond $c^r(i) = c^{r-q}(j)$. This gives us an inclusion $i\colon R_j^{r-q} \rightarrowtail R_j^p$ and a projection $p\colon R_i^r \twoheadrightarrow R_{c^q(i)}^{r-q} = R_j^{r-q}$. The existence of the projection is shown by using induction to check that the top of R_i^r is $R_{c^{r-1}(i)}$. Hence, if the arcs cross, we have a morphism $f = i \circ p\colon R_i^r \to R_j^p$. This is not zero since otherwise p would be zero since i is a monomorphism. But this is certainly not the case.

Now let $f\colon R_i^r \to R_j^p$ be a non-zero morphism. Then $\mathrm{Im}(f)$ is a non-zero subobject of R_j^p which is a brick as well and with socle R_j and length q with $1 \le q \le p$. Also it is a quotient of R_i^r, thus there is a number $0 \le t < r$ with $\mathrm{Im}(f) = R_{c^t(i)}^{r-t}$. Hence, $R_j^q = \mathrm{Im}(f) = R_{c^t(i)}^{r-t}$ which implies $c^t(i) = j$ and $q = r - t$. Therefore, $j = c^t(i)$ lies between i and $c^r(i)$ (not coinciding with $c^r(i)$). Moreover, $p \ge q = r - t \Rightarrow p + t \ge r$ and hence $c^p(j) = c^{p+t}(i)$ lies beyond $c^r(i)$. Altogether, $(i, c^r(i))$ and $(j, c^p(j))$ cross. □

REMARK 6.11. We changed Dichev's definition of noncrossing arcs on the circle a bit. This will enable us to deal also with extensions on the level of the arcs in the next section.

REMARK 6.12. Behind the idea of the noncrossing arcs on the circle there is covering theory. We could also consider arcs between integers on the number line. Identifying each simple object S with $\tau^n(S)$ gives us the circle.

THEOREM 6.13. *Let A be a tame hereditary algebra with regular part $\mathcal{R}$ decomposing as $\coprod_{j\in J} \mathcal{H}_j \times \coprod_{i=1}^s \mathcal{U}_{n_i}$. Then, there is a bijective correspondence between*

- *the set* $\mathrm{Th}_{\mathrm{reg}}(A)$ *of thick subcategories of* $\mathcal{D}^b(\mathrm{mod}(A))$ *consisting only of regular objects, and*
- *the set* $\{(p, x_1, \dots, x_s) \mid p \in 2^J, x_i \in \mathrm{NA}(n_i)\}$.

PROOF. As mentioned above the set $\mathrm{Th}_{\mathrm{reg}}(A)$ is in correspondence with the set of wide subcategories of $\mathcal{R}$. Then, the definition of the correspondence is obvious in consideration of Proposition 6.10. Additionally, we have to recognise that each of the direct factors of $\mathcal{R}$ is exact abelian extension-closed and that there are no non-zero morphisms between different direct factors. The latter is true since objects from different factors do not have composition factors in common.

Moreover, observe that uniserial categories with one simple object do not admit proper thick subcategories and therefore, either the whole factor $\mathcal{H}_j$ is part of the thick subcategory or none of it is. This explains the appearance of the power set 2^J. □

3. Thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$

Finally, we bring the previous results together and state the main classification theorem.

In the previous sections, we have seen the classification of thick subcategories which are generated by exceptional sequences on the one hand and the classification of thick subcategories which only consist of regular objects on the other hand.

The next proposition explains how these two fit together. It is due to Dichev [**34**] in the case that the algebra is a tame hereditary algebra over an algebraically closed field. We adapt the proof to the not algebraically closed case without greater difficulties.

PROPOSITION 6.14. *Let A be a tame hereditary algebra over a field k. Let $\mathcal{C}$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$. Then, at least one of the following holds.*

(1) *The thick subcategory $\mathcal{C}$ is generated by an exceptional sequence in* $\mathrm{mod}(A)$*;*
(2) *each object in $\mathcal{C}$ is regular.*

PROOF. Since in this proof we want to work with concepts like projective objects and extensions, instead of a thick subcategory $\mathcal{C}$ of $\mathcal{D}^b(\mathrm{mod}(A))$ we will now and then consider its corresponding wide subcategory $\{H^0(C) \mid C \in \mathcal{C}\}$ and denote it by the same letter.

Let (Γ, d) be the valued graph of the algebra A. This is an extended Dynkin diagram since the algebra is tame. Let $\mathcal{C}$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$.

We assume that the second statement does not hold and claim that then $\mathcal{C}$ is generated by an exceptional sequence.

If $\mathcal{C}$ does not only consist of regulars, there is an indecomposable preprojective or preinjective module X which is simple in $\mathcal{C}$. Without loss of generality we may assume that X is preprojective, i.e. $X \cong \tau^{-l}(P_m)$ where l is a positive integer, $m \in \Gamma$ and P_m is the indecomposable projective object associated with m.

Let $\mathcal{D} := \tau^l(\mathcal{C})$. We show that $\mathcal{D}$ is generated by an exceptional sequence which implies that $\mathcal{C}$ is generated by the τ^{-l}-shift of this sequence which is exceptional as well since τ is an auto-equivalence on $\mathcal{D}^b(\mathrm{mod}(A))$.

Let $\mathcal{S}_{\mathcal{P}}$ denote the set of preprojective objects which are simple in $\mathcal{D}$, $\mathcal{S}_{\mathcal{Q}}$ those objects which are preinjective and simple in $\mathcal{D}$ and $\mathcal{S}_{\mathcal{R}}$ those objects which are regular and simple in $\mathcal{D}$. Note that $\mathcal{D} = \mathrm{Thick}(\mathcal{S}_{\mathcal{Q}} \cup \mathcal{S}_{\mathcal{R}} \cup \mathcal{S}_{\mathcal{P}})$. The modules in $\mathcal{S}_{\mathcal{Q}} \cup \mathcal{S}_{\mathcal{R}} \cup \mathcal{S}_{\mathcal{P}}$ are pairwise orthogonal since they are simple in $\mathcal{D}$. Then, P_m is an element in $\mathcal{S}_{\mathcal{P}}$.

Within $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ we can order the modules in the following manner to an exceptional sequence. Let $A, B \in \mathcal{S}_{\mathcal{P}}$ with $A = \tau^{-i}(P_1)$ and $B = \tau^{-j}(P_2)$ where P_1 and P_2 are indecomposable projective. There are no non-zero morphisms between A and B in either direction because they are simple. If

$j \geq i$, then we put B in front of A in our sequence and with this we make sure that $\mathrm{Ext}^1(A,B) \cong \mathrm{Ext}^1(P_1, \tau^{i-j}(P_2)) = 0$. The same works within $\mathcal{S}_{\mathcal{Q}}$.

Now we try to do the same in $\mathcal{S}_{\mathcal{R}}$. Let R be a non-zero object in $\mathcal{S}_{\mathcal{R}}$. If such an object does not exist, $\mathcal{S}_{\mathcal{Q}} \cup \mathcal{S}_{\mathcal{P}}$ gives us immediately an exceptional sequence which generates $\mathcal{D}$. Otherwise as mentioned in Chapter 4.1 we have

$$\dim_{\mathrm{End}(S_m)} \mathrm{Hom}(P_m, R) = (\underline{\dim}(R))_m.$$

On the other hand, $\mathrm{Hom}(P_m, R) = 0$ since P_m and R are simple objects in $\mathcal{D}$. Hence, $\underline{\dim}(R)$ is not sincere. Moreover, the regular length of R is strictly smaller than the rank n of the tube in which R lies. This is because all objects in that tube of regular length $\geq n$ are sincere since the objects of regular length n already are sincere because they are homogeneous. This implies that $\mathcal{S}_{\mathcal{R}}$ is finite since there are only finitely many tubes of rank strictly greater than one. Also we get that all $R \in \mathcal{S}_{\mathcal{R}}$ are exceptional and hence by Theorem 4.6 $\underline{\dim}(R)$ is a positive real root for all $R \in \mathcal{S}_{\mathcal{R}}$.

Summing up, for all $R \in \mathcal{S}_{\mathcal{R}}$ we have $(\underline{\dim}(R))_m = 0$ for a fixed vertex $m \in \Gamma$.

Let (Γ', d') be the valued graph which we get from (Γ, d) if we remove the vertex m from Γ and adjust the valuation d. Since (Γ, d) is extended Dynkin, the result is either one Dynkin diagram (the one corresponding to the extended Dynkin diagram (Γ, d)) or it splits up into two Dynkin diagrams. To see this, check the lists of diagrams in Chapter 4.

Thus, we can consider each $R \in \mathcal{S}_{\mathcal{R}}$ as an indecomposable representation of Γ' since $\underline{\dim}(R)$ is a positive real root for Γ'. In this setting we can order the objects in $\mathcal{S}_{\mathcal{R}}$ to an exceptional sequence as above since in the Dynkin case all representations are preprojective. The exceptionality carries over to $\mathrm{mod}(A)$ since if there was a non-trivial extension of two objects in $\mathrm{mod}(A)$, this would also be a non-trivial extension in the category of representations of Γ'.

Hence, within $\mathcal{S}_{\mathcal{R}}$ we can order the objects to an exceptional sequence.

Moreover, by Proposition 4.16 we have that $\mathrm{Ext}^1(\mathcal{S}_{\mathcal{P}}, \mathcal{S}_{\mathcal{Q}})$, $\mathrm{Ext}^1(\mathcal{S}_{\mathcal{P}}, \mathcal{S}_{\mathcal{R}})$ and $\mathrm{Ext}^1(\mathcal{S}_{\mathcal{R}}, \mathcal{S}_{\mathcal{Q}})$ are zero.

Altogether, we have shown that we can order the simple objects in $\mathcal{D}$ to an exceptional sequence. □

The above proposition does not yet complete the classification since there are thick subcategories which are both generated by an exceptional sequence and in which all objects are regular. For example, suppose there is a uniserial subcategory of $\mathrm{mod}(A)$ with three regular simple objects R_1, R_2, R_3. Then (R_2, R_1) forms an exceptional sequence which generates a thick subcategory only consisting of regular objects.

Therefore, we need to refine the definition of noncrossing arcs on a circle.

Definition 6.15. We call an object $x \in \mathrm{NA}(n)$ *exceptional* if we cannot arrange a subfamily of arcs of x to a sequence $(a_1, \ldots, a_r)$ such that $s(a_{i+1}) = e(a_i)$ for $1 \leq i < r$ and $s(a_1) = e(a_r)$. We denote the set of these elements by

$$\mathrm{NA}^{\mathrm{exc}}(n) := \{x \in \mathrm{NA}(n) \mid x \text{ exceptional}\}.$$

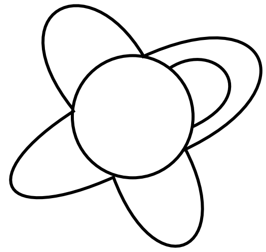


FIGURE 3. A possible $x \in \mathrm{NA}(n) \setminus \mathrm{NA}^{\mathrm{exc}}(n)$.

In the sequel we use the notations and concepts introduced in Section 2 of this chapter, in particular in the proof of Proposition 6.10.

LEMMA 6.16. *Let $\mathcal{U}_n$ be a connected uniserial category with n simple objects. Let $x = (a_1, \dots, a_r) \in \mathrm{NA}(n)$ and let $(E_1, \dots, E_r)$ be the corresponding sequence of pairwise orthogonal bricks in $\mathcal{U}_n$. Then, $x \in \mathrm{NA}^{\mathrm{exc}}(n)$ if and only if we can rearrange $(E_1, \dots, E_r)$ to an exceptional sequence.*

PROOF. Again we denote by $R_1, \dots, R_n$ the simple objects in $\mathcal{U}_n$.

Start with the following observation. Let $(i, c^r(i))$ and $(j, c^p(j))$ be two arcs such that $c^r(i) = j$. Then, we have for the corresponding bricks

$$\mathrm{Ext}^1(R_j^p, R_i^r) \cong D\,\mathrm{Hom}(R_i^r, \tau(R_j^p)) = D\,\mathrm{Hom}(R_i^r, R^p_{c^{-1}(j)})$$

and this is non-zero since $(i, c^r(i))$ and

$$(c^{-1}(j), c^p(c^{-1}(j))) = (c^{r-1}(i), c^{r+p-1}(i))$$

cross.

Now let $(E_1, \dots, E_r)$ and $(a_1, \dots, a_r)$ be as in the statement.

Assume first that $(a_1, \dots, a_r)$ is not exceptional, i.e. we can arrange parts of it to a sequence as in the definition of exceptionality. Without loss of generality this arrangement is again denoted by $(a_1, \dots, a_r)$ and $(E_1, \dots, E_r)$ is the corresponding sequence of bricks. Let us try to order this sequence to an exceptional sequence. We are free to start with E_1. By the preliminary observation we know that $\mathrm{Ext}^1(E_2, E_1) \neq 0$ and hence we have to put E_2 in front of E_1. Then, $\mathrm{Ext}^1(E_3, E_2) \neq 0$ forces us to put E_3 in front of E_2 and therefore also in front of E_1. This goes on and we are forced to the following order $(E_r, E_{r-1}, \dots, E_2, E_1)$ which is not exceptional since also $\mathrm{Ext}^1(E_1, E_r) \neq 0$.

Next, assume that $(a_1, \dots, a_r)$ is exceptional. Then, we find an index $i \in \{1, \dots, n\}$ such that $(i, c(i))$ is not *surrounded* by any arc of $(a_1, \dots, a_r)$. Here not being surrounded means formally that there is no arc of the form $(j, c^p(j))$ with $i = c^q(j)$ for some $0 \leq q < p$. If such an i did not exist, it would contradict the assumption that $(a_1, \dots, a_r)$ is exceptional.

The fact that none of the arcs surrounds $(i, c(i))$ corresponds to the fact that none of the bricks contains R_i as a composition factor. Without loss of generality we may assume $i = n$. The full subcategory of $\mathcal{U}_n$ in which all objects do not have R_n as a composition factor is equivalent to

$\mathrm{mod}(k\vec{A}_{n-1})$ with $\vec{A}_{n-1}$ linearly oriented. See for instance [**63**, Proposition 5.2]. But in the Dynkin case we can order the bricks to an exceptional sequence since they are anyway pairwise orthogonal. Moreover, the bricks do not have self-extensions since otherwise the corresponding arc would not be exceptional. □

Altogether this gives the following classification theorem.

THEOREM 6.17. *Let A be a tame hereditary k-algebra with regular part $\mathcal{R}$ decomposing as $\coprod_{j\in J}\mathcal{H}_j \times \coprod_{i=1}^{s}\mathcal{U}_{n_i}$. The poset of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ is given by the union of posets*

$$\mathrm{Th}_{\mathrm{exc}}(A) \cup \mathrm{Th}_{\mathrm{reg}}(A)$$

where one has bijective correspondences

$$\begin{aligned}
\mathrm{Th}_{\mathrm{exc}}(A) &\longleftrightarrow \mathrm{NC}(W,c),\\
\mathrm{Th}_{\mathrm{reg}}(A) &\longleftrightarrow \{(p,x_1,\dots,x_s) \mid p \in 2^J, x_i \in \mathrm{NA}(n_i)\},\\
\mathrm{Th}_{\mathrm{exc}}(A)\cap \mathrm{Th}_{\mathrm{reg}}(A) &\longleftrightarrow \{(x_1,\dots,x_s) \mid x_i \in \mathrm{NA}^{\mathrm{exc}}(n_i)\}.
\end{aligned}$$

PROOF. That the union is in bijection with the poset of thick subcategories, follows from Proposition 6.14.

The correspondences for $\mathrm{Th}_{\mathrm{exc}}(A)$ and $\mathrm{Th}_{\mathrm{reg}}(A)$ are stated in Theorem 6.2 and Theorem 6.13. The classification of the intersection follows immediately from Lemma 6.16. □

Unfortunately, it is not possible to formulate this adequately as a sum of lattices. This fails already for the reason that $\mathrm{NC}(W,c)$ is not a lattice in most cases.

CHAPTER 7

Finite algebraic triangulated categories

One of the main goals of this thesis is to classify thick subcategories of algebraic triangulated categories of finite type. This chapter provides the necessary information about these categories.

Throughout this chapter we assume that k is an algebraically closed field.

Recall that a triangulated category is called algebraic if it is triangulated equivalent to the stable category $\underline{\mathcal{F}}$ of a Frobenius category $\mathcal{F}$. By Theorem 2.10 such categories are triangulated.

The main example for us is the stable module category $\underline{\text{mod}}(A)$ of a self-injective algebra A. In fact, the whole classification of algebraic triangulated categories of finite type is a generalisation of the classification of self-injective algebras of finite representation type due to Riedtmann [**70**] and Asashiba [**5**]. Although the theory for self-injective algebras is covered by this general theory, we will present the important results for self-injective algebras in the next chapter.

A further important example is the stable category $\underline{\text{CM}}(R)$ of maximal Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity R.

Definition 7.1. A triangulated category $\mathcal{T}$ is called *locally finite* (after [**85**]) if for each indecomposable X of $\mathcal{T}$ there are at most finitely many isomorphism classes of indecomposables Y such that $\text{Hom}_{\mathcal{T}}(X, Y) \neq 0$. If there are only finitely many indecomposable objects at all the category is called *finite*.

The structure of a locally finite triangulated category is classified via the shape of its Auslander-Reiten quiver. In order to describe this shape, we recall some definitions and notations concerning translation quivers and automorphism groups of quivers.

Let $\vec{\Delta} = (\vec{\Delta}_0, \vec{\Delta}_1, s, t)$ be a quiver. For a vertex $x \in \vec{\Delta}_0$ we denote by x^+ the set of direct successors of x and by x^- the set of direct predecessors of x.

In Chapter 2 we already got to know the repetition $\mathbb{Z}\Delta$ of a quiver $\vec{\Delta}$.

Definition 7.2. A group of automorphisms G of a quiver $\vec{\Delta}$ is said to be *admissible* [**70**] if no orbit of G intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It is said to be *weakly admissible* [**35**] if, for each $g \in G \setminus \{1\}$ and for each $x \in \vec{\Delta}_0$ we have $x^+ \cap (gx)^+ = \emptyset$. Note that admissible implies weakly admissible.

