Coalitional and Strategic Market Games

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Coalitional and Strategic Market Games

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Co-tutelle

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and
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Contents

Introduction

This thesis studies so called market games. Ideas from two important fields of economic theory are combined: **General Equilibrium Theory** and **Game Theory**.

Several types of economies are considered in *general equilibrium theory*. Of particular interest for this thesis are pure exchange economies, exchange economies with production and finite as well as infinite horizon exchange economies combined with financial markets. In the basic set up, the economic agents exchange their initially owned endowments and possibly trade financial assets. Moreover, firms, if there are some, are allowed to produce within a given production possibility set. A widely used solution concept is the notion of a competitive equilibrium defined for example in Debreu (1959). The basic idea is the following: Given a price system the agents of an economy maximize their utility taking their individual budget constraint into consideration while at the same time firms maximize their profits from production and a market clearing condition is satisfied.

Game theory is divided into two main branches, non-cooperative and cooperative game theory. A non-cooperative game is usually described by a set of players, a set of strategies or actions of each player and payoff or utility functions, that map strategy or action profiles into payoffs or utilities. In non-cooperative game theory the players individually choose their strategies and cooperation is not allowed or differently even if cooperation might be profitable, it cannot be enforced. Some well known solution concepts for non-cooperative games are the Nash equilibrium (for normalform games, Nash 1951) or its refinement the subgame perfect Nash equilibrium (for extensive form games, Selten 1965). For games with incomplete information the perfect Bayesian equilibrium is a widely used concept. In a Nash equilibrium for each player the action he chooses individually maximizes his payoff considering the actions of the other players as given. Differently from non-cooperative game theory, in a cooperative game the players are allowed to form coalitions and to choose joint actions. In general, a cooperative game is described by a set of players, from which the set of possible coalitions is derived as its power set, and a coalitional function, that defines the worth a coalition of players can achieve through cooperation (without giving specific shares to each player). One solution concept for cooperative games is the core (due to Gillies 1953). The idea of the core is the following: A utility allocation is in the core, if it is affordable by the grand coalition, consisting of all players, and if there is no coalition that can improve upon this allocation. Other solution concepts used in this thesis are a refinement of the core, namely the inner core (for example Shubik 1984, p.681–682), or for bargaining games, a special class of cooperative games, the Nash bargaining solution (Nash 1950, 1953).

My thesis includes different types of economies and non-cooperative games as well as cooperative games. As already mentioned, I look at a special class of games, so called *market games*. There is a non-cooperative version usually referred to as strategic market games introduced by Shapley and Shubik (1977). Furthermore, there are cooperative or coalitional market games, often just called market games, that go back to Shapley and Shubik (1969). The idea behind *strategic market games* is to model the price formation in an economy as a non-cooperative game and to abstain from the usual assumption of price taking behavior in general equilibrium models. Whereas in *coalitional market games* the relationship between cooperative games and markets, as a special kind of economies, and their solution concepts are investigated. My thesis consists of two main parts: The first one is on coalitional market games whereas the second one is on strategic market games. The goal of this introduction is to give a short overview on the main literature and the research questions investigated in this thesis including a short description of the formal results. Each of the different chapters of the two parts can be read independently. The different chapters are self-containing and include a more detailed introduction into the subject and its underlying literature as well as a complete description of the model.

Part I: Coalitional Market Games

In coalitional market games the relationship between cooperative games and markets or economies is investigated. From the cooperative point of view market games are cooperative games, with transferable (TU) or non-transferable utility (NTU), that generate or induce markets in a certain sense. Shapley and Shubik (1969) consider TU market games. They prove that every totally balanced TU game is a market game. Furthermore, Shapley and Shubik (1975) show that starting with a TU game every payoff vector in the core of that game is competitive in its direct market and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector. The idea of market games was applied to NTU games by Billera and Bixby (1974). Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced game, that is compactly convexly generated, is a market game. Qin (1993) compares the inner core of NTU market games with the competitive payoff vectors of markets that represent this game. He obtains the analogous results for NTU games as Shapley and Shubik (1975) conjectured.

Part I on coalitional market games is divided into three independent chapters, that were established in joint work with Jan-Philip Gamp.

First, we study TU market games in chapter 1. Based on Shapley and Shubik (1975) we investigate the relationship between certain subsets of the core for TU market games and competitive payoff vectors of certain markets linked to that game. This can be considered as the case in between the two extreme cases of Shapley and Shubik (1975). They already remark that their results can be extended to any closed convex subset of the core, but they omit the details of the proof which we present here. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

More precisely, let $N = \{1, 2..., n\}$ be a set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. A *cooperative game with* *transferable utility (TU)* is given by the pair (N, v) where N is the player set and $v : \mathcal{N} \to \mathbb{R}$ is the characteristic or coalitional function. A *market* is given by $\mathcal{E} = (X^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $\omega^i \in X^i$ is the initial endowment vector,
- $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Combining these two concepts in a certain sense we obtain TU market games, meaning that for a TU market game there exists a market such that the value a coalition S can reach according to the coalitional function coincides with the joint utility that is generated by feasible S-allocations in the market, resulting from redistributions of the initial endowment vectors within the coalition S. Having established the link between TU games and markets we look at the relationship between their solution concepts. Here, we elaborate on the details to prove the following theorem:

> **Theorem.** *Let (N, v) be a TU market game and let A be a closed, convex subset of the core. Then there exists a market such that this market represents the game (N, v) and such that the set of competitive payoff vectors of this market is the set A.*

Second, in chapter 2 we consider NTU market games. The extension of the results of Qin (1993) to subsets of the inner core remained so far an open problem. We extend his results to a large class of closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of the inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely and thus we obtain a class of markets with the desired property. We have some freedom in different aspects of our construction. First, to define our market we use an auxiliary NTU game where we enlarge the given NTU game. For this enlargement we use for every inner core point one of its normal vectors.

This normal vector is not always unique. Second, for the auxiliary game we define a mapping using a 'projection'. This projection can be chosen in different ways. Third, we add to the utility function an ε-term, that needs to be between certain bounds and hence is not defined uniquely. Thus, we do not obtain a single market but a whole class of markets.

Formally, again let $N = \{1, 2..., n\}$ be a set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. An *NTU (non-transferable utility) game* is a pair (N, V), that consists of a player set N and a coalitional function V, which defines for every coalition the utility allocations this coalition can reach, regardless of what the other players outside this coalition do. Hence, define the coalitional function V from the set of coalitions, $\mathcal{N}$, to the set of non-empty subsets of $\mathbb{R}^n$, such that for every coalition $S \in \mathcal{N}$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S)$ is non-empty and $V(S)$ is S-comprehensive, meaning $V(S) \supseteq V(S) - \mathbb{R}^S_+$. A *market (with production)* is given by $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,

- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,

- $\omega^i \in X^i - Y^i$, the initial endowment vector,

- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Combining these two concepts, in a similar as in the TU case, we obtain NTU market games. For an NTU market game there exists a market such that the set of utility allocations a coalition S can reach according to the coalitional function coincides with the set of utility allocations that is generated by feasible S-allocations in the market, resulting from redistributions of the initial endowment vectors within the coalition S and production plans in a joint production set of the coalition S. Having established the link between NTU games and markets we look at the relationship between their solution concepts. Let A be a closed subset of the inner core of a given compactly convexly generated NTU market game (N, V) such that (N, V) together with the set A satisfy a certain property, called strict positive separability. We establish the following result:

> **Theorem.** *Let $[(N, V), A]$ satisfy strict positive separability. Then there exists a market such that this market represents the game (N, V) and such that the set of competitive payoff vectors of this market is the set A.*

Third, in Chapter 3 we investigate the relationship between the inner core and asymmetric Nash bargaining solutions for n-person bargaining games with complete information. We show that the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core if all points in the underlying bargaining set are strictly positive. Furthermore, we prove that every bargaining game is a market game. By using the results of Qin (1993) we conclude that for every possible vector of weights of the asymmetric Nash bargaining solution there exists an economy that has this asymmetric Nash bargaining solution as its unique competitive payoff vector. We relate the literature of Trockel (1996, 2005) with the ideas of Qin (1993). Our result can be seen as a market foundation for every asymmetric Nash bargaining solution in analogy to the results on non-cooperative foundations of cooperative games.

More detailed, we consider NTU bargaining games as a special class of NTU games where smaller coalitions than the grand coalition do not gain from cooperation. They cannot reach higher utility levels as the singleton coalitions for all its members simultaneously. Only in the grand coalition every individual can be made better off. The *asymmetric Nash bargaining solution with a vector of weights* $\theta = (\theta_1, ..., \theta_n) \in \Delta^n_{++}$, for short θ-asymmetric, for a n-person NTU bargaining game (N, V) with disagreement point 0 is defined as the maximizer of the θ-asymmetric Nash product given by $\prod_{i=1}^n u_i^{\theta_i}$ over the set $V(N)$. To obtain a market foundation for the asymmetric Nash bargaining solutions we establish first the following Proposition:

> **Proposition.** *Let (N, V) be a n-person NTU bargaining game with disagreement point* 0 *and underlying bargaining set from* $\mathbb{R}^n_+$.

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the asymmetric Nash bargaining solution a^θ for θ is in the inner core of (N, V).*

Second after having argued that NTU bargaining games are market games we show:

> **Proposition.** *Given a n-person NTU bargaining game (N, V) (with disagreement point 0 and generating set from $\mathbb{R}^n_+$) and a vector of weights $\theta \in \Delta^n_{++}$, there is market that represents (N, V) and where additionally the unique competitive payoff vector of this market coincides with the θ-asymmetric Nash bargaining solution a^θ of the NTU bargaining game (N, V).*

Part II: Strategic Market Games

The idea of strategic market games goes back to Shapley and Shubik (1977). They use a non-cooperative game to describe the price formation in an exchange economy. Every player is asked to place a bid and an offer for every commodity. Afterwards the price of the commodity is computed as the ratio of the total bid to the total offer of that commodity. Strategic market games enable to study the feedback effect of trading strategies in illiquid markets when individual trades may have an impact on prices. An overview about strategic market games and related contributions can be found in Giraud (2003). In this thesis the departing point is the model in Giraud and Weyers (2004). They consider a strategic market game with finite horizon and (possibly) incomplete asset markets. Their main result is that generically every sequentially strictly individually rational and default-free stream of allocations can be approximated by a full subgame-perfect equilibrium.

Part II on strategic market games is divided into two independent chapters, where the second one was established in joint work with Gaël Giraud and presents a work that is still ongoing.

First, in chapter 4, I study a strategic market game with finite horizon, incomplete markets and the possibility of default. This is modeled by using collaterals. The model of a strategic market game with finite horizon and incomplete markets of Giraud and Weyers (2004) is extended by introducing the possibility of default. In order to avoid bankruptcy a collateral requirement for financial assets similarly as in Araujo et al. (2002) is introduced. I show that a given allocation that clears the markets and satisfies the budget constraints can be induced by defining appropriate, almost full strategies. Furthermore, I look at the set of sequentially strictly individually rational allocations and study the existence of approximately subgame perfect Nash equilibria. It turns out that the analogue of a perfect folk theorem similarly to the one in Giraud and Weyers (2004) holds. Hence, even with collateral requirements almost everything is possible as soon as people are sufficiently patient, since almost every feasible, affordable, sequentially strictly individually rational consumption stream can be obtained by means of some almost full approximate subgame perfect Nash equilibrium.

Formally, the following theorem is established:

> **Theorem.** *For any N, there exists an open and dense subset $\Omega^*(N)$ of initial endowments and an integer $T^0(N)$ and R such that for every finite horizon $T \geq T^0(N) \geq R$: if the initial endowments belong to $\Omega^*_T(N)$ and if the issuing nodes of all financial assets are in the first $T - R - 1$ periods, then every consumption stream $(\bar{x}^i)_{i\in\mathcal{N}}$, that is feasible, affordable and sequentially strictly individually rational in the first $T - T^0(N)$ periods, is an approximate subgame perfect Nash equilibrium of the strategic market game with finite horizon T.*

In a second contribution, chapter 5, coauthored by Gaël Giraud, we study a strategic market game with finitely many traders, infinite horizon and real assets. To this standard framework (see, e.g. Giraud and Weyers, 2004) we add two key ingredients: First, default is allowed at equilibrium by means of some collateral requirement for financial assets; second, information among players about the structure of uncertainty is incomplete. We focus on learning equilibria, at the end of which no player shares incorrect beliefs — not because those players with heterogeneous beliefs were eliminated from the market (although default is possible at equilibrium) but because they have taken time to update their prior belief. We then prove a partial Folk theorem à la Wiseman (2011) of the following form: For any function that maps each state of the world to a sequence of feasible and sequentially strictly individually rational allocations (for short SSIRF), and for any degree of precision, there is a perfect Bayesian equilibrium (PBE) in which patient players learn the realized state with this degree of precision and achieve a payoff close to the one specified for each state.

More precisely, the uncertain state of the world is a transition matrix that gives the probabilities with which a succeeding node in a tree-like time structure is reached. The sets of players and actions are common knowledge, but the distribution of initial endowments and one-period utility levels conditional on action profiles is chosen randomly in each period, and the players do not observe nature's choice. Neither do they observe each player's action. The probability distribution according to which uncertainty realizes in each period is a (stationary) Markov chain. This Markov distribution itself is chosen at random once and for all at the start of play, and, again, the investors do not observe nature's choice. The players have a common prior over the finite set of possible Markov chains (states of the world), and they have various ways of learning the state of the world over time. We make the following assumptions:

- *Assumption G:*
  The set of consumption goods is partitioned into two distinct subsets. Only commodities in the one set can be used as collateral, and assets' promises deliver only in commodities that belong to the other subset.

- *Informativeness Assumption (IA):*

  (1) For any pair of nodes $(t, s_{t-1}, s) = \xi \neq \xi' = (t, s_{t-1}, s')$, any player i, and any strategy profile, σ, that induces an SSIRF allocation at both states, the vectors of signals,

  $$(u^i_\xi(x^i_\xi(\sigma)), x^i_\xi(\sigma), w^i_\xi, A_j(\xi))$$

  and

  $$(u^i_{\xi'}(x^i_{\xi'}(\sigma)), x^i_{\xi'}(\sigma), w^i_{\xi'}, A_j(\xi'))$$

differ.

(2) Every ω is irreducible, aperiodic and admits an invariant measure, μ_ω. Moreover, for any pair ω, ω', if μ_ω and $\mu_{\omega'}$ are two corresponding invariant measures, then $\mu_\omega = \mu_{\omega'} \Rightarrow \omega \neq \omega'$.

Additionally, we introduce in the strategic market games the restriction that collaterals need to be established from the initial endowments of that period and we impose a condition avoiding self-trades on the asset markets. We show:

Theorem. *Under (G) and (IA), let* $\varepsilon > 0$ *and* $(x^{*i}[\omega])_{i \in \mathcal{N}, \omega \in \Omega}$ *be a* SSIRF *allocation in consumption goods, and let* $\mathbb{P}$ *be a prior belief that assigns strictly positive probability to each state of the world. Then there exists* $\lambda(\mathbb{P}) < 1$ *such that for all* $\lambda > \lambda(\mathbb{P})$*, there is a* PBE *that with probability at least* $1 - \varepsilon$*, conditional on any state* ω *being realized, yields a payoff vector within* ε *of* $\left(U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \sigma, \omega)\right)_i$*. In equilibrium, conditional on* ω*, each player i's* interim *private belief converges to the truth:* $\lim_{t \to \infty} \mathbb{P}^i_{\xi=(t, s_{t-1}, s)}(h^i_\xi)[\omega] = 1$ *with probability* 1.

Introduction en Français

Nous nous intéressons dans cette thèse de doctorat aux jeux de marchés. Des idées de deux domaines importants de la théorie économique y sont combinés : celles de la **théorie de l'équilibre général** et celles de la **théorie des jeux**.

Plusieurs modèles économiques sont considérés dans la *théorie de l'équilibre général*. Dans cette thèse on s'intéresse surtout aux économies d'échange pures, aux économies d'échange avec production et aux économies d'échange combinées à des marchés financiers à horizon fini ou infini. Dans le modèle de base les agents économiques échangent les dotations dont ils disposent au départ et peuvent également agir sur les marchés financiers lorsque ceux-ci existent. De plus, les entreprises, s'il en existe, peuvent produire des biens dans un ensemble de production donné. Un concept de solution largement répandu est la notion d'équilibre compétitif définie dans Debreu (1959) par exemple. L'idée principale est la suivante : étant donné un système de prix, les agents économiques maximisent leurs utilités tout en satisfaisant leurs contraintes budgétaires respectives, tandis que les entreprises maximisent leurs profits et une condition de liquidation du marché est satisfaite.

La *théorie des jeux* comporte deux domaines principaux, celui des jeux non-coopératifs et celui des jeux coopératifs. Un jeu non-coopératif peut être caractérisé par un ensemble de joueurs, un ensemble de stratégies ou d'actions possibles pour chacun des joueurs et chaque joueur est doté d'une fonction de paiement ou d'utilité. Chaque fonction associe à un profil de stratégies un paiement ou une utilité pour le joueur auquel elle correspond.

Dans un jeu non-coopératif les joueurs choisissent leurs stratégies individuellement et la coopération est interdite (même si la coopération pourrait être rentable, on suppose qu'il n'y a pas de mécanisme permettant qu'elle ait lieu). Les concepts de solution les plus connus pour les jeux non-coopératifs sont l'équilibre de Nash (pour les jeux sous forme normale, Nash 1951) et un raffinement de cet équilibre, à savoir, l'équilibre de Nash parfait en sous-jeux (pour les jeux dynamiques, Selten 1965). Dans le cas de jeux à information incomplète un des concepts les plus utilisés est celui d'équilibre Bayésien parfait. Dans l'équilibre de Nash chaque joueur choisit ses actions individuellement afin de maximiser son paiement en considérant les actions des autres joueurs comme étant données. A l'opposé des jeux non-coopératifs, les joueurs peuvent former des coalitions et choisir des actions collectivement dans les jeux coopératifs. En général un jeu coopératif est décrit par un ensemble de joueurs et une fonction caractéristique, qui associe à chaque coalition une valeur. Cette valeur correspond à ce que les joueurs de la coalition peuvent obtenir par coopération (sans spécifier les gains de chaque joueur dans la coalition). Un concept de solution pour les jeux coopératifs est celui de cœur (voir Gillies 1953). L'idée sous-jacente est la suivante : une allocation est dans le cœur du jeu si la grande coalition formée par l'ensemble de tous les joueurs peut obtenir cette allocation et si aucune autre coalition ne peut améliorer le gain de ses membres en quittant la grande coalition. Les autres concepts de solution utilisés dans cette thèse sont ceux du cœur interne, qui correspond à une amélioration du cœur (voir par exemple Shubik 1984, p.681-682) et, pour les jeux de négociation (qui constituent une classe particulière de jeux coopératifs), la solution de négociation de Nash (Nash 1950, 1953).

Ma thèse traite de différents types d'économies et de jeux non-coopératifs ainsi que de jeux coopératifs. J'étudie une classe particulière de jeux, appelés jeux de marchés. Lorsqu'ils sont non-coopératifs, ces jeux introduits par Shapley et Shubik (1977) sont généralement dénommés jeux de marchés stratégiques. Il existe également des jeux de marchés coopératifs, souvent appelés jeux de marchés, introduits par Shapley et Shubik (1969). L'idée des *jeux de marchés stratégiques* est d'utiliser un jeu non-coopératif afin de décrire la formation des prix dans une économie d'échange où l'individu a une influence stratégique sur le prix. Alors que dans le cas des *jeux de marchés coopératifs*, le lien entre jeux coopératifs et marchés (des types particuliers d'économies) et les concepts de solution associés sont étudiés. Ma thèse comporte deux parties : La première partie porte sur les jeux de marchés coopératifs et la deuxième sur les jeux de marchés stratégiques. L'objectif de cette introduction est de présenter les problèmes étudiés, la littérature associée ainsi que les principaux résultats obtenus. Chaque chapitre de la thèse inclut une description plus détaillée du problème traité et de la littérature associée ainsi qu'une description complète des modèles et des preuves.

Partie I : Les Jeux de Marchés Coopératifs

Dans les *jeux de marchés coopératifs* la relation entre jeux coopératifs et marchés ou économies est étudiée. Du point de vue coopératif, les jeux de marchés sont des jeux coopératifs (à utilité transférable (TU) ou utilité non-transférable (NTU)) qui peuvent, dans un certain sens, être représentés par des marchés. Shapley et Shubik (1969, 1975) considèrent des jeux de marchés coopératifs TU. Ils montrent que chaque jeu totalement équilibré est un jeu de marché. De plus, ces auteurs prouvent que chaque vecteur dans le cœur du jeu est un vecteur de paiement compétitif dans son marché direct et que pour chaque vecteur dans le cœur, il existe au moins un marché ayant ce vecteur comme unique vecteur de paiement compétitif. L'idée des jeux de marchés a été appliquée aux jeux de marchés coopératifs NTU par Billera et Bixby (1974). Analogiquement au résultat de Shapley et Shubik (1969), ils montrent que les jeux totalement équilibrés, qui sont générés compactement et convexement, sont des jeux de marchés. Qin (1993) compare le cœur interne des jeux de marchés NTU avec des vecteurs de paiement compétitifs de marchés représentant des jeux de marchés NTU. Il obtient un résultat analogue pour les jeux de marchés NTU comme conjecturé par Shapley et Shubik (1975).

La partie I sur les jeux de marchés coopératifs est subdivisée en trois chapitres indépendants dont les résultats ont été établis en commun avec Jan-Philip Gamp.

Tout d'abord, nous étudions les jeux de marchés coopératifs TU dans le chapitre 1. Partant de Shapley et Shubik (1975), nous examinons les relations entre certains sous-ensembles du cœur des jeux de marchés TU et les vecteurs de paiement compétitif des marchés associés à ces jeux. On peut considérer que ce type de relation a déjà été étudié dans les approches de Shapley et Shubik (1975). Leurs approches constituent en fait des cas extrêmes de notre approche. Ces auteurs ont déjà remarqué que leurs résultats peuvent être généralisés à des sous-ensembles convexes et fermés du coeur, mais ils n'en ont pas donné de preuves détaillées. Nous présentons ici une preuve dans un cadre plus général et obtenons ainsi un résultat plus fort dont les deux résultats de Shapley et Shubik (1975) sont des cas particuliers.

Plus précisément, soit $N = \{1, 2..., n\}$ un ensemble de joueurs. Soit $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ l'ensemble des coalitions non vides. Un *jeu coopératif avec utilité transférable (TU)* est défini par une paire (N, v) où N est l'ensemble des joueurs et $v : \mathcal{N} \to \mathbb{R}$ est une fonction de coalition, ou fonction caractéristique. Un marché est défini par $\mathcal{E} = (X^i, \omega^i, u^i)_{i \in N}$ avec pour chaque individu $i \in N$:

- $X^i \subseteq \mathbb{R}^\ell_+$ est un ensemble non vide, convexe et fermé, l'ensemble de la consommation, où $\ell \geq 1$, $\ell \in \mathbb{N}$ est le nombre de biens,
- $\omega^i \in X^i$ est le vecteur des dotations initiales,
- $u^i : X^i \to \mathbb{R}$ est une fonction continue et concave, la fonction d'utilité.

En combinant ces deux concepts d'une certaine manière, nous obtenons des jeux de marchés coopératifs TU. Dans un jeux de marché coopératif TU il existe un marché tel que la valeur $v(S)$ d'une coalition S coïncide avec l'utilité commune générée par les allocations réalisables pour S résultant d'une redistribution des dotations initiales au sein de la coalition S. Après avoir établi ce lien entre les jeux coopératifs TU et les marchés nous étudions les relations entre leurs différents concepts de solution. Dans ce chapitre 1, nous présenterons les détails nécessaires à la preuve du théorème suivant :

> **Théorème.** *Soit (N, v) un jeu de marché coopératif et soit A un sous-ensemble convexe et fermé du cœur. Alors il existe un marché qui représente le jeu (N, v) tel que l'ensemble des vecteurs de paiement compétitif de ce marché est l'ensemble A.*

Ensuite, dans le chapitre 2, nous considérons des jeux de marchés coopératifs NTU. L'extension des résultats de Qin (1993) était restée jusqu'à maintenant un problème ouvert. Nous étendons ses résultats à une large classe de sous-ensembles fermés du cœur interne : étant donné un jeu de marchés coopératifs NTU, nous construisons un marché qui dépend d'un ensemble fermé donné du cœur interne. Ce marché représente le jeu et de plus l'ensemble fermé correspond à l'ensemble des paiements des équilibres compétitifs du marché. Il se trouve que ce marché n'est pas déterminé de manière unique et nous obtenons ainsi une classe des marchés avec les propriétés voulues. Nous disposons d'une marge de liberté par rapport à certains aspects de notre construction. Premièrement, pour définir notre marché nous utilisons un jeu NTU auxiliaire dans lequel nous étendons le jeu NTU considéré. Pour cette extension, nous prenons pour chaque point dans le cœur interne un de ses vecteurs normaux. Ce vecteur normal n'est donc pas toujours unique. Deuxièmement, en ce qui concerne le jeu auxiliaire nous définissons une 'projection'. Cette projection peut être choisie de différentes manières. Troisièmement, nous ajoutons à la fonction d'utilité un terme ε qui est compris entre deux valeurs et qui n'est par conséquent pas défini de manière unique. Pour ces raisons, nous n'obtenons pas un seule marché mais une classe entière de marchés.

D'un point de vue formel, soit $N = \{1, 2..., n\}$ l'ensemble des joueurs. L'ensemble des coalitions non vides est donné par $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Un *jeu coopératif avec utilité non-transférable (NTU)* est défini par un couple (N, V), où N est l'ensemble des joueurs et $V : \mathcal{N} \to \mathbb{R}$ est une fonction de coalition, ou fonction caractéristique définissant les allocations d'utilité possibles pour les coalitions. La fonction caractéristique est définie de l'ensemble des coalitions $\mathcal{N}$ vers l'ensemble des ensembles non vides de $\mathbb{R}^n$, et est telle que pour chaque coalition $S \in \mathcal{N}$ nous avons $V(S) \subseteq \mathbb{R}^S$, $V(S)$ non vide et $V(S)$ est S-compréhensif, c'est-à-dire $V(S) \supseteq V(S) - \mathbb{R}^S_+$. Un *marché (avec production)* est défini par $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ avec pour chaque individu $i \in N$:

- $X^i \subseteq \mathbb{R}^\ell_+$ est un ensemble non vide, convexe et fermé, l'ensemble de la consommation, où $\ell \geq 1$, $\ell \in \mathbb{N}$ est le nombre de biens,

- $Y^i \subseteq \mathbb{R}^\ell$ est un ensemble non vide, convexe et fermé, l'ensemble de la production, tel que $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,

- $\omega^i \in X^i$ est le vecteur des dotations initiales,

- $u^i : X^i \to \mathbb{R}$ est une fonction continue et concave, la fonction d'utilité.

De manière similaire au cas des jeux coopératifs TU, la combinaison des deux concepts nous donne des jeux de marchés coopératifs NTU. Dans un jeux de marchés coopératifs NTU il existe un marché tel que l'ensemble des allocations d'utilité $V(S)$ d'une coalition S coïncide avec l'ensemble des allocations d'utilité générée par les allocations réalisables pour S, qui résultent d'une redistribution des dotations initiales et des plans de production dans l'ensemble de production de la coalition S. Après avoir établi un lien entre les jeux coopératifs NTU et les marchés, nous étudions les liens entre leurs concepts de solutions. Soit A un sous-ensemble fermé du cœur interne d'un jeu de marché coopératif NTU (N, V) à génération convexe et compacte, tel que (N, V) et le sous-ensemble A vérifient une propriété de séparabilité strictement positive. Nous établissons le résultat suivant :

> **Théorème.** *Soit $[(N, V), A]$ tel que la séparabilité strictement positive soit satisfaite. Alors il existe un marché qui représente le jeu (N, V) et tel que l'ensemble des vecteurs de paiement compétitif de ce marché soit l'ensemble A.*

Dans le chapitre 3 nous étudions la relation entre le cœur interne et les solutions de négociation asymétriques de Nash pour les jeux de négociation à n personnes avec information complète. Nous prouvons que l'ensemble des solutions de négociation asymétriques de Nash relatives pour différents vecteurs de poids strictement positifs coïncide avec le cœur interne si tous les points de l'ensemble de négociation sont strictement positifs. De plus, nous montrons que les jeux de négociation sont des jeux de marchés. En utilisant les résultats de Qin (1993) nous concluons que, pour chaque vecteur de poids possible de la solution de négociation asymétrique de Nash, il existe une économie ayant cette solution de négociation asymétrique comme unique vecteur de paiement compétitif. Nous établissons également des liens entre les travaux de Trockel (1996, 2005) et de Qin (1993). Notre résultat montre que la théorie des marchés peut être perçu comme un fondement de la solution de négociation asymétrique de Nash, par analogie avec les résultats concernant les fondements non-coopératifs des jeux coopératifs.

Plus précisément, nous considérons les jeux de négociation NTU comme une sous-classe des jeux coopératifs NTU dans lesquels les coalitions différentes de la grande coalition ne gagnent rien en coopérant. Elles ne peuvent pas obtenir plus que la somme des utilités des coalitions singletons correspondant à leurs membres. En revanche, dans la grande coalition, chaque individu peut obtenir plus que ce qu'il obtiendrait seul. La *solution de négociation asymétrique de Nash, pour un vecteur de poids* $\theta = (\theta_1, ..., \theta_n) \in \Delta^n_{++}$ (*i.e.* θ-asymétrique) et pour un jeu de négociation à n personnes (N, V) ayant pour point de désaccord 0, est définie comme le maximiseur du produit θ-asymétrique de Nash donné par $\prod_{i=1}^n u_i^{\theta_i}$ sur l'ensemble $V(N)$. Pour faire le lien entre la théorie des jeux de marché et la solution de négociation asymétrique de Nash, nous établissons d'abord la proposition suivante :

> **Proposition.** *Soit* (N, V) *un jeu de négociation à n-personnes ayant pour point de désaccord* 0 *et avec un ensemble de négociation fondamental dans* $\mathbb{R}^n_+$*.*
>
> - *Etant donné un vecteur de poids* $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$*, alors la solution de négociation asymétrique de Nash* a^θ *pour* θ *est dans le cœur interne de* (N, V)*.*

Ensuite, après avoir établi que les jeux de négociation sont des jeux de marchés NTU, nous prouvons :

> **Proposition.** *Etant donné un jeu de négociation à n-personnes (ayant pour point de désaccord* 0 *et un ensemble de négociation fondamental dans* $\mathbb{R}^n_+$*), il existe un marché qui représente le jeu* (N, V)*, de plus, l'unique vecteur de paiement compétitif de ce marché est la solution de négociation* θ*-asymétrique de Nash du jeu* (N, V)*.*

Partie II : Les Jeux de Marchés Stratégiques

L'idée de *jeux de marché stratégiques* remonte à Shapley et Shubik (1977). Ils utilisent un jeu non-coopératif afin de décrire la formation des prix dans une économie d'échange. On demande à chaque joueur de faire une demande et une offre pour chaque bien. Ensuite, le prix du bien est calculé comme étant le rapport entre la demande totale et l'offre totale de ce bien. Les jeux de marchés stratégiques permettent d'étudier les effets des stratégies individuelles sur les prix lorsque les marchés sont illiquides. Une introduction aux jeux de marchés stratégiques ainsi qu'un résumé de la littérature existante peuvent être trouvés dans Giraud (2003). Dans cette thèse le point de départ est le modèle de Giraud et Weyers (2004). Ils étudient un jeu de marché stratégique à horizon fini en présence de marchés financiers (qui peuvent être) incomplets. Leur résultat principal est que chaque allocation séquentiellement strictement individuellement rationnelle et sans défaut peut être approximée par un équilibre de Nash parfait de sous-jeux à stratégies complètes.

La partie II se compose de deux chapitres indépendants. Les résultats du deuxième chapitre correspondent à un travail commun avec Gaël Giraud.

Tout d'abord, dans le chapitre 4, j'étudie un jeu de marchés stratégiques à horizon fini, en présence de marchés financiers incomplets et j'introduis la possibilité de défaut utilisant des collatéraux. Le modèle de Giraud et Weyers (2004) présentant un jeu de marchés stratégique à horizon fini avec marchés financiers incomplets est enrichi par la prise en compte de la possibilité de défaut. Pour éviter la faillite, une obligation de collatéral pour les actifs financiers est introduite comme dans Araujo et al. (2002). Je montre qu'une allocation donnée des biens qui équilibre les marchés et satisfait les contraintes budgétaires peut être induite avec des stratégies appropriées, qui sont presque complètes. Ensuite, je regarde l'ensemble des allocations séquentiellement strictement individuellement rationnelles et j'étudie l'existence des équilibres de Nash parfaits en sous-jeux dans un sens approximatif. Comme dans Giraud et Weyers (2004), il est alors possible de prouver un théorème analogue à un théorème de folk. Ainsi, même en présence d'obligation de collatéral, presque tout est possible tant que les joueurs sont assez patients, puisque presque toutes les allocations réalisables, abordable et séquentiellement strictement individuellement rationnelle, peuvent être obtenues approximativement par un équilibre de Nash parfait en sous-jeux presque complet.

Formellement le théorème suivant est établi :

> **Théorème.** *Pour chaque N, il existe un sous-ensemble ouvert et dense $\Omega^*(N)$ des dotations initiales, et $T^0(N)$ et R tel que, pour tout horizon fini $T \geq T^0(N) \geq R$, si les dotations initiales sont dans $\Omega^*_T(N)$ et si les nœuds d'émission des actifs financiers sont dans les premières $T-R-1$ périodes, alors chaque allocation des biens $(\bar{x}^i)_{i\in\mathcal{N}}$ réalisable, abordable et séquentiellement strictement individuellement rationnelle dans les premières $T-T^0(N)$ périodes est une approximation de l' équilibre de Nash parfait en sous-jeux presque complet de jeu de marchés stratégique à horizon fini T.*

Dans une deuxième contribution, co–écrite avec Gaël Giraud, nous étudions un jeu de marché stratégique avec un nombre fini de joueurs, à horizon fini et avec de l'incertitude. Nous ajoutons dans le modèle standard (par exemple Giraud et Weyers 2004) les ingrédients principaux suivants : Premièrement, le défaut est possible dans l'équilibre avec exigence de collatéral pour les actifs financiers ; deuxièmement, l'information entre les joueurs à propos de la structure de l'incertitude est incomplète. Nous nous concentrons sur les équilibres avec apprentissage à l'issue desquels aucun des joueurs n'a de convictions incorrectes — non pas parce que les joueurs ayant des convictions hétérogènes ont été éliminés (quoique le défaut est possible dans l'équilibre) mais parce qu'ils ont pris du temps pour ajuster leurs convictions initiales. Nous prouvons alors un thèoreme de folk partiel à la Wiseman (2011) de la manière suivante : Pour chaque fonction associant à chaque état du monde une suite d'allocations admissibles et séquentiellement strictement individuellement rationnelles (SSIRF), et pour chaque degré de précision arbitraire, il existe un équilibre Bayésien parfait (PBE) dans lequel les joueurs apprennent l'état du monde avec ce degré de précision et ils obtiennent un paiement proche du paiement spécifié pour chaque état.

Plus précisément, l'état incertain du monde est une matrice de transition qui donne les probabilités avec lesquelles les nœuds suivants d'un arbre sont réalisés. L'ensemble des joueurs et des actions sont connus, mais les distributions des dotations initiales et les niveaux d'utilités conditionnels aux actions sont choisis de façon aléatoire à chaque période, et les joueurs n'observent pas le choix de la nature. Il n'ont pas non plus la possibilité d'observer les actions des autres joueurs. La distribution des probabilités avec laquelle l'incertitude se réalise à chaque période est une chaine de Markov stationnaire. Cette chaine de Markov est choisie au hasard une fois au début du jeu, et les investisseurs ne l'observent pas. Les joueurs commencent avec la même probabilité de départ sur l'ensemble fini des matrices de Markov (les états du monde) possibles, et ont plusieurs possibilités de découvrir le vrai état du monde. Nous faisons les hypothèses suivantes :

- *Hypothèse G :*
  L'ensemble des biens est partitionné en deux sous-ensembles distincts. Seuls les biens dans un sous-ensemble donné peuvent être utilisés comme collatéral et les promesses d'actifs se font en termes de biens appartenant à l'autre sous-ensemble.
- *Hypothèse sur les informations (IA) :*
  (1) Pour toute paire de nœuds $(t, s_{t-1}, s) = \xi \neq \xi' = (t, s_{t-1}, s')$, chaque joueur i et chaque profil de stratégies, σ, qui induit une allocation SSIRF dans les deux états, les vecteurs de signaux,
  $$(u^i_\xi(x^i_\xi(\sigma)), x^i_\xi(\sigma), w^i_\xi, A_j(\xi))$$
  et
  $$(u^i_{\xi'}(x^i_{\xi'}(\sigma)), x^i_{\xi'}(\sigma), w^i_{\xi'}, A_j(\xi'))$$
  sont différents.

  (2) Chaque ω est irréductible, apériodique et admet une mesure invariante, μ_ω. En plus, pour chaque paire ω, ω', si μ_ω et $\mu_{\omega'}$ sont

deux mesures invariantes, alors $\mu_\omega = \mu_{\omega'} \Rightarrow \omega \neq \omega'$.

En plus, nous introduisons dans les jeux de marchés stratégiques la restriction que les collatéraux doivent être établis en utilisant les dotations initiales à chaque période, et nous imposons aussi une condition pour éviter les ventes à soi-même sur les marchés d'actifs. Nous montrons :

Théorème. *Sous (G) et (IA), soit* $\varepsilon > 0$ *et soit* $(x^{*i}[\omega])_{i\in\mathcal{N},\omega\in\Omega}$ *une allocation* SSIRF *des biens de consommation, et soit* $\mathbb{P}$ *une conviction a priori assignant une probabilité strictement positive à chaque état du monde. Alors il existe* $\lambda(\mathbb{P}) < 1$ *tel que, pour tout* $\lambda > \lambda(\mathbb{P})$*, il y a un* PBE *ayant une probabilité supérieure ou égale à* $1 - \varepsilon$ *conditionnelle à chaque état* ω *réalisé, permettant d'obtenir un vecteur de paiement dans* ε *de* $\left(U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \sigma, \omega)\right)_i$. *A l'équilibre, conditionnel à* ω*, la conviction* interim *privée de chaque joueur* i *converge vers la verité*

$$\lim_{t\to\infty} \mathbb{P}^i_{\xi=(t,s_{t-1},s)}(h^i_\xi)[\omega] = 1$$

avec une probabilité égale à 1.

Part I

Coalitional Market Games

Chapter 1

Competitive Outcomes and the Core of TU Market Games

1.1 Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. Furthermore, Shapley and Shubik (1975) show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector. Moreover, they claim that an analogous result holds also for closed convex subsets of the core. Shapley and Shubik (1975) give a hint how this can be shown but they omit the details of the proof. By following this remark of Shapley and Shubik (1975) we give a detailed proof how their two main results can be extended to any closed convex subset of the core. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

Similarly to the approach of Shapley and Shubik (1969, 1975), Inoue (2010c) uses coalition production economies as in Sun et al. (2008) instead of markets. Inoue (2010c) shows that every TU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy whose set of competitive payoff vectors coincides with the core of the balanced cover of the original TU game.

A different extension of Shapley and Shubik (1969, 1975) is Garratt and Qin (2000). They consider time-constrained market games, where the agents are supposed to supply one unit of time to the market. Their main result is that a TU game is a time-constrained market game if and only if it is superadditive. This result of Garratt and Qin (2000) was again extended by

Bejan and Gómez (2011) introducing additionally location and free disposal constraints. They show that in this sense the entire class of TU games can be considered as market games.

For NTU market games Brangewitz and Gamp (2011a) extend the NTU analogue to Shapley and Shubik (1975), namely Qin (1993), to closed subsets of the inner core. Hereby, the techniques used to show the results in the TU and the NTU case are notably different.

1.2 TU market games

In this section we state the main definitions and results on TU market games. The following introduction for TU market games is mainly based on Shapley and Shubik (1969) and Shapley and Shubik (1975).

Let $N = \{1, 2..., n\}$ be a set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Thus, a coalition is a non-empty subset of players. A *cooperative game with transferable utility (TU)* is given by the pair (N, v) where N is the player set and $v : \mathcal{N} \to \mathbb{R}$ is the *characteristic* or *coalitional function.*[1] A *subgame* (T, v_T) of a TU game (N, v) is a subset of players $T \in \mathcal{N}$ and the characteristic function v_T with $v_T(S) = v(S)$ for $S \subseteq T$, $S \neq \emptyset$. A *payoff vector* for a TU game (N, v) is a vector $x \in \mathbb{R}^n$. The payoff of a coalition $S \in \mathcal{N}$ is given by $x(S) = \sum_{i \in S} x_i$. The *core* $C(v)$ of a TU game (N, v) is the set of payoff vectors where the value $v(N)$, the grand coalition N can achieve, is distributed and no coalition can improve upon,

$$C(v) = \{x \in \mathbb{R}^n |\, x(N) = v(N),\, x(S) \geq v(S) \quad \forall\, S \in \mathcal{N}\}.$$

Given a set of players $N = \{1, 2..., n\}$, a family $\mathcal{B} \subseteq \mathcal{N}$ is a *balanced family* if there exist weights $\{\gamma_S\}_{S \in \mathcal{B}}$, with $\gamma_S \geq 0$, such that for all $i \in N$ we have

$$\sum_{S \in \mathcal{B},\, S \ni i} \gamma_S = 1.$$

[1] Shapley and Shubik (1969) define the characteristic function as well for the empty set with $v(\emptyset) = 0$. Others, for example Billera and Bixby (1974), exclude the empty set from this definition.

The weights γ_S do not depend on the individual players but on the coalition $S \in \mathcal{N}$. The above condition can be as well written as

$$\sum_{S \in \mathcal{N}} \gamma_S e^S = e^N$$

where $e^S \in \mathbb{R}^n$ is the vector with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Let the set of weights be denoted by $\Gamma(e^N)$. The balancing weights can be interpreted as the intensity with which player i participates in a coalition or the fraction of time he spends to be in this coalition.

A TU game (N, v) is *balanced* if for every balanced family $\mathcal{B}$ with weights $\{\gamma_S\}_{S \in \mathcal{B}}$ we have

$$\sum_{S \in \mathcal{B}} \gamma_S v(S) \leq v(N).$$

A TU game (N, v) is *totally balanced* if all its subgames are balanced. The *totally balanced cover* of a TU game (N, v) is the smallest TU game $(N, \bar{v})$ that is totally balanced and contains the game (N, v).

Shapley and Shubik (1969) recall the following result of Shapley (1965):

Theorem 1.1 (Shapley and Shubik (1969))**.** *A game has a non-empty core if and only if it is balanced.*

In oder to define a TU market game we first need to introduce the notion of a market. For the TU case it is sufficient to consider markets without production.

Definition 1.1 (market)**.** Let $N = \{1, 2..., n\}$ be the set of agents (or players). A *market* is given by $\mathcal{E} = (X^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $\omega^i \in X^i$ is the initial endowment vector,
- $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in the case with non-transferable utility (NTU) usually markets with production are considered, see for example Billera and Bixby (1974) or Qin (1993).

Let $S \in \mathcal{N}$ be a coalition. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \,\middle|\, x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} x^i = \sum_{i \in S} \omega^i \right\}.$$

Elements of $F(S)$ are often denoted for short by x^S. The feasible S-allocations are those allocations the coalition S can achieve by redistributing their initial endowments within the coalition.

Now we define a TU market game in the following way:

Definition 1.2 (TU market game)**.** A TU game (N, v) that is representable by a market is a *TU market game.* This means there exists a market $\mathcal{E}$ such that $(N, v_{\mathcal{E}}) = (N, v)$ with

$$v_{\mathcal{E}}(S) = \max_{x^S \in F(S)} \sum_{i \in S} u^i(x^i) \quad \text{for all } S \in \mathcal{N}.$$

For a TU market game there exists a market such that the value a coalition S can reach according to the coalitional function coincides with the joint utility that is generated by feasible S-allocations in the market.

Given a TU game we can generate a market from this game in a "natural" way. Shapley and Shubik (1969) call this market a direct market.

Definition 1.3 (direct market)**.** A TU game (N, v) generates a *direct market* $\mathcal{D}_v = (X^i, \omega^i, u^i)_{i \in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+$,
- the initial endowment $\omega^i = e^{\{i\}}$ with $e^{\{i\}}_i = 1$ and $e^{\{i\}}_j = 0$ for $j \neq i$,
- the utility function $u^i(x) = \max \left\{ \sum_{S \in \mathcal{N}} \gamma_S v(S) \,\middle|\, \gamma_S \geq 0 \,\forall\, S \in \mathcal{N}, \sum_{S \in \mathcal{N}} \gamma_S e^S = x \right\}$.

The utility function $u^i(\cdot)$ of the direct market $\mathcal{D}_v$ is identical for every individual $i \in N$ and is homogeneous of degree 1, concave and continuous. Note that in a direct market every consumer owns initially his own (private) good or interpreted differently every player "is" himself a good. Using the direct market $\mathcal{D}_v$, Shapley and Shubik (1969) obtain the following characterization of TU market games.

Theorem 1.2 (Shapley and Shubik (1969))**.** *A game is a market game if and only if it is totally balanced.*

This means that in order to consider TU market games it is sufficient to consider just those TU games that are totally balanced. To obtain the above result Shapley and Shubik (1969) start by looking at an arbitrary TU game and its direct market. Hereafter, they consider the TU game of the direct market and show that it is equal to the totally balanced cover of the TU game they started with.

In a second paper Shapley and Shubik (1975) investigate the relationship between competitive payoffs, that arise from a competitive solution in the market, and the core of TU market games.

Definition 1.4 (competitive solution)**.** A *competitive solution* is an ordered pair $(p^*, (x^{*i})_{i \in N})$, where p^* is an arbitrary n-vector of prices and x^{*N} is a feasible N-allocation, such that

$$u^i(x^{*i}) - p^* \cdot x^{*i} = \max_{x^i \in \mathbb{R}^l_+} [u^i(x^i) - p \cdot x^i] \quad \text{for all } i \in N.$$

We are in a setting with transferable utility. Thus, there is implicitly the additional commodity money, that makes the transfer of utility possible. Suppose ξ_0^i are the initial money holdings of agent i. Then his "true" maximization problem is

$$\max_{x^i \in \mathbb{R}^l_+} [u^i(x^i) + \xi_0^i - p \cdot \left(x^i - \omega^i\right)].$$

Since the solution of the maximization problem is independent of the initial money holdings and the initial endowment, it is equivalent to solve the in the definition above stated maximization problem.

Definition 1.5 (competitive payoff vector)**.** A vector α^* is a *competitive payoff vector* if it arises from a competitive solution $(p^*, (x^{*i})_{i \in N})$ such that

$$\alpha^{*i} = u^i(x^{*i}) - p^* \cdot (x^{*i} - \omega^i).$$

Shapley and Shubik (1975) show the following two relationships between the core and competitive payoff vectors.

Theorem 1.3 (1, Shapley and Shubik (1975))**.** *Every payoff vector in the core of a TU market game is competitive in the direct market of that game.*

Theorem 1.4 (2, Shapley and Shubik (1975))**.** *Among the markets that generate a given totally balanced TU game, there exists a market having any given core point as its unique competitive payoff vector.*

These two theorems represent the two extreme cases where on the one hand the whole core equals the set of competitive payoff vectors of the direct market and one the other hand a given core point is the unique competitive payoff vector of a certain other market. The main ideas to prove the above two theorems are the following: For the first result Shapley and Shubik (1975) use the direct market to show that its competitive payoff vectors coincide with the core of the TU market game. To prove the second theorem they introduce a second game with a modified coalitional function for the grand coalition N. Afterwards they look at the direct market of the original game with a modified utility function depending on a given core point. Finally they show that this market represents the original TU game and has a given core point as its unique competitive payoff vector.

