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Product and Quality Innovations: An Optimal Control Approach

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Product and Quality Innovations: An Optimal Control Approach

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Abstract

In the suggested paper an attempt to combine two different aspects of innovative activity which are known as product and process innovations is made. The main objective of the paper is to demonstrate the importance of interdependence between these two types of innovative activity through means of a simple one agent model. Research questions which are studied in the paper are as following:

1. Whether it is possible to analyze simultaneously two different aspects of innovations on a continious and dynamical basis. To our knowledge there are no models which would suggest the dynamical analysis of such type.
2. What is the interaction between product and quality innovations and how it may influence both processes.
3. What is the effect of the degree of diversity of products on the innovative activity.

The suggested model contains no uncertainty and only one agent (monopolist) which may invest in expansion of variety of products available on the market and into the development of quality of every product which already exists. To develop such a model in continious time I make use of Infinitedimensional Optimal Control methods. Main finding of the paper is that interaction between different types of innovative activity significsantly changes the dynamics of product emergence to the market as well as the quality of products on the market as incentives of investor are changed. Another important finding is that heterogeneity of products and its nature are fundamental for the outcome of the model. The complementarity of product and process innovations changes our view on technology management and competition in this field.

1 Introduction

One of the basic sources of economic growth is technological progress, as it is argued by economic growth theory. Technological progress happens due to innovative activity of economic agents. Thats why modeling innovations is one of the key areas of modern economics. Starting from 1960s there have been a lot of attempts of such modeling and incorporation of innovations to macroeconomic models. This strand of literature concentrated oh the effects of technological progress on the economic growth, rather then on the nature and source of this progress. That's why nowadays these theories are referred to as theories of exogenous technological progress and/or exogenous economic growth. However it was soon recognized, that it is important to model explicitly the process of innovations themselves.

At the beginning of 1990's two new approaches to innovations emerged, namely, Romer's (1990) model of expanding variety of products and Aghion & Howitt's (1993) model of quality ladders. Each of them addressed different aspects of innovative activity, but unlike previous models, they endogenized technological change. First of them explained technological progress as the process of invention of new goods. This process expands the variety of products available on the market, thus stimulating growth through increase in consumer demand (Dixit-Stigliz theory) or through increase of productivity (Romer's model). However, quality (or productivity in the case of investment goods) was assumed to be constant. It is this approach which is fundamental in New Economic Geography literature.

Second approach explained technological change as the process of creative destruction of products. Every product is assumed to have varying quality, which may be increased through investments outcome of which is assumed to be uncertain. However, quantity or variety of goods available on the market is assumed to be constant. This approach gave birth to a vast strand of literature starting from early 1990's. Here main subject of study is competition between innovative firms and thus it is somehow related to IO literature.

It is argued, that both these approaches are complementary in nature, describing two aspects of the same process, which are going on simultaneously. At the same time, there is no a unified model, which would take into account both these aspects in the dynamical framework.

However some attempts has been made to bring together process and product innovations. Frequently this purpose is achieved through the construction of the 2-stage static game, where on the first stage the decision upon the introduction of new good is being made and on the second - how much investment to put into the development of quality of this new introduced product (conditional upon the successful introduction of it on the first stage). One example of such papers is the paper [8] which is mainly devoted not to the interaction between both types of innovations itself, but to the relation between organizational structure of the firm and its innovative decisions. Paper is mainly devoted to the correspondence between the organizational structure of a firm and its choice of the level of innovations of both types (which is made only once in the 2-nd period). It is shown, that the complementarity between process and product innovations is the direct consequence of the complementarity between firm's manufacturing flexibility and its research capabilities. Current paper correlates with this kind of literature in the idea of simultaneous decision making upon innovations of both types. However, I consider the situation with multiple products to be introduced on the market with some speed, which is controlled by the innovating firm.

One other paper which corresponds in some area to the suggested analysis is [9]. Here the process of innovations is also formulated as the 2-stage game, but the paper tackles mainly with incentives to innovate and their relation to the particular characteristics of the profit function. In the current model I abstract from the market characteristics of the firm (e.g. profit function characteristics) and simply assume some linear and constant return from the increase in quality and range of products. Both these examples are static in nature and they do not handle multiproduct situations.

Later on it has been noted, that real innovative companies are often multiproduct monopolies. Papers by Lambertini (2001,2003) study the equilibrium characteristics of the investments of such a monopoly. Here he allows for multiproduct investments, and the number of existing products may also increase in the result of product innovations. However the whole model is static because it handles only the equlibrium points of innovative policy of a monopolist. Author does not study any dynamical characteristics of product and process innovations but only the equilibrium distribution of investments. In the second paper Lambertini claims that equilibrium level of quality investments is higher for the monopolist then the social optimum. However, more recent paper by Lin [12] claims that this heavily depends on the level of economies of scope for the monopolist. In general to be able to answer this question one has to account for dynamical perspective and the evolution of the product space. Such an attempt is being made in the current paper. Current paper tries to fill this gap. It concentrates on two questions,namely:

- How expansion of goods variety influences quality innovations to already existing and new products?
- What is the role of structural characteristics of these products by themselves in innovative process?

To answer these and related questions the model is built using optimal control theory methods. The basic framework discussed below allows for a very general formulation of innovative activities. However in the current paper the simplified approach is chosen as the main goal is just to demonstrate the importance of such a unified dynamical approach to innovations. For that we assume no uncertainty in the model despite the fact that the importance of uncertainty in innovations is widely acknowledged now. More then this, we model only one agent and do not take into account any possible strategic interactions here. In a sense the model is a dynamic verswion of the model of multiproduct monopolist innovative agent of [10], where additionally range of potential products is assumed to be continuum.

The structure of the paper is as following. First we discuss the basic framework suggested and importance of the assumptions being made in the process. Second we build up the general model in the form of distributed parameter optimal control one. After that we state some necessary theoretical properties of such a construction in the 4th section. In the 5th section we describe general solution and its properties. It turns out that not much may be stated on this stage concerning the model's dynamics. That's why in 6th section we simplify a model by allowing all products to be identical. This trivial case helps us to analyze more general and complicated properties of the model. 7th section is devoted to heterogeneous case. however, we allow only for some special type of heterogeneity which yields linear differential system as the result. In 8th section we discuss the overall dynamics of quality investments and reconstruct the dynamics of the whole model, combining results of previous sections. In 9th section we analyze influence of various parameters changes on the dynamics of the system. In the last section we discuss results and understanding achieved upon the unified process of innovation and mention some possible future extensions and refinements of the model suggested here.

2 Assumptions and Basic Framework

To model the process of expansion of products' variety and quality growth simultaneously we introduce the notion of the products' space. This space contains as elements all products which are already invented as well as potential products that may be invented in the future. Every product has its own characteristic - its quality. We identify every product with its quality function. From this point of view, space of products is the functional space and its elements are quality functions for every product i. Then products themselves (both already invented and potential ones) are dimensions of such a space. More than this, we assume that this number is the real number, so dimensionality of the product space is uncountable. In such a framework it is no longer correct to speak about number of products but rather about the range of them. This range is assumed to be bounded from above by some maximal range of products which can be invented in a given system (economy, market).