Theorem 7.3 (Xiao/Zhu [**85**]). *Let $\mathcal{T}$ be a Krull-Schmidt locally finite connected triangulated category. Then, the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of*

$\mathcal{T}$ is isomorphic to $\mathbb{Z}\Delta/\Phi$ where Δ is Dynkin of type A, D or E and Φ is a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. The underlying graph Δ is unique up to isomorphism, and the group Φ is unique up to conjugacy. □

EXAMPLE 7.4. Let $G = \langle x \mid x^p = 1\rangle$ be the cyclic group of prime order p. Let k be a field with $\mathrm{char}(k) = p$. Then, the group algebra kG (for the definition of kG see Chapter 8) is of the form $k[X]/(X-1)^p$. The indecomposable kG-modules are given by $k[X]/(X-1)^i$ for $1 \le i \le p$ where this is projective and injective for $i = p$. The Auslander-Reiten sequences look as follows.

$$k[X]/(X-1)^i \to k[X]/(X-1)^{i+1} \oplus k[X]/(X-1)^{i-1} \to k[X]/(X-1)^i$$

for $1 < i \le p-1$. Hence, the Auslander-Reiten quiver of $\underline{\mathrm{mod}}(kG)$ is of the form $\mathbb{Z}A_{p-1}/\langle\tau\rangle$.

The result of Xiao und Zhu generalises Riedtmann's Theorem [**70**] which says that the stable Auslander-Reiten quiver of a representation-finite algebra is isomorphic to $\mathbb{Z}\Delta/\Phi$ where Δ is Dynkin and Φ is an admissible group of automorphisms. Riedtmann also gives a complete list of possible admissible groups of automorphisms of $\mathbb{Z}\Delta$. Xiao and Zhu as well as Amiot [**1**] extend this list to a list of possible weakly admissible groups of automorphisms as follows.

THEOREM 7.5 (Xiao/Zhu [**85**]). *Let Δ be a Dynkin graph and let Φ be a non trivial weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. This is a list of its possible generators.*

- *$\Delta = A_n$ with n odd: possible generators are τ^r and $\phi\tau^r$ with $r \ge 1$ where ϕ is the reflection at the 'central line' of $\mathbb{Z}A_n$ which is given by the vertices $\{(i, \frac{n+1}{2}) \mid i \in \mathbb{Z}\}$. Here we take as a basis of $\mathbb{Z}A_n$ the linearly oriented $\vec{A}_n$.*
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$
- *$\Delta = A_n$ with n even: then possible generators are $\phi\tau^r$ with $r \ge 1$ where $\phi(p,q) = (p+q-\frac{n}{2}-1, n+1-q)$. Here $\phi^2 = \tau$.*
- *$\Delta = D_n$ with $n \ge 5$: possible generators are τ^r and $\phi\tau^r$ with $r \ge 1$ where ϕ exchanges $(i, n-1)$ and (i,n) $\forall i \in \mathbb{Z}$ and fixes the other vertices of $\mathbb{Z}D_n$. Here we take as a basis of $\mathbb{Z}D_n$ the linearly oriented $\vec{D}_n$.*
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-3 \longrightarrow n-2 \begin{array}{l} \nearrow n \\ \searrow n-1 \end{array}$$
- *$\Delta = D_4$: possible generators are τ^r and $\phi\tau^r$ with $r \ge 1$ and where ϕ is either defined as for $n \ge 5$ or as follows. Again with the linearly oriented $\vec{D}_4$, $\phi((i,3)) = (i,4), \phi((i,4)) = (i-1,1), \phi((i-1,1)) = (i,3)$ and $\phi((i,2)) = (i,2)$ $\forall i \in \mathbb{Z}$.*
- *$\Delta = E_6$: possible generators are τ^r and $\phi\tau^r$ with $r \ge 1$ and where ϕ is the reflection at the central line of $\mathbb{Z}E_6$ which is given by $\{(i,3),(i,4) \mid i \in \mathbb{Z}\}$. Here we assume the following orientation*

and numbering.

$$\begin{array}{ccccccccccc} & & & & 4 & & & & \\ & & & & \uparrow & & & & \\ 1 & \longleftarrow & 2 & \longleftarrow & 3 & \longrightarrow & 5 & \longrightarrow & 6 \end{array}$$

- $\Delta = E_7, E_8$: *possible generators are* τ^r *with* $r \geq 1$. □

REMARK 7.6. The only weakly admissible group of automorphisms which is not admissible occurs for A_n with n even and is generated by ϕ.

REMARK 7.7. If $\mathcal{T}$ is locally finite, but not finite, then the group Φ of automorphisms is trivial.

Now we know the shape of the Auslander-Reiten quiver of our categories. But the classification goes even farther as we are able to classify them up to triangulated equivalence. This is particularly convenient for our purposes of classifying thick subcategories.

THEOREM 7.8 (Amiot [**1**]). *If $\mathcal{T}$ is a finite triangulated category which is connected, algebraic and standard, then there exists a Dynkin diagram Δ of type A, D or E and an auto-equivalence Φ of $\mathcal{D}^b(\text{mod}(k\vec{\Delta}))$ such that $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}^b(\text{mod}(k\vec{\Delta}))/\Phi$.* □

REMARK 7.9. Since $\mathcal{T}$ is standard and since by Proposition 2.28 the category of indecomposable objects in $\mathcal{D}^b(\text{mod}(k\vec{\Delta}))$ is equivalent to the mesh category $k(\mathbb{Z}\Delta)$, Δ and Φ are those induced by Δ and the group of automorphisms coming from the Auslander-Reiten quiver of $\mathcal{T}$. We denote the automorphism of $\mathcal{D}^b(\text{mod}(k\vec{\Delta}))$ induced by the group of automorphisms Φ of $\mathbb{Z}\Delta$ by the same character Φ.

Note that the construction of the orbit category requires an automorphism on $\mathcal{T}$. A standard construction allows one to replace a category with auto-equivalence by a category with automorphism.

Finally, one may ask whether each Dynkin type Δ and each weakly admissible group of automorphisms Φ give rise to a locally-finite triangulated category with Auslander-Reiten quiver $\mathbb{Z}\Delta/\Phi$. Xiao and Zhu [**85**] point out that this is actually true. Just take the orbit category $\mathcal{D}^b(\text{mod}(k\vec{\Delta}))/\Phi$. To show this one checks the assumptions in Keller's Theorem 3.3 on triangulated orbit categories.

However, it is not possible to realise each possible Auslander-Reiten quiver $\mathbb{Z}\Delta/\Phi$ via the stable module category of a representation-finite self-injective algebra. Thus, the generalisation to an arbitrary algebraic triangulated category really leads to a greater class of categories.

In order to name our categories precisely, we introduce the *type* of a finite connected triangulated category. This is adopted from Asashiba's approach [**5**] for self-injective algebras.

DEFINITION 7.10. Let $\mathcal{T}$ be a finite connected triangulated category with Auslander-Reiten quiver $\Gamma_{\mathcal{T}} \cong \mathbb{Z}\Delta/\Phi$. As we have seen, Φ is always generated by an element $\phi\tau^r$ with $r \geq 1$ where ϕ is an automorphism of order $t = 1, 2, 3$ or of infinite order (which only appears for A_n with n even). Then we define the *type* of $\mathcal{T}$ to be $\text{typ}(\mathcal{T}) = (\Delta, r, t)$ where $t \in \{1, 2, 3, \infty\}$.

Note that the type does not depend on the orientation of Δ.

The set of types of finite connected triangulated categories is equal to the disjoint union of the following sets.

- $\{(A_n, r, 1) \mid n, r \in \mathbb{N}\}$;
- $\{(A_{2n+1}, r, 2) \mid n, r \in \mathbb{N}\}$;
- $\{(A_{2n}, r, \infty) \mid n, r \in \mathbb{N}\}$;
- $\{(D_n, r, 1) \mid n, r \in \mathbb{N}, n \geq 4\}$;
- $\{(D_n, r, 2) \mid n, r \in \mathbb{N}, n \geq 4\}$;
- $\{(D_4, r, 3) \mid r \in \mathbb{N}\}$;
- $\{(E_n, r, 1) \mid r \in \mathbb{N}, n = 6, 7, 8\}$;
- $\{(E_6, r, 2) \mid r \in \mathbb{N}\}$.

CHAPTER 8

Self-injective algebras

Every algebra is projective as a module over itself. If the algebra is also injective, it is called *self-injective*.

The stable module category of a self-injective algebra is our most important example of an algebraic triangulated category. Hence, in this chapter, we collect some knowledge about self-injective algebras.

Most importantly, the stable module category $\underline{\mathrm{mod}}(A)$ of a self-injective algebra is a triangulated category. We have already discussed this in Chapter 2.2. Recall that the shift Σ in the triangulated category $\underline{\mathrm{mod}}(A)$ is given by $X \mapsto \mathrm{Coker}(X \hookrightarrow E(X))$. In literature, this equivalence is often called Ω^{-1} with inverse $\Omega(X)$ the kernel of the projective cover of X, also known as *Heller's syzygy functor*.

Next to the definition above there are other criteria to check whether an algebra is self-injective. An algebra is self-injective if and only if its projective and injective modules coincide. Also there is the following connection.

DEFINITION 8.1. A finite-dimensional algebra A over a field k is called *Frobenius* if A is equipped with a nondegenerate bilinear form $(-,-)\colon A \times A \to k$ with $(ab, c) = (a, bc)$ for all $a, b, c \in A$.

A Frobenius algebra is called *symmetric* if $(-,-)$ is symmetric.

PROPOSITION 8.2. *A Frobenius algebra is self-injective.* □

Moreover, being symmetric has a consequence which is important for the appearance of thick subcategories.

PROPOSITION 8.3 ([**38**],I.7.5). *If the algebra A is symmetric, then $\tau \cong \Omega^2$.* □

COROLLARY 8.4. *Let A be a symmetric algebra and let $\mathcal{C}$ be a thick subcategory of $\underline{\mathrm{mod}}(A)$. Let Γ_0 be a connected component of the stable Auslander-Reiten quiver of A. If an indecomposable X corresponding to a vertex in Γ_0 is in $\mathcal{C}$, this holds for all indecomposables corresponding to Γ_0.*

In particular, if $\underline{\mathrm{mod}}(A)$ is connected, it does not contain non-zero proper thick subcategories.

PROOF. If $X \in \mathcal{C}$, then also $\tau(X) = \Omega^2(X) \in \mathcal{C}$ since $\mathcal{C}$ is thick. Hence, the middle term of the Auslander-Reiten sequence ending in X belongs to $\mathcal{C}$. Thus, starting with X, we get everything corresponding to the connected component Γ_0. □

Examples for self-injective algebras are preprojective algebras of representation-finite hereditary algebras (see [**54**]), finite-dimensional Hopf algebras (see [**62**]), Hecke algebras (see [**39**]), and group algebras of finite groups. Let us recall some details concerning this last class of examples.

1. Group algebras

DEFINITION 8.5. Let G be a finite group and let k be a field. The *group algebra* kG consists of linear combinations of the form $\sum_{g\in G} a_g g$ with $a_g \in k \ \forall g \in G$. Addition in kG is given by

$$\sum_{g\in G} a_g g + \sum_{g\in G} a_g g = \sum_{g\in G} (a_g + b_b) g,$$

multiplication by a scalar $a \in k$ is defined as

$$a \sum_{g\in G} a_g g = \sum_{g\in G} a a_g g,$$

and finally, multiplication in kG is given by

$$\sum_{g\in G} a_g g \sum_{g\in G} a_g g = \sum_{g\in G, h\in G} (a_g b_h) gh.$$

PROPOSITION 8.6. *Let G be a finite group. The group algebra kG is self-injective.*

PROOF. One can show that the following bilinear form turns kG into a symmetric Frobenius algebra. Namely, for $a, b \in G$ define $(a, b) :=$ the coefficients of $1 \in G$ in ab if ab is expressed in the group basis. For instance see [**38**, I.3.2]. □

The following statement shows that the characteristic of the field has a great influence on the category of modules over the group algebra.

THEOREM 8.7 (Maschke). *Let G be a finite group and let k be a field. Then, the group algebra kG is semisimple if and only if the characteristic of k does not divide the order of G.* □

Hence, if $\mathrm{char}(k) \nmid |G|$, the module category of kG is not particularly interesting. Concerning thick subcategories of $\underline{\mathrm{mod}}(kG)$, that would mean that they are classified by collections of the simple kG-modules. Thus, if we deal with p-groups for example, we will always assume that $\mathrm{char}(k) = p$.

In this thesis, we are mostly interested in finite algebraic triangulated categories and we have already seen how they look in Theorem 7.3 and Theorem 7.8. Hence, we should think about what holds for group algebras.

First of all, the finiteness condition. For group algebras there is a characterisation of those of finite representation type.

THEOREM 8.8 (Higman). *Let G be a finite group and let k be a field of characteristic p dividing the order of G. The group algebra kG is representation-finite if and only if the Sylow p-subgroups of G are cyclic.*

PROOF. For a proof see for example [**7**, Theorem 5.6]. □

Moreover, we want to know of which type in the sense of Definition 7.10 the categories $\underline{\text{mod}}(kG)$ are for a representation finite group algebra kG. The answer is given in the following theorem.

THEOREM 8.9 ([**12**], Corollary 6.3.5, Theorem 6.5.5). *Let k be an algebraically closed field. Let B be a block of kG with defect group D. Then, B has finite representation type if and only if D is cyclic.*

If this is the case, i.e. D is cyclic of order p^n, then the stable Auslander-Reiten quiver of B is of the form $\mathbb{Z}A_{p^n-1}/\langle\tau^e\rangle$ where e is the inertial index of kG. □

For the definitions of block, defect group and inertial index see [**12**]. The details are not important for our purpose. What is in particular important is that representation-finite group algebras for p-groups are of type A. This insight enables us to compare (and distinguish) our classification of thick subcategories to that of Benson, Carlson and Rickard [**14**] for the stable module category of the group algebra of a p-group. We will do this in Chapter 11.

2. Finite representation type

There is an elaborate classification of representation-finite self-injective algebras. As said, by now the theory is covered by the general theory discussed in Chapter 7. Nevertheless, it is worthwhile to present it here in order to see which types actually occur for self-injective algebras. Moreover, historically this is the guideline for the general theory.

Everything starts with Riedtmann's Theorem on the shape of the Auslander-Reiten quiver.

THEOREM 8.10 (Riedtmann [**70**]). *Let A be a finite-dimensional representation-finite algebra over an algebraically closed field k. Then, a connected component of the stable Auslander-Reiten quiver of A is of the form $\mathbb{Z}\Delta/\Phi$ where Δ is Dynkin of type A, D or E and Φ is admissible.* □

We have already discussed admissible automorphism groups in Chapter 7.

Note that the theorem does not assume A to be self-injective. But if A is self-injective, the theorem yields the most possible information about the module category. Namely, in this case, we get the stable Auslander-Reiten quiver from the original quiver by only deleting projective vertices. Thus, one only has to study where in the quiver the projectives might be. This is done by Riedtmann (and others) in her further work on self-injective algebras for the different Dynkin types; see [**71**], [**73**], [**72**], [**26**]. In this thesis, this is not needed since we are interested in the stable module category and although we do not specify the position of the projectives, we do know the triangulated structure by Theorem 7.8.

As indicated in Chapter 7, not each possible admissible automorphism group occurs for self-injective algebras. The classification of Asashiba [**5**] points out which ones do occur.

Let Λ be a representation-finite self-injective standard algebra with stable Auslander-Reiten quiver ${}_s\Gamma_\Lambda \cong \mathbb{Z}\Delta/\langle\phi\tau^r\rangle$. Here τ is the translation on $\mathbb{Z}\Delta$ and ϕ is an automorphism of order 1, 2 or 3 defined in 7.5 such that $\langle \phi\tau^r \rangle$ is an admissible group of automorphisms of $\mathbb{Z}\Delta$. Define the type $\mathrm{typ}(\Lambda) = (T(\Lambda), f(\Lambda), t(\Lambda))$ of Λ where $T(\Lambda) = \Delta$ is the tree class, $f(\Lambda) = r/m_\Delta$ is the *frequency*, $t(\Lambda)$ is the *order* of ϕ and $m_\Delta = h_\Delta - 1$ where h_Δ is the *Coxeter number* associated to Δ, i.e. the order of the Coxeter element.

REMARK 8.11. Note that the definition of *type* in Chapter 7 is not the generalisation of this one since the second entry is different. For instance, if we have an algebra Λ with $\mathrm{typ}(\Lambda) = (A_5, 2, 2)$, then the type of the corresponding triangulated category is given by $\mathrm{typ}(\underline{\mathrm{mod}}(\Lambda)) = (A_5, 10, 2)$.

The type determines Λ up to stable equivalence (i.e. a triangle equivalence between the stable module categories).

THEOREM 8.12 (Asashiba [**5**]). *Let Λ, Λ' representation-finite and self-injective algebras.*

(1) *If Λ is standard and Λ' is non-standard, then Λ and Λ' are not stably equivalent.*
(2) *If both Λ and Λ' are standard, then Λ and Λ' are stably equivalent if and only if $\mathrm{typ}(\Lambda) = \mathrm{typ}(\Lambda')$.*
(3) *If both Λ and Λ' are non-standard, then Λ and Λ' are stably equivalent if and only if $\mathrm{typ}(\Lambda) = \mathrm{typ}(\Lambda')$.* □

Moreover, Asashiba gives a complete list of these types for standard algebras.

THEOREM 8.13 (Asashiba [**5**]). *The set of types of standard self-injective algebras of finite representation type is equal to the disjoint union of the following sets*

- $\{(A_n, s/n, 1) \mid n, s \in \mathbb{N}\}$;
- $\{(A_{2p+1}, s, 2) \mid p, s \in \mathbb{N}\}$;
- $\{(D_n, s, 1) \mid n, s \in \mathbb{N}, n \geq 4\}$;
- $\{(D_{3m}, s/3, 1) \mid m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$;
- $\{(D_n, s, 2) \mid n, s \in \mathbb{N}, n \geq 4\}$;
- $\{(D_4, s, 3) \mid s \in \mathbb{N}\}$;
- $\{(E_n, s, 1) \mid n = 6, 7, 8, s \in \mathbb{N}\}$;
- $\{(E_6, s, 2) \mid s \in \mathbb{N}\}$. □

Completing the classification, in the appendix of [**6**] Asashiba gives a representative for each of the above types. These representatives are given by quivers and relations.

One could also give them as *orbit algebras* of the *repetitive algebras* of the respective Dynkin type.

3. Repetitive algebras and covering theory

Repetitive algebras are an important concept in the study of self-injective algebras. For one thing they are self-injective and constructing repetitive algebras provides a method to build self-injective algebras out of other algebras.