1.3 Results on TU market games

Shapley and Shubik (1975) already remark that for TU market games a extension of their proof for their second theorem leads to the following result.

Theorem 1.5. *Let (N, v) be a totally balanced TU game and let A be a closed, convex subset of the core. Then there exists a market such that this* *market represents the game (N, v) and such that the set of competitive payoff vectors of this market is the set A.*

Shapley and Shubik (1975) omit the details of the proof. We elaborate on them here. They remark that it is enough to change the definition of the utility function.

In the following we first define the according market and show afterwards in two steps that this market satisfies the properties we require.

Let (N, v) be a totally balanced TU game with $N = \{1, ..., n\}$ the set of players and the coalitional function v. Let $\mathcal{D}_v$ be its direct market as defined before. For $d \in \mathbb{R}_{++}$ define the TU game (N, v_d) by

$$v_d(S) = v(S) \quad \text{for all } S \subset N$$

and

$$v_d(N) = v(N) + d.$$

Since $d > 0$ the game (N, v_d) is totally balanced. Analogously let $\mathcal{D}_{v_d}$ be the direct market of (N, v_d). Let $(u^i_d)_{i \in N}$ denote the utility functions of $\mathcal{D}_{v_d}$, i.e.

$$u^i_d(x) = \max \left\{ \sum_{S \in \mathcal{N}} \gamma_S v_d(S) \middle| \gamma_S \geq 0 \, \forall \, S \in \mathcal{N}, \sum_{S \in \mathcal{N}} \gamma_S e^S = x \right\}.$$

As the utility functions u^i_d in the direct market $\mathcal{D}_{v_d}$ are identical for every individual $i \in N$, we write for short u_d.

Let A be a any non-empty closed convex subset of the core. For $\alpha \in A$ let $u_{d,\alpha}$ be defined as

$$u_{d,\alpha}(x) = \min(u_d(x), \alpha \cdot x).$$

Then define the function $u_{d,A}$ by

$$u_{d,A}(x) = \min_{\alpha \in A} u_{d,\alpha}(x).$$

Since $u_{d,A}$ is continuous and concave we can define a market by

$$\mathcal{E}_{v_d} = \left(\mathbb{R}^n_+, e^{\{i\}}, u^i_{d,A}\right)_{i \in N}.$$

with $u^i_{d,A} = u_{d,A}$ for all $i \in N$. It is easy to see that $u_{d,A}$ is homogeneous of degree 1.

Next, we show first that the market game of this market is (N, v) and second that the set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ is exactly the set A.

Proposition 1.1. *The market $\mathcal{E}_{v_d}$ represents the game (N, v).*

Proof. Recall that for the market $\mathcal{E}_{v_d}$ the set

$$F(S) = \left\{ x^S \in \mathbb{R}^{n \cdot S}_+ | \sum_{i \in S} x^i = \sum_{i \in S} e^{\{i\}} \right\}$$

is the set of feasible allocations for a coalition $S \in \mathcal{N}$.

Looking at the market game generated by the market $\mathcal{E}_{v_d}$ we obtain

$$\begin{aligned}
v_{\mathcal{E}_{v_d}}(S) &= \max_{x^S \in F(S)} \sum_{i \in S} u^i_{d,A}(x^i) \\
&= |S| \max_{x^S \in F(S)} \sum_{i \in S} \frac{1}{|S|} u_{d,A}(x^i) \\
&\overset{(1)}{=} |S| \max_{x^S \in F(S)} u_{d,A}\left(\frac{e^S}{|S|}\right) \\
&= |S| u_{d,A}\left(\frac{e^S}{|S|}\right) \\
&\overset{(2)}{=} u_{d,A}(e^S) \\
&= \min_{\alpha \in A} u_{d,\alpha}(e^S) \\
&= \min_{\alpha \in A}(\min(u_d(e^S), \alpha \cdot e^S)) \\
&\overset{(3)}{=} \min_{\alpha \in A}(\min(v_d(S), \alpha \cdot e^S)) \\
&= \min_{\alpha \in A}(v_d(S), \alpha \cdot e^S)
\end{aligned}$$

$$\overset{(4)}{=} v(S)$$

The detailed arguments are the following:

(1) First observe that $\sum_{i\in S}\frac{1}{|S|}u_{d,A}(x^i) \leq u_{d,A}\left(\sum_{i\in S}\frac{x^i}{|S|}\right) = u_{d,A}\left(\frac{e^S}{|S|}\right)$ from the concavity of $u_{d,A}$ and the market clearing condition. We take the maximum on both sides over the feasible S-allocations $F(S)$ and we observe that $\bar{x}^i = \frac{1}{|S|}e^S$ for all $i \in S$ is a feasible S-allocation. Therefore, we obtain that setting $(\bar{x}^i)_{i\in S}$ maximizes the expression on the left side and hence we get equality.

(2) The equality follows from the homogeneity of degree 1 of $u_{d,A}$.

(3) Using the totally balancedness of the game (N, v_d) we obtain

$$u_d(e^S) = \max\left\{\sum_{T\in\mathcal{N}}\gamma_T v_d(T)\middle|(\gamma_T)\geq 0, \sum_{T\in\mathcal{N}}\gamma_T e^T = e^S\right\} = v_d(S).$$

(4) For $S \subset N$ this minimum is equal to $v(S)$, since α is in the core of the TU game (N, v) and therefore $\alpha \cdot e^S \geq v(S) = v_d(S)$. For $S = N$ the minimum is equal to $\alpha' \cdot e^N$ for some $\alpha' \in A$ and since α' is in the core of (N, v) we have $\alpha' \cdot e^N = v(N)$. As $d > 0$ we have $v(N) < v_d(N)$.

Thus $v_{\mathcal{E}_{v_d}} = v$ and hence the market $\mathcal{E}_{v_d}$ generates the game (N, v). □

Proposition 1.2. *The set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ are coincides with the set A.*

Proof. The proof is divided into five parts:

1. First, suppose $((x^{*i})_{i\in N}, p^*)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then competitive payoffs are of the form $\left(p^* \cdot e^{\{i\}}\right)_{i\in N}$.

   From the definition of a competitive solution it follows that $(x^{*i})_{i\in N}$ clears the markets,

$$\sum_{i=1}^{n} x^{*i} = \sum_{i=1}^{n} e^{\{i\}} = e^N$$

and maximizes for each trader i his trading profit given by

$$u_{d,A}(x^i) - p \cdot x^i.$$

Moreover, we have from the existence of a maximum and the fact that the trading profit as a function of the consumption bundle is homogeneous of degree 1 that

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} = 0.$$

Looking at the competitive payoffs of competitive solutions we observe

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

2. Second, suppose $((x^{*i})_{i \in N}, p^*)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then $\left(\left(\frac{1}{n}e^N\right)_{i \in N}, p^*\right)$ is as well a competitive solution in the market $\mathcal{E}_{v_d}$. In addition the competitive payoffs coincide.

   From the fact that the trading profit equals zero we obtain

$$\begin{aligned} u_{d,A}\left(\frac{1}{n}e^N\right) - p^* \cdot \frac{1}{n}e^N &= u_{d,A}\left(\frac{1}{n}\sum_{i=1}^n x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^n x^{*i} \\ &\overset{(1)}{=} \frac{1}{n}\sum_{i=1}^n u_{d,A}\left(x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^n x^{*i} \\ &= \frac{1}{n}\left(\sum_{i=1}^n u_{d,A}\left(x^{*i}\right) - p^* \cdot \sum_{i=1}^n x^{*i}\right) \\ &= \frac{1}{n}\left(\sum_{i=1}^n \left(u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i}\right)\right) \\ &= 0. \end{aligned}$$

   The detailed argument is the following:

   (1) Using the concavity of $u_{d,A}$ gives us "$\geq$" and from maximality of x^{*i} we obtain the equality.

   As already seen in 1., looking at the competitive payoffs of these competitive solutions we observe

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = u_{d,A}\left(\frac{1}{N}e^N\right) - p^* \cdot \left(\frac{1}{N}e^N\right) + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

To summarize these results mean that looking for competitive solutions and their competitive payoffs we can focus on possible equilibrium prices of the allocation $\left(\frac{1}{N}e^N\right)_{i\in N}$. Then those competitive solutions give us all possible competitive payoffs.

3. Third, as in the proof of Proposition 1.1, equality (3)

$$u_d\left(\frac{1}{N}e^N\right) = \frac{1}{N}v_d(N) > \frac{1}{N}v(N) = u_{d,A}\left(\frac{1}{N}e^N\right)$$

and furthermore

$$u_{d,A}\left(\frac{1}{N}e^N\right) = \alpha' \cdot \left(\frac{1}{N}e^N\right)$$

for all $\alpha' \in A$. Because of the continuity of $u_d(\cdot)$ it follows for all $\alpha' \in A$ that $u_d(x) > \alpha' \cdot x$ for x in a small neighborhood of $\frac{1}{N}e^N$. Thus, in a neighborhood of $\frac{1}{N}e^N$, $u_{d,A}(x) = \min_{\alpha' \in A}(\alpha' \cdot x)$.

4. Forth, it remains to check for which prices p^* the pair $\left(\left(\frac{1}{N}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. In a first step we show that each $p^* \in A$ can be chosen as an equilibrium price vector, in a second step we show that any $p^* \notin A$ cannot be an equilibrium price vector. For the second step it is enough to concentrate on $p^* \in C(v) \setminus A$ as we have seen in 1. that the equilibrium price vector determines the competitive payoff vector, which are necessarily in the core.

   Step 1: Suppose $p^* \in A$. Then for all $x^i \in \mathbb{R}^n_+$ we have

$$\min_{\alpha' \in A}\left(\alpha' \cdot x^i\right) - p^* \cdot x^i \leq p^* \cdot x^i - p^* \cdot x^i = 0$$

and furthermore

$$\min_{\alpha' \in A}\left(\alpha' \cdot \left(\frac{1}{N}e^N\right)\right) - p^* \cdot \left(\frac{1}{N}e^N\right) = 0.$$

Hence, $x^i = \frac{1}{N}e^N$ maximizes the trading profit of agent i. Furthermore, the markets clear, as $\sum_{i \in N} \frac{1}{N}e^N = e^N$.

So, the pair $\left(\left(\frac{1}{N}e^N\right)_{i \in N}, p^*\right)$ is a competitive solution.

Step 2: Suppose $p^* \in C(v) \setminus A$. Recall that the set A is compact and convex. Hence, we can apply the separating hyperplane theorem[2] and obtain that there exists $\bar{x} \in \mathbb{R}^n_+$ such that for all $\alpha \in A$

$$\alpha \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

Therefore we conclude that

$$\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

Now, for sufficiently small $\varepsilon > 0$ we have that $\frac{1}{N}e^N + \varepsilon\bar{x}$ is in a neighborhood of $\frac{1}{N}e^N$ where we have $u_{d,A}(x) = \min_{\alpha' \in A}(\alpha' \cdot x)$. But

$$\begin{aligned}&\min_{\alpha' \in A}\left(\alpha' \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right)\right) - p^* \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right)\\ &= \varepsilon\left(\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x}\right) > 0.\end{aligned}$$

This implies that $\frac{1}{N}e^N$ does not maximize agent i's trading profit for $p^* \notin A$.

5. To summarize the line of argument:

If $((x^{*i})_{i \in N}, p^*)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then by 2. we have that $\left(\left(\frac{1}{n}e^N\right)_{i \in N}, p^*\right)$ is a competitive solution. By 4. we show that $p^* \in A$ and by 1. we know that its competitive payoff vector

[2] See for example Mas-Colell et al. (1995, Theorem M.G.2, p.948).

is equal to p^*.
On the other hand if $p^* \in A$ then by 4. we have that $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. The competitive payoff vector is equal to p^*.

□

1.4 Concluding Remarks

Shapley and Shubik (1975) investigate the relationship between competitive payoffs of markets that represent a cooperative game and their relation to solution concepts for cooperative games. We presented the details of the proof of Shapley and Shubik (1975), that extends their two main results to closed, convex subsets of the core. This shows also the two theorems of Shapley and Shubik (1975). In a further contribution (Brangewitz and Gamp, 2011a) we establish an analogue result for NTU market games.

Chapter 2

Competitive Outcomes and the Inner Core of NTU Market Games

2.1 Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. Furthermore, Shapley and Shubik (1975) show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector.

Cooperative games with non-transferable utility (NTU) can be considered as a generalization of TU games, where the transfer of the utility within a coalition does not take place at a fixed rate. In this paper we consider NTU market games. After Shapley and Shubik (1969), Billera and Bixby (1974) investigated the NTU case and obtained similar results for compactly convexly generated NTU games. Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced NTU game, that is compactly convexly generated, is a market game. The inner core is a refinement of the core for NTU games. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector.

Similarly to the approach of Billera and Bixby (1974), Inoue (2010b) uses coalition production economies as in Sun et al. (2008) instead of markets. Inoue (2010b) shows that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy whose set of competitive payoff vectors coincides with the inner core of the balanced cover of the original NTU game.

Here we consider the classical approach using markets. We investigate the case in between the two extreme cases of Qin (1993), where on the one hand there exists a market that has the complete inner core as its set of competitive payoff vectors and on the other hand there is a market that has a given inner core point as its unique competitive payoff vector. We extend the results of Qin (1993) to closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of the inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely. Several parameters in our construction can be chosen in different ways. Thus, we obtain a class of markets with the desired property.

Shapley and Shubik (1975) remark that in the TU case their result can be extended to any closed and convex subset of the core. Whether a similar result analogously to the one of Shapley and Shubik (1975) holds for NTU market games, was up to now not clear. Our result shows, that in the NTU case it is even possible to focus on closed, typically non-convex, subsets of the inner core.

The inner is one solution concept for NTU games. Extending the results of Qin (1993) to closed subsets of the inner core means in particular to show such a result for all solution concepts selecting closed subsets of the inner core.

2.2 NTU market games

Let $N = \{1, ..., n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ be a set of players. Let $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ be the set of coalitions. Define for a coalition $S \in \mathcal{N}$ the following sets $\mathbb{R}^S = \{x \in \mathbb{R}^n | x_i = 0 \text{ if } i \notin S\} \subseteq \mathbb{R}^n$, $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S | x_i \geq 0 \text{ for all } i \in S\} \subseteq \mathbb{R}^n_+$, $\mathbb{R}^S_{++} = \{x \in \mathbb{R}^S | x_i > 0 \text{ for all } i \in S\} \subseteq \mathbb{R}^n_{++}$. For a vector $a \in \mathbb{R}^n$ and a coalition $S \in \mathcal{N}$ let a^S denote the vector, where for $i \in S$ we have $a^S_i = a_i$ and $a^S_j = 0$ for $j \notin S$. Moreover, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ denote the inner product by $a \cdot b = \sum_{i=1}^n a_i b_i$ and the Hadamard product by $a \circ b = (a_1 b_1, ..., a_n b_n)$.

An *NTU (non-transferable utility) game* is a pair (N, V), that consists of a player set $N = \{1, ..., n\}$ and a coalitional function V, which defines for every coalition the utility allocations this coalition can reach, regardless of what the other players outside this coalition do. Hence, define the coalitional function V from the set of coalitions, $\mathcal{N}$, to the set of non-empty subsets of $\mathbb{R}^n$, such that for every coalition $S \in \mathcal{N}$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S)$ is non-empty and $V(S)$ is S-comprehensive, meaning $V(S) \supseteq V(S) - \mathbb{R}^S_+$.

The literature on NTU market games, as for example Billera and Bixby (1974) and Qin (1993), considers NTU games that are compactly and convexly generated. An NTU game (N, V) is *compactly (convexly) generated* if for all coalitions $S \in \mathcal{N}$ there exists a compact (convex) set $C^S \subseteq \mathbb{R}^S$ such that the coalitional function has the form $V(S) = C^S - \mathbb{R}^S_+$.

Given a player set $N = \{1, ..., n\}$ the set of balancing weights is defined by $\Gamma(e^N) = \left\{(\gamma_S)_{S \subseteq N} | \gamma_S \geq 0 \ \forall \ S \subseteq N, \sum_{S \subseteq N} \gamma_S e^S = e^N \right\}$. The balancing weights can be interpreted in the following way: Every player i has one unit of time that he can split over all the coalitions, he is a member of, with the constraint that a coalition has to agree on a common weight. Thereby, each player has to spend all his time. The weight γ_S can be seen as well as the intensity with which each player participates in the coalition $S \in \mathcal{N}$. In particular, if we have a partition of the player set into a coalition S and its complement $N \setminus S$ a balancing weight can be defined by $\gamma_S = \gamma_{N \setminus S} = 1$ and $\gamma_T = 0$ for all other coalitions T except for S and $N \setminus S$. An NTU game (N, V) is *balanced* if for all balancing weights $\gamma \in \Gamma(e^N)$ we have $\sum_{S \subseteq N} \gamma_S V(S) \subseteq V(N)$. Moreover, an NTU game (N, V) is *totally balanced* if it is balanced in all subgames. This means for all coalitions $T \in \mathcal{N}$ and for all balancing weights $\gamma \in \Gamma(e^T) = \left\{(\gamma_S)_{S \subseteq T} | \gamma_S \geq 0 \ \forall \ S \subseteq T, \sum_{S \subseteq T} \gamma_S e^S = e^T\right\}$ we have $\sum_{S \subseteq T} \gamma_S V(S) \subseteq V(T)$.

In order to define an NTU market game we first consider the notion of a market which is less general than the notion of an economy according to for example Arrow and Debreu (1954). In a market the number of consumers coincides with the number of producers. Each consumer has his own private production set. In contrast to the usual notion of an economy a market is assumed to have concave and not just quasi concave utility functions.

Definition 2.1 (market)**.** A *market* is given by $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,
- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

As pointed out before in a market each consumer is assumed have his own private production set. This assumption is not as restrictive as it appears to be. A given private ownership economy can be transformed into an economy with the same number of consumers and producers without changing the set of competitive equilibria or possible utility allocations, see for example Qin and Shubik (2009, section 4).

In the following, we often consider markets where $X^i \subseteq \mathbb{R}^{kn}_+$ with $k, n \in \mathbb{N}$. Then, consumption vectors are usually written as $x^i = \left(x^{(1)i}, ..., x^{(k)i}\right) \in X^i$ where $x^{(m)i} \in \mathbb{R}^n_+$ for $m = 1, ..., k$. In a sense, we divide the kn consumption goods in k consecutive groups of n goods. The vector $x^{(m)i}$ is the m^{th} group of n consumption goods of the consumption vector x^i. We use an analogous notation for the production goods and price vectors.

Given a market we define which allocations are considered as feasible for some coalition $S \in \mathcal{N}$. An S-allocation is a tuple $(x^i)_{i \in S}$ such that $x^i \in X^i$ for each $i \in S$. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

Hence, an S-allocation is feasible if there exist for all $i \in S$ production plans $y^i \in Y^i$ such that $\sum_{i \in S}(x^i - \omega^i) = \sum_{i \in S} y^i$. We refer to a feasible S-allocation in the following together with suitable production plans as a feasible S-allocation $(x^i)_{i \in S}$ with $(y^i)_{i \in S}$.

In the definition of feasibility it is implicitly assumed that by forming a coalition the available production plans are the sum of the individually available production plans. This approach is different from the idea to use coalition production economies, where every coalition has already in the definition of the economy its own production possibility set. Nevertheless, a market can be "formally" transformed into a coalition production economy by defining the production possibility set of a coalition as the sum of the individual production possibility sets.

Given the notion of a market and of feasible allocations for coalitions $S \in \mathcal{N}$ we define an NTU market game in the following way:

Definition 2.2 (NTU market game)**.** An NTU game (N, V) that is representable by a market is an *NTU market game*. This means there exists a market $\mathcal{E}$ such that $(N, V_{\mathcal{E}}) = (N, V)$ with

$$V_{\mathcal{E}}(S) = \left\{ u \in \mathbb{R}^S \,|\, \exists\, (x^i)_{i \in S} \in F(S), u_i \leq u^i(x^i), \,\forall\, i \in S \right\}.$$

For an NTU market game there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible S-allocations in the market or that give less utility than some feasible S-allocation.

One of the main results on NTU market games in Billera and Bixby (1974) is the following:

Theorem 2.1 (2.1, Billera and Bixby (1974))**.** *An NTU game (N, V) is an NTU market game if and only if it is totally balanced and compactly convexly generated.*

Hence, in order to study NTU market games, it is sufficient to look at those NTU games that are totally balanced and compactly convexly generated.

For the succeeding analysis, it will be useful to shift a given NTU game in the following way (compare Billera and Bixby (1973b, Proposition 2.2)): Given a vector $c \in \mathbb{R}^n$ define the coalitional function $(V + c)$ via $(V + c)(S) = V(S) + \sum_{i \in S} c_i$. To represent a shifted game by a market we have to shift the utility function of agent i by c_i. Hence, the shifted game with coalitional function $(V + c)$ is again a market game. Furthermore, shifting the utility functions of the agents does not change the set of competitive equilibria. Having this idea of shifting in mind we will focus in some proofs on games where for every coalition $S \in \mathcal{N}$ we have $C^S \subseteq \mathbb{R}^S_{++}$.

To prove the above result Billera and Bixby (1974) introduce the notion of an induced market that arises from a compactly convexly generated NTU game.

Definition 2.3 (induced market)**.** Let (N, V) be a compactly convexly generated NTU game. The *induced market* of the game (N, V) is defined by

$$\mathcal{E}_V = (X^i, Y^i, u^i, \omega^i)_{i \in N}$$

with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{2n}$,
- the production set $Y^i = \textit{convexcone} \left[\bigcup_{S \in \mathcal{N}} \left(C^S \times \{-e^S\} \right) \right] \subseteq \mathbb{R}^{2n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i(x^i) = x_i^{(1)i}$.

It can easily be seen that this is a market according to the previous definition. Note that in an induced market we have input and output goods. Initially every consumer owns one unit of his personal input good that can only be used for the production process. By using his input good the consumer can get utility just from his personal output good. The consumption and production set are the same for every player. Just the utility functions and the initial endowments are dependent on the player.

The individual production sets in an induced market are convex cones and identical for all agents. In this situation taking the sum over production sets of some agents leads to the same production set. Setting $Y = \sum_{i\in N} Y^i$ the condition for feasibility of S-allocations reduces to $\sum_{i\in S}(x^i - \omega^i) \in Y$. Furthermore, for convex-cone technologies the competitive equilibrium profits are equal to 0. This means that in equilibrium we do not have to specify shares of the production as it usually done in private ownership economies. Thus, as long as the individual production sets are convex cones and identical for all agents, we could alternatively consider a model for the production where we have only one production set for all agents and possible coalitions without specifying the shares. This model could be used instead of the production setup in the definition of a market.

In the definition of the induced market it is assumed that every individual has already the production possibilities, that become available if coalitions form, included in his personal production set. This means he already knows everything that can be produced in the different coalitions, even if he does not possess the necessary input commodities himself. Starting with an NTU game the utility allocations a coalition can reach in the derived induced market are not described by defining production sets individually for every coalition but by using input and output commodities. A utility allocation, that is reachable in the NTU game by a coalition S, is reachable in the induced market by the same coalition if the individuals pool their initial endowments using "one general" production possibility set. Utility allocations that require the cooperation of individuals outside the coalition S are technologically possible but can actually not be produced as the input commodities of these individuals are needed. In contrast to this interpretation in coalition production economies every coalition has its own production set.

The main proof of the above theorem from Billera and Bixby (1974) relies on Billera (1974). In a similar manner as Shapley and Shubik (1969), he starts with an NTU game, (N, V), and looks at the induced market of that game, $\mathcal{E}_V$, and afterwards at the NTU game that is induced by the induced market, $V_{\mathcal{E}_V}$. He shows that this game coincides with the totally balanced cover of the game (N, V).

The next step is to investigate the existing literature on and to study the relationship between solution concepts in cooperative game theory, as the inner core, and those in general equilibrium theory, as the notion of a competitive equilibrium. Analogously to the TU case of Shapley and Shubik (1975), Qin (1993) shows that the inner core of an NTU market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector. Before we extend the results of Qin (1993) we recall the basic definitions and state his main results. We start with the definition of the inner core and the notion of competitive payoff vectors in the context of NTU market games. Afterwards, we state the main results of Qin (1993) and comment on the ideas he uses to prove them.

In order to define the inner core we first consider a game that is related to a compactly generated NTU game, called the λ-transfer game. Fix a transfer rate vector $\lambda \in \mathbb{R}^n_+$. Define $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$ as the maximal sum of weighted utilities that coalition S can achieve given the transfer rate vector λ. The λ-*transfer game*, denoted as (N, V_λ), of (N, V) is defined by taking the same player set N and the coalitional function $V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$. Qin (1994, p.433) gives the following interpretation of the λ-transfer game: “The idea of the λ-transfer game may be captured by thinking of each player as representing a different country. The utilities are measured in different currencies, and the ratios λ_i/λ_j are the exchange rates between the currencies of i and j." As for the λ-transfer game only proportions matter we can assume without loss of generality that λ is normalized, i.e. $\lambda \in \Delta = \left\{\lambda \in \mathbb{R}^n_+ | \sum_{i=1}^n \lambda_i = 1\right\}$. Define the positive unit simplex by $\Delta_{++} = \left\{\lambda \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n \lambda_i = 1\right\}$.

The inner core is a refinement of the core. The *core* $C(V)$ of an NTU game (N, V) is defined as the set of utility allocations that are achievable by the grand coalition N such that no coalition S can improve upon this allocation. Thus,

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i\}.$$

A utility allocation is in the *inner core* $IC(V)$ of a compactly generated game (N, V) if it is achievable by the grand coalition N and if additionally there exists a transfer rate vector $\lambda \in \Delta$ such that this utility allocation is in the core of the λ-transfer game. More precisely:

Definition 2.4 (inner core)**.** The *inner core* of a compactly generated NTU game (N, V) is given by

$$IC(V) = \{u \in V(N) | \exists \lambda \in \Delta \text{ such that } u \in C(V_\lambda)\}.$$

Qin (1993, Remark 1, p. 337) remarks that if the NTU game is compactly convexly generated the vectors of supporting weights for a utility vector in the inner core must all be strictly positive. This can be seen by the following argument: If for one player $i \in N$ λ_i is equal to 0, then the core of the λ-transfer game is empty, because player i can improve upon any $u \in V_\lambda(N)$ by forming the singleton coalition $\{i\}$.

Qin (1994) considers sufficient conditions for the inner core to be non-empty. In particular he shows that a compactly generated NTU game (N, V), where $V(N)$ is convex, has a non-empty inner core if it is balanced with slack, meaning that for balancing weights $(\gamma_S)_{S \subseteq N}$ with $\gamma_N = 0$ we have $\sum\limits_{S \subset N} \gamma_S V(S) \subset \text{int}_{\mathbb{R}^n} V(N)$ where $\text{int}_{\mathbb{R}^n} V(N)$ is the interior of $V(N)$ relative to $\mathbb{R}^n$. Other contributions related to the non-emptiness of the inner core can be found for example in Iehlé (2004), Bonnisseau and Iehlé (2007) or Inoue (2010a).

We now define a competitive equilibrium for a market $\mathcal{E}$.

Definition 2.5 (competitive equilibrium)**.** A *competitive equilibrium* for a market $\mathcal{E}$ is a tuple

$$\left((\hat{x}^i)_{i\in N},(\hat{y}^i)_{i\in N},\hat{p}\right)\in\mathbb{R}^{\ell n}_+\times\mathbb{R}^{\ell n}_+\times\mathbb{R}^{\ell}_+$$

such that

(i) $\sum_{i\in N}\hat{x}^i=\sum_{i\in N}(\hat{y}^i+\omega^i)$ (market clearing),

(ii) for all $i\in N$, $\hat{y}^i$ solves $\max_{y^i\in Y^i}\hat{p}\cdot y^i$ (profit maximization),

(iii) and for all $i\in N$, $\hat{x}^i$ is maximal with respect to the utility function u^i in the budget set $\{x^i\in X^i|\hat{p}\cdot x^i\leq\hat{p}\cdot(\omega^i+\hat{y}^i)\}$ (utility maximization).

Given a competitive equilibrium its *competitive payoff vector* is defined as $\left(u^i\left(\hat{x}^i\right)\right)_{i\in N}$.

Qin (1993) investigates the relationship between the inner core of an NTU market game and the set of competitive payoff vectors of a market that represents this game. He establishes, following a conjecture of Shapley and Shubik (1975), the two theorems below analogously to the TU-case of Shapley and Shubik (1975).

Theorem 2.2 (1, Qin (1993))**.** *The inner core of an NTU market game coincides with the set of competitive payoff vectors of the induced market by that game.*

Theorem 2.3 (3, Qin (1993))**.** *For every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector.*

To show his first result Qin (1993) uses the notion of the induced market of a compactly convexly generated NTU game as it was already used by Billera and Bixby (1974). It turns out that the set of competitive equilibrium payoff vectors of the induced market coincides with the inner core.

For his second result Qin (1993) fixes an inner core point, denoted by u^* [1], and chooses one transfer rate vector $\lambda^*_{u^*}$ from an associated λ-transfer game. He modifies the given NTU game by applying a suitable strictly monotonic transformation on the utility allocations a coalition can reach. In this modified game the given inner core point u^* can be strictly separated from the set of utility allocations the grand coalition can reach (excluding u^*). Denote the modified game by $(N, \bar{V})$ and the convex compact sets generating this game by $(\bar{C}^S)_{S\in\mathcal{N}}$. A market to prove Theorem 3 of Qin (1993) can be defined as follows:

Define for all coalitions $S \in \mathcal{N}$

$$\begin{aligned} A^1_S &= \left\{ \left(u^S, -e^S, -e^S, -e^S, 0\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}, \\ A^2_S &= \left\{ \left(u^S, 0, -e^S, 0, -e^S\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}, \\ A^3_S &= \left\{ \left(u^S, 0, 0, -e^S, -e^S\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}. \end{aligned}$$

Let $\mathcal{E}_{\bar{V},u^*} = (X^i, Y^i, \omega^i, u^i)_{i\in N}$ be the market with for every individual $i \in N$

- the consumption set $X^i = X = \mathbb{R}^n_+ \times \{(0,0,0)\} \times \mathbb{R}^n_+ \subseteq \mathbb{R}^{5n}_+$,
- the production set $Y^i = Y = convexcone\left[\bigcup_{S\subseteq N}\left(A^1_S \cup A^2_S \cup A^3_S\right)\right] \subseteq \mathbb{R}^{5n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}\right) \in \mathbb{R}^{5n}_+$,
- the utility function $u^i(x^i) = \min\left\{x^{(1)i}_i, \frac{(\lambda^*_{u^*}\circ u^*)\cdot x^{(5)i}}{\lambda^*_{u^*i}}\right\}$ with $x^i = (x^{(1)i}, 0, 0, 0, x^{(5)i}) \in X^i$ and $x^{(1)i}_k$ is the k^{th} entry of $x^{(1)i}$.

Note that, similarly to the induced market, all individuals have the same consumption sets and the same production sets. The individuals differ in their initial endowment vectors and their utility functions. Qin (1993) introduces the sets A^1_S, A^2_S, A^3_S in order to be able to show that the equilibrium price vector for the 5^{th} group of n goods, $\hat{p}^{(5)}$, is strictly positive. The i^{th} consumer obtains utility from the i^{th} component of the vector of the 1^{st}

[1] Qin (1993) considers only NTU games where for all coalitions $S \in \mathcal{N}$ the generating sets satisfy $C^S \subseteq \mathbb{R}^S_+$ and $C^S \cap \mathbb{R}^S_{++} \neq \emptyset$ and hence has $u^* \gg 0$.

group of n goods and from all the 5^{th} n goods. The dependence of the utility function on all components of the 5^{th} group of n goods is crucial to show the positiveness of $\hat{p}^{(5)}$. To prove his result Qin (1993) shows that the market $\mathcal{E}_{\bar{V},u^*}$ represents the modified game and that the given inner core point is the unique competitive payoff vector of this economy. By applying the inverse strictly monotonic transformation to the utility functions he obtains his result.

In order to extend the results of Qin (1993) to a large class of closed subsets of the inner core we make use of the fact that for compactly convexly generated NTU games competitive payoff vectors need necessarily to be in the inner core. To see this we use a modified version of Proposition 1 from de Clippel and Minelli (2005).

Let $N = \{1, ..., n\}$ be the set of agents and $\{1, .., \ell\}$ be the set of commodities. Let $X^i \subseteq \mathbb{R}^\ell_+$ be a convex set containing 0, the consumption set of agent i. Each individual has a continuous, concave, (weakly) increasing and locally non-satiated utility function $u^i : \mathbb{R}^\ell_+ \to \mathbb{R}$ and an initial endowment vector $\omega^i \in \mathbb{R}^\ell_+ \setminus \{0\}$. Let $Y^i \subseteq \mathbb{R}^\ell$ be a non-empty and closed convex cone, the production set of agent i's firm.

Lemma 2.1. *Let* $\left((\hat{x}^i)_{i\in N}, (\hat{y}^i)_{i\in N}, \hat{p}\right)$ *be a competitive equilibrium such that* $\hat{p} \cdot \omega^i > 0$ *for all individuals* $i \in N$. *Then* $\left(u^i\left(\hat{x}^i\right)\right)_{i\in N}$ *is in the inner core of the game induced by the economy.*

The proof of Lemma 2.1 can be found in Appendix 2.5.1.

2.3 An extension of the Results of Qin (1993)

In the above two theorems Qin (1993) considers on the one hand the whole inner core and on the other hand a single point in the inner core. In this section we extend the results of Qin (1993) by showing a similar result for closed subsets of the inner core. In the following we consider NTU market games and closed subsets of the inner core with certain properties. We want to ensure that for every point in a subset of the inner core, denoted by A, of a given NTU market game (N, V) we can find a normal vector such that this point is strictly separated from the set $V(N)$ without the point by the hyperplane using this normal vector. If we assume that the individual rational part of $V(N)$ is strictly convex, then this property is satisfied. Moreover, we want to assume that this set of normal vectors, where each normal vector corresponds to one point of the set A, is bounded below by a strictly positive vector. This means that the exchange rates, represented by the normal vectors, within the set A cannot be too extreme. We make the following definition:

Definition 2.6 (strict positive separability)**.** A pair $[(N, V), A]$ consisting of a compactly, convexly generated and totally balanced NTU game (N, V) and a closed subset A of its inner core satisfies *strict positive separability [SPS]* if the following condition holds:

> There exists an $\varepsilon > 0$ and a mapping $\lambda : A \to \Delta_{++}$, that associates to every point $x \in A$ a normal vector $\lambda(x) = \lambda^x$, such that
>
> – every point $x \in A$ can be strictly separated from the set $V(N) \setminus \{x\}$ using this normal vector λ^x, i.e.
>
> $$\lambda^x \cdot x > \lambda^x \cdot y \quad \text{for all } y \in V(N) \setminus \{x\},$$
>
> – for all $x \in A$ every coordinate of the normal vector λ^x is strictly greater than ε, i.e.
>
> $$\lambda_i^x > \varepsilon \quad \text{for all } i \in N.$$

For a pair $[(N, V), A]$ satisfying strict positive separability there might exist more than on mapping λ and more than one ε. In the following we always consider one fixed mapping λ together with one fixed ε satisfying the conditions. Whenever λ or ε appear we mean the ones we fixed knowing that we might have chosen different ones.

The assumption of strict positive separability is not as restrictive as it might appear. It is satisfied for example if the individual rational part of

$V(N)$ is strictly convex and A is a closed subset of the interior of the inner core.

Note that from $\varepsilon < \lambda_i^x = \frac{\lambda_i^x}{1} \leq \frac{\lambda_i^x}{\lambda_j^x}$ it follows that

$$\varepsilon < \min_{i,j \in N} \frac{\lambda_i^x}{\lambda_j^x} \quad \text{for all } \lambda^x,\ x \in A.$$

Figure 2.1 illustrates the idea of strict positive separability with some examples. Assume that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 2.1.

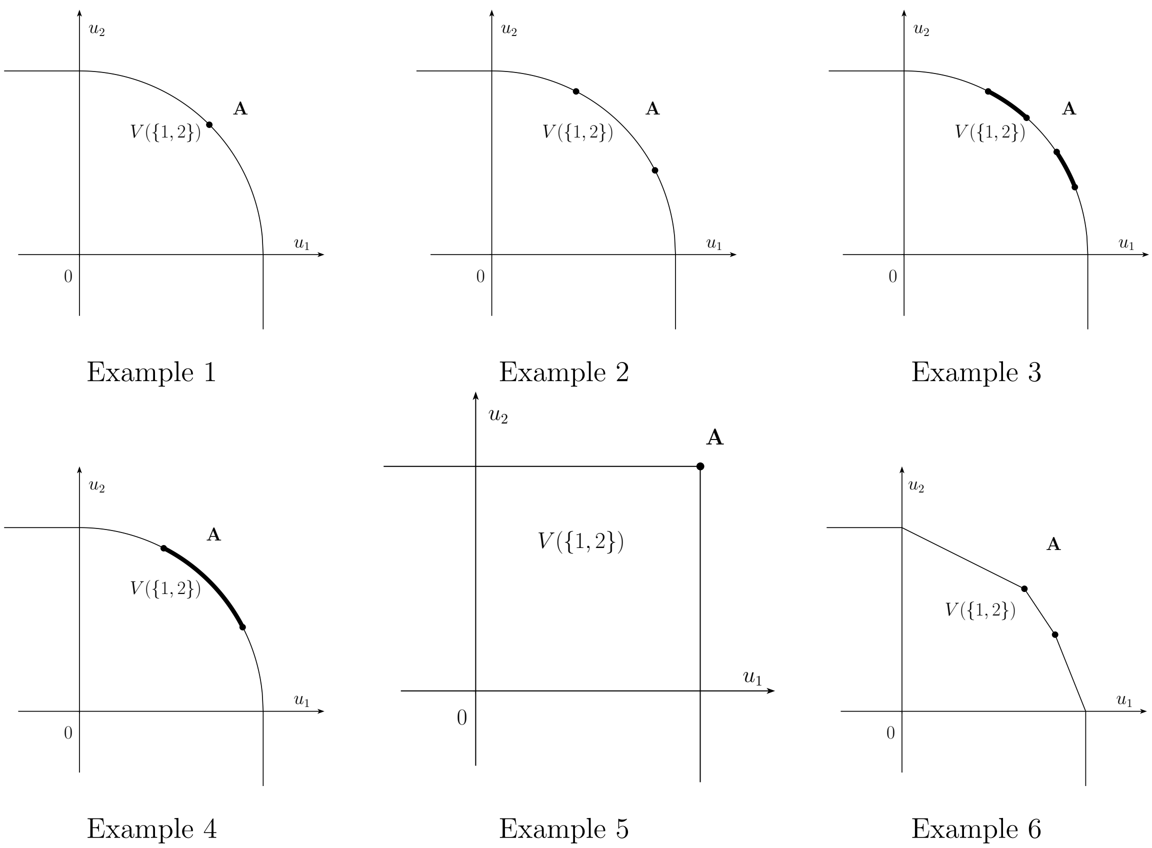

Figure 2.1: Examples where SPS is satisfied.

In Examples 1, 2, 3 and 4 the set $V(\{1,2\})$ is strictly convex. Here the inner core is given by all points on the efficient boundary without the two points on the axes. Thus, the NTU game together with every closed subset of its inner core satisfies SPS. This holds in particular for single points, finite sets, closed and connected sets or finite unions of closed sets.

Example 5 illustrates the case where the set $V(\{1,2\})$ is generated by a square and thus the inner core consists only of the corner point. In this case all the vectors in the strictly positive two-dimensional simplex support this inner core point. In order to establish SPS we just take one of these supporting vectors.

In Example 6 the set $V(\{1,2\})$ is generated by a polyhedron. The set A is a finite set, consisting of some corner points of the polyhedron. For each of these corner points there exists a strictly positive normal vector that strictly separates it from $V(\{1,2\})$ without this corner point. The NTU game (N,V) and this choice of the set A satisfy SPS.

Figure 2.2 shows some examples that do not satisfy strict positive separability. As before assume that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 2.2.

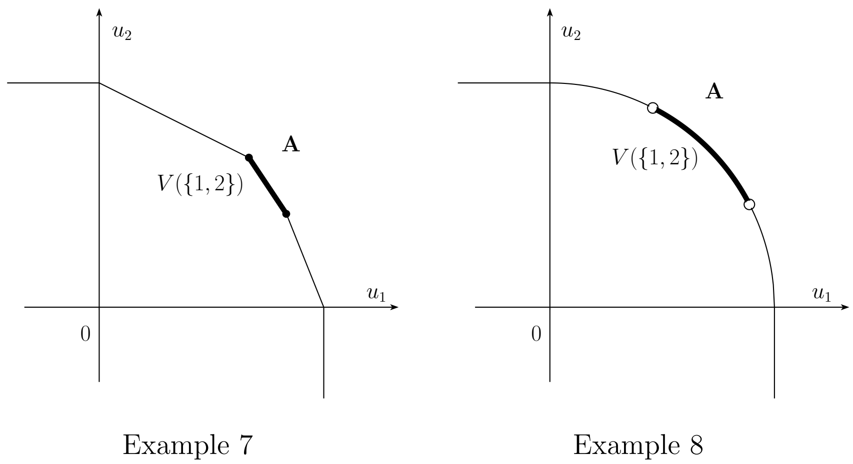

Figure 2.2: Examples where SPS is not satisfied.

In contrast to Example 6, in Example 7 the set A is chosen to be the line segment connecting two neighboring corner points of a polyhedron. Hence, all points in the set A have a common normal vector. Thus, each of this points cannot be strictly separated from the polyhedron without this point. Therefore, SPS is not satisfied. In Example 8 each point in the set A can be strictly separated from $V(\{1,2\})$ without the point. Nevertheless SPS is not satisfied, as the set A is not closed.

The properties, that we require at this point by considering only $[(N,V),A]$ satisfying SPS, are stronger than the properties, that we really need. For example it is sufficient if we can strictly separate each point in the boundary of A from A without it. Nevertheless, we choose to consider $[(N,V),A]$ which satisfy SPS, because they allow for an easy interpretation. After the presentation of the main results we discuss the question, how this can be weakened such that cases as in Example 6 are included in our results.

Now we prove the following result:

Theorem 2.4. *Let $[(N,V),A]$ satisfy strict positive separability. Then there exists a market such that this market represents the game (N,V) and such that the set of competitive payoff vectors of this market is the set A.*

We show this result for NTU games where for every coalition $S \in \mathcal{N}$ we have $C^S \subseteq \mathbb{R}^S_{++}$. Due to the remark on page 43 this is not a restriction as we can shift an arbitrary given NTU game such that this condition is satisfied. After having applied our results we shift back the obtained economies such that they represent the original game. Hence, in the following if we consider an NTU game, we always assume for every coalition $S \in \mathcal{N}$ that we have $C^S \subseteq \mathbb{R}^S_{++}$.

Before beginning with the construction of a market satisfying the properties mentioned above, we introduce an auxiliary game and some notation.

Let $[(N,V),A]$ satisfy SPS. Let $(N,\tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap\limits_{a \in A} \{z \in \mathbb{R}^n | \lambda^a \cdot z \leq \lambda^a \cdot a\} & \text{if } S = N \end{cases}$$

where λ^a is as in the definition of SPS.

Note that to define the game $(N, \tilde{V})$ we use for every point of the set $a \in A$ just one normal vector that strictly separates this point from $V(N) \setminus \{a\}$. The games (N, V) and $(N, \tilde{V})$ are equal except for the grand coalition N. For the coalition N we extend the set $V(N)$ depending on the normal vectors of the set A. For illustration purposes figure 2.3 shows as an example for two players the sets $V(\{1,2\})$ and $\tilde{V}(\{1,2\})$.

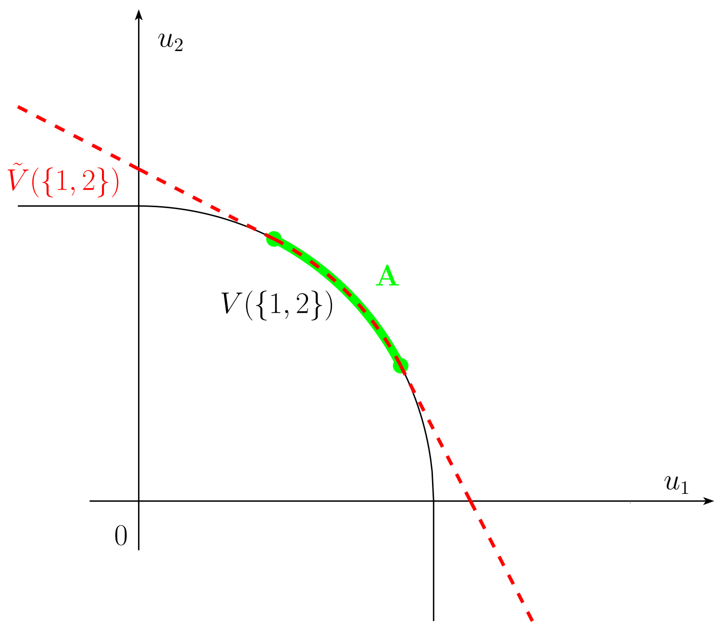

Figure 2.3: Example: The sets $V(\{1,2\})$ and $\tilde{V}(\{1,2\})$ for $N = \{1,2\}$.

To describe the relation between $(N, \tilde{V})$ and (N, V) we introduce the following notation: Let $z \in \tilde{V}(N)$ and

$$\bar{t}^z = \min \left\{ t \in \mathbb{R}_+ | z - te^N \in V(N) \right\}.$$

Define

$$\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| \exists t \in \mathbb{R}_+ \text{ such that } z - te^N \in C^N \right\}.$$

Then we also have $\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| z - \bar{t}^z e^N \in C^N \right\}$.

The following remark is easy to verify:

Remark.

1. The game (N, V) is contained in the game $(N, \tilde{V})$. This means we have $V(S) \subseteq \tilde{V}(S)$ for all $S \subseteq N$.

2. The set $\tilde{C}^N$ is convex and furthermore, $C^N \subseteq \tilde{C}^N$.

3. The game $(N, \tilde{V})$ is a convexly generated and totally balanced NTU-game, but it is not compactly generated. In particular we have $\tilde{V}(N) \neq \tilde{C}^N - \mathbb{R}^n_+$.

4. SPS ensures in particular: If we take x in $V(N)$ outside from A, then x is in the interior of $\tilde{V}(N)$,

$$x \in V(N) \setminus A \;\Rightarrow x \in \operatorname{int}\left(\tilde{V}(N)\right).$$

The second point of the remark can be seen as follows: Take $z_1, z_2 \in \tilde{C}^N$ and $\alpha \in [0,1]$. Then there exist t^{z_1} and t^{z_2} such that $z_1 - t^{z_1}e^N \in C^N$ and $z_2 - t^{z_2}e^N \in C^N$. As C^N is per assumption convex $\alpha\left(z_1 - t^{z_1}e^N\right) + (1-\alpha)\left(z_2 - t^{z_2}e^N\right) \in C^N$. As well the set $\tilde{V}(N)$, as an intersection of halfspaces, is convex and hence $\alpha z_1 + (1-\alpha)z_2 \in \tilde{V}(N)$. Thus taking $t^{\alpha z_1 + (1-\alpha)z_2} = \alpha t^{z_1} + (1-\alpha)t^{z_2}$ shows that $(\alpha z_1 + (1-\alpha)z_2) - t^{\alpha z_1 + (1-\alpha)z_2}e^N = \alpha\left(z_1 - t^{z_1}e^N\right) + (1-\alpha)\left(z_2 - t^{z_2}e^N\right) \in C^N$. Therefore, we have $\alpha z_1 + (1-\alpha)z_2 \in \tilde{C}^N$. Hence, $\tilde{C}^N$ is convex.