The space of products is thus characterized by the infinite-dimensional vector-valued functions $Q(t)$, which describe evolution of all products' qualities over time. Such a space is hard to analyze in general and we put additional structure on it to make in manageable. That is,we assume that quality growth of every product does not explicitly depend on other products. Then every function $Q(t)$ may be represented by the infinite dimensional system of real-valued functions $q_i(t)$ or, equivalently, by the function of two arguments $q(i,t)$, where i is the number of a product and t is time. In the last representation it may be shown, that the space of functions $q(i,t)$ is an $\mathbf{L}^2$ space, provided it has a compact support. So we assume that the range of product as well as time are compact subsets of $\mathbb{R}$:

$$\begin{aligned} t \in [0,T] = \mathbf{T} \subset \mathbb{R}_+; \\ i \in [0,N] = \mathbf{I} \subset \mathbb{R}_+. \end{aligned} \tag{1}$$

We also assume boundedness of Lebesgue integrals over that space. This assumption is not essential for the analysis, and is taken to simplify the gtheoretical analysis. We denote the space of such functions by $\mathbf{L}^2(\mathbf{T}\times\mathbf{I};\mathbf{Q})$, where $\mathbf{Q} \subset \mathbb{R}$ as well.

Process of the expansion of variety of actual products is described by a one-dimensional function of time, $n(t)$, which takes values in the space $\mathbf{I}$. So effective range of products is constantly changing over time. It is then natural to require that quality for products which are not yet invented, cannot change from its initial level. Then $n(t)$ dynamics represents the motion in the space of potential products along the subspace represented by $q(i,t)$ functions. It describes effective coordinates of this function. Analytically the last requirement can be expressed as a constraint upon the function $q(i,t)$:

$$\begin{aligned} \dot{q}(i,t) = 0 \mid_{i>n(t)}; \\ \forall i \in \mathbf{I} \end{aligned} \tag{2}$$

The whole process of innovations is then described by continuous expansion of the range of products available and by simultaneous growth of quality of all products which are already invented in some infinite dimensional product space $\mathbf{Q}$, while quality growth process for each product is independent from other processes and is launched at the time when this product is actually invented, (2).

To impose control over these processes we introduce investments being made in the variety expansion and in quality growth of every product separately, $u(t)$ and $g(i,t)$ respectively. We assume these two process to be positive real valued and bounded from above. Dynamics of variety expansion and quality growth are determined through investment policy, that is the actual choice of investments at each point in time in both directions. To simplify the analysis we assume no uncertainty in the model.

3 Model

Under the basic framework consider in previous section to cast the model into the Optimal Control framework, we assume the scheme of so-called 'planned' innovation: there is only one agent (social planner) which maximizes the output of innovations in any given period of time over the fixed time horizon according to some objective functional. It is defined as:

$$J \stackrel{\text{def}}{=} \int_0^T e^{-rt} \left(\int_0^{n(t)} q(i,t) - \frac{1}{2} g(i,t)^2 di - \frac{1}{2} u(t)^2 \right) dt \to \mathbf{max} \tag{3}$$

Planner is maximizing integral sum of qualities of all products invented until each time$t$ minus investments being made to every invented product's quality and to the overall expansion process over the planning horizon.

Dynamics of quality growth and expansion process are governed by subsequent dynamic equations:

$$\begin{aligned} \dot{n(t)} &= \alpha u(t), \\ \dot{q(i,t)} &= \gamma(i) g(i,t) - \beta(i) q(i,t), \\ \forall i &\in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+ \\ \forall t &\in [0,..,T] = \mathbf{T} \subset \mathbb{R}_+ \end{aligned} \tag{4}$$

and static constraints:

$$\begin{aligned} u(t) &\geq 0; \\ g(i,t) &\geq 0; \\ n(t) &\leq N; \\ q(i,t) \mid_{i=n(t)} &= 0. \end{aligned} \tag{5}$$

We assume zero initial quality for all products and some fixed initial range of products available. Observe, that the last constraint in (5) is equivalent to (2), provided investments to quality growth are nonnegative. Next observation concerns $\gamma(i)$ and $\beta(i)$ functions. These are functions of efficiency of investments to every products quality and rate of quality decay in the absence of investments depending on the product's number i respectively. These two functions represent structural characteristics of the products space being considered as a whole, as they define relative differences in products as functions.

Equations (3),(4),(5) constitute a distributed parameter control system. In general such systems may be hard to solve, but making use of the assumption upon the independence of quality growth processes for every i it can be equivalently written in the form of infinite-dimensional optimal control system with respect to $q_i(t), g_i(t)$ functions. Moreover, optimal solutions of such a system correspond to optimal solution of distributed parameter system, provided that convergence with respect to the norm in $\mathbf{Q}$ space (which is understood as the space of functions $q(i,t)$ with an L^2 structure) is equivalent to convergence in components. This last follows from the fact that $q_i(t)$ functions are continuous projections of $q(i,t)$ function and they form a basis in the $\mathbf{Q}$ space, since they are independent of each other. In such a representation $\gamma(i)$ and $\beta(i)$ functions correspond to coefficients γ_i and β_i for each product i.

Infinite dimensional representation of the system (3),(4),(5) allows the application of Maximum Principle, while each $q_i(t)$ function represents a separate state variable. As a result, one have system of $N+1$ first order conditions as well as adjoint equations.

4 Theoretical results

We begin with theoretical results concerning model (3),(4),(5). One of the reasons to transform the model from distributed parameter form to infinite dimensional one is that Maximum principle is easier to prove for infinite dimensional problem. Note, that any distributed parameter system may be transformed into the infinite dimensional ODE system. However,preservation of optimality results is not granted. That is, solution to infinite dimensional model may be not an optimal solution to the subsequent distributed parameter problem due to the form of objective functional (3). In formal terms this means simply that convergence with respect to $\mathbf{L}^2(\mathbf{T} \times \mathbf{I}; \mathbf{Q})$ norm implies but is not implied by convergence in components (e.g., projection spaces for each $i \in \mathbf{I}$) in general. However in the case of the model (3),(4),(5) one may show such an equivalence. For that it is sufficient to show the continuity of all projections of the functional (3) along the $\mathbf{I}$ index space.

Proposition 1 *Objective functional (3) is continuous with respect to* $\boldsymbol{I}$ *and* $\boldsymbol{T}$ *spaces.*

Proof. One need only to show the continuity of $\int_0^{n(t)} q(i,t) - \frac{1}{2}g(i,t)^2 di$ term in (3). For every given $n(t)$ it is a function of the interval in the subsequent $\mathbf{L}^2$ space, generated by $q(i,t)$ and $g(i,t)$ functions By definition ([2])this function of an interval is continuous if the generating function of a point has countable number of discontinuities. This is true for the function given, as $q(i,t)$ is a continuous function (since its a state variable) and $g(i,t)$ has finite number of discontinuities in t for every i. These implies continuity with respect to $\mathbf{T}$ space. The only problem is possible discontinuity along $\mathbf{I}$ space, since this index space is a subset of real numbers.
Consider projections of the form $g_t(i), q_t(i)$. These are functions of the index for every t. Note that functions $q_t(i)$ are continuous as the last condition in (5) is exactly the continuity requirement. Regularity of $g_t(i)$ functions depends on the regularity of parameter functions $\gamma(i), \beta(i)$ only which are arbitrary at the moment. We conclude that it is sufficient to assume some regularity conditions on functions $\gamma(i), \beta(i)$ to grant objective functional continuity. In current model we assume them to be continuous functions of i. Then functions $g_t(i)$ can have at most countable number of discontinuities and this is sufficient for functions of interval $\int_0^{n(t)} q(i,t) - \frac{1}{2}g(i,t)^2 di$ to be continuous. ■
With continuity of the objective functional one may freely transform the problem (3),(4),(5) to the infinite dimensional framework.