In order to construct these algebras, we need to introduce the practice of identifying algebras with categories and the other way round. This will also enable us to define orbit algebras by just using the concept of orbit categories we already know. Let k be a field. Following [**21**] and [**44**], we call a k-category R *locally bounded* if

- distinct objects of R are not isomorphic,
- for each object x of R, $R(x,x)$ is a local k-algebra, and
- for each object x of R, we have

$$\sum_{y \in \mathrm{obj}(R)} (\dim_k R(x,y) + \dim_k R(y,x)) < \infty.$$

If additionally, R has only finitely many objects, then it is called *bounded*. Now to a bounded k-category R we associate a finite-dimensional k-algebra with underlying vector space $\bigoplus_{x,y \in \mathrm{obj}(R)} R(x,y)$. We denote the associated algebra by the same letter R.

For a locally bounded category R we define $\mathrm{Mod}(R)$ (or $\mathrm{mod}(R)$) to be the category of (finitely generated) contravariant functors from R to $\mathrm{Mod}(k)$. This is of course defined in this way so that it coincides with the category of modules for the associated algebra R.

A well-known example of this concept is the path category of a finite acyclic quiver. The associated algebra is just the path algebra of the quiver.

Let G be an *admissible* group of k-linear automorphisms of the category R, i.e. the action of G on the objects of R is free and there are only finitely many orbits. Then we define the orbit category R/G as in Chapter 3 and we may as well associate an algebra to this, the *orbit algebra*. It is finite-dimensional since G is admissible.

Moreover, it is important to mention that a group G of automorphisms on R defines a group of automorphism on $\mathrm{Mod}(R)$. To be specific, this works as follows. Let $g\colon R \to R$ be in G and let $M \in \mathrm{Mod}(R)$, i.e. a functor $M\colon R^{\mathrm{op}} \to \mathrm{mod}(k)$. Then, $g(M)$ is defined to be the functor $M(g^{-1}(-))\colon R^{\mathrm{op}} \to \mathrm{mod}(k)$.

The whole construction comes along with a projection

$$F\colon R \to R/G.$$

By definition this is a *covering functor* in the sense of Gabriel [**44**]. Hence, we may associate functors on the level of the corresponding module categories, namely the *pull-up functor* $F_\bullet\colon \mathrm{Mod}(R/G) \to \mathrm{Mod}(R)$ defined by $M \mapsto M \circ F$ and the *push-down functor*

$$F_\lambda\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R/G)$$

as the left adjoint functor to $F_\bullet$. In [**21**, Section 3.2] one finds an explicit definition of F_λ. For example for modules $M \in \mathrm{Mod}(R)$ one has

$$(F_\lambda M)(a) = \bigoplus_{F(x)=a} M(x) \text{ for } a \in R/G.$$

Using this one can show the following about the composition of the two functors.

Proposition 8.14 (Gabriel [**43**], Lemma 3.2). *For $M \in \mathrm{Mod}(R)$ there is an isomorphism*

$$\bigoplus_{g\in G} g(M) \cong F_\bullet F_\lambda(M).$$

□

The push-down functor can help to relate the category $\mathrm{Mod}(R/G)$ to $\mathrm{Mod}(R)$. For example for morphisms one gets what one expects.

Proposition 8.15 (Gabriel [**43**], Theorem 3.6 c). *Suppose that R is locally bounded and that G acts freely on R. Then,*

$$\mathrm{Hom}_{R/G}(F_\lambda M, F_\lambda N) = \bigoplus_{g\in G} \mathrm{Hom}_R(M, g(N))$$

for all $M, N \in \mathrm{ind}(R)$. □

Originally, Gabriel assumes in this statement that G acts freely on $\mathrm{ind}(R)$, but with [**65**] it is sufficient to assume G acting freely on R.

It would be convenient if one could describe the category $\mathrm{mod}(R/G)$ in terms of $\mathrm{mod}(R)$. Together with the proposition above this would be possible if the push-down functor restricted to $\mathrm{mod}(R)$ was dense. This is not always true and we need to assume a further condition on R. For any R-module M we denote by $\mathrm{supp}(M)$ the full subcategory of R consisting of all objects $a \in R$ such that $M(a) \neq 0$.

Moreover, for an object $a \in R$ we denote by R_a the full subcategory of R formed by

$$\bigcup_{M\in \mathrm{ind}(R), a\in \mathrm{supp}(M)} \mathrm{supp}(M).$$

Then, R is called *locally support-finite* if R_a is finite for each object a of R.

Proposition 8.16 (Dowbor-Lenzing-Skowronski [**37**]). *Let R be locally support-finite and let G be a group of automorphisms on R acting freely on the objects of R. Then, the push-down functor induces a bijection between the G-orbits of isomorphism classes of objects in $\mathrm{ind}(R)$ and the isomorphism classes of objects in $\mathrm{ind}(R/G)$. In particular, $F_\lambda\colon \mathrm{mod}(R) \to \mathrm{mod}(R/G)$ is dense.* □

Now we turn to specific examples of the covering theory above. The definition of a repetitive algebra goes back to Hughes and Waschbüsch [**50**]. Let A be a finite-dimensional k-algebra and let $1_A = e_1 + \ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal primitive idempotents. Moreover, we assume that A is basic, i.e. $Ae_i \ncong Ae_j$ for all $i \neq j$. Firstly, we associate to A the *repetitive category* $\hat{A}$ whose objects are $e_{m,i}$ with $m \in \mathbb{Z}$ and $i \in \{1, \ldots, n\}$, and whose morphism spaces are

$$\hat{A}(e_{m,i}, e_{r,j}) = \begin{cases} e_j A e_i & \text{if } r = m, \\ D(e_i A e_j) & \text{if } r = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\hat{A}$ associated with the repetitive category is called the *repetitive algebra* of A.

Theorem 8.17 (Happel [**46**], Lemma 2.2). *Let A be a finite-dimensional algebra and let $\hat{A}$ be the associated repetitive algebra. Then, $\hat{A}$ is self-injective and accordingly,* $\operatorname{mod}(\hat{A})$ *is a Frobenius category.* □

If G is an admissible automorphism group, this implies that the orbit algebra $\hat{A}/G$ is self-injective, too.

At this point we should also define an important automorphism of $\hat{A}$, namely $\nu_{\hat{A}}\colon \hat{A} \to \hat{A}$ mapping $e_{m,i}$ to $e_{m+1,i}$. This induces an automorphism on $\operatorname{mod}(\hat{A})$ (and on $\underline{\operatorname{mod}}(\hat{A})$) which coincides with the Nakayama automorphism $\nu_{\hat{A}} = D \operatorname{Hom}_{\hat{A}}(-, \hat{A})$. Note that for a self-injective algebra Λ, on $\underline{\operatorname{mod}}(\Lambda)$ we have

$$\nu_\Lambda \cong \tau_\Lambda \Omega_\Lambda^{-2}.$$

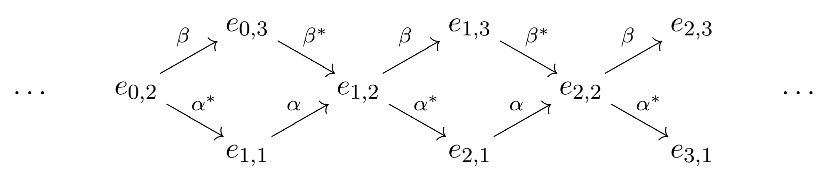
This is by the definitions of all the appearing functors. Observe also that τ and Ω are triangle equivalences on $\underline{\operatorname{mod}}(\Lambda)$ which makes ν_Λ a triangle equivalence, too.

With the help of $\nu_{\hat{A}}$ we may define the following important concept.

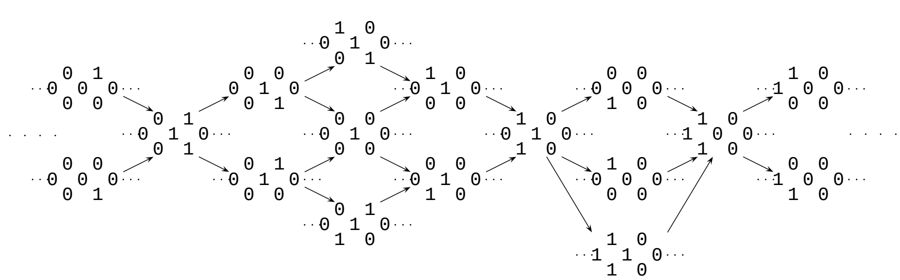
Definition 8.18. Let A be a basic connected algebra, $\hat{A}$ its repetitive algebra and $\nu_{\hat{A}}$ the Nakayama automorphism. Then $T(A) = \hat{A}/\langle \nu_{\hat{A}} \rangle$ is called the *trivial extension algebra* of A. One can show that $T(A)$ is the same as $A \ltimes D(A)$ which is the more usual definition of the trivial extension algebra.

Similarly, we define the *r-fold trivial extension algebra* $T(A)^r = \hat{A}/\langle \nu_{\hat{A}}^r \rangle$ as in [**81**, Section 6].

Example 8.19. Let $A = k(\vec{A}_3)$ with $\vec{A}_3\colon 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. The identity of A decomposes into $e_1 + e_2 + e_3$. Then, the repetitive category is determined by the quiver



with respect to the relations $\alpha^*\alpha = 0$, $\beta\beta^* = 0$ and $\alpha\alpha^* = \beta^*\beta$. The repetitive algebra $\hat{A}$ is the path algebra of the above infinite quiver with respect to these relations. The following figure is an extract from the Auslander-Reiten quiver of $\hat{A}$.



Deleting the projective/injective representations, we see that the stable Auslander-Reiten quiver is $\mathbb{Z}A_3$. Consider the orbit algebra $T(A) = \hat{A}/\nu_{\hat{A}}$. This is given by the quiver

$$1 \leftrightarrows 2 \leftrightarrows 3$$

with respect to the inherited relations from $\hat{A}$. Checking what $\nu_{\hat{A}}$ does on $\mathrm{mod}(\hat{A})$, we get that the stable Auslander-Reiten quiver of $\hat{A}/\nu_{\hat{A}}$ is $\mathbb{Z}A_3/\langle\tau^3\phi\rangle$ where ϕ is the 'reflection' of $\mathbb{Z}A_3$ defined in Theorem 7.5. Thus, $\mathrm{typ}(\underline{\mathrm{mod}}(T(A))) = (A_3, 3, 2)$ in the sense of Definition 7.10.

As the example suggests, for algebras like the above one, $\underline{\mathrm{mod}}(\hat{A})$ is closely related to the bounded derived category of $\mathrm{mod}(A)$.

THEOREM 8.20 (Happel [**46**], Theorem 4.9). *If A is a finite-dimensional algebra of finite global dimension, then $\underline{\mathrm{mod}}(\hat{A})$ and $\mathcal{D}^b(\mathrm{mod}(A))$ are equivalent as triangulated categories.* □

CHAPTER 9

Thick subcategories of finite algebraic triangulated categories

The purpose of this chapter is to classify the thick subcategories of a finite triangulated category which is connected, algebraic and standard, i.e. exactly the kind of category discussed in Chapter 7. As there we assume in this chapter that the field k is algebraically closed unless it is stated otherwise.

1. The classification

We have seen in Chapter 7 that each such triangulated category is triangulated equivalent to an orbit category $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle\phi\tau^r\rangle$ where $\vec{\Delta}$ is a Dynkin quiver and $\langle\phi\tau^r\rangle$ is an admissible group of automorphisms. Thus, we want to classify thick subcategories of these orbit categories. By Theorem 3.4 we know that the thick subcategories of the orbit category are in bijective correspondence with the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ which are invariant under $\langle\phi\tau^r\rangle$. By Theorem 6.2 and since Δ is Dynkin, we get that the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ are in correspondence with $\mathrm{NC}(W_\Delta, c)$ where $W_\Delta = W$ is the associated Weyl group and $c \in W_\Delta$ is the Coxeter transformation with respect to an admissible numbering of $\vec{\Delta}$. Hence, we need to explain what the automorphism $\phi\tau^r$ means for $\mathrm{NC}(W,c)$.

As long as we are only concerned with τ this is fairly evident. We formulate the statement in a greater generality than necessary in this chapter since we will need this at least for tame hereditary algebras later in this thesis.

Recall that we denote by cox the bijection from the set $\mathrm{Th}_{\mathrm{exc}}(A)$ of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ which are generated by an exceptional sequence to $\mathrm{NC}(W,c)$.

Proposition 9.1. *Let A be a hereditary algebra over an arbitrary field k (not necessarily algebraically closed). Let c be the Coxeter transformation in the associated Weyl group. Let $\mathcal{S} \in \mathrm{Th}_{\mathrm{exc}}(A)$. Then,*

$$\mathrm{cox}(\tau(\mathcal{S})) = c\,\mathrm{cox}(\mathcal{S})c^{-1}.$$

Proof. Let $E_1, \dots, E_r$ be an exceptional sequence in $\mathrm{mod}(A)$ generating $\mathcal{S}$. Thus, $\mathrm{cox}(\mathcal{S}) = s_{E_1}\cdots s_{E_r}$. Since it is more convenient here we will work with the wide subcategory which corresponds to $\mathcal{S}$ by Theorem 2.29 and call it also $\mathcal{S}$. The wide subcategory $\mathcal{S}$ is also generated by the above exceptional sequence. Moreover, we are allowed to choose $E_1, \dots, E_r$ in such a way that they are simple in the wide subcategory $\mathcal{S}$. In fact, this is then a complete list of simple objects in the wide subcategory.

We can reorder this sequence in such way that the possible indecomposable projective modules lie at the end of the sequence:

$$(E_1, \dots, E_{s-1}, E_s = P_{i_s}, \dots, E_r = P_{i_r}).$$

Just preserve the order within the projectives and the non-projectives, respectively. After that, we only have to take care that $\mathrm{Hom}(P, E) = 0$ and $\mathrm{Ext}^1(P, E) = 0$ for P one of the projective modules and E one of the non-projective modules. The second equation is clear and the first one holds by Schur's lemma.

Consider the sequence

$$(\tau(E_1), \dots, \tau(E_{s-1}), I_{i_s}, \dots, I_{i_r})$$

in $\mathrm{mod}(A)$.

Clearly, this generated $\tau(\mathcal{S})$ since $\mathrm{Thick}(\tau(E_1), \dots, \tau(E_{s-1}), I_{i_s}, \dots, I_{i_r})$ equals

$$\tau(\mathrm{Thick}(E_1, \dots, E_{s-1}, \Sigma E_s, \dots, \Sigma E_r)) = \tau(\mathcal{S}).$$

We show that it is an exceptional sequence in $\mathrm{mod}(A)$. By definition of the sequence it actually lies in $\mathrm{mod}(A)$.

Next, we have to check the exceptionality of the sequence. Using the fact that Σ, ν $\tau = \Sigma^{-1}\nu$ are equivalences on $\mathcal{D}^b(\mathrm{mod}(A)) = \mathcal{D}^b(A)$, we get for $i < j$

$$\begin{aligned}
\mathrm{Hom}_A(\tau(E_j), \tau(E_i)) &= \mathrm{Hom}_{\mathcal{D}^b(A)}(\tau(E_j), \tau(E_i)) = \mathrm{Hom}_{\mathcal{D}^b(A)}(E_j, E_i) = 0 \\
\mathrm{Ext}^1(\tau(E_j), \tau(E_i)) &\cong \mathrm{Hom}_{\mathcal{D}^b(A)}(\tau(E_j), \tau(\Sigma E_i)) \cong \mathrm{Hom}_{\mathcal{D}^b(A)}(E_j, \Sigma E_i) \\
&\cong \mathrm{Ext}^1(E_j, E_i) = 0.
\end{aligned}$$

There are no non-zero morphisms from one of the injective modules I_i to one of the modules $\tau(E_j)$ since $\tau(E_j)$ is not injective. Moreover,

$$\begin{aligned}
\mathrm{Ext}^1(I_i, \tau(E_j)) &\cong \mathrm{Hom}_{\mathcal{D}^b(A)}(I_i, \Sigma\tau(E_j)) \cong \mathrm{Hom}_{\mathcal{D}^b(A)}(\Sigma^{-1} I_i, \tau(E_j)) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(A)}(\tau^{-1}(\Sigma^{-1} I_i), E_j) \cong \mathrm{Hom}_A(P_i, E_j) = 0.
\end{aligned}$$

And lastly, for $k < l$

$$\mathrm{Hom}_A(I_{i_l}, I_{i_k}) \cong \mathrm{Hom}_A(P_{i_l}, P_{i_k)}) = 0,$$

and clearly, there are no non-trivial extensions between two injectives.

As mentioned in Chapter 4.2 we know that $c\,\underline{\dim}(M) = \underline{\dim}(\tau(M))$ for $M \in \mathrm{mod}(A)$ indecomposable not projective and $c\,\underline{\dim}(P_i) = -\underline{\dim}(I_i)$. Hence,

$$s_{\tau(E_i)} = c s_{E_i} c^{-1} \quad \text{and} \quad s_{I_i} = c s_{P_i} c^{-1}.$$

Altogether, we have

$$\mathrm{cox}(\tau(\mathcal{S})) = s_{\tau(E_1)} \cdots s_{\tau(E_{s-1})} s_{I_{i_s}} \cdots s_{I_{i_r}} = c\,\mathrm{cox}(\mathcal{S}) c^{-1}.$$

□

The above proposition controls the finite algebraic triangulated categories $\mathcal{T} \cong \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle \phi\tau^r \rangle$ with $\mathrm{typ}(\mathcal{T}) = (\Delta, r, 1)$ but it does not help yet if the order of ϕ is strictly greater than 1. Fortunately, in most cases one may express ϕ in terms of τ and the shift functor Σ. This helps since each thick subcategory is invariant under the shift functor and therefore, the shift has no influence on the question of whether the thick subcategory is invariant under the automorphism group. The following Proposition describes Σ in terms of τ and the respective automorphism ϕ. We conclude this from [**43**] and [**18**].