Definition 2.7. Define the mapping $P_A : \tilde{V}(N) \longrightarrow V(N)$ via

$$P_A(x) = x - \bar{t}^x e^N.$$

The following figure illustrates the mapping P_A for the example from figure 2.3.

Note, that if $x \in V(N)$ then $\bar{t}^x = 0$ and $P_A(x) = x$.

Remark.

1. The mapping P_A is continuous and its image is $V(N)$.

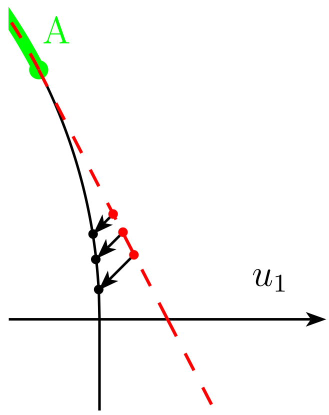

Figure 2.4: Illustration of the mapping P_A for the example from figure 2.3.

2. The set $\tilde{C}^N$ can be written as

$$\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| P_A(z) \in C^N \right\} = P_A^{-1}\left(C^N\right),$$

thus we have $P_A\left(\tilde{C}^N\right) = C^N$.

2.3.1 The basic idea

First, we present an intermediate result, which is interesting in itself. For $[(N,V),A]$ satisfying SPS we construct a market such that this market represents the given game and such that the set of payoff vectors of competitive equilibria with strictly positive price vectors coincides with the given set A. In the last chapter we show, how we deal with the case, when the equilibrium price vectors are not necessarily strictly positive, using a more complicated market with a similar structure.

Definition 2.8. Let $[(N,V),A]$ satisfy SPS. Then the market $\mathcal{E}^0_{V,A}$ is defined by

$$\mathcal{E}^0_{V,A} = \left(X^i, Y^i, u^i, \omega^i\right)_{i\in N}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{4n}$,

- the production set

$$Y^i = convexcone \left[\left(\bigcup_{S \in \mathcal{N} \setminus \{N\},\ c^S \in C^S} \left(c^S, -e^S, c^S, -e^S \right) \right) \cup \left(\bigcup_{\tilde{c}^N \in \tilde{C}^N} \left(P_A \left(\tilde{c}^N \right), -e^N, \tilde{c}^N, -e^N \right) \right) \right] \subseteq \mathbb{R}^{4n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}\right)$,

- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i \left((x^{(1)}, 0, x^{(3)}, 0) \right) = \min \left(x_i^{(1)}, x_i^{(3)} \right)$.

Note that this market has the same consumption and production set for every individual $i \in N$. The individuals differ in their initial endowment vectors and their utility functions. There are input and output commodities. The 2^{nd} group and the 4^{th} group of n commodities are the input commodities and every individual $i \in N$ owns one unit of his personal input commodity in the i^{th} component of the 2^{nd} and the 4^{th} group of n goods. The 1^{st} and the 3^{rd} group of n goods are the output commodities, from whose i^{th} component player $i \in N$ obtains utility. The construction of this market is based on the idea of the induced market in Billera and Bixby (1974) or Qin (1993).

We now need to establish first that the market $\mathcal{E}^0_{V,A}$ is indeed a market for the NTU market game (N, V).

Lemma 2.2. *The market $\mathcal{E}^0_{V,A}$ represents the game (N, V).*

The proof of Lemma 2.2 is inspired by Billera (1974).

Proof.

- As $V(S) = C^S - \mathbb{R}^S_+$ it is enough to show, that for all $S \in \mathcal{N}$ the payoff vectors in the set C^S can be achieved by coalition S in the market $\mathcal{E}^0_{V,A}$. Let $z \in C^S$. We show, that there exists a feasible S-allocation $\left(x^i\right)_{i \in S}$ with $\left(y^i\right)_{i \in S}$ such that $u^i \left(x^i\right) = z_i$ for all $i \in S$.

Define for $i \in S$ the consumption plan

$$x^i = \left(z^{\{i\}}, 0, z^{\{i\}}, 0\right)$$

and let

$$y^i = \frac{1}{|S|}\left(z, -e^S, z, -e^S\right)$$

be the production plan for all $i \in S$. By the definition of the consumption sets we observe $x^i \in X^i$ for all $i \in S$. With regard to the production sets for $S \neq N$ we have immediately $y^i \in Y^i$ for all $i \in S$. For $S = N$ note that $z \in V(N) \subseteq \tilde{V}(N)$ and thus $P_A(z) = z$. Hence, we have $y^i \in Y^i$ for all $i \in N$. Observe that

$$\sum_{i \in S}\left(x^i - \omega^i\right) = \sum_{i \in S} y^i.$$

Hence, $\left(x^i\right)_{i \in S}$ is a feasible S-allocation and

$$u^i\left(x^i\right) = z_i \quad \text{for all } i \in S.$$

- Let $\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0\right)_{i \in S}$ be a feasible S-allocation with $\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}\right)_{i \in S}$ in the market $\mathcal{E}^0_{V,A}$.
  The feasibility implies

$$\left(\sum_{i \in S}\bar{x}^{(1)i}, -e^S, \sum_{i \in S}\bar{x}^{(3)i}, -e^S\right) = \sum_{i \in S}\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}\right).$$

Each production set is a convex cone of a union of convex sets. Hence, an arbitrary production plan can be written in the following way: Choose one suitable element from each of the convex sets and build a linear combination (with non-negative coefficients) of these elements. For the 1^{st} and the 2^{nd} group of n commodities we obtain, that there exist $\alpha^i_R \in \mathbb{R}_+$ for all $R \in \mathcal{N}$, $z^i_R \in C^R$ for all $R \in \mathcal{N} \setminus \{N\}$ and

$\tilde{z}_N^i \in \tilde{C}^N$, such that

$$\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right) = \sum_{R \in \mathcal{N} \setminus \{N\}} \alpha_R^i \left(z_R^i, -e^R\right) + \alpha_N^i \left(P_A\left(\tilde{z}_N^i\right), -e^N\right).$$

As $P_A\left(\tilde{C}^N\right) = C^N$ there exists $z_N^i \in C^N$ such that $P_A\left(\tilde{z}_N^i\right) = z_N^i$ and hence we have

$$\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right) = \sum_{R \in \mathcal{N}} \alpha_R^i \left(z_R^i, -e^R\right).$$

As feasibility implies $\left(\sum_{i \in S} \bar{x}^{(1)i}, -e^S\right) = \sum_{i \in S} \left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right)$, for the 2^{nd} group of n coordinates we have that

$$\begin{aligned} e^S &= \sum_{i \in S} \sum_{R \in \mathcal{N}} \alpha_R^i e^R \\ &= \sum_{R \in \mathcal{N}} \left(\sum_{i \in S} \alpha_R^i\right) e^R. \end{aligned}$$

Thus $\alpha_R^i > 0$ implies $R \subseteq S$ and if we define $\alpha(R) = \sum_{i \in S} \alpha_R^i$, then $(\alpha(R))_{R \subseteq S}$ is a balanced family for the coalition S. Looking at the 1^{st} group of n coordinates we have

$$\begin{aligned} \sum_{i \in S} \bar{x}^{(1)i} &= \sum_{R \subseteq S} \sum_{i \in S} \alpha_R^i z_R^i \\ &= \sum_{\{R \subseteq S | \alpha(R) > 0\}} \alpha(R) \left(\frac{1}{\alpha(R)} \sum_{i \in S} \alpha_R^i z_R^i\right). \end{aligned}$$

Since C^R is convex we have

$$\frac{1}{\alpha(R)} \sum_{i \in S} \alpha_R^i z_R \in C^R$$

and hence, using totally balancedness, $\sum_{i\in S} \bar{x}^{(1)i} \in V(S)$.

From the definition of the utility function we obtain $u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0\right) \leq \bar{x}_i^{(1)i}$. Since $\left(\bar{x}_i^{(1)i}\right)_{i\in S} \leq \sum_{i\in S} \bar{x}^{(1)i} \in V(S)$ we have by the S-comprehensiveness of $V(S)$ that $\left(u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0\right)\right)_{i\in S} \in V(S)$.

□

We verify that the payoff vectors in the set A are indeed competitive payoff vectors of the market $\mathcal{E}^0_{V,A}$:

Proposition 2.1. *Every point in the set A is equilibrium payoff vector of the market $\mathcal{E}^0_{V,A}$.*

Proof. Let $a \in A$ and $\lambda^a \in \Delta$ be a normal vector such that a is in the core of the λ^a-transfer game. We know that λ^a is strictly positive (compare the remark on page 46). By the assumption that $C^N \subseteq \mathbb{R}^N_{++}$ we know that a is strictly positive. To prove the proposition, we show that the consumption and production plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(a^{\{i\}}, 0, a^{\{i\}}, 0\right)\right)_{i\in N}$$

and

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a, -e^N, a, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^a, \lambda^a \circ a, \lambda^a, \lambda^a \circ a\right)$$

constitute a competitive equilibrium in the market $\mathcal{E}^0_{V,A}$.

First note that as $P_A(a) = a$ we have $\hat{y}^i \in Y^i$ for all $i \in N$. According to the remark above, the price system $\hat{p}$ is strictly positive. As we have a convex-cone-technology maximum profits are zero. We observe

$$\hat{p} \cdot \hat{y}^i = \frac{1}{n}\left(\lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N + \lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N\right) = 0.$$

Hence, the production plan $\hat{y}^i$ is profit maximizing.

As we have a min-type or Leontief utility function, it is optimal for each agent i to spend his budget in a way such that $\hat{x}_i^{(1)i} = \hat{x}_i^{(3)i}$ and that he does not consume anything of the other commodities. Furthermore, he has to spend all his budget, because the preferences are locally non-satiated and continuous. The budget constraint is satisfied with equality,

$$\hat{p} \cdot \hat{x}^i = \lambda^a \cdot \left(a^{\{i\}} + a^{\{i\}}\right) = (\lambda^a \circ a) \cdot \left(e^{\{i\}} + e^{\{i\}}\right) = \hat{p} \cdot \omega^i$$

and

$$\hat{x}^{(1)i} = a^{\{i\}} = \hat{x}^{(3)i}.$$

Hence, the consumption vector $\hat{x}^i$ is utility maximizing on the budget set of agent i.

Furthermore, the market clearing condition

$$\sum_{i \in N} \hat{x}^i = \sum_{i \in N} \omega^i + \sum_{i \in N} \hat{y}^i$$

is satisfied.
Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i \left(\hat{x}^i\right)\right)_{i \in N} = a.$$

□

Looking again at the competitive equilibrium price vectors in the proof of Proposition 2.1 note: For a competitive equilibrium with payoff vector $a \in A$ the equilibrium price vector for the 1^{st} (respectively 3^{rd}) group of n goods, the output goods, is the normal vector λ^a separating the point a from $V(N)$. The transfer rate vectors coincide with the equilibrium prices for the output goods of the market. The input goods are priced by $\lambda^a \circ a$. This is the transfer rate vector weighted by the according point of the set A. Interpreted differently: The input goods are first weighted by the point a of the set A and afterwards they are priced by the transfer rate vector λ^a. The relationship of the transfer rate vectors and the prices of competitive equilibria was observed in several publications discussing the relation between NTU games and economies. Examples are Shubik (1985), Shapley (1987), Trockel (1996) and Qin (1993). Shapley (1987, p. 192) states: "There is a strong analogy though no formal equivalence that we know of between the comparison weights that we must introduce in order to obtain a feasible transfer value and the prices in a competitive market." Here we obtain a formal equivalence for the prices of the output goods and an indirect link for the prices of the input goods. Trockel (1996) investigated this equivalence for NTU bargaining games and Qin (1993) obtained very similar equilibrium prices as we have here.

Next, we consider the utility allocations outside the set A. Using Lemma 2.1 it is sufficient to consider those vectors in the inner core.

Proposition 2.2. *Any payoff vector of a competitive equilibrium of the market $\mathcal{E}^0_{V,A}$ with a strictly positive equilibrium price vector is an element of the set A.*

Proof. Lemma 2.1 ensures that every competitive equilibrium payoff vector is in the inner core. Assume that there exists a competitive equilibrium $((x^i)_{i\in N}, (y^i)_{i\in N}, p)$ such that its payoff vector $(u^i(x^i))_{i\in N}$ is in the inner core but not in the set A and such that the equilibrium price vector is strictly positive, $p \gg 0$.

Then, there exists an element c^N in the inner core outside A such that $u^i(x^i) = c^N_i$ for all player $i = 1, ..., n$. Let $x^i = (x^{(1)i}, x^{(2)i}, x^{(3)i}, x^{(4)i})$. By the definition of the consumption set we know $x^{(2)i} = x^{(4)i} = 0$ and by the definition of the utility function we obtain $x^{(1)i}_i \geq c^N_i$ and $x^{(3)i}_i \geq c^N_i$ for all $i = 1, ..., n$.

<u>Claim 1:</u> From the utility maximization and the strict positivity of the price vector it follows that we need to have

$$x^{(1)i}_i = c^N_i = x^{(3)i}_i.$$

The proof of Claim 1 can be found in Appendix 2.5.2.

We get by the market clearing condition:

$$y = \sum_{i \in N} \left(x^i - \omega^i\right) = \left(c^N, -e^N, c^N, -e^N\right).$$

But the production plan $y = (c^N, -e^N, c^N, -e^N)$ is not profit maximizing.[2]

To see this notice the following: As c^N is in the inner core but outside the set A there exists a $\tilde{c}^N$ with $P_A\left(\tilde{c}^N\right) = c^N$ and $\tilde{c}^N \gg c^N$. Consider the production plan $\left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right)$. Looking at the profits and using the strict positivity of the price vector we observe

$$\begin{aligned} p \cdot y &= p^{(1)} \cdot c^N - p^{(2)} \cdot e^N + p^{(3)} \cdot c^N - p^{(4)} \cdot e^N \\ &< p^{(1)} \cdot c^N - p^{(2)} \cdot e^N + p^{(3)} \cdot \tilde{c}^N - p^{(4)} \cdot e^N \\ &= p^{(1)} \cdot P_A\left(\tilde{c}^N\right) - p^{(2)} \cdot e^N + p^{(3)} \cdot \tilde{c}^N - p^{(4)} \cdot e^N \\ &\leq 0. \end{aligned}$$

Thus, we have found a production plan that has strictly higher profits than y. This is a contradiction, since y needs to be profit maximizing.

It follows that with strictly positive price vectors the allocations outside the set A but in the inner core cannot be competitive equilibrium payoff vectors. □

Combining the two propositions above we obtain the following theorem:

Theorem 2.5. *Let $[(N, V), A]$ satisfy strict positive separability. The set of payoff vectors of competitive equilibria with a strictly positive equilibrium price vector of the market $\mathcal{E}^0_{V,A}$ coincides with the set A.*

Positive equilibrium price vectors are required to obtain the above results

Up to now we always considered competitive equilibria with only strictly positive equilibrium price vectors. This was indeed necessary. If we also

[2]Since the individual production sets are convex cones, to check profit maximization it is sufficient to consider the joint production plans. We have $\sum_{i=1}^n Y^i = Y^j$ for any $j \in N$.

allow for price vectors that are not strictly positive, then we can construct a competitive equilibrium with competitive payoff vectors outside the given set A. To see this fix $a \notin A$ but in the inner core. Then there exists $\tilde{a} \in \tilde{C}^N$ such that $P_A(\tilde{a}) = a$ and $\tilde{a} \gg a$. Consider

$$\hat{x}^i = \left((P_A(\tilde{a}))^{\{i\}}, 0, \tilde{a}^{\{i\}}, 0 \right) = \left(a^{\{i\}}, 0, \tilde{a}^{\{i\}}, 0 \right) \text{ for all } i \in N,$$
$$\hat{y}^i = \left(\frac{1}{n} \left(P_A(\tilde{a}), -e^N, \tilde{a}, -e^N \right) \right) = \left(\frac{1}{n} \left(a, -e^N, \tilde{a}, -e^N \right) \right) \text{ for all } i \in N,$$
$$\hat{p} = (\lambda^a, \lambda^a \circ a, 0, 0)$$

where λ^a is one normal vector from a λ^a-transfer game and $(P_A(\tilde{a}))^{\{i\}}$ is the vector that has as its i^{th} coordinate the i^{th} coordinate of $P_A(\tilde{a})$ and zero coordinates otherwise. Analogously define $\tilde{a}^{\{i\}}$.

We show that $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ constitutes a competitive equilibrium with the payoff vector $a \notin A$.

- First note that $u^i(\hat{x}^i) = \min\{a_i, \tilde{a}_i\} = a_i$, since we have $\tilde{a} \gg a$.
- For the profit maximization we obtain

$$\hat{p} \cdot \hat{y}^i = \frac{1}{n} \left(\lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N \right) = 0.$$

  Since the maximum profits are zero, $\hat{y}^i$ is profit maximizing.
- For the utility maximization we obtain that the budget constraint is satisfied with equality,

$$\hat{p} \cdot \hat{x}^i = \lambda^a \cdot a^{\{i\}} = (\lambda^a \circ a) \cdot e^{\{i\}} = \hat{p} \cdot \omega^i,$$

  and furthermore individual i spends all his budget for the i^{th} commodity in the 1^{st} group of n goods. Since the prices are equal to zero for the 3^{rd} and 4^{th} group of n goods he can consume $\hat{x}_i^{(3)i} = \tilde{a}_i$ without using any of his budget. Thus, $\hat{x}^i$ is utility maximizing.

- Moreover, the market clearing condition is satisfied

$$\sum_{i\in N} \hat{x}^i = \sum_{i\in N} \omega^i + \sum_{i\in N} \hat{y}^i.$$

Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i\left(\hat{x}^i\right)\right)_{i\in N} = a \notin A.$$

2.3.2 The main results

In order to deal with the general case without assuming the strict positivity of price vectors, we modify the market from the previous section in an appropriate way. This modification allows us to show, that the prices of the 3^{rd} group of n commodities are strictly positive, $p^{(3)} \gg 0$. For the rest of this section let $[(N,V),A]$ satisfy SPS. To simplify the notation of the market, we introduce some sets before:

For the definition of the production sets define for all coalitions $S \in \mathcal{N} \setminus \{N\}$

$$\begin{aligned}
A_S^1 &= \left\{\left(c^S, -e^S, c^S, -e^S, -e^S\right) | c^S \in C^S\right\}, \\
A_S^2 &= \left\{\left(c^S, 0, c^S, -e^S, 0\right) | c^S \in C^S\right\}, \\
A_S^3 &= \left\{\left(c^S, 0, c^S, 0, -e^S\right) | c^S \in C^S\right\}
\end{aligned}$$

and for the grand coalition N define

$$\begin{aligned}
A_N^1 &= \left\{\left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right) | \tilde{c}^N \in \tilde{C}^N\right\}, \\
A_N^2 &= \left\{\left(P_A\left(\tilde{c}^N\right), 0, \tilde{c}^N, -e^N, 0\right) | \tilde{c}^N \in \tilde{C}^N\right\}, \\
A_N^3 &= \left\{\left(P_A\left(\tilde{c}^N\right), 0, \tilde{c}^N, 0, -e^N\right) | \tilde{c}^N \in \tilde{C}^N\right\}.
\end{aligned}$$

In order to obtain the result without the assumption of strictly positive price vectors, we modify the utility functions, the production and consumption sets. The utility functions do not depend anymore only on the two personal output commodities but also on the whole second group of output commodities. For that we add 'a little bit' of utility from the other players output goods. This 'little bit' is described by using the $\varepsilon > 0$ from the definition of SPS.

Definition 2.9 (induced A-market)**.** Let $[(N,V),A]$ satisfy strict positive separability. Let $\varepsilon > 0$ such that $\varepsilon < \min_{i,j\in N} \frac{\lambda_i^a}{\lambda_j^a}$ for all $a \in A$. The *induced A-market* of the game (N,V) and the set A is defined by

$$\mathcal{E}_{V,A,\varepsilon} = (X^i, Y^i, u^i, \omega^i)_{i\in N}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S\in\mathcal{N}} (A_S^1 \cup A_S^2 \cup A_S^3)\right] \subseteq \mathbb{R}^{5n}$
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with

$$u^i\left(x^{(1)}, 0, x^{(3)}, 0, 0\right) = \min\left(x_i^{(1)}, x_i^{(3)} + \varepsilon \sum_{j\neq i} x_j^{(3)}\right).$$

Note that this market is very similar to the market we defined in the previous section. We change the definition of the production and consumption sets slightly by introducing a further input commodity. Moreover, the utility functions here depend on all coordinates of the 3^{rd} group of n goods.

Having defined the induced A-market we prove the following theorem, which is the main result of this paper:

Theorem 2.6. *Let $[(N,V),A]$ satisfy strict positive separability. Then there exists a market such that this market represents the game (N,V) and such that the set of competitive payoff vectors of this market is the set A.*

To prove the above theorem we use the induced A-market $\mathcal{E}_{V,A,\varepsilon}$ as defined before. We divide the proof of this Theorem into 3 parts: First we show, that $\mathcal{E}_{V,A,\varepsilon}$ represents the game (N,V), in the second part we prove, that every vector in the set A is a competitive payoff vector, and in the third part we show that competitive payoff vectors always belong to the set A.

Lemma 2.3. *The induced A-market $\mathcal{E}_{V,A,\varepsilon}$ represents the game (N,V).*

The proof of Lemma 2.3 is inspired by Billera (1974).

Proof.

- As $V(S) = C^S - \mathbb{R}^S_+$ it is enough to show, that the payoffs in the set C^S can be achieved by coalition S in the market $\mathcal{E}_{V,A,\varepsilon}$. Let $z \in C^S$. We show, that there exists a feasible S-allocation $(x^i)_{i\in S}$ with $(y^i)_{i\in S}$ such that $u^i(x^i) = z_i$ for all $i \in S$.
  Define for $i \in S$ the consumption plan
  $$x^i = \left(z^{\{i\}}, 0, z^{\{i\}}, 0, 0\right)$$
  and let
  $$y^i = \frac{1}{|S|}\left(z, -e^S, z, -e^S, -e^S\right)$$
  be the production plan for all $i \in S$. By the definition of the consumption sets we observe $x^i \in X^i$ for all $i \in S$. With regard to the production sets for $S \neq N$ we have immediately $y^i \in Y^i$ for all $i \in S$. For $S = N$ note that $z \in V(N) \subseteq \tilde{V}(N)$ and thus $P_A(z) = z$. Hence, we have $y^i \in Y^i$ for all $i \in N$. Observe that
  $$\sum_{i\in S}\left(x^i - \omega^i\right) = \sum_{i\in S} y^i.$$
  Hence, $(x^i)_{i\in S}$ is a feasible S-allocation and
  $$u^i\left(x^i\right) = z_i \quad \text{for all } i \in S.$$

- Let $\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right)_{i\in S}$ be a feasible S-allocation with $\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}, \bar{y}^{(5)i}\right)_{i\in S}$ in the market $\mathcal{E}_{V,A,\varepsilon}$.

The feasibility implies

$$\left(\sum_{i\in S}\bar{x}^{(1)i},-e^S,\sum_{i\in S}\bar{x}^{(3)i},-e^S,-e^S\right)=\sum_{i\in S}\left(\bar{y}^{(1)i},\bar{y}^{(2)i},\bar{y}^{(3)i},\bar{y}^{(4)i},\bar{y}^{(5)i}\right).$$

Each production set is a convex cone of a union of convex sets. Hence, an arbitrary production plan can be written in the following way: Choose one suitable element from each of the convex sets and build a linear combination (with non-negative coefficients) of these elements. For the 1^{st} and the 2^{nd} group of n commodities we obtain, that there exist $\alpha_R^i \in \mathbb{R}_+$ for all $R \in \mathcal{N}$, $z_R^i \in C^R$ for all $R \in \mathcal{N} \setminus \{N\}$ and $\tilde{z}_N^i \in \tilde{C}^N$, such that

$$\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)=\sum_{R\in\mathcal{N}\setminus\{N\}}\alpha_R^i\left(z_R^i,-e^R\right)+\alpha_N^i\left(P_A\left(\tilde{z}_N^i\right),-e^N\right).$$

As $P_A\left(\tilde{C}^N\right)=C^N$ there exists $z_N^i \in C^N$ such that $P_A\left(\tilde{z}_N^i\right)=z_N^i$ and hence we have

$$\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)=\sum_{R\in\mathcal{N}}\alpha_R^i\left(z_R^i,-e^R\right).$$

As feasibility implies $\left(\sum\limits_{i\in S}\bar{x}^{(1)i},-e^S\right)=\sum\limits_{i\in S}\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)$, for the 2^{nd} group of n coordinates we have that

$$\begin{aligned} e^S &= \sum_{i\in S}\sum_{R\in\mathcal{N}}\alpha_R^i e^R \\ &= \sum_{R\in\mathcal{N}}\left(\sum_{i\in S}\alpha_R^i\right)e^R. \end{aligned}$$

Thus $\alpha_R^i > 0$ implies $R \subseteq S$ and if we define $\alpha(R)=\sum\limits_{i\in S}\alpha_R^i$, then $(\alpha(R))_{R\subseteq S}$ is a balanced family for the coalition S. Looking at the 1^{st} group of n coordinates we have

$$\sum_{i\in S}\bar{x}^{(1)i} = \sum_{R\subseteq S}\sum_{i\in S}\alpha_R^i z_R^i$$
$$= \sum_{\{R\subseteq S|\alpha(R)>0\}}\alpha(R)\left(\frac{1}{\alpha(R)}\sum_{i\in S}\alpha_R^i z_R^i\right).$$

Since C^R is convex we have

$$\frac{1}{\alpha(R)}\sum_{i\in S}\alpha_R^i z_R \in C^R$$

and hence, using totally balancedness, $\sum_{i\in S}\bar{x}^{(1)i} \in V(S)$.

From the definition of the utility function we obtain $u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right) \leq \bar{x}_i^{(1)i}$. Since $\left(\bar{x}_i^{(1)i}\right)_{i\in S} \leq \sum_{i\in S}\bar{x}^{(1)i} \in V(S)$ we have by the S-comprehensiveness of $V(S)$ that $\left(u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right)\right)_{i\in S} \in V(S)$.

□

Proposition 2.3. *Every point in A is an equilibrium payoff vector of the market $\mathcal{E}_{V,A,\varepsilon}$.*

Proof. The above proposition holds by an argument similar to the one used in the proof of Proposition 2.1. Let $a \in A$ and $\lambda^a \in \Delta$ an associated normal vector. We know that λ^a is strictly positive (compare the remark on page 46). Note that the consumption and production plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(a^{\{i\}}, 0, a^{\{i\}}, 0, 0\right)\right)_{i\in N}$$

and

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a, -e^N, a, -e^N, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^a, \frac{2}{3}\left(\lambda^a \circ a\right), \lambda^a, \frac{2}{3}\left(\lambda^a \circ a\right), \frac{2}{3}\left(\lambda^a \circ a\right)\right)$$

constitute a competitive equilibrium in the market $\mathcal{E}_{V,A,\varepsilon}$. The equilibrium price vector is strictly positive since a and λ^a are strictly positive.

As we have a convex-cone-technology maximum profits are zero. We observe

$$\hat{p}\cdot\hat{y}^i = \frac{1}{n}\left(\lambda^a \cdot a - \frac{2}{3}\left(\lambda^a \circ a\right)\cdot e^N + \lambda^a \cdot a - \frac{2}{3}\left(\lambda^a \circ a\right)\cdot e^N - \frac{2}{3}\left(\lambda^a \circ a\right)\cdot e^N\right) = 0.$$

Hence, the production plan $\hat{y}^i$ is profit maximizing.

Next we show that the consumption vector x^i is utility maximizing on the budget set of agent i.

- First notice that the budget constraint is satisfied with equality,

$$\hat{p}\cdot\hat{x}^i = \lambda^a \cdot \left(a^{\{i\}} + a^{\{i\}}\right) = \frac{2}{3}\left(\lambda^a \circ a\right)\cdot\left(e^{\{i\}} + e^{\{i\}} + e^{\{i\}}\right) = \hat{p}\cdot\omega^i.$$

- Second the consumption vector of agent i satisfies

$$\hat{x}_i^{(1)i} = \hat{x}_i^{(3)i} + \varepsilon \sum_{j\neq i} \hat{x}_j^{(3)i}.$$

  This means agent i consumes in a way such that he receives the "same amount of utility" from the 1^{st} group of n goods and the 3^{rd} group of n goods. For an agent with a min-type or Leontief utility function it is a necessary condition for utility maximization to consume in such a way (as long as we have strictly positive prices). This can be seen by similar arguments like in the proof of Claim 1.

- Third, it remains to check that $\hat{x}^i$ is indeed utility maximizing for agent i on his budget set. Hereby, the crucial point to see is, that agent i only consumes his personal output goods, and not the output goods of the other agents. In particular, this means for the 3^{rd} group of n commodities $\hat{x}_j^{(3)i} = 0$ for $j \neq i$.

  First look at the consumption of the 3^{rd} group of n goods when half of the wealth, $\lambda^a \cdot a^{\{i\}}$, is used for these goods.

  If agent i spends the wealth only for his personal output commodity, he consumes $\hat{x}^{(3)i} = a^{\{i\}}$. Then we have $\hat{p}^{(3)} \cdot \hat{x}^{(3)i} = \lambda^a \cdot a^{\{i\}}$. Suppose now agent i changes his consumption plan for the 3^{rd} group of n commodities to a plan $\tilde{x}^{(3)i}$, where he consumes as well one of the other agents output goods, meaning $\tilde{x}_j^{(3)i} > 0$ for one $j \neq i$. To do this agent i needs to decrease the consumption in his personal output good and hence $\hat{x}_i^{(3)i} > \tilde{x}_i^{(3)i}$. Set $\delta = \hat{x}_i^{(3)i} - \tilde{x}_i^{(3)i}$. Then this δ he consumes less gives him an available budget of $\lambda_i^a \delta$, that he can now use to spend for the other agents commodity j. If agent i now spends $\lambda_i^a \delta$ for good j, he can purchase $\frac{\lambda_i^a}{\lambda_j^a}\delta$ units of good j which gives him an additional level of "utility" in good j of the 3^{rd} group of n goods.

Look at

$$\begin{aligned}
&\hat{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \hat{x}_j^{(3)i} - \left(\tilde{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \tilde{x}_j^{(3)i} \right) \\
&= \hat{x}_i^{(3)i} - \left(\hat{x}_i^{(3)i} - \delta + \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \cdot \delta \right) \\
&= \delta - \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \cdot \delta \\
&= \delta \left(1 - \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \right).
\end{aligned}$$

The above expression is positive since $\varepsilon < \frac{\lambda_j^a}{\lambda_i^a}$ for all $i, j \in N$ and hence $\varepsilon \frac{\lambda_i^a}{\lambda_j^a} < \frac{\lambda_j^a}{\lambda_i^a} \frac{\lambda_i^a}{\lambda_j^a} = 1$. Thus we have

$$\hat{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \hat{x}_j^{(3)i} > \tilde{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \tilde{x}_j^{(3)i}.$$

The potential loss of utility from consuming less of his personal output commodity is higher than the potential gain from consuming agent j's output commodity given a fixed wealth.

A similar argument also holds true, when agent i changes the consumption in a way such that he consumes output goods of several other agents.

Thus agent i cannot increase his utility by changing his consumption plan for the 3^{rd} group of n commodities from $\hat{x}^{(3)i}$ to $\tilde{x}^{(3)i}$ and consuming output commodities of the other agents $j \neq i$ instead of his own output commodities.

Now it is easy to see, that spending half of the total wealth for each of the two groups of output commodities leads to the same amount of utility in both arguments of the min-type utility function and is hence utility maximizing.

Furthermore, the market clearing condition

$$\sum_{i \in N} \hat{x}^i = \sum_{i \in N} \omega^i + \sum_{i \in N} \hat{y}^i$$

is satisfied.
Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i\left(\hat{x}^i\right)\right)_{i \in N} = a.$$

□

In the above proof the competitive equilibrium price vectors are linked to the transfer rate vectors of points in the set A similarly as in the proof of Proposition 2.1. The output goods are directly priced by the transfer rate vectors and the input goods are priced by the transfer rate vectors weighted by the according point of the set A (multiplied by $\frac{2}{3}$).

It remains to show, that vectors not belonging to the set A cannot be competitive payoff vectors. The crucial point is to show, that $p^{(3)}$ is strictly positive.

Lemma 2.4. *Let $((x^i)_{i \in N}, (y^i)_{i \in N}, p)$ be any competitive equilibrium for the induced A-market. Then $p^{(3)}$ is strictly positive.*

Proof. Let $((x^i)_{i \in N}, (y^i)_{i \in N}, p)$ be a competitive equilibrium for the induced A-market. By the market clearing condition we have

$$\sum_{i \in N} x^i = \sum_{i \in N} y^i + \left(0, e^N, 0, e^N, e^N\right)$$

and by profit maximization $p \cdot y^i = 0$ for all $i \in N$. By the definition of the production set for each $i \in N$ there exist $\gamma_S^{i1},\ \gamma_S^{i2},\ \gamma_S^{i3} \geq 0$ for all $S \in \mathcal{N}$, $u_S^{i1},\ u_S^{i2},\ u_S^{i3} \in C^S$ for all $S \in \mathcal{N} \setminus \{N\}$ and $\tilde{u}_N^{i1},\ \tilde{u}_N^{i2},\ \tilde{u}_N^{i3} \in \tilde{C}^N$ such that

$$\begin{aligned} y^i = & \sum_{S \in \mathcal{N} \setminus \{N\}} \left(\sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\gamma_S^{i1} e^S,\ \sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\left(\gamma_S^{i1} + \gamma_S^{i2}\right) e^S,\ -\left(\gamma_S^{i1} + \gamma_S^{i3}\right) e^S \right) \\ & + \left(\sum_{j=1}^{3} \gamma_N^{ij} P_A\left(\tilde{u}_N^{ij}\right),\ -\gamma_N^{i1} e^N,\ \sum_{j=1}^{3} \gamma_N^{ij} \tilde{u}_N^{ij},\ -\left(\gamma_N^{i1} + \gamma_N^{i2}\right) e^N,\ -\left(\gamma_N^{i1} + \gamma_N^{i3}\right) e^N \right). \end{aligned}$$

As $P_A\left(\tilde{C}^N\right) = C^N$ there exist $u_N^{ij} \in C^N$ such that $P_A\left(\tilde{u}_N^{ij}\right) = u_N^{ij}$ for $j = 1, 2, 3$. Thus, we have for all $i \in N$

$$\begin{aligned} y^i = & \sum_{S \in \mathcal{N} \setminus \{N\}} \left(\sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\gamma_S^{i1} e^S,\ \sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\left(\gamma_S^{i1} + \gamma_S^{i2}\right) e^S,\ -\left(\gamma_S^{i1} + \gamma_S^{i3}\right) e^S \right) \\ & + \left(\sum_{j=1}^{3} \gamma_N^{ij} u_N^{ij},\ -\gamma_N^{i1} e^N,\ \sum_{j=1}^{3} \gamma_N^{ij} \tilde{u}_N^{ij},\ -\left(\gamma_N^{i1} + \gamma_N^{i2}\right) e^N,\ -\left(\gamma_N^{i1} + \gamma_N^{i3}\right) e^N \right). \end{aligned}$$

By the definition of the consumption set we need to have $x^{(2)i} = x^{(4)i} = x^{(5)i} = 0$ for all $i \in N$. Hence, for all $i \in N$, we obtain, using the market clearing condition and the definition of the production sets, for all coalitions $S \in \mathcal{N}$

$$\begin{aligned} \sum_{T \subseteq N} \gamma_T^{i1} e^T &= e^S, \\ \sum_{T \subseteq N} \left(\gamma_T^{i1} + \gamma_T^{i2}\right) e^T &= e^S, \\ \sum_{T \subseteq N} \left(\gamma_T^{i1} + \gamma_T^{i3}\right) e^T &= e^S. \end{aligned}$$

It follows that $\gamma_S^{i2} = \gamma_S^{i3} = 0$ for all $i \in N$ and for all $S \in \mathcal{N}$ and that for some $i \in N$ and some $S \in \mathcal{N}$ we have $\gamma_S^{i1} > 0$.

Suppose now, that $p_i^{(3)} = 0$ for at least one $i \in N$. We show, that this leads to a contradiction.

First observe: If $p_i^{(3)} = 0$ for one $i \in N$, then $p_k^{(3)} = 0$ for all $k \in N$.

To see this suppose $p_k^{(3)} > 0$ for some $k \in N$. For every individual $j \in N$ the consumption bundle x^j maximizes his utility function over his budget set $\{\hat{x}^j \in X^j | p \cdot \hat{x}^j \leq p \cdot \omega^j\}$. This implies, if $p_i^{(3)} = 0$ that agent j does not consume any good that has a positive price. If he did so, this would decrease his available budget whereas he can reach the same utility from consuming good i that is for free. Precisely $p_i^{(3)} = 0$ implies $x_k^{(3)j} = 0$ for all $j \in N$ and for all $k \in N$ such that $k \neq i$ and $p_k^{(3)} > 0$.
However, the market clearing condition and the definition of the production set require

$$\sum_{j \in N} x^{(3)j} = \sum_{S \in \mathcal{N} \setminus \{N\}} \gamma_S^{i1} u_S^{i1} + \gamma_N^{i1} \tilde{u}_N^{i1} \gg 0,$$

since $u_S^{i1} \in C^S \subseteq \mathbb{R}_{++}^S$ and $\tilde{u}_N^{i1} \geq u_N^{i1} \in C^N \subseteq \mathbb{R}_{++}^N$. Hence, we obtain a contradiction and thus $p^{(3)} = 0$.

Since $u^j(\check{x}^j) > u^j(\bar{x}^j)$ whenever $\check{x}_j^{(1)j} > \bar{x}_j^{(1)j}$ and $\check{x}^{(3)j} > \bar{x}^{(3)j}$, it follows from $p^{(3)} = 0$ that $p_j^{(1)}$ must be positive. This holds for all $j \in N$, thus $p^{(1)} \gg 0$.
Since $C^S \subseteq \mathbb{R}_{++}^S$, it follows that $p^{(1)} \cdot u_S^{i1} > 0$. Since the maximal profits are equal to zero because of the convex-cone-technology, it must be true that

$$p^{(1)} \cdot u_S^{i1} - p^{(2)} \cdot e^S - p^{(4)} \cdot e^S - p^{(5)} \cdot e^S = 0. \qquad (\star)$$

For any $j \in N$ choose $u \in C^{\{j\}} \cap \mathbb{R}_{++}^{\{j\}}$ and $\gamma > 0$. Then

$$\left(\gamma u, 0, \gamma u, -\gamma e^{\{j\}}, 0\right) \in Y^j$$

and

$$p \cdot \left(\gamma u, 0, \gamma u, -\gamma e^{\{j\}}, 0\right) = \gamma \left(p_j^{(1)} u - p_j^{(4)}\right).$$

Since $p^{(1)} \gg 0$, $p_j^{(4)}$ must be positive, because otherwise this would contradict the fact, that maximal profits are 0. Thus, $p^{(4)} \gg 0$. Similarly $p^{(5)} \gg 0$. Therefore, from the equation $(\star)$ above we obtain using $-p^{(5)} \cdot e^S < 0$ and

$-p^{(2)} \cdot e^S \leq 0$

$$p^{(1)} \cdot u_S^{i1} - p^{(4)} \cdot e^S > 0.$$

Hence, we have

$$p \cdot \left(u_S^{i1}, 0, u_S^{i1}, -e^S, 0\right) = p^{(1)} \cdot u_S^{i1} + p^{(3)} \cdot u_S^{i1} - p^{(4)} \cdot e^S = p^{(1)} \cdot u_S^{i1} - p^{(4)} \cdot e^S > 0.$$

But $\left(u_S^{i1}, 0, u_S^{i1}, -e^S, 0\right) \in Y^i$ as it is of the form as points in the set A_S^2. This is a contradiction to the fact, that the maximal profits are zero. Thus $p^{(3)} \gg 0$. □

We use this result to show the remaining Proposition that completes the proof of the theorem:

Proposition 2.4. *Any payoff vector of a competitive equilibrium of the market $\mathcal{E}_{V,A,\varepsilon}$ is an element of the set A.*

Proof. Suppose there exists a competitive equilibrium $\left((x^i)_{i \in N} (y^i)_{i \in N}, p\right)$, such that $(u^i(x^i))_{i \in N} = c^N$ with $c^N \notin A$.

From Lemma 2.1 we know that c^N is in the inner core.

> That Lemma 2.1 is applicable can be seen as follows: We know that $p \cdot \omega^i > 0$. Otherwise agent i would have a budget of 0 and we needed to have $p_i^{(2)} = p_i^{(4)} = p_i^{(5)} = 0$. This would mean that the production plan $\left(c^{\{i\}}, -e^{\{i\}}, c^{\{i\}}, -e^{\{i\}}, -e^{\{i\}}\right)$ with $c^{\{i\}} \in C^{\{i\}}$ has strictly positive profits. This would be a contradiction. Thus, for all individuals $i \in N$ we have $p \cdot \omega^i > 0$.

By Lemma 2.4 we know $p^{(3)} \gg 0$. Furthermore we know

$$y = \sum_{i \in N} y^i = \left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right)$$

for some $\tilde{c}^N \in \tilde{C}^N$ satisfying $P_A\left(\tilde{c}^N\right) = c^N$ as any other production would contradict the market clearing condition in the 1^{st} group of n coordinates. From the profit maximization we know that $\tilde{c}^N$ has to be chosen on the boundary of $\tilde{C}(N)$ and hence, since $c^N \notin A$, we have $\tilde{c}^N \gg c^N$. By the market clearing condition (for the 3^{rd} group of n coordinates) we have

$$\sum_{i \in N} x^{(3)i} = \tilde{c}^N. \qquad (\star\star)$$

Furthermore, by utility maximization we obtain

$$c_i^N = x_i^{(3)i} + \varepsilon \sum_{j \neq i} x_j^{(3)i}. \qquad (\star\star\star)$$

As $c^N \ll \tilde{c}^N$, equation $(\star\star\star)$ implies, that we have $x_i^{(3)i} < \tilde{c}_i^N$ for all $i \in N$.

Hence, for every $i \in N$ we have $\sum_{j \neq i} x_i^{(3)j} > 0$. Thus, for every $i \in N$ there exists $j \neq i$ satisfying $x_i^{(3)j} > 0$. Define a mapping $M : N \longrightarrow N$ in the following way: Every $i \in N$ is mapped to one $j \neq i$ satisfying $x_i^{(3)j} > 0$. Then, we can find $k \in N$ and $t \in \mathbb{N}$ such that $M^t(k) = k$.

We use these results to show some constraints on the equilibrium prices: As $x_k^{(3)M(k)} > 0$, the utility maximization of agent $M(k)$ implies, that we have $p_k^{(3)} \leq \varepsilon p_{M(k)}^{(3)}$. Otherwise, agent $M(k)$ would not consume good k, but instead more of good $M(k)$. In the same way, we can show similar equations for other prices and obtain

$$p_k^{(3)} \leq \varepsilon p_{M(k)}^{(3)} \leq \varepsilon^2 p_{M^2(k)}^{(3)} \leq ... \leq \varepsilon^t p_{M^t(k)}^{(3)} = \varepsilon^t p_k^{(3)}.$$

But $\varepsilon^t < 1$. This is a contradiction.

□

As already mentioned before, assuming SPS is more restrictive than actually needed. Requiring the strict separation property for all points in the set A can be weakened to requiring it only for the boundary points of the set A. In fact, we need for the construction of the auxiliary game $(N, \tilde{V})$ that outside the set A the efficient boundary is strictly enlarged. This means the property that if we take $x \in V(N) \setminus A$, then x being in the interior of $\tilde{V}(N)$ is the crucial property to eliminate equilibria with a payoff vector outside the set A. Using this weaker assumption allows a choice of the set A as in Example 7. An example, where even this weaker version of the strict positive separability property is violated, and where our approach cannot be applied can be found in Figure 2.5. Assume as before that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 2.5.

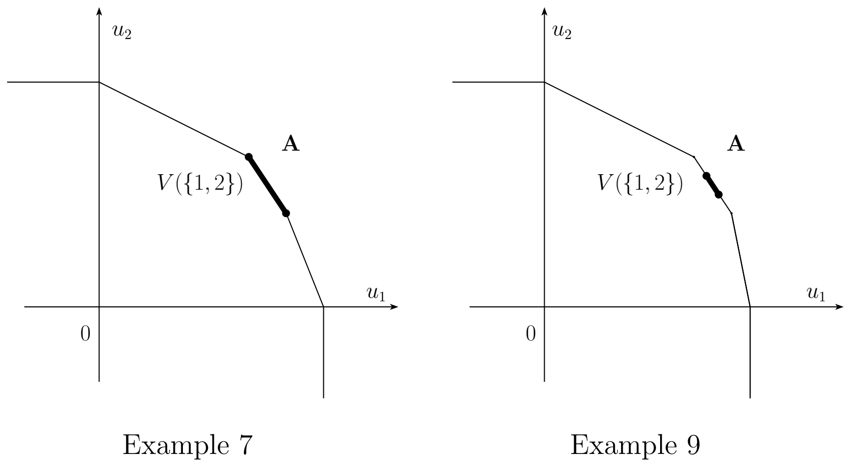

Figure 2.5: Examples where SPS is not satisfied.

In contrast to Example 7, in Example 8 the set A is chosen in such a way that it is a closed interval of a line segment connecting two neighboring corner points, but not the whole line segment. Because of the polyhedral structure none of the points in the set A can be strictly separated from the set $V(\{1,2\})$ without the point.

Another important aspect of our result is the fact that the induced A-market is not determined uniquely. We have some freedom in different aspects of our construction and obtain a whole class of markets, that can be used to prove our main theorem:

- First, to define the induced A-market we use the auxiliary NTU game $(N, \tilde{V})$ where we enlarge the given NTU game (N, V). For this enlargement we use for every inner core point one of its normal vectors. This normal vector is not always unique.

- Second, for the auxiliary game $(N, \tilde{V})$ we define the mapping P_A which can be chosen in different ways. The important property is that for the points outside the given subset of the inner core, A, we have $P_A(z) \gg z$ for all $z \in IC(A) \setminus A$. Moreover, for points in the given set A we require $P_A(z) = z$ for all $z \in A$.

- Third, we add to the utility function of the induced A-market an ε-term, that needs to be between certain bounds and hence is not determined uniquely. Moreover, we can choose different ε for different players.

2.4 Concluding Remarks

In this paper we have continued the work of Shapley and Shubik (1975) and Qin (1993) to investigate competitive payoff vectors of markets that represent a cooperative game and their relation to solution concepts for cooperative games.

We extend the results of Qin (1993) to a large class of closed subsets of the inner core: Given an NTU market game we construct the induced A-market depending on a given closed subset of its inner core. This market represents the game and further has the given set as the set of payoff vectors of competitive equilibria. More precisely, inspired by the construction of the induced market of Billera and Bixby (1974) and by the markets that Qin (1993) uses to prove his two main results, we define a market in an appropriate way to generalize the results of Qin (1993) to a large class of closed subsets of the inner core. It turns out that this market is not determined uniquely and thus we obtain a whole class of markets that has the given closed subset of the inner core as the set of payoff vectors of competitive equilibria.

In the literature it was already known that one game can be represented by several markets, see Billera and Bixby (1974) or Qin (1993). Our work confirms that going from NTU games to markets some structural information is added that is not present in the NTU game. To a given NTU market game we can associate a huge class of markets that represents the NTU game. In particular, by choosing the structure, that we add, we can control the set of payoffs of competitive equilibria.

