SDE and BSDE on Hilbert spaces: applications to quasi-linear evolution equations and the asymptotic properties of the stochastic quasi-geostrophic equation

Rongchan Zhu

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Rongchan Zhu

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Abstract

In this thesis the following three related problems are considered.

1. We consider the following quasi-linear parabolic system of backward partial differential equations

$$(\partial_t + L)u + f(\cdot, \cdot, u, \nabla u\sigma) = 0 \text{ on } [0, T] \times \mathbb{R}^d \qquad u_T = \phi,$$

where L is a possibly degenerate second order differential operator with merely measurable coefficients. We solve this system in the framework of generalized Dirichlet forms and employ the stochastic calculus associated to the Markov process with generator L to obtain a probabilistic representation of the solution u by solving the corresponding backward stochastic differential equation. The solution satisfies the corresponding mild equation which is equivalent to being a generalized solution of the PDE. A further main result is the generalization of the martingale representation theorem using the stochastic calculus associated to the generalized Dirichlet form given by L. The nonlinear term f satisfies a monotonicity condition with respect to u and a Lipschitz condition with respect to ∇u.

2. We consider the following quasi-linear parabolic system of backward partial differential equations on a Banach space E

$$(\partial_t + L)u + f(\cdot, \cdot, u, A^{1/2}\nabla u) = 0 \text{ on } [0, T] \times E, \qquad u_T = \phi,$$

where L is a possibly degenerate second order differential operator with merely measurable coefficients. The results in 1 can be concluded in this case.

3. We study the 2D stochastic quasi-geostrophic equation in $\mathbb{T}^2$ for general parameter $\alpha \in (0, 1)$ and multiplicative noise. We prove it is uniquely ergodic provided the noise is non-degenerate for $\alpha > \frac{2}{3}$. In this case, the convergence to the (unique) invariant measure is exponentially fast. In the general case, we prove the existence of Markov selections.

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Contents

Chapter 0

Introduction

This thesis is devoted to stochastic differential equations (SDE) and backward stochastic differential equations (BSDE) on Hilbert spaces. In the mid 1940s Itô introduced the stochastic integral and stochastic integral equations. Since then, motivated by the demand from modern applications (e.g. physics, chemistry, biology and control theory), the theory of SDE has been well developed.

Roughly speaking, the solution of a stochastic differential equation is an adapted process X satisfying

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t; \quad X_0 = \xi,$$

where W is a Brownian motion. This is similar to the Cauchy problem of an ordinary differential equation. However, if we consider the terminal value problem for this stochastic equation and just take the time reversal of the solution of the SDE as a solution, the main problem lies in the adaptedness of the solution, which is essential to the definition of stochastic integral with respect to Brownian motion. This does not happen in the deterministic case. To solve this problem, Pardoux and Peng in [PP90] introduced the solution of a BSDE, which consists of a pair of adapted processes (Y, Z) satisfying

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t; \quad Y_T = \xi,$$

where ξ is the terminal condition. Since this type of equation appears in numerous problems in finance, the subject has become increasingly important and popular.

The existence and uniqueness of the solution of the BSDE with Lipschitz coefficients has been obtained by Pardoux and Peng in [PP90]. Later on, there have been a series of papers (c.f. [Pa99], [BDHPS03], [FT02], [BC08] and the references therein) extending their results for more general coefficients and more general state spaces. Important results concerning the link between those BSDEs and PDEs are also stated in Pardoux and Peng ([PP92]) (see below). The first aim of this thesis is to generalize their results in the framework of generalized Dirichlet forms.

BSDE and generalized Dirichlet forms: finite dimensional case

In Chapter 2 we consider the following quasi-linear parabolic system of backward partial differential equations

$$(\partial_t + L)u + f(\cdot,\cdot,u,\nabla u\sigma) = 0 \text{ on } [0,T]\times\mathbb{R}^d \qquad u_T = \phi, \tag{1.1}$$

where L is a second order linear differential operator and f is monotone in u and Lipschitz in ∇u and σ is the diffusion coefficient for the process associated with L. If L has sufficiently regular coefficients there is a well-known theory to obtain a probabilistic representation of the solutions to (1.1), using corresponding backward stochastic differential equations (BSDE) and also to solve BSDE with the help of (1.1), originally due to E. Pardoux and S. Peng ([PP92]). The main aim of this chapter is to implement this approach for a very general class of linear operators L, which are possibly degenerate, have merely measurable cofficients and are in general non-symmetric. Solving (1.1) for such general L is the first main task of this chapter (see Theorem 2.2.8). The second main contribution is to prove the martingale representation theorem (Theorem 2.3.8) for the underlying reference diffusions generated by such general operators L.

If f and the coefficients of the second-order differential operator L are sufficiently smooth, the PDE has a classical solution u. Consider $Y_t^{s,x} := u(t, X_t^{s,x})$, $Z_t^{s,x} := \nabla u\sigma(t, X_t^{s,x})$ where $X_t^{s,x}, s \le t \le T$, is the diffusion process with infinitesimal generator L which starts from x at time s and σ is the diffusion coefficient of X. Then, using Itô's formula one checks that $(Y_t^{s,x}, Z_t^{s,x})_{s\le t\le T}$ solves the BSDEs

$$Y_t^{s,x} = \phi(X_T^{s,x}) + \int_t^T f(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})dr - \int_t^T Z_r^{s,x} dB_r. \tag{1.2}$$

Conversely, by standard methods one can prove that (1.2) has a unique solution $(Y_t^{s,x}, Z_t^{s,x})_{s\le t\le T}$ and then $u(s,x) := Y_s^{s,x}$ is a solution to PDE (1.1). If f and the coefficients of L are Lipschitz continuous then a series of papers (e.g. [BPS05], [Pa99] and the reference therein) prove that the above relation between PDE (1.1) and BSDE (1.2) remains true, if one considers viscosity solutions to PDE (1.1). In both these approaches, since the coefficients are Lipschitz continuous, the Markov process X with infinitesimal operator L is a diffusion process which satisfies an SDE and so one may use its associated stochastic calculus.

In [BPS05] Bally, Pardoux and Stoica consider a semi-elliptic symmetric second-order differential operator L (which is written in divergence form) with measurable coefficients. They prove that the above system of PDE has a unique solution u in some functional space. Then using the theory of *symmetric* Dirichlet forms and its associated stochastic calculus, they prove that the solution $Y^{s.x}$ of the BSDE yields a precised version of the solution u so that, moreover, one has $Y_t^{s,x} = u(t, X_{t-s})$, P^x-a.s. In [S09], the analytic part of [BPS05] has been generalized to a non-symmetric case with L satisfying the *weak sector condition.* Here the weak sector condition means

$$((1-L)u, v) \le K((1-L)u, u)^{1/2}((1-L)v, v)^{1/2}, \text{ for } u, v \in \mathcal{D}(L),$$

for some constant $K > 0$, i.e. the non-symmetric part of the operator L can be dominated by the symmetric part. In [L01], A.Lejay considers the generator $L = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}) + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$ for bounded a, b. In [ZR11], T.S. Zhang and Q.K.Ran (see also [Z]) consider L of a more general form, but $a = (a_{ij})$ is required to be uniformly elliptic and $b \in L^p$ for $p > d$. Anyway, since L satisfies the weak sector condition in these cases, it generates a sectorial (i.e. a small perturbation of a symmetric) Dirichlet form, so the theory of Dirichlet forms from [MR92] can be applied in [L01], [Z], [ZR11].

In [St2] Stannat extends the known framework of Dirichlet forms to the class of generalized Dirichlet forms. By this we can analyze differential operators where the second order part may be degenerate and at the same time the first order part may be unbounded satisfying no global L^p-condition for $p \ge d$. The motivation for the first chapter is to extend the results in [BPS05] to the case, where L generates a generalized Dirichlet form so that we can allow the coefficients of L to be more general.

In Chapter 2, we consider PDE (1.1) for a non-symmetric second order differential operator L, which is associated to the bilinear form

$$\begin{aligned}\mathcal{E}(u,v) := &\sum_{i,j=1}^d \int a_{ij}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial v}{\partial x^j}(x)m(dx) + \int c(x)u(x)v(x)m(dx)\\ &+\sum_{i=1}^d \int \sum_{j=1}^d a_{ij}(x)(b_j(x) + \hat{b}_j(x))\frac{\partial u}{\partial x^i}v(x)m(dx) \; \forall u, v \in C_0^\infty(\mathbb{R}^d).\end{aligned} \tag{1.3}$$

where $C_0^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions with compact support. We stress that (a_{ij}) is not necessarily assumed to be (locally) strictly positive definite, but may be degenerate in general. When $b \equiv 0$, the bilinear form $\mathcal{E}$ satisfies the weak sector condition. For the perturbation term given by b we need

$b\sigma \in L^2(\mathbb{R}^d;\mathbb{R}^d,m)$, where $\sigma\sigma^* = a$ and σ^* is the transpose of the matrix of σ. That implies that we do not have the weak sector condition for the bilinear form. We use the theory of generalized Dirichlet forms and its associated stochastic calculus (cf [St1, St2, Tr1, Tr2]) to generalize the results in [BPS05]. Here m is a finite measure or Lebesgue measure on $\mathbb{R}^d$. If D is a bounded open domain, we choose m as $1_D(x)dx$. Then in certain cases the solution of PDE (1.1) satisfies the Neumann boundary condition. If we replace $C_0^\infty(\mathbb{R}^d)$ by $C_0^\infty(D)$, the solution of PDE (1.1) satisfies the Dirichlet boundary condition.

In the analytic part of Chapter 2, we do not need $\mathcal{E}$ to be a generalized Dirichlet form. We start from a semigroup (P_t) satisfying conditions (A1)-(A4), specified in Section 2.1 below. Such a semigroup can, however, be constructed from a generalized Dirichlet form. It can also be constructed by other methods (see e.g. [DR02]). Under conditions (A1)-(A4), the coefficients of L may be quite singular and only very broad assumptions on a and b are needed (see the examples in Sections 2.3 and 2.4).

Chapter 2 is organized as follows. In Sections 2.1 and 2.2, we use functional analytical methods to solve PDE (1.1) (see Theorems 2.2.8 and 2.2.11) in the sense of Definition 2.1.5, i.e. there are sequences $\{u^n\}$ which are strong solutions with data (ϕ^n, f^n) such that

$$\|u^n - u\|_T \to 0, \|\phi^n - \phi\|_2 \to 0, \lim_{n\to\infty} f^n = f \text{ in } L^1([0,T];L^2).$$

Here $\|\cdot\|_T := (\sup_{t\leq T}\|\cdot\|_2^2 + \int_0^T \mathcal{E}^{a,\hat{b}}_{c_2+1}(\cdot)dt)^{1/2}$, where $\mathcal{E}^{a,\hat{b}}$ is the summand in the left hand side of (1.3) with $b\equiv 0$. The above definition for the solution is equivalent to that of the following mild equation in L^2-sense

$$u(t,x) = P_{T-t}\phi(x) + \int_t^T P_{s-t} f(s,\cdot,u_s,D_\sigma u_s)(x)ds,$$

(see Proposition 2.1.9). If we use the definition of weak solution to define our solution as in [BPS05], uniqueness of the solution cannot be obtained since only $|b\sigma| \in L^2(\mathbb{R}^d;m)$. Furthermore, the function f in PDE (1.1) need not to be Lipschitz continuous with respect to the third variable; monotonicity suffices. And μ which appears in the monotonicity conditions (see condition (H2) in Section 2.2.2 below) can depend on t. f is, however, assumed to be Lipschitz continuous with respect to the last variable. We emphasize that the first order term of L cannot be incorporated into f without the condition that b is bounded. Hence we are forced to take it as part of L and hence have to consider a diffusion process X in (1.2) which is generated by an operator L associated with a (in general non-sectorial) generalized Dirichlet form. We also emphasize that under our conditions, PDE (1.1) cannot be tackled by standard monotonicity methods (see e.g. [Ba10]) because of the lack of a suitable Gelfand triple $V \subset H \subset V^*$ with V being a reflexive Banach space.

In Section 2.3, we extend the stochastic calculus of generalized Dirichlet forms in order to generalize the martingale representation theorem. In order to treat BSDE, we show in Theorem 2.3.8 that there exists a set of null capacity $\mathcal{N}$ outside of which the following representation result holds : for every bounded $\mathcal{F}_\infty$-measurable random variable ξ, there exists a predictable process $(\phi_1, ..., \phi_d) : [0,\infty) \times \Omega \to \mathbb{R}^d$, such that for each probability measure ν, supported by $\mathbb{R}^d \setminus \mathcal{N}$, one has

$$\xi = E^\nu(\xi|\mathcal{F}_0) + \sum_{i=0}^{d} \int_0^\infty \phi_s^i dM_s^{(i)} \qquad P^\nu - a.s..$$

where $M^i, i = 1, ..., d$ are the coordinate martingales associated with the process X. As a result, one can choose the exceptional set $\mathcal{N}$ such that if the process X starts from a point of $\mathcal{N}^c$, it remains always in this set. As a consequence we deduce the existence of solutions for the BSDE using the existence for PDE (1.1) in the usual way, however, only under P^m, because of our general coefficients of L (c.f. Theorem 2.3.12).

In Section 2.4, we employ the martingale representation to deduce existence and uniqueness for the solutions of BSDE (1.2). As a consequence, in Theorem 2.4.7, the existence and uniqueness of solutions for PDE (1.1), not covered by our analytic results in Section 2.2, is obtained by $u(s, x) = Y_s^s$, where Y_t^s is the solution of the BSDE. Moreover we have, $Y_t^s = u(t, X_{t-s}), P^x$-a.s., $x \in \mathbb{R}^d \setminus \mathcal{N}$. Further examples are given in Section 2.5.

BSDE and generalized Dirichlet form: infinite dimensional case

In Chapter 3, we consider the following quasi-linear parabolic system of backward partial differential equations on a (real) Banach space E

$$(\partial_t + L)u + f(\cdot, \cdot, u, A^{1/2}\nabla u) = 0 \text{ on } [0, T] \times E, \qquad u_T = \phi, \tag{1.4}$$

where L is a second order differential operator with measurable coefficients, ∇u is the H-gradient of u and $(H, \langle \cdot, \cdot \rangle_H)$ is a separable real Hilbert space such that $H \subset E$ densely and continuously. A is a symmetric, positive-definite and bounded operator on H. This equation is also called *nonlinear Kolmogorov equation* on an infinite dimensional space. In fact, in this chapter we study systems of PDE of type (1.4), i.e. u takes values in $\mathbb{R}^l$ for some fixed $l \in \mathbb{N}$. For simplicity, in this introductory section we explain our results in the case $l = 1$.

Various concepts of solution are known for (linear and) nonlinear parabolic equations in infinite dimensions. In Chapter 3 we will consider solutions in the sense of Definition 3.1.4, i.e. there is a sequence $\{u^n\}$ of strong solutions with data (ϕ^n, f^n) such that

$$\|u^n - u\|_T \to 0, \|\phi^n - \phi\|_2 \to 0 \text{ and } \lim_{n\to\infty} f^n = f \text{ in } L^1([0,T]; L^2).$$

We will prove the above definition for solution is equivalent to being a solution of the following mild equation in L^2 sense

$$u(t,x) = P_{T-t}\phi(x) + \int_t^T P_{s-t} f(s, \cdot, u_s, A^{1/2}\nabla u_s)(x) ds, \tag{1.5}$$

(see Proposition 3.1.7). This formula is meaningful provided u is even only once differentiable with respect to x. Thus, the solutions we consider are in a sense intermediate between classical and viscosity solutions.

The notion of viscosity solution, developed by many authors, in particular M. Crandall and P. L. Lions and their collaborators, is not discussed here. Generally speaking, the class of equations that can be treated by this method (c.f. [L88,L89,L92]) is much more general than those considered in this paper: it includes fully nonlinear operators. However, none of these results are applicable to our situation because the coefficients of the operator L are only measurable in our case.

In [FT02], mild solutions of the above PDE (1.4) have been considered, and a probabilistic technique, based on backward stochastic differential equations, has been used to prove the existence and uniqueness for the mild solution. Furthermore, their results has been extended in [BC08] and [M11]. All these results need some regular conditions for the coefficients of L and f to make sure that the process X has regular dependence on parameters, which are not required for our results. In Chapter 3, we will prove the existence and uniqueness of a solution u of (1.4) for a general non-symmetric operator L by methods from functional analysis (Theorem 3.2.8). In fact Chapter 3 is an extension of Chapter 2 to the infinite dimensional case. Though Chapter 2 serves as guideline, serious obstacles appear at various places if E is infinite dimensional, which we overcome in this work.

The connection between backward stochastic equations and nonlinear partial differential equations was proved for the finite dimensional case e.g. in [BPS05], [PP92] (see also the references therein). A further motivation of Chapter 3 is to give a probabilistic interpretation for the solutions of the above PDE's, i.e. in this infinite dimensional case.

If E is a Hilbert space, which equals to H, f and the coefficients of the second-order differential operator L are sufficiently regular, then PDE (1.4) has a classical solution and one may construct the pair of processes $Y_s^{t,x} := u(s, X_s^{t,x}), Z_s^{t,x} := A^{1/2}\nabla u(s, X_s^{t,x})$ where $X_s^{t,x}, t \leq s \leq T$, is the diffusion process with infinitesimal operator L which starts from x at time t and A is the diffusion coefficient for X. Then, using Itô's formula one checks that $(Y_s^{t,x}, Z_s^{t,x})_{t\leq s\leq T}$ solves the BSDE

$$Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \langle Z_r^{t,x}, dW_r\rangle_H, \tag{1.6}$$

where W_r is a cylindrical Wiener process in H. Conversely, for regular coefficients by standard methods one can prove that (1.6) has a unique solution $(Y_t^{s,x}, Z_t^{s,x})_{s\leq t\leq T}$ and then $u(s,x) := Y_s^{s,x}$ is a solution to PDE (1.4). If f and the coefficients of L are Lipschitz continuous then in [FT02] the authors prove that the probabilistic interpretation above remains true, if one considers mild solutions to PDE (1.4). There are many papers that study forward-backward systems in infinite dimension (cf [FT02], [FH07] and the references therein). In these approaches, since the coefficients are Lipschitz continuous, the Markov process X with infinitesimal operator L is a diffusion process which satisfies an SDE and so one can use its associated stochastic calculus to conclude the results.

In Chapter 3, we consider PDE (1.4) for a non-symmetric second order differential operator L in infinite dimensions, which is associated to the bilinear form

$$\mathcal{E}(u,v) = \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H d\mu(z) + \int \langle A(z)b(z), \nabla u(z)\rangle_H v(z)d\mu(z), u, v \in \mathcal{F}C_b^\infty,$$

where $\mathcal{F}C_b^\infty$ will be defined in Section 3.1. Here we only need $|A^{1/2}b|_H \in L^2(E;\mu)$. That is to say, in general the above bilinear form $\mathcal{E}$ does not satisfy any weak sector condition. We use the theory of generalized Dirichlet forms and the associated stochastic calculus(cf. [St1, St2, Tr1, Tr2]) to generalize the results in [BPS05].

In the analytic part of Chapter 3, we don't need $\mathcal{E}$ to be a generalized Dirichlet form. We start from a semigroup (P_t) satisfying conditions (A1)-(A3), specified in Section 3.1 below. Such a semigroup can e.g. be constructed from a generalized Dirichlet form. It can also be constructed by other methods (see e.g. [DR02]). Under conditions (A1)-(A3), the coefficients of L may be quite singular and only very broad assumptions on A and b are needed.