First note that (5) imply compactness of state space for every i and for $n(t)$ both. Second observation is that the whole system of differential equations (4) may be decoupled into equations for $n(t)$ and for $q(i,t)$. The only link between the two components of the system is through the (5). The system for quality growth is then written in the form of infinite dimensional system of ODEs:

$$\dot{q_i(t)} = \gamma_i g_i(t) - \beta_i q_i(t),$$
$$\forall i \in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+. \tag{6}$$

Which may be written in operator form:

$$\dot{q(t)} = Aq(t) + Bg(t) \tag{7}$$

This system of controlled equations is linear and as such is subject of Hille-Yosida theory. To grant implementation of Maximum Principle one has to be sure that the uncontrolled part of the system (6) is a well-posed Cauchy problem. This can be done through Hille-Yosida theorem [3]. For the system (7) it is particularly simple, since operator A has diagonal form and does not depend on t explicitly. We are not giving the formal proof here and confine ourselves to the observation that it is sufficient to proof that operator A has a full rank and its invert is bounded. This obviously depends on the choice of $\gamma(i), \beta(i)$ functions, so we assume these to be not zero everywhere and with bounded inverses. Note, that this is the second regularity assumption imposed on parameter functions.

Next we show that optimal controls exist. First observe that control space possess product topology and can be decoupled into controls over variety expansion process and quality growth. First of these is one dimensional and bounded, second is infinite dimensional and bounded in each coordinate. Define control space as

$$\mathcal{J} = \mathcal{U} \times \mathcal{G} \tag{8}$$

Taking into account constraints (5) one may define admissible control set as

$$\mathcal{J}_{ad} = \mathcal{U}_{ad} \times \mathcal{G}_{ad} \tag{9}$$

where both subspaces are compacts since they are bounded from above and below and are close subsets of real line and real plane subsequently. Then the whole admissible control space is compact also. This yields the existence result.

Proposition 2 *Optimal controls $u(t)^*$, $g(t)^*$ exist.*

Proof follows from compactness of admissible control space.

Now one may make use of Maximum Principle approach to obtain optimal controls. For that some standard properties of control space and control system have to be fulfilled, including regularity of the B operator in (7) and completeness of the control space with respect to perturbations. These two are straightforward to show and we do not stop to give rigorous proofs. it is sufficient to note that completeness of the control space with respect to spike perturbations is the consequence of compactness of admissible control space and regularity of operator B follows from the fact that it is diagonal and does not depend on time. Then the only regularity requirement is on the functions $\gamma(i)$, which is already assumed above.

The difficulty of infinite dimensional problem is that the existence of non zero optimal set of adjoint variables is not granted. It has to be proved separately. For that we make use of the following lemma:

Lemma 3 *[3] Let $\{t_n\}, \{\tilde{j}^n\}, \{\tilde{y}^n\}$ be the sequences of time, controls and states converging to optimal solution of the control problem. Assume that there exists such $\rho > 0$ and a precompact sequence $\{Q_n\}, Q_n \subseteq E$, such that:*

$$\bigcap_{n=n_0}^{\infty} \{t_n^{-1} R(0, t_n; \boldsymbol{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho) + Q_n\} \tag{10}$$

contains an interior point for n_0 large enough. Then the multiplier z is not zero.

Here z is the set of adjoint variables, E is some Banach space, ρ is the distance in this space. It can be shown that requirements of this lemma hold for the given problem due to the special structure of the control and state spaces (they are compact and possess product topology). Full proof may be found in Appendix. In conclusion note that all existence and regularity results presented here directly depend on the regularity assumptions on $\gamma(i), \beta(i)$ functions. Up to now we did not specify them explicitly but assumed that:

- These are continuous functions of i;
- Inverse functions $\gamma(i)^{-1}, \beta(i)^{-1}$ exist.

No monotonicity or differentiability requirements are necessary. Observe, that these functions represent measure of heterogeneity of the product space $\mathbf{Q}$: the more regular these functions are, the more comparable different products are to each other in their structural characteristics.

5 General Solution

General solution to the problem (3),(4),(5) is obtained through application of the Maximum principle to the subsequent infinite dimensional system. Hamiltonian function is given by

$$\mathcal{H} = \int_0^{n(t)} q_i(t) - \frac{1}{2} g_i(t)^2 di - \frac{1}{2} u(t)^2 +$$
$$\lambda(t) \times \alpha u(t) + \int_0^N \psi_i(t) \times (\gamma_i g_i(t) - \beta_i q_i(t)) di \tag{11}$$

For each i one may derive from first order conditions equations for optimal control and state trajectories for quality of every product:

$$g_i(t) = \gamma_i \psi_i(t);$$
$$\dot{\psi_i}(t) = (r + \beta_i)\psi_i(t) - 1;$$
$$\dot{q_i}(t) = \gamma_i g_i(t) - \beta_i q_i(t). \tag{12}$$

This system yield optimal control and state trajectories as functions of γ_i and β_i:

$$g_i(t) = \gamma_i \left(\frac{1 - e^{(r+\beta_i)(t-T)}}{r + \beta_i} \right);$$
$$q_i(t) = \frac{\left((e^{-rT-\beta_i(t+T)} - e^{(r+\beta_i)(T-t)})\beta_i + (1 - e^{-\beta_i t})(2\beta_i + r)\right)\gamma_i^2}{(r + \beta_i)(2\beta_i + r)\beta_i}. \tag{13}$$

However, taking into account (5) these solutions are effective only for $i > n(t)$ as investments cannot be positive until that time and quality level remains zero. This means quality dynamics has a piecewise form:

$$g_i(t) = \begin{cases} \gamma_i \left(\frac{1-e^{(r+\beta_i)(t-T)}}{r+\beta_i} \right), n(t) \geq i \\ 0, n(t) < i \end{cases}$$

$$q_i(t) = \begin{cases} \frac{\left((e^{-rT-\beta_i(t+T)} - e^{(r+\beta_i)(T-t)})\beta_i + (1-e^{-\beta_i t})(2\beta_i + r)\right)\gamma_i^2}{(r+\beta_i)(2\beta_i+r)\beta_i}, n(t) \geq i, \\ 0, n(t) < i \end{cases} \tag{14}$$

First order conditions for $n(t)$ yield the system of two differential equations on $n(t)$ and its costate:

$$\begin{aligned} \dot{\lambda(t)} &= r\lambda(t) + \frac{1}{2} g_{n(t)}(t)^2 - \psi_{n(t)}(t)\gamma_{n(t)} g_{(n(t)}(t); \\ \dot{n(t)} &= \alpha\lambda(t). \end{aligned} \tag{15}$$

with boundary conditions

$$\begin{aligned} n(0) &= n_0; \\ \lambda(T) &= 0. \end{aligned} \tag{16}$$

Variety expansion process however, is the process across all these states q_i. Hence, to obtain dynamics of $n(t)$ one should aggregate quality dynamics across states to return to the distributed parameter form of the problem. In that way system (15) depends on functions $\gamma(i)$ and $\beta(i)$. After substitution for $g_{n(t)}(t)$ from (14) it can be seen that system (15) is not time invariant for any specification of $\gamma(i)$ and $\beta(i)$ functions. Dynamics of this system depends at each point in time from the value of investments efficiency to the growth of quality of the next product to be invented and do not depend on any other ones. This system represents a recurrence relation in the space of products $\mathbf{Q}$, as $\gamma(i)$ function depends on i, which is also the value of $n(t)$ function. That means that at $i = n(t)$ $\gamma(i)$ function is the function of $n(t)$. Actual shape of dynamics of the process of variety expansion heavily depends on the shape of this function then.
Through direct integration one may obtain general equation of motion for $n(t)$, as an integro-differential equation:

$$n(t) = \int_0^t \int_T^s \gamma(n(\tau)) * e^{\beta(n(\tau)) \times f(\tau,T)} d\tau ds \tag{17}$$

This is hard to analyze and existence of the solution is not granted for any $\gamma(\bullet), \beta(\bullet)$ functions.