PROPOSITION 9.2. *Let Δ be a Dynkin graph, let $h_\Delta = m_\Delta + 1$ be the Coxeter number and let $\Sigma\colon \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta})) \to \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ be the shift functor.*

(1) *Let Δ be of the form $A_n (n \geq 3$ odd$), D_n(n$ odd$)$ or E_6, then Σ is isomorphic to $\phi\tau^{-\frac{h_\Delta}{2}}$. Here ϕ is the automorphism of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ of order 2 induced by the automorphism ϕ of $\mathbb{Z}\Delta$ defined in Theorem 7.5.*
(2) *Let Δ be of the form $A_1, D_n(n$ even$), E_7$ or E_8, then $\Sigma \cong \tau^{-\frac{h_\Delta}{2}}$.*
(3) *Let Δ be of the form $A_n(n$ even$)$, then $\Sigma \cong \phi\tau^{-\frac{m_\Delta}{2}}$ where ϕ is the automorphism of infinite order discussed in Theorem 7.5.*

In particular, in all cases $\Sigma^2 \cong \tau^{-h_\Delta}$. □

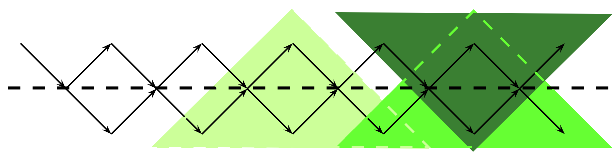


FIGURE 1. Illustration of the shift Σ of $\mathcal{D}^b(\mathrm{mod}(\vec{A}_3))$ as the composition of $\tau^{-\frac{4}{2}}$ and ϕ where ϕ is the reflection at the dashed line.

Before we apply this to the classification, we insert one proposition which simplifies later calculations.

PROPOSITION 9.3. (1) *Let $s := \gcd(h_\Delta, p)$ with $p \in \mathbb{N}$. Then, a thick subcategory $\mathcal{S}$ of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ is $\langle\tau^p\rangle$-invariant if and only if it is $\langle\tau^s\rangle$-invariant.*
(2) *Let $s := r \bmod h_\Delta$ with $r \in \mathbb{N}$ and let*

$$\phi\colon \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta})) \to \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$$

be an automorphism. A thick subcategory $\mathcal{S}$ of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ is $\langle\phi\tau^r\rangle$-invariant if and only if it is $\langle\phi\tau^s\rangle$-invariant.

PROOF. Let $\gcd(h_\Delta, p) = s$. Then, there are $m, n \in \mathbb{Z}$ with $s = mh_\Delta + np$. Let $\mathcal{S} \neq 0$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ which is $\langle\tau^p\rangle$-invariant. Let $X \neq 0$ be indecomposable in $\mathcal{S}$. Then, by Proposition 9.2

$$X = \tau^0(X) = \tau^{-mh_\Delta - np + s}(X) \cong \tau^{-np}(\Sigma^{2m})\tau^s(X).$$

Then, $\Sigma^{2m}\tau^s(X) \cong \tau^{np}(X) \in \mathcal{S}$ since $\mathcal{S}$ is $\langle\tau^p\rangle$-invariant. This implies $\tau^s(X) \in \mathcal{S}$ since $\mathcal{S}$ is thick.

Conversely, a $\langle\tau^s\rangle$-invariant subcategory is clearly $\langle\tau^p\rangle$-invariant.

The second part works similarly applying $\tau^{-h_\Delta} = \Sigma^2$. □

REMARK 9.4. In particular, there are no proper thick subcategories in the case p and h_Δ are coprime since a $\langle\tau^1\rangle$-invariant thick subcategory is either 0 or the whole ambient category.

Finally, we obtain the following classification.

THEOREM 9.5. *Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (Δ, r, t) excluding the cases $(D_n, r, 2)$ for n even and $(D_4, r, 3)$. Let $c \in W_\Delta$ be the Coxeter transformation with respect to an admissible numbering of some fixed orientation $\vec{\Delta}$ of Δ. Put*

$$p = \begin{cases} r & \text{if } \operatorname{typ}(\mathcal{T}) = (\Delta, r, 1), \\ \frac{h_\Delta}{2} + r & \text{if } \operatorname{typ}(\mathcal{T}) = (A_{2n+1}, r, 2), (D_n, r, 2) \text{ with } n \text{ odd or } (E_6, r, 2), \\ \frac{m_\Delta}{2} + r & \text{if } \operatorname{typ}(\mathcal{T}) = (A_{2n}, r, \infty), \end{cases}$$

and $s = \gcd(h_\Delta, p)$. Then, there is a bijective correspondence between

- *the set of thick subcategories of $\mathcal{T}$, and*
- *the set $(\mathrm{NC}(W_\Delta, c))^{c^s}$ of noncrossing partitions w satisfying*

$$w = c^s w c^{-s}.$$

PROOF. Let $\mathcal{T}$ be as above. Then, $\mathcal{T}$ is triangulated equivalent to $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle \phi\tau^r \rangle$ where ϕ and τ are induced by the respective weakly admissible groups of automorphisms of $\mathbb{Z}\Delta$ and $\vec{\Delta}$ is the oriented diagram for some fixed orientation of Δ.

By Theorem 3.4 the thick subcategories of $\mathcal{T}$ are in bijective correspondence with the thick $\langle \phi\tau^r \rangle$-invariant subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$.

Using Proposition 9.2, a thick subcategory $\mathcal{S}$ of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ is $\langle \phi\tau^r \rangle$-invariant if and only if it is $\langle \tau^p \rangle$-invariant. This is because $\Sigma(\mathcal{S}) = \mathcal{S}$.

By Proposition 9.3, Theorem 6.2 and Proposition 9.1 the thick $\langle \tau^p \rangle$-invariant subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ are in correspondence with the elements of $\mathrm{NC}(W_\Delta, c)$ which are invariant under s-fold conjugation by c. □

REMARK 9.6. The cases $(D_n, r, 2)$ for n even and $(D_4, r, 3)$ are excluded in this general theorem since in these cases it is not possible to express the automorphism ϕ of order 2 or 3 in terms of τ and Σ. We consider these cases separately in Proposition 9.12 and Proposition 9.13.

2. The A_n-case

If $\Delta = A_n$ we can formulate the classification theorem in terms of the alternative description $\mathrm{NC}^A(n+1)$ of noncrossing partitions introduced in Chapter 5.1.

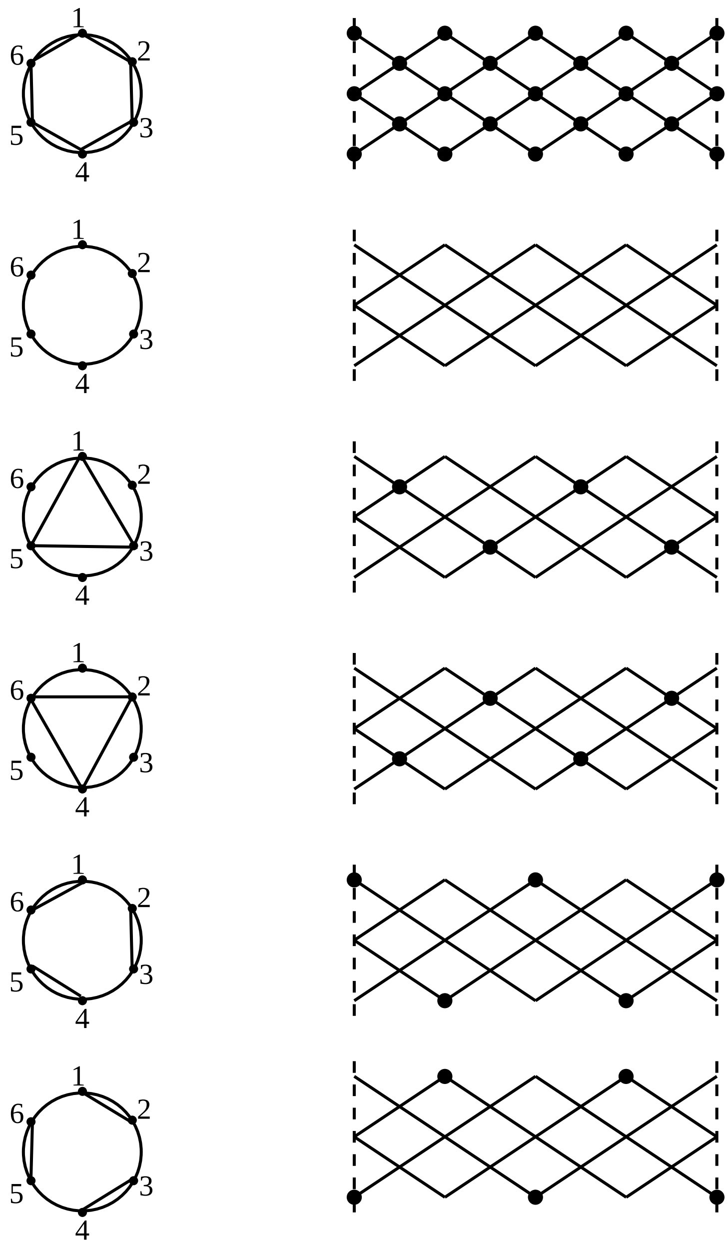
PROPOSITION 9.7. *Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (A_n, r, t). Again let $s = \gcd(h_{A_n}, p)$ where p depends on the type as discussed in Theorem 9.5. Then, there is a bijective correspondence between*

- *the set of thick subcategories of $\mathcal{T}$, and*
- *the set of elements of $\mathrm{NC}^A(n+1)$ invariant under a clockwise rotation by $s\frac{2\pi}{n+1}$.*

PROOF. By Proposition 5.11 conjugation by the Coxeter transformation corresponds to rotation. Then, use Theorem 9.5. □

Example 9.8. Let Λ the Nakayama algebra with ${}_s\Gamma_\Lambda = \mathbb{Z}A_5/\langle\tau^4\rangle$. Then, $h_{A_5} = 6$ and $s = \gcd(4,6) = 2$.

Hence, we are interested in the noncrossing partitions of $\{1,\dots,6\}$ which are invariant under rotation by $\frac{2}{3}\pi$. They are listed here together with their corresponding thick subcategories of $\underline{\mathrm{mod}}(\Lambda)$ indicated by their indecomposables arranged in the Auslander-Reiten quiver ${}_s\Gamma_\Lambda = \mathbb{Z}A_5/\langle\tau^{-4}\rangle$.



3. The D_n-case

Analogously, we can formulate the classification for $\Delta = D_n$. Recall the maps $\rho, \sigma\colon \mathrm{NC}^D(n) \to \mathrm{NC}^D(n)$ from Definition 5.13.

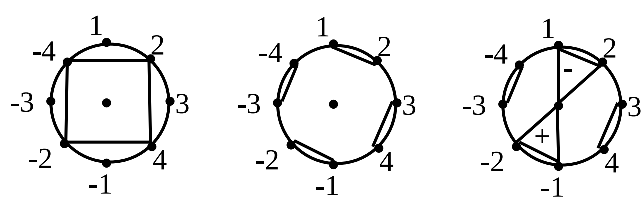
PROPOSITION 9.9. *Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (D_n, r, t) excepting $(D_n, r, 2)$ for n even and $(D_4, r, 3)$. Let $s = \gcd(h_{D_n}, r)$ or $s = \gcd(h_{D_n}, r + \frac{h_\Delta}{2})$ depending on t (see Theorem 9.5). Then, there is a bijective correspondence between*

- *the set of thick subcategories of $\mathcal{T}$, and*
- *the set of elements of $\mathrm{NC}^D(n)$ invariant under $(\sigma\rho)^s$.*

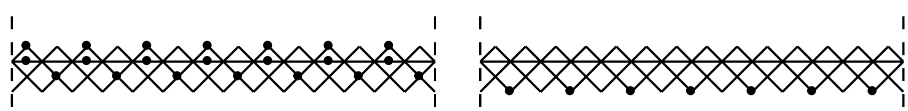
Note that $\sigma^s = \mathrm{id}$ if s is even and $\sigma^s = \sigma$ if s is odd, and that ρ and σ commute.

PROOF. Apply Theorem 9.5 and Lemma 5.14. □

EXAMPLE 9.10. Let Λ be a self-injective representation-finite algebra with $\mathrm{typ}(\Lambda) = (D_5, 2, 2)$, i.e. its stable Auslander-Reiten quiver is of the form $\mathbb{Z}D_5/\langle\phi\tau^{14}\rangle$ or $\mathrm{typ}(\underline{\mathrm{mod}}(\Lambda)) = (D_5, 14, 2)$. We have $s = \gcd(8, 14 + \frac{8}{2}) = 2$ and hence, applying Proposition 9.9, we look for the ρ^2-invariant elements of $\mathrm{NC}^D(5)$. Here ρ^2 is a rotation by $\frac{\pi}{2}$. The invariance works for instance for the first two of the following partitions, but not for the third.



The corresponding thick subcategories in the first two cases are the following thick subcategories.

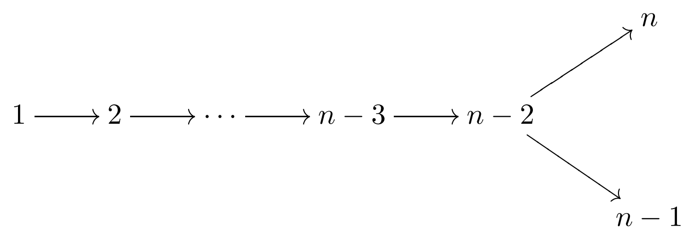


Since we cannot express ϕ in terms of Σ and τ if n is even, we have to study this case separately. The following lemma and the following proposition work for arbitrary n but we only have to use it for even n.

LEMMA 9.11. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_n))$. Let ϕ be the automorphism of $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_n))$ induced by the automorphism ϕ of order 2 of $\mathbb{Z}D_n$ defined in Theorem 7.5. Then,*

$$\mathrm{cox}(\phi(\mathcal{S})) = \sigma(\mathrm{cox}(\mathcal{S})).$$

PROOF. Recall that $\phi\colon \mathbb{Z}D_n \to \mathbb{Z}D_n$ exchanges $(i, n-1)$ and (i, n) $\forall i \in \mathbb{Z}$ and fixes the other vertices. According to Riedtmann [**74**] we call the vertices $(i, n-1)$ and (i, n) for $i \in \mathbb{Z}$ *high* vertices and the others *low* vertices. Accordingly, we will call the indecomposable objects of $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_n))$ high or low. The map ϕ induces an automorphism on $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_n))$ and hence on $\mathrm{mod}(k\vec{D}_n)$. Fix the following orientation and numbering of D_n.



Determining the Auslander-Reiten quiver of $\mathrm{mod}(k\vec{D}_n)$ with respect to this orientation one sees that ϕ maps the indecomposable high representation

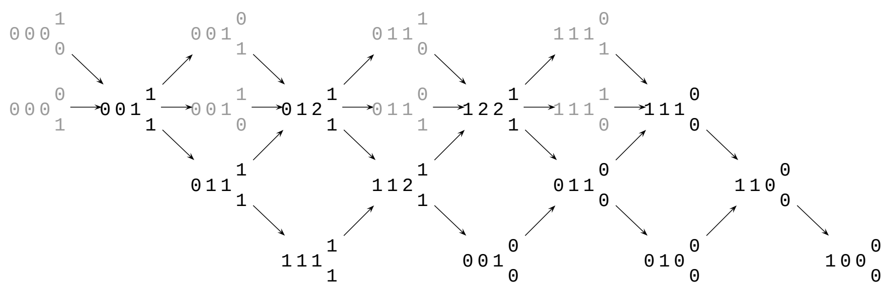


FIGURE 2. Auslander-Reiten quiver of $\mathrm{mod}(k\vec{D}_5)$. *High* representations are coloured grey.

E^{n-1}_{n-t} of $\vec{D}_n$ represented by the dimension vector $e_{n-1} + e_{n-2} + \ldots + e_{n-t}$ to the high representation E^n_{n-t} representated by $e_n + e_{n-2} + \ldots + e_{n-t}$ and vice versa. The low representations are fixed.

We have

$$s_{E^n_{n-t}} = s_n s_{n-2} \cdots s_{n-t} \cdots s^{-1}_{n-2} s^{-1}_n = ((-(n-t), n))$$

and

$$s_{E^{n-1}_{n-t}} = s_{n-1} s_{n-2} \cdots s_{n-t} \cdots s^{-1}_{n-2} s^{-1}_{n-1} = ((n-t, n)).$$

Now let $\mathcal{S}$ be a thick subcategory and let w be the corresponding D_n-partition. Again we consider w block by block.

A pair of non-zero blocks $\{i_1, \ldots, i_k\} \uplus \{-i_1, \ldots, -i_k\}$ corresponds to a paired cycle $(i_1, \ldots, i_k)(-i_1, \ldots, -i_k)$. This is a product of reflections

$$((i_1, i_2))((i_2, i_3)) \cdots ((i_{k-1}, i_k)) = s_{E_1} \cdots s_{E_{k-1}}$$

corresponding to an exceptional sequence $E_1, \ldots, E_{k-1}$ in $\mathrm{mod}(k\vec{D}_n)$.

If n and $-n$ are not contained in the paired cycle, then none of the representations $E_1, \ldots, E_{k-1}$ are high representations and hence applying ϕ does not change w. If n is contained, we have the product of reflections

$$(i_1, \ldots, n)(-i_1, \ldots, -n) = ((i_1, i_2)) \cdots ((i_{k-1}, n))$$

corresponding to an exceptional sequence $E_{i_1}, \ldots, E_{i_{k-1}} = E^{n-1}_{i_{k-1}}$. A consideration of the morphisms in $k(\mathbb{Z}D_n)$ as in [**74**] shows that the sequence $\phi(E_{i_1}) = E_{i_1}, \ldots, \phi(E^{n-1}_{i_{k-1}}) = E^n_{i_{k-1}}$ is exceptional. Hence,

$$s_{\phi(E_1)} \cdots s_{\phi(E_{i_{k-1}})} = ((i_1, i_2)) \cdots ((-i_{k-1}, n)) = (i_1, \ldots, -n)(-i_1, \ldots, n).$$

Finally, consider a zero-block represented by

$$\begin{aligned}(i_1,\dots,i_k,-i_1,\dots,-i_k)(n,-n) &= ((i_1,i_2))\cdots((i_{k-1},i_k))((i_k,n))((i_k,-n))\\ &= s_{E_{i_1}}\cdots s_{E_{i_k}^{n-1}}s_{E_{i_k}^{n}}.\end{aligned}$$

Applying ϕ to the exceptional sequence, yields the same term since $((i_k,n))$ and $((i_k,-n))$ commute. $\square$

PROPOSITION 9.12. *Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type $(D_n,r,2)$. Let $s=r \bmod h_{D_n}$. There is a bijective correspondence between*

- *the set of thick subcategories of $\mathcal{T}$, and*
- *the set of elements of $\mathrm{NC}^D(n)$ invariant under $\sigma^{s+1}\rho^s$.*

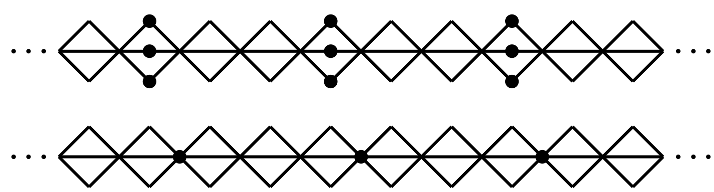
PROOF. Use the second part of Proposition 9.3 for the reduction to $s=r \bmod h_{D_n}$. Then, apply Lemma 5.14 and Lemma 9.11. $\square$

Finally, the case of a category of type $(D_4,r,3)$ is still missing. Easy observations allow us to classify the thick subcategories in this case by hand.