Chapter 3 is organized as follows. In Sections 3.1 and 3.2, we use functional analytical methods to solve PDE (1.4) (see Theorem 3.2.8) in the sense of Definition 3.1.4 or equivalently in the sense of (1.5). Here the function f need not to be Lipschitz continuous with respect to y; monotonicity suffices. And μ which appears in the monotonicity conditions (see condition (H2) in Section 3.2.2 below) can depend on t. f is, however, assumed to be Lipschitz continuous with respect to the last variable. We emphasize that the first order term with coefficient Ab of L cannot be incorporated into f unless it is bounded. Hence we are forced to take it as a part of L and hence we have to consider a diffusion process X which is generated by an operator L which is the generator of a (in general non-sectorial) generalized Dirichlet form. We also emphasize that under our conditions PDE (1.4) cannot be tackled by standard monotonicity methods (see e.g. [Ba10]) because of lack of a suitable Gelfand triple $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ with $\mathcal{V}$ being a reflexive Banach space.

In Section 3.3, we assume that $\mathcal{E}$ is a generalized Dirichlet form and is associated with a strong Markov process $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P^x)$. Such a process can be constructed if $\mathcal{E}$ is quasi-regular. We extend the stochastic calculus for the Markov process in order to generalize the martingale representation theorem. More precisely, in order to treat BSDE's, in Theorem 3.3.3 we show that there is a set $\mathcal{N}$ of null capacity outside of which the following representation theorem holds : for every bounded $\mathcal{F}_\infty$-measurable random variable ξ, there exists a predictable process ϕ : $[0,\infty) \times \Omega \to H$, such that for each probability measure ν, supported by $E \setminus \mathcal{N}$, one has

$$\xi = E^\nu(\xi|\mathcal{F}_0) + \sum_{i=0}^{\infty} \int_0^\infty \phi_s^i dM_s^i \qquad P^\nu - a.e.,$$

where $M^i, i \in \mathbb{N}$ are the coordinate martingales associated with the process X. In fact, one may choose the exceptional set $\mathcal{N}$ such that if the process X starts from a point of $\mathcal{N}^c$, it remains always in $\mathcal{N}^c$. As a consequence we deduce the existence of solutions for the BSDE using the existence of solutions for PDE (1.4) in the usual way, however, only under P^μ, because of our very general coefficients of L (c.f. Theorem 3.3.7).

In Section 3.4, we employ the above results to deduce existence and uniqueness for the solutions of the BSDE under P^x for $x \in \mathcal{N}^c$. As a consequence, in Theorem 3.4.4 one finds a version of the solution to PDE (1.4) which satisfies the mild equation pointwise, i.e. for the solution Y^s of the BSDE, we have $Y_t^s = u(t, X_{t-s})$, P^x-a.s. In particular, Y_t^t is P^x-a.s. equal to $u(t,x)$.

In Section 3.5, we give some examples of the operator L satisfying our general conditions (A1)-(A5). In Section 3.6, we consider an application of our results to a control problem. An admissible control $\theta(t,\omega)$ is a progressively measurable process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and takes values in some metric space K. Given a measurable function $c : [0,T] \times E \times K \to H$ and a admissible control θ, we define $N_t^\theta = \int_0^t c_s(X_s, \theta_s).dM_s, \Gamma_t^\theta = \exp(N_t^\theta - \frac{1}{2}\langle N^\theta \rangle_t)$, and $P^{\theta,x} = \Gamma^\theta.P^x$. The aim is to choose a control process θ, within a set of admissible controls, to minimize a cost functional of the form:

$$J^\theta(x) = E^{\theta,x}[\phi(X_T) + \int_0^T h(s, X_s, \theta_s)ds],$$

where ϕ and h are measurable functions and $E^{\theta,x}$ means taking expectation under $P^{\theta,x}$. There is a vast literature on such control problems in infinite dimensions if X is a solution of an SDE on a Hilbert space (c.f. [FT02] [G96] and the reference therein). In our case, the process X is generated by a linear operator L with merely measurable coefficients as above and X does not need to satisfy an SDE. As the coefficients of L are very general, X doesnot have regular dependence on parameters, which is essential in [FT02]. Moreover, we also donot need that ϕ and h are Gâteaux differentiable with respect to x. By the results in Sections 3 and 5, we directly provide a mild solution of the Hamilton-Jacobi-Bellman equation.

Ergodicity of the stochastic quasi-geostrophic equation

Up to the early 1960s, most works on SDE has been confined to ordinary differential equation. Later on, a large number of models were found that could be described by partial differential equations with random parameters, such as the coefficients or the forcing term. As a result, the study of SDE in infinite dimensional space has begun to attract a lot of attention of many researchers. In this thesis, we are concerned with the long time behavior of the stochastic quasi-geostrophic equation, which is an interesting SDE in infinite dimensional space.

In Chapter 4, we study the long time behavior of the stochastic partial differential equation by proving the uniqueness of invariant measures and strong asymptotic stability, i.e. the law of the process converges to the invariant measure in total variation norm. In order to have uniqueness of the invariant measure, the Markov process should satisfy some irreducibility property, together with some regularity. Here we prove the strong Feller property and the irreducibility of the associated Markov process. Then the classical results in the ergodic theory of Markov processes, as developed by Doob, Khas'minskii and others, can be applied to obtain the uniqueness of invariant measures as well as the strong asymptotic stability (see e.g. [DZ96]).

Consider the following 2D stochastic quasi-geostrophic equation in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\frac{\partial\theta(t,\xi)}{\partial t} = -u(t,\xi)\cdot\nabla\theta(t,\xi) - \kappa(-\triangle)^\alpha\theta(t,\xi) + (G(\theta)\eta)(t,\xi), \tag{1.7}$$

with initial condition

$$\theta(0,\xi) = \theta_0(\xi), \tag{1.2}$$

where $\theta(t,\xi)$ is a real-valued function of $\xi \in \mathbb{T}^2$ and $t \geq 0$, $0 < \alpha < 1, \kappa > 0$ are real numbers. u is determined by θ through a stream function ψ via the following relations:

$$u = (u_1, u_2) = (-R_2\theta, R_1\theta). \tag{1.9}$$

Here R_j is the j-th periodic Riesz transform and $\eta(t,\xi)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below. The case $\alpha = \frac{1}{2}$ is called the critical case, the case $\alpha > \frac{1}{2}$ sub-critical and the case $\alpha < \frac{1}{2}$ super-critical.

This equation is an important model in geophysical fluid dynamics. The case $\alpha = 1/2$ exhibits similar features (singularities) as the 3D Navier-Stokes equations and can therefore serve as a model case for the latter. In the deterministic case this equation has been intensively investigated because of both its mathematical importance and its background in geophysical fluid dynamics, (see for instance [CV06], [Re95], [CW99], [Ju03], [Ju04], [KNV07] and the references therein). In the deterministic case, the global existence of weak solutions has been obtained in [Re95] and one most remarkable result by [CV06] proves the existence of a classical solution for $\alpha = 1/2$ and the other by [KNV07] proves solutions for $\alpha = 1/2$ with periodic C^∞ data remain C^∞ for all the time.

In Chapter 4 we study the 2D stochastic quasi-geostrophic equation in $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and multiplicative noise. First using an abstract result for obtaining Markov selections from [GRZ09], we prove the existence of an a.s. Markov family for general parameter $\alpha \in (0,1)$ (see Theorem 4.2.5).

Then we prove the ergodicity of the solution in the subcritical case, provided that the noise is non-degenerate and regular (see Theorem 4.3.10). The proof follows from employing the weak-strong uniqueness principle in [FR08] (Theorem 4.3.4) and as usual first establishing the strong Feller property (Theorem 4.3.3). Though one would expect to get ergodicity for $\alpha > \frac{1}{2}$, surprisingly it turns out that one needs $\alpha > \frac{2}{3}$. As the dynamics exists only in the martingale sense and standard tools of stochastic analysis are not available, the computations are made for an approximating cutoff dynamics, which is equal to the original dynamics on a small random time interval. As the noise is non-degenerate, we can use the Bismut-Elworthy-Li formula to prove the strong Feller property. Since in our case $\alpha < 1$, it is more difficult to use the H^α-norm to control the nonlinear term even though the equation is on $\mathbb{T}^2$. To prove the weak-strong uniqueness principle we need some regularity for the trajectories of the noise. Therefore, we need conditions on G so that it is enough regularizing. However, in order to apply the Bismut-Elworthy-Li formula, we also need G^{-1} to be regularizing enough. As a result, $\alpha > 2/3$ is required (see Remark 4.3.2 below). It seems difficult to use the Kolmogorov equation method as in [DD03], [DO06] or a coupling approach as in [O08] in our situation (see Remark 4.3.2 below).

In order to prove the exponential convergence, we need to show decay of the solution's L^p-norm for suitable p. To prove it, we have to improve the crucial positivity lemma from [Re95] (see Lemma 4.4.1 below).

Chapter 4 is organized as follows. For the general case the existence of Markov selections is obtained in Section 4.2. In Section 4.3, we prove the ergodicity of the solution for $\alpha > 2/3$ provided the noise is non-degenerate. The exponential convergence to the (unique) invariant measure is shown in Section 4.4 (Theorem 4.4.5). We also consider the ergodicity of the equation driven by the mildly degenerate noise following the idea of [EH01] in Section 4.5 (Theorem 4.5.17).

Chapter 1

Preliminaries

In this chapter, we collect some results about the generalized Dirichlet form and the associated stochastic calculus for the following chapters. We omit all proofs and refer the reader to [St2, Tr1, Tr2] for details. In the first part, we recall the definitions of a generalized Dirichlet form and a quasi-regular generalized Dirichlet form. In the second part, we collect some useful results about the stochastic calculus associated with the generalized Dirichlet form, such as the Fukushima decomposition.

1.1 Some basic concepts for Generalized Dirichlet forms

Let us recall the definition of a generalized Dirichlet form from [St2]. Let E be a Hausdorff topological space and assume that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $C(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} := L^2(E, m)$ is a separable (real) Hilbert space. Let $(\mathcal{A}, \mathcal{V})$ be *a coercive closed form* on $\mathcal{H}$ in the sense of [MR92], i.e. $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $\mathcal{A} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a positive definite bilinear map, $\mathcal{V}$ is a Hilbert space with inner product $\widetilde{\mathcal{A}}_1(u, v) := \frac{1}{2}(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + (u, v)_{\mathcal{H}}$, and $\mathcal{A}$ satisfies the *weak sector condition*

$$|\mathcal{A}_1(u, v)| \leq K \mathcal{A}_1(u, u)^{1/2} \mathcal{A}_1(v, v)^{1/2},$$

$u, v \in \mathcal{V}$, with sector constant K. We will always denote the corresponding norm by $\|\cdot\|_{\mathcal{V}}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain that $\mathcal{V} \to \mathcal{H} \cong \mathcal{H}' \to \mathcal{V}'$ densely and continuously.

Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following assumptions:

(i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup of contractions $(U_t)_{t \geq 0}$ on $\mathcal{H}$.

(ii) $\mathcal{V}$ is Λ-admissible, i.e. $(U_t)_{t\geq}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Let $(\Lambda, \mathcal{F})$ with corresponding norm $\|\cdot\|_{\mathcal{F}}$ be the closure of $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ as an operator from $\mathcal{V}$ to $\mathcal{V}'$ and $(\hat{\Lambda}, \hat{\mathcal{F}})$ be its dual operator.

Let

$$\mathcal{E}(u,v) = \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in \mathcal{V}, v \in \hat{\mathcal{F}}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$ and $\langle \cdot, \cdot \rangle$ coincides with the inner product $(\cdot, \cdot)_H$ in H when restricted to $H \times \mathcal{V}$. Define $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$. We call $\mathcal{E}$ the bilinear form associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$.

Definition 1.1 The bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called *a generalized Dirichlet form*, if

$$u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathcal{E}(u, u - u^+ \wedge 1) \geq 0.$$

We also recall the definition of semi-Dirichlet form from [MOR95]. For the closed coercive form $(\mathcal{A}, \mathcal{V})$ is called a *semi-Dirichlet form* if $u \in \mathcal{V}$, $u^+ \wedge 1 \in \mathcal{V}$ and $\mathcal{A}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$.

Suppose the adjoint semigroup $(\hat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can also be restricted to a C_0-semigroup on $\mathcal{V}$. Let $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$ denote the generator of $(\hat{U}_t)_{t\geq 0}$ on $\mathcal{H}$, $\hat{\mathcal{A}}(u,v) := \mathcal{A}(v,u), u, v \in \mathcal{V}$ and let the coform $\hat{\mathcal{E}}$ be defined as the bilinear form associated with $(\hat{\mathcal{A}}, \mathcal{V})$ and $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$.

In [St2, Section I.3], they construct the resolvent $(G_\alpha)_{\alpha>0}$ such that for all $\alpha > 0$, $\mathcal{E}(G_\alpha f, v) = (f, v)_{\mathcal{H}}, \forall f \in \mathcal{H}, v \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the *resolvent associated with* $\mathcal{E}$. Let $(\hat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$. $(\hat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with* $\mathcal{E}$. By [St2, Proposition 3.6] $(G_\alpha)_{\alpha>0}$ is a strongly continuous contraction resolvent on $\mathcal{H}$.

For the generalized Dirichlet form, we also have the concept of the quasi-regular generalized Dirichlet form. By this we can construct a strong Markov process associated with it. This will be used in the probabilistic part of this chapter (see Section 2.3). We recall the definition of the quasi-regular generalized Dirichlet form here. For this reason we introduce some useful notations.

An element of u of $\mathcal{H}$ is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \leq u$ (resp. $\beta \hat{G}_{\beta+1} u \leq u$) for all $\beta \geq 0$. Let $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$.

Definition 1.2 (i) An increasing sequence of closed subset $(F_k)_{k\geq 1}$ is called an $\mathcal{E}$-nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$, $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$, where $u_{F_k^c} := e_{u \cdot 1_{F_k^c}}$ is the 1-reduced function defined in [St2, Definition III.1.8].

(ii) A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E\backslash F_k$.

(iii) A property of points in E holds $\mathcal{E}$-quasi-everywhere($\mathcal{E}-q.e.$) if the property holds outside some $\mathcal{E}$-exceptional set.

(iv) A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.) if there exists an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$, such that $\cup_{k\geq 1} F_k \subset E\backslash N$ and $f|_{F_k}$ is continuous for all k.

Definition 1.3 The generalized Dirichlet form $\mathcal{E}$ is called *quasi-regular* if:

(i) There exists an $\mathcal{E}$-nest consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exist $u_n \in \mathcal{F}, n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E\backslash N$.

1.2 Stochastic calculus associated with Generalized Dirichlet forms

In this section we assume that an m-tight special standard process ([MR92, IV Definition 1.13]) $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P^x)$ is properly associated in the resolvent sense with the quasi-regular generalized Dirichlet form $\mathcal{E}$, i.e. $R_\alpha f := E^x \int_0^\infty e^{-\alpha t} f(X_t)dt$ is an $\mathcal{E}$-quasi-continuous m-version of $G_\alpha f$, where $G_\alpha, \alpha > 0$ is the resolvent of $\mathcal{E}$ and $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; m)$. The coform $\hat{\mathcal{E}}$ introduced in Section 1.1 is a generalized Dirichlet form with the associated resolvent $(\hat{G}_\alpha)_{\alpha>0}$ and there exists an m-tight special standard process properly associated in the resolvent sense with $\hat{\mathcal{E}}$. In this section we will obtain the results under this assumption.

Now we recall [Tr2, Theorem 1.9], which give a description of the $\mathcal{E}$-exceptional set and will be used for the proof of the martingale representation theorem. $\hat{P}_{\hat{G}_1 b\mathcal{H}^+}$ denotes the set of all 1-coexcessive elements in $\mathcal{V}$ which are dominated by elements of $\hat{G}_1 b\mathcal{H}^+$, where $\hat{G}_1 b\mathcal{H}^+ := \{\hat{G}_1 h | h \in b\mathcal{H}^+\}$. $\widetilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all the $\mathcal{E}$-q.e. m-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$.

By [Tr2, Theorem 1.4], we obtain for $\hat{u} \in \hat{P}_{\hat{G}_1 b\mathcal{H}^+}$, there exists a unique σ-finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set such that for all $\widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$,

$$\int \widetilde{f} d\mu_{\hat{u}} = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha \hat{G}_{\alpha+1}\hat{u}).$$

Define

$$\hat{S}_{00} := \{\mu_{\hat{u}} | \hat{u} \in \hat{P}_{\hat{G}_1 b\mathcal{H}^+} \text{ and } \mu_{\hat{u}}(E) < \infty\}.$$

Then we have the following results from [Tr2, Theorem 1.9].

Theorem 1.4 For $B \in \mathcal{B}(E)$, B is $\mathcal{E}$-exceptional if and only if $\nu(B) = 0, \forall \nu \in \hat{S}_{00}$.

Definition 1.5 A positive measure μ on $(E, \mathcal{B}(E))$ is said to be of finite 1-order co-energy integral if there exists $\hat{U}_1\mu \in \mathcal{V}$, such that

$$\int_E \widetilde{G_1 h} d\mu = \mathcal{E}_1(G_1 h, \hat{U}_1\mu),$$

for all $h \in \mathcal{H}$ and for all $\mathcal{E}$-q.c. m-versions $\widetilde{G_1 h}$ of $G_1 h$. The measures of finite 1-order co-energy integral are denoted by $\hat{S}_0$.

By [Tr2, Section 1.3], $\hat{S}_{00} \subset \hat{S}_0$.

Now we introduce the spaces which will be relevant for our further investigations.

Definition 1.6 A family $(A_t)_{t\geq 0}$ of extended real valued functions on Ω is called an *additive functional* of X if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a defining set $\Lambda \in \mathcal{F}_\infty$ and an $\mathcal{E}$-exceptional set $N \subset E$, such that $P^z[\Lambda] = 1$ for all $z \in E\backslash N, \theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0, |A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

Define

$$\begin{aligned}\mathcal{M} := \{M | M &\text{ is a finite additive functional, } E^z[M_t^2] < \infty, E^z[M_t] = 0 \\ &\text{for } \mathcal{E} - q.e.z \in E \text{ and all } t \geq 0\}.\end{aligned}$$

$M \in \mathcal{M}$ is called a *martingale additive functional*(MAF). Furthermore, define

$$\dot{\mathcal{M}} = \{M \in \mathcal{M} | e(M) < \infty\}.$$

Here $e(M) = \frac{1}{2}\lim_{\alpha\to\infty}\alpha^2 E^m[\int_0^\infty e^{-\alpha t} M_t^2 dt]$. The elements of $\dot{\mathcal{M}}$ are called martingale additive functional's (MAF) of finite energy. Let $M \in \mathcal{M}$. There exists an $\mathcal{E}$-exceptional set N, such that $(M_t, \mathcal{F}_t, P_z)_{t\geq 0}$ is a square integrable martingale for all $z \in E\backslash N$. Moreover, there exists a unique (up to equivalence) positive continuous additive functional $\langle M\rangle$, called the sharp bracket of M, such that $(M_t^2 - \langle M\rangle_t, \mathcal{F}_t, P_z)_{t\geq 0}$ is a martingale for all $z \in E\backslash N$. By [Tr1, Theorem 2.10] $\dot{\mathcal{M}}$ is a real Hilbert space with inner product e. It now follows that one half of the total mass of the Revuz measure $\mu_{\langle M\rangle}$ associated to the sharp bracket of $M \in \mathcal{M}$ is equal to the energy of M, i.e.

$$e(M) = \frac{1}{2}\int d\mu_{\langle M\rangle}.$$

For $M, L \in \dot{\mathcal{M}}$ let

$$\langle M, L\rangle_t = \frac{1}{2}(\langle M+L\rangle_t - \langle M\rangle_t - \langle L\rangle_t).$$

Then the finite signed measure $\mu_{\langle M,L\rangle}$ defined by $\mu_{\langle M,L\rangle} = \frac{1}{2}(\mu_{\langle M+L\rangle} - \mu_{\langle M\rangle} - \mu_{\langle L\rangle})$ is the Revuz measure related to $\langle M, L\rangle$. We also define

$$e(M,L) = \frac{1}{2}\lim_{\alpha\to\infty}\alpha^2 E^m[\int_0^\infty e^{-\alpha t}M_t L_t dt].$$

Define the following space:

$$\mathcal{N}_c := \{N|N \text{ is a finite continuous additive functional, } e(N) = 0, E^z[|N_t|] < \infty \\ \text{for } \mathcal{E} - q.e. z \in E \text{ and all } t \geq 0\}.$$

Now we recall the well-known Fukushima decomposition in the framework of generalized Dirichlet forms.