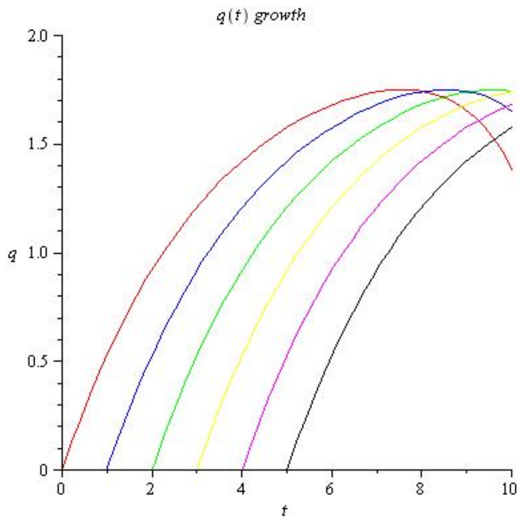
Some general properties of the system may be captured from the form of the general solution though. Quality growth for all products have similar form and differ from each other only by values of $\gamma(i)$ and $\beta(i)$ functions. Starting from the time when the product is invented, its growth process is independent of any other variables of the system. However, time when this process starts is defined through the expansion process, $n(t)$. This last one depends heavily on the parameters' functions of the product space. This means that no single product nor its quality process affects variety expansion. Instead this last one depends on some aggregate characterization of the product space, which is given by $\gamma(i)$ and $\beta(i)$ functions. Mere existence of solution to the problem itself depends on the properties of these functions which may be viewed as fundamental characteristics of the product space itself. That is, the more regularity requirements one puts on them, the more homogeneous space of products in terms of their diversity one is considering (e.g. smoothness assumptions upon $\gamma(i)$ would mean smooth transition across different products' quality characteristics). Note that this diversity is different from the range of products N, which simply denotes the size of the maximal range of products (size of the space). Parameter functions are measures of efficiency of investments across products. They may or may not depend on the maximal range of potential products. Here we confine ourselves to the case when they do depend on N. That means the measure of diversity of efficiency of investments into each product's quality is linked with the maximal range: the bigger is the range the more is the maximal quality level for each product, as it will be seen later. However the general framework is not limited to these particular specification and more general structure may be considered (for example, one may consider the case where effciency function $\gamma(i)$ depends only on relative positions of products but not on the total range of them).

To obtain some particular solution to the problem one have to specify the form of these parameter functions. We consider as an example the simplest case which linearizes the system (15) in the next section.

6 Homogeneous Products

To demonstrate the role of diversity of the product space, first we account for the homogeneous version of it. For that we assume both γ_i, β_i functions to be constant across different products. Note, that this would not mean that products are identical in their consumption characteristics, but only that efficiency of investments into the quality of different products does not change with the expansion of variety of products. Thus from the investments point of view, such a product space may be called 'homogeneous'.

In such a homogeneous case every product's quality has essentially the same shape of dynamics. The only difference is in the starting date of investments (being defined from variety expansion). Then the only source for identifying different products and separating them from each other lies in the $n(t)$ space, since the piecewise form of quality functions (14). That is in the area of their piecewise definition. Actual projections of quality functions in $q-t$ plane are plotted below for some arbitrary underlying process of variety expansion (it defines the $q=0$ point at t axis for all solution curves).



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All product's quality has some maximal attainable level, which is never reached in the finite time. This level is:

$$q_i^* = \frac{\gamma_i^2}{(r+\beta_i)\beta_i} \tag{18}$$

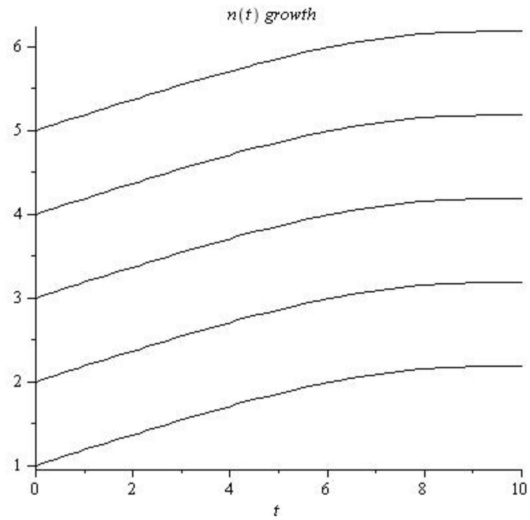
It can be obtained by equation to zero lefthandside of the system (12). With constant β, γ functions this level is unique and the same for all products being invented. This level is also the saddle-point of the system, since the characteristic equation has two distinct real roots of different signs. This means that asymptotically all products' qualities would reach that point and stay there in infinite time horizon with given initial conditions. With the assumption of homogeneity between products' investment characteristics assumption the system (15) is reduced to

$$\begin{aligned} \dot{\lambda(t)} &= r\lambda(t) - \frac{1}{2(r+\beta)^2}\gamma^2 \times (1 - e^{(r+\beta)(t-T)})^2; \\ \dot{n(t)} &= \alpha\lambda(t). \end{aligned} \tag{19}$$

This is still a non-autonomous system but can be readily solved in elementary functions.

$$\begin{aligned} n(t) &= \frac{1}{(\beta+r)^3(2\beta+r)\beta r^2} \times \left(C_1 - C_2 e^{2(r+\beta)(t-T)} + C_3 e^{(r+\beta)(t-T)} - C_3 e^{r(t-T)} + C_4 t\right); \\ \lambda(t) &= \frac{\gamma^2}{(\beta+r)^2(2\beta+r)\beta r} \times \left(C_5 - C_6 e^{2(r+\beta)(t-T)} + C_7 e^{(r+\beta)(t-T)} - C_8 e^{r(t-T)}\right). \end{aligned} \tag{20}$$

This solution is an increasing function of time in $n(t)$ and decreasing in $\lambda(t)$ parts. With solution at hand, one can easily analyze the influence of different parameters on the systems's behavior. Note first, that the qualitative dynamics of every product's quality does not change irrespectively of the chosen specification of the parameter functions. Thus, it is exactly the same as in the general model with exception that $\gamma(i), \beta(i)$ functions are now constants. Below several solution curves corresponding to (20) with different initial range values are plotted.



