$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules

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Introduction

We fix a prime number p. In this introduction we always denote by $\mathcal{O}$ and $\mathcal{O}'$ rings of integers of non-Archimedean local fields of characteristic zero, π and π' are uniformizing elements of $\mathcal{O}$ and $\mathcal{O}'$ and p is the characteristic of the residue fields of $\mathcal{O}$ and $\mathcal{O}'$, which have q resp. q' elements. All rings and algebras over a commutative ring are assumed to be commutative. Unless otherwise stated, R is a unitary $\mathcal{O}$-algebra resp. unitary ring.

The easiest approach to formal groups over a p-adic ring R might be to classify them by reduced Cartier modules over the ring $\mathbb{E}_R$ (see [Zin84]). In [Dri76], Drinfeld generalized this equivalence to formal $\mathcal{O}$-modules over R and reduced Cartier modules over the ring $\mathbb{E}_{\mathcal{O},R}$ for each $\mathcal{O}$ and $\mathcal{O}$-algebra R.

In the case that R is a perfect field of characteristic p, Dieudonné modules over R can be considered as reduced Cartier modules over R and by demanding certain nilpotence conditions concerning the operator V of the Dieudonné modules it is possible to show that these Dieudonné modules are equivalent to the category of p-divisible formal groups over R. Zink generalized the concept of a Dieudonné module in [Zin02], obtained the display structure ($3n$-display in the original source) for general rings R and constructed a BT functor from the category of displays over R to the category of formal groups over R. For rings R with p nilpotent in R, we get, by considering only nilpotent displays (displays in [Zin02]), that the restriction of the BT functor to the category of nilpotent displays over R has its image in the category of p-divisible formal groups over R. Zink was able to show that this restriction functor is an equivalence in many important cases and Lau finally showed in [Lau08] that this restriction functor is an equivalence for all rings R with p nilpotent in R. So we can basically describe p-divisible formal groups with structures from linear algebra. The task of this thesis is now to generalize this equivalence to nilpotent $\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules for $\mathcal{O}$-algebras R with π nilpotent in R. For this purpose we investigate the idea of Drinfeld's proof in [Dri76] for the generalized Cartier equivalence and obtain our generalized equivalence in a similar manner. Hence, we do not obtain the equivalence by generalizing every result needed for establishing the equivalence of nilpotent displays over R and p-divisible formal groups over R (even though we still have to generalize many results), but we use the already established equivalence for the $\mathcal{O} = \mathbb{Z}_p$ case. One advantage is that we better understand the relations between the different display structures for varying $\mathcal{O}$. Some parts of this generalization of the theory are already utilized in [Hed, Chapter 9].

Now until the end of the paragraph following Proposition 2, R is a not necessarily unitary $\mathcal{O}$-algebra resp. $\mathcal{O}'$-algebra. For an $\mathcal{O}$-algebra R we define an $\mathcal{O}$-algebra structure on the set

$$W_{\mathcal{O}}(R) = \{\, (b_0, b_1, \dots) \,|\, b_i \in R \,\},$$

which is uniquely determined by demanding:

- For every $\mathcal{O}$-algebra morphism $R \to R'$ (of not necessarily unitary $\mathcal{O}$-algebras) the induced morphism $W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R')$ is an $\mathcal{O}$-algebra morphism.
- The maps

$$\begin{array}{rcl} \mathrm{w}_n : W_{\mathcal{O}}(R) & \to & R \\ \underline{b} = (b_0, b_1, \dots) & \mapsto & b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^n b_n \end{array}$$

  are $\mathcal{O}$-algebra morphisms.

We will call this the *$\mathcal{O}$-algebra of ramified Witt vectors over R*, its elements *ramified Witt vectors* and the map w_n the *n-th Witt polynomial.* The construction of $W_{\mathcal{O}}(R)$ clearly depends on the choice of π, but if we choose any other uniformizing element and consider the $\mathcal{O}$-algebra of ramified Witt vectors with respect to this element, we obtain that both $\mathcal{O}$-algebras of ramified Witt vectors are canonically isomorphic. We can state the following Lemma:

Lemma 1. Let B be a π-torsion free $\mathcal{O}$-algebra and $\tau : B \to B$ an $\mathcal{O}$-algebra morphism with $\tau(x) \equiv x^q \mod \pi$. Then there is a unique $\mathcal{O}$-algebra morphism $\kappa : B \to W_{\mathcal{O}}(B)$, such that $\mathrm{w}_n(\kappa(b)) = \tau^n(b)$ holds for each $b \in B$ and $n \geq 0$.

This Lemma is particularly important, when we consider a nonramified extension of non-Archimedean local fields of characteristic zero $\mathcal{O} \to \mathcal{O}'$. If we denote by σ the relative Frobenius of this extension, then there is a unique $\mathcal{O}$-algebra morphism

$$\kappa : \mathcal{O}' \to W_{\mathcal{O}}(\mathcal{O}'), \tag{1}$$

such that $\mathrm{w}_n(\kappa(a)) = \sigma^n(a)$ holds for each $a \in \mathcal{O}'$ and $n \geq 0$, where the $\mathcal{O}$-algebra structure of $W_{\mathcal{O}}(\mathcal{O}')$ has been established with respect to a fixed prime element

of $\mathcal{O}$.
Next we introduce the $\mathcal{O}$-module morphism ${}^V : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ and the $\mathcal{O}$-algebra morphism ${}^F : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ for all $\mathcal{O}$-algebras R, the first is called the Verschiebung and the second one is the Frobenius. They are defined by functoriality in R and the relations, for all $n \geq 0$,

$$\begin{aligned} \mathrm{w}_n({}^F x) &= \mathrm{w}_{n+1}(x), \\ \mathrm{w}_{n+1}({}^V x) &= \pi \mathrm{w}_n(x), \quad \mathrm{w}_0({}^V x) = 0, \end{aligned}$$

where $x \in W_{\mathcal{O}}(R)$ and the equations and multiplications have to be unterstood in R. One easily verifies that

$${}^{FV} = \pi, \quad {}^V({}^F xy) = x{}^V y$$

hold for all $x, y \in W_{\mathcal{O}}(R)$. If we denote the Image of ${}^V : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ by $I_{\mathcal{O},R}$, we obtain that $I_{\mathcal{O},R}$ is the ideal of ramified Witt vectors, whose first component is zero, which is the same to say that $I_{\mathcal{O},R} = \ker(\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R)$ holds, because

$${}^V(b_0, b_1, \ldots) = (0, b_0, b_1, \ldots)$$

holds for all $(b_0, b_1, \ldots) \in W_{\mathcal{O}}(R)$.

We define the *Teichmüller representant* $[a] \in W_{\mathcal{O}}(R)$ by $(a, 0, 0 \ldots)$ for R an $\mathcal{O}$-algebra and $a \in R$. For a nilpotent $\mathcal{O}$-algebra $\mathcal{N}$ we denote by $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ the $\mathcal{O}$-subalgebra of $W_{\mathcal{O}}(\mathcal{N})$, which consists of the ramified Witt vectors with finitely many nonzero entries. For the relations of the different $\mathcal{O}$-algebras of ramified Witt vectors with varying $\mathcal{O}$, we have the following result:

Proposition 2. Let $\mathcal{O} \to \mathcal{O}'$ be an extension of rings of integers of non-Archimedean local fields of characteristic zero, π, π' fixed uniformizing elements of $\mathcal{O}$ resp. $\mathcal{O}'$ and f the degree of extension of the residue fields. Let $\mathrm{Alg}_{\mathcal{O}}$ resp. $\mathrm{Alg}_{\mathcal{O}'}$ denote the category of (not necessarily unitary) $\mathcal{O}$-algebras resp. $\mathcal{O}'$-algebras. Then there exists a unique morphism $u : W_{\mathcal{O}} \to W_{\mathcal{O}'}$ of functors from $\mathrm{Alg}_{\mathcal{O}'}$ to $\mathrm{Alg}_{\mathcal{O}}$, such that ${\mathrm{w}'}_n \circ u = \mathrm{w}_{fn}$ holds. For a nilpotent $\mathcal{O}'$-algebra $\mathcal{N}$ the restriction morphism $u_{\mathcal{N}} : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \to W_{\mathcal{O}'}(\mathcal{N})$ has its image in $\widehat{W_{\mathcal{O}'}}(\mathcal{N})$. Furthermore, for an $\mathcal{O}'$-algebra R we have $u_R([a]) = [a]$ for $a \in R$, $u_R({}^{F^f} x) = {}^{F'}(u_R(x)), u_R({}^V x) = (\pi/\pi')^{V'}(u_R({}^{F^{f-1}} x))$ for $x \in W_{\mathcal{O}}(R)$, where all the objects related to $\mathcal{O}'$ are marked with a dash.

With abuse of notation, we usually replace u_R by u if it is clear which R we consider. The morphism of functors u of the previous Proposition is, up to a canonical isomorphism of functors, independent of the choice of the uniformizing elements π, π' of $\mathcal{O}$ resp. $\mathcal{O}'$. In the following, when we consider the Definition of an f-$\mathcal{O}$-display and the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ resp. $\Gamma_i(\mathcal{O}, \mathcal{O}')$ etc., we sometimes make use of the $\mathcal{O}$-algebra resp. $\mathcal{O}'$-algebra of ramified Witt vectors for a particular choice of the uniformizing element π resp. π' for $\mathcal{O}$ resp. $\mathcal{O}'$. Nevertheless, up to canonical isomorphism, the structures are independent of the choice of π resp. π'.

Unless otherwise stated, until the end of this introduction S, R, R' etc. are now assumed to be unitary $\mathcal{O}$-algebras with π nilpotent in them and if they are assumed to be unitary $\mathcal{O}'$-algebras, then π' should always be nilpotent in them.

Definition 3. Let $f \geq 1$ be a natural number. An *f-$\mathcal{O}$-display* $\mathcal{P}$ over R is a quadruple (P, Q, F, F_1), where P is a finitely generated projective $W_{\mathcal{O}}(R)$-module, Q a submodule of P and $F : P \to P$ and $F_1 : Q \to P$ are ${}^{F^f}$-linear maps, such that the following properties are satisfied:

1. $I_{\mathcal{O},R}P \subset Q$ and P/Q is a direct summand of the R-module $P/I_{\mathcal{O},R}P$.
2. F_1 is an ${}^{F^f}$-linear epimorphism, i.e., its linearisation
$$\begin{array}{rcl} F_1^\sharp : W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} Q & \to & P \\ w \otimes q & \mapsto & wF_1q, \end{array}$$
where $w \in W_{\mathcal{O}}(R)$ and $q \in Q$, is surjective.
3. For $x \in P$ and $w \in W_{\mathcal{O}}(R)$, we have
$$F_1({}^V wx) = {}^{F^{f-1}}wFx.$$

The finite projective R-module P/Q is the *tangential space* of $\mathcal{P}$. If $f = 1$, we call $\mathcal{P}$ just an *$\mathcal{O}$-display.*

Except for the occuring f, this Definition is completely analogous to [Zin02, Definition 1], where the defined structure is called a $3n$-display there. Furthermore, for each f-$\mathcal{O}$-display $\mathcal{P} = (P, Q, F, F_1)$ there exists a unique $W_{\mathcal{O}}(R)$-linear map
$$V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P,$$
which satisfies the following equations for all $w \in W_{\mathcal{O}}(R), x \in P$ and $y \in Q$:
$$\begin{array}{rcl} V^\sharp(wFx) & = & \pi \cdot w \otimes x \\ V^\sharp(wF_1y) & = & w \otimes y \end{array}$$

By $V^{n\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fn}, W_{\mathcal{O}}(R)} P$ we denote the composite map ${}^{F^{f(n-1)}}V^\sharp \circ \ldots \circ {}^{F^f}V^\sharp \circ V^\sharp$, where ${}^{F^{fi}}V^\sharp$ is the $W_{\mathcal{O}}(R)$-linear map
$$\mathrm{id} \otimes_{F^{fi}, W_{\mathcal{O}}(R)} V^\sharp : W_{\mathcal{O}}(R) \otimes_{F^{fi}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R) \otimes_{F^{f(i+1)}, W_{\mathcal{O}}(R)} P.$$

We call $\mathcal{P}$ *nilpotent*, if there is a number N such that the composite map
$$\mathrm{pr} \circ V^{N\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R)/(I_{\mathcal{O},R} + \pi W_{\mathcal{O}}(R)) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P$$

is the zero map. The f-$\mathcal{O}$-displays over R form a category, we call it $(f-\mathrm{disp}_{\mathcal{O}}/R)$ or only $(\mathrm{disp}_{\mathcal{O}}/R)$, when $f=1$ (see section 2.2 for the morphisms between the f-$\mathcal{O}$-displays). The nilpotent f-$\mathcal{O}$-displays over R form a full subcategory, we denote it by $(f-\mathrm{ndisp}_{\mathcal{O}}/R)$ or $(\mathrm{ndisp}_{\mathcal{O}}/R)$, respectively.

Let $\mathcal{N}$ be a nilpotent R-algebra. For a given f-$\mathcal{O}$-display $\mathcal{P}=(P,Q,F,F_1)$ we consider the following $W_{\mathcal{O}}(R)$-modules:

$$\begin{aligned} \widehat{P}_{\mathcal{N}} &= \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P, \\ \widehat{Q}_{\mathcal{N}} &= \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} L \oplus \widehat{I}_{\mathcal{O},\mathcal{N}} \otimes_{W_{\mathcal{O}}(R)} T, \end{aligned}$$

where $P=L\oplus T$ is a normal decomposition, i.e., L and T are $W_{\mathcal{O}}(R)$-submodules of P, such that $Q=L\oplus I_{\mathcal{O},R}T$ holds, and $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ is the $W_{\mathcal{O}}(R)$-subalgebra of $W_{\mathcal{O}}(\mathcal{N})$ as before Proposition 2. We obtain an ${}^{F^f}$-linear map $F_1:\widehat{Q}_{\mathcal{N}}\to\widehat{P}_{\mathcal{N}}$ given by $w\otimes y\mapsto {}^{F^f}w\otimes F_1y$ and ${}^{V}w\otimes x\mapsto^{F^{f-1}} w\otimes Fx$ for $w\in W_{\mathcal{O}}(\mathcal{N})$, $y\in Q$ and $x\in P$. Hence, it is possible to define the formal $\mathcal{O}$-module $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ (see Appendix A for the definitions of (π-divisible) formal $\mathcal{O}$-modules) by the exact sequence of $\mathcal{O}$-modules

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1-\mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N}) \longrightarrow 0$$

for all $\mathcal{N}\in\mathrm{Nil}_R$, where Nil_R denotes the category of nilpotent R-algebras. In case $f=1$, we just write $BT_{\mathcal{O}}(\mathcal{P},-)$ instead of $BT_{\mathcal{O}}^{(1)}(\mathcal{P},-)$. Furthermore, if $\mathcal{P}$ is nilpotent, then $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ is a π-divisible formal $\mathcal{O}$-module, so we obtain a functor

$$BT_{\mathcal{O}}^{(f)}:(f-\mathrm{ndisp}_{\mathcal{O}}/R)\to(\pi-\text{divisible formal }\mathcal{O}-\text{modules}/R),$$

which we want to be an equivalence for $f=1$. In a more general setting, assume that $\mathcal{O}\to\mathcal{O}'$ is nonramified of degree f and R is an $\mathcal{O}'$-algebra with π' nilpotent in R. Then it is not too hard to check with the help of (1) that $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ is a (π'-divisible) formal $\mathcal{O}'$-module for a (nilpotent) f-$\mathcal{O}$-display $\mathcal{P}$. Hence, it also makes sense to ask, whether

$$BT_{\mathcal{O}}^{(f)}:(f-\mathrm{ndisp}_{\mathcal{O}}/R)\to(\pi'-\text{divisible formal }\mathcal{O}'-\text{modules}/R)$$

is an equivalence.

Definition 4. Let $\mathcal{O}\to\mathcal{O}'$ be an extension of rings of integers of non-Archimedean local fields of characteristic zero, R an $\mathcal{O}'$-algebra and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. Then we call an $\mathcal{O}'$-action, i.e., an $\mathcal{O}$-algebra morphism $\iota:\mathcal{O}'\to\mathrm{End}\,\mathcal{P}$, *strict*, iff the induced action $\bar{\iota}:\mathcal{O}'\to P/Q$ coincides with the $\mathcal{O}'$-module structure given by the R-module structure of P/Q and restriction to scalars. We denote by $(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$ the category of nilpotent $\mathcal{O}$-displays over R equipped with a strict $\mathcal{O}'$-action. The objects in this category are $(\mathcal{P}, \alpha)$, where $\mathcal{P}$ is a nilpotent $\mathcal{O}$-display over R and $\alpha : \mathcal{O}' \to \operatorname{End} \mathcal{P}$ the strict $\mathcal{O}'$-action, but if it is clear that we have such an action attached, we write with abuse of notation just $\mathcal{P}$ instead of $(\mathcal{P}, \alpha)$.

After taking Drinfeld's paper [Dri76] as inspiration, we state at first:

Lemma 5. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified extension of degree f, R an $\mathcal{O}'$-algebra and $\mathcal{P} = (P, Q, F, F_1)$ an $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action. Then we may decompose P and Q canonically in $P = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} P_i, Q = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} Q_i$, where each P_i and $Q_i = P_i \cap Q$ are $W_{\mathcal{O}}(R)$-modules, $P_i = Q_i$ for all $i \neq 0$ and $F(P_i), F_1(Q_i) \subseteq P_{i+1}$ hold for all i (where we consider i modulo f).

With the help of this we can construct the functor

$$\Omega_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'} / R) \to (f - \mathrm{ndisp}_{\mathcal{O}} / R)$$

given by sending (P, Q, F, F_1) equipped with a strict $\mathcal{O}'$-action to $(P_0, Q_0, F_1^{f-1} F, F_1^f)$ and restricting a morphism between two f-$\mathcal{O}$-displays to the zeroth component.

Furthermore, for a nonramified extension $\mathcal{O} \to \mathcal{O}'$ of degree f and R an $\mathcal{O}'$-algebra, we define the functor

$$\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{ndisp}_{\mathcal{O}} / R) \to (\mathrm{ndisp}_{\mathcal{O}'} / R)$$

by sending $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{1,0})$ to $\mathcal{P}' = (P', Q', F', F_1')$, where the elements of the quadruple are given by

$$\begin{aligned} P' &= W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_0, \\ Q' &= \ker(W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_0 \to P_0/Q_0 : w \otimes x \mapsto w_0 \operatorname{pr}(x)), \\ F' &= {}^{F'} \otimes_{W_{\mathcal{O}}(R)} F_0, \\ F_1'(w \otimes z) &= {}^{F'} w \otimes_{W_{\mathcal{O}}(R)} F_{1,0}(z), \\ F_1'({}^{V'} w \otimes x) &= w \otimes_{W_{\mathcal{O}}(R)} F_0 x, \end{aligned}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P_0$ and $z \in Q_0$, where we have used the morphism $u : W_{\mathcal{O}}(R) \to W_{\mathcal{O}'}(R)$. Here the operators related to $W_{\mathcal{O}'}(R)$ are marked with a dash. The mapping of the morphisms is simply given by tensoring. We define

$$\Gamma_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'} / R) \to (\mathrm{ndisp}_{\mathcal{O}'} / R)$$

as the composite of $\Omega_2(\mathcal{O},\mathcal{O}')$ and $\Omega_1(\mathcal{O},\mathcal{O}')$.
It can be checked that for each $\mathcal{O}'$-algebra R the diagram

$$\begin{array}{ccc}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \searrow^{BT_{\mathcal{O}}} & \\
\downarrow{\scriptstyle \Omega_1(\mathcal{O},\mathcal{O}')} & & \\
(f-\mathrm{ndisp}_{\mathcal{O}}/R) & \xrightarrow{BT^{(f)}_{\mathcal{O}}} & (\pi'\text{-divisible formal } \mathcal{O}'-\text{modules}/R) \\
\downarrow{\scriptstyle \Omega_2(\mathcal{O},\mathcal{O}')} & & \\
(\mathrm{ndisp}_{\mathcal{O}'}/R) & \nearrow_{BT_{\mathcal{O}'}} &
\end{array}$$

is commutative.
Now let $\mathcal{O}'$ be totally ramified over $\mathcal{O}$ and R an $\mathcal{O}'$ algebra. We define the functor

$$\Gamma_2(\mathcal{O},\mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R).$$

by sending a nilpotent $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action, say $\mathcal{P} = (P, Q, F, F_1)$ (plus the attached $\mathcal{O}'$-action), to

$$\begin{array}{rcl}
P' & = & W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}'\otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P, \\
Q' & = & \ker(W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}'\otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P \to P/Q : w \otimes x \mapsto w_0\,\mathrm{pr}(x)), \\
F'(w \otimes x) & = & {}^{F'}w \cdot y^{-1} \otimes F_1((\pi' - [\pi'])x), \\
F'_1({}^{V'}w \otimes x) & = & y^{-1}w \otimes F_1((\pi' - [\pi'])x), \\
F'_1(w \otimes z) & = & {}^{F'}w \otimes F_1(z),
\end{array}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P$ and $z \in Q$, where we have used the morphism

$$\begin{array}{rcl}
\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) & \to & W_{\mathcal{O}'}(R) \\
a \otimes w & \mapsto & au(w),
\end{array}$$

where $a \in \mathcal{O}'$ and $w \in W_{\mathcal{O}}(R)$, and $y \in W_{\mathcal{O}'}(R)$ is given by ${}^{V'}y = \pi' - [\pi']$.
The diagram

$$\begin{array}{ccc}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{BT_{\mathcal{O}}} & (\pi'\text{-divisible formal } \mathcal{O}'-\text{modules}/R) \\
\downarrow{\scriptstyle \Gamma_2(\mathcal{O},\mathcal{O}')} & \nearrow_{BT_{\mathcal{O}'}} & \\
(\mathrm{ndisp}_{\mathcal{O}'}/R) & &
\end{array}$$

is commutative.
We define the boolean variable $P(\mathcal{O},\mathcal{O}',R)$, for a nonramified extension $\mathcal{O} \to \mathcal{O}'$ of degree f and an $\mathcal{O}'$-algebra R to be true, iff the following assertion is true:

> The $BT^{(f)}_{\mathcal{O}}$ functor is an equivalence between nilpotent f-$\mathcal{O}$-displays over R and π'-divisible formal $\mathcal{O}'$-modules over R.

In case $\mathcal{O}' = \mathcal{O}$, we just write $P(\mathcal{O}, R)$ instead of $P(\mathcal{O}, \mathcal{O}', R)$. As in Drinfeld's argumentation we need that $P(\mathbb{Z}_p, R)$ is true for all rings R with p nilpotent in R. This has been established in [Lau08].

From now on, when we talk about $BT_{\mathcal{O}}^{(f)}$ and $BT_{\mathcal{O}}(= BT_{\mathcal{O}}^{(1)})$ we always consider the functors restricted to nilpotent display structures. Whenever we talk about $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}')$ or $\Gamma_1(\mathcal{O}, \mathcal{O}')$, we always assume $\mathcal{O}'$ to be nonramified over $\mathcal{O}$ of degree f and whenever we talk about $\Gamma_2(\mathcal{O}, \mathcal{O}')$, we always assume $\mathcal{O}'$ to be totally ramified over $\mathcal{O}$.
When we claim assertions like

> For every $\mathcal{O}'$-algebra R with π' nilpotent in R the functors $\Gamma_1(\mathcal{O}, \mathcal{O}')$ and $\Gamma_2(\mathcal{O}, \mathcal{O}')$ are equivalences of categories.

we actually mean that for every nonramified extension $\mathcal{O} \to \mathcal{O}'$ and every $\mathcal{O}'$-algebra R with π' nilpotent in R the functor $\Gamma_1(\mathcal{O}, \mathcal{O}')$ is an equivalence of categories and the analogous assertion for every totally ramified extension and $\Gamma_2(\mathcal{O}, \mathcal{O}')$.
Now let $\mathcal{O} \to \mathcal{O}'$ be a nonramified/totally ramified extension and R an $\mathcal{O}'$-algebra with π' nilpotent in R. If we assume that $P(\mathcal{O}, R)$ respectively $P(\mathcal{O}, \mathcal{O}', R)$ (in the nonramified case) is true, then $\Omega_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ respectively $\Omega_2(\mathcal{O}, \mathcal{O}')$ is faithful, which follows from the above diagrams. So assuming $P(\mathcal{O}, R)$ respectively $P(\mathcal{O}, \mathcal{O}', R)$ to be true, one only has to show, in order to obtain all desired equivalences, that $\Omega_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ respectively $\Omega_2(\mathcal{O}, \mathcal{O}')$ is full and essentially surjective.

Now let $\mathfrak{a} \subseteq R$ be an ideal. An *$\mathcal{O}$-pd-structure* is a map $\gamma : \mathfrak{a} \to \mathfrak{a}$, such that

- $\pi \cdot \gamma(x) = x^q$,
- $\gamma(r \cdot x) = r^q \cdot \gamma(x)$ and
- $\gamma(x + y) = \gamma(x) + \gamma(y) + \sum_{0<i<q}(\binom{q}{i}/\pi) \cdot x^i \cdot y^{q-i}$

hold for all $r \in R$ and $x, y \in \mathfrak{a}$. Let us denote by γ^n the n-fold iterate of γ. If we define
$$\alpha_n = \pi^{q^{n-1}+q^{n-2}+\ldots+q+1-n} \cdot \gamma^n : \mathfrak{a} \to \mathfrak{a},$$
we may define for each n a map
$$\begin{aligned} \mathrm{w}'_n : W_{\mathcal{O}}(\mathfrak{a}) &\to \mathfrak{a} \\ (x_0, x_1, \ldots, x_n, \ldots) &\to \alpha_n(x_0) + \alpha_{n-1}(x_1) + \ldots + \alpha_1(x_{n-1}) + x_n, \end{aligned}$$
which should not be confused with the n-th Witt polynomial of $W_{\mathcal{O}'}(S)$ for some $\mathcal{O}'$ and some $\mathcal{O}'$-algebra S. The map w'_n is w_n-linear, this means that beside additivity $\mathrm{w}'_n(rx) = \mathrm{w}_n(r)\mathrm{w}'_n(x)$ holds for all $n \in \mathbb{N}$, $x \in W_{\mathcal{O}}(\mathfrak{a})$ and $r \in W_{\mathcal{O}}(R)$.

The main application of this structure is the following: We define on $\mathfrak{a}^{\mathbb{N}}$ a $W_{\mathcal{O}}(R)$-module structure by setting

$$\xi[a_0, a_1, \ldots] = [\mathrm{w}_0(\xi)a_0, \mathrm{w}_1(\xi)a_1, \ldots]$$

for all $\xi \in W_{\mathcal{O}}(S)$ and $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$ and get an isomorphism of $W_{\mathcal{O}}(S)$-modules

$$\begin{array}{rcl} \log : W_{\mathcal{O}}(\mathfrak{a}) & \to & \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, \ldots) & \mapsto & [\mathrm{w}'_0(\underline{a}), \mathrm{w}'_1(\underline{a}), \ldots]. \end{array}$$

Since ^{F} acts on the right hand side by

$${}^F[a_0, a_1, \ldots] \quad = \quad [\pi a_1, \pi a_2, \ldots, \pi a_i, \ldots]$$

for all $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$, we obtain for the ideal $\mathfrak{a} \subset W_{\mathcal{O}}(\mathfrak{a})$, defined by

$$\log^{-1}(\,[a, 0, 0, \ldots]\,| \quad \text{for all } a \in \mathfrak{a}),$$

that ${}^F\mathfrak{a} = 0$ holds.

Now we turn our focus to deformation theory. A surjection $S \to R$ of $\mathcal{O}$-algebras with π nilpotent in S, such that the kernel $\mathfrak{a}$ may be equipped with an $\mathcal{O}$-pd-structure, is called an *$\mathcal{O}$-pd-thickening*. Let us consider such an $\mathcal{O}$-pd-thickening and a nilpotent f-$\mathcal{O}$-display $\mathcal{P} = (P, Q, F, F_1)$ over R. A *$\mathcal{P}$-triple* $\mathcal{T} = (\widetilde{P}, F, F_1)$ over S consists of a finitely generated projective $W_{\mathcal{O}}(S)$-module $\widetilde{P}$, which lifts P, and ${}^{F^f}$-linear morphisms $F : \widetilde{P} \to \widetilde{P}$ and $F_1 : \widehat{Q} \to \widetilde{P}$, where $\widehat{Q}$ denotes the inverse image of Q by the surjection $\widetilde{P} \to P$ (which has kernel $W_{\mathcal{O}}(\mathfrak{a})\widetilde{P}$). Furthermore, the following equations are required:

$$\begin{array}{rcl} F_1({}^V wx) & = & {}^{F^{f-1}}wFx \\ F_1(\mathfrak{a}\widetilde{P}) & = & 0, \end{array}$$

with $w \in W_{\mathcal{O}}(R)$, $x \in \widetilde{P}$ and $\mathfrak{a} \subset W_{\mathcal{O}}(R)$ as above. F_1 is uniquely determined by these requirements.
Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism between nilpotent f-$\mathcal{O}$-displays over R and $\mathcal{T}_i$ be a $\mathcal{P}_i$-triple over S for $i = 1, 2$. Then an *α-morphism* $\widetilde{\alpha} : \widetilde{P_1} \to \widetilde{P_2}$ is a morphism of $W_{\mathcal{O}}(S)$-modules which lifts α and commutes with the F and F_1 maps, which only makes sense since $\widetilde{\alpha}(\widehat{Q_1}) \subset \widehat{Q_2}$. For triples we have the following assertion:

Proposition 6. Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism between two nilpotent f-$\mathcal{O}$-displays over R. For $\mathcal{P}_i$-triples $\mathcal{T}_i$ over S there is a unique α-morphism of triples $\widetilde{\alpha} : \mathcal{T}_1 \to \mathcal{T}_2$.

The *Hodge filtration of an f-$\mathcal{O}$-display* $\mathcal{P}$ over an $\mathcal{O}$-algebra R is the R-submodule $Q/I_{\mathcal{O},R}P \subseteq P/I_{\mathcal{O},R}P$.

Proposition 7. Let $S \to R$ be an $\mathcal{O}$-pd-thickening. Then nilpotent f-$\mathcal{O}$-displays over S are equivalent to nilpotent f-$\mathcal{O}$-displays $\mathcal{P}'$ over R plus a lift of the Hodge filtration to a direct summand of $P/I_{\mathcal{O},S}P$, where (P, F, F_1'') is the unique $\mathcal{P}'$-triple over S.

With the help of this result we can prove the following Proposition:

Proposition 8. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension, $S \to R$ a surjection of $\mathcal{O}'$-algebras with π' nilpotent in S and nilpotent kernel. If one of the functors $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}'), \Gamma_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is essentially surjective over R, then this is also true for the respective functor over S.

The last Proposition enables us to show that, given a nonramified / totally ramified extension $\mathcal{O} \to \mathcal{O}'$ with ramification index f, $BT_{\mathcal{O}}^{(f)}, \Gamma_i(\mathcal{O}, \mathcal{O}'), \Omega_i(\mathcal{O}, \mathcal{O}')$ are equivalences of categories for all $\mathcal{O}'$-algebras R, which are complete local rings with perfect residue field, nilpotent nilradical and π' nilpotent in R. This is particularly important for $\mathcal{O} = \mathcal{O}'$, since we obtain then that $BT_{\mathcal{O}}$ is an equivalence for all $\mathcal{O}$-algebras R with the above properties.
By using stack theory, we obtain the following Proposition:

Proposition 9. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension. Assume that $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}'), \Gamma_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is fully faithful for all $\mathcal{O}'$-algebras with π' nilpotent in them, then the respective functor is an equivalence for all such algebras.

The proof of the last Proposition eventually reduces to the fact that we already know that the functors right before the Proposition are equivalences for these $\mathcal{O}'$-algebras.
Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension with ramification index f. By the last Proposition, what remains to show that $BT_{\mathcal{O}}^{(f)}, \Gamma_i(\mathcal{O}, \mathcal{O}'), \Omega_i(\mathcal{O}, \mathcal{O}')$ are equivalences for all $\mathcal{O}'$-algebras R with π' nilpotent in R is, assuming that $P(\mathcal{O}, R)$ resp. $P(\mathcal{O}, \mathcal{O}', R)$ is true for all $\mathcal{O}'$-algebras R with π' nilpotent in R, that for all $\mathcal{O}'$-algebras R with π' nilpotent in R the functor $BT_{\mathcal{O}'}$ is faithful resp. $BT_{\mathcal{O}}^{(f)}$ is faithful when we restrict to the full subcategory of the nilpotent f-$\mathcal{O}$-displays over R consisting of the objects which lie in the image of $\Omega_1(\mathcal{O}, \mathcal{O}')$.
For this we are going to construct, for a fixed nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R, a crystal of $\mathcal{O}^{\text{crys}}$ -modules on $\operatorname{Spec} R$ (see Definition 5.1.1 for our definition of the crystalline site). It suffices, to give the value of the crystal $\mathcal{D}_{\mathcal{P}}$ for $\mathcal{O}$-pd-thickenings $\operatorname{Spec} R' \to \operatorname{Spec} S$, where $\operatorname{Spec} R' \hookrightarrow \operatorname{Spec} R$ is an affine open neighbourhood. When the triple over S associated to $\mathcal{P}_{R'}$ looks like $(\widetilde{P}, F, V^{-1})$, we define

$$\mathcal{D}_{\mathcal{P}}(\operatorname{Spec} R' \to \operatorname{Spec} S) := S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} \widetilde{P}.$$

If the setting is clear, we just write $\mathcal{D}_{\mathcal{P}}(S)$ instead of $\mathcal{D}_{\mathcal{P}}(\operatorname{Spec} R' \to \operatorname{Spec} S)$.
Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$. We now introduce the category

$\mathrm{Ext}_{1,S\to R}$ (for the basic definitions of (generalized) Cartier theory we refer to section 2.4 in this thesis). For S an $\mathcal{O}$-algebra and L an S-module, we may define the group $C(L) = \prod_{i\geq 0} V^i L$, which becomes an $\mathbb{E}_{\mathcal{O},S}$-module by the equations

$$\begin{aligned} \xi(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 0} V^i \mathrm{w}_n(\xi) l_i, \\ V(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 0} V^{i+1} l_i, \\ F(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 1} V^{i-1} \pi l_i \end{aligned}$$

for all $\xi \in W_{\mathcal{O}}(S)$ and $l_i \in L$. Let G be a (π-divisible) formal $\mathcal{O}$-module over R with Cartier module M, which we consider as an $\mathbb{E}_{\mathcal{O},S}$-module. Then an *extension* (L, N, M) *of* M *by the* S*-module* L is an exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(L) \to N \to M \to 0,$$

with N a reduced $\mathbb{E}_{\mathcal{O},S}$-module and $\mathfrak{a}N \subset V^0 L$, where $\mathfrak{a} \subset W_{\mathcal{O}}(S) \subset \mathbb{E}_{\mathcal{O},S}$ is as above.
Now let G, G' be two formal $\mathcal{O}$-modules over R, $M (= M_G), M' (= M_{G'})$ their Cartier modules and $\beta : M \to M'$ a morphism between them over R. Furthermore, let (L, N, M) and (L', N', M') be extensions of M and M'. Then a morphism of extensions $(L, N, M) \to (L', N', M')$ consists of a morphism of S-modules $\varphi : L \to L'$, a morphism of $\mathbb{E}_{\mathcal{O},S}$-modules $u : N \to N'$ and the $\mathbb{E}_{\mathcal{O},R}$-linear morphism β, such that the diagram of $\mathbb{E}_{\mathcal{O},S}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(L) & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C(\varphi)} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & C(L') & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

is commutative, where $C(\varphi)$ is given by sending $V^i l$ to $V^i \varphi(l)$ for each $i \geq 0$ and $l \in L$.

Definition 10. With the above notation, we define the category $\mathrm{Ext}_{1,S\to R}$ by the objects (L, N, M), such that M is the Cartier module of a π-divisible formal $\mathcal{O}$-module over R. The morphisms are those previously described.

We show the equivalence of $\mathrm{Ext}_{1,S\to R}$ with a second category $\mathrm{Ext}_{2,S\to R}$ when $\mathfrak{a}$ is nilpotent. Since we deal only with π-divisible formal $\mathcal{O}$-modules and not the more general π-divisible $\mathcal{O}$-modules, we find in (the generalization of) [Zin, Universal extension, Theorem 3] a stronger result than in [Mes72, Chapt. 4 Theorem 2.2.] for p-divisible formal groups or [FGL07, Theoreme B.6.3.] for π-divisible formal $\mathcal{O}$-modules, where the results are only stated with respect to nilpotent ($\mathcal{O}$-)pd-thickenings, but continue to hold for all p-divisible groups or π-divisible $\mathcal{O}$-modules, respectively. We utilize this result for $\mathrm{Ext}_{2,S\to R}$ and obtain with the help of the association to $\mathrm{Ext}_{1,S\to R}$ the following result:

Theorem 11. If $S \to R$ is an $\mathcal{O}$-pd-thickening with nilpotent kernel and G a π-divisible formal $\mathcal{O}$-module over R, then there is a universal extension $(L^{\mathrm{univ}}, N^{\mathrm{univ}}, M_G) \in \mathrm{Ext}_{1,S\to R}$. Here the universality means, for any π-divisible formal $\mathcal{O}$-module G' over R, any morphism of $\mathbb{E}_{\mathcal{O},R}$-modules $\beta : M_G \to M_{G'}$ and any extension $(L, N, M_{G'}) \in \mathrm{Ext}_{1,S\to R}$, there is a unique morphism

$$(\varphi, u, \beta) : (L^{\mathrm{univ}}, N^{\mathrm{univ}}, M_G) \to (L, N, M_{G'}).$$

Definition 12. We define the *crystal of Grothendieck-Messing on the nilpotent ideal crystalline site* (see Defintion 5.1.1) by

$$\mathbb{D}_G(S) = \mathrm{Lie}\, N^{\mathrm{univ}}.$$

It is now very interesting to associate $\mathcal{D}_{\mathcal{P}}$ and $\mathbb{D}_{BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)}$ with each other. Let $S \to R$ be an $\mathcal{O}$-pd-thickening and $\mathcal{P}$ a nilpotent f-$\mathcal{O}$-display over R. Then we verify that the exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}/U \to M(\mathcal{P}) \to 0 \tag{2}$$

lies in $\mathrm{Ext}_{1,S\to R}$. Here $(\widetilde{P}, F, F_1)$ is the unique $\mathcal{P}$-triple over S, the second arrow maps $y \in \widehat{Q}$ to $V^f \otimes F_1 y - 1 \otimes y$, the third arrow is given by the canonical map $\widetilde{P} \to P$ and U is the $\mathbb{E}_{\mathcal{O},S}$-submodule of $\mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$ generated by $(F \otimes x - V^{f-1} \otimes Fx)_{x\in\widetilde{P}}$.

Proposition 13. In case $f = 1$ and the kernel of the $\mathcal{O}$-pd-thickening $S \to R$ is nilpotent, the previous extension is the universal one.

Theorem 14. For a nilpotent $\mathcal{O}$-display $\mathcal{P}$ over R and the associated π-divisible formal $\mathcal{O}$-module G we obtain a canonical isomorphism of crystals on the nilpotent ideal crystalline site over $\mathrm{Spec}\, R$:

$$\mathcal{D}_{\mathcal{P}} \simeq \mathbb{D}_G$$

It respects the Hodge filtration on $\mathcal{D}_{\mathcal{P}}(R)$ and $\mathbb{D}_G(R)$, respectively.

If we consider a morphism $W_{\mathcal{O}}(R) \to S$ of (topological) $\mathcal{O}$-pd-thickenings (see Definition 3.2.1) over R, we obtain that

$$\mathcal{D}_{\mathcal{P}}(S) \simeq S \otimes_{W_{\mathcal{O}}(R)} P$$

holds. We mainly consider $S = W_{\mathcal{O},n}(R)$. Given a morphism $\alpha : \mathcal{P} \to \mathcal{P}'$ of $\mathcal{O}$-displays over R, we obtain a morphism $G \to G'$ of the associated π-divisible formal $\mathcal{O}$-modules G and G'. By the universality of $\mathbb{D}_G$ and $\mathbb{D}_{G'}$, we obtain a morphism

$$W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P = \mathbb{D}_G(W_{\mathcal{O},n}(R)) \to \mathbb{D}_{G'}(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P',$$

which must be given by $1 \otimes \alpha$. Since we clearly obtain a morphism of the inverse systems $(W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P)_n$ and $(W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P')_n$, we get α back by passing to the projective limit. So we can state:

Proposition 15. Let R be an $\mathcal{O}$-algebra with π nilpotent in R. Then $BT_{\mathcal{O}}$ is faithful.

From the faithfulness of $BT_{\mathcal{O}}$ we can deduce together with Proposition 9, applied to $\Gamma_i(\mathcal{O}, \mathcal{O}')$, the generalized main Theorem of display theory:

Theorem 16. For every $\mathcal{O}$ and every $\mathcal{O}$-algebra R with π nilpotent in R, the $BT_{\mathcal{O}}$ functor is an equivalence of categories between the category of nilpotent $\mathcal{O}$-displays over R and the category of π-divisible formal $\mathcal{O}$-modules over R.

Furthermore, the following result holds:

Proposition 17. Let $\mathcal{O} \to \mathcal{O}'$ be nonramified (of degree f) and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then $\Omega_1(\mathcal{O}, \mathcal{O}')$ is fully faithful.

As mentioned above, since $P(\mathcal{O}, R)$ is true, we only need to show for the previous Proposition that $BT_{\mathcal{O}}^{(f)}$ is faithful when we restrict to the full subcategory of the category of nilpotent f-$\mathcal{O}$-displays over R consisting of the objects which lie in the image of $\Omega_1(\mathcal{O}, \mathcal{O}')$. For this purpose, we let $\mathcal{P}$ be an $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action and denote by $\mathcal{P}_0$ its image via $\Omega_1(\mathcal{O}, \mathcal{O}')$. Now let $S \to R$ be an $\mathcal{O}'$-algebra morphism, which is also an $\mathcal{O}$-pd-thickening. We consider (2) for $\mathcal{P}$ and $\mathcal{P}_0$ and are able to write down the unique morphism of extensions from the extension for $\mathcal{P}$ to $\mathcal{P}_0$ explicitly. With the help of this the result follows easily.
So we obtain with the help of Proposition 17, Proposition 9 and Theorem 16 that the following assertions hold (in fact the assertion for the $\Gamma_i(\mathcal{O}, \mathcal{O}')$ can be established without using Proposition 17) :

Corollary 18. Let $\mathcal{O}'$ over $\mathcal{O}$ be nonramified of degree f and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then the following functors are equivalences of categories:

- $\Omega_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (f - \mathrm{ndisp}_{\mathcal{O}}/R)$
- $BT_{\mathcal{O}}^{(f)} : (f - \mathrm{ndisp}_{\mathcal{O}}/R) \to (\pi' - \text{divisible formal } \mathcal{O}' - \text{modules}/R)$
- $\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{ndisp}_{\mathcal{O}}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R)$
- $\Gamma_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R)$

Let $\mathcal{O}'$ be totally ramified over $\mathcal{O}$ and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then

- $\Gamma_2(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'} /R) \to (\mathrm{ndisp}_{\mathcal{O}'} /R)$

is an equivalence of categories.

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Chapter 1

$\mathcal{O}$-algebras of ramified Witt vectors

From now on, we fix a prime number p and all rings and algebras over a commutative ring are assumed to be commutative. In this chapter we first define a special ring structure, a so-called RRS, in Definition 1.1.1, which should be considered as a generalization of the rings of integers of a non-Archimedean local field of characteristic zero and construct to each RRS $\mathcal{O}$ and each (not necessarily unitary) $\mathcal{O}$-algebra R an $\mathcal{O}$-algebra of ramified Witt vectors $W_{\mathcal{O}}(R)$. If there is given a suitable kind of ring morphism $\mathcal{O} \to \mathcal{O}'$, we will be able to construct a morphism of functors $W_{\mathcal{O}} \to W_{\mathcal{O}'}$ from the category of $\mathcal{O}'$-algebras to the category of $\mathcal{O}$-algebras. After restricting to the rings of integers of non-Archimedean local fields of characteristic zero for $\mathcal{O}$, we will consider the relations of $W_{\mathcal{O}}(l)$ to local field theory, where l is a perfect field extending the residue field of $\mathcal{O}$. With the help of these structures we will be able to define and to work on f-$\mathcal{O}$-displays in the next chapters.

1.1 The $\mathcal{O}$-algebra of ramified Witt vectors $W_{\mathcal{O}}(R)$

Definition 1.1.1. Let $\mathcal{O}$ be a commutative unitary ring, $0 \neq \pi \in \mathcal{O}$ not a zero-divisor and q a power of p. If additionally $p \in \pi\mathcal{O}$ and $x \equiv x^q \mod \pi$ holds for all $x \in \mathcal{O}$, we call the triple $(\mathcal{O}, \pi, q)$ a *ramification ring structure*, short *RRS*. If all the other attachments are clear or only of a theoretical use (where the exact structure is not needed), we usually just write $\mathcal{O}$. An *excellent* morphism μ of RRSs between $(\mathcal{O}, \pi, q = p^f)$ and $(\mathcal{O}', \pi', q' = p^g)$ is a ring morphism $\mu : \mathcal{O} \to \mathcal{O}'$, such that $0 \neq \mu(\pi) \in \pi'\mathcal{O}'$ is not a zero-divisor and $\frac{g}{f} \in \mathbb{N}$ holds.

Even though the structure is defined quite generally here, we are most interested in taking $\mathcal{O}$ to be the ring of integers of a non-Archimedean local field of characteristic zero, so, generally, this should be the case one has in mind. Here one has $(\mathcal{O}, \pi, q)$, where π is a uniformizing element of $\mathcal{O}$ and q is the order of the residue field of $\mathcal{O}$. Let $\mathcal{O}$ be an RRS. Our aim is now to introduce for an $\mathcal{O}$-algebra R an $\mathcal{O}$-algebra structure on the set

$$W_{\mathcal{O}}(R) = \{ (b_0, b_1, \dots) \mid b_i \in R \},$$

which is uniquely determined by certain additional properties. We will call this the *$\mathcal{O}$-algebra of ramified Witt vectors over R*, its elements the *ramified Witt vectors* and the map

$$\begin{array}{rcl} \mathrm{w}_n : W_{\mathcal{O}}(R) & \to & R \\ \underline{b} = (b_0, b_1, \dots) & \mapsto & b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^n b_n \end{array}$$

the *n-th Witt polynomial*.

Theorem 1.1.2. Let $\mathcal{O}$ be an RRS. Then for any $\mathcal{O}$-algebra R, there exists a unique $\mathcal{O}$-algebra structure on $W_{\mathcal{O}}(R)$ with the following properties:

1. For every $\mathcal{O}$-algebra morphism $\nu : R \to R'$ the induced morphism $\overline{\nu} : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R')$ given by $\underline{b} = (b_0, b_1, \dots) \mapsto (\nu(b_0), \nu(b_1), \dots)$ for all $\underline{b} \in W_{\mathcal{O}}(R)$ is an $\mathcal{O}$-algebra morphism.

2. The maps $\mathrm{w}_n : W_{\mathcal{O}}(R) \to R$ are $\mathcal{O}$-algebra morphisms.

In order to prove this Theorem, we first have to establish the following Lemma.

Lemma 1.1.3. Let B be a π-torsion free $\mathcal{O}$-algebra, $\tau : B \to B$ an $\mathcal{O}$-algebra morphism with

$$\tau(x) \equiv x^q \mod \pi.$$

Consider a sequence $u_0, u_1, \dots$ of elements of B. There is a vector $\underline{b} \in W_{\mathcal{O}}(B)$ with $\mathrm{w}_n(\underline{b}) = u_n$, iff

$$\tau(u_{n-1}) \equiv u_n \mod \pi^n \tag{1.1}$$

is fulfilled for every n. Furthermore, the vector $\underline{b}$ is unique.

Proof: Let x and y be elements of B. If $x \equiv y \mod \pi^n$ is satisfied, then $x^q \equiv y^q \mod \pi^{n+1}$ holds. Especially we get $\tau(x^{q^r}) \equiv x^{q^{r+1}} \mod \pi^{r+1}$ for all $r \geq 0$. Now suppose we have a vector $\underline{b}$ which satisfies $\mathrm{w}_n(\underline{b}) = u_n$ for every n. Then we obtain

$$\begin{array}{rcl} \tau(u_{n-1}) & = & \tau(b_0^{q^{n-1}} + \pi b_1^{q^{n-2}} + \dots + \pi^{n-1} b_{n-1}) \\ & \equiv & b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^{n-1} b_{n-1}^q \\ & = & b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^{n-1} b_{n-1}^q + \pi^n b_n - \pi^n b_n \\ & = & u_n - \pi^n b_n \equiv u_n \mod \pi^n. \end{array}$$

Hence, we have shown the forward direction. To prove that (1.1) is sufficient, we construct $\underline{b}$ inductively (and in a unique way, so we see as well that $\underline{b}$ is unique). Let $b_0, b_1, \ldots, b_{n-1}$ be already constructed. Now we search for a b_n, such that

$$b_0^{q^n} + \pi b_1^{q^{n-1}} + \ldots + \pi^{n-1} b_{n-1}^q + \pi^n b_n = u_n$$

is satisfied. By above calculations we have

$$u_n \equiv \tau(u_{n-1}) \equiv b_0^{q^n} + \pi b_1^{q^{n-1}} + \ldots + \pi^{n-1} b_{n-1}^q \mod \pi^n,$$

where we have used the congruence (1.1). So we have $u_n - (b_0^{q^n} + \pi b_1^{q^{n-1}} + \ldots + \pi^{n-1} b_{n-1}^q) = \pi^n k$ for a suitable $k \in B$. Hence it is possible to take $b_n = k$. The uniqueness follows, since B is π-torsion free. □

Now we turn to the proof of Theorem 1.1.2.
Proof: We first consider $B = \mathcal{O}[X_0, Y_0, X_1, Y_1, \ldots]$ with its obvious $\mathcal{O}$-algebra structure. We then define $\tau : B \to B$ to be the $\mathcal{O}$-algebra morphism given by $\tau(X_i) = X_i^q$ and $\tau(Y_i) = Y_i^q$ for all $i \geq 0$ and denote by $\underline{X}, \underline{Y}$ the ramified Witt vectors $(X_0, X_1, \ldots), (Y_0, Y_1, \ldots) \in W_{\mathcal{O}}(B)$.
We define the elements $\underline{X} + \underline{Y}, \underline{X} \cdot \underline{Y}, a \cdot \underline{X} \in W_{\mathcal{O}}(B)$ for each $a \in \mathcal{O}$ by

$$\begin{aligned} \mathrm{w}_n(\underline{X} + \underline{Y}) &= \mathrm{w}_n(\underline{X}) + \mathrm{w}_n(\underline{Y}), \\ \mathrm{w}_n(\underline{X} \cdot \underline{Y}) &= \mathrm{w}_n(\underline{X}) \mathrm{w}_n(\underline{Y}), \\ \mathrm{w}_n(a \cdot \underline{X}) &= a \mathrm{w}_n(\underline{X}). \end{aligned}$$

These elements exist and are uniquely determined by Lemma 1.1.3, because B is clearly a π-torsion free $\mathcal{O}$-algebra and τ fulfils the required properties of this Lemma. Now let R be an arbitrary $\mathcal{O}$-algebra and $\underline{b} = (b_0, b_1, \ldots), \underline{c} = (c_0, c_1, \ldots) \in W_{\mathcal{O}}(R)$. We consider the $\mathcal{O}$-algebra morphism $L_{\underline{b},\underline{c}} : B \to R$ given by $L_{\underline{b},\underline{c}}(X_i) = b_i$ and $L_{\underline{b},\underline{c}}(Y_i) = c_i$ for all $i \geq 0$ and define $\underline{b} + \underline{c}, \underline{b} \cdot \underline{c}, a \cdot \underline{b}$ for each $a \in \mathcal{O}$ by

$$\begin{aligned} \underline{b} + \underline{c} &= \overline{L_{\underline{b},\underline{c}}}(\underline{X} + \underline{Y}), \\ \underline{b} \cdot \underline{c} &= \overline{L_{\underline{b},\underline{c}}}(\underline{X} \cdot \underline{Y}), \\ a \cdot \underline{b} &= \overline{L_{\underline{b},\underline{c}}}(a \cdot \underline{X}), \end{aligned}$$

where $\overline{L_{\underline{b},\underline{c}}} : W_{\mathcal{O}}(B) \to W_{\mathcal{O}}(R)$ should denote the by $L_{\underline{b},\underline{c}}$ induced map. It is easily seen that this $\mathcal{O}$-algebra structure on $W_{\mathcal{O}}(R)$ is the only one which can fulfil the required properties of the Theorem - if it is one, but this is easily verified. Furthermore, it is not too hard to check that $\overline{L_{\underline{b},\underline{c}}}$ is an $\mathcal{O}$-algebra morphism.
It remains to verify the required properties in the assertion. Let R and R' be two $\mathcal{O}$-algebras, $\nu : R \to R'$ an $\mathcal{O}$-algebra morphism and $\overline{\nu} : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R')$ the induced map. We have to show that it is an $\mathcal{O}$-algebra morphism. For this

purpose, let $\underline{b}, \underline{c} \in W_{\mathcal{O}}(R)$ and $a \in \mathcal{O}$. Consider the diagram

$$\begin{array}{ccc} W_{\mathcal{O}}(B) & & \\ {\scriptstyle \overline{L_{\underline{b},\underline{c}}}}\downarrow & \searrow^{\overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{c})}}} & \\ W_{\mathcal{O}}(R) & \xrightarrow[\overline{\nu}]{} & W_{\mathcal{O}}(R'). \end{array}$$

It can easily be verified that this diagram is commutative. Hence we obtain

$$\begin{aligned} \overline{\nu}(\underline{b}+\underline{c}) &= \overline{\nu}\overline{L_{\underline{b},\underline{c}}}(\underline{X}+\underline{Y}) \\ &= \overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{c})}}(\underline{X}+\underline{Y}) \\ &= \overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{c})}}(\underline{X}) + \overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{c})}}(\underline{Y}) \\ &= \overline{\nu}(\underline{b}) + \overline{\nu}(\underline{c}). \end{aligned}$$

Similarly, we get $\overline{\nu}(\underline{b} \cdot \underline{c}) = \overline{\nu}(\underline{b}) \cdot \overline{\nu}(\underline{c})$ and $\overline{\nu}(a\underline{b}) = a\overline{\nu}(\underline{b})$. This proves the first requirement. For the second one, we consider the commutative diagram

$$\begin{array}{ccc} W_{\mathcal{O}}(B) & \xrightarrow{\mathrm{w}_n} & B \\ {\scriptstyle \overline{L_{\underline{b},\underline{c}}}}\downarrow & & \downarrow{\scriptstyle L_{\underline{b},\underline{c}}} \\ W_{\mathcal{O}}(R) & \xrightarrow[\mathrm{w}_n]{} & R, \end{array}$$

where $\underline{b}, \underline{c}$ are as above. With similar considerations as above it is easily verified that the $\mathrm{w}_n : W_{\mathcal{O}}(R) \to R$ are $\mathcal{O}$-algebra morphisms. □

We should remark that for each $\mathcal{O}$ a ring of integers of a non-Archimedean local field of characteristic zero and each $\mathcal{O}$-algebra R the $\mathcal{O}$-algebra $W_{\mathcal{O}}(R)$ clearly depends on the choice of π for the RRS $(\mathcal{O}, \pi, q)$, but we will see in Corollary 1.2.3 that this does not make big difficulties for us.
With the help of Lemma 1.1.3 we can deduce:

Lemma 1.1.4. Let $\mathcal{O}$ be an RRS, B a π-torsion free $\mathcal{O}$-algebra and $\tau : B \to B$ an $\mathcal{O}$-algebra morphism with $\tau(x) \equiv x^q \mod \pi$. Then there is a unique $\mathcal{O}$-algebra morphism $\kappa : B \to W_{\mathcal{O}}(B)$, such that $\mathrm{w}_n(\kappa(b)) = \tau^n(b)$ holds for each $b \in B$ and $n \geq 0$.