Theorem 1.7 ([Tr1, Theorem 4.5]) If $\hat{G}_\alpha$ is sub-Markovian and strongly continuous on $\mathcal{V}$, then for $u \in \mathcal{F}$, there exists a unique $M^{[u]} \in \dot{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that

$$u(X) - u(X_0) = M^{[u]} + N^{[u]}.$$

Furthermore, by [Tr2, Lemma 2.12], we obtain that for $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $M \in \dot{\mathcal{M}}$, there exists a unique element denoted by $f \cdot M \in \dot{\mathcal{M}}$ such that for all $L \in \dot{\mathcal{M}}$

$$\frac{1}{2}\int f d\mu_{\langle M,L\rangle} = e(f\cdot M, L).$$

Chapter 2

BSDE and generalized Dirichlet form: finite dimensional case

In this chapter we establish that the relation between PDE (1.1) and BSDE (1.2) mentioned in introduction holds under the condition that the operator L is associated with a generalized Dirichlet form. In Section 2.1 we give some basic assumptions on the operator L and prove some basic relations for linear equation. In Section 2.2, we use analytic methods to solve PDE (1.1). In Section 2.3, we prove the martingale representation theorem for the process associated with the operator L. By this we obtain the existence and uniqueness of solution of BSDE (1.2) in Section 2.4. The relation between PDE and BSDE is also established in this section. Further extensions and examples are given in Section 2.5. The main results of this chapter have already been submitted for publication, see [Zhu a].

2.1 Preliminaries

Let $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^k$ be a measurable map. Then there exists a measurable map $\tau : \mathbb{R}^d \mapsto \mathbb{R}^k \otimes \mathbb{R}^d$ such that

$$\sigma\tau = \tau^*\sigma^*, \qquad \tau\sigma = \sigma^*\tau^*, \qquad \sigma\tau\sigma = \sigma,$$

where σ^* is the transpose of the matrix of σ(see e.g. [BPS05, Lemma A.1]). Then $a := \sigma\sigma^* = (a_{ij})_{1\le i,j\le d}$ takes values in the space of symmetric non-negative definite matrices. Let also $b : \mathbb{R}^d \to \mathbb{R}^d$ be measurable. Assume that the basic measure $m(dx)$ for the generalized Dirichlet form, to be defined below, is a finite measure or Lebesgue measure on $\mathbb{R}^d$.

Denote the Euclidean norm and the scalar product in $\mathbb{R}^d$ by $|\cdot|$, $\langle\cdot,\cdot\rangle$ respectively, while on the space of matrices $\mathbb{R}^d \otimes \mathbb{R}^k$ we use the trace scalar product and its associated norm, i.e., for $z = (z_{ij}) \in \mathbb{R}^d \otimes \mathbb{R}^k$, $\langle z_1, z_2 \rangle = \mathrm{trace}(z_1 z_2^*)$, $|z| = (\sum_{i=1}^d \sum_{j=1}^k z_{ij}^2)^{1/2}$. Let L^2, $L^2(\mathbb{R}^d;\mathbb{R}^k)$ denote $L^2(\mathbb{R}^d, m)$, $L^2(\mathbb{R}^d, m; \mathbb{R}^k)$ respectively. $(\cdot,\cdot)$ denotes the L^2-inner product. For $1 \leq p \leq \infty$, $\|\cdot\|_p$ denotes the usual norm in $L^p(\mathbb{R}^d; m)$. If W is a function space, we use bW to denote the bounded function in W.

Furthermore, let $a_{ij}, \sum_{j=1}^d a_{ij}b_j, \sum_{j=1}^d a_{ij}\hat{b}_j \in L^1_{\mathrm{loc}}(\mathbb{R}^d, m)$ and $c \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^+; m)$. We introduce the bilinear form

$$\begin{aligned}\mathcal{E}(u,v) :=& \sum_{i,j=1}^d \int a_{ij}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial v}{\partial x^j}(x)m(dx) + \int c(x)u(x)v(x) \\ &+ \sum_{i=1}^d \int \sum_{j=1}^d a_{ij}(x)(b_j(x)+\hat{b}_j(x))\frac{\partial u}{\partial x^i}v(x)m(dx)\ \forall u,v \in C_0^\infty(\mathbb{R}^d).\end{aligned}$$

Consider the following conditions:

(A1) The bilinear form

$$\mathcal{E}^a(u,v) = \sum_{i,j=1}^d \int a_{ij}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial v}{\partial x^j}(x)m(dx)\ \forall u,v \in C_0^\infty(\mathbb{R}^d),$$

is closable on $L^2(\mathbb{R}^d, m)$.

Define $\mathcal{E}^a_1(\cdot,\cdot) := \mathcal{E}^a(\cdot,\cdot) + (\cdot,\cdot)$. The closure of $C_0^\infty(\mathbb{R}^d)$ with respect to $\mathcal{E}^a_1$ is denoted by F^a. Then $(\mathcal{E}^a, F^a)$ is a well-defined symmetric Dirichlet form on $L^2(\mathbb{R}^d, m)$.

For the bilinear form

$$\begin{aligned}\mathcal{E}^{a,\hat{b}}(u,v) :=& \sum_{i,j=1}^d \int a_{ij}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial v}{\partial x^j}(x)m(dx) + \int c(x)u(x)v(x) \\ &+ \sum_{i=1}^d \int \sum_{j=1}^d a_{ij}(x)\hat{b}_j(x)\frac{\partial u}{\partial x^i}v(x)m(dx),\end{aligned}$$

we consider the following conditions:

(A2) There exists a constant $c_2 \geq 0$ such that $\mathcal{E}^{a,\hat{b}}_{c_2}(\cdot,\cdot) := \mathcal{E}^{a,\hat{b}}(\cdot,\cdot) + c_2(\cdot,\cdot)$ is a semi-Dirichlet form (see Section 1.1) with domain $F := \overline{C_0^\infty(\mathbb{R}^d)}^{\tilde{\mathcal{E}}^{a,\hat{b}}_{c_2+1}}$, and there exist constants $c_1, c_3 > 0$ such that for $u \in C_0^\infty(\mathbb{R}^d)$

$$c_1\mathcal{E}^a(u,u) \leq \mathcal{E}_{c_2}^{a,\hat{b}}(u,u), \tag{2.1.1}$$

and

$$\int cu^2 dm \leq c_3 \mathcal{E}_{c_2+1}^{a,\hat{b}}(u,u), \tag{2.1.2}$$

where $\tilde{\mathcal{E}}^{a,\hat{b}}(u,v) := \mathcal{E}^{a,\hat{b}}(u,v) + \mathcal{E}^{a,\hat{b}}(v,u)$.

By (2.1.1) we have that $F \subset F^a$ and that for $u \in F$ (2.1.1) and (2.1.2) are satisfied.

(A3) $|b\sigma| \in L^2(\mathbb{R}^d; m)$ and there exists $\alpha \geq 0$ such that

$$\int \langle b\sigma, (\nabla u^2)\sigma \rangle dm \geq -\alpha \|u\|_2^2, \quad u \in C_0^\infty(\mathbb{R}^d). \tag{2.1.3}$$

(A4) There exists a positivity preserving C_0-semigroup P_t on $L^1(\mathbb{R}^d; m)$ such that for any $t \in [0,T], \exists C_T > 0$

$$\|P_t f\|_\infty \leq C_T \|f\|_\infty.$$

Then for $0 \leq t \leq T$, P_t extends to a semigroup on $L^p(\mathbb{R}^d; m)$ for all $p \in [1, \infty)$ by the Riesz-Thorin Interpolation Theorem (denoted by P_t for simplicity) which is strongly continuous on $L^p(\mathbb{R}^d; m)$. We denote its L^2-generator by $(L, \mathcal{D}(L))$ and assume that $b\mathcal{D}(L) \subset bF$ and for any $u \in bF$ there exists uniformly bounded $u_n \in \mathcal{D}(L)$ such that $\tilde{\mathcal{E}}_{c_2+1}^{a,\hat{b}}(u_n - u) \to 0$ and that it is associated with the bilinear form in the sense that $\mathcal{E}(u,v) = -(Lu, v)$ for $u, v \in b\mathcal{D}(L)$.

We emphasize that in contrast to previous work P_t in (A4) is no longer analytic on $L^2(\mathbb{R}^d; m)$. By (A4) there exist constants M_0, c_0 such that

$$\|P_t f\|_2 \leq M_0 e^{c_0 t} \|f\|_2, \ \forall f \in L^2(\mathbb{R}^d; m). \tag{2.1.4}$$

To obtain a semigroup P_t satisfying the above conditions, we can use generalized Dirichlet forms (Definition 1.1).

Remark 2.1.1 (i) Some general criteria imposing conditions on a in order that $\mathcal{E}^a$ to be closable are e.g. given in [FOT94, Section 3.1] and [MR92, Chap II, Section 2].

(ii) There are examples considered in [MR92, Chap. II, Subsection 2d] satisfying

(A2). Assume the Sobolev inequality

$$\|u\|_q \leq C(\mathcal{E}^a(u,u) + \|u\|_2^2)^{1/2}, \ \forall u \in C_0^\infty(\mathbb{R}^d),$$

is satisfied, where $\frac{1}{q} + \frac{1}{d} = \frac{1}{2}$ and $\|\cdot\|_q$ denotes the usual norm in L^q. If $|\hat{b}\sigma| \in L^d(\mathbb{R}^d;m) + L^\infty(\mathbb{R}^d;m)$ and $c \in L^{d/2}(\mathbb{R}^d;m) + L^\infty(\mathbb{R}^d;m)$, then (A2) is satisfied (see [MOR95]). In [ZR11] they consider the bilinear form $Q(u,v) = \mathcal{E}^{a,\hat{b}}(u,v) + \int \langle d_1(x), \nabla v(x)\rangle u(x)dm$, where $d_1 \in L^q(\mathbb{R}^d), q > d$. In their case, the result for the existence of the solution of the nonlinear PDE can be obtained by [PR07, Theorem 4.2.4] since the nonlinear part is Lipschitz in u and ∇u. In our case, we have more general conditions on b and f, so that we cannot find a suitable Gelfand triple $V \subset H \subset V^*$ with V being a reflexive Banach space and use monotonicity methods as in [PR07].

(iii) We can construct a semigroup P_t satisfying (A4) by the theory of generalized Dirichlet forms. More precisely, suppose there exists a constant $\hat{c} \geq 0$ such that $\mathcal{E}_{\hat{c}}(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + \hat{c}(\cdot,\cdot)$ is a generalized Dirichlet form with domain $\mathcal{F} \times \mathcal{V}$ in one of the following three senses:

(a)$(E, \mathcal{B}(E), m) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m)$,

$(\mathcal{E}^{a,\hat{b}}_{c_2}, F) = (\mathcal{A}, \mathcal{V})$,

$-\langle \Lambda u, v\rangle - (\hat{c} - c_2)(u,v) = \sum_i^d \int \sum_{j=1}^d a_{ij}(x) b_j(x) \frac{\partial u}{\partial x^i} v(x) m(dx)$ for $u, v \in C_0^\infty(R^d)$;

(b)$(E, \mathcal{B}(E), m) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m)$,

$\mathcal{A} \equiv 0$ and $\mathcal{V} = L^2(\mathbb{R}^d, m)$,

$-\langle \Lambda u, v\rangle = \mathcal{E}_{\hat{c}}(u,v)$ for $u, v \in C_0^\infty(\mathbb{R}^d)$ and $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$;

(c) $\mathcal{E}_{\hat{c}} = \mathcal{A}$, $\Lambda \equiv 0$ (In this case $(\mathcal{E}_{\hat{c}}, \mathcal{V})$ is a sectorial Dirichlet form in the sense of [MR92]).

Then there exists a sub-Markovian C_0-semigroup of contractions $P_t^{\hat{c}}$ associated with the generalized Dirichlet form $\mathcal{E}_{\hat{c}}$. Define $P_t := e^{\hat{c}t} P_t^{\hat{c}}$. If it is a C_0-semigroup on L^1 then it satisfies (A4). Then we have

$$\mathcal{D}(L) \subset \mathcal{F} \subset F.$$

(iv) The semigroup can be also constructed by other methods. (see e.g. [DR02], [BDR09]).

(v) By (A3) we have that $\mathcal{E}$ is positivity preserving i.e.

$$\mathcal{E}(u, u^+) \geq 0 \ \forall u \in \mathcal{D}(L),$$

which can be obtained by the same arguments as [St2, Proposition 4.4].

(vi) The condition that for any $u \in bF$ there exists uniformly bounded $u_n \in \mathcal{D}(L)$ such that $\tilde{\mathcal{E}}^{a,\hat{b}}_{c_2+1}(u_n - u) \to 0$ is satisfied if $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$. It can also be satisfied in the case of (iii) by the theory of generalized Dirichlet form.

(vii) All the conditions are satisfied by the bilinear form considered in [DR02], [L01], [St1, Section 1 (a)] and the following example which is considered in [St2].

Example 2.1.2 Let $b_i \in L^2(\mathbb{R}^d; dx)$, $1 \le i \le d$. Consider the bilinear form

$$\mathcal{E}(u,v) := \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^{d} \int b_i \frac{\partial u}{\partial x_i} v dx; u, v \in C_0^\infty(\mathbb{R}^d)$$

Assume there exist constants $c, L \ge 0$ such that

$$\int \langle b, \nabla u \rangle dx \le 2c\|u\|_1 \text{ for all } u \in C_0^\infty(\mathbb{R}^d), u \ge 0,$$

$$-\sum_{i,j=1}^{d} \int b_i \frac{\partial u}{\partial x_j} dx h_i h_j \le L\|u\|_1 |h|^2,$$
$$\text{for all } u \in C_0^\infty(\mathbb{R}^d), u \ge 0, h \in \mathbb{R}^d,$$

(or equivalently, b is quasi-monotone, i.e.

$$\langle b(x) - b(y), x - y \rangle \le L|x-y|^2, \forall x, y \in \mathbb{R}^d,)$$

and for some continuous, monotone increasing function $f : [0,\infty) \to [1,\infty)$ with $\int_0^\infty \frac{dr}{f(r)} = \infty$ we have that

$$|b(x)| \le f(|x|), x \in \mathbb{R}^d.$$

Then in [St2, Subsection II.2] it is proved that there exists a generalized Dirichlet form in $L^2(\mathbb{R}^d)$ extending $\mathcal{E}_c$. We denote the semigroup associated with $\mathcal{E}_c$ by P_t^c. If we define $P_t := e^{ct} P_t^c$, then it is the semigroup associated with $\mathcal{E}$. By the computation in [St2, Subsection II.2], P_t is sub-Markovian. So it satisfies the conditions (A1)-(A4).

Further examples are presented in Section 2.3 (see Examples 2.3.2 and 2.3.3) and Sections 2.4, 2.5.

Then we use the same notations $\hat{F}, \mathcal{C}_T, \|\cdot\|_T$ associated with $\mathcal{E}^{a,\hat{b}}$ as in [BPS05]: $\mathcal{C}_T = C^1([0,T]; L^2) \cap L^2([0,T]; F)$, which turns out to be the appropriate space of test functions, i.e.

$$\mathcal{C}_T = \{\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R} | \varphi_t \in F \text{ for almost each } t, \int_0^T \mathcal{E}^{a,\hat{b}}(\varphi_t)dt < \infty,$$
$$t \to \varphi_t \text{ is differentiable in } L^2 \text{and } t \to \partial_t \varphi_t \text{ is } L^2 - \text{continuous on } [0,T]\}.$$

Here and below we set $\mathcal{E}^{a,\hat{b}}(u)$ for $\mathcal{E}^{a,\hat{b}}(u,u)$. We also set $\mathcal{C}_{[a,b]} = C^1([a,b];L^2) \cap L^2([a,b];F)$. For $\varphi \in \mathcal{C}_T$, we define

$$\|\varphi\|_T := (\sup_{t \leq T} \|\varphi_t\|_2^2 + \int_0^T \mathcal{E}^{a,\hat{b}}_{c_2}(\varphi_t)dt)^{1/2}.$$

$\hat{F}$ is the completion of $\mathcal{C}_T$ with respect to $\|\cdot\|_T$. By [BPS05], $\hat{F} = C([0,T];L^2) \cap L^2(0,T;F)$. Define the space $\hat{F}^a$ w.r.t. $\mathcal{E}^a_1$ analogous to $\hat{F}$. Then we have $\hat{F} \subset \hat{F}^a$. We also introduce the following space

$$W^{1,2}([0,T];L^2(\mathbb{R}^d)) = \{u \in L^2([0,T];L^2); \partial_t u \in L^2([0,T];L^2)\},$$

where $\partial_t u$ is the derivative of u in the weak sense (see e.g. [Ba10]).

2.1.1 Linear Equations

Consider the linear equation

$$(2.1.5) \qquad \begin{aligned} (\partial_t + L)u + f = 0, &\qquad 0 \leq t \leq T, \\ u_T(x) = \phi(x), &\qquad x \in \mathbb{R}^d, \end{aligned}$$

where $f \in L^1([0,T];L^2), \phi \in L^2$.

As in [BPS05] we set $D_\sigma \varphi := (\nabla \varphi)\sigma$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, define $V_0 = \{D_\sigma \varphi : \varphi \in C_0^\infty(\mathbb{R}^d)\}$, and let V be the closure of V_0 in $L^2(\mathbb{R}^d;\mathbb{R}^k)$. Then we have the following results:

Proposition 2.1.3 Assume (A1)-(A3) hold. Then:

(i) For every $u \in F^a$ there is a unique element of V, which we denote by $D_\sigma u$, such that

$$\mathcal{E}^a(u) = \int \langle D_\sigma u(x), D_\sigma u(x) \rangle m(dx).$$

(ii) Furthermore, if $u \in \hat{F}^a$, then there exists a measurable function $\phi : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $|\phi\sigma| \in L^2((0,T) \times \mathbb{R}^d)$ and $D_\sigma u_t = \phi_t \sigma$ for almost all $t \in [0,T]$.

(iii)Let $u^n, u \in \hat{F}^a$ be such that $u^n \to u$ in $L^2((0,T) \times \mathbb{R}^d)$ and $(D_\sigma u^n)_n$ is Cauchy in $L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^k)$. Then $D_\sigma u^n \to D_\sigma u$ in $L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^k)$, i.e. D_σ is closed as an operator from $\hat{F}^a$ into $L^2((0,T)\times\mathbb{R}^d)$.

Proof See [BPS05, Proposition 2.3]. □

For $u \in F, v \in bF$, we define

$$\mathcal{E}(u,v) := \mathcal{E}^{a,\hat{b}}(u,v) + \int \langle b\sigma, D_\sigma u\rangle v m(dx).$$

Notation We denote by $\tilde{\nabla} u$ the set of all measurable functions $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\phi\sigma = D_\sigma u$ as elements of $L^2(\mathbb{R}^d, \mathbb{R}^k)$.

2.1.2 Solution of the Linear Equation

We recall the following standard notions.

