Yosida Approximations for Multivalued Stochastic Differential Equations on Banach Spaces via a Gelfand Triple

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Abstract

In this thesis, we study multivalued stochastic differential equations on a Gelfand triple (V, H, V^*) of the following type:

$$\begin{cases} dX(t) \in b(t, X(t))\, dt - A(t, X(t))\, dt + \sigma(t, X(t))\, dL(t), \\ X(0) = X_0. \end{cases}$$

The drift operator is divided into a single-valued Lipschitz part b and a multivalued random, time–dependent, maximal monotone part A with full domain V and image sets in the dual space V^*. The Banach space V as well as its dual space V^* are assumed to be uniformly convex.

The results of the thesis consist of two main parts. In the first part, we prove the existence and uniqueness of solutions to the multivalued stochastic differential equations with multiplicative Wiener noise where the multivalued maximal monotone operator A admits an additional coercivity and boundedness assumption. The proof is based on the Yosida approximation approach. We establish L^2-convergence of solutions for the stochastic partial differential equations

$$dX_\lambda(t) = (b(t, X_\lambda(t)) - A_\lambda(t, X_\lambda(t)))\, dt + \sigma(t, X_\lambda(t))\, dN(t)$$

as $\lambda \searrow 0$, where A_λ is the Yosida approximation of the maximal monotone operator A.

In the second part, we extend the framework of the first part by adding Poisson noise and replacing the differential dt of the drift by a more general measure $dN(t)$ induced by a non-decreasing cádlág process $N(t)$. Using the Yosida approximation approach, we obtain analogous existence and uniqueness results to the Wiener case.

As examples of multivalued maximal monotone operators, we discuss the subdifferential of a lowersemicontinuous, convex proper function as well as the multivalued porous media operator.

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Chapter 0

Introduction

Multivalued Stochastic Differential Equations

There are several reasons to examine a special kind of dynamical systems, namely those that have velocities not uniquely determined by the state of the system, but depending loosely upon it. Those equations can be generalized by replacing their discontinuous drift operator with their so-called *essential extension*. This is achieved by, roughly speaking, filling the gaps of the graph at points of discontinuity (cf. Section 3.4). The essential extension is multivalued, i.e. it is a map that associates to any point in its domain a set. As a consequence, one obtains a (deterministic) multivalued differential equation of the type

$$dX(t) \in A(t, X(t))dt,$$

considered on a Hilbert space H, where A is a multivalued operator.

Examples of dynamical systems with random phenomena in Physics, Economics and Biology give rise to the consideration of differential equations perturbed by random noise. Taking into account such random phenomena, one arrives at multivalued stochastic differential equations on H of the following type:

$$dX(t) \in A(t, X(t))\ dt + \sigma(t, X(t))\ dL(t). \tag{0.1}$$

Here, the stochastic process $L(t)$ may be a Wiener process, a Lévy process or a fractional Brownian motion. One concrete example of such a multivalued equation is given by the multivalued stochastic porous media equation (cf. Section 6.3).

Substantial research on the theory of deterministic multivalued differential equations has already been done. The monographs [Bré73], [AC84], [Sho97] and [Zei90a]/[Zei90b] should be mentioned as especially significant works.

P. Krée was the first to introduce the stochastic notion of multivalued differential equations in finite dimensions, i.e. $H = \mathbb{R}^d$, of the following type:

$$\begin{cases} dX(t) \in b(t, X(t))\, dt - A(X(t))\, dt + \sigma(t, X(t))\, dW(t), \\ X(0) = X_0. \end{cases} \tag{0.2}$$

(cf. [Kré82] and [KS86, Chapter XIV]). Here, X_0 is the initial condition and the drift coefficient is separated into a Lipschitz part b and a time–independent, deterministic (multivalued) maximal monotone part A. The diffusion coefficient σ is assumed to be Lipschitz. In [Kré82], P. Krée showed pathwise uniqueness and uniqueness in a product situation. The work of E. Cépa on this subject should also be mentioned (cf. [Cép95] and [Cép98]). In [Cép98], he investigated the deterministic Skorohod problem with a multivalued maximal monotone operator in the finite dimensional case $H = \mathbb{R}^d$. The deterministic result is used to give a new proof of existence and uniqueness for the stochastic case. The core element of the proof is a contraction argument (cf. [Cép98, Théorème 5.1]).

Yosida Approximation Approach

One possible strategy of solving multivalued differential equations of type (0.2) is based upon the Yosida approximation approach. In his work [Pet95], R. Petterson proved the existence of multivalued stochastic differential equations of type (0.2) in the finite dimensional case $H = \mathbb{R}^d$ with a time–independent, deterministic maximal monotone operator A via the Yosida approximation method for the first time. The basic idea of the Yosida approximation approach can be summarized as follows:

Let H be a Hilbert space and assume that the multivalued operator $A : H \to 2^H$ is maximal monotone. Then on the basis of the deterministic theory (cf. [Bar93], [Sho97]), it is known that one can define the Yosida approximation A_λ, $\lambda > 0$, of A by

$$A_\lambda x := \frac{1}{\lambda}(x - J_\lambda x), \quad x \in H, \tag{0.3}$$

where the resolvent J_λ of A is defined by

$$J_\lambda x := (I + \lambda A)^{-1} x, \quad x \in H. \tag{0.4}$$

The main advantage of the Yosida approximation A_λ is that it is single-valued and (at least in the Hilbert space case) Lipschitz continuous. Now one can consider the family of stochastic differential equations

$$dX_\lambda(t) = b(t, X_\lambda(t))\, dt - A_\lambda(X_\lambda(t))\, dt + \sigma(t, X_\lambda(t))\, dW(t), \qquad \lambda > 0, \tag{0.5}$$

which arise from replacing the multivalued maximal monotone operator A in (0.2) by its (single-valued) Yosida approximation A_λ. By the Lipschitz continuity of A_λ, the differential equation (0.5) becomes solvable by means of standard Picard–Lindelöf iteration methods. Therefore, the problem of finding solutions to (0.2) is reduced to the study of convergence of the approximating equations to the initial multivalued differential equations. In this respect, the Yosida approximation turns out to be useful because of its convergence to the minimal selection A_0 of A.

In a nutshell: By use of the Yosida approximation approach one can extend the existence result of a given single-valued framework to the multivalued case, provided a solution to the single-valued case exists.

Variational Framework

In [Pet95] as well as in [Cép98], the framework is restricted to the finite–dimensional case. An interesting question is whether the arguments hold in the case of infinite dimensions as well. There are various applications justifying this generalization. An example of a multivalued differential equation in infinite dimensions is the multivalued stochastic porous media equation, which is a stochastic partial differential equation (cf. Section 6.3).

Multivalued stochastic differential equations with Wiener noise of type (0.1) in infinite dimensions have been examined in [Ras96], [BR97] and [Zha07] where the multivalued operator A acts on a Hilbert space H. In [Ras96], the existence and uniqueness of equations of type (0.1) have been proved for an α-maximal monotone $A : H \to 2^H$ that satisfies a certain growth condition (cf. [Ras96, Condition H_1]).

However, some monotone operators on function spaces do not leave the underlying Hilbert space invariant. Consider e.g. the monotone operator $Au := u^3$ on L^2. Indeed, $Au \notin L^2$ for some $u \in L^2$. This problem leads to the so called *variational framework*. In this framework, one defines the Gelfand triple

$$V \subset H \subset V^*,$$

where V is a separable Banach space with its dual space V^*. A separable Hilbert space H, identified with its dual space H^* via the Riesz isomorphism, is continuously and densely embedded between these two Banach spaces.

In this thesis, we consider multivalued stochastic differential equations on a Gelfand triple, i.e. as in (0.2), with a drift that is split up into a multivalued maximal monotone operator A and a single-valued Lipschitz continuous operator b taking values in H. The operator A is maximal monotone, random and time–dependent and is defined on the whole space V, but takes values in the larger space V^*.

The variational approach was first used by Pardoux (cf. [Par75], [Par72]) to study stochastic partial differential equations, then this technique was further developed by Krylov and Rozovoskii [KR79]. In the monograph [PR07], a new presentation of a general existence and uniqueness result under certain monotonicity– and coercivity– assumptions based upon [KR79] was developed.

A general question the variational approach encounters is why one does not redefine the operator A by restricting its domain in a way such that the image sets of A are, once again, in H, i.e. one considers the operator $A_H : H \to H$, $A_H := A|_{\mathcal{D}(A_H)}$, where

$$\mathcal{D}(A_H) := \left\{ x \in V \;\middle|\; A(x) \subset H \right\}.$$

In the multivalued case, this would lead back to the framework of [Ras96]. However, since the initial value X_0 always requires to be contained in the closure of the domain of A, $X_0 \in \overline{\mathcal{D}(A_H)}$, and since it is not at all clear whether the restricted domain $\mathcal{D}(A_H)$ is dense in H, this method would imply that one cannot consider differential equations starting at an arbitrary vector X_0 in H. This is a quite notable restriction. Furthermore, if one considers a time–dependent, random operator A, the domain $\mathcal{D}(A_H)$ also becomes time–dependent and random, which is by no means desirable. This motivates and justifies the extended variational framework with the operator A acting from V to the larger space V^* used in this thesis.

Diffusion with Jumps

The research community is currently developing an increasing interest in stochastic partial differential equations that are driven by noise which is discontinuous in time. In finance, for example, cases in which the noise may have jumps play an increasingly important role in the modelling of risk factors. Conclusions based on such models have a higher degree of congruence with empirical data than those based on the traditional model of a Brownian motion (cf. e.g. [App04], [IP06]).

In order to meet these concerns in the context of a multivalued drift, one may consider multivalued stochastic differential equations of type (0.1), where the diffusion $dL(t)$ is induced by a Lévy process. Thanks to the Lévy–Itô decomposition (cf. Theorem D.4), the class of stochastic differential equations with Hilbert space–valued Lévy-noise may be reduced to differential equations where the stochastic diffusion term is the sum of a Wiener process and a compensated Poisson measure. More precisely, one considers multivalued differential equations of the type

$$
\begin{cases}
dX(t) \in b(t, X(t))\, dt - A(X(t))\, dt + \sigma(t, X(t))\, dW(t), \\
\qquad\qquad\qquad + \int_Z G(t, X(t), z)\bar{\mu}(dt, dz), \\
X(0) = X_0,
\end{cases}
\tag{0.6}
$$

where W is a Wiener process and $\bar{\mu}$ is a compensated Poisson measure on some measure space $(Z, \mathcal{Z}, \mu)$.

There is a lot of recent progress in this field for the existence and uniqueness of single-valued stochastic differential equations with jumps (0.6) (See e.g. [LR04], [Kno05], [BH09], [BM09], [Pré10], [MZ10], [MPR10], [MR10a], [MR10b], [Mar10] as well as the monographs [PZ07] and [App09]). In the single-valued case, the variational framework with respect to continuous martingales as integrators in [KR79] has been extended to general discontinuous martingales by I. Gyöngy and N. Krylov (cf. [GK81], [GK82], [Gyö82]). In finite dimensions, the multivalued case with jumps (0.6) has been studied in [RWZ10] and [Wu11].

Main Results and Structure of this Thesis

To keep this work reasonably self-contained, we recall some necessary fundamentals in the theory of stochastic processes in Chapter 1. We collect some well-known facts on stochastic processes and define martingales on general Banach spaces. We introduce the Q-Wiener process as well as the Poisson random measure.

Chapter 2 is devoted to the stochastic integral on general Hilbert spaces. We introduce the stochastic integral with respect to general (discontinuous) square integrable martingales as integrators and deduce, as a special case, stochastic integration with respect to cylindrical Wiener processes. Furthermore, we construct the stochastic integral with respect to compensated Poisson random measures and identify this Poisson integral as a stochastic integral with respect to a square integrable martingale.

In Chapter 3, we introduce the general analytic concepts needed to study multivalued differential equations with maximal monotone drift. We define maximal monotone operators on Banach spaces, introduce their Yosida approximation and state some of its important properties. Furthermore, we define the measurability of a multivalued operator and prove the measurability of the Yosida approximation.

Chapter 4 contains one of the two main results of this thesis. We consider multivalued stochastic differential equations on a Gelfand triple (V, H, V^*) perturbed by multiplicative Wiener noise of the type (0.2). The drift operator is divided into a Lipschitz part b and a random and time–dependent (multivalued) maximal monotone part A with full domain V and image sets in the dual space V^*. Here, the Banach space V as well as its dual space V^* are assumed to be uniformly convex. Based on some additional coercivity assumptions (cf. Hypotheses (H1) - (H5)) we prove the existence and uniqueness for multivalued stochastic partial differential equations of the type (0.2). The central result of this chapter is stated in Theorem 4.5. Its proof is based on the Yosida approximation approach.

The second main result is proved in Chapter 5. We extend the results developed in Chapter 4 by adding Poisson noise to the multivalued stochastic differential equation as in (0.6) and replacing the differential dt of the drift with a more general measure $dN(t)$ induced by a non-decreasing cádlág process $N(t)$. Under conditions (H1)-(H5), we prove the existence and uniqueness of multivalued stochastic differential equations on a Gelfand triple of the following type:

$$\begin{cases} dX(t) \in [B(t,X(t)) - A(t,X(t))]\, dN(t) + D(t,X(t-))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z G(t,X(t-),z)\bar{\mu}(dt,dz) \\ X(0) = X_0. \end{cases} \tag{0.7}$$

The main existence result is stated in Theorem 5.5.

In Chapter 6, we discuss some applications of the results of Chapters 4 and 5. In particular, we compare them to the single-valued case. Furthermore, we consider the subdifferential of a lowersemicontinuous, convex proper function as an example of a maximal monotone operator. Finally, we examine the multivalued stochastic porous media equation and generalize the results in [BDPR09b]. In [BDPR09b], weak solutions of stochastic porous media equations are considered, whereas in this thesis we obtain strong solutions (cf. Remark 6.6).

Further Concepts and Problem Discussion

The Wiener Case

The existence proofs in the Wiener case (cf. Chapter 4) as well as in the Poisson case (cf. Chapter 5) are based upon the Yosida approximation approach. The first step of this method is devoted to the proof of the existence and uniqueness of the approximating equations (0.5). In the case of a Hilbert space, the Yosida approximation is Lipschitz continuous and consequently sufficiently regular to obtain the existence of a solution via standard methods.

Since in this thesis, the multivalued maximal monotone operator A is defined on a Banach space V with set values in its dual space V^*, a more general definition of the Yosida approximation is needed than the one for Hilbert spaces. In Banach spaces, the Yosida approximation $A_\lambda : V \to V^*$ of A is defined by

$$A_\lambda x := \frac{1}{\lambda} J(x - J_\lambda x), \quad \lambda > 0,\, x \in V,$$

where $J : V \to V^*$ is the duality mapping and the resolvent $J_\lambda x := x_\lambda$ is defined to be the (unique) solution to the resolvent equation

$$0 \in J(x_\lambda - x) + \lambda A x_\lambda \tag{0.8}$$

(cf. Section 3.2). In the resolvent equation (0.8), the duality map J takes over the role of the Riesz-isomorphism for the Hilbert space case (cf. (0.4)). The duality map J is homogeneous, but in general not linear. Hence, one cannot separate the terms inside J in (0.8) in order to obtain an explicit definition as in (0.4). This is, however, necessary to derive the Lipschitz continuity of J_λ and A_λ. Therefore, the Lipschitz continuity property of the Yosida approximation does not necessarily hold in this more general setting (for a counter example see Example 3.22). In order to overcome this difficulty, additional regularizing assumptions on A are needed to obtain the existence and uniqueness for the approximating equations.

Since A_λ is not Lipschitz, standard fix point arguments do not apply to solve the approximating equation (0.5). To overcome this difficulty, the central idea is to apply the main result in [PR07] that yields the existence and uniqueness for (single-valued) stochastic differential equations driven by Wiener noise in the variational framework to (0.5). To this end, the following type of boundedness– and coercivity– assumptions on the multivalued operator A is needed (cf. Hypotheses (H4) and (H5) in Chapter 4):

There exist $\alpha \in]1, \infty[$, $C \in]0, \infty[$ such that

$${}_{V^*}\langle v, x\rangle_V \geq C \left(\|x\|_V^\alpha + 1\right) \tag{0.9}$$

and

$$\left\|A^0(x)\right\|_{V^*} \leq C \left(\|x\|_V^{\alpha-1} + 1\right) \tag{0.10}$$

for all $x \in V$ and all $v \in A(x)$, where A^0 denotes the minimal selection of A (cf. Definition 3.8).

A central question is whether these conditions are carried forward to the Yosida approximation A_λ so that [PR07] is applicable yielding the existence and uniqueness for the approximating solutions for (0.5). Since the Yosida approximation A_λ is bounded by the minimal selection A^0 of A (cf. Proposition 3.19.ii)), this question is easily answered for the boundedness condition. The answer for the coercivity condition turns out to be more difficult. Lemma 3.21 assures that this is the case, but only for the exponent $\alpha \in]1, 2]$ in (0.9). This is the main reason why we have to restrict the exponent α to the interval $]1, 2[$ in Chapter 4.

The drift operator in (0.2) also has a Lipschitz continuous component b. However, the boundedness condition in [PR07] of the form (0.10) is not satisfied for b with $\alpha \in]1,2]$. That is the reason why [LR10, Theorem 1.1] (generalizing [PR07, Theorem 4.2.4]) is used in Chapter 4. In [LR10], arbitrary exponents of the H-norm on the right-hand side of (0.10) are admitted as well (cf. Remark 4.8). Accordingly, the proof of the a priori estimate (cf. Proposition 4.9) uses techniques similar to the proof of [LR10, Lemma 2.2].

A further, more technical question that has to be answered as part of the proof is that of the measurability of the Yosida approximation. Since the multivalued drift operator A is assumed to be time–dependent and random, measurability of the Yosida approximation is not a trivial question. Indeed, the definition of the Yosida approximation involves the resolvent which is only implicitly defined by (0.8). The search of measurable solutions to equations of the type (0.8), where the operator A is time–dependent and random, leads to the theory of random inclusions (cf. Section 3.3.1 as well as [Han57], [BR72], [Ito78], [Kra86]). Proposition 3.25 ensures the measurability of the Yosida approximation.

The a priori estimate (Proposition 4.9) permits a weak compactness argument to obtain weak convergence of the approximating solution X_λ. In the final step, the limit is identified as a solution to the originating equation (0.2) (cf. Proof of Theorem 4.5). Some aspects of this proof are inspired by the proof of [PR07, Theorem 4.2.4].

Another central problem in the theory of multivalued differential equations that the proof of Theorem 4.5 encounters is the following: One has to make sure that the weak limit η of the Yosida approximation A_λ applied to the approximating solution X_λ is contained in the image of A applied to the weak limit X of the sequence $(X_\lambda)_{\lambda>0}$, i.e. for $A_\lambda(X_\lambda) \rightharpoonup \eta$ one has to prove that

$$\eta \in A(X). \tag{0.11}$$

Every maximal monotone operator is weakly-strongly closed (cf. Proposition 3.2.i)). Under certain circumstances, a maximal monotone operator is even weakly-weakly closed (cf. Proposition 3.3). In Step 3 of the proof of Theorem 4.5, we apply this weakly–weakly–closedness result to the multivalued operator

$$\mathcal{A} : L^\alpha([0,T] \times \Omega, V) \to 2^{L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,V^*)}$$

defined by $x \mapsto A(\cdot, x)$ in order to prove (0.11).

The Poisson Case

The basis of the considerations in Chapter 5 is taken from [Gyö82]. Its main result [Gyö82, Theorem 2.10] provides the existence and uniqueness for single-valued stochastic differential equations for general square integrable martingales. Following the Yosida approximation approach, replacing the result from [PR07] in the Wiener case, this result is used to prove the existence and uniqueness of the approximating equations. This is possible because the compensated Poisson random measure can be identified with a square integrable martingale (cf. Theorem 2.24).

However, since the result in [Gyö82] only covers the case where the exponent α in Hypotheses (H4) and (H5) of Chapter 4 equals 2, the setting of Chapter 5 is reduced to the case $\alpha = 2$. Therefore, in this respect, the result in Chapter 5 does not fully generalize the one in Chapter 4.

Furthermore, the result in [Gyö82] requires the condition $\Delta N \cdot C < 1$ in (H2) of Chapter 5 for the non-decreasing cádlág process N. As a consequence, the jumps of N are bounded.

For the identification of weak limits in the proof of Theorem 4.5, we used the integration by parts–formula applied to the product of the exponential and the square of the H-norm of the limit X of the approximating solutions X_λ (cf. (4.16)). In the discontinuous case, the Dolean exponential takes over the role of the conventional exponential (cf. Lemma 5.16). Accordingly, we apply the integration by parts–formula for (discontinuous) semimartingales (cf. Theorem D.16) in the discontinuous case (cf. (5.34)).

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Chapter 1

Fundamentals on Stochastic Analysis

In this chapter, we recall the necessary fundamentals in the theory of stochastic processes. We gather some well-known facts on stochastic processes and define the martingale on general Banach spaces. These statements are valid for cádlág processes. Of course, this covers the special case of continuous processes. Furthermore, we introduce the Q-Wiener process as well as the Poisson random measure. In Chapters 4 and 5, these processes will serve as integrators for the stochastic integral. For more details on the theory of stochastic processes, we refer to [Kno05], [IW81], [Pro05], [PR07], [App09].

1.1 Stochastic Processes

Let $(E, \|\cdot\|)$ be a separable Banach space and $\mathcal{B}(E)$ its Borel σ-algebra and let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \geq 0$, i.e. it is right-continuous and $\mathcal{F}_0$ contains all P-null-sets of $\mathcal{F}$. A process X on $(\Omega, \mathcal{F}, P)$ is *adapted* to the filtration $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$-measurable for each $t \in [0, T]$. X is called *progressively measurable* with respect to filtration $\mathcal{F}_t$ if, for every $t \in [0, T]$, the map $(s, \omega) \mapsto X_s(\omega)$ from $[0, t] \times \Omega$ into $(E, \mathcal{B}(E))$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable. Clearly, a progressively measurable process is adapted. Conversely, any adapted process with right- or left-continuous paths is progressively measurable. If X is progressively measurable and τ is a stopping time (with respect to the same filtration $(\mathcal{F}_t)$), then X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$.

An E-valued right-continuous process $X(t),\ t \geq 0$, with paths having left limits is called cádlág. Accordingly, an E-valued left-continuous process $X(t),\ t \geq 0$, with paths having right limits is called cáglád. For an E-

valued process $X(t),\ t \geq 0$, with paths having left limits we define

$$X(t-) := \begin{cases} \lim_{s \nearrow t, s<t} X(s) & t > 0 \\ 0 & t = 0 \end{cases}$$

and

$$\Delta X(t) := \begin{cases} X(t) - X(t-) & t > 0 \\ X(0) & t = 0. \end{cases}$$

An real-valued process $N(t),\ t \geq 0$, is called *non-decreasing process* if it is $(\mathcal{F}_t)$-adapted and has P-a.s. positive, non-decreasing, finite paths.

Let us gather some facts about cádlág processes which can be proved analogously to the case of continuous processes.

Proposition 1.1. *Let $X_t,\ t \geq 0$ be an E-valued cádlág process. Fix some $\omega \in \Omega$ and let $0 \leq a < b < \infty$. Then,*

i. *$t \mapsto X_t(\omega)$ is bounded in $[a, b]$ and attains its bounds there.*

ii. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of E-valued cádlág processes such that it converges uniformly to some X. Then X is cádlág.*

iii. $\sup_{t\in[a,b]} \|X_t(t-)\| \leq \sup_{t\in[a,b]} \|X_t(\omega)\|$.

Analogous statements are valid for cáglád processes.

1.2 Martingales in General Banach Spaces

Let $L^p(\Omega, \mathcal{F}, P; E),\ 1 \leq p < \infty$, be the space of all E-valued Bochner p-integrable functions on $(\Omega, \mathcal{F}, P)$ as introduced in Section C.1.

Definition 1.2. *Let $M(t),\ t \geq 0$, be an E-valued stochastic process on $(\Omega, \mathcal{F}, P)$.*

i. *The process M is called an $\mathcal{F}_t$-*martingale *if:*

- $E[\|M(t)\|] < \infty$ *for all* $t \geq 0$,
- $M(t)$ *is* $\mathcal{F}_t$*-measurable for all* $t \geq 0$,
- $E\left[M(t) \mid \mathcal{F}_s\right] = M(s)$ *P-a.s. for all* $0 \leq s \leq t < \infty$.

ii. *The process M is called* local martingale *(up to T) if there exists a* localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$, *of $(\mathcal{F}_t)$-stopping times such that*

- $(M_{t\wedge\tau_n})_{t\in[0,T]}$ *is a martingale for all* $n \in \mathbb{N}$,
- $\sup_{n\in\mathbb{N}} \tau_n = T$ *P-a.s..*

Remark 1.3. *For the existence and uniqueness of the conditional expectation, we refer to [PR07, Proposition 2.2.1].*

Definition 1.4. *A real-valued process $M(t)$, $t \geq 0$, is called $(\mathcal{F}_t)$–submartingale if it is an integrable $(\mathcal{F}_t)$-adapted process such that for all $0 \leq s \leq t < \infty$*

$$M(s) \leq E\left[M(t) \;\middle|\; \mathcal{F}_s\right] \quad P\text{-a.s.}.$$

For right-continuous E-valued $\mathcal{F}_t$-martingales, the following theorem provides the equivalence of the standard norms on $L^p(\Omega, \mathcal{F}, P; C([0,T]; E))$ and $C([0,T]; L^p(\Omega, \mathcal{F}, P; E))$.

Theorem 1.5 (Maximal inequality)**.** *Let $p > 1$ and let E be a separable Banach space. If $M(t)$, $t \in [0,T]$, is a right-continuous E-valued $\mathcal{F}_t$-martingale, then*

$$\begin{aligned}\left(E\left[\sup_{t\in[0,T]} \|M(t)\|^p\right]\right)^{\frac{1}{p}} &\leq \frac{p}{p-1} \sup_{t\in[0,T]} \left(E\left[\|M(t)\|^p\right]\right)^{\frac{1}{p}} \\ &= \frac{p}{p-1} \left(E\left[\|M(T)\|^p\right]\right)^{\frac{1}{p}}.\end{aligned}$$

Proof. See [PR07, Theorem 2.2.7, p.20]. □

Definition 1.6. *By $\mathcal{M}_T^2(E)$, we denote the space of all E-valued, square integrable, cádlág martingales $M(t)_{t\in[0,T]}$ with the norm*

$$\begin{aligned}\|M\|_{\mathcal{M}_T^2} &:= \sup_{t\in[0,T]} \left(E\left[\|M(t)\|^2\right]\right)^{\frac{1}{2}} = \left(E\left[\|M(T)\|^2\right]\right)^{\frac{1}{2}} \\ &\leq \left(E\left[\sup_{t\in[0,T]} \|M(t)\|^2\right]\right)^{\frac{1}{2}} \leq 2 \cdot \left(E\left[\|M(T)\|^2\right]\right)^{\frac{1}{2}}.\end{aligned}$$

Proposition 1.7. *The space $\left(\mathcal{M}_T^2(E), \|\cdot\|_{\mathcal{M}_T^2}\right)$ is a Banach space.*

Proof. See [Kno05, Proposition 1.13, p.10]. □

1.3 The Wiener Process

Definition 1.8 (Q-Wiener process)**.** *A U-valued stochastic process $W(t)$, $t \in [0,T]$, on $(\Omega, \mathcal{F}, P)$ starting in* 0 *is called a* (standard) Q-Wiener *process if:*

- *W has P-a.s. continuous trajectories,*

- *the increments of W are independent, i.e. the random variables*

$$W(t_1),\ W(t_2) - W(t_1), \dots,\ W(t_n) - W(t_{n-1})$$

are independent for all $0 \le t_1 < \cdots < t_n \le T$, $n \in \mathbb{N}$,

- *the increments have Gaussian laws, i.e.*

$$P \circ \big(W(t) - W(s)\big)^{-1} = N\big(0, (t-s)Q\big) \quad \textit{for all } 0 \le s \le t \le T,$$

where $Q \in L(U)$ is non-negative and symmetric.

The process Q is called the covariance *of the Wiener process.*

Definition 1.9. *A Q-Wiener process $W(t)$, $t \in [0,T]$, is called a Q-Wiener process with respect to a filtration $\mathcal{F}_t$, $t \in [0,T]$, if:*

- *$W(t)$, $t \in [0,T]$, is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and*
- *$W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t \le T$.*

Note that, in fact, any U-valued Q-Wiener process $W(t)$ on $(\Omega, \mathcal{F}, P)$ may be regarded as a Q-Wiener process with respect to a normal filtration (see [PR07, Proposition 2.1.13, p.16]).

Proposition 1.10. *A U-valued Q-Wiener process $W(t)$ with respect to a normal filtration $\mathcal{F}_t$, $t \in [0,T]$ is a square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.*

Proof. See [PR07, Proposition 2.2.10, p.21]. □

1.4 Poisson Random Measures and Poisson Point Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(S, \mathcal{S})$ a measurable space. Let $\mathbb{M}$ be the space of $\mathbb{Z}_+ \cup \{+\infty\}$-valued measures on $(S, \mathcal{S})$ and

$$\mathcal{B}_{\mathbb{M}} := \sigma(\mathbb{M} \ni \mu \mapsto \mu(B) \mid B \in \mathcal{S}).$$

Definition 1.11 (Poisson random measure)**.** *A random variable $\mu : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ is called* Poisson random measure *if the following conditions hold:*

i. For all $B \in \mathcal{S}$, $\mu(B) : \Omega \to Z_+ \cup \{+\infty\}$ is Poisson distributed with parameter $E[\mu(B)]$, i.e.

$$P(\mu(B) = n) = e^{-E[\mu(B)]} \frac{(E[\mu(B)])^n}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

If $E[\mu(B)] = \infty$, then $\mu(B) = \infty$ P-a.s..

ii. *If $B_1, \dots, B_m \in \mathcal{S}$ are pairwise disjoint, then $\mu(B_1), \dots, \mu(B_m)$ are independent.*

Let $(Z, \mathcal{Z})$ be another measurable space and set

$$(S, \mathcal{S}) := ([0, \infty[\times Z, \mathcal{B}([0, \infty[) \otimes \mathcal{Z}).$$

Definition 1.12. *A* point function *p on Z is a mapping $p : D_p \subset]0, \infty[\to Z$ where the domain D_p of p is countable.*

Remark 1.13. *The point function p induces a measure $\mu(dt, dy)$ on $([0, \infty[\times Z, \mathcal{B}([0, \infty[) \otimes \mathcal{Z})$ in the following way:*
Define $\tilde{p} : D_p \to]0, \infty[\times Z,\ t \mapsto (t, p(t))$ and denote by c the counting measure on $(D_p, \mathcal{P}(D_p))$, i.e. $c(A) := \#A$ for all $A \in \mathcal{P}(D_p)$. Here, $\mathcal{P}(D_p)$ denotes the power set of D_p. For $(A \times B) \in \mathcal{B}([0, \infty[) \otimes \mathcal{Z}$, define the measure

$$\mu(A \times B) := c(\tilde{p}^{-1}(A \times B)).$$

Then, in particular, for all $A \in \mathcal{B}([0, \infty[)$ and $B \in \mathcal{Z}$ we obtain

$$\mu(A \times B) = \#\{t \in D_p \mid t \in A, p(t) \in B\}.$$

For $t \geq 0$, $B \in \mathcal{Z}$ we write

$$\mu(t, B) := \mu(]0, t] \times B).$$

Let $\mathcal{P}_Z$ be the space of all point functions on Z and

$$\mathcal{B}_{\mathcal{P}_Z} := \sigma(\mathcal{P}_Z \ni p \mapsto \mu(t, B) | t > 0,\ B \in \mathcal{Z}).$$

Definition 1.14. *i.* *A* point process *on Z and $(\Omega, \mathcal{F}, P)$ is a random variable $p : (\Omega, \mathcal{F}) \to (\mathcal{P}_Z, \mathcal{B}_{\mathcal{P}_Z})$.*

ii. *A point process p is called* stationary *if for every $t > 0$, p and $\theta_t p$ have the same probability law. Here, θ_t is given by $\theta_t :]0, \infty[\to]0, \infty[,\ s \mapsto s + t$.*

iii. *A point process p is called* σ-finite *if there exists $(B_n)_{n \in \mathbb{N}} \subset \mathcal{Z}$ such that $B_n \nearrow Z$ as $n \to \infty$ and $E[\mu(t, B_n)] < \infty$ for all $t > 0$ and $n \in \mathbb{N}$.*

iv. *A point process p on Z is called* Poisson point process *if there exists a Poisson random measure $\tilde{\mu}$ on $(]0, \infty[\times Z, \mathcal{B}(]0, \infty[) \otimes \mathcal{Z})$ such that there exists a P-zero set $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and all $(A \times B) \in \mathcal{B}(]0, \infty[) \otimes \mathcal{Z}$*

$$\mu(\omega)(A \times B) = \tilde{\mu}(\omega)(A \times B).$$

Proposition 1.15. *Let p be a σ-finite Poisson point process on Z and $(\Omega, \mathcal{F}, P)$. Then, p is stationary if and only if there exists a σ-finite measure m on $(Z, \mathcal{Z})$ such that*

$$E[\mu(dt, dy)] = dt \otimes m(dz),$$

where dt denotes the Lebesgue-measure on $]0, \infty[$. In that case, the measure m is uniquely determined.

Proof. See [Kno05, Proposition 2.10, p.24]. □

The measure m in Proposition 1.15 is called the *characteristic measure* of μ.

Definition 1.16. *Let $\mathcal{F}_t$, $t \geq 0$, be a filtration on $(\Omega, \mathcal{F}, P)$ and p a point process on Z and $(\Omega, \mathcal{F}, P)$.*

i. The process p is called $(\mathcal{F}_t)$-adapted *if for every $t \geq 0$ and $B \in \mathcal{Z}$ $\mu(t, B)$ is $(\mathcal{F}_t)$-measurable.*

ii. The process p is called an $(\mathcal{F}_t)$-Poisson point process *if it is an $(\mathcal{F}_t)$-adapted, σ-finite Poisson point process such that $\{\mu(]t, t+h] \times B) | h > 0, B \in \mathbb{Z}\}$ is independent of $\mathcal{F}_t$ for all $t \geq 0$.*

We define the set $\Gamma_\mu := \{B \in \mathcal{Z} \mid E[\mu(t, B)] < \infty \quad \forall t > 0\}$.

Definition 1.17. *Let $\mathcal{F}_t$ be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ and p a point process on Z. The process p is said to be of class (QL) with respect to $\mathcal{F}_t$ if it is $(\mathcal{F}_t)$-adapted and σ-finite and for all $B \in \mathcal{Z}$ there exists a process $\hat{\mu}(t, B)$, $t \geq 0$, such that*

i. for $B \in \Gamma_\mu$, $\hat{\mu}(t, B)$, $t \geq 0$, is a continuous $(\mathcal{F}_t)$-adapted increasing process with $\hat{\mu}(0, B) = 0$ P-a.s.,

ii. for all $t \geq 0$ and P-a.e. $\omega \in \Omega$, $\hat{\mu}(\omega)(t, \cdot)$ is a σ-finite measure on $(Z, \mathcal{Z})$.

iii. for $B \in \Gamma_\mu$,

$$\bar{\mu}(t, B) := \mu(t, B) - \hat{\mu}(t, B),\ t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale.

$\hat{\mu}$ is called compensator *of μ and $\bar{\mu}$ is called* compensated Poisson random measure *of μ.*

Proposition 1.18. *Let $\mathcal{F}_t$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ and let m be a σ-finite measure on $(Z, \mathcal{Z})$ and p a stationary $(\mathcal{F}_t)$-Poisson point process on Z with characteristic measure m. Then p is quasi-left-continuous with respect to $\mathcal{F}_t$ with compensator $\hat{\mu}(t, B) = t \cdot m(B)$, $t \geq 0$, $B \in \mathcal{Z}$.*

Proof. See [Kno05, Corollary 2.18, p.31]. □

Chapter 2

Stochastic Integration

In this chapter, we will introduce the stochastic integral on general Hilbert spaces needed to define the stochastic partial differential equations occurring in Chapters 4 and 5. First, we construct the stochastic integral with respect to general (discontinuous) square integrable martingales as integrators and give a characterization of the space of integrands. From this general theory we deduce the stochastic integration with respect to cylindrical Wiener processes and gather some important properties of this type of the stochastic integral.

Secondly, we construct the stochastic integral with respect to compensated Poisson random measures where the random measure is induced by a stationary Poisson point process. Finally, we identify the Poisson integral as a stochastic integral with respect to a square integrable martingale. This is due to the fact that a Poisson random measure can be treated as a square integrable Levy martingale (cf. Theorem 2.24).

For all notions in the operator theory of Hilbert spaces that are used but not explained here, we refer to Appendix B.

2.1 The Stochastic Integral with respect to a (Discontinuous) Martingale

The main reference of this section is [PZ07].

Let U be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \in [0, \infty[$, and let $\mathcal{M}_T^2(U)$ be the space of all cádlág square integrable martingales in U with respect to $(\mathcal{F}_t)$.