PROPOSITION 9.13. *Let $\mathcal{T}$ be a finite triangulated category which is connected algebraic and standard of type $(D_4,r,3)$.*

Put $s=r \bmod 3$. So we may assume $s\in\{0,1,2\}$.

If $s=0$, the only proper thick subcategories of $\mathcal{T}$ are in correspondence with the following thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_4))$.



In the cases $s=1$ and $s=2$ there are no proper thick subcategories.

PROOF. Let ϕ be the automorphism of order 3 described in Theorem 7.5. Since $\Sigma\cong\tau^3$, we have that a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{D}_4))$ is $\langle\phi\tau^r\rangle$-invariant if and only if it is $\langle\phi\tau^s\rangle$-invariant.

In case $s=0$, adding further indecomposables to the described proper thick subcategories would yield the whole category because of thickness.

The same argument shows that in case $s=1$ and $s=2$ there are no proper thick subcategories. $\square$

4. The number of thick subcategories

Thinking of Chapter 5.5, there is a polynomial determining the number of thick subcategories. Namely, the classifying set $(\mathrm{NC}(W_\Delta,c))^{c^s}$ is the fixed point set appearing in the cyclic sieving phenomenon for noncrossing partitions.

THEOREM 9.14. *Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (Δ,r,t) excluding the cases $(D_n,r,2)$*

for n even and $(D_4, r, 3)$. Let $c \in W_\Delta$ be the Coxeter transformation with respect to an admissible numbering of $\vec{\Delta}$. Put

$$p = \begin{cases} r & \text{if } \operatorname{typ}(\mathcal{T}) = (\Delta, r, 1), \\ \frac{h_\Delta}{2} + r & \text{if } \operatorname{typ}(\mathcal{T}) = (A_{2n+1}, r, 2), (D_n, r, 2) \text{ with } n \text{ odd or } (E_6, r, 2), \\ \frac{m_\Delta}{2} + r & \text{if } \operatorname{typ}(\mathcal{T}) = (A_{2n}, r, \infty), \end{cases}$$

and $s = \gcd(h_\Delta, p)$. Then, the number of thick subcategories of $\mathcal{T}$ equals

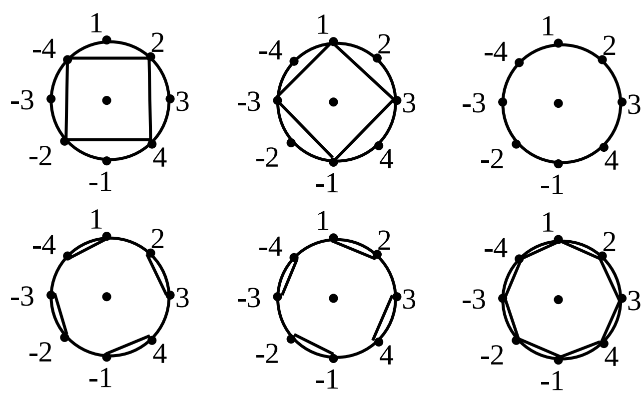
$$\operatorname{Cat}(W_\Delta, z_{o(c^s)}) = \operatorname{Cat}(W_\Delta, e^{2\pi i s/h_\Delta}).$$

PROOF. Use Theorem 9.5 and Theorem 5.10. Moreover, the order of c^s equals $\frac{h_\Delta}{s}$ since s divides h_Δ. □

EXAMPLE 9.15. We return to the previous example of a self-injective algebra Λ with $\operatorname{typ}(\underline{\operatorname{mod}}(\Lambda)) = (D_5, 14, 2)$. We have $s = \gcd(8, 18) = 2$. The order of c^2 equals 4, and hence the number of thick subcategories of $\underline{\operatorname{mod}}(\Lambda)$ is equal to

$$\operatorname{Cat}(W_{D_5}, e^{\frac{\pi i}{2}}) = 6.$$

Let us check this number. The thick subcategories correspond by Proposition 9.9 to the elements of $\mathrm{NC}^D(5)$ which are invariant under rotation by $\frac{\pi}{2}$ and these are the following six elements of $\mathrm{NC}^D(5)$.



Again there are two cases which are not covered by the theorem. If $\operatorname{typ}(\mathcal{T}) = (D_4, r, 3)$, then we can count the thick subcategories by hand according to Theorem 9.13. If $\operatorname{typ}(\mathcal{T}) = (D_n, r, 2)$ for an even n, then we can use the combinatorial description $\mathrm{NC}^D(n)$ to count. This is possible in general for types (D_n, r, t) with $t \leq 2$ and (A_n, r, t), and for the sake of completeness we present the observations for all these types. Also this approach contains a description or even an algorithm to construct the relevant elements in $\mathrm{NC}^A(n)$ or $\mathrm{NC}^D(n)$.

PROPOSITION 9.16. *Let $h = xs$ with $x, s \in \mathbb{N}$ and $x > 1$. Then, there is a surjective map*

$$F\colon \{\textit{elements of } \mathrm{NC}^A(h) \textit{ invariant under rotation by } (s\frac{2\pi}{h})\} \to \mathrm{NC}^A(s).$$

Moreover, for each $w \in \mathrm{NC}^A(s)$ there are exactly $s+1$ elements in the preimage of w under F. Hence, the number of elements of $\mathrm{NC}^A(h)$ invariant under rotation by $s\frac{2\pi}{h}$ equals $(s+1)C_s = \binom{2s}{s}$.

PROOF. The map F is given as follows. Let $w = w_1 \uplus \ldots \uplus w_r$ be an invariant noncrossing partition of $\{1, \ldots, h\}$. Then,

$$F(w) = F(w_1) \cup \ldots \cup F(w_r)$$

with

$$F(w_i) = F(\{p_1, \ldots, p_m\}) = \{[p_1], \ldots, [p_m]\}$$

where $[p_j] = p_j \bmod s$ if $s \nmid p_j$ and $[p_j] = s$ otherwise.

To a partition in $\mathrm{NC}^A(s)$ we construct all partitions in its preimage. This construction requires the Kreweras complement $K(w)$ for $w \in \mathrm{NC}^A(n)$ defined in Chapter 5.4.

Now let $w \in \mathrm{NC}^A(s)$. We list all $s+1$ partitions in the preimage of w. In particular, this provides a method to construct all $(s\frac{2\pi}{h})$-invariant noncrossing partitions of $\{1, \ldots, h\}$ from the noncrossing partitions of $\{1, \ldots, s\}$.

Let $w = w_1 \uplus \ldots \uplus w_r$ be a noncrossing partition of $\{1, \ldots, s\}$. A block w_i of w corresponds in the preimage

- either to a big block of cardinality $x|w_i|$
- or to x disjoint blocks of cardinality $|w_i|$.

Otherwise the partition in the preimage would be crossing or not invariant.

To each block w_i of w we define a partition in the preimage which contains a big block of cardinality $x|w_i|$. All other blocks of this partition are uniquely determined by this. Otherwise the partition would be crossing. This construction yields r partitions in the preimage.

Next we define to each block of $K(w)$ a partition in the preimage of w. Let v be a partition in the preimage of $K(w)$ under F containing a big block. There is a partition $u \in \mathrm{NC}^A(h)$ with $v = K(u)$. Thus,

$$F(u) = F(K(v)) = K(F(v)) = K(K(w)) = w,$$

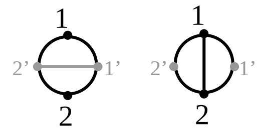
and u does not contain a big block since otherwise $K(u) \uplus u$ would be crossing. Therefore, u is different from the partitions defined in the first step. Moreover, the partitions constructed from different blocks of $K(w)$ are different since the assignment K is bijective. Simion and Ullman show in [**79**] that $K(w)$ has $s - r + 1$ blocks and hence, we obtain $s - r + 1$ further partitions in the preimage.

Altogether, we have defined $r + s - r + 1 = s + 1$ partitions in the preimage.

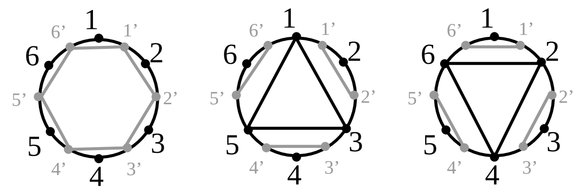
It remains to show that there are no further. Let $v \in \mathrm{NC}^A(h)$ be a partition in $F^{-1}(w)$. If v contains a big block, we have counted v in the first step. If not, v contains exactly xr blocks, hence $K(v)$ contains $h - (xr) + 1$ blocks. Consider $K(v)$. If $K(v)$ contains a big block, we have counted v in the second step. If not, $K(v)$ contains $x(s + 1 - r)$ blocks. Thus, $h - xr + 1 = x(s + 1 - r)$ and this implies $x = 1$, a contradiction. □

REMARK 9.17. The above proposition counts the number of thick subcategories for a finite algebraic triangulated category of type (A_n, r, t). Just put $h = h_{A_n}$ and $s = \gcd(h, p)$ according to Theorem 9.5.

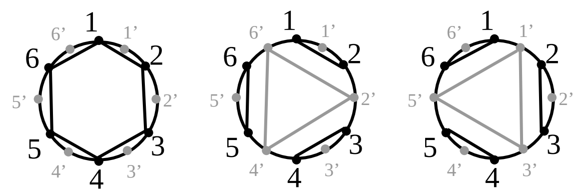
EXAMPLE 9.18. Let $s = 2$ and $h = 3 \cdot 2 = 6$. The noncrossing partitions of $\{1, 2\}$ are the following black coloured partitions.



The elements of $\mathrm{NC}^A(h)$ invariant under rotation by $s\frac{2\pi}{h}$ are given by the following partitions. First we construct the preimage of the first partition.



And this is the preimage of the second partition. The grey coloured parti-



tions represent the partitions of the primed numbers as used in the proof of Proposition 9.16.

PROPOSITION 9.19. *Let s be an integer with $0 \leq s \leq h_{D_n} = 2n-2$. Then,*

$$\#(\mathrm{NC}^D(n))^{(\sigma\rho)^s} = \begin{cases} \binom{2p}{p} \text{ with } p = \gcd(n-1,s) & \text{if } s \notin \{0, n-1, 2n-2\}, \\ \mathrm{Cat}(D_n) & \text{if } s = 0, 2n-2, \text{ or if} \\ & s = n-1 \text{ with } n \text{ even}, \\ \mathrm{Cat}(D_{n-1}) & \text{if } s = n-1 \text{ with } n \text{ odd}. \end{cases}$$

Here $\mathrm{Cat}(D_n) = \binom{2n}{n} - \binom{2n-2}{n-1}$.

PROOF. Denote by $\mathrm{NC}^D_\pm(n)$ the elements of $\mathrm{NC}^D(n)$ with a non-zero block containing n or $-n$.

Assume $s \notin \{0, n-1, 2n-2\}$. The elements of $\mathrm{NC}^D_\pm(n)$ are not $(\sigma\rho)^s$-invariant: Let $w \in \mathrm{NC}^D_\pm(n)$ and let B be a block of w with $n \in B$. Then $C := (\sigma\rho)^s(B)$ is a block containing n or $-n$ which is neither B itself nor $-B = \sigma\rho^{n-1}(B)$. If the partition was $(\sigma\rho)^s$-invariant, C would be a block of w, but this is not possible since C and B or $-B$ are neither equal nor disjoint.

Hence, we have a correspondence between

- $(\mathrm{NC}^D(n))^{(\sigma\rho)^s}$, and
- the elements of $\mathrm{NC}^D(n) \backslash \mathrm{NC}^D_\pm(n)$ invariant under rotation by $s\frac{\pi}{n-1}$.

Let $p = \gcd(n-1, s)$. Since each element of $\mathrm{NC}^D(n) \setminus \mathrm{NC}^D_\pm(n)$ is invariant under rotation by $(n-1)\frac{\pi}{n-1}$, we can add a correspondence with

- the elements of $\mathrm{NC}^D(n)\backslash\mathrm{NC}^D_{\pm}(n)$ invariant under rotation by $p\frac{\pi}{n-1}$.

These elements, in turn, are canonically in correspondence with

- the elements of $\mathrm{NC}^A(2n-2)$ invariant under rotation by $p\frac{\pi}{n-1}$.

For this purpose, just forget the centroid. Note that a zero-block containing only a single pair i and $-i$ — which is not allowed for $\mathrm{NC}^D(n)$ — does not appear since $p \neq n-1$ by assumption.

Since p is a non-trivial divisor of $2n-2$, by Proposition 5.5 the above elements correspond to

- $\mathrm{NC}^B(p)$

and the number of elements in that is equal to $\binom{2p}{p}$.

Next, assume $s = 2n-2$ or $s = 0$. Then, the number equals the number of elements in $\mathrm{NC}^D(n)$ invariant under rotation by 2π. These are all, and as mentioned in Chapter 5 their number equals the type D Catalan number $\mathrm{Cat}(D_n) = \binom{2n}{n} - \binom{2n-2}{n-1}$.

If $s = n-1$ where n is even, we have to count the elements of $\mathrm{NC}^D(n)$ invariant under $\sigma\rho^{n-1}$. This is again all of $\mathrm{NC}^D(n)$ by definition.

Finally, let $s = n-1$ and let n be odd. We look for the $(\sigma\rho)^{n-1} = \rho^{n-1}$-invariant elements of $\mathrm{NC}^D(n)$. These are the elements $\mathrm{NC}^D(n) \setminus \mathrm{NC}^D_{\pm}(n)$. As above we compare this with the elements of $\mathrm{NC}^A(2n-2)$ which are invariant under rotation by $(n-1)\frac{\pi}{n-1} = \pi$. But this time, this set, which consists of $\binom{2(n-1)}{n-1}$ elements, is too big. We have to ignore the elements which would correspond to a single pair $\{i, -i\}$ in $\mathrm{NC}^D(n)$. With the help of Proposition 9.16 we can count these elements. Recall the surjective map

$$F\colon \{\text{elements of } \mathrm{NC}^A(2n-2) \text{ inv. under rotation by } \pi\} \to \mathrm{NC}^A(n-1).$$

Let w be a partition of $\mathrm{NC}^A(2n-2)$ which we want to ignore, i.e. w contains exactly one block of the form $\{i, i+(n-1)\}$ where $1 \le i \le n-1$. Then, $F(w)$ contains a singleton $\{i\}$, and $F(w) \setminus \{i\}$ can be understood as an arbitrary element of $\mathrm{NC}^A(n-2)$. Hence, the preimage of $F(w)$ consists of $C_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2}$ elements. There are $n-1$ possibilities to place the 'forbidden' block in w and thus, we subtract $(n-1)\frac{1}{n-1}\binom{2(n-2)}{n-2} = \binom{2(n-2)}{n-2}$. □

Remark 9.20. The above proposition counts the number of thick subcategories of a finite algebraic triangulated category of type $(D_n, r, 1)$ or $(D_n, r, 2)$ if n is odd. Just put $s = \gcd(h_{D_n}, r)$ or $s = \gcd(h_{D_n}, r + \frac{h_{D_n}}{2})$, respectively.

Proposition 9.21. *Let s be an integer with $0 \le s \le h_{D_n} = 2n-2$. Then,*

$$\#(\mathrm{NC}^D(n))^{\sigma^{s+1}\rho^s} = \begin{cases} \binom{2p}{p} \text{ with } p = \gcd(n-1, s) & \text{if } s \notin \{0, n-1, 2n-2\}, \\ \mathrm{Cat}(D_{n-1}) & \text{if } s = 0, 2n-2, \text{ or if} \\ & s = n-1 \text{ with } n \text{ even}, \\ \mathrm{Cat}(D_n) & \text{if } s = n-1 \text{ with } n \text{ odd}. \end{cases}$$

Proof. The arguments are completely analogous to the arguments in Proposition 9.19 taking into account the change from σ^s to σ^{s+1}. □

Remark 9.22. This counts in particular the thick subcategories of a finite algebraic triangulated category of type $(D_n, r, 2)$ for even n. Put $s = r \bmod h_{D_n}$. This is the case which was not yet covered by Theorem 9.14.

5. Overview

Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (Δ, r, t). The following table gives an overview of the classification of the thick subcategories of $\mathcal{T}$ and of the number of thick subcategories of $\mathcal{T}$. Moreover, one can see where to find the respective information within this thesis.

Note that each row contains at least one entry concerning classification and one entry concerning counting. Hence, the classification is complete.