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This solution does not have any fixed points, as it can be readily seen from system (19)and it's characteristic equation, which has one real positive root and one zero root, which means unstable dynamical system. To observe that consider the reduction of the system (19) to the autonomous system:

$$\dot{\lambda(t)} = r\lambda(t);$$
$$\dot{n}(t) = \alpha\lambda(t). \quad (21)$$

This linear ODE system has two eigenvalues, one zero and another positive real:

$$|\theta I - J| = (\theta - r)\theta;$$
$$\theta_1 = 0, \theta_2 = r > 0. \quad (22)$$

Observe that $\frac{1}{2(r+\beta)^2}\gamma^2 \times (1 - e^{(r+\beta)(t-T)})^2$ is an expansion of autonomous sytem (21) along the λ axis. Then the dynamics of initial non-autonomous syten is the expansion of the subsequent autonomous one with preservation of fixed points structure ([4]). If the autonomous system does not have fixed points, then the non-autonomous also doesn't have them. One has the unbounded (but rather slow) growth of $n(t)$ function then.

Next observe parameter influence:

- Range of products N does not influence dynamics of the system in the case of interior solution as long as parameter values do not depend on it. However if $\gamma = \gamma(N), \beta = \beta(N)$, as it was assumed earlier, its influence is equivalent to the influence of these parameters;

- Length of the planning horizon, T positively influences the range of product invented until this horizon. This is the direct consequence of the monotonicity of solution with respect to time. Shadow price of investments, $\lambda(t)$ has lesser initial values for shorter horizons;

- Rate of decay of the quality, β which is now constant across products, significantly reduces the speed of variety expansion. However, even for $\beta > \gamma$ variety expansion has a positive dynamics;

- Efficiency of investments into products qualities, γ positively influences the dynamics of variety expansion the same is true for efficiency of investment into variety expansion itself. The last one has more significant influence.

All the parameters influence dynamics in a quiet trivial way. Note however, that as long as γ, β are functions of the range of products, N, no boundary solutions may occur with $n(T) = N$.

Now observe the character of interdependence between variety expansion and quality growth processes. In this homogeneous case the only link is one noted above, that is, quality investments to every product starts only when it is invented. At the same time all investments to quality if they occur, are identical in their speed. Variety expansion process does depend on the characteristics of the product space. However this dependence is also quiet fragile: the only difference between the independent variety expansion process and the one accounted here is in constants γ, β. Obviously, this does not change dynamical characteristics of the process, but only the mass of inventions, acquired at each point in time. Recall that in general formulation we assume parameters' functions $\beta(i), \gamma(i)$ as dependent on N. Then in the absence of heterogeneity between products and provided constants β, γ depend on range N, this last remains the only characteristic of the product space, since γ, β do not change across products and the differences between products characteristics do not play any role. More then this, this characteristic is not related to quality growth, but to the expansion process itself. So one may conclude, that qualitative behavior of the system (19) does not differ substantially from independent development of quality and product innovations. Observe that the variety expansion process will not evolve if investments to all products qualities would be zero all the time, that is if $\gamma = 0$. So there is some but very weak dependence of variety expansion from investments to quality growth. This observation lead to the conclusion that some sufficient measure of heterogeneity of the product space is essential for non-trivial interdependence of quality growth and variety expansion processes. On the other hand, the simple version of the model studied in this section demonstrates, that standard models of innovations which treat both processes independently may be casted into the suggested framework as special cases for homogeneous (in the sense defined above) space of products. If one want to describe evolution of range and quality of products which are not that identical in their investment characteristics, one have to explicitly model both processes simultaneously.

7 Linear Model

From now on we assume specific form of parameter functions. We assume $\gamma(\bullet)$ function does depend on N and is monotonic and decreasing in i, while we assume $\beta(\bullet)$ function constant for simplicity:

$$\begin{aligned} \beta(i) &= \beta; \\ \gamma(i) &= \sqrt{N-i} \times \gamma. \end{aligned} \tag{23}$$

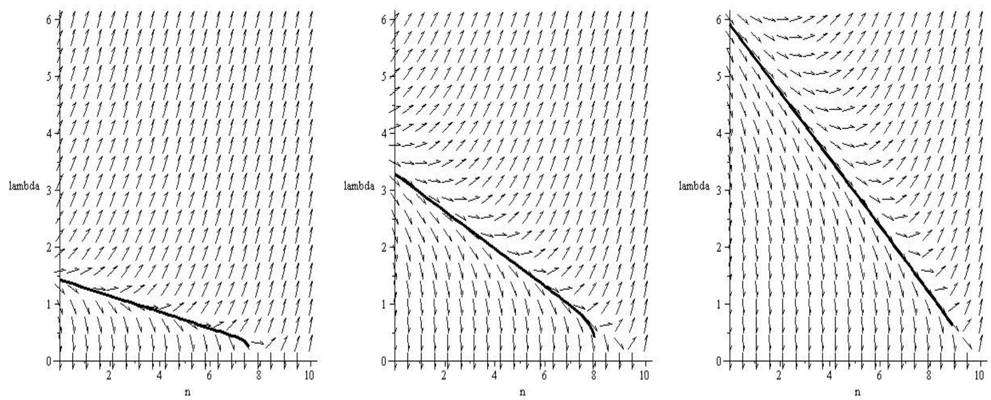
With such a form of parameter functions system (15) is a linear non-autonomous system:

$$\begin{aligned} \dot{\lambda(t)} &= r\lambda(t) - \frac{\gamma^2 (N - n(t))(1 - e^{(\beta+r)(t-T)})^2}{(r+\beta)^2}; \\ \dot{n(t)} &= \alpha\lambda(t). \end{aligned} \tag{24}$$

This system does not have solutions in elementary functions [4] but it does have real-valued solutions according to the general Sturm-Luiville theory [5]. So one can analyze dynamics of this system. For that one may employ the method of analysis, which is valid for linear dynamical systems (non-autonomous) only [4].
That is, one may introduce some artificial variable, $x(t)$, which would make the system autonomous:

$$\begin{aligned} \dot{\lambda(t)} &= r\lambda(t) - \frac{\gamma^2 (N - n(t)) x(t)}{(r+\beta)^2}; \\ \dot{n(t)} &= \alpha\lambda(t); \\ x(t) &= (1 - e^{(\beta+r)(t-T)})^2. \end{aligned} \tag{25}$$

In t In this way by fixing certain levels of $x(t)$ variables one has autonomous linear system in $\lambda(t), n(t)$ which has usual solution and can be analyzed by standard techniques. $x(t)$ variable is the increasing function of time, varying from zero to one. It can be viewed as a contraction operator in the system (25),which transforms the phase space of the system. In our case this operator spans the phase space along the $\lambda(t)$ direction. To view this, consider gradient fields for different levels of x (0.2, 0.5, 1) for some parameter values:



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The only locus of the phase space which is consistent with boundary value of the costate variable ($\lambda(T) = 0$) is the locus constrained by the solid black line from above. During the evolution of a system, this locus is spanned along the lambda axis.
Then one may move this system and its resulting solutions along the $x(t)$ axis with the given speed to reconstruct the dynamics of initial non-autonomous system. Observe also, that in terminal time, $t = T$ the original system converges to the system (25). Given arguments reveal, the action of the map $x(t)$ on the system preserves its fixed points and changes the dynamics with respect to shadow price movements. So one may obtain stability results through investigation of the 25
For autonomous sytem one may compute eigenvalues of the system matrix:

$$|\theta I - J| = \theta(\theta - r) - \frac{\alpha\gamma^2}{(r+\beta)^2}x = \theta^2 - r\theta - \frac{\alpha\gamma^2}{(r+\beta)^2}x;$$

$$\theta_{1,2} = \frac{r}{2} \pm \frac{\sqrt{r^2 + 4\frac{\alpha\gamma^2}{(r+\beta)^2}x}}{2}. \tag{26}$$

It is clear, that both eigenvalues are real and of different signs.