2.1.1 The Operator-valued Angle Bracket Process

Let $\langle M \rangle_t$ be the angle bracket of M as defined in Proposition D.9. Let $L_1(U)$ denote the space of all nuclear operators on U equipped with the nuclear norm. Let $L_1^+(U) \subset L_1(U)$ be the space of symmetric non-negative nuclear operators. Let $x \otimes y$ be the tensor product on U defined by

$$x \otimes y(z) := \langle y, z\rangle_U x \quad \forall\, x, y, z \in U.$$

Recall that $x \otimes y \in L^1(U)$ and $\|x \otimes y\|_{L_1(U)} = \|x\|_U \|y\|_U$ (cf. Section B.3.4). Consequently, for $M \in \mathcal{M}_T^2(U)$ the process $(M(t) \otimes M(t))_{t\geq 0}$ is an $L_1(U)$-valued right-continuous process such that

$$E\left[\|M(t) \otimes M(t)\|_{L_1(U)}\right] = E\left[\|M(t)\|_U^2\right] < \infty, \quad t \geq 0.$$

Theorem 2.1. *Let $M \in \mathcal{M}_T^2(U)$. Then there exists a unique right–continuous $L_1^+(U)$-valued increasing predictable process $(\langle\!\langle M, M\rangle\!\rangle_t)_{t\geq 0}$ such that $\langle\!\langle M, M\rangle\!\rangle_0 = 0$ and the process $(M(t) \otimes M(t) - \langle\!\langle M, M\rangle\!\rangle_t)_{t\geq 0}$ is an $L_1(U)$-valued martingale. Moreover, there exists a predictable $L_1^+(U)$-valued process $(\mathbb{Q}_t)_{t\geq 0}$ such that*

$$\langle\!\langle M, M\rangle\!\rangle_t = \int_0^t \mathbb{Q}_s \, d\langle M, M\rangle_s, \quad \forall\, t \geq 0. \tag{2.1}$$

Proof. See [PZ07, Theorem 8.2, p.109]. □

Definition 2.2. *We call the $L_1^+(U)$-valued process $\mathbb{Q}_t$ satisfying* (2.1) *the* martingale covariance *of M.*

Note that in this general framework, the martingale covariance Q_t could in fact depend on ω and t.

2.1.2 Construction of the Stochastic Integral

Definition 2.3. *Let $L(U, H)$ be the Banach space of continuous linear operators from U into H. An $L(U, H)$-valued stochastic process Ψ is said to be* elementary *if there exists a sequence of non-negative numbers $0 = t_0 < t_1 < \ldots < t_m$, a sequence of operators $\Psi_j \in L(U, H)$, $j = 1, \ldots, m$, and a sequence of events $A_j \in \mathcal{F}_{t_j}$, $j = 0, \ldots, m-1$, such that*

$$\Psi(s) = \sum_{j=0}^{m-1} \mathbb{I}_{A_j} \mathbb{I}_{]t_j, t_{j+1}]}(s) \Psi_j, \quad s \geq 0.$$

We shall denote by $\mathcal{E} := \mathcal{E}(U, H)$ the class of all elementary processes with values in $L(U, H)$. For an elementary process Ψ, we set

$$\int_0^t \Psi(s)\, dM_s := \sum_{j=0}^{m-1} \mathbb{I}_{A_j} \Psi_j(M(t_{j+1} \wedge t) - M(t_j \wedge t)), \quad t \geq 0.$$

Let $L_2(U, H)$ be the space of all Hilbert-Schmidt operators from U into H equipped with the Hilbert-Schmidt norm $\|\cdot\|_{L_2(U,H)}$.

Proposition 2.4. *For any $\Psi \in \mathcal{E}(U, H)$,*

$$E\left[\left\|\int_0^t \Psi(s)\, dM_s\right\|_H^2\right] = E\left[\int_0^t \left\|\Psi(s)\mathbb{Q}_s^{1/2}\right\|_{L_2(U,H)}^2 \, d\langle M, M\rangle_s\right], \quad t \geq 0. \tag{2.2}$$

Proof. See [PZ07, Proposition 8.6, p.112] □

Let $T < \infty$. Equip the class of all elementary processes $\mathcal{E}$ with the semi-norm

$$\|\Psi\|_{M,T}^2 := E\left[\int_0^T \left\|\Psi(s)\mathbb{Q}_s^{1/2}\right\|_{L_2(U,H)}^2 \, d\langle M, M\rangle_s\right], \quad \Psi \in \mathcal{E}. \tag{2.3}$$

For Φ, $\Psi \in \mathcal{E}$, we identify Ψ with Φ if $\|\Psi - \Phi\|_{M,T} = 0$. Let $\mathcal{L}_{M,T}^2(H)$ be the completion of $(\mathcal{E}, \|\cdot\|_{M,T})$ with respect to $\|\cdot\|_{M,T}$. Let $\mathcal{L}_{M,T,U}^2(H)$ be the class of all $L(U, H)$-valued processes belonging to $\mathcal{L}_{M,T}^2(H)$. Note that for $\Psi \in \mathcal{L}_{M,T,U}^2(H)$, the $\mathcal{L}_{M,T}^2(H)$-norm is given by (2.3).

Theorem 2.5. *i. For any $t \in [0, T]$, there is a unique extension of $\int_0^t \Psi(s)\ dM_s$ to a continuous linear operator, also denoted by $\int_0^t \Psi(s)\, dM_s$, from $(\mathcal{L}_{M,T}^2(H), \|\cdot\|_{M,T})$ into $L^2(\Omega, \mathcal{F}, P; H)$. Moreover, for any $\Psi \in \mathcal{L}_{M,T}^2(H)$,*

$$E\left[\left\|\int_0^T \Psi(s)\, dM_s\right\|_H^2\right] = \|\Psi\|_{M,T}^2 .$$

ii. For all $\Psi \in \mathcal{L}_{M,T}^2(H)$ and $0 \leq s \leq t \leq T$, we have $\mathbb{I}_{]s,t]}\Psi \in \mathcal{L}_{M,T}^2(H)$ and

$$E\left[\left\|\int_0^t \Psi(s)\, dM_s - \int_0^s \Psi(s)\, dM_s\right\|_H^2\right] = \left\|\mathbb{I}_{]s,t]}\Psi\right\|_{M,T}^2 \leq \|\Psi\|_{M,T}^2 .$$

iii. For any $\Psi \in \mathcal{L}_{M,T}^2(H)$, $\left(\int_0^t \Psi(s)\, dM_s\right)_{t\in[0,T]}$ is an H-valued martingale. It is square integrable and mean-square continuous.

iv. For any $\Phi, \Psi \in \mathcal{L}_{M,T,U}^2(H)$ and any $t \in [0, T]$,

$$\left\langle \int_0^{\cdot} \Psi(s)\, dM_s, \int_0^{\cdot} \Phi(s)\, dM_s \right\rangle_t = \int_0^t \langle \Psi(s)\mathbb{Q}_s^{1/2}, \Phi(s)\mathbb{Q}_s^{1/2}\rangle_{L_2(U,H)} \, d\langle M, M\rangle_s.$$

v. *Let $\tilde{H}$ be another Hilbert space and $L \in L(H, \tilde{H})$. Then, for every $\Psi \in \mathcal{L}^2_{M,T}(H)$, $L(\Psi) \in \mathcal{L}^2_{M,T}(H)$ and*

$$L\left(\int_0^t \Psi(s)\, dM_s\right) = \int_0^t L(\Psi(s))\, dM_s, \quad t \geq 0.$$

Proof. See [PZ07, Theorem 8.7, p.113]. □

2.1.3 Space of Integrands

Let M be a U-valued right-continuous square integrable martingale with martingale covariance Q being independent of t and ω and let $\langle M, M\rangle_t = ct$ for some $c \geq 0$.

Remark 2.6. *If M is a Lévy process, then $\langle M, M\rangle_t = t \cdot \operatorname{Tr} Q$ and $\langle\langle M, M\rangle\rangle_t = t \cdot Q$ (see [PZ07, Theorem 4.49]), where Q is the covariance operator of M (cf. Definition D.3). Hence, by Theorem 2.1 for the martingale covariance of M we have*

$$t \cdot Q = \operatorname{Tr} Q \int_0^t \mathbb{Q}_s ds,$$

yielding $\mathbb{Q} = \frac{Q}{\operatorname{Tr} Q}$. Thus, in case of M being a Lévy process, $\mathbb{Q}$ is independent of t and ω.

Let $Q^{\frac{1}{2}}$ be the square root of Q (cf. Proposition B.21) and $Q^{-\frac{1}{2}}$ be its pseudo inverse (cf. Chapter B.3.3). Then, the space $U_0 := Q^{-\frac{1}{2}}(U)$ with the inner product defined by

$$\langle x, y\rangle_{U_0} := \langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y\rangle_U, \quad \forall x, y \in Q^{\frac{1}{2}}(U),$$

is a Hilbert space (cf. Proposition B.23). Furthermore, let $L_2(U_0, H)$ be the space of Hilbert-Schmidt operators from U_0 to H which is a separable Hilbert space (cf. Proposition B.20).

We define the σ-field of predictable sets.

Definition 2.7. *The predictable σ-field is defined by*

$$\begin{aligned}\mathcal{P}_T :=& \sigma\left(\{]s,t] \times F_s \;\middle|\; 0 \leq s < t \leq T,\ F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \;\middle|\; F_0 \in \mathcal{F}_0\}\right)\\ =& \sigma\left(g : [0,T] \times \Omega \to \mathbb{R} \;\middle|\; g \text{ is } (\mathcal{F}_t)\text{-adapted and left-continuous}\right).\end{aligned}$$

A process $X : [0,T] \times \Omega \to \mathbb{R}$ is called predictable *if is measurable with respect to $\mathcal{P}_T$.*

Clearly, every $(\mathcal{F}_t)$-adapted, left-continuous process is predictable.

Now, we are able to identify $\mathcal{L}^2_{M,T}(H)$.

Theorem 2.8. *Let M be a U-valued right-continuous square integrable martingale with martingale covariance Q independent of t and ω. Then*

$$\begin{aligned}\mathcal{L}^2_{M,T}(H) =& L^2(\Omega \times [0,T], \mathcal{P}_T, P \otimes dt; L_2(U_0, H)) \\ =& \Big\{ \Psi : [0,T] \times \Omega \to L_2(U_0,H) \;\Big|\; \Psi \text{ is predictable and} \\ & \|\Psi\|^2_{M,T} = E\left[\int_0^T \|\Psi(s)\|^2_{L_2(U_0,H)}\, ds\right] < \infty \Big\}\end{aligned}$$

and for $\Psi \in \mathcal{L}^2_{M,T}(H)$, the stochastic integral is a square integrable martingale with

$$E\left[\left\|\int_0^T \Psi(s)\, dM_s\right\|^2_H\right] = E\left[\int_0^T \|\Psi(s)\|^2_{L_2(U_0,H)}\, ds\right]$$

and

$$\left\langle \int_0^\cdot \Psi(s)\, dM_s, \int_0^\cdot \Psi(s)\, dM_s \right\rangle_t = \int_0^t \|\Psi(s)\|^2_{L_2(U_0,H)}\, ds, \quad t \in [0,T].$$

Proof. See [PZ07, Corollary 8.17]. □

2.2 Stochastic Integration with respect to a Cylindrical Wiener Process

As a reference, we cite [PR07].

Let $W(t)$ be a Q-Wiener process on U with respect to $\mathcal{F}_t$, $t \in [0,T]$ with covariance $Q \in L(U)$ being a non-negative, symmetric operator with finite trace. By Proposition 1.10, $W(t)$ is a square integrable $\mathcal{F}_t$-martingale and since $W(t)$ is a Lévy process (cf. Theorem D.4), its martingale covariance Q is independent of t and ω (cf. Remark 2.6). Hence, Theorem 2.8 directly provides the stochastic integral with respect to a standard Q-Wiener process. Define $\mathcal{N}^2_W := \mathcal{L}^2_{W,T}(H)$.

Via localization, we can extend the definition of the stochastic integral to the linear space of *stochastically integrable* processes on $[0,T]$ with respect to a Q-Wiener process given by

$$\begin{aligned}\mathcal{N}_W([0,T];H) :=& \Big\{ \Phi : [0,T] \times \Omega \to L_2(U_0,H) \;\Big|\; \Phi \text{ is predictable} \\ & \text{and } P\Big(\int_0^T \|\Phi(s)\|^2_{L_2^0}\, ds < \infty\Big) = 1 \Big\}.\end{aligned}$$

By the following lemma, we conclude that the stochastic integral on $\mathcal{N}_W$ is a continuous H-valued local martingale.

Lemma 2.9. *Let $\Phi \in \mathcal{N}_W^2$ and τ be a $\mathcal{F}_t$-stopping time such that $P(\tau \leq T) = 1$. Then there exists a set $N \in \mathcal{F}$, independent of $t \in [0,T]$, such that $P(N) = 0$ and*

$$\int_0^t 1_{]0,\tau]}(s)\Phi(s)\, dW(s) = \int_0^{\tau\wedge t} \Phi(s)\, dW(s) \quad \text{on } \Omega\backslash N,\ \forall t \in [0,T].$$

Proof. See [PR07, Lemma 2.3.9, p.31]. □

Let us extend the definition of the stochastic integral to the case where Q is not necessarily of finite trace. To this end, let $Q \in L(U)$ be non-negative and symmetric, but not necessarily with finite trace. We need a further Hilbert space $(U_1, \langle\ ,\ \rangle)$ and a Hilbert–Schmidt embedding

$$J : (U_0, \langle\ ,\ \rangle) \to (U_1, \langle\ ,\ \rangle).$$

Note that $(U_1, \langle\ ,\ \rangle)$ and J as above always exists (see [PR07, Remark 2.5.1]). Then the process given by the following proposition is called a *cylindrical Q-Wiener process* in U.

Proposition 2.10. *Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of $U_0 = Q^{\frac{1}{2}}(U)$ and β_k, $k \in \mathbb{N}$, a family of independent real-valued Brownian motions. Define $Q_1 := JJ^*$, where J^* denotes the adjoint operator of J. Then $Q_1 \in L(U_1)$ is a non-negative and symmetric operator with finite trace. The series*

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0,T], \tag{2.4}$$

converges in $\mathcal{M}_T^2(U_1)$ and defines a Q_1-Wiener process on U_1. Moreover, it satisfies $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and for all $u_0 \in U_0$

$$\|u_0\|_0 = \left\| Q_1^{-\frac{1}{2}} J u_0 \right\|_1 = \|Ju_0\|_{Q_1^{\frac{1}{2}} U_1},$$

i.e. $J : U_0 \to Q_1^{\frac{1}{2}} U_1$ is an isometry.

Proof. See [PR07, Proposition 2.5.2, p.40]. □

In this case we have $\Phi \in L_2(Q^{\frac{1}{2}}(U), H)$ if and only if $\Phi\circ J^{-1} \in L_2(Q_1^{\frac{1}{2}}(U_1), H)$. Furthermore,

$$\|\Phi\|^2_{L_2(Q^{\frac{1}{2}}(U),H)} = \|\Phi \circ J^{-1}\|^2_{L_2(Q_1^{\frac{1}{2}}(U_1),H)}.$$

We define

$$\int_0^t \Phi(s)\, dW(s) := \int_0^t \Phi(s) \circ J^{-1}\; dW(s), \quad t \in [0,T]. \tag{2.5}$$

Then, the class of all integrable processes is given by

$$\Big\{\Phi : \Omega_T \to L_2(Q^{\frac{1}{2}}(U), H) \mid \Phi \text{ is predictable} \\ \text{and } P\Big(\int_0^T \|\Phi(s)\|^2_{L_2(Q^{\frac{1}{2}}(U),H)}\; ds < \infty\Big) = 1\Big\},$$

as in the case where $W(t)$, $t \in [0,T]$, is a standard Q-Wiener process in U.

If $Q \in L(U)$ is non-negative, symmetric and with finite trace, the standard Q-Wiener process may also be considered as a cylindrical Q-Wiener process by setting $J = I : U_0 \to U$ where I is the identity map. In this case, both definitions of the stochastic integral coincide.

2.2.1 Properties of the Wiener Integral

Lemma 2.11. *Let $\Phi \in \mathcal{N}_W(0,T;\tilde{H})$ and let $(\tilde{H}, \|\cdot\|_{\tilde{H}})$ be another separable Hilbert space and $L \in L(H,\tilde{H})$. Then $L(\Phi(t))$, $t \in [0,T] \in \mathcal{N}_W(0,T;\tilde{H})$ and*

$$L\Big(\int_0^T \Phi(t)\, dW(t)\Big) = \int_0^T L\big(\Phi(t)\big)\, dW(t) \quad P\text{-a.s.}.$$

Proof. See [PR07, Lemma 2.4.1, p.35]. □

Lemma 2.12. *Let $\Phi \in \mathcal{N}_W(0,T)$ and let f be an $(\mathcal{F}_t)$-adapted cáglád process with values in H. Set*

$$\int_0^T \langle f(t), \Phi(t)\, dW(t)\rangle := \int_0^T \tilde{\Phi}_f(t)\, dW(t) \tag{2.6}$$

with $\tilde{\Phi}_f(t)(u) := \langle f(t), \Phi(t)u\rangle$, $u \in U_0$. Then the stochastic integral in (2.6) *is well-defined as a continuous real-valued local martingale.*

Proof. (cf. [PR07, Lemma 2.4.2, p.36]) Since f as an $(\mathcal{F}_t)$-adapted cáglád process is predictable and $\Phi \in \mathcal{N}_W(0,T)$, $\tilde{\Phi}_f : [0,T] \times \Omega \to L_2(U_0,\mathbb{R})$ is predictable. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of U_0. Then, for all $(t,\omega) \in [0,T] \times \Omega$

$$
\begin{aligned}
\left\|\tilde{\Phi}_f(t,\omega)\right\|^2_{L_2(U_0,\mathbb{R})} &= \sum_{k=1}^{\infty}\langle f(t,\omega), \Phi(t,\omega)e_k\rangle^2 \\
&= \sum_{k=1}^{\infty}\langle \Phi^*(t,\omega)f(t,\omega), e_k\rangle^2_{U_0} \\
&= \|\Phi^*(t,\omega)f(t,\omega)\|^2_{U_0} \\
&\le \|\Phi^*(t,\omega)\|^2_{L(H,U_0)} \|f(t,\omega)\|^2_H \\
&\le \|\Phi^*(t,\omega)\|^2_{L_2(H,U_0)} \|f(t,\omega)\|^2_H \\
&= \|\Phi(t,\omega)\|^2_{L_2(U_0,H)} \|f(t,\omega)\|^2_H ,
\end{aligned}
$$

where we used Remark B.19(i) in the last step. Since f is cáglád, $\sup_{t\in[0,T]} \|f(t)\|_H < \infty$. Hence,

$$
\int_0^T \left\|\tilde{\Phi}_f(t)\right\|^2_{L_2(U_0,\mathbb{R})} dt \le \sup_{t\in[0,T]} \|f(t)\|_H \int_0^T \|\Phi(t)\|^2_{L_2(U_0,H)} dt < \infty \quad \text{P-a.e..}
$$

□

The following lemma provides the existence of the quadratic variation of the stochastic integral.

Lemma 2.13. *Let $\Phi \in \mathcal{N}_W([0,T])$ and $M(t) := \int_0^t \Phi(s)\, dW(s)$, $t \in [0,T]$. Define*

$$
\langle M\rangle_t := \int_0^t \|\Phi(s)\|^2_{L_2(U_0,H)} ds,\ t \in [0,T].
$$

Then $\langle M\rangle$ is the quadratic variation of M. If $\Phi \in \mathcal{N}^2_W(0,T)$, then for any sequence $I_l := \{0 = t^l_0 < t^l_1 < \ldots < t^l_{k_l} = T\}$, $l \in \mathbb{N}$, of partitions with $\max_i(t^l_i - t^l_{i-1}) \to 0$ as $l \to \infty$

$$
\lim_{l\to\infty} E\left(\left| \sum_{t^l_{j+1}\le t} \left\|M(t^l_{j+1}) - M(t^l_j)\right\|^2 - \langle M\rangle_t \right|\right) = 0.
$$

Proof. See [PR07, Lemma 2.4.3, p.37]. □

2.3 Stochastic Integration with respect to a Poisson Point Process

In this section, we are going to construct the stochastic integral with respect to the compensated Poisson measure induced by a Poisson point process.

The main reference is [Kno05]. For the stochastic integral with respect to compensated Poisson measures on Banach spaces, we refer to [AR05].

Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space, $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \geq 0$, and $(Z, \mathcal{Z})$ be a measure space with a σ-finite measure m. Furthermore, let p be a stationary $(\mathcal{F}_t)$-Poisson point process Z with characteristic measure m. For a detailed definition, see Section 1.4.

The Poisson point process p induces a Poisson random measure μ on $[0,T] \times Z$ (cf. Remark 1.13) and by Proposition 1.18, the compensator of μ is given by $dt \otimes m$. The measure $\bar{\mu} := \mu - dt \otimes m$ is called the compensated Poisson measure of μ.

Remark 2.14. *The integration theory in [Kno05] is developed with respect to an $(\mathcal{F}_t)$-Poisson point process of class (QL) (cf. Definition 1.17). However, by Proposition 1.18, a stationary process is automatically of class (QL) and therefore, all results of [Kno05] apply to this special case. Throughout this thesis, we always assume p being a stationary $(\mathcal{F}_t)$-Poisson point process.*

Set

$$\Gamma := \{B \in \mathcal{Z} \mid m(B) < \infty\}$$

and define the predictable σ-field

$$\begin{aligned}\mathcal{P}_T(Z) :=& \sigma(g : [0,T] \times \Omega \times Z \to \mathbb{R} \mid g \text{ is } (\mathcal{F}_t \otimes \mathcal{Z})\text{-adapted} \\ &\qquad \text{and left–continuous}) \\ =& \sigma\big(\{]s,t] \times F_s \times B \mid 0 \leq s \leq t \leq T,\ F_s \in \mathcal{F}_s,\ B \in \mathcal{Z}\} \\ &\quad \cup \{\{0\} \times F_0 \times B \mid F_0 \in \mathcal{F}_0,\ B \in \mathcal{Z}\}\big).\end{aligned}$$

In the first step, we define the stochastic integral with respect to $\bar{\mu}$ for elementary processes.

Definition 2.15. *i. An H-valued process $\Phi(t) : \Omega \times Z \to H$, $t \in [0,T]$, is said to be* elementary *if there exists a partition $0 = t_0 < t_1 < \ldots < t_k = T$ and for $m \in \{0, \ldots, k-1\}$ there exist $B_1^m, \ldots, B_n^m \in \Gamma$, pairwise disjoint, such that*

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{n} \Phi_i^m 1_{]t_m, t_{m+1}] \times B_i^m},$$

where $\Phi_i^m \in L^2(\Omega, \mathcal{F}_{t_m}, P; H)$, $1 \leq i \leq n$, $0 \leq m \leq k-1$.

ii. The linear space of all elementary processes is denoted by $\mathcal{E}$.

For $\Phi \in \mathcal{E}$ and $t \in [0,T]$, we define the stochastic integral by

$$\begin{aligned}\mathrm{Int}(\Phi)(t) &:= \int_{]0,t]} \int_Z \Phi(s,z)\, \bar{\mu}(ds,dz) \\ &:= \sum_{m=0}^{k-1} \sum_{i=1}^{n} \Phi_i^m (\bar{\mu}(t_{m+1} \wedge t, B_i^m) - \bar{\mu}(t_m \wedge t, B_i^m)).\end{aligned} \tag{2.7}$$

Then $\mathrm{Int}(\Phi)$ is P-a.s. well-defined and Int is linear in $\Phi \in \mathcal{E}$. For $\Phi \in \mathcal{E}$, we define

$$\|\Phi\|_T^2 := E\left[\int_{]0,T]} \int_Z \|\Phi(s,z)\|_H^2\, m(dz)\, ds\right].$$

Proposition 2.16. *If $\Phi \in \mathcal{E}$ then $\mathrm{Int}(\Phi) \in \mathcal{M}_T^2(H)$, $\mathrm{Int}(\Phi)(0) = 0$ P-a.s. and for all $t \in [0,T]$*

$$E\left[\|\mathrm{Int}(\Phi)(t)\|_H^2\right] = E\left[\int_{]0,t]} \int_Z \|\Phi(s,z)\|_H^2\, m(dz)\, ds\right].$$

In particular, $\mathrm{Int} : (\mathcal{E}, \|\cdot\|_T^2) \to (\mathcal{M}_T^2(H), \|\cdot\|_{\mathcal{M}_T^2})$ *is an isometry,*

$$\|\mathrm{Int}(\Phi)\|_{\mathcal{M}_T^2} = \|\Phi\|_T^2.$$

Proof. See [Kno05, Proposition 2.22, p.33]. □

In order to get a norm on $\mathcal{E}$ one has to consider equivalence classes of elementary processes with respect to $\|\cdot\|_T$. For simplicity, the space of equivalence classes is again denoted by $\mathcal{E}$. Since $\mathcal{E}$ is dense in the abstract completion $\overline{\mathcal{E}}^{\|\cdot\|_T}$ of $\mathcal{E}$ with respect to $\|\cdot\|_T$, there exists a unique isometric extension of Int to $\overline{\mathcal{E}}^{\|\cdot\|_T}$. In particular, the isometric formula in Proposition 2.16 does also hold for every process in $\overline{\mathcal{E}}^{\|\cdot\|_T}$.

The completion of $\mathcal{E}$ with respect to $\|\cdot\|_T$ can be characterized as follows:

Proposition 2.17. *Let $\mathcal{P}_T(Z)$ be the predictable σ-field on $[0,T] \times \Omega \times Z$ and*

$$\begin{aligned}\mathcal{N}_{\bar{\mu}}^2(T,Z;H) :=& \Big\{ \Phi : [0,T] \times \Omega \times Z \to H \;\Big|\; \Phi \textit{ is } \mathcal{P}_T(Z)/\mathcal{B}(H)\textit{-measurable} \\ & \textit{and } \|\Phi\|_T = E\Big[\int_{]0,T]} \int_Z \|\Phi(s,z)\|_H^2\, m(dz) ds\Big]^{1/2} < \infty \Big\} \\ =& L^2([0,T] \times \Omega \times Z, \mathcal{P}_T(Z), dt \otimes P \otimes m; H).\end{aligned}$$

Then

$$\overline{\mathcal{E}}^{\|\cdot\|_T} = \mathcal{N}_{\bar{\mu}}^2(T,Z;H).$$

Proof. See [Kno05, Proposition 2.24, p.39]. □

2.3.1 Properties of the Poisson Integral

Let us gather some important properties of the stochastic integral with respect to a compensated Poisson measure.

Proposition 2.18. *Assume that $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$ and that τ is an $(\mathcal{F}_t)$-stopping time such that $P(\tau \le T) = 1$. Then $1_{]0,\tau]}\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$ and*

$$\int_{]0,t]} \int_Z 1_{]0,\tau]}(s)\Phi(s,z)\, \bar{\mu}(ds,dz) = \int_{]0,t\wedge\tau]} \int_Z \Phi(s,z)\, \bar{\mu}(ds,dz) \quad P\text{-a.s.}$$

for all $t \in [0, T]$.

Proof. See [Kno05, Proposition 3.5, p.47]. □

Proposition 2.19. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, \mathbb{R})$ and define*

$$\mathrm{Int}_\Phi(t) := \int_{]0,t]} \int_U \Phi(s,z)\, \bar{\mu}(ds,dz), \quad t \in [0,T].$$

Then Int_Φ is cádlág and $\mathrm{Int}_\Phi(t) = \mathrm{Int}_\Phi(t-)$ P-a.s. for all $t \in [0, T]$.

Proof. See [Kno05, Proposition 3.6, p.49]. □

Proposition 2.20. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$, $\tilde{H}$ be a further Hilbert space and $L \in L(H, \tilde{H})$. Then $L(\Phi) \in \mathcal{N}^2_{\bar{\mu}}(T, Z, \tilde{H})$ and*

$$L\left(\int_{]0,t]} \int_Z \Phi(s,z)\, \bar{\mu}(ds,dz)\right) = \int_{]0,t]} \int_Z L(\Phi(s,z))\, \bar{\mu}(ds,dz) \quad P\text{-a.s.}$$

for all $t \in [0, T]$.

Proof. See [Kno05, Proposition 3.7, p.50]. □

Proposition 2.21. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$. Then for all $t \in [0, T]$*

$$E\left[\int_{]0,t]} \int_Z \Phi(s,z)\, \mu(ds,dz)\right] = E\left[\int_{]0,t]} \int_Z \Phi(s,z)\, m(dz)ds\right],$$

Proof. For $\Phi \in \mathcal{E}$, we deduce

$$\begin{aligned}
&E\left[\int_{]0,t]}\int_Z \Phi(s,z)\ \mu(ds,dz)\right]\\
=&E\left[\sum_{m=0}^{k-1}\sum_{i=1}^{n}\Phi_i^m(\bar{\mu}(t_{m+1}\wedge t, B_i^m)-\bar{\mu}(t_m\wedge t, B_i^m))\right]\\
=&\sum_{m=0}^{k-1}\sum_{i=1}^{n}E\left[\Phi_i^m\right]E\left[\bar{\mu}(t_{m+1}\wedge t, B_i^m)-\bar{\mu}(t_m\wedge t, B_i^m)\right] \qquad (2.8)\\
=&\sum_{m=0}^{k-1}\sum_{i=1}^{n}E\left[\Phi_i^m\right]m(B_i^m)\left(t_{m+1}\wedge t - t_m\wedge t\right)\\
=&E\left[\int_{]0,t]}\int_Z \Phi(s,z)\ m(dz)ds\right].
\end{aligned}$$

By a monotone class argument, (2.8) is also valid for every $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T,Z,H)$ (cf. [Kno05, Proposition 3.1, p.43]). □

Let us denote the square bracket of an H-valued process $X(t)$ by $[X]_t$ (cf. Definition D.11).

Proposition 2.22. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T,Z,\mathbb{R})$. Then*

$$(I_\Phi(t))_{t\geq 0} := \left(\int_{]0,t]}\int_Z \Phi(s,z)\ \bar{\mu}(ds,dz)\right)_{t\geq 0} \in \mathcal{M}^2_T(\mathbb{R})$$

and

$$[I_\Phi]_t = \int_{]0,t]}\int_Z |\Phi(s,z)|^2\ \mu(ds,dz).$$

Proof. See [Kno05, Proposition 3.9, p.52] or [PZ07, Theorem 8.23.iv)]. □

Proposition 2.22 can be generalized to the case where Φ is an H-valued process, as the following corollary shows.

Corollary 2.23. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T,Z,H)$ and define*

$$I_\Phi(t) := \int_{]0,t]}\int_Z \Phi(s,z)\ \bar{\mu}(ds,dz), \quad t\geq 0.$$

Then

$$[I_\Phi]_t = \int_{]0,t]}\int_Z \|\Phi(s,z)\|_H^2\ \mu(ds,dz).$$

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H and τ_n be a partition as in Proposition D.10. Then, by the Parseval equality, the linearity of the Poisson integral (cf. Proposition 2.20) and the dominated convergence theorem, we obtain

$$\begin{aligned}
[I_\Phi]_t &= \lim_{n\to\infty} \sum_{i=1}^{k_n-1} \left\| \int_{]t_i^n\wedge t, t_{i+1}^n\wedge t]} \int_Z \Phi(s,z)\ \bar{\mu}(ds,dz) \right\|_H^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n-1} \left\| \sum_m \left(\int_{]t_i^n\wedge t, t_{i+1}^n\wedge t]} \int_Z \langle \Phi(s,z), e_m\rangle_H \ \bar{\mu}(ds,dz) \right) e_m \right\|_H^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n-1} \sum_m \left| \int_{]t_i^n\wedge t, t_{i+1}^n\wedge t]} \int_Z \langle \Phi(s,z), e_m\rangle_H \ \bar{\mu}(ds,dz) \right|^2
\end{aligned}$$

where we have used the Pythagoras theorem in the last step. Hence,

$$[I_\Phi]_t = \sum_m \left[\int_{]0,t]} \int_Z \langle \Phi(s,z), e_m\rangle_H \ \bar{\mu}(ds,dz) \right]$$

and we can apply Proposition 2.22 to obtain

$$\begin{aligned}
[I_\Phi]_t &= \sum_m \int_{]0,t]} \int_Z |\langle \Phi(s,z), e_m\rangle_H|^2 \ \mu(ds,dz) \\
&= \int_{]0,t]} \int_Z \|\Phi(s,z)\|_H^2 \ \mu(ds,dz).
\end{aligned}$$

by the dominated convergence theorem. □

2.4 Comparison of Integrals

Let $\bar{\mu}$ be a compensated Poisson measure on $[0,T]\times Z$ and define $U_0 := L^2(Z,\mathcal{Z},m)$. Let U be a separable Hilbert space such that the embedding $U_0 \hookrightarrow U$ is dense and Hilbert-Schmidt.

Theorem 2.24. *The compensated Poisson random measure $\bar{\mu}$ on $[0,T]\times Z$ can be identified with a square integrable martingale $M_{\bar{\mu}}$ on U such that $U_0 = Q_{M_{\bar{\mu}}}^{\frac{1}{2}}(U)$, where $Q_{M_{\bar{\mu}}}$ is the covariance operator of $M_{\bar{\mu}}$. Furthermore, $M_{\bar{\mu}}$ is a Lévy process.*

Proof. See [PZ07, Theorem 7.28]. □

The following Proposition identifies the Poisson integral as a stochastic integral with respect to a square integrable martingale .

Proposition 2.25. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z; H)$ and* $\mathrm{Int}(\Phi)(t)$, $t \in [0, T]$, *be the stochastic integral in* (2.7). *Define*

$$I^H_\Phi(t)(\varphi) := \int_Z \Phi(t,z)\varphi(z)\, m(dz), \quad \varphi \in L^2(Z, \mathcal{Z}, m).$$

Then $I^H_\Phi \in \mathcal{L}^2_{\bar{\mu},T}(H)$ (cf. Definition in Theorem 2.8) and

$$\mathrm{Int}(\Phi)(t) = \int_0^t I^H_\Phi(s)\, dM_{\bar{\mu}}(s), \quad t \in [0,T].$$

Proposition 2.25 can be deduced from the real-valued case:

Proposition 2.26. *Let $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z; \mathbb{R})$. Then $I^{\mathbb{R}}_\Phi \in \mathcal{L}^2_{\bar{\mu},T}(\mathbb{R})$ (where $I^{\mathbb{R}}_\Phi$ is defined in Proposition 2.25) and*

$$\mathrm{Int}(\Phi)(t) = \int_0^t I^{\mathbb{R}}_\Phi(s)\, dM_{\bar{\mu}}(s), \quad t \in [0,T].$$

Proof. See [PZ07, Proposition 8.24]. □

Proof of Proposition 2.25. Let $\{e_n\}_{n\in\mathbb{N}}$ be a orthonormal basis of $U_0 = L^2(Z, \mathcal{Z}, m)$. Then, by the Parseval equality

$$\begin{aligned}
\left\|I^H_\Phi\right\|_{L_2(U_0,H)} &= \sum_{n=1}^\infty \left\|I^H_\Phi(e_n))\right\|_H^2 \\
&= \sum_{n=1}^\infty \left\|\int_Z \Phi(\cdot, z)e_n(z)m(dz)\right\|_H^2 \\
&\le \sum_{n=1}^\infty \int_Z \|\Phi(\cdot,z)\|_H\, |e_n(z)|\, m(dz) \\
&= \sum_{n=1}^\infty \langle \|\Phi\|_H, e_n\rangle_{L^2(Z,\mathcal{Z},m)} \\
&= \|\Phi\|^2_{L^2(Z,\mathcal{Z},m;H)} < \infty \quad P\otimes dt\text{-a.s..}
\end{aligned} \tag{2.9}$$

Furthermore, from (2.9) it follows that

$$E\left[\int_0^T \left\|I^H_\Phi(s)\right\|_{L_2(U_0,H)}\, ds\right] \le E\left[\int_0^T \int_Z \|\Phi(s,z)\|_H^2\, m(dz)\, ds\right] < \infty,$$

since $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$. Since I^H_Φ is predictable, we have proved that $I^H_\Phi \in \mathcal{L}^2_{\bar{\mu},T}(H)$.

Now, let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H. Then for $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z, H)$, we have $\langle \Phi, e_n\rangle_H \in \mathcal{N}^2_{\bar{\mu}}(T, Z; \mathbb{R})$ for all $n \in \mathbb{N}$. Thus, by Proposition 2.26,

$$\mathrm{Int}(\langle \Phi, e_n\rangle_H)(t) = \int_0^t I^{\mathbb{R}}_{\langle \Phi, e_n\rangle_H}(s)\, d\tilde{q}(s), \quad t \in [0,T].$$

Using this together with Proposition 2.5(v), Proposition 2.20 and the dominated convergence theorem, we deduce

$$
\begin{aligned}
\operatorname{Int}(\Phi)(t) &= \int_0^t \int_Z \Phi(s,z)\ \bar{\mu}(ds,dz) \\
&= \int_0^t \int_Z \sum_{n=1}^{\infty} \langle \Phi(s,z), e_n\rangle_H e_n\ \bar{\mu}(ds,dz) \\
&= \left(\sum_{n=1}^{\infty} \int_0^t \int_Z \langle \Phi(s,z), e_n\rangle_H\ \bar{\mu}(ds,dz) \right) e_n \\
&= \left(\sum_{n=1}^{\infty} \int_0^t I^{\mathbb{R}}_{\langle \Phi, e_n\rangle_H}(s)\ dM_{\bar{\mu}}(s) \right) e_n \\
&= \left(\sum_{n=1}^{\infty} \int_Z \int_0^t \langle \Phi(s,z), e_n\rangle_H(s)\ dM_{\bar{\mu}}(s)\ m(dz) \right) e_n \\
&= \int_Z \int_0^t \Phi(s,z)\ dM_{\bar{\mu}}(s)\ m(dz) \\
&= \int_0^t I^H_{\Phi}(s)\ dM_{\bar{\mu}}(s).
\end{aligned}
$$

□

Chapter 3

Maximal Monotone Operators on Banach Spaces

In this chapter, we introduce the general analytic framework needed to study multivalued differential equations with maximal monotone drift. We are going to define maximal monotone operators on Banach spaces, introduce its Yosida approximation and prove some necessary properties. Furthermore, we consider the measurability of a multivalued operator and the measurability of the Yosida approximation of a maximal monotone operator. This chapter is mainly based upon [Bar93] and [Bar10].

General notions for multivalued maps are gathered in Appendix A. For all unexplained concepts in the theory of nonlinear operators on Banach spaces, we refer to Appendix B. Throughout this chapter, let X be a Banach space and X^* its dual space. Let $\mathcal{G}(A)$ denote the graph of the operator A.