In all cases denote by $c \in W_\Delta$ the respective Coxeter transformation with respect to an admissible numbering of some fixed orientation of Δ.

type	classifying poset	alternative description	number as W-q-Catalan number	number directly
$(A_n, r, 1)$	$(\mathrm{NC}(W_{A_n}, c))^{c^s}$, $s = \gcd(n+1, r)$, see Theorem 9.5	elements of $\mathrm{NC}^A(n+1)$ invariant under rotation by $s\frac{2\pi}{n+1}$, $s = \gcd(n+1, r)$, see Proposition 9.7	$\mathrm{Cat}(W_{A_n}, e^{\frac{2\pi i s}{n+1}})$ Theorem 9.14	C_s if $s = n+1$; $\binom{2s}{s}$ else; Proposition 9.16
$(A_n, r, 2)$ $n \geq 3$ odd	$(\mathrm{NC}(W_{A_n}, c))^{c^s}$, $s = \gcd(n+1, \frac{n+1}{2} + r)$, see Theorem 9.5	elements of $\mathrm{NC}^A(n+1)$ invariant under rotation by $s\frac{2\pi}{n+1}$, $s = \gcd(n+1, \frac{n+1}{2} + r)$, see Proposition 9.7	$\mathrm{Cat}(W_{A_n}, e^{\frac{2\pi i s}{n+1}})$ Theorem 9.14	C_s if $s = n+1$; $\binom{2s}{s}$ else; Proposition 9.16
(A_n, r, ∞) n even	$(\mathrm{NC}(W_{A_n}, c))^{c^s}$, $s = \gcd(n+1, \frac{n}{2} + r)$, see Theorem 9.5	elements of $\mathrm{NC}^A(n+1)$ invariant under rotation by $s\frac{2\pi}{n+1}$, $s = \gcd(n+1, \frac{n}{2} + r)$, see Proposition 9.7	$\mathrm{Cat}(W_{A_n}, e^{\frac{2\pi i s}{n+1}})$ Theorem 9.14	C_s if $s = n+1$; $\binom{2s}{s}$ else; Proposition 9.16
$(D_n, r, 1)$	$(\mathrm{NC}(W_{D_n}, c))^{c^s}$, $s = \gcd(2n-2, r)$, see Theorem 9.5	elements of $\mathrm{NC}^D(n)$ invariant under $(\sigma\rho)^s$ $s = \gcd(2n-2, r)$, see Proposition 9.9	$\mathrm{Cat}(W_{D_n}, e^{\frac{\pi i s}{n-1}})$ Theorem 9.14	$\mathrm{Cat}(D_n)$ if $s = 2n-2$ or $s = n-1$ odd; $\mathrm{Cat}(D_{n-1})$ if $s = n-1$ even; $\binom{2p}{p}$ else where $p = \gcd(n-1, s)$; Proposition 9.19
$(D_n, r, 2)$ n odd	$(\mathrm{NC}(W_{D_n}, c))^{c^s}$, $s = \gcd(2n-2, \frac{2n-2}{2} + r)$, see Theorem 9.5	elements of $\mathrm{NC}^D(n)$ invariant under $(\sigma\rho)^s$ $s = \gcd(2n-2, \frac{2n-2}{2} + r)$, see Proposition 9.9	$\mathrm{Cat}(W_{D_n}, e^{\frac{\pi i s}{n-1}})$ Theorem 9.14	$\mathrm{Cat}(D_n)$ if $s = 2n-2$; $\mathrm{Cat}(D_{n-1})$ if $s = n-1$; $\binom{2p}{p}$ else where $p = \gcd(n-1, s)$; Proposition 9.19
$(D_n, r, 2)$ n even		elements of $\mathrm{NC}^D(n)$ invariant under $\sigma^{s+1}\rho^s$, $s = r \bmod (2n-2)$, see Proposition 9.12		$\mathrm{Cat}(D_{n-1})$ if $s = 0$ or $s = n-1$; $\binom{2p}{p}$ else where $p = \gcd(n-1, s)$; Proposition 9.21
$(D_4, r, 3)$		$s = r \bmod 3$, $s = 0$: six distinguished proper thick subcategories, $s = 1, 2$: no proper ones, see Proposition 9.13		8 2
$(E_n, r, 1)$ $n = 6, 7, 8$	$(\mathrm{NC}(W_{E_n}, c))^{c^s}$, $s = \gcd(h_{E_n}, r)$, see Theorem 9.5		$\mathrm{Cat}(W_{E_n}, e^{2\pi i s/h_{E_n}})$ Theorem 9.14	
$(E_6, r, 2)$	$(\mathrm{NC}(W_{E_6}, c))^{c^s}$, $s = \gcd(12, r+6)$, see Theorem 9.5		$\mathrm{Cat}(W_{E_6}, e^{\frac{\pi i s}{6}})$ Theorem 9.14	

6. The cyclic sieving phenomenon for thick subcategories

We have seen above that the W-q-Catalan number appearing in the cyclic sieving phenomenon for noncrossing partitions helps us counting thick subcategories.

But we can also view it the other way round. The set of thick subcategories yields a correspondent of this combinatorial phenomenon in the algebraic or category theoretical world. To make this clear, we formulate it explicitly.

Let $\vec{\Delta}$ be a Dynkin quiver and denote by $T(\mathcal{D}^b(k\vec{\Delta}))$ the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$. The Auslander-Reiten translation τ acts canonically on this set. Note that then $\langle\tau\rangle$ is a finite group since $\tau^{h_\Delta} = \Sigma^{-2}$ and this is the identity on $T(\mathcal{D}^b(k\vec{\Delta}))$. Then, the triple

$$(T(\mathcal{D}^b(k\vec{\Delta})), \langle\tau\rangle, \mathrm{Cat}(W_\Delta, q))$$

exhibits the cyclic sieving phenomenon.

Here the fixed point set

$$(T(\mathcal{D}^b(k\vec{\Delta})))^{\tau^s}$$

for $1 \leq s \leq h_\Delta$ corresponds to the set of thick subcategories of the orbit category $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle\tau^s\rangle$. As mentioned in Chapter 7, after an observation of Xiao und Zhu [**85**] the orbit category $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\langle\tau^s\rangle$ is actually a triangulated category for each s.

7. The lattice structure

Let $\mathcal{T}$ be a finite triangulated category which is connected, algebraic and standard of type (Δ, r, t). The set $T(\mathcal{T})$ of thick subcategories of $\mathcal{T}$ forms a lattice as it does for any triangulated category. In fact, we can view it as a *sublattice* of the lattice of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ by identifying a thick subcategory $\mathcal{S}$ in $\mathcal{T}$ with its preimage $\pi^{-1}(\mathcal{S})$ where π is the projection $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta})) \to \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/\Phi \cong \mathcal{T}$. Analogously, $T(\mathcal{T})$ corresponds to a sublattice of $\mathrm{NC}(W, c)$. Denote by

$$\widetilde{\mathrm{cox}}\colon T(\mathcal{T}) \to \mathrm{NC}(W, c)$$

the associated map. This is given by $\mathcal{S} \mapsto \mathrm{cox}(\pi^{-1}(\mathcal{S}))$. It is order-preserving since π and cox are. Also it is compatible with taking lattice complements.

Proposition 9.23. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. Then,*

$$\widetilde{\mathrm{cox}}({}^\perp\mathcal{S}) = K^c(\widetilde{\mathrm{cox}}(\mathcal{S})).$$

Proof. First we need to show that $\pi^{-1}(\mathcal{S}^\perp) = (\pi^{-1}(\mathcal{S})^\perp)$. Observe that we usually identify the objects in the orbit category with the objects in the original category but for clarity we will speak for instance of $\pi(X)$ indicating that the object is considered in the orbit category. Pick an element $Z \in \pi^{-1}(\mathcal{S}^\perp)$. Then, for each $R \in \pi^{-1}(\mathcal{S})$ we have

$$0 = \mathrm{Hom}_{\mathcal{T}}(\pi(R), \pi(Z)) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(k\vec{\Delta})}(R, \Phi^n Z).$$

In particular, this implies $\mathrm{Hom}_{\mathcal{D}^b(k\vec{\Delta})}(R, Z) = 0$ for each $R \in \pi^{-1}(\mathcal{S})$ and hence, $Z \in (\pi^{-1}(\mathcal{S}))^\perp$.

Now pick $Z \in (\pi^{-1}(\mathcal{S}))^\perp$. Consider an arbitrary object of $\mathcal{S}$. This is of the form $\pi(R)$ for some $R \in \pi^{-1}(\mathcal{S})$ since π is dense. Then,

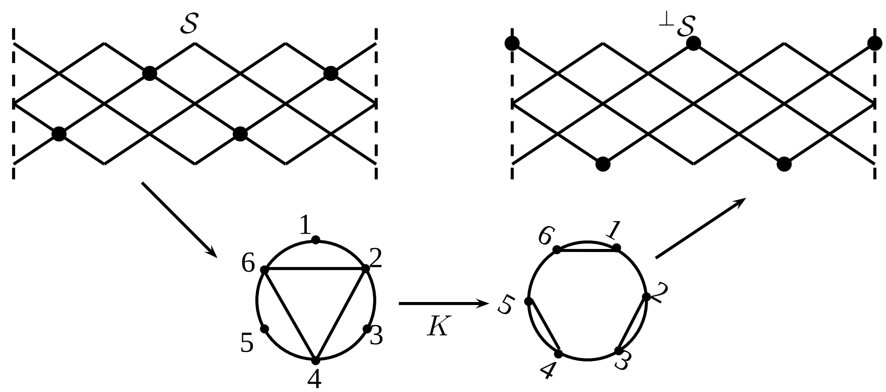
$$\mathrm{Hom}_{\mathcal{T}}(\pi(R), \pi(Z)) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(k\vec{\Delta})}(\Phi^n(R), Z).$$

Observe that $\pi^{-1}(\mathcal{S})$ is naturally $\langle\Phi\rangle$-invariant and therefore, $\Phi^n(R) \in \pi^{-1}(\mathcal{S})$ for each $n \in \mathbb{Z}$. Hence, we have that $\mathrm{Hom}_{\mathcal{D}^b(k\vec{\Delta})}(\Phi^n(R), Z)$ equals zero for each $n \in \mathbb{Z}$ and so does the above direct sum. Thus, $\pi(Z) \in \mathcal{S}^\perp$ and so $Z \in \pi^{-1}(\mathcal{S}^\perp)$.

The assertion for the left perpendicular category is analogous.

An application of Proposition 6.5 now proves the statement. □

Example 9.24. As in Section 2 of this chapter we consider a triangulated category $\mathcal{T}$ of type $(A_5, 4, 1)$. The following diagram shows exemplary how we may determine ${}^\perp\mathcal{S}$ from a thick subcategory $\mathcal{S}$ of $\mathcal{T}$ via the Kreweras complement K in $NC^A(6)$.



CHAPTER 10

Thick subcategories for orbit algebras of repetitive algebras

1. The classification

In Chapter 8.3 we introduced repetitive algebras and their orbit algebras. This is an important class of self-injective algebras. Self-injective algebras of finite representation type are of this form. Namely they are isomorphic to an orbit algebra $\hat{A}/G$ where A is a representation-finite hereditary algebra and G is an admissible group of automorphisms. The stable module categories of these algebras are classified by Amiot's Theorem 7.8 as orbit categories of derived categories of representation-finite hereditary algebras and therefore, we could give a complete classification of the thick subcategories by the last chapter.

If A is an arbitrary hereditary finite-dimensional algebra and G is an admissible group of automorphism on $\hat{A}$, there is no such classification of $\underline{\mathrm{mod}}(\hat{A}/G)$ similar to Amiot's. Let

$$H\colon \mathcal{D}^b(\mathrm{mod}(A)) \to \underline{\mathrm{mod}}(\hat{A})$$

be the triangulated equivalence of Theorem 8.20 and let H' be its quasi-inverse. In the sequel we refer to H as the *Happel functor*. Recall that $g \in G$ induces an automorphism on $\underline{\mathrm{mod}}(\hat{A})$ which we denote by the same letter. Then, via the Happel functor $g \in G$ induces an automorphism g' on $\mathcal{D}^b(\mathrm{mod}(A))$ according to the following diagram.

$$\begin{array}{ccc} \underline{\mathrm{mod}}(\hat{A}) & \xleftarrow{H} & \mathcal{D}^b(\mathrm{mod}(A)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \underline{\mathrm{mod}}(\hat{A}) & \xrightarrow[H']{} & \mathcal{D}^b(\mathrm{mod}(A)) \end{array}$$

If we assume that the induced automorphism group G' is generated by one automorphism on $\mathcal{D}^b(\mathrm{mod}(A))$, we can consider the orbit category $\mathcal{D}^b(\mathrm{mod}(A))/G'$. Now in general it is not clear whether $\underline{\mathrm{mod}}(\hat{A}/G)$ is triangulated equivalent to $\mathcal{D}^b(\mathrm{mod}(A))/G'$ where G' is the induced automorphism group. If one does not assume certain properties for $\hat{A}$ (like that it is locally bounded and locally support-finite), the categories might not even be equivalent as additive categories.

Nevertheless, we can classify the thick subcategories of $\underline{\mathrm{mod}}(\hat{A}/G)$ in the following setting. We assume that the push-down functor $F_\lambda\colon \underline{\mathrm{mod}}(\hat{A}) \to \underline{\mathrm{mod}}(\hat{A}/G)$ associated to the projection $F\colon \hat{A} \to \hat{A}/G$ is dense. Moreover, we assume that the induced automorphism group G' on $\mathcal{D}^b(\mathrm{mod}(A))$ fulfils the assumptions in Keller's Theorem 3.3. Note that this means in particular that we need to assume that G' and therefore also G is a cyclic group generated by one automorphism of $\mathcal{D}^b(\mathrm{mod}(A))$.

THEOREM 10.1. *Let A be a hereditary finite-dimensional k-algebra, let $\hat{A}$ be its repetitive algebra and let G be a cyclic admissible group of automorphism on $\hat{A}$. Assume that F_λ and G' fulfil the assumptions described above. Then, there is a bijective correspondence between*

- *the set of thick subcategories of* $\underline{\mathrm{mod}}(\hat{A}/G)$,
- *the set of G-invariant thick subcategories of* $\underline{\mathrm{mod}}(\hat{A})$, *and*
- *the set of G'-invariant thick subcategories of* $\mathcal{D}^b(\mathrm{mod}(A))$.

PROOF. Let $\mathcal{C}$ be a thick subcategory of $\underline{\mathrm{mod}}(\hat{A}/G)$. In [**21**, Section 3.2] it is shown that F_λ is an exact functor which sends projectives to projectives. Hence, taking into account the construction of triangles for the two Frobenius categories, one can show that it is a triangle functor. Hence, $F_\lambda^{-1}(\mathcal{C})$ is a thick subcategory of $\underline{\mathrm{mod}}(\hat{A})$ and it is clear that it is G-invariant.

Now let $\mathcal{C}$ be a thick and G-invariant subcategory of $\underline{\mathrm{mod}}(\hat{A})$. We need to show that $F_\lambda(\mathcal{C})$ is thick in $\underline{\mathrm{mod}}(\hat{A}/G)$. We proceed very similarly to the proof of Theorem 3.4 and actually use parts of the proof.

Consider a triangle in $\underline{\mathrm{mod}}(\hat{A}/G)$. The push-down functor is dense and hence, we can write the objects of the triangle as images of objects in $\underline{\mathrm{mod}}(\hat{A})$. Thus, the triangle has the form

$$F_\lambda(X) \to F_\lambda(Y) \to F_\lambda(Z) \to \Sigma F_\lambda(X).$$

Assume $F_\lambda(X), F_\lambda(Y) \in F_\lambda(\mathcal{C})$. This implies $X, Y \in \mathcal{C}$. Applying the pull-up functor $F_\bullet$ to the above triangle yields by Proposition 8.14 to a sequence in $\underline{\mathrm{Mod}}(\hat{A})$ of the form

$$\bigoplus_{g\in G} g(X) \to \bigoplus_{g\in G} g(Y) \to \bigoplus_{g\in G} g(Z) \to \Sigma \bigoplus_{g\in G} g(X).$$

The pull-up functor is exact since the push-down functor is (see for instance [**20**]). Moreover, it is easily seen in this setting that $F_\bullet$ sends projectives to projectives. Therefore, this is a triangle again. In [**60**] Krause and Le extend the Happel functor to a functor

$$H'\colon\ \underline{\mathrm{Mod}}(\hat{A}) \to \mathcal{K}(\mathrm{Inj}(A))$$

which also preserves triangles. Apply this to the last triangle and get, since H' commutes with direct sums, a triangle of the form

$$\bigoplus_{g\in G} H'(g(X)) \to \bigoplus_{g\in G} H'(g(Y)) \to \bigoplus_{g\in G} H'(g(Z)) \to \Sigma \bigoplus_{g\in G} H'(g(X)).$$

Since $H'g = g'H'$ for each $g \in G$ this leads to a triangle

$$\bigoplus_{g'\in G'} g'(H'(X)) \to \bigoplus_{g'\in G'} g'(H'(Y)) \to \bigoplus_{g'\in G'} g'(H'(Z)) \to \Sigma \bigoplus_{g'\in G'} g'(H'(X))$$

in $\mathcal{D}(\mathrm{Mod}(A))$. Since G' fulfils the assumptions of Keller's Theorem, this is actually in $\mathcal{D}(\mathrm{mod}(A))$ as in the proof of Theorem 3.3. Now consider $H'(\mathcal{C})$ where H' is the quasi-equivalence of the original Happel functor here. This is a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$ and it is G'-invariant. We have $H'(X), H'(Y) \in H'(\mathcal{C})$. Hence, we are exactly in the same situation as in the proof of Theorem 3.4. As there we can show that $H'(Z)$ lies in $H'(\mathcal{C})$. Thus, $Z \in \mathcal{C}$ and $F_\lambda(Z) \in F_\lambda(\mathcal{C})$. This shows that $F_\lambda(\mathcal{C})$ is closed under triangles. Moreover, one can easily show that it also fulfils the other properties of a thick subcategory.

Finally, the correspondence with the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ comes for free using the Happel functor. □

Note that the theorem yields an applicable method to classify thick subcategories of $\underline{\mathrm{mod}}(\hat{A}/G)$ in the described setting since we know the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ well by Chapter 6. If A is of tame or finite representation type, we know them completely by Theorem 6.17. Thus, one only needs to check which of them are invariant under the action of the automorphism group.

In the sequel we execute this classification in two important cases.

2. Thick subcategories for trivial extension algebras of tame hereditary algebras

In Chapter 8 we introduced the r-fold trivial extension algebra $T(A)^r = \hat{A}/\langle \nu_{\hat{A}}^r \rangle$. We will use the classification of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ of Chapter 6 here and therefore, we refer to this chapter concerning notations et cetera. However, we introduce one new notation here. Recall the set $\mathrm{NA}(n)$ of noncrossing arcs. We denote by $(\mathrm{NA}(n))^s$ the set of noncrossing arcs which are invariant under rotation by $s\frac{2\pi}{n}$.