This Lemma is particularly important, when we consider a nonramified extension of non-Archimedean local fields of characteristic zero $\mathcal{O} \to \mathcal{O}'$. If we denote by σ the relative Frobenius of this extension, then there is a unique $\mathcal{O}$-algebra morphism

$$\kappa : \mathcal{O}' \to W_{\mathcal{O}}(\mathcal{O}'), \tag{1.2}$$

such that $\mathrm{w}_n(\kappa(a)) = \sigma^n(a)$ holds for each $a \in \mathcal{O}'$ and $n \geq 0$. Here, the $\mathcal{O}$-algebra structure of $W_{\mathcal{O}}(\mathcal{O}')$ has been established with respect to a fixed prime element of $\mathcal{O}$. Our next aim is to introduce the $\mathcal{O}$-module morphism ${}^{V} : W_{\mathcal{O}}(R) \to$

$W_{\mathcal{O}}(R)$ and the $\mathcal{O}$-algebra morphism ${}^F : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ for all $\mathcal{O}$-algebras R, which should be similar to those mappings defined in [Zin02]. The first is called the Verschiebung and the second one is the Frobenius. They are defined by functoriality in R and the relations, for all $n \geq 0$,

$$\mathrm{w}_n({}^F x) = \mathrm{w}_{n+1}(x), \tag{1.3}$$
$$\mathrm{w}_{n+1}({}^V x) = \pi \mathrm{w}_n(x), \quad \mathrm{w}_0({}^V x) = 0, \tag{1.4}$$

where $x \in W_{\mathcal{O}}(R)$ and the equations and multiplications have to be unterstood in R. It is very important to remark that in contrast to the original Definition we have a π here instead of a p. We have to show that we can construct in both cases for every R and every element of $W_{\mathcal{O}}(R)$ a unique image, hence the maps are well-defined, and that they are $\mathcal{O}$-algebra morphisms resp. $\mathcal{O}$-module morphisms.

With the same notation as in the proof of the Theorem, we receive with the help of Lemma 1.1.3 that this is the case for $B = \mathcal{O}[X_0, Y_0, X_1, Y_1, \ldots]$. It should be remarked that

$${}^F(\underline{X} + \underline{Y}) = {}^F\underline{X} + {}^F\underline{Y}, \tag{1.5}$$
$${}^F(\underline{X} \cdot \underline{Y}) = {}^F\underline{X} \cdot {}^F\underline{Y}, \tag{1.6}$$
$${}^F(a \cdot \underline{X}) = a \cdot {}^F\underline{X} \tag{1.7}$$

hold for all $a \in \mathcal{O}$ and (1.5) and (1.7) are true for ^{V} instead of ^{F}.

Now consider a general $\mathcal{O}$-algebra R. We define for $\underline{b} \in W_{\mathcal{O}}(R)$ the Frobenius and the Verschiebung by

$${}^F\underline{b} = \overline{L_{\underline{b},\underline{0}}}({}^F\underline{X}), \tag{1.8}$$
$${}^V\underline{b} = \overline{L_{\underline{b},\underline{0}}}({}^V\underline{X}), \tag{1.9}$$

where $\underline{0} = (0, 0, \ldots) \in W_{\mathcal{O}}(R)$ and $\overline{L_{\underline{b},\underline{0}}}$ is as in the proof of the Theorem. It is not too hard to check that

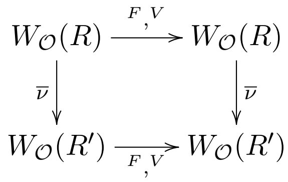
$${}^F, {}^V : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$$

are $\mathcal{O}$-algebra morphisms resp. $\mathcal{O}$-module morphisms with the help of the equations (1.5) to (1.7) for ^{F} and (1.5) and (1.7) for ^{V} instead of ^{F}. It remains to show that ^{F} and ^{V} are functorial and that the defining equations hold.

For the first aspect consider an $\mathcal{O}$-algebra morphism $\nu : R \to R'$, which in turn induces the $\mathcal{O}$-algebra morphism $\overline{\nu} : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R')$. For $\underline{b} \in W_{\mathcal{O}}(R)$ we have by construction

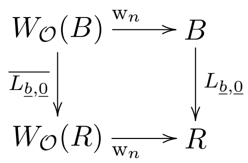
$$\overline{L_{\underline{b},\underline{0}}}({}^F\underline{X}) = {}^F\overline{L_{\underline{b},\underline{0}}}(\underline{X}), \tag{1.10}$$
$$\overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{0})}}({}^F\underline{X}) = {}^F\overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{0})}}(\underline{X}), \tag{1.11}$$
$$\overline{L_{\underline{b},\underline{0}}}({}^V\underline{X}) = {}^V\overline{L_{\underline{b},\underline{0}}}(\underline{X}), \tag{1.12}$$
$$\overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{0})}}({}^V\underline{X}) = {}^V\overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{0})}}(\underline{X}). \tag{1.13}$$

To show the functoriality, we assert that the diagram

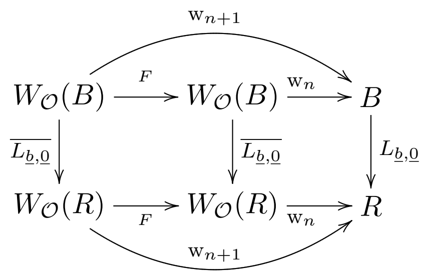


is commutative.* This means that $\overline{\nu}(^F\underline{b}) =^F \overline{\nu}(\underline{b})$ and $\overline{\nu}(^V\underline{b}) =^V \overline{\nu}(\underline{b})$ must hold for all $\underline{b} \in W_{\mathcal{O}}(R)$. Since $\overline{\nu}\overline{L_{\underline{b},\underline{0}}} = \overline{L_{\overline{\nu}(\underline{b}),\overline{\nu}(\underline{0})}}$, we easily obtain the claimed equations by the equations (1.10)-(1.13).

It remains to show that the equations (1.3) and (1.4) hold. For this, we remark that for every $n \geq 0$ and every $\underline{b} \in W_{\mathcal{O}}(R)$ the diagram



is commutative. If in addition we consider for every $\underline{b} \in W_{\mathcal{O}}(R)$ the diagram



and use of which parts of the diagram we already know that they are commutative, we obtain, by utilizing the definition of $^F\underline{b}$, that the equations for ^{F} are fulfilled. Analogous considerations lead us to establish the equations for ^{V}.

Concerning these two morphisms, we need to mention two elementary relations:

$$^{FV} = \pi \tag{1.14}$$

$$^V(^Fxy) = x^Vy \quad x, y \in W_{\mathcal{O}}(R) \tag{1.15}$$

The equations can be obtained by considering the values of the Witt polynomials in a suitable universal case. We denote the image of $^V : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ by $I_{\mathcal{O},R}$ and we obtain easily that

$$^V(b_0, b_1, \ldots) = (0, b_0, b_1, \ldots)$$

*Here we consider the diagram only for ^{V} and ^{F}, respectively, and not in the way that we set, for instance, ^{V} in the first line and ^{F} in the second.

holds for all $(b_0, b_1, \ldots) \in W_{\mathcal{O}}(R)$. Hence we can say that $I_{\mathcal{O},R}$ is the ideal of ramified Witt vectors, whose first component is zero, or, equivalently said, $I_{\mathcal{O},R} = \ker(\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R)$. This ideal will become important, for example, for the definition of an f-$\mathcal{O}$-display over $W_{\mathcal{O}}(R)$, which we will introduce in the next chapter.

1.2 The morphism u and some basic results

Let $\mu : \mathcal{O} = (\mathcal{O}, \pi, q = p^k) \to \mathcal{O}' = (\mathcal{O}', \pi', q' = p^l)$ be an excellent morphism of RRSs. We denote by $\mathrm{Alg}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras. When we consider the Witt functor $W_{\mathcal{O}}$ from $\mathrm{Alg}_{\mathcal{O}}$ to $\mathrm{Alg}_{\mathcal{O}}$, we study the interaction between the two functors for $\mathcal{O}$ and $\mathcal{O}'$.

The following proposition will first become essential, when we consider reduced Cartier modules and their equivalence to formal $\mathcal{O}$-modules, where $\mathcal{O}$ is the ring of integers of a non-Archimedean local field of characteristic zero. We define the *Teichmüller representant* $[a] \in W_{\mathcal{O}}(R)$ by $(a, 0, 0 \ldots)$ for $\mathcal{O}$ an RRS, R an $\mathcal{O}$-algebra and $a \in R$. For an RRS $\mathcal{O}$ and a nilpotent $\mathcal{O}$-algebra $\mathcal{N}$, we denote by $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ the $\mathcal{O}$-subalgebra of $W_{\mathcal{O}}(\mathcal{N})$, which consists of the ramified Witt vectors with finitely many nonzero entries.

Proposition 1.2.1. Let $\mathcal{O} = (\mathcal{O}, \pi, q = p^k)$ and $\mathcal{O}' = (\mathcal{O}', \pi', q' = p^l)$ be two RRSs with $g := \frac{l}{k} \in \mathbb{N}_{\geq 1}$ and μ as above. Then there is a unique functor morphism $u : W_{\mathcal{O}} \to W_{\mathcal{O}'}$, such that $\mathrm{w}'_n \circ u = \mathrm{w}_{gn}$ holds (where the w_i and w'_i belong to the obvious structures), with both functors considered as functors from $\mathrm{Alg}_{\mathcal{O}'}$ to $\mathrm{Alg}_{\mathcal{O}}$. For a nilpotent $\mathcal{O}'$-algebra $\mathcal{N}$ the restriction morphism $u_{\mathcal{N}} : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \to W_{\mathcal{O}'}(\mathcal{N})$ has its image in $\widehat{W_{\mathcal{O}'}}(\mathcal{N})$. Furthermore, for an $\mathcal{O}'$-algebra R we have $u_R([a]) = [a]$ for $a \in R$, $u_R({}^{F^g}x) = {}^{F'}(u_R(x)), u_R({}^{V}x) = (\mu(\pi)/\pi')^{V'}(u_R({}^{F^{g-1}}x))$ for $x \in W_{\mathcal{O}}(R)$, where all the objects related to $\mathcal{O}'$ are marked with a dash.

With abuse of notation, we usually denote u_R by u if it is clear which R we consider.

Proof: As usual, we first consider a special $\mathcal{O}'$-algebra, which is in this case $B = \mathcal{O}'[X_0, Y_0, \ldots]$. We define the $\mathcal{O}'$-algebra morphism τ on B by $\tau(X_i) = X_i^{q^g}$ and $\tau(Y_i) = Y_i^{q^g}$. With the help of Lemma 1.1.3 we want to define u_B and show that this u_B is unique. Let $\underline{b} \in W_{\mathcal{O}}(B)$. Consider the sequence $(\mathrm{w}_{gn}(\underline{b}))_n$; because

$$\tau(\mathrm{w}_{g(n-1)}(\underline{b})) \equiv \mathrm{w}_{gn}(\underline{b}) \mod \pi'^n$$

for all $n \geq 1$, where we have used μ implicitly, there is a unique $\underline{b'} \in W_{\mathcal{O}'}(B)$, such that $\mathrm{w}_{gn}(\underline{b}) = \mathrm{w}'_n(\underline{b'})$ for all n. Hence it is sensible, and also the only way, to define $u_B(\underline{b}) = \underline{b'}$, so we get the unique map u_B. Now we have to show that the relations

$$
\begin{aligned}
u_B(\underline{b}+\underline{c}) &= u_B(\underline{b})+u_B(\underline{c}), &(1.16)\\
u_B(\underline{b}\cdot\underline{c}) &= u_B(\underline{b})u_B(\underline{c}), &(1.17)\\
u_B(a\underline{b}) &= au_B(\underline{b}), &(1.18)
\end{aligned}
$$

hold for all $\underline{b},\underline{c}\in W_{\mathcal{O}}(B)$ and $a\in\mathcal{O}$. Since

$$
\begin{aligned}
\mathrm{w}'_n(u_B(\underline{b}+\underline{c})) = \mathrm{w}_{gn}(\underline{b}+\underline{c}) &= \mathrm{w}_{gn}(\underline{b})+\mathrm{w}_{gn}(\underline{c})\\
&= \mathrm{w}'_n(u_B(\underline{b}))+\mathrm{w}'_n(u_B(\underline{c}))
\end{aligned}
$$

holds, we have established (1.16) and the equations (1.17) and (1.18) follow analogously.
Similarly to our ^{F} and ^{V} considerations, we can pass from B to any $\mathcal{O}'$-algebra R, and establish the map u_R. To show that u is functorial and $\mathrm{w}'_n\circ u=\mathrm{w}_{gn}$ holds, we also refer to the discussion concerning ^{F} and ^{V}, which follows then easily by the construction of u_R. For the assertions for the nilpotent $\mathcal{O}$-algebras we also consider B at first, make the calculations in the Witt polynomials there, from which we finally obtain, by passing to the respective nilpotent $\mathcal{O}$-algebra, the result. The equations are easily verified by considering the universal situation B, where we just need to consider the Witt polynomials, and then by passing to any $\mathcal{O}'$-algebra R as usual by considering only special elements $x,[a]\in W_{\mathcal{O}}(R)$, where $a\in R$. $\square$

For many considerations in the next chapter we need a Corollary, which can be found in [Dri76] and is a direct consequence of Proposition 1.2.1.

Corollary 1.2.2. (cf. [Dri76, Proposition 1.2]) Let $\mathcal{O}\to\mathcal{O}'$ be an extension of rings of integers of non-Archimedean local fields of characteristic zero, π,π' fixed uniformizing elements of $\mathcal{O}$ resp. $\mathcal{O}'$ and f the degree of extension of the residue fields. Then there exists a unique morphism $u: W_{\mathcal{O}}\to W_{\mathcal{O}'}$ of functors from $\mathrm{Alg}_{\mathcal{O}'}$ to $\mathrm{Alg}_{\mathcal{O}}$, such that ${\mathrm{w}'}_n\circ u=\mathrm{w}_{fn}$ holds. For a nilpotent $\mathcal{O}'$-algebra $\mathcal{N}$ the restriction morphism $u_{\mathcal{N}}:\widehat{W_{\mathcal{O}}}(\mathcal{N})\to W_{\mathcal{O}'}(\mathcal{N})$ has its image in $\widehat{W_{\mathcal{O}'}}(\mathcal{N})$. Furthermore, for an $\mathcal{O}'$-algebra R we have $u_R([a])=[a]$ for $a\in R$, $u_R({}^{F^f}x)={}^{F'}(u_R(x)), u_R({}^Vx)=(\pi/\pi')^{V'}(u_R({}^{F^{f-1}}x))$ for $x\in W_{\mathcal{O}}(R)$, where all the objects related to $\mathcal{O}'$ are marked with a dash.

The assertion for the nilpotent $\mathcal{O}'$-algebras will first get important in section 2.4 and 2.5.

Corollary 1.2.3. Let $\mathcal{O}$ be a ring of integers of a non-Archimedean local field of characteristic zero, π_0,π_1 two uniformizing elements of $\mathcal{O}$ and q the order of the residue field. Then the excellent morphism of RRS $\mathcal{O}_0=(\mathcal{O},\pi_0,q)\to\mathcal{O}_1=(\mathcal{O},\pi_1,q)$, given by the identity, induces a morphism of functors u, which is for all

$\mathcal{O}$-algebras R an isomorphism $u_R : W_{\mathcal{O}_0}(R) \simeq W_{\mathcal{O}_1}(R)$. Hence, the functor $W_{\mathcal{O}}$ is, up to a canonical in R functorial isomorphism, independent of the particular choice of the uniformizing element.

By the previous two corollaries we also obtain that the morphism of functors u in Corollary 1.2.2 is, up to a canonical isomorphism of morphisms of functors, independent of the choice of the uniformizing elements π, π' of $\mathcal{O}$ resp. $\mathcal{O}'$. Hence, given an extension $\mathcal{O} \to \mathcal{O}'$, we will often just make assertions for the morphism $W_{\mathcal{O}} \to W_{\mathcal{O}'}$ without particularly referring to any uniformizing element of $\mathcal{O}$ and $\mathcal{O}'$.

Corollary 1.2.4. The canonical excellent morphism $(\mathbb{Z}, p, p) \to (\mathbb{Z}_p, p, p)$ of RRSs induces a morphism u, which is for all $\mathbb{Z}_p$-algebras R an isomorphism $u_R : W_{\mathbb{Z}}(R) \simeq W_{\mathbb{Z}_p}(R)$.

Lemma 1.2.5. Let $\mathcal{O}$ be an RRS and k an $\mathcal{O}$-algebra with $\pi k = 0$, which is a perfect field of characteristic p. Then $W_{\mathcal{O}}(k)$ is a principal ideal domain and all ideals are of the form ${}^{V^n}W_{\mathcal{O}}(k)$ or 0.

Proof: We consider an ideal $0 \neq J \subseteq W_{\mathcal{O}}(k)$ (for $J = 0$, this is trivial). Hence, there is a $w \in J$, such that $w_l \neq 0$ for a natural number l and so there is an n, such that for all $i < n$ and $w' \in J$ we have $w'_i = 0$ and it exists an element $x \in J$, such that $x_n \neq 0$. If we show that $({}^{V^n}1)W_{\mathcal{O}}(k) \subseteq xW_{\mathcal{O}}(k)$ holds, or which is the same to say that ${}^{V^n}1 = x\widehat{w}$ for a $\widehat{w} \in W_{\mathcal{O}}(k)$, we then have

$$({}^{V^n}1)W_{\mathcal{O}}(k) \subseteq xW_{\mathcal{O}}(k) \subseteq J \subseteq^{V^n} W_{\mathcal{O}}(k) = ({}^{V^n}1)W_{\mathcal{O}}(k),$$

which are then in fact identities. We leave it to the reader to show the existence of $\widehat{w}$. □

Lemma 1.2.6. Let $\mathcal{O}$ be an RRS and R an $\mathcal{O}$-algebra with $\pi R = 0$. Then we get that

$${}^{F}(x_0, x_1, \ldots) = (x_0^q, x_1^q, \ldots)$$

holds for all $(x_0, x_1, \ldots) \in W_{\mathcal{O}}(R)$. Hence, if R is perfect, ^{F} is an isomorphism.

Proof: By going to the universal situation $W_{\mathcal{O}}(B)$ with $B = \mathcal{O}[X_0, X_1, X_2 \ldots]$ and $\underline{X}$ as usual, it can be verified that ${}^{F}\underline{X} = (b_0, b_1, \ldots)$ holds with $b_i = X_i^q + \pi P_i$, where P_i is an element of $\mathcal{O}[X_0, X_1, X_2, \ldots]$ and so, if we consider the $\mathcal{O}$-algebra morphism $\varphi : B \to R$ given by $X_i \mapsto x_i$, we get ${}^{F}(x_0, x_1, \ldots) = (\varphi(b_0), \varphi(b_1), \ldots) = (x_0^q, x_1^q, \ldots)$. The last assertion is clear. □

1.3 Generalized results concerning rings of integers of non-Archimedean local fields of characteristic zero

We now restate some basic facts of rings of integers of non-Archimedean local fields of characteristic zero from a more general point of view, which is very helpful especially when the interaction of the $W_{\mathcal{O}}$ is of interest. We mainly refer to Serre's book over local fields [Ser79]. In this section all rings are assumed to be unitary.

Lemma 1.3.1. (cf. [Ser79, Chapter II, Proposition 8]) Let $\mathcal{O} = (\mathcal{O}, \pi, q)$ be a ring of integers of a non-Archimedean local field of characteristic zero and A a complete and separated $\mathcal{O}$-algebra in the π-adic topology, such that $A/\pi A$ is a perfect ring of characteristic p. Then there exists exactly one system of representatives $f : A/\pi A \to A$, for which $f(\lambda^q) = f(\lambda)^q$. In order for $a \in A$ to be an element of $f(A/\pi A)$, it is necessary and sufficient that a is a q^n-th power for all $n \geq 0$; we also note that $f(\lambda\mu) = f(\lambda)f(\mu)$ holds for all $\lambda, \mu \in A/\pi A$. Finally, if π is not a zero-divisior of A, every element of A may be uniquely expressed by

$$\sum_{i=0}^{\infty} f(a_i)\pi^i$$

for suitable $a_i \in A/\pi A$.

Let $\omega : A \to A'$ be an $\mathcal{O}$-algebra morphism between two π-adically complete and separated $\mathcal{O}$-algebras, such that $A/\pi A$ and $A'/\pi A'$ are perfect rings of characteristic p. Then ω commutes with multiplicative representatives, i.e., $\omega(f_A(a)) = f'_A(\overline{\omega}(a))$ for all $a \in A/\pi A$, where the indices of the f's have their obvious meaning and $\overline{\omega}$ is the induced map from $A/\pi A$ to $A'/\pi A'$, because, by the previous Lemma, we know that it is necessary and sufficient for an element of a π-adic complete and separated $\mathcal{O}$-algebra to be a multiplicative representative that it is a q^n-th power for all n.
Let X_i, Y_i, for $i \geq 0$, be a family of variables. Then we denote by $S = \mathcal{O}[X_i^{q^{-\infty}}, Y_i^{q^{-\infty}}]$ the union of all rings $\mathcal{O}[X_i^{q^{-n}}, Y_i^{q^{-n}}]$ for all n. It is obvious that S is complete and separated in the π-adic topology. If $k = \mathcal{O}/\pi\mathcal{O}$, then $S/\pi S = k[X_i^{q^{-\infty}}, Y_i^{q^{-\infty}}]$ is perfect of characteristic p. The X_i, Y_i are multiplicative representatives in S, since they are q^n-th powers for all n. Now consider $x = \sum_{i=0}^{\infty} X_i\pi^i$ and $y = \sum_{i=0}^{\infty} Y_i\pi^i$. For $\star = +, \times$ or $-$, we obtain that $x \star y = \sum_{i=0}^{\infty} f(Q_i^{\star})\pi^i$ holds, where $Q_i^{\star} \in k[X_i^{q^{-\infty}}, Y_i^{q^{-\infty}}]$. These $Q_i^{\star}$ determine the structure of a π-adic complete and separated $\mathcal{O}$-algebra with perfect residue ring of characteristic p:

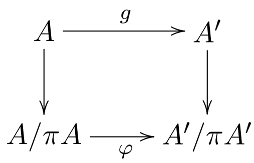
Lemma 1.3.2. Let A be as above and $f : A/\pi A \to A$ as in Lemma 1.3.1. Let $\{a_i\}$ and $\{b_i\}$ be two sequences of elements of $A/\pi A$. Then

$$\sum_{i=0}^{\infty} f(a_i)\pi^i \star \sum_{i=0}^{\infty} f(b_i)\pi^i \quad = \quad \sum_{i=0}^{\infty} f(c_i)\pi^i$$

where $c_i = Q_i^\star(a_0, b_0, a_1, b_1, \ldots)$.

Proof: This is the obvious generalization of [Ser79, Chapter II, Proposition 9]. □

Proposition 1.3.3. (cf. [Ser79, Chapter II, Proposition 10]) Let A, A' be two π-adically complete and separated $\mathcal{O}$-algebras, such that $A/\pi A$ and $A'/\pi A'$ are perfect of characteristic p and π is not a zero-divisor in A. Then we may lift every $\mathcal{O}$-algebra morphism $\varphi : A/\pi A \to A'/\pi A'$ uniquely to an $\mathcal{O}$-algebra morphism $g : A \to A'$, such that

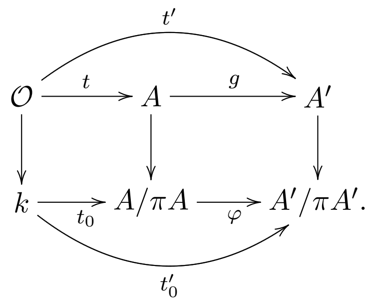


commutes.

Proof: Since every $\mathcal{O}$-algebra morphism from A to A' commutes with multiplicative representatives, we must have for an element $a \in A$ with coordinates $\{a_i\}$

$$g(a) = \sum_{i=0}^{\infty} g(f_A(a_i))\pi^i = \sum_{i=0}^{\infty} f_{A'}(\varphi(a_i))\pi^i,$$

so the uniqueness follows and by Lemma 1.3.2 we get that g, when defined by the above equation, is a ring morphism. In order to show that it is an $\mathcal{O}$-algebra morphism we consider the ring morphisms $t : \mathcal{O} \to A$ and $t' : \mathcal{O} \to A'$ which define the $\mathcal{O}$-algebra structure, which are the unique lifts of $t_0 : k \to A/\pi A$ and $t_0' : k \to A'/\pi A'$, with k the residue field of $\mathcal{O}$, and obtain the diagram of ring morphisms



This diagram must be commutative, because $t_0' = \varphi t_0$ must hold, since φ is a $\mathcal{O}$-algebra morphism and since the squares in this diagram must commute. We obtain that gt is the unique lift of t_0' and hence must be equal to t', which then shows that g is an $\mathcal{O}$-algebra morphism. □

Corollary 1.3.4. Let $\mathcal{O}$ be a ring of integers of a non-Archimedean local field of characteristic zero and k its residue field. Then there is a unique isomorphism of $\mathcal{O}$-algebras between $\mathcal{O}$ and $W_{\mathcal{O}}(k)$, such that

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & W_{\mathcal{O}}(k) \\ \downarrow & \swarrow & \\ k & & \end{array}$$

is commutative. Hence, this isomorphism is given by the ordinary $\mathcal{O}$-algebra structure $\mathcal{O} \to W_{\mathcal{O}}(k)$.

Proof: This follows easily by the previous Proposition by remarking that $W_{\mathcal{O}}(k)$ is π-adic by obvious reasons and that $W_{\mathcal{O}}(k)/\pi W_{\mathcal{O}}(k) = k$ holds, since we have

$$x = \sum_{n=0}^{\infty} {}^{V^n}[x_n] = \sum_{n=0}^{\infty} {}^{V^n F^n}[x_n^{q^{-n}}] = \sum_{n=0}^{\infty} \pi^n [x_n^{q^{-n}}]$$

for each $x \in W_{\mathcal{O}}(k)$ □

Lemma 1.3.5. Let $\mathcal{O} \to \mathcal{O}'$ be a totally ramified extension of rings of integers of non-Archimedean local fields of characteristic zero and k the residue field of $\mathcal{O}'$ and $\mathcal{O}$,which has q elements. Then $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$ and $W_{\mathcal{O}'}(l)$, where l is a perfect field extending k, are canonically isomorphic as $\mathcal{O}'$-algebras. This morphism is obtained by sending $a \otimes w$ to $au(w)$ and is $W_{\mathcal{O}}(l)$-linear as well.

Proof: Since it is easily seen that π' is not a zero divisor in $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$ and $W_{\mathcal{O}'}(l)$ and that both rings are π'-adic, we just need to confirm that $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)/\pi'\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$ and $W_{\mathcal{O}'}(l)/\pi' W_{\mathcal{O}'}(l)$ equal l. Then we can utilize Proposition 1.3.3. By the analogous calculation as in the proof of the previous Corollary we obtain $W_{\mathcal{O}'}(l)/\pi' W_{\mathcal{O}'}(l) = l$. We now consider the exact sequence

$$0 \to \pi'\mathcal{O}' \to \mathcal{O}' \to k \to 0.$$

After tensoring these $\mathcal{O}$-modules with $W_{\mathcal{O}}(l)$ we obtain the exact sequence

$$0 \to \pi'\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \to \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \to k \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \to 0.$$

Since $\pi'\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$ is the maximal ideal of $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$ we get that the residue field is

$$\mathcal{O} \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)/\pi'\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \cong k \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) = k \otimes_{\mathcal{O}} l \cong l,$$

hence first assertion follows. Because of the uniqueness of this (iso)morphism we also obtain the last assertion of the Lemma. □

Lemma 1.3.6. Let $\mathcal{O}$ and $\mathcal{O}'$ be rings of integers of non-Archimedean local fields of characteristic zero, k the residue field of $\mathcal{O}'$ and l a perfect field extending k. If $\mathcal{O}'$ is nonramified over $\mathcal{O}$, then $u_l : W_{\mathcal{O}}(l) \to W_{\mathcal{O}'}(l)$ is an isomorphism. If $\mathcal{O}'$ is totally ramified over $\mathcal{O}$ with ramification index e, where π, π' are fixed uniformizing elements of $\mathcal{O}$ resp. $\mathcal{O}'$, then $u_l : W_{\mathcal{O}}(l) \to W_{\mathcal{O}'}(l)$ is injective and turns $W_{\mathcal{O}'}(l)$ into a free $W_{\mathcal{O}}(l)$-module of rank e obtained by adjoining π' to $W_{\mathcal{O}}(l)$, which satisfies an Eisenstein equation $\pi'^e + a_1\pi'^{e-1} + \ldots + a_e = 0$, i.e., $a_i \in \pi\mathcal{O}$ and $a_e \notin \pi^2\mathcal{O}$ holds.

Proof: The assertion for the nonramified case is easily seen by computing u_l directly. In the totally ramified case, we obtain, because of $W_{\mathcal{O}}(k) = \mathcal{O}$ and $W_{\mathcal{O}'}(k) = \mathcal{O}'$, that

$$W_{\mathcal{O}}(k)[\pi']/(P(\pi')) = W_{\mathcal{O}'}(k)$$

holds, where $P(x) = x^e + a_1x^{e-1} + \ldots + a_e$. Hence, by Lemma 1.3.5 we obtain the isomorphism

$$W_{\mathcal{O}}(l)[\pi']/(P(\pi')) = \mathcal{O}' \otimes_{W_{\mathcal{O}}(k)} W_{\mathcal{O}}(l) \simeq W_{\mathcal{O}'}(l).$$

□

Chapter 2

f-$\mathcal{O}$-Display theory

Unless otherwise stated, from now on, $\mathcal{O}$ is always an RRS and the rings resp. $\mathcal{O}$-algebras denoted by R, R', S, S', R_0 etc. are always assumed to be unitary. Let $f \geq 1$ be a natural number. In this chapter we introduce the basic definitions and assertions of f-$\mathcal{O}$-display theory generalized to our situation. Once we finished this, we are going to introduce for each $\mathcal{O}$-algebra R (with π nilpotent in R) the $BT_{\mathcal{O}}^{(f)}$ functor, which associates to each (nilpotent) f-$\mathcal{O}$-display over R a (π-divisible) formal $\mathcal{O}$-module over R. Furthermore, we will revisit Drinfeld's equivalence for reduced $\mathbb{E}_{\mathcal{O},R}$-modules and formal $\mathcal{O}$-modules and let us inspire by this in order to introduce functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_i(\mathcal{O}, \mathcal{O}')$ for nonramified/ totally ramified extensions $\mathcal{O} \to \mathcal{O}'$ of rings of integers of non-Archimedean local fields of characteristic zero in the last section, of which we will show in the end that they are equivalences which will in turn be important to establish the generalized main Theorem of display theory. Furthermore, from now on, when we consider rings of integers of non-Archimedean local fields of characteristic zero $\mathcal{O}, \mathcal{O}'$ etc., then π, π' etc. are uniformizing elements of $\mathcal{O}, \mathcal{O}'$ etc. and p is the characteristic of the residue fields of $\mathcal{O}, \mathcal{O}'$ etc., which have q, q' etc. elements. For the constructions made for Drinfeld's generalized Cartier equivalence and the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ resp. $\Gamma_i(\mathcal{O}, \mathcal{O}')$ etc., we sometimes make use of the $\mathcal{O}$-algebra resp. $\mathcal{O}'$-algebra of ramified Witt vectors for a particular choice of the uniformizing element π resp. π' for the rings of integers of non-Archimedean local fields of characteristic zero $\mathcal{O}$ resp. $\mathcal{O}'$. Nevertheless, up to canonical isomorphism, the structures are independent of the choice of π resp. π'.

2.1 f-$\mathcal{O}$-Displays

In this section we introduce the definition of an f-$\mathcal{O}$-display and some aspects concerning Zink's display theory with respect to an $\mathcal{O}$-algebra of ramified Witt vectors.

Definition 2.1.1. Let $\mathcal{O}$ be an RRS, $f \geq 1$ a natural number and R an $\mathcal{O}$-algebra. An *f-$\mathcal{O}$-display* $\mathcal{P}$ over R is a quadruple (P, Q, F, F_1), where P is a finitely generated projective $W_{\mathcal{O}}(R)$-module, Q a submodule of P and $F : P \to P$ and $F_1 : Q \to P$ are ${}^{F^f}$-linear maps, such that the following properties are satisfied:

1. $I_{\mathcal{O},R}P \subset Q$ and there is as normal decomposition of P, i.e., there is a direct sum decomposition of $W_{\mathcal{O}}(R)$-modules $P = L \oplus T$, such that $Q = L \oplus I_{\mathcal{O},R}T$ holds.
2. F_1 is an ${}^{F^f}$-linear epimorphism, i.e., its linearisation
$$\begin{aligned} F_1^\sharp : W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} Q &\to P \\ w \otimes q &\mapsto wF_1 q, \end{aligned}$$
where $w \in W_{\mathcal{O}}(R)$ and $q \in Q$, is surjective.
3. For $x \in P$ and $w \in W_{\mathcal{O}}(R)$, we have
$$F_1({}^V wx) = {}^{F^{f-1}} wFx.$$

The finite projective R-module P/Q is the *tangential space* of $\mathcal{P}$. If $f = 1$, we call $\mathcal{P}$ just an *$\mathcal{O}$-display.*

This definition is very similar to [Zin02, Definition 1]. Furthermore, as in Zink's article we should remark that
$$F_1({}^V 1x) = Fx$$
holds for all $x \in P$, hence F is uniquely determined by F_1. When we apply this equation to an element $y \in Q$ we get
$$Fy = \pi \cdot F_1 y$$
by (1.14). In the definition of an f-$\mathcal{O}$-display we demanded the existence of a normal decomposition. However, we may state an equivalent form for π-adically complete and separated R, which will be needed later.

Proposition 2.1.2. (cf. [Zin02, Remark after Lemma 21]) Let $S \to R$ be a surjection of rings, such that S is complete and separated in the adic topology of the kernel, and P a finitely generated projective R-module. Then there is a tuple consisting of a finitely generated projective S-module $\widetilde{P}$ and an isomorphism $\phi : R \otimes_S \widetilde{P} \to P$. This tuple is uniquely determined up to isomorphism.

Now if we consider $\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R$ for a π-adically complete and separated $\mathcal{O}$-algebra R, where $\mathcal{O}$ is an RRS, then the kernel is $I_{\mathcal{O},R}$ and $W_{\mathcal{O}}(R)$ is complete and separated in the $I_{\mathcal{O},R}$-adic topology by the obvious generalization of [Zin02, Proposition 3]. Let us consider for an f-$\mathcal{O}$-display over R the finitely generated projective R-module $P_0 = P/I_{\mathcal{O},R}P$. The R-modules $L_0 = Q/I_{\mathcal{O},R}P$ and $T_0 = P/Q$ form a direct sum decomposition of P_0. By lifting the finitely generated projective R-module L_0 to a finitely generated projective $W_{\mathcal{O}}(R)$-module L we obtain by the universal property of projective modules a morphism of the lifts $L \to P$. If we lift T_0 in the same way we obtain a morphism $L \oplus T \to P$, which is an isomorphism by the lemma of Nakayama. By these considerations the following Corollary is easily seen:

Corollary 2.1.3. Let $\mathcal{O}$ be an RRS and R a π-adically complete and separated $\mathcal{O}$-algebra. Then the first property of Definition 2.1.1 is equivalent to the assertion that $I_{\mathcal{O},R}P \subseteq Q$ holds and P/Q is a finitely generated projective R-module.

Next we introduce an operator $V^\sharp$, which reminds us of the usual operator V in Cartier and Dieudonné theory:

Lemma 2.1.4. (cf. [Zin02, Lemma 10]) Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. There exists a unique $W_{\mathcal{O}}(R)$-linear map

$$V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P,$$

which satisfies the following equations for all $w \in W_{\mathcal{O}}(R), x \in P$ and $y \in Q$:

$$\begin{aligned} V^\sharp(wFx) &= \pi \cdot w \otimes x, \\ V^\sharp(wF_1y) &= w \otimes y \end{aligned}$$

Furthermore, we get $F^\sharp V^\sharp = \pi \, \mathrm{id}_{\mathcal{P}}$ and $V^\sharp F^\sharp = \pi \, \mathrm{id}_{W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P}$.

The proof in Zink's paper is easily generalized to $W_{\mathcal{O}}(R)$ and the ${}^{F^f}$-linearity, so we do not state it here.

By $V^{n\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fn}, W_{\mathcal{O}}(R)} P$ we mean the composite map ${}^{F^{f(n-1)}}V^\sharp \circ \ldots \circ {}^{F^f} V^\sharp \circ V^\sharp$, where ${}^{F^{fi}}V^\sharp$ is the $W_{\mathcal{O}}(R)$-linear map

$$\mathrm{id} \otimes_{F^{fi}, W_{\mathcal{O}}(R)} V^\sharp : W_{\mathcal{O}}(R) \otimes_{F^{fi}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R) \otimes_{F^{f(i+1)}, W_{\mathcal{O}}(R)} P.$$

Now we are able to introduce the Definition of a nilpotent f-$\mathcal{O}$-display.

Definition 2.1.5. Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra with π nilpotent in R and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We call $\mathcal{P}$ *nilpotent*, if there is a number N such that the composite map

$$\mathrm{pr} \circ V^{N\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R)/(I_{\mathcal{O},R} + \pi W_{\mathcal{O}}(R)) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P$$

is the zero map.

2.2 Morphisms, base changes and descent data for f-$\mathcal{O}$-displays

Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra and $\mathcal{P} = (P, Q, F, F_1), \mathcal{P}' = (P', Q', F', F_1')$ two f-$\mathcal{O}$-displays over R.

Definition 2.2.1. A *morphism* $\alpha : (P, Q, F, F_1) \to (P', Q', F', F_1')$ between two f-$\mathcal{O}$-displays is a morphism of $W_{\mathcal{O}}(R)$-modules

$$\alpha_P : P \to P',$$

such that the image of $\alpha_Q := \alpha_P|_Q$ is contained in Q' and that the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\alpha_P} & P' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ P & \xrightarrow[\alpha_P]{} & P' \end{array}$$

and

$$\begin{array}{ccc} Q & \xrightarrow{\alpha_Q} & Q' \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_1'} \\ P & \xrightarrow[\alpha_P]{} & P' \end{array}$$

commute.

Together with these morphisms, the f-$\mathcal{O}$-displays over R form a category, we call it $(f - \mathrm{disp}_{\mathcal{O}}/R)$ or only $(\mathrm{disp}_{\mathcal{O}}/R)$, when $f = 1$. For π nilpotent in R, the nilpotent f-$\mathcal{O}$-displays over R form a full subcategory, we denote it by $(f - \mathrm{ndisp}_{\mathcal{O}}/R)$ or $(\mathrm{ndisp}_{\mathcal{O}}/R)$, respectively.

Another very similar (and in fact categorial equivalent) structure compared to f-$\mathcal{O}$-displays are the f-$\mathcal{O}$-Dieudonné modules over an $\mathcal{O}$-algebra R with $\pi R = 0$, which is a perfect field.

Definition 2.2.2. Let $\mathcal{O}$ be an RRS and R as above. Then an f-$\mathcal{O}$-Dieudonné module is a finitely generated free $W_{\mathcal{O}}(R)$-module M together with two maps, an ${}^{F^f}$-linear map $F : M \to M$ and an ${}^{F^{-f}}$-linear map $V : M \to M$, such that $FV = VF = \pi$. A morphism between two such modules M and M' is as usual, i.e., a $W_{\mathcal{O}}(R)$-linear map $M \to M'$ such that the two corresponding pairs F, F' and V, V' respect this mapping.

The following proposition may be considered as an extension of Proposition 15 in [Zin02].

Proposition 2.2.3. With $\mathcal{O}$ and R as above, the category of f-$\mathcal{O}$-displays over R is equivalent to the category of f-$\mathcal{O}$-Dieudonné modules over R. Nilpotent

f-$\mathcal{O}$-displays correspond to f-$\mathcal{O}$-Dieudonné modules, where V is topologically nilpotent in the π-adic topology, i.e., for all $r \in \mathbb{N}$, there is an $n \in \mathbb{N}$, such that $V^n M \subseteq \pi^r M$.

Since this fact is not very hard to prove, we only give a sketch of proof which is based on what has been done in [Zin02]. For a given ramified f-$\mathcal{O}$-Dieudonné module (M, F, V), we get an f-$\mathcal{O}$-display (P, Q, F, F_1) by defining $P = M$, $Q = VM$, $F : M \to M$ and $F_1 = V^{-1} : VM \to M$.
If we start with an f-$\mathcal{O}$-display (P, Q, F, F_1) we get an f-$\mathcal{O}$-Diedonné module (P, F, V) by setting as the composite $V = \iota V^\sharp$, where $V^\sharp : P \to W_{\mathcal{O}}(k) \otimes_{F^f, W_{\mathcal{O}}(k)} P$ is as usual and $\iota : W_{\mathcal{O}}(k) \otimes_{F^f, W_{\mathcal{O}}(k)} P \to P$ is given by $w \otimes x \mapsto {}^{F^{-f}} wx$. Implicitly, we have used Lemma 1.2.6 here, since it justifies to use ${}^{F^{-f}}$ here and in the above Definition. The equivalence of the nilpotent structures is left to the reader.

To introduce the notion of a base change, we need an $\mathcal{O}$-algebra morphism $R \to S$.

Definition 2.2.4. (cf. [Zin02, Definition 20]) We define the *f-$\mathcal{O}$-display obtained by base change* $\mathcal{P}_S = (P_S, Q_S, F_S, F_{1,S})$ to consist of

- $P_S := W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P$,
- $Q_S := \ker(\mathrm{w}_0 \otimes \mathrm{pr})$, where
$$\mathrm{w}_0 \otimes \mathrm{pr} : W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P \to S \otimes_R P/Q,$$
- $F_S :={}^{F^f} \otimes F$ and
- $F_{1,S} : Q_S \to P_S$ to be the unique ${}^{F^f}$-linear morphism which satisfies
$$\begin{aligned} F_{1,S}(w \otimes y) &= {}^{F^f} w \otimes F_1 y, \\ F_{1,S}({}^V w \otimes x) &= {}^{F^{f-1}} w \otimes Fx \end{aligned}$$
for all $w \in W_{\mathcal{O}}(S)$, $x \in P$ and $y \in Q$.

Because the uniqueness of $F_{1,S}$ is clear, we choose a normal decomposition $P = L \oplus T$ and get an isomorphism

$$Q_S \simeq W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} L \oplus I_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(R)} T,$$

with which the existence is easily verified. Hence, the definition is sensible.

We should remark a very important case of base change, which will be needed for obtaining that $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module over R for a nilpotent f-$\mathcal{O}$-display over R, where R is an $\mathcal{O}$-algebra with π nilpotent in R (cf. [Zin02, Example 23]). Let R be an $\mathcal{O}$-algebra, such that $\pi R = 0$. Let $Frob_q$

denote the Frobenius endomorphism defined by $Frob_q(r) = r^q$ for all $r \in R$ and $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R. The Frobenius ^{F} on $W_{\mathcal{O}}(R)$ is given by $W_{\mathcal{O}}(Frob_q)$. Hence, if we set

$$\begin{aligned} P^{(q)} &= W_{\mathcal{O}}(R) \otimes_{F, W_{\mathcal{O}}(R)} P, \\ Q^{(q)} &= I_{\mathcal{O},R} \otimes_{F, W_{\mathcal{O}}(R)} P + \operatorname{Im}(W_{\mathcal{O}}(R) \otimes_{F, W_{\mathcal{O}}(R)} Q) \end{aligned}$$

and define the operators $F^{(q)}$ and $F_1^{(q)}$ in a unique way by

$$\begin{aligned} F^{(q)}(w \otimes x) &= {}^{F^f} w \otimes Fx, \\ F_1^{(q)}({}^V w \otimes x) &= {}^{F^{f-1}} w \otimes Fx, \\ F_1^{(q)}(w \otimes y) &= {}^{F^f} w \otimes F_1 y \end{aligned}$$

for all $w \in W_{\mathcal{O}}(R)$, $x \in P$ and $y \in Q$, we obtain that the f-$\mathcal{O}$-display obtained by base change with respect to $Frob_q$ is $\mathcal{P}^{(q)} = (P^{(q)}, Q^{(q)}, F^{(q)}, F_1^{(q)})$. It is essential to demand $\pi R = 0$ here, otherwise $Q^{(q)}/I_{\mathcal{O},R}P^{(q)}$ would not necessarily be a direct summand of $P^{(q)}/I_{\mathcal{O},R}P^{(q)}$. Let us denote the k-fold iterate of this construction by $P^{(q^k)}$ and consider the map $V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P$ of Lemma 2.1.4 and $F^\sharp : W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P \to P$. $V^\sharp$ maps P into $Q^{(q^f)}$ and $F^\sharp$ maps $Q^{(q^f)}$ into $I_{\mathcal{O},R}P$. Both commute with the pairs $(F, F^{(q^f)})$ and $(F_1, F_1^{(q^f)})$ respectively, so $V^\sharp$ induces the so called Frobenius morphism of $\mathcal{P}$, which is a morphism of f-$\mathcal{O}$-displays

$$\mathrm{Fr}_{\mathcal{P}} : \mathcal{P} \ \to \ \mathcal{P}^{(q^f)}, \tag{2.1}$$

and $F^\sharp$ induces a map of f-$\mathcal{O}$-displays

$$\mathrm{Ver}_{\mathcal{P}} : \mathcal{P}^{(q^f)} \ \to \ \mathcal{P},$$

which is called the Verschiebung. By using Lemma 2.1.4 we obtain two analogous relations to the ones cited in this Lemma

$$\mathrm{Fr}_{\mathcal{P}} \mathrm{Ver}_{\mathcal{P}} = \pi \cdot \mathrm{id}_{\mathcal{P}^{(q^f)}} \ \text{ and } \ \mathrm{Ver}_{\mathcal{P}} \mathrm{Fr}_{\mathcal{P}} = \pi \cdot \mathrm{id}_{\mathcal{P}}.$$

In order to overcome the nilpotence requirement of π in Definition 2.1.5, we can extend it now to the following case:

Let R be a topological $\mathcal{O}$-algebra, where the linear topology is given by the ideals $R = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \ldots \supset \mathfrak{a}_n \ldots$, such that $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$ holds. Furthermore, we demand the nilpotence of π in $R/\mathfrak{a}_1$ (and hence in all $R/\mathfrak{a}_i$) and that R is complete and separated with respect to this filtration.

Definition 2.2.5. With R as above an f-$\mathcal{O}$-display over R is called nilpotent, if the f-$\mathcal{O}$-display obtained by base change to $R/\mathfrak{a}_1$ is nilpotent in the sense of Definition 2.1.5.

Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. We denote by $\mathcal{P}_i$ the f-$\mathcal{O}$-display over $R/\mathfrak{a}_i$ obtained by base change. Then $\mathcal{P}_i$ is a nilpotent f-$\mathcal{O}$-display in the sense of Definition 2.1.5. There are obvious transition isomorphisms (see Definition 2.2.1)

$$\phi_i : (\mathcal{P}_{i+1})_{R/\mathfrak{a}_i} \to \mathcal{P}_i.$$

Conversely, assume we are given for each index i a nilpotent f-$\mathcal{O}$-display $\mathcal{P}_i$ over the discrete $\mathcal{O}$-algebra $R/\mathfrak{a}_i$ and transition isomorphisms ϕ_i as above. It is easily seen that the system $(\mathcal{P}_i, \phi_i)$ is obtained from a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R. Hence, we obtain, after considering the morphisms of the category of systems of nilpotent f-$\mathcal{O}$-displays $(\mathcal{P}_i, \phi_i)$ and the morphisms of the category of nilpotent f-$\mathcal{O}$-displays over R, that both categories are equivalent by the above association. This equivalence fits well to [Mes72, Chapter II, Lemma (4.16)].

We need to introduce descent theory for f-$\mathcal{O}$-displays, i.e., we need to find out, given a faithfully flat $\mathcal{O}$-algebra morphism $R \to S$, what structure do we have to require addtitionally to an f-$\mathcal{O}$-display over S to lift it uniquely to an f-$\mathcal{O}$-display over R.

Lemma 2.2.6. (cf. [Zin02, 1.3. Descent]) Let $R \to S$ be a faithfully flat $\mathcal{O}$-algebra morphism. Then we get the exact sequence

$$R \to S \underset{q_2}{\overset{q_1}{\rightrightarrows}} S \otimes_R S \overset{q_{12}}{\underset{q_{13}}{\overset{\xrightarrow{q_{23}}}{\rightrightarrows}}} S \otimes_R S \otimes_R S, \tag{2.2}$$

where q_i is the map, which sends an element of S to the i-th factor of $S \otimes_R S$ and q_{ij} is given by sending the first component of $S \otimes_R S$ to the i-th component of $S \otimes_R S \otimes_R S$ and the second one to the j-th component of it.

Definition 2.2.7. With $\mathcal{O}$, $R \to S$, q_i and q_{ij} as above and π nilpotent in R we denote for an f-$\mathcal{O}$-display over S, say $\mathcal{P}$, the f-$\mathcal{O}$-display over $S \otimes_R S$ obtained by base change via q_i by $q_i^\star \mathcal{P}$ and similarly for f-$\mathcal{O}$-displays over $S \otimes_R S \otimes_R S$ and q_{ij}. A descend datum for $\mathcal{P}$ relative to $R \to S$ is an isomorphism of f-$\mathcal{O}$-displays $\alpha : q_1^\star \mathcal{P} \to q_2^\star \mathcal{P}$, such that the cocycle condition holds, i.e., the diagram

$$\begin{array}{ccc} q_{12}^\star q_1^\star \mathcal{P} & \xrightarrow{q_{12}^\star \alpha} & q_{12}^\star q_2^\star \mathcal{P} \\ \| & & \| \\ q_{13}^\star q_1^\star \mathcal{P} & & q_{23}^\star q_1^\star \mathcal{P} \\ \downarrow{\scriptstyle q_{13}^\star \alpha} & & \downarrow{\scriptstyle q_{23}^\star \alpha} \\ q_{13}^\star q_2^\star \mathcal{P} & = & q_{23}^\star q_2^\star \mathcal{P} \end{array}$$

is commutative.