Definition 3.1. *i. A multivalued operator $A : X \to 2^{X^*}$ is said to be* monotone *if*

$$ {}_{X^*}\langle y_1 - y_2, x_1 - x_2 \rangle_X \geq 0, \quad \forall [x_i, y_i] \in \mathcal{G}(A),\ i = 1, 2. \tag{3.1}$$

ii. A monotone operator $A : X \to 2^{X^*}$ is said to be maximal monotone *if there exists no other proper monotone extension $\tilde{A}$ of A, i.e.*

$$\mathcal{G}(A) \subsetneq \mathcal{G}(\tilde{A}).$$

Proposition 3.2. *Let A be maximal monotone. Then:*

i. A is weakly-strongly closed in $X \times X^*$, i.e. if $[x_n, y_n] \in \mathcal{G}(A)$, $x_n \to x$ weakly in X and $y_n \to y$ strongly in X^*, then $[x, y] \in \mathcal{G}(A)$.

ii. A^{-1} is maximal monotone in $X^* \times X$.

iii. For each $x \in \mathcal{D}(A)$, $A(x)$ is a closed, convex subset of X^*.

Proof. See [Bar93, Section 2.1, Proposition 1.1] . □

Under certain circumstances, a maximal monotone operator is even weakly-weakly closed as the following proposition shows.

Proposition 3.3. *Let X be a reflexive Banach space and let $A : X \to 2^{X^*}$ be a maximal monotone operator. Let $[u_n, v_n] \in \mathcal{G}(A)$, $n \in \mathbb{N}$, be such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, and either*

$$\limsup_{n,m\to\infty} {}_{X^*}\langle v_n - v_m, u_n - u_m\rangle_X \le 0$$

or

$$\limsup_{n\to\infty} {}_{X^*}\langle v_n, u_n\rangle_X \le {}_{X^*}\langle v, u\rangle_X .$$

Then $[u, v] \in \mathcal{G}(A)$.

Proof. See [Bar10, Lemma 2.3, p.38] and [Bar10, Corollary 2.4, p.41]. □

We will make use of the following characterizations of maximal monotonicity.

Theorem 3.4. *Let X be a reflexive Banach space and let $A : X \to X^*$ be a (single-valued) monotone hemicontinuous operator. Then A is maximal monotone in $X \times X^*$.*

Proof. See [Bar93, Section 2.1, Theorem 1.3]. □

Theorem 3.5. *Let X be a reflexive Banach space and let A and B be maximal monotone operators from X to 2^{X^*} such that*

$$(int\, D(A)) \cap D(B) \neq \emptyset.$$

Then $A + B$ is maximal monotone in $X \times X^*$.

Proof. See [Bar93, Section 2.1, Theorem 1.5]. □

Corollary 3.6. *Let X be a reflexive Banach space, B be a monotone hemicontinuous operator from X to X^* and $A : X \to 2^{X^*}$ be a maximal monotone operator. Then $A + B$ is maximal monotone.*

Proof. Apply Theorem 3.4 and Theorem 3.5. □

Proposition 3.7. *Let X be a reflexive Banach space and let A be a coercive, maximal monotone operator from X to X^*. Then A is surjective, i.e. $\mathcal{R}(A) = X^*$.*

Proof. See [Bar93, Section 2.1, Corollary 1.2]. □

We are especially interested in the selection of a maximal monotone operator with respect to its minimal norm:

Definition 3.8. *The* minimal selection $A^0 : \mathcal{D}(A) \subset X \to 2^{X^*}$ *of a maximal monotone operator* A *is defined by*

$$A^0(x) := \left\{ y \in A(x) \;\middle|\; \|y\| = \min_{z \in A(x)} \|z\| \right\}, \quad x \in \mathcal{D}(A).$$

Remark 3.9. *If* X *is strictly convex, then* A^0 *is single-valued.*

Proof. Let $x \in \mathcal{D}(A)$. Assume that $y_1, y_2 \in A^0(x)$, $y_1 \neq y_2$. Define $\delta := \|A^0(x)\|$. If $\delta = 0$, then $y_1 = y_2 = 0$. Thus, $\delta > 0$. By Proposition 3.2.iii), $A(x)$ is a closed, convex set. Hence, $\frac{1}{2}(y_1 + y_2) \in A(x)$ which implies $\left\|\frac{1}{2}(y_1 + y_2)\right\| \geq \delta$. On the other hand, for $\tilde{y}_1 := \frac{1}{\delta} y_1$ and $\frac{1}{\delta} y_2$, we have $\|\tilde{y}_1\| = \|\tilde{y}_2\| = 1$. Since X is strictly convex, it follows that $1 > \frac{1}{2}\|\tilde{y}_1 + \tilde{y}_2\| = \frac{1}{2\delta}\|y_1 + y_2\|$, which is a contradiction. Hence, $y_1 = y_2$. □

3.1 The Duality Mapping

The duality mapping as a map from X to X^* represents an important auxiliary tool in the theory of maximal monotone operators on Banach spaces.

Definition 3.10. *The duality mapping* $J : X \to 2^{X^*}$ *is defined by*

$$J(x) := \{x^* \in X^* |\; {}_{X^*}\langle x^*, x\rangle_X = \|x\|^2 = \|x^*\|^2\} \quad \forall x \in X.$$

Remark 3.11. *By the Hahn-Banach theorem, for every* $x \in X$ *there exists* $x_0^* \in X^*$ *such that* $\|x_0^*\| = 1$ *and* ${}_{X^*}\langle x_0^*, x\rangle_X = \|x\|$. *Setting* $u := \|x\| x_0^*$, *it follows that* ${}_{X^*}\langle u, x\rangle_X = \|x\|^2 = \|x_0^*\| \|x\| = \|u\|^2$. *Therefore,* $u \in J(x)$ *and, indeed,* $\mathcal{D}(J) = X$.

The properties of the duality mapping are closely related to the convexity of the underlying space. In general, the duality mapping is multivalued. But the following theorem is valid:

Theorem 3.12. *Let* X *be a Banach space. If* X^* *is strictly convex, then the duality mapping* $J : X \to X^*$ *is single-valued.*

Proof. See [Bar93, Chapter 1, Theorem 1.2]. □

Now, we want to state some features of the duality mapping.

Proposition 3.13. *Let* X *and* X^* *be uniformly convex. Then:*

i. The duality map $J : X \to X^*$ *is linearly bounded, 2-coercive, continuous and odd.*

ii. The operator J *is bijective and if we identify* X^{**} *with* X*, the inverse operator*

$$J^{-1} : X^* \to X$$

is equal to the duality map of the dual space X^* *and single-valued.*

iii. J is strictly monotone, i.e. it is monotone and

$$_{X^*}\langle Ju - Jv, u - v\rangle_X = 0 \quad \Rightarrow \quad u = v.$$

Proof. See [Zei90b, Proposition 32.22]. □

The following fundamental result in the theory of maximal monotone operators due to G. Minty and F. Browder provides a very useful characterization of maximal monotonicity.

Theorem 3.14. *Let X and X^* be reflexive and strictly convex. Let $A : X \to 2^{X^*}$ be a monotone operator and let $J : X \to X^*$ be the duality mapping of X. Then A is maximal monotone if and only if, for any $\lambda > 0$ (equivalently, for some $\lambda > 0$),*

$$\mathcal{R}(A + \lambda J) = X^*.$$

Proof. See [Bar93, Section 2.1, Theorem 1.2]. □

Corollary 3.15. *If A is maximal monotone, then μA is maximal monotone for all $\mu > 0$.*

Proof. For a fixed $\mu > 0$, we set $A^\mu := \mu A$ and take $x, y \in \mathcal{D}(A)$. For $v^\mu \in A^\mu(x)$, there exists $v \in A(x)$ such that $\mu v = v^\mu$. Since A is monotone, for $x, y \in \mathcal{D}(A)$ and $v^\mu \in A^\mu(x), w^\mu = A^\mu(y)$ we have $_{X^*}\langle v^\mu - w^\mu, x - y\rangle_X = \mu^2 \, _{X^*}\langle v - w, x - y\rangle_X \geq 0$. By the maximal monotonicity of A and Theorem 3.14 with $\lambda := 1$, we conclude that $\mathcal{R}(\mu A + \mu J) = \mu\mathcal{R}(A + J) = X^*$. □

Remark 3.16. *Let us emphasize that every uniformly convex Banach space is automatically strictly convex and reflexive (cf. Remark B.8 and Proposition B.9). Consequently, all results above do also hold for uniformly convex Banach spaces.*

3.2 Yosida Approximation on Banach Spaces

We are now going to introduce the Yosida approximation of a maximal monotone operator on Banach spaces. Subsequently, let us assume that X is uniformly convex with uniformly convex dual X^*. Hence, the dualization mapping J is single-valued.

For every $x \in X$ and $\lambda > 0$ we consider the following *resolvent equation*:

$$0 \in J(x_\lambda - x) + \lambda A x_\lambda. \tag{3.2}$$

Proposition 3.17. *For all $x \in X$ there exists a unique solution x_λ to* (3.2).

Proof. By Corollary 3.15, λA is maximal monotone. By Proposition 3.13.i), J is monotone and demicontinuous, in particular hemicontinuous. Furthermore, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim\limits_{n\to\infty} \|x_n\| = \infty$. Since

$$ {}_{X^*}\langle J(x-y), x-y\rangle_X = \|x-y\|^2 \quad \forall x, y \in X, $$

we have

$$ \lim_{n\to\infty} \frac{{}_{X^*}\langle J(x_n-\tilde{x}), x_n-\tilde{x}\rangle_X}{\|x_n\|} = \lim_{n\to\infty} \frac{\|x_n-\tilde{x}\|^2}{\|x_n\|} = \infty. $$

Therefore, the map $y \mapsto J(y-\tilde{x})$ is coercive. Hence, applying Corollary 3.6 we conclude that the mapping $\bar{A} : X \to 2^{X^*}$ defined by $x_\lambda \mapsto J(x_\lambda - x) + \lambda A x_\lambda$ is maximal monotone.

Claim. For $x_0 \in \mathcal{D}(A)$ the mapping $\bar{A} : x_\lambda \mapsto J(x_\lambda - x_0) + \lambda A x_\lambda$ is coercive.

Proof. Take a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ such that $\lim\limits_{n\to\infty} \|x_n\| = \infty$ and fix $y_n \in \bar{A}(x_n)$, i.e. $y_n = J(x_n - x_0) + \lambda v_n$ for some $v_n \in A(x_n)$. Then

$$ \begin{aligned} &\frac{{}_{X^*}\langle y_n, x_n - x_0\rangle_X}{\|x_n\|} \\ =&\frac{{}_{X^*}\langle J(x_n-x_0), x_n-x_0\rangle_X}{\|x_n\|} + \lambda \frac{{}_{X^*}\langle v_n, x_n-x_0\rangle_X}{\|x_n\|} \\ =&\frac{\|x_n-x_0\|^2}{\|x_n\|} + \lambda \frac{{}_{X^*}\langle v_n - w, x_n-x_0\rangle_X}{\|x_n\|} + \frac{{}_{X^*}\langle w, x_n-x_0\rangle_X}{\|x_n\|}, \end{aligned} $$

for $w \in A(x_0)$.
Obviously, $\frac{\|x_n-x_0\|^2}{\|x_n\|} \stackrel{n\to\infty}{\longrightarrow} \infty$. By the monotonicity of A we have

$$ \lambda \frac{{}_{X^*}\langle v_n - w, x_n-x_0\rangle_X}{\|x_n\|} \geq 0. $$

Furthermore, the third summand is bounded:

$$ \frac{|{}_{X^*}\langle w, x_n-x_0\rangle_X|}{\|x_n\|} \leq \frac{\|w\|\,\|x_n-x_0\|}{\|x_n\|} < \infty. $$

Hence, $\lim\limits_{n\to\infty} {}_{X^*}\langle y_n, x_n - x_0\rangle_X \|x_n\|^{-1} = \infty$. □

By Proposition 3.7, we obtain surjectivity of the map $x_\lambda \mapsto J(x_\lambda - \tilde{x}) + \lambda A x_\lambda$. Thus, there exists a solution x_λ to (3.2).

Now we want to show the uniqueness of the solution. To this end, let x_1, x_2 be two solutions of (3.2), i.e. $0 = J(x_i - \tilde{x}) + \lambda v_i$, for some $v_i \in A(x_i)$, $i = 1, 2$. Setting $\tilde{x}_i := x_i - \tilde{x}$, $i = 1, 2$, by monotonicity of A and J we obtain

$$ \begin{aligned} 0 =& {}_{X^*}\langle J(\tilde{x}_1) - J(\tilde{x}_2), \tilde{x}_1 - \tilde{x}_2\rangle_X + \lambda\, {}_{X^*}\langle v_1 - v_2, x_1 - x_2\rangle_X \\ \geq& {}_{X^*}\langle J(\tilde{x}_1) - J(\tilde{x}_2), \tilde{x}_1 - \tilde{x}_2\rangle_X \geq 0, \end{aligned} $$

thus ${}_{X^*}\langle J(\tilde{x}_1) - J(\tilde{x}_2), \tilde{x}_1 - \tilde{x}_2 \rangle_X = 0$. Since J is strictly monotone (cf. Proposition 3.13.iii)), we conclude that $\tilde{x}_1 = \tilde{x}_2$ or equivalently, $x_1 = x_2$. $\square$

Proposition 3.17 justifies the following definition.

Definition 3.18. *i. The* resolvent $J_\lambda : X \to X$ *of a maximal monotone operator A is defined by $J_\lambda x := x_\lambda$, where x_λ is the unique solution to* (3.2).

ii. The Yosida approximation $A_\lambda : X \to 2^{X^*}$ *is given by*

$$A_\lambda x := \frac{1}{\lambda} J(x - J_\lambda x), \quad \lambda > 0, \quad x \in X.$$

The following properties of the resolvent and the Yosida approximation are valid:

Proposition 3.19.

i. A_λ is single-valued, maximal monotone, bounded on bounded subsets and demicontinuous from X to X^*.

ii. $\|A_\lambda x\| \leq \|A^0 x\|$ for every $x \in \mathcal{D}(A)$, $\lambda > 0$.

iii. J_λ is bounded on bounded subsets, demicontinuous and

$$\lim_{\lambda \to 0} J_\lambda x = x, \quad \forall x \in co\ \{\mathcal{D}(A)\},$$

where $co\ \{\cdot\}$ denotes the closed convex hull of $\{\cdot\}$.

iv. For $\lambda \to 0$, $A_\lambda x \to A^0 x$ for all $x \in \mathcal{D}(A)$.

v. For all $x \in X$, we have

$$A_\lambda(x) \in A(J_\lambda(x)).$$

vi. If $\lambda_n \to 0$, $x_n \rightharpoonup x$, $A_{\lambda_n} x_n \rightharpoonup y$ and

$$\limsup_{n,m\to\infty} {}_{X^*}\langle A_{\lambda_n} x_n - A_{\lambda_m} x_m, x_n - x_m \rangle_X \leq 0,$$

then $[x, y] \in \mathcal{G}(A)$ and

$$\lim_{n,m\to\infty} {}_{X^*}\langle A_{\lambda_n} x_n - A_{\lambda_m} x_m, x_n - x_m \rangle_X = 0.$$

Proof. (i). According to [Bar93, Section 2.1, Proposition 1.3], A_λ is single-valued, monotone, bounded on bounded subsets and demicontinuous. Applying Theorem 3.4, it follows that A_λ is maximal monotone.

(ii)-(iv), (vi). See [Bar93, Proposition 1.3].

(v). From (3.2) and the definition of J_λ, we conclude that

$$-J(J_\lambda(x) - x) \in \lambda A(J_\lambda(x)) \quad \forall x \in X.$$

Since J is odd, by the definition of A_λ we obtain

$$A_\lambda(x) = \frac{1}{\lambda} J(x - J_\lambda(x)) = -\frac{1}{\lambda} J(J_\lambda(x) - x) \in A(J_\lambda(x)) \quad \forall x \in X.$$

□

Instead of the implicit definition of the Yosida approximation as an operator depending on the resolvent which is implicitly defined via the resolvent equation (3.2), one can explicitly express the Yosida approximation in the following way.

Lemma 3.20. *Let A_λ be the Yosida approximation of A. Then*

$$A_\lambda(x) = \left(A^{-1} + \lambda J^{-1}\right)^{-1} x, \quad x \in X.$$

Proof. Fix $x \in X$ and let $J_\lambda(x)$ be the resolvent of A defined by (3.2). Then, by the definition of the Yosida approximation and the homogeneity of the duality mapping J^{-1}, we have $J_\lambda(x) = x - \lambda J^{-1}(A_\lambda(x))$. Inserting this into the resolvent equation (3.2), we obtain $A_\lambda(x) \in A(x - \lambda J^{-1}(A_\lambda(x)))$ or equivalently,

$$x \in \left(A^{-1} + \lambda J^{-1}\right)(A_\lambda(x)).$$

Since A_λ is single-valued, we conclude that $A_\lambda(x) = \left(A^{-1} + \lambda J^{-1}\right)^{-1} x$. □

The following lemma plays a fundamental role in the proof of existence and uniqueness of multivalued stochastic differential equations. It states that the coercivity of a maximal monotone operator is carried forward to its Yosida approximation:

Lemma 3.21. *Let $\alpha \in]1, 2]$, $A : X \to 2^{X^*}$ be a maximal monotone operator and A_λ its Yosida approximation. If for some constants $C_1 > 0$ and $C_2 \in \mathbb{R}$*

$${}_{X^*}\langle v, x \rangle_X \geq C_1 \|x\|^\alpha + C_2 \quad \forall x \in \mathcal{D}(A), \forall v \in A(x),$$

then there exist $\lambda_0 > 0$ and $C > 0$ such that for all $0 < \lambda < \lambda_0$

$${}_{X^*}\langle A_\lambda x, x \rangle_X \geq C_1 2^{-\alpha} \|x\|^\alpha + C \quad \forall x \in X.$$

Proof. Fix $x \in X$. By the definition of A_λ and the property of J we have

$$\begin{aligned}
&{}_{X^*}\langle A_\lambda x, x - J_\lambda x\rangle_X \\
=&\frac{1}{\lambda}\, {}_{X^*}\langle J(x - J_\lambda x), x - J_\lambda x\rangle_X \\
=&\frac{1}{\lambda}\, \|x - J_\lambda x\|^2 .
\end{aligned}$$

Hence, since $A_\lambda(x) \in A(J_\lambda x)$ (cf. Proposition 3.19.v)) and since A is coercive, we deduce

$$\begin{aligned}
{}_{X^*}\langle A_\lambda x, x\rangle_X &= {}_{X^*}\langle A_\lambda x, J_\lambda x\rangle_X + \frac{1}{\lambda}\, \|x - J_\lambda x\|^2 \\
&\geq C_1 \|J_\lambda x\|^\alpha + \frac{1}{\lambda}\, \|x - J_\lambda x\|^2 + C_2 \\
&\geq C_1 \|J_\lambda x\|^\alpha + \frac{1}{\lambda}\, \|x - J_\lambda x\|^\alpha + C
\end{aligned}$$

for some $C > 0$, since $\alpha \in]1, 2]$. Furthermore, for $\lambda_0 := \frac{1}{C_1}$ we have $(\frac{1}{\lambda} - C_1) \geq 0$ for all $0 < \lambda < \lambda_0$. Hence, we obtain

$$\begin{aligned}
{}_{X^*}\langle A_\lambda x, x\rangle_X &= C_1 \|J_\lambda x\|^\alpha + \left(\frac{1}{\lambda} - C_1\right) \|x - J_\lambda x\|^\alpha + C_1 \|x - J_\lambda x\|^\alpha + C \\
&\geq C_1 \left(\|J_\lambda x\|^\alpha + \|x - J_\lambda x\|^\alpha\right) + C \\
&\geq C_1 2^{-\alpha+1} \|x\|^\alpha + C, \quad \forall \lambda < \lambda_0.
\end{aligned}$$

In the last step, we have used $2^{\alpha-1}(a^\alpha + b^\alpha) \geq (a+b)^\alpha$ for $\alpha > 1,\ a, b \geq 0$. □

Let us note that in the Hilbert space case, the Yosida approximation is Lipschitz continuous. However, in the Banach space case this is not necessarily true as the following example shows:

Example 3.22. *Let $A := J$. Using Lemma 3.20, we derive its Yosida approximation:*

$$\begin{aligned}
A_\lambda(x) &= (J^{-1} + \lambda J^{-1})^{-1} x \\
&= \left\{ y \in X^* \mid y = \left((1+\lambda) J^{-1}\right)^{-1} x \right\} \\
&= \left\{ y \in X^* \mid (1+\lambda) J^{-1} y = x \right\} \\
&= \left\{ y \in X^* \mid y = J\left(\frac{x}{1+\lambda}\right) \right\} = \frac{1}{1+\lambda} J(x).
\end{aligned}$$

Since the duality map J is generally not Lipschitz continuous (consider e.g. J on $X = L^p$ and $X^* = L^{\frac{p}{p-1}}$), $p \in]1, 2[$), so is its Yosida approximation.

3.3 Random Multivalued Operators

Let us introduce the following notion of measurability for multivalued operators taken from [CV77].

Definition 3.23. *Let $(S, \mathcal{S})$ be a measurable space and $(E, \mathcal{E})$ be a Polish space. A multivalued operator $A : S \to 2^E$ is called* Effros-measurable *if*

$$\{x \in S|\, A(x) \cap G \neq \varnothing\} \in \mathcal{S}$$

for each open set $G \subset \mathcal{E}$.

Every multivalued Effros-measurable operator A can be characterized as the closure of a countable set of measurable selections, as the next proposition shows.

Proposition 3.24. *Let A be a multivalued operator. Then the following statements are equivalent:*

i. The operator A is Effros-measurable.

ii. There exists a sequence ξ_n of measurable selections of A such that

$$A = \overline{\{\xi_n, n \in \mathbb{N}\}}.$$

Proof. See [CV77, Chapter III] or [Mol05, Theorem 2.3]. □

In the theory of stochastic differential equations with a time-dependent random drift operator, the question of measurability of the resolvent as well as the Yosida approximation is of particular importance. The following proposition generalizes the proof of measurability of the Yosida approximation in [KK92, Theorem 3.2] to the multivalued case.

Proposition 3.25. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete, σ-finite measure space, X be a separable uniformly convex Banach space with its dual X^* and $D \subset X$. Let $A : \Omega \times D \to 2^{X^*}$ be an $\mathcal{F} \otimes \mathcal{B}(X)/\mathcal{B}(X^*)$-Effros-measurable, maximal monotone operator. Then, the resolvent $J_\lambda : \Omega \times X \to X$ and the Yosida approximation $A_\lambda : \Omega \times X \to X^*$ of A are $\mathcal{F} \otimes \mathcal{B}(X)/\mathcal{B}(X)$-measurable and $\mathcal{F} \otimes \mathcal{B}(X)/\mathcal{B}(X^*)$-measurable, respectively.*

For the proof, we need the following result.

Proposition 3.26. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, X be a separable Banach space, and $F : \Omega \to X$ a mapping such that $\mathcal{G}(F) \in \mathcal{F} \times \mathcal{B}(X)$. Then F is $\mathcal{F}/\mathcal{B}(X)$-measurable.*

Proof. See [Him75, Theorem 3.4]. □

Proof of Proposition 3.25. Let us write $A(\omega)(\cdot) = A(\omega, \cdot)$, $\omega \in \Omega$ and fix $x \in X$. By Lemma 3.20, we obtain

$$\begin{aligned}
\mathcal{G}(A_\lambda(\cdot, x)) &= \left\{(\omega, y) \in \Omega \times X^* \mid y = \left(A(\omega)^{-1} + \lambda J^{-1}\right)^{-1} x\right\} \\
&= \left\{(\omega, y) \in \Omega \times X^* \mid x \in A(\omega)^{-1} y + \lambda J^{-1} y\right\} \\
&= \left\{(\omega, y) \in \Omega \times X^* \mid \left(x - \lambda J^{-1} y\right) \in A(\omega)^{-1} y\right\} \\
&= \left\{(\omega, y) \in \Omega \times X^* \mid y \in A(\omega)\left(x - \lambda J^{-1} y\right)\right\} \\
&= \left\{(\omega, y) \in \Omega \times X^* \mid 0 \in A(\omega)\left(x - \lambda J^{-1} y\right) - y\right\}
\end{aligned}$$

Since X is reflexive, J^{-1} is the duality mapping from X^* to X. Since then, J^{-1} is demicontinuous and X is separable, Pettis Theorem implies that J^{-1} is $\mathcal{B}(X^*)/\mathcal{B}(X)$-measurable. Consequently, the mapping $y \mapsto x - \lambda J^{-1} y$ is $\mathcal{B}(X^*)/\mathcal{B}(X)$-measurable. Hence, the mapping $(\omega, y) \to (\omega, x - \lambda J^{-1} y)$ is $\mathcal{F} \otimes \mathcal{B}(X^*)/\mathcal{B}(X)$-measurable. Composing this and A, it follows that $(\omega, y) \to A(\omega)(\omega, x - \lambda J^{-1} y) - y$ is $\mathcal{F} \otimes \mathcal{B}(X)/\mathcal{B}(X)$-Effros-measurable. By the definition of Effros-measurability we obtain $\mathcal{G}(A_\lambda(\cdot, x)) \in \mathcal{F} \otimes \mathcal{B}(X^*)$. Now, Proposition 3.26 implies that $A_\lambda(\cdot, x)$ is $\mathcal{F}/\mathcal{B}(X^*)$-measurable. Demicontinuitity of A_λ in x yields that A_λ is $\mathcal{F} \otimes \mathcal{B}(X)/\mathcal{B}(X^*)$-measurable. The second assertion follows directly by noting that $J_\lambda(\omega, x) = x - \lambda J^{-1}(A_\lambda(\omega, x))$ for $(\omega, x) \in \Omega \times X$. $\square$

3.3.1 Random Inclusions

Let X be a separable Banach space. The existence of solutions $y(x)$ for inclusions of the form

$$y(x) \in A(x), \quad x \in X, \tag{3.3}$$

where A is a multivalued operator, has been extensively studied for the deterministic case. (See for example Proposition 3.7.) However, it is a non-trivial generalization to consider inclusions of type (3.3), where the multivalued operator does depend on an additional variable ω in a measurable space Ω. The solution x of

$$y(\omega, x(\omega)) \in A(\omega, x(\omega)), \quad x \in X, \ \omega \in \Omega,$$

does not necessarily need to be measurable even if there exists an ω-wise solution (See eg. [BR72, Chapter 3], [Han57], [Ito78], [Kra86]). The following result generalizes [Kra86, Theorem 3.2] by dropping the lower-semicontinuity-assumption.

Proposition 3.27. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, X be a separable uniformly convex Banach space and $D \subset X$. Let $A : \Omega \times D \to 2^{X^*}$ be an operator such that $A(\omega, \cdot)$ is maximal monotone for every $\omega \in \Omega$, $A(\cdot, x)$ is $\mathcal{F}$-Effros-measurable for every $x \in D$ and $0 \in A(\omega, 0)$ for all $\omega \in \Omega$. Let*

$L : \Omega \times X \to X^*$ be a single-valued bounded, coercive and maximal monotone operator such that $L(\cdot, x)$ is $\mathcal{F}$-Effros-measurable for every $x \in X$. Then for each $\mathcal{F}$-measurable, bounded operator $y(\cdot) \in X^*$, there exists an $\mathcal{F}$-measurable, bounded operator $x(\cdot) \in X$, such that

$$y(\omega) \in A(\omega, x(\omega)) + L(\omega, x(\omega)) \quad \forall\, \omega \in \Omega.$$

Proof. W.l.o.g. we assume $y(\omega) = 0$ for all $\omega \in \Omega$. (Otherwise consider $\tilde{A}(\omega, \cdot) := A(\omega, \cdot) - y(\omega)$. Again, $\tilde{A}$ is maximal monotone and Effros-measurable.) We consider the equation

$$A_\lambda(\omega, x_\lambda(\omega)) + L(\omega, x_\lambda(\omega)) = 0, \quad \forall \omega \in \Omega, \tag{3.4}$$

where A_λ is the Yosida approximation of A. By Proposition 3.25, A_λ is $\mathcal{F}$-measurable. Since A_λ is demicontinuous and maximal monotone, the operator $A_\lambda + L$ satisfies the assumptions of [Ito78, Theorem 6.2]. Hence, there exists an $\mathcal{F}$-measurable bounded operator $x_\lambda(\cdot) \in X$ that solves (3.4). The rest of the proof is analogous to [Kra86, Theorem 3.2]. □

Remark 3.28. *Note that in the proof of [Kra86, Theorem 3.2], the operator A is assumed to be lower-semicontinuous to be able to prove the measurability of the Yosida approximation. However, the lower-semicontinuity of A is obsolete, as Proposition 3.25 shows.*

3.4 Extension of Maximal Monotone Operators

In some cases, it may be convenient to replace a monotone operator by its maximal monotone extension. The following general lemma assures that such an extension can always be found.

Lemma 3.29. *The graph of any monotone multivalued map A is contained in the graph of a maximal monotone multivalued map.*

Proof. Consider the set of all monotone maps denoted by M_A which extend A. We introduce an order on M_A by

$$B \leq C \;:\overset{\text{Def.}}{\Leftrightarrow}\; \mathcal{G}(B) \subset \mathcal{G}(C), \; B, C \in M_A.$$

Thus, M_A is partially ordered. Take a chain K in M_A, i.e. $B \leq C$ or $C \leq B$ for all $B, C \in K$. The set

$$\mathcal{F} := \bigcup_{B \in K} \mathcal{G}(B)$$

is obviously the graph of a monotone multivalued map $\tilde{A}$. Clearly, $\mathcal{G}(A) \subset \mathcal{F} = G(\tilde{A})$. Hence, $\tilde{A} \in M_A$. Consequently, we have found an upper bound for all elements in K which belongs to K. According to Zorn's lemma, M_A has maximal elements. □

Now, let $X = \mathbb{R}^d$ and let us assume that $A : \mathbb{R}^d \to \mathbb{R}^d$ is a single-valued monotone operator defined on all of $\mathbb{R}^d$. If A is continuous, then Theorem 3.4 implies that A is maximal monotone.

In case of A being discontinuous, it can be extended to a (multivalued) maximal monotone operator, as motivated by the following proposition.

Proposition 3.30. *Let A be a (multivalued) monotone operator defined on all of $\mathbb{R}^d$ such that its graph is closed in $\mathbb{R}^d \times \mathbb{R}^d$ and the set Ax is convex for every $x \in \mathbb{R}^d$. Then A is maximal monotone in $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof. See [Bar10, Proposition 2.4, p.45]. □

The explicit construction of such an extension performs as follows.

Definition 3.31. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a monotone operator. The* essential extension *$\bar{A}$ of A is defined by*

$$\bar{A}(x) := \bigcap_{\delta > 0} A^\delta(x),$$

where

$$A^\delta(x) := \bigcap_{m(N)=0} co\,\{A(y) : \|y - x\| \leq \delta,\, y \notin N\}. \tag{3.5}$$

Here, $co\,\{\cdot\}$ denotes the closed convex hull of $\{\cdot\}$ and $m(N)$ is the d–dimensional Lebesgue-measure of $N \subset \mathbb{R}^d$.

The essential extension plays an important role in many applications (cf. e.g. Section 6.3). In the case of $d = 1$, the essential extension $\bar{A}$ can be obtained, roughly speaking, via "filling the gaps" of the graph at points of discontinuity, i.e.

$$\bar{A}(x) = [A(x-), A(x+)], \quad \forall x \in \Delta(A),$$

where $A(x-) := \lim_{y \nearrow x} A(y)$ and $A(x+) := \lim_{y \searrow x} A(y)$.

Proposition 3.32. *The essential extension $\bar{A}$ is maximal monotone in $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof. At first, let us show that $\bar{A}$ is monotone. To this end, fix $x, y \in \mathbb{R}^d$ and let $v \in A(x)$ and $w \in A(y)$. Note that $\bar{A}(x) \subset A^\delta(x)$ for all $\delta > 0$, where A^δ is as in (3.5). Thus, we can find $\lambda_1, \lambda_2 \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ satisfying $\|x_1 - x\| \wedge \|x_2 - x\| \wedge \|y_1 - y\| \wedge \|y_2 - y\| \leq \delta$, such that

$$v = \lambda_1 A(x_1) + (1 - \lambda_1)A(x_2) \quad \text{and} \quad w = \lambda_2 A(y_1) + (1 - \lambda_2)A(y_2).$$

A short calculation yields

$$
\begin{aligned}
\langle v-w, x-y\rangle = {} & \lambda_1\lambda_2\langle A(x_1)-A(y_1), x-y\rangle \\
& + \lambda_1(1-\lambda_2)\langle A(x_1)-A(y_2), x-y\rangle \\
& + (1-\lambda_1)\lambda_2\langle A(x_2)-A(y_1), x-y\rangle \\
& + (1-\lambda_1)(1-\lambda_2)\langle A(x_2)-A(y_2), x-y\rangle.
\end{aligned}
\tag{3.6}
$$

Let us estimate the first summand. By the monotonicity of A and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
& \lambda_1\lambda_2\langle A(x_1)-A(y_1), x-y\rangle \\
= & \lambda_1\lambda_2\langle A(x_1)-A(y_1), x_1-y_1\rangle \\
& \quad + \lambda_1\lambda_2\langle A(x_1)-A(y_1), x-x_1\rangle + \lambda_1\lambda_2\langle A(x_1)-A(y_1), y_1-y\rangle \\
\geq & \lambda_1\lambda_2\langle a(x_1)-a(y_1), x-x_1\rangle + \lambda_1\lambda_2\langle A(x_1)-A(y_1), y_1-y\rangle \\
\geq & -\lambda_1\lambda_2 \left\|A(y_1)-A(x_1)\right\| \left(\|x-x_1\| + \|y_1-y\|\right) \nearrow 0,
\end{aligned}
$$

as $\delta \to 0$. Estimating the other summands in (3.6) in a similar way, we arrive at $\langle v-w, x-y\rangle \geq 0$, i.e. $\bar{A}$ is monotone.

Furthermore, the graph of $\bar{A}$ is closed and convex as an intersection of the closed convex sets A^δ. Thus, Proposition 3.30 applies and ensures that $\bar{A}$ is maximal monotone. $\square$

Chapter 4

Multivalued Stochastic Partial Differential Equations with Wiener Noise

In this chapter, we consider multivalued stochastic differential equations on a Gelfand triple (V, H, V^*) perturbed by multiplicative Wiener noise. The drift operator is divided into a Lipschitz part b and a random and time-dependent (multivalued) maximal monotone part A with full domain V and image sets in the dual space V^*. The proof of the existence of a solution is based on the Yosida approximation approach as presented in Chapter 3. The corresponding framework for the single-valued case is presented in [PR07]. The main result therein is used in order to obtain the existence and uniqueness of the occurring approximating solutions.

4.1 Variational Framework

Let H be a separable real Hilbert space with inner product $\langle\ ,\ \rangle_H$. We identify H with its dual space H^* via the Riesz isomorphism R. Let V be a uniformly convex Banach space with a uniformly convex dual space V^* such that $V \subset H$ continuously and densely. We obtain the Gelfand triple (V, H, V^*) (cf. Definition B.10). As usual, the Borel σ-algebra $\mathcal{B}(V)$ is generated by V^* and $\mathcal{B}(H)$ by H^*.

Furthermore, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \in [0, \infty[$. Fix some $T \in [0, \infty[$ and $\alpha \in]1, 2]$. Throughout this chapter, let $C > 0$ be a universal constant which may vary from line to line.

We consider multivalued stochastic partial differential equations of the following type:

$$\begin{cases} dX(t) \in (b(t, X(t)) - A(t, X(t)))\, dt + \sigma(t, X(t))\, dW(t), \\ X(0) = X_0. \end{cases} \tag{4.1}$$

Here, X_0 is an $\mathcal{F}_0$-measurable random variable with $X_0 \in L^{\frac{\alpha}{\alpha-1}}(\Omega, \mathcal{F}_0, P; H)$ and $W(t)$ is a cylindrical Q-Wiener process with covariance $Q = I$ on an additional separable Hilbert space $(U, \langle\, ,\, \rangle_U)$.

We consider operators

$$\begin{aligned} A :&[0,T] \times V \times \Omega \to 2^{V^*}, \\ b :&[0,T] \times V \times \Omega \to H, \\ \sigma :&[0,T] \times V \times \Omega \to L_2(U, H), \end{aligned}$$

i.e. we assume A being multivalued with domain $\mathcal{D}(A) = V$ and (multivalued) image sets in V^*. Here, $L_2(U, H)$ denotes the space of Hilbert-Schmidt operators from U to H. For shorthand, by $b(t, x)$ we mean the mapping $\omega \mapsto b(t, x, \omega)$ and analogously for $\sigma(t, x)$ and $A(t, x)$. The operators b and σ are assumed to be progressively measurable, i.e. for every $t \in [0, T]$, these maps, restricted to $[0, t] \times V \times \Omega$, are $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable. The multivalued operator A is assumed to be progressively Effros-measurable, i.e. for every $t \in [0, T]$, A is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t/\mathcal{B}(V^*)$-Effros-measurable.