Definition 2.1.4 (*strong solutions*) A function $u \in \hat{F}\cap L^1((0,T);\mathcal{D}(L))$ is called a strong solution of equation (2.1.5) with data ϕ, f, if $t \mapsto u_t = u(t,\cdot)$ is L^2-differentiable on $[0,T], \partial_t u_t \in L^1((0,T);L^2)$ and the equalities in (2.1.5) hold m-a.e..

Definition 2.1.5 (*generalized solutions*) A function $u \in \hat{F}$ is called a generalized solution of equation (2.1.5), if there are sequences $\{u^n\}$ which are strong solutions with data (ϕ^n, f^n) such that

$$\|u^n - u\|_T \to 0, \|\phi^n - \phi\|_2 \to 0, \lim_{n\to\infty} f^n = f \text{ in } L^1([0,T];L^2).$$

Proposition 2.1.6 Assume (A3)-(A4) hold.

(i) Let $f \in C^1([0,T];L^p)$ for $p \in [1,\infty)$. Then

$$w_t := \int_t^T P_{s-t} f_s ds \in C^1([0,T];L^p),$$

and

$$\partial_t w_t = -P_{T-t} f_T + \int_t^T P_{s-t}\partial_s f_s ds.$$

(ii) Assume that $\phi \in \mathcal{D}(L)$, $f \in C^1([0,T];L^2)$ and for each $t \in [0,T]$, $f_t \in \mathcal{D}(L)$. Define

$$u_t := P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Then u is a strong solution of (2.1.5) and, moreover, $u \in C^1([0,T];L^2)$.

Proof By the same arguments as in the proof of [BPS05, Proposition 2.6] the results follow. □

Remark 2.1.7 Compared to [BPS05, Proposition 2.6], in (ii) we add the assumption $\phi \in \mathcal{D}(L)$ and $f_t \in \mathcal{D}(L)$, $t \in [0,T]$, as we cannot deduce $P_t\phi \in \mathcal{D}(L)$ for $\phi \in L^2$, since (P_t) might not be analytic.

Proposition 2.1.8 Suppose (A4) holds. If u is a strong solution for (2.1.5), it is a mild solution for (2.1.5) i.e.

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Proof For fixed t, $\varphi \in \mathcal{D}(\hat{L})$

$$(u_T, \hat{P}_{T-t}\varphi) - (u_t, \varphi) = \int_t^T (-Lu_s - f_s, \hat{P}_{s-t}\varphi)ds + \int_t^T (u_s, \hat{L}\hat{P}_{s-t}\varphi)ds,$$

where $\hat{L}, \hat{P}_t$ denote the adjoints on $L^2(\mathbb{R}^d, m)$ of L, P_t respectively. As u is a strong solution, we can deduce that

$$(u_t, \varphi) = (P_{T-t}\phi + \int_t^T P_{s-t} f_s ds, \varphi).$$

Since $\mathcal{D}(\hat{L})$ is dense in L^2, the result follows. □

Proposition 2.1.9 Suppose that conditions (A1)-(A4) hold, $f \in L^1([0,T]; L^2)$ and $\phi \in L^2$. Then equation (2.1.5) has a unique generalized solution $u \in \hat{F}$ and

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds. \tag{2.1.6}$$

The solution satisfies the three relations:

(2.1.7)

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds \le 2\int_t^T (f_s, u_s)ds + \|\phi\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds, \qquad 0 \le t \le T.$$

$$\|u\|_T^2 \le M_T(\|\phi\|_2^2 + (\int_0^T \|f_t\|_2 dt)^2). \tag{2.1.8}$$

(2.1.9)
$$\int_{t_0}^{T}((u_t,\partial_t\varphi_t)+\mathcal{E}^{a,\hat{b}}(u_t,\varphi_t)+\int\langle b\sigma,D_\sigma u_t\rangle\varphi_t dm)dt=\int_{t_0}^{T}(f_t,\varphi_t)dt+(\phi,\varphi_T)-(u_{t_0},\varphi_{t_0}),$$

for any $\varphi\in b\mathcal{C}_T, t_0\in[0,T]$. M_T is a constant depending on T. (2.1.9) can be extended easily for $\varphi\in bW^{1,2}([0,T];L^2)\cap L^2([0,T];F)$.

Moreover, if $u\in\hat{F}$ is bounded and satisfies (2.1.9) for any $\varphi\in b\mathcal{C}_T$ with bounded f,ϕ, then u is a generalized solution given by (2.1.6).

Proof [Existence] Define u by (2.1.6). First assume that ϕ, f are bounded and satisfy the conditions of Proposition 2.1.6 (ii). Then, since u is bounded and by Proposition 2.1.6 we know that u is a strong solution of (2.1.5), hence it obviously satisfies (2.1.9). Furthermore, $u\in C^1([0,T];L^2)$. Hence, actually $u\in b\mathcal{C}_T$ and consequently, for $t\in[0,T]$

$$\int_{t}^{T}((u_s,\partial_t u_s)+\mathcal{E}^{a,\hat{b}}(u_s,u_s)+\int\langle b\sigma,D_\sigma u_s\rangle u_s dm)ds=\int_{t}^{T}(f_s,u_s)ds+(\phi,u_T)-(u_t,u_t).$$

By (2.1.3) we have $\int\langle b\sigma,D_\sigma u_s\rangle u_s dm\geq-\alpha\|u_s\|_2^2$. Hence

$$\|u_t\|_2^2+2\int_t^T\mathcal{E}^{a,\hat{b}}(u_s)ds\leq 2\int_t^T(f_s,u_s)ds+\|\phi\|_2^2+2\alpha\int_t^T\|u_s\|_2^2ds,\qquad 0\leq t\leq T.$$

As

$$\begin{aligned}\int_t^T(f_s,u_s)ds=&\int_t^T((f_s,P_{T-s}\phi)+(f_s,\int_s^T P_{r-s}f_r dr))ds\\ \leq&\int_t^T\|f_s\|_2\|P_{T-s}\phi\|_2ds+\int_t^T\|f_s\|_2\|\int_s^T P_{r-s}f_rdr\|_2ds\\ \leq&M_0e^{T-t}(\|\phi\|_2\int_t^T\|f_s\|_2ds+\int_t^T(\|f_s\|_2\int_s^T\|f_r\|_2dr)ds),\end{aligned}$$

and
$$\int_t^T\|u_s\|_2^2ds\leq M_{T-t}(\|\phi\|_2^2+(\int_0^T\|f_t\|_2dt)^2),$$
we get
$$\|u_t\|_2^2+\int_t^T\mathcal{E}^{a,\hat{b}}(u_s)ds\leq M_{T-t}(\|\phi\|_2^2+(\int_0^T\|f_t\|_2dt)^2).$$
Hence, it follows that

(2.1.10)
$$\|u\|_T^2\leq M_T(\|\phi\|_2^2+(\int_0^T\|f_t\|_2dt)^2).$$

Here M_{T-t} can change from line to line and is independent of f, ϕ. Now we will prove the result for general data ϕ and f. Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of bounded function in $C^1([0,T];L^2)$ such that $f_t \in \mathcal{D}(L)$ for a.e. $t \in [0,T]$ and $\int_0^T \|f_t^n - f_t\|_2 dt \to 0$ (This sequence can be obtained since $\{\alpha_t g(x); \alpha_t \in C_0^\infty[0,T], g \in b\mathcal{D}(L)\}$ is dense in $L^1([0,T];L^2)$). Take bounded functions $(\phi^n)_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ such that $\phi^n \to \phi$ in L^2. Let u^n denote the solution given by (2.1.6) with $f = f^n, \phi = \phi^n$.

By linearity, $u^n - u^m$ is associated with $(\phi^n - \phi^m, f^n - f^m)$. Since by (2.1.10)

$$\|u^n - u^m\|_T^2 \le M_T(\|\phi^n - \phi^m\|_2^2 + (\int_0^T \|f_t^n - f_t^m\|_2 dt)^2),$$

we deduce that $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}$. Hence $u = \lim_{n\to\infty} u^n$ in $\|\cdot\|_T$ is the generalized solution of (2.1.5) and we have

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Next we prove (2.1.7)(2.1.8) (2.1.9) for u. We have (2.1.9) for u^n with f^n, ϕ^n and $\varphi \in b\mathcal{C}_T$, i.e.

$$\int_0^T ((u_t^n, \partial_t\varphi_t) + \mathcal{E}^{a,\hat{b}}(u_t^n, \varphi_t) + \int \langle b\sigma, D_\sigma u_t^n\rangle \varphi_t dm)dt = \int_0^T (f_t^n, \varphi_t)dt + (\phi^n, \varphi_T) - (u_0^n, \varphi_0).$$

Since we have

$$\begin{aligned} |\int_0^T \mathcal{E}^{a,\hat{b}}(u_t^n - u_t, \varphi_t)dt| \le & K(\int_0^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(u_t^n - u_t)dt)^{\frac{1}{2}}(\int_0^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(\varphi_t)dt)^{\frac{1}{2}} \\ & + \int_0^T (c_2+1)(u_t^n - u_t, \varphi_t)dt \\ \to & 0, \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} |\int_0^T \int \langle b\sigma, D_\sigma(u_t^n - u_t)\rangle \varphi_t dmdt| &\le \|\varphi\|_\infty (\int_0^T \int |b\sigma|^2 dmdt)^{\frac{1}{2}} (\int_0^T \int |D_\sigma(u_t^n - u_t)|^2 dmdt)^{\frac{1}{2}} \\ &= \|\varphi\|_\infty (\int_0^T \int |b\sigma|^2 dmdt)^{\frac{1}{2}} (\int_0^T \mathcal{E}^a(u_t^n - u_t)dt)^{\frac{1}{2}} \\ &\to 0, \text{ as } n \to \infty, \end{aligned}$$

we deduce that

$$\int_0^T ((u_t, \partial_t\varphi_t) + \mathcal{E}^{a,\hat{b}}(u_t, \varphi_t) + \int \langle b\sigma, D_\sigma u_t\rangle \varphi_t dm)dt = \int_0^T (f_t, \varphi_t)dt + (\phi, \varphi_T) - (u_0, \varphi_0),$$

for any $\varphi \in b\mathcal{C}_T$.

The relations (2.1.7), (2.1.8) hold for the approximating functions:

$$\|u_t^n\|_2^2+2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s^n)ds \leq 2\int_t^T (f_s^n, u_s^n)ds+\|\phi^n\|_2^2+2\alpha\int_t^T \|u_s^n\|_2^2 ds, \qquad 0\leq t\leq T,$$

and

$$\|u^n\|_T^2 \leq M_T(\|\phi^n\|_2^2+(\int_0^T \|f_t^n\|_2 dt)^2).$$

Since $\|u_t^n\|_T \to \|u_t\|_T$, $n\to\infty$, we conclude

$$\lim_{n\to\infty}\int_0^T \mathcal{E}^{a,\hat{b}}(u_t^n)dt = \int_0^T \mathcal{E}^{a,\hat{b}}(u_t)dt.$$

It is easy to see that $\lim_{n\to\infty}\int_t^T (f_s^n, u_s^n)ds = \int_t^T (f_s, u_s)ds$. Then by passing to the limit in the above relations, (2.1.7) and (2.1.8) hold for u.

[Uniqueness] Let $v \in \hat{F}$ be another generalized solution of (2.1.5) and $(v^n)_{n\in\mathbb{N}}$, $(\tilde{\phi}^n)_{n\in\mathbb{N}}, (\tilde{f}^n)_{n\in\mathbb{N}}$ be the corresponding approximating sequences in the definition of generalized solutions. By Proposition 2.1.8

$$\sup_{t\in[0,T]}\|u_t^n - v_t^n\|_2^2 \leq M_T(\|\phi^n-\tilde{\phi}^n\|_2^2+(\int_0^T \|f_t^n-\tilde{f}_t^n\|_2 dt)^2).$$

Letting $n\to\infty$, this implies $u=v$.

For the last result we note that $\forall t_0\in[0,T], \varphi\in b\mathcal{C}_T$

(2.1.11)

$$\int_{t_0}^T ((u_t,\partial_t\varphi_t)+\mathcal{E}^{a,\hat{b}}(u_t,\varphi_t)+\int\langle b\sigma, D_\sigma u_t\rangle\varphi_t dm)dt = \int_{t_0}^T (f_t,\varphi_t)dt+(\phi,\varphi_T)-(u_{t_0},\varphi_{t_0}).$$

For $t\geq \frac{1}{n}$, define

$$u_t^n := n\int_0^{\frac{1}{n}} u_{t-s}ds, \qquad f_t^n := n\int_0^{\frac{1}{n}} f_{t-s}ds, \qquad \phi^n := n\int_0^{\frac{1}{n}} u_{T-s}ds.$$

Let us check that each u^n also fulfills (2.1.11) with f^n, ϕ^n. We set $\varphi_r^s := \varphi_{r+s}$ for $0\leq s+r\leq T$. Then for fixed $t_0\in(0,T]$, and $n\geq\frac{1}{t_0}$,

$$\begin{aligned}&\int_{t_0}^T ((u_t^n,\partial_t\varphi_t)+\mathcal{E}^{a,\hat{b}}(u_t^n,\varphi_t)+\int\langle b\sigma, D_\sigma u_t^n\rangle\varphi_t dm)dt\\ =&n\int_0^{\frac{1}{n}}\int_{t_0}^T (u_{t-s},\partial_t\varphi_t)+\mathcal{E}^{a,\hat{b}}(u_{t-s},\varphi_t)+\int\langle b\sigma, D_\sigma u_{t-s}\rangle\varphi_t dmdtds\end{aligned}$$

$$
\begin{aligned}
=&n\int_0^{\frac{1}{n}}\int_{t_0-s}^{T-s}(u_t,\partial_t\varphi_t^s)+\mathcal{E}^{a,\hat{b}}(u_t,\varphi_t^s)+\int\langle b\sigma,D_\sigma u_t\rangle\varphi_t^s dmdtds\\
=&n\int_0^{\frac{1}{n}}[\int_{t_0-s}^{T-s}(f_t,\varphi_t^s)dt+(u_T,\varphi_{T-s}^s)-(u_{t_0-s},\varphi_{t_0-s}^s)]ds\\
=&n\int_0^{\frac{1}{n}}[\int_{t_0}^{T}(f_{t-s},\varphi_t)dt+(u_{T-s},\varphi_T)-(u_{t_0-s},\varphi_{t_0})]ds\\
=&\int_{t_0}^{T}(f_t^n,\varphi_t)dt+(\phi^n,\varphi_T)-(u_{t_0}^n,\varphi_{t_0}).
\end{aligned}
$$

For the mild solution v associated with f,ϕ, the above relation also holds with v^n replacing u^n. Hence we have

$$
\begin{aligned}
&\int_{t_0}^{T}(((u-v)_t^n,\partial_t\varphi_t)+\mathcal{E}^{a,\hat{b}}((u-v)_t^n,\varphi_t)+\int\langle b\sigma,D_\sigma(u-v)_t^n\rangle\varphi_t dm)dt\\
=&-((u-v)_{t_0}^n,\varphi_{t_0}).
\end{aligned}
$$

Since $(u-v)_t^n\in b\mathcal{C}_{[\frac{1}{n},T]}$ the above equation holds with $(u-v)_t^n$ as a test function, i.e. for $n\geq\frac{1}{t_0}$

$$
\begin{aligned}
&\int_{t_0}^{T}(((u-v)_t^n,\partial_t(u-v)_t^n)+\mathcal{E}^{a,\hat{b}}((u-v)_t^n,(u-v)_t^n)+\int\langle b\sigma,D_\sigma(u-v)_t^n\rangle(u-v)_t^n dm)dt\\
=&-((u-v)_{t_0}^n,(u-v)_{t_0}^n).
\end{aligned}
$$

So we have

$$
\|(u-v)_{t_0}^n\|_2^2+2\int_t^T\mathcal{E}^{a,\hat{b}}((u-v)_t^n,(u-v)_t^n)dt\leq 2\alpha\int_{t_0}^T\|(u-v)_t^n\|_2^2dt.
$$

By Gronwall's lemma it follows that

$$
\|(u-v)_{t_0}^n\|_2^2=0.
$$

Letting $n\to\infty$, we have $\|u_{t_0}-v_{t_0}\|_2=0$. Then letting $t_0\to 0$, we have $\|u_0-v_0\|=0$. Then $u_t=P_{T-t}\phi+\int_t^T P_{s-t}f_sds$ is a generalized solution for (2.1.5). $\square$

2.1.3 Basic Relations for the Linear Equation

In this section we assume that (A1)-(A4) hold.

Lemma 2.1.10 If u is a bounded generalized solution of equation (2.1.5) with some function $\phi\geq 0$, $\phi\in L^2\cap L^\infty$, then u^+ satisfies the following relation for

$0 \le t_1 < t_2 \le T$

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_s, u_s^+)ds + \|u_{t_2}^+\|_2^2.$$

Proof Choose the approximation sequence u^n for u as in the existence proof of Proposition 2.1.9. Denote its related data by f^n, ϕ^n .

We have the following equations:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - u_t\|_2 = 0, \qquad \lim_{n\to\infty} \int_0^T \mathcal{E}^{a,\hat{b}}(u_t^n - u_t)dt = 0,$$

$$\lim_{n\to\infty} \int_0^T \|f_t^n - f_t\|_2 dt = 0, \qquad \lim_{n\to\infty} \|\phi^n - \phi\|_2 = 0.$$

Suppose that the following holds

$$\|(u_{t_1}^n)^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)ds + \|(u_{t_2}^n)^+\|_2^2, \tag{2.1.12}$$

where $0 \le t_1 \le t_2 \le T$. Since $\|u^n\|_2$ are uniformly bounded, we obtain

$$\lim_{n\to\infty} \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)ds = \int_{t_1}^{t_2} (f_s, u_s^+)ds.$$

By passing n to the limit in equation (2.1.12) we get for $0 \le t_1 \le t_2 \le T$,

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_s, u_s^+)ds + \|u_{t_2}^+\|_2^2.$$

Therefore, the problem is reduced to the case that u belongs to $b\mathcal{C}_T$; in the remainder we assume $u \in b\mathcal{C}_T$. (2.1.9), written with $u^+ \in bW^{1,2}([0,T]; L^2) \cap L^2([0,T]; F)$ as test function, takes the form

$$\begin{aligned} &\int_{t_1}^{t_2} (u_t, \partial_t(u_t^+))dt + \int_{t_1}^{t_2} \mathcal{E}^{a,\hat{b}}(u_t, u_t^+)dt + \int_{t_1}^{t_2} \int \langle b\sigma, D_\sigma u_t\rangle u_t^+ dmdt \\ &= \int_{t_1}^{t_2} (f_t, u_t^+)dt + (u_{t_2}, u_{t_2}^+)) - (u_{t_1}, u_{t_1}^+)). \end{aligned} \tag{2.1.13}$$

By [Ba10, Theorem 1.19] we obtain

$$\int_{t_1}^{t_2} (u_t, \partial_t(u_t^+))dt = \frac{1}{2}(\|u_{t_2}^+\|_2^2 - \|u_{t_1}^+\|_2^2).$$

Then

$$\begin{aligned}
&\|u_{t_1}^+\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}^{a,\hat{b}}(u_t, u_t^+)dt + 2\int_{t_1}^{t_2}\int \langle b\sigma, D_\sigma u_t\rangle u_t^+ dmdt \\
=&2\int_{t_1}^{t_2}(f_t, u_t^+)dt + \|u_{t_2}^+\|_2^2.
\end{aligned} \tag{2.1.14}$$

Next we prove for $u \in bF$

$$\mathcal{E}(u, u^+) \geq 0. \tag{2.1.15}$$

By Remark 2.1.1 (v), we have the above relation for $u \in \mathcal{D}(L)$. For $u \in bF$, by (A4) we can choose a uniformly bounded sequence $\{u_n\} \subset \mathcal{D}(L)$ such that $\mathcal{E}^{a,\hat{b}}_{c_2+1}(u_n - u) \to 0$. Then we have

$$\begin{aligned}
&|\int \langle b\sigma, D_\sigma u\rangle u^+ dm - \int \langle b\sigma, D_\sigma u_n\rangle u_n^+ dm| \\
\leq&|\int \langle b\sigma, D_\sigma u_n - D_\sigma u\rangle u_n^+ dm| + |\int \langle b\sigma, D_\sigma u\rangle (u_n^+ - u^+)dm| \\
\leq&M(\int |D_\sigma u_n - D_\sigma u|^2 dm)^{\frac{1}{2}} + |\int \langle b\sigma, D_\sigma u\rangle (u_n^+ - u^+)dm| \\
\to&0.
\end{aligned}$$

By (A2) and [MOR95] $\mathcal{E}^{a,\hat{b}}(u^+) \leq 4K^2\mathcal{E}^{a,\hat{b}}(u)$, $\sup_n \mathcal{E}^{a,\hat{b}}(u_n^+) \leq 4K^2 \sup_n \mathcal{E}^{a,\hat{b}}(u_n) < \infty$, we also have

$$\begin{aligned}
&|\mathcal{E}^{a,\hat{b}}(u_n, (u_n)^+) - \mathcal{E}^{a,\hat{b}}(u, u^+)| \\
\leq&|\mathcal{E}^{a,\hat{b}}_{c_2+1}(u_n - u, (u_n)^+) + \mathcal{E}^{a,\hat{b}}_{c_2+1}(u, (u_n)^+ - u^+)| \\
&+ (c_2+1)|(u_n - u, (u_n)^+)| + (c_2+1)|(u, (u_n)^+ - u^+)| \\
\leq&K(\mathcal{E}^{a,\hat{b}}_{c_2+1}(u_n - u))^{\frac{1}{2}}(\mathcal{E}^{a,\hat{b}}_{c_2+1}((u_n)^+))^{\frac{1}{2}} + |\mathcal{E}^{a,\hat{b}}_{c_2+1}(u, (u_n)^+ - u^+)| \\
&+ (c_2+1)(\|(u_n)^+\|_2\|u_n - u\|_2 + (c_2+1)\|(u_n)^+ - u^+\|_2\|u\|_2) \\
\to&0.
\end{aligned}$$

As a result, we obtain (2.1.15) for $u \in bF$. Then the assertion follows. □

To extend the class of solutions we are working with to allow $f \in L^1(dt \times dm)$, we need the following proposition. It is a modified version of the above lemma.