It is obvious that we obtain for any f-$\mathcal{O}$-display $\mathcal{P}$ over R a canonical descent datum $\alpha_{\mathcal{P}}$ for the base change $\mathcal{P}_S$ over S relative to $R \to S$.

Theorem 2.2.8. (cf. [Zin02, Theorem 37]) With the terminology as in Definition 2.2.7, we get that the functor $\mathcal{P} \mapsto (\mathcal{P}, \alpha_{\mathcal{P}})$ from the category of f-$\mathcal{O}$-displays over R to the category of f-$\mathcal{O}$-displays over S equipped with a descent datum relative to $R \to S$ is an equivalence of categories. We also obtain an equivalence, when we restrict to nilpotent f-$\mathcal{O}$-display structures.

Definition 2.2.9. Let $\mathcal{O}$ be an RRS, S an $\mathcal{O}$-algebra, R an S-algebra and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We call an S-action, i.e., an $\mathcal{O}$-algebra morphism $\iota : S \to \operatorname{End}\mathcal{P}$, *strict*, iff the induced action $\bar{\iota} : S \to P/Q$ coincides with the S-module structure given by the R-module structure of P/Q and restriction to scalars.

Now we try to utilize this assertion to prove a result, which will become important in chapter four, when we are dealing with algebraic stacks.

Proposition 2.2.10. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of RRSs, i.e., not necessarily excellent (see Definition 1.1.1), $R \to S$ a faithfully flat morphism of $\mathcal{O}'$-algebras and f, f' two natural numbers ≥ 1. Let G_A be a functor between the category of (nilpotent) f-$\mathcal{O}$-displays over A and the category of (nilpotent) f'-$\mathcal{O}'$-displays over A for $A = R, S, S \otimes_R S, S \otimes_R S \otimes_R S$. Assume that these functors are compatible with the base change functors induced by q_i, q_{ij} (with the obvious notation) and $R \to S$, that $G_{S\otimes_R S}$ is fully faithful and $G_{S\otimes_R S\otimes_R S}$ is faithful. Now let $\mathcal{P}'$ be a (nilpotent) f'-$\mathcal{O}'$-display over R, such that the base change ${\mathcal{P}'}_S$ lies in the image of G_S. Then $\mathcal{P}'$ lies in the image of G_R. The same assertion is true, when the domain of G_A is the category of (nilpotent) f-$\mathcal{O}$-displays over A equipped with a strict $\mathcal{O}'$-action for each A as above.

Proof: Let $\mathcal{P}$ be a (nilpotent) f-$\mathcal{O}$-display over S, such that $G_S(\mathcal{P}) = {\mathcal{P}'}_S$. It is our aim to construct for $\mathcal{P}$ a descent datum relative $R \to S$, so we would obtain by Theorem 2.2.8 a (nilpotent) f-$\mathcal{O}$-display over R, which has the image $\mathcal{P}'$. Since we obtain a descent datum for ${\mathcal{P}'}_S$, we may lift the isomorphism $\alpha' : q_1^\star {\mathcal{P}'}_S \cong q_2^\star {\mathcal{P}'}_S$ to $\alpha : q_1^\star \mathcal{P} \cong q_2^\star \mathcal{P}$, since $G_{S\otimes_R S}$ is fully faithful. Now we may establish the cocycle diagram for α, where we need to show its commutativity, but this follows from the faithfulness of $G_{S\otimes_R S\otimes_R S}$ and the compatibility of the G's with the base change functors. The last assertion follows from the same argumentation as above by attaching a strict $\mathcal{O}'$-action to the objects of the categories of the equivalence of Theorem 2.2.8. □

2.3 The formal $\mathcal{O}$-module $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$

Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra and $\mathcal{N}$ a nilpotent R-algebra. Then by restriction to scalars $\mathcal{N}$ can be considered as a nilpotent $\mathcal{O}$-algebra. We get that $W_{\mathcal{O}}(\mathcal{N})$ is a $W_{\mathcal{O}}(R)$-algebra. As in the previous chapter, we denote by $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ the $W_{\mathcal{O}}(R)$-subalgebra of $W_{\mathcal{O}}(\mathcal{N})$ consisting of the ramified Witt vectors with finitely many non-zero entries. For a given f-$\mathcal{O}$-display $\mathcal{P} = (P,Q,F,F_1)$ we consider the following $W_{\mathcal{O}}(R)$-modules, which can obviously be considered as $\mathcal{O}$-modules by restriction to scalars via $\mathcal{O} \to W_{\mathcal{O}}(R)$:

$$\widehat{P}_{\mathcal{N}} = \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \tag{2.3}$$

$$\widehat{Q}_{\mathcal{N}} = \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} L \oplus \widehat{I}_{\mathcal{O},\mathcal{N}} \otimes_{W_{\mathcal{O}}(R)} T \tag{2.4}$$

Here $P = L \oplus T$ is a normal decomposition. Let S be the unitary R-algebra $R|\mathcal{N}| = R \oplus \mathcal{N}$ with an addition in the obvious way and a multiplication given by

$$(r_1,n_1)(r_2,n_2) = (r_1r_2, r_1n_2 + r_2n_1 + n_1n_2) \tag{2.5}$$

for all $n_i \in \mathcal{N}$ and $r_i \in R$. If we denote by $\mathcal{P}_S = (P_S,Q_S,F_S,F_{1,S})$ the f-$\mathcal{O}$-display over S obtained from $\mathcal{P}$ via base change $R \to S$, we can consider $\widehat{P}_{\mathcal{N}}$ as a submodule of P_S and obtain $\widehat{Q}_{\mathcal{N}} = \widehat{P}_{\mathcal{N}} \cap Q_S$. By restricting $F_S : P_S \to P_S$ and $F_{1,S} : Q_S \to P_S$, we obtain operators

$$F : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}},$$
$$F_1 : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}.$$

Now we are able to associate to an f-$\mathcal{O}$-display $\mathcal{P}$ a finite dimensional formal $\mathcal{O}$-module $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$, for the basic definitions of formal groups / formal $\mathcal{O}$-modules we refer to Appendix A. In the case that $f = 1$, we will just refer to $BT_{\mathcal{O}}(\mathcal{P},-)$. Our aim is to use the $\mathcal{O}$-module structure of the just introduced modules, such that every group $BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N})$ becomes an $\mathcal{O}$-module and for every R-algebra morphism $\mathcal{N} \to \mathcal{N}'$ the induced group morphism $BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N}) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N}')$ becomes an $\mathcal{O}$-module morphism. For this purpose we formulate a theorem, which is a modified version of theorem 81 given by Zink in [Zin02]. The proof is very similar, but because of its importance for the understanding of the $BT_{\mathcal{O}}^{(f)}$ functor we will write it down here fully.

Theorem 2.3.1. Let $\mathcal{P} = (P,Q,F,F_1)$ be an f-$\mathcal{O}$-display over R. Then the functor from Nil_R (the category of nilpotent R-algebras) to the category of $\mathcal{O}$-modules, which associates to any $\mathcal{N} \in \mathrm{Nil}_R$ the cokernel of the morphism of abelian groups

$$F_1 - \mathrm{id} : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}} \tag{2.6}$$

where id is the natural inclusion, is a finite dimensional formal $\mathcal{O}$-module, when considered as a functor to abelian groups equipped with a natural $\mathcal{O}$-action. This functor is called $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$. We obtain an exact sequence of $\mathcal{O}$-modules

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1-\mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N}) \longrightarrow 0.$$

In the proof, we have to make use of something similar to divided powers as has been done in [Zin02, 1.4. Rigidity.].

Definition 2.3.2. (cf. [Fal02, Definition 14]) Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra and $\mathfrak{a} \subseteq R$ an ideal. An *$\mathcal{O}$-pd-structure* is a map $\gamma : \mathfrak{a} \to \mathfrak{a}$, such that

- $\pi \cdot \gamma(x) = x^q$,
- $\gamma(r \cdot x) = r^q \cdot \gamma(x)$ and
- $\gamma(x+y) = \gamma(x) + \gamma(y) + \sum_{0<i<q}(\binom{q}{i}/\pi) \cdot x^i \cdot y^{q-i}$

hold for all $r \in R$ and $x, y \in \mathfrak{a}$. If γ^n denotes the n-fold iterate of γ, we call γ *nilpotent*, if $\mathfrak{a}^{[n]} = 0$ for all $n \gg 0$, where $\mathfrak{a}^{[n]} \subseteq \mathfrak{a}$ is generated by all products $\prod \gamma^{a_i}(x_i)$ with $x_i \in \mathfrak{a}$ and $\sum q^{a_i} \geq n$.

If we define for each n a map

$$\alpha_n = \pi^{q^{n-1}+q^{n-2}+\ldots+q+1-n} \cdot \gamma^n : \mathfrak{a} \to \mathfrak{a}, \tag{2.7}$$

we can define

$$\begin{array}{rcl} \mathrm{w}'_n : W_{\mathcal{O}}(\mathfrak{a}) & \to & \mathfrak{a} \\ (x_0, x_1, \ldots, x_n, \ldots) & \to & \alpha_n(x_0) + \alpha_{n-1}(x_1) + \ldots + \alpha_1(x_{n-1}) + x_n, \end{array} \tag{2.8}$$

which should not be confused with the n-th Witt polynomial of $W_{\mathcal{O}'}(S)$ for some $\mathcal{O}'$ and some $\mathcal{O}'$-algebra S. The map w'_n is w_n-linear, this means that beside additivity $\mathrm{w}'_n(rx) = \mathrm{w}_n(r)\mathrm{w}'_n(x)$ holds for all $n \in \mathbb{N}$, $x \in W_{\mathcal{O}}(\mathfrak{a})$ and $r \in W_{\mathcal{O}}(R)$. The main application of this structure is the following: We define on $\mathfrak{a}^{\mathbb{N}}$ a $W_{\mathcal{O}}(R)$-module structure by setting

$$\xi[a_0, a_1, \ldots] = [\mathrm{w}_0(\xi)a_0, \mathrm{w}_1(\xi)a_1, \ldots]$$

with $\xi \in W_{\mathcal{O}}(S)$ and $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$. It is not too hard to check that we then get an isomorphism of $W_{\mathcal{O}}(S)$-modules

$$\begin{array}{rcl} \log : W_{\mathcal{O}}(\mathfrak{a}) & \to & \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, \ldots) & \mapsto & [\mathrm{w}'_0(\underline{a}), \mathrm{w}'_1(\underline{a}), \ldots]. \end{array} \tag{2.9}$$

We should also remark how ^{F},^{V} and multiplication are described on the right hand side (by passing to a suitable universal situation):

$$[a_0, a_1, \ldots][b_0, b_1, \ldots] = [a_0 b_0, \pi a_1 b_1, \ldots, \pi^i a_i b_i, \ldots], \tag{2.10}$$

$${}^F[a_0, a_1, \ldots] = [\pi a_1, \pi a_2, \ldots, \pi a_i, \ldots], \tag{2.11}$$

$${}^V[a_0, a_1, \ldots] = [0, a_0, a_1, \ldots, a_i, \ldots] \tag{2.12}$$

hold for all $[a_0, a_1, \ldots], [b_0, b_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$. We define the ideal $\mathfrak{a} \subset W_{\mathcal{O}}(\mathfrak{a})$ by

$$\log^{-1}(\,[a, 0, 0, \ldots]\,| \quad \text{for all } a \in \mathfrak{a}). \tag{2.13}$$

It should be remarked that ${}^F\mathfrak{a} = 0$ holds. We will use this ideal in the following simple Lemma:

Lemma 2.3.3. (cf. [Zin02, Lemma 38]) Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R and $\mathfrak{a} \subseteq R$ an ideal equipped with an $\mathcal{O}$-pd-structure. Then there is a unique extension of F_1 to

$$F_1 : W_{\mathcal{O}}(\mathfrak{a})P + Q \to P,$$

such that $F_1\mathfrak{a}P = 0$ holds.

Proof: If we choose a normal decomposition $P = L \oplus T$, then

$$W_{\mathcal{O}}(\mathfrak{a})P + Q = \mathfrak{a}T \oplus L \oplus I_{\mathcal{O},R}T.$$

We define $F_1 : W_{\mathcal{O}}(\mathfrak{a})P + Q \to P$ with the help of this decomposition. We need to verify that $F_1\mathfrak{a}L = 0$ holds, which follows, since ${}^F\mathfrak{a} = 0$. □

Furthermore, if $\alpha_n(\mathfrak{a}) = 0$ for all $n \gg 0$, we get a map

$$\log : \widehat{W_{\mathcal{O}}}(\mathfrak{a}) \to \mathfrak{a}^{(\mathbb{N})}, \tag{2.14}$$

which becomes an isomorphism if γ is nilpotent. Now we turn to the proof of Theorem 2.3.1.

Proof: If $\mathcal{N}^2 = 0$, then $\mathcal{N}$ has a trivial $\mathcal{O}$-pd structure $\gamma = 0$, and we can consider all the results of 1.4. of [Zin02] without fearing our new situation here. Of course, all the argumentations dealing with the $\mathcal{N}^2 = 0$ case and $\gamma = 0$ can be extended to arbitrary nilpotent R-algebras with a nilpotent $\mathcal{O}$-pd structure. We can extend $F_1 : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$ to a map

$$F_1 : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \tag{2.15}$$

by applying Lemma 2.3.3 to $F_1 : Q_S \to P_S$ first (with $S = R \oplus \mathcal{N}$) and then restricting to $\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P$. So now we first show that (2.6) is injective. The functors $\mathcal{N} \mapsto \widehat{P}_{\mathcal{N}}$ and $\mathcal{N} \mapsto \widehat{Q}_{\mathcal{N}}$ from Nil_R to $\mathrm{Mod}_{\mathcal{O}}$ are exact in the sense that if we apply any of these two functors to any short exact sequence in Nil_R we obtain a short exact sequence in $\mathrm{Mod}_{\mathcal{O}}$, where we establish the fact for $\widehat{Q}_{\mathcal{N}}$ by considering its decomposition (2.4). Any nilpotent $\mathcal{N}$ admits a filtration

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \ldots \subset \mathcal{N}_r = \mathcal{N}$$

with $\mathcal{N}_i^2 \subset \mathcal{N}_{i-1}$, so we are allowed to reduce our observations to $\mathcal{N}^2 = 0$ and we then may equip $\mathcal{N}$ with the trivial $\mathcal{O}$-pd structure again. The map (2.6) can be seen as the restriction of

$$F_1 - \mathrm{id} : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \quad \to \quad \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P, \tag{2.16}$$

where F_1 is the map (2.15). We obtain the injectivity of (2.6), when we prove that (2.16) is an isomorphism by showing that this F_1 is nilpotent, what we will do now.
Because the divided powers are nilpotent, we get an isomorphism

$$\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \bigoplus_{i\geq 0} \mathcal{N} \otimes_{\mathrm{w}_i, W_{\mathcal{O}}(R)} P.$$

We want to describe, what happens if we let F_1 act on the right hand side induced by this isomorphism. We define the operators K_i for all $i \geq 0$ by

$$\begin{array}{rcl} K_i : \mathcal{N} \otimes_{\mathrm{w}_{f+i}, W_{\mathcal{O}}(R)} P & \longrightarrow & \mathcal{N} \otimes_{\mathrm{w}_i, W_{\mathcal{O}}(R)} P \\ a \otimes x & \longmapsto & \pi^{f-1} a \otimes Fx. \end{array}$$

Then it is easily checked that F_1 is given on the right side by

$$F_1[u_0, u_1, \ldots] = [K_1 u_f, K_2 u_{f+1}, \ldots]$$

and the nilpotence follows.
Hence, we may define $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N})$ by the exact sequence

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N}) \longrightarrow 0.$$

There is an obvious $\mathcal{O}$-module structure on $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N})$. For an R-algebra morphism $\eta : \mathcal{N} \to \mathcal{M}$ with $\mathcal{N}, \mathcal{M} \in \mathrm{Nil}_R$ we receive an $\mathcal{O}$-module morphism $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \eta) : BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N}) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{M})$ by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{Q}_{\mathcal{N}} & \xrightarrow{F_1 - \mathrm{id}} & \widehat{P}_{\mathcal{N}} & \longrightarrow & BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta''} & & \downarrow{\scriptstyle BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \eta)} & & \\ 0 & \longrightarrow & \widehat{Q}_{\mathcal{M}} & \xrightarrow{F_1 - \mathrm{id}} & \widehat{P}_{\mathcal{M}} & \longrightarrow & BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{M}) & \longrightarrow & 0, \end{array}$$

where η'' is the induced morphism $\overline{\eta} \otimes \mathrm{id} : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{\mathcal{M}}$ and η' is the restriction of η'' to $\widehat{Q}_{\mathcal{N}}$. It is easily seen that the image of η' is contained in $\widehat{Q}_{\mathcal{M}}$ by using $\overline{\eta}(\widehat{I_{\mathcal{O},\mathcal{N}}}) \subset \widehat{I_{\mathcal{O},\mathcal{M}}}$ and the decomposition for $\widehat{Q}_{\mathcal{N}}$ and $\widehat{Q}_{\mathcal{M}}$. We need to verify that the conditions of Definition A.0.1 hold. The first two points are clear because we already remarked that the functors $\mathcal{N} \mapsto \widehat{P}_{\mathcal{N}}$ and $\mathcal{N} \mapsto \widehat{Q}_{\mathcal{N}}$ are exact. For the remaining points we need to look at $t_{BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)}$. Because we only consider Mod_R in Nil_R, we have for any $\mathcal{N} \in \mathrm{Mod}_R$ that $\mathcal{N}^2 = 0$. Hence, we can equip $\mathcal{N}$ with the trivial $\mathcal{O}$-pd structure again. We define an isomorphism

$$\exp_{\mathcal{P}} : \mathcal{N} \otimes_R P/Q \longrightarrow BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N})$$

by the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{Q}_{\mathcal{N}} & \xrightarrow{\mathrm{id}} & \widehat{P}_{\mathcal{N}} & \longrightarrow & \mathcal{N} \otimes_R P/Q & \longrightarrow & 0 \\
 & & \Vert & & \downarrow {\scriptstyle F_1-\mathrm{id}} & & \downarrow {\scriptstyle \exp_{\mathcal{P}}} & & \\
0 & \longrightarrow & \widehat{Q}_{\mathcal{N}} & \xrightarrow{F_1-\mathrm{id}} & \widehat{P}_{\mathcal{N}} & \longrightarrow & BT_{\mathcal{O}}^{(f)}(\mathcal{P},\mathcal{N}) & \longrightarrow & 0.
\end{array}$$

One can easily deduce from this diagram that $\exp_{\mathcal{P}}$ is an isomorphism. We see furthermore that $t_{BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)}$ is isomorphic to $M \mapsto M \otimes_R P/Q$ via this exponential mapping. Hence, it suffices to consider the latter functor and we obtain easily that the last two points of Definition A.0.1 are satisfied. We conclude that $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ is a formal group (with $\mathcal{O}$-action), but since it is easily seen that the two $\mathcal{O}$-actions on the tangential space coincide, it is also a formal $\mathcal{O}$-module. □

Let $\alpha : R \to S$ be an $\mathcal{O}$-algebra morphism and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We get an f-$\mathcal{O}$-display $\alpha_\star\mathcal{P}$ over S by base change and obtain a formal $\mathcal{O}$-module $BT_{\mathcal{O}}^{(f)}(\alpha_\star\mathcal{P},-)$ over S. On the other hand, we obtain a formal $\mathcal{O}$-module $\alpha_\star BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ over S by considering Nil_S as a subcategory of Nil_R and restricting $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ to it. The following Corollary says that the functor $\mathcal{P} \to BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ from the category of f-$\mathcal{O}$-displays to the category of formal $\mathcal{O}$-modules commutes with base change.

Corollary 2.3.4. (cf. [Zin02, Corollary 86]) With the conditions as above we get an isomorphism of formal $\mathcal{O}$-modules over S

$$\alpha_\star BT_{\mathcal{O}}^{(f)}(\mathcal{P},-) \cong BT_{\mathcal{O}}^{(f)}(\alpha_\star\mathcal{P},-).$$

We have for all $\mathcal{N} \in \mathrm{Nil}_S$ the obvious isomorphism

$$\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \cong \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(S)} W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P = \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(S)} \alpha_\star P,$$

which induces the isomorphism of the Corollary. We want to cite two Propositions of [Zin02], from which we deduce that $BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ is a π-divisible formal $\mathcal{O}$-module for all nilpotent f-$\mathcal{O}$-displays $\mathcal{P}$. The proofs are omitted here, because apart from some obvious changes we would only copy them.

Proposition 2.3.5. (cf. [Zin02, Proposition 87]) Let $\mathcal{O} = (\mathcal{O}, \pi, q = p^m)$ be an RRS, R an $\mathcal{O}$-algebra, such that $\pi R = 0$, and $\mathcal{P}$ a nilpotent f-$\mathcal{O}$-display over R. Furthermore, let $\mathrm{Fr}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^{(q^f)}$ be the Frobenius endomorphism (see (2.1)) and $G = BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)$ resp. $G^{(q^f)} = BT_{\mathcal{O}}^{(f)}(\mathcal{P}^{(q^f)},-)$ be the formal $\mathcal{O}$-module associated to $\mathcal{P}$ resp. $\mathcal{P}^{(q^f)}$. We obtain, because $BT_{\mathcal{O}}^{(f)}$ commutes with base change by Corollary 2.3.4, a morphism of formal $\mathcal{O}$-modules

$$BT_{\mathcal{O}}^{(f)}(\mathrm{Fr}_{\mathcal{P}}) : G \to G^{(q^f)},$$

which is the Frobenius morphism of the formal $\mathcal{O}$-module G (with respect to $x \mapsto x^q$) iterated f times Fr_G^f. (This Frobenius is the obvious generalization of [Zin84, Kapitel V]).

Proposition 2.3.6. (cf. [Zin02, Proposition 88]) With the setting as in Proposition 2.3.5, we obtain that there is a number N and a morphism of nilpotent f-$\mathcal{O}$-displays

$$\gamma : \mathcal{P} \to \mathcal{P}^{(q^{fN})},$$

such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\pi} & \mathcal{P} \\ {\scriptstyle \mathrm{Fr}_{\mathcal{P}}^N}\downarrow & \swarrow {\scriptstyle \gamma} & \\ \mathcal{P}^{(q^{fN})} & & \end{array}$$

is commutative.

Corollary 2.3.7. (cf. [Zin02, Proposition 89]) Let $\mathcal{O}$ be an RRS, R an $\mathcal{O}$-algebra with π nilpotent in R and $\mathcal{P}$ a nilpotent f-$\mathcal{O}$-display over R. Then $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module (cf. Definition A.1.2).

Proof: First we consider the case, when $\pi R = 0$. Then we may apply $BT_{\mathcal{O}}^{(f)}$ to the diagram of Proposition 2.3.6 and we obtain that some iteration of the Frobenius on $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ factors through π and some other morphism. By [Zin84, 5.18 Lemma] and [Zin84, 5.10 Satz], we obtain that π is an isogeny. Hence, $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module over R.
If π is nilpotent in R, then a formal $\mathcal{O}$-module is π-divisible, iff its reduction modulo π is π-divisible (cf. [Zin84, 5.12 Korollar]). Hence, we know that $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is π-divisible. □

2.4 Drinfeld's equivalence of formal $\mathcal{O}$-modules and reduced Cartier modules revisited

Let $\mathcal{O}$ be an RRS and R an $\mathcal{O}$-algebra. First we will introduce the Cartier ring $\mathbb{E}_{\mathcal{O},R}$ as it has been done in [Dri76]. In this article the not necessarily commutative $\mathcal{O}$-algebra $A_{\mathcal{O},R}$ is considered, which is generated by $W_{\mathcal{O}}(R)$ and the elements F and V in which the following relations are demanded to hold for all $a \in W_{\mathcal{O}}(R)$

$$VaF = {}^{V}a, \tag{2.17}$$
$$Fa = {}^{F}aF, \tag{2.18}$$
$$aV = V^{F}a, \tag{2.19}$$
$$FV = \pi. \tag{2.20}$$

The right ideals, spanned by V^l, i.e.,

$$V^l A_{\mathcal{O},R} = \{ \sum_{n,m\geq 0} V^n[a_{n,m}]F^m \in A_{\mathcal{O},R} \,|\, a_{n,m} \in R, a_{k,m} = 0 \text{ if } k < l \},$$

where we have the elements in the usual presentation, give us a topology and we define

$$\mathbb{E}_{\mathcal{O},R} = \varprojlim A_{\mathcal{O},R}/V^l A_{\mathcal{O},R}.$$

We may embed $W_{\mathcal{O}}(R)$ in $\mathbb{E}_{\mathcal{O},R}$ by

$$x \;=\; \sum_{i=0}^{\infty} V^i[x_i]F^i \in \mathbb{E}_{\mathcal{O},R} \tag{2.21}$$

for each $x \in W_{\mathcal{O}}(R)$. Furthermore, each element $e \in \mathbb{E}_{\mathcal{O},R}$ may be written in a unique way as

$$e = \sum_{n,m\geq 0} V^n[a_{n,m}]F^m, \tag{2.22}$$

where $a_{n,m} \in R$ and for fixed n the coefficients $a_{n,m}$ are zero for m large enough. We now come to the definition of a special kind of Cartier module (this means an $\mathbb{E}_{\mathcal{O},R}$-module here), for which we will show that the category of all those modules is equivalent to the category of formal $\mathcal{O}$-modules over R (see Definition A.0.2), when $\mathcal{O}$ is the ring of integers of non-Archimedean local field of characteristic zero:

Definition 2.4.1. (cf. [Dri76]) With R and $\mathcal{O}$ as above, a Cartier module over R and $\mathcal{O}$, i.e., an $\mathbb{E}_{\mathcal{O},R}$-module, say M, is called *reduced*, if the action of V is injective, $M = \varprojlim M/V^kM$ and M/VM is a finite projective R-module. M/VM is called the *tangential space* of M.

Definition 2.4.2. Let $\mathcal{O}$ be an RRS, $\mathcal{O} \to S$ a ring morphism, R an S-algebra and M a (reduced) $\mathbb{E}_{\mathcal{O},R}$-module. Then we call an S-action, i.e., an $\mathcal{O}$-algebra morphism $\iota : S \to \operatorname{End} M$, *strict*, iff the induced action $\bar{\iota} : S \to M/VM$ coincides with the S-module structure given by the R-module structure of M/VM and restriction to scalars.

Unless otherwise stated, we will assume for the rest of this section that $\mathcal{O}$ and $\mathcal{O}'$ are rings of integers of non-Archimedean local fields of characteristic zero. This is important, when, for example, one wants to show that the element y defined by (2.24) is a unit in $W_{\mathcal{O}}(R)$, which would not be the case for each RRS, e.g., $(\mathbb{Z}, p, p)$. Now let R be an $\mathcal{O}$-algebra and M a reduced $\mathbb{E}_{\mathcal{O},R}$-module.
If we assume that the R-module M/VM is free, we may choose as in [BC91, (1.5)] a V-basis of M, say $m_1, \ldots, m_d \in M$. This means that the reductions of the m_i

modulo V form an R-module basis of M/VM. We may write each element of M in a unique way as

$$\sum_{n\geq 0}\sum_{i=1}^{d} V^n[c_{n,i}]m_i$$

with $c_{n,i} \in R$. Furthermore, we get that F is uniquely described by

$$F(m_i) = \sum_{n\geq 0}\sum_{j=1}^{d} V^n[c_{n,j,i}]m_j$$

for each $i = 1, \ldots, d$ with $c_{n,j,i} \in R$. Conversely, if we are given $c_{n,j,i} \in W_{\mathcal{O}}(R)$ for each $n \geq 0$ and $1 \leq i, j \leq d$, there exists up to unique isomorphism a reduced $\mathbb{E}_{\mathcal{O},R}$-module M with M/VM free over R and a V-basis $m_1, \ldots, m_d \in M$, such that

$$F(m_i) = \sum_{n\geq 0}\sum_{j=1}^{d} V^n c_{n,j,i}m_j$$

holds for each $i = 1, \ldots, d$.
The same argumentation holds for the action of π. This means, given a reduced Cartier module M and a V-basis $m_1, \ldots, m_d \in M$, there are uniquely determined elements $d_{m,j,i} \in R$ for $m \geq 1$ and $1 \leq i, j \leq d$, such that

$$\pi m_i = [\pi]m_i + \sum_{m\geq 1}\sum_{j=1}^{d} V^m[d_{m,j,i}]m_j$$

holds for all $i = 1, \ldots, d$. Conversely, if we are given $d_{m,j,i} \in W_{\mathcal{O}}(R)$ for each $m \geq 1$ and $1 \leq i, j \leq d$, we obtain up to unique isomorphism a reduced $\mathbb{E}_{\mathcal{O},R}$-module M with M/VM free over R and a V-basis $m_1, \ldots, m_d \in M$, such that

$$\pi m_i = [\pi]m_i + \sum_{m\geq 1}\sum_{j=1}^{d} V^m d_{m,j,i}m_j$$

holds for each $i = 1, \ldots, d$. We obtain F by the fact that

$$\pi - [\pi] = VyF \tag{2.23}$$

holds with $y \in W_{\mathcal{O}}(R)$, which can be considered as the image of an element $y \in W_{\mathcal{O}}(\mathcal{O})$ (with a little misuse in the notation) given by

$$\mathrm{w}_n(y) = 1 - \pi^{q^{n+1}-1} \tag{2.24}$$

via the obvious morphism $W_{\mathcal{O}}(\mathcal{O}) \to W_{\mathcal{O}}(R)$. Hence, since $y_0 = \mathrm{w}_0(y)$ is a unit in $\mathcal{O}$, y is a unit in $W_{\mathcal{O}}(\mathcal{O})$ and so a unit in $W_{\mathcal{O}}(R)$ and we obtain

$$\begin{aligned} Fm_i &= y^{-1}\sum_{m\geq 1}\sum_{j=1}^{d} V^{m-1}d_{m,j,i}m_j \\ &= \sum_{m\geq 1}\sum_{j=1}^{d} V^{m-1}({}^{F^{m-1}}(y^{-1})d_{m,j,i})m_j, \end{aligned}$$

so by the aforementioned, the exists a unique reduced Cartier module, satisfying the structural equations for F and hence the equations for π as well.

Theorem 2.4.3. (cf. [Dri76]) Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero and R an $\mathcal{O}$-algebra. Then the category of formal $\mathcal{O}$-modules over R is equivalent to the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules.

The proof, which now follows, is essentially the same, as the one that can be found in the article of Drinfeld, but we have taken an extended form of reduced Cartier modules, since we consider an extended form of formal $\mathcal{O}$-modules, at least compared to the Definition of formal $\mathcal{O}$-modules in the article of Drinfeld, which rely on the Definition of formal groups in the sense of [Laz75], where the tangential space is finite and free. The second part of the proof is due to Zink. Here the proof of the equivalence is a bit different from Drinfeld's. Nevertheless, the functor is the same. We will give the full proof here, since in Drinfeld's article the proof was very short and needs explanation at many points.
Proof: When $\mathcal{O} = \mathbb{Z}_p$, the theorem is established in [Zin84, 4.23 Satz]. Hence, it suffices to prove that if $\mathcal{O} \to \mathcal{O}'$ is an extension and the theorem is true for $\mathcal{O}$, then it is true for $\mathcal{O}'$ as well.
First we assume $\mathcal{O}'$ to be nonramified over $\mathcal{O}$ of degree f and R to be an $\mathcal{O}'$-algebra. Formal $\mathcal{O}'$-modules over R are equivalent to reduced $\mathbb{E}_{\mathcal{O},R}$-modules equipped with a strict $\mathcal{O}'$-action. The morphism $\mathcal{O}' \to R$ induces a morphism $\mathcal{O}' \to W_{\mathcal{O}}(R)$ which is obtained by the composition of $\kappa : \mathcal{O}' \to W_{\mathcal{O}}(\mathcal{O}')$ (see (1.2)) and $W_{\mathcal{O}}(\mathcal{O}') \to W_{\mathcal{O}}(R)$. From this we obtain that $\mathcal{O}' \otimes_{\mathcal{O}} \mathbb{E}_{\mathcal{O},R}$ is isomorphic to a product of f copies of $\mathbb{E}_{\mathcal{O},R}$. This can be seen in the following way: Let σ denote the relative Frobenius of the extension $\mathcal{O} \to \mathcal{O}'$. Then $\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}'$ is isomorphic to $\mathcal{O}'^f$ via the map $x \otimes y \mapsto (xy, x\sigma(y), \ldots, x\sigma^{f-1}(y))$. So we get

$$\begin{aligned}\mathcal{O}' \otimes_{\mathcal{O}} \mathbb{E}_{\mathcal{O},R} = \mathcal{O}' \otimes_{\mathcal{O}} (\mathcal{O}' \otimes_{\mathcal{O}'} \mathbb{E}_{\mathcal{O},R}) &= (\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}') \otimes_{\mathcal{O}'} \mathbb{E}_{\mathcal{O},R} = \mathcal{O}'^f \otimes_{\mathcal{O}'} \mathbb{E}_{\mathcal{O},R} \\ &= (\mathcal{O}' \otimes_{\mathcal{O}'} \mathbb{E}_{\mathcal{O},R})^f = \mathbb{E}^f_{\mathcal{O},R}.\end{aligned}$$

We obtain a $\mathbb{Z}/f\mathbb{Z}$-grading on M via $M = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} M_i$, where

$$M_i = \{\, m \in M \mid \iota(a)m = ^{F^{-i}} am \text{ for all } a \in \mathcal{O}' \,\}.$$

Here $\iota : \mathcal{O}' \to \operatorname{End} M$ is the strict $\mathcal{O}'$-action and the ${}^{F^{-i}}a$ comes from $\mathcal{O}' \to W_{\mathcal{O}}(R)$. Since $a(M_i) \subseteq M_i$ for all $a \in W_{\mathcal{O}}(R)$, $V(M_i) \subseteq M_{i+1}$ and $F(M_i) \subseteq M_{i-1}$ by (2.17)-(2.20), we have $\deg a = 0$ for all $a \in W_{\mathcal{O}}(R)$, $\deg V = 1$ and $\deg F = -1$. $V : M_i \to M_{i+1}$ is an isomorphism for $i \neq -1$, and the actions of $\mathcal{O}'$ on M_0, i.e., ι and the other action obtained by $\mathcal{O}' \to W_{\mathcal{O}}(R) \to \mathbb{E}_{\mathcal{O},R}$, coincide. We define $U := V^{1-f}F : M_0 \to M_0$. The image lies in M_0, since the map has degree zero. To the element $\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}) F'^m$ of the Cartier ring $\mathbb{E}_{\mathcal{O}',R}$ in the

usual representation, i.e., $x_{m,n} \in W_{\mathcal{O}}(R)$, such that $x_{m,n} = 0$ for fixed n and almost all m, where u is taken from Corollary 1.2.2, we associate the operator $\sum_{n,m=0}^{\infty} V^{fn} x_{m,n} U^m$. We have to verify that this association is well-defined, so by this we would have turned M_0 into an $\mathbb{E}_{\mathcal{O}',R}$-module, which is reduced, and may take this functor as an equivalence.
Obviously it suffices to show that, if $\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}) F'^m = 0$, then the associated operator operates as zero on M_0. We should bear in mind that

$$\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}) F'^m = \sum_{n,m,k=0}^{\infty} V'^{n+k}[u(x_{m,n})_k] F'^{k+m}$$

by means of (2.21). We define $x_{m,n}^{(0)} := x_{m,n}$ for all m, n. By reducing modulo V' we get

$$\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}^{(0)}) F'^m \equiv \sum_{m=0}^{\infty} [u(x_{m,0}^{(0)})_0] F'^m \equiv 0 \mod V',$$

from which it follows that $u(x_{m,0}^{(0)})_0 = (x_{m,0}^{(0)})_0 = 0$ for all $m \geq 0$. Hence we can write $x_{m,0}^{(0)} = {}^V y_{m,0}^{(1)}$ for all m and obtain $u(x_{m,0}^{(0)}) = {}^{V'} u({}^{F^{f-1}} y_{m,0}^{(1)})$. So we have $\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}) F'^m = V' u(x_{0,1}^{(1)}) + \sum_{m>0}^{\infty} V' u(x_{m,1}^{(1)}) F'^m + \sum_{n>1,m=0}^{\infty} V'^n u(x_{m,n}^{(1)}) F'^m$, where $x_{0,1}^{(1)} := x_{0,1}^{(0)}$, $x_{m,1}^{(1)} = {}^{F^{f-1}} y_{m,0}^{(1)} + x_{m,1}^{(0)}$ for all $m > 0$ and $x_{m,n}^{(1)} := x_{m,n}^{(0)}$ for all $n \geq 2, m \geq 0$. By reducing modulo V'^2 we get that $(x_{m,1}^{(1)})_0 = 0$ for all m. Inductively, we get $(x_{m,j}^{(j)})_0 = 0$ for all $j \geq 0$ and m, where for $j \geq 1$ we define $x_{0,j}^{(j)} := x_{0,j}^{(j-1)}$, $x_{m,n}^{(j)} := x_{m,n}^{(j-1)}$ for all $n \geq j+1, m \geq 0$ and $x_{m,j}^{(j)} := x_{m,j}^{(j-1)} + {}^{F^{f-1}} y_{m-1,j-1}^{(j)}$ for $m > 0$ with $x_{m-1,j-1}^{(j-1)} = {}^V y_{m-1,j-1}^{(j)}$. Hence, we see for $t \in M_0$ and $t_m := U^m t$ for all $m \in \mathbb{N}_0$

$$\begin{aligned}
\sum_{n,m=0}^{\infty} V^{fn} x_{m,n} U^m t &= \sum_{m=0}^{\infty} x_{m,0}^{(0)} t_m + \sum_{n=1,m=0}^{\infty} V^{fn} x_{m,n}^{(0)} t_m \\
&= \sum_{m=0}^{\infty} V y_{m,0}^{(0)} F t_m + \sum_{n=1,m=0}^{\infty} V^{fn} x_{m,n}^{(0)} t_m \\
&= \sum_{m=0}^{\infty} V y_{m,0}^{(0)} V^{f-1} t_{m+1} + \sum_{n=1,m=0}^{\infty} V^{fn} x_{m,n}^{(0)} t_m \\
&= \sum_{m=0}^{\infty} V^f {}^{F^{f-1}} y_{m,0}^{(0)} t_{m+1} + \sum_{n=1,m=0}^{\infty} V^{fn} x_{m,n}^{(0)} t_m \\
&= V^f x_{0,1}^{(1)} t_0 + \sum_{m=1}^{\infty} V^f x_{m,1}^{(1)} t_m + \sum_{n=2,m=0}^{\infty} V^{fn} x_{m,n}^{(1)} t_m
\end{aligned}$$

By repeating this step inductively, we get that $\sum_{n,m=0}^{\infty} V^{fn} x_{m,n} U^m t \in V^k M$ for all k and because M is separated, $\sum_{n,m=0}^{\infty} V^{fn} x_{m,n} U^m t = 0$ must hold. Hence, the association is well-defined. We obtain a functor from the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules with a strict $\mathcal{O}'$-action to the category of reduced $\mathbb{E}_{\mathcal{O}',R}$-modules by passing to the M_0-modules and restricting the morphisms to the zeroth component. It is easily seen that this functor is an equivalence.

Now we consider the case, where $\mathcal{O} \to \mathcal{O}'$ is a totally ramified extension, and assume R to be an $\mathcal{O}'$-algebra . Let us consider the unique continuous ring morphism $\mu : \mathbb{E}_{\mathcal{O},R} \to \mathbb{E}_{\mathcal{O}',R}$, which is obtained by

$$\begin{aligned} \mu|_{W_{\mathcal{O}}(R)} &= u, \\ \mu(V) &= V', \\ \mu(F) &= (\pi/\pi')F'. \end{aligned}$$

With the help of this morphism we can build an obvious functor G from the category of reduced $\mathbb{E}_{\mathcal{O}',R}$-modules in the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules equipped with a strict $\mathcal{O}'$-action. First we show that this functor is fully faithful. We define $\varphi \in \mathbb{E}_{\mathcal{O}',R}$ by $V'\varphi = \pi' - [\pi']$ and get that φ must be of the form yF' with $y \in W_{\mathcal{O}'}(R)$ as in equation (2.23) (with V' and F' instead of V and F). From equation (2.24) (with π' instead of π), we get that y is a unit. It follows that all elements ${}^{F^k}y$ are units in $W_{\mathcal{O}'}(R)$ for all $k \geq 0$ and hence, we may write each element of $\mathbb{E}_{\mathcal{O}',R}$ in the form

$$\sum_{n,m=0}^{\infty} V'^m u(x_{m,n})\varphi^m,$$

where $x_{m,n} \in W_{\mathcal{O}}(R)$. Let M, M' be two reduced $\mathbb{E}_{\mathcal{O}',R}$-modules and $\psi : G(M) \to G(M')$ an $\mathbb{E}_{\mathcal{O},R}$-linear morphism between them which respects the $\mathcal{O}'$-action. We need to show that ψ is $\mathbb{E}_{\mathcal{O}',R}$-linear. But this follows from $V'\psi = \mu(V)\psi = \psi\mu(V) = \psi V'$, $u(w)\psi = \mu(w)\psi = \psi\mu(w) = \psi u(w)$ for each $w \in W_{\mathcal{O}}(R)$ and $\varphi\psi = \psi\varphi$, which in turn follows from $V'\varphi\psi = (\pi' - [\pi'])\psi = \psi(\pi' - [\pi']) = \psi V'\varphi = V'\psi\varphi$. Hence, G is fully faithful.
In order to show the essential surjectivity of G, we first consider the case, where the tangential space of the reduced $\mathbb{E}_{\mathcal{O},R}$-module equipped with a strict $\mathcal{O}'$-action is free. Let M be such a module and $m_1, \ldots, m_d$ a V-basis of M. If the action of π' is described by

$$\pi' m_i = [\pi']m_i + \sum_{n\geq 1}\sum_{j=1}^{d} V^n[c_{n,j,i}]m_j \tag{2.25}$$

with $c_{n,j,i} \in R$ for $i = 1, \ldots, n$, we define the reduced $\mathbb{E}_{\mathcal{O}',R}$-module M' with $M'/V'M'$ free by

$$\pi' m'_i = [\pi']m'_i + \sum_{n\geq 1}\sum_{j=1}^{d} V'^n[c_{n,j,i}]m'_j, \tag{2.26}$$

where the m'_i should be a V'-basis of M' and the Teichmüller representants are elements of $W_{\mathcal{O}'}(R)$. Since $u([a]_{\mathcal{O}}) = [a]_{\mathcal{O}'}$ for each $a \in R$ (where the indices should indicate in which $\mathcal{O}$-algebra resp. $\mathcal{O}'$-algebra of ramified Witt vectors we consider the Teichmüller representants), we obtain that we may rewrite (2.26) as

$$\pi' m'_i = u([\pi'])m'_i + \sum_{n\geq 1}\sum_{j=1}^{d} V'^n u([c_{n,j,i}])m'_j. \tag{2.27}$$

By iterating the equation (2.25) for each $k \geq 0$, we obtain in M that

$$\pi'^k m_i = [\pi'^k]m_i + \sum_{n\geq 1}\sum_{j=1}^{d} V^n \xi^{(k)}_{n,j,i} m_j$$

holds with $\xi^{(k)}_{n,j,i} \in W_{\mathcal{O}}(R)$. Now let

$$\pi = \sum_{k=1}^{e} a_k \pi'^k \tag{2.28}$$

be obtained from the Eisenstein equation of degree e, when e is the ramification index of the extension $\mathcal{O} \to \mathcal{O}'$, which π' satisfies, so $a_k \in \pi\mathcal{O}$ for all $k < e$ and $a_e \in \mathcal{O}^\times$. From the above equations, we obtain

$$\begin{aligned} \pi m_i &= (\sum_{k=1}^{e} a_k \pi'^k) m_i \\ &= \sum_{k=1}^{e} a_k[\pi'^k]m_i + \sum_{n\geq 1}\sum_{j=1}^{d} V^n(\sum_{k=1}^{e} a_k \xi^{(k)}_{n,j,i})m_j. \end{aligned} \tag{2.29}$$

This yields

$$\begin{aligned} \sum_{n\geq 1}\sum_{j=1}^{d} V^n(\sum_{k=1}^{e} a_k \xi^{(k)}_{n,j,i})m_j &= (\pi - \sum_{k=1}^{e} a_k[\pi'^k])m_i \\ &= V\alpha F m_i \end{aligned} \tag{2.30}$$

with

$$^V\alpha = \pi - \sum_{k=1}^{e} a_k[\pi'^k]. \tag{2.31}$$

We need to verify that α_0 is a unit in R. Then we would get the structural equations by

$$F m_i = \alpha^{-1} \sum_{n\geq 1}\sum_{j=1}^{d} V^{n-1}(\sum_{k=1}^{e} a_k \xi^{(k)}_{n,j,i})m_j.$$

By some calculation, we obtain that

$$\alpha_0 = 1 - \sum_{k=1}^{e-1}(a_k/\pi)\pi'^{kq} + \pi^{q-1} \cdot a_e^{-q+1} \cdot S,$$

with

$$S = \sum_{0 \le i_1,\dots i_{e-1}, j \le q, i_1+\dots+i_{e-1}+j=q} \binom{q}{i_1,\dots,i_{e-1},j} \prod_{k=1}^{e-1}(-a_k/\pi)^{i_k}\pi'^{i_k}.$$

We see that α_0 is clearly a unit when considered as an element of $\mathcal{O}'$ and hence a unit in R.
We obtain in M' by the iteration of π' given by (2.27) and addition, the analogue to (2.29)

$$\begin{aligned} \pi m_i' &= (\sum_{k=1}^{e} a_k \pi'^k) m_i' \\ &= \sum_{k=1}^{e} a_k u([\pi'^k]) m_i' + \sum_{n\ge1}\sum_{j=1}^{d} V'^n (\sum_{k=1}^{e} a_k u(\xi_{n,j,i}^{(k)})) m_j'. \end{aligned}$$

From this we obtain the analogue to (2.30)

$$\begin{aligned} \sum_{n\ge1}\sum_{j=1}^{d} V'^n (\sum_{k=1}^{e} a_k u(\xi_{n,j,i}^{(k)})) m_j' &= (\pi - \sum_{k=1}^{e} a_k[\pi'^k]) m_i' \\ &= u(\pi - \sum_{k=1}^{e} a_k[\pi'^k]) m_i' \\ &= V'u(\alpha)(\pi/\pi')F'm_i', \end{aligned}$$

with $\alpha \in W_{\mathcal{O}}(R)^\times$ given by (2.31). Since we may consider M' as an $\mathbb{E}_{\mathcal{O},R}$-module via the map μ, we obtain that the structural equations are the same as those for M. Hence, M and M' are isomorphic in a canonical way as $\mathbb{E}_{\mathcal{O},R}$-modules. Clearly, the π'-action of both modules is respected by this isomorphism and so the essential surjectivity is clear for reduced $\mathbb{E}_{\mathcal{O},R}$-modules with strict $\mathcal{O}'$-action and free tangential space. Now let M be an arbitrary reduced $\mathbb{E}_{\mathcal{O},R}$-module with a strict $\mathcal{O}'$-action, i.e., M/VM is just projective and not necessarily free. Let

$$P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} M/VM \to 0 \tag{2.32}$$

be an exact sequence of R-modules, with P_1 and P_2 finite and free. Let $e_1, \dots, e_d$, be a basis of P_1 and m_i be liftings of the $\beta(e_i)$. We find equations

$$\pi' m_i = [\pi'] m_i + \sum_{n\ge1}\sum_{j=1}^{d} V^n [c_{n,j,i}] m_j$$

for each $i = 1, \dots, d$ and define the reduced $\mathbb{E}_{\mathcal{O}',R}$-module L_1 with tangential space P_1 by these equations (with some V-basis $\widetilde{e}_i$ instead of m_i), where we consider the Teichmüller representants as elements of $W_{\mathcal{O}'}(R)$ and V' in place of V. It is not too hard to verify that the obvious surjective mapping $L_1 \stackrel{\widetilde{\beta}}{\to} M$, given by

$$\sum_{n,i} V'^n[c_{n,i}]\widetilde{e}_i \mapsto \sum_{n,i} V^n[c_{n,i}]m_i,$$

with $c_{n,i} \in R$, is an $\mathbb{E}_{\mathcal{O},R}$-linear morphism (where L_1 is considered as an $\mathbb{E}_{\mathcal{O},R}$-module via the map μ) respecting the $\mathcal{O}'$-action and may be identified modulo V with β. The kernel K of $\widetilde{\beta}$ is a reduced $\mathbb{E}_{\mathcal{O},R}$-module equipped with a strict $\mathcal{O}'$-action and K/VK equals the image of α. When we repeat this procedure for K instead of M and $P_2 \to \operatorname{Im}\alpha$ instead of β, we obtain a reduced $\mathbb{E}_{\mathcal{O}',R}$-module L_2 with tangential space P_2 and an $\mathbb{E}_{\mathcal{O},R}$-linear morphism $\widetilde{\alpha} : L_2 \to L_1$ respecting the $\mathcal{O}'$-action, which equals α modulo $V = V'$. Hence, we may represent M by an exact sequence of reduced $\mathbb{E}_{\mathcal{O},R}$-modules with strict $\mathcal{O}'$-actions

$$L_2 \stackrel{\widetilde{\alpha}}{\to} L_1 \to M \to 0, \tag{2.33}$$

such that the tangential spaces of L_1 and L_2 are free (and the morphisms respect the $\mathcal{O}'$-actions). Since the functor G is fully faithful, $\widetilde{\alpha}$ is also a $\mathbb{E}_{\mathcal{O}',R}$-linear morphism of reduced $\mathbb{E}_{\mathcal{O}',R}$-modules and we obtain an exact sequence of (at first not necessarily reduced) $\mathbb{E}_{\mathcal{O}',R}$-modules

$$L_2 \stackrel{\iota}{\to} L_1 \to \operatorname{Coker}(\widetilde{\alpha}) =: M' \to 0.$$

Clearly, when we consider this sequence as a sequence of $\mathbb{E}_{\mathcal{O},R}$-modules with strict $\mathcal{O}'$-actions, we get (2.33) back. But since μ maps V to V' it is clear that M' is also a reduced $\mathbb{E}_{\mathcal{O}',R}$-module. Hence, the equivalence is established in the totally ramified case as well. □

We should remark that for each $\mathcal{O}$ an RRS, R an $\mathcal{O}$-algebra, we may consider $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ for each $\mathcal{N} \in \mathrm{Nil}_R$ as an $\mathbb{E}_{\mathcal{O},R}$-module. For $e \in \mathbb{E}_{\mathcal{O},R}$ as in (2.22), the action is written as a right multiplication and is defined by

$$we = \sum_{n,m \geq 0} {}^{V^m}([a_{n,m}]({}^{F^n}w)),$$

where w is an element of $\widehat{W_{\mathcal{O}}}(\mathcal{N})$. It is left to the reader that this association defines indeed a module structure. This generalizes [Zin02, Equation (166)]. Clearly, an morphism between $\mathcal{N} \to \mathcal{N}'$ in Nil_R induces a morphism of $\mathbb{E}_{\mathcal{O},R}$-modules $\widehat{W_{\mathcal{O}}}(\mathcal{N}) \to \widehat{W_{\mathcal{O}}}(\mathcal{N}')$.
Now let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero again. By considering $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ as an $\mathbb{E}_{\mathcal{O},R}$-module we will be able to see how the functor from reduced $\mathbb{E}_{\mathcal{O},R}$-modules to formal $\mathcal{O}$-modules over R described in Theorem 2.4.3 looks precisely, but first we need two Lemmas.

Lemma 2.4.4. (cf. [Zin86, (2.10) Lemma]) Let R be an $\mathcal{O}$-algebra, $\mathcal{N}$ a nilpotent R-algebra with a nilpotent $\mathcal{O}$-pd structure and M a reduced $\mathbb{E}_{\mathcal{O},R}$-module. Then we have an isomorphism of $\mathcal{O}$-modules

$$\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M \simeq \mathcal{N} \otimes_R M/VM,$$

given by $\underline{n} \otimes m \mapsto \sum_i \mathrm{w}'_i(\underline{n}) \otimes \overline{F^i m}$ for all $\underline{n} \in \widehat{W_{\mathcal{O}}}(\mathcal{N})$ and $m \in M$, where the w'_i are given by (2.8). The inverse mapping is given by $n \otimes \overline{m} \mapsto \log^{-1}(n, 0, \ldots) \otimes m$ for all $n \in \mathcal{N}$ and $\overline{m} \in M/VM$, where m is any lifting of $\overline{m}$ and log is given by (2.14).

The proof is left as an easy exercise.

Lemma 2.4.5. (cf. [Zin84, 4.41 Satz]) Let M be a reduced $\mathbb{E}_{\mathcal{O},R}$-module and $\mathcal{N}$ a nilpotent R-algebra. Then $\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M) = 0$ holds for each $i \geq 1$.