Definition 4.1. *A solution to* (4.1) *on the interval* $[0, T]$ *is a couple* (X, η) *of processes such that* $X \in L^\alpha([0, T] \times \Omega, V)$ *and* $\eta \in L^1([0, T] \times \Omega, V^*)$ *and for* P*-a.e.* $\omega \in \Omega$

i. X_t *is continuous,*

ii. the processes X_t *and* $\int_0^t \eta(s)\, ds$ *are* $(\mathcal{F}_t)$*-adapted,*

iii. for almost all $t \in [0, T]$

$$\eta(t) \in A(t, X(t)),$$

iv. for all $t \in [0, T]$*, the following equation holds:*

$$\begin{aligned} &X(t) = X_0 + \int_0^t b(s, X(s))\, ds - \int_0^t \eta(s)\, ds + \int_0^t \sigma(s, X(s))\, dW(s), \\ &X(0) = X_0. \end{aligned} \tag{4.2}$$

Remark 4.2. *The notion of the solution in Definition 4.1 refers to* $dt \otimes P$*-equivalence classes. More exactly, for the equivalence class* $\hat{X}$ *of* X *as in Definition 4.1, we have* $\hat{X} \in L^\alpha([0, T] \times \Omega, V)$ *and* P*-a.s.*

$$X(t) = X_0 + \int_0^t b(s, \bar{X}(s))\, ds - \int_0^t \eta(s)\, ds + \int_0^t \sigma(s, \bar{X}(s))\, dW(s)$$

holds, where $\bar{X}$ is any V-valued, progressively measurable $dt \otimes P$-version of $\hat{X}$. Accordingly, we will always consider our notion of the solution with respect to $dt \otimes P$-equivalence classes, but will henceforth use our usual notion without mentioning this explicitly.

The existence and uniqueness of a solution in terms of Definition 4.1 holds, supposed that the following conditions on b, σ and A are valid:

Let f be an $(\mathcal{F}_t)$-adapted process with $f \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega)$ and recall that $\alpha \in]1,2]$.

(H1) **(Maximal monotonicity)** For all $x, y \in V$ and for all $(t,\omega) \in [0,T] \times \Omega$ we have

$$ {}_{V^*}\langle v - w, x - y \rangle_V \geq 0 \quad \forall v \in A(t,x), \, \forall w \in A(t,y) \tag{4.3}$$

and $x \mapsto A(t,x)$ is maximal.

(H2) **(Lipschitz continuity)** There exists $C_L \in [0, \infty[$ such that

$$\|b(t,x) - b(t,y)\|_H + \|\sigma(t,x) - \sigma(t,y)\|_{L_2(U,H)} \leq C_L \|x - y\|_H \quad \text{on } \Omega$$

for all $t \in [0,T]$ and $x, y \in V$.

(H3) **(Boundedness in** 0**)**

$$\|b(t,0)\|_H + \|\sigma(t,0)\|_{L_2(U,H)} \leq f(t) \quad \text{on } \Omega,$$

for all $t \in [0,T]$.

(H4) **(Coercivity)** There exists $C_C \in]0, \infty[$ such that

$$ {}_{V^*}\langle v, x \rangle_V \geq C_C \|x\|_V^\alpha + f(t)$$

for all $(t,\omega) \in [0,T] \times \Omega$, $x \in V$ and $v \in A(t,x)$.

(H5) **(Boundedness)** There exists $C_B \in]0, \infty[$ such that

$$\left\|A^0(t,x)\right\|_{V^*} \leq C_B \|x\|_V^{\alpha-1} + f(t)$$

for all $x \in V, \ t \in [0,T]$ on Ω.

Remark 4.3. *i. Since V is a dense subset of H and since b is uniformly continuous by (H2), the domain of b may be directly extended to H.*

ii. Conditions (H2) and (H3) imply linear growth of b and σ, i.e.

$$\|b(t,x)\|_H^2 + \|\sigma(t,x)\|_{L^2(U,H)}^2 \leq C \|x\|_H^2 + f^2$$

for all $x \in H$ and $(t,\omega) \in [0,T] \times \Omega$.

iii. Instead of assuming that A is monotone and that it satisfies (H4) and b satisfies (H2) and (H3), it is easy to show that one may consider the stochastic partial differential equation

$$dX(t) \in \tilde{A}(t, X(t))\, dt + \sigma(t, X(t))\, dW(t), \tag{4.4}$$

where the multivalued operator $\tilde{A}$ satisfies a one-sided Lipschitz condition *as well as a coercivity condition. More precisely, for all $x, y \in V$, $\tilde{v} \in \tilde{A}(\cdot, x)$ and $\tilde{w} \in \tilde{A}(\cdot, y)$ there exist $\tilde{C}_1$, $\tilde{C}_2 \in [0, \infty[$, $\tilde{C}_3 \in]0, \infty[$ and an $(\mathcal{F}_t)$-adapted process $\tilde{f} \in L^1([0, T] \times \Omega)$ such that*

$$_{V^*}\langle \tilde{v} - \tilde{w}, x - y\rangle_V \leq \tilde{C}_1 \|x - y\|_H^2 \quad \text{on } [0, T] \times \Omega \tag{A1}$$

and

$$_{V^*}\langle \tilde{v}, x\rangle_V \leq \tilde{C}_2 \|x\|_H^2 - \tilde{C}_3 \|x\|_V^\alpha + \tilde{f}(t) \quad \text{on } [0, T] \times \Omega. \tag{A2}$$

4.2 The Yosida Approximation Approach

Now, we want to apply the Yosida approximation approach to the multivalued stochastic differential equation (4.1). Since the multivalued operator A is maximal monotone, we can define the resolvent J_λ and the Yosida approximation A_λ as in Section 3. Note that in the variational framework both J_λ and A_λ are time-dependent and random. However, for fixed $(t, \omega) \in [0, T] \times \Omega$ all results of Section 3 are applicable.

Let us consider the family of approximating equations

$$\begin{cases} dX_\lambda(t) = (b(t, X_\lambda(t)) - A_\lambda(t, X_\lambda(t)))\, dt + \sigma(t, X_\lambda(t))\, dW(t), \\ X_\lambda(0) = X_0. \end{cases} \tag{4.5}$$

which arise from replacing the multivalued maximal monotone operator A in (4.1) by its (single-valued) Yosida approximation A_λ.

Remark 4.4. *Proposition 3.25 ensures that the resolvent J_λ as well as the Yosida approximation A_λ are progressively measurable.*

Now we turn to the main theorem of this chapter:

Theorem 4.5. *Let A, b and σ satisfy Conditions (H1)-(H5). Then, there exists a solution to Problem* (4.1) *in the sense of Definition 4.1 being the weak limit of $\{X_\lambda\}_{\lambda>0}$ in $L^\alpha([0, T] \times \Omega; V)$.*

The proof of Theorem 4.5 is divided into several steps. At first, we will prove that the approximating equations (4.5) are uniquely solvable for every $\lambda > 0$ (cf. Proposition 4.7). Knowing about the existence of solutions to the approximating equations, we will then verify that the sequence of solutions satisfies an a priori estimate (cf. Proposition 4.9 below). Finally, it will be proved that the weak limit is, in fact, a solution to problem (4.1).

4.2.1 Existence and Uniqueness of the Approximating Solution

The main result in [PR07, Chapter 4.2] ensures the existence and uniqueness of the approximating solutions to (4.5). However, the growth condition [PR07, H4] is not sufficient for our framework (cf. Remark 4.8 below). Therefore, we need to apply the result [LR10, Theorem 1.1], which corresponds to the existence and uniqueness result in [PR07, Theorem 4.2.4], but its growth condition [PR07, H4] has been generalized in a way that suits our framework. Let us state the main result in [LR10]:

Theorem 4.6. *Let $T \in [0, \infty[$ be fixed, $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $W(t)$ a cylindrical Q-Wiener process with $Q = I$, $X_0 \in L^{\frac{\alpha}{\alpha-1}}(\Omega, \mathcal{F}_0, P; H)$ and $A : [0,T] \times V \times \Omega \to V^*$ as well as $B : [0,T] \times V \times \Omega \to L_2(U,H)$ be progressively measurable. Furthermore, assume that A, B satisfy the following conditions:*

(LR1) For all $u, v, x \in V$, $\omega \in \Omega$ and $t \in [0,T]$, the map

$$\mathbb{R} \ni \lambda \mapsto {}_{V^*}\langle A(t, u + \lambda v, \omega), x\rangle_V$$

is continuous.

(LR2) There exist $\alpha \in]1, \infty[$, $\beta \in [0, \infty[$ and $c \in \mathbb{R}$ such that for all $u, v \in V$,

$$\begin{aligned}&2\,{}_{V^*}\langle A(\cdot, u) - A(\cdot, v), u - v\rangle_V + \|B(\cdot, u) - B(\cdot, v)\|^2_{L_2(U,H)}\\ &\leq (c + \varrho(v))\|u - v\|^2_H\end{aligned}$$

on $[0,T] \times \Omega$, where $\varrho : V \to [0, \infty[$ is a measurable function and locally bounded in V such that

$$\varrho(v) \leq C(1 + \|v\|^\alpha_V)(1 + \|v\|^\beta_H), \quad v \in V.$$

(LR3) There exist $c_1 \in \mathbb{R}$, $c_2 \in]0, \infty[$ and an $(\mathcal{F}_t)$-adapted process $f \in L^1([0,T] \times \Omega,\ dt \otimes P)$ such that for all $v \in V, t \in [0,T]$,

$$2\,{}_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|^2_{L_2(U,H)} \leq c_1\|v\|^2_H - c_2\|v\|^\alpha_V + f(t) \quad \text{on } \Omega,$$

where α and β are as in (LR2).

(LR4) There exist $c_3 \in [0, \infty[$ and an $(\mathcal{F}_t)$-adapted process $g \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega,\ dt \otimes P)$ such that for all, $v \in V$, $t \in [0,T]$

$$\|A(t,v)\|^{\frac{\alpha}{\alpha-1}}_{V^*} \leq (g(t) + c_3\|v\|^\alpha_V)\left(1 + \|v\|^\beta_H\right) \quad \text{on } \Omega,$$

where α and β are as in (LR2).

Then, there exists a process $X \in L^\alpha([0,T] \times \Omega,\ dt \otimes P; V) \cap L^2([0,T] \times \Omega,\ dt \otimes P; H)$ *such that* P*-a.s.*

$$X(t) = X(0) + \int_0^t A(s, X(s))\, ds + \int_0^t B(s, X(s))\, dW(s), \qquad t \in [0,T],$$

in the sense of $dt \otimes P$*-equivalence classes (cf. Remark 4.2). Additionally,*

$$E\left[\sup_{t\in[0,T]} \|X(t)\|_H^2\right] < \infty.$$

Proof. See [LR10, Theorem 1.1]. □

This theorem will provide us with the existence and uniqueness of the approximating equation:

Proposition 4.7. *Suppose assumptions (H1) - (H5) hold, then there exists a unique process* $X_\lambda \in L^\alpha([0,T] \times \Omega, V)$ *such that* P*-a.s. for all* $t \in [0,T]$

$$X_\lambda(t) = X_\lambda(0) + \int_0^t (b(s, X_\lambda(s)) - A_\lambda(s, X_\lambda(s)))\ ds + \int_0^t \sigma(s, X_\lambda(s))\, dW(s) \tag{4.6}$$

in the sense of $dt \otimes P$*-equivalence classes (cf. Remark 4.2). Furthermore, for all* $\lambda > 0$

$$E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^2\right] < \infty. \tag{4.7}$$

Proof. Taking $A := b - A_\lambda$ and $B := \sigma$, [PR07, Problem (4.2.1)] corresponds to Problem (4.5) (cf. Remark 4.3.iii)). In order to apply Theorem 4.6, we have to check that Conditions (LR1) - (LR4) are valid:

i. By the demicontinuity of A_λ (cf. Proposition 3.19.i)) and by (H2), $b - A_\lambda$ is hemicontinuous. Hence, (LR1) holds.

ii. By the monotonicity of A_λ (cf. Proposition 3.19.i)) and by (H2), (LR2) is satisfied (for the non-local case $\rho = 0$).

iii. By (H4) and Lemma 3.21, the Yosida approximation A_λ, $\lambda < \lambda_0$, is coercive with constant α as in (H4). Thus, by Conditions (H2) and (H3) we deduce

$$\begin{aligned} &2\, {}_{V^*}\langle b(\cdot, v) - A_\lambda(\cdot, v), v\rangle_V + \|\sigma(\cdot, v)\|_{L_2(U,H)}^2 \\ \le & C\left(\|v\|_H^2 + f^2\right) - C_C \|v\|_V^\alpha \end{aligned}$$

on $[0,T] \times \Omega$. Hence, we obtain (LR3).

iv. Furthermore, since $\mathcal{D}(A) = V$ and $\|A_\lambda(\cdot, x)\|_{V^*} \leq \left\|A^0(\cdot, x)\right\|_{V^*}$ $x \in \mathcal{D}(A)$ on $[0, T]$ (cf. Proposition 3.19.ii)), (H2), (H3) and (H5) imply

$$\|b(\cdot, v) - A_\lambda(\cdot, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq C \left(\|v\|_V^\alpha + \|v\|_H^{\frac{\alpha}{\alpha-1}} + f^{\frac{\alpha}{\alpha-1}} \right). \tag{4.8}$$

Thus, (LR4) holds with $\beta := \frac{\alpha}{\alpha-1}$.

Consequently, we can apply Theorem 4.6 and obtain a solution to (4.5). Additionally, (4.7) holds. □

Remark 4.8. *In* (4.8)*, the exponent of the H-norm of v is given by $\frac{\alpha}{\alpha-1}$, which is bigger than* 2 *since $\alpha \in]1, 2[$. Therefore, the growth condition [PR07, H4] (where the exponent of the V-norm is equal to $\alpha < 2$) is not satisfied. That is the reason why [LR10, Theorem 1.1] with the generalized growth condition is used.*

4.2.2 A Priori Estimate

We want to apply a weak compactness argument to the approximating solution X_λ. To this end, we need the following a priori estimate.

Proposition 4.9. *Let $p \in [2, \frac{\alpha}{\alpha-1}]$. Assuming (H1) - (H5), then*

$$\begin{aligned} &E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^p\right] + E\left[\int_0^T \|X_\lambda(t)\|_H^{p-2} \|X_\lambda(t)\|_V^\alpha \, dt\right] \\ \leq &C\left(E\left[\|X_0\|_H^p\right] + E\left[\int_0^T f^{\frac{p}{2}}(s) \, ds\right]\right) \end{aligned} \tag{4.9}$$

for all $\lambda > 0$. In particular, for $p = 2$

$$E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^2\right] + \int_0^T E\left[\|X_\lambda(t)\|_V^\alpha\right] \, dt \leq C. \tag{4.10}$$

Some preparations leading up to the proof have to be made. Subsequently, the following Itô formula will be crucial.

Theorem 4.10. *Let $\alpha \in]1, 2]$, $X_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ and $Y \in L^{\frac{\alpha}{\alpha-1}}([0, T] \times \Omega, V^*)$, $Z \in L^2([0, T] \times \Omega, L_2(U, H))$, both progressively measurable. Define the continuous V^*-valued process*

$$X(t) := X_0 + \int_0^t Y(s) \, ds + \int_0^t Z(s) \, dW(s), \, t \in [0, T].$$

If $X \in L^\alpha([0, T] \times \Omega, V)$, then X is an H-valued continuous $(\mathcal{F}_t)$-adapted process,

$$E\left[\sup_{t\in[0,T]} \|X(t)\|_H^2\right] < \infty$$

and the following Itô-formula holds for the square of its H-norm P-a.s.

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2\,{}_{V^*}\langle Y(s), X(s)\rangle_V + \|Z(s)\|_{L_2(U,H)}^2\right) ds \\ + 2\int_0^t \langle X(s), Z(s)\,dW(s)\rangle_H \quad \textit{for all } t \in [0,T] \tag{4.11}$$

in the sense of $dt \otimes P$-equivalence classes (cf. Remark 4.2).

Proof. See [PR07, Theorem 4.2.5]. □

Remark 4.11. *i. Though it is not explicitly mentioned in [PR07, Theorem 4.2.5], in the case $\alpha \in]1,2]$ the following additional conditition has to be satisfied:*

$$E[\|X(t)\|_H^2] < \infty \quad \textit{for } dt\textit{-a.e. } t \in [0,T].$$

However, in our situation this is automatically the case due to Proposition 4.7.

ii. By the proof of [PR07, Theorem 4.2.4] (as a special case of [LR10, Theorem 1.1]) it follows that in our situation Theorem 4.10 is applicable.

Corollary 4.12. *In the situation of Theorem 4.10, for all $t \in [0,T]$ we have*

$$E[\|X(t)\|_H^2] = E[\|X_0\|_H^2] + \int_0^t E\left[2\,{}_{V^*}\langle Y(s), \bar{X}(s)\rangle_V + \|Z(s)\|_{L_2(U,H)}^2\right] ds.$$

Proof. See [PR07, Remark 4.2.8]. □

Proof of Proposition 4.9. For fixed $\lambda > 0$, we apply Theorem 4.10 to the continuous unique solution X_λ of Proposition 4.7 with $Y := b(\cdot, X_\lambda) - A_\lambda(\cdot, X_\lambda)$ and $Z := \sigma(\cdot, X_\lambda)$.
In particular, it follows that $\|X_\lambda(t)\|_H^2$ is a real-valued semi-martingale. Consequently, we can apply the one-dimensional Itô-formula with the C^2-function $F(r) := (r+\varepsilon)^{\frac{p}{2}}$, $p \geq 2$, $\varepsilon > 0$, to $\|X_\lambda(t)\|_H^2$ and obtain

$$\begin{aligned}
&\left(\|X_\lambda(t)\|_H^2 + \varepsilon\right)^{\frac{p}{2}} - \left(\|X_\lambda(0)\|_H^2 + \varepsilon\right)^{\frac{p}{2}} \\
=&\frac{p(p-2)}{4}\int_0^t \left(\|X_\lambda(s)\|_H^2 + \varepsilon\right)^{\frac{p-4}{2}} \|\sigma(s, X_\lambda(s))^* X_\lambda(s)\|_U^2 \, ds \\
&+ \frac{p}{2}\int_0^t \left(\|X_\lambda(s)\|_H^2 + \varepsilon\right)^{\frac{p-2}{2}} \Big[2\,{}_{V^*}\langle b(s, X_\lambda(s)) - A_\lambda(s, X_\lambda(s)), X_\lambda(s)\rangle_V \\
&\qquad + \|\sigma(s, X_\lambda(s))\|_{L_2(U,H)}^2\Big] \, ds \\
&+ p\int_0^t \left(\|X_\lambda(s)\|_H^2 + \varepsilon\right)^{\frac{p-2}{2}} \langle X(s), \sigma(s, X_\lambda(s))\,dW(s)\rangle_H,
\end{aligned}$$

where σ^* is the adjoint operator of σ. Using $\|\sigma^*\|_{L_2(H,U)} = \|\sigma\|_{L_2(U,H)}$, estimating the first and the second summand by use of (H2), (H3) and (H4) and letting $\varepsilon \to 0$, we obtain

$$
\begin{aligned}
&\|X_\lambda(t)\|_H^p \\
\leq &\|X_0\|_H^p + \frac{p}{2}\int_0^t \|X(s)\|_H^{p-2}\left[-C_2\|X_\lambda(s)\|_V^\alpha + C\|X_\lambda(s)\|_H^2 + f(s)\right] ds \\
&+ p\int_0^t \|X_\lambda(s)\|_H^{p-2}\langle X(s), \sigma(s, X_\lambda(s))\, dW(s)\rangle_H \\
\leq &\|X_0\|_H^p - \frac{p}{2}C_2\int_0^t \|X(s)\|_H^{p-2}\|X_\lambda(s)\|_V^\alpha\, ds \\
&+ C\int_0^t \left(\|X_\lambda(s)\|_H^p + f^{\frac{p}{2}}(s)\right) ds \\
&+ p\int_0^t \|X_\lambda(s)\|_H^{p-2}\langle X(s), \sigma(s, X_\lambda(s))\, dW(s)\rangle_H, \quad t \in [0,T].
\end{aligned}
$$

In the last step, we have used $ab \leq a^{\frac{p}{2}} + b^{\frac{p}{p-2}}$, $a, b \geq 0$.

We introduce the localizing sequence τ_N by

$$
\tau_N := \inf\left\{t \in [0,T] \;\middle|\; \|X_\lambda(t)\|_H > N\right\} \wedge T \quad \forall N \in \mathbb{N}. \tag{4.12}
$$

Note that $\lim_{N\to\infty}\tau_N = T$ P-a.s. for all $N \in \mathbb{N}$. By the Burkholder–Davis–Gundy inequality (cf. Theorem D.14) and Young's inequality, we obtain

$$
\begin{aligned}
&p\,E\left[\sup_{r\in[0,\tau_N]}\left|\int_0^r \|X_\lambda(s)\|_H^{p-2}\langle X_\lambda(s), \sigma(s, X_\lambda(s))\, dW(s)\rangle_H\right|\right] \\
\leq &3p\,E\left[\left(\int_0^{\tau_N}\|X_\lambda(s)\|_H^{2(p-2)+2}\|\sigma(s, X_\lambda(s))\|_{L_2(U,H)}^2\, ds\right)^{\frac{1}{2}}\right] \\
\leq &C\,E\left[\sup_{s\in[0,\tau_N]}\|X_\lambda(s)\|_H^{p-1}\left(\int_0^{\tau_N}\left(\|X_\lambda(s)\|_H^2 + f(s)\right) ds\right)^{\frac{1}{2}}\right] \\
\leq &E\left[\frac{1}{2}\sup_{s\in[0,\tau_N]}\|X_\lambda(s)\|_H^p + C\left(\int_0^{\tau_N}\left(\|X_\lambda(s)\|_H^2 + f(s)\right) ds\right)^{\frac{p}{2}}\right] \\
\leq &\frac{1}{2}E\left[\sup_{s\in[0,\tau_N]}\|X_\lambda(s)\|_H^p\right] + C\int_0^{\tau_N}\left(E\left[\|X_\lambda(s)\|_H^p\right] + f^{\frac{p}{2}}(s)\right) ds.
\end{aligned}
$$

Altogether, we obtain

$$
\begin{aligned}
&E\left[\sup_{s\in[0,\tau_N]} \|X_\lambda(s)\|_H^p\right] \\
\leq &2\, E\left[\|X_0\|_H^p\right] - pC_2\, E\left[\int_0^{\tau_N} \|X_\lambda(s)\|_H^{p-2} \|X_\lambda(s)\|_V^\alpha \; ds\right] \\
&+ C\left(\int_0^{\tau_N} \sup_{r\in[0,s]} E\left[\|X_\lambda(r)\|_H^p\right]\; ds + E\left[\int_0^T f^{\frac{p}{2}}(s)\; ds\right]\right).
\end{aligned}
$$

Note that the subtracted term $\frac{1}{2}E\left[\sup_{s\in[0,\tau_N]} \|X_\lambda(s)\|_H^p\right]$ is finite by the choice of τ_N. Applying the Bellman-Gronwall inequality and Lebesgue's dominated convergence theorem (C is independent of λ), we finally arrive at (4.10). □

Corollary 4.13. *In the situation of Proposition 4.7, we have*

$$
\begin{aligned}
\limsup_{\lambda\to 0} \int_0^T E\bigg[&\|A_\lambda(s, X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \|b(s, X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}} \\
&+ \|\sigma(s, X_\lambda(s))\|_{L_2(U,H)}^2 \bigg]\; ds < \infty.
\end{aligned}
\tag{4.13}
$$

Proof. Because the operator A has full domain, $\mathcal{D}(A) = V$, and by Proposition 3.19.ii), $\|A_\lambda(\cdot, x)\|_{V^*} \leq \left\|A^0(\cdot, x)\right\|_{V^*}$ for all $x \in \mathcal{D}(A)$ on $[0, T]$, Conditions (H2), (H3) and (H5) imply that

$$
\begin{aligned}
&\|A_\lambda(s, X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \|b(s, X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \|\sigma(s, X_\lambda(s))\|_{L_2(U,H)}^2 \\
\leq &C\left(\|X_\lambda(s)\|_V^\alpha + \|X_\lambda(s)\|_H^{\frac{\alpha}{\alpha-1}} + f^{\frac{\alpha}{\alpha-1}}(s)\right).
\end{aligned}
$$

Applying Proposition 4.9 for $p := \frac{\alpha}{\alpha-1}$, we obtain (4.13). □

4.3 Existence and Uniqueness

Proof of Theorem 4.5. By Proposition 4.9, we have

$$
\limsup_{\lambda\to 0}\left(\|X_\lambda\|_{L^\alpha([0,T]\times\Omega;V)} + \sup_{t\in[0,T]} \|X_\lambda\|_{L^2(\Omega;H)}\right) < \infty.
$$

Furthermore, by Corollary 4.13 we have

$$
\begin{aligned}
\limsup_{\lambda\to 0}\bigg(&\|A_\lambda(\cdot, X_\lambda)\|_{L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega;V^*)} + \|b(\cdot, X_\lambda)\|_{L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega;V^*)} \\
&+ \|\sigma(\cdot, X_\lambda)\|_{L^2([0,T]\times\Omega;L_2(U,H))}\bigg) < \infty.
\end{aligned}
$$

Since by Corollary C.8, the spaces $L^2([0,T] \times \Omega; L^2(U,H))$, $L^\alpha([0,T] \times \Omega; V)$, $L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega; V^*)$ are reflexive, the Banach-Alaoglu-Theorem yields the following convergences along some subsequence:

i. $X_\lambda \to X$ weakly in $L^\alpha([0,T] \times \Omega; V)$ and weakly star in $L^2(\Omega, L^\infty([0,T], H))$,

ii. $b(\cdot, X_\lambda) \to \bar{b}$ weakly in $L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega; V^*)$,

iii. $A_\lambda(\cdot, X_\lambda) \to \eta$ weakly in $L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega; V^*)$, in particular, $\eta \in L^{\frac{\alpha}{\alpha-1}}([0,T], V^*)$ P-a.s.,

iv. $\sigma(\cdot, X_\lambda) \to \bar{\sigma}$ weakly in $L^2([0,T] \times \Omega; L_2(U,H))$ and therefore,

$$\int_0^\cdot \sigma(s, X_\lambda(s))\, dW(s) \to \int_0^\cdot \bar{\sigma}(s)\, dW(s)$$

weakly in $L^\infty([0,T], L^2(\Omega; H))$, since the stochastic integral is a continuous linear operator, hence weakly continuous.

Step 1. $\int_0^t \eta(s)\, ds$ is an $(\mathcal{F}_t)$-adapted process.

Let $\varphi \in L^\infty(\Omega, V)$. Since $A_\lambda(\cdot, X_\lambda)$ converges weakly in $L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega, V^*)$ and, since $\varphi(\cdot)1_{[0,t]}(\cdot) \in L^\infty([0,T] \times \Omega, V^*)$, we have

$$\begin{aligned}
&{}_{L^{\frac{\alpha}{\alpha-1}}(\Omega,V^*)}\left\langle \int_0^t A_\lambda(s, X_\lambda(s))\, ds, \varphi \right\rangle_{L^\alpha(\Omega,V)} \\
&= \int_{[0,T]\times\Omega} 1_{[0,t]}(s)\, {}_{V^*}\langle A_\lambda(s, X_\lambda(s), \omega), \varphi(\omega)\rangle_V \; ds \otimes P(d\omega) \\
&\overset{\lambda_n \to 0}{\longrightarrow} \int_{[0,T]\times\Omega} 1_{[0,t]}(s)\, {}_{V^*}\langle \eta(s, \omega), \varphi(\omega)\rangle_V \; ds \otimes P(d\omega) \\
&= {}_{L^{\frac{\alpha}{\alpha-1}}(\Omega,V^*)}\left\langle \int_0^t \eta(s)\, ds, \varphi \right\rangle_{L^\alpha(\Omega,V)}
\end{aligned} \tag{4.14}$$

Since $L^\infty(\Omega, V^*)$ is dense in $L^{\frac{\alpha}{\alpha-1}}(\Omega, V^*)$ and since by (4.13), the integral $\int_0^t A_\lambda(s, X_\lambda(s))\, ds$ is bounded in $L^{\frac{\alpha}{\alpha-1}}(\Omega, V^*)$ uniformly in λ, we conclude that $\int_0^t \eta(s)\; ds$ is the weak limit of $\int_0^t A_\lambda(s, X_\lambda(s))\; ds$ in $L^{\frac{\alpha}{\alpha-1}}(\Omega, V^*)$ (cf. [Zei90a, Proposition 21.23(g)]). Hence, by the Theorem of Mazur (cf. [Zei90a, Proposition 21.23(e)]), for some $\{\lambda_n\}_{n\in\mathbb{N}}$ there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \in \text{co}\,\{\int_0^t A_{\lambda_n}(s, X_{\lambda_n}(s))\; ds\}$, where co denotes the closed convex hull, and $v_n \overset{n\to\infty}{\longrightarrow} \int_0^t \eta(s)\; ds$ in $L^{\frac{\alpha}{\alpha-1}}(\Omega, V^*)$. Since $\int_0^t A_{\lambda_n}(s, X_{\lambda_n}(s))\; ds$ is $(\mathcal{F}_t)$-adapted and, consequently, v_n is a linear combination of $(\mathcal{F}_t)$-adapted processes, the limit point $\int_0^t \eta(s)\; ds$ is also $\mathcal{F}_t$-adapted.

Note that by an analogous argumentation, $\bar{b}$ and $\bar{\sigma}$ are progressively measurable since the approximants are progressively measurable.

Step 2. (X, η) satisfy (4.2) P-a.s..

Following [PR07, Proof of Theorem 4.2.4], we define $\varphi \in L^\infty([0,T] \times \Omega)$ and $v \in V$. By (4.5), using (i)-(iv) and Fubini's theorem, along some subsequence we obtain

$$
\begin{aligned}
&E\left[\int_0^T {}_{V^*}\langle X(t), \varphi(t)v\rangle_V \ dt\right] \\
&= \lim_{\lambda\to 0} E\left[\int_0^T {}_{V^*}\langle X_\lambda(t), \varphi(t)v\rangle_V \ dt\right] \\
&= \lim_{\lambda\to 0} E\Bigg[\int_0^T {}_{V^*}\langle X_0, \varphi(t)v\rangle_V \ dt \\
&\quad + \int_0^T \int_0^t {}_{V^*}\langle b(s, X_\lambda(s)) - A_\lambda(s, X_\lambda(s)), \varphi(t)v\rangle_V \ ds\, dt \\
&\quad + \int_0^T \left\langle \int_0^t \sigma(s, X_\lambda(s))\, dW(s), \varphi(t)v \right\rangle_H \ dt\Bigg] \\
&= \lim_{\lambda\to 0} \Bigg(E\left[\int_0^T {}_{V^*}\langle X_0, \varphi(t)v\rangle_V \ dt\right] \\
&\quad + E\left[\int_0^T {}_{V^*}\left\langle b(s, X_\lambda(s)) - A_\lambda(s, X_\lambda(s)), \int_s^T \varphi(t)\, dt\, v \right\rangle_V \ ds\right] \\
&\quad + \int_0^T E\left[\varphi(t) {}_{V^*}\left\langle \int_0^t \sigma(s, X_\lambda(s))\, dW(s), v \right\rangle_V\right] \ dt\Bigg) \\
&= E\left[\int_0^T {}_{V^*}\left\langle X_0 + \int_0^t \big(\bar{b}(s) - \eta(s)\big)\ ds + \int_0^t \bar{\sigma}(s)\, dW(s), \varphi(t)v \right\rangle_V \ dt\right].
\end{aligned}
\tag{4.15}
$$

Since $\varphi \in L^\infty(\Omega)$ and $v \in V$ are arbitrary, defining

$$
\bar{X}(t) := X_0 + \int_0^t \big(\bar{b}(s) - \eta(s)\big)\ ds + \int_0^t \bar{\sigma}(s)\, dW(s), \quad t \in [0,T],
$$

we obtain

$$
\bar{X} = X \quad dt \otimes P\text{-a.e.}.
$$

In the next step, we show that

$$
\bar{b} = b(\cdot, X) \quad \text{and} \quad \bar{\sigma} = \sigma(\cdot, X) \quad dt \otimes P\text{-a.e.}
$$

To this end, we first observe that by Corollary 4.12 and the product rule applied to $\|X(t)\|_H^2$, we obtain

$$
\begin{aligned}
&E\left[e^{-\beta t}\,\|X(t)\|_H^2\right] - E\left[\|X_0\|_H^2\right] \\
=&\int_0^t E\left[\|X(s)\|_H^2\right]\, d\left(e^{-\beta s}\right) + \int_0^t e^{-\beta s}\, d\left(E\left[\|X(s)\|_H^2\right]\right) \\
=&E\left[\int_0^t e^{-\beta s}\left(2\,{}_{V^*}\langle \bar{b}(s) - \eta(s), X(s)\rangle_V + \|\ \bar{\sigma}(s)\|_{L_2(U,H)}^2 - \beta\,\|X(s)\|_H^2\right)\, ds\right].
\end{aligned} \tag{4.16}
$$

Note that by i)-iv), we are meeting the conditions of Theorem 4.10 and thus, Corollary 4.12 is applicable.

On the other hand, we apply Corollary 4.12 and the product rule to $\|X_\lambda(t)\|_H^2$ and obtain

$$
\begin{aligned}
&E\left[e^{-\beta t}\,\|X_\lambda(t)\|_H^2\right] - E\left[\|X_0\|_H^2\right] \\
=&E\left[\int_0^t e^{-\beta s}\,(2\,{}_{V^*}\langle b(s,X_\lambda(s)) - A_\lambda(s,X_\lambda(s)), X_\lambda(s)\rangle_V\right. \\
&\qquad\left. +\ \|\sigma(s,X_\lambda(s))\|_{L_2(U,H)}^2 - \beta\,\|X_\lambda(s)\|_H^2\right)\, ds\Big] \\
=&E\left[\int_0^t e^{-\beta s}\,(2\,{}_{V^*}\langle b(s,X_\lambda(s)) - b(s,\phi(s)), X_\lambda(s) - \phi(s)\rangle_V\right. \\
&\qquad\left. +\ \|\sigma(s,X_\lambda(s)) - \sigma(s,\phi(s))\|_{L_2(U,H)}^2 - \beta\,\|X_\lambda(s) - \phi(s)\|_H^2\right)\, ds\Big] \\
&+E\left[\int_0^t e^{-\beta s}\,(2\,{}_{V^*}\langle b(s,\phi(s)), X_\lambda(s)\rangle_V + {}_{V^*}\langle b(s,X_\lambda(s)) - b(s,\phi(s)), \phi(s)\rangle_V\right. \\
&\qquad - \|\sigma(s,\phi(s))\|_{L_2(U,H)}^2 + 2\langle \sigma(s,X_\lambda(s)), \sigma(s,\phi(s))\rangle_{L_2(U,H)} \\
&\qquad\left. -\ 2\beta\langle X_\lambda(s), \phi(s)\rangle_H + \beta\,\|\phi(s)\|_H^2\right)\, ds\Big] \\
&-E\left[\int_0^t e^{-\beta s}\,(2\,{}_{V^*}\langle A_\lambda(s,X_\lambda(s)), X_\lambda(s)\rangle_V)\, ds\right]
\end{aligned} \tag{4.17}
$$

where we have used $a^2 = (a-b)^2 - b^2 + 2ab$ in the last step. By (H2), the first summand of the right-hand side in (4.17) is negative for $\beta := 2C_L + C_L^2$.