Another point of view on our results is to analyze situations where we start with given markets and consider the induced games. Looking at competitive equilibria and how they appear in the game, we observe that almost everything is possible. Depending on the specific market the set of competitive equilibrium payoff vectors might fill up the whole inner core or be almost any closed subset, in particular any single point. Hence, our result demonstrates that we can not expect to observe more game theoretic properties of competitive equilibria than knowing that competitive payoffs are in the inner core. Only by imposing additional structural assumptions on the markets, for example restricting the class of utility functions, we may observe additional game theoretic properties.

We establish a link between closed subsets of the inner core and competitive payoffs of certain economies. Extending the results of Qin (1993) to closed subsets of the inner core means in particular to establish a link for all solution concepts selecting closed subsets of the inner core. Therefore, our results can be seen as a market foundation of game theoretic solution concepts that select closed subsets of the inner core. For the particular class of bargaining games a more precise presentation of the idea of a market foundation can be found in Trockel (1996, 2005) and Brangewitz and Gamp (2011b).

The result presented here includes the result of Qin (1993) for a single point in the inner core. This holds also in a very general setup by using monotone transformations of utilities in the same way as it was done in Qin (1993). Nevertheless, if we consider closed subsets of the inner core that contain more than a single point, the idea to transform the utilities seems not to work. Due to this fact we assume some separation properties on the game and the given closed subset of its inner core.

Furthermore, by investigating the NTU case we realized that a simple generalization of the approach of Shapley and Shubik (1975) in the framework of Qin (1993) does not work and we need to stay closer to the results on NTU games. More precisely, changing the utility function in the market, that Qin (1993) uses to prove his second result, in analogy to the TU case of Shapley and Shubik (1975) to

$$u^i(x^i) = \min\left\{x_i^{(1)i}, \min_{u^* \in A}\left\{\frac{\left(\lambda^{u^*} \circ u^*\right) \cdot x^{(5)i}}{\lambda_i^{u^*}}\right\}\right\}$$

does not lead to markets with the desired properties.

Having our result in mind there remains the open question if we can further weaken our assumptions such that the results can be proved for more general cases. Another interesting related line of research is to continue to look at the class of games that are linked to coalition production economies as analyzed by Inoue (2010b). Given a balanced NTU game Inoue (2010b) defines a coalition production economy such that this economy represents the game and has moreover the whole inner core as the set of competitive equilibrium payoff vectors. It remains an open question if one can find analogously to Qin (1993) and to this work a coalition production economy such that one inner core point or a certain subset of the inner core are competitive equilibrium payoff vectors in this coalition production economy. Moreover, it is interesting to compare the set of competitive equilibrium allocations of different market representations of a given NTU market game. Does there exist a general and more simple method to obtain desired competitive payoffs? Can we characterize a class of NTU games where this is possible? What happens if we restrict our attention for example to bargaining games?

2.5 Appendix

2.5.1 Proof of Lemma 2.1

For the proof of Lemma 2.1 we follow the idea of de Clippel and Minelli (2005).

Proof. Let $(\hat{x}^i)_{i\in N}$ and $(\hat{y}^i)_{i\in N}$ be a competitive equilibrium allocation at a price $\hat{p} \in \mathbb{R}^\ell_+ \setminus \{0\}$. For each individual $i \in N$ define the set

$$C^i = \left\{(u,m) \in \mathbb{R}^2 | \exists z^i \in X^i : u \leq u^i\left(z^i\right) - u^i\left(\hat{x}^i\right), m \leq \hat{p} \cdot \left(\omega^i + \hat{y}^i - z^i\right)\right\}.$$

By the concavity of u^i, this set is convex. On the other hand, $C^i \cap \mathbb{R}^2_{++} = \emptyset$, as $\hat{x}^i$ is optimal for individual i in his budget set.

> Suppose $(u,m) \in C^i$ and $(u,m) \gg 0$, then there exists $z^i \in X^i$ with $u(\hat{x}^i) < u^(z^i)$ and $\hat{p} \cdot z^i < \hat{p} \cdot (\omega^i + \hat{y}^i)$ which means z^i gives individual i a higher utility as $\hat{x}^i$ and is affordable under the price system $\hat{p}$. This is in contradiction to the optimality of $\hat{x}^i$.

By the separating hyperplane theorem there exists a non-zero, non-negative vector $(\alpha^i, \beta^i) \in \mathbb{R}^2_+$ such that we can separate 0 from C^i and obtain

$$\alpha^i u^i\left(\hat{x}^i\right) \geq \alpha^i u^i\left(z^i\right) - \beta^i \hat{p} \cdot \left(z^i - \omega^i - \hat{y}^i\right)$$

for all $z^i \in X^i$.
As $\hat{p} \cdot \omega^i > 0$, it follows from the above inequality that we have $\alpha^i > 0$.

> To see this suppose $\alpha^i = 0$ $(\beta^i > 0)$. Then, as in equilibrium $\hat{p} \cdot \hat{y}^i = 0$, we obtain from the above inequality
>
> $$0 \leq \hat{p} \cdot \left(z^i - \omega^i - \hat{y}^i\right) \quad \text{for all } z^i \in X^i,$$
>
> which is not true, as $0 \in X^i$ and $\hat{p} \cdot \hat{y}^i = 0$. Thus $\alpha^i > 0$.

We can assume $\alpha^i = 1$ without the loss of generality. Moreover, monotonicity and locally non-satiation of the utility function imply that $\beta^i > 0$. Let

$\lambda^i = \frac{1}{\beta^i}$. Summing up over all $i \in S$ we obtain

$$\sum_{i \in S} \lambda^i u^i \left(\hat{x}^i\right) \geq \sum_{i \in S} \lambda^i u^i \left(z^i\right) - \hat{p} \cdot \sum_{i \in S} \left(z^i - \omega^i - \hat{y}^i\right)$$

for all $S \subseteq N$ and for all $z^i \in \mathbb{R}^\ell_+$ with $i \in S$.

If a coalition S could λ-improve on x with $\left(\bar{x}^i\right)_{i \in S}$ (with the production plan $\bar{y}^i \in Y^i$), then the previous inequality would be violated, because we have, due to feasibility,

$$\sum_{i \in S} \left(\bar{x}^i - \omega^i - \bar{y}^i\right) \leq 0$$

and thus we obtain a contradiction by

$$\begin{aligned}
\sum_{i \in S} \lambda^i u^i \left(\bar{x}^i\right) &> \sum_{i \in S} \lambda^i u^i \left(\hat{x}^i\right) \\
&\geq \sum_{i \in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i \in S} \left(\bar{x}^i - \omega^i - \hat{y}^i\right) \\
&\geq \sum_{i \in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i \in S} \left(\bar{x}^i - \omega^i\right) \\
&\geq \sum_{i \in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i \in S} \bar{y}^i \\
&\geq \sum_{i \in S} \lambda^i u^i \left(\bar{x}^i\right).
\end{aligned}$$

□

2.5.2 Proof of Claim 1

Proof. We show

$$x_i^{(1)i} = x_i^{(3)i}$$

by contradiction. Then it immediately follows from $u^i(x^i) = c_i^N$ that

$$x_i^{(1)i} = x_i^{(3)i} = c_i^N.$$

Suppose $x_i^{(3)i} > x_i^{(1)i}$. This cannot be utility maximizing in the presence of strictly positive prices. If player i consumes a little bit less of the i^{th} good of the 3^{rd} group of n goods and invests the - not anymore used - additional budget in the i^{th} good of the 1^{st} group of n goods, then he can strictly increase his utility.
Precisely, from the assumption $u^i(x^i) = c_i^N$ and $x_i^{(3)i} > x_i^{(1)i}$ it follows that $x_i^{(1)i} = c_i^N$. For δ sufficiently small, i.e. $0 < \delta < x_i^{(3)i} - x_i^{(1)i}$, player i can increase his utility by consuming δ less of the i^{th} good of the 3^{rd} group of n goods and increasing the consumption in the i^{th} good of the 1^{st} group of n goods by $\frac{p_i^{(3)}}{p_i^{(1)}}\delta$. To consume $\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta e^{\{i\}}, 0, x^{(3)i} - \delta e^{\{i\}}, 0\right)$ is still budget feasible for player i, because

$$p^{(1)}\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta e^{\{i\}}\right) + p^{(3)}\left(x^{(3)i} - \delta e^{\{i\}}\right) = p^{(1)}x^{(1)i} + p^{(3)}x^{(3)i} \leq p \cdot \omega^i.$$

Hereby, the last inequality follows from the budget feasibility of x^i. Moreover, the utility of consumer i strictly increases, since

$$u^i\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta, 0, x^{(3)i} - \delta, 0\right) > x_i^{(1)i} = u^i\left(x^{(1)i}, 0, x^{(3)i}, 0\right)$$

by the choice of δ. This is a contradiction to the assumption that x^i is utility maximizing. Hence, we have $x_i^{(3)i} \leq x_i^{(1)i}$.

By exchanging the roles of $x_i^{(1)i}$ and $x_i^{(3)i}$ we can analogously show $x_i^{(3)i} \geq x_i^{(1)i}$. Therefore, we have $x_i^{(3)i} = x_i^{(1)i}$. □

Chapter 3

Inner Core, Asymmetric Nash Bargaining Solution and Competitive Payoffs

3.1 Introduction

The inner core and asymmetric Nash bargaining solutions represent solution concepts for cooperative games. The inner core is defined for cooperative games whereas asymmetric Nash bargaining solutions are usually only applied to a subclass of cooperative games, namely bargaining games. A recent contribution of Compte and Jehiel (2010) generalizes the symmetric Nash bargaining solution to other cooperative games (with transferable utility). In this paper we consider the relationship between the inner core and asymmetric Nash bargaining solutions for bargaining games. Moreover, as an application of these results we show how asymmetric Nash bargaining solutions can be justified in a general equilibrium framework as a competitive payoff vector of a certain economy.

In the first section we give a literature overview to motivate our ideas. In the second section we recall the definitions of the inner core, a bargaining game and asymmetric Nash bargaining solutions. Afterwards, we investigate for bargaining games the relationship between the inner core and the set of asymmetric Nash bargaining solutions. Finally, we apply these results to market games and obtain by this a market foundation of asymmetric Nash bargaining solutions.

3.2 Motivation and Background

The inner core is a refinement of the core for cooperative games with non-transferable utility (NTU). For cooperative games with transferable utility (TU) the inner core coincides with the core. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector.

The Nash bargaining solution for bargaining games, a special class of cooperative games, where just the singleton and the grand coalition are allowed to form, goes back to Nash (1950, 1953). The (symmetric) Nash bargaining solution is defined as the maximizer of the product of the utilities over the individual rational bargaining set or as the unique solution that satisfies the following axioms: Invariance to affine linear Transformations, Pareto Optimality, Symmetry and Independence of Irrelevant Alternatives. If the bargaining power of the players is different an asymmetric Nash bargaining solution can be defined as the maximizer of an accordingly weighted Nash product. Concerning the axiomatization this means that the Symmetry axiom is replaced by an appropriate Asymmetry axiom, see Roth (1979). In addition to the axiomatic approach the literature studies non-cooperative foundations to justify cooperative solutions like the (asymmetric) Nash bargaining solution. The idea is to find an appropriate non-cooperative game whose equilibrium outcomes coincide with a given cooperative solution (see for example Bergin and Duggan (1999), Trockel (2000)). Here, we study the foundation of the asymmetric Nash bargaining solution by having this solution as a payoff vector of a competitive equilibrium in a certain economy.

There are different approaches to consider the relationship between cooperative games and economies or markets. On the one hand for example Shapley (1955), Shubik (1959) Debreu and Scarf (1963) and Aumann (1964) consider economies as games. On the other hand there is the approach to start with a cooperative game and to consider related economies as it was introduced by Shapley and Shubik (1969, 1975).

Starting with a market Shapley (1955) considers markets as cooperative games with two kinds of players, seller and buyer. He introduces in this context the general notion of an 'abstract market game'. This is a cooperative game with certain conditions on the characteristic function. Shubik

(1959) extends the ideas of Edgeworth (from 1881) and studies 'Edgeworth market games'. In particular he shows that if the number of players of both sides in an Edgeworth market game is the same, then the set of imputations coincides with the contract curve of Edgeworth. Furthermore, he considers non-emptiness conditions for the core of this class of games. Debreu and Scarf (1963) show that under certain assumptions a competitive allocation is in the core. Aumann (1964) investigates, based among others on the oceanic games from Milnor and Shapley (1978)[1], economies with a continuum of traders and obtains that in this case the core equals the set of equilibrium allocations.

Starting with a cooperative game Shapley and Shubik (1969) look at these problems from a different viewpoint and study which class of cooperative games can be represented by a market. A market represents a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. Shapley and Shubik (1969) call any game that can be represented by a market a 'market game'. In the TU-case it turns out that every totally balanced TU game is a market game. Furthermore, Shapley and Shubik (1975) start with a TU game and show that every payoff vector in the core of that game is competitive in a certain market, the direct market. The direct market has a nice structure: Besides a numeraire commodity there are as many goods as players and initially every player owns one unit of 'his personal commodity'. Moreover, Shapley and Shubik (1975) show that for a given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector.

The idea of market games was applied to NTU games by Billera and Bixby (1974). Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced game, that is compactly convexly generated, is an NTU market game. Qin (1993) compares the inner core of NTU market games with the competitive payoff vectors of markets that represent this game. He shows that for a given NTU market game there exists a market such that the set of equilibrium payoff vectors coincides with the

[1]The reference Milnor and Shapley (1978) is based on the Rand research memoranda from the early 1960's.

inner core of the game. In a second result, he shows that given an inner core point there exists a market, which represents the game and has this given inner core point as its unique competitive equilibrium payoff. Brangewitz and Gamp (2011a) extend the results of Qin (1993) to a large class of closed subsets of the inner core.

Apart from this literature Trockel (1996, 2005) considers bargaining games directly as Arrow-Debreu or coalition production economies. One difference to other literature is that he allows to obtain output in the production without requiring input. In contrast to Shapley and Shubik (1969, 1975), Trockel (1996, 2005) considers NTU games rather than TU games. Motivated by the approach of Sun et al. (2008) and the approach of Billera and Bixby (1974), Inoue (2010b) uses coalition production economies instead of markets. Inoue (2010b) shows that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy such that its set of competitive payoff vectors coincides with the inner core of the balanced cover of the original NTU game.

Here, we show that we can apply the main results of Qin (1993) to a special class of NTU games, namely bargaining games. By that we obtain a market foundation of the asymmetric Nash bargaining solution. In contrast to Trockel (1996, 2005) we do not use Arrow-Debreu or coalition production economies directly but we consider bargaining games as market games by using the economies of Qin (1993). By this we relate the approach of Trockel (1996, 2005) on the one hand with the ideas of Qin (1993) on the other hand. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions in analogy to the results on non-cooperative foundations of cooperative games (see Trockel (2000), Bergin and Duggan (1999)).

3.3 Inner Core and Asymmetric Nash Bargaining Solution

3.3.1 NTU Games and the Inner Core

Let $N = \{1, ..., n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ be the set of players. Let $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ be the set of non-empty coalitions and $\mathcal{P}(\mathbb{R}^n) = \{A | A \subseteq \mathbb{R}^n\}$ be the set of all subsets of $\mathbb{R}^n$. Define $\mathbb{R}^S_+ = \left\{x \in \mathbb{R}^n_+ | x_i = 0, \forall i \notin S\right\}$.

Definition 3.1 (NTU game)**.** An *NTU game* is a pair (N, V), where the coalitional function is defined as

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

such that for all non-empty coalitions $S \subseteq N$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$ and $V(S)$ is S-comprehensive.

Definition 3.2 (compactly (convexly) generated)**.** An NTU game (N, V) is *compactly (convexly) generated* if for all $S \in \mathcal{N}$ there exists a compact (convex) $C^S \subseteq \mathbb{R}^S$ such that the coalitional function can be written as $V(S) = C^S - \mathbb{R}^S_+$.

In order to define the inner core we first consider a game that is related to a compactly generated NTU game. Given a compactly generated NTU game we define for a given transfer rate vector $\lambda \in \mathbb{R}^N_+$ the λ-transfer game.

Definition 3.3 (λ-transfer game)**.** Let (N, V) be a compactly generated NTU game and let $\lambda \in \mathbb{R}^N_+$. Define the *$\lambda$-transfer game* of (N, V) by (N, V_λ) with

$$V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$$

where $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$.

Qin (1994, p.433) gives the following interpretation of the λ-transfer game: "The idea of the λ-transfer game may be captured by thinking of each player as representing a different country. The utilities are measured in different currencies, and the ratios λ_i / λ_j are the exchange rates between the currencies of i and j." As for the λ-transfer game only proportions matter we can assume without loss of generality that λ is normalized, i.e. $\lambda \in \Delta^n = \left\{\lambda \in \mathbb{R}^n_+ | \sum_{i=1}^n \lambda_i = 1\right\}$. Define the positive unit simplex by $\Delta^n_{++} = \left\{\lambda \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n \lambda_i = 1\right\}$.

The inner core is a refinement of the core. The *core* $C(V)$ of an NTU game (N, V) is defined as those utility allocations that are achievable by the grand coalition N such that no coalition S can improve upon this allocation. Thus,

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i\}.$$

Definition 3.4 (inner core, Shubik (1984))**.** The *inner core* $IC(V)$ of a compactly generated NTU game (N, V) is

$$IC(V) = \{u \in V(N) | \exists \lambda \in \Delta \text{ such that } u \in C(V_\lambda)\}$$

where $C(V_\lambda)$ denotes the core of the λ-transfer game of (N, V).

This means a vector u is in the inner core if and only if u is affordable by the grand coalition N and if u is in the core of an appropriately chosen λ-transfer game. If a utility allocation u is in the inner core, then u is as well in the core.

For compactly convexly generated NTU games we have the following remark:

Remark (Qin (1993), Remark 1, p. 337)**.** The vectors of supporting weights for a utility vector in the inner core must all be strictly positive.

3.3.2 NTU Bargaining Games and Asymmetric Nash Bargaining Solutions

We consider a special class of NTU games, where only the singleton or the grand coalition can form, namely NTU bargaining games. Two-person bargaining games with complete information and the (symmetric) Nash bargaining solution were originally defined by Nash (1950).

Alternatively to the notion based on Nash (1950)[2] we adapt the notation and interpret bargaining games here as a special class of NTU games where only the grand coalition can profit from cooperation. Smaller coalitions are theoretically possible but there are no incentives to form them as everybody obtains the same utility as being in a singleton coalition. Starting from the definition of a bargaining game based on Nash (1950) we define an NTU bargaining game. Let $B \subseteq \mathbb{R}^n$ be a compact, convex set and assume that there exists at least one $b \in B$ with $b \gg 0$. For normalization purposes we assume here that the disagreement outcome is 0 and that $B \subseteq \mathbb{R}^n_+$. Nevertheless the results presented here can easily be generalized to the case that the disagreement point is not equal to 0.

Definition 3.5 (NTU bargaining game)**.** Define an *NTU bargaining game*[3] (N, V) with the generating set B using the player set N and the coalitional function

$$V : \mathcal{N} \longrightarrow \mathcal{P}\left(\mathbb{R}^n\right)$$

defined by

$$\begin{aligned} V\left(\{i\}\right) : &= \{b \in \mathbb{R}^n | b_i \leq 0, b_j = 0, \forall j \neq i\} = \{0\} - \mathbb{R}^{\{i\}}_+, \\ V\left(S\right) : &= \{0\} - \mathbb{R}^S_+ \quad \text{for all S with } 1 < |S| < n, \\ V\left(N\right) : &= \{b \in \mathbb{R}^n | \exists\, b' \in B : b \leq b'\} = B - \mathbb{R}^n_+. \end{aligned}$$

The definition of an NTU bargaining game reflects the idea that smaller coalitions than the grand coalition do not gain from cooperation. They can-

[2]Following the idea of Nash (1950) a *n-person bargaining game* with complete information is defined as a pair (B, d) with the following properties:

1. $B \subseteq \mathbb{R}^n$,
2. B is convex and compact,
3. $d \in B$ and there exists at least one element $b \in B$ such that $d \ll a$.

($d \ll b$ if and only if $d_i < b_i$ for all $i = 1, ..., n$. This means that there is a utility allocation in B that gives every player a strictly higher utility than the disagreement point.)
B is called the feasible or decision set and d is called the status quo, conflict or disagreement point.

[3]Billera and Bixby (1973a, Section 4) modeled bargaining games in the same way.

not reach higher utility levels as the singleton coalitions for all its members simultaneously. Only in the grand coalition every individual can be made better off. In the further analysis we use the above comprehensive version of an n-person NTU bargaining game.

One solution concept for bargaining games with complete information is that of an asymmetric Nash bargaining solution. To define this solution we take as the set of possible vectors of weights or bargaining powers the strictly positive n-dimensional unit simplex Δ^n_{++}.

Definition 3.6 (asymmetric Nash bargaining solution)**.** The *asymmetric Nash bargaining solution with a vector of weights* $\theta = (\theta_1, ..., \theta_n) \in \Delta^n_{++}$, for short θ-asymmetric, for a n-person NTU bargaining game (N, V) with disagreement point 0 is defined as the maximizer of the θ-asymmetric Nash product given by $\prod_{i=1}^n u_i^{\theta_i}$ over the set $V(N)$.[4]

Hereby, we consider the symmetric Nash bargaining solution as one particular asymmetric Nash bargaining solution, namely the one with the vector of weights $\theta = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$. Hence, the correct interpretation of "asymmetric" in this sense is "not necessarily symmetric".

As the NTU bargaining game (N, V) is compactly convexly generated, the set $V(N)$ is closed and convex and hence the maximizer above exists. Note that the assumption that the vectors of weights are from Δ^n_{++} instead of $\mathbb{R}^n_{++}$ can be made without loss of generality.

The asymmetric Nash bargaining solution is a well-known solution concept for bargaining games. Similarly to the symmetric Nash bargaining solution the asymmetric Nash bargaining solution satisfies the axioms Invariance to affine linear Transformations, Pareto Optimality and Independence of Irrelevant Alternatives. As for example shown in Roth (1979, p.20), these axioms together with an appropriate asymmetry assumption fixing the vector of weights characterize an asymmetric Nash bargaining solution. Dropping only the Symmetry axiom without making an appropriate

[4]For bargaining games with a general threat point $d \in \mathbb{R}^n$ the θ-asymmetric Nash product is given by $\prod_{i=1}^n (u_i - d_i)^{\theta_i}$.

asymmetry assumption is not sufficient to characterize the set of asymmetric Nash bargaining solutions. Peters (1992, p.17–25) shows that one needs to consider so called "bargaining solutions corresponding to weighted hierarchies" which include as a special case the asymmetric Nash bargaining solutions.

3.3.3 Relationship between the Inner Core and Asymmetric Nash Bargaining Solutions

Having introduced the concept of the inner core and the asymmetric Nash bargaining solution, we investigate the relationship of these concepts for NTU bargaining games. As in NTU bargaining games only the grand coalition can profit from cooperation, looking at the inner core only transfer possibilities within the grand coalition need to be considered. Hereby, it turns out that there is a close connection between the inner core and asymmetric Nash bargaining solutions:

Proposition 3.1. *Let (N, V) be a n-person NTU bargaining game with disagreement point 0 and generating set $B \subseteq \mathbb{R}^n_{++}$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the θ-asymmetric Nash bargaining solution, a^θ, is in the inner core of (N, V).*
- *For any given inner core point a^θ we can find an appropriate vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$ such that a^θ is the maximizer of the θ-asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$.*

Proof.

"$\Rightarrow$" Suppose a^θ is the θ-asymmetric Nash bargaining solution. To prove that a^θ is in the inner core of (N, V), we need to show that a^θ is in the core of the NTU bargaining game (N, V) and that there exists a transfer rate vector $\lambda^\theta \in \Delta^n_+$ such that a^θ is in the core of the λ^θ-transfer game (N, V_{λ^θ}).

a^θ is the maximizer of the θ-asymmetric Nash product

$$\prod_{i=1}^{n} u_i^{\theta_i}$$

over $V(N)$. Since there exists at least one $u \gg 0$ in $V(N)$ the θ-asymmetric Nash product is strictly positive and thus a^θ is as well the maximizer of the logarithm of the θ-asymmetric Nash product

$$g(u) = \sum_{i=1}^{n} \theta_i log(u_i).$$

Since a^θ is the maximizer of the θ-asymmetric Nash product, a^θ is Pareto optimal. Thus, there is no coalition S that can improve upon a^θ. Remember that we are considering bargaining games. Thus in particular no singleton coalition can improve upon a^θ. We conclude that a^θ has to be in the core of the bargaining game (N, V).

Next, we show that a^θ is as well in the core of an appropriately chosen λ-transfer game. The gradient of the function $g(u)$ at a^θ is given by $\frac{\partial g}{\partial x}(a^\theta) = \left(\frac{\theta_1}{a_1^\theta}, ..., \frac{\theta_n}{a_n^\theta}\right)$. We show now, that we have

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \leq \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta$$

for all $x \in V(N)$.[5] To see this, let $x \in V(N)$ and $t \in [0, 1]$ and define $x^t = tx + (1-t)a^\theta$. Observe that $x^t \in V(N)$ since $V(N)$ is convex. Now we get using the maximality of a^θ and by applying Taylor's Theorem that

$$0 \geq g(x^t) - g(a^\theta) = (x^t - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}\left(|x^t - a^\theta|^2\right) = t(x - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}(t^2).$$

This means that we have

$$\frac{\partial g}{\partial x}(a^\theta)(x - a^\theta) \leq 0$$

[5] Compare for the idea of this argument Rosenmüller (2000, p. 549).

and hence

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \le \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta.$$

Taking the normalized gradient, defining

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, ..., \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right)$$

and observing that $\lambda^\theta \gg 0$ we obtain that a^θ is in the core of the λ^θ-transfer game (N, V_{λ^θ}).

"$\Leftarrow$" If $a \in \mathbb{R}^n_+$ is some given vector in the inner core of (N, V), then a is in the core of (N, V) and there exists a transfer rate vector $\lambda \in \Delta^n_+$ such that a is in the core of the λ-transfer game (N, V_λ). Since a is in the core of the λ-transfer game and the NTU bargaining game (N, V) is compactly generated, we know that λ needs to be strictly positive in all coordinates. Otherwise at least one coalition could still improve upon a. We have $a \gg 0$, because a is in the inner core. If we now take the vector of weights of the asymmetric Nash bargaining solution equal to

$$\theta = (\theta_1, .., \theta_n) = \left(\frac{\lambda_1 a_1}{\sum_{i=1}^n \lambda_i a_i}, ..., \frac{\lambda_n a_n}{\sum_{i=1}^n \lambda_i a_i} \right)$$

then a is the maximizer of the asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$ over $V(N)$. Hereby, similar arguments as in the first step can be used to show that this is the appropriate choice of θ. Hence, a is the asymmetric Nash bargaining solution with weights θ of the bargaining game (N, V).

□

One direction of Proposition 3.1 can be generalized to the case where the generating set is a subset of $\mathbb{R}^n_+$ but not a subset of $\mathbb{R}^n_{++}$. The set of asymmetric Nash bargaining solutions is always contained in the inner core, but the inner core might be strictly larger then the set of asymmetric Nash bargaining solutions. This can be seen in the following two-player example with disagreement point $(0, 0)$:

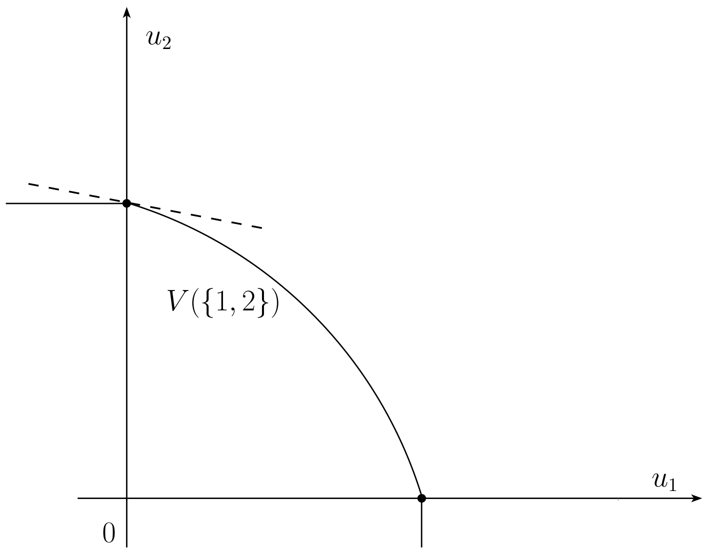

Figure 3.1: Example.

The two points on the axis are in this example in the inner core, as there exits a strictly positive transfer rate vector λ, such that they are in the core of the λ-transfer game. But they cannot result from an asymmetric Nash bargaining solution as any of these solutions chooses only points that are strictly larger than the disagreement point in all coordinates. Thus, the inner core is in this example strictly larger than the set of asymmetric Nash bargaining solutions.

Hence, in general for underlying bargaining sets from $\mathbb{R}^n_+$ and not necessarily from $\mathbb{R}^n_{++}$ Proposition 3.1 reduces to the following statement:

Proposition 3.2. *Let (N, V) be a n-person NTU bargaining game with disagreement point 0 and underlying bargaining set from $\mathbb{R}^n_+$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the asymmetric Nash bargaining solution a^θ for θ is in the inner core of (N, V).*

3.4 Application to Market Games

3.4.1 Market Games

In this section we use the result from the preceding section to investigate the relationship between asymmetric Nash bargaining solutions and competitive payoffs of a market that represents the n-person NTU bargaining game. We start by showing that every NTU bargaining game is a market game. Afterwards, we apply the results of Qin (1993) and Brangewitz and Gamp (2011a) to our results from the previous section.

Definition 3.7 (market)**.** A *market* is given by $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$ is the number of commodities,
- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,
- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in a market the number of consumers coincides with the number of producers. Each consumer has his own private production set. This assumption is not as restrictive as it appears to be. A given private ownership economy can be transformed into an economy with the same number of consumers and producers without changing the set of competitive equilibria or possible utility allocations, see for example Qin and Shubik (2009, section 4). Differently from the usual notion of an economy a market is assumed to have concave and not just quasi-concave utility functions.

Let $S \in \mathcal{N}$ be a coalition. The feasible S-allocations are those allocations that the coalition S can achieve by redistributing their initial endowments and by using the production possibilities within the coalition.

Definition 3.8 (feasible S-allocation)**.** The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

Hence, an S-allocation is feasible if there exist for all $i \in S$ production plans $y^i \in Y^i$ such that $\sum_{i \in S}(x^i - \omega^i) = \sum_{i \in S} y^i$.

In the definition of feasibility it is implicitly assumed that by forming a coalition the available production plans are the sum of the individually available production plans. This approach is different from the idea to use coalition production economies, where every coalition has already in the definition of the economy its own production possibility set. Nevertheless, a market can be "formally" transformed into a coalition production economy by defining the production possibility set of a coalition as the sum of the individual production possibility sets.

Definition 3.9 (NTU market game)**.** An NTU game that is representable by a market is a *NTU market game*, this means there exists a market $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ such that $(N, V_{\mathcal{E}}) = (N, V)$ with

$$V_{\mathcal{E}}(S) = \{u \in \mathbb{R}^S | \, \exists \, (x^i)_{i \in S} \in F(S), u_i \leq u^i(x^i), \, \forall \, i \in S\}.$$

For an NTU market game there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible S-allocations in the market or that give less utility than some feasible S-allocation.

In order to show that every NTU bargaining game is a market game we use the following result from Billera and Bixby (1974):

Theorem 3.1 (2.1, Billera and Bixby (1974))**.** *An NTU game is an NTU market game if and only if it is totally balanced and compactly convexly generated.*

Proposition 3.3. *Every bargaining game* (N, V) *is a market game.*[6]

Proof. We show that every bargaining game is totally balanced. Suppose we have an n-person NTU bargaining game. For totally balancedness we need to check that for every coalition $T \subseteq N$ and for all balancing weights

$$\gamma \in \Gamma(e^T) = \left\{ (\gamma_S)_{S \subseteq T} \in \mathbb{R}_+ | \sum_{S \subseteq T} \gamma_S e^S = e^T \right\}$$

we have

$$\sum_{S \subseteq T} \gamma_S V(S) \subseteq V(T).$$

Since the worth each coalition $S \subsetneq N$ can achieve is $V(S) = \{0\} - \mathbb{R}_+$ and since the grand coalition N can achieve $V(N) = B - \mathbb{R}^n_+$ with at least one element $b \in B$ with $b \gg 0$, we have for all $S \subseteq N$ that $V(S) \subseteq V(N)$ holds. Since for all $S \subseteq N$ we have for the balancing weights $0 \leq \gamma_S \leq 1$ and $\sum_{S \subseteq T} \gamma_S e^S = e^T$ the balancedness condition is satisfied. Thus, the bargaining game is totally balanced and hence a market game. □

We now define a competitive equilibrium for a market $\mathcal{E}$.

Definition 3.10 (competitive equilibrium)**.** A *competitive equilibrium* for a market $\mathcal{E}$ is a tuple

$$\big((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p}\big) \in \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell}_+$$

such that

(i) $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} (\hat{y}^i + \omega^i)$ (market clearing),

(ii) for all $i \in N$, $\hat{y}^i$ solves $\max_{y^i \in Y^i} \hat{p} \cdot y^i$ (profit maximization),

(iii) and for all $i \in N$, $\hat{x}^i$ is maximal with respect to the utility function u^i in the budget set $\{x^i \in X^i | \hat{p} \cdot x^i \leq \hat{p} \cdot (\omega^i + \hat{y}^i)\}$ (utility maximization).

[6]This result was already observed by Billera and Bixby (1973a, Theorem 4.1). In their proof they define a market representation of a bargaining game with $m \leq n^2$ commodities and nondecreasing utility functions.

Given a competitive equilibrium $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ its *competitive payoff vector* is defined as $(u^i(\hat{x}^i))_{i \in N}$.

Qin (1993) investigates the relationship between the inner core of an NTU market game and the set of competitive payoff vectors of a market that represents this game. He establishes, following a conjecture of Shapley and Shubik (1975), the two theorems below analogously to the TU-case of Shapley and Shubik (1975).

Theorem 3.2 (3, Qin (1993))**.** *For every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector.*

Theorem 3.3 (1, Qin (1993))**.** *For every NTU market game, there is a market that represents the game and further has the whole inner core as its competitive payoff vectors.*[7]

3.4.2 Results

Now we apply Theorem 3 of Qin (1993) to prove the existence of an economy corresponding to some vector of weights $\theta \in \Delta^n_{++}$, such that the unique competitive payoff vector of this economy coincides with the θ-asymmetric Nash bargaining solution of the n-person NTU bargaining game.

Proposition 3.4. *Given a n-person NTU bargaining game (N, V) (with disagreement point 0 and generating set from $\mathbb{R}^n_+$) and a vector of weights $\theta \in \Delta^n_{++}$, there is market that represents (N, V) and where additionally the unique competitive payoff vector of this market coincides with the θ-asymmetric Nash bargaining solution a^θ of the NTU bargaining game (N, V).*

Proof. (N, V) is a market game by Proposition 3.3. Moreover, Proposition 3.1 (or Proposition 3.2 respectively) shows, that the θ-asymmetric Nash bargaining solution a^θ is an element of the inner core. Thus, we can apply Theorem 3 from Qin (1993). □

[7]A market that satisfies this property is the so called "induced market" introduced by Billera and Bixby (1974). Its definition can be found in Qin (1993).

The market behind Proposition 3.4 can be taken from Qin (1993) or Brangewitz and Gamp (2011a) taking necessary monotone transformations of the original game as done in Qin (1993) into consideration. A version of these markets for NTU bargaining games can be found in Appendix 2.1 and 2.2.

An Alternative Market for Proposition 3.4

The two markets from Qin (1993) or Brangewitz and Gamp (2011a) have a quite complicated structure. In the following we give a simpler version a market, where strictly positive prices are required. This market is a modification from Brangewitz and Gamp (2011a).

Given a n-person NTU bargaining game (N, V) and a vector of weights $\theta \in \Delta_{++}$. Let a^θ be the θ-asymmetric bargaining solution. From Proposition 3.1 (or Proposition 3.2 respectively) we know that the corresponding λ^θ-transfer game is (N, V_{λ^θ})

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Figure 3.2 illustrates as an example for $N = \{1, 2\}$ the sets $V(\{1, 2\})$ and $V_{\lambda^\theta}(\{1, 2\})$ for an NTU bargaining game with disagreement point $(0, 0)$.

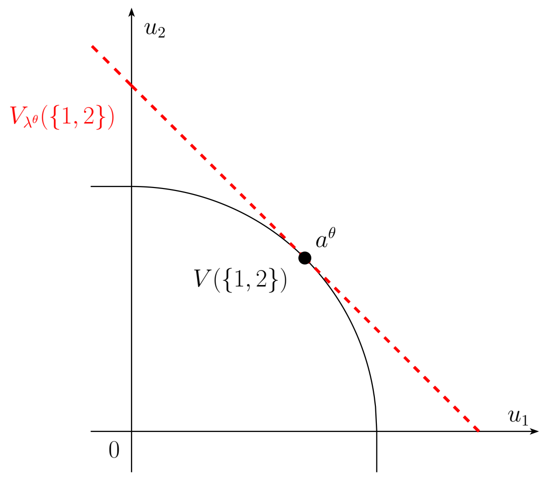

Figure 3.2: Illustration of the sets $V(\{1, 2\})$ and $V_{\lambda^\theta}(\{1, 2\})$.

Let $z \in V_{\lambda^\theta}(N)$ and $\bar{t}^z = \min\left\{t \in \mathbb{R}_+ | z - te^N \in V(N)\right\}$. Define the mapping P_θ by $P_\theta : V_{\lambda^\theta}(N) \longrightarrow V(N)$ via $P_\theta(z) = z - \bar{t}^z e^N$. Figure 3.3 illustrates for the same example as in Figure 3.2 the mapping P_θ.

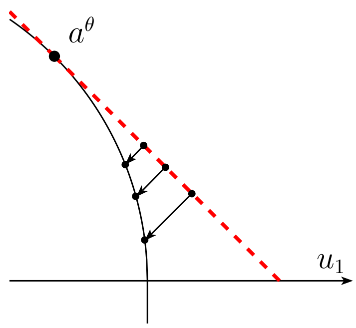

Figure 3.3: Illustration of the mapping P_θ.

The market for the NTU bargaining game (N, V) and vector of weights θ, denoted by $\mathcal{E}_{V,\theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{3n}$,
- the production set

$$Y^i = convexcone\left[\left(\bigcup_{S \in \mathcal{N}\setminus\{N\}} \left\{\left(0, 0, -e^S\right)\right\}\right) \bigcup \left(\bigcup_{c \in \left(V_{\lambda^\theta}(N) \cap \mathbb{R}^n_+\right)} \left\{\left(P_\theta(c), c, -e^N\right)\right\}\right)\right] \subseteq \mathbb{R}^{3n},$$

- the initial endowment vector $\omega^i = \left(0, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(2)i}\right)$ where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(2)i}$ and $x_j^{(2)i}$.

It can be shown using the arguments of Brangewitz and Gamp (2011a) that the market $\mathcal{E}_{V,\theta}$ represents the NTU bargaining game (N, V) and has as its unique competitive equilibrium payoff vector (assuming strictly positive

equilibrium price vectors) the θ-asymmetric Nash bargaining solution a^θ. For the method of proof and the details we refer to Brangewitz and Gamp (2011a). Here, we only state how the competitive equilibria of the market $\mathcal{E}_{V,\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, \left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta, 2\ \lambda^\theta \circ\ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\theta}$.

Considering NTU bargaining games as NTU market games there is a market such that the same sets of utility allocations are reachable in the game and the market. Proposition 3.4 shows that in the class of markets representing a given NTU bargaining game there is a market that has a given asymmetric Nash bargaining solution (with a fixed vector of weights) as its unique competitive payoff vector. We establish a link between utility allocations coming from asymmetric Nash bargaining in NTU bargaining games and payoffs arising from competitive equilibria in certain markets. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions. Instead of considering non-cooperative games to give foundations of cooperative solutions, we link cooperative behavior described by asymmetric Nash bargaining with competitive behavior in markets.

In addition a similar interpretation holds true for the whole inner core and certain of its subsets. Combining Proposition 3.1 with Theorem 1 of Qin (1993) we obtain:

Proposition 3.5. *Let (N, V) be a n-person NTU bargaining game with disagreement point* 0 *and generating set from $\mathbb{R}^n_{++}$. Then there is market that represents (N, V) and where additionally the set of asymmetric Nash solutions of (N, V) coincides with the set of competitive payoff vectors of the market.*

Proof. (N, V) is a market game by Proposition 3.3 and the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core of (N, V) by Proposition 3.1. Thus, we can apply Theorem 1 of Qin (1993). □

The two results of Qin (1993) we use above represent two extreme cases. On the one hand he uses the whole inner core and on the other hand he uses only one single point from the inner core. Brangewitz and Gamp (2011a) show how the results of Qin (1993) can be extended to a large class of closed subsets of the inner core. Using their results we obtain:

Proposition 3.6. *Given a n-person NTU bargaining game (N, V) (with disagreement point* 0 *and generating set from $\mathbb{R}^n_+$) and a closed set $\Theta \subset \Delta^n_{++}$ of strictly positive vectors of weights. Moreover, assume that every θ-asymmetric Nash bargaining solution a^θ with vector of weights $\theta \in \Theta$ can be strictly separated from the set $V(N) \setminus \{a^\theta\}$.*[8] *Then there is market that represents the NTU bargaining game (N, V) and the set of competitive payoff vectors of this market coincides with the set of θ-asymmetric Nash bargaining solutions with vectors of weights $\theta \in \Theta$, $\{a^\theta | \theta \in \Theta\}$, of the NTU bargaining game (N, V).*

Proof. (N, V) is a market game by Proposition 3.3. Moreover, Proposition 3.1 (or Proposition 3.2 respectively) shows, that the θ-asymmetric Nash bargaining solution with a vector of weights $\theta \in \Delta^n_{++}$ is an element of the inner core. Furthermore, note that the set of vectors of weights Θ is assumed to be closed. If we take now as a parameter the vectors of bargaining weights θ and consider the function that associates to every vector of weights θ the θ-asymmetric Nash bargaining solution a^θ, we observe that this function is

[8] More details concerning this assumptions and how they might be weakened can be found in Brangewitz and Gamp (2011a).

continuous in θ.[9] Moreover, as continuous functions map compact sets into compact sets, we know that if we take a closed set of vectors of weights Θ that the set of θ-asymmetric Nash bargaining solutions $\{a^\theta | \theta \in \Theta\}$ is closed. Therefore, the assumptions in Brangewitz and Gamp (2011a) are satisfied and their result can be applied. □

Proposition 3.5 can be regarded as the other extreme case in contrast to the result in Proposition 3.4. Knowing that competitive payoff vectors are under weak assumptions always in the inner core (compare de Clippel and Minelli (2005), Brangewitz and Gamp (2011a)), in the class of markets representing a game the market behind Proposition 3.5 is the market with the largest set of possible competitive payoff vectors.

Proposition 3.6 has the following interpretation: If the vector of weights or interpreted differently the bargaining power is not exactly known, then as an approximation using Proposition 3.6 we obtain the coincidence of the set of asymmetric Nash bargaining solutions with a closed subset of weight vectors and the set of competitive payoff vectors of a certain market.

3.5 Concluding Remarks

The results above show that asymmetric Nash bargaining solutions as solution concepts for bargaining games are linked via the inner core to competitive payoff vectors of certain markets. Thus, our result can be seen as a market foundation of the asymmetric Nash bargaining solutions. This result holds for bargaining games in general as any asymmetric Nash bargaining solution is always in the inner core (Proposition 3.2). The idea of a market foundation parallels the one that is used in implementation theory. Here, rather than giving a non-cooperative foundation for solutions of cooperative games, we provide a market foundation. Our result may be seen as an existence result.

Another interesting related line of research, that we do not follow here, is to consider the recent definition of Compte and Jehiel (2010) of the coali-

[9] To see this we use Theorem 2.4 of Fiacco and Ishizuka (1990) applied to maximization problems.

tional Nash bargaining solution. They consider cooperative games with transferable utility (TU) and define the coalitional Nash bargaining solution as the point in the core that maximizes the Nash product (with equal weights). Thus, using Theorem 2 of Shapley and Shubik (1975) for TU market games, where they define for any given core point a market that has this point as its unique competitive payoff vector, gives a market foundation as well for the symmetric coalitional Nash bargaining solution by choosing the symmetric coalitional Nash bargaining solution as this given core point. It seems interesting to study how this idea can be generalized for asymmetric coalitional Nash bargaining solutions or for (asymmetric) coalitional Nash bargaining solutions for NTU games.

Our approach parallels the one in Trockel (1996, 2005). Trockel (1996) is based on a direct interpretation of a n-person bargaining game as an Arrow-Debreu economy with production and private ownership, a so called bargaining economy. He shows that, given a bargaining economy, the consumption vector of the unique stable Walrasian equilibrium coincides with the asymmetric Nash bargaining solution with the vector of weights corresponding to the shares in the production of the bargaining economy. The main difference between our result and his is that Trockel (1996) did not consider markets in the sense of Billera and Bixby (1974) or Qin (1993) and thus his bargaining economies do not constitute the kind of market representation as defined in Billera and Bixby (1974) or Qin (1993). Similarly Trockel (2005) uses coalition production economies to establish a core equivalence of the Nash bargaining solution. By using the markets of Qin (1993) we obtained a market foundation of the asymmetric Nash bargaining solution. This can be seen as a link between the literature on market games (as in Billera and Bixby (1974), Qin (1993)) and the ideas of Trockel (1996, 2005).

3.6 Appendix

3.6.1 The Market behind Proposition 3.4 from Qin (1993)

Qin (1993) considers NTU games in general and does not restrict his attention to NTU bargaining games. The market behind Proposition 3.4 from Qin (1993) has a simpler structure if we restrict our attention to NTU bargaining games. The difference lies in the description of the private production sets.

To show his result Qin (1993) modifies the given NTU game by applying a strictly monotonic transformation to the utility functions. This allows him to assume that the given inner core point can be strictly separated in the modified NTU game. Qin (1993) shows that this market represents the modified game and that the given inner core point is the unique competitive payoff vector of this economy. By applying the inverse strictly monotonic transformation to the utility functions he obtains his result. As we do not want to restrict our attention to bargaining games with strictly convex generating sets, a similar transformation need to be applied to the NTU bargaining game to use the market defined below.

The transformed bargaining game is denoted by $(N, \bar{V})$ with generating set $\bar{C}^N$. Define for the grand coalition N the following sets

$$\begin{aligned} A_N^1 &= \left\{ \left(u^N, -e^N, -e^N, -e^N, 0\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\ A_N^2 &= \left\{ \left(u^N, 0, -e^N, 0, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\ A_N^3 &= \left\{ \left(u^N, 0, 0, -e^N, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \end{aligned}$$

and for the remaining coalitions

$$\begin{aligned} A_S^1 &= \left\{ \left(0, -e^S, -e^S, -e^S, 0\right) \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^2 &= \left\{ \left(0, 0, -e^S, 0, -e^S\right) \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^3 &= \left\{ \left(0, 0, 0, -e^S, -e^S\right) \right\} \subseteq \mathbb{R}^{5n}, \end{aligned}$$

Let $\theta \in \Theta$ be a given vector of weights and a^θ the θ-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Let $\mathcal{E}_{\bar{V},\theta} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ be the market with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{(0,0,0)\} \times \mathbb{R}_+^n \subseteq \mathbb{R}_+^{5n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S \subseteq N} \left(A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}\right) \in \mathbb{R}_+^{5n}$,
- the utility function $u^i(x^i) = \min \left\{ x_i^{(1)i}, \frac{\sum_{j=1}^n \lambda_j^\theta a_j^\theta x_j^{(5)i}}{\lambda_i} \right\}$
  where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(5)i}$ and $x_j^{(5)i}$.