Proof: First we consider the case, where $\mathcal{N}^2 = 0$. If we take an exact sequence of reduced $\mathbb{E}_{\mathcal{O},R}$-modules

$$0 \to L \to P \to M \to 0$$

with P a finite free $\mathbb{E}_{\mathcal{O},R}$-module, we obtain by tensoring with $\widehat{W_{\mathcal{O}}}(\mathcal{N})$ by Lemma 2.4.4 an exact sequence (since M/VM is a projective, hence flat R-module) of $\mathcal{O}$-modules

$$0 \to \mathcal{N} \otimes_R L/VL \to \mathcal{N} \otimes_R P/VP \to \mathcal{N} \otimes_R M/VM \to 0.$$

This shows that the sequence of $\mathbb{E}_{\mathcal{O},R}$-modules

$$0 \to \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} L \to \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} P \to \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M \to 0$$

is exact. Thus, by considering the long exact sequence

$$0 \leftarrow \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M \leftarrow \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} P \leftarrow \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} L \leftarrow \mathrm{Tor}_1^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M)$$

$$\leftarrow \mathrm{Tor}_1^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), P) \leftarrow \mathrm{Tor}_1^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), L) \leftarrow \mathrm{Tor}_2^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M) \leftarrow \ldots$$

we first conclude that $\mathrm{Tor}_1^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M) = 0$ and since $\mathrm{Tor}_1^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), P) = 0$ for each $i \geq 1$, we obtain

$$\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), L) = \mathrm{Tor}_{i+1}^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M)$$

for each $i \geq 1$. Inductively, since L is also a reduced $\mathbb{E}_{\mathcal{O},R}$-module, we obtain that $\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}), M) = 0$ for each $i \geq 1$.

For general $\mathcal{N}$ we proceed inductively as well. Assume the assertion has been shown for each $\mathcal{N}$ with $\mathcal{N}^{r-1} = 0$ and let $\mathcal{N}'$ be a nilpotent R-algebra with $\mathcal{N}'^{r} = 0$. By considering the exact sequence

$$0 \to \mathcal{N}'^{r-1} \to \mathcal{N}' \to \mathcal{N}'/\mathcal{N}'^{r-1} \to 0$$

and considering the long exact sequence analogue to the above one only with the variation in the first argument, we obtain by $\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}'^{r-1}), M) = \mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}'/\mathcal{N}'^{r-1}), M) = 0$ for each $i \geq 1$ that $\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W_{\mathcal{O}}}(\mathcal{N}'), M) = 0$ holds for each $i \geq 1$. □

Proposition 2.4.6. For each reduced $\mathbb{E}_{\mathcal{O},R}$-module M the functor from Nil_R to $\mathrm{Mod}_{\mathcal{O}}$ given by $\widehat{W_{\mathcal{O}}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ is a formal $\mathcal{O}$-module. Furthermore, the equivalence functor from the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules to the category of formal $\mathcal{O}$-modules as constructed in Theorem 2.4.3 is given by this functor.

Intuitively, the Proposition says that the construction of Drinfeld of the equivalence is the obvious generalization of the classical equivalence for $\mathbb{Z}_p$.
Proof: The only fact which is nontrivial in order to establish that $\widehat{W_{\mathcal{O}}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ is a formal $\mathcal{O}$-module, is that the tangential space is a finite projective R-module and that it preserves exact sequences. But this follows from Lemmas 2.4.4 and 2.4.5.
The second assertion is already confirmed for the $\mathbb{Z}_p$-case (cf. [Zin84, 4.23 Satz]). Hence, as in Drinfeld's proof, it suffices to show that if the assertion is true for some $\mathcal{O}$, for any extension $\mathcal{O} \to \mathcal{O}'$ the assertion then follows for $\mathcal{O}'$. So we first consider the case, in which $\mathcal{O}'$ is nonramified over $\mathcal{O}$, and then the case, in which $\mathcal{O}'$ is totally ramified over $\mathcal{O}$. We will only focus on the objects and leave it to the reader to verify that the assertion holds for the morphisms as well. Let $\mathcal{O}'$ be nonramified over $\mathcal{O}$, R an $\mathcal{O}'$-algebra and M a reduced $\mathbb{E}_{\mathcal{O},R}$-module equipped with a strict $\mathcal{O}'$-action. By construction we obtain an $\mathbb{E}_{\mathcal{O}',R}$-module M_0. We need to show now that $\widehat{W_{\mathcal{O}}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ and $\widehat{W_{\mathcal{O}'}}(-) \otimes_{\mathbb{E}_{\mathcal{O}',R}} M_0$ are isomorphic as formal $\mathcal{O}'$-modules. We consider the $\mathcal{O}'$-module morphism $\tau_{\mathcal{N}}$, which is obtained by the commutative diagram of $\mathcal{O}'$-modules

$$\begin{array}{ccc}
\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M & = & \bigoplus_{i=0}^{f-1} (\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M)_i \\
\downarrow \tau_{\mathcal{N}} & & \downarrow \mathrm{pr} \\
\widehat{W_{\mathcal{O}'}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O}',R}} M_0 & \xleftarrow{\omega_{\mathcal{N}}} & (\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M)_0,
\end{array}$$

where $\omega_{\mathcal{N}}$ is obtained by sending $a \otimes m$ to $u(a) \otimes m$ with u as usual. This map makes sense, because

$$(\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M)_i = \{ \sum_{j \in J} a_j \otimes m_j \mid J \text{ finite }, a_j \in \widehat{W_{\mathcal{O}}}(\mathcal{N}), m_j \in M_i \}$$

holds for all $i = 0, \ldots, f-1$, where $M = \bigoplus_{i=0}^{f-1} M_i$ is the graduation of M from the proof of Theorem 2.4.3. Since each $(\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M)_i$ may be considered in a canonical way as an $\mathbb{E}_{\mathcal{O}',R}$-module, we obtain that $\omega_{\mathcal{N}}$ and the projection are in fact an $\mathbb{E}_{\mathcal{O}',R}$-module morphisms, hence $\mathcal{O}'$-linear. In order to show that $\tau_{\mathcal{N}}$ is an isomorphism of $\mathcal{O}'$-modules for each $\mathcal{N}$, it suffices to reduce to the case $\mathcal{N}^2 = 0$, which is rather obvious by Lemma 2.4.4 since M (as an $\mathbb{E}_{\mathcal{O},R}$-module) and M_0 (as an $\mathbb{E}_{\mathcal{O}',R}$-module) have the same tangential spaces.
Now let $\mathcal{O} \to \mathcal{O}'$ be totally ramified and R an $\mathcal{O}'$-algebra. We start with a reduced $\mathbb{E}_{\mathcal{O}',R}$-module M' and consider the $\mathbb{E}_{\mathcal{O},R}$-module M equipped plus a strict $\mathcal{O}'$-action, which is obtained by restriction to scalars. We get an $\mathcal{O}'$-module morphism

$$\gamma_{\mathcal{N}} : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M = \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} \mathbb{E}_{\mathcal{O}',R} \otimes_{\mathbb{E}_{\mathcal{O}',R}} M' \overset{\lambda_{\mathcal{N}} \otimes \mathrm{id}_{M'}}{\to} \widehat{W_{\mathcal{O}'}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O}',R}} M',$$

where $\lambda_{\mathcal{N}} : \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} \mathbb{E}_{\mathcal{O}',R} \to \widehat{W_{\mathcal{O}'}}(\mathcal{N})$ is obtained by sending $a \otimes e$ to $u(a)e$. Hence, by reducing to the $\mathcal{N}^2 = 0$ case and the fact that the tangential spaces of M and M' are the same, we obtain by Lemma 2.4.4 again that $\gamma_{\mathcal{N}}$ is an isomorphism for each $\mathcal{N}$. □

It is of course interesting to ask how we may describe the reduced Cartier module of a formal $\mathcal{O}$-module associated to an f-$\mathcal{O}$-display over an $\mathcal{O}$-algebra R, where $\mathcal{O}$ is the ring of integers of a non-Archimedean local field of characteristic zero.

Proposition 2.4.7. (cf. [Zin02, Proposition 90]) Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over an $\mathcal{O}$-algebra R. The reduced $\mathbb{E}_{\mathcal{O},R}$-module associated to the formal $\mathcal{O}$-module $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ associated to $\mathcal{P}$ is given by

$$M(\mathcal{P}) = M_{BT_{\mathcal{O}}^{(f)}(\mathcal{P},-)} = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P / (F \otimes x - V^{f-1} \otimes Fx, V^f \otimes F_1 y - 1 \otimes y)_{x \in P, y \in Q}.$$

If $\mathcal{O}'$ is the nonramified extension of degree f of $\mathcal{O}$ and R an $\mathcal{O}'$-algebra, we obtain a naturally arising strict $\mathcal{O}'$-action on $\mathcal{P}$ by the usual map $\mathcal{O}' \to W_{\mathcal{O}}(\mathcal{O}') \to W_{\mathcal{O}}(R)$ (see (1.2)), from which we also get a strict $\mathcal{O}'$-action on $M(\mathcal{P})$.

Proof: First we need to verify that $M = M(\mathcal{P})$ is indeed a reduced $\mathbb{E}_{\mathcal{O},R}$-module. By setting $N = (F \otimes x - V^{f-1} \otimes Fx, V^f \otimes F_1 y - 1 \otimes y)_{x \in P, y \in Q}$ we obtain a diagram of $\mathbb{E}_{\mathcal{O},R}$-modules

$$\begin{array}{ccccccccc} & & \ker V & & \ker V & & \ker V & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow V & & \downarrow V & & \downarrow V & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha} & \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \beta & & \downarrow & & \\ & & \operatorname{Coker} V & & \operatorname{Coker} V = \bigoplus_{i \geq 0} RF^i \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(R)} P & & \operatorname{Coker} V, & & \end{array}$$

whose rows and columns are exact. Because it is obvious that $V : \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \to \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P$ is injective, is suffices by the snake lemma to show for the injectivity of $V : M \to M$ that the obvious map

$$\beta \circ \alpha : N \to \bigoplus_{i \geq 0} RF^i \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(R)} P$$

has kernel VN, which is not too difficult to verify. By generalizing the above diagram from V to V^k, we obtain exact sequences

$$0 \to N/V^k N \to \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/V^k \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \to M/V^k M \to 0$$

and since

$$N/V^{k+1} N \to N/V^k N$$

is surjective, we get by a standard argument (cf. [Liu06, Chapter I, Lemma 3.1.]) an exact sequence

$$0 \to \varprojlim N/V^k N \to \varprojlim \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/V^k \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \to \varprojlim M/V^k M \to 0.$$

Since

$$\varprojlim \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/V^k \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P$$

and

$$\varprojlim N/V^k N = N$$

holds, it is clear that $M = \varprojlim M/V^k M$ holds. Furthermore, the tangential space of M is P/Q, which is a finite projective R-module. Hence, M is a reduced $\mathbb{E}_{\mathcal{O},R}$-module. By representing $BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N})$ by the usual sequence, we have an obvious morphism from $\widehat{P}_{\mathcal{N}}$ to $\widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ and the image of $F_1 - \mathrm{id} : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$ lies in the kernel of this mapping, so there is a canonical $\mathcal{O}$-module morphism

$$BT_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N}) \to \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$$

and by reducing to the $\mathcal{N}^2 = 0$ case we obtain the isomorphism, since the tangential spaces are the same. If R is an $\mathcal{O}'$-algebra, with $\mathcal{O}'$ nonramified over $\mathcal{O}$ of degree f, then the assertion for the strict $\mathcal{O}'$-action is obvious. $\square$

In case that $\mathcal{P}$ is nilpotent and R a perfect field extending the residue field of $\mathcal{O}$, the reduced Cartier module can be described by

$$M = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/(V^f \otimes x - 1 \otimes Vx, F \otimes x - V^{f-1} \otimes Fx)_{x \in P},$$

where the operator $V : P \to P$ on the right hand side of the tensor product in the expression $V^f \otimes x - 1 \otimes Vx$ is constructed after Proposition 2.2.3. Since this V is topological nilpotent, we obtain, if $V^k P$ is a subset of πP for a fixed k, that for each $x \in P$

$$V^{knf} \otimes x = 1 \otimes V^{kn} x = 1 \otimes \pi^n x'$$

holds for each n and some $x' \in P$. We can represent M by the following simpler module structure given by P^f, where the i-th component x_i of an element $x = (x_0, \ldots, x_{f-1}) \in P^f$ corresponds to $V^i \otimes x_i$. The actions of F, V and an element $w \in W_{\mathcal{O}}(k)$ are given by

$$\begin{aligned} Fx &= (\pi x_1, \pi x_2, \ldots, \pi x_{f-1}, F x_0), \\ Vx &= (V x_{f-1}, x_0, \ldots, x_{f-2}), \\ wx &= (w x_0, {}^F w x_1, \ldots, {}^{F^i} x_i, \ldots, {}^{F^{f-1}} w x_{f-1}) \end{aligned}$$

where the F and V on the right hand side are the operators of the f-$\mathcal{O}$-Dieudonné module. This does indeed define an $\mathbb{E}_{\mathcal{O},R}$-modules structure because of the above described nilpotence of V. Furthermore, if R is an $\mathcal{O}'$-algebra, where $\mathcal{O}'$ is the nonramified extension of $\mathcal{O}$ of degree f, we have an obvious strict $\mathcal{O}'$-action on $M = P^f$ by $\mathcal{O}' \to W_{\mathcal{O}}(\mathcal{O}') \to W_{\mathcal{O}}(R)$.

2.5 Introducing $\Gamma_i(\mathcal{O}, \mathcal{O}')$ and $\Omega_i(\mathcal{O}, \mathcal{O}')$

In this section, each $\mathcal{O}$ and $\mathcal{O}'$ are assumed to be rings of integers of a non-Archimedean local fields of characteristic zero and R an $\mathcal{O}'$-algebra. Assume now for this abstract that π' is nilpotent in R. We will construct four functors: For $\mathcal{O} \to \mathcal{O}'$ nonramified of degree f, we define functors $\Omega_1(\mathcal{O}, \mathcal{O}')$ from nilpotent $\mathcal{O}$-displays over R equipped with a strict $\mathcal{O}'$-action (see Definition 2.2.9) to nilpotent f-$\mathcal{O}$-displays over R and $\Omega_2(\mathcal{O}, \mathcal{O}')$ from nilpotent f-$\mathcal{O}$-displays over R to nilpotent $\mathcal{O}'$-displays over R and we will consider the composition $\Gamma_1(\mathcal{O}, \mathcal{O}')$. For a totally ramified extension $\mathcal{O} \to \mathcal{O}'$, we set up a functor $\Gamma_2(\mathcal{O}, \mathcal{O}')$ from nilpotent $\mathcal{O}$-displays over R equipped with a strict $\mathcal{O}'$-action to nilpotent $\mathcal{O}'$-displays over R. Their motivation arises from the previous section by considering Drinfeld's method of proving the equivalence between formal $\mathcal{O}$-modules and reduced $\mathbb{E}_{\mathcal{O},R}$-modules. In the end, showing that $BT_{\mathcal{O}}$ is an equivalence of categories for each $\mathcal{O}$ and each $\mathcal{O}$-algebra R with π nilpotent in R is equivalent to show that $\Gamma_1(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ are equivalences for all cases.

2.5.1 The functors $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_1(\mathcal{O},\mathcal{O}')$

Unless otherwise stated, in this subsection we only consider nonramified extensions. Here we introduce the functors $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_1(\mathcal{O},\mathcal{O}')$.

Lemma 2.5.1. Let $\mathcal{O}\to\mathcal{O}'$ be nonramified of degree f, R a π'-adic $\mathcal{O}'$-algebra and $\mathcal{P}=(P,Q,F,F_1)$ an $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action ι. Then we may decompose P and Q canonically in $P=\bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}}P_i, Q=\bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}}Q_i$, where each P_i and $Q_i=P_i\cap Q$ are $W_{\mathcal{O}}(R)$-modules, $P_i=Q_i$ for all $i\neq 0$, $F(P_i),F_1(Q_i)\subseteq P_{i+1}$ for all i (where we consider i modulo f) and

$$\mu_{i,j}:W_{\mathcal{O}}(R)\otimes_{F^i,W_{\mathcal{O}}(R)}P_j \quad\to\quad P_{i+j},$$

given by $w\otimes p_j\mapsto wF_1^ip_j$ is an isomorphism for all $i+j\leq f$ and $j\neq 0$.

Proof: First we need to remark that we have got two actions of $\mathcal{O}'$ on $\mathcal{P}$; one is obtained by the given action ι and the other one is obtained by the composite map of $\mathcal{O}'\to W_{\mathcal{O}}(\mathcal{O}')\to W_{\mathcal{O}}(R)$, where the first map is (1.2). If we denote the relative Frobenius of the extension $\mathcal{O}\to\mathcal{O}'$ by σ, we obtain that P and Q are $\mathcal{O}'\otimes_{\mathcal{O}}W_{\mathcal{O}}(R)=W_{\mathcal{O}}(R)^f$-modules and that we may decompose them as follows:

$$\begin{aligned} P &= \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} P_i,\\ Q &= \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} Q_i, \end{aligned}$$

where

$$P_i \quad=\quad \{\, x\in P \,|\, (a\otimes 1)x=(1\otimes\sigma^i(a))x \text{ for all } a\in\mathcal{O}'\,\}$$

and

$$Q_i=Q\cap P_i.$$

The elements $a\otimes 1$ and $1\otimes\sigma^i(a)$ in the construction of the P_i are elements of $\mathcal{O}'\otimes_{\mathcal{O}}W_{\mathcal{O}}(R)$. Since

$$P/Q=\bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}}P_i/Q_i,$$

we get, because of the strictness of the attached $\mathcal{O}'$-action on $\mathcal{P}$, that $P_i=Q_i$ for all $i\neq 0$. It is easily verified that $F(P_i),F_1(Q_i)\subseteq P_{i+1}$ hold for all i.

To show that $\mu_{i,j} : W_{\mathbb{Z}_p}(R) \otimes_{F^i, W_{\mathbb{Z}_p}(R)} P_j \to P_{i+j}$ is an isomorphism, we just need to consider the obvious generalization of [Zin02, Lemma 9], which says that for a given normal decomposition $L \oplus T = P$ we obtain an ^{F}-linear isomorphism

$$L \oplus T \xrightarrow{F_1 \oplus F} L \oplus T.$$

Since each normal decomposition looks like

$$L = L_0 \oplus P_1 \oplus \ldots \oplus P_{f-1}$$

$$T = T_0 \oplus 0 \oplus \ldots \oplus 0$$

the result is easily seen. □

Definition 2.5.2. Let $f \geq 1$ be an integer, $\mathcal{O} \to \mathcal{O}'$ a (not necessarily non-ramified / totally ramified) extension of rings of integers of non-Archimedean local fields of characteristic zero and R an $\mathcal{O}'$-algebra. Then the category $(f - \operatorname{disp}_{\mathcal{O},\mathcal{O}'}/R)$ is defined by the f-$\mathcal{O}$-displays $\mathcal{P}$ over R equipped with a strict $\mathcal{O}'$-action as objects and those morphisms between f-$\mathcal{O}$-displays respecting the attached $\mathcal{O}'$-actions as morphisms. Now let R be π'-adic. The category $(f - \operatorname{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$ is the full subcategory of $(f - \operatorname{disp}_{\mathcal{O},\mathcal{O}'}/R)$, whose objects are the nilpotent f-$\mathcal{O}$-displays over R equipped with a strict $\mathcal{O}'$-action. The objects in the categories $(f - \operatorname{disp}_{\mathcal{O},\mathcal{O}'}/R)$ resp. $(f - \operatorname{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$ are $(\mathcal{P},\alpha)$, where $\mathcal{P}$ is a (nilpotent) f-$\mathcal{O}$-display over R and $\alpha : \mathcal{O}' \to \operatorname{End}\mathcal{P}$ the strict $\mathcal{O}'$-action, but if it is clear that we have such an action attached, we write with abuse of notation just $\mathcal{P}$ instead of $(\mathcal{P},\alpha)$.

Definition 2.5.3. With the setting as in the previous Proposition we are able to define a functor

$$\Omega_1(\mathcal{O},\mathcal{O}') : (\operatorname{disp}_{\mathcal{O},\mathcal{O}'}/R) \to (f - \operatorname{disp}_{\mathcal{O}}/R)$$

given by sending (P, Q, F, F_1) equipped with a strict $\mathcal{O}'$-action to $(P_0, Q_0, F_1^{f-1}F, F_1^f)$ and restricting a morphism between two f-$\mathcal{O}$-displays respecting the attached $\mathcal{O}'$-actions to the zeroth component. Furthermore, we obtain by restriction the functor

$$\Omega_1(\mathcal{O},\mathcal{O}') : (\operatorname{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (f - \operatorname{ndisp}_{\mathcal{O}}/R).$$

It is easily checked that the functors commute with base change.
At first glance, it appears to be not too difficult to show that $\Omega_1(\mathcal{O},\mathcal{O}')$ is an equivalence of categories in both cases (i.e., for the nilpotent and not neccesarily nilpotent case) as in Drinfeld's proof of the previous chapter. Unfortunately, we can only deduce $P_0 = W_{\mathcal{O}}(R) \otimes_{F^{f-i}, W_{\mathcal{O}}(R)} P_i$ from the previous Lemma for $i \neq 0$, so we can only show the essential surjectivity directly for cases when

${}^{F}: W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ is an automorphism (see Proposition 3.3.3 for the case of perfect fields, which extend the residue field of $\mathcal{O}'$).
Furthermore, it is not too hard to convince oneself by (1.2) that for any $\mathcal{O}'$-algebra R, the $BT_{\mathcal{O}}^{(f)}$ functor from f-$\mathcal{O}$-displays over R to formal $\mathcal{O}$-modules defines a functor to formal $\mathcal{O}'$-modules. Analogously, if π' is nilpotent in R, the $BT_{\mathcal{O}}^{(f)}$ functor restricted to nilpotent f-$\mathcal{O}$-displays over R to π-divisible formal $\mathcal{O}$-modules defines in fact a functor to π'-divisible formal $\mathcal{O}'$-modules. Hence, the following Proposition makes sense.

Proposition 2.5.4. Let $\mathcal{O} \to \mathcal{O}'$ be nonramified of degree f and R a π'-adic $\mathcal{O}'$-algebra. Then the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{BT_{\mathcal{O}}} & (\text{formal } \mathcal{O}' - \text{modules}/R) \\ \Big\downarrow{\scriptstyle \Omega_1(\mathcal{O},\mathcal{O}')} & \nearrow{\scriptstyle BT_{\mathcal{O}}^{(f)}} & \\ (f - \mathrm{disp}_{\mathcal{O}}/R) & & \end{array}$$

If π' is nilpotent in R, then the restriction of the above diagram

$$\begin{array}{ccc} (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{BT_{\mathcal{O}}} & (\pi'\text{-divisible formal } \mathcal{O}' - \text{modules}/R) \\ \Big\downarrow{\scriptstyle \Omega_1(\mathcal{O},\mathcal{O}')} & \nearrow{\scriptstyle BT_{\mathcal{O}}^{(f)}} & \\ (f - \mathrm{ndisp}_{\mathcal{O}}/R) & & \end{array}$$

is commutative.

Proof: Let $\mathcal{P}$ be a (nilpotent) $\mathcal{O}$-display $\mathcal{P}$ over R and $\mathcal{P}_0$ its image via $\Omega_1(\mathcal{O}, \mathcal{O}')$. In order to show the commutativity on the objects, we just have to construct a morphism

$$BT_{\mathcal{O}}(\mathcal{P}, -) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$$

and to show that this morphism is in fact an isomorphism. For this purpose consider for a nilpotent R-algebra $\mathcal{N}$ the sequence

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0$$

and the one defining $BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$. By using the $\mathbb{Z}/f\mathbb{Z}$-grading of Q and P, we obtain for the above sequence

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \widehat{Q}_{i,\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i\mathcal{N}} \longrightarrow BT_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0,$$

where $\widehat{P}_{i,\mathcal{N}}$ and $\widehat{Q}_{i,\mathcal{N}}$ have their obvious meaning, and it should be remarked that $\widehat{P}_{i,\mathcal{N}} = \widehat{Q}_{i,\mathcal{N}}$ holds for all $i \neq 0$, which is important for the reason, why we may apply the map F_1 on $\widehat{P}_{i,\mathcal{N}}$ for $i \neq 0$. There is a map θ from $\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i,\mathcal{N}} = \widehat{P}_{\mathcal{N}}$ to

$\widehat{P}_{0,\mathcal{N}}$ defined by $\theta(x_0, x_1, \dots, x_{f-1}) = \sum_{j=1}^{f} F_1^{f-j} x_j$ (with indices taken modulo f) and we want to show that the image of θ of the image of $F_1 - \mathrm{id}$ is contained in the image of $F_1^f - \mathrm{id}$, which would establish a map $BT_{\mathcal{O}}(\mathcal{P},\mathcal{N}) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,\mathcal{N})$. An element $(x_0,\dots,x_{f-1})$ of $\widehat{P}_{\mathcal{N}} = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i,\mathcal{N}}$ is contained in $(F_1 - \mathrm{id})(\widehat{Q}_{\mathcal{N}})$, iff there is a $(q_0,\dots,q_{f-1}) \in \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{Q}_{i,\mathcal{N}} = \widehat{Q}_{\mathcal{N}}$, such that

$$x_i = F_1 q_{i-1} - q_i \tag{2.34}$$

hold for all i, where the indices have to be considered modulo f again. Inductively we obtain for such an element $(x_0,\dots,x_{f-1})$ that

$$q_i = F_1^i q_0 - \sum_{j=1}^{i} F_1^{i-j} x_j \tag{2.35}$$

holds for all $i = 0,\dots,f$. So we get

$$F_1^f q_0 - q_0 = \sum_{j=1}^{f} F_1^{f-j} x_j, \tag{2.36}$$

from which we can deduce $\theta(F_1 - \mathrm{id})(\widehat{Q}_{\mathcal{N}}) \subseteq (F_1^f - \mathrm{id})(\widehat{Q}_{0,\mathcal{N}})$. Hence the induced map

$$\overline{\theta} : BT_{\mathcal{O}}(\mathcal{P},\mathcal{N}) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,\mathcal{N})$$

is well-defined. It is obvious that $\overline{\theta}$ is a morphism respecting the $\mathcal{O}'$-module structure and that $\overline{\theta} = \overline{\theta}(\mathcal{N})$ is functorial in $\mathcal{N}$. Furthermore, $\overline{\theta}$ is injective, since, if $\overline{(x_0,\dots,x_{f-1})} \in BT_{\mathcal{O}}(\mathcal{P},\mathcal{N})$ is mapped to zero, i.e., (2.36) holds for some $q_0 \in \widehat{Q}_{0,\mathcal{N}}$, we get that $(q_0,\dots,q_{f-1})$ with q_0 as right above and q_i given by (2.35) for $i = 1,\dots,f-1$ fulfils (2.34) relative to $(x_0,\dots,x_{f-1})$ and hence $\overline{(x_0,\dots,x_{f-1})}$ is zero. We obtain the surjectivity, since for $\overline{x_0} \in BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,\mathcal{N})$ we may take the element $\overline{(x_0,0,\dots,0)} \in BT_{\mathcal{O}}(\mathcal{P},\mathcal{N})$, which is mapped to $\overline{x_0}$. Hence, $BT_{\mathcal{O}}(\mathcal{P},-)$ and $BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,-)$ are isomorphic. The commutativity on the morphism sets is left to the reader. □

In order to obtain a functor from nilpotent $\mathcal{O}$-displays over R equipped with a strict $\mathcal{O}'$-action to nilpotent $\mathcal{O}'$-displays over R, it would suffice to give a suitable functor from nilpotent f-$\mathcal{O}$-displays to nilpotent $\mathcal{O}'$-displays over R.

Definition 2.5.5. With $\mathcal{O} \to \mathcal{O}'$ nonramified of degree f and R an $\mathcal{O}'$-algebra, we define a functor

$$\Omega_2(\mathcal{O},\mathcal{O}') : (f - \mathrm{disp}_{\mathcal{O}}/R) \to (\mathrm{disp}_{\mathcal{O}'}/R)$$

by sending $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{1,0})$ with a normal decomposition $L_0 \oplus T_0 = P_0$ to $\mathcal{P}' = (P', Q', F', F_1')$, where the elements of the quadruple are given by

$$\begin{aligned} P' &= W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_0, \\ Q' &= I_{\mathcal{O}',R} \otimes_{W_{\mathcal{O}}(R)} T_0 \oplus W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} L_0, \\ F' &= {}^{F'} \otimes_{W_{\mathcal{O}}(R)} F_0, \\ F_1'(w \otimes z) &= {}^{F'} w \otimes_{W_{\mathcal{O}}(R)} F_{1,0}(z), \\ F_1'({}^{V'} w \otimes x) &= w \otimes_{W_{\mathcal{O}}(R)} F_0 x, \end{aligned}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P_0$ and $z \in Q_0$, where we have used the morphism $u : W_{\mathcal{O}}(R) \to W_{\mathcal{O}'}(R)$. Here the operators related to $W_{\mathcal{O}'}(R)$ are marked with a dash. The mapping of the morphisms is simply given by tensoring. Of course, for π'-adic $\mathcal{O}'$-algebras R this defines a functor

$$\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \text{ndisp}_{\mathcal{O}} / R) \to (\text{ndisp}_{\mathcal{O}'} / R)$$

and we define

$$\Gamma_1(\mathcal{O}, \mathcal{O}') : (\text{disp}_{\mathcal{O},\mathcal{O}'} / R) \to (\text{disp}_{\mathcal{O}'} / R)$$

as the composite of $\Omega_2(\mathcal{O}, \mathcal{O}')$ and $\Omega_1(\mathcal{O}, \mathcal{O}')$ and analogously for the nilpotent case for all π'-adic $\mathcal{O}'$-algebras R.

It is easily checked that the definition of Q' is in fact independent of the normal decomposition of P_0 and that F_1' exists. It should be remarked that this functor looks very similar to the usual base change and that it is rather obvious that the functors commute with base change.
Furthermore, it gets now apparent why it was necessary to define f-$\mathcal{O}$-displays, since if we would simply tensor an $\mathcal{O}$-display over R by $W_{\mathcal{O}'}(R)$ as above via the morphism u, we would not obtain sensible mappings F' and F_1' in general, because we only know $u({}^{F^f} x) = {}^{F'} u(x)$ (see Corollary 1.2.2).

Proposition 2.5.6. Let $\mathcal{O} \to \mathcal{O}'$ be nonramified of degree f and R an $\mathcal{O}'$-algebra. Then the following diagram is commutative:

$$\begin{array}{ccc} (f - \text{disp}_{\mathcal{O}} / R) & \xrightarrow{BT_{\mathcal{O}}^{(f)}} & (\text{formal } \mathcal{O}' - \text{modules}/R) \\ {\scriptstyle \Omega_2(\mathcal{O},\mathcal{O}')} \downarrow & \nearrow_{BT_{\mathcal{O}'}} & \\ (\text{disp}_{\mathcal{O}'} / R) & & \end{array}$$

If π' is nilpotent in R, then the restriction of the above diagram

$$\begin{array}{ccc} (f - \text{ndisp}_{\mathcal{O}} / R) & \xrightarrow{BT_{\mathcal{O}}^{(f)}} & (\pi'\text{-divisible formal } \mathcal{O}' - \text{modules}/R) \\ {\scriptstyle \Omega_2(\mathcal{O},\mathcal{O}')} \downarrow & \nearrow_{BT_{\mathcal{O}'}} & \\ (\text{ndisp}_{\mathcal{O}'} / R) & & \end{array}$$

is commutative.

Proof: Let $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{1,0})$ be a (nilpotent) f-$\mathcal{O}$-display over R and $\mathcal{P}' = (P', Q', F', F_1')$ its image via $\Omega_2(\mathcal{O},\mathcal{O}')$. We need to show that $BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$ and $BT_{\mathcal{O}'}(\mathcal{P}', -)$ are isomorphic in the category of (π'-divisible) formal $\mathcal{O}'$-modules over R. For $\mathcal{N} \in \mathrm{Nil}_R$ the equations

$$\begin{aligned} \widehat{P'}_{\mathcal{N}} &= \widehat{W_{\mathcal{O}'}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P_0 \\ \widehat{Q'}_{\mathcal{N}} &= \widehat{I_{\mathcal{O}'\mathcal{N}}} \otimes_{W_{\mathcal{O}}(R)} T_0 \oplus \widehat{W_{\mathcal{O}'}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} L_0 \end{aligned}$$

hold (for a normal decomposition $L_0 \oplus T_0$ of P_0) and we define a map

$$\mu = u_{\mathcal{N}} \otimes \mathrm{id} : \widehat{P}_{0,\mathcal{N}} \to \widehat{P'}_{\mathcal{N}},$$

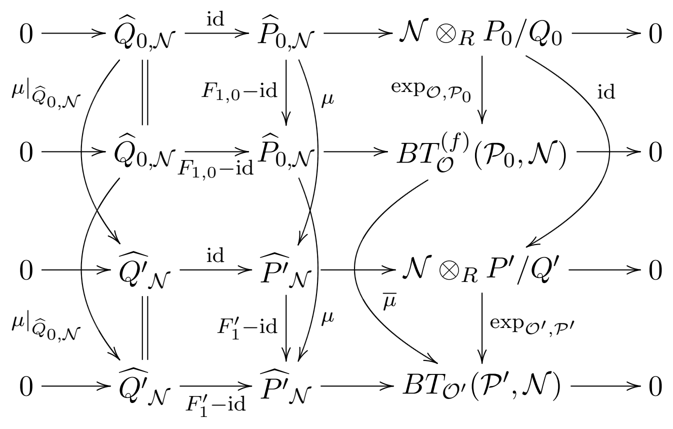
where $u_{\mathcal{N}}$ is the map defined in Proposition 1.2.1. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{Q}_{0,\mathcal{N}} & \xrightarrow{F_{1,0}-\mathrm{id}} & \widehat{P}_{0,\mathcal{N}} & \longrightarrow & BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,\mathcal{N}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu|_{\widehat{Q}_{0,\mathcal{N}}}} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \overline{\mu}} & & \\ 0 & \longrightarrow & \widehat{Q'}_{\mathcal{N}} & \xrightarrow{F_1'-\mathrm{id}} & \widehat{P'}_{\mathcal{N}} & \longrightarrow & BT_{\mathcal{O}'}(\mathcal{P}',\mathcal{N}) & \longrightarrow & 0, \end{array}$$

where $\overline{\mu}$ is the induced map, which makes sense, because it is easily verified that the image of $\mu|_{\widehat{Q}_{0,\mathcal{N}}}$ is contained in $\widehat{Q'}_{\mathcal{N}}$. In order to show that $\overline{\mu}$ is in fact an isomorphism, we may reduce to the case that $\mathcal{N}^2 = 0$. If we consider the exact sequence

$$0 \longrightarrow Q' \longrightarrow P' = W_{\mathcal{O}}(R) \otimes_{W_{\mathbb{Z}_p}(R)} P_0 \xrightarrow{\omega} R \otimes_R P_0/Q_0 = P_0/Q_0 \longrightarrow 0,$$

where $\omega = \mathrm{w}_0' \otimes \mathrm{pr}$, we get that P'/Q' is isomorphic to P_0/Q_0 as R-modules. It is easily seen that the diagram



is commutative, where the upper two rows and the lower two rows are as in the diagram at the end of the proof of Theorem 2.3.1 for the construction of $\exp_{\mathcal{O},\mathcal{P}_0}$

and $\exp_{\mathcal{O}',\mathcal{P}'}$, respectively. Since both exp mappings are isomorphisms, we get that $\overline{\mu} : BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, \mathcal{N}) \to BT_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$ is an isomorphism. The commutativity on the morphism sets is easy. □

2.5.2 The functor $\Gamma_2(\mathcal{O}, \mathcal{O}')$

After establishing the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_1(\mathcal{O}, \mathcal{O}')$ in the nonramified case, we will now construct $\Gamma_2(\mathcal{O}, \mathcal{O}')$ in the totally ramified case. As in the non-ramified case, we took the construction of Drinfeld's functors, which helped to establish the equivalence of reduced $\mathbb{E}_{\mathcal{O},R}$-modules and formal $\mathcal{O}$-modules over an $\mathcal{O}$-algebra R, as inspiration. In this section $\mathcal{O} \to \mathcal{O}'$ is always assumed to be a totally ramified extension.

Definition 2.5.7. With $\mathcal{O} \to \mathcal{O}'$ totally ramified and R an $\mathcal{O}'$-algebra with π' nilpotent in R, we define a functor

$$\Gamma_2(\mathcal{O}, \mathcal{O}') : (\mathrm{disp}_{\mathcal{O},\mathcal{O}'} / R) \to (\mathrm{disp}_{\mathcal{O}'} / R)$$

by sending the $\mathcal{O}$-display $\mathcal{P}$ equipped with a strict $\mathcal{O}'$-action to the $\mathcal{O}'$-display $\mathcal{P}'$, which is defined by

$$\begin{aligned}
P' &= W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P, \\
Q' &= \ker(W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P \to P/Q : w \otimes x \mapsto w_0 \operatorname{pr}(x)), \\
F'(w \otimes x) &= {}^{F'}w \cdot y^{-1} \otimes F_1((\pi' - [\pi'])x), \\
F'_1({}^{V'}w \otimes x) &= y^{-1} w \otimes F_1((\pi' - [\pi'])x), \qquad (2.37) \\
F'_1(w \otimes z) &= {}^{F'}w \otimes F_1(z), \qquad (2.38)
\end{aligned}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P$ and $z \in Q$, where we have used the morphism

$$\begin{aligned}
\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) &\to W_{\mathcal{O}'}(R) \\
a \otimes w &\mapsto a u(w),
\end{aligned}$$

where $a \in \mathcal{O}'$ and $w \in W_{\mathcal{O}}(R)$, and $y \in W_{\mathcal{O}'}(R)$ is given by ${}^{V'}y = \pi' - [\pi']$. Here P is considered as an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module. The mappings of the morphisms should be the obvious ones. The functor

$$\Gamma_2(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'} / R) \to (\mathrm{ndisp}_{\mathcal{O}'} / R)$$

is defined by restriction.

We can deduce (2.37) by the equation for F', since $wF'(x) = F'_1({}^{V'}wx)$ must hold for each $w \in W_{\mathcal{O}'}(R)$ and $x \in P'$. Furthermore, we can deduce the equation for F' by (2.38), because for each $w \in W_{\mathcal{O}'}(R)$ and $x \in P$

$$\begin{array}{rcl} y^{-1}{}^{F'}w \otimes F_1((\pi' - [\pi'])x) & \overset{(2.38)}{=} & y^{-1}F_1'(w \otimes (\pi' - [\pi'])x) \\ & = & y^{-1}F_1'(w^{V'}y \otimes x) \\ & = & y^{-1}F_1'(({}^{V'}({}^{F'}wy) \otimes 1)(1 \otimes x)) \\ & = & y^{-1}(({}^{F'}wy) \otimes 1)F'(1 \otimes x) = F'(w \otimes x) \end{array}$$

must hold. One can easily verify that the functors commute with base change. It is not all obvious that this definition makes sense: We have to check that P' is a finite projective module over $W_{\mathcal{O}'}(R)$, that the map F_1' exists (it is clear that it is unique, if it exists) and is an ${}^{F'}$-linear epimorphism and that the nilpotence condition is preserved. First of all it is clear that P is finite over $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$, hence P' is finite over $W_{\mathcal{O}'}(R)$. The fact that P' is projective over $W_{\mathcal{O}'}(R)$ follows with the next Proposition. But first we need a small Lemma:

Lemma 2.5.8. Let $(S, \mathfrak{m}) \hookrightarrow (\overline{S}, \overline{\mathfrak{m}})$ be an embedding of local rings and P a finite S-module. If $\overline{P} = \overline{S} \otimes_S P$ is free over $\overline{S}$, then P is free over S.

Proof: Since $P/\mathfrak{m}P$ is free over the field $S/\mathfrak{m}$, we may take a basis $\overline{x_1}, \dots, \overline{x_d}$ of $P/\mathfrak{m}P$ and consider liftings $x_1, \dots, x_d \in P$, which lift the corresponding $\overline{x_i}$. Because

$$\overline{S}/\overline{\mathfrak{m}} \otimes_{S/\mathfrak{m}} P/\mathfrak{m}P = \overline{P}/\overline{\mathfrak{m}}\overline{P}$$

holds, the elements $1 \otimes \overline{x_i} \in \overline{S}/\overline{\mathfrak{m}} \otimes_{S/\mathfrak{m}} P/\mathfrak{m}P$ form a basis of $\overline{P}/\overline{\mathfrak{m}}\overline{P}$. If we consider now the elements $1 \otimes x_i \in \overline{S} \otimes_S P = \overline{P}$, we obtain a basis of $\overline{P}$. This can be seen as follows: First we get by the Lemma of Nakayama that

$$\begin{array}{rcl} \alpha : \overline{S}^d & \to & \overline{P} \\ \overline{e_i} & \mapsto & 1 \otimes x_i \end{array}$$

is surjective and then that it is injective, because $\overline{P}$ is free, so the kernel is finitely generated and by the Lemma of Nakayama zero. By defining

$$\begin{array}{rcl} \beta : S^d & \to & P \\ e_i & \mapsto & x_i \end{array}$$

we obtain the commutative diagram of S-modules

$$\begin{array}{ccc} S^d & \xrightarrow{\beta} & P \\ \big\downarrow & & \big\downarrow \\ \overline{S}^d & \xrightarrow{\alpha} & \overline{P}. \end{array}$$

The injectivity of β follows, since α is an isomorphism of $\overline{S}$-modules. The surjectivity follows by Nakayama again, hence β is an isomorphism and P is free. □

Proposition 2.5.9. Let R and $\mathcal{O} \to \mathcal{O}'$ be as in the previous Definition and P a finite projective $W_{\mathcal{O}}(R)$-module equipped with an $\mathcal{O}$-algebra morphism $\mathcal{O}' \to \operatorname{End}_{W_{\mathcal{O}}(R)} P$. Then P is a finite and projective $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module.

Proof: First we consider the case, where $R = k$ is a perfect field of characteristic p, which extends the residue field of $\mathcal{O}$ and $\mathcal{O}'$. Then $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$ is isomorphic to $W_{\mathcal{O}'}(k)$ by Lemma 1.3.5, hence a PID by Lemma 1.2.5. Since P is finite and torsion free over $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$, it must be free.
Now let $R = k'$ be an arbitrary field extending the residue fields of $\mathcal{O}$ and $\mathcal{O}'$. We consider the algebraic closure k of k' and the result follows with Lemma 2.5.8 if we take $S = W_{\mathcal{O}}(k') \otimes_{\mathcal{O}} \mathcal{O}'$ and $\overline{S} = W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$.
Next we assume that $(R, \mathfrak{m})$ is local with residue field k. The $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$-module $W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(R)} P$ is free, so there is a basis of the form $1 \otimes y_1, \ldots, 1 \otimes y_d$ with $y_i \in P$. We claim that the y_i form a basis of the $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-module P. Let us consider the morphism of $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-modules

$$\begin{aligned} \gamma : (W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}')^d &\to P \\ e_i &\mapsto y_i. \end{aligned}$$

Clearly the cokernel B of γ is finitely generated as an $W_{\mathcal{O}}(R)$-module and $W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(R)} B$ is zero. Since R is local, we obtain that $W_{\mathcal{O}}(R)$ is local with the maximal ideal $\overline{M} = W_{\mathcal{O}}(\mathfrak{m}) + I_{\mathcal{O},R}$. By the above we get $\overline{M}B = B$ and so $B = 0$ by Nakayama. Hence, γ is surjective. Since P is finite and projective as a $W_{\mathcal{O}}(R)$-module and $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$ is finitely generated over $W_{\mathcal{O}}(R)$, the kernel of γ is also finitely generated over $W_{\mathcal{O}}(R)$. By tensoring with $W_{\mathcal{O}}(k)$ we obtain the zero module, hence the kernel of γ is zero by Nakayama again and P is free over $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$.
Now let R be a general $\mathcal{O}'$-algebra with π' nilpotent in R. P is projective over $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$, iff $P_n := W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P$ is projective over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$ for each $n \geq 1$, where $W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/^{V^n} W_{\mathcal{O}}(R)$.
We first show that P_n is finitely presented over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$. For any collection $x_1, \ldots, x_k$ of generators of P_n over $W_{\mathcal{O},n}(R)$, the kernel of the $W_{\mathcal{O},n}(R)$-linear surjection

$$\begin{aligned} W_{\mathcal{O},n}(R)^k &\to P_n \\ e_i &\mapsto x_i \end{aligned}$$

is finitely generated. Now for a fixed choice of generators $y_1, \ldots, y_d$ of generators of P_n over $W_{\mathcal{O},n}(R)$, we consider the $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-linear surjection

$$\begin{aligned} \delta : (W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}')^d &\to P_n \\ e_i &\mapsto y_i. \end{aligned}$$

Clearly, the $y_i \pi'^j$ also form a generating system over $W_{\mathcal{O},n}(R)$, hence we obtain by the above that the $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-module $\ker \delta$ is finitely generated over

$W_{\mathcal{O},n}(R)$, hence also over $W_{\mathcal{O},n}(R)\otimes_{\mathcal{O}}\mathcal{O}'$, which establishes the fact that P_n is finitely presented over $W_{\mathcal{O},n}(R)\otimes_{\mathcal{O}}\mathcal{O}'$.*
It suffices to show that for each maximal ideal of $W_{\mathcal{O},n}(R)\otimes_{\mathcal{O}}\mathcal{O}'$ the localization of P_n at this ideal is free over the localized ring. It is not too hard to verify that the maximal ideals of $A := W_{\mathcal{O},n}(R)\otimes_{\mathcal{O}}\mathcal{O}'$ are of the form

$$\overline{M} = \pi'^0(W_{\mathcal{O},n}(\mathfrak{m}) + I_{\mathcal{O},n,R}) + \pi' W_{\mathcal{O},n}(R) + \ldots + \pi'^{e-1}W_{\mathcal{O},n}(R),$$

where $\mathfrak{m}$ runs through the maximal ideals of R and $I_{\mathcal{O},n,R}\subseteq W_{\mathcal{O},n}(R)$ has its intuitive meaning. We claim that

$$A_{\overline{M}} \;\simeq\; W_{\mathcal{O},n}(R_{\mathfrak{m}})\otimes_{\mathcal{O}}\mathcal{O}' \tag{2.39}$$

holds. First one sees that every element of the image of $A\backslash\overline{M}$ via the obvious morphism $W_{\mathcal{O},n}(R)\otimes_{\mathcal{O}}\mathcal{O}'\to W_{\mathcal{O},n}(R_{\mathfrak{m}})\otimes_{\mathcal{O}}\mathcal{O}'$ is a unit. Now let B be any A-algebra such that $A\backslash\overline{M}\subset B^{\times}$. By considering $W_{\mathcal{O},n}(R)$ as a subring of A in the canonical way we get that there is a unique morphism of $W_{\mathcal{O},n}(R)$-algebras $g: W_{\mathcal{O},n}(R_{\mathfrak{m}})\to B$, since $W_{\mathcal{O},n}(R_{\mathfrak{m}})$ is the localization of $W_{\mathcal{O},n}(R)$ at $W_{\mathcal{O},n}(\mathfrak{m}) + I_{\mathcal{O},n,R}$. By considering the value z of $\pi'\in A$ in B we get a unique morphism of A-algebras $W_{\mathcal{O},n}(R_{\mathfrak{m}})\otimes_{\mathcal{O}}\mathcal{O}'\to B$ given by g and $\pi'\mapsto z$. By the universal property of localizations the isomorphism (2.39) is established. Since $(W_{\mathcal{O},n}(R_{\mathfrak{m}})\otimes_{\mathcal{O}}\mathcal{O}')\otimes_A P_n$ is clearly free over $W_{\mathcal{O},n}(R_{\mathfrak{m}})\otimes_{\mathcal{O}}\mathcal{O}'$ by the assertion for local rings, the general assertion follows. □

Our next aim is to show that F_1' exists. Let $L\oplus T = P$ be a normal decomposition of $\mathcal{P}$. We define M_0 for each $W_{\mathcal{O}}(R)$-module M by $R\otimes_{\mathrm{w}_0,W_{\mathcal{O}}(R)} M$. Let us now consider the exact sequence of $\mathcal{O}'\otimes_{\mathcal{O}} R$-modules

$$0\to L_0\to P_0\to T_0\to 0,$$

where the $\mathcal{O}'$-action on L_0 is induced by the action of $\mathcal{O}'$ on Q and the $\mathcal{O}'$-action on T_0 is given as the action on the cokernel of the map $L_0\to P_0$. By tensoring this sequence with $R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} -$ we get the canonical morphism of R-modules $R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} P_0 \to R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} T_0$. If we consider the canonical $\mathcal{O}'\otimes_{\mathcal{O}} R$-linear morphism $T_0\to R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} P_0$, we obtain the commutative diagram

$$\begin{array}{ccc} T_0 & \longrightarrow & R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} P_0 \\ & \diagdown\!\!\diagdown & \downarrow \\ & & R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} T_0, \end{array}$$

where the equality follows from the strictness of the $\mathcal{O}'$-action on the tangent space. Hence, we get the injectivity of $T_0\to R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} P_0$ and obtain for the exact sequence of R-modules

$$0\to\Delta\to P_0 = L_0\oplus T_0\to R\otimes_{\mathcal{O}'\otimes_{\mathcal{O}}R} P_0\to 0$$

*The same method can be applied to show that P is finitely presented over $W_{\mathcal{O}}(R)\otimes_{\mathcal{O}}\mathcal{O}'$.

that $\Delta \subseteq L_0$. Since the sequence splits (because $R \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} R} P_0$ is projective over R by the previous Proposition), we obtain an R-module decomposition of L_0 into $\Delta \oplus L_{\Delta,0}$, such that $L_{\Delta,0} \oplus T_0 \simeq R \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} R} P_0$ in P_0. Now we can lift $\Delta \oplus L_{\Delta,0} = L_0$ and T_0 to the projective $W_{\mathcal{O}}(R)$-modules $\Delta^\star \oplus L_\Delta = L$ and T. The obvious morphism of $W_{\mathcal{O}'}(R)$-modules

$$W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} L_\Delta \oplus W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} T \to W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P$$

is an isomorphism, which can be seen be reducing from $W_{\mathcal{O}'}(R)$ to R and utilizing the construction above. Hence

$$Q' = W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} L_\Delta \oplus I_{\mathcal{O}',R} \otimes_{W_{\mathcal{O}}(R)} T$$

holds and we define the map

$$F'_{10} : Q' \quad \to \quad W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P \tag{2.40}$$

by

$$\begin{aligned} F'_{10}(w \otimes l_\Delta) &= {}^{F'}w \otimes F_1(l_\Delta) \\ F'_{10}({}^{V'}w \otimes t) &= wy^{-1} \otimes F_1((\pi' - [\pi'])t) \end{aligned}$$

for $w \in W_{\mathcal{O}'}(R)$, $l_\Delta \in L_\Delta$ and $t \in T$.
We have to check that

$$F'_{10}(1 \otimes \delta^\star) \quad = \quad 1 \otimes F_1(\delta^\star) \tag{2.41}$$

holds for each $\delta^\star \in \Delta^\star$ and with this we are going to establish that the relation (2.38) defining F'_1 holds for our construction of F'_{10}. Hence, the existence of F'_1 would follow.
If we declare on R the $\mathcal{O}' \otimes_{\mathcal{O}} R$-module structure by the product mapping ε, then

$$0 \to \ker(\varepsilon) \to \mathcal{O}' \otimes_{\mathcal{O}} R \xrightarrow{\varepsilon} R \to 0$$

is an exact sequence of $\mathcal{O}' \otimes_{\mathcal{O}} R$-modules. By tensoring this sequence with the projective $\mathcal{O}' \otimes_{\mathcal{O}} R$-module P_0, we obtain the exact sequence

$$0 \to \ker(\varepsilon)P_0 = \Delta \to P_0 \to R \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} R} P_0 \to 0.$$

Now let us have a look at

$$\ker(\varepsilon) = \{ \sum_{i=0}^{e-1} r_i z^i \mid \sum_{i=0}^{e-1} r_i \pi'^i = 0 \text{ in } R \},$$

where we have considered $\mathcal{O}' \otimes_{\mathcal{O}} R$ as $R[z]/P(z)$ and P is the Eisenstein polynomial of degree e, where e is the ramification index of the extension $\mathcal{O} \to \mathcal{O}'$, for which $P(\pi') = 0$.

Lemma 2.5.10. $\ker\varepsilon \subseteq \mathcal{O}' \otimes_{\mathcal{O}} R = R[z]/P(z)$ is generated as an ideal by $z\otimes 1 - 1\otimes\pi'$. This element is nilpotent, hence $\ker(\varepsilon)$ is contained in the nilradical.