Letting $\lambda \to 0$ and using (i)-(iv), (4.17) turns into

$$\begin{aligned}
&\liminf_{\lambda\to 0} E\left[e^{-\beta t}\,\|X_\lambda(t)\|_H^2\right] - E\left[\|X_0\|_H^2\right]\\
&+\limsup_{\lambda\to 0} 2E\left[\int_0^t e^{-\beta s}\,{}_{V^*}\langle A_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V\;ds\right]\\
\le& E\left[\int_0^t e^{-\beta s}\left(2\,{}_{V^*}\langle b(s,\phi(s)),X(s)\rangle_V + {}_{V^*}\langle \bar{b}(s)-b(s,\phi(s)),\phi(s)\rangle_V\right.\right.\\
&\quad+\|\sigma(s,\phi(s))\|_{L_2(U,H)}^2 + 2\langle\bar{\sigma}(s),\sigma(s,\phi(s))\rangle_{L_2(U,H)}\\
&\quad\left.\left.-\,2\beta\langle X(s),\phi(s)\rangle_H + \beta\,\|\phi(s)\|_H^2\right)\;ds\right].
\end{aligned}\tag{4.18}$$

Note that for any non-negative $\psi \in L^\infty([0,T],dt)$ it follows from (i) that

$$\begin{aligned}
&E\left(\int_0^T \psi(t)\|X(t)\|_H^2\;dt\right)\\
=&\lim_{\lambda\to 0} E\left(\int_0^T \langle\psi(t)X(t),X_\lambda(t)\rangle_H\;dt\right)\\
\le&\left(E\int_0^T \psi(t)\|X(t)\|_H^2\;dt\right)^{1/2}\liminf_{\lambda\to 0}\left(E\int_0^T \psi(t)\|X_\lambda(t)\|_H^2\;dt\right)^{1/2} < \infty.
\end{aligned}$$

This implies

$$E\left(\int_0^T \psi(t)\|X(t)\|_H^2\;dt\right) \le \liminf_{\lambda\to 0} E\left(\int_0^T \psi(t)\|X_\lambda(t)\|_H^2\;dt\right). \tag{4.19}$$

Hence, combining (4.16), (4.18) and (4.19) we arrive at

$$\begin{aligned}
&\limsup_{\lambda\to 0} 2E\Bigg[\int_0^T \psi(t)\Bigg(\int_0^t e^{-\beta s}\,{}_{V^*}\langle A_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V\\
&\quad-\,{}_{V^*}\langle\eta(s),X(s)\rangle_V\;ds\Bigg)\;dt\Bigg]\\
\le& E\left[\int_0^T \psi(t)\left(\int_0^t e^{-\beta s}\left(-2\,{}_{V^*}\langle\bar{b}(s)-b(s,\phi(s)),X(s)-\phi(s)\rangle_V\right.\right.\right.\\
&\quad\left.\left.\left.-\;\|\sigma(s,\phi(s))-\bar{\sigma}(s)\|_{L_2(U,H)}^2 + \beta\,\|X(s)-\phi(s)\|_H^2\right)\;ds\right)\;dt\right]
\end{aligned}\tag{4.20}$$

Since $\eta(s)$ is the weak limit of $A_\lambda(s,X_\lambda(s))$, by the monotonicity of A_λ we obtain

$$\begin{aligned}
&\liminf_{\lambda\to 0} E\left[\int_0^t e^{-\beta s}\ {}_{V^*}\langle A_\lambda(s, X_\lambda(s)), X_\lambda(s)\rangle_V - {}_{V^*}\langle \eta(s), X(s)\rangle_V\ ds\right]\\
=&\liminf_{\lambda\to 0} E\left[\int_0^t e^{-\beta s}\ {}_{V^*}\langle A_\lambda(s, X_\lambda(s)), X_\lambda(s) - X(s)\rangle_V\ ds\right]\\
\geq&\liminf_{\lambda\to 0} E\left[\int_0^t e^{-\beta s}\ {}_{V^*}\langle A_\lambda(s, X_\lambda(s)) - A_\lambda(s, X(s)), X_\lambda(s) - X(s)\rangle_V\ ds\right]\\
&+\liminf_{\lambda\to 0} E\left[\int_0^t e^{-\beta s}\ {}_{V^*}\langle A_\lambda(s, X(s)), X_\lambda(s) - X(s)\rangle_V\ ds\right]\\
\geq&\liminf_{\lambda\to 0} E\left[\int_0^t e^{-\beta s}\ {}_{V^*}\langle A_\lambda(s, X(s)), X_\lambda(s) - X(s)\rangle_V\ ds\right].
\end{aligned} \tag{4.21}$$

Recall that $A_\lambda(x) \to A^0(x)$ (strongly) $\forall x \in \mathcal{D}(A)$ where A^0 denotes the minimal selection of A (cf. Proposition 3.19.iv)). Since $\mathcal{D}(A) = V$ and $X_\lambda \to X$ weakly in $L^\alpha([0,T]\times\Omega; V)$, the right-hand side of (4.21) converges to 0 as $\lambda \to 0$ (cf. [Zei90a, Proposition 21.23(j)]). Hence, (4.20) turns into

$$\begin{aligned}
0 \geq &E\left[\int_0^T \psi(t)\left(\int_0^t e^{-\beta s}\left(2\ {}_{V^*}\langle \bar{b}(s) - b(s,\phi(s)), X(s) - \phi(s)\rangle_V\right.\right.\right.\\
&\left.\left.\left. + \ \|\sigma(s,\phi(s)) - \bar{\sigma}(s)\|^2_{L_2(U,H)} - \beta\,\|X(s) - \phi(s)\|^2_H\right)\ ds\right)\ dt\right]
\end{aligned} \tag{4.22}$$

Taking $\phi = X$, we conclude that $\bar{\sigma}(s) = \sigma(s, X(s))$. Inserting $\phi = X - \varepsilon\tilde{\phi}$, $\varepsilon > 0$, $\tilde{\phi} \in L^\infty([0,T]\times\Omega; V)$ into (4.22), dropping the second integrand and dividing both sides by ε, we obtain

$$\begin{aligned}
0 \geq &E\Bigg[\int_0^T \psi(t)\Bigg(\int_0^t e^{-\beta s} 2\Bigg(\ {}_{V^*}\Big\langle \bar{b}(s) - b(s, X(s) - \varepsilon\tilde{\phi}(s)), \tilde{\phi}(s)\Big\rangle_V\\
&- \beta\varepsilon\left\|\tilde{\phi}(s)\right\|_H^2\Bigg)\ ds\Bigg)\ dt\Bigg].
\end{aligned}$$

By (H2) and Lebesgue's convergence theorem, letting $\varepsilon \to 0$ yields

$$0 \geq E\left[\int_0^T \psi(t)\left(\int_0^t e^{-\beta s}\left(\ {}_{V^*}\Big\langle \bar{b}(s) - b(s, X(s)), \tilde{\phi}(s)\Big\rangle_V\right)\ ds\right)\ dt\right].$$

Since $\tilde{\phi}$ and ψ have been chosen arbitrarily, we conclude $\bar{b} = b(\cdot, X)$.

Now we are able to apply Theorem 4.10 and conclude that X is an $(\mathcal{F}_t)$-adapted process, continuous in H.

Step 3. $\eta(t) \in A(t, X(t))$ for almost all $t \in [0,T]$ and $\omega \in \Omega$.

We want to apply Proposition 3.3.

Claim 1. The (multivalued) operator

$$\mathcal{A} : L^{\alpha}([0,T] \times \Omega, V) \to 2^{L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,V^*)}$$

defined by

$$x \mapsto A(\cdot, x)$$

is maximal monotone.

Proof. Let $x_1,\ x_2 \in L^{\alpha}([0,T] \times \Omega, V)$ and $v_i \in A(\cdot, x_i),\ i = 1, 2$. Then, by the monotonicity of A we have

$$\begin{aligned}
&{}_{L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,V^*)}\langle v_1 - v_2, x_1 - x_2\rangle_{L^{\alpha}([0,T]\times\Omega,V)} \\
=&E\left[\int_0^T {}_{V^*}\langle v_1(t) - v_2(t), x_1(t) - x_2(t)\rangle_V \ dt\right] \geq 0.
\end{aligned}$$

Hence $\mathcal{A}$ is monotone.

Since for every $(t, \omega) \in [0,T] \times \Omega$ the operator $A(t, \cdot, \omega)$ is maximal monotone and J is coercive and maximal monotone, Proposition 3.27 implies that for any $y \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega, V^*)$ there exists a progressively measurable process $x(t) \in V$ such that

$$y(t) \in A(t, x(t)) + \lambda J(x(t)),$$

on Ω for all $t \in [0,T]$ and $\lambda > 0$. Let $v(\cdot) \in A(\cdot, x)$ such that

$$y = v + \lambda J(x) \tag{4.23}$$

on $[0,T] \times \Omega$. Taking the dualization product with $x(\cdot)$ in (4.23), by (H4) we obtain

$$\begin{aligned}
{}_{V^*}\langle y, x\rangle_V =& {}_{V^*}\langle v, x\rangle_V + \lambda\, {}_{V^*}\langle J(x), x\rangle_V \\
&\geq C_C \|x\|_V^{\alpha} + \lambda \|x\|_V^2 + f \\
&\geq C_C \|x\|_V^{\alpha} + f
\end{aligned}$$

on $[0,T] \times \Omega$ for some $f \in L^1([0,T] \times \Omega)$. Thus, by Young's inequality, $x \in L^{\alpha}([0,T] \times \Omega, V)$ since $y \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega, V^*)$. Now, Theorem 3.14 applies and we conclude that $\mathcal{A}$ is maximal monotone. □

Claim 2. $J_\lambda(X_\lambda)$ converges weakly along some sequence $\lambda \to 0$ to X in $L^{\alpha}([0,T] \times \Omega, V)$.

Proof. Since by the definition of the Yosida approximation, $\lambda A_\lambda(X_\lambda) = J(X_\lambda - J_\lambda(X_\lambda))$, it follows that

$$\|J_\lambda(X_\lambda) - X_\lambda\|_V = \lambda \|A_\lambda(X_\lambda)\|_{V^*}.$$

Hence, by (4.13)

$$
\begin{aligned}
&E\left[\int_0^T \|J_\lambda(X_\lambda)-X_\lambda\|_V^{\frac{\alpha}{\alpha-1}}\ ds\right]\\
\leq&\lambda^{\frac{\alpha}{\alpha-1}}\limsup_{\lambda\to 0}\int_0^T E\left[\|A_\lambda(s,X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right]\ ds \xrightarrow{\lambda\to 0} 0.
\end{aligned}
$$

Since $\alpha \leq \frac{\alpha}{\alpha-1}$, we conclude

$$
\lim_{\lambda\to 0}\|J_\lambda(X_\lambda)-X_\lambda\|_{L^\alpha([0,T]\times\Omega;V)}=0. \tag{4.24}
$$

In particular, for $\varphi \in L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,V^*)$, we have $E\left[\int_0^T {}_{V^*}\langle\varphi, J_\lambda(X_\lambda)-X_\lambda\rangle_V\ ds\right]\xrightarrow{\lambda\to 0} 0$. Since $X_\lambda\to X$ weakly in $L^\alpha([0,T]\times\Omega,V)$, we deduce

$$
\begin{aligned}
&E\left[\int_0^T {}_{V^*}\langle\varphi, J_\lambda(X_\lambda)-X\rangle_V\ ds\right]\\
=&E\left[\int_0^T {}_{V^*}\langle\varphi, J_\lambda(X_\lambda)-X_\lambda\rangle_V\ ds\right]+E\left[\int_0^T {}_{V^*}\langle\varphi, X_\lambda-X\rangle_V\ ds\right]\xrightarrow{\lambda\to 0} 0.
\end{aligned}
$$

□

Claim 3.

$$
\begin{aligned}
&\limsup_{\lambda\to 0}E\left[\int_0^T {}_{V^*}\langle A_\lambda(s,X_\lambda(s)), J_\lambda(X_\lambda(s))\rangle_V\ ds\right]\\
\leq&E\left[\int_0^T {}_{V^*}\langle \eta(s), X(s)\rangle_V\ ds\right].
\end{aligned}
$$

Proof. In (4.20), taking $\phi = X$ and recalling that $\bar{b} = b(\cdot, X)$ and $\bar{\sigma} = \sigma(\cdot, X)$, we obtain

$$
\begin{aligned}
&\limsup_{\lambda\to 0}E\left[\int_0^T {}_{V^*}\langle A_\lambda(s,X_\lambda(s)), X_\lambda(s)\rangle_V\ ds\right]\\
\leq&E\left[\int_0^T {}_{V^*}\langle \eta(s), X(s)\rangle_V\ ds\right].
\end{aligned} \tag{4.25}
$$

Thus, by Hölder's inequality, (4.24) and (4.25) we conclude

$$
\begin{aligned}
&\limsup_{\lambda\to 0}E\left[\int_0^T {}_{V^*}\langle A_\lambda(s,X_\lambda(s)), J_\lambda(X_\lambda(s))\rangle_V\ ds\right]\\
\leq&\limsup_{\lambda\to 0}\left(\int_0^T E\left[\|A_\lambda(s,X_\lambda(s))\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right]\ ds\cdot\|J_\lambda(X_\lambda)-X_\lambda\|_{L^\alpha([0,T]\times\Omega;V)}\right)\\
&\quad+\limsup_{\lambda\to 0}E\left[\int_0^T {}_{V^*}\langle A_\lambda(s,X_\lambda(s)), X_\lambda(s)\rangle_V\ ds\right]\\
\leq&E\left[\int_0^T {}_{V^*}\langle \eta(s), X(s)\rangle_V\ ds\right].
\end{aligned}
$$

□

By (iii), $A_\lambda(X_\lambda)$ converges weakly to η in $L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega, V^*)$. Therefore, due to Claim 1, Claim 2, Claim 3 and the fact that $A_\lambda(s,x) \in A(J_\lambda(s,x))\ \forall x \in V$ on $[0,T]\times\Omega$ (cf. Proposition 3.19.v)), all conditions of Proposition 3.3 are fulfilled and we can conclude that $\eta \in \mathcal{A}(X)\quad dt\otimes P$-a.e., especially $\eta(t,\omega) \in A(t, X(t,\omega),\omega)$ for almost all $(t,\omega) \in [0,T]\times\Omega$. □

On account of the Itô-formula, we have the following uniqueness result.

Proposition 4.14. *The solution of* (4.1) *is path-wise unique in the following sense: For every two solutions X_1 and X_2 of* (4.1) *and some constant $C > 0$ we have*

$$E\left[\|X_1(t) - X_2(t)\|_H^2\right] \le e^{Ct} E\left[\|X_1(0) - X_2(0)\|_H^2\right] \quad \forall t \in [0,T].$$

Proof. Let (X_1,η_1), (X_2,η_2) be two solutions of (4.1). We apply Corollary 4.12 to $X_1 - X_2$ and obtain

$$\begin{aligned}
&E\left[\|X_1(t) - X_2(t)\|_H^2\right]\\
=&E\left[\|X_1(0) - X_2(0)\|_H^2\right]\\
&+2\int_0^t E\left[{}_{V^*}\langle b(s,X_1(s)) - b(s,X_2(s)), X_1(s) - X_2(s)\rangle_V\right]\, ds\\
&-2\int_0^t E\left[{}_{V^*}\langle \eta_1(s) - \eta_2(s), X_1(s) - X_2(s)\rangle_V\right]\, ds\\
&+\int_0^t E\left[\|\sigma(s,X_1(s)) - \sigma(s,X_2(s))\|_{L_2(U,H)}^2\right]\, ds.
\end{aligned}$$

Since $\eta_i(t) \in A_i(t, X(t))$, $i \in \{1,2\}$, by the monotonicity of A we obtain

$$-2\int_0^t E\left[{}_{V^*}\langle \eta_1(s) - \eta_2(s), X_1(s) - X_2(s)\rangle_V\right]\, ds \le 0.$$

Now, (H2) and (H3) yields

$$\begin{aligned}
&E\left[\|X_1(t) - X_2(t)\|_H^2\right]\\
\le&E\left[\|X_1(0) - X_2(0)\|_H^2\right] + C\int_0^t E\left[\|X_1(s) - X_2(s)\|_H^2\right]\, ds.
\end{aligned}$$

Hence, applying Bellman-Gronwall inequality, we obtain the assertion. □

L^2-Convergence

Proposition 4.15. *Suppose that conditions (H1)-(H5) hold, then for any sequences (λ), (μ) such that $\lambda,\mu \to 0$ and for some $C > 0$, we have*

$$E\left[\sup_{s\in[0,T]} \|X_\lambda(s) - X_\mu(s)\|_H^2\right] \le C\cdot(\lambda+\mu).$$

Proof. Let $C > 0$ be a universal constant. Applying Theorem 4.10 to $X_\lambda(t) - X_\mu(t)$, we obtain

$$\begin{aligned}
&\|X_\lambda(t) - X_\mu(t)\|_H^2 \\
=&\|X_0 - X_0\|_H^2 \\
&+ 2\int_0^t {}_{V^*}\langle b(s, X_\lambda(s)) - b(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \, ds \\
&- 2\int_0^t {}_{V^*}\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \, ds \\
&+ 2\int_0^t \langle X_\lambda(s) - X_\mu(s), (\sigma(s, X_\lambda(s)) - \sigma(s, X_\mu(s)))\, dW(s)\rangle_H \\
&+ \int_0^t \|\sigma(s, X_\lambda(s)) - \sigma(s, X_\mu(s))\|_{L_2(U,H)}^2 \, ds.
\end{aligned} \tag{4.26}$$

By (H2) and (H3) we obtain

$$\begin{aligned}
&{}_{V^*}\langle b(s, X_\lambda(s)) - b(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \\
&\quad + \|\sigma(s, X_\lambda(s)) - \sigma(s, X_\mu(s))\|_{L_2(U,H)}^2 \\
=&\langle b(s, X_\lambda(s)) - b(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_H \\
&\quad + \|\sigma(s, X_\lambda(s)) - \sigma(s, X_\mu(s))\|_{L_2(U,H)}^2 \\
\le& C\,\|X_\lambda(s) - X_\mu(s)\|_H^2 .
\end{aligned}$$

By the definition of A_λ and the bijectivity of J we have $I = J^{-1}(\lambda A_\lambda) + J_\lambda$. Hence,

$$\begin{aligned}
&- {}_{V^*}\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \\
=&- {}_{V^*}\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), J_\lambda X_\lambda(s) - J_\mu X_\mu(s)\rangle_V \\
&- {}_{V^*}\big\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), J^{-1}(\lambda A_\lambda(s, X_\lambda(s)))\big\rangle_V . \\
&+ {}_{V^*}\big\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), J^{-1}(\mu A_\mu(s, X_\mu(s)))\big\rangle_V .
\end{aligned} \tag{4.27}$$

By Proposition 3.19.v), we have $A_\lambda(s, X_\lambda(s)) \in A(J_\lambda(s, X_\lambda(s)))$ and $A_\mu(s, X_\mu(s)) \in A(J_\mu(s, X_\mu(s)))$. Using the monotonicity of A and the fact that J^{-1} is the dualization map from V^* to $V^{**} = V$, (4.27) yields

$$\begin{aligned}
&- {}_{V^*}\langle A_\lambda(s, X_\lambda(s)) - A_\mu(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \\
\leq &- \frac{1}{\lambda}\, {}_{V^*}\big\langle \lambda A_\lambda(s, X_\lambda(s)), J^{-1}(\lambda A_\lambda(s, X_\lambda(s)))\big\rangle_V \\
&\quad + \big|\, {}_{V^*}\big\langle A_\lambda(s, X_\lambda(s)), J^{-1}(\mu A_\mu(s, X_\mu(s)))\big\rangle_V \big| \\
&- \frac{1}{\mu}\, {}_{V^*}\big\langle \mu A_\mu(s, X_\mu(s)), J^{-1}(\mu A_\mu(s, X_\mu(s)))\big\rangle_V \\
&\quad + \big|\, {}_{V^*}\big\langle A_\mu(s, X_\mu(s)), J^{-1}(\lambda A_\lambda(s, X_\lambda(s)))\big\rangle_V \big| \\
\leq &- \lambda \, \|A_\lambda(s, X_\lambda(s))\|_{V^*}^2 + \mu \, \|A_\lambda(s, X_\lambda(s))\|_{V^*} \, \|A_\mu(s, X_\mu(s))\|_{V^*} \\
&- \mu \, \|A_\mu(s, X_\mu(s))\|_{V^*}^2 + \lambda \, \|A_\lambda(s, X_\lambda(s))\|_{V^*} \, \|A_\mu(s, X_\mu(s))\|_{V^*} \\
\leq &\frac{\lambda}{4} \, \|A_\mu(s, X_\mu(s))\|_{V^*}^2 + \frac{\mu}{4} \, \|A_\lambda(s, X_\lambda(s))\|_{V^*}^2
\end{aligned} \tag{4.28}$$

where we have used the elementary inequality $ab \leq \frac{1}{4}a^2 + b^2$ in the last step. Again, we localize the Itô-integral and estimate it by the Burkholder–Davis–Gundy inequality similarly to the proof of Proposition 4.15. After taking the supremum over $[0, T]$ and the expectation in (4.26), we finally arrive at

$$\begin{aligned}
E\left[\sup_{s\in[0,T]} \left\|X_\lambda^N(s) - X_\mu^N(s)\right\|_H^2\right] \leq C \int_0^T E\left[\sup_{r\in[0,s]} \left\|X_\lambda^N(r) - X_\mu^N(r)\right\|_H^2\right] ds \\
+ \mu C_H \int_0^T E\left[\|A_\lambda(s, X_\lambda(s))\|_{V^*}^2\right] ds \\
+ \lambda C_H \int_0^T E\left[\|A_\mu(s, X_\mu(s))\|_{V^*}^2\right] ds.
\end{aligned}$$

Since $\frac{\alpha}{\alpha-1} \geq 2$, by Corollary 4.13 we have

$$\limsup_{\lambda\to 0} \int_0^T E\left[\|A_\lambda(s, X_\lambda(s))\|_{V^*}^2\right] ds < \infty.$$

Hence, Bellman-Gronwall inequality implies that

$$E\left[\sup_{s\in[0,T]} \left\|X_\lambda^N(s) - X_\mu^N(s)\right\|_H^2\right] \leq C \cdot (\lambda + \mu). \tag{4.29}$$

Note that the constant C in (4.29) is independent of N, λ and μ. Now let $N \to \infty$ to get (4.29) without N. □

As a direct consequence of Proposition 4.15, the following convergence result holds:

Corollary 4.16. *There exists a process $X \in L^2([0, T] \times \Omega; H))$, being P-a.s. continuous in H, such that*

$$\lim_{\lambda\to 0} E\left[\sup_{t\in[0,T]} \|X_\lambda(t) - X(t)\|_H^2\right] = 0. \tag{4.30}$$

Chapter 5

Multivalued Stochastic Partial Differential Equations Driven by Poisson Noise

In this chapter, we extend the results developed in Chapter 4 by adding Poisson noise to the multivalued stochastic differential equation. Thanks to the Lévy–Itô decomposition (cf. Theorem D.4), this covers a large class of equations driven by Hilbert space-valued Lévy noise. Furthermore, we replace the differential dt of the drift with a more general measure $dN(t)$ induced by a non-decreasing cádlág process $N(t)$.

Again, existence and uniqueness are established via the Yosida approximation approach. The single-valued counterpart in [Gyö82] will provide us with the existence and uniqueness of the solution to the approximating equations.

5.1 Variational Framework

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \in [0, \infty[$. Let (V, H, V^*) be a Gelfand triple (cf. Definition B.10) where $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and V is a uniformly convex Banach space with a uniformly convex dual space V^*. Let $(U, \langle \cdot, \cdot \rangle_U)$ be an additional separable Hilbert space. $L_2(U, H)$ denotes the space of Hilbert-Schmidt operators from U to H.

In addition, let $(Z, \mathcal{Z}, m)$ be a measure space with a σ-finite measure m, p be a stationary $(\mathcal{F}_t)$-Poisson point process Z with the characteristic measure m and let μ be the Poisson random measure on $[0, T] \times Z$ induced by p with compensator $dt \otimes m$ (cf. Section 1.4). Let $\bar{\mu} := \mu - dt \otimes m$ denote the compensated Poisson measure of μ. Here and in the following, we shall denote the Lebesgue measure on $\mathbb{R}$ by dt.

Furthermore, let $W(t)$ be a cylindrical Q-Wiener process on U with the covariance $Q = I$ and let $N(t)$ and $V(t)$ be predictable non-decreasing real-valued cádlág processes such that $dV(t) \geq dN(t)$ and $dV(t) \geq dt$. (For example, the process $V(t) := N(t) + t$ satisfies these conditions.) Fix some $T \in [0, \infty[$. Throughout this chapter, let $C > 0$ be a universal constant which may vary from line to line.

We consider multivalued stochastic partial differential equations of the following type:

$$\begin{cases} dX(t) \in [B(t, X(t)) - A(t, X(t))]\, dN(t) + D(t, X(t-))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z G(t, X(t-), z)\bar{\mu}(dt, dz) \\ X(0) = X_0. \end{cases} \tag{5.1}$$

Here, X_0 is an $\mathcal{F}_0$-measurable random variable with $X_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$.

Remark 5.1. *Setting $G \equiv 0$ and $N(t) := t$, we return to the situation in Chapter 4.*

We consider operators

$$\begin{aligned} A :&[0, T] \times V \times \Omega \to 2^{V^*}, \\ B :&[0, T] \times V \times \Omega \to H, \\ D :&[0, T] \times V \times \Omega \to L_2(U, H), \\ G :&[0, T] \times V \times Z \times \Omega \to H, \end{aligned}$$

such that B, D and G are progressively measurable. The multivalued operator A is assumed to be progressively Effros-measurable, i.e. for every $t \in [0, T]$, A is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t / \mathcal{B}(V^*)$-Effros-measurable.

Definition 5.2. *A solution to* (5.1) *on the interval* $[0, T]$ *is a couple* (X, η) *of processes such that* $X \in L^2([0, T] \times \Omega,\ dV \otimes P; V)$ *and* $\eta \in L^1([0, T] \times \Omega, dN \otimes P; V^*)$ *and for* P*-a.e.* $\omega \in \Omega$

i. X is cádlág,

ii. the processes X and $\int_{]0,T]} \eta(s)\, dN(s)$ are $(\mathcal{F}_t)$-adapted,

iii. for almost all $t \in [0, T]$

$$\eta(t) \in A(t, X(t)),$$

iv. for all $t \in [0,T]$, the following equation holds:

$$
\begin{aligned}
X(t) = & X_0 + \int_{]0,t]} (B(s, X(s)) - \eta(s)) \ dN(s) \\
& + \int_{]0,t]} D(s, X(s-)) \, dW(s) \\
& + \int_{]0,t]} \int_Z G(s, X(s-), z) \bar{\mu}(ds, dz), \\
X(0) = & X_0.
\end{aligned}
\tag{5.2}
$$

We define

$$
\varrho_N(t) := \frac{dN(t)}{dV(t)} \text{ and } \varrho_t(t) := \frac{dt}{dV(t)}.
$$

Remark 5.3. *Note that by Theorem D.7, ϱ_N and ϱ_t are well-defined as the Radon-Nikodym derivatives satisfying*

$$
N(t) - N(0) = \int_{]0,t]} \varrho_N(s) \, dV(s) \quad \textit{and} \quad t = \int_{]0,t]} \varrho_t(s) \, dV(s)
$$

and $\varrho_t, \varrho_N \in [0,1]$ dV-a.s.. Furthermore, the predictable processes N and V induce the measures dN and dV on $([0,T] \times \Omega, \mathcal{P}_T)$ (cf. Definition 2.7). Therefore, the Radon-Nikodym Theorem implies that ϱ_N and ϱ_t are predictable.

Set

$$
\mathbb{A} := \varrho_N A, \quad \mathbb{B} := \varrho_N B, \quad \mathbb{D} := \varrho_t^{1/2} D, \quad \text{and} \quad \mathbb{G} := \varrho_t^{1/2} G.
$$

Let f be an $(\mathcal{F}_t)$-adapted process with $f \in L^2([0,T] \times \Omega, \ dV \otimes P)$. We impose the following conditions:

(H1) (Maximal monotonicity) For all $x, y \in V$ and all $(t, \omega) \in [0,T] \times \Omega$ we have

$$
{}_{V^*}\langle v - w, x - y \rangle_V \geq 0 \quad \forall v \in \mathbb{A}(t,x), \, \forall w \in \mathbb{A}(t,y) \tag{5.3}
$$

and $x \mapsto \mathbb{A}(t,x)$ is maximal.

(H2) (Lipschitz continuity) There exists $C_L \in [0, \infty[$ such that $\Delta V \cdot C_L < 1$ and

$$
\begin{aligned}
& \|\mathbb{B}(t,x) - \mathbb{B}(t,y)\|_H + \|\mathbb{D}(t,x) - \mathbb{D}(t,y)\|_{L_2(U,H)} \\
& + \left(\int_Z \|\mathbb{G}(t,x,z) - \mathbb{G}(t,y,z)\|_H^2 \, m(dz) \right)^{1/2} \leq C_L \|x - y\|_H \quad \text{on } \Omega
\end{aligned}
$$

for all $t \in [0,T]$ and $x, y \in V$.

(H3) **(Boundedness in** 0**)**

$$\|\mathbb{B}(t,0)\|_H + \|\mathbb{D}(t,0)\|_{L_2(U,H)} + \left(\int_Z \|\mathbb{G}(t,0,z)\|_H^2 \, m(dz)\right)^{1/2} \leq f(t)$$

on Ω for all $t \in [0,T]$.

(H4) **(Coercivity)** There exists $C_C \in]0,\infty[$ such that

$$_{V^*}\langle v,x\rangle_V \geq C_C \|x\|_V^2 + f(t)$$

for all $(t,\omega) \in [0,T] \times \Omega$, $x \in V$ and $v \in \mathbb{A}(t,x)$.

(H5) **(Boundedness)** There exists $C_B \in]0,\infty[$ such that

$$\left\|\mathbb{A}^0(t,x)\right\|_{V^*} \leq C_B \|x\|_V + f(t)$$

on Ω for all $x \in V,\ t \in [0,T]$.

Remark 5.4. *i. Conditions (H2) and (H3) imply a linear growth condition on $\mathbb{B}$, $\mathbb{D}$ and $\mathbb{G}$, i.e.*

$$\|\mathbb{B}(t,x)\|_H^2 + \|\mathbb{D}(t,x)\|_{L_2(U,H)}^2 + \int_Z \|\mathbb{G}(t,x,z)\|_H^2 \, m(dz) \leq C \|x\|_H^2 + f^2$$

for all $x \in V$ on $[0,T] \times \Omega$.

ii. We want to emphasize that the condition $\Delta V \cdot C_L < 1$ in (H2) implies that the jumps of V and consequently the jumps of N are bounded. This restriction is necessary in order to satisfy Conditions (G2) and (G3) in Theorem 5.7 below.

5.2 The Yosida Approximation Approach

As in Chapter 4, we consider the family of approximating equations

$$\begin{cases} dX_\lambda(t) = [\mathbb{B}(t,X_\lambda(t)) - \mathbb{A}_\lambda(t,X_\lambda(t))]\, dV(t) + D(t,X_\lambda(t-))\, dW(t) \\ \qquad\qquad + \int_Z G(t,X_\lambda(t-),z)\bar{\mu}(dt,dz) \\ X_\lambda(0) = X_0 \end{cases} \tag{5.4}$$

with corresponding solutions $\{X_\lambda\}_{\lambda>0}$, where $\mathbb{A}_\lambda$ is the Yosida approximation of $\mathbb{A}$.

The main result of this chapter is stated in the following theorem.

Theorem 5.5. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$ and $\mathbb{G}$ satisfy Conditions (H1)-(H5). Then, there exists a solution to Problem* (5.1) *in the sense of Definition 5.2 being the weak limit of $\{X_\lambda\}_{\lambda>0}$ in $L^2([0,T] \times \Omega,\ dV \otimes P; V)$.*

Again, we begin with the proof of the existence and uniqueness of the solution to the approximating equation (5.4).

Proposition 5.6. *Suppose assumptions (H1) - (H5) hold, then there exists a unique solution* $X_\lambda \in L^2([0,T]\times\Omega,\ dV\otimes P; V)$ *to Problem* (5.4) *such that* $t \mapsto X_\lambda(t)$ *is cádlág in* H *and*

$$E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^2\right] < \infty$$

for every $\lambda > 0$.

In order to prove this proposition, we will apply the main result in [Gyö82]:

Theorem 5.7. *Let* $V \subset H \subset V^*$ *be a Gelfand triple and* U *be a further separable Hilbert space. Let* M *be a quasi-left-continuous square integrable martingale taking values in* U. *Furthermore, let* N *and* V *be adapted non-decreasing real-valued cádlág processes such that* $dV(t) \geq dN(t)$ *and* $dV(t) \geq d\langle M\rangle_t$. *Define* $Q_t := \tilde{Q}_t \frac{d\langle M\rangle_t}{dV(t)}$ *where* $\tilde{Q}_t$ *is the martingale covariance of* M. *Let*

$$\begin{aligned} &A : [0,T]\times\Omega\times V \to V^*,\\ &B : [0,T]\times\Omega, \times V \to L_2(Q_t^{\frac{1}{2}}(U), H), \end{aligned}$$

such that the operators $\mathcal{A} := A\frac{dN}{dV}$ *and* $\mathcal{B} := BQ_t^{\frac{1}{2}}$ *are progressively measurable and for* $L,\ R,\ \varepsilon > 0$ *and* $g \in L^1([0,T]\times\Omega, dV\otimes P)$, $X_0 \in L^2(\Omega, P; H)$ *such that* $\Delta V L < 1$ *(for every* $\omega\in\Omega$, $t\in[0,T]$*), the following assumptions are satisfied* $P\otimes dt$*-a.s.:*

(G1) $\mathcal{A}$ *is demicontinuous,*

(G2) For every $v_1,\ v_2 \in V$

$$\begin{aligned} &{}_{V^*}\langle \mathcal{A}(v_1) - \mathcal{A}(v_2), v_1 - v_2\rangle_V + \|\mathcal{B}(v_1) - \mathcal{B}(v_2)\|_{L_2(U,H)}^2\\ \leq&2L\,\|v_1 - v_2\|_H^2, \end{aligned}$$

(G3) For every $v \in V$

$$2\,{}_{V^*}\langle \mathcal{A}(v), v\rangle_V + \|\mathcal{B}(v)\|_{L_2(U,H)}^2 \leq g + 2L\,\|v\|_H^2 - \varepsilon\,\|v\|_V^2,$$

(G4) For every $v \in V$

$$\|\mathcal{A}(v)\|_{V^*}^2 \leq g + R\,\|v\|_V^2.$$

Then, the equation for all $t \in [0,T]$

$$X(t) = X_0 + \int_{]0,t]} A(s, X(s))\, dN(s) + \int_{]0,t]} B(s, X(s))\, dM(s) \quad P\text{-a.s.}$$

admits a unique strong solution $X \in L^2([0,T] \times \Omega, dV \otimes P; V)$ such that $t \mapsto X(t)$ is $(\mathcal{F})_t$-adapted, cádlág in H.

Proof. The uniqueness is given by [Gyö82, Theorem 2.9]. The existence follows by [Gyö82, Theorem 2.10]. □

Remark 5.8. *The existence and uniqueness result in [Gyö82] only covers the case where the exponent α in Hypotheses (H4) and (H5) of Chapter 4 equals to 2. As a consequence, in this chapter we are restricted to examination of the case $\alpha = 2$.*

Proof of Proposition 5.6. By Theorem 2.24, the compensated Poisson random measure $\bar{\mu}$ can be identified with a square integrable martingale $M_{\bar{\mu}}$ on a separable Hilbert space $\tilde{U}$ such that $\tilde{U}_0 = Q^{\frac{1}{2}}(\tilde{U})$, where Q is the covariance operator of $M_{\bar{\mu}}$. By Theorem 2.24, $M_{\bar{\mu}}$ is also a Lévy process. Thus, by Proposition D.6, $M_{\bar{\mu}}$ is quasi-left-continuous. Since by Proposition 1.10, $W \in \mathcal{M}_T^2(U)$, it follows that the process $M := W + M_{\bar{\mu}}$ is a square integrable martingale.

Furthermore, we define $\mathcal{A} := \mathbb{B} - \mathbb{A}_\lambda$ and $\mathcal{B} := \mathbb{D} + I_{\mathbb{G}}^H$, where

$$I_{\mathbb{G}}^H(t,x,\omega)(\varphi) := \int_Z \mathbb{G}(t,x,z,\omega)\varphi(z)\, m(dz), \quad \varphi \in L^2(Z, \mathcal{Z}, m)$$

(cf. Proposition 2.25). In analogy to Proposition 4.7, it follows that assumptions (H1) – (H5) imply (G1) – (G4) of Theorem 5.7. Hence, Theorem 5.7 is applicable and (5.4) admits a unique solution such that $t \mapsto X_\lambda(t)$ is cádlág in H. □

For the calculation of an a priori estimate, we will need the following Itô-formula based upon [GK82].

Theorem 5.9. *Let V be an adapted non-decreasing real-valued cádlág process, $\alpha \in]1,\infty[$, $X_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ and let $A \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega, dV \otimes P; V^*)$, $D \in L^2([0,T] \times \Omega, dt \otimes P; L_2(U,H))$ and $G \in L^2([0,T] \times \Omega \times Z, dt \otimes P \otimes m; H)$ all be progressively measurable. Define*

$$X(t) := X_0 + \int_{]0,t]} A(s)\, dV(s) + \int_{]0,t]} D(s)\, dW(s) + \int_{]0,t]} \int_Z G(s,z) \bar{\mu}(ds,dz)$$

such that for a $dV(t) \otimes P$-version we have $X \in L^\alpha([0,T] \times \Omega, dV(t) \otimes P; V)$. Then X is an H-valued cádlág $(\mathcal{F}_t)$-adapted process,

$$E\left[\sup_{t \in]0,T]} \|X(t)\|_H^2\right] < \infty$$

and P-a.s., the following Itô-formula holds:

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_{]0,t]} \left(2\,{}_{V^*}\langle A(s), X(s)\rangle_V - \Delta V(s)\, \|A(s)\|_H^2\right)\, dV(s)$$
$$+ \int_{]0,t]} \|D(s)\|_{L_2(U,H)}^2\, ds + \int_{]0,t]}\int_Z \|G(s,z)\|_H^2\, \mu(ds,dz) + 2M(t), \tag{5.5}$$

where

$$M(t) := \int_{]0,t]} \langle X(s-), D(s)\, dW(s)\rangle_H + \int_{]0,t]}\int_Z \langle X(s-), G(s,z)\rangle_H \bar{\mu}(ds,dz)$$

is a cádlág real-valued local martingale.

Proof. We apply [GK82, Theorem 2] with

$$h(s) := \int_{]0,t]} D(s)\, dW(s) + \int_{]0,t]}\int_Z G(s,z)\bar{\mu}(ds,dz).$$

and obtain that X is an H-valued cádlág process and P-a.s. the following Itô formula holds:

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_{]0,t]} \left(2\,{}_{V^*}\langle A(s), X(s)\rangle_V - \Delta V(s)\, \|A(s)\|_H^2\right)\, dV(s)$$
$$+ \int_{]0,t]} \langle X(s-),\ dh(s)\rangle_H + [h]_t,$$

where $[h]_t$ is the square bracket of h (cf. Definition D.11). Since $D \in \mathcal{N}_W(0,T)$ and $X(t-)$ as an $(\mathcal{F}_t)$-adapted cádlág process is predictable, it follows from Lemma 2.12 that the stochastic integral $\int_{]0,t]} \langle X(s-), D(s)\, dW(s)\rangle_H$ is well-defined as a continuous real-valued local martingale. Furthermore, since $G \in \mathcal{N}_{\bar{\mu}}^2(T, Z; H)$, the process $\Phi_G : t \mapsto \langle X(t-), G(t,\cdot)\rangle_H$ is predictable and

$$E\left[\int_{]0,T]}\int_Z |\langle X(s-), G(s,z)\rangle_H|^2\, m(dz)ds\right]$$
$$\le \sup_{t\in[0,T]} \|X(t)\|_H^2\, E\left[\int_{]0,T]}\int_Z \|G(s,z)\|_H^2\ m(dz)ds\right] < \infty.$$

Hence, $\Phi_G \in \mathcal{N}_{\bar{\mu}}^2(T, Z; \mathbb{R})$. By Proposition 2.20, we obtain that the integral

$$\int_{]0,t]} \langle X(s-), dh(s)\rangle_H$$
$$= \int_{]0,t]} \langle X(s-), D(s)\, dW(s)\rangle_H + \int_{]0,t]}\int_Z \langle X(s-), G(s,z)\rangle_H \bar{\mu}(ds,dz),$$

is well-defined and, by Proposition 2.16 and Proposition 2.19, is a real-valued square integrable cádlág martingale.