THEOREM 10.2. *Let A be a tame hereditary k-algebra with regular part $\mathcal{R}$ decomposing as $\coprod_{j \in J} \mathcal{H}_j \times \coprod_{i=1}^{s} \mathcal{U}_{n_i}$. The poset of thick subcategories of $\underline{\mathrm{mod}}(T(A)^r)$ corresponds bijectively to the union of posets*

$$\mathrm{Th}_{\mathrm{exc}}(A)^{\mathrm{inv}} \cup \mathrm{Th}_{\mathrm{reg}}(A)^{\mathrm{inv}}$$

where one has bijective correspondences

$$\begin{aligned} \mathrm{Th}_{\mathrm{exc}}(A)^{\mathrm{inv}} &\longleftrightarrow (\mathrm{NC}(W,c))^{c^r} \\ &\qquad = \{w \in \mathrm{NC}(W,c) \mid c^r w c^{-r} = w\}, \\ \mathrm{Th}_{\mathrm{reg}}(A)^{\mathrm{inv}} &\longleftrightarrow \{(p, x_1, \ldots, x_s) \mid p \in 2^J, x_i \in (\mathrm{NA}(n_i))^r\}, \\ \mathrm{Th}_{\mathrm{exc}}(A)^{\mathrm{inv}} \cap \mathrm{Th}_{\mathrm{reg}}(A)^{\mathrm{inv}} &\longleftrightarrow \{(x_1, \ldots, x_s) \mid x_i \in (\mathrm{NA}^{\mathrm{exc}}(n_i))^r\}. \end{aligned}$$

PROOF. As mentioned in Chapter 8.3, $\nu_{\hat{A}}$ induces the Nakayama automorphism on $\underline{\mathrm{mod}}(\hat{A})$ and this is isomorphic to $\tau\Omega^{-2}$. Since the Happel functor is a triangle functor which in particular preserves Auslander-Reiten triangles, the induced automorphism of $\nu_{\hat{A}}$ on $\mathcal{D}^b(\mathrm{mod}(A))$ is isomorphic to $\tau\Sigma^2$. Since A is tame, $\hat{A}$ is locally support-finite and hence, by Proposition 8.16 the push-down functor is dense. Moreover, it is easy to check that $\tau^r\Sigma^{2r}$ fulfils the assumptions of Keller's Theorem. Thus, we can apply Theorem 10.1 and get that the thick subcategories of $\underline{\mathrm{mod}}(T(A)^r)$ correspond bijectively to the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ which are invariant under $\langle \tau^r\Sigma^{2r} \rangle$ and since each thick subcategory is invariant under Σ, these are the thick subcategories which are invariant under $\langle \tau^r \rangle$.

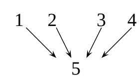
The thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ in turn correspond to $\mathrm{Th}_{\mathrm{exc}}(A) \cup \mathrm{Th}_{\mathrm{reg}}(A)$ as stated in Theorem 6.17.

A thick subcategory $\mathcal{C} \in \mathrm{Th}_{\mathrm{exc}}(A)$ corresponds to $\mathrm{cox}(\mathcal{C}) \in \mathrm{NC}(W, c)$ and it is invariant under $\langle \tau^r \rangle$ if and only if $\mathrm{cox}(\mathcal{C}) = \mathrm{cox}(\tau^r(\mathcal{C}))$. By Proposition 9.1 this holds if and only if

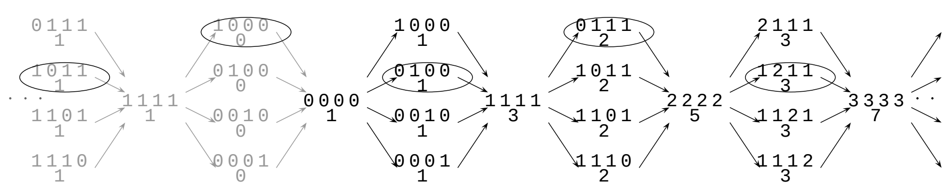
$$\mathrm{cox}(\mathcal{C}) = c^r \,\mathrm{cox}(\mathcal{C}) c^{-r}.$$

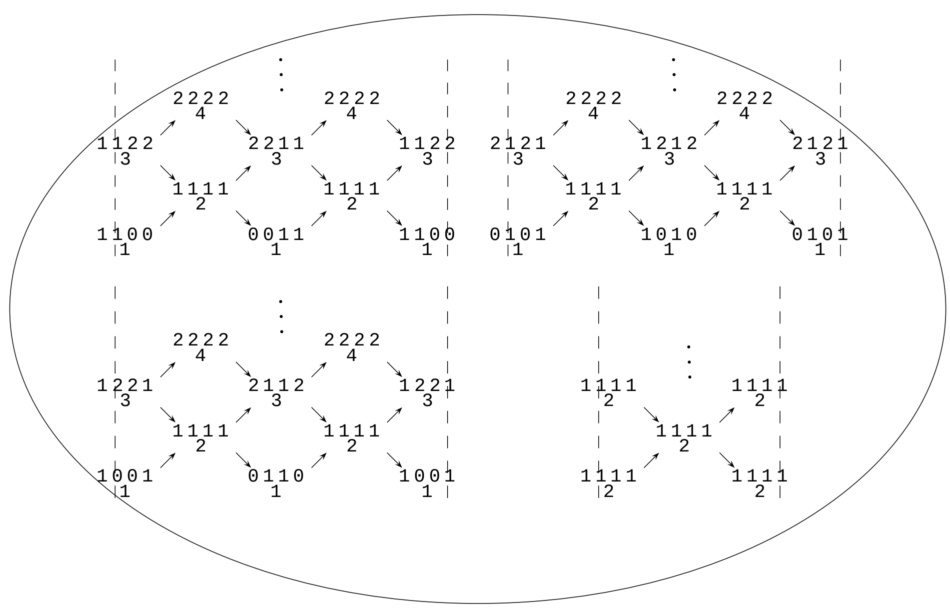
A thick subcategory $\mathcal{C} \in \mathrm{Th}_{\mathrm{reg}}(A)$ corresponds to $\{(p, x_1, \ldots, x_s) \mid p \in 2^J, x_i \in \mathrm{NA}(n_i)\}$. By definition of $\mathrm{NA}(n_i)$ the τ-action on $\mathcal{C}$ corresponds to a rotation of the $x_i \in \mathrm{NA}(n_i)$ by $\frac{2\pi}{n_i}$. Moreover, each homogeneous tube is invariant under $\langle \tau^r \rangle$. □

Example 10.3. Let $A = kQ$ where Q is the following quiver with underlying graph $\tilde{D}_4$.



Let $r = 2$. The author has written a `gap` program to compute noncrossing partitions invariant under r-fold conjugation with the Coxeter element (using braid group actions). The result is that there actually are proper thick subcategories in $\mathrm{Th}_{\mathrm{exc}}(A)$. A $\langle \tau^2 \rangle$-invariant thick subcategory of $\mathcal{D}^b(\mathrm{mod}(A))$ is illustrated in the following figure.





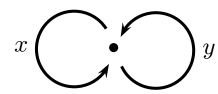
As the figure suggests already, the Auslander-Reiten quiver of kQ admits three exceptional tubes of rank 2 plus the usual infinite family of rank 1 tubes. Clearly, all thick subcategories of $\mathrm{Th}_{\mathrm{reg}}(A)$ are invariant under $\langle\tau^2\rangle$ and $(\mathrm{NA}(2))^2 = \mathrm{NA}(2)$.

3. Thick subcategories for $k\langle X,Y\rangle/(X^2,Y^2,XY-qYX)$

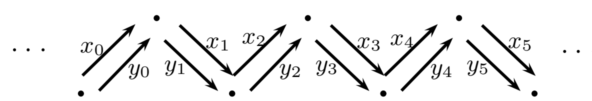
Let k be an algebraically closed field and let $q \in k^*$. We classify the thick subcategories of $\underline{\mathrm{mod}}(\Lambda_q)$ where Λ_q is the algebra

$$k\langle X,Y\rangle/(X^2,Y^2,XY-qYX).$$

The algebra is isomorphic to the path algebra of the following quiver. with

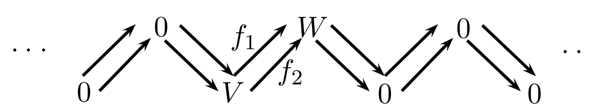


respect to relations $x^2 = y^2 = xy - qyx = 0$. This is isomorphic to the orbit algebra $\hat{B}/\langle F_q\rangle$ where $B = k(1 \rightrightarrows 2)$ is the Kronecker algebra, consequently $\hat{B}$ is the path algebra of the following infinite quiver with respect to relations

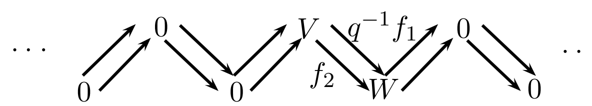


$x_{i+1}x_i = y_{i+1}y_i = x_{i+1}y_i - y_{i+1}x_i = 0$, and F_q sends vertices i to $i+1$, arrows y_i to y_{i+1} and x_i to qx_{i+1}.

The objects in $\underline{\mathrm{mod}}(\hat{B})$ are locally representations of the Kronecker quiver, i.e. representations of the following form in some degree. Such a



representation is sent by $F_q\colon \underline{\mathrm{mod}}(\hat{B}) \to \underline{\mathrm{mod}}(\hat{B})$ (the induced automorphism of $F_q\colon \hat{B} \to \hat{B}$) to a representation of the following form.



The representations of the Kronecker quiver in turn are classified as follows.

- The indecomposable preprojective representations are uniquely determined by the dimension vectors $(n, n+1)$ for $n \geq 0$. The Auslander-Reiten translation τ of a representation with dimension vector $(n, n+1)$ is determined by the dimension vector $(n-2, n-1)$ (for $n \geq 2$).
- The indecomposable preinjective representations are uniquely determined by the dimension vectors $(n+1, n)$ for $n \geq 0$. The Auslander-Reiten translation τ of a representation with dimension vector $(n+1, n)$ is determined by the dimension vector $(n+3, n+2)$.

- There is a $P_1(k)$-family of tubes of rank one where the indecomposable representations of regular length n have dimension vectors (n, n). Denote an indecomposable regular representation with dimension vector (n, n) paramatrised by $[\lambda : 1] \in P_1(k)$ by $R(\lambda, n)$.

Recall that indecomposable objects of $\mathcal{D}^b(\mathrm{mod}(B))$ are just given by shifts $\Sigma^m(M)$ of the modules M described above.

We use the explicit construction of the Happel functor in [**11**] to determine the induced automorphism F_q on $\mathcal{D}^b(\mathrm{mod}(B))$ and get the following on indecomposable objects.

- $\Sigma^m(k^n \rightrightarrows k^{n+1}) \mapsto \Sigma^{m-1}(k^{n+1} \rightrightarrows k^{n+2})$ for $n \geq 1$,
- $\Sigma^m(0 \rightrightarrows k) \mapsto \Sigma^{m-1}(k \rightrightarrows k^2)$,
- $\Sigma^m(k^{n+1} \rightrightarrows k^n) \mapsto \Sigma^{m-1}(k^n \rightrightarrows k^{n-1})$ for $n \geq 2$,
- $\Sigma^m(k^2 \rightrightarrows k) \mapsto \Sigma^{m-1}(k \rightrightarrows 0)$,
- $\Sigma^m(k \rightrightarrows 0) \mapsto \Sigma^{m-1}(0 \rightrightarrows k)$,
- $\Sigma^m(R(\lambda, n)) \mapsto \Sigma^{m-1}(R(q^{-1}\lambda, n))$

for $m \in \mathbb{Z}$.

Observe that $F_q^2(X) = \Sigma^{-2}\tau^{-1}(X)$ for $X \in \mathcal{D}^b(\mathrm{mod}(B))$ not regular.

By Theorem 10.1 the thick subcategories of $\underline{\mathrm{mod}}(\Lambda_q)$ correspond to the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(B))$ which are invariant under $\langle F_q \rangle$. Again we may apply this since F_q obviously fulfils the assumptions of Keller's Theorem and since the push-down functor is dense since B is of tame representation type.

According to Theorem 6.17 the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(B))$ correspond to $\mathrm{Th}_{\mathrm{exc}}(B) \cup \mathrm{Th}_{\mathrm{reg}}(B)$ where $\mathrm{Th}_{\mathrm{exc}}(B)$ corresponds to $\mathrm{NC}(W, s_1 s_2)$ and $\mathrm{Th}_{\mathrm{reg}}(B)$ corresponds to $2^{P_1(k)}$. Moreover, $\mathrm{Th}_{\mathrm{exc}}(B) \cap \mathrm{Th}_{\mathrm{reg}}(B) = \{0\}$.

One can easily see (use for instance braid group actions on $\mathrm{NC}(W, s_1 s_2)$) that a proper thick subcategory $\mathcal{S} \in \mathrm{Th}_{\mathrm{exc}}(B)$ is of the form $\mathcal{S} = \mathrm{add}(\Sigma^n X \mid n \in \mathbb{Z})$ for an indecomposable not regular object $X \in \mathcal{D}^b(\mathrm{mod}(B))$. Then, if $\mathcal{S}$ is $\langle F_q \rangle$-invariant, it contains X as well as $F_q^2(X) = \Sigma^{-2}\tau^{-1}(X)$. Thus, it also contains $\tau^{-1}(X)$. This would imply that $\mathcal{S} = \mathcal{D}^b(\mathrm{mod}(B))$.

Finally, denote by

$$\varphi_q \colon P_1(k) \to P_1(k), [\lambda, 1] \mapsto [q^{-1}\lambda, 1].$$

A thick subcategory $\mathcal{S} \in \mathrm{Th}_{\mathrm{reg}}(B)$ corresponds to $V \in 2^{P_1(k)}$. Hence, the $\langle F_q \rangle$-invariant thick subcategories of $\mathrm{Th}_{\mathrm{reg}}(B)$ correspond to $2^{P_1(k)/\varphi_q}$.

Altogether, we can record the following.

PROPOSITION 10.4. *The set of thick subcategories of* $\underline{\mathrm{mod}}(\Lambda_q)$ *is given by*

$$\{\underline{\mathrm{mod}}(\Lambda_q), \{0\}\} \cup \mathrm{Th}_{\mathrm{reg}}(B)$$

where $\mathrm{Th}_{\mathrm{reg}}(B)$ *corresponds bijectively to* $2^{P_1(k)/\varphi_q}$*. Hence, proper thick subcategories of* $\underline{\mathrm{mod}}(\Lambda_q)$ *are collections of rank one tubes parametrised by orbits of* $P_1(k)$ *under* φ_q*.* □

REMARK 10.5.
(1) If $q = 1$, then Λ_q is isomorphic to the group algebra of the Klein four group and $P_1(k)/\varphi_q = P_1(k)$.
(2) If q is a root of unity, then the orbits in $P_1(k)/\varphi_q$ are finite.

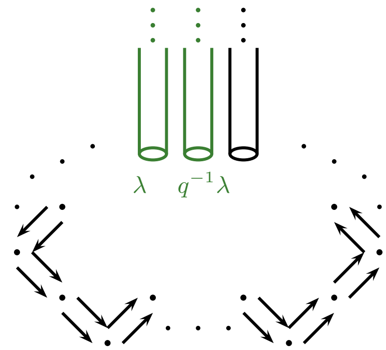


Figure 1. A proper thick subcategory of $\underline{\mathrm{mod}}(\Lambda_q)$ marked in the Auslander-Reiten quiver of $\underline{\mathrm{mod}}(\Lambda_q)$

(3) If $k = \mathbb{C}$ and $|q| = 1$, then we can regard $P_1(k)$ as the Riemann sphere and φ_q describes a rotation of the sphere.

CHAPTER 11

Other classification methods - history and comparison

The problem of classifying thick subcategories was approached before in various mathematical fields. The methods used there are in principle different from the combinatorial methods of this thesis. These different methods from commutative algebra and algebraic topology apply mainly to triangulated categories which are different from the triangulated categories considered here. But there are also intersections. In this chapter, we briefly present the most famous classification theorems, and compare them to our classifications.

The first result is due to Hopkins [**47**] and Neeman [**66**] and classifies the thick subcategories of the *derived category of bounded complexes of finitely generated projective modules* $\mathcal{D}^{\mathrm{per}}(R)$ over a commutative noetherian ring R which is also known as the *category of perfect complexes*.

THEOREM 11.1 (Hopkins, Neeman). *Let R be a commutative noetherian ring. There is a bijective correspondence between*

- *the set of thick subcategories of* $\mathcal{D}^{\mathrm{per}}(R)$*, and*
- *the set of subsets of* $\mathrm{Spec}(R)$ *which are closed under specialisation.*

□

Here $\mathrm{Spec}(R)$ denotes the *spectrum* of the ring R, i.e. the set of all prime ideals of R.

A subset $\mathcal{V}$ of $\mathrm{Spec}(R)$ is called *specialisation closed* if

$$\mathfrak{p} \in \mathcal{V} \text{ and } \mathfrak{p} \subseteq \mathfrak{q} \Rightarrow \mathfrak{q} \in \mathcal{V}.$$

In [**66**] Neeman generalises this to the the unbounded derived category $\mathcal{D}(R)$ of all R-modules where R is commutative and noetherian. Namely, in this case there is a bijective correspondence between the set of localising subcategories of $\mathcal{D}(R)$ and the set of all subsets of $\mathrm{Spec}(R)$. A subcategory of $\mathcal{D}(R)$ is called *localising* if it is a full triangulated subcategory and if it is closed under the formation of arbitrary direct sums.

Due to the fact that R is supposed to be a commutative ring, this approach in general does not apply to our classification in Chapter 6 of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ where A is a hereditary algebra.

But we can compare the classification of Chapter 9 with the classification of the thick subcategories of $\underline{\mathrm{mod}}(kG)$ for a p-group G due to Benson, Carlson and Rickard [**14**]. Therefore, we consider this classification in a bit more detail. For the background of this theory we refer to [**13**, Chapter 5].

From now on let k be an algebraically closed field of characteristic p and let G be a finite p-group. In order to use the commutative algebra methods here we consider the *group cohomology ring*

$$H^*(G,k) := \operatorname{Ext}^*_{kG}(k,k).$$

The multiplication is induced by the *Yoneda composition* of two homogeneous elements. One can show that $H^*(G,k)$ is a graded commutative ring. If $p = 2$, this is commutative, and if p is an odd prime, we modify the ring by considering only even degrees.

Denote by

$$\operatorname{Proj}(H^*(G,k))$$

the set of all homogeneous non-maximal prime ideals of $H^*(G,k)$.

Note that in certain situations the group cohomology ring is well-known. For example from [**12**, Corollary 3.5.7] we get that for $G = (\mathbb{Z}/p\mathbb{Z})^r$ and $\operatorname{char}(k) = p$ it is just the polynomial ring

$$k[x_1, \ldots, x_r]$$

on r variables if $p = 2$, and it is the the tensor product of a polynomial ring and an exterior algebra if p is odd.