Proof: The elements

$$1\otimes 1, z\otimes 1, \dots, z^{e-1}\otimes 1$$

form a basis of $\mathcal{O}'\otimes_{\mathcal{O}} R$ considered as a free R-module. From this we obtain a new R-module basis

$$1\otimes 1, z\otimes 1 - 1\otimes\pi', \dots, z^{e-1}\otimes 1 - 1\otimes \pi'^{e-1}.$$

The elements $z\otimes 1 - 1\otimes\pi', \dots, z^{e-1}\otimes 1 - 1\otimes\pi'^{e-1}$ are all elements of $\ker\varepsilon$, so we obtain an R-module surjection

$$\mathcal{O}'\otimes_{\mathcal{O}} R/(z\otimes 1 - 1\otimes\pi', \dots, z^{e-1}\otimes 1 - 1\otimes\pi'^{e-1}) \to R$$

given by the product morphism. Since the left hand side of this morphism is isomorphic to R, it is in fact an isomorphism. It is obvious that the elements $z^i\otimes 1 - 1\otimes\pi'^i$ are for each $i\geq 1$ elements of the ideal generated by $z\otimes 1 - 1\otimes\pi'$. Since the $z\otimes 1$ and $1\otimes\pi'$ are both nilpotent, $z\otimes 1 - 1\otimes\pi'$ is nilpotent as well. □

Now let $\delta_1,\dots,\delta_{n_1}$ and $l_{\Delta,1},\dots,l_{\Delta,n_2}$ and $t_1,\dots,t_{n_3}$ be generating systems for the R-modules $\Delta, L_{\Delta,0}$ and T_0, respectively. Since each element in Δ may be represented by a finite sum of elements $c_i p_i$ with $c_i\in\ker(\varepsilon)$ and $p_i\in P_0$, we get the following system of equations

$$\begin{aligned} \delta_1 &= \sum_{i=1}^{n_1} b_{i1}^{(0)}\delta_i + \sum_{i=1}^{n_2} c_{i1}^{(0)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i1}^{(0)} t_i \\ &\vdots \\ \delta_{n_1} &= \sum_{i=1}^{n_1} b_{i,n_1}^{(0)}\delta_i + \sum_{i=1}^{n_2} c_{i,n_1}^{(0)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,n_1}^{(0)} t_i, \end{aligned}$$

where the $b_{ij}^{(0)}, c_{ij}^{(0)}$ and $d_{ij}^{(0)}$ are all in $\ker(\varepsilon)$. Now we subtract from both sides of the first equation $b_{1,1}^{(0)}\delta_1$ and obtain

$$\kappa_1\delta_1 = \sum_{i=2}^{n_1} b_{i,1}^{(0)}\delta_i + \sum_{i=1}^{n_2} c_{i,1}^{(0)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,1}^{(0)} t_i,$$

where $\kappa_1 = 1 - b_{1,1}^{(0)}$ is a unit in $\mathcal{O}'\otimes_{\mathcal{O}} R$, since $b_{1,1}^{(0)}$ is contained in the Jacobson radical by the previous Lemma. After multiplying with κ_1^{-1} we obtain an equation

$$\delta_1 = \sum_{i=2}^{n_1} b_{i,1}^{(1)}\delta_i + \sum_{i=1}^{n_2} c_{i,1}^{(1)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,1}^{(1)} t_i,$$

with $b_{i,1}^{(1)}, c_{i,1}^{(1)}$ and $d_{i,1}^{(1)}$ in $\ker(\varepsilon)$. Inserting this for δ_1 in the other $n_1 - 1$ equations, we obtain the system of equations

$$\begin{aligned}\delta_1 &= \sum_{i=2}^{n_1} b_{i1}^{(1)}\delta_i + \sum_{i=1}^{n_2} c_{i1}^{(1)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i1}^{(1)} t_i \\ &\vdots \\ \delta_{n_1} &= \sum_{i=2}^{n_1} b_{i,n_1}^{(1)}\delta_i + \sum_{i=1}^{n_2} c_{i,n_1}^{(1)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,n_1}^{(1)} t_i\end{aligned}$$

with $b_{i,j}^{(1)}, c_{i,j}^{(1)}$ and $d_{i,j}^{(1)}$ in $\ker(\varepsilon)$. Hence, we could express every δ_i without using δ_1. Now let us consider

$$\delta_2 = \sum_{i=2}^{n_1} b_{i2}^{(1)}\delta_i + \sum_{i=1}^{n_2} c_{i2}^{(1)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i2}^{(1)} t_i.$$

By subtracting $b_{22}^{(1)}\delta_2$ from both sides, we receive

$$\kappa_2\delta_2 = \sum_{i=3}^{n_1} b_{i2}^{(1)}\delta_i + \sum_{i=1}^{n_2} c_{i2}^{(1)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i2}^{(1)} t_i,$$

where $\kappa_2 = 1 - b_{2,2}^{(1)}$ is a unit in $\mathcal{O}' \otimes_{\mathcal{O}} R$. After multiplying with κ_2^{-1} we obtain an equation

$$\delta_2 = \sum_{i=3}^{n_1} b_{i,1}^{(2)}\delta_i + \sum_{i=1}^{n_2} c_{i,1}^{(2)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,1}^{(2)} t_i,$$

with $b_{i,1}^{(2)}, c_{i,1}^{(2)}$ and $d_{i,1}^{(2)}$ in $\ker(\varepsilon)$. Inserting this for δ_2 in the other $n_1 - 1$ equations, we obtain the system of equations

$$\begin{aligned}\delta_1 &= \sum_{i=3}^{n_1} b_{i1}^{(2)}\delta_i + \sum_{i=1}^{n_2} c_{i1}^{(2)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i1}^{(2)} t_i \\ &\vdots \\ \delta_{n_1} &= \sum_{i=3}^{n_1} b_{i,n_1}^{(2)}\delta_i + \sum_{i=1}^{n_2} c_{i,n_1}^{(2)} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,n_1}^{(2)} t_i\end{aligned}$$

with $b_{i,j}^{(2)}, c_{i,j}^{(2)}$ and $d_{i,j}^{(2)}$ in $\ker(\varepsilon)$. Hence, we could express every δ_i without using δ_1, δ_2. We could repeat this for δ_3, δ_4, etc. until reaching the equation for δ_{n_1} and in the end we obtain a system of equations

$$\begin{aligned}\delta_1 &= \sum_{i=1}^{n_2} c_{i1} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i1} t_i \\ &\vdots \\ \delta_{n_1} &= \sum_{i=1}^{n_2} c_{i,n_1} l_{\Delta,i} + \sum_{i=1}^{n_3} d_{i,n_1} t_i\end{aligned}$$

with $c_{i,j}$ and $d_{i,j}$ in $\ker(\varepsilon)$, where no δ_i is needed in order to express a δ_j. Let $c_{i,j}$ be of the form $\sum_{k=0}^{e-1} r_{ijk}z^k$ and $d_{i,j}$ be of the form $\sum_{k=0}^{e-1} s_{ijk}z^k$ with r_{ijk} and $s_{ijk} \in R$. Then we define $c_{ij}^\star := \sum_{k=0}^{e-1}[r_{ijk}]\pi'^k$ and $d_{ij}^\star := \sum_{k=0}^{e-1}[s_{ijk}]\pi'^k$ in $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$. Let $l_{\Delta,i}^\star \in L_\Delta$ be liftings of $l_{\Delta,i}$ and $t_i^\star \in T$ be liftings of t_i and consider for each j the element

$$\omega_j = \sum_{i=1}^{n_2} c_{i,j}^\star l_{\Delta,i}^\star + \sum_{i=1}^{n_3} d_{i,j}^\star t_i^\star$$

of P. When projected to $\Delta^\star$ we get that the collection of all these projected elements must generate $\Delta^\star$ over $W_{\mathcal{O}}(R)$, because the reductions generate Δ and we can apply the Lemma of Nakayama then. Let $l_j \in L_\Delta$ be the projection of ω_j to L_Δ and $\sum_{i=1}^{n_3} {}^V w_{ij} t_i^\star$ with $w_{ij} \in W_{\mathcal{O}}(R)$ be the projection of ω_j to T, where have used the strictness of the $\mathcal{O}'$-action in order to get that the zeroth entries in the scalar factors can be chosen to be zero. If we define now

$$\begin{aligned}\delta_j^\star &= \omega_j - l_j - \sum_{i=1}^{n_3} {}^V w_{ij} t_i^\star \\ &= \sum_{i=1}^{n_2} c_{i,j}^\star l_{\Delta,i}^\star + \sum_{i=1}^{n_3} d_{i,j}^\star t_i^\star - l_j - \sum_{i=1}^{n_3} {}^V w_{ij} t_i^\star\end{aligned}$$

we get for each j an element of $\Delta^\star$, whose collection generates this module over $W_{\mathcal{O}}(R)$. In order to show that (2.41) holds for each $\delta^\star \in \Delta^\star$, it suffices to verify this equation for each $\delta_j^\star$. Let us consider $1 \otimes \delta_j^\star \in W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P$. After applying (the above constructed map) F'_{10} to this element, we obtain with the above description

$$\begin{aligned}F'_{10}(1 \otimes \delta_j^\star) &= \sum_{i=1}^{n_2}(\sum_{k=0}^{e-1}[r_{ijk}^q]\pi'^k) \otimes F_1(l_{\Delta,i}^\star) + \sum_{i=1}^{n_3} y^{-1}m_{ij} \otimes F_1((\pi' - [\pi'])t_i^\star) \\ &- 1 \otimes F_1(l_j) - \sum_{i=1}^{n_3} y^{-1}(\pi/\pi')u(w_{ij}) \otimes F_1((\pi' - [\pi'])t_i^\star),\end{aligned}$$

where m_{ij} is given by ${}^{V'}m_{ij} = \sum_{k=0}^{e-1}[s_{ijk}]\pi'^k \in W_{\mathcal{O}'}(R)$. Furthermore, we obtain

$$\begin{aligned}1 \otimes F_1(\delta_j^\star) &= \sum_{i=1}^{n_2}(\sum_{k=0}^{e-1}[r_{ijk}^q]\pi'^k) \otimes F_1(l_{\Delta,i}^\star) + \sum_{i=1}^{n_3} 1 \otimes F_1((\sum_{k=0}^{e-1}[s_{ijk}]\pi'^k)t_i^\star) \\ &- 1 \otimes F_1(l_j) - \sum_{i=1}^{n_3} u(w_{ij}) \otimes F(t_i^\star),\end{aligned}$$

so to verify (2.41) it suffices to confirm

$$y^{-1}m_{ij} \otimes F_1((\pi' - [\pi'])t_i^\star) = 1 \otimes F_1((\sum_{k=0}^{e-1}[s_{ijk}]\pi'^k)t_i^\star) \qquad (2.42)$$

$$y^{-1}(\pi/\pi') \otimes F_1((\pi' - [\pi'])t_i^\star) = 1 \otimes F(t_i^\star) \qquad (2.43)$$

for each i and j. We make the following definitions for elements in $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ for $k \geq 0$:

$$\begin{aligned}
X_k &= \pi'^k + [\pi'^q]\pi'^{k-1} + \ldots + [\pi'^{(k-1)q}]\pi' + [\pi'^{kq}] \\
X_k^\star &= \pi'^k + [\pi']\pi'^{k-1} + \ldots + [\pi'^{k-1}]\pi' + [\pi'^k] \\
X &= \sum_{k=0}^{e-1} a_{k+1} X_k \\
X^\star &= \sum_{k=0}^{e-1} a_{k+1} X_k^\star
\end{aligned}$$

Here $\pi = \sum_{k=1}^{e} a_k \pi'^k$ is as in (2.28). We need for the outcome the equation in $W_{\mathcal{O}'}(R)$

$$u(\alpha)\pi/\pi' = yX \tag{2.44}$$

in $W_{\mathcal{O}'}(R)$, where the unit $\alpha \in W_{\mathcal{O}}(R)$ is defined as in (2.31) by ${}^V\alpha = \pi - \sum_{k=1}^{e} a_k [\pi'^k]$. The equation can be checked by considering the equation in $W_{\mathcal{O}'}(\mathcal{O}')$ and then by evaluating the Witt polynomials and applying Lemma 1.1.3.

Now we turn to solving equation (2.43):

$$\begin{aligned}
y^{-1}(\pi/\pi') \otimes F_1((\pi' - [\pi'])t_i^\star) &= u(\alpha)^{-1} X \otimes F_1((\pi' - [\pi'])t_i^\star) \\
&= u(\alpha)^{-1} \otimes X F_1((\pi' - [\pi'])t_i^\star) \\
&= u(\alpha)^{-1} \otimes F_1(X^\star(\pi' - [\pi'])t_i^\star) \\
&= u(\alpha)^{-1} \otimes F_1({}^V\alpha t_i^\star) \\
&= 1 \otimes F(t_i^\star)
\end{aligned}$$

Here we have used (2.44) and $X^\star(\pi' - [\pi']) = \pi - \sum_{k=1}^{e} a_k[\pi'^k] =^V \alpha$.

Next we turn our focus to (2.42). At first we will reorganize the right hand side of (2.42) by

$$\begin{aligned}
1 \otimes F_1((\sum_{k=0}^{e-1} [s_{ijk}]\pi'^k)t_i^\star) &= 1 \otimes F_1((\sum_{k=1}^{e-1} [s_{ijk}](\pi'^k - [\pi'^k]))t_i^\star) \\
&+ 1 \otimes F_1((\sum_{k=0}^{e-1} [s_{ijk}][\pi'^k])t_i^\star) \\
&= \sum_{k=1}^{e-1} [s_{ijk}^q] X_{k-1} \otimes F_1((\pi' - [\pi'])t_i^\star) + 1 \otimes F_1({}^V\kappa t_i^\star) \\
&= \sum_{k=1}^{e-1} [s_{ijk}^q] X_{k-1} \otimes F_1((\pi' - [\pi'])t_i^\star) + u(\kappa) \otimes F(t_i^\star),
\end{aligned}$$

where $\kappa \in W_{\mathcal{O}}(R)$ is defined by ${}^V\kappa = \sum_{k=0}^{e-1}[s_{ijk}][\pi'^k]$. After subtracting $\sum_{k=1}^{e-1}[s_{ijk}^q]X_{k-1} \otimes F_1((\pi' - [\pi'])t_i^\star)$ from both sides of (2.42), we obtain that we have to show

$$(y^{-1}m_{ij} - \sum_{k=1}^{e-1}[s_{ijk}^q]X_{k-1}) \otimes F_1((\pi' - [\pi'])t_i^\star) \quad = \quad u(\kappa) \otimes F(t_i^\star). \quad (2.45)$$

If we assume now that

$$m_{ij} - y\sum_{k=1}^{e-1}[s_{ijk}^q]X_{k-1} \quad = \quad u(\kappa)\pi/\pi' \quad (2.46)$$

holds in $W_{\mathcal{O}'}(R)$, we get (2.45) with the same method we used to show (2.43), so we need only to confirm (2.46). For this purpose we first pass to the universal situation $R_0 = \mathcal{O}'[Y_1, \ldots, Y_{e-1}]$ and have the map $R_0 \to R$, which sends Y_k to s_{ijk} and hence $-\sum_{i=1}^{e-1} Y_i\pi'^i$ to s_{ij0}, in mind. If we define now, with abuse of notation, m_{ij},κ and X, X_j etc. in the same manner in $W_{\mathcal{O}'}(R_0)$, resp. $W_{\mathcal{O}}(R_0)$, resp. $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R_0)$, than we did in the original case in $W_{\mathcal{O}'}(R)$, resp. $W_{\mathcal{O}}(R)$, resp. $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ (i.e., we replace s_{ijk} by Y_k for $k \neq 0$ and s_{ij0} by $Y_0 := -\sum_{i=1}^{e-1} Y_i\pi'^i$), it suffices for the verification of (2.46) in $W_{\mathcal{O}'}(R)$, to verify the equation in $W_{\mathcal{O}'}(R_0)$, which in turn can be done by Lemma 1.1.3 only by considering the values of the Witt polynomials of both sides. Let us define $Z \in R_0$ by $-\sum_{i=1}^{e-1} Y_i\pi'^{i-1}$. We get by

$$\begin{aligned} \mathrm{w}'_n({}^{V'}m_{ij}) &= \sum_{k=0}^{e-1} Y_k^{q^n}\pi'^k \\ &= \pi'\mathrm{w}'_{n-1}(m_{ij}) \end{aligned}$$

that

$$\mathrm{w}'_n(m_{ij}) \quad = \quad \sum_{k=1}^{e-1} Y_k{}^{q^{n+1}}\pi'^{k-1} + \pi'^{q^{n+1}-1}Z^{q^{n+1}}$$

holds. Furthermore

$$\begin{aligned} \mathrm{w}'_n(y\sum_{k=1}^{e-1}[Y_k^q]X_{k-1}) &= \sum_{k=1}^{e-1} Y_k^{q^{n+1}}(1 - \pi'^{q^{n+1}-1})(\sum_{\nu=0}^{k-1}\pi'^{k-1}\pi'^{\nu(q^{n+1}-1)}) \\ &= \sum_{k=1}^{e-1} Y_k^{q^{n+1}}(\pi'^{k-1} - \pi'^{kq^{n+1}-1}) \end{aligned}$$

holds, which yields

$$\mathrm{w}'_n(m_{ij} - y\sum_{k=1}^{e-1}[Y_k^q]X_{k-1}) \quad = \quad \pi'^{q^{n+1}-1}Z^{q^{n+1}} + \sum_{k=1}^{e-1} Y_k^{q^{n+1}}\pi'^{kq^{n+1}-1}.$$

Because of

$$\begin{aligned} \mathrm{w}_n({}^V\kappa) &= \sum_{k=0}^{e-1} Y_k^{q^n} \pi'^{kq^n} \\ &= \pi \mathrm{w}_{n-1}(\kappa) \\ &= \pi \mathrm{w}'_{n-1}(u(\kappa)) \end{aligned}$$

we get

$$(\pi/\pi')\mathrm{w}'_n(u(\kappa)) = \pi'^{q^{n+1}-1} Z^{q^{n+1}} + \sum_{k=1}^{e-1} Y_k^{q^{n+1}} \pi'^{kq^{n+1}-1}$$

and we obtain that (2.46) holds, which in turn verifies (2.43) and this finally confirms (2.41).
Now we are able to show that (2.38) holds for F'_{10}. For this purpose let $z = l_\Delta^\star + \delta^\star + \sum_{i=0}^{n_3} {}^V w_i t_i^\star$ be an arbitrary element of Q, where $l_\Delta^\star \in L_\Delta$, $\delta^\star \in \Delta^\star$, the $t_i^\star \in T$ form a generating system of T over $W_{\mathcal{O}}(R)$ and w_i are elements of $W_{\mathcal{O}}(R)$. Then we obtain for $w \in W_{\mathcal{O}'}(R)$ with (2.41) and (2.44)

$$\begin{aligned} F'_{10}(w \otimes z) &= F'_{10}(w \otimes l_\Delta^\star) + F'_{10}(w \otimes \delta^\star) + \sum_{i=1}^{n_3} F'_{10}({}^{V'}(({}^{F'}w)(\pi/\pi')u(w_i)) \otimes t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} y^{-1}({}^{F'}w)(\pi/\pi')u(w_i) \otimes F_1((\pi' - [\pi'])t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} u(\alpha)^{-1}({}^{F'}w)Xu(w_i) \otimes F_1((\pi' - [\pi'])t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} u(\alpha)^{-1}({}^{F'}w)u(w_i) \otimes F_1(X^\star(\pi' - [\pi'])t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} u(\alpha)^{-1}({}^{F'}w)u(w_i) \otimes F_1({}^V\alpha t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} ({}^{F'}w)u(w_i) \otimes F(t_i^\star) \\ &= {}^{F'}w \otimes F_1(l_\Delta^\star + \delta^\star) + \sum_{i=1}^{n_3} {}^{F'}w \otimes F_1({}^V w_i t_i^\star) \\ &= {}^{F'}w \otimes F_1(z), \end{aligned}$$

so (2.38) continues to hold. Hence, the existence of F'_1 follows, since it is the map F'_{10}. With the above results it is an easy exercise to show that F'_1 is an ${}^{F'}$-linear epimorphism.
Now we should also have a look on the fact that $\Gamma_2(\mathcal{O}, \mathcal{O}')$ preserves the nilpotence

condition. Let $\mathcal{P}$ be a nilpotent $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action. It can be easily verified that for each $n \geq 0$ the diagram

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow{\scriptstyle V^{n\sharp}} & & \downarrow{\scriptstyle V'^{n\sharp}} \\ W_{\mathcal{O}}(R) \otimes_{F^n, W_{\mathcal{O}}(R)} P & \longrightarrow & W_{\mathcal{O}'}(R) \otimes_{F'^n, W_{\mathcal{O}'}(R)} P' \\ \downarrow & & \downarrow \\ R/\pi R \otimes_{\mathrm{pr}\circ \mathrm{w}_0, W_{\mathcal{O}}(R)} W_{\mathcal{O}}(R) \otimes_{F^n, W_{\mathcal{O}}(R)} P & \twoheadrightarrow & R/\pi' R \otimes_{\mathrm{pr}\circ \mathrm{w}'_0, W_{\mathcal{O}'}(R)} W_{\mathcal{O}'}(R) \otimes_{F'^n, W_{\mathcal{O}'}(R)} P' \end{array}$$

is commutative. We have to show that the composite map μ_n of the two vertical arrows on the right half is zero for some n. But if the composite map of the two vertical arrows on the left half is zero for some n_0, then $\mu_{n_0}(1 \otimes x) = 0$ for each $x \in P$. Since the elements $1 \otimes x$ generate the $W_{\mathcal{O}'}(R)$-module P', we get that $\mu_{n_0} = 0$ holds.

Proposition 2.5.11. Let $\mathcal{O} \to \mathcal{O}'$ be totally ramified and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{BT_{\mathcal{O}}} & (\text{ formal } \mathcal{O}' - \text{modules}/R) \\ \downarrow{\scriptstyle \Gamma_2(\mathcal{O},\mathcal{O}')} & \nearrow{\scriptstyle BT_{\mathcal{O}'}} & \\ (\mathrm{disp}_{\mathcal{O}'}/R) & & \end{array}$$

Also the restriction of the above diagram

$$\begin{array}{ccc} (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{BT_{\mathcal{O}}} & (\pi'\text{-divisible formal } \mathcal{O}' - \text{modules}/R) \\ \downarrow{\scriptstyle \Gamma_2(\mathcal{O},\mathcal{O}')} & \nearrow{\scriptstyle BT_{\mathcal{O}'}} & \\ (\mathrm{ndisp}_{\mathcal{O}'}/R) & & \end{array}$$

is commutative.

Proof: Let $\mathcal{P} = (P, Q, F, F_1)$ be a (nilpotent) $\mathcal{O}$-display over R with a strict $\mathcal{O}'$-action and $\mathcal{P}' = (P', Q', F', F_1')$ be its image via $\Gamma_2(\mathcal{O},\mathcal{O}')$. We need to show that $BT_{\mathcal{O}}(\mathcal{P}, -)$ and $BT_{\mathcal{O}'}(\mathcal{P}', -)$ are isomorphic in the category of (π'-divisible) formal $\mathcal{O}'$-modules over R. For a nilpotent R-algebra $\mathcal{N}$ we have $\widehat{\mathcal{P}'}_{\mathcal{N}} \simeq \widehat{W_{\mathcal{O}'}}(\mathcal{N}) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P$ and we may define

$$\mu = u_{\mathcal{N}} \otimes \mathrm{id} : \widehat{P}_{\mathcal{N}} = \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \widehat{\mathcal{P}'}_{\mathcal{N}}.$$

We have to show that

$$\begin{array}{ccc} \widehat{Q}_{\mathcal{N}} & \xrightarrow{F_1 - \mathrm{id}} & \widehat{P}_{\mathcal{N}} \\ \downarrow{\scriptstyle \mu|_{\widehat{Q}_{\mathcal{N}}}} & & \downarrow{\scriptstyle \mu} \\ \widehat{Q'}_{\mathcal{N}} & \xrightarrow{F_1' - \mathrm{id}} & \widehat{P'}_{\mathcal{N}} \end{array}$$

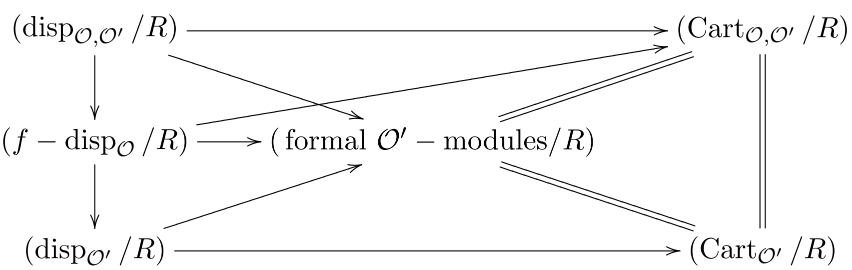
is commutative, which would induce an $\mathcal{O}'$-module morphism $\overline{\mu} : BT_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \to BT_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$. This follows easily for $w \otimes l \in \widehat{Q}_{\mathcal{N}}$ with $w \in \widehat{W_{\mathcal{O}}}(\mathcal{N})$ and $l \in L$. For ${}^V w \otimes t$ with $w \in \widehat{W_{\mathcal{O}}}(\mathcal{N})$ and $t \in T$ we have to utilize that $y^{-1}(\pi/\pi')u(w) \otimes F_1((\pi' - [\pi'])t) = u(w) \otimes F(t)$ holds in $\widehat{P'}_{\mathcal{N}}$ by (2.43). To show that $\overline{\mu}$ is an isomorphism of $\mathcal{O}'$-modules, we can reduce to $\mathcal{N}^2 = 0$ and proceed in a similar manner as showing that $BT_{\mathcal{O}}(\mathcal{P}_0, \mathcal{N})$ and $BT_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$ are isomorphic in Proposition 2.5.6. The commutativity on the morphism sets is left to the reader. □

2.5.3 Concluding remarks

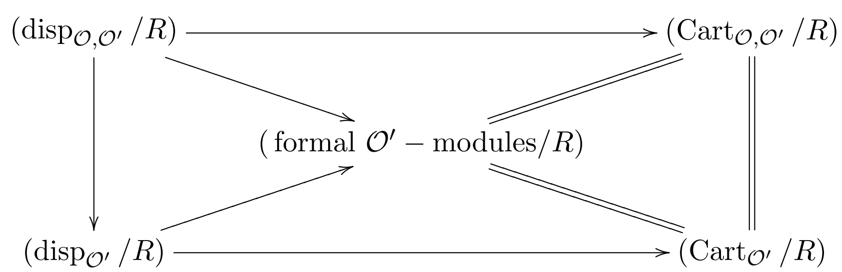
Definition 2.5.12. Let $\mathcal{O} \to \mathcal{O}'$ be a (not necessarily nonramified / totally ramified) extension of rings of integers of non-Archimedean local fields of characteristic zero and R an $\mathcal{O}'$-algebra. We denote by $(\mathrm{Cart}_{\mathcal{O}'}/R)$ the category of reduced $\mathbb{E}_{\mathcal{O}',R}$-modules and by $(\mathrm{Cart}_{\mathcal{O},\mathcal{O}'}/R)$ the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules equipped with a strict $\mathcal{O}'$-action.

After considering the constructions made by Drinfeld concerning Cartier modules, Proposition 2.4.7 and the construction of the $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_i(\mathcal{O}, \mathcal{O}')$, the assertions of the following Proposition is obvious:

Proposition 2.5.13. Let $\mathcal{O} \to \mathcal{O}'$ be nonramified of degree f and R a π'-adic $\mathcal{O}'$-algebra. Then the diagram



is commutative, where the arrows follow by the previous constructions. Let $\mathcal{O} \to \mathcal{O}'$ be totally ramified and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then the diagram



is commutative, where the arrows follow by the previous constructions again.

We need to make some conventions, which should hold troughout the rest of this thesis.

Conventions 2.5.14. Whenever we talk about $\Omega_1(\mathcal{O},\mathcal{O}')$, $\Omega_2(\mathcal{O},\mathcal{O}')$ or $\Gamma_1(\mathcal{O},\mathcal{O}')$, we always assume $\mathcal{O}'$ to be nonramified over $\mathcal{O}$ of degree f and whenever we talk about $\Gamma_2(\mathcal{O},\mathcal{O}')$, we always assume $\mathcal{O}'$ to be totally ramified over $\mathcal{O}$.
When we claim assertions like

> For any $\mathcal{O}'$-algebra R with nilpotent nilradical and π' nilpotent in R the functors $\Gamma_1(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ are equivalences of categories.

we actually mean

> Let $\mathcal{O} \to \mathcal{O}'$ be nonramified. Then for any $\mathcal{O}'$-algebra R with nilpotent nilradical and π' nilpotent in R the functor $\Gamma_1(\mathcal{O},\mathcal{O}')$ is an equivalence of categories.
> Let $\mathcal{O} \to \mathcal{O}'$ be totally ramified. Then for any $\mathcal{O}'$-algebra R with nilpotent nilradical and π' nilpotent in R the functor $\Gamma_2(\mathcal{O},\mathcal{O}')$ is an equivalence of categories.

Unless otherwise stated, when we talk about $BT_{\mathcal{O}}^{(f)}, BT_{\mathcal{O}}(= BT_{\mathcal{O}}^{(1)}), \Gamma_i(\mathcal{O},\mathcal{O}')$, $\Omega_i(\mathcal{O},\mathcal{O}')$ we always consider the functors restricted to nilpotent display structures.

Definition 2.5.15. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified extension of degree f. For an $\mathcal{O}'$-algebra R with π' nilpotent in R we define the boolean variable $P(\mathcal{O},\mathcal{O}',R)$ to be true, iff the following assertion is true:

> The $BT_{\mathcal{O}}^{(f)}$ functor is an equivalence between nilpotent f-$\mathcal{O}$-displays over R and π'-divisible formal $\mathcal{O}'$-modules over R.

In case $\mathcal{O}' = \mathcal{O}$, we just write $P(\mathcal{O},R)$ instead of $P(\mathcal{O},\mathcal{O}',R)$.

Theorem 2.5.16. (cf. [Lau08, Theorem 1.1.]) The functor $BT_{\mathbb{Z}_p}$ is an equivalence of categories for all rings with p nilpotent in it.

Hence, the previous Theorem, the main Theorem of (classical) display theory, says that $P(\mathbb{Z}_p,R)$ is true for each ring R with p nilpotent in R. This is particularly important, since we need a starting point in order to argue in the analogous way as Drinfeld did.
The following Lemma only presents very basic facts, all of which are obvious but need to be noted, so we will not prove them.

Lemma 2.5.17. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then:

- Let $\{ i,j \} = \{ 1,2 \}$. If $\Gamma_1(\mathcal{O},\mathcal{O}')$ and $\Omega_i(\mathcal{O},\mathcal{O}')$ are equivalences of categories, then the same is true for $\Omega_j(\mathcal{O},\mathcal{O}')$

Now we assume that $P(\mathcal{O},R)$ is true. Then the following assertions are true:

- $\Gamma_1(\mathcal{O},\mathcal{O}')$ and $\Omega_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ are faithful.
- $BT_{\mathcal{O}}^{(f)}$ resp. $BT_{\mathcal{O}'}$ is essentially surjective.
- If $BT_{\mathcal{O}'}$ is faithful, then $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ is fully faithful.
- If $BT_{\mathcal{O}}^{(f)}$ is faithful on the image of $\Omega_1(\mathcal{O},\mathcal{O}')$, then $\Omega_1(\mathcal{O},\mathcal{O}')$ is fully faithful.
- $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ is an equivalence of categories, iff $BT_{\mathcal{O}'}$ is one.
- $\Omega_1(\mathcal{O},\mathcal{O}')$ is an equivalence of categories, iff $BT_{\mathcal{O}}^{(f)}$ is one.

Let us now only assume that $P(\mathcal{O},\mathcal{O}',R)$ is true for a nonramified extension $\mathcal{O} \to \mathcal{O}'$ (i.e., $P(\mathcal{O},R)$ is not necessarily true).

- $\Omega_2(\mathcal{O},\mathcal{O}')$ is faithful.
- If $BT_{\mathcal{O}'}$ is faithful, then $\Omega_2(\mathcal{O},\mathcal{O}')$ is fully faithful.
- $BT_{\mathcal{O}'}$ is an equivalence of categories, iff $\Omega_2(\mathcal{O},\mathcal{O}')$ is one.

Unfortunately, we cannot see directly, under the assumption that $P(\mathcal{O},R)$ resp. $P(\mathcal{O},\mathcal{O}',R)$ is true, that $BT_{\mathcal{O}}^{(f)}$ or $BT_{\mathcal{O}'}$ (both with respect to $P(\mathcal{O},R)$) resp. $BT_{\mathcal{O}'}$ (with respect to $P(\mathcal{O},\mathcal{O}',R)$) is full, since we only know this fact for the full subcategory of nilpotent f-$\mathcal{O}$-displays over R resp. nilpotent $\mathcal{O}'$-displays over R whose objects are the images of $\Omega_1(\mathcal{O},\mathcal{O}')$ resp. $\Omega_2(\mathcal{O},\mathcal{O}')$ or $\Gamma_1(\mathcal{O},\mathcal{O}')$ or $\Gamma_2(\mathcal{O},\mathcal{O}')$, and for these functors we do not know so far that they are essentially surjective in general.
To prove that $BT_{\mathcal{O}}$ is an equivalence of categories, is eventually equivalent to show that $\Gamma_1(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ are equivalences for all $\mathcal{O}'$-algebras R with π' nilpotent in R, where $\mathcal{O} \to \mathcal{O}'$ is a nonramified / totally ramified extension.
The in the end established equivalence of nilpotent f-$\mathcal{O}$-displays over R and nilpotent $\mathcal{O}'$-displays over R is nontrivial, since the equivalence of $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $BT_{\mathcal{O}'}$ does not tell much about nilpotent f-$\mathcal{O}$-displays and to which category their category is equivalent to. Hence, we also obtain that $BT_{\mathcal{O}}^{(f)}$ is an equivalence between nilpotent f-$\mathcal{O}$-displays and π'-divisible formal $\mathcal{O}'$-modules over R. This equivalence is particularly interesting, when $\mathcal{O} = \mathbb{Z}_p$ and $\mathcal{O} \to \mathcal{O}'$ nonramified, since the ramified f-$\mathbb{Z}_p$-displays over R are closely related to the classical displays.

Chapter 3

Deformation theory

Let $\mathcal{O}$ be an RRS and $S \to R$ a surjection of $\mathcal{O}$-algebras. A lift for a fixed (nilpotent) f-$\mathcal{O}$-display over R is a (nilpotent) f-$\mathcal{O}$-display over S, for which the (nilpotent) f-$\mathcal{O}$-display obtained base change to R is isomorphic to the original f-$\mathcal{O}$-display over R. In this chapter we show for some special cases that lifts of nilpotent f-$\mathcal{O}$-displays exist, and what information we need to obtain unique lifts. With these results we are able to show the equivalence of $BT_{\mathcal{O}}$ for each $\mathcal{O}$ a ring of integers of a non-Archimedean local field of characteristic zero and R a complete local $\mathcal{O}$-algebra with perfect residue field, nilpotent nilradical and π nilpotent in R. Throughout this chapter our standard source of reference will be [Lau10], in which Lau established deformation theory for frames and windows, which in turn are introduced in [Zin01]. These structures generalize the concept of (nilpotent) displays over p-adic rings, but not in a way which would contain our f-$\mathcal{O}$-displays, hence we have to do a slight generalization of frames and windows as well. So, just like Lau, we get results, which are valid for more general structures then just for f-$\mathcal{O}$-displays over π-adic $\mathcal{O}$-algebras.

3.1 $\mathcal{O}$-frames and f-$\mathcal{O}$-windows

Definition 3.1.1. (cf. [Lau10, Definition 2.1.]) An $\mathcal{O}$-frame is a quintuple $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$, where $\mathcal{O} = (\mathcal{O}, \pi, q)$ is an RRS, S an $\mathcal{O}$-algebra, $I \subseteq S$ an ideal, $R = S/I$ together with an $\mathcal{O}$-algebra morphism $\sigma : S \to S$ and a σ-linear morphism of S-modules $\sigma_1 : I \to S$, which satisfy the following properties:

1. $I + \pi S \subseteq \mathrm{Rad}(S)$,
2. $\sigma(a) \equiv a^q \mod \pi S$ for all $a \in S$ and
3. $\sigma_1(I)$ generates S as an S-module.

A special situation one should have in mind for an $\mathcal{O}$-frame is the so called Witt $\mathcal{O}$-frame $(W_{\mathcal{O}}(R), I_{\mathcal{O},R}, W_{\mathcal{O}}(R)/I_{\mathcal{O},R} = R, ^F, ^{V^{-1}})$ for a π-adic complete and separated $\mathcal{O}$-algebra R. We will denote this $\mathcal{O}$-frame by $W_{\mathcal{O},R}$.
Now let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. We declare a morphism of $\mathcal{O}$-frames $\alpha : \mathcal{F} \to \mathcal{F}'$ by an $\mathcal{O}$-algebra morphism $\alpha : S \to S'$, such that $\alpha(I) \subseteq I'$, $\sigma'\alpha = \alpha\sigma$ and $\sigma_1'\alpha = \alpha\sigma_1$ hold. We could extend the definition of a morphism by demanding that just $\sigma_1'\alpha = u\alpha\sigma_1$ holds with $u \in S'$ a unit. This the definition of Lau in [Lau10, Definition 2.6.], where our morphisms would be strict morphisms in his notation. Nevertheless, these general morphisms are not important for us.
Nearly all assertions from [Lau10] can be rewritten such that they fit to our situation here. If a proof is essentially the same (beside some obvious changes) and the idea behind it not used here any further, we omit it. Let $\rho : A \to B$ be a ring morphism. We define for any A-module M the B-module $M^{(\rho)}$ by $B \otimes_{\rho,A} M$. For any B-module N and ρ-linear map $g : M \to N$, we define the B-linear map $g^\sharp : M^{(\rho)} \to N$ by $b \otimes m \mapsto bg(m)$. The following Lemma is easy, but nonetheless very important.

Lemma 3.1.2. (cf. [Lau10, Lemma 2.2.]) Let $\mathcal{F}$ be an $\mathcal{O}$-frame. Then there is as unique $\theta \in S$, such that $\sigma(a) = \theta\sigma_1(a)$ holds for all $a \in I$.

Proof: The third condition of Definition 3.1.1 says that the linearisation $\sigma_1^\sharp : I^{(\sigma)} \to S$ is surjective. If $b \in I^{(\sigma)}$ satisfies $\sigma_1^\sharp(b) = 1$, then necessarily $\theta = \sigma^\sharp(b)$. For $a \in I$ we obtain $\sigma(a) = \sigma_1^\sharp(b)\sigma(a) = \sigma_1^\sharp(ba) = \sigma^\sharp(b)\sigma_1(a)$, which confirms the assertion. □

Definition 3.1.3. (cf. [Lau10, Definition 2.3.]) An f-$\mathcal{O}$-window over an $\mathcal{O}$-frame $\mathcal{F}$ is a quadruple $\mathcal{P} = (P, Q, F, F_1)$, where P is a finitely generated projective S-module, $Q \subseteq P$ a submodule, $F : P \to P$ and $F_1 : Q \to P$ are σ^f-linear morphisms of S-modules, with the following properties:

1. There is a decomposition $P = L \oplus T$ with $Q = L \oplus IT$, where L, T are S-submodules of P,
2. $F_1(ax) = \sigma^{f-1}(\sigma_1(a))F(x)$ for $a \in I$ and $x \in P$ and
3. $F_1(Q)$ generates P as an S-module.

If we have $f = 1$, then we just denote f-$\mathcal{O}$-windows by $\mathcal{O}$-windows.

If we are now given a Witt $\mathcal{O}$-frame for a π-adic $\mathcal{O}$-algebra R, where $\mathcal{O}$ is an RRS, then the f-$\mathcal{O}$-windows are precisely the f-$\mathcal{O}$-displays over R. We need to remark that, as in the usual display theory, F is uniquely determined by F_1: If $b \in I^{(\sigma)}$ satisfies $\sigma_1^\sharp(b) = 1$, then we obtain by the second condition of the Definition of an f-$\mathcal{O}$-window $F(x) = F_1^\sharp(b'x)$ for all $x \in P$, where

$F_1^\sharp : S \otimes_{\sigma^f,S} Q \to P$ is the σ^f-linearisation of F_1 and b' is the image of b via the map $I^{(\sigma)} \stackrel{1\otimes \mathrm{id}}{\to} S \otimes_{\sigma^{f-1},S} I^{(\sigma)} = I^{(\sigma^f)}$. In particular we have $F(x) = \sigma^{f-1}(\theta)F_1(x)$ for all $x \in Q$, see the proof of Lemma 3.1.2.
Similar to what we have done in the previous chapter there is for an f-$\mathcal{O}$-window over an $\mathcal{O}$-frame $\mathcal{F}$ a unique morphism of S-modules

$$V^\sharp : P \to S \otimes_{\sigma^f,S} P$$

satisfying $V^\sharp(wF_1y) = w \otimes y$ for all $y \in Q$ and $w \in S$. We define $V^{n\sharp} : P \to S \otimes_{\sigma^{fn},S} P$ as usual, i.e., as the composite of the S-linear maps

$$\mathrm{id} \otimes_{\sigma^i,S} V^\sharp : S \otimes_{\sigma^{fi},S} P \to S \otimes_{\sigma^{f(i+1)},S} P,$$

from $i = 0, \ldots, n-1$. The nilpotence condition in the f-$\mathcal{O}$-display case is defined relative to $I_{\mathcal{O},R} + \pi W_{\mathcal{O}}(R)$, we will do it more general. For this purpose we call an ideal J of S with $\sigma(J) + I + \theta S \subseteq J$, where θ is obtained from Lemma 3.1.2, an *ideal of definition for* $\mathcal{F}$. The ideal $I + \pi S$ is always an ideal of definition, since θ is an element of this ideal, which follows from the argumentation of Lemma 3.1.2.

Definition 3.1.4. For an ideal of definition J for an $\mathcal{O}$-frame $\mathcal{F}$, we call an f-$\mathcal{O}$-window over $\mathcal{F}$ *nilpotent* (with respect to J), if there is a number N, such that $V^{N\sharp} \equiv 0$ modulo J.

Lemma 3.1.5. (cf. [Lau10, Lemma 2.5.]) Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be an $\mathcal{O}$-frame, $P = L \oplus T$ a finitely generated projective S-module and $Q = L \oplus IT$, where L, T are S-submodules of P. Then the set of f-$\mathcal{O}$-window structures (P, Q, F, F_1) over $\mathcal{F}$ corresponds bijectively to the set of σ^f-linear isomorphisms $\Psi : L \oplus T \to P$ given by $\Psi(l + t) = F_1(l) + F(t)$ for $l \in L$ and $t \in T$. Conversly, if we start with a Ψ, we obtain an f-$\mathcal{O}$-window over $\mathcal{F}$ by $F(l + t) = \sigma^{f-1}(\theta)\Psi(l) + \Psi(t)$ and $F_1(l + at) = \Psi(l) + \sigma^{f-1}(\sigma_1(a))\Psi(t)$ for $l \in L, t \in T$ and $a \in I$.

We call the triple (L, T, Ψ) a *normal decomposition of* (P, Q, F, F_1).

Let $\mathcal{O}$ be an RRS, $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be a morphism between two $\mathcal{O}$-frames and $\mathcal{P} = (P, Q, F, F_1), \mathcal{P}' = (P', Q', F', F_1')$ be f-$\mathcal{O}$-windows over $\mathcal{F}$ and $\mathcal{F}'$, respectively. We declare an α-morphism g between $\mathcal{P}$ and $\mathcal{P}'$ as a morphism of S-modules $P \to P'$ with $g(Q) \subseteq Q'$, which fulfils $F'g = gF$ and $F_1'g = gF_1$. A morphism of f-$\mathcal{O}$-windows over $\mathcal{F}$ is an $\mathrm{id}_{\mathcal{F}}$-morphism.
We are allowed to define a base change for f-$\mathcal{O}$-windows, which is a lot like the base change defined in section 2.2 of the previous chapter and is of course an extension for f-$\mathcal{O}$-displays over π-adic $\mathcal{O}$-algebras, when considering the f-$\mathcal{O}$-windows over the Witt $\mathcal{O}$-frame for this $\mathcal{O}$-algebra.

With α as above, we associate an f-$\mathcal{O}$-window $\alpha_\star\mathcal{P} =: \mathcal{P}' = (P', Q', F', F_1')$ over $\mathcal{F}'$ to an f-$\mathcal{O}$-window $\mathcal{P}$ over $\mathcal{F}$ in the following way:

$$\begin{aligned} P' &= S' \otimes_S P \\ Q' &= S' \otimes_S L \oplus I' \otimes_S T \\ F' &= \sigma'^f \otimes F \\ F_1'(s' \otimes q) &= \sigma'^f(s') \otimes F_1 y \\ F_1'(i' \otimes p) &= \sigma'^{f-1}\sigma_1'(i') \otimes Fx \end{aligned}$$

Here $P = L \oplus T$ is a normal decomposition and $s' \in S'$, $i' \in I'$, $y \in Q$ and $x \in P$. There is an obvious mapping $\mathrm{Hom}_{\mathcal{F}'}(\alpha_\star\mathcal{P}, \widetilde{\mathcal{P}}) \to \mathrm{Hom}_\alpha(\mathcal{P}, \widetilde{\mathcal{P}})$ for all f-$\mathcal{O}$-windows $\widetilde{\mathcal{P}}$ over $\mathcal{F}'$ given by composing maps, which is in fact an isomorphism (cf. [Lau10, Lemma 2.9.]). This property determines $\alpha_\star\mathcal{P}$ uniquely.

Definition 3.1.6. Let $\mathcal{F}$ and $\mathcal{F}'$ be two $\mathcal{O}$-frames and $\alpha : \mathcal{F} \to \mathcal{F}'$ a morphism between them. We say that α is crystalline if it induces an equivalence of categories between f-$\mathcal{O}$-windows over $\mathcal{F}$ and f-$\mathcal{O}$-windows over $\mathcal{F}'$. If we are given two ideals of definition $J \subset S$ and $J' \subset S'$ such that $\alpha(J) \subseteq J'$, then $\alpha_\star$ sends nilpotent f-$\mathcal{O}$-windows over $\mathcal{F}$ with respect to J to nilpotent f-$\mathcal{O}$-windows over $\mathcal{F}'$ with respect to J'. We call α nilcrystalline if it induces an equivalence of categories between the nilpotent f-$\mathcal{O}$-windows.

We now come to the central assertions of this section. The proofs of them are generally omitted, since they are essentially the same as the ones given in [Lau10], one has only to observe that adding $\mathcal{O}$ in the frames, the changes induced by it and the occurrence of f does not make big differences. Since we demand in the next Theorem a different condition than in the referred source in order to obtain the nilcrystalline property, will give an outline what changes in the proofs of the referred source in the generalized setting. This new setting is helpful to deduce deformation assertions for $\mathcal{O}$-pd-thickening more directly (see the next section) than with the naturally generalized condition for J of [Lau10, Theorem 10.3.]. The equivalence property for nilpotent f-$\mathcal{O}$-window structures in Lemma 3.1.8 cannot be found in the reference, but this is easily seen.

Theorem 3.1.7. (cf. [Lau10, Theorem 3.2., Theorem 10.3.]) Let $\mathcal{O}$ be an RRS and $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ a morphism between two $\mathcal{O}$-frames, such that it induces $R \cong R'$ and a surjection $S \to S'$ with kernel $\mathfrak{b} \subset I$. If there is a finite sequence $\mathfrak{b} = \mathfrak{b}_0 \supseteq \ldots \supseteq \mathfrak{b}_n = 0$ with $\sigma(\mathfrak{b}_i) \subseteq \mathfrak{b}_{i+1}$ and $\sigma_1(\mathfrak{b}_i) \subseteq \mathfrak{b}_i$ such that σ_1 is elementwise nilpotent on $\mathfrak{b}_i/\mathfrak{b}_{i+1}$ and finitely generated projective S'-modules lift to projective S-modules, then α is crystalline. If we drop the elementwise-nilpotence condition and are given an ideal of definition $J \subset S$ with $(\prod_{i=0}^n \sigma^i(J))\mathfrak{b} = 0$ for large n, then α is nilcristalline with respect to the ideals J and $J' = J/\mathfrak{b} \subset S'$.

We now only show what changes in the proof of [Lau10, Theorem 10.3.] in the generalized setting with the different condition for J.
Proof: Let $\mathcal{P}$ be an f-$\mathcal{O}$-window over $\mathcal{F}$ with normal decomposition (L, T, Ψ) and let the elements $x_1, \ldots, x_r \in L$ be generators of the S-module L. We consider the σ^f-linear map

$$\lambda : L \subseteq L \oplus T \overset{(\Psi^\sharp)^{-1}}{\to} L^{(\sigma^f)} \oplus T^{(\sigma^f)} \overset{\mathrm{pr}}{\to} L^{(\sigma^f)}.$$

$\mathcal{P}$ is nilpotent with respect to J, iff λ is nilpotent modulo J (see [Lau10, Remark 10.2.]). Hence, since we consider nilpotent $\mathcal{P}$, there is a $k \geq 1$, such that for the composite map

$$\lambda^k : L \overset{\lambda}{\to} L^{(\sigma^f)} = S \otimes_{\sigma^f, S} L \overset{1 \otimes \lambda}{\to} S \otimes_{\sigma^f, S} L^{(\sigma^f)} = L^{(\sigma^{2f})} \to \ldots \to L^{(\sigma^{kf})}$$

and each generator $x_m \in L$ the element $\lambda^k(x_m)$ is of the form $\sum_{i=1}^r j_{im} \otimes x_i$ with $j_{im} \in J$. With the analogous setting as in the proofs of [Lau10, Theorem 10.3.] and [Lau10, Theorem 3.2.] (where we have $\mathfrak{b}$ and $\mathfrak{b}_i$ here in place of $\mathfrak{a}$ and $\mathfrak{a}_i$ there), we have to show that the endomorphism U of the group $\mathcal{H} = \mathrm{Hom}_{\sigma^f\text{-linear}}(L, \mathfrak{b}P)$ given by $U(w_L) = F_1' w_L^\sharp \lambda$ is nilpotent. For $x \geq 1$ the operator U^x equals

$$L \overset{\lambda^x}{\to} L^{(\sigma^{xf})} = S \otimes_{\sigma^{(x-1)f}, S} S \otimes_{\sigma^f, S} L \overset{1 \otimes w_L^\sharp}{\to} (\mathfrak{b}P)^{(\sigma^{(x-1)f})} \overset{h_{x-1}}{\to} \mathfrak{b}P \overset{F_1'}{\to} \mathfrak{b}P,$$

where h_{x-1} is given by the composite map

$$(\mathfrak{b}P)^{(\sigma^{(x-1)f})} = S \otimes_{\sigma^{(x-2)f}, S} S \otimes_{\sigma^f, S} \mathfrak{b}P \overset{1 \otimes F_1'^\sharp}{\to} (\mathfrak{b}P)^{(\sigma^{(x-2)f})} \overset{1 \otimes F_1'^\sharp}{\to} \ldots \to \mathfrak{b}P.$$

It is easily seen that for any fixed $y \geq 1$ the condition $(\prod_{i=0}^n \sigma^i(J))\mathfrak{b} = 0$ for large n is equivalent to $(\prod_{i=0}^n \sigma^{yi}(J))\mathfrak{b} = 0$ for large n. Now let a fixed $n \geq 1$ be chosen that large, such that $(\prod_{i=0}^n \sigma^{fki}(J))\mathfrak{b} = 0$ holds. We claim that the map U^{kn} equals zero. It is not too hard to verify that for each generator $x_m \in L$ the element $\lambda^{kn}(x_m)$ is of the form

$$\sum_{i=1}^r \sum_{z \in Z_i} \prod_{c=0}^n \sigma^{fkc}(j_{icz}) \otimes 1 \otimes \ldots 1 \otimes x_i$$

where $j_{icz} \in J$ and Z_i are finite index sets. After applying $1 \otimes w_L^\sharp$ and h_{kn-1} to this element we obtain that the image is zero by assumption, hence U^{kn} is zero. □

The *Hodge filtration of an f-$\mathcal{O}$-window* $\mathcal{P}$ over an $\mathcal{O}$-frame $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ is the R-submodule $Q/IP \subseteq P/IP$.

Lemma 3.1.8. (cf. [Lau10, Lemma 4.2.]) Let $\mathcal{O}$ be an RRS and $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ a morphism between two $\mathcal{O}$-frames, such that $S \cong S'$ holds. Then $R \to R'$ is surjective and $I \subseteq I'$ holds. The f-$\mathcal{O}$-windows $\mathcal{P}$ over $\mathcal{F}$ are equivalent to a pair consisting of an f-$\mathcal{O}$-window $\mathcal{P}'$ over $\mathcal{F}'$ together with a lift of its Hodge filtration to a direct summand $V \subseteq P/IP$. If J is an ideal of definition for $\mathcal{F}'$, then we have the equivalence for nilpotent f-$\mathcal{O}$-window structures with respect to the ideal J for $\mathcal{F}$ and $\mathcal{F}'$.