Lemma 2.1.11 Let $u \in b\hat{F}$ and $f \in L^1(dt \times dm)$ satisfying the weak relation (2.1.9) with test functions in $b\mathcal{C}_T$ and some function $\phi \geq 0$, $\phi \in L^2 \cap L^\infty$. Then u^+

satisfies the following relation with $0 \le t_1 < t_2 \le T$

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_s, u_s^+)ds + \|u_{t_2}^+\|_2^2.$$

Proof First note that we can prove analogously to the proof of Lemma 2.1.10 that for each $u \in b\mathcal{C}_T$ satisfying the weak relation (2.1.9) with data (ϕ, f) over the interval $[t_1, t_2]$, where $\varepsilon \le t_1 \le t_2 \le T$ for $\varepsilon > 0$, the following holds

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_t, u_t^+)dt + \|u_{t_2}^+\|_2^2.$$

For $u \in \hat{F}$ we take approximating functions u^n and (ϕ^n, f^n) as in the last part of the proof of Proposition 2.1.9 . Then we have that u^n satisfies the weak relation (2.1.9) for the data ϕ^n, f^n with test functions in $b\mathcal{C}_T$ over the interval $[\varepsilon, t_2]$ and $\frac{1}{n} \le \varepsilon \le t_2 \le T$. Note

$$\lim_{n\to\infty} \int_{\varepsilon}^{T} \|f_t^n - f_t\|_1 dt = 0.$$

Then we obtain

$$\|(u_{t_1}^n)^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_t^n, (u_t^n)^+)dt + \|(u_{t_2}^n)^+\|_2^2,$$

where $\varepsilon \le t_1 \le t_2 \le T$ for $\varepsilon > 0$. The convergence of all terms, which does not depend on f, follows by the same arguments as the proof of Lemma 2.1.10. Since u is bounded, it is easy to see that u^n is uniformly bounded. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} |\int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)ds - \int_{t_1}^{t_2} (f_s, u_s^+)ds| \\
\le& M \lim_{n\to\infty} \int_{t_1}^{t_2} \|f_s^n - f_s\|_1 ds + \lim_{n\to\infty} \int_{t_1}^{t_2} (f_s, (u_s^n)^+ - u_s^+)ds \\
=& 0.
\end{aligned}$$

Finally, we obtain

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2} (f_t, u_t^+)dt + \|u_{t_2}^+\|_2^2,$$

where $\varepsilon \le t_1 \le t_2 \le T$ for $\varepsilon > 0$. Letting $\varepsilon \to 0$, the assertion follows. □

The next proposition is a modification of [BPS05, Proposition 2.9]. It represents a version of the maximum principle.

Proposition 2.1.12 Let $u \in b\hat{F}$ and $f \in L^1(dt \times dm), f \ge 0$, satisfying the weak relation (2.1.9) with test functions in $b\mathcal{C}_T$ and some function $\phi \geq 0$, $\phi \in L^2 \cap L^\infty$. Then $u \geq 0$ and it is represented by the following relation:

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Here we use P_t is a C_0-semigroup on $L^1(\mathbb{R}^d; m)$ to make $P_{s-t}f_s$ meaningful.

Proof Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of bounded functions such that

$$0 \leq f^n \leq f^{n+1} \leq f, \qquad \lim_{n\to\infty} f^n = f.$$

Since f^n is bounded, we have $f^n \in L^1([0,T]; L^2)$. Define

$$u_t^n := P_{T-t}\phi + \int_t^T P_{s-t} f_s^n ds.$$

Then by Proposition 2.1.9, $u^n \in \hat{F}$ is a unique generalized solution for the data (ϕ, f^n). Clearly $0 \leq u^n \leq u^{n+1}$ for $n \in \mathbb{N}$. Define $y := u^n - u$ and $\tilde{f} := f^n - f$. Then $\tilde{f} \leq 0$ and y satisfies the weak relation (2.1.9) for the data $(0, \tilde{f})$. Therefore by Lemma 2.1.11, we have for $t_1 \in [0,T]$

$$\|y_{t_1}^+\|_2^2 \leq 2\int_{t_1}^T (\tilde{f}_s, y_s^+) ds \leq 0.$$

We conclude that $\|y_{t_1}^+\|_2^2 = 0$. Therefore, $u \geq u^n \geq 0$ for $n \in \mathbb{N}$. Set $v := \lim_{n\to\infty} u^n$. By (2.1.7) we have

$$\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s^n) ds \leq 2\int_t^T (f_s^n, u_s^n) ds + \|\phi\|_2^2 + 2\alpha \int_t^T \|u_s^n\|_2^2 ds,$$

which implies that

$$\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s^n) ds \leq 2M \int_t^T \int |f_s^n| dm ds + \|\phi\|_2^2 + 2\alpha \int_t^T \|u_s^n\|_2^2 ds.$$

By Gronwall's lemma, we have $\sup_n \sup_{t\in[0,T]} \|u_t^n\|_2^2 \leq$const. We obtain $\lim_{n\to\infty} \|u_t^n - v_t\|_2^2 = 0$ and

$$\lim_{n\to\infty} |\int_t^T \int (f_s^n u_s^n - f_s v_s) dm ds| = 0.$$

By [MR92, Lemma 2.12] we obtain

$$\int_t^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(v_s) ds \leq \int_t^T \liminf_{n\to\infty} \mathcal{E}_{c_2+1}^{a,\hat{b}}(u_s^n) ds \leq \liminf_{n\to\infty} \int_t^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(u_s^n) ds.$$

Finally, for $t \in [0,T]$ we get

$$\begin{aligned}\|v_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(v_s)ds &\le \lim_{n\to\infty} \|u_t^n\|_2^2 + \liminf_{n\to\infty} \int_t^T \mathcal{E}^{a,\hat{b}}(u_s^n)ds \\ &\le \lim_{n\to\infty}(2\int_t^T (f_s^n, u_s^n)ds + \|\phi\|_2^2) + \lim_{n\to\infty} 2\alpha \int_t^T \|u_s^n\|_2^2 ds \\ &= 2\int_t^T (f_s, v_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|v_s\|_2^2 ds.\end{aligned}$$

Since the right hand side of this inequality is finite and $t \mapsto v_t$ is L^2-continuous, it follows that $v \in \hat{F}$.

Now we show that v satisfies the weak relation (2.1.9) for the data (ϕ, f). As $\varphi^n(t) := \|u_t^n - v_t\|_2$ is continuous and decreasing, we conclude by Dini's theorem

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - v_t\|_2 = 0,$$

and therefore

$$\lim_{n\to\infty} \int_0^T \|u_t^n - v_t\|_2^2 = 0.$$

Furthermore, there exists $K \in \mathbb{R}_+$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$|\int_0^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(u_s^{n_k})ds| \le K \qquad \forall k \in \mathbb{N}.$$

in particular

$$\int_0^T \int |D_\sigma u_s^{n_k}|^2 dm ds \le \frac{K}{c_1} \qquad \forall k \in \mathbb{N}.$$

We obtain

$$\lim_{k\to\infty} \int_0^T \mathcal{E}^{a,\hat{b}}(u_s^{n_k}, \varphi_s)ds = \int_0^T \mathcal{E}^{a,\hat{b}}(v_s, \varphi_s)ds,$$

and

$$\lim_{k\to\infty} \int_0^T \int \langle b\sigma, D_\sigma u_s^{n_k}\rangle \varphi_s dm ds = \int_0^T \int \langle b\sigma, D_\sigma v_s\rangle \varphi_s dm ds,$$

which implies (2.1.9) for v associated to (ϕ, f). Clearly $u - v$ satisfies (2.1.9) with data $(0,0)$ for $\varphi \in b\mathcal{C}_T$. By Proposition 2.1.9 we have $u - v = 0$. Since

$$v_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds,$$

the assertion follows. □

Corollary 2.1.13 Let $u \in b\hat{F}$ and $f \in L^1(dt\times dm)$ satisfy the weak relation (2.1.9) with test functions in $b\mathcal{C}_T$ and some function $\phi \in L^2 \cap L^\infty$. Let $g \in L^1(dt \times dm)$ be a bounded function such that $f \le g$. Then u is represented by the following relation:

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Proof Define $f^n := (f \vee (-n)) \wedge g, n \in \mathbb{N}$. Then $(f^n)_{n\in\mathbb{N}}$ is a sequence of bounded functions such that $f^n \downarrow f$ and $f^n \le g$ then by the same arguments as the proof of Proposition 2.1.12, the assertion follows. □

The following proposition is a modification of [BPS05, Proposition 2.10] . It is essential for the analytic treatment of the non-linear equation (1.1) which is done in the next section.

Proposition 2.1.14 Let $u = (u^1, ..., u^l)$ be a vector valued function where each component is a generalized solution of the linear equation (2.1.5) associated to certain data f^i, ϕ^i, which are bounded and satisfy the conditions in Proposition 2.1.6 (ii) for $i = 1, ..., l$. Denote by ϕ, f the vectors $\phi = (\phi^1, ..., \phi^l), f = (f^1, ..., f^l)$ and by $D_\sigma u$ the matrix whose rows consist of the row vectors $D_\sigma u^i$. Then the following relations hold m-almost everywhere

$$|u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2)ds = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle ds. \tag{2.1.16}$$

$$|u_t| \le P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle ds. \tag{2.1.17}$$

Here we write $\hat{x} = x/|x|$, for $x \in \mathbb{R}^l$, $x \neq 0$ and $\hat{x} = 0$, if $x = 0$.

Proof By Proposition 2.1.6 (ii) we have $u \in b\mathcal{C}_T$.

First we assume $l = 1$. If we can check that u^2 satisfies (2.1.9) with data $(2uf - 2|D_\sigma u|^2 - cu^2, \phi^2)$ for $\varphi \in b\mathcal{C}_T$, then (2.1.16) will follow by Corollary 2.1.13. We have the following relations:

$$\int_0^T (u_t^2, \partial_t\varphi_t)dt = 2\int_0^T (u_t, \partial_t(u_t\varphi_t))dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T),$$

$$\mathcal{E}^{a,\hat{b}}(u_t^2, \varphi_t) = 2\mathcal{E}^{a,\hat{b}}(u_t, u_t\varphi_t) - (2|D_\sigma u_t|^2 + cu_t^2, \varphi_t),$$

and

$$\int \langle b\sigma, D_\sigma(u_t^2)\rangle \varphi_t dm = 2\int \langle b\sigma, D_\sigma u_t\rangle u_t\varphi_t dm.$$

For the second relation we use (2.1.2). Since u is a generalized solution of (2.1.5), we obtain

$$\int_0^T (u_t, \partial_t(u_t\varphi_t))dt - (u_T, u_T\varphi_T) + (u_0, u_0\varphi_0) - \int_0^T (f_t, u_t\varphi_t)dt$$
$$= - \int_0^T \mathcal{E}^{a,\hat{b}}(u_t, u_t\varphi_t)dt - \int_0^T \int \langle b\sigma, D_\sigma u_t\rangle u_t\varphi_t dmdt.$$

By the above relations we have
(2.1.18)
$$\int_0^T (u_t^2, \partial_t\varphi_t)dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T) + \int_0^T (\mathcal{E}^{a,\hat{b}}(u_t^2, \varphi_t) + \int \langle b\sigma, D_\sigma(u_t^2)\rangle\varphi_t dm)dt$$
$$=2\int_0^T (f_t u_t, \varphi_t)dt - \int_0^T (2|D_\sigma u_t|^2 + c|u_t|^2, \varphi_t)dt.$$

Hence, by Corollary 2.1.13 (2.1.16) holds in the case $l = 1$. To deduce this relation in the case $l > 1$, it suffices to add the relations corresponding to the components $|u_t^i|^2, i = 1, ..., l$. For (2.1.17), define for $\varepsilon > 0$, $h_\varepsilon(t) := \sqrt{t+\varepsilon} - \sqrt{\varepsilon}$ for $t \geq 0$. Then by integration by parts we have

$$\mathcal{E}^{a,\hat{b}}(h_\varepsilon(|u|^2), \varphi) =\mathcal{E}^{a,\hat{b}}(|u|^2, h'_\varepsilon(|u|^2)\varphi) - (h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2, \varphi)$$
$$+ (c(h_\varepsilon(|u|^2) - |u|^2 h'_\varepsilon(|u|^2)), \varphi),$$

and

$$\int_0^T (h_\varepsilon(|u_t|^2), \partial_t\varphi_t)dt = \int_0^T (|u_t|^2, \partial_t(\varphi_t h'_\varepsilon(|u_t|^2)))dt - (|u_T|^2, \varphi_T h'_\varepsilon(|u_T|^2))$$
$$+ (|u_0|^2, \varphi_0 h'_\varepsilon(|u_0|^2)) + (h_\varepsilon(|u_T|^2), \varphi_T) - (h_\varepsilon(|u|_0^2), \varphi_0).$$

If we choose $\varphi h'_\varepsilon(|u|^2)$ as test function in (2.1.18), we obtain

$$\int_0^T (|u_t|^2, \partial_t(\varphi_t h'_\varepsilon(|u_t|^2)))dt + (|u_0|^2, \varphi_0 h'_\varepsilon(|u_0|^2)) - (|u_T|^2, \varphi_T h'_\varepsilon(|u_T|^2))$$
$$+ \int_0^T (\mathcal{E}^{a,\hat{b}}(|u_t|^2, \varphi_t h'_\varepsilon(|u_t|^2)) + \int \langle b\sigma, D_\sigma(|u_t|^2)\rangle \varphi_t h'_\varepsilon(|u_t|^2)dm)dt$$
$$=2\int_0^T (\langle f_t, u_t\rangle, \varphi_t h'_\varepsilon(|u_t|^2))dt - \int_0^T (2|D_\sigma u_t|^2 + c|u_t|^2, \varphi_t h'_\varepsilon(|u_t|^2))dt.$$

By the above relations we have

$$\begin{aligned}&\int_0^T (h_\varepsilon(|u_t|^2), \partial_t\varphi_t)dt - (h_\varepsilon(|u_T|^2), \varphi_T) + (h_\varepsilon(|u_0|^2), \varphi_0)\\&+\int_0^T (\mathcal{E}^{a,\hat{b}}(h_\varepsilon(|u_t|^2), \varphi_t) + \int \langle b\sigma, D_\sigma(h_\varepsilon(|u_t|^2))\rangle\varphi_t dm)dt\\=&\int_0^T -(h''_\varepsilon(|u_t|^2)|D_\sigma(|u_t|^2)|^2, \varphi_t) + (c(h_\varepsilon(|u|^2) - |u|^2 h'_\varepsilon(|u|^2)), \varphi)dt\\&+2\int_0^T (\langle f_t, u_t\rangle h'_\varepsilon(|u_t|^2), \varphi_t)dt - \int_0^T (h'_\varepsilon(|u_t|^2)(2|D_\sigma u_t|^2 + c|u_t|^2, \varphi_t)dt.\end{aligned}$$

As

$$|D_\sigma(|u|^2)|^2 = 4\langle u, D_\sigma u(D_\sigma u)^* u\rangle,$$

we deduce

$$\begin{aligned}&2\langle f, u\rangle h'_\varepsilon(|u|^2) - 2h'_\varepsilon(|u|^2)|D_\sigma u|^2 - h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2\\=&\frac{\langle f, u\rangle - |D_\sigma u|^2}{(|u|^2+\varepsilon)^{\frac{1}{2}}} + \frac{|u|^2\langle \hat{u}, D_\sigma u(D_\sigma u)^*\hat{u}\rangle}{(|u|^2+\varepsilon)^{\frac{3}{2}}}\\=&\frac{\langle f, u\rangle}{(|u|^2+\varepsilon)^{\frac{1}{2}}} - \frac{\varepsilon|D_\sigma u|^2 + |u|^2(|D_\sigma u|^2 - \langle \hat{u}, D_\sigma u(D_\sigma u)^*\hat{u}\rangle)}{(|u|^2+\varepsilon)^{\frac{3}{2}}}\\\leq&\frac{\langle f, u\rangle}{(|u|^2+\varepsilon)^{\frac{1}{2}}}.\end{aligned}$$

By Proposition 2.1.14 and since $c(h_\varepsilon(|u_s|^2) - 2|u_s|^2 h'_\varepsilon(|u_s|^2)) \leq 0$, we deduce

$$h_\varepsilon(|u_t|^2) \leq P_{T-t}h_\varepsilon(|\phi|^2) + \int_t^T P_{s-t}\frac{\langle f_s, u_s\rangle}{(|u_s|^2+\varepsilon)^{\frac{1}{2}}}ds.$$

Letting $\varepsilon \to 0$ the results follow. □

The next corollary is a version of the above proposition for general data, where we use P_t is a C_0-semigroup on L^1.