Now let M be a finitely generated kG-module. Consider the map

$$H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k) \xrightarrow{-\otimes_k M} \operatorname{Ext}^*_{kG}(M,M).$$

The kernel of this map is a homogeneous ideal I_M in $H^*(G,k)$ and we denote by $V_G(M)$ the set

$$V(I_M) = \{\mathfrak{p} \in \operatorname{Proj}(H^*(G,k)) \mid I_M \subseteq \mathfrak{p}\}.$$

Theorem 11.2 (Benson-Carlson-Rickard). *Let G be a p-group. There is a bijective correspondence between*

- *the set of thick subcategories of* $\underline{\operatorname{mod}}(kG)$*, and*
- *the set of subsets of* $\operatorname{Proj}(H^*(G,k))$ *which are closed under specialisation.*

Here a specialisation closed subset $\mathcal{V}$ corresponds to the thick subcategory consisting of modules $M \in \underline{\operatorname{mod}}(kG)$ with $V_G(M) \subseteq \mathcal{V}$. □

Before we continue we should mention that there is also a generalisation of this. In [**15**] Benson, Iyengar and Krause give a classification of the localising subcategories of $\underline{\operatorname{Mod}}(kG)$.

In Chapter 9 we classified thick subcategories of finite algebraic triangulated categories. The category $\underline{\operatorname{mod}}(kG)$ is a category of this kind if kG is representation-finite. Let G be a p-group as in the Benson-Carlson-Rickard classification which is additionally representation-finite. By Theorem 8.9 the type of $\underline{\operatorname{mod}}(kG)$ equals $(A_{p^n-1}, e, 1)$. By [**12**, Section 6.5] the inertial index e arising here divides $p-1$. Since p is prime, this implies $\gcd(h_{A_{p^n-1}}, e) = \gcd(p^n, e) = 1$. Thus, our approach implies that such a category does not admit non-zero proper thick subcategories.

Example 11.3. Let G be a cyclic group of order p. Then,

$$\operatorname{Proj}(H^*(G,k)) = \operatorname{Proj}(k[x])$$

and this only consists of one point, the generic point. The generic point corresponds to the whole category $\underline{\operatorname{mod}}(kG)$ while the empty set corresponds to the thick subcategory $\{0\}$.

This consideration shows that already in the finite case the classification of this thesis complements the existing methods. Namely, for arbitrary finite groups the Benson-Carlson-Rickard classification does not capture all thick subcategories but only those which are tensor ideal.

Finally, we apply both approaches to a tame example.

Example 11.4. Let

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

be the Klein four group. Here specialisation closed subsets of

$$\mathrm{Proj}(H^*(G,k)) = \mathrm{Proj}(k[x_1, x_2])$$

correspond to arbitrary subsets of $P_1(k)$ plus all of $\mathrm{Proj}(k[x_1, x_2])$. This fits together with our combinatorial classification of the thick subcategories of $\underline{\mathrm{mod}}(\Lambda_q)$ for $q = 1$ in Proposition 10.4.

CHAPTER 12

Hom-Vanishing

In the study of triangulated categories it is helpful to know whether there are non-zero morphisms between two given objects of the category. Thick subcategories are invariants of the category which can decide this question. We will use the previous classifications of thick subcategories to formulate an equivalent condition to the fact that the morphism space between two objects vanishes. Precisely, we formulate this concerning the vanishing of the following modified space of morphisms.

DEFINITION 12.1. Given a pair of objects X, Y in a triangulated category $\mathcal{T}$. The *graded* Hom-*space* is given as

$$\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y).$$

Recall that we assigned in Chapter 2.2 to a thick subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ the perpendicular categories ${}^{\perp}\mathcal{S}$ and $\mathcal{S}^{\perp}$. It is important that these are thick subcategories again.

PROPOSITION 12.2. *Let $\mathcal{T}$ be a triangulated category. Let X, Y be two objects in $\mathcal{T}$. Then,*

$$\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = 0 \Longleftrightarrow \mathrm{Thick}(X) \subseteq {}^{\perp}\,\mathrm{Thick}(Y).$$

PROOF. We show

$$X \in {}^{\perp}\,\mathrm{Thick}(Y) \Longleftrightarrow \mathrm{Hom}^*_{\mathcal{T}}(X,Y) = 0.$$

This implies the assertion since $\mathrm{Thick}(X)$ is the smallest thick subcategory containing X and is thus contained in each thick subcategory which contains X, in particular then in ${}^{\perp}\,\mathrm{Thick}(Y)$.

If $X \in {}^{\perp}(\mathrm{Thick}(Y))$, then $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$ for all $n \in \mathbb{Z}$ and hence, $\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = 0$.

Now assume that $\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = 0$. Since $\mathrm{Hom}(X,-)$ is a homological functor,

$$\mathrm{Ker}(\mathrm{Hom}^*_{\mathcal{T}}(X,-)) = \{Z \in \mathcal{T} \mid \mathrm{Hom}^*(X,Z) = 0\}$$

is closed under suspensions, sums, summands and triangles and hence, it is a thick subcategory of $\mathcal{T}$. By assumption $Y \in \mathrm{Ker}(\mathrm{Hom}^*_{\mathcal{T}}(X,-))$ and therefore, $\mathrm{Thick}(Y) \subseteq \mathrm{Ker}(\mathrm{Hom}^*_{\mathcal{T}}(X,-))$. This implies $X \in {}^{\perp}\,\mathrm{Thick}(Y)$. $\square$

PROPOSITION 12.3. *Let A be a hereditary k-algebra, let* cox, *c, W be as in Theorem 6.2. Let X, Y be two objects in $\mathcal{D}^b(\mathrm{mod}(A))$ such that* $\mathrm{Thick}(X), \mathrm{Thick}(Y)$ *are generated by an exceptional sequence. Then,*

$$\mathrm{Hom}^*_{\mathcal{D}^b(A)}(X,Y) = 0 \Longleftrightarrow \mathrm{cox}(\mathrm{Thick}(X)) \leq K^c(\mathrm{cox}(\mathrm{Thick}(Y))).$$

PROOF. Use the above Proposition 12.2, the fact that cox is order-preserving and Proposition 6.5. □

Now to the other class of triangulated categories we are concerned with in this thesis, the orbit categories. Up to the end of this chapter we assume that A is a hereditary k-algebra and $\Phi\colon \mathcal{D}^b(\mathrm{mod}(A)) \to \mathcal{D}^b(\mathrm{mod}(A))$ is an automorphism such that the assumptions in Keller's Theorem 3.3 hold, i.e. $\mathcal{T} := \mathcal{D}^b(\mathrm{mod}(A))/\Phi$ is triangulated. To avoid inconveniently long formulations we introduce the notation $\mathrm{Th}_{\mathrm{exc}}(\mathcal{T})$ for the set of thick subcategories $\mathcal{S}$ of $\mathcal{T}$ such that $\pi^{-1}(\mathcal{S})$ is generated by an exceptional sequence. Here π denotes again the projection from the derived category to the orbit category. Note that if $\mathcal{T}$ is a finite algebraic triangulated category, then $\mathrm{Th}_{\mathrm{exc}}(\mathcal{T})$ coincides with the set of all thick subcategories of $\mathcal{T}$. As for those in Chapter 9.7 we denote by

$$\widetilde{\mathrm{cox}}\colon\ \mathrm{Th}_{\mathrm{exc}}(\mathcal{T}) \to \mathrm{NC}(W, c)$$

the map sending a thick subcategory $\mathcal{S}$ to $\mathrm{cox}(\pi^{-1}(\mathcal{S}))$.

PROPOSITION 12.4. *Let* $\mathcal{S} \in \mathrm{Th}_{\mathrm{exc}}(\mathcal{T})$. *Then,*

$$\widetilde{\mathrm{cox}}({}^{\perp}\mathcal{S}) = K^c(\widetilde{\mathrm{cox}}(\mathcal{S})).$$

PROOF. The proof of Proposition 9.23 is not restricted to the case where A is representation-finite. □

PROPOSITION 12.5. *Let X and Y be objects in $\mathcal{T}$ such that* $\mathrm{Thick}(X)$ *and* $\mathrm{Thick}(Y)$ *are in* $\mathrm{Th}_{\mathrm{exc}}(\mathcal{T})$. *Then,*

$$\mathrm{Hom}^*_{\mathcal{T}}(X, Y) = 0 \iff \widetilde{\mathrm{cox}}(\mathrm{Thick}(X)) \le K^c(\widetilde{\mathrm{cox}}(\mathrm{Thick}(Y))).$$

PROOF. Use Propositions 12.4 and 12.2 and the fact that $\widetilde{\mathrm{cox}}$ is order-preserving. □

CHAPTER 13

Cluster categories, thick subcategories and noncrossing partitions

There are several connections between classification problems in cluster theory, the classification of thick subcategories and noncrossing objects. This chapter intends to present these connections and distinguish the approach of this thesis from other concepts which look similar but are different.

In fact, cluster theory was the motivation to study noncrossing partitions from the representation theory point of view in the first place.

For the basics about cluster categories we keep with [**28**].

In this chapter we assume that k is an algebraically closed field.

DEFINITION 13.1. Let $\vec{\Delta}$ be a quiver without oriented cycles. The *cluster category* $\mathcal{C}_{\vec{\Delta}}$ associated to $\vec{\Delta}$ is by definition the orbit category

$$\mathcal{C} = \mathcal{C}_{\vec{\Delta}} = \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/(\tau^{-1} \circ \Sigma).$$

This is a triangulated category by Theorem 3.3.

1. Cluster tilting objects and noncrossing partitions

The introduction of the cluster category was motivated by the study of the *cluster algebra* introduced by Fomin and Zelevinsky in [**40**]. For the definition of the cluster algebra and associated notions we refer to [**28**]. In a certain way one can associate to a cluster algebra a quiver and the cluster algebras of finite type are again classified by quivers of Dynkin type. For this finite type by [**41**, Theorem 4.5] the *clusters* of the cluster algebra are in bijective correspondence with the cluster tilting objects in the cluster category of the associated Dynkin type. In order to define tilting in this cluster context, we introduce the Ext-space in $\mathcal{C}$ in the following way:

$$\mathrm{Ext}^1_{\mathcal{C}_{\vec{\Delta}}}(X,Y) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{D}^b(k\vec{\Delta})}((\tau^{-1}\circ\Sigma)^j(X),Y)$$

where $\mathrm{Ext}^1_{\mathcal{D}^b(k\vec{\Delta})}(X,Y) := \mathrm{Hom}_{\mathcal{D}^b(k\vec{\Delta})}(X,\Sigma(Y))$.

DEFINITION 13.2. An object T in $\mathcal{C}$ is called a *cluster tilting object* if $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$ and if it is maximal with this property, i.e. if there is an object U with $\mathrm{Ext}^1_{\mathcal{C}}(T,U) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(U,T)$, then U is a direct summand of T. A cluster tilting object is called *basic* if all its direct summands are pairwise not isomorphic.

THEOREM 13.3 (Ingalls/Thomas [**53**]). *Let $\vec{\Delta}$ be a quiver without oriented cycles. There is a bijective correspondence between*

- *the set of basic cluster tilting objects in $\mathcal{C}_{\vec{\Delta}}$, and*

- *the set of wide subcategories of* $\mathrm{mod}(k\vec{\Delta})$ *which are generated by an exceptional sequence.*

PROOF. The correspondence is a composition of the correspondences of Proposition 2.26, Theorem 2.11, Corollary 2.17 in [**53**]. □

REMARK 13.4. The bijection is not natural since it is a composition of correspondences as mentioned above. There does not seem to be a way to construct it directly.

Note that by Chapter 6 we hence have a bijective correspondence between

- the set of basic cluster tilting objects in $\mathcal{C}_{\vec{\Delta}}$, and
- the set of thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ generated by an exceptional sequence, and
- $\mathrm{NC}(W_\Delta, c)$.

2. Thick subcategories of the cluster category

Since $\mathcal{C}_{\vec{\Delta}}$ is a triangulated category by itself, we can also ask for its thick subcategories. It is an orbit category of the bounded derived category of a hereditary algebra and thus we can use Theorem 3.4 to determine the thick subcategories.

LEMMA 13.5. *Let* $\vec{\Delta}$ *be a quiver without oriented cycles. Let* $\mathcal{X}$ *be a connected component of the Auslander-Reiten quiver of* $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ *and let* $\mathcal{S}$ *be a thick subcategory of* $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ *with* $\tau(\mathcal{S}) = \mathcal{S}$. *If there is an indecomposable object which lies as well in* $\mathcal{X}$ *as in* $\mathcal{S}$, *then* $\mathrm{add}(\mathcal{X}) \subseteq \mathcal{S}$.

PROOF. Let $X \in \mathcal{X} \cap \mathcal{S}$ be indecomposable. Since $\mathcal{S}$ is τ-invariant, all τ-translates of X are contained in $\mathcal{S}$. Since $\mathcal{S}$ is thick, all middle terms of the corresponding Auslander-Reiten sequences lie in $\mathcal{S}$. Then, the τ-translates of these middle terms are again contained in $\mathcal{S}$. Continuing like this, we get that all of $\mathcal{X}$ lies in $\mathcal{S}$ since $\mathcal{X}$ is connected. □

THEOREM 13.6.
(1) *Let* $\vec{\Delta}$ *be a Dynkin quiver. Then the cluster category* $\mathcal{C}_{\vec{\Delta}}$ *admits no non-zero proper thick subcategories.*
(2) *Let* $\vec{\Delta}$ *be an extended Dynkin quiver. Let* $\mathcal{R} = \coprod_{t\in T} \mathcal{R}_t$ *be the decomposition of the regular part of* $\mathrm{mod}(k\vec{\Delta})$ *as in Theorem 4.13. The only proper thick subcategories of* $\mathcal{C}_{\vec{\Delta}}$ *are of the form*

$$\pi(\mathrm{Thick}(\coprod_{s\in S} \mathcal{R}_s)) \cong \coprod_{s\in S} \mathcal{R}_s$$

where $S \in 2^T$ *(not empty) and* π *denotes the projection.*

PROOF. By Theorem 3.4 the thick subcategories of $\mathcal{C}$ correspond to the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ which are invariant under $\langle \tau^{-1}\Sigma \rangle$. This is equivalent to being invariant under $\langle \tau \rangle$.

Now let $\vec{\Delta}$ be Dynkin. Let $\mathcal{S}$ be a thick subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ of this form. By Proposition 2.26 the Auslander-Reiten quiver of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ is of the form $\mathbb{Z}\Delta$ and hence it is connected. From Lemma 13.5 we get that if there is a non-zero object in $\mathcal{S}$, then automatically, $\mathcal{S} = \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$.

Let $\vec{\Delta}$ be extended Dynkin. Let $\mathcal{S}$ be a non-zero thick and $\langle\tau\rangle$-invariant subcategory of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$. If $\mathcal{S}$ only consists of regular objects, then as discussed in Chapter 6, it is of the form

$$\mathrm{Thick}(\coprod_{s\in S}\mathcal{S}_s) = \coprod_{n\in\mathbb{Z}}\Sigma^n(\coprod_{s\in S}\mathcal{S}_s)$$

where $\mathcal{S}_s$ is a wide non-zero subcategory of $\mathcal{R}_s$ and $S \in 2^T$. This is $\langle\tau\rangle$-invariant if and only if $\mathcal{S}_s$ is $\langle\tau\rangle$-invariant for each $s \in S$. Since $\mathcal{R}_s$ is connected, by Lemma 13.5 $\mathcal{S}_s = \mathcal{R}_s$ for each $s \in S$. Hence,

$$\mathcal{S} = \mathrm{Thick}(\coprod_{s\in S}\mathcal{R}_s) = \coprod_{n\in\mathbb{Z}}\Sigma^n(\coprod_{s\in S}\mathcal{R}_s).$$

Now we have an isomorphism

$$\Sigma^n\coprod_{s\in S}\mathcal{R}_s = (\Sigma\tau^{-1})^n\coprod_{s\in S}\mathcal{R}_s \cong \coprod_{s\in S}\mathcal{R}_s$$

in the orbit category $\mathcal{C}$ for all $n \in \mathbb{Z}$. Thus, a thick subcategory $\mathcal{S}$ only consisting of regular objects corresponds to

$$\pi(\mathcal{S}) \cong \coprod_{s\in S}\mathcal{R}_s.$$

Now assume that $\mathcal{S}$ contains a non-zero object X which is not regular. Denote by $\mathcal{P}[0]$ and $\mathcal{Q}[0]$ the preprojective and preinjective components of $\mathrm{mod}(k\vec{\Delta})$. As seen in Chapter 4 and in Proposition 2.26, the connected components of $\mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$ including non-regulars are given by $\mathcal{Q}[n-1]\cup\mathcal{P}[n]$ and are of the form $\mathbb{Z}\Delta$ for $n \in \mathbb{Z}$. Say $X \in \mathcal{P}[0]$. By Lemma 13.5 this implies $\mathcal{Q}[-1]\cup\mathcal{P}[0] \subseteq \mathcal{S}$. Hence, $\mathcal{Q}[n], \mathcal{P}[n] \subseteq \mathcal{S}$ for all $n \in \mathbb{Z}$ since $\mathcal{S}$ is thick. For each $t \in T$ any morphism from $\mathcal{P}[0]$ to $\mathcal{Q}[0]$ factors through a module in $\mathcal{R}_t[0]$ (see [**80**, Theorem 3.4]). In particular, this holds for the embedding of X into its injective hull $E(X) \in \mathcal{Q}[0]$. Thus, for each $t \in T$ there is a a module $R \in \mathcal{R}_t[0]$ and a monomorphism $X \to R$. The cokernel of this lies in $\mathcal{Q}[0]$ since it is not possible that it lies in $\mathcal{R}[0]$ because $\mathcal{R}[0]$ is a wide subcategory of $\mathrm{mod}(k\vec{\Delta})$. Hence, $R \in \mathcal{S}$ and since $\mathcal{S}$ is invariant under $\langle\tau\rangle$, $\mathcal{R}_t[n] \subseteq \mathcal{S}$ for all $t \in T$ and all $n \in \mathbb{Z}$. Altogether, $\mathcal{S} = \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))$. □

REMARK 13.7. More generally, one could also consider the *m-cluster category*

$$\mathcal{C}^m_{\vec{\Delta}} := \mathcal{D}^b(\mathrm{mod}(k\vec{\Delta}))/(\tau^{-1}\Sigma^m)$$

for $m \in \mathbb{Z}$ as introduced by Keller in [**56**].

If $\vec{\Delta}$ is Dynkin we have the same result as above. In the extended Dynkin case the proper thick subcategories appear as

$$\coprod_{0\leq n<m}\Sigma^n\coprod_{s\in S}\mathcal{R}_s.$$

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