3.2 Applications to triples

To show how these two results are useful, we consider a morphism of $\mathcal{O}$-frames $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$, where $\mathcal{O}$ is an RRS, $S \to S'$ surjective with the kernel $\mathfrak{b}$ and $I' = IS'$. We would like to factor α into morphisms

$$(S, I, R, \sigma, \sigma_1) \overset{\alpha_1}{\to} \mathcal{F}'' = (S, I'', R', \sigma, \sigma_1'') \overset{\alpha_2}{\to} (S', I', R', \sigma', \sigma_1'), \tag{3.1}$$

in such a way that α_2 fulfils the first or the second part of the hypotheses of Theorem 3.1.7, i.e., for the crystalline or the nilcrystalline property (with respect to an ideal of definition J for $\mathcal{F}''$). We must have $I'' = I + \mathfrak{b}$. So all what remains is to define $\sigma_1'' : I'' \to S$, or which is the same as to define a σ-linear morphism $\sigma_1'' : \mathfrak{b} \to \mathfrak{b}$ with $\sigma_1'' = \sigma_1$ on $I \cap \mathfrak{b}$, such that the hypotheses of Theorem 3.1.7 are fulfilled. Then by using the above Lemma and Theorem we get that (nilpotent) f-$\mathcal{O}$-windows over $\mathcal{F}$ (with respect to J) are equivalent to (nilpotent) f-$\mathcal{O}$-windows $\mathcal{P}'$ over $\mathcal{F}'$ (with respect to $J/\mathfrak{b}$) together with a lift of the Hodge filtration to a direct summand of P/IP, where $\mathcal{P}'' = (P, Q'', F'', F_1'')$ is the unique lift of $\mathcal{P}'$ under α_2.
With the help of the isomorphism defined in (2.14) it is possible for us to define σ_1'' in cases, which are important for us, which will be helpful in Proposition 3.3.4. We need to define the notion of an $\mathcal{O}$-pd-thickening:

Definition 3.2.1. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras, such that the kernel $\mathfrak{a}$ may be equipped with an $\mathcal{O}$-pd-structure (see Definition 2.3.2). If π is nilpotent in S we call $S \to R$ an *$\mathcal{O}$-pd-thickening*. If the $\mathcal{O}$-pd-structure on $\mathfrak{a}$ is nilpotent, we call $S \to R$ a *nilpotent $\mathcal{O}$-pd-thickening*. We call $S \to R$ a *topological $\mathcal{O}$-thickening*, if there is a sequence of subideals $\mathfrak{a}_n$ of $\mathfrak{a}$, such that π is nilpotent in $S/\mathfrak{a}_n$, S is complete and separated in the linear topology defined by the $\mathfrak{a}_n$ and each $\mathfrak{a}_n$ may be equipped with an $\mathcal{O}$-pd-structure.

To apply these structures we take (3.1) in a more concrete term:

$$W_{\mathcal{O},S} \overset{\alpha_1}{\to} W_{\mathcal{O},S/R} = (W_{\mathcal{O}}(S), \tilde{I}, R, \sigma, \sigma_1'') \overset{\alpha_2}{\to} W_{\mathcal{O},R}, \tag{3.2}$$

where $S \to R$ is an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$. Let $J \subseteq W_{\mathcal{O}}(S)$ be $I_{\mathcal{O},S} + \pi W_{\mathcal{O}}(S) + W_{\mathcal{O}}(\mathfrak{a})$. Since $\tilde{I} = I_{\mathcal{O},S} + W_{\mathcal{O}}(\mathfrak{a})$, we are able to define with the

help of log (cf. (2.9)) the map σ_1'' in the way that $\sigma_1''[a_0, a_1, \ldots] = [a_1, a_2, \ldots]$ in logarithmic coordinates on $W_{\mathcal{O}}(\mathfrak{a})$. We can take J as an ideal of definition for $W_{\mathcal{O},S}$ and $W_{\mathcal{O},S/R}$ and we may apply Lemma 3.1.8 then. Now consider the filtration $\mathfrak{b}_i = \pi^i\mathfrak{b}$ on $\mathfrak{b} = W_{\mathcal{O}}(\mathfrak{a})$, which is zero for large n by considering (2.11) and (2.12) and ${}^{F}V = \pi$. Furthermore, $(\prod_{i=0}^{n} \sigma^i(J))\mathfrak{b} = 0$ holds for large n, since $\sigma^n(j) \in I_{\mathcal{O},S} + \pi W_{\mathcal{O}}(S)$ for all $n > 0$ and $j \in J$, $(I_{\mathcal{O},S} + \pi W_{\mathcal{O}}(S))^n \subseteq I_{\mathcal{O},S}$ for large n, $I_{\mathcal{O},S}^{n+1} \subseteq \pi^n W_{\mathcal{O}}(S)$ for all $n \geq 0$ and $\pi^n W_{\mathcal{O}}(\mathfrak{a})$ is zero for n large enough. Hence, we may apply Theorem 3.1.7 to α_2 to obtain that is nilcristalline with respect to $J \subset W_{\mathcal{O}}(S)$ and $J' = J/\mathfrak{b} \subset W_{\mathcal{O}}(R)$.
Since it is easily seen that nilpotent f-$\mathcal{O}$-windows over $W_{\mathcal{O},S}$ with respect to J and $I_{\mathcal{O},S} + \pi W_{\mathcal{O}}(S)$ are the same, we obtain:

Proposition 3.2.2. With $S \to R$ an $\mathcal{O}$-pd-thickening, we obtain that nilpotent f-$\mathcal{O}$-displays over S are equivalent to nilpotent f-$\mathcal{O}$-displays $\mathcal{P}'$ over R plus a lift of the Hodge filtration to a direct summand of $P/I_{\mathcal{O},S}P$, where (P, Q'', F, F_1'') is the unique lift of $\mathcal{P}'$ under α_2.

From this result we can deduce rigidity assertions:

Corollary 3.2.3. Let $S \to R$ be an $\mathcal{O}$-pd-thickening or a surjection of $\mathcal{O}$-algebras with nilpotent kernel and π nilpotent in S and $\mathcal{P}, \mathcal{P}'$ be two f-$\mathcal{O}$-displays over S. Then

$$\mathrm{Hom}_S(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_R(\mathcal{P}_R, \mathcal{P}'_R)$$

is injective.

Proof: First let $S \to R$ be an $\mathcal{O}$-pd-thickening. By Proposition 3.2.2 we get that the nilpotent f-$\mathcal{O}$-displays over S are equivalent to nilpotent f-$\mathcal{O}$-displays over R plus the lift of the Hodge filtration, which must be respected by the morphisms in the category for R. This shows that $\mathrm{Hom}_S(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_R(\mathcal{P}_R, \mathcal{P}'_R)$ is injective.
Now let $S \to R$ be a surjection of $\mathcal{O}$-algebras with nilpotent kernel $\mathfrak{a}$ and π nilpotent in S and n chosen that large, such that $\mathfrak{a}^n = 0$. If we define $S_i = S/\mathfrak{a}^i$ for $i = 1 \ldots n$, we can consider the obvious surjections of $\mathcal{O}$-algebras

$$S = S_n \to S_{n-1} \to \ldots \to S_1 = R.$$

For $S_{i+1} \to S_i$ the kernel $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ can be equipped with the trivial $\mathcal{O}$-pd structure, so the injectivity of $\mathrm{Hom}_S(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_R(\mathcal{P}_R, \mathcal{P}'_R)$ follows inductively by the above assertion for $\mathcal{O}$-pd-thickenings. □

As in [Zin02, 2.2. Triples and Crystals], we should now have a look at $\mathcal{P}$-triples, where $\mathcal{P}$ is a nilpotent f-$\mathcal{O}$-display over R.

Definition 3.2.4. Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$. A *$\mathcal{P}$-triple* $\mathcal{T} = (\widetilde{P}, F, F_1)$ *over* S consists of a finitely generated projective $W_{\mathcal{O}}(S)$-module

$\widetilde{P}$, which lifts P, and ${}^{F^f}$-linear morphisms $F : \widetilde{P} \to \widetilde{P}$ and $F_1 : \widehat{Q} \to \widetilde{P}$, where $\widehat{Q}$ denotes the inverse image of Q by the surjection $\widetilde{P} \to P$ (which has kernel $W_{\mathcal{O}}(\mathfrak{a})\widetilde{P}$). Furthermore, the following equations are required:

$$\begin{aligned} F_1({}^V wx) &= {}^{F^{f-1}} wFx \\ F_1(\mathfrak{a}\widetilde{P}) &= 0, \end{aligned}$$

with $w \in W_{\mathcal{O}}(R)$ and $x \in \widetilde{P}$. Here $\mathfrak{a} \subset W_{\mathcal{O}}(R)$ is given by the logarithm (see (2.13)).

F_1 is uniquely determined by these requirements (by choosing any lifting of $\mathcal{P}$ to a nilpotent f-$\mathcal{O}$-display over S and applying Lemma 2.3.3.

A morphism between triples is as follows: Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism between nilpotent f-$\mathcal{O}$-displays over R and $\mathcal{T}_i$ be a $\mathcal{P}_i$-triple S for $i = 1, 2$. Then an *α-morphism* $\widetilde{\alpha} : \widetilde{P_1} \to \widetilde{P_2}$ is a morphism of $W_{\mathcal{O}}(S)$-modules which lifts α and commutes with the F and F_1 maps, which only makes sense since $\widetilde{\alpha}(\widehat{Q_1}) \subset \widehat{Q_2}$. We need to define base change of triples. For this purpose let $S \to R$, $S' \to R'$ be $\mathcal{O}$-pd-thickenings, respectively, and let $\varphi : R \to R'$ be an $\mathcal{O}$-algebra morphism. Assume we are given a morphism of $\mathcal{O}$-pd-thickenings, i.e., an $\mathcal{O}$-algebra morphism $S \to S'$, such that

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi} & R' \end{array}$$

commutes. Now for a $\mathcal{P}$-triple $\mathcal{T}$ over S, we define the $\mathcal{P}_{R'}$-triple $\mathcal{T}_{S'}$ over S' by setting

$$\mathcal{T}_{S'} = (W_{\mathcal{O}}(S') \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}, \widetilde{F}, \widetilde{F_1}),$$

where $\widetilde{F}$ is the ${}^{F^f}$-linear extension of F and $\widetilde{F_1}$ on $\widehat{Q'}$ is uniquely determined by

$$\begin{aligned} \widetilde{F_1}(w \otimes y) &= {}^{F^f} w \otimes F_1 y, \\ \widetilde{F_1}({}^V w \otimes x) &= {}^{F^{f-1}} w \otimes Fx, \\ \widetilde{F_1}(a \otimes x) &= 0, \end{aligned}$$

for $x \in \widetilde{P}, y \in \widehat{Q}, w \in W_{\mathcal{O}}(S')$ and $a \in \mathfrak{a}' \subset W_{\mathcal{O}}(\mathfrak{a}')$, where $\mathfrak{a}'$ is the kernel of $S' \to R'$.

Now let $S \to R$ be an $\mathcal{O}$-pd-thickening. It is rather obvious that nilpotent f-$\mathcal{O}$-windows over $W_{\mathcal{O},S/R}$ with respect to $J = I_{\mathcal{O},S} + \pi W_{\mathcal{O}}(S) + W_{\mathcal{O}}(\mathfrak{a})$ and $\mathcal{P}$-triples for all nilpotent f-$\mathcal{O}$-displays over S are practically the same. For this purpose consider the category f-$\mathcal{O}$-$\mathcal{C}_{S/R}$ consisting of the objects $(\mathcal{P}, \mathcal{T})$, where $\mathcal{P}$ is a nilpotent f-$\mathcal{O}$-display over S and $\mathcal{T}$ a $\mathcal{P}$-triple over S. A morphism between two objects of f-$\mathcal{O}$-$\mathcal{C}_{S/R}$, say $(\mathcal{P}, \mathcal{T}) \to (\mathcal{P}', \mathcal{T}')$, consists of a morphism of displays $\alpha : \mathcal{P} \to \mathcal{P}'$ and an α-morphism $\widetilde{\alpha} : \mathcal{T} \to \mathcal{T}'$. We obtain an equivalence of categories between the category of nilpotent f-$\mathcal{O}$-windows over $W_{\mathcal{O},S/R}$ with respect to J and the category f-$\mathcal{O}$-$\mathcal{C}_{S/R}$, such that the diagram

$$\begin{array}{ccc} (\text{nilpotent } f\text{-}\mathcal{O}\text{-windows over } W_{\mathcal{O},S/R} \text{ with respect to } J) & \longrightarrow & f\text{-}\mathcal{O}\text{-}\mathcal{C}_{S/R} \\ \downarrow{\scriptstyle \alpha_{2\star}} & \swarrow & \\ (f - \text{ndisp}_{\mathcal{O}} / R) & & \end{array}$$

commutes (the upper categories lie over the lower one), where $\alpha_{2\star}$ is induced by the map α_2 of (3.2). This equivalence is given by sending a nilpotent f-$\mathcal{O}$-windows over $W_{\mathcal{O},S/R}$ with respect to J, say $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, \widetilde{F}, \widetilde{F_1})$, to $(\alpha_{2\star}\widetilde{\mathcal{P}}, (\widetilde{P}, \widetilde{F}, \widetilde{F_1}))$, and morphism between to f-$\mathcal{O}$-windows over $W_{\mathcal{O},S/R}$ with respect to J, say $\tau : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}'}$, to $(\alpha_{2\star}\tau, \tau)$. The inverse functor is easily constructed. Hence, it follows that we could also work with nilpotent f-$\mathcal{O}$-windows over $W_{\mathcal{O},S/R}$ with respect to J in place of $\mathcal{P}$-triples, but since we try to follow the notation of [Zin02], we take $\mathcal{P}$-triples. Since $\alpha_{2\star}$ is an equivalence of categories by Theorem 3.1.7 we obtain with the above notation:

Proposition 3.2.5. (cf. [Zin02, Theorem 46]) Let $S \to R$ be an $\mathcal{O}$-pd-thickening and $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ a morphism between two nilpotent f-$\mathcal{O}$-displays over R. For $\mathcal{P}_i$-triples $\mathcal{T}_i$ over S there is a unique α-morphism of triples $\widetilde{\alpha} : \mathcal{T}_1 \to \mathcal{T}_2$.

Hence, given an $\mathcal{O}$-pd-thickening $S \to R$ and a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R, it makes sense to talk about "the" $\mathcal{P}$-triple over S, since by the previous association we can always find a $\mathcal{P}$-triple and it is uniquely determined up to unique isomorphism by the previous Proposition.

3.3 Applications to f-$\mathcal{O}$-displays

In this section every $\mathcal{O}, \mathcal{O}'$, etc. is assumed to be a ring of integers of a non-Archimedean local field of characteristic zero. We want to prove that $P(\mathcal{O}, \mathcal{O}', R)$ is true for as many $\mathcal{O}'$-algebras R as possible. All these proofs have a very similar framework and the following Definition is helpful to simplify the proofs.

Definition 3.3.1. Let R be a fixed ring with p nilpotent in R. We say that the boolean variable $A(R)$ is true, iff the following three assertions hold:

- Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $P(\mathcal{O}, R)$ is true, then $\Omega_1(\mathcal{O}, \mathcal{O}')$ is an equivalence of categories. (Hence, $P(\mathcal{O}, \mathcal{O}', R)$ is true.)

- Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $P(\mathcal{O}, \mathcal{O}', R)$ is true, then $\Omega_2(\mathcal{O}, \mathcal{O}')$ is an equivalence of categories. (Hence, $P(\mathcal{O}', R)$ is true.)

- Let $\mathcal{O} \to \mathcal{O}'$ be a totally ramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $P(\mathcal{O}, R)$ is true, then $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is an equivalence of categories. (Hence, $P(\mathcal{O}', R)$ is true.)

Lemma 3.3.2. Let R be a ring with p nilpotent in R, such that $A(R)$ is true. Then for each nonramified extension $\mathcal{O} \to \mathcal{O}'$ and each $\mathcal{O}'$-algebra structure on R, $P(\mathcal{O}, \mathcal{O}', R)$ is true. In particular, $P(\mathcal{O}, R)$ is true for each $\mathcal{O}$ and each $\mathcal{O}$-algebra structure on R.

Proof: First we prove that $P(\mathcal{O}, R)$ is true with $\mathcal{O}$ arbitrary and R equipped with an $\mathcal{O}$-algebra structure. We choose $\mathcal{O}_0$, such that $\mathcal{O}_0$ is nonramified over $\mathbb{Z}_p$ and $\mathcal{O}$ is totally ramified over $\mathcal{O}_0$. We can establish with Theorem 2.5.16 and the three points of Definition 3.3.1 that $P(\mathcal{O}, R)$ is true by first considering the step $\mathbb{Z}_p \to \mathcal{O}_0$ and then $\mathcal{O}_0 \to \mathcal{O}$. By using the first part of Definition 3.3.1 again we obtain that $P(\mathcal{O}, \mathcal{O}', R)$ is true for each nonramified extension $\mathcal{O} \to \mathcal{O}'$ and each $\mathcal{O}'$-algebra structure on R. □

Proposition 3.3.3. Let l be a perfect field of characteristic p. Then $A(l)$ is true. Hence by Lemma 3.3.2 we obtain that $P(\mathcal{O}, \mathcal{O}', l)$ is true for each nonramified extension $\mathcal{O} \to \mathcal{O}'$, such that l is a perfect field of characteristic p extending the residue field of $\mathcal{O}'$.

Proof: We need to confirm the points defining $A(l)$, assuming that $P(\mathcal{O}, l)$ resp. $P(\mathcal{O}, \mathcal{O}', l)$ is true. First we consider the case that $\mathcal{O}'$ is nonramified over $\mathcal{O}$ and l extends the residue field of $\mathcal{O}'$. We first show the essential surjectivity of $\Omega_1(\mathcal{O}, \mathcal{O}')$. Let $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{10})$ be a nilpotent* f-$\mathcal{O}$-display over l. We define for each $i = 1, \ldots, f-1$

$$P_i = W_{\mathcal{O}}(l) \otimes_{F^{i-f}} P_0$$

and consider

$$\begin{aligned} P &= \bigoplus_{i=0}^{f-1} P_i \\ Q &= Q_0 \oplus \bigoplus_{i=1}^{f-1} P_i. \end{aligned}$$

*We could use the same arguments for the not necessarily nilpotent case to establish the essential surjectivity there.

The operators F and F_1 are given by

$$\begin{aligned} F(x_0, 1\otimes x_1, \ldots, 1\otimes x_{f-1}) &= (x_{f-1}, 1\otimes F_0x_0, 1\otimes x_1, \ldots, 1\otimes x_{f-2}) \\ F_1(y_0, 1\otimes x_1 \ldots, 1\otimes x_{f-1}) &= (x_{f-1}, 1\otimes F_{10}y_0, 1\otimes x_1, \ldots, 1\otimes x_{f-2}) \end{aligned}$$

with $x_i \in P_0$ and $y_0 \in Q_0$. Then $\mathcal{P} = (P, Q, F, F_1)$ is a nilpotent $\mathcal{O}$-display over l. By letting the $\mathcal{O}'$-action of P_0 act on the second factors of the tensor products of the P_i we obtain a strict $\mathcal{O}'$-action of $\mathcal{P}$. It is clear that $\mathcal{P} + \mathcal{O}'$ is mapped via $\Omega_1(\mathcal{O}, \mathcal{O}')$ to $\mathcal{P}_0$ and an easy exercise to show the fully faithfulness.
Now we have a look at $\Omega_2(\mathcal{O}, \mathcal{O}')$. Since $u_l : W_{\mathcal{O}}(l) \to W_{\mathcal{O}'}(l)$ is an isomorphism by Lemma 1.3.6, it is easily seen that $\Omega_2(\mathcal{O}, \mathcal{O}')$ is essentially surjective. Because we assume $P(\mathcal{O}, \mathcal{O}', l)$ to be true we need for the fully faithfulness only to show that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{P}_{\star 0}) \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{P}', \mathcal{P}'_{\star})$$

is surjective, where $\mathcal{P}_0, \mathcal{P}_{\star 0}$ are nilpotent f-$\mathcal{O}$-displays over l and $\mathcal{P}', \mathcal{P}'_{\star}$ are the respective associated $\mathcal{O}'$-displays over l, but this is again fairly obvious.
Now let $\mathcal{O}'$ be totally ramified over $\mathcal{O}$ and let l extend the residue field of $\mathcal{O}'$ and $\mathcal{O}$. We consider $\Gamma_2(\mathcal{O}, \mathcal{O}')$ and assume $P(\mathcal{O}, l)$ to be true. Let $\mathcal{P} = (P, Q, F, F_1)$ be an $\mathcal{O}$-display over l equipped with a strict $\mathcal{O}'$-action and $\mathcal{P}' = (P', Q', F', F_1')$ its image via $\Gamma_2(\mathcal{O}, \mathcal{O}')$. By considering Lemma 1.3.6, we obtain the isomorphism of rings $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \simeq W_{\mathcal{O}'}(l)$ and so the module P' is P interpreted as $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$-module from which the essential surjectivity follows easily and it is left as an exercise to the reader to verify that $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is full for l, which would establish that it is an equivalence. □

Proposition 3.3.4. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified extension of rings of integers of non-Archimedean local fields of characteristic zero of degree f and R an $\mathcal{O}'$-algebra with nilpotent nilradical and π' nilpotent in R. Then $BT_{\mathcal{O}}^{(f)}$ is faithful. In particular, $BT_{\mathcal{O}}$ is faithful for each $\mathcal{O}$ an each $\mathcal{O}$-algebra R with nilpotent nilradical and π nilpotent in R.

The last assertion of the Proposition is only a partial result compared to the fact that we prove the faithfulness for all $\mathcal{O}'$-algebras with π' nilpotent in them in Chapter 5. Nevertheless, there we need crystal theory, so it seems sensible to state this result on its own, since we need only deformation theory.

Proof: Let k' be the residue field of $\mathcal{O}'$.
If $R = l$ is a perfect field extending k', the fully faithfulness of $BT_{\mathcal{O}}^{(f)}$ follows from Proposition 3.3.3.
Now let k be any field extending k' and l the algebraic closure of k. If $\mathcal{P}, \mathcal{P}_{\star}$ are two nilpotent f-$\mathcal{O}$-displays over k, $\mathcal{P}_l, \mathcal{P}_{\star,l}$ the corresponding nilpotent f-$\mathcal{O}$-displays over l obtained by base change and $X, X_{\star}, X_l, X_{\star,l}$ the corresponding

π'-divisible formal $\mathcal{O}'$-modules, then the faithfulness of the $BT^{(f)}_{\mathcal{O},k}$ functor follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O},k}(\mathcal{P},\mathcal{P}_\star) & \xrightarrow{BT^{(f)}_{\mathcal{O},k}} & \mathrm{Hom}_{\mathcal{O}',k}(X,X_\star) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\mathcal{O},l}(\mathcal{P}_l,\mathcal{P}_{\star,l}) & \overset{BT^{(f)}_{\mathcal{O},l}}{\hookrightarrow} & \mathrm{Hom}_{\mathcal{O}',l}(X_l,X'_{\star,l}), \end{array}$$

where the indices of the Hom-sets should indicate over which $\mathcal{O}'$-algebra we consider them. Now let R be a reduced $\mathcal{O}'$-algebra with $\pi' R = 0$ and $\mathcal{P},\mathcal{P}_\star$ two nilpotent f-$\mathcal{O}$-displays over R. Hence, we may embed R into a product $\prod_{i\in I} K_i$ of fields, each extending k', and with the help of the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O},R}(\mathcal{P},\mathcal{P}_\star) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}',R}(X,X_\star) \\ \big\downarrow & & \big\downarrow \\ \prod_{i\in I}\mathrm{Hom}_{\mathcal{O},K_i}(\mathcal{P}_{K_i},\mathcal{P}_{\star,K_i}) & \hookrightarrow & \prod_{i\in I}\mathrm{Hom}_{\mathcal{O}',K_i}(X_{K_i},X_{\star,K_i}) \end{array}$$

the faithfulness follows for this case. Now we may assume that R is an $\mathcal{O}'$-algebra with π nilpotent in R and nilpotent nilradical $\mathfrak{a}$. Let $R_1 = R/\mathfrak{a}$ and $\mathcal{P},\mathcal{P}_\star$ be nilpotent f-$\mathcal{O}$-displays over R. We obtain the injectivity of $\mathrm{Hom}_{\mathcal{O},R}(\mathcal{P},\mathcal{P}_\star) \to \mathrm{Hom}_{\mathcal{O},R_1}(\mathcal{P}_{R_1},\mathcal{P}_{\star,R_1})$ by Corollary 3.2.3. With the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O},R}(\mathcal{P},\mathcal{P}_\star) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}',R}(X,X_\star) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\mathcal{O},R_1}(\mathcal{P}_{R_1},\mathcal{P}_{\star,R_1}) & \hookrightarrow & \mathrm{Hom}_{\mathcal{O}',R_1}(X_{R_1},X_{\star,R_1}) \end{array}$$

the result follows. □

Proposition 3.3.5. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension, $S \to R$ a surjection of $\mathcal{O}'$-algebras with π' nilpotent in S and nilpotent kernel and $\widehat{\mathcal{P}}$ a nilpotent f-$\mathcal{O}$-display over S (for $\Omega_1(\mathcal{O},\mathcal{O}')$) resp. a nilpotent $\mathcal{O}'$-display over S (for $\Omega_2(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$), such that $\widehat{\mathcal{P}}_R$ lies in the image of $\Omega_1(\mathcal{O},\mathcal{O}')_R$ resp. $\Omega_2(\mathcal{O},\mathcal{O}')_R$ resp. $\Gamma_1(\mathcal{O},\mathcal{O}')_R$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')_R$. Then $\widehat{\mathcal{P}}$ lies in the image of the respective functor over S. In particular, if one of the functors $\Omega_1(\mathcal{O},\mathcal{O}'),\Omega_2(\mathcal{O},\mathcal{O}'),\Gamma_1(\mathcal{O},\mathcal{O}')$ or $\Gamma_2(\mathcal{O},\mathcal{O}')$ is essentially surjective over R, then this is also true for the respective functor over S.

Proof: The assertions for $\Gamma_1(\mathcal{O},\mathcal{O}')$ follows from the assertions for $\Omega_i(\mathcal{O},\mathcal{O}')$, so we will only consider $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$. Let $\mathfrak{a}$ be the kernel and $\mathfrak{a}^n = 0$ for an integer $n \geq 0$. By considering the sequence $S/\mathfrak{a}^i$ for $i = 0,\dots,n$ and the $\mathcal{O}'$-algebra surjections $S/\mathfrak{a}^i \to S/\mathfrak{a}^{i-1}$, we obtain that we may reduce for each functor to the case, where $\mathfrak{a}^2 = 0$. By taking the trivial $\mathcal{O}$-pd structure on $\mathfrak{a}$, we may construct the morphisms of $\mathcal{O}$-frames (see (3.2))

$$W_{\mathcal{O},S} \overset{\alpha_1}{\to} (W_{\mathcal{O}}(S), \tilde{I}, R, \sigma, \sigma_1) \overset{\alpha_2}{\to} W_{\mathcal{O},R}.$$

With the help of Theorem 3.1.7 and Lemma 3.1.8 we get that the category of nilpotent (f-)$\mathcal{O}$-displays over S is equivalent to the category of nilpotent (f-)$\mathcal{O}$-displays over R equipped with a lift of the Hodge filtration. Of course the same is true for $\mathcal{O}'$. Additionally, the equivalence assertions over $\mathcal{O}$ continue to hold, if we add a strict $\mathcal{O}'$-action to each object and consider only those morphisms respecting the $\mathcal{O}'$-actions. Hence, we obtain commutative diagrams

$$\begin{array}{ccc}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/S) & \xrightarrow{\Omega_{1,S}} & (f-\mathrm{ndisp}_{\mathcal{O}}/S) \\
\| & & \| \\
(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{\alpha_1} & (f-\mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R) \\
\downarrow & & \downarrow \\
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{\Omega_{1,R}} & (f-\mathrm{ndisp}_{\mathcal{O}}/R),
\end{array}
\qquad
\begin{array}{ccc}
(f-\mathrm{ndisp}_{\mathcal{O}}/S) & \xrightarrow{\Omega_{2,S}} & (\mathrm{ndisp}_{\mathcal{O}'}/S) \\
\| & & \| \\
(f-\mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R) & \xrightarrow{\alpha_2} & (\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R) \\
\downarrow & & \downarrow \\
(f-\mathrm{ndisp}_{\mathcal{O}}/R) & \xrightarrow{\Omega_{2,S}} & (\mathrm{ndisp}_{\mathcal{O}'}/R)
\end{array}$$

and

$$\begin{array}{ccc}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/S) & \xrightarrow{\Gamma_{2,S}} & (\mathrm{ndisp}_{\mathcal{O}'}/S) \\
\| & & \| \\
(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{\alpha'} & (\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R) \\
\downarrow & & \downarrow \\
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) & \xrightarrow{\Gamma_{2,R}} & (\mathrm{ndisp}_{\mathcal{O}'}/R),
\end{array}$$

where the dagger at each category in the middle of each diagram should indicate the further structure (i.e., the lift of the Hodge filtration) and the horizontal maps are $\Omega_1(\mathcal{O},\mathcal{O}'), \Omega_2(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ (over S and R) or at least induced by it (for the α-arrows in the middle of each diagram). We need to know what happens with the liftings in the middle left categories of the diagrams, when α_1, α_2 or α' are applied. With the help of this it easily shown that $\widehat{\mathcal{P}}$ lies in the image of the respective functors over S, since we only need to show that the to $\widehat{\mathcal{P}}$ corresponding element in the middle right categories of the respective above diagram lies in the image of α_1 resp. α_2 resp. α'.
First we consider α_1: Let $\mathcal{P}$ be a nilpotent $\mathcal{O}$-display over S equipped with a strict $\mathcal{O}'$-action. The element $(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R)$ corresponding to $\mathcal{P}$ is $(\mathcal{P}_R, S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L)$ with L as usual and the induced strict $\mathcal{O}'$-action. The element of $(f - \mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R)$ corresponding to $\Omega_1(\mathcal{O},\mathcal{O}')_S(\mathcal{P})$ is

$$(\Omega_1(\mathcal{O},\mathcal{O}')_S(\mathcal{P})_R, S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L_0) = (\Omega_1(\mathcal{O},\mathcal{O}')_R(\mathcal{P}_R), S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L_0),$$

where L_0 is obtained as in Lemma 2.5.1. Hence, α_1 is given by sending $(\mathcal{P}_R, S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L)$ to $(\Omega_1(\mathcal{O}, \mathcal{O}')_R(\mathcal{P}_R), (S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L)_0)$, where the last zero in the index should indicate one takes only the zeroth component of the obvious direct sum decomposition of $S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L$ (see the proof of Lemma 2.5.1). Let $(\widehat{\mathcal{P}}_R, M_0)$ be the element of $(f - \mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R)$ corresponding to $\widehat{\mathcal{P}}$ and $\mathcal{P}_\star \in (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$ chosen, such that $\Omega_1(\mathcal{O}, \mathcal{O}')_R(\mathcal{P}_\star) = \widehat{\mathcal{P}}_R$ holds. Let (P, F, F_1) be the $\mathcal{P}_\star$-triple over S. By Proposition 3.2.5, we can lift the $\mathcal{O}'$-action of $\mathcal{P}_\star$ uniquely, so P becomes an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(S)$-module and we obtain the usual grading (see Lemma 2.5.1)

$$P = \bigoplus_{i=0}^{f-1} P_i.$$

Since the lifted $\mathcal{O}'$-action leaves the P_i invariant, we get that with the S-module

$$M = M_0 \oplus \bigoplus_{i=1}^{f-1} S \otimes_{W_{\mathcal{O}}(S)} P_i$$

we obtain a lifting respecting the $\mathcal{O}'$-action, hence $(\mathcal{P}_\star, M)$ is an element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R)$ and

$$\alpha_1(\mathcal{P}_\star, S) = (\widehat{\mathcal{P}}_R, M_0)$$

holds, so $\widehat{\mathcal{P}}$ lies in the image of $\Omega_1(\mathcal{O}, \mathcal{O}')_S$.
We get that for a nilpotent f-$\mathcal{O}$-display $\mathcal{P}_0$ over S the corresponding element in $(f - \mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R)$ is $(\mathcal{P}_{0,R}, S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L_0)$ with L_0 as usual (cf. Definition 2.5.5). Because of the construction of $\Omega_2(\mathcal{O}, \mathcal{O}')$ we obtain that the element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R)$ corresponding to $\Omega_2(\mathcal{O}, \mathcal{O}')_S(\mathcal{P}_0)$ is

$$(\Omega_2(\mathcal{O}, \mathcal{O}')_S(\mathcal{P}_0)_R, S \otimes_{\mathrm{w}'_0, W_{\mathcal{O}'}(S)} L_{0,\star}) = (\Omega_2(\mathcal{O}, \mathcal{O}')_R(\mathcal{P}_{0,R}), S \otimes_{\mathrm{w}'_0, W_{\mathcal{O}'}(S)} L_{0,\star}),$$

where $L_{0,\star} = W_{\mathcal{O}'}(S) \otimes_{W_{\mathcal{O}}(S)} L_0$. Because of

$$\begin{aligned} S \otimes_{\mathrm{w}'_0, W_{\mathcal{O}'}(S)} L_{0,\star} &= S \otimes_{\mathrm{w}'_0, W_{\mathcal{O}'}(S)} W_{\mathcal{O}'}(S) \otimes_{W_{\mathcal{O}}(S)} L_0 \\ &= S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} L_0 \end{aligned}$$

we get that α_2 is given by sending $(\mathcal{P}_0, M)$ to $(\Omega_2(\mathcal{O}, \mathcal{O}')_R(\mathcal{P}_0), M)$, where $\mathcal{P}_0 \in (f - \mathrm{ndisp}_{\mathcal{O}}/R)$ and M is a lifting of the Hodge filtration. Hence, the element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R)$ corresponding to $\widehat{\mathcal{P}}$ lies in image of α_2 by obvious reasons and so $\widehat{\mathcal{P}}$ lies in image of $\Omega_2(\mathcal{O}, \mathcal{O}')_S$.
Now we have a look at α': Let $\mathcal{P}$ be a nilpotent $\mathcal{O}$-display over S equipped with a strict $\mathcal{O}'$-action. With the notation as right after the proof of Proposition 2.5.9 the element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R)$ corresponding to $\mathcal{P}$ is $(\mathcal{P}_R, L_{\Delta,0} \oplus \Delta)$ plus the induced strict $\mathcal{O}'$-action. The element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R)$ corresponding to $\Gamma_2(\mathcal{O}, \mathcal{O}')_S(\mathcal{P})$ is

$$(\Gamma_2(\mathcal{O}, \mathcal{O}')_S(\mathcal{P})_R, L_{\Delta,0}).$$

Hence, α' is given by sending $(\mathcal{P}_R, L_{\Delta,0} \oplus \Delta)$ to $(\Gamma_2(\mathcal{O},\mathcal{O}')_R(\mathcal{P}_R), L_{\Delta,0})$. Let $(\widehat{\mathcal{P}}_R, M_0)$ be the element of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R)$ corresponding to $\widehat{\mathcal{P}}$ and $\mathcal{P}_\star \in (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$ chosen, such that $\Gamma_2(\mathcal{O},\mathcal{O}')_R(\mathcal{P}_\star) = \widehat{\mathcal{P}}_R$ holds. Let (P, F, F_1) be the $\mathcal{P}_\star$-triple over S. By Proposition 3.2.5, we can lift the $\mathcal{O}'$-action of $\mathcal{P}_\star$ uniquely. If we define the S-module P_0 by $S \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(S)} P$ and the R-module $P_{\star,0}$ by $R \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(R)} P_\star$, we get a commutative diagram of S-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_S & \longrightarrow & P_0 & \longrightarrow & S \otimes_{\mathcal{O}'\otimes_{\mathcal{O}} S} P_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta_R & \longrightarrow & P_{\star,0} & \longrightarrow & R \otimes_{\mathcal{O}'\otimes_{\mathcal{O}} R} P_{\star,0} & \longrightarrow & 0, \end{array}$$

where the upper line lifts the lower line via $S \to R$. Clearly Δ_S is $\mathcal{O}'$-invariant for the lifted $\mathcal{O}'$-action and for the module M_0, considered as a submodule of P_0 (the sequence splits), holds $\kappa(M_0) \subseteq M_0 \oplus \Delta_S$ for each $\kappa \in \mathcal{O}'$ by assumption, since M_0 respected the $\mathcal{O}'$-action when we considered the element $(\widehat{\mathcal{P}}_R, M_0)$ of $(\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R)$. Since M_0 lifts the module $L_{\Delta,0,R} \subset R \otimes_{\mathrm{w}_0, W_{\mathcal{O}}(R)} P_\star$ we get that $(\mathcal{P}_\star, \Delta_S \oplus M_0)$ is in $(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R)$ and is mapped to $(\widehat{\mathcal{P}}_R, M_0)$ via α'. Hence, $\widehat{\mathcal{P}}$ lies in the image of $\Gamma_2(\mathcal{O},\mathcal{O}')_S$. □

Proposition 3.3.6. Let R be a complete local ring with maximal ideal $\mathfrak{m}$, perfect residue field, nilpotent nilradical and p nilpotent in R. Then $A(R)$ is true. Hence by Lemma 3.3.2 we obtain that $P(\mathcal{O},\mathcal{O}',R)$ is true for each nonramified extension $\mathcal{O} \to \mathcal{O}'$ and each $\mathcal{O}'$-algebra structure on R. Furthermore, $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_i(\mathcal{O},\mathcal{O}')$ over R are equivalences of categories for nonramified/ totally ramified extensions $\mathcal{O} \to \mathcal{O}'$ and each $\mathcal{O}'$-algebra structure on R.

Proof: Let us assume that R is equipped with an $\mathcal{O}'$-algebra structure and $P(\mathcal{O},R)$ resp. $P(\mathcal{O},\mathcal{O}',R)$ is true. Then by Proposition 3.3.4 the functors $\Omega_1(\mathcal{O},\mathcal{O}'), \Gamma_2(\mathcal{O},\mathcal{O}')$ resp. $\Omega_2(\mathcal{O},\mathcal{O}')$ are fully faithful, so we only have to show that they are essentially surjective. With the help of the previous Proposition we may consider from now on only reduced R in the proof. By considering Proposition 3.3.3 this is immediate for the case, when R is a perfect field of characteristic p extending the residue field of $\mathcal{O}'$. For general reduced complete local R with perfect residue field and p nilpotent in R, equipped with an $\mathcal{O}'$-algebra structure we obtain, by using the previous Proposition again, that the equivalences are established for $R/\mathfrak{m}^n$ for each n.
Now we take a look at $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ for the whole R. Since these functors are compatible with base change, we may take a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ resp. a nilpotent $\mathcal{O}'$-display $\mathcal{P}$ over R, make a base change to $R/\mathfrak{m}^n$ for each n and we obtain a nilpotent f-$\mathcal{O}$-display resp. a nilpotent $\mathcal{O}'$-display $\mathcal{P}_{R/\mathfrak{m}^n}$. These nilpotent displays now correspond to nilpotent $\mathcal{O}$-displays over $R/\mathfrak{m}^n$ with strict $\mathcal{O}'$-actions resp. to nilpotent f-$\mathcal{O}$-displays over $R/\mathfrak{m}^n$ and they form an

inverse system. By building the projective limit we obtain an $\mathcal{O}$-display over R with a strict $\mathcal{O}'$-action resp. an f-$\mathcal{O}$-display over R, say $\mathcal{P}_\star$, which is mapped via $\Omega_i(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ to $\mathcal{P}$, when the functors are considered as functors from general display structures, i.e., not necessarily nilpotent ones. To show the essentially surjectivity of $\Omega_i(\mathcal{O},\mathcal{O}',)$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$, when restricted to nilpotent display structures again, it remains to show that these display structures $\mathcal{P}_\star$ are nilpotent. For this we may utilize the fact that R is reduced and may be embedded into a product of algebraic closed fields of characteristic p. Hence we may restrict ourselves to the case, when R is an algebraically closed field of characteristic p which extends the residue field of $\mathcal{O}'$. First we treat $\Omega_1(\mathcal{O},\mathcal{O}')$ and consider the commutative diagram of $W_{\mathcal{O}}(R)$-modules

$$\begin{array}{ccc}
P_\star & \longrightarrow & P = P_{\star,0} \\
\downarrow {\scriptstyle V_\star^{fN\sharp}} & & \downarrow {\scriptstyle V^{N\sharp}} \\
W_{\mathcal{O}}(R) \otimes_{F^{fN},W_{\mathcal{O}}(R)} P_\star & \longrightarrow & W_{\mathcal{O}}(R) \otimes_{F^{fN},W_{\mathcal{O}}(R)} P_{\star,0} \\
\downarrow & & \downarrow \\
R \otimes_{\mathrm{w}_{fN},W_{\mathcal{O}}(R)} P_\star & \twoheadrightarrow & R \otimes_{\mathrm{w}'_{fN},W_{\mathcal{O}}(R)} P_{\star,0},
\end{array}$$

where N is chosen that large, such that the right vertical composite map is zero. The nilpotence of $\mathcal{P}_\star$ follows, since $P_{\star,i} = F_1^i(Q_{\star,0})$ holds for each $i = 1,\dots,f-1$ with the usual graduation and so the composite map

$$P_\star \stackrel{V^{f(N+1)\sharp}}{\longrightarrow} W_{\mathcal{O}}(R) \otimes_{F^{f(N+1)},W_{\mathcal{O}}(R)} P_\star \to R \otimes_{\mathrm{w}_{f(N+1)},W_{\mathcal{O}}(R)} P_\star$$

is zero.
Let us now consider $\Omega_2(\mathcal{O},\mathcal{O}')$. Here we obtain the commutative diagram of $W_{\mathcal{O}}(R)$-modules

$$\begin{array}{ccc}
P_\star & \longrightarrow & P = W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_\star \\
\downarrow {\scriptstyle V_\star^{N\sharp}} & & \downarrow {\scriptstyle V^{N\sharp}} \\
W_{\mathcal{O}}(R) \otimes_{F^{fN},W_{\mathcal{O}}(R)} P_\star & \longrightarrow & W_{\mathcal{O}'}(R) \otimes_{F'^N,W_{\mathcal{O}'}(R)} W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_\star \\
\downarrow & & \downarrow \\
R \otimes_{\mathrm{w}_{fN},W_{\mathcal{O}}(R)} P_\star & \longrightarrow & R \otimes_{\mathrm{w}'_N,W_{\mathcal{O}'}(R)} W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_\star,
\end{array}$$

where N is chosen as above. The lower horizontal map is an isomorphism, from which we can deduce the nilpotence of $\mathcal{P}_\star$.
Now we shift our focus towards the totally ramified case. We get a commutative

diagram of $W_{\mathcal{O}}(R)$-modules

$$\begin{array}{ccc}
P_\star & \longrightarrow & P \\
\downarrow {\scriptstyle V_\star^{N\sharp}} & & \downarrow {\scriptstyle V^{N\sharp}} \\
W_{\mathcal{O}}(R) \otimes_{F^N, W_{\mathcal{O}}(R)} P_\star & \longrightarrow & W_{\mathcal{O}'}(R) \otimes_{F'^N, W_{\mathcal{O}'}(R)} P \\
\downarrow & & \downarrow \\
R \otimes_{\mathrm{w}_N, W_{\mathcal{O}}(R)} P_\star & \longrightarrow & R \otimes_{\mathrm{w}'_N, W_{\mathcal{O}'}(R)} P
\end{array}$$

with N as above. Since the lower horizontal map is an isomorphism the nilpotence of $\mathcal{P}_\star$ follows. □

Chapter 4

The stack of truncated f-$\mathcal{O}$-displays

In this chapter we assume that the reader is familiar with the basic terminology of stacks, as it can be found in [LMB91]. We take the ideas of [Lau08], but apply them not to the functors $BT_{\mathcal{O}}$ resp. $BT_{\mathcal{O}}^{(f)}$, but to the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_i(\mathcal{O}, \mathcal{O}')$, where $\mathcal{O} \to \mathcal{O}'$ is a nonramified / totally ramified extension of rings of integers of non-Archimedean local fields of characteristic zero. Unless otherwise stated, if we just talk about $\mathcal{O}$ (with no reference to an $\mathcal{O}'$) then we just mean any ring of integers of a non-Archimedean local field of characteristic zero; for given $f \geq 1$ and $\mathcal{O}$, k is the residue field of $\mathcal{O}'$, where $\mathcal{O}'$ is the nonramified extension of $\mathcal{O}$ of degree f, and R is an k-algebra. The primary ideas are essentially taken from [Lau08], but with the definition of a truncated f-$\mathcal{O}$-display inspired from [Lau, Chapter 3].

4.1 Truncated f-$\mathcal{O}$-displays

If we denote for an π-adic $\mathcal{O}$-algebra R and a positive integer n the ring of truncated ramified Witt vectors of length n by $W_{\mathcal{O},n}(R)$ and the kernel of w_0 by $I_{\mathcal{O},R,n}$ then we have an $\mathcal{O}$-algebra morphism ${}^{F_n} : W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R)$ induced by the Frobenius on $W_{\mathcal{O}}(R)$ and the inverse of the Verschiebung of $W_{\mathcal{O}}(R)$ induces a ^{F_n}-linear bijective map ${}^{V_n^{-1}} : I_{\mathcal{O},R,n+1} \to W_{\mathcal{O},n}(R)$. If $\pi R = 0$, the Frobenius induces an $\mathcal{O}$-algebra endomorphism ^{F_n} of $W_{\mathcal{O},n}(R)$ and the ideal $I_{\mathcal{O},R,n+1}$ of $W_{\mathcal{O},n+1}(R)$ is a $W_{\mathcal{O},n}(R)$-module. Since this ^{F_n} is obtained by the map ${}^{F_n} : W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R)$ because the $(n+1)$-th entry has no influence on the value, this abuse of notation seems to be tolerable. A similar argumentation establishes that $I_{\mathcal{O},R,n+1}$ is a $W_{\mathcal{O},n}(R)$-module, since for every lift of a fixed element of $W_{\mathcal{O},n}(R)$ to an element of $W_{\mathcal{O},n+1}(R)$ the multiplication with a fixed element of $I_{\mathcal{O},R,n+1}$ has the same value.

Definition 4.1.1. Let $f \geq 1$, $\mathcal{O}$ as usual and R a k-algebra. An *f-$\mathcal{O}$-pre-display over R* is a sextuple $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$, where P and Q are $W_{\mathcal{O}}(R)$-modules with morphisms

$$I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P,$$

and $F : P \to P$ and $F_1 : Q \to P$ are ${}^{F^f}$-linear maps, such that $\iota\varepsilon : I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \to P$ and $\varepsilon(1 \otimes \iota) : I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} Q \to Q$ are the multiplication morphisms and $F_1\varepsilon = {}^{F^{f-1}V^{-1}} \otimes F$ holds. If P and Q are $W_{\mathcal{O},n}(R)$-modules, we call $\mathcal{P}$ an f-$\mathcal{O}$-pre-display of level n.

A morphism between two f-$\mathcal{O}$-pre-displays $\mathcal{P}, \mathcal{P}'$ consists of a tuple of morphisms (α_0, α_1), such that

$$\begin{array}{ccccc} I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P & \xrightarrow{\varepsilon} & Q & \xrightarrow{\iota} & P \\ \downarrow{\scriptstyle 1\otimes\alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} \\ I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P' & \xrightarrow{\varepsilon'} & Q' & \xrightarrow{\iota'} & P' \end{array}$$

commutes and $\alpha_1 \circ F_1 = F_1' \circ \alpha_0$ and $\alpha_1 \circ F = F' \circ \alpha_1$ hold. It is easily seen that we obtain an abelian category, named $(f - \text{pre-disp}_{\mathcal{O}} / R)$, which contains $(f - \text{disp}_{\mathcal{O}} / R)$ as a full subcategory. We denote the abelian subcategory of f-$\mathcal{O}$-pre-displays of level n by $(f - \text{pre-disp}_{\mathcal{O},n} / R)$.

Definition 4.1.2. A *truncated pair of level n over R* is a quadruple $\mathcal{B} = (P, Q, \iota, \varepsilon)$, where P and Q are $W_{\mathcal{O},n}(R)$-modules with module morphisms

$$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$$

such that

- $\iota\varepsilon : I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \to P$ and $\varepsilon(1 \otimes \iota) : I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} Q \to Q$ are the multiplication maps, i.e., they coincide with

  $$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \to I_{\mathcal{O},n,R} \otimes_{W_{\mathcal{O},n}(R)} P \xrightarrow{\text{mult}} P$$

  and

  $$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} Q \to I_{\mathcal{O},n,R} \otimes_{W_{\mathcal{O},n}(R)} Q \xrightarrow{\text{mult}} Q,$$

  respectively, where $I_{\mathcal{O},n+1,R} \to I_{\mathcal{O},n,R}$ is the restriction map and mult the multiplication map,

- P is projective and of finite type over $W_{\mathcal{O},n}(R)$,

- $\text{Coker}(\iota)$ is projective over R and

- We have an exact sequence

$$0 \to J_{R,n+1} \otimes_R \operatorname{Coker}(\iota) \xrightarrow{\overline{\varepsilon}} Q \xrightarrow{\iota} P \to \operatorname{Coker}(\iota) \to 0,$$

  where $J_{R,n+1}$ is defined as the kernel of the restriction map $W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R)$ and $\overline{\varepsilon}$ is induced by ε.

A *normal decomposition* for a truncated pair is a pair of projective $W_{\mathcal{O},n}(R)$-modules (L, T) with $L \subseteq Q$ and $T \subseteq P$, such that

$$L \oplus T \xrightarrow{\iota+1} P \text{ and } L \oplus (I_{\mathcal{O},R,n+1} \otimes_{W_{\mathcal{O},n}(R)} T) \xrightarrow{1+\varepsilon} Q$$

are bijective morphisms. By the obvious generalization of [Lau, Lemma 3.3.] every ramified truncated pair admits a normal decomposition .

Definition 4.1.3. A *truncated f-$\mathcal{O}$-display of level n over R* is an f-$\mathcal{O}$-pre-display $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$ of level n over R, such that $(P, Q, \iota, \varepsilon)$ is a truncated pair of level n and the image of F_1 generates P as a $W_{\mathcal{O},n}(R)$-module.

The *rank* of $\mathcal{P}$ is defined as the rank of P over $W_{\mathcal{O},n}(R)$. We denote the category of truncated f-$\mathcal{O}$-displays of level n over R by $(f - \operatorname{disp}_{\mathcal{O},n}/R)$. This is a full subcategory of the category of f-$\mathcal{O}$-pre-displays of level n over R.
If we are given a truncated pair $(P, Q, \iota, \varepsilon)$ with normal decomposition (L, T), then we have a bijection between the set of pairs (F, F_1) such that $(P, Q, \iota, \varepsilon, F, F_1)$ is a truncated f-$\mathcal{O}$-display and the set of F_n^f-linear isomorphisms $\Psi : L \oplus T \to P$, such that $\Psi|_L = F_1|_L$ and $\Psi|_T = F|_T$. If L and T are free $W_{\mathcal{O},n}(R)$-modules, then Ψ is described by an invertible matrix with coefficients in $W_{\mathcal{O},n}(R)$. The proof of the bijection is an obvious variation of [Zin02, Lemma 9] and the case, when L and T are free, is a variation of the explanation after this Lemma. We call (L, T, Ψ) a *normal decomposition of* $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$.
Furthermore, we need to remark that morphisms (α_0, α_1) between two truncated f-$\mathcal{O}$-displays over level n, say $\mathcal{P}, \mathcal{P}'$, may be described in a reduced way. If we are given a normal decomposition (L, T) of $\mathcal{P}$, it suffices to know $(\alpha_0|_L, \alpha_1|_T)$, since we obtain by the definition of a morphism that $\alpha_1|_{\iota L} = \iota' \circ \alpha_0|_L$ and $\alpha_0|_{\varepsilon(I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} T)} = \varepsilon'(1 \otimes \alpha_1|_T)$ must hold.
All assertions from Lemma 3.5. to Proposition 3.14. in [Lau] are true in their obvious generalization, and their proofs will be essentially the same, so we omit most of them here. We will only prove Lemma 3.6. and Lemma 3.10., since we need to know what truncation means.

Lemma 4.1.4. (cf. [Lau, Lemma 3.6.]) Let $f \geq 1$, $\mathcal{O}$ and a morphism of k-algebras $\beta : R \to R'$ be given. Then there is a unique base change functor

$$\beta_\star : (f - \operatorname{disp}_{\mathcal{O},n}/R) \quad \to \quad (f - \operatorname{disp}_{\mathcal{O},n}/R')$$

together with a natural isomorphism

$$\mathrm{Hom}_{(f-\text{pre-disp}_{\mathcal{O},n}/R)}(\mathcal{P},\beta^\star\mathcal{P}') \cong \mathrm{Hom}_{(f-\text{disp}_{\mathcal{O},n}/R')}(\beta_\star\mathcal{P},\mathcal{P}'),$$

for all truncated f-$\mathcal{O}$-displays $\mathcal{P}$ of level n over R resp. $\mathcal{P}'$ of level n over R'. Here $\beta^\star$ is the functor $(f-\text{pre-disp}_{\mathcal{O},n}/R') \to (f-\text{pre-disp}_{\mathcal{O},n}/R)$ obtained by restriction to scalars.