Corollary 2.1.15 Let $u = (u^1, ..., u^l)$ be a vector-valued function, where each component is a generalized solution of the linear equation (2.1.5) associated to certain data $f^i \in L^1([0,T]; L^2), \phi^i \in L^2$ for $i = 1, ..., l$. Denote by ϕ, f the vectors $\phi = (\phi^1, ..., \phi^l), f = (f^1, ..., f^l)$ and by $D_\sigma u$ the matrix whose rows consist of the row vectors $D_\sigma u^i$. Then the following relations hold m-almost everywhere

$$(2.1.19) \quad |u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u|^2 + \frac{1}{2}c|u_s|^2)ds = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle ds.$$

$$|u_t| \leq P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle ds. \tag{2.1.20}$$

Proof Analogously to the proof of Proposition 2.1.14 it is enough to verify (2.1.19) for $l = 1$. For $\phi \in L^2, f \in L^1([0,T], L^2)$, take ϕ_n, f_n as in the proof of Proposition 2.1.9, then we have

(a). $u_{n,t} := P_{T-t}\phi_n + \int_t^T P_{s-t}f_{n,s}ds$ is a generalized solution ,

(b). $\lim_{n\to\infty}\int_t^T \|f_{n,s} - f_s\|_2 ds = 0$,

(c). $\lim_{n\to\infty}\|\phi_n - \phi\|_2 = 0$,

(d). $\lim_{n\to\infty}\|u_n - u\|_T = 0$.

By Proposition 2.1.14 we have
(2.1.21)

$$|u_{n,t}|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_{n,s}|^2 + \frac{1}{2}c|u_{n,s}|^2)ds = P_{T-t}|\phi_n|^2 + 2\int_t^T P_{s-t}\langle u_{n,s}, f_{n,s}\rangle ds.$$

By (b) and (d) we obtain

$$\begin{aligned}
&\|\int_t^T P_{s-t}((u_{n,s}, f_{n,s}) - (u_s, f_s))ds\|_1 \\
\leq& C\int_t^T (\|u_{n,s}\|_2\|f_{n,s} - f_s\|_2 + \|f_s\|_2\|u_{n,s} - u_s\|_2)ds \\
\leq& C(\sup_{s\in[0,T]}\|u_{n,s}\|_2\int_t^T \|f_{n,s} - f_s\|_2 ds + \sup_{s\in[0,T]}\|u_{n,s} - u_s\|\int_t^T \|f_s\|_2 ds) \\
\to& 0, \text{ as } n\to\infty.
\end{aligned}$$

Here we used that P_t is a C_0-semigroup on $L^1(\mathbb{R}^d; m)$. By (d) we conclude that

$$\begin{aligned}
&\int_t^T \||D_\sigma u_{n,s}|^2 - |D_\sigma u_s|^2\|_1 ds \\
\leq&((\int_t^T \|D_\sigma u_{n,s}\|_2^2 ds)^{\frac{1}{2}} + (\int_t^T \|D_\sigma u_s\|_2^2 ds)^{\frac{1}{2}})(\int_t^T \|D_\sigma u_{n,s} - D_\sigma u_s\|_2^2 ds)^{\frac{1}{2}} \\
=&((\int_t^T \mathcal{E}^a(u_{n,s})ds)^{\frac{1}{2}} + (\int_t^T \mathcal{E}^a(u_s)ds)^{\frac{1}{2}})(\int_t^T \mathcal{E}^a(u_{n,s} - u_s)ds)^{\frac{1}{2}} \\
\to&0, \text{ as } n\to\infty,
\end{aligned}$$

and that

$$\begin{aligned}
&\int_t^T \||cu_{n,s}|^2-|cu_s|^2\|_1 ds\\
\leq&((\int_t^T \|c^{1/2}u_{n,s}\|_2^2 ds)^{\frac{1}{2}}+(\int_t^T \|c^{1/2}u_s\|_2^2 ds)^{\frac{1}{2}})(\int_t^T \|c^{1/2}u_{n,s}-c^{1/2}u_s\|_2^2 ds)^{\frac{1}{2}}\\
\leq&M((\int_t^T \mathcal{E}^{a,\hat{b}}_{c_2+1}(u_{n,s})ds)^{\frac{1}{2}}+(\int_t^T \mathcal{E}^{a,\hat{b}}_{c_2+1}(u_s)ds)^{\frac{1}{2}})(\int_t^T \mathcal{E}^{a,\hat{b}}_{c_2+1}(u_{n,s}-u_s)ds)^{\frac{1}{2}}\\
\to&0,\ \text{as } n\to\infty,
\end{aligned}$$

where we used (2.1.1) and (2.1.2) in the second inequality. Thus, we obtain

$$\lim_{n\to\infty}\int_t^T P_{s-t}(|D_\sigma u_{n,s}|^2)ds=\int_t^T P_{s-t}|D_\sigma u_s|^2 ds,$$

and

$$\lim_{n\to\infty}\int_t^T P_{s-t}(|cu_{n,s}|^2)ds=\int_t^T P_{s-t}|cu_s|^2 ds.$$

Passing to the limit $n\to\infty$ in equation (2.1.21) (2.1.19) follows. (2.1.20) also follows by using the same method. □

Lemma 2.1.16 If $f,g\in L^1([0,T];L^2)$ and $\phi\in L^2$, then:

$$\int_t^T P_{s-t}(f_sP_{T-s}\phi)ds\leq\frac{1}{2}P_{T-t}\phi^2+\int_t^T\int_s^T P_{s-t}(f_sP_{r-s}f_r)drds,\ m-a.e. \tag{2.1.22}$$

Proof Define

$$h_t:=P_{T-t}\phi,\qquad v_t:=\int_t^T P_{s-t}f_s ds.$$

By (2.1.19) we deduce

$$h_t^2+2\int_t^T P_{s-t}(|D_\sigma h_s|^2+\frac{1}{2}c|h_s|^2)ds=P_{T-t}\phi^2,$$

$$v_t^2+2\int_t^T P_{s-t}(|D_\sigma v_s|^2+\frac{1}{2}c|v_s|^2)ds=2\int_t^T P_{s-t}(f_s\int_s^T P_{r-s}f_r dr)ds,$$

and

$$h_tv_t+2\int_t^T P_{s-t}(\langle D_\sigma h_s,D_\sigma v_s\rangle+\frac{1}{2}ch_sv_s)ds=\int_t^T P_{s-t}(f_sP_{T-s}\phi)ds.$$

So, we obtain

$$\begin{aligned}\int_t^T P_{s-t}(f_s P_{T-s}\phi)ds =& h_t v_t + 2\int_t^T P_{s-t}(\langle D_\sigma h_s, D_\sigma v_s\rangle + \frac{1}{2}c h_s v_s)ds\\ \leq& \frac{1}{2}(h_t^2 + v_t^2) + \int_t^T P_{s-t}(|D_\sigma h_s|^2 + |D_\sigma v_s|^2 + \frac{1}{2}c|v_s|^2 + \frac{1}{2}c|h_s|^2)ds\\ =& \frac{1}{2}P_{T-t}\phi^2 + \int_t^T\int_s^T P_{s-t}(f_s P_{r-s} f_r)drds.\end{aligned}$$

□

2.2 The Non-linear Equation

In this section, we solve the non-linear equation (2.2.1). In the case of non-linear equations, we are going to consider systems of equations, with the unknown functions and their first-order derivatives mixed in the non-linear term of the equation. The non-linear term is a given measurable function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l$, $l \in \mathbb{N}$. We are going to treat the following system of equations.

$$(\partial_t + L)u + f(\cdot,\cdot,u,D_\sigma u) = 0 \qquad u_T = \phi. \tag{2.2.1}$$

Here $\phi \in L^2(\mathbb{R}^d, dm; \mathbb{R}^l)$.

Definition 2.2.1 (*Generalized solutions of the nonlinear equation*) A generalized solution of equation (2.2.1) is a system $u = (u^1, u^2, ..., u^l)$ of l elements in $\hat{F}$ with the property that $f^i(\cdot,\cdot,u,D_\sigma u)$ belongs to $L^1([0,T]; L^2)$ and there are sequences $\{u_n\}$ which are strong solutions of (2.2.1) with data (ϕ_n, f_n) such that

$$\|u_n - u\|_T \to 0, \|\phi_n - \phi\|_2 \to 0, \text{ and } \lim_{n\to\infty} f_n(\cdot,\cdot,u_n,D_\sigma u_n) = f(\cdot,\cdot,u,D_\sigma u) \text{ in } L^1([0,T]; L^2).$$

Definition 2.2.2 (*Mild equation*) A mild solution of equation (2.2.1) is a system $u = (u^1, u^2, ..., u^l)$ of l elements in $\hat{F}$, which has the property that each function $f^i(\cdot,\cdot,u,D_\sigma u)$ belongs to $L^1([0,T]; L^2(m))$ and such that for every $i \in \{1, ..., l\}$,

$$u^i(t,x) = P_{T-t}\phi^i(x) + \int_t^T P_{s-t} f^i(s,\cdot,u_s,D_\sigma u_s)(x)ds, m-a.e.. \tag{2.2.2}$$

Lemma 2.2.3 u is a generalized solution of the nonlinear equation (2.2.1) if and only if it solves the mild equation (2.2.2).

Proof The assertion follows by Proposition 2.1.9. □

We will use the following notations:

$$\|\phi\|_2^2 = \sum_{i=1}^{l} \|\phi^i\|_2^2, \phi \in L^2(\mathbb{R}^d; \mathbb{R}^l),$$

$$\mathcal{E}(u,v) = \sum_{i=1}^{l} \mathcal{E}(u^i, v^i), \mathcal{E}^a(u,v) = \sum_{i=1}^{l} \mathcal{E}^a(u^i, v^i), u, v \in F^l,$$

$$\mathcal{E}^{a,\hat{b}}(u,v) = \sum_{i=1}^{l} \mathcal{E}^{a,\hat{b}}(u^i, v^i), u, v \in F^l,$$

$$\|u\|_T^2 := \sup_{t\leq T} \|u_t\|_2^2 + \int_0^T \mathcal{E}_{c_2+1}^{a,\hat{b}}(u_t)dt, u \in \hat{F}^l.$$

2.2.1 The Case of Lipschitz Conditions

In this subsection we consider a measurable function $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^l\times\mathbb{R}^l\otimes\mathbb{R}^k \to \mathbb{R}^l$ such that

$$|f(t,x,y,z) - f(t,x,y',z')| \leq C(|y-y'| + |z-z'|), \tag{2.2.3}$$

with t, x, y, y', z, z' arbitrary and C is a constant independent of t, x. Set $f^0(t,x) := f(t,x,0,0)$.

Proposition 2.2.4 Suppose that the conditions (A1)-(A4) hold and that f satisfies condition (2.2.3), $f^0 \in L^2([0,T]\times\mathbb{R}^d, dt\times dm; \mathbb{R}^l)$ and $\phi \in L^2(\mathbb{R}^d;\mathbb{R}^l)$. Then the equation (2.2.1) admits a unique generalized solution $u \in \hat{F}^l$ and it satisfies the following estimate

$$\|u\|_T^2 \leq e^{T(1+2C+\frac{C^2}{c_1}+2\alpha+c_2)}(\|\phi\|_2^2 + \|f^0\|^2_{L^2([0,T]\times\mathbb{R}^d)}).$$

Proof If $u \in \hat{F}^l$, then by relation (2.2.3) we have

$$\begin{aligned}|f(\cdot,\cdot,u,D_\sigma u)| &\leq |f(\cdot,\cdot,u,D_\sigma u) - f(\cdot,\cdot,0,0)| + |f(\cdot,\cdot,0,0)| \\ &\leq C(|u| + |D_\sigma u|) + |f^0|.\end{aligned}$$

As $f^0 \in L^2([0,T]\times\mathbb{R}^d, dt\times dm; \mathbb{R}^l)$ and $|D_\sigma u|$ is an element of $L^2([0,T]\times\mathbb{R}^d)$, we get $f(\cdot,\cdot,u,D_\sigma u) \in L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^l)$.

Now we define the operator $A : \hat{F}^l \to \hat{F}^l$ by

$$(Au)^i_t := P_{T-t}\phi^i + \int_t^T P_{s-t} f^i(s, \cdot, u_s, D_\sigma u_s) ds, \qquad i = 1, ..., l.$$

Then Proposition 2.1.9 implies that $Au \in \hat{F}^l$. In the following we write $f^i_{u,s} := f^i(s, \cdot, u_s, D_\sigma u_s)$. Since $(Au)^i_t - (Av)^i_t = \int_t^T P_{s-t}(f^i_{u,s} - f^i_{v,s})ds$ is the mild solution with data $(f^i_u - f^i_v, 0)$, by the same arguments as in the proof of Proposition 2.1.9 we have

$$\begin{aligned}
\| \int_t^T P_{s-t}(f^i_{u,s} - f^i_{v,s}) ds \|^2_{[t,T]} &\le M_T (\int_t^T \|f_{u,s} - f_{v,s}\|_2 ds)^2 \\
&\le M_T (T - t) \int_t^T \|f_{u,s} - f_{v,s}\|_2^2 ds \\
&\le M_T (T - t) \int_t^T (\|u_s - v_s\|_2^2 + \|D_\sigma u_s - D_\sigma v_s\|_2^2) ds \\
&\le M_T (T - t) \|u - v\|^2_{[t,T]},
\end{aligned}$$

where M_T can change from line to line. Here $\|u\|_{[T_a,T_b]} := (\sup_{t\in[T_a,T_b]} \|u_t\|_2^2 + \int_{T_a}^{T_b} \mathcal{E}^{a,\hat{b}}_{c_2+1}(u_t)dt)^{\frac{1}{2}}$, where $0 \le T_a \le T_b \le T$. Fix T_1 sufficiently small such that $\gamma := M_T(T - T_1) < 1$. Then we have :

$$\|Au - Av\|^2_{[T_1,T]} \le \gamma \|u - v\|^2_{[T_1,T]}.$$

Then there exists a unique $u_1 \in \hat{F}_{[T_1,T]}$ such that $Au_1 = u_1$ where $\hat{F}_{[T_a,T_b]} := C([T_a, T_b]; L^2) \cap L^2((T_a, T_b); F)$ for $T_a \in [0, T]$ and $T_b \in [T_a, T]$.

We define the operator $A^1 : \hat{F}^l \to \hat{F}^l$ by

$$(A^1 u)^i_t := P_{T_1 - t} u^i_{1,T_1} + \int_t^{T_1} P_{s-t} f^i(s, \cdot, u_s, D_\sigma u_s) ds, \qquad i = 1, ..., l.$$

Then by the same method as above, we get

$$\|A^1 u - A^1 v\|^2_{[t,T_1]} \le M_T(T_1 - t)\|u - v\|^2_{[t,T_1]}.$$

Now we choose $T_2 < T_1$ such that $M_T(T_1 - t) < 1$. We obtain that there exists a unique $u_2 \in \hat{F}_{[T_2,T_1]}$ such that $A^1 u_2 = u_2$. If we define $u := u_1 1_{[T_1,T]} + u_2 1_{[T_2,T_1)}$, then for $T_2 \le t \le T_1$

$$P_{T-t}\phi^i + \int_t^T P_{s-t} f^i(s, \cdot, u_s, D_\sigma u_s) ds$$

$$
\begin{aligned}
&= P_{T-t}\phi^i + \int_t^{T_1} P_{s-t}f^i(s,\cdot,u_s,D_\sigma u_s)ds + \int_{T_1}^{T} P_{s-t}f^i(s,\cdot,u_{1,s},D_\sigma u_{1,s})ds \\
&= P_{T-t}\phi^i + \int_t^{T_1} P_{s-t}f^i(s,\cdot,u_s,D_\sigma u_s)ds + P_{T_1-t}(u^i_{1,T_1} - P_{T-T_1}\phi^i) \\
&= P_{T-t}\phi^i + \int_t^{T_1} P_{s-t}f^i(s,\cdot,u_s,D_\sigma u_s)ds + P_{T_1-t}u^i_{1,T_1} - P_{T-t}\phi^i \\
&= u^i_{2,t}.
\end{aligned}
$$

If $t > T_1$,

$$
\begin{aligned}
(Au)^i_t &= P_{T-t}\phi^i + \int_t^{T} P_{s-t}f^i(s,\cdot,u_{1,s},D_\sigma u_{1,s})ds \\
&= u^i_{1,t},
\end{aligned}
$$

Therefore, we can construct a solution over the interval $[T_2, T]$. Clearly there exists $n \in \mathbb{N}$ such that $T < n(T - T_1)$. Hence, the construction is done after n steps.

In order to obtain the estimate in the statement, we write

$$
\begin{aligned}
&|\int_t^T (f_{u,s}, u_s)ds| \\
\le& \int_t^T \|f^0_s\|_2\|u_s\|_2 ds + C\int_t^T \|u_s\|_2^2 ds + C\int_t^T \|D_\sigma u_s\|_2\|u_s\|_2 ds \\
\le& \frac{1}{2}\int_t^T \|f^0_s\|_2^2 ds + (\frac{1}{2} + C + \frac{1}{2c_1}C^2)\int_t^T \|u_s\|_2^2 ds + \frac{c_1}{2}\int_t^T \mathcal{E}^a(u_s)ds.
\end{aligned}
$$

By relation (2.1.7) of Proposition 2.1.9 it follows that

$$
\begin{aligned}
\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds \le& 2\int_t^T (f_{u,s}, u_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s\|_2^2 ds \\
\le& \|\phi\|_2^2 + \int_t^T \|f^0_s\|_2^2 ds + (1 + 2C + \frac{C^2}{c_1} + 2\alpha + c_2)\int_t^T \|u_s\|_2^2 ds \\
&+ \int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds.
\end{aligned}
$$

Now by Gronwall's lemma the desired estimate follows.

[Uniqueness] Let u_1 and u_2 be two solutions of equation (2.2.1). By using (2.1.7) for the difference $u_1 - u_2$ we get

$$
\begin{aligned}
&\|u_{1,t} - u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_{1,s} - u_{2,s})ds \\
\le& 2\int_t^T (f(s,\cdot,u_{1,s},D_\sigma u_{1,s}) - f(s,\cdot,u_{2,s},D_\sigma u_{2,s}), u_{1,s} - u_{2,s})ds + 2\alpha\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds
\end{aligned}
$$

$$\leq 2\int_t^T C(|D_\sigma u_{1,s} - D_\sigma u_{2,s}|, |u_{1,s} - u_{2,s}|)ds + (2\alpha + C)\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds$$
$$\leq (\frac{C^2}{c_1} + c_2 + 2\alpha + C)\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_{1,s} - u_{2,s})ds.$$

By Gronwall's lemma it follows that

$$\|u_{1,t} - u_{2,t}\|_2^2 = 0,$$

hence $u_1 = u_2$. □

2.2.2 The Case of Monotonicity Conditions

Let $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^l\times\mathbb{R}^l\otimes\mathbb{R}^k \to \mathbb{R}^l$ be a measurable function and $\phi \in L^2(\mathbb{R}^d, m; \mathbb{R}^l)$ be the final condition of (2.2.1). We impose the following conditions:

(H1) (*Lipschitz condition in z*) There exists a fixed constant $C > 0$ such that for t, x, y, z, z' arbitrary

$$|f(t,x,y,z) - f(t,x,y,z')| \leq C|z - z'|.$$

(H2) (*Monotonicity condition in y*) For x, y, y', z arbitrary, there exists a function $\mu_t \in L^1([0,T];\mathbb{R})$ such that

$$\langle y - y', f(t,x,y,z) - f(t,x,y',z)\rangle \leq \mu_t|y - y'|^2.$$

We set $\alpha_t := \int_0^t \mu_s ds$.

(H3) (*Continuity condition in y*) For t, x and z fixed, the map

$$y \mapsto f(t,x,y,z)$$

is continuous.

We need the following notations:

$$f^0(t,x) := f(t,x,0,0), \qquad f'(t,x,y) := f(t,x,y,0) - f(t,x,0,0),$$

$$f'^{,r}(t,x) := \sup_{|y|\leq r} |f'(t,x,y)|.$$

(H4) For each $r > 0$, $f'^{,r} \in L^1([0,T]; L^2)$.

(H5) $\|\phi\|_\infty < \infty, \|f^0\|_\infty < \infty, |\phi| \in L^2, |f^0| \in L^2([0,T]; L^2)$.