Furthermore, since the stochastic integral $I_D(t) := \int_{]0,t]} D(s)\, dW(s)$ is continuous, from Lemma 2.13 we deduce

$$[I_D(t)] = \langle I_D(t) \rangle = \int_{]0,t]} \|D(s)\|^2_{L_2(U,H)}\, ds.$$

Therefore, by Corollary 2.23 we obtain

$$[h] = \int_{]0,t]} \|D(s)\|^2_{L_2(U,H)}\, ds + \int_{]0,t]} \int_Z \|G(s,z)\|^2_H\, \mu(ds,dz).$$

Thus, (5.5) is valid.

It remains to prove that

$$E\left[\sup_{t\in]0,T]} \|X(t)\|^2_H\right] < \infty.$$

To this end, we observe that since $\Delta V(s) \geq 0$, using Hölder's inequality from (5.5) we infer that

$$\begin{aligned}
&\|X(t)\|^2_H \\
&\leq \|X_0\|^2_H + 2\left(\int_{]0,T]} \|A(s)\|^{\frac{\alpha}{\alpha-1}}_{V^*}\, dV(s)\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{]0,T]} \|X(s)\|^{\alpha}_V\, dV(s)\right)^{\frac{1}{\alpha}} \\
&\quad + 2\int_{]0,t]} \langle X(s-), D(s)\, dW(s)\rangle_H + 2\int_{]0,t]} \int_Z \langle X(s-), G(s,z)\rangle_H \bar{\mu}(ds,dz) \\
&\quad + \int_{]0,T]} \|D(s)\|^2_{L_2(U,H)}\, ds + \int_{]0,T]} \int_Z \|G(s,z)\|^2_H\, \mu(ds,dz).
\end{aligned} \tag{5.6}$$

Defining

$$\tau_n := \inf\left\{t \in [0,T] \;\middle|\; \|X(t)\|_H > n\right\} \wedge T \quad \forall n \in \mathbb{N},$$

Theorem D.13 implies that τ_n is a stopping time. Note that $\tau_n \to T$ for $n \to \infty$ P-a.s.. By the Burkholder–Davis–Gundy inequality (cf. Theorem

D.14), Lemma 2.13 and Young's inequality, we obtain

$$
\begin{aligned}
&2\, E\left[\sup_{t\in[0,\tau_n]}\left|\int_{]0,t]}\langle X(s-), D(s)\; dW(s)\rangle_H\right|\right]\\
\leq &C\, E\left[\left[\int_{]0,t]}\langle X(s-), D(s)\; dW(s)\rangle_H\right]_{\tau_n}^{1/2}\right]\\
\leq &C\, E\left[\left(\int_{]0,\tau_n]}\|X(s-)\|_H^2\,\|D(s)\|_{L_2(U,H)}^2\; ds\right)^{1/2}\right]\\
\leq &\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X(s)\|_H^2\right]+CE\left[\int_{]0,T]}\|D(s)\|_{L_2(U,H)}^2\; ds\right],
\end{aligned}
\tag{5.7}
$$

where we have used $\sup_{s\in[0,t]}\|X(s-)\|_H^2\leq\sup_{s\in[0,t]}\|X(s)\|_H^2\,,\ t\in[0,T]$, in the last step.
By Theorem D.14, Young's inequality, Corollary 2.23 and Proposition 2.21, we deduce

$$
\begin{aligned}
&2E\left[\sup_{t\in[0,\tau_n]}\left|\int_{]0,t]}\int_Z\langle X(s-), G(s,z)\rangle_H\bar{\mu}(ds,dz)\right|\right]\\
\leq &CE\left[\left[\int_{]0,\cdot]}\int_Z\langle X(s-), G(s,z)\rangle_H\;\bar{\mu}(ds,dz)\right]_{\tau_n}^{\frac{1}{2}}\right]\\
\leq &CE\left[\sup_{s\in[0,\tau_n]}\|X(s)\|_H\left[\int_{]0,\cdot]}\int_Z\|G(s,z)\|_H\;\bar{\mu}(ds,dz)\right]_{\tau_n}^{\frac{1}{2}}\right]\\
\leq &\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X(s)\|_H^2\right]+CE\left[\left[\int_{]0,\cdot]}\int_Z\|G(s,z)\|_H\;\bar{\mu}(ds,dz)\right]_{\tau_n}\right]\\
= &\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X(s)\|_H^2\right]+CE\left[\int_{]0,\tau_n]}\int_Z\|G(s,z)\|_H^2\;\mu(dz,\;ds)\right]\\
\leq &\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X(s)\|_H^2\right]+CE\left[\int_{]0,T]}\int_Z\|G(s,z)\|_H^2\; m(dz)\; ds\right].
\end{aligned}
\tag{5.8}
$$

Furthermore, by Proposition 2.21 we have

$$
E\left[\int_{]0,T]}\int_Z\|G(s,z)\|_H^2\;\mu(ds,dz)\right]=E\left[\int_{]0,T]}\int_Z\|G(s,z)\|_H^2\; m(dz)ds\right].
\tag{5.9}
$$

In (5.6), we take the supremum over $[0,\tau_n]$ and then apply the expectation. After using (5.7), (5.8) and (5.9), and the integrability assumptions of A,

D, G and X, we finally arrive at

$$E\left[\sup_{s\in[0,\tau_n]} \|X(s)\|_H^2\right] \leq C,$$

where C is independent of τ_n. Letting $n \to \infty$, we conclude the proof.

□

Corollary 5.10. *In the situation of Theorem 5.9, we have*

$$\begin{aligned}
&E\left[\|X(t)\|_H^2\right] - E\left[\|X_0\|_H^2\right] \\
&= \int_{]0,t]} E\left[2\,{}_{V^*}\langle A(s), X(s)\rangle_V - \Delta V(s)\,\|A(s)\|_H^2\right] dV(s) \\
&\quad + \int_{]0,t]} E\left[\|D(s)\|_{L_2(U,H)}^2 + \int_Z \|G(s,X(s),z)\|_H^2\, m(dz)\right] ds, \quad t \in]0,T].
\end{aligned}$$

Proof. At first, we note that by Lemma 2.12 and Proposition 2.16 the process

$$M(t) := \int_{]0,t]} \langle X(s-), D(s)\, dW(s)\rangle_H + \int_{]0,t]}\int_Z \langle X(s-), G(s,z)\rangle_H \bar{\mu}(ds,dz)$$

is a real-valued local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of M. Then for all $n \in \mathbb{N}$ and $t \in [0,T]$, by Theorem 5.9 we have

$$\begin{aligned}
&E\left[\|X(t\wedge\tau_n)\|_H^2\right] - E\left[\|X_0\|_H^2\right] \\
&= \int_{]0,t]} E\left[1_{]0,\tau_n]}(s)\left(2\,{}_{V^*}\langle A(s), X(s)\rangle_V - \Delta V(s)\,\|A(s)\|_H^2\right)\right] dV(s) \\
&\quad + \int_{]0,t]} E\left[1_{]0,\tau_n]}(s)\left(\|D(s)\|_{L_2(U,H)}^2 + \int_Z \|G(s,X(s),z)\|_H^2\, m(dz)\right)\right] ds.
\end{aligned} \tag{5.10}$$

Since by Theorem 5.9, $E\left[\sup_{t\in]0,T]} \|X(t)\|_H^2\right] < \infty$, and since the integrands on the right-hand side of (5.10) are $dt \otimes P$-integrable, we can apply Lebesgue's dominated convergence theorem to obtain the assertion. □

Applying Theorem 5.9 to the unique solution X_λ of the approximating equation, we obtain the following corollary:

Corollary 5.11. *For every $\lambda > 0$, we have*

$$E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^2\right] < \infty.$$

Now, we turn to the calculation of an a priori estimate. In the following, let us abbreviate $\zeta_\lambda := \mathbb{B} - \mathbb{A}_\lambda$.

Proposition 5.12. *Assuming (H1) - (H5), then there exists a constant $C > 0$ (independent of λ) such that*

$$\begin{aligned} E\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_H^2\right] + \int_{]0,T]} E\Big[\|X_\lambda(t)\|_V^2 \\ + \Delta V(t) \, \|\zeta_\lambda(t, X_\lambda(t))\|_H^2 \Big] \, dV(t) < C \end{aligned} \tag{5.11}$$

for all $\lambda > 0$.

Proof. Let X_λ be the unique solution of Problem (5.4). From Theorem 5.9 we deduce

$$\begin{aligned} \|X_\lambda(t)\|_H^2 = \|X_0\|_H^2 &+ \int_{]0,t]} 2 \, {}_{V^*}\langle \mathbb{B}(s, X_\lambda(s)) - \mathbb{A}_\lambda(s, X_\lambda(s)), X_\lambda(s)\rangle_V \, dV(s) \\ &- \int_{]0,t]} \Delta V(s) \, \|\zeta_\lambda(s, X_\lambda(s))\|_H^2 \, dV(s) \\ &+ 2 \int_{]0,t]} \langle X_\lambda(s), D(s, X_\lambda(s)) \, dW(s)\rangle_H \\ &+ 2 \int_{]0,t]} \int_Z \langle X_\lambda(s-), G(s, X_\lambda(s), z)\rangle_H \bar{\mu}(ds, dz) \\ &+ \int_{]0,t]} \|D(s, X_\lambda(s))\|_{L_2(U,H)}^2 \, ds \\ &+ \int_{]0,t]} \int_Z \|G(s, X_\lambda(s), z)\|_H^2 \, \mu(ds, dz). \end{aligned} \tag{5.12}$$

Let

$$\tau_n := \inf\left\{t \in [0,T] \;\middle|\; \|X_\lambda(t)\|_H > n\right\} \wedge T, \quad \forall n \in \mathbb{N},$$

be a stopping time (cf. Theorem D.13) such that $\lim_{n\to\infty} \tau_n = T$ P-a.s. for all $n \in \mathbb{N}$. In analogy to (5.7), by Theorem D.14, Lemma 2.13, Young's inequality and Remark 5.4, we obtain

$$\begin{aligned} &2\, E\left[\sup_{r\in[0,\tau_n]} \left| \int_{]0,r]} \langle X_\lambda(s-), D(s, X_\lambda(s)) \, dW(s)\rangle_H \right|\right] \\ \le& \frac{1}{4} E\left[\sup_{s\in[0,\tau_n]} \|X_\lambda(s)\|_H^2\right] + C \int_{]0,\tau_n]} E\left[\|\mathbb{D}(s, X_\lambda(s))\|_{L_2(U,H)}^2\right] dV(s) \\ \le& \frac{1}{4} E\left[\sup_{s\in[0,\tau_n]} \|X_\lambda(s)\|_H^2\right] + C \int_{]0,\tau_n]} E\left[\|X_\lambda(s)\|_H^2 + f^2(s)\right] dV(s), \end{aligned} \tag{5.13}$$

where $f \in L^2([0,T], dV)$.

In analogy to (5.8), by Theorem D.14, Young's inequality, Corollary 2.23, Proposition 2.21 and Remark 5.4, we deduce

$$\begin{aligned}
&2E\left[\sup_{t\in[0,\tau_n]}\left|\int_{]0,t]}\int_Z \langle X_\lambda(s-), G(s,X_\lambda(s),z)\rangle_H \bar{\mu}(ds,dz)\right|\right] \\
\leq&\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X_\lambda(s)\|_H^2\right] + C\int_{]0,\tau_n]}\int_Z E\left[\|G(s,X_\lambda(s),z)\|_H^2\right]\, m(dz)\, ds \\
\leq&\frac{1}{4}E\left[\sup_{s\in[0,\tau_n]}\|X_\lambda(s)\|_H^2\right] + C\int_{]0,\tau_n]} E\left[\|X_\lambda(s)\|_H^2 + f^2(s)\right]\, dV(s).
\end{aligned} \tag{5.14}$$

Furthermore, by Proposition 2.21 and Remark 5.4 we have

$$\begin{aligned}
&E\left[\int_{]0,\tau_n]}\int_Z \|G(s,X_\lambda(s),z)\|_H^2\, \mu(ds,dz) + \int_{]0,\tau_n]}\|D(s,X_\lambda(s))\|_{L_2(U,H)}^2\, ds\right] \\
=&E\left[\int_{]0,\tau_n]}\int_Z \|G(s,X_\lambda(s),z)\|_H^2\, m(dz)\, ds + \int_{]0,\tau_n]}\|D(s,X_\lambda(s))\|_{L_2(U,H)}^2\, ds\right] \\
=&E\left[\int_{]0,\tau_n]}\left(\int_Z \|\mathbb{G}(s,X_\lambda(s),z)\|_H^2\, m(dz) + \|\mathbb{D}(s,X_\lambda(s))\|_{L_2(U,H)}^2\right)\, dV(s)\right] \\
\leq&C\int_{]0,\tau_n]} E\left[\|X_\lambda(s)\|_H^2 + f^2(s)\right] dV(s).
\end{aligned} \tag{5.15}$$

By (H3) and Remark 5.4, we infer that

$$\begin{aligned}
&{}_{V^*}\langle \mathbb{B}(s,X_\lambda(s)) - \mathbb{A}(s,X_\lambda(s)), X_\lambda(s)\rangle_V \\
\leq& -C_C\,\|X_\lambda(s)\|_V^2 + C\,\|X_\lambda(s)\|_H^2 + f^2(s)
\end{aligned} \tag{5.16}$$

for a.e. $(\omega,t) \in \Omega\times[0,T]$. Applying (5.13)-(5.16) to (5.12), we finally arrive at

$$\begin{aligned}
&E\left[\sup_{s\in[0,\tau_n]}\|X_\lambda(s)\|_H^2\right] + C_C\, E\left[\int_{]0,\tau_n]}\|X_\lambda(s)\|_V^2\, dV(s)\right] \\
\leq&2\, E\left[\|X_0\|_H^2\right] - \int_{]0,\tau_n]} E\left[\Delta V(s)\,\|\zeta_\lambda(s,X_\lambda(s))\|_H^2\right]\, dV(s) \\
&+C\left(\int_{]0,\tau_n]}\sup_{r\in[0,s]} E\left[\|X_\lambda(r)\|_H^2\right]\, dV(s) + \int_{]0,T]} E\left[f^2(s)\right]\, dV(s)\right).
\end{aligned}$$

Note that the subtracted term $\frac{1}{2}E\left[\sup_{s\in[0,\tau_n]}\|X_\lambda(s)\|_H^2\right]$ is finite by the choice of τ_n. Applying the Bellman-Gronwall inequality and Lebesgue's dominated convergence theorem for $n \to \infty$, we finish the proof. $\square$

Corollary 5.13.

$$\limsup_{\lambda\to 0}\int_{]0,T]} E\Bigg[\|\mathbb{A}_\lambda(s,X_\lambda(s))\|_{V^*}^2 + \|\mathbb{B}(s,X_\lambda(s))\|_{V^*}^2 \\ + \|\mathbb{D}(s,X_\lambda(s))\|_{L_2(U,H)}^2 + \int_Z \|\mathbb{G}(s,X_\lambda(s),z)\|_H^2\, m(dz)\Bigg]\, dV(s) < \infty.$$

Proof. Because $\mathcal{D}(\mathbb{A}) = V$ and $\|\mathbb{A}_\lambda(\cdot,x)\|_{V^*} \le \left\|\mathbb{A}^0(\cdot,x)\right\|_{V^*}$ for all $x \in \mathcal{D}(\mathbb{A})$ on $[0,T]$, by (H5) and Remark 5.4 we have

$$\|\mathbb{A}_\lambda(s,X_\lambda(s))\|_{V^*}^2 + \|\mathbb{B}(s,X_\lambda(s))\|_{V^*}^2 + \|\mathbb{D}(s,X_\lambda(s))\|_{L_2(U,H)}^2 \\ + \int_Z \|\mathbb{G}(s,X_\lambda(s),z)\|_H^2\, m(dz) \le C\left(\|X_\lambda(s)\|_V^2 + f^2(s)\right).$$

Hence, Proposition 5.12 implies the assertion. $\square$

5.3 Existence, Uniqueness and Convergence

In the existence proof, we are going to use the integration by parts–formula for (discontinuous) semimartingales (see Step 2 in the proof of Theorem 5.5 below). In order to control the jump part of the solution, which occurs in (5.36) below, we need the following strong convergence result.

Proposition 5.14. *Suppose conditions (H1)-(H5) hold, then for any sequences (λ), (μ) such that $\lambda, \mu \to 0$ there exists $C > 0$ such that*

$$E\left[\sup_{s\in[0,T]}\|X_\lambda(s) - X_\mu(s)\|_H^2\right] \le C\cdot(\lambda+\mu).$$

Proof. Applying Theorem 5.9 to $X_\lambda(t) - X_\mu(t)$ yields

$$\begin{aligned}
&\|X_\lambda(t) - X_\mu(t)\|_H^2 - \|X_0 - X_0\|_H^2 \\
=&2\int_{]0,t]} {}_{V^*}\langle \mathbb{B}(s, X_\lambda(s)) - \mathbb{B}(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \; dV(s) \\
&- 2\int_{]0,t]} {}_{V^*}\langle \mathbb{A}_\lambda(s, X_\lambda(s)) - \mathbb{A}_\mu(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \; dV(s) \\
&- 2\int_{]0,t]} \Delta V(s) \, \|\zeta_\lambda(s, X_\lambda(s)) - \zeta_\mu(s, X_\mu(s))\|_H^2 \; dV(s) \\
&+ \int_{]0,t]} \|\mathbb{D}(s, X_\lambda(s)) - \mathbb{D}(s, X_\mu(s))\|_{L_2(U,H)}^2 \; dV(s) \\
&+ \int_{]0,t]} \int_Z \|\mathbb{G}(s, X_\lambda(s), z) - \mathbb{G}(s, X_\mu(s), z)\|_H^2 \; m(dz) \; dV(s) + M(t),
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
M(t) :=& \int_{]0,t]} \langle X_\lambda(s-) - X_\mu(s-), \mathbb{D}(s, X_\lambda(s)) - \mathbb{D}(s, X_\mu(s)) \, dW(s)\rangle_H \\
&+ \int_{]0,t]} \int_Z \langle X_\lambda(s-) - X_\mu(s-), \mathbb{G}(s, X_\lambda(s)) - \mathbb{G}(s, X_\mu(s))\rangle_H \; \bar{\mu}(ds, dz).
\end{aligned}$$

By Remark 5.4, we have

$$\begin{aligned}
&{}_{V^*}\langle \mathbb{B}(s, X_\lambda(s)) - \mathbb{B}(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \\
&+ \|\mathbb{D}(s, X_\lambda(s)) - \mathbb{D}(s, X_\mu(s))\|_{L_2(U,H)}^2 \\
&+ \int_Z \|\mathbb{G}(s, X_\lambda(s), z) - \mathbb{G}(s, X_\mu(s), z)\|_H^2 \; m(dz) \\
\leq& C \|X_\lambda(s) - X_\mu(s)\|_H^2 .
\end{aligned}$$

Furthermore, by (4.28) we have

$$\begin{aligned}
&- {}_{V^*}\langle \mathbb{A}_\lambda(s, X_\lambda(s)) - \mathbb{A}_\mu(s, X_\mu(s)), X_\lambda(s) - X_\mu(s)\rangle_V \\
\leq& \frac{\lambda}{4} \|\mathbb{A}_\mu(s, X_\mu(s))\|_{V^*}^2 + \frac{\mu}{4} \|\mathbb{A}_\lambda(s, X_\lambda(s))\|_{V^*}^2 .
\end{aligned}$$

Again, we localize $M(t)$ with the stopping time

$$\tau_n := \inf \left\{ t \in [0, T] \;\middle|\; \|X(t)\|_H > n \right\} \wedge T, \quad n \in \mathbb{N},$$

and estimate it by the Burkholder–Davis–Gundy inequality in a similar way as we did in the proof of Proposition 5.12 and obtain

$$\begin{aligned}
2E\left[\sup_{t\in]0,\tau_n]} |M(t)|\right] \leq& \frac{1}{2} E\left[\sup_{s\in]0,\tau_n]} \|X_\lambda(s) - X_\mu(s)\|_H^2\right] \\
&+ C\int_{]0,\tau_n]} E\left[\|X_\lambda(s) - X_\mu(s)\|_H^2 + f^2(s)\right] \; dV(s)
\end{aligned}$$

for some $f \in L^2([0,T] \times \Omega, dV \otimes P)$. Let $X^n(t) := X(\tau_n \wedge t),\ t \in [0,T]$. Since

$$\Delta V(s) \, \|\zeta_\lambda(s, X_\lambda(s)) - \zeta_\mu(s, X_\mu(s))\|_H^2 \geq 0,$$

after taking the supremum over $[0,T]$ and expectation in (5.17), we finally arrive at

$$\begin{aligned}
&E\left[\sup_{s\in[0,T]} \left\|X_\lambda^n(s) - X_\mu^n(s)\right\|_H^2\right] \\
\leq &C\bigg(\int_{]0,T]} E\left[\sup_{r\in[0,s]} \left\|X_\lambda^n(r) - X_\mu^n(r)\right\|_H^2\right] \, dV(s) \\
&+ \mu \int_{]0,T]} E\left[\|\mathbb{A}_\lambda(s, X_\lambda(s))\|_{V^*}^2\right] \, dV(s) \\
&+ \lambda \int_{]0,T]} E\left[\|\mathbb{A}_\mu(s, X_\mu(s))\|_{V^*}^2\right] \, dV(s)\bigg).
\end{aligned}$$

By Corollary 5.13, we have

$$\limsup_{\lambda\to 0} \int_0^T E\left[\|A_\lambda(s, X_\lambda(s))\|_{V^*}^2\right] \, dV(s) < \infty.$$

Consequently, Bellman-Gronwall inequality implies that

$$E\left[\sup_{s\in[0,T]} \left\|X_\lambda^n(s) - X_\mu^n(s)\right\|_H^2\right] \leq C \cdot (\lambda + \mu). \tag{5.18}$$

Note that the constant C in (5.18) is independent of n, λ and μ. Now let $n \to \infty$ to get (5.18) without n. □

Corollary 5.15. *There exists a process $X \in L^2([0,T] \times \Omega; H)$ such that*

$$\lim_{\lambda\to 0} E\left[\sup_{t\in[0,T]} \|X_\lambda(t) - X(t)\|_H^2\right] = 0. \tag{5.19}$$

For the proof of Theorem 5.5, we also need the following lemma:

Lemma 5.16. *Let $\beta \in [0, C_L]$ with C_L as defined in (H2). Then, there exists a unique non-increasing process θ such that*

$$d\theta(t) = -\beta\theta(t-) \, dV(t), \quad \theta(0) = 1,$$

i.e.

$$\theta(t) = 1 - \beta \int_{]0,t]} \theta(s-) \, dV(s). \tag{5.20}$$

Furthermore, $\theta(t) \in [0,1]$ for all $t \in [0,T]$.

Proof. Let $X := -\beta V$. Since V is an increasing process, the process $X := -\beta V$ is clearly a semimartingale. Thus, Theorem D.15 implies that its Dolean exponential defined by

$$\theta(t) := \exp(X(t) - \frac{1}{2}[X](t)) \prod_{s \le t} (1 + \Delta X(s)) e^{-\Delta X(s) + \frac{1}{2}(\Delta X(s))^2} \tag{5.21}$$

satisfies

$$\theta(t) = 1 + \int_{]0,t]} \theta(s-)\, dX(s).$$

Furthermore, since X is decreasing, (5.21) implies that θ is also decreasing. Since $\theta(0) = 1$, we conclude that $\theta \le 1\ \forall t \ge 0$. Since $C_L \Delta V < 1$, from (5.21), we infer that $\theta \ge 0\ \forall t \ge 0$. □

Proof of Theorem 5.5

We divide the proof into several steps.
Step 1 (Weak convergence).

By Corollary C.8, the spaces $L^2([0,T] \times \Omega,\ dV \otimes P; L^2(U,H)),\ L^2([0,T] \times \Omega,\ dV \otimes P; V),\ L^2([0,T] \times \Omega,\ dV \otimes P; V^*)$ are reflexive. Hence, by Proposition 5.12, Corollary 5.13 and the Banach-Alaoglu-Theorem, yields the following convergences along some subsequence:

$$\begin{aligned} X_\lambda \to X \text{ weakly in } L^2([0,T] \times \Omega,\ dV \otimes P; V) \\ \text{and weakly star in } L^2(\Omega, P; L^\infty([0,T], dV; H)), \end{aligned} \tag{5.22}$$

$$\mathbb{A}_\lambda(\cdot, X_\lambda) \to \eta^* \text{ weakly in } L^2([0,T] \times \Omega, dV \otimes P; V^*), \tag{5.23}$$

$$\mathbb{B}(\cdot, X_\lambda) \to \bar{\mathbb{B}} \text{ weakly in } L^2([0,T] \times \Omega, dV \otimes P; V^*), \tag{5.24}$$

$$\mathbb{D}(\cdot, X_\lambda) \to \bar{\mathbb{D}} \text{ weakly in } L^2([0,T] \times \Omega, dV \otimes P; L_2(U,H)), \tag{5.25}$$

$$\mathbb{G}(\cdot, X_\lambda, \cdot) \to \bar{\mathbb{G}} \text{ weakly in } L^2([0,T] \times \Omega \times Z,\ dV \otimes P \otimes m; H), \tag{5.26}$$

$$\Delta V \zeta_\lambda(\cdot, X_\lambda) \to \Delta V \bar{\zeta} \text{ weakly in } L^2([0,T] \times \Omega,\ dV \otimes P; H). \tag{5.27}$$

Claim 1.

$$\bar{\zeta} = \bar{\mathbb{B}} - \eta^* \quad dV \otimes P\text{-a.e..}$$

Proof. By the convergences (5.23), (5.24) and (5.27), for any $\varphi \in L^\infty([0,T]\times \Omega, V)$ and some subsequence λ we have

$$
\begin{aligned}
&E\left[\int_{]0,T]} {}_{V^*}\langle \Delta V(t)\left(\bar\zeta(t) - (\bar{\mathbb{B}}(t) - \eta^*(t))\right), \varphi(t)\rangle_V \, dV(t)\right] \\
=&\lim_{\lambda\to 0} E\left[\int_{]0,T]} \langle \Delta V(t)\zeta_\lambda(t, X_\lambda(t)), \varphi(t)\rangle_H \, dV(t)\right] \\
&\quad - \lim_{\lambda\to 0} E\left[\int_{]0,T]} \Delta V(t)\, {}_{V^*}\langle \mathbb{B}(t, X_\lambda(t)) - \mathbb{A}_\lambda(t, X_\lambda(t), \varphi(t)\rangle_V \, dV(t)\right] \\
=&0.
\end{aligned}
$$

□

Claim 2. There exist processes $\bar D \in L^2([0,T]\times\Omega, dt\otimes P; L_2(U,H))$, $\bar G \in L^2([0,T]\times\Omega\times V, dt\otimes P\otimes m; H)$ and $\bar B, \eta \in L^2([0,T]\times\Omega, dN\otimes P, V^*)$ such that $\bar D \varrho_t^{\frac12} = \bar{\mathbb{D}}$, $\bar{\mathbb{G}} = \varrho_t^{\frac12}\bar G$, $\bar{\mathbb{B}} = \varrho_N \bar B$ and $\eta^* = \varrho_N \eta$ $dV\otimes P$-a.s..

Proof. Let $\varphi \in L^\infty([0,T]\times\Omega, dV(t)\otimes P, V)$. By (5.24), we have

$$
\begin{aligned}
&E\left[\int_{]0,T]} {}_{V^*}\langle \bar{\mathbb{B}}(t), \varphi(t)\rangle_V \, dV(t)\right] \\
=&\lim_{\lambda\to 0} E\left[\int_{]0,T]} {}_{V^*}\langle \varrho_N(t) B(t, X_\lambda(t)), \varphi(t)\rangle_V \, dV(t)\right] \\
=&\lim_{\lambda\to 0} E\left[\int_{]0,T]} {}_{V^*}\langle B(t, X_\lambda(t)), \varphi(t)\rangle_V \, dN(t)\right].
\end{aligned}
\tag{5.28}
$$

Since the left-hand side of (5.28) is finite, there exists some $\bar B \in L^2([0,T]\times \Omega,\ dN(t)\otimes P, H)$ such that $B(\cdot, X_\lambda) \rightharpoonup \bar B$ along some subsequence in $L^2([0,T]\times\Omega, dN(t)\otimes P, V)$. Therefore, using (5.28), it follows that

$$
\begin{aligned}
E\left[\int_{]0,T]} {}_{V^*}\langle \bar{\mathbb{B}}(t), \varphi(t)\rangle_V \, dV(t)\right] =& E\left[\int_{]0,T]} {}_{V^*}\langle \bar B(t), \varphi(t)\rangle_V \, dN(t)\right] \\
=& E\left[\int_{]0,T]} {}_{V^*}\langle \varrho_N(t)\bar B(t), \varphi(t)\rangle_V \, dV(t)\right].
\end{aligned}
$$

Hence, $\bar{\mathbb{B}} = \varrho_N \bar B$ $dV\otimes P$-a.s..

Similarly, by (5.26), for $\varphi \in L^2([0,T]\times\Omega\times Z, dV(t)\otimes P\otimes m; H)$ we have

$$
\begin{aligned}
&E\left[\int_{]0,T]}\int_Z \langle \bar{\mathbb{G}}(t),\varphi(t)\rangle_H \; m(dz)dV(t)\right]\\
&=\lim_{\lambda\to 0} E\left[\int_{]0,T]}\int_Z \langle G(t,X_\lambda(t)),\varphi(t)\rangle_H \varrho_t(t)^{\frac12}\; m(dz)dV(t)\right]\\
&\geq \lim_{\lambda\to 0} E\left[\int_{]0,T]}\int_Z \langle G(t,X_\lambda(t)),\varphi(t)\rangle_H \varrho_t(t)\; m(dz)dV(t)\right]\\
&=\lim_{\lambda\to 0} E\left[\int_{]0,T]}\int_Z \langle G(t,X_\lambda(t)),\varphi(t)\rangle_H \; m(dz)\; dt\right],
\end{aligned}
$$

where the inequality is valid since $\varrho_t(t)\in[0,1]$, $t\in[0,T]$. Therefore, there exists some $\bar{G}\in L^2([0,T]\times\Omega\times Z, dt\otimes P\otimes m; H)$ such that along some subsequence

$$G(\cdot,X_\lambda)\to\bar{G}\quad\text{weakly in } L^2([0,T]\times\Omega\times Z, dt\otimes P\otimes m; H) \tag{5.29}$$

and

$$
\begin{aligned}
&E\left[\int_{]0,T]}\int_Z \langle \bar{G}(t),\varphi(t)\rangle_H \varrho_t(t)dV(t)\right]\\
&=\lim_{\lambda\to 0} E\left[\int_{]0,T]}\int_Z \langle G(t,X_\lambda(t)),\varphi(t)\rangle_H \varrho_t(t)dV(t)\right]\\
&=\lim_{\lambda\to 0} E\left[\int_{]0,T]}\int_Z \langle \mathbb{G}(t,X_\lambda(t)),\varphi(t)\rangle_H \varrho_t(t)^{\frac12}dV(t)\right]\\
&=E\left[\int_{]0,T]}\int_Z \langle \bar{\mathbb{G}}(t)\varrho_t(t)^{\frac12},\varphi(t)\rangle_H dV(t)\right].
\end{aligned}
$$

Hence $\bar{G}\varrho_t = \bar{\mathbb{G}}\varrho_t^{\frac12}\; dV(t)\otimes P$-a.s.. Since $\varrho_t\neq 0$ dt-a.s., it follows that $\bar{G}\varrho_t^{\frac12} = \mathbb{G}\; dV(t)\otimes P$-a.s.. The rest of the claim can be proved analogously.

□

Claim 3.

$$\int_{]0,\cdot]}\int_Z G(s,X_\lambda(s),z)\bar{\mu}(ds,dz)\to\int_{]0,\cdot]}\int_Z \bar{G}(s,z)\bar{\mu}(ds,dz) \tag{5.30}$$

and

$$\int_{]0,\cdot]} D(s,X_\lambda(s))\,dW(s)\to\int_{]0,\cdot]}\bar{D}(s)\,dW(s), \tag{5.31}$$

both weakly in $L^\infty([0,T],dt;L^2(\Omega,P;H))$.

Proof. Defining

$$\Phi(g)(t) := \int_{]0,t]} \int_Z g(s,z)\bar{\mu}(ds,dz),$$

for any $g \in L^2([0,T] \times \Omega \times Z, dV \otimes P \otimes m; H)$, by the isometric property of the Poisson integral (cf. Proposition 2.16) it follows that

$$\begin{aligned}
\|\Phi(g)\|^2_{L^\infty([0,T],dt;L^2(\Omega,P;H))} &= \sup_{t\in[0,T]} E\left[\left\|\int_{]0,t]}\int_Z g(s,z)\bar{\mu}(ds,dz)\right\|_H^2\right] \\
&= E\left[\int_{]0,T]}\int_Z \|g(s,z)\|_H^2 \, m(dz)ds\right] \\
&= \|g\|^2_{L^2([0,T]\times\Omega\times Z, dt\otimes P\otimes m;H)}.
\end{aligned}$$

Hence Φ is continuous from $L^\infty([0,T], dt; L^2(\Omega, P; H))$ to $L^2([0,T] \times \Omega \times Z, dt \otimes P \otimes m; H)$. In particular, it is weakly continuous, since it is linear (cf. Proposition 2.20). Consequently, from (5.29) it follows that $\Phi(G(\cdot, X_\lambda)) \to \Phi(\bar{G})$ weakly in $L^\infty([0,T], dt; L^2(\Omega, P; H))$. Using the isometric property of the Wiener integral (cf. Theorem 2.8) and its linearity (cf. Lemma 2.11), we obtain the second assertion. □

Note that the $(\mathcal{F}_t)$-adaptedness of $\int_0^t \eta(s)\, ds$ is obtained in the same way as in the proof of Theorem 4.5.

Step 2. (X, η) satisfy (5.2) P-a.s..

Take $\varphi \in L^\infty([0,T] \times \Omega)$ and $v \in V$. Using (5.4), (5.22) - (5.26), (5.30), (5.31), Claim 2 and Fubini's theorem, we obtain

$$
\begin{aligned}
&E\left[\int_{]0,T]} {}_{V^*}\langle X(t), \varphi(t)v\rangle_V \, dV(t)\right] \\
&= \lim_{\lambda\to 0} E\left[\int_{]0,T]} {}_{V^*}\langle X_\lambda(t), \varphi(t)v\rangle_V \, dV(t)\right] \\
&= \lim_{\lambda\to 0} E\Big[\int_{]0,T]} {}_{V^*}\langle X_0, \varphi(t)v\rangle_V \, dV(t) \\
&\quad + \int_{]0,T]}\int_{]0,t]} {}_{V^*}\langle \mathbb{B}(s, X_\lambda(s)) - \mathbb{A}_\lambda(s, X_{\lambda_n}(s)), \varphi(t)v\rangle_V \, dV(s)dV(t) \\
&\quad + \int_{]0,T]} \left\langle \int_{]0,t]} D(s, X_\lambda(s)) \, dW(s), \varphi(t)v \right\rangle_H dV(t) \\
&\quad + \int_{]0,T]} \left\langle \int_{]0,t]}\int_Z G(s, X_\lambda(s), z)\bar{\mu}(ds, dz), \varphi(t)v \right\rangle_H dV(t)\Big] \\
&= E\Big[\int_{]0,T]} {}_{V^*}\Big\langle X_0 + \int_{]0,t]} \big(\bar{B}(s) - \eta(s)\big) \, dN(s) \\
&\qquad + \int_{]0,t]} \bar{D}(s) \, dW(s) + \int_{]0,t]}\int_Z \bar{G}(s,z)\bar{\mu}(ds,dz), \varphi(t)v \Big\rangle_V dV(t)\Big].
\end{aligned}
\tag{5.32}
$$

Consequently,

$$
\begin{aligned}
X(t) = X_0 &+ \int_{]0,t]} \big(\bar{B}(s) - \eta(s)\big) \, dN(s) + \int_{]0,t]} \bar{D}(s) \, dW(s) \\
&+ \int_{]0,t]}\int_Z \bar{G}(s,z)\bar{\mu}(ds,dz) \quad P-\text{a.s.}.
\end{aligned}
\tag{5.33}
$$

Now, Theorem 5.9 applies and we obtain that X_t is an H-valued, $(\mathcal{F}_t)$-adapted cádlág process.