Qin (1993) shows that the market $\mathcal{E}_{\bar{V},\theta}$ represents the modified NTU bargaining game $(N, \bar{V})$ and has as its unique competitive payoff vector a^θ, a given inner core point. For the method of proof and the details we refer to Qin (1993). Here, we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_{\bar{V},\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, 0, 0, e^{\{i\}} \right) \right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n} \left(a^\theta, -e^N, -e^N, -e^N, 0\right) \right) \right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \lambda^\theta \circ a^\theta \right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute the unique competitive equilibrium in the market $\mathcal{E}_{\bar{V},\theta}$.

3.6.2 The Market behind Proposition 3.5 from Qin (1993)

Similarly to Proposition 3.4 the market behind Proposition 3.5 from Qin (1993), called the induced market of an NTU market game, simplifies for NTU bargaining games to:

Definition 3.11 (induced market)**.** Let (N, V) be NTU bargaining game. The *induced market* of the game (N, V) is defined by $\mathcal{E}_V = (X^i, Y^i, u^i, \omega^i)_{i \in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{2n}$,

- the production set

$$Y^i = convexcone \left[\bigcup_{S \in \mathcal{N} \setminus N} \{(0, -e^S)\} \cup \left(C^N \times \{-e^N\} \right) \right] \subseteq \mathbb{R}^{2n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}\right)$,

- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i(x^i) = x_i^{(1)i}$
  where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate.

Qin (1993) shows that the market $\mathcal{E}_V$ represents the NTU bargaining game (N, V) and has as its set of competitive payoff vectors the whole inner core. For the method of proof and the details we refer to Qin (1993). Here, we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_V$ look like:

Let $\theta \in \Theta$ be a given vector of weights and a^θ the θ-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0 \right) \right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_V$.

3.6.3 The Market behind Proposition 3.6 from Brangewitz and Gamp (2011a)

Similarly to Proposition 3.4 and Proposition 3.5 the market behind Proposition 3.6 from Brangewitz and Gamp (2011a), called the induced A-market of an NTU market game, can be simplified for NTU bargaining games (under the assumptions of Proposition 3.6). For $\theta \in \Theta$ define

$$\lambda^{\theta} = \left(\frac{\frac{\theta_1}{a_1^{\theta}}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^{\theta}}}, \ldots, \frac{\frac{\theta_n}{a_n^{\theta}}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^{\theta}}} \right).$$

Let $(N, \tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap\limits_{\theta \in \Theta} \left\{ u \in \mathbb{R}^n | \lambda^{\theta} \cdot u \leq \lambda^{\theta} \cdot a^{\theta} \right\} & \text{if } S = N \end{cases}$$

where a^{θ} denotes the θ-asymmetric Nash bargaining solution.

Define the mapping $P_{\Theta} : \tilde{V}(N) \longrightarrow V(N)$ via

$$P_{\Theta}(x) = x - \bar{t}^{x} e^{N}.$$

Define

$$\tilde{C}^{N} = \left\{ z \in \tilde{V}(N) \middle| \exists t \in \mathbb{R}_{+} \text{ such that } z - te^{N} \in C^{N} \right\}.$$

Then we also have $\tilde{C}^{N} = \left\{ z \in \tilde{V}(N) \middle| z - \bar{t}^{z} e^{N} \in C^{N} \right\}$.

For the definition of the production sets define for all coalitions $S \in \mathcal{N} \setminus \{N\}$

$$\begin{aligned} A_S^1 &= \left\{ \left(0, -e^S, 0, -e^S, -e^S\right) \right\}, \\ A_S^2 &= \left\{ \left(0, 0, 0, -e^S, 0\right) \right\}, \\ A_S^3 &= \left\{ \left(0, 0, 0, 0, -e^S\right) \right\} \end{aligned}$$

and for the grand coalition N define

$$A_N^1 = \left\{ \left(P_\Theta \left(\tilde{c}^N \right), -e^N, \tilde{c}^N, -e^N, -e^N \right) | \tilde{c}^N \in \tilde{C}^N \right\},$$
$$A_N^2 = \left\{ \left(P_\Theta \left(\tilde{c}^N \right), 0, \tilde{c}^N, -e^N, 0 \right) | \tilde{c}^N \in \tilde{C}^N \right\},$$
$$A_N^3 = \left\{ \left(P_\Theta \left(\tilde{c}^N \right), 0, \tilde{c}^N, 0, -e^N \right) | \tilde{c}^N \in \tilde{C}^N \right\}.$$

The market $\mathcal{E}_{V,\Theta}$ using the closed set of weights Θ of the NTU bargaining game is defined by

$$\mathcal{E}_{V,\Theta} = (X^i, Y^i, u^i, \omega^i)_{i \in N}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{0\} \times \mathbb{R}_+^n \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S \in \mathcal{N}} \left(A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}} \right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with

$$u^i \left(x^i \right) = \min \left(x_i^{(1)i}, x_i^{(3)i} + \varepsilon \sum_{j \neq i} x_j^{(3)i} \right)$$

  where ε is chosen such that $\varepsilon < \lambda_i^\theta = \frac{\lambda_i^\theta}{1} \leq \frac{\lambda_i^\theta}{\lambda_j^\theta}$ for all $\theta \in \Theta$ and $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(3)i}$ and $x_j^{(3)i}$.

Using Brangewitz and Gamp (2011a) it can be shown that the market $\mathcal{E}_{V,\Theta}$ represents the NTU bargaining game (N, V) and its set of competitive equilibrium payoff vectors coincides with the set $\{a^\theta | \theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011a).

The competitive equilibria of the market $\mathcal{E}_{V,\Theta}$ are of the following form: Let $\theta \in \Theta$ be the vector of weights and a^θ the θ-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i \right)_{i \in N} = \left(\left(\left(a^\theta \right)^{\{i\}}, 0, \left(a^\theta \right)^{\{i\}}, 0, 0 \right) \right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, a^\theta, -e^N, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right)\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\Theta}$.

In addition to the market $\mathcal{E}_{V,\Theta}$ Brangewitz and Gamp (2011a) define a market where the set of payoff vectors of competitive equilibria with a strictly positive equilibrium price vectors coincides with the set $\{a^\theta | \theta \in \Theta\}$. This market, denoted by $\mathcal{E}^0_{V,\Theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{4n}$,
- the production set

$$Y^i = convexcone\left[\left(\bigcup_{S\in\mathcal{N}\setminus\{N\}} \left\{\left(0, -e^S, 0, -e^S\right)\right\}\right) \cup \left(\bigcup_{\tilde{c}^N\in\tilde{C}^N} \left(P_\Theta\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right)\right)\right] \subseteq \mathbb{R}^{4n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(3)i}\right)$.

Similarly as for the market presented before, it can be shown using Brangewitz and Gamp (2011a) that the market $\mathcal{E}^0_{V,\Theta}$ represents the NTU bargaining game (N, V) and its set of competitive equilibrium payoff vectors with strictly positive prices coincides with the set $\{a^\theta | \theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011a).

Here, we only state how the competitive equilibria of the market $\mathcal{E}^0_{V,\theta}$ look like:
Let $\theta \in \Theta$ be the vector of weights and a^θ the θ-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, \left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta \circ a^\theta, \lambda^\theta, \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda^\theta_i a^\theta_i$, constitute a competitive equilibrium in the market $\mathcal{E}^0_{V,\Theta}$.

Part II

Strategic Market Games

Chapter 4

Finite Horizon Strategic Market Games with Collateral

4.1 Introduction

Default and collateral have been important in the subprime crisis in 2008. The crisis showed that the liquidity of the market may suddenly vanish. We study financial strategic market games that incorporate the market illiquidity. Geanakoplos (1990, page 11) argues

> "The main problem is that competitive equilibrium does not provide an explanation of the process of price formation. The single period condenses a sequence of exchanges over which information is revealed. It is probably worthwhile to consider an explicit model of price formation such as the Shapley-Shubik mechanism, in which agents act before they know the prices. Agents might then not be aware of all the spot prices before they decided on their bids and offers."

By choosing their actions the individuals in the economy may have an influence on the prices of goods and financial assets. The idea of strategic market games goes back to Shapley and Shubik (1977). They use a non-cooperative game to describe the price formation in an exchange economy. Every player is asked to place a bid and an offer for every commodity. Afterwards the price of the commodity is computed as the ratio of the total bid to the total offer of that commodity. Strategic market games enable to study the feedback effect of trading strategies in illiquid markets when individual trades may have an impact on prices. An overview about strategic market games and related contributions can be found in Giraud (2003).

With this contribution we extend the model of Giraud and Weyers (2004) by introducing the possibility of default. We change the structure of the financial market and introduce a collateral requirement for financial assets as it is for example done in Araujo et al. (2002). We show that we can induce a given allocation that clears the markets and satisfies the budget constraints by defining appropriate, almost full strategies. Furthermore, we look at the set of sequentially strictly individually rational allocations and study the existence of approximate subgame perfect Nash equilibria. It turns out that we obtain an analogue of a perfect folk theorem similarly to the one in Giraud and Weyers (2004). Hence, even with collateral requirements almost everything is possible as soon as people are sufficiently patient, since almost every feasible, affordable, sequentially strictly individually rational consumption stream can be obtained by means of some almost full approximate subgame perfect Nash equilibrium.

4.2 Literature Overview

In this paper we combine the literature from different areas. This includes in particular the literature on economies with incomplete financial markets, modeling default by using collaterals and strategic market games.

In this section we present a literature overview on these areas. We start with the literature on incomplete markets and collateral and afterwards give an overview on strategic market games.

Incomplete Financial Markets

Models with incomplete financial markets incorporate that there are not always enough financial assets to make current contracts for all future events. There are some future events (or dates) for which no contracts can be made contingent on those events. Radner (1972) studies the existence of equilibria with incomplete financial markets in an economy with finite horizon. He defines an equilibrium as "[...] a set of prices at the first date, a set of common price expectations for the future, and a consistent set of individual plans for consumers and producers such that, given the current prices and price expectations, each individual agent's plan is optimal for him, subject to an appropriate sequence of budget constraints." (Radner, 1972, page 289). To show his existence result Radner (1972) imposes a bound on forward transactions. Similarly to Radner (1972), but without this bound, Hart (1975) addresses the question of optimality and of existence of equilibria in a three period pure exchange general equilibrium model with incomplete markets, finitely many consumers, goods and securities for contingent future commodities. He shows by simple examples that equilibria are not generally Pareto optimal if the market structure is incomplete and that opening new markets might make everybody worse off. Moreover, he establishes that in the presence of incomplete markets and under the standard convexity and continuity assumptions an equilibrium might fail to exist, without the additional assumption Radner (1972) made, and moreover he gives conditions for the existence. Werner (1985) establishes, making some assumptions on the preferences and the initial endowments, the existence of an equilibrium in a model with purely financial, incomplete future markets. Another, generic, existence result, where securities are claims to future commodity bundles, can be found in Duffie and Shafer (1985, 1986). Geanakoplos (1990) and Magill and Schafer (1991) give an overview of general equilibrium models with incomplete asset markets. Magill and Quinzii (1994) study the existence of equilibria in an infinite horizon version of this model imposing debt constraints or transversality conditions to avoid Ponzi schemes.

Modeling Default

There are different approaches in the literature how default can be modeled: **Collateral** or **Default Penalties**.

The first possibility is to introduce for every financial asset a collateral requirement. This means each time an individual sells a financial asset a specified amount of certain goods, called collateral, is needed to ensure that this individual keeps its promise. If the financial asset does not deliver the promised amount, then there still remains the collateral for the buyer of the asset, assuming implicitly some kind of durability of the collateral. Depending on the future prices the asset defaults or not. There is no further punishment if the promise is not fulfilled. Geanakoplos (2003a), Geanakoplos (2003b) and Geanakoplos and Zame (2007) study a two-period general equilibrium model with durable goods and collateralized assets viewing individuals as price-takers. They analyze the impact of the presence of the collateral on prices, allocations and efficiency of markets. Araujo et al. (2002) study infinite horizon economies with incomplete markets and collateral structure. They assume finitely many agents, commodities and assets. Their main result is the existence of an equilibrium in such a model without a further assumption on debt constraints or transversality conditions to avoid the possibility of Ponzi schemes. The collateral structure implies the presence of a uniform borrowing constraint in equilibrium (Araujo et al., 2002, Remark 2).

A second possibility to model default is allowing the individuals to default whenever they want to and introducing at the same time a default penalty if they do so. This concept was for example considered by Shubik and Wilson (1977), Zame (1993) or more recently by Dubey et al. (2005). An early contribution containing already default penalties is Shubik and Wilson (1977). They look at a strategic market game with an outside bank to borrow money. In case of default an individual suffers a default penalty. Zame (1993) argues that default may play an important positive role in an economy with incomplete markets. He shows that introducing default penalties and hence allowing the individuals strategically to default, may enhance the efficiency compared to the no default scenario. Dubey et al. (2005) study a general equilibrium model with incomplete markets in which they model default using default penalties. They consider assets as pools and introduce expected delivery rates for the assets. One of their results shows the existence of an equilibrium (according to their specified notion).

An interesting contribution studying the relationship between economies with collateral and economies with default penalties is Maldonado and Orrillo (2007). They compare the equilibria in a two period economy for these two classical ways modeling default. Their main result is that starting from an equilibrium for an economy with collateral requirements they show this equilibrium is as well an equilibrium with default penalties by defining an appropriate economy. For the converse to be true they need some additional assumptions on the payoff functions and the initial endowments of the agents.

Strategic Market Games

The idea of strategic market games goes back to Shapley and Shubik (1977):

> "A general model of non-cooperative trading equilibrium is described in which prices depend in a natural way on the buying and selling decisions of the traders, avoiding the classical as-

> sumption that individuals must regard prices as fixed. The key to the approach is the use of a single, specified commodity as "cash," which may or may not have intrinsic value."

Shapley and Shubik (1977) use a non-cooperative game to describe the price formation in an exchange economy. Every player is asked to place a bid and an offer for every commodity. Afterwards the price of the commodity is computed as the ratio of the total bid to the total offer of that commodity. Shapley and Shubik (1977) make the assumption that the individuals offer all their initial endowments. This assumption was relaxed for example by Shapley (1976) or Peck et al. (1992), who study the dimensionality of Nash equilibrium allocations. Concerning the "money" Shapley and Shubik (1977) take an explicit numeraire commodity, called money. The feasible bids in this case are constrained by the endowments in money. An other possibility is to take inside or fiat money and to introduce a budget constraint that depends on the prices and hence on the strategies of the other players as in Postlewaite and Schmeidler (1978). An overview on the existing literature concerning strategic market games can be found in Giraud (2003). As already mentioned the model here relies on Giraud and Weyers (2004) and builds on the investigation of subgame-perfect equilibria of a strategic market game with a finite horizon. Compared to Giraud and Weyers (2004) we change the structure of a no default financial market to a structure with default by introducing collateral requirements.

4.3 Finite Horizon Strategic Market Games with Collateral

We look at a strategic market game with finitely many discrete time periods, $t = 1, ..., T$. We take a tree-like time structure and assume that the individuals play at every point in time a strategic market game in the economy, taking into consideration the trades on the financial markets from the previous periods. Hence, the game we consider is not a repeated game in the usual sense, as part of the available budget in the actual period is a result of the actions chosen in the previous periods. Nevertheless, the mechanism of price formation remains the same. By using a strategic market game we drop the assumption of price-taking behavior from the usual competitive framework. The individuals have the possibility to influence the prices in the economy by choosing their actions strategically.

4.3.1 The Economy

The Time Structure

The set of finitely many time periods is denoted by $\mathcal{T} = \{1, ..., T\}$. Let $\mathcal{S} = \{1, ..., S\}$ be the set of possible states of nature. We assume that, when the game starts, no information about the state of nature is available and when the game ends the uncertainty is resolved. This is modeled by an increasing family of information partitions (in the sense as defined below), denoted by $(\mathcal{F}_t)_{t\in\mathcal{T}}$, with the property $\mathcal{F}_1 = \{\mathcal{S}\}$ and $\mathcal{F}_T = \{\{1\}, ..., \{S\}\}$. For each date $t \in \mathcal{T}$ and each $\varsigma \in \mathcal{F}_t$, we call the pair $\xi = (t, \varsigma)$ a date-event or a node. Let $\mathbf{D} = \bigcup_{t\in\mathcal{T}} (\{t\} \times \mathcal{F}_t)$ be the finite set of all nodes. Define a partial order $\geq$ $(>)$ on $\mathbf{D}$ by $\xi = (t, \varsigma) \geq (>)\xi' = (t', \varsigma')$ if and only if $t \geq (>)t'$ and $\varsigma \subseteq \varsigma'$. The pair $(\mathbf{D}, \geq)$ is called a tree and its root is $\xi_0 = (1, \mathcal{S})$. The terminal nodes are elements $(T, \varsigma) \in \{T\} \times \mathcal{F}_T$ and the set of terminal nodes is denoted by $\mathbf{D}_T$. Let $\mathbf{D}^- = \mathbf{D} \setminus \mathbf{D}_T$ be the set of non-terminal nodes. Each non-terminal node $\xi = (t, \varsigma) \in \mathbf{D}^-$ has a finite set of immediate successors

$$\xi^+ = \{\xi' \in \mathbf{D} | \xi' = (t + 1, \varsigma'), \varsigma' \subseteq \varsigma\}.$$

Moreover every node $\xi = (t, \varsigma)$, except the root, has a unique predecessor $\xi^- = (t - 1, \varsigma')$ with $\varsigma \subseteq \varsigma'$. For any node $\xi \in \mathbf{D}$ the set of all nodes with $\xi' \geq (>)\xi$ is denoted by $\mathbf{D}(\xi)$ $(\mathbf{D}(\xi)^+)$ and is itself a tree with root ξ. Let $d = |\mathbf{D}|$ denote the number of nodes in the tree and let $\tau(\xi)$ be the time at which node ξ is reached, i.e. $\tau : \mathbf{D} \to \mathcal{T}$ such that $\xi = (t, \varsigma) \mapsto t$.

Consumption Goods and Financial Assets

We consider a pure exchange economy with finitely many individuals and L consumption goods combined with a financial market where the players are

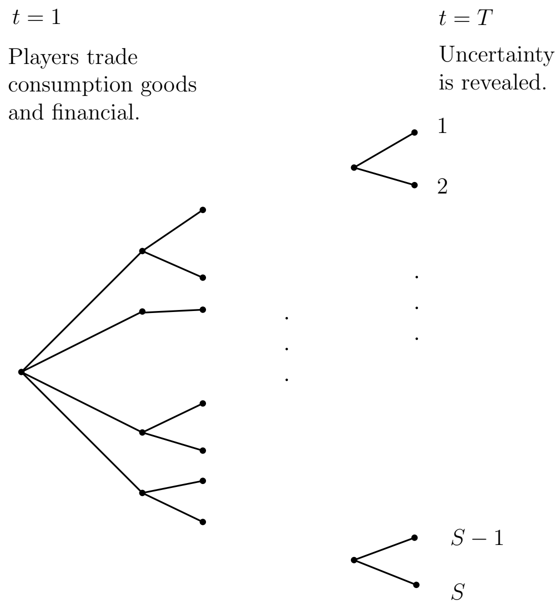

Figure 4.1: The T-period strategic market game.

allowed to purchase or sell financial assets. There is a finite set of players, denoted $\mathcal{N} = \{1, ..., N\}$, and a finite set of consumption goods, denoted by $\mathcal{L} = \{1, ..., L\}$. Consumption goods need to be consumed or can be used as a collateral on the financial markets. There is no possibility to store a consumption good for the next period except for using it as a collateral.

We assume that there are J possible short-term assets. The set of assets is given by $\mathcal{J} = \{1, ..., J\}$. The available financial assets are supposed to be exogenously given. A financial asset $j \in \mathcal{J}$ is characterized by a tuple (ξ^j, A_j, C_j) consisting of three elements: an issuing node, a promised amount of goods and collateral requirement. The issuing node is a node in the tree $\mathbf{D}$ and for short denoted by ξ^j. The promised amount of goods is described by a function $A_j : \mathbf{D} \to \mathbb{R}^L_+$ such that $A_j(\xi) = 0$ for all $\xi \in \mathbf{D} \setminus (\xi^j)^+$. For $\xi' \in (\xi^j)^+$ the promises $A_j(\xi')$ are the amounts of goods that a seller of asset j promises to deliver to a buyer of asset j in the next period after the issuing node ξ^j. The delivery, $p_\xi \cdot A_j(\xi)$, is assumed to be made in fiat money using spot prices, p_ξ. We only consider short-term assets. Therefore, for other nodes before the issuing node and at least two periods after the asset was issued, we the promised amount is zero. The collateral requirements

$C_j \in \mathbb{R}^L_+$ are to ensure that the promised amounts of goods A_j is delivered. The consumption goods serve as a collateral. We assume that the individuals are not allowed to consume the collateral. The collateral is stored in a warehouse for one period and is required at the issuing node ξ^j of asset j. For $\xi' \in (\xi^j)^+$ the collateral, stored in a warehouse, underlies a depreciation $Y^s_{\xi'}$. Tobacco is one example of a consumption good that possess the required properties to serve as a collateral. We assume that in the last period T no assets are traded.

The Players

Every player $i \in \mathcal{N}$ is characterized by a family of twice continuously differentiable, strictly increasing and concave utility functions. For each date $t \in \mathcal{T}$ and each player $i \in \mathcal{N}$ there is a time-independent utility function $u^i_t : \mathbb{R}^L_+ \to \mathbb{R}$. Moreover every player $i \in \mathcal{N}$ possesses a strictly positive initial endowment in consumption goods $w^i(\xi) \in \mathbb{R}^L_{++}$ at every node $\xi \in \mathbf{D}$. Player i is supposed to maximize the discounted sum of expected utilities. We assume that every player i has a discount factor $\lambda^i \in [0, 1]$. Thus, his utility function is of the form

$$u^i(x^i) = (1 - \lambda^i) \sum_{t=1}^{T} (\lambda^i)^{t-1} E[u^i_t(x^i_t)]$$

where the expected values at all dates $t \in \mathcal{T}$ are taken using given probability distributions on the respective succeeding nodes,

$$E[u^i_t(x^i_t)] = \sum_{\xi:\tau(\xi)=t} p_t(\xi) u^i_t(x^i_t(\xi))$$

with $\sum_{\xi:\tau(\xi)=t} p_t(\xi) = 1$ and $p_t(\xi) > 0$ for all ξ with $\tau(\xi) = t$.

4.3.2 The T-Period Strategic Market Game

In the strategic market game, based on the idea of Shapley and Shubik (1977), each player places for every consumption good $\ell \in \mathcal{L}$ at every node $\xi \in \mathbf{D}$ a bid $b^i_\ell(\xi)$ and an offer $q^i_\ell(\xi)$. The bid $b^i_\ell(\xi)$ signals how much (in terms of fiat money) player i is willing to pay for the purchase of good ℓ and the offer $q^i_\ell(\xi)$ (in terms of physical commodities) is the amount he wants to sell. The price of good ℓ is then computed as the ratio of the total bid to the total offer, that is

$$p_\ell(\xi) = \begin{cases} \frac{\sum_{i=1}^N b^i_\ell(\xi)}{\sum_{i=1}^N q^i_\ell(\xi)} & \text{if } \sum_{i=1}^N q^i_\ell(\xi) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence, if there are no offers on the market the price is set equal to 0. Likewise if there are no bids, the price is as well equal to 0. A market without trade is called closed.[1]

Similarly to trade assets on financial markets at every non-terminal node $\xi \in \mathbf{D}^-$ each player places a bid $\beta^i_j(\xi)$ containing the amount of money he wants to spend for buying the real asset $j \in \mathcal{J}$ and an offer $\gamma^i_j(\xi)$ containing the amount of assets he wants to sell. The price of the real asset is given by

$$\pi_j(\xi) = \begin{cases} \frac{\sum_{i=1}^N \beta^i_j(\xi)}{\sum_{i=1}^N \gamma^i_j(\xi)} & \text{if } \sum_{i=1}^N \gamma^i_j(\xi) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, price vectors are given by $\left((p_\ell(\xi))_{\ell\in\mathcal{L}}, (\pi_j(\xi))_{j\in\mathcal{J}}\right)$.

When the promises are settled a seller of the asset $j \in \mathcal{J}$ compares the value of the promise with the value of the collateral and pays back the minimal value, either $p(\xi')Y_{\xi'}C_j$ or $p(\xi')A_j(\xi')$, at the node $\xi' \in (\xi^j)^+$. Hence, if default appears or not depends on the value of the promise compared to the value of the collateral. This again depends on the price $p_\ell(\xi')$ which is determined by the bids and offers at the node ξ' on the market for consumption goods.

Define for every asset $j \in \mathcal{J}$ at the nodes $\xi' \in (\xi^j)^+$

$$D_j(\xi') = \min\left\{p(\xi')A_j(\xi'), p(\xi')Y_{\xi'}C_j\right\}.$$

[1]Defining the price as zero when there are no offers on the market we follow here for example Amir et al. (1990, p.128). Similar assumptions can be found in Postlewaite and Schmeidler (1978, p.128), Peck et al. (1992, p.275) or Giraud and Weyers (2004, p.474).

Feasible Bids and Offers

There are some physical and budgetary restrictions on the bids and offers the individuals can choose. In this section we define the feasible bids and offers at every node $\xi \in \mathbf{D}$. We take into consideration that for every asset $j \in \mathcal{J}$ player i sells, he has to have the required amount of collateral. The amount of collateral needed depends on the asset sales and not on the net trades. Suppose we are at node ξ and asset $j \in \mathcal{J}$ is issued at this node. Assume player i offers to sell $\gamma^i_j(\xi)$ units of asset j, which means that he promises to deliver $\gamma^i_j(\xi)A_j(\xi')$ at the nodes $\xi' \in (\xi^j)^+$. In this case he needs to have an amount of $\gamma^i_j(\xi)C_j \in \mathbb{R}^L_+$ goods as a collateral. In oder to fulfill the obligations from the asset markets player i is allowed to offer his collateral from the previous period on the goods market. The money needed to pay his obligations is included in the individual budget constraint on feasible bids and offers.

The feasible bids and offers are given by:

$$q^i_\ell(\xi) + \sum_{j=1}^{J} \gamma^i_j(\xi) C_{j\ell} \leq w^i_\ell(\xi) + \sum_{j=1}^{J} \gamma^i_j(\xi^-) \left(Y_\xi C_j\right)_\ell \qquad (F1\xi)$$

for all $\ell \in \mathcal{L}$,

$$q^i_\ell(\xi), b^i_\ell(\xi), \beta^i_j(\xi), \gamma^i_j(\xi) \geq 0 \qquad (F2\xi)$$

for all $\ell \in \mathcal{L}$, $j \in \mathcal{J}$ and

$$\beta^i_j(\xi) = \gamma^i_j(\xi) = 0 \text{ if } \xi \neq \xi(j) \qquad (F3\xi)$$

for all $j \in \mathcal{J}$. We assume that the initial holdings of assets are equal to 0.

The offered amount of goods plus the amount of goods that is needed as a collateral cannot exceed the initial endowment of player i at node $\xi \in \mathbf{D}$ plus the depreciated collateral, he put aside in the previous period to fulfill his promises, condition $(F1\xi)$. Furthermore, the bids and offers cannot be negative, condition $(F2\xi)$, and assets can only be traded at their issuing node, condition $(F3\xi)$, as the assets are assumed to be short-term assets.

Remark. Condition $(F3\xi)$ is needed in the proof of our main theorem. In order to give the agents incentives to play certain strategies there is a final reward phase with closed asset markets. Condition $(F3\xi)$ ensures that financial assets can only be traded at their issuing date. Therefore, the individuals are not allowed to trade financial assets in the following way: Suppose all individuals play a strategy bidding and offering zero for asset j. Then, say individual i could unilaterally deviate and bid and offer a strictly positive amount for asset j and trade in this way with "himself" as there is no other player trading asset j. This action has the consequence that the collateral need to be put up and is be stored until next period. Therefore, even if the all agents play a Nash equilibrium on the goods markets and no trade on the asset markets, there might be a profitable deviation using this kind of "self-trading" actions. These strategies are excluded by condition $(F3\xi)$ together with an assumption on the issuing nodes of an exogenously given financial structure. A different condition avoiding "self-trade" actions is made in Brangewitz and Giraud (2011) where an appropriate condition is directly added to the feasibility constraints. We can adopt this way here as well.

Moreover, player i faces the following budget constraints on fiat money when placing bids and offers:

$$\sum_{\ell=1}^{L} b_\ell^i(\zeta) + \sum_{j=1}^{J} \beta_j^i(\zeta) \leq \sum_{\ell=1}^{L} p_\ell(\zeta) q_\ell^i(\zeta) + \sum_{j=1}^{J} \pi_j(\zeta)\gamma_j^i(\zeta) + \sum_{j=1}^{J} \left(\frac{\beta_j^i(\zeta^-)}{\pi_j(\zeta^-)} - \gamma_j^i(\zeta^-) \right) D_j(\zeta) \quad (*_\xi^i)$$

for all $\zeta \leq \xi$. Thus, by condition $(*_\xi^i)$ the total value of bids for consumption goods and promises cannot exceed the total monetary amount of consumption goods and of promises that is offered including the dividends. This needs to hold for the actual node and all nodes before. If one of the budget constraints in $(*_\xi^i)$ is violated, then individual i is removed from the economy for all nodes $\mathbf{D}^+(\xi)$ and all his goods are confiscated for those nodes.

After trading took place player i's allocation of good $\ell \in \mathcal{L}$ available for consumption at the end of the current period, time t, is

$$x_\ell^i(\xi) = \begin{cases} w_\ell^i(\xi) + \sum_{j=1}^{J} \gamma_j^i(\xi^-)\left(Y_\xi C_j\right)_\ell - q_\ell^i(\xi) + \frac{b_\ell^i(\xi)}{p_\ell(\xi)} - \sum_{j=1}^{J} \gamma_j^i(\xi) C_{j\ell} & \text{if } (*_\xi^i) \text{ holds and } p_\ell(\xi) > 0 \\ w_\ell^i(\xi) + \sum_{j=1}^{J} \gamma_j^i(\xi^-)\left(Y_\xi C_j\right)_\ell - q_\ell^i(\xi) - \sum_{j=1}^{J} \gamma_j^i(\xi) C_{j\ell} & \text{if } (*_\xi^i) \text{ holds and } p_\ell(\xi) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, his holdings of asset $j \in \mathcal{J}$ are given by his sales

$$\varphi_j^i(\xi) = \begin{cases} \gamma_j^i(\xi) & \text{if } (*_\xi^i) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

and his purchases

$$\theta_j^i(\xi) = \begin{cases} \frac{\beta_j^i(\xi)}{\pi_j(\xi)} & \text{if } (*_\xi^i) \text{ holds and } \pi_j(\xi) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if $\theta_j^i(\xi) - \varphi_j^i(\xi) < 0$ then player i sold more of the financial asset $j \in \mathcal{J}$ than he bought. Analogously for $\theta_j^i(\xi) - \varphi_j^i(\xi) > 0$ he is a net buyer.

Allowable Actions and Strategies

The *actions* of the players are the choices of bids and offers. The action set of player i at node ξ is defined as

$$A^i(\xi) = \left\{ \left(q_\ell^i(\xi), b_\ell^i(\xi)\right)_{\ell \in \mathcal{L}}, \left(\gamma_j^i(\xi), \beta_j^i(\xi)\right)_{j \in \mathcal{J}} \in \mathbb{R}_+^{2L} \times \mathbb{R}_+^{2J} \middle| (F1\xi),\ (F2\xi) \text{ and } (F3\xi) \text{ are satisfied} \right\}.$$

Let $A(\xi) = \times_{i=1}^{N} A^i(\xi)$. Note that the definition of the action sets includes actions that possibly violate the budget constraint $(*_\xi^i)$. As the prices for consumption goods and financial assets depend on the actions of all players, including the budget constraint $(*_\xi^i)$ as a further restriction into the definition of the action sets would make the action sets dependent on the other players' actions. And hence, we would obtain a generalized game as introduced by Debreu (1952).[2] To avoid this we define the feasible strategies using only the conditions $(F1\xi)$, $(F2\xi)$ and $(F3\xi)$ and assume as stated before the removal of individuals from the economy that violate the budget

[2]For more information on generalized Nash equilibrium problems see for example Debreu (1952), Harker (1991) or Facchinei and Kanzow (2010).

constraint $(*^i_\xi)$. The stage-payoff of player i at node ξ is given by the utility he obtains from consumption.

In a strategic market game the allocation an individual finally obtains depends on his own actions and the prices, which represent an aggregate of the own and the other individual's actions. Because of this it is enough for an individual to know the prices of the commodities and assets rather than the particular actions of the other individuals.

A *strategy* for the T-period strategic market game consists of choosing actions at every node $\xi \in \mathbf{D}$. Let $H^i(\xi)$ denote the set of possible histories for individual i at node ξ given by

$$H^i(\xi) = \left\{ p(\xi'), \pi(\xi'), \varphi^i(\xi'), \theta^i(\xi') \middle| \text{ for all } \xi' < \xi \right\}.$$

The history at the root ξ_0 is given by $H^i(\xi_0) = \emptyset$. Therefore, a strategy of individual i is given by $\sigma^i : \bigcup_{\xi \in \mathbf{D}} H^i(\xi) \to \left(\mathbb{R}^L_+\right)^2 \times \left(\mathbb{R}^J_+\right)^2$ such that

$$\sigma^i(h) \in A^i(\xi)$$

for all $\xi \in \mathbf{D}$ and for all $h \in H^i(\xi)$.

Definition 4.1 (full, almost full strategy profiles)**.** A strategy profile for the strategic market game with N player and finite horizon T

$$\left(\left(q^i_\ell(\xi), b^i_\ell(\xi)\right)_{\ell \in \mathcal{L}}, \left(\gamma^i_j(\xi), \beta^i_j(\xi)\right)_{j \in \mathcal{J}} \right)_{i \in \mathcal{N}, \xi \in \mathbf{D}}$$

is called *full* if

$$\sum_{i=1}^N q^i_\ell(\xi) > 0, \quad \sum_{i=1}^N b^i_\ell(\xi) > 0, \quad \sum_{i=1}^N \gamma^i_j(\xi) > 0, \quad \sum_{i=1}^N \beta^i_j(\xi) > 0$$

hold for all $\ell \in \mathcal{L}$, $j \in \mathcal{J}$, $\xi \in \mathbf{D}$. A subgame perfect Nash equilibrium is called *full* if it is a full strategy profile. We define a strategy profile as *almost full* if the strict inequality holds on the goods markets. This means

$$\sum_{i=1}^N q^i_\ell(\xi) > 0, \quad \sum_{i=1}^N b^i_\ell(\xi) > 0$$

hold for all $\ell \in \mathcal{L}$, $\xi \in \mathbf{D}$. Analogously a subgame perfect Nash equilibrium is called *almost full* if it is an almost full strategy profile.

Next, we study the existence of approximate subgame perfect Nash equilibria of the T-period strategic market game. First we show that any given feasible and affordable allocation can be induced by some trading strategies that not equal to zero.

Definition 4.2 (feasible and affordable allocation)**.** An allocation $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}} \in \mathbb{R}_+^{L\cdot d\cdot N} \times \mathbb{R}_+^{J\cdot d\cdot N} \times \mathbb{R}_+^{J\cdot d\cdot N}$ is said to be *feasible and affordable*, if there exists a price system $(\bar{p}, \bar{\pi}) \in \mathbb{R}_+^{L\cdot d} \times \mathbb{R}_+^{J\cdot d}$ such that the following conditions are satisfied at every node $\xi \in \mathbf{D}$:

- *Individual budget restriction for every player $i \in \mathcal{N}$:*[3]
  $\sum_{\ell=1}^{L} \bar{p}_\ell(\xi) \left(\bar{x}_\ell^i(\xi) + \sum_{j=1}^{J} \bar{\varphi}_j^i(\xi) C_{j\ell}\right) + \sum_{j=1}^{J} \bar{\pi}_j(\xi) \left(\bar{\theta}_j^i(\xi) - \bar{\varphi}_j^i(\xi)\right)$
  $= \sum_{\ell=1}^{L} \bar{p}_\ell(\xi) \left(w_\ell^i(\xi) + \sum_{j=1}^{J} \bar{\varphi}_j^i(\xi^-) \left(Y_\xi C_j\right)_\ell\right) + \sum_{j=1}^{J} \left(\bar{\theta}_j^i(\xi^-) - \bar{\varphi}_j^i(\xi^-)\right) D_j(\xi),$
- *market clearing on spot markets for every good $\ell \in \mathcal{L}$:*
  $\sum_{i=1}^{N} \left(\bar{x}_\ell^i(\xi) + \sum_{j=1}^{J} \bar{\varphi}_j^i(\xi) C_{j\ell}\right) = \sum_{i=1}^{N} \left(w_\ell^i(\xi) + \sum_{j=1}^{J} \bar{\varphi}_j^i(\xi^-) \left(Y_\xi C_j\right)_\ell\right),$
- *and market clearing on financial markets for every asset $j \in \mathcal{J}$:*
  $\sum_{i=1}^{N} \bar{\theta}_j^i(\xi) = \sum_{i=1}^{N} \bar{\varphi}_j^i(\xi),$

$(\bar{x}^i)_{i\in\mathcal{N}}$ is called a *feasible and affordable consumption stream.*

The following remark is easy to verify.

Remark. Convexity of the set of feasible and affordable consumption streams: Let $\delta \in [0, 1]$. Let $(\bar{x}^i)_{i\in\mathcal{N}}$ and $(\bar{x}'^i)_{i\in\mathcal{N}}$ be two feasible and affordable consumption streams.

1. Then
$$\left(\delta\bar{x}^i + (1-\delta)\bar{x}'^i\right)_{i\in\mathcal{N}}$$
   is a feasible and affordable consumption stream as well.

[3] We define $\bar{\varphi}_j^i(\xi_0^-) = \bar{\theta}_j^i(\xi_0^-) = 0$.

2. Since the utility functions are assumed to be concave we have for every node $\xi = (t, \varsigma) \in \mathbf{D}^-$

$$\delta u_t^i(\bar{x}_t^i(\xi)) + (1-\delta) u_t^i(\bar{x}_t'^i(\xi)) \leq u_t^i(\delta \bar{x}_t^i(\xi) + (1-\delta)\bar{x}_t'^i(\xi))$$

and thus

$$\delta E[u_t^i(\bar{x}_t^i)] + (1-\delta) E[u_t^i(\bar{x}_t'^i)] \leq E[u_t^i(\delta \bar{x}_t^i + (1-\delta)\bar{x}_t'^i)].$$

Definition 4.3 (sequentially strictly individually rational)**.** An consumption stream $(\bar{x}^i)_{i\in\mathcal{N}} \in \mathbb{R}_+^{L\cdot d\cdot N}$ is said to be *sequentially strictly individually rational* up to time T^*, if

$$E[u_t^i(\bar{x}_t^i)] > E[u_t^i(w_t^i)]$$

for all $i \in \mathcal{N}$ and all $t \leq T^*$.

We obtain for our model an analogous Lemma as in Giraud and Weyers (2004).

Lemma 4.1. *If the initial allocations $(w^i(\xi))_i \gg 0$ are Pareto-inefficient in the L-good spot economy at each node $\xi \in \mathbf{D}$ then for every terminal date T there exists a sequentially strictly individually rational and feasible, affordable allocation.*

The proof can be found in the Appendix 4.5.1. Analogously to Lemma 2 in Giraud and Weyers (2004) we obtain:

Lemma 4.2. *Let $(\bar{x}^i)_{i\in\mathcal{N}} \in \mathbb{R}_+^{L\cdot d\cdot N}$ be a feasible, affordable and sequentially strictly individually rational consumption stream. Let $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ be the according feasible and affordable allocation with the price system $(\bar{p}, \bar{\pi})$. Then the following strategies result in $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$:*

For all $\xi \in \mathbf{D}$, $i \in \mathcal{N}$, $\ell \in \mathcal{L}$ and $j \in \mathcal{J}$ let

$$\begin{aligned}
q^i_\ell(\xi) &= w^i_\ell(\xi) + \sum_{j=1}^{J} \bar{\varphi}^i_j(\xi^-)\,(Y_\xi C_j)_\ell\,,\\
b^i_\ell(\xi) &= \bar{p}_\ell(\xi)\left(\bar{x}^i_\ell(\xi) + \sum_{j=1}^{J} \bar{\varphi}^i_j(\xi) C_{j\ell}\right),\\
\gamma^i_j(\xi) &= \bar{\varphi}^i_j(\xi),\\
\beta^i_j(\xi) &= \bar{\pi}_j(\xi)\bar{\theta}^i_j(\xi).
\end{aligned}$$

Moreover, the above strategies are almost full.

The proof can be found in the Appendix 4.5.2. The idea Lemma 4.2 is that every feasible, affordable and sequentially strictly individually rational consumption stream can be achieved through some trading strategies. The individuals offer all their initial endowments plus the collaterals needed in the previous period and bid exactly the value of the given feasible and affordable allocation plus the collateral requirements using the price system corresponding to this allocation. Similarly, on the asset markets the individuals offer their asset sales and bid the value of their purchases. Note that we do not make an statement if the strategies defined in Lemma 4.2 form a Nash equilibrium. Even if there is no trade on the asset markets, the one stage actions are not necessarily Nash equilibrium actions.

Remark. Note that we cannot always use almost full strategies to induce a given feasible, affordable and sequentially strictly individually rational consumption stream. If there are no transactions on the asset markets necessary to induce such a given allocation, then the strategies on the asset markets are no longer full. Forcing individuals to offer some (even very small) amount of financial assets in order to obtain full strategies will for them have the consequence that they need to possess the required amount of collateral and moreover this additional amount of collateral can no longer be consumed, such that the utility of the actual period decreases. This problem might be solved by imposing the collateral requirements only on the net trades. In this case buying and selling the same amount of the same financial asset does not require some additional collateral. Another point is that we assumed in ($F3\xi$) that assets can only be traded at their issuing node. We decided to stay with the collateral requirement on the asset sales and to have only almost full strategies to induce a given feasible, affordable and sequentially strictly individually rational consumption stream. The question of netting asset sales and purchases is for example for monetary equilibria discussed in Dubey and Geanakoplos (2003). In a model with netting the asset sales are not constrained by the available amount of money and hence very large short sales are possible. In our model we explicitly constrain the asset sales by the collateral requirements.

If we consider an economy without trade on the financial markets or without collateral requirements, the lower bound for the utility a player can obtain in the whole T-period strategic market game is the utility from his initial endowment. If we introduce collateral requirements and if there is trade in financial markets, this situation might change. The following lemma shows that this is not the case if we consider the whole T-period strategic market game.

Lemma 4.3. *There is no Nash equilibrium and hence no subgame perfect Nash equilibrium of the T-period strategic market game at which at least one player has strictly less utility from this allocation than from his initial endowment.*

The idea of the proof is to construct a strategy that insures a player at least the utility from his initial endowment. This utility can always be obtained using the no-trade equilibrium strategy by bidding zero for every good and every asset. It turns out even with forcing <u>all</u> the players to place strictly positive bids and offers on the goods markets, we can define appropriate strategies. The proof can be found in the Appendix 4.5.3.

Remark. Due to the fact that the individuals consider expected utilities the observation in Lemma 4.3 does not exclude that at single nodes the utility obtained is less than the one from the according initial endowment. In Lemma 4.3 we only consider the total utility from the T-period strategic market game. It might happen that very low consumption in one state of nature is compensated by high consumption in an other state, such that in expectation the individual is better off than consuming his initial endowment. We discuss this issue again at the end of this section when we make the comparison with the no default setting.

Definition 4.4 (approximate subgame perfect Nash equilibrium)**.** A consumption stream $(\bar{x}^i)_{i\in\mathcal{N}}$ is an *approximate subgame perfect Nash equilibrium* of the strategic market game (as defined above) with N players and time horizon T if for every $\varepsilon^* > 0$ there is a discount factor λ^0 such that if $\min_{i\in\mathcal{N}} \lambda^i \geq \lambda^0$, then there is an almost full subgame perfect Nash equilibrium with strategies σ such that

$$|u^i(x^i(\sigma)) - u^i(\bar{x}^i)| < \varepsilon^*$$

for all $i \in \mathcal{N}$.

We can parametrize the T-period strategic market games by its time horizon T, the discount rates $\lambda = (\lambda^i)_{i\in\mathcal{N}}$ and the initial endowments $w = (w^i)_{i\in\mathcal{N}}$. Denote the associated economies by $\mathcal{E}(T, \lambda, w)$. Every strategic market game with finite horizon and the associated economy $\mathcal{E}(T, \lambda, w)$ can be as well considered as a truncation of an infinite horizon game with the associated economy $\mathcal{E}^\infty(\lambda, w)$, that becomes stationary after time T. Let $\Omega(N)$ denote the set of all allocations that redistribute the initial endowments in the economy $\mathcal{E}^\infty(\lambda, w)$ somehow such that the market clearing condition at every node is satisfied and let $\Omega_T(N)$ be its restriction until period T. Thus,

$$\Omega_T(N) = \left\{ \bar{w} \in \mathbb{R}^{L\cdot d\cdot N} \middle| \sum_{i=1}^{N} \bar{w}^i = \sum_{i=1}^{N} w^i \right\}.$$

Theorem 4.1. *For any N, there exists an open and dense subset $\Omega^*(N)$ of initial endowments and an integer $T^0(N)$ and R such that for every finite horizon $T \geq T^0(N) \geq R$: if the initial endowments belong to $\Omega^*_T(N)$ and if the issuing nodes of all financial assets are in the first $T - R - 1$ periods, then every consumption stream $(\bar{x}^i)_{i\in\mathcal{N}}$, that is feasible, affordable and sequentially strictly individually rational in the first $T - T^0(N)$ periods, is an* *approximate subgame perfect Nash equilibrium of the strategic market game with finite horizon T.*

The proof of the above theorem is very similar to the one given in Giraud and Weyers (2004). The main idea is to apply the Lemma 4.4 below to obtain existence of equilibria in the one shot strategic market games (with closed financial markets) and to use the previous results to construct strategies that induce a given feasible and affordable allocation (Lemma 4.2).

For the proof of the above theorem we use the following lemma as stated in Giraud and Weyers (2004) [Lemma 3], that was proven by Peck et al. (1992).

Lemma 4.4. *For each node $\xi \in \mathbf{D}$ there exists an open and dense subset of initial allocations of the one-period economy at node ξ with closed asset markets, such that the set of Nash equilibrium allocations of the one shot strategic market game on the commodity markets with actions, where all bids and offers are strictly positive, is a smooth sub-manifold of dimension $L(N-1)$ of the set of allocations where markets clear exactly.*

The proof of Theorem 4.1 can be found in the Appendix 4.5.4.

Remark. As already indicated in Remark 4.3.2 condition $(F3\xi)$ together with the assumption that there are no financial assets issued at the last part of the T-period strategic market game, avoids “self trading” actions and excludes with that storage possibilities of the collaterals at this part of the game. Lemma 4.4 ensures the existence of one shot Nash equilibria (with closed asset markets) for appropriately chosen initial endowments. So far it remains an open question if this choice of the initial endowments is compatible with storage possibilities of the collateral.

4.3.3 Comparison with the no Default Situation

Giraud and Weyers (2004) impose a no default condition on the sequentially strictly individually rational allocations they consider. This condition implies that the value of the consumption goods plus the changes in the asset portfolio need to be financed by the initial endowments taking asset market obligations and dividends into consideration. Asset market obligations are always met, even if the initial endowments might be used to fulfill them. We argue that in this model without default there exists a sequentially strictly individually rational and default free allocation in which there exists a state where one consumer is driven out of the market. This means there exists a state where the consumption of one consumer comes arbitrarily close to zero. This is possible as the sequential strict individual rationality is defined by taking the expectation over future states according to a fixed probability distribution. Hence, using the folk theorem like result from Giraud and Weyers (2004) there exist approximate subgame perfect equilibria of the finite horizon strategic market game without default with the property that at least one consumer is driven out of the market.