If $m(\mathbb{R}^d) < \infty$ the last two conditions in (H5) are ensured by the boundedness of ϕ and f^0. The conditions (H1), (H4), and (H5) imply that if $u \in b\hat{F}$, then

$|f(u, D_\sigma u)| \in L^1([0,T]; L^2)$. It seems impossible to apply general monotonicity methods to the map $\mathcal{V} \ni u \mapsto f(t,\cdot,u(\cdot), D_\sigma u) \in \mathcal{V}'$ because of lack of a suitable reflexive Banach space $\mathcal{V}$ such that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. Therefore, also here we proceed developing a hands-on approach to prove existence and uniqueness of solutions for equation (2.2.1) as done in [BPS05].

Lemma 2.2.5 In (H2) without loss of generality we can assume that $\mu_t \equiv 0$.

Proof Let us make the change $u_t^* = \exp(\alpha_t) u_t$ and

$$\phi^* = \exp(\alpha_T)\phi, \qquad f_t^*(y,z) = \exp(\alpha_t) f_t(\exp(-\alpha_t)y, \exp(-\alpha_t)z) - \mu_t y,$$

for the data. Next we will prove that u is a generalized solution associated to the data (ϕ, f) if and only if u^* is a solution associated to the data (ϕ^*, f^*). Hence we can write

$$u_t^i = P_{T-t}\phi^i(x) + \int_t^T P_{s-t} f^i(s,\cdot,u_s, D_\sigma u_s)(x) ds,$$

equivalently as

$$\begin{aligned}
u_t^{i,*} =& \exp(\alpha_t) P_{T-t}\phi^i(x) + \exp(\alpha_t)\int_t^T P_{s-t} f^i(s,\cdot,u_s,D_\sigma u_s)(x)ds \\
=& \exp(\alpha_T) P_{T-t}\phi^i(x) + (\exp(\alpha_t) - \exp(\alpha_T)) P_{T-t}\phi^i(x) \\
&+ \int_t^T (\exp(\alpha_t) - \exp(\alpha_s)) P_{s-t} f^i(s,\cdot,u_s,D_\sigma u_s)(x)ds \\
&+ \int_t^T \exp(\alpha_s) P_{s-t} f^i(s,\cdot,u_s,D_\sigma u_s)(x)ds \\
=& P_{T-t}\phi^{i,*}(x) + \int_t^T P_{s-t}(\exp(\alpha_s) f^i(s,\cdot,u_s,D_\sigma u_s)(x))ds \\
&- \int_t^T \mu_s \exp(\alpha_s) P_{T-t}\phi^i(x) ds - \int_t^T \int_t^l \mu_s \exp(\alpha_s) P_{l-t} f^i(l,\cdot,u_l,D_\sigma u_l) ds dl \\
=& P_{T-t}\phi^{i,*}(x) + \int_t^T P_{s-t}(\exp(\alpha_s) f^i(s,\cdot,\exp(\alpha_s)u_s^*, \exp(\alpha_s) D_\sigma u_s^*)(x))ds \\
&- \int_t^T P_{s-t}(\mu_s \exp(\alpha_s) P_{T-s}\phi^i(x) + \int_s^T \mu_s \exp(\alpha_s) P_{l-s} f^i(l,\cdot,u_l,D_\sigma u_l)dl)ds \\
=& P_{T-t}\phi^{i,*}(x) + \int_t^T P_{s-t} f^{i,*}(s,\cdot,u_s^*,D_\sigma u_s^*)(x)ds.
\end{aligned}$$

Next we prove f^* satisfies (H1)-(H5). It is obvious that (H1), (H3)-(H5) are satisfied.

Let us prove that f^* satisfies (H2) with $\mu_t \equiv 0$. We have

$$\begin{aligned}
&\langle y-y', f^*(t,x,y,z) - f^*(t,x,y',z)\rangle \\
=&\langle y-y', \mu_t y' - \mu_t y\rangle \\
&+ (\exp(\alpha_t))^2 \langle \exp(-\alpha_t)y - \exp(-\alpha_t)y', f(t,x,\exp(-\alpha_t)y, \exp(-\alpha_t)z)\rangle \\
&- (\exp(\alpha_t))^2 \langle \exp(-\alpha_t)y - \exp(-\alpha_t)y', f(t,x,\exp(-\alpha_t)y', \exp(-\alpha_t)z)\rangle \\
\le& -|y-y'|^2\mu_t + \mu_t(\exp(\alpha_t))^2 |\exp(-\alpha_t)y - \exp(-\alpha_t)y'|^2 \\
=&0.
\end{aligned}$$

Thus, by making the transformation $f \to f^*$, we can assume that $\mu_t \equiv 0$. $\square$

Lemma 2.2.6 Suppose that conditions (A1)-(A4), (H1) and the following weaker form of condition (H2) (with $\mu_t \equiv 0$) hold:

(H2') $\langle y, f'(t,x,y)\rangle \le 0$ for all t, x, y.

If u is a generalized solution of (2.2.1), then there exists a constant K depending on C, μ_t, T, α such that

$$\|u\|_T^2 \le K(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt). \tag{2.2.4}$$

Proof Since u is a solution of (2.2.1), by Proposition 2.1.9 we have

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds \le 2\int_t^T (f_s, u_s)ds + \|u_T\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds.$$

Conditions (H1) and (H2') yield

$$\begin{aligned}
\langle f_s(u_s, D_\sigma u_s), u_s\rangle =&\langle f_s(u_s, D_\sigma u_s) - f_s(u_s, 0) + f_s'(u_s) + f_s^0, u_s\rangle \\
\le& |f_s(u_s, D_\sigma u_s) - f_s(u_s,0)||u_s| + \langle f_s'(u_s), u_s\rangle + |f_s^0||u_s| \\
\le& (C|D_\sigma u_s| + |f_s^0|)|u_s|.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds \le& 2\int_t^T \int (C|D_\sigma u_s| + |f_s^0|)|u_s| dm ds + \|u_T\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds \\
\le& \int_t^T \mathcal{E}^{a,\hat{b}}(u_s)ds + (\frac{C^2}{c_1} + 1 + 2\alpha + c_2)\int_t^T \|u_s\|_2^2 ds + \int_t^T \|f_s^0\|_2^2 ds \\
&+ \|u_T\|_2^2.
\end{aligned}$$

Then Gronwall's lemma yields

$$\|u\|_T^2 \leq K(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt).$$

□

By a modification of the arguments in [BPS05, Lemma 3.3] we obtain the following estimates.

Lemma 2.2.7 Suppose that conditions (A1)-(A4), (H1) and (H2') hold. If u is a generalized solution of (2.2.1), there exists a constant K, which depends on C, μ and T, such that

$$\|u\|_\infty \leq K(\|\phi\|_\infty + \|f^0\|_\infty). \tag{2.2.5}$$

Proof By Corollary 2.1.15 we have

$$|u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u|^2) \leq P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s(u_s, D_\sigma u_s)\rangle ds. \tag{2.2.6}$$

Following the same arguments as the proof of Lemma 2.2.6 we deduce

$$\langle f_s(u_s, D_\sigma u_s), u_s\rangle \leq (C|D_\sigma u_s| + |f_s^0|)|u_s|.$$

By Corollary 2.1.15 (2.1.20) we obtain

$$|u_s| \leq P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)dr.$$

Then we have

$$\begin{aligned}
&\int_t^T P_{s-t}\langle f_s(u_s, D_\sigma u_s), u_s\rangle ds \\
\leq &\int_t^T P_{s-t}[(P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)dr)(C|D_\sigma u_s| + |f_s^0|)]ds.
\end{aligned}$$

So, by (2.2.6) and Lemma 2.1.16 we obtain

$$\begin{aligned}
&|u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_s|^2)ds \\
\leq &P_{T-t}|\phi|^2 + 2(\int_t^T P_{s-t}[(P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)dr)(C|D_\sigma u_s| + |f_s^0|)]ds)
\end{aligned}$$

$$\leq 3P_{T-t}|\phi|^2 + 2C^2\int_t^T\int_s^T P_{s-t}(|D_\sigma u_s|P_{r-s}|D_\sigma u_r|)drds + 2\int_t^T\int_s^T P_{s-t}(|f_s^0|P_{r-s}|f_r^0|)drds$$
$$+2\int_t^T\int_s^T P_{s-t}[P_{r-s}(C|D_\sigma u_r| + |f_r^0|)(C|D_\sigma u_s| + |f_s^0|)]drds.$$

Since

$$\begin{aligned}
&\int_t^T\int_s^T P_{s-t}[P_{r-s}(C|D_\sigma u_r| + |f_r^0|)(C|D_\sigma u_s| + |f_s^0|)]drds \\
\leq&\frac{1}{2}\int_t^T\int_s^T [P_{s-t}(C|D_\sigma u_s| + |f_s^0|)^2] + P_{s-t}[(P_{r-s}(C|D_\sigma u_r| + |f_r^0|))^2]drds \\
\leq&\int_t^T\int_s^T C^2P_{s-t}|D_\sigma u_s|^2 + P_{s-t}|f_s^0|^2 + \frac{1}{2}P_{r-t}(C|D_\sigma u_r| + |f_r^0|)^2drds \\
\leq&2C^2(T-t)\int_t^T P_{s-t}|D_\sigma u_s|^2ds + 2(T-t)\int_t^T P_{s-t}|f_s^0|^2ds,
\end{aligned}$$

and by Schwartz's inequality one has

$$\begin{aligned}
&\int_t^T\int_s^T P_{s-t}(|D_\sigma u_s|P_{r-s}|D_\sigma u_r|)drds \\
\leq&\int_t^T\int_s^T \frac{1}{2}(P_{s-t}|D_\sigma u_s|^2)drds + \int_t^T\int_s^T \frac{1}{2}(P_{r-t}|D_\sigma u_r|^2)drds \\
\leq&(T-t)\int_t^T P_{s-t}|D_\sigma u_s|^2ds,
\end{aligned}$$

we conclude

$$\begin{aligned}
&|u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_s|^2)ds \\
\leq&3P_{T-t}|\phi|^2 + 6C^2(T-t)\int_t^T P_{s-t}|D_\sigma u_s|^2ds + 6(T-t)\int_t^T P_{s-t}|f_s^0|^2ds.
\end{aligned}$$

So we can deduce by iteration the estimate over the interval $[0,T]$ and obtain

$$\begin{aligned}
|u_t|^2 \leq& \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\tilde{K}(P_{T-t}|\phi|^2 + (T-t)\int_t^T P_{s-t}|f_s^0|^2ds) \\
\leq& \sup_{t\in[0,T]}\tilde{K}(\|\phi^2\|_\infty + T^2\|f^0\|_\infty^2) \\
\leq& K^2(\|\phi\|_\infty^2 + \|f^0\|_\infty^2),
\end{aligned}$$

which implies (2.2.5). $\square$

By the same methods as in [BPS05, Theorem 3.2], we obtain the following results. As the method is similar as in the proof of [BPS05, Theorem 3.2], we will give the proof in the Appendix A.

Theorem 2.2.8 Suppose that $m(dx)$ is a finite measure and that conditions (A1)-(A4), (H1)-(H5) hold. Then there exists a unique generalized solution of equation (2.2.1) and it satisfies the following estimates for some K_1 and K_2 independent of u, ϕ, f

$$\|u\|_T^2 \leq K_1(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt).$$

$$\|u\|_\infty \leq K_2(\|\phi\|_\infty + \|f^0\|_\infty).$$

The following lemma is essential to the case that $m(dx) = dx$.

Lemma 2.2.9 Assume conditions (A1)-(A4),(H1)-(H5) hold. If $u \in \hat{F}$ is bounded and for $\varphi \in b\mathcal{C}_T$ satisfies

$$\int_0^T \mathcal{E}(u_t, \varphi_t) + (u_t, \partial_t \varphi_t) dt = \int_0^T (f_t(u_t, D_\sigma u_t), \varphi_t) dt + (u_T, \varphi_T) - (u_0, \varphi_0).$$

Then we have

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s) ds \leq 2\int_t^T (f_s(u_s, D_\sigma u_s), u_s) ds + \|\phi\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds, \quad 0 \leq t \leq T.$$

Proof Define $u_t^h = \frac{1}{2h}\int_{t-h}^{t+h} u_s ds$. Choose $\phi_t = u_t^h$, then we have for $t_0 \in [0, T]$

$$\int_{t_0}^T \mathcal{E}(u_t, u_t^h) + \frac{1}{2h}(u_t, u_{t+h}) - \frac{1}{2h}(u_t, u_{t-h}) dt = \int_{t_0}^T (f_t(u_t, D_\sigma u_t), u_t^h) dt + (u_T, u_T^h) - (u_{t_0}, u_{t_0}^h).$$

That is to say,

$$\begin{aligned} &\frac{1}{2h}\int_{T-h}^T (u_t, u_{t+h}) dt - \frac{1}{2h}\int_{t_0-h}^{t_0} (u_t, u_{t+h}) dt + \int_{t_0}^T \mathcal{E}(u_t, u_t^h) dt \\ &= \int_{t_0}^T (f_t(u_t, D_\sigma u_t), u_t^h) dt + (u_T, u_T^h) - (u_{t_0}, u_{t_0}^h). \end{aligned} \tag{2.2.7}$$

Letting $h \to 0$ in (2.2.7), the assertion follows . □

For the case $m(dx) = dx$, we will use a weight function of the form $\pi(x) = \exp[-\rho\theta(x)]$, with $\theta \in C^1(\mathbb{R}^d)$ being a fixed function such that $0 \leq \theta(x) \leq |x|$, and $\theta(x) = |x|$ if $|x| \geq 1$, and $\rho \in \mathbb{R}^+$. If one chooses $\rho > 0$, then clearly one has $m(\mathbb{R}^d) < \infty$. We denote the generalized Dirichlet form, function spaces and the generator associated with $\rho > 0$ by $\mathcal{E}^\rho$, $\hat{F}^\rho$, $\mathcal{C}_T^\rho$, L_ρ respectively. In the case $\rho = 0$, we drop ρ in the notation, i.e. $\mathcal{E} = \mathcal{E}^0$. And for the case $\rho = 0$, we need the following condition.

(A2') (*Sobolev inequality*) For $\rho = 0$, σ is a bounded measurable field in $\mathbb{R}^d$ and

$$\|u\|_q \leq C\mathcal{E}^a(u,u)^{1/2},\ \forall u \in C_0^\infty(\mathbb{R}^d),$$

where $\frac{1}{q} + \frac{1}{d} = \frac{1}{2}$ and $\|\cdot\|_q$ denotes the usual norm in L^q. And $|\hat{b}\sigma| \in L^d(\mathbb{R}^d; dx) + L^\infty(\mathbb{R}^d; dx)$, $c \in L^{d/2}(\mathbb{R}^d; dx) + L^\infty(\mathbb{R}^d; dx)$.

If (A2'), (A3) are satisfied, for $u, v \in bF$, we have

$$\mathcal{E}^\rho(u,v) = \int \langle D_\sigma u, D_\sigma v\rangle dm + \int cuvdm + \int \langle (b\sigma + \hat{b}\sigma, D_\sigma u\rangle vdm.$$

If $\rho = 0$, we additionally have

$$\mathcal{E}^{a,\hat{b}}(u,u) \leq C\mathcal{E}_1^a(u,u),$$

and that $F = F^a$. We consider the following condition,which is a technical condition for our proof:

(H6). $\mathcal{E}^a(u) < \infty, u \in L^2 \Rightarrow u \in F$.

The Sobolev inequality and (H6) are satisfied if a is uniformly elliptic. By [S09, Lemma 4.20] we have:

Lemma 2.2.10 Assume conditions (A2'), (A3) and (H6) hold. Let $\rho > 0$. Then it holds

$$\mathcal{E}^\rho(u,\varphi) = \mathcal{E}(u, \varphi\exp(-\theta\rho)) + (M_\rho u, \varphi)_\rho,$$

for $u \in F_\rho, \varphi \in bF_\rho$, where $M_\rho u = \rho\langle D_\sigma\theta, D_\sigma u\rangle$.

Theorem 2.2.11 Suppose that $m(dx) = dx$ and that the conditions (A1),(A2'),(A3), (A4) (H1)-(H5), (H6) hold. Then there exists a unique generalized solution of equation (2.2.1) and it satisfies the following estimates with constants K_1 and K_2 independent of u, ϕ, f

$$\|u\|_T^2 \leq K_1(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt).$$

$$\|u\|_\infty \leq K_2(\|\phi\|_\infty + \|f^0\|_\infty).$$

Proof Set for $\rho > 0$

$$f^\rho(t,x,y,z) := f(t,x,y,z) + \rho \sum_{l=1}^{k} \sum_{i=1}^{d} \sigma_l^i(x)\partial_i\theta(x) z_l(x),$$

and consider

$$(\partial_t + L_\rho)u + f^\rho(u, D_\sigma u) = 0, \qquad u_T = \phi. \tag{2.2.8}$$

The associated weak equation has the form $\forall \varphi \in b\mathcal{C}_T^\rho$

$$\int_0^T \mathcal{E}^\rho(u_t,\varphi_t) + (u_t,\partial_t\varphi_t)_\rho dt = \int_0^T (f_t^\rho(u_t, D_\sigma u_t),\varphi_t)_\rho dt + (u_T,\varphi_T)_\rho - (u_0,\varphi_0)_\rho. \tag{2.2.9}$$

As f^ρ satisfies conditions (H1)-(H5), we have a generalized solution u^ρ of (2.2.8).

Fix $\rho > 0$ and take $f_n \in C_0^\infty(\mathbb{R}^d)$ such that $f_n(x) = 1$ for $x \in B_n(0)$, $f_n(x) = 0$ for $x \in B_{2n}^c(0)$, $\partial_{x_i} f_n(x)$ are uniformly bounded and $\partial_{x_i} f_n(x) \to 0$ as $n \to \infty$. If $\varphi \in b\mathcal{C}_T$, then $\varphi f_n \exp(\theta\rho) \in b\mathcal{C}_T^\rho$. As

$$\begin{aligned}&\int_0^T \mathcal{E}^\rho(u_t^\rho, \varphi_t f_n \exp(\theta\rho)) + (u_t^\rho, \partial_t\varphi_t f_n) dt\\ =&\int_0^T (f_t^\rho(u_t^\rho, D_\sigma u_t^\rho), f_n\varphi_t)dt + (u_T^\rho, f_n\varphi_T) - (u_0^\rho, f_n\varphi_0),\end{aligned}$$

by Lemma 2.2.10 we have

$$\int_0^T \mathcal{E}(u_t^\rho,\varphi_t f_n)+(u_t^\rho,\partial_t\varphi_t f_n)dt = \int_0^T (f_t(u_t^\rho, D_\sigma u_t^\rho), f_n\varphi_t)dt+(u_T^\rho, f_n\varphi_T)-(u_0^\rho, f_n\varphi_0). \tag{2.2.10}$$

If $u \in \hat{F}_{\tilde\rho}$ satisfies (2.2.10) for fixed $\tilde\rho$ with test function $\varphi \in b\mathcal{C}_T$, then u satisfies (2.2.9) for $\rho \geq \tilde\rho$, with test functions φ where $\varphi \in b\mathcal{C}_T^\rho$.

Now fix $\rho_1 > 0$. Then there exists a solution u^{ρ_1} of (2.2.8) associated to ρ_1. We conclude that u^{ρ_1} satisfies the weak equation (2.2.9) for all $\rho > \rho_1$ with $\varphi \in b\mathcal{C}_T^\rho$. Then by Lemma 2.2.9 and the same arguments as the uniqueness proof of Theorem 2.2.8 we have $u^{\rho_1} = u^\rho$ for all $\rho > \rho_1$.