It remains to show that

$$\bar{\mathbb{B}} = \mathbb{B}(\cdot, X), \bar{\mathbb{D}} = \mathbb{D}(\cdot, X) \text{ and } \bar{\mathbb{G}}(\cdot, z) = \mathbb{G}(\cdot, X, z) \quad \forall z \in Z,\ dV(t)\otimes P-\text{a.e.}.$$

To this end we first observe that by Corollary 5.10, we have

$$
\begin{aligned}
& E\left[\|X(t)\|_H^2\right] - E\left[\|X_0\|_H^2\right] \\
=& E\bigg[\int_{]0,t]} \Big(2\,{}_{V^*}\langle \bar{\mathbb{B}}(s) - \eta^*(s), X(s)\rangle_V - \Delta V(s) \left\|\bar{\mathbb{B}}(s) - \eta^*(s)\right\|_H^2\Big)\, dV(s) \\
& + \int_{]0,t]} \left(\left\|\bar{D}(s)\right\|_{L_2(U,H)}^2 + \int_Z \left\|\bar{G}(s,z)\right\|_H^2 m(dz) \right)\, ds\bigg] \\
=& E\bigg[\int_{]0,t]} 2\,{}_{V^*}\langle \bar{\mathbb{B}}(s) - \eta^*(s), X(s)\rangle_V - \Delta V(s) \left\|\bar{\mathbb{B}}(s) - \eta^*(s)\right\|_H^2 \\
& + \left\|\bar{\mathbb{D}}(s)\right\|_{L_2(U,H)}^2 + \int_Z \left\|\bar{\mathbb{G}}(s,z)\right\|_H^2 m(dz) dV(s)\bigg]
\end{aligned}
$$

Let θ be the solution of the equation $d\theta(t) = -\beta\theta(t-)dV(t)$ with $\theta(0) = 1$ (cf. Lemma 5.16) such that $\theta(t) \in [0,1]$ for all $t \in [0,T]$. By (5.33), $\|X(t)\|_H^2$ is a semimartingale. Furthermore, θ is a process of finite variation. Thus, we can apply the integration by parts–formula for discontinuous processes (cf. Proposition D.16) for $M := \|X\|_H^2$ and $A := \theta$ and obtain

$$
\begin{aligned}
& E\left[\theta(t)\,\|X(t)\|_H^2 - \|X_0\|_H^2\right] \\
=& E\left[\int_{]0,t]} \|X(s-)\|_H^2\, d\theta(s) + \int_{]0,t]} \theta(s)\, d\left(\|X(s)\|_H^2\right)\right] \\
=& E\bigg[\int_{]0,t]} \Big(- \beta\theta(s-)\,\|X(s-)\|_H^2 \\
& + \theta(s)\Big(2\,{}_{V^*}\langle \bar{\mathbb{B}}(s) - \eta^*(s), X(s)\rangle_V - \Delta V(s) \left\|\bar{\mathbb{B}}(s) - \eta^*(s)\right\|_H^2 \\
& + \left\|\bar{\mathbb{D}}(s)\right\|_{L_2(U,H)}^2 + \int_Z \left\|\bar{\mathbb{G}}(s,z)\right\|_H^2 m(dz)\Big)\Big) dV(s)\bigg] \\
=& E\bigg[\int_{]0,t]} \theta(s)\Big(2\,{}_{V^*}\langle \bar{\mathbb{B}}(s) - \eta^*(s), X(s)\rangle_V - \Delta V(s) \left\|\bar{\mathbb{B}}(s) - \eta^*(s)\right\|_H^2 \\
& + \left\|\bar{\mathbb{D}}(s)\right\|_{L_2(U,H)}^2 + \int_Z \left\|\bar{\mathbb{G}}(s,z)\right\|_H^2 m(dz) - \beta\,\|X(s-)\|_H^2 \Big) \\
& + \Delta\theta(s)\beta\,\|X(s-)\|_H^2\, dV(s)\bigg],
\end{aligned}
\tag{5.34}
$$

where we have used $\theta(t-) = \theta(t) - \Delta\theta(t)$ in the last step. Analogously, we obtain

$$
\begin{aligned}
&E\left[\theta(t)\left\|X_\lambda(t)\right\|_H^2\right]-E\left[\left\|X_0\right\|_H^2\right] \\
=&E\bigg[\int_{]0,t]}\Big(\theta(s)\Big(2\,{}_{V^*}\langle\mathbb{B}(s,X_\lambda(s))-\mathbb{A}_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V \\
&\quad-\Delta V(s)\left\|\mathbb{B}(s,X_\lambda(s))-\mathbb{A}_\lambda(s,X_\lambda(s))\right\|_H^2+\left\|\mathbb{D}(s,X_\lambda(s))\right\|_{L_2(U,H)}^2 \\
&\quad+\int_Z\left\|\mathbb{G}(s,X_\lambda(s),z)\right\|_H^2 m(dz)-\beta\left\|X_\lambda(s-)\right\|_H^2\Big) \\
&\quad+\Delta\theta(s)\beta\left\|X_\lambda(s-)\right\|_H^2\Big)dV(s)\bigg].
\end{aligned}
$$

Let $\phi\in L^2([0,T]\times\Omega,\ dV\otimes P;V)$. Using $a^2=(a-b)^2-b^2+2ab$, we deduce

$$
\begin{aligned}
&E\left[\theta(t)\left\|X_\lambda(t)\right\|_H^2\right]-E\left[\left\|X_0\right\|_H^2\right] \\
&\quad+2E\bigg[\int_{]0,t]}\theta(s)\,{}_{V^*}\langle\mathbb{A}_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V\,dV(s)\bigg] \\
=&I_1+I_2+I_3+I_4,
\end{aligned}
\tag{5.35}
$$

where

$$
\begin{aligned}
I_1:=&E\bigg[\int_{]0,t]}\theta(s)\Big(2\,{}_{V^*}\langle\mathbb{B}(s,X_\lambda(s))-\mathbb{B}(s,\phi(s)),X_\lambda(s)-\phi(s)\rangle_V \\
&\quad+\left\|\mathbb{D}(s,X_\lambda(s))-\mathbb{D}(s,\phi(s))\right\|_{L_2(U,H)}^2 \\
&\quad+\int_Z\left\|\mathbb{G}(s,X_\lambda(s),z)-\mathbb{G}(s,\phi(s),z)\right\|_H^2 m(dz) \\
&\quad-\beta\left\|X_\lambda(s-)-\phi(s-)\right\|_H^2\Big)dV(s)\bigg],
\end{aligned}
$$

$$
\begin{aligned}
I_2:=&E\bigg[\int_{]0,t]}\theta(s)\Big(2\,{}_{V^*}\langle\mathbb{B}(s,\phi(s)),X_\lambda(s)\rangle_V \\
&\quad+{}_{V^*}\langle\mathbb{B}(s,X_\lambda(s))-\mathbb{B}(s,\phi(s)),\phi(s)\rangle_V-\left\|\mathbb{D}(s,\phi(s))\right\|_{L_2(U,H)}^2 \\
&\quad+2\langle\mathbb{D}(s,X_\lambda(s)),\mathbb{D}(s,\phi(s))\rangle_{L_2(U,H)} \\
&\quad-\int_Z\left\|\mathbb{G}(s,\phi(s),z)\right\|_H^2+2\langle\mathbb{G}(s,X_\lambda(s),z),\mathbb{G}(s,\phi(s),z)\rangle_H\,m(dz) \\
&\quad-2\beta\langle X_\lambda(s-),\phi(s-)\rangle_H+\beta\left\|\phi(s-)\right\|_H^2\Big)dV(s)\bigg],
\end{aligned}
$$

$$
I_3:=-E\bigg[\int_{]0,t]}\theta(s)\Delta V(s)\left\|\mathbb{B}(s,X_\lambda(s))-\mathbb{A}_\lambda(s,X_\lambda(s))\right\|_H^2dV(s)\bigg]
$$

and

$$I_4 := E\bigg[\int_{]0,t]} \Delta\theta(s)\beta \left\|X_\lambda(s-)\right\|_H^2 dV(s)\bigg].$$

Since

$$-\beta \left\|X_\lambda(s-) - \phi(s-)\right\|_H^2 \leq -\beta \left\|X_\lambda(s) - \phi(s)\right\|_H^2 + \beta \left\|\Delta X_\lambda(s) - \Delta\phi(s)\right\|_H^2,$$

Remark 5.4 implies that

$$I_1 \leq E\left[\int_{]0,t]} \theta(s)\beta \left\|\Delta X_\lambda(s) - \Delta\phi(s)\right\|_H^2 \ dV(s)\right]. \tag{5.36}$$

By the L^2-convergence of X_λ (cf. Corollary 5.15), it follows that

$$\liminf_{\lambda\to 0} I_1 \leq E\left[\int_{]0,t]} \theta(s)\beta \left\|\Delta X(s) - \Delta\phi(s)\right\|_H^2 \ dV(s)\right].$$

Using the convergence properties (5.22) - (5.26), we deduce

$$\begin{aligned}
&\liminf_{\lambda\to 0} I_2 \\
&\leq E\bigg[\int_{]0,t]} \theta(s)\Big(2\,{}_{V^*}\langle \mathbb{B}(s,\phi(s)), X(s)\rangle_V + {}_{V^*}\big\langle \bar{\mathbb{B}}(s) - \mathbb{B}(s,\phi(s)), \phi(s)\big\rangle_V \\
&\qquad - \left\|\mathbb{D}(s,\phi(s))\right\|_{L_2(U,H)}^2 + 2\langle \bar{\mathbb{D}}(s), \mathbb{D}(s,\phi(s))\rangle_{L_2(U,H)} \\
&\qquad - \int_Z \left\|\mathbb{G}(s,\phi(s),z)\right\|_H^2 + 2\langle \bar{\mathbb{G}}(s,z), \mathbb{G}(s,\phi(s),z)\rangle_H \ m(dz) \\
&\qquad - 2\beta\langle X(s-), \phi(s-)\rangle_H + \beta \left\|\phi(s-)\right\|_H^2 \Big) dV(s)\bigg],
\end{aligned}$$

Furthermore, (5.27) and The Banach-Steinhaus theorem imply that

$$\liminf_{\lambda\to 0} I_3 \leq -E\bigg[\int_{]0,t]} \theta(s)\Delta V(s) \left\|\mathbb{B}(s) - \eta^*(s)\right\|_H^2 dV(s)\bigg].$$

Since $\theta(s)$ is an non-increasing process, we have $\Delta\theta(s) \leq 0$. Thus, (5.22) and the Banach-Steinhaus theorem imply that

$$\liminf_{\lambda\to 0} I_4 \leq E\bigg[\int_{]0,t]} \Delta\theta(s)\beta \left\|X(s-)\right\|_H^2 dV(s)\bigg].$$

Altogether, we arrive at

$$
\begin{aligned}
&\liminf_{\lambda\to 0} E\left[\theta(t)\left\|X_\lambda(t)\right\|_H^2\right] - E\left[\left\|X_0\right\|_H^2\right] \\
&\quad + \limsup_{\lambda\to 0} E\bigg[\int_{]0,t]}\theta(s)\Big(2\,{}_{V^*}\langle \mathbb{A}_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V\Big)dV(s)\bigg] \\
\le & E\bigg[\int_{]0,t]}\theta(s)\Big(2\,{}_{V^*}\langle \mathbb{B}(s,\phi(s)),X(s)\rangle_V + {}_{V^*}\big\langle \bar{\mathbb{B}}(s)-\mathbb{B}(s,\phi(s)),\phi(s)\big\rangle_V \\
&\qquad - \left\|\mathbb{D}(s,\phi(s))\right\|_{L_2(U,H)}^2 + 2\langle \bar{\mathbb{D}}(s),\mathbb{D}(s,\phi(s))\rangle_{L_2(U,H)} \\
&\qquad - \int_Z \left\|\mathbb{G}(s,\phi(s),z)\right\|_H^2 + 2\langle \bar{\mathbb{G}}(s,z),\mathbb{G}(s,\phi(s),z)\rangle_H\, m(dz) \\
&\qquad - 2\beta\langle X(s-),\phi(s-)\rangle_H + \beta\left\|\phi(s-)\right\|_H^2 \\
&\qquad - \Delta V(s)\left\|\bar{\mathbb{B}}(s)-\eta^*(s)\right\|_H^2\Big) \\
&\quad + \theta(s)\beta\left\|\Delta X(s)-\Delta\phi(s)\right\|_H^2 + \Delta\theta(s)\beta\left\|X(s-)\right\|_H^2 dV(s)\bigg].
\end{aligned}
\tag{5.37}
$$

In analogy to the derivation of (4.19), for $\psi \in L^\infty([0,T],dV(t))$ we obtain

$$
E\left[\int_{]0,T]}\psi(t)\left\|X(t)\right\|_H^2 dV(t)\right] \le \liminf_{\lambda\to 0} E\left[\int_{]0,T]}\psi(t)\left\|X_\lambda(t)\right\|_H^2 dV(t)\right].
\tag{5.38}
$$

Combining (5.34) and (5.37) with (5.38), we deduce

$$
\begin{aligned}
&\limsup_{\lambda\to 0} 2E\bigg[\int_{]0,T]}\psi(t)\bigg(\int_{]0,t]}\theta(s)\,{}_{V^*}\langle \mathbb{A}_\lambda(s,X_\lambda(s)),X_\lambda(s)\rangle_V \\
&\qquad - {}_{V^*}\langle \eta(s),X(s)\rangle_V\, dV(s)\bigg)dV(t)\bigg] \\
\le & E\bigg[\int_{]0,T]}\psi(t)\bigg(\int_{]0,t]}\theta(s)\bigg(-2\,{}_{V^*}\big\langle \bar{\mathbb{B}}(s)-\mathbb{B}(s,\phi(s)),X(s)-\phi(s)\big\rangle_V \\
&\qquad - \left\|\mathbb{D}(s,\phi(s))-\bar{\mathbb{D}}(s)\right\|_{L_2(U,H)}^2 + \beta\left\|X(s)-\phi(s)\right\|_H^2 \\
&\qquad - \int_Z \left\|\mathbb{G}(s,\phi(s),z)-\bar{\mathbb{G}}(s,z)\right\|_H^2 m(dz) \\
&\qquad + \beta\left\|\Delta X(s)-\Delta\phi(s)\right\|_H^2\bigg)dV(s)\bigg)dV(t)\bigg].
\end{aligned}
\tag{5.39}
$$

Following the argumentation towards (4.22), we see that the left-hand side of (5.39) is greater or equal to 0. Hence, we arrive at

$$\begin{aligned}0 \geq & E\Bigg[\int_{]0,T]} \psi(t) \Bigg(\int_{]0,t]} \theta(s) \Big(2\, {}_{V^*}\langle \bar{\mathbb{B}}(s) - \mathbb{B}(s,\phi(s)), X(s) - \phi(s) \rangle_V \,. \\ & \quad + \left\| \mathbb{D}(s,\phi(s)) - \bar{\mathbb{D}}(s) \right\|^2_{L_2(U,H)} - \beta \left\| X(s) - \phi(s) \right\|^2_H \\ & \quad + \int_Z \left\| \mathbb{G}(s,\phi(s),z) - \bar{\mathbb{G}}(s,z) \right\|^2_H m(dz) \\ & - \beta \left\| \Delta X(s) - \Delta \phi(s) \right\|^2_H \Big) dV(s) \Bigg) dV(t) \Bigg]\end{aligned} \tag{5.40}$$

Setting $\phi = X$, we conclude that $\bar{\mathbb{D}}(s) = \mathbb{D}(s, X(s))$ and $\bar{\mathbb{G}}(s,z) = \mathbb{G}(s, X(s), z)$. Setting $\phi = X - \varepsilon\tilde{\phi}$, $\varepsilon > 0$, $\tilde{\phi} \in L^\infty([0,T] \times \Omega,\ dV \otimes P, V)$, by Lebesgue's dominated convergence theorem, we deduce that $\bar{\mathbb{B}} = \mathbb{B}(\cdot, X)$ (cf. Step 2 of the proof of Theorem 4.5).

Step 3.

$$\eta \in A(\cdot, X) \quad dV \otimes P\text{-a.s..}$$

Proof. The proof that $\eta^* \in \mathbb{A}(\cdot, X)$ is analogous to Step 3 of the proof of Theorem 4.5. Note that setting $\phi = X$ in (5.39), we have that

$$\begin{aligned}&\limsup_{\lambda \to 0} E\left[\int_0^T {}_{V^*}\langle A_\lambda(s, X_\lambda(s)), X_\lambda(s) \rangle_V \ dV(s)\right] \\ \leq & E\left[\int_0^T {}_{V^*}\langle \eta(s), X(s) \rangle_V \ dV(s)\right].\end{aligned} \tag{5.41}$$

Thus, by Hölder's inequality, Claim 3 of Step 3 in the proof of Theorem 4.5 follows. □

We have finally proved Theorem 5.5. □

Analogously to Proposition 4.14, we obtain the following uniqueness result.

Proposition 5.17. *The solution of* (5.1) *is path-wise unique in the following way: For every two solutions* X_1 *and* X_2 *of* (5.1) *and some constant* $C > 0$ *we have*

$$E\left[\left\| X_1(t) - X_2(t) \right\|^2_H\right] \leq e^{Ct} E\left[\left\| X_1(0) - X_2(0) \right\|^2_H\right] \quad \forall t \in [0,T].$$

Chapter 6

Applications

We present some applications to the existence and uniqueness results of the solution to multivalued stochastic differential equations discussed in Chapter 4 and Chapter 5.

6.1 Single-valued Case

The case of single-valued stochastic differential equations is covered by the multivalued framework: Let (V, H, V^*) be a Gelfand triple and A be a single-valued hemicontinuous monotone operator defined on the whole domain V. Then (5.1) turns into the following equation:

$$\begin{cases} dX(t) = [B(t, X(t)) - A(t, X(t))]\, dN(t) + D(t, X(t-))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z G(t, X(t-), z)\bar{\mu}(dt, dz) \\ X(0) = X_0. \end{cases}$$

By Corollary 3.6, the hemicontinuity and monotonicity assumptions of the single-valued operator A imply that A is maximal monotone. Consequently, we are in the framework of Chapters 4 and 5.

Let us compare the framework for the Wiener case developed in Chapter 4 with the results in [PR07] in more detail. Suppose [PR07, Hypotheses (H1)-(H4)] are valid for an appropriate exponent $\alpha \in]1, 2]$, the diffusion operator σ obeys a Lipschitz condition and additionally $\|b(\cdot, x)\|_{V^*} \leq C\left(\|x\|_V^{\alpha-1} + 1\right)$, $C > 0$, is valid. Then, conditions (H1)-(H5) are satisfied. In particular, all examples of the single-valued case in [PR07, Section 4.1], such as the stochastic reaction diffusion equation and the (single-valued) stochastic porous media equation, are covered by the multivalued framework. However, in this framework we are only able to treat the case in which the exponent is $\alpha \in]1, 2]$ in (H4), (H5) compared to the single-valued case [PR07] where $\alpha \in]1, \infty[$.

6.2 The Subdifferential Operator

Let (V, H, V^*) be a Gelfand triple. A function $\varphi : V \to \bar{\mathbb{R}} :=] - \infty, \infty]$ is called *proper* and *convex* on V if it is not identically $+\infty$ and satisfies the inequality

$$\varphi((1-\lambda)v + \lambda w) \leq (1-\lambda)\varphi(v) + \lambda\varphi(w) \quad \forall v, w \in V,\ \forall \lambda \in [0,1].$$

The function $\varphi : V \to \bar{\mathbb{R}}$ is said to be *lower semicontinuous* on V if

$$\liminf_{u \to v} \varphi(u) \geq \varphi(v), \quad \forall v \in V.$$

For a lower semicontinuous, convex, proper function $\varphi : V \to \bar{\mathbb{R}}$, the mapping $\partial\varphi : V \to 2^{V^*}$ defined by

$$\partial\varphi(v) = \{v^* \in V^* | \varphi(v) \leq \varphi(w) + {}_{V^*}\langle v^*, v - w\rangle_V,\ \forall w \in X\}$$

is called the *subdifferential* of φ.

Proposition 6.1. *For a lower semicontinuous, convex, proper function φ the subdifferential $\partial\varphi : V \to 2^{V^*}$ is maximal monotone.*

Proof. See [Bar10], Theorem 2.8]. □

Consequently, assuming appropriate boundedness and coercivity conditions (H4) and (H5) for $\partial\varphi$, the main results in Chapters 4 and 5 yield the existence and uniqueness of the solution to the following equation:

$$\begin{cases} dX(t) \in (b(t, X(t)) - \partial\varphi(X(t)))\, dt + \sigma(t, X(t))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z G(t, X(t-), z)\bar{\mu}(dt, dz), \\ X(0) = X_0. \end{cases}.$$

If φ is Gâtaux differentiable on V with the Gâtaux differential $\nabla\varphi$, then $\partial\varphi = \nabla\varphi$ (see [Bar10, Section 1.2, Example 3]). In this case, (H4) and (H5) turn into

$${}_{V^*}\langle \nabla\varphi(x), x\rangle_V \geq c_1 \|x\|_V^\alpha + c_2 \quad c_1 \in]0, \infty[,\ c_2 \in [0, \infty[,\ \forall x \in V$$

and

$$\|\nabla\varphi(x)\|_{V^*} \leq c_3 \|x\|_V^{\alpha-1} + c_4 \quad c_3 \in]0, \infty[,\ c_4 \in [0, \infty[,\ \forall x \in V,$$

where $\alpha \in]1, 2]$ ($\alpha = 2$ for Chapter 5 respectively).

Remark 6.2. *The reflection case where the drift is only of bounded variation (see eg. [Cép98], [Zha07]) is not covered by the developed framework. The reason is that for a reflection on a closed convex subset $K \subsetneq V$, the corresponding subdifferential of the indicator function I_K defined by*

$$I_K(x) := \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$$

is only defined on K (cf. [Bar10, Section 1.2 Example 2]) and not on the whole space V. However, this assumption is explicitly needed in the framework of Chapters 4 and 5.

6.3 Multivalued Stochastic Porous Media Equation

Finally, we want to examine the multivalued stochastic porous media equation as an explicit example of a multivalued stochastic partial differential equation. First, we motivate the stochastic porous media equation from a physical point of view. Then, we introduce the necessary mathematical framework and particularly define an appropriate Gelfand triple and finally apply the existence and uniqueness result of Chapters 4 and 5.

6.3.1 Motivation

The porous media equation originally describes the flow of an ideal gas in a homogeneous porous medium (cf. [Aro86], [Váz07]). In a macroscopic model, this flow can be formulated in terms of the density $\varrho(x,t)$, the pressure $p(x,t)$ and velocity $\vec{v}(x,t)$. The quantities are related by the following three laws:

i. The law of mass balance relates the change of density in time to the velocity,
$$\varepsilon \frac{\partial \varrho}{\partial t} + \text{div } (\varrho \vec{v}) = 0,$$
where $\varepsilon \in]0,1[$ is the porosity of the medium, i.e. the volume fraction available to the gas.

ii. *Darcy's law* describes the dynamics of flows through porous media
$$\vec{v} = -\frac{k}{\mu} \nabla p,$$
where k is the permeability of the medium and μ is the dynamic viscosity.

iii. The *state equation* of an ideal gas describes the relation between the pressure p and the density ϱ,

$$p = \frac{R}{M} T \varrho^{\gamma}.$$

Here, T is the temperature, R is the ideal gas constant, M the molar mass and $\gamma \geq 1$ the so-called polytropic exponent. (One assumes $\gamma = 1$ for isotermal processes and $\gamma > 1$ for adiabatic ones.)

Combining these three laws, we arrive at the classical porous media equation

$$\frac{\partial \varrho}{\partial t} = C\Delta(\varrho^m) \tag{6.1}$$

with exponent $m := \gamma + 1$, where C is a constant independent of x and t.

Scaling out the constant by defining a new time $t' = Ct$ and taking into account certain random phenomena, we arrive at the following prototype of the stochastic porous media equation:

$$dX(t) = \Delta(|X(t)|^{m-1}X(t))\, dt + \sigma(t, X(t))\, dN(t). \tag{6.2}$$

Here, the random forcing term $\sigma(t, X(t))\, dN(t)$ can be Wiener noise or more generally have certain jumps (cf. [BM09]).

Extension of the Stochastic Porous Media Equation

The stochastic porous media equation (6.2) can be generalized in several reasonable ways:

i. In the above model, the exponent m is always equal to or larger than 2. However, from a mathematical point of view, the difference to considering equations of type (6.1) with $m \geq 1$ is very small. One hase to consider, though, that if $m \in]0, 1[$, the diffusion coefficient $D(x) = |x|^{m-1}$ becomes singular at the point $x = 0$. The so-called *fast diffusion equation* owes its name to its behavior in the neighborhood of 0 (cf. [BDPR09a]).

ii. A further extension is established by the so called *generalized stochastic porous media equation*:

$$dX(t) = \Delta(\Psi(X(t)))\, dt + \sigma(t, X(t))\, dN(t).$$

Here, the growth behavior of the drift is enveloped by an arbitrary continuous and increasing function $\Psi : \mathbb{R} \to \mathbb{R}$.

iii. In some cases, the growth function Ψ reveals certain discontinuities, as occurring in the case of self-organized criticality (cf. [BDPR09b]). In order to overcome such a difficulty, one can "fill the gaps" of the graph of the growth function at points of discontinuity. The rigorous method is to replace the growth function Ψ by its *essential extension* $\bar{\Psi} : \mathbb{R} \to 2^{\mathbb{R}}$ (cf. Section 3.4). We finally arrive at the following multivalued stochastic differential equation:

$$dX(t) \in \Delta(\Psi(X(t)))\, dt + \sigma(t, X(t))\, dN(t).$$

6.3.2 Mathematical Background

As the main reference, we cite [PR07].

Let $n \in \mathbb{N}$, $n \geq 3$, $\Lambda \subset \mathbb{R}^n$ be open, let $d\xi$ be the Lebesgue measure on Λ and let $p \in \left[\frac{2n}{n+2}, 2\right]$. We assume that

$$|\Lambda| := \int_{\mathbb{R}^n} \mathbb{I}_\Lambda(\xi)\, d\xi < \infty.$$

Let $C_0^\infty(\Lambda)$ denote the set of all infinitely differentiable real-valued functions on Λ with compact support. We define the *Sobolev space* $H_0^{1,p}(\Lambda)$ of order 1 in $L^p(\Lambda)$ with *Dirichlet boundary conditions* by the completion of $C_0^\infty(\Lambda)$ with respect to the norm

$$\|u\|_{1,p} := \left(\int_\Lambda (|u(\xi)|^p + |\nabla u(\xi)|^p)\, d\xi \right)^{\frac{1}{p}}, \quad \forall u \in C_0^\infty(\xi).$$

We set $H_0^1(\Lambda) := H_0^{1,2}(\Lambda)$ and denote the dual space of $H_0^1(\Lambda)$ by H^{-1}.

By the Sobolev embedding theorem (cf. [Eva10, Chapter 5.6, Theorem 2]), we have

$$H_0^{1,2}(\Lambda) \subset L^{\frac{2n}{n-2}}(\Lambda)$$

continuously and densely and, since $\frac{p}{p-1} \in \left[2, \frac{2n}{n-2}\right]$, we obtain

$$L^p(\Lambda) \equiv \left(L^{\frac{p}{p-1}}(\Lambda)\right)^* \subset H^{-1}(\Lambda)$$

continuously and densely.

We define the Laplace operator $\Delta : H_0^1(\Lambda) \to H^{-1}(\Lambda)$ by

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial \xi_i^2}.$$

Lemma 6.3. *The map $(-\Delta)^{-1} : H^{-1} \to H_0^1(\Lambda)$ is the Riesz isomorphism for H^{-1}, i.e. for every $x \in H^{-1}$*

$$\langle x, \cdot \rangle_{H^{-1}} = {}_{H_0^1}\langle (-\Delta)^{-1} x, \cdot \rangle_{H^{-1}}. \tag{6.3}$$

Proof. See [PR07, Lemma 4.1.12]. $\square$

Therefore, we can identify $H^{-1}(\Lambda)$ with its dual $(H^{-1}(\Lambda))^* = H_0^1(\Lambda)$ via the Riesz map $(-\Delta)^{-1} : H^{-1} \to H_0^1$, hence defining

$$V := L^p(\Lambda),\ H := H^{-1}(\Lambda) \text{ and } V^* := (L^p(\Lambda))^* \left(= \Delta(L^{\frac{p}{p-1}})\right)$$

we obtain

$$V \subset H \subset V^* \tag{6.4}$$

continuously and densely. Note that for $n = 1, 2$, even stronger Sobolev embeddings hold and, therefore, we obtain the above triple directly. (In that case $p \in]1, 2]$.)
In fact, the domain of Δ can be extended to $L^{\frac{p}{p-1}}(\Lambda)$, as the next proposition shows.

Lemma 6.4. *The map*

$$\Delta : H_0^1(\Lambda) \to (L^p(\Lambda))^*$$

extends to a linear isometry

$$\Delta : L^{\frac{p}{p-1}}(\Lambda) \to (L^p(\Lambda))^* = V^*$$

and for all $u \in L^{\frac{p}{p-1}}(\Lambda)$, $v \in L^p(\Lambda)$

$$_{V^*}\langle -\Delta u, v\rangle_V = {}_{L^{\frac{p}{p-1}}}\langle u, v\rangle_{L^p} = \int_\Lambda u(\xi)v(\xi)\, d\xi. \tag{6.5}$$

Proof. See [PR07, Lemma 4.1.13]. $\square$

6.3.3 Existence of the Solution

Let $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ be a function having the following properties:

(Ψ1) Ψ is maximal monotone, i.e. for all $s, t \in \mathbb{R}$

$$(s-t)(x-y) \geq 0 \quad \forall x \in \Psi(s),\ y \in \Psi(t)$$

and Ψ is maximal (in the sense of Definition 3.1.ii).

(Ψ2) There exist $c_1 \in]0, \infty[,\ c_2 \in [0, \infty[$ such that

$$s \cdot x \geq c_1|s|^p - c_2 \quad \forall s \in \mathbb{R},\ \forall x \in \Psi(s).$$

(Ψ3) There exist $c_3,\ c_4 \in]0, \infty[$ such that

$$|x| \leq c_3 + c_4|s|^{p-1} \quad \forall s \in \mathbb{R},\ \forall x \in \Psi(s).$$

Then, the *(multivalued) porous media operator* $A : V(= L^p(\Lambda)) \to 2^{V^*}$ is defined by

$$A(u) := -\Delta\Psi(u), \quad u \in L^p(\Lambda).$$

Note that ($\Psi3$) implies that

$$\Psi(v) \subset L^{\frac{p}{p-1}}(\Lambda) \text{ for all } v \in L^p(\Lambda). \tag{6.6}$$

Hence, by Lemma 6.4 the operator A is well-defined.

Example 6.5. *For* $\Psi(s) := \mathrm{sign}\,(\mathrm{s})(\varrho + |\mathrm{s}|^{\mathrm{p}-1})$, $\varrho > 0$, *(*$\Psi1$*) – (*$\Psi3$*) are satisfied.*

Now let us check whether Conditions (H1), (H4) and (H5) are valid:

(H4): Let $x \in L^p(\Lambda)$, $\tilde{v} \in \Psi(x)$ and $v := -\Delta\tilde{v}$. Then, by (6.5),

$${}_{V^*}\langle v, x\rangle_V = \int_\Lambda \tilde{v}(\xi)x(\xi)\, d\xi \geq \int_\Lambda (c_1|x|^p - c_2)\, d\xi.$$

Hence, (H4) holds with $C_2 := c_1$, $\alpha = p$ and $g := -c_2|\Lambda|$.

(H5): Let $x \in L^p(\Lambda)$ and $\tilde{v} \in \Psi(x)$ such that $-\Delta\tilde{v} = A^0(x)$. Then, by the isometry property of Δ (see Lemma 6.4) and ($\Psi3$),

$$\begin{aligned}\left\|A^0(x)\right\|_{V^*} &= \|\Delta\tilde{v}\|_{V^*} = \|\tilde{v}\|_{L^{\frac{p}{p-1}}} \\ &\leq c_3\left(\int |x(\xi)|^p\, d\xi\right)^{\frac{p-1}{p}} + c_4|\Lambda|^{\frac{p-1}{p}} \\ &= c_3\,\|x\|_V^{p-1} + c_4|\Lambda|^{\frac{p-1}{p}},\end{aligned}$$

so (H5) is satisfied with $C_3 := c_3$, $\alpha = p$ and $h := c_4|\Lambda|^{\frac{p-1}{p}}$.

(H1): Let $x, y \in L^p(\Lambda)$, $\tilde{v} \in \Psi(x)$, $\tilde{w} \in \Psi(y)$ and $v := -\Delta\tilde{v}$, $w := -\Delta\tilde{w}$. Then by (6.5)

$$\begin{aligned}{}_{V^*}\langle v - w, x - y\rangle_V &= {}_{V^*}\langle -\Delta(\tilde{v} - \tilde{w}), u - v\rangle_V \\ &= \int (\tilde{v}(\xi) - \tilde{w}(\xi))(x(\xi) - y(\xi))\, d\xi \geq 0\end{aligned}$$

where we have used ($\Psi1$) in the last step.

Claim. $-\Delta\Psi$ is maximal monotone.

Proof. By Theorem 3.14, we have to show that for arbitrary but fixed $y \in V^*$ there exists $x \in V$ such that

$$J(x) - \Delta v = y \quad \text{on } \Lambda, \tag{6.7}$$

where $v \in \Psi(x)$ and J is the duality mapping from V to V^*. To this end, we consider the approximating equation

$$J(x) - \Delta\Psi_\lambda(x) = y, \tag{6.8}$$

where $\Psi_\lambda : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of Ψ. Since

$$\begin{aligned}
&{}_{V^*}\langle -\Delta\Psi_\lambda(x) + \Delta\Psi_\lambda(y), x - y\rangle_V \\
={}& \int_\Lambda (\Psi_\lambda(x(\xi)) - \Psi_\lambda(y(\xi))(x(\xi) - y(\xi))\, d\xi \geq 0,
\end{aligned}$$

the operator $-\Delta\Psi_\lambda$ is monotone. Furthermore, it is continuous. Hence, by Theorem 3.4 it is maximal monotone. Since J is maximal monotone, by Theorem 3.5 the operator $x \mapsto J(x) - \Delta\Psi_\lambda(x)$ is maximal monotone.

By Lemma 3.21, we have $s \cdot \Psi_\lambda(s) \geq c_1|s|^p - c_2$, for some $c_1 \in]0, \infty[$, $c_2 \in [0, \infty[$. Thus,

$${}_{V^*}\langle -\Delta\Psi_\lambda(x), x\rangle_V \geq C(\|x\|_V^p - 1). \tag{6.9}$$

It follows that

$$\lim_{n\to\infty} \frac{{}_{V^*}\langle J(x_n) - \Delta\Psi_\lambda(x_n), x_n\rangle_V}{\|x_n\|_V} \geq \lim_{n\to\infty} \frac{\|x_n\|_V^2 + C(\|x_n\|_V^p - 1)}{\|x_n\|_V} = \infty$$

for arbitrary $\{x_n\}_n \subset V$ such that $\|x_n\|_V \to \infty$. Now, Proposition 3.7 implies that there exists a solution x_λ of (6.8).

Let $A_\lambda := -\Delta\Psi_\lambda$. By (6.9) and (6.7), we obtain

$$\begin{aligned}
C\,\|x_\lambda\|_V^p &\leq {}_{V^*}\langle A_\lambda(x_\lambda), x_\lambda\rangle_V + C \\
&= {}_{V^*}\langle y - J(x_\lambda), x_\lambda\rangle_V + C \\
&\leq {}_{V^*}\langle y, x_\lambda\rangle_V + C \\
&\leq \frac{C}{2}\,\|x_\lambda\|_V^p + \tilde{C}\left(\|y\|_{V^*}^{\frac{p}{p-1}} + 1\right).
\end{aligned}$$

Therefore, $\sup_{\lambda>0} \|x_\lambda\|_V < C$. Consequently, by the definition of J, Proposition 3.19.iv) and (H5) we obtain

$$\begin{aligned}
&\sup_{\lambda>0} (\|A_\lambda(x_\lambda)\|_{V^*} + \|J(x_\lambda)\|_{V^*}) \\
\leq& \sup_{\lambda>0} \left(\left\|A^0(x_\lambda)\right\|_{V^*} + \|x_\lambda\|_V\right) \\
\leq& C \cdot \sup_{\lambda>0} \left(\|x_\lambda\|_V^{p-1} + \|\Lambda\|^{\frac{p-1}{p}} + \|x_\lambda\|_V\right) \\
<& \infty.
\end{aligned}$$

Hence, $x_\lambda \to x$ weakly in V and $A_\lambda(x_\lambda) \to x_A$ and $J(x_\lambda) \to x_J$ weakly in V^*, along some subsequence. Now, we have to show that $x_A \in A(x)$ and $x_J = J(x)$. To this end, by (6.8), we deduce

$$\begin{aligned}
&{}_{V^*}\langle J(x_{\lambda_1}) - J(x_{\lambda_2}), x_{\lambda_1} - x_{\lambda_2}\rangle_V \\
&+ {}_{V^*}\langle A_{\lambda_1}(x_{\lambda_1}) - A_{\lambda_2}(x_{\lambda_2}), x_{\lambda_1} - x_{\lambda_2}\rangle_V \\
=& {}_{V^*}\langle y - y, x_{\lambda_1} - x_{\lambda_2}\rangle_V \\
=&0.
\end{aligned} \tag{6.10}$$

Since J is monotone, (6.10) implies that

$$\limsup_{\lambda_1,\lambda_2>0} {}_{V^*}\langle A_{\lambda_1}(x_{\lambda_1}) - A_{\lambda_2}(x_{\lambda_2}), x_{\lambda_1} - x_{\lambda_2}\rangle_V \leq 0.$$

Hence, by Proposition 3.19.vi), we obtain $x_A \in A(x)$ and

$$\lim_{\lambda_1,\lambda_2\to 0} {}_{V^*}\langle A_{\lambda_1}(x_{\lambda_1}) - A_{\lambda_2}(x_{\lambda_2}), x_{\lambda_1} - x_{\lambda_2}\rangle_V = 0.$$

Combining this and 6.10, it follows that

$$\limsup_{\lambda_1,\lambda_2>0} {}_{V^*}\langle J(x_{\lambda_1}) - J(x_{\lambda_2}), x_{\lambda_1} - x_{\lambda_2}\rangle_V \leq 0.$$

Thus, $x_J = J(x)$. Hence, $x \in V$ solves problem (6.7) and Theorem 3.14 applies. Thus, $-\Delta\Psi$ is maximal monotone. □

Summing up, we can establish the existence and uniqueness for the following multivalued stochastic porous media equation:

$$\begin{cases} dX(t) \in \Delta\Psi(X(t))\,dt + \sigma(t, X(t))\,dW(t) \\ \qquad\qquad + \displaystyle\int_Z G(t, X(t-), z)\bar{\mu}(dt, dz), \\ X(0) = X_0, \end{cases} \tag{6.11}$$

with σ and G satisfying (H2) and (H3).