Proof: In terms of normal decompositions $\beta_\star$ is given by

$$(L,T,\Psi) \mapsto (W_{\mathcal{O},n}(R') \otimes_{W_{\mathcal{O},n}(R)} L, W_{\mathcal{O},n}(R') \otimes_{W_{\mathcal{O},n}(R)} T, ^{F_n^f} \otimes \Psi).$$

The rest is obvious. □

Lemma 4.1.5. (cf. [Lau, Lemma 3.10.]) Let $f \geq 1$, $\mathcal{O}$ and a k-algebra R be given. Then there are unique truncation functors

$$\begin{aligned} \tau_n : (f-\text{disp}_{\mathcal{O}}/R) &\to (f-\text{disp}_{\mathcal{O},n}/R) \\ \tau_n : (f-\text{disp}_{\mathcal{O},n+1}/R) &\to (f-\text{disp}_{\mathcal{O},n}/R) \end{aligned}$$

together with a natural isomorphism

$$\mathrm{Hom}_{(f-\text{pre-disp}_{\mathcal{O}}/R)}(\mathcal{P},\mathcal{P}') \cong \mathrm{Hom}_{(f-\text{disp}_{\mathcal{O},n}/R)}(\tau_n\mathcal{P},\mathcal{P}'),$$

if $\mathcal{P}$ is an f-$\mathcal{O}$-display or a truncated f-$\mathcal{O}$-display of level $n+1$ over R and $\mathcal{P}'$ a truncated f-$\mathcal{O}$-display of level n over R. These truncation functors are compatible with base change.

Proof: In terms of normal decompositions τ_n is given by

$$(L,T,\Psi) \mapsto (W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} L, W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} T, ^{F_n^f} \otimes \Psi).$$

The rest is obvious again. □

We now fix some integers $h \geq 0, f \geq 1$ and the ring $\mathcal{O}$ and denote by $f-\mathrm{Disp}_{\mathcal{O},n} \to \operatorname{Spec} k$ the fibered category of truncated f-$\mathcal{O}$-displays of level n and rank h. Hence, $f-\mathrm{Disp}_{\mathcal{O},n}(\operatorname{Spec} R)$ is the groupoid of truncated f-$\mathcal{O}$-displays of level n and rank h over R. There is an obvious morphism $\tau_{\mathcal{O},n} : f-\mathrm{Disp}_{\mathcal{O},n+1} \to f-\mathrm{Disp}_{\mathcal{O},n}$ induced by the truncation functors.

Lemma 4.1.6. (cf. [Lau, Proposition 3.15.]) The fibered category $f-\mathrm{Disp}_{\mathcal{O},n}$ is a smooth Artin algebraic stack with affine diagonal. The truncation morphism $f-\mathrm{Disp}_{\mathcal{O},n+1} \to f-\mathrm{Disp}_{\mathcal{O},n}$ is smooth and surjective.

Proof: By the generalization of [Lau, Proposition 3.14.], we know that $f-\mathrm{Disp}_{\mathcal{O},n}$ is an fpqc stack. To clarify the affineness of the diagonal, we have to show that for truncated f-$\mathcal{O}$-displays $\mathcal{P}_1$ and $\mathcal{P}_2$ of level n and rank h over a k-algebra R

the sheaf $\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is represented by an affine scheme. By passing to an open cover of $\operatorname{Spec} R$, we may assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ have normal decompositions with free modules. The homomorphisms of the underlying truncated pairs are clearly represented by an affine scheme. Commuting with F and F_1 is a closed condition and a homomorphism of truncated pairs is an isomorphism iff it induces isomorphisms on $\operatorname{Coker}(\iota)$ and $\operatorname{Coker}(\varepsilon)$, which is equivalent to demand that two determinants are invertible. Hence, $\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is represented by an affine scheme.
For each integer integer d with $0 \leq d \leq h$, let $f - \mathrm{Disp}_{\mathcal{O},n,d}$ be the substack of $f - \mathrm{Disp}_{\mathcal{O},n}$ where $\operatorname{Coker}(\iota)$ has rank d. We define the functor $X_{\mathcal{O},n,d}$ from the category of affine k-schemes to (Sets) by defining $X_{\mathcal{O},n,d}(\operatorname{Spec} R)$ as the set of invertible $W_{\mathcal{O},n}(R)$-matrices of rank h. Hence, $X_{\mathcal{O},n,d}$ is an affine open subscheme of the affine space of dimension nh^2 over k. We now define the morphism $\pi_{\mathcal{O},n,d} : X_{\mathcal{O},n,d} \to f - \mathrm{Disp}_{\mathcal{O},n,d}$ in the way that $\pi_{\mathcal{O},n,d}(M)$ is the truncated f-$\mathcal{O}$-display given by the normal representation (L, T, Ψ), where $L = W_{\mathcal{O},n}(R)^{h-d}$, $T = W_{\mathcal{O},n}(R)^d$ and M is the matrix representation of Ψ. We define the sheaf of groups $G_{\mathcal{O},n,d}$ by associating to each k-algebra R the group of invertible matrices $\binom{AB}{CD}$ with $A \in \operatorname{Aut}(L)$, $B \in \operatorname{Hom}(T, L)$, $C \in \operatorname{Hom}(L, I_{\mathcal{O},R,n+1} \otimes_{W_{\mathcal{O},n}(R)} T)$ and $D \in \operatorname{Aut}(T)$, where L and T are as above. $G_{\mathcal{O},n,d}$ is an affine open subscheme of the affine space of dimension nh^2 over k and $\pi_{\mathcal{O},n,d}$ is a $G_{\mathcal{O},n,d}$-torsor. So we see that $f - \mathrm{Disp}_{\mathcal{O},n,d}$ and $f - \mathrm{Disp}_{\mathcal{O},n}$ are smooth algebraic stacks over k. The truncation morphism $\tau_{\mathcal{O},n}$ is smooth and surjective because it commutes with the obvious projection $X_{\mathcal{O},n+1,d} \to X_{\mathcal{O},n,d}$, which is smooth and surjective. $\square$

For a truncated f-$\mathcal{O}$-display $\mathcal{P}$ of level n over a k-algebra R there is a unique morphism $V^\sharp : P \to P^{(1)} = W_{\mathcal{O},n}(R) \otimes_{F_n^f, W_{\mathcal{O},n}(R)} P$ with $V^\sharp(F_1(x)) = 1 \otimes x$ for all $x \in Q$. The proof of this is fairly similar to the one of Lemma 2.1.4. $V^\sharp$ is compatible with truncation. We call $\mathcal{P}$ nilpotent, if there is an m, such that the m-th fold iterate of $V^\sharp$, i.e., the composite morphism $P \to P^{(1)} \to \ldots \to P^{(m)}$, is zero. Because $I_{\mathcal{O},R,m}$ is nilpotent, $\mathcal{P}$ is nilpotent, iff its truncation to level 1 is nilpotent. An f-$\mathcal{O}$-display over R is nilpotent iff all its truncations are nilpotent.

Lemma 4.1.7. (cf. [Lau, Lemma 3.17.]) There is a unique reduced closed substack $f - \mathrm{nDisp}_{\mathcal{O},n} \subset f - \mathrm{Disp}_{\mathcal{O},n}$ such that the geometric points of $f - \mathrm{nDisp}_{\mathcal{O},n}$ are precisely the nilpotent truncated f-$\mathcal{O}$-displays of level n. We have the cartesian diagram

$$\begin{array}{ccc} f - \mathrm{nDisp}_{\mathcal{O},n+1} & \longrightarrow & f - \mathrm{nDisp}_{\mathcal{O},n} \\ \downarrow & & \downarrow \\ f - \mathrm{Disp}_{\mathcal{O},n+1} & \longrightarrow & f - \mathrm{Disp}_{\mathcal{O},n} . \end{array}$$

In particular, $f - \mathrm{nDisp}_{\mathcal{O},n+1} \to f - \mathrm{nDisp}_{\mathcal{O},n}$ is smooth and essentially surjective on R-valued points for every R.

Proof: Apart from the last assertion, this is the obvious generalization of a partial result made in [Lau, Lemma 3.17.]. The smoothness and essential surjectivity follow easily. $\square$

Is is sensible to ask if it is possible to establish the analogous results (compared to [Lau08, Chapter 1 and 2]) for truncated π-divisible $\mathcal{O}$-modules of level n and height h and then proceed as in the rest of [Lau08]. These results are indeed true (see [Fal02]). Nevertheless, it is possible to use our established arguments in the codomains of $\Omega_1(\mathcal{O},\mathcal{O}'), \Omega_2(\mathcal{O},\mathcal{O}'), \Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ in order to establish that these functors are equivalences for all $\mathcal{O}'$-algebras with π' nilpotent in them, under the assumption that the respective functor is fully faithful for all $\mathcal{O}'$-algebras with π' nilpotent in them. For the fully faithfulness of the functors we will establish a generalized version of Zink's universal extensions in the next chapter.

4.2 Applications to f-$\mathcal{O}$-displays

Proposition 4.2.1. (cf. [Lau08, Proposition 1.2.]) Let $f \geq 1$ and $\mathcal{O}$ be given. For any positive integer h there is a sequence of finitely generated reduced k-algebras $B_1 \to B_2 \to \dots$ with faithfully flat smooth maps and a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ of rank h over $B = \bigcup B_i$ with the property that for any other nilpotent f-$\mathcal{O}$-display $\mathcal{P}'$ over a reduced k-algebra R and of rank h, there are is a sequence $R \to S_1 \to S_2 \to \dots$ of faithfully flat étale k-algebra morphisms and a k-algebra morphism $B \to S = \bigcup S_i$ such that $\mathcal{P}_S \cong \mathcal{P}'_S$.

Proof: We construct recursively an infinite commutative diagram

$$\begin{array}{ccccccc}
Y_1 & \longleftarrow & Y_2 & \longleftarrow & Y_3 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
f-\mathrm{nDisp}_{\mathcal{O},1} & \longleftarrow & f-\mathrm{nDisp}_{\mathcal{O},2} & \longleftarrow & f-\mathrm{nDisp}_{\mathcal{O},3} & \longleftarrow & \cdots,
\end{array}$$

where $Y_m = \operatorname{Spec} B_m$ for a finitely generated k-algebra B_m, such that $Y_1 \to f-\mathrm{nDisp}_{\mathcal{O},1}$ and $Y_{m+1} \to \chi_{m+1} = f-\mathrm{nDisp}_{\mathcal{O},m+1} \times_{f-\mathrm{nDisp}_{\mathcal{O},m}} Y_m$ are smooth presentations. By Lemma 4.1.7 the morphisms $B_m \to B_{m+1}$ are faithfully flat and smooth. We have a canonical nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over $B = \varinjlim B_m$.
A nilpotent f-$\mathcal{O}$-display $\mathcal{P}'$ over a reduced k-algebra R is equivalent to a compatible system of morphisms $\operatorname{Spec} R \to f-\mathrm{nDisp}_{\mathcal{O},m}$. For $\operatorname{Spec} S_1 = \operatorname{Spec} R \times_{f-\mathrm{nDisp}_{\mathcal{O},1}} Y_1$, there is a natural map $\operatorname{Spec} S_1 \to \chi_2$ and for $m \geq 2$ we have got for $\operatorname{Spec} S_m = \operatorname{Spec} S_{m-1} \times_{\chi_m} Y_m$ that there is a natural map $\operatorname{Spec} S_m \to \chi_{m+1}$. Hence we obtain compatible isomorphisms $\tau_n(\mathcal{P})_S \cong \tau_n(\mathcal{P}')_S$ over $S = \bigcup S_n$, where τ_n should be the truncation morphisms, hence we obtain $\mathcal{P}_S \cong \mathcal{P}'_S$. Because a surjective smooth morphism has a section étale locally, we may replace the S_n by an étale system. $\square$

Definition 4.2.2. (cf. [Lau08, Definition 5.4.]) A nilpotent f-$\mathcal{O}$-display over a k-algebra R is called of reduced type if all its truncations are in $f - \mathrm{nDisp}_{\mathcal{O},m}$.

Proposition 4.2.3. (cf. [Lau08, Lemma 5.5.]) A nilpotent f-$\mathcal{O}$-display over a k-algebra R is of reduced type, iff there are k-algebra morphisms $R \to S \leftarrow A$ with A reduced, $S = \bigcup S_i$ for a system of étale faithfully flat k-algebra morphisms $R \to S_1 \to S_2 \to \ldots$, and the base change of this f-$\mathcal{O}$-display to S descends to A.

Proof: While the backward direction is immediate, we need, in order to prove the forward direction, the proof of Proposition 4.2.1. But here, we may drop the equivalence condition in the first line of the second part, since we already demand that our f-$\mathcal{O}$-display is of reduced type. □

Unless otherwise stated, from now on until the end of this chapter, R, S, etc. are just $\mathcal{O}'$-algebras and not necessarily k-algebras.

Definition 4.2.4. We call a faithfully flat morphism of $\mathcal{O}'$-algebras $R \to S$ an *admissible covering*, if $S \otimes_R S$ is reduced.

The use of this Definition is the following: Let us assume that $\Omega_1(\mathcal{O}, \mathcal{O}')$, $\Omega_2(\mathcal{O}, \mathcal{O}'), \Gamma_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is fully faithful for all $\mathcal{O}'$-algebras with π' nilpotent in them. If $R \to S$ is an admissible covering over $\mathcal{O}'$ with π' nilpotent in R, we may apply Proposition 2.2.10. So if we get that for a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R resp. nilpotent $\mathcal{O}'$-display $\mathcal{P}$ over R the nilpotent f-$\mathcal{O}$-display over S resp. nilpotent $\mathcal{O}'$-display over S obtained by base change lies in the image of the corresponding functor over S, then $\mathcal{P}$ does so as well. All assertions we will need about admissible coverings, can be found in [Lau08, Chapter 3], where the ring morphisms have to be replaced by $\mathcal{O}'$-algebra morphisms. The proof of [Lau08, Proposition 3.4.] depends on [Lau08, Lemma 3.3.], which is not correct. In [Lau, 8.2.] it is clarified, how to prove the Proposition without using this Lemma.

Proposition 4.2.5. (cf. [Lau08, Proposition 4.4.,Lemma 6.1.]) Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension. Assume that $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}')$, $\Gamma_1(\mathcal{O}, \mathcal{O}')$ or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is fully faithful for all $\mathcal{O}'$-algebras with π' nilpotent in them, then the respective functor is an equivalence for all such algebras.

Proof: It remains to show that $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}'), \Gamma_1(\mathcal{O}, \mathcal{O}')$ resp. $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is essentially surjective for all $\mathcal{O}'$-algebras R with π' nilpotent in R. We treat only the $\Omega_1(\mathcal{O}, \mathcal{O}')$-case, since the others follow analogously. At first we show the assertion for all reduced k-algebras R, where k is always the residue field of $\mathcal{O}'$ here. Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. With $R \to S \leftarrow B$ given as in Proposition 4.2.1, $\mathcal{P}_S$ descends to B. Since $R \to S$ is an admissible covering, it is enough to show that $\Omega_1(\mathcal{O}, \mathcal{O}')$ is essentially surjective over B. When k'

is an uncountable algebraically closed field of characteristic p extending k, then $B \to B \otimes_k k'$ is an admissible covering and we may apply [Lau08, Proposition 3.2.] to $B \otimes_k k' = \bigcup B_i \otimes_k k'$, so we may reduce to the base ring $\prod (B \otimes_k k')_{\mathfrak{m}}$, where the product runs through all maximal ideals $\mathfrak{m}$ of $B \otimes_k k'$. We may reduce to $(B \otimes_k k')_{\mathfrak{m}}$, since nilpotent f-$\mathcal{O}$-displays are compatible with arbitrary products of reduced local $\mathcal{O}'$-algebras. The residue field of $(B \otimes_k k')_{\mathfrak{m}}$ is k' by [Lau08, Lemma 4.3.] and we may apply [Lau08, Proposition 3.4.] to consider just the completion of $(B \otimes_k k')_{\mathfrak{m}}$, for which the assertion is already known by Proposition 3.3.6.

Now we consider general $\mathcal{O}'$-algebras R with π' nilpotent in R. By Proposition 3.3.5 it suffices to treat the case, where R is a k-algebra. Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. Because $f-\mathrm{Disp}_{\mathcal{O},1}$ is of finite type, we obtain that $f-\mathrm{nDisp}_{\mathcal{O},1} \to f-\mathrm{Disp}_{\mathcal{O},1}$ is finitely presented. Since $\mathcal{P}$ is modulo a nilpotent ideal of reduced type, we may assume by Proposition 3.3.5 that $\mathcal{P}$ is of reduced type. Now let $R \to S \leftarrow A$ be as in Proposition 4.2.3. Because $\Omega_1(\mathcal{O}, \mathcal{O}')$ fully faithful, it suffices to show that $\mathcal{P}_S$ lies in the image of $\Omega_1(\mathcal{O}, \mathcal{O}')$, which holds, since $\mathcal{P}_S$ descends to A, which is reduced, and the result follows by the first part of the proof. □

Chapter 5

Crystals

In this section we associate to each nilpotent $\mathcal{O}$-display and to each corresponding π-divisible formal $\mathcal{O}$-module a crystal and show that they are isomorphic on the nilpotent ideal crystalline site of $\mathcal{O}$-pd-thickenings. From this we can deduce that $BT_{\mathcal{O}}$ is faithful for all $\mathcal{O}$-algebras with π nilpotent in them. For nilpotent f-$\mathcal{O}$-displays we will construct an extension with which we can establish the fully faithfulness of $\Omega_1(\mathcal{O}, \mathcal{O}')$. Combining these results we obtain all desired equivalences. The rings $\mathcal{O}, \mathcal{O}', \mathcal{O}_0$ are always assumed to be rings of integers non-Archimedean local fields of characteristic zero.

5.1 The crystal associated to f-$\mathcal{O}$-displays

We need to have a look at the different types of crystalline sites (see [Zin02, Remark after Theorem 46]).

Definition 5.1.1. Let X be a scheme over $\operatorname{Spec}\mathcal{O}$ with π locally nilpotent in $\mathcal{O}_X$ (which should not be confused with $\mathcal{O}$).

- The *crystalline site* consists of objects (U, T, δ), with $U \subset X$ an open subscheme, $U \to T$ a closed immersion over $\operatorname{Spec}\mathcal{O}$, with π locally nilpotent on T, defined by an ideal $\mathcal{J} \subset \mathcal{O}_T$ and δ an $\mathcal{O}$-pd structure on $\mathcal{J}$ (where we extend Definition 2.3.2 trivially from an ideal of an $\mathcal{O}$-algebra to $\mathcal{J}$), which has to be compatible with the canonical $\mathcal{O}$-pd structure on $\pi\mathcal{O} \subset \mathcal{O}$.

- The *nilpotent ideal crystalline site* consists of those objects of the crystalline site, for which the ideal $\mathcal{J}$ is (locally) nilpotent.

The nilpotent ideal crystalline site should not be confused with the (more common) nilpotent crystalline site, which consists of those objects in the crystalline site, where the $\mathcal{O}$-pd structure on $\mathcal{J}$ is (locally) nilpotent, which is a stronger condition than to demand that $\mathcal{J}$ itself is (locally) nilpotent.
We let $W_{\mathcal{O}}(\mathcal{O}_X^{\text{crys}})$ be the sheaf on the crystalline site, which associates to (U, T, δ) the $\mathcal{O}$-algebra $W_{\mathcal{O}}(\Gamma(T, \mathcal{O}_T))$. We call a crystal in $W_{\mathcal{O}}(\mathcal{O}_X^{\text{crys}})$-modules a *Witt crystal*. For an f-$\mathcal{O}$-display $\mathcal{P}$ over an $\mathcal{O}$-algebra R with π nilpotent in R we will now define a Witt crystal $\mathcal{K}_{\mathcal{P}}$ on the crystalline site over $\operatorname{Spec} R$. It suffices, to give the value of $\mathcal{K}_{\mathcal{P}}$ for $\mathcal{O}$-pd-thickenings $\operatorname{Spec} R' \to \operatorname{Spec} S$, where $\operatorname{Spec} R' \hookrightarrow \operatorname{Spec} R$ is an affine open neighbourhood. If the $\mathcal{P}$-triple over S associated to $\mathcal{P}_{R'}$ looks like $(\widetilde{P}, F, F_1)$ (see section 3.2) we define

$$\mathcal{K}_{\mathcal{P}}(\operatorname{Spec} R' \to \operatorname{Spec} S) = \widetilde{P},$$

which we will also denote by $\mathcal{K}_{\mathcal{P}}(S)$ if the setting is clear.

Definition 5.1.2. The sheaf $\mathcal{K}_{\mathcal{P}}$ on the crystalline site over $\operatorname{Spec} R$ is called the *Witt crystal associated to* $\mathcal{P}$. We define the *Dieudonné crystal* by

$$\mathcal{D}_{\mathcal{P}}(S) = \mathcal{K}_{\mathcal{P}}(S)/I_{\mathcal{O},S}\mathcal{K}_{\mathcal{P}}(S),$$

which is a crystal in $\mathcal{O}_{\operatorname{Spec} R}^{\text{crys}}$-modules on the crystalline site.

We define for any topological $\mathcal{O}$-pd-thickening $(S, \mathfrak{a}_n) \to R'$ the crystals by

$$\mathcal{K}_{\mathcal{P}}(S) = \varprojlim_n \mathcal{K}_{\mathcal{P}}(S/\mathfrak{a}_n)$$
$$\mathcal{D}_{\mathcal{P}}(S) = \varprojlim_n \mathcal{D}_{\mathcal{P}}(S/\mathfrak{a}_n).$$

It can be easily verified, that we can formulate the main assertions for triples for topological $\mathcal{O}$-pd-thickenings in an obviously generalized manner. Both crystals are compatible with base change: If we consider a morphism of $\mathcal{O}$-pd-thickenings as in Section 3.2

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

we obtain

$$\begin{aligned} \mathcal{K}_{\mathcal{P}_{R'}}(S') &\simeq W_{\mathcal{O}}(S') \otimes_{W_{\mathcal{O}}(S)} \mathcal{K}_{\mathcal{P}}(S), \\ \mathcal{D}_{\mathcal{P}_{R'}}(S') &\simeq S' \otimes_S \mathcal{D}_{\mathcal{P}}(S). \end{aligned}$$

These isomorphisms are by obvious reasons also true, when we consider topological $\mathcal{O}$-pd-thickenings. Now let us consider the canonical morphism

$$\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R.$$

The kernel $I_{\mathcal{O},R}$ may be equipped with an $\mathcal{O}$-pd-structure

$$\gamma({}^V w) = \pi^{q-2}\, {}^V(w^q) \tag{5.1}$$

for all $w \in W_{\mathcal{O}}(R)$. One easily verifies that this is indeed such a structure by going to a suitable universal situation. The morphism $\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R$ is a topological $\mathcal{O}$-pd-thickening, since $\mathrm{w}_0 : W_{\mathcal{O},n}(R) \to R$ are $\mathcal{O}$-pd-thickenings with an $\mathcal{O}$-pd-structure given by γ. If $S \to R$ is an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$, then $\mathfrak{a}$ considered as an ideal of $W_{\mathcal{O}}(S)$ (by (2.13)) has the same $\mathcal{O}$-pd-structure as considered as an ideal of S. The kernel of the composite map $W_{\mathcal{O}}(S) \to S \to R$ is $I_{\mathcal{O},S} \oplus \mathfrak{a}$, where on both summands we have $\mathcal{O}$-pd-structures, so this follows for the whole kernel. Hence, $W_{\mathcal{O}}(S) \to R$ is a topological $\mathcal{O}$-pd-thickening by considering $W_{\mathcal{O},n}(S) \to R$ for each n. For the following Theorem we need to introduce the Cartier morphism

$$\Delta : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(W_{\mathcal{O}}(R)),$$

which is uniquely determined for every $\mathcal{O}$-algebra by functoriality and

$$\widehat{\mathrm{w}}_n(\Delta(\xi)) = {}^{F^n} \xi$$

for all $\xi \in W_{\mathcal{O}}(R)$, where $\widehat{\mathrm{w}}_n : W_{\mathcal{O}}(W_{\mathcal{O}}(R)) \to W_{\mathcal{O}}(R)$ should denote the n-th Witt polynomial for $W_{\mathcal{O}}(W_{\mathcal{O}}(R))$. Furthermore, the following relations hold for each n and $\xi \in W_{\mathcal{O}}(R)$ (the operators belonging to $W_{\mathcal{O}}(W_{\mathcal{O}}(R))$ are marked with a hat):

$$W_{\mathcal{O}}(\mathrm{w}_n)(\Delta(\xi)) = {}^{F^n} \xi$$

$$\Delta({}^F \xi) = {}^{\widehat{F}} (\Delta(\xi)) = W_{\mathcal{O}}({}^F)(\Delta(\xi))$$

$$\Delta({}^V \xi) - {}^{\widehat{V}} (\Delta(\xi)) = [{}^V \xi, 0, 0, \ldots]$$

By passing to a suitable universal situation, these equations are easily verified.

Theorem 5.1.3. (cf. [Zin02, Proposition 53, Corollary 56]) Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$ and $\mathcal{P} = (P, Q, F, F_1)$ be a nilpotent f-$\mathcal{O}$-display over R. Let $\mathcal{T} = (\widetilde{P}, F, F_1)$ be the unique $\mathcal{P}$-triple over S and $\overline{\mathcal{T}}$ the unique $\mathcal{P}$-triple related to the topological $\mathcal{O}$-pd-thickening $W_{\mathcal{O}}(S) \to R$ with kernel $I_{\mathcal{O},S} \oplus \mathfrak{a}$. Then

$$\overline{\mathcal{T}} = (W_{\mathcal{O}}(W_{\mathcal{O}}(S)) \otimes_{\Delta, W_{\mathcal{O}}(S)} \widetilde{P}, F, F_1)$$

holds, where F and F_1 are uniquely determined by the equations

$$\begin{aligned} F(\widehat{\xi} \otimes x) &= {}^{\widehat{F}^f} \widehat{\xi} \otimes Fx \\ F_1(\widehat{\xi} \otimes y) &= {}^{\widehat{F}^f} \widehat{\xi} \otimes F_1 y \\ F_1({}^{\widehat{V}} \xi \otimes x) &= {}^{\widehat{F}^{f-1}} \widehat{\xi} \otimes Fx \end{aligned}$$

for all $\widehat{\xi} \in W_{\mathcal{O}}(W_{\mathcal{O}}(S)), x \in \widetilde{P}$ and $y \in \widehat{Q}$, where $\widehat{Q}$ is the inverse image of Q by the map $\widetilde{P} \to P$. Then we have

$$\mathcal{K}_{\mathcal{P}}(S) = \widetilde{P} = W_{\mathcal{O}}(S) \otimes_{\widehat{\mathrm{w}}_0} (W_{\mathcal{O}}(W_{\mathcal{O}}(S)) \otimes_{\Delta, W_{\mathcal{O}}(S)} \widetilde{P}) = \mathcal{D}_{\mathcal{P}}(W_{\mathcal{O}}(S)).$$

If $W_{\mathcal{O}}(R) \to S$ is a morphism of (topological) $\mathcal{O}$-pd-thickenings over R, we obtain that

$$\begin{aligned} \mathcal{K}_{\mathcal{P}}(S) &\simeq W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} \mathcal{K}_{\mathcal{P}}(R) \\ \mathcal{D}_{\mathcal{P}}(S) &\simeq S \otimes_{W_{\mathcal{O}}(R)} \mathcal{K}_{\mathcal{P}}(R) \end{aligned}$$

hold, where $W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(S)$ is given by

$$W_{\mathcal{O}}(R) \xrightarrow{\Delta} W_{\mathcal{O}}(W_{\mathcal{O}}(R)) \to W_{\mathcal{O}}(S).$$

The proof of [Zin02, Proposition 53] is absolutely analogous to the situation here, so we omit it. The last assertion of the Theorem follows easily from the first one by considering the trivial $\mathcal{O}$-pd-thickening $R \to R$ and then making a base change with respect to $W_{\mathcal{O}}(R) \to S$. The most important situations, in which we will use this fact, are for $S = W_{\mathcal{O},n}(R)$.

5.2 Universal extensions and the crystal of Grothendieck-Messing

In this section we want to introduce more general (universal) extensions compared to the ones introduced by Zink and show the existence of universal extensions. For S an $\mathcal{O}$-algebra and L an S-module, we may define the group $C(L) = \prod_{i \geq 0} V^i L$. We may turn $C(L)$ into an $\mathbb{E}_{\mathcal{O},S}$-module by the equations

$$\begin{aligned} \xi(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 0} V^i \mathrm{w}_n(\xi) l_i, \\ V(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 0} V^{i+1} l_i, \\ F(\sum_{i\geq 0} V^i l_i) &= \sum_{i\geq 1} V^{i-1} \pi l_i, \end{aligned}$$

for all $\xi \in W_{\mathcal{O}}(S)$ and $l_i \in L$. We may interpret $C(L)$ as the Cartier module of the additive group of L. If $\widehat{L}^+$ denotes the functor from Nil_S to $(\,\mathcal{O} - \text{modules}\,)$ defined by

$$\widehat{L}^+(\mathcal{N}) = (\mathcal{N} \otimes_S L)^+$$

for each $\mathcal{N} \in \mathrm{Nil}_S$, then there is a functor isomorphism

$$\mathcal{N} \otimes_S L \simeq \widehat{W_{\mathcal{O}}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},S}} C(L) \tag{5.2}$$

given by $n \otimes l \mapsto [n] \otimes V^0 l$ for $n \in \mathcal{N}$ and $l \in L$. The inverse mapping is given by sending $w \otimes \sum_{i\geq 0} V^i l_i$ to $\sum_{i\geq 0} \mathrm{w}_i(w) \otimes l_i$ for $w \in \widehat{W_{\mathcal{O}}}(\mathcal{N})$ and $l_i \in L$ for all $i \geq 0$ (cf. [Zin86, (2.1) Lemma.]).

Definition 5.2.1. Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$ and G a (π-divisible) formal $\mathcal{O}$-module over R with Cartier module M, which we consider as an $\mathbb{E}_{\mathcal{O},S}$-module. Then an *extension* (L, ι, N, κ, M) *of* M *by the* S*-module* L is an exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(L) \overset{\iota}{\to} N \overset{\kappa}{\to} M \to 0,$$

with N a reduced $\mathbb{E}_{\mathcal{O},S}$-module and $\mathfrak{a}N \subset V^0 L$, where $\mathfrak{a} \subset W_{\mathcal{O}}(S) \subset \mathbb{E}_{\mathcal{O},S}$ is given by (2.13). For simplicity, we just write (with abuse of notation) (L, N, M) instead of (L, ι, N, κ, M).

Now let G, G' be two formal $\mathcal{O}$-modules over R, M, M' their Cartier modules and $\beta : M \to M'$ a morphism between them over R. Furthermore, let (L, N, M) and (L', N', M') be extensions of M and M'. Then a morphism of extensions $(L, N, M) \to (L', N', M')$ consists of a morphism of S-modules $\varphi : L \to L'$, a morphism of $\mathbb{E}_{\mathcal{O},S}$-modules $u : N \to N'$ and the $\mathbb{E}_{\mathcal{O},R}$-linear morphism β, such that the diagram of $\mathbb{E}_{\mathcal{O},S}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(L) & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C(\varphi)} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & C(L') & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

is commutative, where $C(\varphi)$ is given by sending $V^i l$ to $V^i \varphi(l)$ for each $i \geq 0$ and $l \in L$.

Definition 5.2.2. With the above notation, we define the category $\mathrm{Ext}_{1,S\to R}$ by the objects (L, N, M), such that M is the Cartier module of a π-divisible formal $\mathcal{O}$-module over R. The morphisms are those previously described.

In [Zin02, 3.2. The universal extension] we received a geometric interpretation of the extensions in Zink's sense in order to utilize [Mes72, Chapt. IV Theorem (2.2)]. The generalization of Messing's result can be found in [FGL07, Theoreme B.6.3.]. In these Theorems the divided power respectively the $\mathcal{O}$-pd structure on the kernel $\mathfrak{a}$ of the surjection of rings $S \to R$ was required to be nilpotent. Luckily, since we deal only with p-divisible formal groups respectively π-divisible formal $\mathcal{O}$-modules we can overcome the nilpotence condition of the $\mathcal{O}$-pd structure (cf. [Zin, Die Universelle Erweiterung nach Grothendieck und Messing, Theorem 3]). First we establish the existence of a universal extension over an $\mathcal{O}$-algebra R with π nilpotent in R, for which we consider [FGL07, Annexe B.2]. We should remark that in this book it is only referred to coherent sheaves in this particular section, nevertheless all assertions also work when we take the sheaves to be quasi-coherent (which is in fact more natural as a generalization of [Mes72]). From now on, if we write $\mathcal{O}_F$ we mean the usual $\mathcal{O}$, where the F should indicate the non-Archimedean local field of characteristic zero. This convention is sometimes necessary in order to stress that this $\mathcal{O}$ is not the structure sheaf of a scheme. In the following Definition and Proposition and the discussion after them, we consider π-divisible formal $\mathcal{O}_F$-modules in the sense of Messing / Fargues, i.e., fppf-sheaves in $\mathcal{O}_F$-modules with additional conditions.

Definition 5.2.3. (cf. [FGL07, Definition B.3.2.]) Let H be a π-divisible formal $\mathcal{O}_F$-module over $S = \operatorname{Spec} R$, with R an $\mathcal{O}_F$-algebra and $\pi^N R = 0$. An *$\mathcal{O}_F$-vector extension of H (by $\underline{V}$)* is an extension

$$0 \to \underline{V} \to E \to H \to 0$$

of sheaves of $\mathcal{O}_F$-modules over S_{fppf}, such that V is a quasi-coherent $\mathcal{O}_S$-module, $\underline{V}$ is the associated $fppf$-sheaf and the induced action of $\mathcal{O}_F$ on $\operatorname{Lie} E$ is strict. Here $\operatorname{Lie} E$ is defined as the kernel of $f_\star f^\star E \stackrel{\varepsilon=0}{\to} E$, where $f : \operatorname{Spec}(R[\varepsilon]) \to \operatorname{Spec} R = S$.

As in [Mes72], we have $\operatorname{Hom}(H, \underline{W}) = 0$ for each quasi-coherent $\mathcal{O}_S$-module W. Hence, any extension of H by $\underline{W}$ is uniquely determined by its class in $\operatorname{Ext}^1(H, \underline{W})$, because the extensions do not admit automorphisms. Therefore, we may introduce the notion of a *universal $\mathcal{O}$-vector extension*: This is an $\mathcal{O}$-vector extension

$$0 \to \underline{V_{\mathcal{O}}(H)} \to E_{\mathcal{O}}(H) \to H \to 0,$$

such that for any morphism of π-divisible formal $\mathcal{O}$-modules $u : H \to H'$ and any $\mathcal{O}$-vector extension

$$0 \to \underline{W} \to E \to H' \to 0$$

there are unique morphisms $E_{\mathcal{O}}(H) \to E$ and $\underline{V_{\mathcal{O}}(H)} \to \underline{W}$, which is induced by an R-linear morphism $V_{\mathcal{O}}(H) \to W$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{V_{\mathcal{O}}(H)} & \longrightarrow & E_{\mathcal{O}}(H) & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle u} & & \\ 0 & \longrightarrow & \underline{W} & \longrightarrow & E & \longrightarrow & H' & \longrightarrow & 0 \end{array}$$

is commutative.

Proposition 5.2.4. (cf. [FGL07, Proposition B.3.3., Remarque B.3.6.]) With the notation as above, there exists a universal $\mathcal{O}$-vector extension. Furthermore, $E_{\mathcal{O}}(H)$ is a formal $\mathcal{O}$-module, $V_{\mathcal{O}}(H)$ and $\operatorname{Lie} E_{\mathcal{O}}(H)$ are corresponding to finite projective R-modules and there is an exact sequence

$$0 \to \underline{V_{\mathcal{O}}(H)} \to \operatorname{Lie} E_{\mathcal{O}}(H) \to \operatorname{Lie} H \to 0.$$

The proof of Fargues first uses that there is a universal vector extension for the case $\mathcal{O}_F = \mathbb{Z}_p$. We consider $I = \ker(\mathcal{O}_S \otimes_{\mathbb{Z}_p} \mathcal{O}_F \to \mathcal{O}_S)$ and construct

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{V_{\mathbb{Z}_p}(H)} & \longrightarrow & E_{\mathbb{Z}_p}(H) & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \underline{V_{\mathbb{Z}_p}(H)}/I \cdot \operatorname{Lie} E_{\mathbb{Z}_p}(H) & \longrightarrow & \widetilde{E} & \longrightarrow & H & \longrightarrow & 0,
\end{array}$$

where $\widetilde{E} = \underline{V_{\mathbb{Z}_p}(H)}/I \cdot \operatorname{Lie} E_{\mathbb{Z}_p}(H) \coprod_{\underline{V_{\mathbb{Z}_p}(H)}} E_{\mathbb{Z}_p}(H) = E_{\mathbb{Z}_p}(H)/I \cdot E_{\mathbb{Z}_p}(H)$. This is possible since the $\mathcal{O}_F$-action on H as a p-divisible formal group induces $\mathcal{O}_F$-actions on $\underline{V_{\mathbb{Z}_p}(H)}$ and $E_{\mathbb{Z}_p}(H)$ (cf. [Mes72, Chapt. IV Proposition (1.15)]). Then the lower horizontal sequence is the universal one as in [FGL07]. We only have to check that in the proof of the universality of the constructed sequence, we may use quasi-coherent modules as well as coherent ones and that we can consider morphisms of π-divisible formal $\mathcal{O}$-modules $H \to H'$ and $\mathcal{O}$-vector extensions of H' than just the identity morphism $H \to H$ and $\mathcal{O}$-vector extensions of H (where we use [Mes72, Chapt. IV Proposition (1.15)] again), so the proof works completely in the same manner.

We need to remark that formal $\mathcal{O}$-modules in Zink's sense and in Messing's/ Fargues' sense are not the same, i.e., the first ones are functors from Nil_R to the category of abelian groups equipped with a strict $\mathcal{O}$-action, while the second ones are fppf-sheaves over $\operatorname{Spec} R$ in $\mathcal{O}$-modules, such that the $\mathcal{O}$-action is strict. However, we can overcome this problem by associating to a formal $\mathcal{O}$-module over R, say G, in Zink's sense an fppf-sheaf in the following way: Let S be an R-algebra with nilradical $\mathcal{N}$. Then we define

$$G'(S) = \varinjlim_{B=(x_1,\ldots,x_n)\subset\mathcal{N}} G(B),$$

where the colimes runs over each finitely generated ideal B contained in $\mathcal{N}$. It is obvious that each such B is in Nil_R, hence the definition makes sense. Conversely, given a formal $\mathcal{O}$-module G' in Messing's / Fargues' sense over $\operatorname{Spec} R$, we associate a functor $G : \mathrm{Nil}_R \to (\textit{abelian groups})$ by defining

$$G(\mathcal{N}) = \ker(G'(R \oplus \mathcal{N}) \to G'(R)),$$

where $\mathcal{N} \in \mathrm{Nil}_R$ and the $\mathcal{O}$-algebra structure on $R \oplus \mathcal{N}$ is given by (2.5). The strict $\mathcal{O}$-action on G is obtained by obvious arguments. It needs to be checked that these associations do indeed deliver a formal $\mathcal{O}$-module in the sense of the other definition and that they are inverse to each other, which is left to the reader. Hence, after considering the morphisms we obtain an equivalence of the categories of formal $\mathcal{O}$-modules in both senses. The π-divisible formal $\mathcal{O}$-modules correspond to each other.

Definition 5.2.5. Let R be an $\mathcal{O}$-algebra with π nilpotent in R. A *vector group* associated to an R-module M is the functor $\underline{M} : \mathrm{Nil}_R \to (\mathcal{O} - modules)$ by $\mathcal{N} \to M \otimes_R \mathcal{N}$. A *morphism of vector groups* $\underline{M} \to \underline{N}$ for two R-modules M, N is a morphism of functors induced by an R-module morphism $M \to N$. Let G be a π-divisible formal $\mathcal{O}$-module (in the sense of Zink) over R. An *$\mathcal{O}$-extension of G by the finite projective R-module M* is an exact sequence of formal $\mathcal{O}$-modules over R

$$0 \to \underline{M} \to E \to G \to 0. \tag{5.3}$$

By the previous Proposition and the above translation we obtain:

Proposition 5.2.6. (cf. [Zin, Die Universelle Erweiterung nach Grothendieck und Messing, Theorem 2]) Let G be a π-divisible formal $\mathcal{O}$-module (in the sense of Zink) over an $\mathcal{O}$-algebra R with π nilpotent in R. Then there is a universal $\mathcal{O}$-extension of formal $\mathcal{O}$-modules over R

$$0 \to \underline{V_{\mathcal{O}}(G)} \to E_{\mathcal{O}}(G) \to G \to 0. \tag{5.4}$$

This means, given a morphism $f : G \to H$ of π-divisible formal $\mathcal{O}$-modules over R and an $\mathcal{O}$-extension

$$0 \to \underline{M} \to E \to H \to 0, \tag{5.5}$$

there is a unique morphism of R-modules $V_{\mathcal{O}}(f) : V_{\mathcal{O}}(G) \to M$ (inducing $\underline{V_{\mathcal{O}}(G)} \to \underline{M}$) and a unique morphism of formal $\mathcal{O}$-modules $E_{\mathcal{O}}(G) \to E$ over R, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{V_{\mathcal{O}}(G)} & \longrightarrow & E_{\mathcal{O}}(G) & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & E & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

commutes.

It is also possible to apply the argumentation Fargues used to establish Proposition 5.2.4 to [Zin, Die Universelle Erweiterung nach Grothendieck und Messing, Theorem 2] and one would obtain this result for π-divisible formal $\mathcal{O}$-modules in Zink's sense directly. From now on, (π-divisible) formal $\mathcal{O}$-modules are only considered in the sense of Zink (i.e., as functors from Nil_R to the category of abelian groups with an attached strict $\mathcal{O}$-action).

Now we come to the construction of the exponential, for which the following Proposition will be essential.

Proposition 5.2.7. Let S be an $\mathcal{O}$-algebra and $\mathfrak{a} \subset S$ an ideal equipped with an $\mathcal{O}$-pd structure γ. Then for any nilpotent S-algebra $\mathcal{N}$ the algebra $\mathfrak{a} \otimes_S \mathcal{N}$ inherits a nilpotent $\mathcal{O}$-pd structure $\widetilde{\gamma}$ from $\mathfrak{a}$ which is uniquely determined by $\widetilde{\gamma}(a \otimes n) = \gamma(a) \otimes n^q$ for $a \in \mathfrak{a}$ and $n \in \mathcal{N}$.

Proof: We only need to refer to the proof of [Mes72, Chapter III, Lemma (1.8)], where we define the map $\varphi : S^{(\mathfrak{a}\times\mathcal{N})} \to \mathfrak{a}\otimes\mathcal{N}$ by the formula

$$\varphi(\sum_{i=1}^{l} s_i(a_i,n_i)) \quad = \quad \sum_{i=1}^{l}\gamma(a_i)\otimes(s_in_i)^q + \sum(\binom{q}{i_1,\dots,i_l}/\pi)\prod_{j=1}^{l}(s_ja_j\otimes n_j)^{i_j}.$$

Here the last sum runs over all l-tuples $(i_1,\dots,i_l)$ with $i_j>0$ and $\sum_{j=1}^{l} i_j = q$. A similar argumentation, compared to the one there, for our φ establishes the map $\widetilde{\gamma}$. □

Before we can construct the exponential, we first need a reformulated statement of Lemma 2.4.4. Let S be an $\mathcal{O}$-algebra. By utilizing that Drinfeld's functor between reduced $\mathbb{E}_{\mathcal{O},S}$-modules and formal $\mathcal{O}$-modules over S is given by sending M to $\widehat{W_{\mathcal{O}}}(-)\otimes_{\mathbb{E}_{\mathcal{O},S}} M$ by Proposition 2.4.6, we get for each formal $\mathcal{O}$-module G over S and each nilpotent S-algebra $\mathcal{N}$ equipped with a nilpotent $\mathcal{O}$-pd structure an isomorphism

$$\log_G(\mathcal{N}) : G(\mathcal{N}) \to \operatorname{Lie} G\otimes_S\mathcal{N}.$$

Definition 5.2.8. Let G be a formal $\mathcal{O}$-module over an $\mathcal{O}$-algebra S and $\mathfrak{a}\subseteq S$ be an ideal equipped with an $\mathcal{O}$-pd structure. We define the *exponential*

$$\exp_G : \underline{\mathfrak{a}\otimes\operatorname{Lie} G} \to G$$

by

$$\underline{\mathfrak{a}\otimes\operatorname{Lie} G}(\mathcal{N}) = \mathfrak{a}\otimes_S\mathcal{N}\otimes_S\operatorname{Lie} G \overset{\log_G^{-1}(\mathfrak{a}\otimes_S\mathcal{N})}{\longrightarrow} G(\mathfrak{a}\otimes_S\mathcal{N}) \to G(\mathcal{N})$$

for each $\mathcal{N}\in\operatorname{Nil}_S$, where $\log_G$ is defined as right above and the last map is induced by the product morphism $\mathfrak{a}\otimes_S\mathcal{N}\to\mathcal{N}$.

The Definition makes sense, since $\mathfrak{a}\otimes_S\mathcal{N}$ inherits by Proposition 5.2.7 a nilpotent $\mathcal{O}$-pd structure and $\log_G(\mathfrak{a}\otimes_S\mathcal{N})$ is an isomorphism.

Theorem 5.2.9. (cf. [Zin, Die Universelle Erweiterung nach Grothendieck und Messing, Theorem 3]) Let S be an $\mathcal{O}$-algebra with π nilpotent in S, $\mathfrak{a}\subset S$ an ideal equipped with an $\mathcal{O}$-pd structure and H_1, H_2 two π-divisible formal $\mathcal{O}$-modules over S with reductions to $S/\mathfrak{a}=R$, say $H_{1,R}, H_{2,R}$. Let

$$0\to\underline{V_2}\to E_2\to H_2\to 0$$

be a (not necessarily universal) $\mathcal{O}$-extension of H_2. For a given morphism $f : H_{1,R}\to H_{2,R}$, there exists a unique morphism $g : E_{\mathcal{O}}(H_1)\to E_2$, such that for each morphism $u : \underline{V_{\mathcal{O}}(H_1)}\to\underline{V_2}$ of vector groups, which lifts $\underline{V_{\mathcal{O}}(f)} : \underline{V_{\mathcal{O}}(H_{1,R})}\to\underline{V_{2,R}}$, we obtain, with the morphism given as in the diagram

$$\begin{array}{ccc} \underline{V_{\mathcal{O}}(H_1)} & \overset{i_1}{\hookrightarrow} & E_{\mathcal{O}}(H_1) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ \underline{V_2} & \underset{i_2}{\hookrightarrow} & E_2, \end{array}$$

that $g \circ i_1 - i_2 \circ u$ factors as

$$\underline{V_{\mathcal{O}}(H_1)} \to \underline{\mathfrak{a} \otimes \operatorname{Lie} E_2} \overset{\exp_{E_2}}{\to} E_2,$$

where the first map is induced by an S-module morphism $V_{\mathcal{O}}(H_1) \to \mathfrak{a} \otimes \operatorname{Lie} E_2$.

We omit the proof, since it is the obvious generalization of the referenced source.
Now we are going to introduce a category of extensions, which is similar to the one explained in [RZ96, 5.19]. Let $S \to R$ be an $\mathcal{O}$-pd-thickening. We consider sextuples $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G)$, where $\widetilde{G}$ is a π-divisible formal $\mathcal{O}$-module over S, G its base change to R, E a formal $\mathcal{O}$-module over S and $\mathbb{W}$ a vector group associated to a finite projective S-module, such that $\iota : \mathbb{W} \to E$ and $\rho : E \to \widetilde{G}$ induce an $\mathcal{O}$-extension of $\widetilde{G}$

$$0 \to \mathbb{W} \to E \to \widetilde{G} \to 0.$$

A morphism $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G) \to (\mathbb{W}', \iota', E', \rho', \widetilde{G'}, G')$ is a tuple (v, β), where $v : E \to E'$ is a morphism of formal $\mathcal{O}$-modules over S and β a morphism of formal $\mathcal{O}$-modules $G \to G'$ over R, which gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{W}_R & \xrightarrow{\iota_R} & E_R & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_R} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathbb{W}'_R & \xrightarrow{\iota'_R} & E'_R & \longrightarrow & G' & \longrightarrow & 0, \end{array}$$

where v_0 is required to be a morphism of vector groups. Furthermore, we require that for each lifting of v_0 to a morphism of vector groups $\widetilde{v}_0 : \mathbb{W} \to \mathbb{W}'$ the map

$$\iota' \circ \widetilde{v}_0 - v \circ \iota : \mathbb{W} \to E'$$

factors over

$$\mathbb{W} \overset{\xi}{\to} \underline{\mathfrak{a} \otimes_S \operatorname{Lie} E'} \overset{\exp_{E'}}{\to} E',$$

where ξ is a morphism of vector groups.

Definition 5.2.10. We define the category $\operatorname{Ext}_{2,S\to R}$ by the above objects and by the above morphisms.

It is essential to know how to switch between the extensions of Definition 5.2.2 and the extensions of Definition 5.2.10 precisely in order to utilize Theorem 5.2.9 for the extensions in $\operatorname{Ext}_{1,S\to R}$. But before we can give the Theorem which explains this to us, we need to understand the exponential mapping in [Zin86].

Proposition 5.2.11. (cf. [Zin86, (2.3) Satz., (2.11) Satz.]) Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$, M' a reduced $\mathbb{E}_{\mathcal{O},S}$-module and $M = \mathbb{E}_{\mathcal{O},R} \otimes_{\mathbb{E}_{\mathcal{O},S}} M'$. Then there is an exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(\mathfrak{a} \otimes_S M'/VM') \stackrel{\exp}{\to} M' \to M \to 0.$$

Here the map exp is given by sending $V^i(a \otimes m)$ to $V^i \log^{-1}(a, 0, \ldots)m$, where log is given by (2.9). By passing over to functors from Nil_R to $(\,\mathcal{O} - \text{modules}\,)$ via $\widehat{W_{\mathcal{O}}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} K$ for $K = C(\mathfrak{a} \otimes_S M'/VM'), M'$, we obtain the map

$$\exp_{G'} : \underline{\mathfrak{a} \otimes_S \mathrm{Lie}\, G'} \to G',$$

where G' is the formal $\mathcal{O}$-module over S associated to M', where we have utilized (5.2) for obtaining $\widehat{W_{\mathcal{O}}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} C(\mathfrak{a} \otimes_S M'/VM') \simeq \underline{\mathfrak{a} \otimes_S \mathrm{Lie}\, G'}$.

Theorem 5.2.12. Let $S \to R$ be an $\mathcal{O}$-pd-thickening with nilpotent kernel $\mathfrak{a}$. Then there is an equivalence of $\mathrm{Ext}_{1,S\to R}$ and $\mathrm{Ext}_{2,S\to R}$, such that

$$\begin{array}{ccc} \mathrm{Ext}_{1,S\to R} & \longrightarrow & \mathrm{Ext}_{2,S\to R} \\ \downarrow & \swarrow & \\ (\pi - \text{divisible formal } \mathcal{O} - \text{modules}/R) & & \end{array}$$

is commutative. (The $\mathrm{Ext}_{i,S\to R}$ lie over the category of π-divisible formal $\mathcal{O}$-modules over R.)