It is also clear, that this autonomous system has the unique singular point at

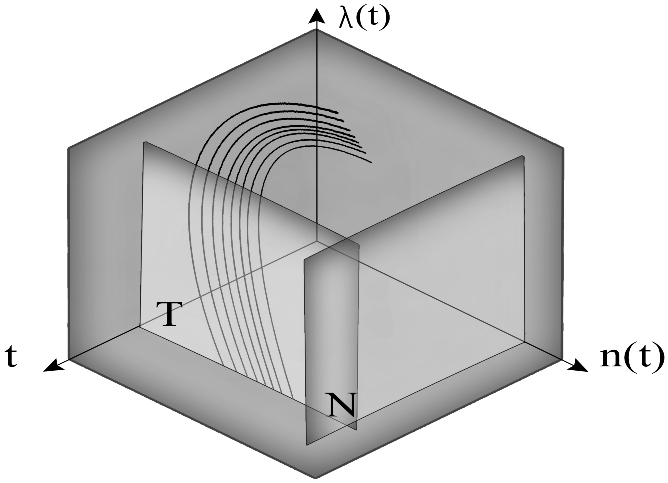
$$\bar{\lambda(t)} = 0;$$
$$\bar{n(t)} = N. \tag{27}$$

which may be reached only at the terminal time.

This yield some system of 2 linearly independent solutions that has rather simple dynamics in the $\lambda(t) - n(t)$ phase space. Given boundary conditions (16) one have monotonic motion of the system. $n(t)$ is growing steadily with decreasing speed, while $\lambda(t)$ - shadow price of investments - is decreasing until zero. This is the standard dynamics of the capital accumulation problem with one singular point.

Observe however, that the original non-autonomous system (24) will reach this level much more slowly, since the $x(t)$ is increasing function of time from zero to one. This would push $n(t)$ level constantly away from its steady-state level, as shadow price of investments will decrease much slower, then in the (25), due to the action of $x(t)$. However, at $T \to \infty$ this $x(t)$ term would eventually go close to 1 for all t and hence asymptotically at long time horizons $n(t)$ dynamics may be described by the means of autonomous system 25.

It also has to be noted, that the type of dynamics is different if all products are identical in terms of investments efficiency (that is, when $\gamma(\bullet)$ function is also constant across products). It has been shown, that the homogeneous system (19) do not have any fixed points and $n(t)$ growth is unbounded. However our main interest is to analyze the differences in the behavior of the system in the presence of heterogeneity of products being developed. This is what $\gamma(\bullet)$ function accounts for as well as $x(t)$ term in the (25). In the simplest case being studied here with parameters specification like in (23) there is no very much structural difference in the system (25) behavior in comparison to autonomous system. Movement along the $x(t)$ axis with some exponential speed brings possibility for temporary shadow price increases, while expansion process is speeding up in comparison with homogeneous case. Shadow price of investments to $n(t)$ may have temporary growth period in the beginning of the planning horizon. At times close to $t = \frac{1}{2}T$ shadow price reaches its maximum and begins to decrease steadily until zero. With longer planning horizons there might be more then longer fluctuations of shadow price of investments. Below is the schematical reconstruction of the 3-dimensional system movement.



.. More importantly, heterogeneity of products brings heterogeneity of quality investments into the model and through that some significant changes to the overall system's behavior.

8 Quality Investments

Now we take a closer look on the 2nd part of the dynamical system - motions of investments to the quality growth. First note, that principal shape of dynamics is independent on the parameter functions specification. For any given i dynamics of quality has a saddle-type. This can be established both graphically and analytically.

For that observe that the system (12) take the form:

$$\dot{\psi_i}(t) = (r + \beta_i)\psi_i(t) - 1$$
$$\dot{q_i}(t) = \gamma_i^2 \psi_i(t) - \beta_i q_i(t) \tag{28}$$

after substitution of optimal controls $g_i(t)$ for each i into the system.
This is the usual system of 2 1-st order ODE's, which may be analyzed through conventional methods [4]. For that we change variables in such a way as to make the system homogeneous:

$$\hat{\psi_i}(t) = (r + \beta_i)\psi_i(t) - 1;$$
$$\hat{q_i}(t) = \beta_i q_i(t) - \frac{\gamma_i^2}{(r + \beta_i)}. \tag{29}$$

Such defined homogeneous system has two eigenvalues which are real, distinct and have different signs:

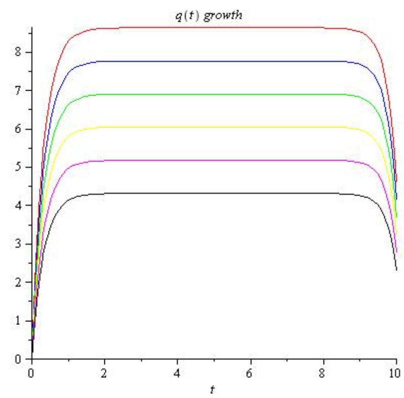
$$|J - \theta_i I| = \theta_i^2 - r\theta_i - \beta_i(\beta_i + r);$$
$$\theta_i = \frac{r}{2} \pm \frac{\sqrt{r^2 + \beta_i(\beta_i + r)}}{2} \tag{30}$$

Obviously, these roots are of different signs, since expression under square root is bigger then $\frac{r}{2}$. This means exactly the saddle -type dynamics of the system. One can easily compute singular points of all these systems for each i.For that just equate left had side of the system (28) to zero to get fixed point values of quality level and shadow price of investments:

$$\bar{\psi_i}(t) = \frac{1}{r + \beta i};$$
$$\bar{q_i}(t) = \frac{\gamma_i^2}{(r + \beta_i)\beta_i} \tag{31}$$

For every product i there is a maximal attainable level of quality corresponding to the saddle point of the dynamical system in quality growth and its shadow price of investments, represented by the co-state variables. This maximal level is defined from $\gamma(i)$ function specification. Following our assumption concerning the structure of the space of products $\mathbf{Q}$, one come to the conclusion that range of products available, $\mathbf{N}$ influences directly this maximal quality. Solution curves for every i fixed has 3 distinct stages: initial rapid growth of investments, then asymptotic approach to the maximal quality level and some decrease of the quality in the end. Note, that in finite time system never reaches its fixed point at the maximal quality level.

Exact specification of the parameter functions define relative positions of qualities of different products. With constant parameters, as in homogeneous case, maximal quality is the same for all products and they are identical in this aspect. However, any other specification of parameter functions generate specific distribution of fixed points for different products. We account only for the linear specification (23) as it is sufficient to demonstrate the role of these functions in the system. In this specification $\gamma(i)$ is the decreasing function of the product's position in the potential products' space. Then maximal attainable quality for every next product is lower, then for the preceding one. Moreover, distribution of these fixed points across products is the straight line. So, the choice of parameter functions not only defines the type of $n(t)$ dynamics, but also the exact form of 2-dimensional attractor for the distributed $q(i,t)$ system. With decreasing specification one have the decrease in maximal qualities across products and the greater is the extent of $\gamma(i)$ function decrease over i's, the greater is the decrease of $q(i,t)$ function across products. Below solution curves for different products' qualities for $T = 10, N = 10$ and some reasonable parameter settings are plotted.