Remark 6.6. *Assuming $\Psi \in C^1(\mathbb{R}\backslash\{0\})$ and for some $\delta > 0$*

$$\Psi'(r) \geq \delta \cdot |r|^{p-2}, \quad p \geq 2,\ \forall r \in \mathbb{R}\backslash\{0\}$$

condition ($\Psi 2$) is readily satisfied. Indeed,

$$\Psi(r) - \Psi(1) = \int_1^r \Psi'(s)\,ds \geq \delta\frac{1}{p-1}\left(|r|^{p-1} - 1\right),$$

hence $r \cdot \Psi(r) \geq c_1|r|^p + c_2$ for $c_1 \in]0, \infty[$ and $c_2 \in \mathbb{R}$.

Comparing the existence result for the Wiener case (Theorem 4.5) with [BDPR09b, Theorem 2.2], in case $p = 2$ we obtain a strong notion of the solution (cf. Definition 4.1) instead of the weak notion in [BDPR09b, Equation (2.1)]. Indeed, [BDPR09b, Hypothesis (i) and (ii)] directly imply (Ψ1) and (Ψ3). For $p = 2$, by Remark 6.6, [BDPR09b, Hypothesis (iv)] imply (Ψ2).

In [BDPR09a], the (single-valued) porous media operator $\Delta\Psi$ with

$$\Psi(s) := \varrho \cdot \mathrm{sign}\,(\mathrm{s})|\mathrm{s}|^{\alpha} + \tilde{\Psi}(\mathrm{s}),$$

$\varrho > 0$, $\alpha \in]0, 1[$, where $\tilde{\Psi}$ is a continuous monotonically non-decreasing function of linear growth, was investigated. Additionally assuming that $\tilde{\psi}$ satisfies the coercivity assumption (Ψ3), Theorem 4.5 yields the existence of the solution for this type of fast diffusion equation.

Appendix A

Multivalued Maps

In this appendix, we clarify some elementary notions on multivalued maps. For a detailed overview over this topic, we refer to [AC84].

Let X and Y be two general sets. We denote the power set of Y by 2^Y. A *multivalued map* F from X to Y is a map that associates to any $x \in X$ a (not necessarily non-empty) subset $F(x) \subset Y$. The subset

$$\mathcal{D}(F) := \{x \in X | F(x) \neq \emptyset\}$$

is called the *domain* of F. Unless otherwise noted the domain of F is assumed to be non-empty. We write $F : \mathcal{D}(F) \subset X \to 2^Y$ and say F is a multivalued map on X if $Y = X$. For a multivalued map we can define the *graph* by

$$\mathcal{G}(F) := \{[x, y] \in X \times Y | y \in F(x)\}.$$

The graph of F provides a convenient characterization of a multivalued map. Conversely, a non-empty set $\mathcal{G} \subset X \times Y$ defines a multivalued map by

$$F(x) := \{y \in Y | [x, y] \in \mathcal{G}\}.$$

In that case, $\mathcal{G}$ is the graph of F. As usual, the *range* $\mathcal{R}(F) \subset Y$ is defined by

$$\mathcal{R}(F) := \bigcup_{x \in X} F(x).$$

Every multivalued map has an *inverse* F^{-1}. In general, it is again a multivalued map with the domain $\mathcal{D}(F^{-1}) := \mathcal{R}(F) \subset Y$ and values

$$F^{-1}(y) := \{x \in X |\, y \in F(x)\}, \quad \forall y \in \mathcal{D}(F^{-1}).$$

The map $B : \mathcal{D}(B) \subset X \to 2^Y$ is called *extension* of $A : \mathcal{D}(A) \subset X \to 2^Y$ if $\mathcal{G}(A) \subset \mathcal{G}(B)$. A single-valued function $f : X \to Y$ is called a *selection* of the set-valued map $F : X \to 2^Y$ if for all $x \in X$, $f(x) \in F(x)$. We define the *scalar multiplication* for a multivalued map by

$$(\lambda A)(x) := \{\lambda v | v \in A(x)\}, \quad \forall x \in \mathcal{D}(\lambda A) := \mathcal{D}(A),\ \lambda \in \mathbb{R},$$

and the *sum* of two multivalued maps A and B by

$$(A+B)(x) = \{v+w | v \in A(x),\, w \in B(x)\}$$

for all $x \in \mathcal{D}(A+B) := \mathcal{D}(A) \cap \mathcal{D}(B)$.

Appendix B

Basic Concepts on Infinite–dimensional Spaces

In this appendix, we present some basic concepts on Banach spaces and in particular Hilbert spaces. For a detailed survey of this topic, we refer to [Bar93], [Bar10], [PR07].

Let $\big(X, \|\cdot\|\big)$ be a real Banach space and $(X^*, \|\cdot\|_{X^*})$ be its dual space.

We define the dualization ${}_{X^*}\langle\ ,\ \rangle_X$ between X and X^* by

$$
{}_{X^*}\langle x^*, x\rangle_X := x^*(x) \text{ for } x^* \in X^*,\ x \in X.
$$

Definition B.1. *Let A be a single-valued operator from X to X^* with $\mathcal{D}(A) = X$.*

i. The operator A is said to be hemicontinuous *if, for all $x, u, v \in X$,*

$$
\lim_{\lambda\to 0} {}_{X^*}\langle A(u+\lambda v), x\rangle_X = {}_{X^*}\langle A(u), x\rangle_X\,.
$$

ii. A is said to be demicontinuous *if, for $x_n \to x \in X$*

$$
\lim_{x_n\to x} {}_{X^*}\langle A(x_n), y\rangle_X = {}_{X^*}\langle A(x), y\rangle_X \quad \forall y \in X.
$$

Remark B.2. *Demicontinuity obviously implies hemicontinuity. Moreover, if a single-valued operator is hemicontinuous and monotone, then it is demicontinuous (cf. [PR07, Remark 4.1.1.ii)]). Hence, for a single-valued monotone operator, these two notions of continuity coincide.*

Definition B.3. *Let $A : X \to 2^{X^*}$ be a multivalued operator.*

i. A is said to be coercive *if*

$$
\frac{{}_{X^*}\langle y_n, x_n - x_0\rangle_X}{\|x_n\|} \overset{n\to\infty}{\longrightarrow} \infty
$$

for some $x_0 \in X$ and for all $[x_n, y_n] \in \mathcal{G}(A)$ such that $\lim\limits_{n\to\infty} \|x_n\| = \infty$.

ii. *A is called* coercive with constant $\alpha \in]1, \infty[$ *if there exists a constant* $c > 0$ *such that*

$$_{X^*}\langle y, x\rangle_X \geq c \|x\|^\alpha$$

for all $[x, y] \in \mathcal{G}(A)$.

Remark B.4. *The notion of coercivity in Definition B.3.i) is weaker than coercivity with constant* α*, because choosing* $x_0 = 0$*, by Definition B.3.ii) we find that*

$$\lim_{n\to\infty} \frac{_{X^*}\langle y_n, x_n\rangle_X}{\|x_n\|} \geq \lim_{n\to\infty} c\|x_n\|^{\alpha-1} = \infty,$$

for $\alpha > 0$.

Definition B.5. *i. A multivalued operator A is said to be* bounded on bounded subsets *if it maps bounded sets into bounded sets.*

ii. A multivalued operator A is said to be linearly bounded *if there exists a constant* $C > 0$ *such that for all* $[x, y] \in A$

$$\|y\| \leq C\|x\|.$$

Obviously, linear boundedness implies local boundedness.

B.1 Geometry of Banach Spaces

Definition B.6. *Let* $i_X : X \to X^{**}$ *be the isometric map defined by*

$$_{X^{**}}\langle i_X(x), x^*\rangle_{X^*} := {}_{X^*}\langle x^*, x\rangle_X, \quad \textit{for } x \in X, x^* \in X^*.$$

The Banach space X is called reflexive *if* i_X *is surjective.*

Definition B.7. *i. X is called* strictly convex *if the unit sphere*

$$S := \{x \in X|\, \|x\| = 1\}$$

contains no line segments, i.e. for all $x, y \in S$, $x \neq y$

$$\frac{x+y}{2} \notin S. \tag{B.1}$$

ii. X is said to be uniformly convex *if for each* $0 < \varepsilon < 2$*, there exists* $\delta > 0$ *such that for all* $x, y \in S$ *satisfying* $\|x - y\| \geq \varepsilon$,

$$\|x + y\| \leq 2(1 - \delta).$$

Remark B.8. *Every uniformly convex space is strict-convex:*
Indeed, let $x, y \in S$*, such that* $\frac{x+y}{2} \in S$*, i.e.* $\|x + y\| = 2$*. Assume that* $x \neq y$*. Note that then* $\|x + y\| > 2(1 - \delta)$ *holds for arbitrary* $\delta > 0$*. Setting* $\varepsilon := \frac{\|x-y\|}{2} > 0$*, by uniform convexity we conclude that* $\|x - y\| < \frac{\|x-y\|}{2}$ *for arbitrary* $\varepsilon > 0$*, which is a contradiction. Thus,* $x = y$ *and we obtain strict convexity.*

Proposition B.9. *Every uniformly convex space is reflexive.*

Proof. See [Wer00, Proposition IV.7.8]. □

Typical examples of uniformly convex Banach spaces are Hilbert spaces and L^p-spaces with $p > 1$ (cf. Theorem C.6).

B.2 The Gelfand Triple

Definition B.10. *Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable real Hilbert space identified with its dual space H^* via the Riesz isomorphism R. Let V be a Banach space with dual space V^* such that the embedding $V \subset H$ is continuous, i.e.*

$$\|v\|_H \le C \|v\|_V \quad \text{for all } v \in V \tag{B.2}$$

and V is dense in H. (V, H, V^*) is called the Gelfand triple*.*

It follows that $H^* \subset V^*$ continuously and densely (cf. [Zei90b, Proposition 23.13]). Consequently,

$$V \subset H \overset{R}{\equiv} H^* \subset V^*. \tag{B.3}$$

continuously and densely and

$${}_{V^*}\langle z, v \rangle_V = \langle z, v \rangle_H \quad \text{for all } z \in H,\ v \in V. \tag{B.4}$$

Note that V^* is separable since $H \subset V^*$ continuously and densely, hence this is true for V as well.

B.3 Operators on Hilbert spaces

Let $\big(U, \langle \cdot, \cdot \rangle_U\big)$ and $\big(H, \langle\ ,\ \rangle_H\big)$ be two separable Hilbert spaces.

Proposition B.11. *Let S be an orthonormal basis of H. Then*

i.

$$x = \sum_{e \in S} \langle x, e \rangle_H e \quad \forall x \in H,$$

ii. The Parseval equality *holds:*

$$\|x\|_H^2 = \sum_{e \in S} |\langle x, e \rangle_H|^2 \quad \forall x \in H.$$

Proof. See [Wer00, Satz V.4.9, p.230]. □

Definition B.12. *i. The space of all bounded linear operators from U to H is denoted by $L(U, H)$. For simplicity, we write $L(U)$ instead of $L(U, U)$.*

ii. By $L^* \in L(H,U)$ *we denote the* adjoint operator *of* $L \in L(U,H)$.

iii. An operator $L \in L(U)$ *is called* symmetric *if* $\langle Lu, v\rangle_U = \langle u, Lv\rangle_U$ *for all* $u, v \in U$.

iv. An operator $L \in L(U)$ *is called* non-negative *if* $\langle Lu, u\rangle \geq 0$ *for all* $u \in U$.

B.3.1 Trace Class Operators

Definition B.13 (Nuclear operator)**.** *An operator* $T \in L(U,H)$ *is said to be* nuclear *if it can be represented by*

$$Tx = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_U \quad \textit{for all } x \in U$$

where $(a_j)_{j\in\mathbb{N}} \subset H$ *and* $(b_j)_{j\in\mathbb{N}} \subset U$ *are such that* $\sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_U < \infty$. *The space of all nuclear operators from* U *to* H *is denoted by* $L_1(U,H)$.

Proposition B.14. *The space* $L_1(U,H)$ *equipped with the norm*

$$\|T\|_{L_1(U,H)} := \inf\Big\{\sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_U \;\Big|\; Tx = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_U,\ x \in U\Big\}$$

is a Banach space.

Proof. See [PR07, Proposition B.0.2]. □

Definition B.15. *Let* $T \in L(U)$ *and let* e_k, $k \in \mathbb{N}$, *be an orthonormal basis of* U. *Then we define*

$$\operatorname{Tr} T := \sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle_U$$

if the series is convergent.

One has to notice that this definition could depend on the choice of the orthonormal basis. However, note the following result concerning nuclear operators.

Remark B.16. *If* $T \in L_1(U)$ *then* $\operatorname{Tr} T$ *is well-defined independently of the choice of the orthonormal basis* e_k, $k \in \mathbb{N}$. *Moreover, we have that*

$$|\operatorname{Tr} T| \leq \|T\|_{L_1(U)}.$$

Proof. See [PR07, Remark B.0.4]. □

Definition B.17. *By* $L_1^+(U)$ *we denote the subspace of* $L_1(U)$ *consisting of all symmetric non-negative nuclear operators.*

B.3.2 Hilbert-Schmidt Operators

Definition B.18 (Hilbert-Schmidt operator)**.** *A bounded linear operator $T : U \to H$ is called* Hilbert-Schmidt *if*

$$\sum_{k\in\mathbb{N}} \|Te_k\|^2 < \infty$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of U. The space of all Hilbert-Schmidt operators from U to H is denoted by $L_2(U, H)$.

Remark B.19. *i. The definition of the Hilbert-Schmidt operator and the number*

$$\|T\|^2_{L_2(U,H)} := \sum_{k\in\mathbb{N}} \|Te_k\|^2$$

does not depend on the choice of the orthonormal basis e_k, $k \in \mathbb{N}$, and we know that $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$. For simplicity we write $\|T\|_{L_2}$ instead of $\|T\|_{L_2(U,H)}$.

ii. $\|T\|_{L(U,H)} \le \|T\|_{L_2(U,H)}$.

Proof. See [PR07, Remark B.0.6]. □

Proposition B.20. *Let $S, T \in L_2(U, H)$ and let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U. If we define*

$$\langle T, S\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Se_k, Te_k\rangle_H$$

we obtain that $\big(L_2(U, H), \langle\ ,\ \rangle_{L_2}\big)$ is a separable Hilbert space.
If f_k, $k \in \mathbb{N}$, is an orthonormal basis of H we get that $f_j \otimes e_k := f_j\langle e_k, \cdot\rangle_U$, $j, k \in \mathbb{N}$, is an orthonormal basis of $L_2(U, H)$.

Proof. See [PR07, Proposition B.0.7]. □

Proposition B.21 (Square root)**.** *Let $T \in L(U)$ be a non-negative and symmetric operator. Then, there exists exactly one element $T^{\frac{1}{2}} \in L(U)$ that is non-negative and symmetric such that*

$$T^{\frac{1}{2}} \circ T^{\frac{1}{2}} = T.$$

If $\operatorname{Tr} T < \infty$, then we have that $T^{\frac{1}{2}} \in L_2(U)$ where $\left\|T^{\frac{1}{2}}\right\|^2_{L_2(U)} = \operatorname{Tr} T$ and $L \circ T^{\frac{1}{2}} \in L_2(U, H)$ for all $L \in L(U, H)$.

Proof. See [PR07, Proposition 2.3.4]. □

B.3.3 Pseudo Inverse of Linear Operators

Definition B.22. *Let $T \in L(U, H)$ and* $\mathrm{Ker}(T) := \{x \in U \mid Tx = 0\}$*. The* pseudo inverse *of T is defined by*

$$T^{-1} := \left(T|_{\mathrm{Ker}(T)^\perp}\right)^{-1} : T\left(\mathrm{Ker}(T)^\perp\right) = T(U) \to \mathrm{Ker}(T)^\perp .$$

Proposition B.23. *Let $T \in L(U)$ and T^{-1} the pseudo inverse of T.*

i. If we define an inner product on $T(U)$ by

$$\langle x, y \rangle_{T(U)} := \langle T^{-1}x, T^{-1}y \rangle_U \quad \text{for all } x, y \in T(U),$$

then $\left(T(U), \langle \cdot, \cdot \rangle_{T(U)}\right)$ is a Hilbert space.

ii. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of $(\mathrm{Ker}\, T)^\perp$. Then Te_k, $k \in \mathbb{N}$, is an orthonormal basis of $\left(T(U), \langle \cdot, \cdot \rangle_{T(U)}\right)$.

Proof. See [PR07, Proposition C.0.3]. □

Proposition B.24. *Let $T \in L(U, H)$ and set $Q := TT^* \in L(H)$. Then we have*

$$Q^{\frac{1}{2}}(U) = T(U) \quad \text{and} \quad \left\| Q^{-\frac{1}{2}}x \right\|_H = \left\| T^{-1}x \right\|_U \quad \text{for all } x \in T(U),$$

where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$.

Proof. See [PR07, Corollary C.0.6]. □

B.3.4 The Tensor Product

Let $(U, \langle \cdot, \cdot \rangle_U$ be a separable Hilbert space.

Definition B.25. *The* tensor product *on $U \times U$ is defined by*

$$x \otimes y(z) := \langle y, z \rangle_U x \quad \forall x, y, z \in U.$$

Remark B.26. *For $x, y \in U$ we have $x \otimes y \in L^1(U)$ and*

$$\|x \otimes y\|_{L_1(U)} = \|x\|_U \, \|y\|_U \, .$$

Appendix C

L^p-Spaces

In this appendix, we introduce the Bochner integral and prove that the L^p-space is convex. Furthermore, we describe the duality of L^p-spaces. As references, we cite [DU77] and [Wer00].

C.1 The Bochner Integral

Let $\big(X, \|\ \|\big)$ be a real separable Banach space, $\mathcal{B}(X)$ the Borel σ-field of X and $(\Omega, \mathcal{F}, \mu)$ a measure space with the finite measure μ.

Definition C.1. *i. The set of simple functions is defined by*

$$\mathcal{E} := \Big\{ f : \Omega \to X \ \Big|\ f = \sum_{k=1}^{n} x_k 1_{A_k},\ x_k \in X,\ A_k \in \mathcal{F},\ 1 \le k \le n,\ n \in \mathbb{N} \Big\}$$

ii. A μ-measurable function $f : \Omega \to X$ is called Bochner integrable *if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$ such that*

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\|\ d\mu = 0.$$

Theorem C.2. *A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if*

$$\int_\Omega \|f\|\ d\mu < \infty.$$

Proof. See [DU77, Chapter II, Theorem 2]. □

Theorem C.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Bochner integrable X-valued functions on Ω. If $\lim\limits_{n\to\infty} f_n = f$ in μ-measure, i.e.*

$$\lim_{n\to\infty} \mu(\{\omega \in \Omega |\ \|f_n - f\| \ge \varepsilon\}) = 0 \quad \forall \varepsilon > 0,$$

and if there exists a real-valued Lebesgue integrable function g on Ω with $\|f_n\| \leq g$ μ-a.s., then f is Bochner integrable and

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| \ d\mu = 0.$$

Proof. See [DU77, Chapter II, Theorem 3]. □

We summarize some important properties of the Bochner integral:

Proposition C.4. *Let $f, g \in L^1(\Omega, \mathcal{F}, \mu; X)$. Then*

i. *(cf. [DU77, Chapter II, Theorem 4])*

$$\left\| \int_\Omega f \ d\mu \right\| \leq \int_\Omega \|f\| \ d\mu.$$

ii. *(cf. [DU77, Chapter II, Corollary 5]) If $\int_A f \ d\mu = \int_A g \ d\mu$ for each $A \in \mathcal{F}$, then $f = g$ μ-a.s..*

iii. *(cf. [DU77, Chapter II, Theorem 6])*

$$\int_\Omega L \circ f \ d\mu = L\Big(\int_\Omega f \ d\mu\Big)$$

for all $L \in L(X, Y)$ where Y is another Banach space.

Definition C.5. *Let $1 \leq p < \infty$. Then we define*

$$\begin{aligned}\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X) &:= \mathcal{L}^p(\mu; X) \\ &:= \left\{ f : \Omega \to X \;\middle|\; f \text{ is } \mathcal{F}\text{-measurable and} \int \|f\|^p \ d\mu < \infty \right\}\end{aligned}$$

and the semi-norm

$$\|f\|_{L^p} := \left(\int \|f\|^p \ d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu; X).$$

The space of all equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X)$ with respect to $\|\cdot\|_{L^p}$ is denoted by $L^p(\Omega, \mathcal{F}, \mu; X) := L^p(\mu; X)$.

The space $L^p(\Omega, \mathcal{F}, \mu; X)$ is a Banach space.

C.2 Convexity of $L^p(\Omega, \mathcal{F}, \mu; X)$

Theorem C.6. *Let $\big(X, \|\cdot\|\big)$ be a uniformly convex Banach space and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ. Let $p \in]1, \infty[$. Then, the space $(L^p(\Omega, \mathcal{F}, \mu; X), \|\cdot\|_{L^p})$ is uniformly convex.*

For the proof of Theorem C.6, we need the following lemma.

Lemma C.7. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and let $p \in]1, \infty[$. Then, for every $\varepsilon > 0$ there exists a constant $C_p(\varepsilon) > 0$ such that*

$$\left\| \frac{x+y}{2} \right\|^p \leq (1 - C_p(\varepsilon)) \frac{\|x\|^p + \|y\|^p}{2} \tag{C.1}$$

for all $x, y \in X$ such that $\max\{\|x\|, \|y\|\} \leq 1$ and $\|x - y\| \geq \varepsilon$.

Proof. First, assume that $\|y\| = 1$.

Case $\|x\| = 1$: Since X is uniformly convex, for some $\delta > 0$ we have

$$\left\| \frac{x+y}{2} \right\|^p \leq (1-\delta)^p < 1 = \frac{\|x\|^p + \|y\|^p}{2}.$$

Case $\|x\| < 1$: By use of the elementary inequality

$$\left(\frac{a+1}{2} \right)^p < \frac{a^p + 1}{2}, \quad \forall a \in]0, 1[,$$

we obtain

$$\left\| \frac{x+y}{2} \right\|^p \leq \left(\frac{\|x\| + 1}{2} \right)^p < \frac{\|x\|^p + \|y\|^p}{2}.$$

Consequently,

$$\left\| \frac{x+y}{2} \right\|^p \cdot \left(\frac{\|x\|^p + \|y\|^p}{2} \right)^{-1} < 1 \tag{C.2}$$

on $K := \{x, y \in X | \, \|x\| \leq 1, \|y\| = 1, \|x - y\| \geq \varepsilon\}$. Since the left-hand side in (C.2) is continuous in x and y and since K is closed, there exists $C_p(\varepsilon) > 0$ such that

$$\left\| \frac{x+y}{2} \right\|^p \cdot \left(\frac{\|x\|^p + \|y\|^p}{2} \right)^{-1} \leq 1 - C_p(\varepsilon) < 1.$$

Now, let $\|y\| \leq 1$, $y \neq 0$. W.l.o.g. $\|x\| \leq \|y\| \neq 0$. Thus, for $\tilde{x} := \frac{x}{\|y\|}$ and $\tilde{y} := \frac{y}{\|y\|}$ we have $\|\tilde{x}\| \leq 1$ and $\|\tilde{y}\| = 1$. Therefore, we can apply the result above and obtain

$$\begin{aligned} \frac{1}{\|y\|^p} \left\| \frac{x+y}{2} \right\|^p = \left\| \frac{\tilde{x}+\tilde{y}}{2} \right\|^p &\leq (1 - C_p(\varepsilon)) \frac{\|\tilde{x}\|^p + \|\tilde{y}\|^p}{2} \\ &= (1 - C_p(\varepsilon)) \frac{\|x\|^p + \|y\|^p}{2\,\|y\|^p}. \end{aligned}$$

Multiplying with $\|y\|^p$ concludes the proof. $\square$

Proof of Theorem C.6. (cf. [Wer00, Satz I.V.7.7, p.169] for the real case.) Let $f, g \in L^p(\Omega, \mathcal{F}, \mu; X)$ such that $\|f\|_{L^p} = \|g\|_{L^p} = 1$ and $\|f - g\|_{L^p} \geq \varepsilon$. Set

$$\Omega_0 := \left\{ \omega \in \Omega \;\middle|\; \|f - g\|^p \geq \frac{\varepsilon^p}{4} \left(\|f\|^p + \|g\|^p\right) \right\}$$

and $\Omega_1 := \Omega \backslash \Omega_0$. For

$$\tilde{f}(\omega) := \frac{f(\omega)}{(\|f(\omega)\|^p + \|g(\omega)\|^p)^{1/p}} \quad \text{and} \quad \tilde{g}(\omega) := \frac{g(\omega)}{(\|f(\omega)\|^p + \|g(\omega)\|^p)^{1/p}}$$

we have $\left\|\tilde{f}(\omega)\right\| \leq 1$ and $\|\tilde{g}(\omega)\| \leq 1$ and by the definition of Ω_0, for $\omega \in \Omega_0$,

$$\left\|\tilde{f}(\omega) - \tilde{g}(\omega)\right\| \geq \frac{\varepsilon}{4^{1/p}}.$$

Hence, we can apply Lemma C.7 and obtain

$$\left\|\frac{f(\omega) + g(\omega)}{2}\right\|^p \leq \left(1 - C_p\left(\frac{\varepsilon}{4^{1/p}}\right)\right) \frac{\|f(\omega)\|^p + \|g(\omega)\|^p}{2} \tag{C.3}$$

for every $\omega \in \Omega_0$. Furthermore, we have

$$\begin{aligned}
\int_{\Omega_1} \|f - g\|^p \, d\mu &\leq \int_{\Omega_1} \frac{\varepsilon^p}{4} \left(\|f\|^p + \|g\|^p\right) \, d\mu \\
&\leq \int_{\Omega} \frac{\varepsilon^p}{4} \left(\|f\|^p + \|g\|^p\right) \, d\mu \\
&= \frac{\varepsilon^p}{4} \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p\right) = \frac{\varepsilon^p}{2}.
\end{aligned}$$

Since $\|f - g\|_{L^p} \geq \varepsilon$, we obtain

$$\begin{aligned}
\varepsilon^p &\leq \int_{\Omega} \|f - g\|^p \, d\mu \\
&= \int_{\Omega_0} \|f - g\|^p \, d\mu + \int_{\Omega_1} \|f - g\|^p \, d\mu \\
&\leq \int_{\Omega_0} \|f - g\|^p \, d\mu + \frac{\varepsilon^p}{2}.
\end{aligned}$$

Hence,

$$\frac{\varepsilon}{2^{1/p}} \leq \|1_{\Omega_0}(f - g)\|_{L^p} \leq \|1_{\Omega_0} f\|_{L^p} + \|1_{\Omega_0} g\|_{L^p}.$$

Using this as well as (C.3), we deduce

$$
\begin{aligned}
1-\left\|\frac{f+g}{2}\right\|_{L^p}^p &= \int_\Omega \left(\frac{\|f\|^p+\|g\|^p}{2}-\left\|\frac{f+g}{2}\right\|^p\right)\,d\mu \\
&\geq \int_{\Omega_0} \left(\frac{\|f\|^p+\|g\|^p}{2}-\left\|\frac{f+g}{2}\right\|^p\right)\,d\mu \\
&\geq C_p\left(\frac{\varepsilon}{4^{1/p}}\right)\int_{\Omega_0}\frac{\|f\|^p+\|g\|^p}{2}\,d\mu \\
&\geq C_p\left(\frac{\varepsilon}{4^{1/p}}\right)\frac{\varepsilon^p}{2^{p+1}}.
\end{aligned}
$$

□

Corollary C.8. *For a uniformly convex Banach space X, the space $(L^p(\Omega,\mathcal{F},\mu;X),\|\cdot\|_{L^p})$ is reflexive.*

Proof. Apply Proposition B.9. □

C.3 Duality of $L^p(\Omega,\mathcal{F},\mu;X)$

Let $\big(X,\|\ \|\big)$ be a real Banach space and $(X^*,\|\cdot\|_{X^*})$ be its dual space.

Definition C.9. *A Banach space X has the Radon-Nikodym property with respect to $(\Omega,\mathcal{F},\mu)$ if for each μ-continuous vector measure $G:\mathcal{F}\to X$ of bounded variation there exists $g\in L^1(\Omega,\mathcal{F},\mu;X)$ such that*

$$G(E)=\int_E g\,d\mu$$

for all $E\in\mathcal{F}$.

Proposition C.10. *Every reflexive Banach space has the Radon-Nikodym property.*

Proof. See [DU77, Chapter III, Corollary 13]. □

Theorem C.11. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space with finite measure μ, $1\leq p<\infty$, and X be a Banach space. Then*

$$L^p(\Omega,\mathcal{F},\mu;X)^*=L^q(\Omega,\mathcal{F},\mu;X^*),$$

where $\frac{1}{p}+\frac{1}{q}=1$ if and only if X^* has the Radon-Nikodym property with respect to μ.

Proof. See [DU77, Chapter IV, Theorem 1]. □

Consequently, the dualization between $L^p(\Omega, \mathcal{F}, \mu; X)$ and $L^q(\Omega, \mathcal{F}, \mu; X^*)$ is defined by

$$
{}_{L^q}\langle f, g\rangle_{L^p} := \int_\Omega {}_{X^*}\langle g, f\rangle_X \; d\mu
$$

for $f \in L^p(\Omega, \mathcal{F}, \mu; X)$, $g \in L^q(\Omega, \mathcal{F}, \mu; X^*)$. In particular, for a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ the space $L^2(\Omega, \mathcal{F}, \mu; H)$ is a Hilbert space with the inner product $\langle f, g\rangle_{L^2} := \int_\Omega \langle g, f\rangle_H \; d\mu \quad$ for $f, g \in L^2(\Omega, \mathcal{F}, \mu; H)$.

Appendix D

Addendum to Stochastic Analysis

In this appendix, we collect some necessary results in the theory of stochastic processes.

Let $(U, \langle\cdot,\cdot\rangle_U)$ be a separable Hilbert space.

D.1 Lévy Processes

Definition D.1. *Let X be a stochastic process with values in U.*

i. The process X is said to be stochastically continuous *if for every $t \geq 0$ and $\varepsilon > 0$*

$$\lim_{s \to t} P\left(\|X(s) - X(t)\|_U > \varepsilon\right) = 0.$$

ii. The process X has independent increments *if $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.*

iii. If the distribution of $X(t) - X(s)$ depends only on the difference $t - s$, we say that X has stationary increments.

iv. The process X is called Lévy process, *if it has stationary, independent increments and is stochastically continuous and $X(0) = 0$.*

Theorem D.2 (Lévy-Khinchine formula)**.** *Let X be a cádlág Lévy process on U and let μ_t be the law of $X(t)$. Then, there exists a unique triple (γ, Q, ν) where $\gamma \in U$, $Q \subset L_1^+(U)$ (cf. Definition B.17), ν is a non-negative measure satisfying $\nu(\{0\}) = 0$ and*

$$\int_H (\|y\|_U^2 \wedge 1)\, \nu(dy) < \infty,$$

such that

$$\int_U e^{i\langle x,y\rangle_U}\ \mu_t(dy) = e^{-t\Psi(x)},$$

where

$$\begin{aligned}\Psi(x) := &-i\langle \gamma, x\rangle_U + \frac{1}{2}\langle Qx, x\rangle_U \\ &+ \int_U \left(1 - e^{i\langle x,y\rangle_U} + 1_{\{\|y\|<1\}}(y) i\langle x, y\rangle_U\right)\ \nu(dy).\end{aligned}$$

Proof. See [PZ07, Theorem 4.24, p.56]. □

Definition D.3. *We call the operator Q appearing in Theorem D.2 the* covariance *of X, the measure μ the* jump intensity measure *of X and the triple (γ, Q, ν) the* characteristics *of X.*

Defining

$$N(t, A) := \#\{s \in]0, t] \mid \Delta X(s) \in A\} \quad A \in \mathcal{B}(H\backslash\{0\}),$$

the Lévy process X induces a Poisson random measure (cf. Remark 1.13). We define the corresponding compensated Poisson random measure $\bar{N}(t, A) := N(t, A) - t\nu(A)$, $A \in \mathcal{B}(U\backslash\{0\})$, where ν is the intensity measure of X.

Theorem D.4 (Lévy-Itô decomposition)**.** *Let X be a Lévy process on U with the characteristics (γ, Q, ν). Then, for every $t \geq 0$,*

$$X(t) = t\gamma + B_Q(t) + \int_{\{\|x\|_U<1\}} x\ \bar{N}(t, dx) + \int_{\{\|x\|_U\geq 1\}} x\ N(t, dx),$$

where B_Q is a Brownian motion with covariance Q independent of $N(\cdot, A)$ for all $A \in \mathcal{B}(U\backslash\{0\})$.

Proof. See [AR05, Theorem 4.1]. □

Definition D.5. *A U-valued cádlág process X is called* quasi-left-continuous *if for every increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$*

$$\lim_{n\to\infty} X(\tau_n) = X\left(\lim_{n\to\infty} \tau_n\right) \quad \text{on } \left\{\lim_{n\to\infty} \tau_n < \infty\right\}.$$

Proposition D.6. *Every Lévy process is quasi-left-continuous.*

Proof. See [Bic02, Lemma 4.6.7, p.258]. □

D.2 One-dimensional Integration Theory

Let A be an non-decreasing process. For fixed $\omega \in \Omega$ the function $t \mapsto A_t(\omega)$ is right-continuous and non-decreasing. This function induces a measure $\mu_A(\omega, ds)$ on $\mathbb{R}_+$. For some bounded and jointly measurable $F(s, \omega)$, we can define the integral ω-wise as

$$I(t, \omega) = \int_0^t F(s, \omega)\, dA_s(\omega).$$

Then, I is continuous in t and jointly measurable.

Theorem D.7. *Let A, C be adapted, non-decreasing processes such that $dC \geq dA$. Then there exists a jointly measurable, adapted process H such that $H(t) \in [0, 1]$, $t \geq 0$ and*

$$A_t - A_0 = \int_0^t H_s\, dC_s.$$

Proof. See [Pro05, Chapter 1, Theorem 52, p.40]. □

More generally, we can consider integration with respect to a semimartingale.

Definition D.8. *A* semimartingale *is defined as a right-continuous, adapted process X admitting a decomposition $M + A$ where M is a local martingale and A is a process of finite variation starting at* 0.

For any semimartingales X and Y, the integral

$$\int_{]0,t]} X(s-)\, dY(s)$$

is well-defined (See eg. [Kal02, Chapter 26]).

D.3 Bracket Processes

Proposition D.9. *Let $M \in \mathcal{M}_T^2(U)$. Then there exists a unique predictable process $\langle M \rangle$ of bounded variation such that*

$$\|M(t)\|_U^2 - \langle M \rangle_t, \quad t \geq 0,$$

is a martingale.

Proof. See [PZ07, Remark 3.46, p.36]. □

Proposition D.10. *Let M be a U-valued cádlág local $(\mathcal{F}_t)$-martingale and let τ_n be a sequence of partitions $\{0 \leq t_1^n < t_2^n < \dots t_{k_n}^n\}$ where t_i^n, $1 \leq i \leq k_n$, are $(\mathcal{F}_t)$-stopping times such that $\lim\limits_{n\to\infty} t_{k_n}^n = \infty$ and $\lim\limits_{n\to\infty} \sup_{1\leq i\leq k_n-1} |t_{i+1}^n - t_i^n| = 0$. Then, the process*

$$A_n := \sum_{i=1}^{k_n-1} \left\| M_{t_{i+1}^n \wedge t} - M_{t_i^n \wedge t} \right\|_U^2$$

converges in $L^1(\Omega, \mathcal{F}_t, P)$ to an increasing process A_t.

Proof. See [Mét77, Section 31.4]. □

Definition D.11. *The limiting process A_t in Proposition D.10 is called* square bracket *and is denoted by $[M]_t$.*

Remark D.12. *i. In the real case, for two cádlág local martingales M, N, we have*

$$[M,N]_t = M(t)N(t) - \int_{]0,t]} M(s-)\, dN(s) - \int_{]0,t]} N(s-)\, dM(s)$$

ii. For a continuous processes X_t, we have $\langle X\rangle_t = [X]_t$.

D.4 Some Important Tools

The next theorem assures that hitting times of cádlág processes are in fact stopping times.

Theorem D.13. *Let X be an $(\mathcal{F}_t)$-adapted right-continuous process with values in U. Then*

$$\tau_C = \inf\{t \geq 0 \mid \|X(t)\| > C\}, \quad C > 0,$$

is a stopping time.

Proof. See [Kal02, Theorem 7.7, p.124]. □

Theorem D.14 (Burkholder–Davis–Gundy inequality)**.** *Let $p \geq 1$ and $(M_t)_{t\geq 0}$ be a real-valued cádlág local martingale with $M_0 = 0$. Then, for every stopping time τ, there exist constants c_p, $C_p > 0$ such that*

$$c_p E\,[M]_\tau^{\frac{p}{2}} \leq E\left[\sup_{t\in[0,\tau]} |M_t|^p\right] \leq C_p E\,[M]_\tau^{\frac{p}{2}},$$

where $[M]_t$ is the square bracket of M.

Proof. Apply [Kal02, Theorem 26.12, p.524] to the stopped process $(M_{\tau\wedge t})_{t\geq 0}$. $\square$

Theorem D.15 (Dolean exponential)**.** *Let X be a semimartingale such that $X_0 = 0$. Then the equation*

$$Z(t) = 1 + \int_{]0,t]} Z(s-)\, dX(s)$$

has the a.s. unique solution

$$Z(t) = \exp\left(X(t) - \frac{1}{2}[X]_t\right) \prod_{s\leq t}(1 + \Delta X(s))e^{-\Delta X(s)+\frac{1}{2}(\Delta X(s))^2}, \quad t \geq 0,$$

where the infinite product converges.

Proof. See [Pro05, Chapter 2, Theorem 37, p.84]. $\square$

Theorem D.16 (Integration by parts)**.** *Let M be a semimartingale and A a predictable process of finite variation. Then,*

$$A_t \cdot M_t = \int_{]0,t]} A(s)\, dM(s) + \int_{]0,t]} M(s-)\, dA(s), \quad P\text{-a.s.}.$$

Proof. See [Kal02, Lemma 26.10, p.523]. $\square$

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