Proof: Let $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G)$ be an object of $\mathrm{Ext}_{2,S\to R}$ and W the finite projective S-module associated to $\mathbb{W}$. We consider the $\mathcal{O}$-extension of formal $\mathcal{O}$-modules over S

$$0 \to \mathbb{W} \to E \to \widetilde{G} \to 0, \tag{5.6}$$

Translating this to Cartier modules, we obtain that

$$0 \to C(W) \to M_E \to M_{\widetilde{G}} \to 0$$

is exact. We now consider the exact sequence

$$0 \to C(\mathfrak{a}\, \mathrm{Lie}\, M_{\widetilde{G}}) = \mathbb{E}_{\mathcal{O},\mathfrak{a}} M_{\widetilde{G}} \to M_{\widetilde{G}} \to M_G \to 0$$

of Proposition 5.2.11. The inverse image of $C(\mathfrak{a}\, \mathrm{Lie}\, M_{\widetilde{G}})$ by the morphism $M_E \to M_{\widetilde{G}}$ is $C(W + \mathfrak{a}\, \mathrm{Lie}\, M_E)$. So if we set $L = W + \mathfrak{a}\, \mathrm{Lie}\, M_E$, we obtain that the exact sequence

$$0 \to C(L) \to M_E \to M_G \to 0$$

is an extension in the sense of Definition 5.2.1, if $\mathfrak{a} M_E \simeq \mathfrak{a}\, \mathrm{Lie}\, M_E$ holds, but this follows easily by bearing in mind that $\mathfrak{a} V =^F \mathfrak{a} = 0$ holds. Hence, $\mathfrak{a} M_E \subset V^0 L$

and we obtain an object (L, M_E, M_G) of $\mathrm{Ext}_{1,S\to R}$.
Now assume that we are given an extension

$$0 \to C(L) \to N \to M = M_G \to 0$$

in $\mathrm{Ext}_{1,S\to R}$. Since M/VM is a finitely generated projective R-module, we may lift it by Proposition 2.1.2 uniquely up to isomorphism to a finitely generated projective S-module P. We consider now any map τ which makes the diagram

$$\begin{array}{ccc} N/VN & \longrightarrow & M/VM \\ & \overset{\tau}{\searrow} & \uparrow \\ & & P \end{array} \tag{5.7}$$

commutative. The existence of τ is guaranteed by the universal property of projective modules (for N/VN). With the help of the Nakayama lemma we obtain that τ is surjective, so the sequence

$$0 \to W = \ker\tau \to N/VN \to P \to 0$$

is exact. Furthermore, we have $L = W + \mathfrak{a}(N/VN) \subset N/VN$. We now consider $\widetilde{M} = N/C(W)$ and claim that this module is a reduced $\mathbb{E}_{\mathcal{O},S}$-module. For this purpose we consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \ker(V : \widetilde{M} \to \widetilde{M}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(W) & \longrightarrow & N & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \\
0 & \longrightarrow & C(W) & \longrightarrow & N & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & \mathrm{Lie}\, N & \longrightarrow & \mathrm{Lie}\,\widetilde{M} = P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

where each row and column is exact. Via the snake lemma we obtain that $V : \widetilde{M} \to \widetilde{M}$ is injective. Since $\mathrm{Lie}\,\widetilde{M} = P$ is a finitely generated projective S-module, we only need to show $\varprojlim \widetilde{M}/V^k\widetilde{M} = \widetilde{M}$. By generalizing the previous diagram via taking V^k instead of V, we obtain the exact sequences

$$0 \to C(W)/V^kC(W) \to N/V^kN \to \widetilde{M}/V^k\widetilde{M} \to 0$$

and since $C(W)/V^{k+1}C(W) \to C(W)/V^kC(W)$ is surjective for each $k \geq 0$, we obtain by a standard result that the sequence

$$0 \to C(W) = \varprojlim C(W)/V^kC(W) \to N = \varprojlim N/V^kN \to \varprojlim \widetilde{M}/V^k\widetilde{M} \to 0$$

is exact. But, because of the exactness of

$$0 \to C(W) \to N \to \widetilde{M} \to 0,$$

the canonical morphism $\widetilde{M} \to \varprojlim \widetilde{M}/V^k\widetilde{M}$ must be an isomorphism. Hence, $\widetilde{M}$ corresponds to a formal $\mathcal{O}$-module $\widetilde{G}$ over S, which lifts G. Since $\mathfrak{a}$ is nilpotent, $\widetilde{G}$ is π-divisble by [Zin84, 5.12 Korollar]. Hence, the previous exact sequence corresponds to an $\mathcal{O}$-extension of formal $\mathcal{O}$-modules over S

$$0 \to \mathbb{W} \stackrel{\iota}{\to} E \stackrel{\rho}{\to} \widetilde{G} \to 0$$

and we obtain an object $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G)$ of $\mathrm{Ext}_{2,S\to R}$. It is easily checked that these two associations are inverse to each other.

We now focus on the morphisms of each category. Let $(v, \beta) : (\mathbb{W}, \iota, E, \rho, \widetilde{G}, G) \to (\mathbb{W}', \iota', E', \rho', \widetilde{G'}, G')$ be a morphism in $\mathrm{Ext}_{2,S\to R}$, where $\mathbb{W}$ resp. $\mathbb{W}'$ is associated to the finite projective S-module W resp. W'. From v we obtain a morphism of $\mathbb{E}_{\mathcal{O},S}$-modules $M_E \to M_{E'}$. With the notation as for the definition of a morphism in $\mathrm{Ext}_{2,S\to R}$, we take a lifting of $v_0 : \mathbb{W}_R \to \mathbb{W}'_R$ to a morphism of vector groups $\widetilde{v}_0 : \mathbb{W} \to \mathbb{W}'$. Since the diagram

$$\begin{array}{ccc} \mathbb{W} & \xrightarrow{\iota} & E \\ \downarrow{\scriptstyle \widetilde{v}_0} & & \downarrow{\scriptstyle v} \\ \mathbb{W}' & \xrightarrow{\iota'} & E' \end{array}$$

fails to be commutative by a map

$$\mathbb{W} \to \underline{\mathfrak{a} \otimes_S \mathrm{Lie}\, E'} \stackrel{\exp_{E'}}{\to} E',$$

where $\mathbb{W} \to \underline{\mathfrak{a} \otimes_S \mathrm{Lie}\, E'}$ is induced by an S-module morphism $W \to \mathfrak{a} \otimes_S \mathrm{Lie}\, E' = \mathfrak{a}\,\mathrm{Lie}\, E'$, we obtain, with $\beta_\star : M_G \to M_{G'}$ the morphism of $\mathbb{E}_{\mathcal{O},R}$-modules corresponding to β, that the first vertical morphism in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(L) = C(W + \mathfrak{a}\,\mathrm{Lie}\, E) & \longrightarrow & M_E & \longrightarrow & M_G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta_\star} & & \\ 0 & \longrightarrow & C(L') = C(W' + \mathfrak{a}\,\mathrm{Lie}\, E') & \longrightarrow & M_{E'} & \longrightarrow & M_{G'} & \longrightarrow & 0 \end{array}$$

is induced by maps from $\mathfrak{a}\,\mathrm{Lie}\, E \to \mathfrak{a}\,\mathrm{Lie}\, E'$, $W \to W'$ and the nontrivial $W \to \mathfrak{a}\,\mathrm{Lie}\, E'$ from above. Hence, the first vertical map is induced by a module morphism and we obtain a morphism in $\mathrm{Ext}_{1,S\to R}$.

Conversely, let us start with a morphism in $\mathrm{Ext}_{1,S\to R}$, say

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(L) = C(W + \mathfrak{a}\,\mathrm{Lie}\, E) & \xrightarrow{\kappa} & M_E & \longrightarrow & M_G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C(\varphi)} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \beta_\star} & & \\ 0 & \longrightarrow & C(L') = C(W' + \mathfrak{a}\,\mathrm{Lie}\, E') & \xrightarrow{\kappa'} & M_{E'} & \longrightarrow & M_{G'} & \longrightarrow & 0, \end{array}$$

where we take the sum respresentation of L and L' from the description of the objects (see above). Once we make the base change from S to R, we obtain a commutative diagram of $\mathbb{E}_{\mathcal{O},R}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(W_R) & \longrightarrow & M_{E,R} & \longrightarrow & M_G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C(\varphi_R)} & & \downarrow{\scriptstyle u_R} & & \downarrow{\scriptstyle \beta_\star} & & \\ 0 & \longrightarrow & C(W'_R) & \longrightarrow & M_{E',R} & \longrightarrow & M_{G'} & \longrightarrow & 0, \end{array}$$

where the columns are exact. When we take the morphism of formal $\mathcal{O}$-modules $v : E \to E'$ corresponding to the morphism $u : M_E \to M_{E'}$ and $\beta : G \to G'$ the morphism corresponding to $\beta_\star$, then we claim that we obtain a morphism $(v, \beta) : (\mathbb{W}, \iota, E, \rho, \widetilde{G}, G) \to (\mathbb{W}', \iota', E', \rho', \widetilde{G'}, G')$, where domain and codomain of this morphism correspond to (L, M_E, M_G) and $(L', M_{E'}, M_{G'})$, respectively. First of all, by base change to R, we obtain by the above diagram that the diagram of formal $\mathcal{O}$-modules over R

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{W}_R & \longrightarrow & E_R & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathbb{W}'_R & \longrightarrow & E'_R & \longrightarrow & G' & \longrightarrow & 0 \end{array}$$

is commutative and $\mathbb{W}_R \to \mathbb{W}'_R$ is a morphism of vector groups. Now let $\widetilde{\varphi} : W \to W'$ be any lifting of $\varphi_R : W_R \to W'_R$. We consider the (not necessarily commutative) diagram of $\mathbb{E}_{\mathcal{O},S}$-modules

$$\begin{array}{ccc} C(W) & \xrightarrow{\kappa} & M_E \\ \downarrow{\scriptstyle C(\widetilde{\varphi})} & & \downarrow{\scriptstyle u} \\ C(W') & \xrightarrow{\kappa'} & M_{E'}. \end{array}$$

Since its reduction to R is commutative, we obtain with

$$\alpha = u \circ \kappa - \kappa' \circ C(\widetilde{\varphi}) : C(W) \to M_{E'}$$

that the diagram of $\mathbb{E}_{\mathcal{O},S}$-modules

$$\begin{array}{ccc} C(W) & \xrightarrow{\alpha} & M_{E'} \\ \downarrow & & \downarrow{\scriptstyle \omega} \\ C(W_R) & \xrightarrow[0]{} & M_{E'_R}, \end{array}$$

with ω the base change morphism, is commutative. Hence, α factorizes as

$$C(W) \to C(\mathfrak{a} \otimes \operatorname{Lie} M_{E'}) \stackrel{\exp}{\to} M_{E'},$$

so by Proposition 5.2.11 we obtain the demanded lifting property of a morphism in $\mathrm{Ext}_{2,S\to R}$. Hence, the tuple (v, β) is indeed a morphism in $\mathrm{Ext}_{2,S\to R}$.
This establishes the equivalence and it is obvious that the diagram in the assertion of the Theorem commutes. □

With the help of the previous Theorem we receive a translated statement of Theorem 5.2.9 for the extensions in $\mathrm{Ext}_{1,S\to R}$ when the kernel of $S \to R$ is nilpotent:

Theorem 5.2.13. (c.f. [Zin02, Theorem 92.]) If $S \to R$ is an $\mathcal{O}$-pd-thickening with nilpotent kernel and G a π-divisible formal $\mathcal{O}$-module over R, then there is a universal extension $(L^{\mathrm{univ}}, N^{\mathrm{univ}}, M_G) \in \mathrm{Ext}_{1,S\to R}$. Here the universality means, for any π-divisible formal $\mathcal{O}$-module G' over R, any morphism of $\mathbb{E}_{\mathcal{O},R}$-modules $\beta : M_G \to M_{G'}$ and any extension $(L, N, M_{G'}) \in \mathrm{Ext}_{1,S\to R}$, there is a unique morphism

$$(\varphi, u, \beta) : (L^{\mathrm{univ}}, N^{\mathrm{univ}}, M_G) \to (L, N, M_{G'}).$$

Definition 5.2.14. With the notation as above, we define the *crystal of Grothendieck-Messing on the nilpotent ideal crystalline site* by

$$\mathbb{D}_G(S) = \mathrm{Lie}\, N^{\mathrm{univ}}.$$

It is clear that in order to check the universality of a given extension, say (L, N, M), we only have to verify that there is a unique morphism to each extension (L', N', M), with the morphism $M \to M$ the identity.

5.3 Comparision of the crystals and the generalized main Theorem of display theory

Our next aim is to give an explicit description of the universal extension for $G = BT_{\mathcal{O}}(\mathcal{P}, -)$, where $\mathcal{P}$ is a nilpotent $\mathcal{O}$-display. The proof of Proposition 5.3.4, in which we get such a description, basically reduces to trivial $\mathcal{O}$-pd-thickenings $k \to k$ in the end, where k is a perfect field extending of the residue field of $\mathcal{O}$. In this case we can work fairly well with the obvious generalization of the results made in [Zin86, 2. Liftungen von formalen Gruppen].

Proposition 5.3.1. (cf. [Zin86, (2.5) Satz]) Let k be a perfect $\mathcal{O}$-algebra with $\pi k = 0$ and $\tau : k' \to k$ an $\mathcal{O}$-pd-thickening over $W_{\mathcal{O}}(k)$ (i.e., τ is an $\mathcal{O}$-pd-thickening, where $\ker \tau$ is equipped with an $\mathcal{O}$-pd-structure γ_τ together with an $\mathcal{O}$-algebra morphism $\varphi : W_{\mathcal{O}}(k) \to k'$, such that $\mathrm{w}_0 = \tau \circ \varphi$ holds and $\varphi\gamma(x) = \gamma_\tau\varphi(x)$ is fulfilled for all $x \in I_{\mathcal{O},k}$, where γ is given by (5.1) *). If M is a reduced

*The last condition only makes sense, since necessarily $\varphi(I_{\mathcal{O},k}) \subseteq \ker \tau$.

$\mathbb{E}_{\mathcal{O},k}$-module, such that $F : M \to M$ is an injection and V nilpotent on M/FM, then for any extension

$$0 \to C(L) \to M' \to M \to 0$$

in $\mathrm{Ext}_{1,k' \to k}$ there is a uniquely determined $W_{\mathcal{O}}(k)[F]$-linear section $\sigma : M \to M'$, such that $\sigma(Vm) - V\sigma(m) \in V^0 L$ for each $m \in M$.

With $k' \to k$ as in the Proposition, we define the category of $W_{\mathcal{O}}[F]$-trivialized extensions by the objects (E, σ), where E is any extension

$$0 \to C(L) \to M' \to M \to 0$$

in $\mathrm{Ext}_{1,k' \to k}$, i.e., the conditions in the Proposition for M do not necessarily hold, and $\sigma : M \to M'$ a $W_{\mathcal{O}}(k)[F]$-linear section, such that $\sigma(Vm) - V\sigma(m) \in V^0 L$ for each $m \in M$. The morphisms between the objects are the morphisms between the extensions respecting the sections.

The category $\underline{H}$ consists of objects (M, T, t, φ), where M is a reduced $\mathbb{E}_{\mathcal{O},k}$-module, T a finitely generated projective k'-module and t and φ are k'-linear maps, such that

$$\begin{array}{ccc} k' \otimes_{W_{\mathcal{O}}(k)} M & \xrightarrow{\;t\;} & T \\ \downarrow & & \downarrow \varphi \\ k \otimes_{W_{\mathcal{O}}(k)} M & \longrightarrow & M/VM \end{array}$$

is commutative. A morphism $(M, T, t, \varphi) \to (M', T', t', \varphi)$ between two such objects consists of an $\mathbb{E}_{\mathcal{O},k}$-module morphism $M \to M'$ and a morphism of k'-modules $T \to T'$, such that the obvious compatibility with the above commutative diagram for both objects is fulfilled.

If we are given a $W_{\mathcal{O}}[F]$-trivialized extension (with the notation as above), then the section σ defines a $W_{\mathcal{O}}(k)$-linear map $M \to M'/VM'$ or equivalently a k'-linear map

$$t : k' \otimes_{W_{\mathcal{O}}(k)} M \to M'/VM'.$$

Because σ is a section, we obtain that the diagram

$$\begin{array}{ccc} k' \otimes_{W_{\mathcal{O}}(k)} M & \xrightarrow{\;t\;} & M'/VM' \\ \downarrow & & \downarrow \varphi \\ k \otimes_{W_{\mathcal{O}}(k)} M & \longrightarrow & M/VM \end{array}$$

is commutative, where φ is the obvious map induced by the extension. Hence the following assertion makes sense:

Proposition 5.3.2. (cf. [Zin86, (2.6) Satz]) The functor given by sending (E, σ) to $(M, M'/VM', t, \varphi)$ defines an equivalence of categories between the $W_{\mathcal{O}}[F]$-trivialized extensions and the category $\underline{H}$.

In the original source T needs not to be finitely generated and projective and the M and M' only need to be V-reduced (i.e., they are modules over the Cartier ring where all conditions hold for reduced Cartier modules apart from the conditions on the tangential spaces), but a close look on the proof in the referred source yields that we can require the stronger conditions and this in the generalized setting for $\mathcal{O}$. Since we will only deal with the trivial case that $k = k'$ is a perfect field of characteristic p, which extends the residue field of $\mathcal{O}$, we will assume this for the following discussion (this makes it trivial that $M/\pi M$ is a free/projective k-module for a reduced $\mathbb{E}_{\mathcal{O},k}$-module M). We now consider for $\underline{H}$ the fiber over a reduced $\mathbb{E}_{\mathcal{O},k}$-module M, such that the conditions of Proposition 5.3.1 hold. Since $M/\pi M$ is finitely generated and free over k by considering the exact sequence of k-modules

$$0 \to M/FM \xrightarrow{V} M/\pi M \to M/VM \to 0,$$

we obtain that $k \otimes_{W_{\mathcal{O}}(k)} M$ is finitely generated and free over k. Hence, we obtain that there is an initial object in $\underline{H}(M)$ given by $(M, k \otimes_{W_{\mathcal{O}}(k)} M, \mathrm{id}, \rho)$, where ρ is the obvious mapping. By Proposition 5.3.1, there is an one-to-one correspondence between the extensions of M in $\mathrm{Ext}_{1,k\to k}$ and the $W_{\mathcal{O}}[F]$-trivialized extensions lying over M. Hence, $(M, k \otimes_{W_{\mathcal{O}}(k)} M, \mathrm{id}, \rho)$ corresponds to the universal extension.

Lemma 5.3.3. Let $S \to R$ be an $\mathcal{O}$-pd-thickening and $\mathcal{P} = (P, Q, F, F_1)$ a nilpotent f-$\mathcal{O}$-display over R. By Proposition 3.2.5 there exists a unique $\mathcal{P}$-triple $(\widetilde{P}, F, F_1)$ over S. The exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}/U \to M(\mathcal{P}) \to 0 \tag{5.8}$$

lies in $\mathrm{Ext}_{1,S\to R}$, where the second arrow maps $y \in \widehat{Q}$ to $V^f \otimes F_1 y - 1 \otimes y$, the third arrow is given by the canonical map $\widetilde{P} \to P$ and U is the $\mathbb{E}_{\mathcal{O},S}$-submodule of $\mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$ generated by $(F \otimes x - V^{f-1} \otimes Fx)_{x \in \widetilde{P}}$.

Proof: It is not too hard to verify that the module N in the middle of sequence (5.8) is a reduced Cartier module and from the canonical map $\widetilde{P} \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$ we obtain an isomorphism $\widetilde{P}/I_{\mathcal{O},S}\widetilde{P} \simeq N/VN$. We need to check the well-definedness of the mapping $C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) \to N$. The mapping $\widehat{Q} \to N$ given by $y \mapsto V^f \otimes F_1 y - 1 \otimes y$ is a group morphism. The subgroup $I_{\mathcal{O},S}\widetilde{P}$ of $\widehat{Q}$ is in the kernel since for each $w \in W_{\mathcal{O}}(S)$ and $x \in \widetilde{P}$

$$\begin{aligned} V^f \otimes F_1{}^V wx - 1 \otimes^V wx &= V^f \otimes^{F^{f-1}} wFx - 1 \otimes^V wx = \\ VwV^{f-1} \otimes Fx - 1 \otimes^V wx &= VwF \otimes x - 1 \otimes^V wx =^V w \otimes x - 1 \otimes^V wx = 0 \end{aligned}$$

holds. By representing $\widehat{Q}$ as $\mathfrak{a}\widetilde{T} \oplus I_{\mathcal{O},S}\widetilde{T} \oplus \widetilde{L}$, where $\mathfrak{a}$ is embedded in $W_{\mathcal{O}}(S)$ as usual and $\widetilde{L}$ and $\widetilde{T}$ are liftings of the modules corresponding to a normal decomposition $P = L \oplus T$, we obtain readily that

$$F(V^f \otimes F_1 y - 1 \otimes y) \quad = \quad 0$$

holds for $y \in \widehat{Q}$, so we obtain that the image of $\widehat{Q}$ in N is an S-module morphism in a natural way, i.e., via

$$s \star (V^f \otimes F_1 y - 1 \otimes y) = [s](V^f \otimes F_1 y - 1 \otimes y)$$

for $s \in S$ and $y \in \widehat{Q}$. This makes sense, since for $s, s' \in S$ and $y \in \widehat{Q}$, we have

$$\begin{aligned} s \star (V^f \otimes F_1 y - 1 \otimes y) + s' \star (V^f \otimes F_1 y - 1 \otimes y) &= \\ ([s] + [s'])(V^f \otimes F_1 y - 1 \otimes y) &= \\ ([s + s'] + \sum_{i=1}^{\infty} V^i [a_i] F^i)(V^f \otimes F_1 y - 1 \otimes y) &= \\ (s + s') \star (V^f \otimes F_1 y - 1 \otimes y) & \end{aligned}$$

for some $a_i \in R$, where we have used $F(V^f \otimes F_1 y - 1 \otimes y) = 0$ for the last equation. The induced map $\widehat{Q}/I_{\mathcal{O},S}\widetilde{P} \to N$ is an S-module morphism, which extends in a unique way to an $\mathbb{E}_{\mathcal{O},S}$-module morphism $C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) \to N$, and we get that the sequence (5.8) is a complex of V-reduced Cartier modules (see the discussion after Proposition 5.3.2 for the definition of V-reduced Cartier modules). Hence, in order to show the exactness of the sequence, we only need to check the exactness on the tangent spaces, which is trivial. Furthermore, we need to confirm that $\mathfrak{a}N \subset \widehat{Q}/I_{\mathcal{O},S}\widetilde{P}$ holds, where $\mathfrak{a} \subset W_{\mathcal{O}}(S)$ as usual and $\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}$ should be the submodule of N via the image of the second arrow in (5.8). Let $a \in \mathfrak{a}, x \in \widetilde{P}$ and $\xi = \sum V^i[\xi_{i,j}]F^j \in \mathbb{E}_{\mathcal{O},S}$ be in the usual representation. We obtain that $a\xi \otimes x = \sum_{i,j} V^{i}{}^{F^i}a[\xi_{i,j}]F^j \otimes x = \sum_j a[\xi_{0,j}]F^j \otimes x = \sum_j V^{j(f-1)} \otimes^{F^{j(f-1)}}(a[\xi_{0,j}])F^j x$ holds, which equals $1 \otimes \sum_j a[\xi_{0,j}]F^j x$ for $f = 1$ and $1 \otimes a[\xi_{0,0}]x$ for $f > 2$, so we only need to verify that an element of the form $1 \otimes ax$ lies in the image of $\widehat{Q} \to N$. But this is clear because of $V^f \otimes F_1 ax - 1 \otimes ax = -1 \otimes ax$. □

We now show under which circumstances the sequence (5.8) defines the universal one. It is not too hard to check that, when $S = R$ is a perfect field, we get, with the discussion after the proof of Proposition 2.4.7 and the discussion before the assertion of the previous Lemma that $f = 1$ must hold for f-$\mathcal{O}$-displays in general, since otherwise the module in the middle of the sequence gets too "small". Bearing this in mind, since we will reduce to this perfect field case, we can assert:

Proposition 5.3.4. Let $S \to R$ be an $\mathcal{O}$-pd-thickening with nilpotent kernel and $\mathcal{P} = (P, Q, F, F_1)$ a nilpotent $\mathcal{O}$-display over R (i.e., $f = 1$). Then the universal extension of the formal $\mathcal{O}$-module $BT_{\mathcal{O}}(\mathcal{P}, -)$ is given by the exact sequence (5.8).

Proof: In order to show the universality of the extension, we first reduce to the case, where $S = R$. Consider the universal extension

$$0 \to C(L^{\text{univ}}) \to N^{\text{univ}} \to M(\mathcal{P}) \to 0,$$

whose existence is guaranteed by Theorem 5.2.13. Let $\widetilde{M}$ be a lifting of $M(\mathcal{P})$ to a reduced Cartier module over S. If we then consider the sequence (cf. Proposition 5.2.11)

$$0 \to C(\mathfrak{a} \otimes_S \operatorname{Lie} \widetilde{M}) \to \widetilde{M} \to M(\mathcal{P}) \to 0,$$

there are unique maps $\varphi : L^{\text{univ}} \to \mathfrak{a} \otimes_S \operatorname{Lie} \widetilde{M}$ and $u : N^{\text{univ}} \to \widetilde{M}$, such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(L^{\text{univ}}) & \longrightarrow & N^{\text{univ}} & \longrightarrow & M(\mathcal{P}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle C(\varphi)} & & \Big\downarrow{\scriptstyle u} & & \Big\| & & \\
0 & \longrightarrow & C(\mathfrak{a} \otimes_S \operatorname{Lie} \widetilde{M}) & \longrightarrow & \widetilde{M} & \longrightarrow & M(\mathcal{P}) & \longrightarrow & 0
\end{array}$$

is commutative. If we define $\widetilde{L}$ as the kernel of $L^{\text{univ}} \to \mathfrak{a} \otimes_S \operatorname{Lie} \widetilde{M}$, we obtain by the snake lemma and the lemma of Nakayama (applied to the cokernel of $\operatorname{Lie} N^{\text{univ}} \to \operatorname{Lie} \widetilde{M}$) that

$$0 \to C(\widetilde{L}) \to N^{\text{univ}} \stackrel{u}{\to} \widetilde{M} \to 0 \tag{5.9}$$

is exact. It is not too hard to check that this extension is universal. Conversely, if we start with the previous universal extension of $\widetilde{M}$, we obtain the universal extension of M by

$$0 \to C(\widetilde{L} + \mathfrak{a}N^{\text{univ}}) \to N^{\text{univ}} \to M \to 0,$$

where the sum $\widetilde{L} + \mathfrak{a}N^{\text{univ}}$ is taken in $\operatorname{Lie} N^{\text{univ}}$. Let $(\widetilde{P}, F, F_1)$ be the unique $\mathcal{P}$-triple over S and $\widetilde{Q} \subset \widehat{Q}$ an arbitrary $W_{\mathcal{O}}(S)$-submodule, such that $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, F, F_1)$ is a nilpotent $\mathcal{O}$-display over S. If we can show that

$$0 \to C(\widetilde{Q}/I_{\mathcal{O},S}\widetilde{P}) \to N \to M(\widetilde{\mathcal{P}}) \to 0 \tag{5.10}$$

in $\operatorname{Ext}_{1,S \to S}$ is universal, then we obtain by the above considerations that the assertion is true for the general case, so we are allowed to restrict ourselves to the case $S = R$.

By starting with the universal extension (5.9) for $\widetilde{M} = M(\widetilde{\mathcal{P}})$, we obtain a morphism of finitely generated projective S-modules $\widetilde{L} \to \widetilde{Q}/I_{\mathcal{O},S}\widetilde{P}$. In order to show that this morphism is an isomorphism we first reduce to the localizations of this morphism for each prime ideal of S. With the help of the Nakayama lemma we may reduce to the residue fields and from this we may pass to the algebraic closures. Hence, it suffices to consider the case when $S = R$ is a perfect field and to show that (5.10) is universal. By the discussion following Proposition 2.4.7 we are allowed to identify $M(\widetilde{\mathcal{P}})$ with $\widetilde{\mathcal{P}}$. Since Proposition 5.3.1 can be applied to our extension, we obtain that the map $\widetilde{P} \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$ given by sending x to $1 \otimes x$ induces the unique $W_{\mathcal{O}}(S)[F]$-linear section σ

$$0 \longrightarrow C(\widetilde{Q}/I_{\mathcal{O},S}\widetilde{P}) \longrightarrow N \overset{\sigma}{\longrightarrow} \widetilde{P} \longrightarrow 0,$$

such that $V\sigma(x) - \sigma(Vx) \in \widetilde{Q}/I_{\mathcal{O},S}\widetilde{P}$. Because this section defines a $W_{\mathcal{O}}(S)$-linear map $\widetilde{P} \to N/VN$, which is the natural $\widetilde{P} \to \widetilde{P}/I_{\mathcal{O},S}\widetilde{P} = S \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$, we obtain the universality of this extension by the argumentation prior Lemma 5.3.3. □

Theorem 5.3.5. (cf. [Zin02, Theorem 94]) Let R be an $\mathcal{O}$-algebra with π nilpotent in R. For a nilpotent $\mathcal{O}$-display $\mathcal{P}$ over R and the associated π-divisible formal $\mathcal{O}$-module G we obtain a canonical isomorphism of crystals on the nilpotent ideal crystalline site over $\operatorname{Spec} R$:

$$\mathcal{D}_{\mathcal{P}} \simeq \mathbb{D}_G$$

It respects the Hodge filtration on $\mathcal{D}_{\mathcal{P}}(R)$ and $\mathbb{D}_G(R)$, respectively.

Proof: By Proposition 5.3.4 we obtain $\mathcal{D}_{\mathcal{P}}(S) = \widetilde{P}/I_{\mathcal{O},S}\widetilde{P} = \mathbb{D}_G(S)$. The assertion for the Hodge filtration is also clear by this Proposition. □

Now let $S \to R$ be an $\mathcal{O}$-pd-thickening with nilpotent kernel, $\varphi : W_{\mathcal{O}}(R) \to S$ be a morphism of $\mathcal{O}$-pd-thickenings and $\mathcal{P} = (P, Q, F, F_1)$ a nilpotent f-$\mathcal{O}$-display over R. By Theorem 5.1.3 we conclude that if $\overline{Q_\varphi}$ denotes the inverse image of $Q/I_{\mathcal{O},R}P$ by the map $S \otimes_{W_{\mathcal{O}}(R)} P \to R \otimes_{W_{\mathcal{O}}(R)} P = P/I_{\mathcal{O},R}P$, then the extension (5.8) is given by

$$0 \to C(\overline{Q_\varphi}) \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(R)} P/(F \otimes x - V^{f-1} \otimes Fx)_{x \in P} \to M(\mathcal{P}) \to 0.$$

Here $\mathbb{E}_{\mathcal{O},S}$ is considered as an $W_{\mathcal{O}}(R)$-module by the map $W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(S)$ as in Theorem 5.1.3. We need to describe the second arrow of the extension. For any $\overline{y} \in \overline{Q_\varphi}$ we take a lifting $y \in Q_\varphi \subset W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P$. We obtain with

$$1 \otimes y \in \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} (W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P) = \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(R)} P$$

that the image of $\overline{y}$ by the second arrow is given by $V^f \otimes F_{1,\varphi} y - 1 \otimes y$, where $F_{1,\varphi}$ is obtained by base change of the lifted F_1, which in turn is an element of the $\mathcal{P}$-triple with respect to $W_{\mathcal{O}}(R) \to R$. This extension is universal for $f = 1$ by Proposition 5.3.4.

Proposition 5.3.6. (cf. [Zin02, Proposition 98]) Let R be an $\mathcal{O}$-algebra with π nilpotent in R. Then $BT_{\mathcal{O}}$ is faithful.

Proof: Let $\mathcal{P}$ and $\mathcal{P}'$ be two nilpotent $\mathcal{O}$-displays over R and $\alpha : \mathcal{P} \to \mathcal{P}'$ a morphism between them. If we denote by G and G' the associated π-divisible formal $\mathcal{O}$-modules, then α induces a morphism $a : G \to G'$ and hence a morphism $b : M_G \to M_{G'}$. For each $n \geq 1$, we obtain with $S = W_{\mathcal{O},n}(R)$ and Proposition 5.3.4 that there is a unique morphism of the above described universal extensions lying over b. Since α induces such a morphism of extensions as well, it must be induced by it. By Theorem 5.1.3 and Theorem 5.3.5 we obtain $\mathbb{D}_G(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P$ and $\mathbb{D}_{G'}(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P'$ for each $n \geq 1$. If we now apply a to the functor $\mathbb{D}$ we obtain for each $n \geq 1$ a morphism

$$W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P = \mathbb{D}_G(W_{\mathcal{O},n}(R)) \to \mathbb{D}_{G'}(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P',$$

which is given by $1 \otimes \alpha$. Since we clearly obtain by these morphisms a morphism of the inverse systems $(W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P)_n$ and $(W_{\mathcal{O},n}(R) \otimes_{W_\mathcal{O}(R)} P')_n$, we get α back by passing to the projective limit. Hence, the faithfulness follows. □

Since all our argumentation to establish all desired equivalences relies in the end on stack theory, it seems sensible to ask, whether it is possible to prove the main assertions, i.e., that $BT_\mathcal{O}$ is an equivalence of categories, for a large class of $\mathcal{O}$-algebras without using this theory again, i.e., we only use stack theory implicitly for establishing that $BT_{\mathbb{Z}_p}$ is an equivalence. This is possible for all $\mathcal{O}$-algebras with nilpotent nilradical and π nilpotent in R.

Proposition 5.3.7. Let R be an $\mathcal{O}$-algebra with nilpotent nilradical and π nilpotent in R. Then $BT_\mathcal{O}$ is an equivalence of categories between the nilpotent $\mathcal{O}$-displays over R and the π-divisible formal $\mathcal{O}$-modules over R. Furthermore, $\Gamma_1(\mathcal{O}, \mathcal{O}')$ resp. $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is an equivalence of categories for nonramified / totally ramified extensions $\mathcal{O} \to \mathcal{O}'$ and $\mathcal{O}'$-algebras R with nilpotent nilradical and π' nilpotent in R.

Proof: By [FGL07, Theoreme B.7.1.] and Theorem 5.3.5, we can establish the obvious generalization of [Zin02, Corollary 95]. By Theorem 5.3.5 and Proposition 5.3.6 we can deduce, together the generalization of [Zin02, Corollary 95], the obvious generalization of [Zin02, Proposition 99], i.e., $BT_\mathcal{O}$ is fully faithful for all $\mathcal{O}$ and all $\mathcal{O}$-algebras R with nilpotent nilradical and π nilpotent in R.
For the first assertion we choose $\mathcal{O}_0$, such that $\mathcal{O}_0$ is nonramified over $\mathbb{Z}_p$ and $\mathcal{O}$ is totally ramified over $\mathcal{O}_0$. Since $BT_{\mathbb{Z}_p}$ is an equivalence by Theorem 2.5.16†, we obtain that $BT_{\mathcal{O}_0}$ is an equivalence, since it is fully faithful by the above assertion and essentially surjective by Lemma 2.5.17. Analogously we obtain

†See also [Lau08, Proposition 4.4.], where the equivalence is established particularly for rings with nilpotent nilradical and p nilpotent in them, which is possible to prove with simpler methods than the general assertion for all rings with p nilpotent in them, which in fact relies on this result.

that $BT_{\mathcal{O}}$ is an equivalence. The assertion for $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ also follows by Lemma 2.5.17. □

By using Proposition 4.2.5, which relies on stack theory, we can deduce with the help of Proposition 5.3.6 and Lemma 2.5.17 the generalized main Theorem of display theory:

Theorem 5.3.8. $BT_{\mathcal{O}}$ is an equivalence of categories between the category of nilpotent $\mathcal{O}$-displays over R and the category of π-divisible formal $\mathcal{O}$-modules over R for all $\mathcal{O}$-algebras R with π nilpotent in R.

It should be remarked that we can extend this result to all π-adic $\mathcal{O}$-algebras by taking projective limits,
Proof: We choose $\mathcal{O}_0$, such that $\mathbb{Z}_p \to \mathcal{O}_0$ is a nonramified and $\mathcal{O}_0 \to \mathcal{O}$ is a totally ramified extension. Since we know, that the assertion holds by Theorem 2.5.16 for the $\mathbb{Z}_p$-case, we get by Lemma 2.5.17 and Proposition 5.3.6, that $\Gamma_1(\mathbb{Z}_p,\mathcal{O}_0)$ is fully faithful for all $\mathcal{O}_0$-algebras R with p nilpotent in R. Hence, by Proposition 4.2.5 $\Gamma_1(\mathbb{Z}_p,\mathcal{O}_0)$ is an equivalence for all $\mathcal{O}_0$-algebras R with p nilpotent in R. By Lemma 2.5.17 we obtain that $BT_{\mathcal{O}_0}$ is an equivalence for all $\mathcal{O}_0$-algebras R with p nilpotent in R. The analogous argumentation for the extension $\mathcal{O}_0 \to \mathcal{O}$, $\Gamma_2(\mathcal{O}_0,\mathcal{O})$ and all $\mathcal{O}$-algebras with π nilpotent in it establishes the result. □

Corollary 5.3.9. Let $\mathcal{O} \to \mathcal{O}'$ be a nonramified / totally ramified extension and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then

- $\Gamma_1(\mathcal{O},\mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R)$
- $\Gamma_2(\mathcal{O},\mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R)$

are equivalences of categories.

As for the previous Theorem we can extend these equivalences to all π'-adic $\mathcal{O}'$-algebras, where $\Gamma_2(\mathcal{O},\mathcal{O}')$ is given by $\varprojlim \Gamma_2(\mathcal{O},\mathcal{O}')_{R/\pi'^i}$. ($\Gamma_2(\mathcal{O},\mathcal{O}')$ was originally only defined for the case, where π' is nilpotent in R.)
Proof: Since $P(\mathcal{O},R)$ and $P(\mathcal{O}',R)$ are true by Theorem 5.3.8, the result follows by Lemma 2.5.17. □

To obtain all other equivalences, we are now going to consider the nilpotent f-$\mathcal{O}$-display case. Let $\mathcal{O} \to \mathcal{O}'$ be nonramified of degree f and R an $\mathcal{O}'$-algebra with π' nilpotent in R. We consider the functor $\Omega_1(\mathcal{O},\mathcal{O}')$ over R. Let $\mathcal{P} = (P,Q,F,F_1)$ be a nilpotent $\mathcal{O}$-display over R with a strict $\mathcal{O}'$-action and $\mathcal{P}_0 = (P_0,Q_0,F_0 = F_1^{f-1}F, F_{10} = F_1^f)$ its image via $\Omega_1(\mathcal{O},\mathcal{O}')$. Let $S \to R$ be an $\mathcal{O}'$-algebra morphism, which is also an $\mathcal{O}$-pd-thickening. Since we can lift the $\mathcal{O}'$-action of $\mathcal{P}$ to the $\mathcal{P}$-triple $(\widetilde{P},F,F_1)$ over S uniquely by Proposition 3.2.5, we obtain an f-grading on this triple. The module $\widetilde{P}$ looks like $\bigoplus \widetilde{P}_i$ (compare

Lemma 2.5.1) and we obtain $\widehat{Q} = \widehat{Q}_0 \oplus \bigoplus_{i\neq 0} \widetilde{P}_i$. Hence, the $\mathcal{P}_0$-triple over S looks like $(\widetilde{P}_0, F_1^{f-1}F, F_1^f)$. Since $BT_{\mathcal{O}}(\mathcal{P},-) \simeq BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,-)$ as formal $\mathcal{O}'$-modules over R (and hence as formal $\mathcal{O}$-modules as well), we obtain that the corresponding reduced $\mathbb{E}_{\mathcal{O},R}$-modules are isomorphic. By using the description of reduced Cartier modules by Proposition 2.4.7 and the proof of Proposition 2.5.4 we obtain that the $\mathbb{E}_{\mathcal{O},R}$-linear isomorphism κ between the modules

$$M_{BT_{\mathcal{O}}(\mathcal{P},-)} = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/(F\otimes x - 1\otimes Fx, V\otimes F_1 y - 1\otimes y)_{x\in P, y\in Q}$$

and

$$M_{BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0,-)} = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P_0/(F\otimes x - V^{f-1}\otimes F_0 x, V^f\otimes F_{10} y - 1\otimes y)_{x\in P_0, y\in Q_0}$$

corresponding to the isomorphism of the associated formal $\mathcal{O}$-modules is given by sending $1\otimes x_0$ to $1\otimes x_0$ and $1\otimes x_i$ to $V^{f-i}\otimes F_1^{f-i}x_i$ for $i\neq 0$ with $x_i\in P_i$, where P_i is obtained from the obvious decomposition of P. If we now consider the sequences (5.8) for $\mathcal{P}$ and $\mathcal{P}_0$, we obtain a morphism of sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) & \longrightarrow & \mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)} \widetilde{P}/U & \longrightarrow & M(\mathcal{P}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mu & & \downarrow \kappa & & \\
0 & \longrightarrow & C(\widehat{Q}_0/I_{\mathcal{O},S}\widetilde{P}_0) & \longrightarrow & \mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)} \widetilde{P}_0/U_0 & \longrightarrow & M(\mathcal{P}_0) & \longrightarrow & 0,
\end{array}$$

where U is the $\mathbb{E}_{\mathcal{O},S}$-submodule of $\mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)}\widetilde{P}$ generated by $(F\otimes x - 1\otimes Fx)_{x\in\widetilde{P}}$, U_0 is the $\mathbb{E}_{\mathcal{O},S}$-submodule of $\mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)}\widetilde{P}_0$ generated by $(F\otimes x - V^{f-1}\otimes F_1^{f-1}Fx)_{x\in\widetilde{P}_0}$ and μ is given by $\mu(1\otimes x_0) = 1\otimes x_0$ and $\mu(1\otimes x_i) = V^{f-i}\otimes F_1^{f-i}x_i$ for $i\neq 0$ with $x_i\in\widetilde{P}_i$ in the obvious decomposition of $\widetilde{P}$ as above. In order to show that μ is well-defined, we consider the morphism

$$\tau : \mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)}\widetilde{P} \to \mathbb{E}_{\mathcal{O},S}\otimes_{W_{\mathcal{O}}(S)}\widetilde{P}_0/(F\otimes x - V^{f-1}\otimes F_1^{f-1}Fx)_{x\in\widetilde{P}_0},$$

which is analogously constructed as μ. Is easily seen that $F\otimes x_0 - 1\otimes Fx_0$ is mapped via τ to zero for $x_0\in\widetilde{P}_0$. For the other cases we represent x_i by $\sum_j a_j F_1^i z_j$ for $a_j\in W_{\mathcal{O}}(S)$ and $z_j\in\widehat{Q}_0$ with which we can also show that $F\otimes x_i - 1\otimes Fx_i$ is mapped to zero via τ. Hence μ is well-defined. Furthermore, it is not too hard to check that the first vertical morphism in the diagram is obtained by the projection $\widehat{Q}/I_{\mathcal{O},S}\widetilde{P} \to \widehat{Q}_0/I_{\mathcal{O},S}\widetilde{P}_0$. Hence, the above morphism of extensions is indeed a morphism in $\mathrm{Ext}_{1,S\to R}$ and μ is surjective.

Proposition 5.3.10. Let $\mathcal{O}\to\mathcal{O}'$ be nonramified (of degree f) and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then $\Omega_1(\mathcal{O},\mathcal{O}')$ is fully faithful.

Proof: Since $P(\mathcal{O},R)$ is true by Theorem 5.3.8, it suffices to show that $BT_{\mathcal{O}}^{(f)}$ is faithful when we restrict to the full subcategory of the nilpotent f-$\mathcal{O}$-displays over R consisting of the objects which lie in the image of $\Omega_1(\mathcal{O},\mathcal{O}')$. Let $\mathcal{P}$ and $\mathcal{P}'$ be two $\mathcal{O}$-displays over R equipped with strict $\mathcal{O}'$-actions, $\mathcal{P}_0$ resp. $\mathcal{P}'_0$ their images via $\Omega_1(\mathcal{O}, \mathcal{O}')$ and $\alpha : \mathcal{P}_0 \to \mathcal{P}'_0$ a morphism between them. α induces a morphism of formal $\mathcal{O}$-modules

$$BT_{\mathcal{O}}(\mathcal{P}, -) \simeq BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -) \to BT_{\mathcal{O}}^{(f)}(\mathcal{P}'_0, -) \simeq BT_{\mathcal{O}}(\mathcal{P}', -). \tag{5.11}$$

Now let $S = W_{\mathcal{O},n}(R)$ for $n \geq 1$ and we consider extensions of $\mathrm{Ext}_{1,S\to R}$. If we denote the exact sequences of (5.8) for $\mathcal{P}, \mathcal{P}', \mathcal{P}_0, \mathcal{P}'_0$ by E, E', E_0, E'_0, we obtain a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ E_0 & \longrightarrow & E'_0, \end{array}$$

where the arrows are morphisms in $\mathrm{Ext}_{1,S\to R}$. The above extensions are universal by Proposition 5.3.4 and so the morphisms, except the lower one, are uniquely determined by the isomorphisms $BT_{\mathcal{O}}(\mathcal{P}, -) \simeq BT_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$ and $BT_{\mathcal{O}}(\mathcal{P}', -) \simeq BT_{\mathcal{O}}^{(f)}(\mathcal{P}'_0, -)$ and the morphism $BT_{\mathcal{O}}(\mathcal{P}, -) \to BT_{\mathcal{O}}(\mathcal{P}', -)$ given by (5.11). The lower morphism is induced by α. Furthermore, the vertical morphisms of extensions are obtained by the discussion before this Proposition. We obtain, by passing to the Lie algebras of the modules in the middle of each extension in the diagram, a commutative diagram

$$\begin{array}{ccc} W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P & \longrightarrow & W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P' \\ \downarrow & & \downarrow \\ W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P_0 & \longrightarrow & W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P'_0. \end{array}$$

So we get, because of the surjectivity of the vertical arrows, that there is at most one mapping $W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P_0 \to W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P'_0$ for each $n \geq 1$, which leaves the diagram commutative, and the fully faithfulness of $\Omega_1(\mathcal{O}, \mathcal{O}')$ follows. □

Furthermore, we obtain by Proposition 4.2.5, Lemma 2.5.17 and Theorem 5.3.8:

Corollary 5.3.11. Let $\mathcal{O}'$ over $\mathcal{O}$ be nonramified of degree f and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then the following functors are equivalences of categories:

- $\Omega_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'} /R) \to (f - \mathrm{ndisp}_{\mathcal{O}} /R)$
- $BT_{\mathcal{O}}^{(f)} : (f - \mathrm{ndisp}_{\mathcal{O}} /R) \to (\pi' - \text{divisible formal } \mathcal{O}' - \text{modules}/R)$
- $\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{ndisp}_{\mathcal{O}} /R) \to (\mathrm{ndisp}_{\mathcal{O}'} /R)$

We can extend these results to all π'-adic $\mathcal{O}'$-algebras by taking projective limits.

Appendices

Appendix A

Formal $\mathcal{O}$-modules

Let R be a commutative unitary ring and Nil_R denote the category of nilpotent $R-$algebras. As in [Zin84] we can embed the category of $R-$modules Mod_R in Nil_R by setting $M^2 = 0$ for any $M \in \mathrm{Mod}_R$. In particular, this is the case for the R-module R. If we are given a functor H from Nil_R to the category of abelian groups or sets, we denote by t_H its restriction to Mod_R.

Definition A.0.1. (cf. [Zin84, Chapter 2],[Zin02, Definition 80]) A *(finite dimensional) formal group over* R is a functor $F : \mathrm{Nil}_R \to (\textit{abelian groups})$, where the following properties are fulfilled:

1. $F(0) = 0$,
2. F is exact, i.e., if
$$0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3 \to 0$$
is a sequence of nilpotent $R-$algebras, which is exact as a sequence of R-modules, then
$$0 \to F(\mathcal{N}_1) \to F(\mathcal{N}_2) \to F(\mathcal{N}_3) \to 0$$
is an exact sequence of abelian groups.
3. The functor t_F commutes with infinite direct sums.
4. $t_F(R)$ is a fintely generated projective $R-$module. (By [Zin02, 3.1 The functor BT.] $t_F(M)$ is in a canonical way an R-module for each $M \in \mathrm{Mod}_R$.)

$t_F(R)$ is called the *tangential space* of F. The rank of $t_F(R)$ is called the *dimension* of F. The *morphisms* between two formal groups are the natural transformations between the functors.

Hence, we obtain the category formal groups over R.

Definition A.0.2. Let $\mathcal{O}$ be a unitary ring and R a unitary $\mathcal{O}$-algebra. Then a *formal $\mathcal{O}$-module over R*, is a formal group over R with an action on it by $\mathcal{O}$ (i.e., a ring morphism from $\mathcal{O}$ to the endomorphisms of the formal group), which induces the natural action on the tangential space, i.e., it coincides with the $\mathcal{O}$-module structure obtained by the R-module structure of the tangential space and restriction to scalars. The *morphisms* between two formal $\mathcal{O}$-modules are the natural transformations between the functors respecting the attached $\mathcal{O}$-actions.

Hence, we obtain the category formal $\mathcal{O}$-modules over R.

A.1 π-divisible formal $\mathcal{O}$-modules

Definition A.1.1. (cf. [Zin84, 5.4 Definition]) With $\mathcal{O}$ an RRS and R an $\mathcal{O}$-algebra, a morphism $\varphi : G \to H$ of formal $\mathcal{O}$-modules over R of equal dimension is called an *isogeny* if $\ker \varphi$ is representable (i.e., $\ker \varphi \simeq \operatorname{Spf} A$ with $A \in \mathrm{Nil}_R$, where $\operatorname{Spf} A : \mathrm{Nil}_R \to \mathrm{Sets}$ is given by $\operatorname{Spf} A(\mathcal{N}) = \mathrm{Hom}_{R-\mathrm{Alg}}(A, \mathcal{N})$ for $\mathcal{N} \in \mathrm{Nil}_R$.).

Definition A.1.2. (cf. [Zin84, 5.28 Definition],[FGL07, Definition B.2.1.]) A formal $\mathcal{O}$-module G over an $\mathcal{O}$-algebra R is called $\pi-$divisible, if the multiplication map $\pi : G \to G$ is an isogeny. The category of π-divisible formal $\mathcal{O}$-modules over R is a full subcategory of the category of formal $\mathcal{O}$-modules over R.

Lemma A.1.3. Let $(\mathbb{Z}_p, p, p) \to \mathcal{O} \to \mathcal{O}'$ be excellent morphisms of RRSs and R an $\mathcal{O}'$-algebra. Assume that $\pi^c = \pi' a$ and $\pi'^d = \pi b$ holds for some $a, b \in \mathcal{O}'$ and $c, d \in \mathbb{N}_1$. Then a formal $\mathcal{O}'$-module G is π'-divisible, iff $\pi : G \to G$ is an isogeny.

This Lemma is especially interesting, when $\mathcal{O}$ and $\mathcal{O}'$ are rings of integers of non-Archimedean local fields of characteristic zero. Then this assertion fits well to [FGL07, Remarque B.2.2.].
Proof: This follows easily by [Zin84, 5.10 Satz]. □

Bibliography

[BC91] Jean-Franois Boutot and Henri Carayol. Uniformisation p-adique des courbes de shimura: les theoremes de cerednik et de drinfeld. *Asterisque*, 196-197:45–158, 1991. [cited at p. 29]

[Dri76] Vladimir Gershonovich Drinfeld. Coverings of p-adic symmetric regions. *Funktsional'nyi analiz i Ego Prilozheniya*, 10:29–40, 1976. [cited at p. iii, viii, 8, 28, 29, 31]

[Fal02] Gerd Faltings. Group schemes with strict $\mathcal{O}$-action. *Moscow Mathematical Journal*, 2:249–279, 2002. [cited at p. 24, 86]

[FGL07] Laurent Fargues, Alain Genestier, and Vincent Lafforgue. *L'isomorphisme entre les tours de Lubin-Tate et de Drinfield*, volume 262 of *Progress in Mathematics*. Birkhuser, 2007. [cited at p. xiii, 93, 94, 95, 109, 116]

[Hed] Mohammad Hadi Hedayatzadeh. Exterior powers of barsotti-tate groups. `http://arxiv.org/abs/1009.2460`. [cited at p. iv]

[Lau] Eike Lau. Smoothness of the truncated display functor. `http://arxiv.org/abs/1006.2723`. [cited at p. 81, 83, 84, 85, 86, 87]

[Lau08] Eike Lau. Displays and formal p-divisible groups. *Inventiones Mathematicae*, 171:617–628, 2008. [cited at p. iii, x, 61, 81, 86, 87, 88, 109]

[Lau10] Eike Lau. Frames and finite group schemes over complete regular local rings. *Documenta Mathematica*, 15:545–569, 2010. `http://arxiv.org/abs/0908.4588`. [cited at p. 63, 64, 65, 66, 67, 68]

[Laz75] Michel Lazard. *Commutative Formal Groups*, volume 443. Springer, Berlin, Heidelberg, New York, 1975. [cited at p. 31]

[Liu06] Qing Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics.* Oxford University Presss, Berlin, Heidelberg, New York, 2006. [cited at p. 40]

[LMB91] Gerard Laumon and Laurent Moret-Bailly. *Champs algebrigues.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, Heidelberg, New York, 1991. [cited at p. 81]

[Mes72] William Messing. *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes*, volume 264 of *Lecture Notes in Mathematics.* Springer, Berlin, Heidelberg, New York, 1972. [cited at p. xiii, 21, 93, 94, 95, 97]

[RZ96] Michael Rapoport and Thomas Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies.* Princeton University Press, 1996. [cited at p. 98]

[Ser79] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics.* Springer, 1979. [cited at p. 10, 11]

[Zin] Thomas Zink. Formale gruppen und displays. Homepage: `http://www.math.uni-bielefeld.de/~zink/V-DFG.html`; visited on December 5th 2011. [cited at p. xiii, 93, 96, 97]

[Zin84] Thomas Zink. *Cartiertheorie kommutativer formaler Gruppen*, volume 68. Teubner Texte zur Mathematik, Leipzig, 1984. [cited at p. iii, 28, 31, 37, 38, 101, 115, 116]

[Zin86] Thomas Zink. *Cartiertheorie ber perfekten Ringen. I*, volume Preprint. Akademie der Wissenschaften der DDR, Berlin, 1986. [cited at p. 37, 93, 98, 99, 103, 104]

[Zin01] Thomas Zink. Windows for displays of p-divisible groups. In *Moduli of Abelian Varieties.* Birk 2001. [cited at p. 63]

[Zin02] Thomas Zink. The display of a formal p-divisible group. *Asterisque*, 278:127–248, 2002. [cited at p. iii, vi, 5, 16, 17, 18, 19, 21, 22, 23, 24, 25, 27, 28, 36, 39, 43, 69, 71, 83, 89, 91, 92, 93, 103, 108, 109, 115]