The following example shows this claim:

Example. Suppose there are $\mathcal{N} = \{1, ..., N\}$ individuals and two future states of nature. Assume that there is one future state, denoted by ξ_u, that appears with a high probability of p close to one, and another one that appears with a very low probability of $1-p$, denoted by ξ_d. There are two possible states of nature in the second period and afterwards there is only one possible state of nature at each node. This is illustrated in Figure 4.2.

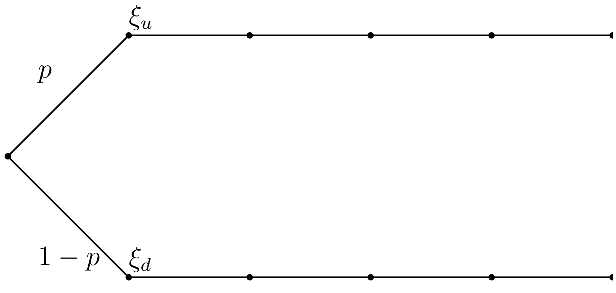

Figure 4.2: Example.

Moreover, assume that the time horizon T and the Pareto inefficient initial endowments, denoted by $(w^i)_{i\in\mathcal{N}}$, are chosen in such a way that the assumptions of Theorem 1 in Giraud and Weyers (2004) hold. Therefore, every sequentially strictly individually rational and default-free allocation is an approximate subgame perfect equilibrium. Let $(\bar{x}^i)_{i\in\mathcal{N}}$ be a sequentially strictly individually rational and default-free allocation. In the following we define a different sequential strictly individually rational and default-free allocation starting from $(\bar{x}^i)_{i\in\mathcal{N}}$ where one agent is driven out of the market. Assume that individual 1 sells a huge amount of a a financial asset such that he has to deliver only something in the unprobable state ξ_d. He sells this amount split equally to the other players. Doing this leaves individual 1 an amount of $\varepsilon > 0$ small in state ξ_d with $w_2^1(\xi_d) \gg \varepsilon$ after the asset market obligations are fulfilled and $\bar{x}_2^1(\xi_u)$ in state ξ_u. Accordingly individuals $j = 2, ..., N$ receive $\bar{x}_2^j(\xi_u)$ and $\bar{x}_2^j(\xi_d) + \frac{1}{N-1}\left(\bar{x}_2^1(\xi_d) - \varepsilon\right)$. This allocation is sequentially strictly individually rational if p is close enough to one as $u_2^1\left(\bar{x}_2^1(\xi_u)\right) \gg u_2^1\left(w_2^1(\xi_u)\right)$, and hence we obtain

$$\begin{aligned}
&p \cdot u_2^1\left(\bar{x}_2^1(\xi_u)\right) + (1-p)\cdot u_2^1(\varepsilon) \\
&\quad > p \cdot u_2^1\left(w_2^1(\xi_u)\right) + (1-p)\cdot u_2^1\left(w_2^1(\xi_d)\right), \\
&p \cdot u_2^j\left(\bar{x}_2^j(\xi_u)\right) + (1-p)\cdot u_2^j\left(\bar{x}_2^j(\xi_d) + \frac{1}{N-1}\left(\bar{x}_2^1(\xi_d) - \varepsilon\right)\right) \\
&\quad > p \cdot u_2^j\left(w_2^j(\xi_u)\right) + (1-p)\cdot u_2^j\left(w_2^j(\xi_d)\right)
\end{aligned}$$

for $j = 2, ..., N$. From period $t \geq 3$ onwards the individuals are assumed to receive $(\bar{x}_t^i)_{i\in\mathcal{N}}$.

This example shows that even without assuming heterogeneous beliefs there are approximate subgame perfect Nash equilibria with nodes where at least one agent is driven out of the market. This cannot happen if we introduce default and collateral requirements. Each time a financial asset is sold the according collateral has to be put aside. If default appears, there still remains the collateral to pay back at least part of the asset market obligations. Hence, in the model with collateral future initial endowments are not used to fulfill asset market obligations from previous periods. This fact avoids the situation that individuals are driven out of the market because of asset market obligations from past periods. This is a crucial difference compared to the model with default.

4.4 Further Research and Concluding Remarks

We model a strategic market game with finite horizon and default. The main results are similar to Giraud and Weyers (2004). It turns out that even if we change the structure of the financial market drastically and allow for default while imposing at the same time a collateral structure for the financial assets, we are still able to obtain an analogue of a perfect folk theorem in a similar way as in Giraud and Weyers (2004). Almost everything is possible as soon as people are sufficiently patient, since almost every feasible, affordable, sequentially strictly individually rational consumption stream can be obtained by means of some almost full approximate subgame perfect Nash equilibrium.

In a setting with incomplete financial markets allowing for default and introducing collateral requirements seems to have at a first glance a negative influence on the economy. We no longer assume that individuals are obliged to fulfill their obligations on the asset markets. Instead we impose that each time a financial asset is sold the according collateral has to be put aside. If an individual does not possess the required collateral, it cannot sell the financial asset. The individuals buying the financial assets anticipate that the seller might default and pay back only the value of the collateral and not the promised amount. Having a closer look at the individual budget restriction (in the definition of a feasible and affordable allocation) shows that by introducing default and collateral there remains for the seller always the collateral to fulfill at least part of his asset market obligations. If he defaults, he has to sell the collateral and use the amount of money he obtains to fulfill his asset market obligations. Thus, with collateral requirements future initial endowments are not touched by asset market obligations from previous periods. We have shown by means of an example that this is not true in the model without default.

Our result in weaker than the one of Giraud and Weyers (2004) in the sense that the strategies we use are only almost full. This means we do not necessarily use full strategies on the asset markets. The collateral requirement makes it impossible to imitate the no-trade equilibrium by bidding and offering at the same time small amounts of financial assets (for example in the punishment phase). The difficulty is that for every asset sale the according collateral has to be put aside. Hence, even if this is possible, as the initial endowments are strictly positive, putting some consumption goods aside means that they cannot be consumed in the actual period. As a result the desired level of utility from consumption cannot be reached exactly. For this reason we decided to accept the fact that the strategies on the asset markets might not be full and hence we obtain an analogue of a perfect folk theorem only in almost full strategies. Another possibility is to take the collateral requirement only on net trades. In this case it is possible to use full strategies on the asset markets. To stay close to the existing literature on collaterals, as for example Araujo et al. (2002) we decided to model the collateral requirements using asset sales and not net trades.

A further feature incorporated in the competitive model with collateral of Araujo et al. (2002) is the possibility to store commodities from one period to an other. The individuals are free to decide if they are willing to consume or to store their commodities. In addition durable goods are included in their analysis. They use a depreciation structure to model the amounts that remain after consumption or storage in the next period. The competitive equilibrium concept is defined taking this into consideration. Neither Giraud and Weyers (2004) nor our model here allows for storage or durable goods. It remains an open question if and how the model can be extended in this direction. Allowing the individuals to store their commodities changes the "initial" endowments over time as everything that was stored is available, maybe depreciated, at the start of the next period. The proof our finite horizon folk theorem like result is crucially relying on the fact that the initial endowments in the reward phase are chosen in such a way that there is an indeterminacy of Nash equilibria in the one shot strategic market games. Applying a Transversality Theorem Peck et al. (1992) obtain their result for a generic choice of initial endowments. They are able to exclude Pareto optimal initial endowments and locally isolated points in their indeterminacy result. Hence, for a model including storage possibilities or durable goods, it remains unclear which assumptions are necessary to ensure a certain dimensionality of the set of Nash equilibria of the strategic market game. Pareto optimal initial endowments were excluded from our analysis here. Peck et al. (1992) show that if the initial endowments are Pareto optimal the Nash equilibrium allocation is unique. For a generic choice of initial endowments that are Pareto inefficient we showed here in a model with collateral almost every feasible, affordable and sequentially strictly individually rational consumption stream can be obtained as an almost full approximate subgame perfect Nash equilibrium of the finite horizon strategic market game if the individuals are patient enough.

Compared to Araujo et al. (2002) we drop the assumption of perfect competition and model the price formation explicitly using strategic market games based on Shapley and Shubik (1977) and obtain by that a huge set of equilibria. Taking into consideration that in reality economies are not always perfectly competitive explains why in certain market allocations arise that are not a competitive equilibrium. For example given the initial endowments in an imperfectly competitive setting modeled by a strategic market game no trade is a Nash equilibrium and as we have seen there are many more in the finite horizon case.

4.5 Appendix

4.5.1 Proof of Lemma 4.1

To show Lemma 4.1 for our model, we modify the proof of Giraud and Weyers (2004) slightly.

Proof. Fix a node $\xi \in \mathbf{D}$ at time $t \leq T$. Since the allocation of initial endowments $(w^i(\xi))_i$ are Pareto-inefficient in the L-good spot economy there exists a consumption stream $(\bar{x}^i(\xi))_i$ that Pareto dominates $(w^i(\xi))_i$ and satisfies for every good $\ell \in \mathcal{L}$

$$\sum_{i=1}^{N} \bar{x}^i_\ell(\xi) = \sum_{i=1}^{N} w^i_\ell(\xi).$$

By the strict monotonicity of the preferences, there exists a consumption stream $(\bar{x}'^i(\xi))_i$ such that

$$u^i_t(\bar{x}'^i(\xi)) > u^i_t(\bar{x}^i(\xi)) \quad i = 1, ..., N$$

and

$$\sum_{i=1}^{N} \bar{x}'^i_\ell(\xi) = \sum_{i=1}^{N} w^i_\ell(\xi).$$

Since the utility functions are strictly increasing, there exists a hyperplane containing $(\bar{x}'^i(\xi))_i$ and $(w^i(\xi))_i$ with a strictly positive price vector $(p_\ell(\xi))_{\ell \in \mathcal{L}}$. Thus the individual budget restriction

$$\sum_{\ell=1}^{L} p_\ell(\xi)\bar{x}'^i_\ell(\xi) = \sum_{\ell=1}^{L} p_\ell(\xi) w^i_\ell(\xi)$$

is satisfied and furthermore

$$E[u^i_t(\bar{x}'^i)] > E[u^i_t(\bar{x}^i_t)] \geq E[u^i_t(w^i_t)]$$

for all $i \in \mathcal{N}$ and $t \leq T$. □

4.5.2 Proof of Lemma 4.2

To show Lemma 4.2 for our model, we modify the proof of Giraud and Weyers (2004).

Proof. The final allocation of good $\ell \in \mathcal{L}$ available for consumption after trading at node $\xi \in \mathbf{D}$ is given by

$$x^i_\ell(\xi) = w^i_\ell(\xi) + \sum_{j=1}^J \gamma^i_j(\xi^-)\left(Y_\xi C_j\right)_\ell - q^i_\ell(\xi) + \frac{b^i_\ell(\xi)}{p_\ell(\xi)} - \sum_{j=1}^J \gamma^i_j(\xi) C_{j\ell}$$

Since $(\bar{x}^i)_{i\in N}$ is a feasible and affordable consumption stream there exists a feasible and affordable allocation $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ such that the asset markets clear at every node $\xi \in \mathbf{D}$. For all $j \in \mathcal{J}$ we have

$$\sum_{i=1}^N \bar{\theta}^i_j(\xi) = \sum_{i=1}^N \bar{\varphi}^i_j(\xi).$$

Using the market clearing condition on the goods markets we obtain from the definition of the strategies

$$\begin{aligned}
\sum_{i=1}^N q^i_\ell(\xi) &= \sum_{i=1}^N \left(w^i_\ell(\xi) + \sum_{j=1}^J \bar{\varphi}^i_j(\xi^-)\left(Y_\xi C_j\right)_\ell \right) \\
&= \sum_{i=1}^N \left(\bar{x}^i_\ell(\xi) + \sum_{j=1}^J \bar{\varphi}^i_j(\xi) C_{j\ell} \right), \\
\sum_{i=1}^N b^i_\ell(\xi) &= p_\ell(\xi) \sum_{i=1}^N \left(\bar{x}^i_\ell(\xi) + \sum_{j=1}^J \bar{\varphi}^i_j(\xi) C_{j\ell} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\bar{p}_\ell(\xi) &= \frac{\sum_{i=1}^n b^i_\ell(\xi)}{\sum_{i=1}^n q^i_\ell(\xi)} \\
&= p_\ell(\xi).
\end{aligned}$$

The final allocation of sales and of purchases for asset $j \in \mathcal{J}$ are given by

$$\varphi_j^i(\xi) = \gamma_j^i(\xi)$$
$$\theta_j^i(\xi) = \frac{\beta_j^i(\xi)}{\pi_j(\xi)}.$$

Hence from the definition of the strategies using the market clearing condition on the asset markets we obtain for the asset prices

$$\begin{aligned}\pi_j(\xi) &= \frac{\sum_{i=1}^N \beta_j^i(\xi)}{\sum_{i=1}^N \gamma_j^i(\xi)} \\ &= \frac{\bar{\pi}_j(\xi)\sum_{i=1}^N \bar{\theta}_j^i(\xi)}{\sum_{i=1}^N \bar{\varphi}_j^i(\xi)} \\ &= \bar{\pi}_j(\xi).\end{aligned}$$

Therefore,

$$\begin{aligned}x_\ell^i(\xi) &= \bar{x}_\ell^i(\xi), \\ \varphi_j^i(\xi) &= \bar{\varphi}_j^i(\xi) \\ \theta_j^i(\xi) &= \bar{\theta}_j^i(\xi).\end{aligned}$$

It remains to check that the budget constraint $(*_\xi^i)$ for the bids and offers is satisfied.

$$\sum_{\ell=1}^L b_\ell^i(\xi) + \sum_{j=1}^J \beta_j^i(\xi) \leq \sum_{\ell=1}^L p_\ell(\xi) q_\ell^i(\xi) + \sum_{j=1}^J \pi_j \gamma_j^i(\xi) + \sum_{j=1}^J \left(\frac{\beta_j^i(\xi^-)}{\pi_j(\xi^-)} - \gamma_j^i(\xi^-) \right) D_j(\xi)$$

Inserting the assumed strategies for $b_l^i(\xi)$, $q_l^i(\xi)$, $\gamma_j^i(\xi)$ and $\beta_j^i(\xi)$ we obtain for $(*_\xi^i)$

$$\begin{aligned}&\sum_{\ell=1}^L \bar{p}_\ell(\xi)\left(\bar{x}_\ell^i(\xi) + \sum_{j=1}^J \bar{\varphi}_j^i(\xi) C_{j\ell}\right) + \sum_{j=1}^J \bar{\pi}_j(\xi)\left(\bar{\theta}_j^i(\xi) - \bar{\varphi}_j^i(\xi)\right) \\ &\quad \leq \sum_{\ell=1}^L \bar{p}_\ell(\xi)\left(w_\ell^i(\xi) + \sum_{j=1}^J \bar{\varphi}_j^i(\xi^-) C_{j\ell}\right)\end{aligned}$$

$$+\sum_{j=1}^{J}\left(\bar{\theta}_j^i(\xi^-)-\bar{\varphi}_j^i(\xi^-)\right)D_j(\xi)$$

which holds since $(\bar{x}^i,\bar{\varphi}^i,\bar{\theta}^i)_{i\in\mathcal{N}}$ was assumed to be a feasible and affordable allocation. As $(w^i(\xi))_i \gg 0$, this strategy profile is almost full. This completes the proof. □

4.5.3 Proof of Lemma 4.3

Proof. We show that there does not even exist a Nash equilibrium that gives at least one player less utility than he can obtain from his initial endowment. This implies that there will not be a subgame perfect Nash equilibrium with this property.

Recall that the utility functions of the players are given by

$$u^i(\bar{x}^i) = (1 - \lambda^i) \sum_{t=1}^{T} (\lambda^i)^{t-1} E[u^i_t(\bar{x}^i_t)].$$

Suppose there exists a Nash equilibrium consumption stream $(\bar{x}'^i)_{i \in \mathcal{N}}$ of the T-period strategic market game such that for at least one player $i \in \mathcal{N}$ we have

$$u^i(\bar{x}'^i) < u^i(w^i).$$

Denote by $\sigma = (\sigma^1, ..., \sigma^N)$ the according strategies. If player i now deviates and plays the strategy $\bar{\sigma}^i = 0$, then the resulting allocation for i is

$$\begin{aligned} x^i_\ell(\xi) &= w^i_\ell(\xi) && \text{for all } \ell \in \mathcal{L}, \\ \varphi^i_j(\xi) &= \theta^i_j(\xi) = 0 && \text{for all } j \in \mathcal{J} \end{aligned}$$

for all nodes $\xi \in \mathbf{D}$. Thus, by deviation player i can ensure himself always a utility of $u^i(w^i)$ which contradicts the assumption that $(\bar{x}'^i)_{i \in \mathcal{N}}$ is a Nash equilibrium allocation of the T-period strategic market game.

For almost full strategies we obtain the following: If we want player i to bid or offer at least $\varepsilon > 0$ (small) on the goods markets where the price $p_\ell(\xi)$ of good $\ell \in \mathcal{L}$ is strictly positive, then his bids and offers for good $\ell \in \mathcal{L}$ need to satisfy

$$q^i_\ell(\xi) \cdot \sum_{k \neq i} b^k_\ell(\xi) = b^i_\ell(\xi) \cdot \sum_{k \neq i} q^k_\ell(\xi)$$

to have the same utility as from his initial endowment.

We distinguish between open and closed markets: If $\sum_{k \neq i} b^k_\ell(\xi) > 0$ and $\sum_{k \neq i} q^k_\ell(\xi) > 0$, the above relation ensures, that if we look at the price $p_\ell(\xi)$

of good $\ell \in \mathcal{L}$, we have

$$\begin{aligned}
p_\ell(\xi) &= \frac{\sum_{i=1}^N b_\ell^i(\xi)}{\sum_{i=1}^N q_\ell^i(\xi)} \\
&= \frac{\sum_{k\neq i} b_\ell^k(\xi) + b_\ell^i(\xi)}{\sum_{i=1}^N q_\ell^i(\xi)} \\
&= \frac{\sum_{k\neq i} b_\ell^k(\xi) + q_\ell^i(\xi) \cdot \frac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)}}{\sum_{i=1}^N q_\ell^i(\xi)} \\
&= \frac{\sum_{k\neq i} b_\ell^k(\xi) \left(1 + \frac{q_\ell^i(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)}\right)}{\sum_{i=1}^N q_\ell^i(\xi)} \\
&= \frac{\sum_{k\neq i} b_\ell^k(\xi) \left(\frac{\sum_{i=1}^N q_\ell^i(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)}\right)}{\sum_{i=1}^N q_\ell^i(\xi)} \\
&= \frac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)}.
\end{aligned}$$

Thus, this action does not influence the price of good $\ell \in \mathcal{L}$. The strategies on the goods markets are

$$\begin{aligned}
b_\ell^i(\xi) &= \tfrac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)} \cdot \varepsilon, \\
q_\ell^i(\xi) &= \varepsilon.
\end{aligned}$$

We have

$$-q_\ell^i(\xi) + \frac{b_\ell^i(\xi)}{p_\ell(\xi)} = -\varepsilon + \frac{\frac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)} \cdot \varepsilon}{\frac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)}} = 0$$

and thus $x_\ell^i(\xi) = w_\ell^i(\xi)$. These strategies are budget feasible for player i since

$$b_\ell^i(\xi) = \frac{\sum_{k\neq i} b_\ell^k(\xi)}{\sum_{k\neq i} q_\ell^k(\xi)} \cdot \varepsilon$$

$$= \frac{\sum_{i=1}^{N} b_\ell^i(\xi)}{\sum_{i=1}^{N} q_\ell^i(\xi)} \cdot \varepsilon$$
$$= p_\ell(\xi) q_\ell^i(\xi).$$

A crucial assumption for the arguments from before is that the markets are open.

For closed markets with $\sum_{k \neq i} b_\ell^k(\xi) = 0$ or/and $\sum_{k \neq i} q_\ell^k(\xi) = 0$ we distinguish different cases.

- $\sum_{k \neq i} b_\ell^k(\xi) = 0$ and $\sum_{k \neq i} q_\ell^k(\xi) = 0$
  In this case by bidding and offering ε on the goods markets and 0 on the asset markets player i opens the goods markets and the price is equal to 1. As the other individuals do not bid or offer in these markets, player i is only trading with himself and keeps his initial endowment.

- $\sum_{k \neq i} b_\ell^k(\xi) > 0$ and $\sum_{k \neq i} q_\ell^k(\xi) = 0$ or $\sum_{k \neq i} b_\ell^k(\xi) = 0$ and $\sum_{k \neq i} q_\ell^k(\xi) > 0$
  In this case the markets are closed, but in each of the two above cases there players that want to trade on one side of the market. If player i places strictly positive offers and bids, then he induces a positive price and opens the markets. If he is alone on the bidding side, then bidding a very small amount will necessarily lead to a strictly positive trade with the other players independently from the amount he might offer. In this case he has strictly more of that good than his initial endowment. The case is different if he is alone on the offer side, then offering a very small amount will necessarily lead to a strictly positive trade with the other players independently from the amount he might to bid.

The case $\sum_{k \neq i} b_\ell^k(\xi) = 0$ or/and $\sum_{k \neq i} q_\ell^k(\xi) = 0$, meaning that just player i places a bid or an offer, does not seem to be reasonable. If we demand from player i to place a bid and an offer of at least ε, the other players should be obliged to bid and offer ε at least as well.

Hence, even if we force all the players to play almost full strategies, by deviation every player can ensure himself always a utility of $u^i(w^i)$ which contradicts the assumption that $(\bar{x}'^i)_{i\in\mathcal{N}}$ is a Nash equilibrium allocation of the T-period strategic market game.

This completes the proof. □

4.5.4 Proof of Theorem 4.1

Proof. To show Theorem 4.1 for our model, we use the idea of Giraud and Weyers (2004).

Note that if the players play the following actions

$$\beta_j^i(\xi) = \gamma_j^i(\xi) = 0$$

on the asset markets at node $\xi \in \mathbf{D}$, the collateral requirement is 0.

The existence of a generic choice of initial endowments $\Omega^*(N)$ at every node $\xi \in \mathbf{D}$ is ensured by Lemma 4.4 assuming that the asset markets are closed. The markets clear at a Nash equilibrium and the budget constraints are binding. For every interior Nash equilibrium of the strategic market game with closed asset markets at node ξ we can find by Lemma 4.4 an interior Nash equilibrium, denoted by $(x^i(\xi, N))_{i\in\mathcal{N}}$, that strictly Pareto-dominates the initial allocations $(w^i(\xi))_{i\in\mathcal{N}}$. At every node $\xi \in \mathbf{D}$ we construct a sequence of $N(T-1)$ Pareto ranked Nash equilibria using Lemma 4.4. This sequence is denoted by $(x_n(\xi, N))_{n=1}^{N(T-1)}$. Denote by $b_\ell^i(\xi, N, n)$ and $q_\ell^i(\xi, N, n)$ the to $(x_n^i(\xi, N))_{i\in\mathcal{N}}$ corresponding bids and offers for commodity $\ell \in \mathcal{L}$ for player $i \in \mathcal{N}$. Define $\varepsilon_n^i(\xi, N)$ as the utility loss of agent i by changing from from the n-th to the $(n+1)$-th Nash equilibrium and $\varepsilon(N)$ is the minimal loss considering all players and all Pareto ranked Nash equilibria.

$$\begin{aligned} \varepsilon_n^i(\xi, N) &= u_t^i\left(x_n^i(\xi, N)\right) - u_t^i\left(x_{n+1}^i(\xi, N)\right) \quad \text{for all } i, n, \xi = (t, \varsigma) \\ \varepsilon_n(N) &= \min_{i\in\mathcal{N}, \xi: t\le\tau} \varepsilon_n^i(\xi, N) \\ \varepsilon(N) &= \min_n \varepsilon_n(N) \end{aligned}$$

Assume that the utilities from feasible and affordable allocations are bounded above by α_t^i for all t and for every player $i \in \mathcal{N}$.[4]

[4]An upper bound for the utility an individual can reach having a given amount of aggregate initial endowments is the utility he obtains if we allocate all the endowments to him.

Define

$$\alpha^i = \max_{t \le \tau} \alpha^i_t,$$
$$\alpha(N) = \max_{i \in \mathcal{N}} \alpha^i$$

and

$$T^0(N) = \lfloor \frac{\alpha(N)}{\varepsilon(N)} + 1 \rfloor.$$

We consider only $T \ge T^0(N)$ and $\frac{\alpha(N)}{\varepsilon(N)} < R \le T$.

Let $(\bar{x}^i)_{i\in\mathcal{N}}$ be a consumption stream, that is feasible, affordable and sequentially strictly individually rational (with respect to initial endowments) in the first $T - T^0(N)$ periods. Denote its associated feasible and affordable allocation by $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$.
Define the strategies as follows:

- for $t \le T - R$: Predefined Play
  Use Lemma 4.2 to construct almost full strategies that exactly achieve $(\bar{x}^i)_{i\in\mathcal{N}}$ in the first $T - R$ periods.

- for $t > T - R$: Punishment and Reward
  - If there was no deviation from the equilibrium path in the first $T - R$ periods, then traders play some actions on the goods markets that yield $x_1(\xi, N)$ and on the asset markets

    $$\beta^i_j(\xi) = \gamma^i_j(\xi) = 0 \quad \text{for all } j \in \mathcal{J},\, i \in \mathcal{N}.$$

  - If there was at least one deviation from the equilibrium path in the first $T - R$ periods, say at time $\tilde{t}$, then the punishment for everyone is to play on the goods markets

    $$b^i_\ell(\xi) = q^i_\ell(\xi) = \frac{\delta}{N} \quad \text{for all } \ell \in \mathcal{L},\, i \in \mathcal{N}.^5$$

[5] Note that it is possible to choose such a $\delta > 0$ (small enough) since the initial endowments are assumed to be strictly positive.

and on the asset markets

$$\beta_j^i(\xi) = \gamma_j^i(\xi) = 0 \quad \text{for all } j \in \mathcal{J},\ i \in \mathcal{N}$$

until period $T - R$ included. Afterwards, if for $t > T - R$ n deviations have been observed, then the further punishment is to play $x_{n+1}(\xi, N)$ on the goods markets and to keep the strategy on the asset markets. Hence, every player occurs at least a utility loss of $\varepsilon_n(N)$ in each subsequent period of the reward phase after the n-th deviation was detected.

We now show that this is an approximate subgame perfect Nash equilibrium, if T is big enough. For the ε-perfection observe that

$$\begin{aligned}
|u^i(x^i(\sigma)) - u^i(\bar{x}^i)| &\leq (1-\lambda^i) \sum_{t=T-R+1}^{T} (\lambda^i)^{t-1} |E[u_t^i(x_t^i)] - E[u_t^i(\bar{x}_t^i)]| \\
&\leq 2\alpha^i (1-\lambda^i) \sum_{t=T-R+1}^{T} (\lambda^i)^{t-1} \\
&= 2\alpha^i (\lambda^i)^{T-R} (1 - (\lambda^i)^R)
\end{aligned}$$

If we choose λ^i appropriately (sufficiently close to 1) this inequality can be made less than $\varepsilon^* > 0$. Thus, there exists $\lambda_1(R, N)$ such that for $\min_{i \in \mathcal{N}} \lambda^i \geq \lambda_1(R, N)$ we have for every $i \in \mathcal{N}$

$$|u^i(x^i(\sigma)) - u^i(\bar{x}^i)| < \varepsilon^*.$$

We further show that nobody has an interest to deviate from the predefined strategies.

Suppose there is a player who deviates in the last R time periods.

- Can a player profitable deviate on the asset markets?
  Each date the bids and offers on the financial markets are equal to 0. Thus, a profitable deviation that includes trading assets with other individuals on the asset markets in the last R time periods is not possible. The only possible deviation on the asset markets a player

can make is to trade with "himself". As a consequence of this action he has to put up the collateral needed which will be stored for the next period. Since we assumed that all assets were issued in the first $T-R-1$ periods and since assets can only be traded in their issuing period, these strategies are not feasible according to condition $(F3\xi)$.

- What is the gain from a deviation on the goods markets?
  According to the prescribed strategies, in the last R time periods they play a Nash equilibrium on the goods markets. Thus no player can profitably deviate.

Suppose player i wants to deviate before the last R periods. Assume he deviates in period $\bar{t} \leq T-R$ at node ξ. Maximally he can reach the upper bound of his utility given by α^i in the period of his deviation. Moreover, the maximal amount of each good he can get in each period afterwards is δ. If he deviates then according to the definition of the strategies above he keeps his initial endowment in every period after the deviation until including $T-R$ and obtains additionally maximally δ (from further deviations) for the periods in between his deviation and the period $T-R$. Therefore his maximal gain, considering that a deviation after period $T-R$ cannot be profitable is

$$d^i = (1-\lambda^i)\left((\lambda^i)^{\bar{t}-1}(\alpha^i - u^i(\bar{x}^i_t(\xi)) + \sum_{t=\bar{t}+1}^{T-R}(\lambda^i)^{t-1}\left(E[u^i_t(w^i_t+\delta\mathbf{1})] - E[u^i_t(\bar{x}^i_t)]\right)\right).$$

We can choose δ small enough such that

$$d^i \leq (1-\lambda^i)(\lambda^i)^{\bar{t}-1}\alpha^i.$$

Moreover, his minimal loss, assuming that his is the n-th deviation, is given by

$$(1-\lambda^i)\sum_{t=T-R+1}^{T}(\lambda^i)^{t-1}\varepsilon_n(N) \geq (1-\lambda^i)R(\lambda^i)^T\varepsilon(N).$$

Thus if

$$d^i \leq (1-\lambda^i)(\lambda^i)^{\bar{t}-1}\alpha^i < (1-\lambda^i)R(\lambda^i)^T\varepsilon(N),$$

player i has no incentive to deviate. It remains to show that

$$(\lambda^i)^{\bar{t}-1}\alpha^i \leq R(\lambda^i)^T\varepsilon(N).$$

For this note that

$$(\lambda^i)^{\bar{t}-1}\alpha^i < \alpha^i \leq \alpha(N)$$

and

$$\begin{aligned} & \alpha(N) < R\varepsilon(N)(\lambda^i)^T \\ \Leftrightarrow\quad & \frac{\alpha(N)}{R\varepsilon(N)} < (\lambda^i)^T \end{aligned}$$

Per assumption we have chosen R such that $R > \frac{\alpha(N)}{\varepsilon(N)}$ and hence the left hand side is smaller than one,

$$\frac{1}{R}\frac{\alpha(N)}{\varepsilon(N)} < \frac{1}{R}R = 1.$$

Therefore, for any T (big enough) we can find R and $\lambda_2(R,N) < 1$ such that

$$(\lambda_2(R,N))^T > \frac{\alpha(N)}{\varepsilon(N)}.$$

If $\min_{i\in\mathcal{N}} \lambda^i \geq \lambda_2(R,N)$, then no player has an interest to deviate in the first $T-R$ periods.

Hence define $\lambda(R,N) := \max\{\lambda_1(R,N), \lambda_2(R,N)\}$ and choose $\min_{i\in\mathcal{N}} \lambda^i \geq \lambda(R,N)$. The proposed strategy profile is an almost full subgame perfect Nash equilibrium that ε^*-approximates $(\bar{x}^i)_{i\in\mathcal{N}}$. □

Chapter 5

Learning in Infinite Horizon Strategic Market Games with Collateral and Incomplete Information

5.1 Introduction

The events leading to the financial crisis 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets also create opportunities for agents with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them. One reason for why there has been relatively little attention, in economic theory, paid to heterogeneity of beliefs and how these interact with financial markets is the market selection hypothesis. The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of, and eventually be driven out of the markets by those with the correct belief. Therefore, agents with incorrect beliefs will have no influence on the economic activity in the long run. This hypothesis has been formalized and extended in recent work by Blume and Easley (2006) and Sandroni (2000). However these authors assume that financial markets are complete, an assumption which plays a central role in allowing agents to pledge all their wealth. By contrast, Cao (2010) presents a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints. Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of those heterogeneous beliefs. This leads to additional leverage and asset price volatility (relative to a model with homogeneous beliefs or relative to the complete markets economy).

In this paper, we explore a middle ground between these two strands of literature, where traders have heterogeneous beliefs, cannot be simply driven out of the market (thanks to the collateral constraints, as in Cao 2010) but strategically learn the true state of the world. The uncertain state of the world is a transition matrix that gives the probabilities with which a succeeding node in a tree-like time structure is reached. The sets of players and actions are common knowledge, but the distribution of initial endowments and one-period utility levels conditional on action profiles is chosen randomly in each period, and the players do not observe nature's choice. Neither do they observe any player's action —hence, markets are assumed to allow anonymous trading. The probability distribution according to which uncertainty realizes in each period is a (stationary) Markov chain. This Markov distribution itself is chosen at random once and for all at the start of play, and, again, the investors do not observe nature's choice. The players have a common prior[1] over the finite set of possible Markov chains (states of the world), and they have various ways of learning the state of the world over time. First, each player observes her own initial endowment and realized payoff in each period —both are realizations of random variables whose distribution depends on the state. Furthermore, each player observes the return of each financial asset she owns in her portfolio (either as a creditor or a debtor) unless this asset defaults on its promise. In the latter case, the collateral is forfeit but the precise delivery of the return remains unknown.

For investors to be able to learn the state, we flesh-out the general equilibrium skeleton with a strategic market game.[2] More precisely, we study a strategic market game with infinite horizon, finitely many long-lived traders, and short-lived real assets. Collateral requirements for financial assets are introduced as in Geanakoplos and Zame (2007) and the subsequent literature. Investors' actions are not observable, so that we stick to the basic anonymity property of large markets. Nevertheless, players can manipulate their opponents' information by influencing publicly announced prices. Despite the risk of information manipulation, however, those traders with incorrect beliefs can realize their mistake along the play of the game, and strategically learn the state of the world. We therefore focus on learning equilibria, at the end of which no player has incorrect beliefs — not because they were eliminated from the market (although default *is* possible at equilibrium) but because they have taken time to cleverly update their

[1]See the footnote on page 177 for an argument of this assumption.
[2]See Giraud (2003) for an introduction.

prior belief. Our main result is a partial Folk theorem à la Wiseman (2011): For any function that maps each state of the world to a sequence of feasible and sequentially strictly individually rational allocations (precise definitions are given in section 5.3), and for any degree of precision, there is a perfect Bayesian equilibrium in which patient players learn the realized state with this degree of precision and achieve a payoff close to the one specified for each state. Hence, within this class of equilibria, no player with incorrect belief stays on the market in the long-run, provided she is patient enough —thus confirming Friedman's (1953) hypothesis but with a completely different argument.

The double role of financial assets

Our model extends the finite horizon case without default considered in Giraud and Weyers (2004) and the finite horizon with default examined in Brangewitz (2011). In both papers, uncertainty is only on future endowments while, here, we allow for uncertainty on endowments, utilities and asset returns.[3] Moreover, the authors restricted themselves to a very specific game-theoretic set-up: one with partial monitoring (players condition their actions on the public history of prices but not on traded quantities, and on the private history of their own individual trades) and ex ante evaluation of each player's payoff — that is, when contemplating a counterfactual, a player considers only the ex ante impact of her deviation with respect to the expectation operator computed thanks to some prior belief over the whole event-tree. Everything being computed ex ante, there was no learning process during the play of the game, and the authors proved the analogue of a perfect Folk theorem.

By contrast, we consider perfect Bayesian equilibria where players can update their belief along the play of the game. This deeply changes the strategic challenges at stake: Players with incorrect beliefs can now learn the state of the world (hence better forecast their future payoffs) through coordinated experimentation, by trying different action profiles, observing the resulting payoff realizations, and updating their beliefs about the region

[3]See Thomas (1995) for an example of general equilibrium model where uncertainty affects consumer's future utilities.

of the event-tree where they are currently located. Financial assets, now, play a double role: On the one hand, they serve as means for reallocating one's resources in face of risky events, on the other they can have a function analogous to that of "arms" in the multiarmed bandit problem (Rothschild, 1974). Since a buyer and a seller of an asset do not know exactly at which node of the tree they are, for each asset, there is a separate unknown probability distribution over returns. Each player's prior beliefs about the return distribution induce subjective payoff expectations for each asset, but the asset with the highest subjective expected payoff may not be the best one to choose: A trader may prefer to sacrifice expected return in the short run to gain some information that will help her in the long run. Since there are several traders meeting on the same market, however, the situation becomes more complicated: Experimentation has to be somehow coordinated to be effective, since each trader must deliver information through a specified action and strategic considerations may interfere with learning.

As an example, suppose that two traders must decide repeatedly whether or not to exchange some given financial asset. In each period, the buyer incurs a cost π (the security's price) but, the next period, the seller incurs the risk of having to pay a return $a > 0$ ("bad state") to the buyer, or to receive $b > 0$ from her ("good state" from the seller's viewpoint). It is worthwhile for the players to trade only if the discounted mean value of the payoff is greater than π for the buyer and the mean value of losses is smaller than π for the seller. But the only way to find out the mean value is to experiment by effectively trading in order to learn across time what the next return of this very asset will be.

The piece of good news provided here is that, as long as it is compatible with our key Informativeness Assumption (IA, to be described in section 5.5 below), market incompleteness does not prevent investors from learning the state. We show, indeed, that, despite price manipulation, infinite-horizon incomplete markets may be fully revealing. This is in the line with the static general equilibrium literature with real assets, where generically, every equilibrium is fully revealing (Radner 1979, Duffie and Shafer 1985). Beyond the difference between our imperfectly competitive approach and the perfect competition hypothesis, the interpretation of our result, however, strongly differs from that of the literature just mentioned. First, we focus only on fully-revealing equilibria where learning enables players to guess the state in the long-run with an arbitrary accuracy: There might exist plenty other — partially revealing or even non-informative— equilibria. Second, we restrict ourselves to real assets for the sake of clarity. A careful reading of our proof, however, shows that our result goes through in the nominal asset case, as well.[4] Therefore, from the point of view adopted in this paper, there is no essential difference between real and nominal assets. This contrasts with the negative results obtained in the perfectly competitive general equilibrium literature with incomplete markets of nominal assets (see Rahi 1995 and the references therein). Third, our (partial) Folk theorem implies a huge indeterminacy of the set of strategic equilibria which also contrasts with the generic determinacy obtained by Duffie and Shafer (1985) in the perfectly competitive set-up with incomplete markets of real assets. Fourth, this indeterminacy delivers an ambivalent message in terms of welfare: Many learning equilibria, although they are fully-revealing, are Pareto-dominated by competitive (Radner) equilibria, while many others Pareto-dominate the perfectly competitive benchmark with incomplete markets.[5]

A last point is worth emphasizing before turning to the strategic aspects of our work. Perfect competition with infinite horizon and incomplete markets faces an important stumbling block for existence, due to the possibility of Ponzi schemes at equilibrium. As a consequence, the literature devoted to this setting usually relies on some transversal budget constraint in order to forbid such Ponzi schemes (see, e.g., Florenzano and Gourdel 1996). On the other hand, when collateral requirements are added, Araujo et al. (2002) show that no Ponzi scheme arises at equilibrium. In our imperfectly competitive set-up, there is no need for such any extra transversal budget

[4]The proof is actually even simpler. This is why we have treated the real asset case.

[5]As a side-consideration, our approach may shed some light on the current debate about dark pools (see Zhu 2011). Dark pools are trading systems that do not display their orders to the public markets. A recent literature investigates whether dark pools harm price discovery. In light of our anonymous trading assumption, our result can be interpreted as showing that, as long as only market orders are allowed, dark pools do not prevent intermediaries from correctly learning the state of the world. Further investigation in this direction would require to refine the market micro-structure and to allow players to send limit-price (not just market) orders to the clearing house.

constraint, even when markets are complete. Due to the finite number of investors, indeed, a Ponzi scheme would require at least one player to borrow money from at least one other player during an infinite number of periods. The lender would clearly better do not to lend her money so many times —hence, participating to a Ponzi scheme cannot be part of everyone's best reply (see, e.g., O'Connell and Zeldes 1988). This is true with and without collateral constraints.

Asymmetric information and markets

The kind of uncertainty under scrutiny in this paper affects each investor's initial endowments, her utility function, and the returns of financial assets. This setting captures many aspects extensively studied in the literature in terms of adverse selection. One key assumption in our approach (the Informativeness Assumption, (IA)) can be stated as follows: Observing the realization of one's (random) initial endowments, one-period (random) utility levels and (strategically determined) final allocations together with *all* the assets' returns suffices for every single trader to learn the true state of the world in the long-run with probability arbitrarily close to 1. Needless to say, this assumption is far from being sufficient to guarantee *a priori* that every player will always learn the true state with arbitrary accuracy: for that purpose, she needs to be able to keep every asset in her portfolio in every period; she may be diverted by the strategic signaling of her opponents; the learning process must remain compatible with the equilibrium conditions, hence should not involve too deep losses. On the other hand, (IA) is verified in a number of important instances:

Arrow securities

(IA) is clearly satisfied when the asset structure is that of Arrow securities, where each security pays off 1 in one single state. In this case, observing assets' returns suffices to identify the Markov chains' realization after each round of trade (even without taking account of prices or of one's private knowledge gained by observing endowments and stage-payoffs). After a sufficiently long time, if every trader succeeds in observing every asset's return, the true state of the world will become common knowledge. Notice, however, that, even in this polar case, full revelation at a strategic equilibrium is not straightforward, and there is something to be proven: Indeed, our argument requires that *every* trader be able to trade *every* Arrow security in *every period*. If one of them fails to observe all the assets' returns in certain periods, then she might not draw the right conclusion about which Markov chain is driving uncertainty, so that players cannot coordinate on any state-dependent equilibrium path. On the other hand, if, say, only the riskless asset (delivering the same return in every state) is marketed, then observing assets' returns does not provide any information.

Akerlof's model

Akerlof's (1970) model of used cars is a static one. Its extension to our intertemporal framework can easily be interpreted as verifying (IA). Suppose, indeed, that the quality index, s, of a car is an integer belonging to $[1, 10]$. s is distributed according to the Markov chain ω. As quality of a car is undistinguishable beforehand by the buyer (due to the asymmetry of information), incentives exist for the seller to pass off low-quality goods as higher-quality ones. The buyer, however, takes this incentive into consideration, and takes the quality of the goods to be uncertain. Only the average quality of the goods will be considered, which, in a one-shot-set-up, will have the side effect that goods that are above average in terms of quality will be driven out of the market. In our multi-period setting, however, this need not occur: Each time t, the seller receives a new (random) endowment of used cars. Each period, the buyers are informed *ex post* (through their stage-payoff) about the actual quality, s, of the car they have bought. Across time, they may learn the transition matrix ω, hence anticipate the distribution of s in the future. Our main result then says that the observation of prices and private knowledge enables actors on the market for used cars to enforce a large set of effective trades. This sharply contrasts with Akerlof's conclusion that the market for used cars should collapse.

Moral hazard.

Since investors take privately observed actions affecting their initial endowments and portfolios, our paper is also linked to the literature on moral hazard. The differences in information and the signaling aspects of the present work are related to, for example, job market signaling model of Spence (1973) or the competitive insurance market considered in Rothschild and Stiglitz (1976). However, we do not consider a classical principal agent model. Every individual may act as a seller or a buyer (or both simultaneously), and this on commodity as well as asset markets. Therefore, we cannot impose, for example, that a seller is always less informed than a buyer or vice versa. Finally, we consider only finitely many players. Our set-up therefore sharply differs from the perfectly competitive case studied in the seminal papers by Prescott and Townsend (1984a,b) or, more recently, by Acemoglu and Simsek (2010). In particular, we get a wide range of equilibria including allocation streams that are Pareto-optimal and others that are dominated. Thus, our result stands at distance both from the generic inefficiency obtained by Greenwald and Stiglitz (1984) or Arnott and Stiglitz (1986, 1990, 1991), and from the more positive results obtained by Acemoglu and Simsek (2010).

The paper is organized as follows: First we describe the infinite horizon economy and its associated strategic market game. Section 5.3 focuses on a particularly important subclass of allocations that plays a key role in the sequel. The next section proves a first (partial) Folk theorem under the simplifying assumption of complete information. Section 5.5 extends the later result to the incomplete information case. The last section concludes.

5.2 The Markov Strategic Market Game with Collateral

5.2.1 The Markov Economy

The environment

Uncertainty about future states is modeled in a Markov set up, following Cao (2010). We assume that in each period, t, the state of nature in the next period is chosen using a Markov transition matrix with a finite set of possible states of nature $\mathcal{S} = \{1, ..., S\}$. Therefore, the state tomorrow only depends on the state today and not the whole history of states that were realized in the past. Nevertheless as in Magill and Quinzii (1994) and the subsequent literature, time, uncertainty and the revelation of information can be described by an event tree, i.e., a directed graph $(\mathbf{D}, \mathcal{A})$ consisting of a set $\mathbf{D}$ of vertices and a set $\mathcal{A} \subset \mathbf{D} \times \mathbf{D}$ of (oriented) arcs.[6] In our Markov set-up, we assume that each node ξ has the same outdegree $S > 1$, and the choice of nodes adjacent from ξ is governed by a Markov chain. A node ξ can be interpreted as a date-event pair (t, s_{t-1}, s), where $t \geq 1$ is the minimal length of a walk between ξ_0 and ξ, $s_{t-1} \in \prod_{t'=1}^{t-1} \mathcal{S}$ is the sequence of realizations of the state of nature up to $t-1$ and $s \in \mathcal{S}$ is the last state in t. Let $\tau(\xi)$ be the time at which node ξ is reached, i.e. $\tau : \mathbf{D} \to \mathbb{N}$ such that $\xi = (t, s_{t-1}, s) \mapsto t$. Define a partial order $\geq$ on $\mathbf{D}$ by $\xi = (t, s_{t-1}, s) \geq \xi' = (t', s_{t'-1}, s')$ if, and only if, there is a walk from ξ' to ξ. Of course, if $\xi \neq \xi'$ and $\xi \geq \xi'$, then $\xi > \xi'$. The unique predecessor of ξ is denoted by $\xi^- = (t-1, s_{t-2}, s')$.[7] The set of immediate successors of ξ, denoted by ξ^+, is the set of nodes that are adjacent from ξ. For any node

[6]The vertex (or node) ξ can be thought of as a particular state of nature and time. If (ξ, η) is an arc, η is a node that directly follows ξ. Formally, ξ is adjacent *to* η and η is adjacent *from* ξ. The number of nodes adjacent to a given vertex ξ is the *indegree* of ξ, i.e. the number of immediate (or direct) predecessors; the number of nodes adjacent from ξ, its *outdegree*, i.e. the number of direct followers. A *walk* from ξ_1 to ξ_k is a sequence $(\xi_1, \xi_2, ..., \xi_k)$ in $\mathbf{D}$ such that ξ_i is adjacent to ξ_{i+1} for $1 \leq i \leq k-1$. There is a unique root ξ_0 (whose indegree is zero). Each node, except the root, has indegree equal to 1, and there is no cycle in $\mathbf{D}$.

[7]We define $s_{-1} = \emptyset$.

$\xi \in \mathbf{D}$, the set of all nodes with $\xi' \geq (>)\xi$ is denoted by $\mathbf{D}(\xi)$ ($\mathbf{D}(\xi)^+$) and is itself a tree with root ξ.

A state of the *world* corresponds to a transition matrix, ω, that is chosen once and for all at time 0, before the start of the play. We assume that there are finitely many states of the world, $\omega \in \Omega$.