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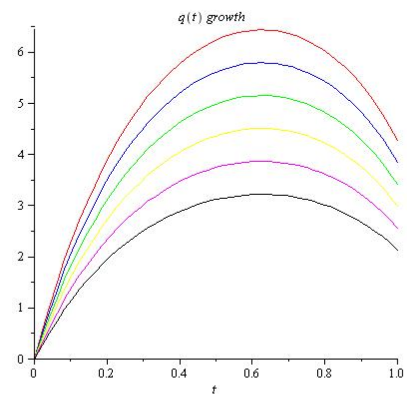

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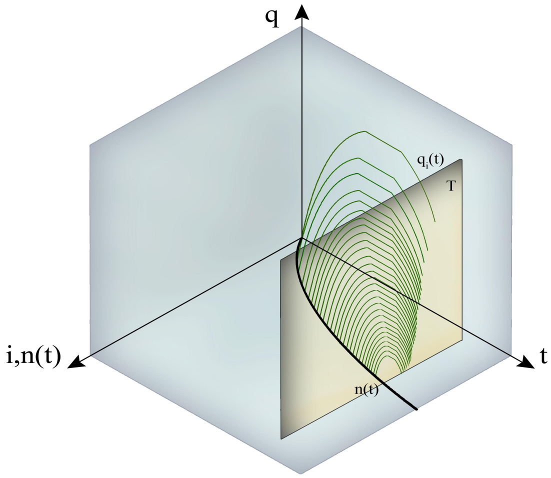
In the case of shorter time horizon (e.g. $T = 1$) qualities never reach their maximal level and actual solution curves are parabolas, as in the last picture. Observe also, that due to the construction of the system, start of investments to some product's quality corresponds to some certain point on the $n(t)$ solution curve. Unfortunately to obtain the exact starting point one have to obtain the exact close-form solution for $n(t)$ process, which is not easy. However, to be able to account for qualitative behavior of the whole $n(t) - q(i,t)$ system one may resort to numerical integration procedures.

Equipped with the knowledge about the behavior of $q(i,t)$ function and about the general properties of the $n(t)$ solution curves, discussed in previous section, it is possible to reconstruct the final combined process. One have a monotonic increase of the variety of products in $n(t) - t$ plane, and this increase is going with slowing speed. This, in turn, means that speed of emergence of new products is decreasing upon the approach of the system to the terminal time, but it never reaches zero with our linear specification. From every point along the $n(t)$ solution curve there is a corresponding process of quality growth for product i. Altogether these quality growth curves generate a generalized function $q(i,t)$ in 3-dimensional space $n - q - t$. Note, that the decreasing speed of expansion means that density of products is higher in the beginning of the process, then in the end. Below is the schematic reconstruction of the system's behavior.



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9 Parametric Analysis

Now consider influence of the system's parameters on the dynamics.

First we claim that the most important parameter is the $\gamma(i)$ function. It governs qualitative features of the system:

- It defines the exact form of the variety expansion process - whether it is monotonic, concave or convex;
- It also defines the distribution of maximal quality levels across the mass of products - the form of 2-dimensional attractor of $q(i,t)$ dynamical system;
- Existence of solution to the general system depends on the regularity of this function.

Economically $\gamma(i)$ is the function of relative efficiency of investments to quality of different products. Then if this efficiency is decreasing along the movement to the boundary of the potential products' space, this naturally mean the increasing difficulty of the quality buildup due to the increase in the complexity of products. It may also be defined the other way around as the assumed ordering in the products' space may be introduced in different ways.

Note, that the above points support our conjecture about $\gamma(i)$ function as the measure of homogeneity of the products' space. The more smooth is this function, the closer are neighboring products to each other in terms of investments efficiency and thus, in their technological nature.

Another technological parameter, β, is assumed to be constant, as it is difficult to analyze the system in the other case. However this assumption seems to be quiet natural, as it is enough to assume varying investments efficiency across products, while β is the rate of decay of qualities in the absence of investments. It can be assumed to be independent on the exact technological characteristics of the given product and is defined on the market.

Increase in this rate of decay shifts down the maximal quality level for all products. This has a clear interpretation. Maximal level of quality is the level at which investments are equal to the depreciation level. Increase in this depreciation without changing investment incentives will slow down each product's development. What is more interesting, increase in this rate of decay also slows down the process of variety expansion, $n(t)$. This fact may be explained by the change in the overall investments policy. Increased rate of the quality decay for all products means that more investment efforts has to be allocated to maintain quality level for all invented products. This means there are less funds available for the ongoing increase of the range of products available. It is now less profitable to increase this range, as the final payoff is obtained in this model only from developed qualities.

Increase in the efficiency of investments to the variety expansion process, α boosts the speed of this expansion and through that, the speed of emergence of new products. This means the higher density of the $q(i,t)$ function at all stages. As the result one would have higher range of products available, but with less developed quality for each of them, since more resources should be allocated to the variety expansion and final payoff increase happens due to the increase of density of products without actual increase in their quality levels.

Increase in the discounting rate, r,shifts the maximal quality level down for each i and also limits possibilities for $n(t)$ growth. This last due to the fact of the shrinkage of feasible set of trajectories for the variety expansion process. It is obvious that the increase in the discounting factor just leads to the decrease of the value of future payoff and thus limits investment capabilities of the planner.

Two parameters left to analyze are the range restriction, $\mathbf{N}$ and fixed time horizon, $\mathbf{T}$. The first one is the measure of the power of potential products' space, $\mathbf{Q}$. Increase in the range of products to be invented leads to the increase in this power. Note, that actual variety expansion process never reaches it's boundary and from formal point of view this restriction is not binding. However it does influence the system dynamics with the given parameter functions specification. As it has been noted, for every i maximal attainable quality is closely related to this range restriction. The wider is this range, the higher is maximal quality for every product in this range, provided the positive dependence of relative investments efficiency on this range. In the other case, with one would assume negative influence (e.g. due to the fact of coordination problem -the higher is the number of products, the harder it is to manage their development), influence of the range of products to be invented on the maximal attainable quality would be negative. Restricting ourselves to the linear specification we note, that asymptotically with $\mathbf{N} \to \infty$, the maximal quality for each product will also increase in that direction. In the limiting case then quality growth will become unconstrained in the sense, that it's dynamics will always remain in the first stage of rapid increase and will never reach the asymptotic approach stage, as the fixed point for each product's i quality level would become infinite. So the saddle-type dynamics of qualities is the direct consequence of range restriction. Economically this means that the wider is the range of potential products one is accounting for, the wider is his set of possibilities for quality development of every product. However, even in the limiting case the ordering of qualities is preserved: every next product has lower quality at each given moment of time, then the preceding one.

One may also account for the influence of the time horizon given. Note, that maximal attainable level of quality for every product is also the steady-state level of quality in the long-run dynamics. Long-run behavior of $q(i, t)$ function is slightly different from the case of the finite time. In the long-run, each product's quality will eventually reach it's steady-state level and, as it is the fixed point of the subsequent dynamical system, will stay there infinitely long. There will not be the final stage of decrease in quality then. In finite time system will not reach its steady state level, but the longer is the time horizon given, the longer is the period of close proximity of quality near it's steady-state level for all i's. while periods of initial rapid growth and final declined would stay unchanged. All these give us the possibility to claim that the majority of time system of qualities dynamics will stay along it's steady state. For $\mathbf{T}$ small enough (lesser then 3 for the reasonable parameters set) qualities' trajectories will not even reach the proximity of their maximal values and start to decline before, that. In this case of short horizon quality dynamics is the set of parabolas.