Consumption goods and financial assets

We consider a pure exchange economy $\mathcal{E}$ with a finite set, $\mathcal{N} = \{1, ..., N\}$, of individuals, L consumption goods, usually indexed by ℓ, and J short-term real assets, indexed by j. The possibility of default is introduced by a collateral requirement as in Araujo et al. (2002). A financial asset $j \in \mathcal{J} := \{1, ..., J\}$ is characterized by a tuple (ξ^j, A_j, C_j) consisting of three elements: an issuing node, promised deliveries and collateral requirements. The issuing node (a node in the tree $\mathbf{D}$) is denoted by ξ^j. The promised amount of goods is described by a function $A_j : \mathbf{D} \to \mathbb{R}^L_+$ such that $A_j(\xi) = 0$ for all $\xi \in \mathbf{D} \setminus (\xi^j)^+$. For $\xi' \in (\xi^j)^+$, the promises $A_j(\xi')$ are the amounts of goods that a seller of asset j promises to deliver to a buyer of asset j in the next period following the issuing node ξ^j. The delivery, $p_\xi \cdot A_j(\xi)$, is assumed to be made in fiat money using spot prices, $p_\xi \in \mathbb{R}^L_+$. We only consider short-term assets. Therefore, for other nodes before the issuing node and at least two periods after the asset was issued, we assume that the promised amounts are zero. The vector $C_j \in \mathbb{R}^L_+$ is the amount of collateral needed at the issuing node, ξ^j, in order to back up the promised delivery A_j. Only consumption goods can serve as collateral.[8] Commodities are assumed to be **perishable**. Thus, they have to be consumed at the very date they enter the economy (as initial endowment), unless they are stored as collateral. Individuals are not allowed to consume a collateral, which is stored in a warehouse for one period. For simplicity, after having been stored one period, a collateral must be consumed, otherwise it gets lost.[9] For our Markov environment, we assume that at each node $\xi \in \mathbf{D}$ the "same" finite

[8]i.e., we do not introduce securities that are backed by other securities: Pyramiding is not allowed.

[9]We could allow for a longer life expectancy of a collateral, of length, say, K, but at the cost of cumbersome notations. We thus take $K = 1$.

number of financial assets is issued. As the time horizon is infinite there will be infinitely many assets in total.

The players

Every player $i \in \mathcal{N}$ is characterized by a twice continuously differentiable, strictly increasing and concave utility function $u^i_\xi : \mathbb{R}^L_+ \to \mathbb{R}$ and a strictly positive initial endowment in consumption goods $w^i_\xi \in \mathbb{R}^L_{++}$ at every node $\xi \in \mathbf{D}$. We assume that $\left(u^i_\xi(\cdot)\right)_\xi$ are uniformly bounded below for all individuals i. Therefore, without loss of generality suppose $u^i_\xi(0) = 0$. Moreover, we assume that individual endowments are uniformly bounded above by some $\overline{w}$, across individuals and periods. Initial holdings of assets are 0. Player i maximizes her expected, discounted utility from consumption. This expectation depends on her subjective beliefs on the state of the world $\omega \in \Omega$, which may themselves vary across time, depending upon the signals sent by other players during the play of the game. We shall therefore define player's i objective function after having recalled the basic structure of the strategic market game.

We also denote by $\mathcal{E}_\xi = \langle w^i_\xi, u^i_\xi(\cdot), (\xi^j, A_j, C_j)_{j \mid \xi^j = \xi} \rangle$ the finite-dimensional *one-shot economy* at node ξ. We denote the infinite horizon economy starting from a certain node ξ, that is not necessarily the root ξ_0, for short *the economy after* ξ, by $\bigcup_{\xi' > \xi} \mathcal{E}_{\xi'}$.

5.2.2 The Strategic Market Game with Collateral

At each period, players take part to a strategic market game *à la* Shapley and Shubik (1977): Each individual places for every consumption good $\ell \in \mathcal{L}$ at every node $\xi \in \mathbf{D}$ a bid $b^i_{\xi,\ell}$ and an offer $q^i_{\xi,\ell}$. The bid $b^i_{\xi,\ell}$ signals how much (in terms of fiat money) player i is willing to pay for the purchase of good ℓ and the offer $q^i_{\xi,\ell}$ (in terms of physical commodities) is the amount she wants to sell. The price of good ℓ is then computed as the ratio of the total bid to the total offer, that is

$$p_{\xi,\ell} = \begin{cases} \frac{\sum_{i=1}^N b^i_{\xi,\ell}}{\sum_{i=1}^N q^i_{\xi,\ell}} & \text{if } \sum_{i=1}^N q^i_{\xi,\ell} > 0 \\ 0 & \text{otherwise} \end{cases}$$

A market without trade is said to be closed.[10]

Similarly, at every node $\xi \in \mathbf{D}$ each player places a bid $\beta^i_{\xi,j}$ stipulating the amount of money she is ready to spend in buying asset j and offers for sale $\gamma^i_{\xi,j}$ units of this very asset. The asset's price is given by:

$$\pi_{\xi,j} = \begin{cases} \frac{\sum_{i=1}^N \beta^i_{\xi,j}}{\sum_{i=1}^N \gamma^i_{\xi,j}} & \text{if } \sum_{i=1}^N \gamma^i_j(\xi) > 0 \\ 0 & \text{otherwise} \end{cases}$$

When the promises are settled, a seller of the financial asset $j \in \mathcal{J}$ compares the value of the promise with the value of the collateral and pays back the minimal value:

$$D_{\xi',j} = \min\left\{p_{\xi'} \cdot A_j(\xi'), p_{\xi'} \cdot C_j\right\} \tag{D}$$

at node $\xi' \in (\xi^j)^+$. Hence, whether default appears or not is not the outcome of a strategic decision but depends upon the commodity price $p_{\xi'}$, which is strategically determined by bids and offers posted at node $\xi' \in (\xi^j)^+$.

Feasible bids and offers

Some physical and budgetary restrictions are put on the bids and offers individuals can choose. At every node $\xi \in \mathbf{D}$ and for every financial asset, player i needs to own the required amount of collateral, which depends on the quantity of asset offered for sales and *not* on the net trades.[11] Assuming player i offers to sell $\gamma^i_{\xi,j}$ units of asset j at node ξ, then she needs to store

[10] Defining the price as zero when there are no offers on the market we follow here for example Amir et al. (1990, p.128). Similar assumptions can be found in Postlewaite and Schmeidler (1978, p.128), Peck et al. (1992, p.275) or Giraud and Weyers (2004, p.474).

[11] As discussed in Dubey and Geanakoplos (2003), netting before imposing the collateral requirement would suppress any constraint on the size of short sales. This would make the proof of our partial Folk theorem only easier.

$\gamma^i_{\xi,j} C_j \in \mathbb{R}^L_+$ as collateral.[12]

Feasible bids and offers must satisfy the following two constraints for all commodities ℓ:

$$\sum_{j=1}^{J} \gamma^i_{\xi,j} C_{j\ell} \le w^i_{\xi,\ell} \tag{$F1\xi$}$$

and

$$q^i_{\xi,\ell} \le \sum_{j=1}^{J} \gamma^i_{\xi^-,j} C_{j\ell} + \Delta(F1\xi), \tag{$F2\xi$}$$

where $\Delta(F1\xi)$ stands for the difference between the right-hand side and the left-hand side of $(F1\xi)$. Inequality $(F1\xi)$ says that the collateral that can be stored by i at node ξ must be taken out of initial endowments. In particular, it cannot consist of commodities that are already inherited from the past as collaterals. This is a way to capture our assumption that every collateral lives at most one period. Either it is consumed at the period it enters into the economy (as initial endowment) or it is stored and consumed one period later. Notice that, in the second period of a collateral's life, it may be traded by its owner, and consumed by another player. Condition $(F2\xi)$ says that the offered amount of goods plus the amount of goods that must be stored as a collateral cannot exceed the initial endowment of player i at node $\xi \in \mathbf{D}$ plus the collateral that was put aside in the previous period. Of course, we impose:

$$q^i_{\xi,\ell}, b^i_{\xi,\ell}, \beta^i_{\xi,j}, \gamma^i_{\xi,j} \ge 0 \tag{$F3\xi$}$$

for all $\ell \in \mathcal{L}$, $j \in \mathcal{J}$.

[12]Later, on page 169 when defining the final allocation in consumption goods, the collateral requirement is taken using the final asset sales, denoted by $\varphi^i_{\xi,j}$ and not directly on the offers $\gamma^i_{\xi,j}$.

The budget constraint

Player i also faces the following budget constraint on fiat money when placing bids and offers:

$$\sum_{\ell=1}^{L} b^i_{\zeta,\ell} + \sum_{j=1}^{J} \beta^i_{\zeta,j} \leq \sum_{\ell=1}^{L} p_{\zeta,\ell} q^i_{\zeta,\ell} + \sum_{j=1}^{J} \pi_{\zeta,j}\gamma^i_{\zeta,j} + \sum_{j=1}^{J} \left(\theta^i_{\zeta^-,j} - \varphi^i_{\zeta^-,j}\right) D_{\zeta,j} \qquad (*^i_\xi 1)$$

for all $\zeta \leq \xi$ where $\theta^i_{\zeta^-,j}$ denotes the final asset purchases and $\varphi^i_{\zeta^-,j}$ the asset sales at node ζ^- (as it will be defined below). Thus, by condition $(*^i_\xi 1)$ the total value of bids cannot exceed the amount of money player i can get given her sales and given the dividends received from her portfolio, $\theta^i_{\zeta^-,j} - \varphi^i_{\zeta^-,j}$. As soon as $(*^i_\xi 1)$ is violated, say at node ξ, individual i is removed from the game for all subsequent nodes $\mathbf{D}^+(\xi)$, and all her goods are confiscated forever.

We shall also need the following condition, for every i:

$$\text{Either} \sum_{k\neq i} \gamma^k_{\xi,j} \neq 0 \quad \text{or} \quad \sum_{k\neq i} \beta^k_{\xi,j} \neq 0, \qquad (*^i_\xi 2)$$

which says that there is at least one other individual on the bidding or on the offering side of the financial markets to trade with i.

Final allocations

After trading took place, player i's holdings of asset $j \in \mathcal{J}$ are given by her sales

$$\varphi^i_{\xi,j} = \begin{cases} \gamma^i_{\xi,j} & \text{if } (*^i_\xi 1) \text{ and } (*^i_\xi 2) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

and her purchases

$$\theta^i_{\xi,j} = \begin{cases} \frac{\beta^i_{\xi,j}}{\pi_{\xi,j}} & \text{if } (*^i_\xi 1) \text{ and } (*^i_\xi 2) \text{ hold and } \pi_{\xi,j} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\theta^i_{\xi,j} - \varphi^i_{\xi,j} < 0$ then player i sold more of the financial asset $j \in \mathcal{J}$ than she bought. Analogously for $\theta^i_{\xi,j} - \varphi^i_{\xi,j} > 0$ she is a net buyer.

Moreover, player i's allocation of good $\ell \in \mathcal{L}$ available for consumption at the end of the current period at node ξ, is

$$x^i_{\xi,\ell} = \begin{cases} w^i_{\xi,\ell} + \sum_{j=1}^J \varphi^i_{\xi^-,j} C_{j\ell} - q^i_{\xi,\ell} + \frac{b^i_{\xi,\ell}}{p_{\xi,\ell}} - \sum_{j=1}^J \varphi^i_{\xi,j} C_{j\ell} & \text{if } (*^i_\xi 1) \text{ holds and } p_{\xi,\ell} > 0 \\ w^i_{\xi,\ell} + \sum_{j=1}^J \varphi^i_{\xi^-,j} C_{j\ell} - q^i_{\xi,\ell} - \sum_{j=1}^J \varphi^i_{\xi,j} C_{j\ell} & \text{if } (*^i_\xi 1) \text{ holds and } p_{\xi,\ell} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. To the best of our knowledge, condition $(*^i_\xi 2)$ is new in the strategic market game literature. It seems to us natural once collateral requirements are introduced. Suppose, indeed, that individual i is the only one who wants to trade on the financial markets, i.e., $\sum_{k \neq i} \gamma^k_{\xi,j} = \sum_{k \neq i} \beta^k_{\xi,j} = 0$. Absent condition $(*^i_\xi 2)$, this individual could open the markets by bidding and offering strictly positive amounts of assets. By doing so, every player could store some collateral until next period just by trading "with herself" today. If for several periods such a strategy is played, while the other players play zero strategies, this would conflict with our assumption that commodities are perishable.

5.3 Feasibility and *interim* individual rationality

Allowable strategies

The action set of player i at node ξ consists in feasible bids and offers:

$$A^i_\xi = \left\{ \left(q^i_{\xi,\ell}, b^i_{\xi,\ell}\right)_{\ell \in \mathcal{L}}, \left(\gamma^i_{\xi,j}, \beta^i_{\xi,j}\right)_{j \in \mathcal{J}} \in \mathbb{R}^{2L}_+ \times \mathbb{R}^{2J}_+ \middle| (F1\xi),\ (F2\xi) \text{ and } (F3\xi) \text{ are satisfied} \right\}.$$

Notice that A^i_ξ depends upon ξ but not upon ω. Let $A_\xi := \prod_{i=1}^N A^i_\xi$. Note that the definition of an action set includes actions that possibly violate the budget constraint $(*^i_\xi 1)$ or $(*^i_\xi 2)$.[13] The stage-payoff of player i at node

[13] An alternative would consist in incorporating these constraints into the very definition of a player's strategy set but this would lead to a generalized game as introduced by

$\xi = (t, s_{t-1}, s)$ is given by the utility, $u^i_\xi(x^i_\xi)$, she obtains from consumption.

Prices are publicly observed by every player. The information transmitted through prices is therefore common knowledge. However, at each node, every player also observes her own initial endowment, her final stage-payoff, her final allocation as well as the returns of the assets present in her portfolio. These observations constitute the *private history* of player i. A *strategy* of player i consists in choosing an action at every node $\xi \in \mathbf{D}$ as a function of her own private history. Let H^i_ξ denote the set of possible private histories for individual i at node ξ, given by

$$H^i_\xi := \left\{ \left(p_{\xi'}, \pi_{\xi'}, \varphi^i_{\xi'}, \theta^i_{\xi'}, x^i_{\xi'}, u^i_{\xi'}(x^i_{\xi'}), w^i_{\xi'}, w^i_\xi \right) \middle| \forall \xi' < \xi \right\}.$$

The history at the root ξ_0 is given by $H^i_{\xi_0} = \left\{ w^i_{\xi_0} \right\}$. Formally, a strategy of player i is a map

$$\sigma^i : \bigcup_{\xi \in \mathbf{D}} H^i_\xi \to \left(\mathbb{R}^L_+\right)^2 \times \left(\mathbb{R}^J_+\right)^2$$

such that $\sigma^i(h) \in A^i_\xi$ for all $\xi \in \mathbf{D}$ and for all $h \in H^i_\xi$. Actions are not observed along the play of the game, which contrasts with the setting considered, e.g., by Wiseman (2011).

Remark. As is well-known, strategic market games exhibit no-trade as a one-shot Nash equilibrium.[14] As we want to prove the analogue of a Folk theorem, we shall therefore need some threats that enforce the equilibrium path. Allowing for punishment phases that consist in playing the autarkic Nash one-shot equilibrium *ad libitum* would make the task rather easy. In order to prove that our result does *not* depend upon this kind of trick (hence is robust to whatever refinement that would allow to get rid of the autarkic one-shot equilibrium[15]), we shall focus on out-of-equilibrium strategies where players effectively trade. A second reason for not relying on the

Debreu (1952) (see also Harker (1991) or Facchinei and Kanzow (2010)).

[14]See Weyers (2004) for the elimination of this autarkic equilibrium after two rounds of elimination of dominated strategies.

[15]Such a refinement has been proposed, e.g. by Weyers (2004). As a consequence, Giraud and Weyers (2004) Folk theorem with complete information was already formulated so as not to rely on the autarkic threat.

heavy hammer of autarkic Nash equilibria is that, as already said, in adverse selection problems, the market collapse has been sometimes predicted as being the unique rational consequence of differential information. Our proof does not depend upon such a global market collapse, even as an out-of-equilibrium threat, and even though default is explicitly allowed along the equilibrium path.

Definition 5.1 (**Full strategy profile**). A strategy profile $\sigma := (\sigma^i)_i$ is called *full* if, the following holds

$$\sum_{i=1}^{N} q^i_{\xi,\ell} > 0, \quad \sum_{i=1}^{N} b^i_{\xi,\ell} > 0, \quad \sum_{i=1}^{N} \gamma^i_{\xi,j} > 0, \quad \sum_{i=1}^{N} \beta^i_{\xi,j} > 0$$

for all $\ell \in \mathcal{L}$, $j \in \mathcal{J}$, $\xi \in \mathbf{D}$.

Private *interim* beliefs

At each node ξ, payoffs are determined as follows: action profile $a_\xi \in A_\xi$ is played; it induces, say, x^i_ξ as a final allocation for player i —which is observed by i only. Then player i's random payoff, $u^i_\xi(x^i_\xi)$, which is also observed by player i only, is drawn according to ω. Notice that, when entering at node ξ, player i may not know for sure that the current node is ξ. Thus, when she takes her action, she considers the expectation of her next payoffs according to her current private belief.

At each time period t, every player i updates her private belief in a Bayesian way, according to her private history. We allow for arbitrary correlation of payoffs in each state across players' utilities, endowments, assets' returns. So player i's belief about player j's private payoff and other higher-order beliefs are unrestricted. Let $\mathbb{P}^i_\xi(h^i_\xi) \in \Delta(\Omega)$ denote player i's *private belief* at node ξ.[16] Together with a strategy profile, σ, such a probability $\mathbb{P}^i_\xi(h^i_\xi)$ induces a distribution $\mathbf{P}^i_\xi(h^i_\xi, \sigma)$ (or $\mathbf{P}^i_\xi(\sigma)$ in short) over the random characteristics of the economy to be selected after ξ, i.e., over $\bigcup_{\xi'>\xi} \mathcal{E}_{\xi'}$. In particular, it provides a distribution over i's future payoffs which, by a

[16]Hereby, $\Delta(\Omega)$ is the set of all probability distributions over the finite set of states of the world.

slight abuse of notations, is also denoted $\mathbf{P}^i_\xi(\sigma)$. At each node, whatever being the past history, individuals are supposed to maximize her expected, discounted utility using their private *interim* belief and a common discount factor $\lambda \in [0,1]$.[17] The objective function of player i is therefore of the form

$$\begin{aligned} U^i_{\mathbf{D}(\xi)}(x^i,\sigma,\omega) &:= (1-\lambda)E_{\mathbf{P}^i_\xi(\sigma)} \sum_{\xi'=(t,s_{t-1},s)>\xi} \lambda^{t-1} u^i_{\xi'}(x^i_{\xi'}) \\ &= (1-\lambda) \sum_{\xi'=(t,s_{t-1},s)>\xi} \lambda^{t-1} E_{\mathbf{P}^i_\xi(\sigma)} \left[u^i_{\xi'}(x^i_{\xi'}) \right] \end{aligned}$$

for each node ξ. (Given the boundedness of the utility function, the last equality is a consequence of Fubini's theorem.)

Feasible allocations and *interim* individual rationality

Without considering explicitly actions or strategies we define feasible allocation as follows:

Definition 5.2 (**Feasible allocation**). An allocation $(\bar{x}^i)_{i\in\mathcal{N}}$ in consumption goods is said to be *feasible*, if there exists a portfolio $(\bar{\varphi}^i,\bar{\theta}^i)_{i\in\mathcal{N}}$ and a price system $(\bar{p},\bar{\pi})$ such that the following conditions are satisfied:

- *Individual budget restriction for every player i and every node $\xi \in \mathbf{D}$:*[18]

$$\begin{aligned} &\textstyle\sum_{\ell=1}^L \bar{p}_{\xi,\ell} \left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{\xi,j} C_{j\ell} \right) + \sum_{j=1}^J \bar{\pi}_{\xi,j} \left(\bar{\theta}^i_{\xi,j} - \bar{\varphi}^i_{\xi,j} \right) \\ &\quad = \textstyle\sum_{\ell=1}^L \bar{p}_{\xi,\ell} \left(w^i_{\xi,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{\xi^-,j} C_{j\ell} \right) + \sum_{j=1}^J \left(\bar{\theta}^i_{\xi^-,j} - \bar{\varphi}^i_{\xi^-,j} \right) D_j(\xi) \end{aligned}$$

- *market clearing on spot markets for every good $\ell \in \mathcal{L}$ and every node:*

$$\textstyle\sum_{i=1}^N \left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{\xi,j} C_{j\ell} \right) = \sum_{i=1}^N \left(w^i_{\xi,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{\xi^-,j} C_{j\ell} \right)$$

- *market clearing on financial markets for every asset $j \in \mathcal{J}$ and every node*:
$\sum_{i=1}^N \bar{\theta}^i_{\xi,j} = \sum_{i=1}^N \bar{\varphi}^i_{\xi,j}$

[17] Allowing for idiosyncratic discount factors would only require notational changes.
[18] We define $\bar{\varphi}^i_{\xi_0^-,j} = \bar{\theta}^i_{\xi_0^-,j} = 0$.

- *and feasible trade in financial assets for every good* $\ell \in \mathcal{L}$*, every node and every player* i*:*
  $\sum_{j=1}^{J} \bar{\varphi}^i_{\xi,j} C_{j\ell} \leq w^i_{\xi,\ell}$

Clearly, for every individual i, the sequence of payoffs resulting from the consumption of initial endowments is bounded from below by a constant, say, $\underline{u}^i$. Define $\underline{u} := \min_{i\in\mathcal{N}} \underline{u}^i$. Since initial endowments are uniformly bounded, the stage-game payoff, $u^i_\xi(\cdot)$, induced by a feasible allocation is also uniformly bounded above by some $\overline{u}^i$ across all action profiles, all states and all periods. Define $\overline{u} := \max_{i\in\mathcal{N}} \overline{u}^i$.

In the next definition, individual rationality is understood according to the *interim* private beliefs shared by players along the play of the game. It is therefore defined *given* some state of the world, ω, and some strategy profile, σ.

Definition 5.3 (**Sequentially strictly individually rational allocation**).

A feasible allocation $(\bar{x}^i)_{i\in\mathcal{N}}$ is said to be *sequentially strictly individually rational (*SSIR*) given* ω, if

$$U^i_{\mathbf{D}(\xi)}(x^i, \sigma, \omega) > (1-\lambda) E_{\mathbf{P}^i_\xi(\sigma)} \sum_{\xi'=(t,s_{t-1},s)>\xi} \lambda^{t-1} u^i_{\xi'}(w^i_{\xi'}).$$

The following Lemma says that our last two definitions generically describe a non-vacuous subset of allocations in the economy $\mathcal{E}$, on which, from now on, we shall focus.

Lemma 5.1. *If the initial allocations* $(w^i_\xi)_i \gg 0$ *are Pareto-inefficient in the* L*-good spot economy at each node* $\xi \in \mathbf{D}$*, then the economy* $\mathcal{E}$ *admits a sequentially strictly individually rational and feasible (*SSIRF*, for short) allocation.*

The next Lemma will prove useful for our main result. It shows that every SSIRF allocation can be enforced by means of some adequate strategy. Such a strategy, however, need not fulfill any equilibrium requirement.

Lemma 5.2. *Let $(\bar{x}^i)_{i\in\mathcal{N}}$ be a* SSIRF *allocation. Let $(\bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ and $(\bar{p}, \bar{\pi})$ be the corresponding portfolio and price system. Then $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ can be implemented through the following strategy profile in the sense that, node-wise, the utility of this strategy profile is arbitrarily close to the node-wise utility of $(\bar{x}^i)_{i\in\mathcal{N}}$. Whatever being the past history, play*

for all $\xi \in \mathbf{D}$, $i \in \mathcal{N}$, $\ell \in \mathcal{L}$ and $j \in \mathcal{J}$

$$q^i_{\xi,\ell} = w^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi^-,j} C_{j\ell}$$

$$b^i_{\xi,\ell} = \bar{p}_{\xi,\ell}\left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi,j} C_{j\ell}\right)$$

$$\gamma^i_{\xi,j} = \begin{cases} \bar{\varphi}^i_{\xi,j} & \text{if } \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases}$$

$$\beta^i_{\xi,j} = \begin{cases} \bar{\pi}_{\xi,j}\bar{\theta}^i_{\xi,j} & \text{if } \sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases}$$

with $\delta > 0$ small. Clearly, the above strategies are full.

If we target a given allocation using the full strategies as defined in Lemma 5.2 and this allocation does not always require trade on the asset markets, then we cannot target the allocation exactly. For the details we refer to the proof in Appendix 5.7.2. This is due to the presence of the collateral constraints. Nevertheless choosing $\delta > 0$ arbitrarily small we reach an allocation that is close to the target allocation.

5.4 Complete Information

We first state our result in the simpler case where information is complete, i.e., the Markov chain ω is known from the beginning by every player.

Theorem 5.1. *Suppose that $\#\Omega = 1$. Every allocation that is* SSIRF *can be approximately enforced as a subgame perfect Nash equilibrium (SPNE).*

Proof. Let $(x^{*i}_\xi)_{i,\xi}$ be a SSIRF allocation for the transition matrix ω with stage-payoffs $(v^{*i}_\xi)_{i,\xi} := \left(u^i_\xi(x^{*i}_\xi)\right)_{i,\xi}$. We denote by E_ω the expectation operator with respect to the beliefs that the state of the world is given by ω. The utility for player i resulting from x^{*i} is then given by

$$U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \sigma, \omega) = (1-\lambda) \sum_{\xi'=(t,s_{t-1},s)>\xi_0} \lambda^{t-1} E_\omega \left[u^i_{\xi'}(x^{*i}_{\xi'}) \right].$$

We construct a sequence of payoff vectors $\left((v^{i,n\text{dev}})_i\right)_{n\in\mathbb{N}}$ that result from SSIRF allocations, and such that: $v^{i,(n+1)\text{dev}}_\xi < v^{i,n\text{dev}}_\xi$ for every integer $n \in \mathbb{N}$ and every node ξ — with $v^{i,0\text{dev}} = v^{*i}$ for each i. These payoffs will be the long-run payoffs after n deviations. They are constructed as follows:

$$v^{i,n\text{dev}}_\xi := u^i_\xi\left(x^{i,n\text{dev}}_\xi\right) \text{ with } x^{i,n\text{dev}}_\xi := \rho_n x^{*i}_\xi + (1-\rho_n) w^i_\xi, \quad \rho_n \in (0,1).$$

Assume that, for every $n \in \mathbb{N}$ and $\xi = (t, s_{t-1}, s) \in \mathbf{D}$:

$$0 < \varepsilon_n < v^{i,n\text{dev}}_\xi - v^{i,(n+1)\text{dev}}_\xi \tag{5.1}$$

Using Lemma 5.2 we construct full strategies that result approximately in the target allocation $(x^{*i})_{i\in\mathcal{N}}$. If there is no deviation from these strategies, then every individual continues to play these strategies. The punishment, if one individual deviates, is to play the following strategies: Every individual bids and offers $\frac{\delta}{N}$ with $\delta > 0$ small on the goods and on the assets markets for the next T_n periods, if the nth deviation had been observed. As all individuals bid and offer the same quantities, these strategies mimic the no trade equilibrium and everybody keeps her initial endowment. On the asset markets however every individual sells $\frac{\delta}{N}$ of every asset and hence needs to have a collateral of $\frac{\delta}{N}C_{j\ell}$. As there is no trade on the goods markets, this additional collateral needs to be established from the initial endowments, which are strictly positive. Thus, δ needs to be small enough such that this is can be done.

After the punishment phase dedicated to the nth deviation there is a reward phase, if no further deviation has occurred. As soon as another deviation occurs, a new punishment phase of length T_{n+1} starts immediately.

Suppose the nth deviation has occurred and there was no further deviation during the punishment phase. Then in the reward phase the individuals play some actions, as defined in Lemma 5.2, leading approximately to a SSIRF allocation with a stage payoff of $v_\xi^{i,n\text{dev}}$. Notice that in order to settle the asset market obligations from the punishment phase and to establish the right asset holdings to reach $v_\xi^{i,n\text{dev}}$ two periods of transition are required to ensure that the individual budget constraint $(*_\xi^i 1)$ is not violated. For the details concerning the transition periods we refer to the proof of Theorem 5.2, page 190. Taking this punishment behavior into consideration we show that there is no incentive to deviate.

Suppose individual i deviates at node $\xi = (t', s_{t'-1}, s)$ and this was the $(n+1)$th deviation observed. We need to compare the gains and losses from the deviation. Individual i can by deviating maximally reach the upper bound of her utility given by $\bar{u}^i$ in the period of her deviation. In the succeeding T_{n+1} periods after the deviation: According to the definition of the strategies above she stays close her initial endowment. The $(n+1)$th deviation payoff is arbitrarily close to

$$(1-\lambda)\Big[\lambda^{t'-1}\overline{u}^i + \sum_{t=t'+1}^{t'+T_{n+1}} \lambda^{t-1} E_\omega[u_\xi^i\left(w_\xi^i\right)] \\ + \sum_{t=T_{n+1}+t'+1}^{T_{n+1}+t'+2} \lambda^{t-1}\overline{u} \\ + \sum_{t \geq T_{n+1}+t'+3} \lambda^{t-1} E_\omega[v_\xi^{i,(n+1)\text{dev}}]\Big]. \tag{5.2}$$

The long-run discounted payoff after the $(n+1)$th deviation consists of once a (maybe) very high payoff from deviating, then the payoff from a punishment phase lasting T_{n+1} periods, two periods of transition with a payoff of maximally $\overline{u}$ and finally the $(n+1)$th reward payoff.

By contrast, if the $(n+1)$th deviation did <u>not</u> take place, i's long-run payoff starting at time t' would be arbitrarily close to:

$$(1-\lambda)\Big[\sum_{t \geq t'} \lambda^{t-1} E_\omega[v_\xi^{i,n\text{dev}}]\Big]. \tag{5.3}$$

Therefore to show that (5.3) - (5.2) is positive it is enough to ensure that:

$$\underline{u} - 3\overline{u} + \sum_{t=t'+1}^{t'+T_{n+1}} \lambda^{t-t'-1} E_\omega[v_\xi^{i,n\text{dev}} - u_\xi^i\left(w_\xi^i\right)] + \varepsilon_n \Big[\sum_{t \geq T_{n+1}+t'+3} \lambda^{t-t'-1} \Big] > 0.$$

Note that since $v^{i,n\text{dev}}$ was assumed to be a payoff that results from a sequentially strictly individually rational allocation we have $E_\omega[v_\xi^{i,n\text{dev}} - u_\xi^i(w_\xi^i)] > 0$ for every $t \in \mathbb{N}$, for every individual $i \in \mathcal{N}$. Therefore define

$$g_\xi := \min_{i \in \mathcal{N}} E_\omega[v_\xi^{i,n\text{dev}} - u_\xi^i\left(w_\xi^i\right)]$$

Therefore, it is sufficient to require that:

$$\underline{u} - 3\overline{u} + \sum_{t=t'+1}^{t'+T_{n+1}} \lambda^{t-t'-1} g_\xi + \varepsilon_n \frac{\lambda^{T_{n+1}+2}}{1-\lambda} > 0.$$

It is easy to see that, whatever being the distance, $\underline{u} - 3\overline{u}$, and for every $\varepsilon_n > 0$, there exists some T_{n+1} big enough so that this last inequality is satisfied. Hence, deviating behavior is not profitable. This completes the proof. □

5.5 Incomplete Information

In this section, we turn to the general case where $\#\Omega \geq 1$. Players observe neither the choice of ω, nor that of ξ. They start with the same prior, $\mathbb{P}$, over Ω, but, along the play, they may (and, in general, they will) have different *interim* beliefs, depending upon the private information they receive.[19] Each household has *five* ways of updating its beliefs about the state ω over time.

- First, at node ξ, each player privately observes her own (random) spot endowment, w_ξ^i, which is chosen by nature according to the transition matrix ω.

[19]This is in accordance with the arguments provided by Heifetz (2006) showing that it makes hardly sense, within a game-theoretic setting, to assume that players start with distinct priors. Of course, Aumann's theorem implies that, along a play of the game it will not be common knowledge that traders have distinct *interim* beliefs.

- Second and third, at every node, after having played her action, each player observes public prices, p_ξ and π_ξ, together with her final allocation, x^i_ξ. Prices and final allocations depend upon the players' actions and vary in informativeness across action profiles: they only reveal the part of the privately hold information that players are ready to transmit through their bids and offers.

- Fourth, a trader may also learn about the state by observing her final stage-payoff, $u^i_\xi(x^i_\xi)$, which is selected according to ξ —given x^i_ξ.

- Finally, the return of the assets she owns in her portfolio (either as a creditor or as a debtor) also provide information about the realization of ξ, hence, about ω.

In order to cope with this differential information set-up, we shall need two key restrictions —Assumptions **G** and **IA**.

> **Assumption G.** The set of L consumption goods is partitioned into two subsets, $\mathcal{L} = \mathcal{L}_a \cup \mathcal{L}_c$ with $\mathcal{L}_a \cap \mathcal{L}_c = \emptyset$. Only commodities in $\mathcal{L}_c$ can be used as collateral, and assets' promises deliver only in commodities that belong to $\mathcal{L}_a$.

In other words, a commodity cannot serve both as a collateral and as a promise. We use this partition of the commodities to ensure that, during the play of the game, a single player cannot prevent the other individuals from learning the true state of the world, ω.

Along a play of the game, while endowments, utility payoffs and asset payoffs are observed privately, prices are publicly revealed. Notice that, given actions a_ξ, prices are entirely determined — i.e., there is no additional randomness on public signals, by contrast with Wiseman (2011) where public signals are random. Of course, the distribution matrix, ω, might be degenerate so that payoffs or returns are non-stochastic conditional on ω. In this case, the realization of payoffs and/or returns perfectly reveals the state of the world. On the contrary, if two distributions do have the same support, players may never be able to learn the true state for sure by just observing their private characteristics and the assets' returns.

Given the state of the world, ω, a strategy profile, $\sigma = (\sigma^i)$, induces a unique probability distribution on the space of sequences $(u^i_\xi(x^i_\xi(\sigma)), x^i_\xi(\sigma), w^i_\xi, A_j(\xi))_{i,j,\xi}$. Let us call this distribution, $\mathbf{P}_{\omega,\sigma}$. This is the distribution over signals from which players try to infer ω. For any two states $\omega \neq \omega'$, there must be at least some player i and some strategy profile $\sigma = (\sigma^i)_i$ such that the distributions induced by (ω, σ) and (ω', σ) over $(u^i_\xi(x^i_\xi(\sigma)), x^i_\xi(\sigma), w^i_\xi, A_j(\xi))_{i,j,\xi}$ differ on a set of positive measure. Two states of the world that yield almost surely the same payoff, final allocation, endowment and return distributions to *every* agent and *whatever* being the strategy played, can be treated as a single state. Therefore, there is no loss of generality in assuming that a complete sequence of stage-payoff profiles, $(u^i_\xi(\cdot))_{i,\xi}$, final allocations, $(x^i_\xi(\sigma))_{i,\xi}$, endowments, $(w^i_\xi)_{i,\xi}$, and asset returns, $(A_j(\xi))_{j,\xi}$, jointly identify the state statistically for at least one well-chosen strategy profile, σ. This does not mean, however, that, by observing her *own private* sequence of realized individual payoffs, endowments and asset returns, a single trader is able to learn the state of the world whatever being the strategy played. Neither need prices suffice to identify by themselves the state.[20] The following assumption is therefore, admittedly, a restriction: it says that, for every "reasonable" strategy profile, stage-payoffs, final allocations and asset returns plus individual endowments contain all the relevant information about ω. Illustrations of textbook models that satisfy this assumption were given in the Introduction of the paper.

Recall that, given some Markov chain ω, $\mu_\omega \in \Delta(\mathcal{S})$ is an *invariant measure* of ω if

$$\mu_\omega(s) = \sum_{s'} \omega_{s's} \mu_\omega(s') \quad \forall s \in \mathcal{S}.$$

Suppose that the Markov chain ω is irreducible and aperiodic.[21] Then, it admits an invariant measure if, and only if, every state of nature $s \in \mathcal{S}$ is

[20] When prices are interpreted as public signals, this generality contrasts with Wiseman (2005) where the sole observation of public signals suffices to identify the state with no ambiguity.

[21] A state $s \in \mathcal{S}$ has period k if any return to state s must occur in multiples of k steps. If $k = 1$, state s is said aperiodic. If every state $s \in \mathcal{S}$ is aperiodic, ω is said *aperiodic.* The Markov chain ω is *irreducible* if it is possible to connect every state $s \in \mathcal{S}$ with any other state $s' \in \mathcal{S}$ with positive probability.

positive recurrent.[22] In this case, μ_ω is unique.

Informativeness Assumption (IA)
(1) For any pair of nodes $(t, s_{t-1}, s) = \xi \neq \xi' = (t, s_{t-1}, s')$, any player i, and any strategy profile, σ, that induces an SSIRF allocation at both states, the vectors of signals, $(u^i_\xi(x^i_\xi(\sigma)), x^i_\xi(\sigma), w^i_\xi, A_j(\xi))$ and $(u^i_{\xi'}(x^i_{\xi'}(\sigma)), x^i_{\xi'}(\sigma), w^i_{\xi'}, A_j(\xi'))$ differ.
(2) Every ω is irreducible, aperiodic and admits an invariant measure, μ_ω. Moreover, for any pair ω, ω', if μ_ω and $\mu_{\omega'}$ are two corresponding invariant measures, then $\mu_\omega = \mu_{\omega'} \Rightarrow \omega \neq \omega'$.

(IA-1) says that, for a reasonable strategy profile, at the end of each period t, each player knows for sure at which node, $\xi = (t, s_{t-1}, s)$, she was playing. Of course, this is far from sufficient in order, for player i, to learn ω. (IA-2) is one way of saying that two states of the world induce different distributions over states of nature in the long-run. Since we are going to consider patient players, two Markov chains ω, ω' that would induce the same asymptotic distribution over signals on the long-run should be identified. The last section of the paper provides some hints about how this assumption can be weakened.

Definition 5.4 (Perfect Bayesian equilibrium). A pair

$$\Big((\sigma)_{i\in\mathcal{N}}, \big(\mathbb{P}^i_\xi(h^i_\xi)\big)_{i\in\mathcal{N}}\Big)$$

consisting of a feasible allocation and a system of private beliefs is a perfect Bayesian equilibrium (PBE) if

- $(\sigma)_{i\in\mathcal{N}}$ is sequentially rational given the private beliefs $\big(\mathbb{P}^i_\xi(h^i_\xi)\big)_{i\in\mathcal{N}}$, i.e., starting at any arbitrary node, given the continuation strategies of the other individuals, no individual can improve her utility by unilaterally changing her strategy profile given her private beliefs $\mathbb{P}^i_\xi(h^i_\xi)$,

[22] A state s is recurrent if, given that the chain starts in s, it will return to s in finite time with probability 1. s is *positive recurrent* if, in addition, the expectation of this hitting time is finite.

- and the private beliefs $\left(\mathbb{P}^i_\xi(h^i_\xi)\right)_{i\in\mathcal{N}}$ are updated via Bayes rule whenever it is possible.[23]

Our main result is that, for any strategy profile that yields an allocation of commodities, assets and collaterals that is SSIRF, there is a PBE in which, with arbitrarily high probability, every player achieves arbitrarily close to the allocation specified for the realized path, as long as households are patient enough. Moreover, along such an equilibrium path, every player learns the realized state with arbitrary precision.

Theorem 5.2. *Under (G) and (IA), let* $\varepsilon > 0$ *and* $(x^{*i}[\omega])_{i\in\mathcal{N},\omega\in\Omega}$ *be a* SSIRF *allocation in consumption goods, and let* $\mathbb{P}$ *be a prior belief that assigns strictly positive probability to each state of the world. Then there exists* $\lambda(\mathbb{P}) < 1$ *such that for all* $\lambda > \lambda(\mathbb{P})$*, there is a* PBE *that with probability at least* $1-\varepsilon$*, conditional on any state* ω *being realized, yields a payoff vector within* ε *of* $\left(U^i_{\mathbf{D}(\xi_0)}(x^{*i},\sigma,\omega)\right)_i$*. In equilibrium, conditional on* ω*, each player* i*'s* interim *private belief converges to the truth:* $\lim_{t\to\infty}\mathbb{P}^i_{\xi=(t,s_{t-1},s)}(h^i_\xi)[\omega] = 1$ *with probability* 1*.*

Proof of Theorem 2

Outline of the proof

The following sketch of the proof may serve as a lighthouse before plunging into the details.

The equilibrium path uses "blocks" of $M+T$ periods each. An *equilibrium block* has a "target allocation" in commodities, denoted by

$$\left((x^{*i}_\xi[\omega])_i\right)_{\tau(\xi)=1,\dots,M+T}$$

[23] Due to our assumptions on the Markov chain, every state of nature is reached with a strictly positive probability. Therefore, given the current state, Bayesian updating is always unambiguous. González-Díaz and Meléndez-Jiménez (2011) discuss the meaning of "whenever it is possible" for general extensive form games with incomplete information. In our special case their notion of a simple perfect Bayesian equilibrium coincides with usual perfect Bayesian equilibrium.

for each state ω. Note that, by definition, there exists a corresponding portfolio

$$\left(\left(\varphi_\xi^i[\omega], \theta_\xi^i[\omega]\right)_i\right)_{\tau(\xi)=1,\ldots,M+T}.$$

Within each equilibrium block, traders follow strategies that rely only on the history since the start of the block. In particular, they do not care about the history that happened before the beginning of the block. Rather, they rely on a *truncated belief*, $\mathbb{P}^i_{\xi\backslash\bar{\xi}}(h^i_\xi) \in \Delta(\Omega)$, defined as follows: Suppose that the block under scrutiny started at node $\bar{\xi} \leq \xi$. Given some history, h^i_ξ, consider the truncated history, $h^i_{\xi\backslash\bar{\xi}}$, containing only the information delivered from $\bar{\xi}$ to ξ^-. The truncated belief, $\mathbb{P}^i_{\xi\backslash\bar{\xi}}\left(h^i_\xi\right) \in \Delta(\Omega)$, is the resulting updated belief starting with prior $\mathbb{P}$ at node $\bar{\xi}$.

The first M periods are used in experimentation to learn the state of the world through assets' returns, initial endowments, final allocations, prices and individual stage-payoffs. The most likely state, $\hat{\omega}$, is identified according to $\mathbb{P}^i_{\xi\backslash\bar{\xi}}\left(h^i_\xi\right)$, and, in the remaining T periods, households choose a full action profile that yields a stream of final allocations close to the target,

$$\left((x_\xi^{*i}[\hat{\omega}])_i\right)_{\tau(\xi)=M+1,\ldots,M+T},$$

with utility payoffs close to

$$\left(u_\xi^i\left(x_\xi^{*i}[\hat{\omega}]\right)_i\right)_{\tau(\xi)=M+1,\ldots,M+T}.$$

If M is large enough to identify the true state with high probability, if $T/(M+T)$ is close to one, so that nearly all of the periods within the block are spent playing (close to) the target action profile, and if players are patient enough, then the expected allocation from the block when the realized state of the world is ω will be very close to the target allocation.

There are 3 types of blocks: an *equilibrium* block, a *punishment* block, and a *post-deviation* block. The initial block is equilibrium, as are all the subsequent blocks until the first deviation. If some deviation occurs during a block, it must impact prices to be profitable. Indeed, deviations that leave prices unchanged cannot modify the final allocation of goods at the end of the period, and hence cannot be profitable —a property which is specific to

Shapley-Shubik games. Since prices are public signals, however, profitable deviations are immediately noticed by all the investors. Of course, a player may also want to deviate not in order to improve her current payoff but with the purpose of modifying the beliefs of her opponents. It turns out that the unique way to achieve this second goal consists in preventing the players from observing the assets' returns by provoking some default. Recall that default is not strategic in this paper. It happens as soon as the value of the collateral falls down below that of the promise (cf. equation (D)). Hence, prices must be (strategically) perturbed by the deviator in order to induce a default that was not agreed upon. We shall see in the proof how to circumvent this difficulty.

As it can be only noticed via prices, in any case, a deviation remains anonymous, even when observed. Hence, punishment blocks cannot be player-specific. The next block starting after a deviation is therefore a collective punishment block. All subsequent blocks are post-deviation blocks, until a new deviation occurs. A deviation is immediately punished by switching to a punishment block.

The target allocation for each player in a punishment block, at node $\xi = (t, s_{t-1}, s)$, is made arbitrarily close to the initial endowment, w^i_ξ, in commodities and no-trade in financial assets. The stage-payoffs of the target allocations in the post-deviation blocks are chosen to be decreasing in the number of deviations so that $u^i_\xi\left(x^{i,n\text{dev}}_\xi[\omega]\right) < u^i_\xi\left(x^{i,(n-1)\text{dev}}_\xi[\omega]\right)$ for each node $\xi = (t, s_{t-1}, s)$ of the post-deviation block and each state of the world, ω —where n is the number of deviations already observed.[24] That is, the payoff to a deviator is lower than she would get in equilibrium, regardless of the state. A patient player, therefore, will not deviate, neither on, nor off the equilibrium path, regardless of her private beliefs.

In order to understand the need for such a block-decomposition, let us draw on an example (inspired from Wiseman, 2011). Suppose that the signals (endowments plus returns, prices, allocations and stage-payoffs) observed by the traders strongly suggest that the state of the world is A; but player's 1 private information at node ξ indicates state B more strongly.

[24]Of course, $u^i_\xi\left(x^{i,0\text{dev}}_\xi[\omega]\right) = u^i_\xi\left(x^{*i}_\xi[\omega]\right)$.

Player 1 believes that eventually everybody's belief will converge to a Dirac mass on state B if players continue to experiment and to learn but:

> 1) in the future, variables selected by equilibrium strategies turn out to yield the same signals in every state, so that no further learning occurs. This happens, for example, if, from ξ on, individual endowments no more depend upon the state, $\xi' > \xi$, selected by nature, while the equilibrium strategy asks traders to trade only, say, a riskless asset whose return does not provide any information at all.
>
> 2) The current market belief may put so little weight on state B that the expected time before convergence is very long, even whenever the equilibrium path does call for further experimentation.

Further, in state B, the equilibrium actions specified for state A may yield a lower stage-payoff to player 1 than her initial endowments in state B, i.e., than the actions designed to punish player 1 for a deviation in state A. And so, player 1 will deviate. In response, however, the other traders may conclude from observing unexpected prices that someone must have believed in state B, so that the market belief may adjust toward state B. Then, such a deviation may be profitable for player 1 *even when her private information is consistent with state* A, provided the punishment profile specified in state B gives her a higher payoff when the actual state is A than does the on-path profile specified in state A. This can occur, again, if player 1's post-deviation payoff in B is higher than the final allocation induced by the equilibrium strategy profile corresponding to state A. So, why should players different from 1 believe the anonymous deviator when she implicitly claims that the state is B by altering prices? Mimicking the colorful argument given by Aumann (1990, p.202) in an analogous context, players different from 1 could say: "Wait; we have a few minutes; let us think this over. Suppose that the deviator —whoever it is— doesn't trust her own claim, and so believes in state A. Then she would still want us to play as if we were in B, because that way she will get a better payoff. And of course, also if she does believe in B, it is better for him that we play as if we were in B. Thus she wants us to believe in B no matter what. It is as if there were no signal that 1 does not believe in A. So we will choose now what we would have chosen without any deviation from him."

The block-construction (borrowed from Wiseman 2011) aims at circumventing this kind of complications. Here are the details.

Proof

The proof consists in total of 8 steps. To give a quick overview, these are:

- Step 1: Given a target allocation we construct a sequence of allocations with utility payoffs below this allocation that will be used to construct a post-deviation payoff.
- Step 2: We define the δ-action profiles for the learning and punishment phase.
- Step 3: In order to start a learning phase, we define pre-M-transition action profiles.
- Step 4: To end a learning phase we define post-M-transition action profiles.
- Step 5: The block construction and the according action profiles are described.
- Step 6: The use of truncated beliefs is described and the choice of the length of the learning phase M is defined.
- Step 7: The length of the targeting period is chosen. In addition it is shown that the actual payoff is close to the target payoff.
- Step 8: It is shown that a deviation from the predescribed strategies is not profitable.