10 Discussion

The main goal of this paper is to demonstrate the importance of unified approach to quality and products' innovations modeling. Although such a model must be more complicated in structure and methods involved, as it requires infinite-dimensional or distributed parameters control methods, but it is clear from the analysis, that this may reveal a lot about interdependencies between these two types of innovative activities.

The most important conceptual feature of the model being discussed is that it allows to reveal the key characteristic of innovations which is related to the distance between different innovative products. This characteristic in the current setting is reflected by relative efficiency of investments to quality growth for different products, $\gamma(i)$. Note, that this is just a feature of interpretation of the model above,but the characteristic of the potential products' space would play the key role in any practical model which would combine two aspects of the process of innovations. to support this claim, just consider the overall picture of the process being described above. As long as one allow for uncountable number of possible products, the process of variety expansion may be treated as the generating function of the space of products, while quality investments as waves, being generated by this function. The overall process of aggregate quality growth is then the distribution over these waves. Then it is straightforward that exact form of this distribution and it's separate waves would depend on the structure of the space, where this process is going on. And the measure of homogeneity of products themselves is exactly such a characteristic. With constant β coefficient it may be claimed also, that this is the only relevant characteristic of this space. Formally this argument follows form the fact, that exact form of generalized distribution (in Schwarzian sense) would depend on the way of definition of measure and distance in the space of products. Clearly, the given $\gamma(i)$ function in the current model is such a measure and it measures the distance between products in terms of investment efficiency. One may imply any other kind of measure of difference between products, such that difference in their consumption properties, in their closeness in terms of industry, etc. What is important, that such type of a model would require by it's mere construction some kind of such a measure. What kind of such measures may be chosen to leave the model consistent and solvable is the interesting question for further research in the field.

Another question of theoretical interest is topological properties of the general products' space $\mathbf{Q}$, with one would keep away from the assumption of the diagonal nature of operators there. In that case, without any restrictions on the nature of interdependence between different products after their invention, little may be said concerning the model structure without imposing some (hopefully weak) restrictions on the topology of the products' space. As it has been seen here, assumption of independence of products after their invention from each other immediately leads us to the separable Hilbert space structure. Intuitively, however, uncountable number of possible products may lead to the endogenous uncertainty in the model and this is one of the reasons of not treating uncertainty here. In general model without assumption of independence $q(i,t)$ distribution would have much more complicated form, and the space of products may turn to be non-separable. In that case one have to treat explicitly probabilistic nature of any action of the planner in such a space. This also may give some formal theoretical foundation for the uncertainty of the innovative process.

There are a lot of obvious more applied question that may be posed within the model's framework. Thus, one may try to account for effects of competition between innovative agents, thus applying Differential Games Theory instead of optimal control. Such an extension would give one a possibility to distinguish between notions of 'new to the market' and 'new to the firm' products and separate different types of innovative agents according to their internal parameters. In such a model as the discussed one, there is at least one more possible degree of freedom for innovative policy: some agents may concentrate more on the process of invention, while others on the quality growth. Another immediate extension is to account for some patent policy effects in the model. In first approximation that would mean every invented product would have it's own life-cycle and then all quality innovations would not be limited by the same terminal time condition, but would possess different ones. This may or may not have a stimulating effect on innovative activities.

11 Appendix

Proposition 4 *The set (10) for the problem (3),(4),(5) contains an interior point for n_0 large enough.*

Proof. First note, that

$$t_n^{-1} R(0, t_n; \mathbf{J}, \tilde{j}^n) \subseteq \mathbf{Der}\ f_n(\tilde{j}^n) \tag{32}$$

This can be shown by writing arbitrary directional derivative $t_n^{-1}\xi(t_n, \tilde{j}^n, v)$ by means of variation-of-constants formula and comparing it with the reachable set. This means that reachable space multiplied by some convergent sequence $\{t_n^{-1}\}$ is just a subset of a set of directional derivatives (which are variations in some sense).
Note further, that since $t_n \to \bar{t} > 0$ and $K_Y(\tilde{y}^n)$ is a cone, one may get rid of t_n^{-1} term in (10). Then, Q_n is a precompact sequence and otherwise arbitrary, while being multiplied by a convergent sequence (t_n) remains precompact. Then (10) may be rewritten:

$$\begin{aligned} &\{t_n^{-1} R(0, t_n; \mathbf{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho) + Q_n\} = \\ &\qquad \{R(0, t_n; \mathbf{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho_2) + X_n\} \end{aligned} \tag{33}$$

where X_n is another precompact sequence.
Consider first two terms of (33) for two fixed n:

$$\begin{aligned} &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2), \\ &R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2). \end{aligned} \tag{34}$$

What we need first is to show that intersection of this sets is not empty. Obviously this depends on the reachable space, since intersection of the cone and ρ-ball is not empty, and intersection of two such sets is also not empty:

$$\begin{aligned} &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2) = \\ &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2) = \\ &\qquad R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap K_Y(\tilde{y}^{n_2}). \end{aligned}$$

So, this depends on the "size" of reachable space for every n. Reachable space includes all variations, possible at time t_n and control sequence value $\tilde{j}^n$. But we know, that admissible control space is spike complete, that is, it includes all possible spike variations of $j(\bullet)$. Then it includes ξ as defined in (**??**) also. What we also know, is that $\mathbf{J}_{ad}$ is "big" enough.
Now take sequences $\{t_n\}, \{\tilde{j}^n\}, \{\tilde{y}^n\}$ such that:

$$\begin{aligned} t_n &\to \bar{t}, \bar{t} - t_n \le \delta_n, \\ d_n(\tilde{j}^n, \bar{j}) &\to 0, d_n(\tilde{j}^n, \bar{j}) \le \delta_n, \\ \|\tilde{y}^n - \bar{y}\| &\to 0, \|\tilde{y}^n - \bar{y}\| \le \delta_n, \\ \delta_n &\to 0, n \in \mathbb{N}. \end{aligned} \tag{35}$$

Now note, that $\mathbf{J}_{opt}$, set of optimal controls, is contained in $\mathbf{J}_{ad}$ by existence of optimal controls. Then taking $\bar{j} = j_{opt}$ one may ensure that there is at least one converging sequence of $\{\tilde{j}^n\}$. Since $\delta_n \to 0$, starting from some n_0 all members of this sequence are contained in $\mathbf{J}_{ad}$. The same is then true for $\{\tilde{y}^n\}$ sequences.
Then, starting at least from some number n_0, reachable space is also there. Then intersection for different close numbers, $n_1 > n_0, n_2 > n_0$ is not empty.
The last question is of intersection of contingent cones. Is it big enough or close to reachable space?
Note, that these cones are decreasing in size with growing n, as y converges to $\bar{y}$. Then there intersections are also decreasing in size. At the same time intersections of reachable spaces is growing in size, so starting from some another n, $n_0^2 \ge n_0$, the difference in (35) is not empty.
The same argument may be repeated for all $n \ge n_0^2$. Adding members of precompact sequence to these sets will not decrease them, so their intersection will remain not empty, thus containing interior point. ∎

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