The details are following.

Step 1.

For each state $\omega \in \Omega$, let $(x_\xi^{*i}[\omega])_{i,\xi}$ be a SSIRF target allocation with stage-payoffs $(v_\xi^{*i}[\omega])_{i,\xi} := \left(u_\xi^i(x_\xi^{*i}[\omega])\right)_{i,\xi}$. Choose a sequence of payoff vectors $\left((v^{i,n\text{dev}}[\omega])_i\right)_{n\in\mathbb{N}}$ that result from SSIRF allocations, and such that: $v_\xi^{i,(n+1)\text{dev}}[\omega] < v_\xi^{i,n\text{dev}}[\omega]$ for every integer $n \in \mathbb{N}$, and every ξ —with $v^{i,0\text{dev}}[\omega] = v^{*i}[\omega]$ for each i. These utility levels will be the long-run payoffs after n deviations and can be constructed as:

$$v_\xi^{i,n\text{dev}}[\omega] := u_\xi^i\left(x_\xi^{i,n\text{dev}}[\omega]\right) \text{ with } x_\xi^{i,n\text{dev}}[\omega] := \rho_n x_\xi^{*i}[\omega] + (1-\rho_n) w_\xi^i, \quad \rho_n \in (0,1).$$

Assume that, for every $n \in \mathbb{N}$ and $\xi = (t, s_{t-1}, s) \in \mathbf{D}$:

$$0 < \varepsilon_n < v_\xi^{i,n\text{dev}}[\omega] - v_\xi^{i,(n+1)\text{dev}}[\omega] \tag{5.4}$$

for every player i and every state $\omega \in \Omega$. Notice that ε_n does not depend upon ξ, while the payoff $v_\xi^{i,n\text{dev}}[\omega]$ does. The sequences $(\rho_n)_n$ and $(\varepsilon_n)_n$ need to be chosen so as to converge sufficiently rapidly towards 0^+ (as $n \to +\infty$) for (5.4) to hold.

Step 2.

Let us now define a δ-*action* profile as follows.

Every player plays some action on the financial markets, so that everybody gets and sells a small quantity, $\delta > 0$, of every security. Consequently, all the commodities that are eligible as collaterals will have to be partially stored. Meanwhile, on the market for consumption goods that serve as a collateral, investors bid very large quantities and offer very small quantities. As a consequence, collateral commodity prices will be large. Let us choose them sufficiently large so that there will be no default along this part of the play. And still, the quantities of commodities that are going to be effectively trade can be made arbitrarily small, as well as the quantities of collaterals they have to put aside because of their trading in securities.

The δ-actions.

Formally, for node $\xi = (t, s_{t-1}, s) \in \mathbf{D}$ in period $\tau(\xi) = t \in \mathbb{N}$, a δ-action is defined as follows: Let $\delta > 0$ be small. Define the actions on the goods markets by

$$b^i_{\xi,\ell} := \begin{cases} \bar{b}_\ell > 0 \text{ large}, & \text{for } \ell \in \mathcal{L}_c \\ \frac{\delta}{N} & \text{for } \ell \in \mathcal{L}_a \end{cases}$$
$$q^i_{\xi,\ell} := \frac{\delta}{N} \quad \text{for } \ell \in \mathcal{L},$$

for all $i \in \mathcal{N}$ and on the asset markets by

$$\beta^i_{\xi,j} := \frac{\delta}{N},$$
$$\gamma^i_{\xi,j} := \frac{\delta}{N}$$

for all $j \in \mathcal{J}$, $i \in \mathcal{N}$.

It can easily be seen that these actions define feasible bids and offers and that the individual budget constraint is satisfied. The collateral requirement is equal to $\frac{\delta}{N} C_{j\ell}$, and hence as $\delta > 0$ is small, condition $(F1\xi)$ and $(F2\xi)$ are satisfied. Condition $(F3\xi)$ is trivially satisfied as well. For the budget feasibility note that, if in period $t-1$, everybody already played a δ-action, for the current period at node ξ the left-hand side of the individual budget constraint $(*^i_\xi 1)$ is equal to

$$\sum_{\ell=1}^{L} b^i_{\xi,\ell} + \sum_{j=1}^{J} \beta^i_{\xi,j} = \sum_{\ell \in \mathcal{L}_c} \bar{b}_\ell + \frac{L_a \delta}{N} + \frac{J\delta}{N}$$

and the right-hand side equals

$$\begin{aligned} &\sum_{\ell=1}^{L} p_{\xi,\ell} q^i_{\xi,\ell} + \sum_{j=1}^{J} \pi_{\xi,j} \gamma^i_{\xi,j} + \sum_{j=1}^{J} \left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right) D_{\xi,j} \\ &= \sum_{\ell \in \mathcal{L}_c} \frac{N \bar{b}_\ell}{\delta} \frac{\delta}{N} + L_a 1 \frac{\delta}{N} + J 1 \frac{\delta}{N} \\ &= \sum_{\ell \in \mathcal{L}_c} \bar{b}_\ell + \frac{L_a \delta}{N} + \frac{J\delta}{N}. \end{aligned}$$

Playing the δ-actions on asset markets every individual sells and offers the same amount of each security. Hence, net trades cancel so that no dividends will actually need to be paid.

Moreover condition $(*^i_\xi 2)$ is satisfied.

Now, what happens if player i deviates from a δ-action profile? She cannot prevent her opponents from observing their own private characteristics. Can she prevent the other players from observing the assets' returns? As she cannot prevent them from trading assets, choosing actions that induce default in all states might stop the learning process of the other players. Acting so as to decrease the price of the collateral commodities while at the same time increasing the price of the non-collateral commodities is the unique way to *cause* default. How can a single player achieve this goal? In order to decrease the price of the collateral commodities at time t, she can increase her offers on the commodity market for these goods. By doing so, she is physically constrained by her (finite) initial endowment: this is constraint $(F2\xi)$. In order to be able to increase the bids for the non-collateral commodities she could first use the money from the additional sales of the collateral commodities and, second, she might have some additional dividends from asset market transactions at time $t-1$. To satisfy the individual budget constraint $(*^i_\xi 1)$ at time $t-1$, hence to finance the asset purchases in that very period, she needs to make some additional asset sales which are again constraint by the availability of (finite) initial endowments that need to be used to put up for the collateral: this is constraint (F2ξ^-) —where ξ^- is the predecessor of node ξ. Hence, player i can neither increase the bids for non-collateral commodities arbitrarily high nor offer arbitrarily large quantities of collateral commodities. The influence on the price of player i is bounded. Thus, for each node ξ, there exists a lower bound on the bids $\bar{b}_\ell$ in the δ-action profile such that, if every trader bids above this bound, player i cannot induce default. From now on, a δ-action will always be understood to be such that every player's bid lies above $\bar{b}_\ell$.

Step 3.

The pre-M-transition actions.

If the asset holdings are strictly positive and if players want to switch to a δ-action profile at node ξ, there needs to be transition period to settle the asset market obligations. Otherwise, the δ-action profile might not be

budget feasible, i.e., might violate condition $(*^i_\xi 1)$.
For the pre-M-transition period at node ξ, define the following actions:

- on the commodity markets

$$b^i_{\xi,\ell} := \begin{cases} \sum_{j=1}^J \theta^i_{\xi^-,j} C_{j\ell} & \text{for } \ell \in \mathcal{L}_c \\ \frac{N}{\delta} & \text{for } \ell \in \mathcal{L}_a \end{cases}$$

$$q^i_{\xi,\ell} := \begin{cases} \sum_{j=1}^J \varphi^i_{\xi^-,j} C_{j\ell} & \text{for } \ell \in \mathcal{L}_c \\ \frac{\delta}{N} & \text{for } \ell \in \mathcal{L}_a \end{cases}$$

for all $\ell \in \mathcal{L}$, $i \in \mathcal{N}$.

- on the asset markets

$$\beta^i_{\xi,j} := \frac{\delta}{N},$$
$$\gamma^i_{\xi,j} := \frac{\delta}{N}$$

for all $j \in \mathcal{J}$, $i \in \mathcal{N}$.

The resulting prices are as follows:

$$p_{\xi,\ell} = \begin{cases} 1 & \text{for } \ell \in \mathcal{L}_c \\ \frac{N^2}{\delta^2} & \text{for } \ell \in \mathcal{L}_a \end{cases}$$
$$\pi_{\xi,j} = 1.$$

for $\ell \in \mathcal{L}$, $j \in \mathcal{J}$. Choose δ sufficiently small so that the prices of commodities used for the promises of assets are so large that all assets default in the transition period.

It is easy to verify that the pre-M-transition actions satisfy the feasibility constraints $(F1\xi)$, $(F2\xi)$ and $(F3\xi)$. For the individual budget constraint $(*^i_\xi 1)$ at node ξ we obtain for the left-hand side

$$\sum_{\ell=1}^L b^i_{\xi,\ell} + \sum_{j=1}^J \beta^i_{\xi,j} = \sum_{\ell \in \mathcal{L}_c} \sum_{j=1}^J \theta^i_{\xi^-,j} C_{j\ell} + \frac{L_a N}{\delta} + \frac{J\delta}{N}$$

and for the right-hand side

$$\begin{aligned}
&\sum_{\ell=1}^{L} p_{\xi,\ell} q^i_{\xi,\ell} + \sum_{j=1}^{J} \pi_{\xi,j}\gamma^i_{\xi,j} + \sum_{j=1}^{J} \left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right) D_{\xi,j} \\
&\quad = \sum_{\ell\in\mathcal{L}_c} 1 \sum_{j=1}^{J} \varphi^i_{\xi^-,j} C_{j\ell} + L_a \frac{N^2}{\delta^2}\frac{\delta}{N} + J1\frac{\delta}{N} + \sum_{j=1}^{J} \left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right)\left(\sum_{\ell\in\mathcal{L}_c} C_{j\ell}\right) \\
&\quad = \frac{L_a N}{\delta} + \frac{J\delta}{N} + \sum_{\ell\in\mathcal{L}_c}\sum_{j=1}^{J} \theta^i_{\xi^-,j} C_{j\ell}.
\end{aligned}$$

Moreover condition $(*^i_\xi 2)$ is satisfied.

From now on, unless otherwise stated, every block of δ-actions will always be preceded by the play of such transition actions.

Step 4.

The post-M-transition actions.

After the experimentation block, when a state, ω, has been identified, the individuals play actions (according to Lemma 5.2) so as to target a given allocation. This target allocation might require some holdings in certain assets which are not budget feasible given the δ-action played in the last experimentation period (e.g., a player may need to have saved much more money than she did according to the δ-action in order to finance her purchases according to the target allocation). Therefore, we add two periods of post-M-transition after the experimentation block where players can settle the asset holdings from the M-block (first post-M-transition period) and build up the necessary asset holdings for the target allocation (second post-M transition period).

Let the identified state be ω with target allocation $x^{*i}_\xi[\omega]$, together with actions, $\varphi^{*i}_\xi[\omega]$ and $\theta^{*i}_\xi[\omega]$, on the asset markets. The first post-M transition period is identical to a pre-M transition period (cf. *supra*). The second post-M-transition period at node ξ can be intuitively described as follows: People who have money from asset sales bid it on the goods markets, people who need money offer a tiny little bit of their endowment in order to get money. Commodity prices resulting from this action profile will be high, as only a little bit of commodity is offered. They turn out to be sufficiently high for every player to fulfill her budget constraint. The only point might be that some player is forced to sell a tiny little bit of her initial endowment while the collateral requirement associated with her asset sales requires her whole endowment vector to be collateralized. This would contradict (F2ξ). Thus, player are actually asked to sell a little bit less of assets than would be needed, were they to mimic exactly the target trades in assets. As a consequence, each player will save a small quantity of collateral that can be sold on the commodity market in order to fulfill her budget constraint. It turns out that the quantity of money lost by selling less assets can be compensated by the addition sale of commodities. More precisely,

- on the commodity markets, play:

$$b^i_{\xi,\ell} := \begin{cases} \frac{1}{\mathcal{L}} \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) & \text{if } \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$q^i_{\xi,\ell} := \begin{cases} -\delta \frac{1}{\mathcal{L}} \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) & \text{if } \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- on the asset markets:

$$\begin{aligned} \beta^i_{\xi,j} &:= \pi^*_{\xi,j}[\omega]\big(\theta^{*i}_{\xi,j}[\omega] - \eta^i_{\xi,j}\big), \\ \gamma^i_{\xi,j} &:= \varphi^{*i}_{\xi,j}[\omega] - \eta^i_{\xi,j} \end{aligned}$$

for all j, i, and where $\eta^i_{\xi,j} := \sum_\ell \frac{q^i_{\xi,\ell}}{C_{j\ell}}$ (with the usual convention $1/0 := 0$).

Since $x^{*i}_\xi[\omega]$ is feasible, the asset markets clear, $\sum_{i=1}^N \varphi^{*i}_\xi[\omega] = \sum_{i=1}^N \theta^{*i}_\xi[\omega]$. Therefore,

$$\begin{aligned} 0 &= \sum_{i=1}^N \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) \\ &= \sum_{\substack{i=1, \\ \varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega] > 0}}^N \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) + \sum_{\substack{i=1, \\ \varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega] < 0}}^N \sum_{j=1}^J \pi^*_{\xi,j}[\omega] \left(\varphi^{*i}_{\xi,j}[\omega] - \theta^{*i}_{\xi,j}[\omega]\right) \end{aligned}$$

Hence,

$$\sum_{\substack{i=1,\\ \varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]>0}}^{N} \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]\right) = -\sum_{\substack{i=1,\\ \varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]<0}}^{N} \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]\right).$$

The resulting prices are as follows:

$$\begin{aligned} p_{\xi,\ell} &= \frac{1}{\delta} \\ \pi_{\xi,j} &= \pi^*_{\xi,j}[\omega]. \end{aligned}$$

for $\ell \in \mathcal{L}$, $j \in \mathcal{J}$. Choose $\delta > 0$ sufficiently small.

It is easy to verify that the transition actions satisfy the feasibility constraints $(F1\xi)$, $(F2\xi)$ and $(F3\xi)$. For the asset trades note that $(F1\xi)$ holds, as $x^{*i}_{\xi}[\omega]$ is feasible.

- Let us check whether the individual budget constraint $(*^i_\xi 1)$ is satisfied at node ξ. If $\sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]\right) \geq 0$, we obtain for the left-hand side

$$\sum_{\ell=1}^{L} b^i_{\xi,\ell} + \sum_{j=1}^{J} \beta^i_{\xi,j} = \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]\right) + \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\theta^{*i}_{\xi,j}[\omega]$$

and for the right-hand side:

$$\begin{aligned} &\sum_{\ell=1}^{L} p_{\xi,\ell} q^i_{\xi,\ell} + \sum_{j=1}^{J} \pi_{\xi,j}\gamma^i_{\xi,j} + \sum_{j=1}^{J}\left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right) D_{\xi,j} \\ &\quad = \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\varphi^{*i}_{\xi,j}[\omega]. \end{aligned}$$

- If $\sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\varphi^{*i}_{\xi,j}[\omega]-\theta^{*i}_{\xi,j}[\omega]\right) \leq 0$, we obtain for the left-hand side:

$$\sum_{\ell=1}^{L} b^i_{\xi,\ell} + \sum_{j=1}^{J} \beta^i_{\xi,j} = \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\left(\theta^{*i}_{\xi,j}[\omega] - \eta^i_{\xi,j}\right),$$

and for the right-hand side:

$$\sum_{\ell=1}^{L} p_{\xi,\ell} q^i_{\xi,\ell} + \sum_{j=1}^{J} \pi_{\xi,j}\gamma^i_{\xi,j} + \sum_{j=1}^{J} \left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right) D_{\xi,j}$$
$$= \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega] \left(\theta^{*i}_{\xi,j}[\omega] - \varphi^{*i}_{\xi,j}[\omega] - \eta^i_{\xi,j}\right) + \sum_{j=1}^{J} \pi^*_{\xi,j}[\omega]\varphi^{*i}_{\xi,j}[\omega].$$

Thus, each budget constraint is satisfied. Finally, It is easy to see that these actions are tailored so that each player verifies the collateral constraint (F2ξ).

Step 5.

We now describe the within-block strategies.

The **equilibrium block** has length $M+T$. Suppose the true state of the world is ω. During the first M periods, play δ-actions (with a transition period if this is not the first equilibrium block of the whole play). During the first M periods of an equilibrium block, every trader is able to observe *all* assets' returns and, by combining this information with her own private initial endowments and stage-payoffs, updates her prior belief, $\mathbb{P}$. According to (IA), by choosing M long enough, the probability that each player puts a weight larger than $1-\varepsilon$ on the true state of the world, ω, can be made arbitrarily close to 1, whatever being $\varepsilon > 0$. More precisely, suppose that there exists a positive integer M such that, conditional on any of the finitely many states $\omega \in \Omega$, updating the prior $\mathbb{P}$ with the M signals that result from the δ-action profile yields a posterior truncated probability, $\mathbb{P}^i_{\xi\backslash\bar{\xi}}(h^i_\xi)$[25], for each player i, that puts weight strictly greater than $1/2$ on $\{\omega\}$ with probability at least $1-\varepsilon$. That such an integer M exists will be proven in Step 6 below.

Let $\hat{\omega}^i$ denote the state given the highest probability by player i under her own belief, $\mathbb{P}^i_{\xi\backslash\bar{\xi}}(h^i_\xi)$ (ties can be broken arbitrarily). Because of the choice of M (see above), the identified state $\hat{\omega}_i$ is identical across players i with probability at least $(1-\varepsilon)^N$. Indeed, this would be the probability

[25]The current equilibrium block is supposed to start at time $\tau(\bar{\xi}) = \bar{t} \in \mathbb{N}$.

according to which every player, having observed her own history, will put a weight greater than 1/2 on the true state, ω, if each history was drawn independently. Even if initial endowments and stage-payoffs were probabilistically independent, the assets' returns are certainly not independent. This correlation among histories can only increase the probability above according to which players reach a consensus on the true state.

Let us denote by $\hat{\omega}$ the state on which, with probability at least $1-\varepsilon$, players put the highest posterior probability at the end of the M-part of the block.[26] For the remaining T periods of the block, players start with a post-M-transition actions, and then play the profile that results in a stage-payoff $u_\xi^i\left(x_\xi^{*i}[\hat{\omega}]\right)$ for $\tau(\xi) = M+2, ..., M+T$ in state $\hat{\omega}$. The actions are constructed using Lemma 5.2 for every node ξ with $\tau(\xi) = M+3, ..., M+T$. Hence, during the first M periods of an equilibrium block, individuals are learning the true state of the world. In the last $T-2$ periods, where T is large relative to M, the target utility allocation is reached. If player i deviates unilaterally, then the equilibrium block ends immediately, and a punishment block begins in the next period. The lengths, M and T, will be chosen more precisely in steps 3 and 6.

After a deviation, a punishment phase is played, made of a certain number, P_n, of punishment blocks, each of length $M+T$, and the end of the current block. The number P_n depends on the number, n, of deviations observed. The construction of a **punishment block** is as follows. Players play *throughout* a δ-action profile as defined earlier (preceded by a transition period). This enables to learn during the punishment phase while keeping the size of net trades arbitrarily tiny. If any player unilaterally deviates from the punishment phase, then the punishment block dedicated to the first deviation ends immediately, and a new punishment phase (consisting in P_{n+1} blocks) begins in the next period. After the P_n punishment blocks, if no further deviation has been detected, players switch to a post-deviation block.

Play in a **post-deviation block** is divided into two parts. First, there are M periods of learning using the δ-action profiles, followed by a post-M-

[26] Ties can be broken by some arbitrary rule.

transition actions, and finally there are $T-2$ periods where action profiles are played, such that the target allocation after the nth deviation is reached. The target allocation in the $T-2$ last periods of a post-deviation block consists in playing a certain sequence of SSIRF allocations. Which allocations are targeted depends on the number of deviations already observed. For example after the first deviation in the $T-2$ last periods, the profile yielding $v_{\xi}^{i,1\text{dev}}[\hat{\omega}]$ in state $\hat{\omega}$ is played, for ξ with $\tau(\xi) = M+3, ..., M+T$. The first two periods after the M block is are post-M-transition actions. Compared to an equilibrium block, a post-deviation block consists as well of a learning phase of M periods and a target allocation in the last $T-2$ periods. The difference is that the second sub-block does not target the equilibrium allocation but rather SSIRF allocations that are strictly worse than the target allocations of the equilibrium block or the previous post-deviation block.

Step 6.

Play begins with an equilibrium block which is followed by a pre-M-transition period (for the settlement of assets' obligations) and another equilibrium block if no unilateral deviation was observed. A post-deviation-n block with no additional deviation is followed similarly by a pre-M-transition period and another post-deviation-n block. A punishment-n block (i.e., a punishment block devoted to the nth deviation) with no unilateral deviation is followed by a post-deviation-n block.

On the equilibrium path, each player's *private* belief, $\mathbb{P}_{\xi}^{i}(h_{\xi}^{i})$, is derived by Bayesian updating the prior, $\mathbb{P}$, using the information of her *private* history, h_{ξ}^{i}. At the same time, each player computes her truncated belief, $\mathbb{P}_{\xi\setminus\bar{\xi}}^{i}(h_{\xi}^{i})$ as defined earlier. This belief serves for the identification of the most likely state of the world, $\hat{\omega}$, according to which the allocation $x_{\xi}^{*i}[\hat{\omega}]$ is targeted during the last $T-2$ periods of the block. By construction, $\mathbb{P}_{\xi\setminus\bar{\xi}}^{i}(h_{\xi}^{i})$ is reset to the prior, $\mathbb{P}$, at the beginning of each block.
For a given ω, let the random variable, T_s^{ω}, be the first return time to state $s \in \mathcal{S}$:

$$T_s^{\omega} := \inf\{n \in \mathbb{N} \mid X_n^{\omega} = s\},$$

where $(X_n^\omega)_n$ stands for the stochastic process (with values in $\mathcal{S}$) corresponding to ω. The number

$$f_s^{(n),\omega} := \Pr(X_s^\omega = n)$$

is the probability that the process returns to state s for the first time after n steps. Since every state s is recurrent, it is easy to prove (and well known) that the expected number of visits to s is infinite, i.e.,

$$\sum_{n\in\mathbb{N}} p_{ss}^{(n),\omega} = \infty,$$

where $p_{ks}^{(n),\omega} := Pr(X_n^\omega = s \mid X_0^\omega = k)$ for any $(k,s) \in \mathcal{S}^2$.

Since the Markov chain, ω, is irreducible and such that all the states in $\mathcal{S}$ are positive recurrent, it admits a unique invariant measure, $\mu_\omega \in \Delta(\mathcal{S})$. The chain ω being aperiodic, the limit of the expected number, p_n, of visits of each state $s \in \mathcal{S}$ verifies:

$$\lim_{n\to+\infty} p_{ks}^{(n),\omega} = \frac{1}{E[T_s^\omega]} = \mu_\omega(s).$$

If M, the length of the experimentation block, is large enough, the probability that, for every i, $\mathbb{P}^i_{\xi\setminus\bar{\xi}}(h^i_\xi)$ puts the maximal probability on the true state, $\{\omega\}$, at the end of the block can be made arbitrarily large: by observing the realization of their random signals, the players can observe the realization of X_ω (IA-1), hence, can compute the empirical mean corresponding to the expected return time M_s of each state s. Hence, they can approximate $p_{ss}^{(n),\omega}$ with arbitrary accuracy. According to (IA-2), two different states of the world, ω, ω' will induce different invariant measures, $\mu_\omega, \mu_{\omega'}$. Thus, for M large enough, all the players will be able to distinguish between state ω and ω' with probability at least $1-\varepsilon$. As a consequence, all the players will learn the true state with probability at least $1-\varepsilon$. Let us denote by M_ε the smallest such integer (whose existence was announced in Step 5 above). The crucial observation is that M_ε is independent from the discount factor λ, since it concerns only the learning process. From now on, we suppose that $M \geq M_\varepsilon$.

Step 7.

It remains to choose T large enough so that each player's welfare loss (with respect to the benchmark $v_\xi^{*i}[\omega]$) can be compensated by a sufficiently long targeting period of length T, provided players are sufficiently patient.

By construction of the δ-actions and by definition of the targeting actions during the T-phase of an equilibrium block, the difference between $v_\xi^{*i}[\omega]$ and the actual payoff that accrues to player i at node ξ can be made lower than ε (for δ sufficiently small). Let us denote by $U^i_{\mathbf{D}(\xi_0)}(\sigma^*, \omega)$ the final overall payoff induced by the equilibrium strategy, and by $U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \omega)$, the final payoff induced by our equilibrium target allocation.[27]

During the learning phase (of length M) and the post-M transition of two periods of each equilibrium block, the maximal stage-utility loss is $\overline{u}$, while during the targeting phase (of length $T-2$), it is ε. Suppose $T-2 = QM$ for some integer Q. One has:

$$U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \omega) - U^i_{\mathbf{D}(\xi_0)}(\sigma^*, \omega) \leq (1-\lambda) \sum_{j=0}^{+\infty} \lambda^{jQM} \Big[\frac{1-\lambda^{M+2}}{1-\lambda} \overline{u} + \lambda^{M+2} \frac{1-\lambda^{QM}}{1-\lambda} \varepsilon \Big],$$

where the sum of the right-hand-side is taken over the sequence (indexed by j) of equilibrium blocks (of length $M+T = (Q+1)M$). Thus,

$$U^i_{\mathbf{D}(\xi_0)}(x^{*i}, \omega) - U^i_{\mathbf{D}(\xi_0)}(\sigma^*, \omega) \leq \frac{1-\lambda^{M+2}}{1-\lambda^{QM}} \overline{u} + \lambda^{M+2} \varepsilon.$$

For every every $\varepsilon > 0$, there exists some $Q_{\varepsilon,\lambda}$ large enough and some λ_ε close enough to 1, so that the right-hand-side of the last inequality is lower than ε. From now on, we assume that $Q \geq Q_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$.

Therefore, along the equilibrium path, players learn the true state with probability at least $1-\varepsilon$ and their final payoff will be within ε of the benchmark. It follows that a patient player prefers not to deviate even if the truncated belief, after a sequence of misleading signals calls for an action profile that she thinks will give her a very low payoff for the duration of the current block: at the start of the next block, the pseudo-belief will revert to

[27] The slight abuse of notations in the arguments of the overall utility should not create any confusion.

the prior, and with high likelihood, experimentation in the next blocks will reveal the true state of the world, and enable the other players to provide her with the equilibrium payoff or to effectively punish her in case of deviation.

Actions may reveal a piece of information about a player's private payoffs. For instance, by deviating, player i may induce a final allocation for player j different from the one that is prescribed at equilibrium. This different allocation may in turn provide j with some information in terms stage-payoff that was out of scope with the equilibrium allocation. And even during a punishment phase, a deviator might be tempted to keep talking with her opponents through the manipulation of their commodity allocations. Nevertheless, (IA) implies that, as long as they still observe every asset's return, all the players will learn the true state with arbitrary precision *whatever* being their stream of stage-payoffs.[28] By manipulating allocations (hence stage-payoffs), a player cannot prevent her opponents from eventually learning the true state of the world, ω.

Step 8.

It remains to choose P_n (the number of punishment blocks after n deviations) large enough so that no player has any incentive to deviate, neither on the equilibrium path, nor off this path, whatever her private belief about ω or her higher order beliefs (about others' beliefs). For this purpose, we need to guarantee that a post-deviation long-run discounted payoff never exceeds the equilibrium long-run discounted payoff. Suppose that the deviation occurs at node $\xi' = (t', s_{t'-1}, s')$, that it is the $(n+1)$th deviation observed during the play and that there are no further deviations at later nodes. It will at most yield $\overline{u}^i$ to player i. Then, the post-deviation payoff can be made ε-close to the following maximum:

$$(1-\lambda)\lambda^{t'-1}\Big[\overline{u}^i + \sum_{t=2}^{M+T} \lambda^{t-1} E_{\mathbf{P}^i_{\xi'}(\sigma)}[u^i_\xi\left(w^i_\xi\right)]$$

[28]In other words, players need to be able to observe the sequence of stage-payoffs resulting from *some* SSIRF allocation, plus asset returns and initial endowments. For a patient player, the choice of the *particular* sequence of SSIRF allocations is irrelevant.

$$+\sum_{k=1}^{P_{n+1}}\Big[\sum_{t=k(M+T)+1}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}[u^i_\xi\left(w^i_\xi\right)]$$
$$+\sum_{k\geq P_{n+1}+1}\Big[\sum_{t=k(M+T)+1}^{k(M+T)+M}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}[u^i_\xi\left(w^i_\xi\right)]$$
$$+\sum_{t=k(M+T)+M+1}^{k(M+T)+M+2}\lambda^{t-1}\overline{u}$$
$$+\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}[v^{i,(n+1)\text{dev}}_\xi[\omega]]\Big]. \tag{5.5}$$

Indeed, the long-run discounted payoff after a deviation consists of once a (maybe) very high payoff from deviating, then the payoff from a punishment during the current block plus P_{n+1} punishment blocks lasting $M+T$ periods and finally the payoff from succeeding post-deviation-$(n+1)$ blocks of $M+T$ periods including possibly a very high payoff in the post-M-transition period. On the other hand, since no deviator can prevent her opponents from learning the state of the world with arbitrary precision (even during the punishment phase and whatever being the behavior of the deviator), the reward payoff, $E_{\mathbf{P}^i_\xi(\sigma)}[v^{i,(n+1)\text{dev}}_\xi[\hat{\omega}]]$, computed with the most likely state, $\hat{\omega}$ (according to the players' truncated belief), can also be made arbitrarily close to $E_{\mathbf{P}^i_\xi(\sigma)}[v^{i,(n+1)\text{dev}}_\xi[\omega]]$.

By contrast, if the $(n+1)$th deviation did not take place, i's long-run discounted payoff would consist in the payoff from post-deviation-n blocks of $M+T$ periods. Therefore, it would be arbitrarily close to:

$$(1-\lambda)\lambda^{t'-1}\sum_{k\geq 0}\Big[\sum_{t=k(M+T)+1}^{k(M+T)+M}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}[u^i_\xi\left(w^i_\xi\right)]$$
$$+\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}[v^{i,n\text{dev}}_\xi[\omega]]\Big]. \tag{5.6}$$

Note that we assumed here payoff of 0 in the post-M-transition periods.

In order to check whether the difference (5.6) - (5.5) is positive, all we need is to ensure that:

$$(1-\lambda)\lambda^{t'-1}\Big[\underline{u}-\overline{u}+\sum_{k=1}^{P_{n+1}}\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{\xi'}(\sigma)}\Big[v^{i,n\mathrm{dev}}_{\xi}[\omega]-u^i_{\xi}\left(w^i_{\xi}\right)\Big]$$
$$+\varepsilon_n\sum_{k\geq P_{n+1}+1}\Big[\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}\Big]-\overline{u}\sum_{k\geq 0}\Big[\sum_{t=k(M+T)+M+1}^{k(M+T)+M+2}\lambda^{t-1}\Big]\Big]>0$$

Note that since $v^{i,n\mathrm{dev}}[\omega]$ results from a SSIRF allocation, we have

$$E_{\mathbf{P}^i_{\xi'}(\sigma)}[v^{i,n\mathrm{dev}}_{\xi}[\omega]-u^i_{\xi}\left(w^i_{\xi}\right)]>0$$

for every node $\xi=(t,s_{t-1},s)$, and every individual i. Let us define

$$g_{\xi}:=\min_{i\in\mathcal{N}}E_{\mathbf{P}^i_{\xi'}(\sigma)}[v^{i,n\mathrm{dev}}_{\xi}[\omega]-u^i_{\xi}\left(w^i_{\xi}\right)].$$

It is sufficient to require that:

$$(1-\lambda)\lambda^{t'-1}\Big[\underline{u}-\overline{u}+\sum_{k=1}^{P_{n+1}}\Big[\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}g_{\xi}\Big]$$
$$+\varepsilon_n\frac{(1-\lambda^{T-2})}{(1-\lambda)(1-\lambda^{M+T})}\lambda^{(P_{n+1}+1)(M+T)+M+2}$$
$$-\overline{u}\frac{(1-\lambda^2)}{(1-\lambda)(1-\lambda^{M+T})}\lambda^{M}\Big]>0,$$

It is easy to see that, whatever being the distance,

$$(1-\lambda)\underline{u}-\left((1-\lambda)+\frac{(1-\lambda^2)}{(1-\lambda^{M+T})}\lambda^{M}\right)\overline{u},$$

and for every $\varepsilon_n>0$, and every $\lambda\geq\lambda_{\varepsilon}$, there exists some integer $P^{\lambda,\varepsilon}_{n+1}$ big enough so that this last inequality is satisfied.

Suppose a deviator keeps deviating. While being punished by δ-actions, the most she can grasp is δ units of each commodity in each period. Immediately a new punishment starts with a punishment phase at least as long as the one before. As the reward in the post-deviation block declines, continuing deviating becomes even less attractive as the payoff in equation (5.5).

This completes the proof that it is in no player's interest to deviate from the prescribed equilibrium strategy, be it on the equilibrium path (i.e., whenever no deviation already occurred), or out of the equilibrium path (i.e., after a deviation occurred), provided: $M \geq M_\varepsilon,\ \lambda \geq \lambda_\varepsilon,\ T = QM$ with $Q \geq Q_\varepsilon$, and $\forall n,\ P_n \geq P_n^{\lambda,\varepsilon}$.

□

5.6 Concluding Comments

In this paper we investigated the general properties of perfect Bayesian equilibria in imperfectly competitive environments with incomplete information. We proved that adding collateral constraints within the rules of trading has an ambiguous effect. Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that players with incorrect beliefs never lose so much as to be driven out of the market. Consequently all agents, regardless of their beliefs, survive in the long run and continue to trade, possibly on the basis of those heterogeneous beliefs. Cao (2010) showed that the presence of heterogeneous beliefs together with collateral lead to additional leverage and asset price volatility (relative to a model with homogeneous beliefs or relative to equilibria in the complete markets economy). Our result suggests that this conclusion is partly due to his narrow (though standard) definition of perfect competition. Indeed, due to imperfect competition, those traders with incorrect beliefs can strategically *learn* the state of the world. We therefore provided a partial characterization of learning equilibria, at the end of which no player shares incorrect beliefs — not because they were eliminated from the market (although default is possible at equilibrium) but because they have taken time to update their prior belief. The striking point is that our (partial) Folk theorem provides us with a wide range of equilibria, many of them being first-best efficient, many others being dominated.

Let us end with a final remark concerning the link of the present work with the perfectly competitive set-up. In Giraud and Weyers (2004), as already mentioned, a first step towards the present Folk theorem had been obtained in the particular setting of *exogenously* incomplete markets (with finite horizon). Here, we get a Folk theorem for economies where missing markets are *endogenously* determined, due both to the presence of collateral constraints and to the lack of complete information. At the end of Giraud and Weyers (2004), however, the asymptotic properties of type-symmetric strategic equilibria were studied when the number of individuals of each type grows to infinity. It was shown that there is a discontinuity at the limit: Indeed, the limit-set of equilibria remains quite large while it is well-known that, at least with real assets, finite-horizon economies with incomplete markets generically admit a finite number of perfectly competitive equilibria (Duffie and Shafer 1985). An analogous remark holds in the present incomplete information set-up. Suppose that each type of player is actually represented by K identical individuals, and let $K \to +\infty$. The same argument as in Giraud and Weyers (2004) allows us to extend our partial Folk theorem to the asymptotic case. Therefore, we get that, at the limit, there is still a continuum of Bayesian perfect equilibria, exhibiting a large variety of efficiency properties (although each individual is negligible). It also suggests that, despite the considerable literature devoted to its foundation, the very concept of perfect competition itself deserves further investigation. In particular, whether it is captured as a price-taking assumption or else as the limit benchmark obtained by letting the weight of each price-maker shrink to zero does not lead to the same conclusion.

Our result and this last observation suggest that considerable care is necessary in invoking the impact of collateral regulation on the inefficiency of equilibria with private information —both in perfectly and imperfectly competitive environments.

5.7 Appendix

5.7.1 Proof of Lemma 5.1

To show Lemma 5.1 for our model, we modify the proof of Giraud and Weyers (2004) slightly.

Proof. Fix a node $\xi \in \mathbf{D}$ at time $t \leq T$. Since the allocation of initial endowments $(w^i_\xi)_i$ are Pareto-inefficient in the L-good spot economy there exists a consumption stream $(\bar{x}^i_\xi)_i$ that Pareto dominates $(w^i_\xi)_i$ and satisfies for every good $\ell \in \mathcal{L}$

$$\sum_{i=1}^{N} \bar{x}^i_{\xi,\ell} = \sum_{i=1}^{N} w^i_{\xi,\ell}.$$

By the strict monotonicity of the preferences, there exists a consumption stream $(\bar{x}'^i{}_\xi)_i$ such that

$$u^i_\xi(\bar{x}'^i_\xi) > u^i_\xi(\bar{x}^i_\xi) \quad i = 1, ..., N$$

and

$$\sum_{i=1}^{N} \bar{x}'^i_{\xi,\ell} = \sum_{i=1}^{N} w^i_{\xi,\ell}.$$

Since the utility functions are strictly increasing, there exists a hyperplane containing $(\bar{x}'^i_\xi)_i$ and $(w^i_\xi)_i$ with a strictly positive price vector p_ξ. Thus the individual budget restriction

$$p_\xi \cdot \bar{x}'^i_\xi = p_\xi \cdot w^i_\xi$$

is satisfied and furthermore

$$E_\omega[u^i_\xi(\bar{x}'^i_\xi)] > E_\omega[u^i_\xi(\bar{x}^i_\xi)] \geq E_\omega[u^i_\xi(w^i_\xi)]$$

for all $i \in \mathcal{N}$ and $t \leq T$. □

5.7.2 Proof of Lemma 5.2

To show Lemma 5.2 for our model, we modify the proof of Giraud and Weyers (2004).

Proof. Since $(\bar{x}^i)_{i\in N}$ is feasible there exist feasible and affordable allocation $(\bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ such that the asset markets clear at every node $\xi \in \mathbf{D}$. For all $j \in \mathcal{J}$ we have

$$\sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} = \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j}.$$

Therefore, if $\sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} = 0$, then $\sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} = 0$ and vice versa.

Using the market clearing condition on the goods markets we obtain from the definition of the actions

$$\begin{aligned} \sum_{i=1}^{N} q^i_{\xi,\ell} &= \sum_{i=1}^{N} \left(w^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi-,j} C_{j\ell} \right) \\ &= \sum_{i=1}^{N} \left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi,j} C_{j\ell} \right), \\ \sum_{i=1}^{N} b^i_{\xi,\ell} &= p_{\xi,\ell} \sum_{i=1}^{N} \left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi,j} C_{j\ell} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \bar{p}_{\xi,\ell} &= \frac{\sum_{i=1}^{n} b^i_{\xi,\ell}}{\sum_{i=1}^{n} q^i_{\xi,\ell}} \\ &= p_{\xi,\ell}. \end{aligned}$$

From the definition of the actions using the market clearing condition on the asset markets we obtain for the asset prices

- for $\sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} = \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} > 0$

$$\pi_{\xi,j} = \frac{\sum_{i=1}^{N} \beta^i_{\xi,j}}{\sum_{i=1}^{N} \gamma^i_{\xi,j}}$$

$$= \frac{\bar{\pi}_{\xi,j} \sum_{i=1}^{N} \bar{\theta}^i_{\xi,j}}{\sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j}}$$
$$= \bar{\pi}_{\xi,j}$$

- for $\sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} = \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} = 0$

$$\begin{aligned} \pi_{\xi,j} &= \frac{\sum_{i=1}^{N} \beta^i_{\xi,j}}{\sum_{i=1}^{N} \gamma^i_{\xi,j}} \\ &= \frac{\bar{\pi}_{\xi,j} \sum_{i=1}^{N} \frac{\delta}{N}}{\sum_{i=1}^{N} \frac{\delta}{N}} \\ &= \bar{\pi}_{\xi,j}. \end{aligned}$$

The final allocation of sales and of purchases for asset $j \in \mathcal{J}$ are given by

$$\begin{aligned} \varphi^i_{\xi,j} &= \gamma^i_{\xi,j}, \\ \theta^i_{\xi,j} &= \frac{\beta^i_{\xi,j}}{\pi_{\xi,j}}. \end{aligned}$$

The final allocation of good $\ell \in \mathcal{L}$ available for consumption after trading at node $\xi \in \mathbf{D}$ is given by

$$x^i_{\xi,\ell} = w^i_{\xi,\ell} + \sum_{j=1}^{J} \varphi^i_{\xi^-,j} C_{j\ell} - q^i_{\xi,\ell} + \frac{b^i_{\xi,\ell}}{p_{\xi,\ell}} - \sum_{j=1}^{J} \varphi^i_{\xi,j} C_{j\ell}$$

Therefore,

$$\begin{aligned} \varphi^i_{\xi,j} &= \begin{cases} \bar{\varphi}^i_{\xi,j} & \text{if } \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases} \\ \theta^i_{\xi,j} &= \begin{cases} \bar{\theta}^i_{\xi,j} & \text{if } \sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases} \\ x^i_{\xi,\ell} &= \begin{cases} \bar{x}^i_{\xi,\ell} & \text{if } \sum_{i=1}^{N} \bar{\varphi}^i_{\xi,j} = \sum_{i=1}^{N} \bar{\theta}^i_{\xi,j} > 0 \\ \bar{x}^i_{\xi,\ell} - \sum_{j=1}^{J} \frac{\delta}{N} C_{j\ell} & \text{otherwise} \end{cases} \end{aligned}$$

It remains to check that the budget constraint $(*^i_\xi 1)$ for the bids and offers is satisfied.

$$\sum_{\ell=1}^{L} b^i_{\xi,\ell} + \sum_{j=1}^{J} \beta^i_{\xi,j} \leq \sum_{\ell=1}^{L} p_{\xi,\ell} q^i_{\xi,\ell} + \sum_{j=1}^{J} \pi_j \gamma^i_{\xi,j} + \sum_{j=1}^{J} \left(\theta^i_{\xi^-,j} - \varphi^i_{\xi^-,j}\right) D_{\xi,j}$$

Inserting the assumed aci for $b^i_{\xi,\ell}$, $q^i_{\xi,\ell}$, $\gamma^i_{\xi,j}$ and $\beta^i_{\xi,j}$ we obtain for $(*^i_\xi 1)$

- for $\sum_{i=1}^N \bar{\theta}^i_{\xi,j} = \sum_{i=1}^N \bar{\varphi}^i_{\xi,j} > 0$

$$\sum_{\ell=1}^{L} \bar{p}_{\xi,\ell} \left(\bar{x}^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi,j} C_{j\ell} \right) + \sum_{j=1}^{J} \bar{\pi}_{\xi,j} \left(\bar{\theta}^i_{\xi,j} - \bar{\varphi}^i_{\xi,j}\right)$$
$$\leq \sum_{\ell=1}^{L} \bar{p}_{\xi,\ell} \left(w^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi^-,j} C_{j\ell} \right) + \sum_{j=1}^{J} \left(\bar{\theta}^i_{\xi^-,j} - \bar{\varphi}^i_{\xi^-,j}\right) D_{\xi,j}$$

  which holds since $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ was assumed to be a feasible allocation.

- for $\sum_{i=1}^N \bar{\theta}^i_{\xi,j} = \sum_{i=1}^N \bar{\varphi}^i_{\xi,j} = 0$

$$\sum_{\ell=1}^{L} \bar{p}_{\xi,\ell} \bar{x}^i_{\xi,\ell} \leq \sum_{\ell=1}^{L} \bar{p}_{\xi,\ell} \left(w^i_{\xi,\ell} + \sum_{j=1}^{J} \bar{\varphi}^i_{\xi^-,j} C_{j\ell} \right) + \sum_{j=1}^{J} \left(\bar{\theta}^i_{\xi^-,j} - \bar{\varphi}^i_{\xi^-,j}\right) D_{\xi,j}$$

As $(w^i_\xi)_i \gg 0$, this strategy profile is full. This completes the proof. □

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Short Curriculum Vitae

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Summary

This thesis consists of two main parts: The first one is on **coalitional market games** whereas the second one is on **strategic market games**. In *coalitional market games* the relationship between cooperative games and markets, and their respective solution concepts are investigated. In joint work with Jan-Philip Gamp we show the following results:

- For coalitional market games with transferable utility we present a detailed proof that extends the results of Shapley and Shubik (1975) to any closed convex subset of the core following a remark of these authors.
- For coalitional market games with non-transferable utility we extend the results of Qin (1993) to a large class of closed subsets of the inner core.
- Afterwards, we investigate the relationship between the inner core and asymmetric Nash bargaining solutions.

A *strategic market game* is a non-cooperative game that is used to describe the price formation in an exchange economy. In this thesis the departing point is the model in Giraud and Weyers (2004).

- For strategic market games with finite horizon, I show proving an analogue of a perfect folk theorem that even with collateral requirements almost everything is possible as soon as people are sufficiently patient.
- Finally, in joint work with Gaël Giraud, for strategic market games with infinite horizon and incomplete information we prove a partial folk theorem à la Wiseman (2011).

Keywords

Market Games, Coalitional Market Games, Competitive Payoffs, Core, Inner Core, Asymmetric Nash Bargaining Solutions, Strategic Market Games, Collateral, Folk Theorem, Finite Horizon, Infinite Horizon, Incomplete Information

Résumé en Français

Cette thèse comporte deux parties : La première partie porte sur les **jeux de marchés coopératifs** et la deuxième sur les **jeux de marchés stratégiques**. Dans le cas des *jeux de marchés coopératifs*, le lien entre jeux coopératifs et marchés et les concepts de solution associés sont étudiés. Établis en commun avec Jan-Philip Gamp nous avons montré les résultats suivants :

- Pour les jeux de marchés coopératifs à utilité transférable nous présentons une preuve qui généralise les résultats de Shapley et Shubik (1975) à des sous-ensembles convexes et fermés du coeur suivant une remarque des auteurs.

- Pour les jeux de marchés coopératifs à utilité non-transférable nous étendons les résultats de Qin (1993) à une large classe de sous-ensembles fermés du cœur interne.

- Ensuite, nous étudions la relation entre le cœur interne et les solutions de négociation asymétriques de Nash pour les jeux de négociation.

Un *jeu de marché stratégique* est un jeu non-coopératif utilisé pour décrire la formation des prix dans une économie d'échange. Dans cette thèse le point de départ est le modèle de Giraud et Weyers (2004).

- Pour les jeux de marchés stratégiques à horizon fini, je montre prouvant un théorème analogue à un théorème de folk, que même en présence d'obligation de collatéral, presque tout est possible tant que les joueurs sont assez patients.

- Finalement, dans un travail commun avec Gaël Giraud, pour les jeux de marché stratégique à horizon infini et avec de l'incertitude nous prouvons un thèoreme de folk partiel à la Wiseman (2011).

Mots clés

Jeux de Marchés, Jeux de Marchés Coopératifs, Paiements Compétitifs, Cœur, Cœur Interne, Solutions de Négociation Asymétriques de Nash, Jeux de Marchés Stratégiques, Collatéraux, Théorème de Folk, Horizon Fini, Horizon Infini, Information Incomplète

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In *coalitional market games* the relationship between cooperative games and markets, and their respective solution concepts are investigated. In joint work with Jan-Philip Gamp we show the following results: For coalitional market games with transferable utility we present a detailed proof that extends the results of Shapley and Shubik (1975) to any closed convex subset of the core following a remark of these authors. For coalitional market games with non-transferable utility we extend the results of Qin (1993) to a large class of closed subsets of the inner core. Afterwards, we investigate the relationship between the inner core and asymmetric Nash bargaining solutions.

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