Finally, we deduce that a solution $u^{\tilde\rho}$ of (2.2.8) associated to $\tilde\rho$ is a solution of (2.2.8) for all $\rho > 0$. Then by Theorem 2.2.8, we have

$$\|u^{\tilde\rho}\|_{T,\rho}^2 \leq K_1(\|\phi\|_{2,\rho}^2 + \int_0^T \|f_t^0\|_{2,\rho}^2 dt).$$

Letting $\rho \to 0$, by Fatou's Lemma, we obtain

$$\begin{aligned}\|u^{\tilde{\rho}}\|_T^2 &= \lim_{\rho\to 0}\|u^{\tilde{\rho}}\|_{T,\rho}^2\\ &\le \lim_{\rho\to 0} K_1(\|\phi\|_{2,\rho}^2 + \int_0^T \|f_t^0\|_{2,\rho}^2 dt)\\ &= K_1(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt),\end{aligned}$$

and

$$\|u^{\tilde{\rho}}\|_\infty \le K_2(\|\phi\|_\infty + \|f^0\|_\infty).$$

By (H6), we have $u^{\tilde{\rho}} \in L^2((0,T),F)$. For $u^{\tilde{\rho}} \in \hat{F}^\rho$ for $\rho > 0$, we obtain for any $h_n \to 0$,

$$\|u^{\tilde{\rho}}_{t+h_n} - u^{\tilde{\rho}}_t\|_{2,\rho} \to 0.$$

Then there exists a subsequence such that $u^{\tilde{\rho}}_{t+h_{n_k}} \to u^{\tilde{\rho}}_t$ for m_ρ-almost every x. Hence, $u^{\tilde{\rho}}_{t+h_{n_k}} \to u^{\tilde{\rho}}_t$ for dx-almost every x. Then by the same arguments as the proof of Lemma 2.2.9, we have

$$\begin{aligned}|\|u_t^{\tilde{\rho}}\|_{2,\rho}^2 - \|u_{t+h}^{\tilde{\rho}}\|_{2,\rho}^2| \le& 2[|\int_t^{t+h}(u_s^{\tilde{\rho}}, f_s^\rho)_\rho ds| + |\int_t^{t+h}\mathcal{E}^{a,\hat{b}}(u_s^{\tilde{\rho}})ds| + \alpha\int_t^{t+h}\|u_s^{\tilde{\rho}}\|_{2,\rho}^2 ds]\\ \le& M\int_t^{t+h}\|f_s\|_{2,\rho}ds + M\int_t^{t+h}\mathcal{E}_{c_2+1}^{a,\hat{b}}(u_s^{\tilde{\rho}})ds.\end{aligned}$$

Letting $\rho \to 0$, we get

$$|\|u_t^{\tilde{\rho}}\|_2^2 - \|u_{t+h}^{\tilde{\rho}}\|_2^2| \le M\int_t^{t+h}\|f_s\|_2 ds + M\int_t^{t+h}\mathcal{E}_{c_2+1}^{a,\hat{b}}(u_s^{\tilde{\rho}})ds.$$

Hence we have $u^{\tilde{\rho}}_{t+h_{n_k}} \to u^{\tilde{\rho}}_t$ in $L^2(\mathbb{R}^d, dx)$. Since this reason holds for every sequence $h_n \to 0$, we have $u^{\tilde{\rho}} \in \mathcal{C}([0,T], L^2)$, hence $u^{\tilde{\rho}} \in \hat{F}$. By the above arguments, we deduce that

$$\int_0^T \mathcal{E}(u_t^{\tilde{\rho}}, \varphi_t f_n) + (u_t^{\tilde{\rho}}, \partial_t\varphi_t f_n)dt = \int_0^T (f_t(u_t^{\tilde{\rho}}, D_\sigma u_t^{\tilde{\rho}}), f_n\varphi_t)dt + (u_T^{\tilde{\rho}}, f_n\varphi_T) - (u_0^{\tilde{\rho}}, f_n\varphi_0).$$

Letting $n \to \infty$, we conclude that

$$\int_0^T \mathcal{E}(u_t^{\tilde{\rho}}, \varphi_t) + (u_t^{\tilde{\rho}}, \partial_t\varphi_t)dt = \int_0^T (f_t(u_t^{\tilde{\rho}}, D_\sigma u_t^{\tilde{\rho}}), \varphi_t)dt + (u_T^{\tilde{\rho}}, \varphi_T) - (u_0^{\tilde{\rho}}, \varphi_0).$$

Since $f_t(u_t^{\tilde{\rho}}, D_\sigma u_t^{\tilde{\rho}}) \in L^1([0,T]; L^2)$, we can choose $(f^n)_{n\in\mathbb{N}} \subset C_0^\infty([0,T]\times\mathbb{R}^d)$ such that $\int_0^T \|f_t^n - f_t(u_t^{\tilde{\rho}}, D_\sigma u_t^{\tilde{\rho}})\|_2 dt \to 0$. Let v^n be the generalized solution associated with (f^n,ϕ). Then v^n is bounded. For

$$v_t := P_{T-t}\phi(x) + \int_t^T P_{s-t}f(s,\cdot,u_s^{\tilde{\rho}},D_\sigma u_s^{\tilde{\rho}})(x)ds,$$

we have $\|v^n - v\|_T \to 0$. On the other hand, by Lemma 2.2.9 we have

$$\begin{aligned}
&\|u_t^{\tilde{\rho}} - v_t^n\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_s^{\tilde{\rho}} - v_s^n)ds \\
\leq& 2\int_t^T (f_s(u_s^{\tilde{\rho}},D_\sigma u_s^{\tilde{\rho}}) - f_s^n, u_s^{\tilde{\rho}} - v_s^n)ds + 2\alpha\int_t^T \|u_s^{\tilde{\rho}} - v_s^n\|_2^2 ds \\
\leq& 2M\int_t^T \|f_s(u_s^{\tilde{\rho}},D_\sigma u_s^{\tilde{\rho}}) - f_s^n\|_2 ds + 2\alpha\int_t^T \|u_s^{\tilde{\rho}} - v_s^n\|_2^2 ds.
\end{aligned}$$

By Gronwall's lemma we obtain $\|v^n - u^{\tilde{\rho}}\|_T \to 0$, as $n \to \infty$. Therefore, we have $u_t^{\tilde{\rho}} = v_t$. That is to say $u^{\tilde{\rho}}$ is a mild solution of (2.2.1). □

2.3 Martingale representation for the processes

The Brownian motion has the *martingale representation property*: any martingale with respect to the filtration generated by the Brownian motion can be expressed as an Itô integral against the Brownian motion. In classic case, this property is essential to the existence of the solution of a BSDE.

The martingale representation property of a family of martingales has been studied in a huge of literature. Several general results have been obtained, for example, Jacod and Yor [JY77] have discovered the equivalence between the martingale representation property and the extremal property of martingale measures. However, when applied to specific situation, further work and hard estimates are often required. Recently, a lot of work (see e.g. [BPS05], [QY10], [Zh]) extend the martingale representation property to Markov processes associated with Dirichlet forms. In this section, we extend the martingale representation theorem under the framework of generalized Dirichlet forms.

2.3.1 Representation under P^x

In order to obtain the results for the probabilistic part, we need $\mathcal{E}$ to be a quasi-regular generalized Dirichlet form (Definition 1.3) in the sense of Remark 2.1.1 (iii) with $c_2, \hat{c} \equiv 0$ and $c \equiv 0$. The Markov process $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P^x)$ with shift operator $(\theta_t)_{t\geq 0}$ is properly associated in the resolvent sense with $\mathcal{E}$, i.e. $R_\alpha f := E^x\int_0^\infty e^{-\alpha t}f(X_t)dt$ is an $\mathcal{E}$-quasi-continuous m-version of $G_\alpha f$, where $G_\alpha, \alpha > 0$

is the resolvent of $\mathcal{E}$ and $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; m)$. The coform $\hat{\mathcal{E}}$ introduced in Section 1.1 is a generalized Dirichlet form with the associated resolvent $(\hat{G}_\alpha)_{\alpha>0}$ and there exists an m-tight special standard process properly associated in the resolvent sense with $\hat{\mathcal{E}}$. We always assume that $(\mathcal{F}_t)_{t\geq 0}$ is the (universally completed) natural filtration of X_t. From now on, we obtain all the results under the above assumption.

For the concepts related to additive functionals that we used in this section, we refer to Section 1.2. We consider the following conditions:

(A5) X is a continuous conservative Hunt process in the state space $\mathbb{R}^d \cup \{\partial\}$. $\hat{G}_\alpha$ is strongly continuous on $\mathcal{V}$ and $\hat{\mathcal{E}}$ is quasi-regular. $C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$ and for $u \in \mathcal{F}$, there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{E}(u_n - u, u_n - u) \to 0, n \to \infty$. $F_k := \{x \in \mathbb{R}^d, |x| \leq k\}$ is an $\mathcal{E}$-nest (Definition 1.2).

Remark 2.3.1 The last two conditions in (A5) are satisfied if $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{F}$. It is easy to verify the condition (A5), if $\mathcal{E}$ satisfies the weak sector condition. The following two examples satisfy (A5) and they don't satisfy the weak sector condition.

Example 2.3.2 Consider $b = (b^i) : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel-measurable vector field. Let us define

$$Lu = \Delta u + \langle b, \nabla u\rangle, \qquad \forall u \in C_b^\infty(\mathbb{R}^d).$$

Assume that

$$\lim_{|x|\to\infty} \langle b(x), x\rangle = -\infty,$$

and that there exist $C_1, C_2, m \in [0, \infty)$ such that

$$|b(x)| \leq C_1 + C_2|x|^m \qquad x \in \mathbb{R}^d.$$

Then by [BR95, Theorem 5.3], there exists a probability measure μ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} Lud\mu = 0 \qquad \forall u \in C_b^\infty(\mathbb{R}^d)$$

and

$$b \in L^2(\mu).$$

By [BR95, Theorem 3.1] we have $d\mu$ is absolutely continuous w.r.t. dx and the density admits a representation φ^2, where $\varphi \in H^{1,2}(\mathbb{R}^d, dx)$. The closure of

$$\mathcal{E}^0(u, v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle d\mu;\ u, v \in C_0^\infty(\mathbb{R}^d),$$

on $L^2(\mathbb{R}^d,\mu)$ is a Dirichlet form. Denote $b^0 := 2\nabla\varphi/\varphi$ and $\beta := b-b^0$. Then we have $\beta \in L^2(\mathbb{R}^d;\mathbb{R}^d,\mu)$. Then by [St1, Proposition 1.10 and Proposition 2.4] $(L, C_0^\infty(\mathbb{R}^d))$ is L^1-unique. Then by the proof of [St1, Proposition 2.4] for $u \in b\mathcal{F}$ there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{E}(u_n-u,u_n-u)\to 0, n\to\infty$.

Consider the bilinear form

$$\mathcal{E}(u,v) = \frac{1}{2}\int\langle\nabla u,\nabla v\rangle d\mu - \int\langle\frac{1}{2}\beta,\nabla u\rangle v d\mu \; u,v\in C_0^\infty(\mathbb{R}^d).$$

Then by the computation in [Tr2, Section 4d] we have that conditions (A1)-(A5) hold for the bilinear form $\mathcal{E}$.

Example 2.3.3 Consider $d \geq 2$, $A = (a^{ij})$ a Borel-measurable mapping on $\mathbb{R}^d$ with values in the non-negative symmetric matrices on $\mathbb{R}^d$, and let $b = (b^i) : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel-measurable vector field. Consider the operator

$$L_{A,b}\psi = \partial_i(a^{ij}\partial_j\psi) + b^i\partial_i\psi, \qquad \forall\psi \in C_0^\infty(\mathbb{R}^d),$$

where we use the standard summation rule for repeated indices. By $H^{1,p}(\mathbb{R}^d,dx)$ we denote the standard Sobolev space of functions on $\mathbb{R}^d$ whose first order derivatives are in $L^p(\mathbb{R}^d,dx)$. Assume that for $p > d$

(C1)$a^{ij} \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d,dx)$, (a^{ij}) is uniformly strictly elliptic in $\mathbb{R}^d$.

(C2)$b^i \in L^p_{\mathrm{loc}}(\mathbb{R}^d,dx)$.

Here by $H^{1,p}_{loc}(\mathbb{R}^d,dx)$ we denote the class of all functions f on $\mathbb{R}^d$ such that $f\chi \in H^{1,p}(\mathbb{R}^d,dx)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. And $L^p_{\mathrm{loc}}(\mathbb{R}^d,dx)$ denotes the class of all functions f on $\mathbb{R}^d$ such that $f\chi \in L^p(\mathbb{R}^d)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. Assume that there exists $V \in C^2(\mathbb{R}^d)$ ("Lyapunov function") such that

$$\lim_{|x|\to\infty} V(x) = +\infty, \qquad \lim_{|x|\to\infty} L_{A,b}V(x) = -\infty.$$

Examples of V can be found in [BRS00] and the reference therein.

Then by [BRS00, Theorem 2.2] there exists a probability measure μ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} L_{A,b}\psi d\mu = 0 \qquad \forall\psi\in C_0^\infty(\mathbb{R}^d).$$

Then by [BRS00, Theorem 2.1] we have $d\mu$ is absolutely continuous w.r.t. dx and that the density admits a representation φ^2, where $\varphi^2 \in H^{1,p}_{loc}(\mathbb{R}^d,dx)$. The closure of

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int\langle\nabla ua,\nabla v\rangle d\mu;\; u,v\in C_0^\infty(\mathbb{R}^d),$$

on $L^2(\mathbb{R}^d,\mu)$ is a Dirichlet form.

If in addition, there is a positive Borel function θ on $[0,\infty)$ such that $\lim_{t\to\infty}\theta(t)=+\infty$ and

$$L_{A,b}V(x)\leq c_1-c_2\theta(|bA^{-\frac{1}{2}}|)|bA^{-\frac{1}{2}}|^2$$

outside some ball, then by [BKR06, Theorem 2.6] $b\in L^2(\mathbb{R}^d;\mathbb{R}^d,\mu)$. Set $b^0=(b^0_1,...,b^0_d)$, where $b^0_i:=2\sum_{j=1}^d a^{ij}\partial_j\varphi/\varphi, i=1,...,d$and $\beta:=b-b^0$. By [BKR06, Theorem 2.6] $\beta\in L^2(\mathbb{R}^d;\mathbb{R}^d,\mu)$. Then by [St1, Proposition 1.10 and Proposition 2.4] $(L,C_0^\infty(\mathbb{R}^d))$ is L^1-unique. Then by the proof of [St1, Proposition 2.4] for $u\in b\mathcal{F}$ there exists a sequence $\{u_n\}\subset C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{E}(u_n-u,u_n-u)\to 0,n\to\infty$.

Consider the bilinear form

$$\mathcal{E}(u,v)=\frac{1}{2}\int\langle\nabla ua,\nabla v\rangle d\mu-\int\langle\frac{1}{2}\beta,\nabla u\rangle vd\mu\ u,v\in C_0^\infty(\mathbb{R}^d).$$

Then by the computation in [Tr2, Section 4d] we have that conditions (A1)-(A5) hold for the bilinear form $\mathcal{E}$.

For an initial distribution $\mu\in\mathcal{P}(\mathbb{R}^d)$, here $\mathcal{P}(\mathbb{R}^d)$ denotes all the probabilities on $\mathbb{R}^d$, we prove the *Fukushima reprensentation property* mentioned in [QY10] holds for X, i.e. there is an algebra $K(\mathbb{R}^d)\subset\mathcal{B}_b(\mathbb{R}^d)$ which generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and is invariant under R_α for $\alpha>0$, and there are finitely many continuous martingales $M^1,...,M^d$ over $(\Omega,\mathcal{F}^\mu,\mathcal{F}^\mu_t,P^\mu)$ such that for any potential $u=R_\alpha f$, where $\alpha>0$ and $f\in K(\mathbb{R}^d)$, the martingale part $M^{[u]}$ of the semimartingale $u(X_t)-u(X_0)$ has the martingale representation in terms of $(M^1,...,M^d)$, that is, there are predictable processes $F_1,...,F_d$ on $(\Omega,\mathcal{F}^\mu,\mathcal{F}^\mu_t)$ such that

$$M^{[u]}_t=\sum_{j=1}^d\int_0^t F^j_s dM^j_s\qquad P^\mu-a.s..$$

Let us first calculate the energy measure related to $\langle M^{[u]}\rangle, u\in C_0^\infty(\mathbb{R}^d)$. By [Tr2, (23)], for bounded $g\in L^1(\mathbb{R}^d,m)$, we have

$$\begin{aligned}
&\int\hat{G}_\gamma g d\mu_{\langle M^{[u]}\rangle}\\
&=\lim_{\alpha\to\infty}\alpha(U^{\alpha+\gamma}_{\langle M^{[u]}\rangle}1,\hat{G}_\gamma g)\\
&=\lim_{\alpha\to\infty}\lim_{t\to\infty}E_{\hat{G}_\gamma g\cdot m}(\alpha e^{-(\gamma+\alpha)t}\langle M^{[u]}\rangle_t)+\lim_{\alpha\to\infty}E_{\hat{G}_\gamma g\cdot m}(\int_0^\infty\langle M^{[u]}\rangle_t\alpha(\gamma+\alpha)e^{-(\gamma+\alpha)t}dt)\\
&=\lim_{\alpha\to\infty}\lim_{t\to\infty}\alpha\langle\mu_{\langle M^{[u]}\rangle},e^{-(\gamma+\alpha)t}\int_0^t\hat{P}_s\hat{G}_\gamma g ds\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\alpha\to\infty}\alpha(\gamma+\alpha)(\int_0^\infty e^{-(\gamma+\alpha)t}E_{\hat{G}_\gamma g\cdot m}((u(X_t)-u(X_0)-N_t^{[u]})^2)dt)\\
=&\lim_{\alpha\to\infty}\alpha(\gamma+\alpha)(\int_0^\infty e^{-(\gamma+\alpha)t}E_{\hat{G}_\gamma g\cdot m}((u(X_t)-u(X_0))^2)dt)\\
=&\lim_{\alpha\to\infty}2\alpha(u-\alpha G_\alpha u,u\hat{G}_\gamma g)-\alpha(u^2,\hat{G}_\gamma g-\alpha\hat{G}_\alpha\hat{G}_\gamma g)\\
=&2(-Lu,u\hat{G}_\gamma g)-(-Lu^2,\hat{G}_\gamma g)\\
=&2\mathcal{E}(u,u\hat{G}_\gamma g)-\mathcal{E}(u^2,\hat{G}_\gamma g)\\
=&2\mathcal{E}^{a,\hat{b}}(u,u\hat{G}_\gamma g)-\mathcal{E}^{a,\hat{b}}(u^2,\hat{G}_\gamma g)+2\int\langle b\sigma,D_\sigma u\rangle u\hat{G}_\gamma gm(dx)-\int\langle b\sigma,D_\sigma u^2\rangle\hat{G}_\gamma gm(dx)\\
=&2\mathcal{E}^{a}(u,u\hat{G}_\gamma g)-\mathcal{E}^{a}(u^2,\hat{G}_\gamma g)\\
=&2\int\langle D_\sigma u,D_\sigma(u\hat{G}_\gamma g)\rangle dm-\int\langle D_\sigma u^2,D_\sigma(\hat{G}_\gamma g)\rangle dm\\
=&2\int\langle D_\sigma u,D_\sigma u\rangle\hat{G}_\gamma gdm.
\end{aligned}
$$

Thus, by [Tr2, Theorem 2.5] we obtain

$$\mu_{\langle M^{[u]}\rangle}=2\langle D_\sigma u,D_\sigma u\rangle\cdot dm.$$

So, for $u,v\in C_0^\infty(\mathbb{R}^d)$, for q.e. x under P^x,

$$\langle M^{[u]},M^{[v]}\rangle_t=2\int_0^t\langle D_\sigma u,D_\sigma v\rangle(X_s)ds. \tag{2.3.1}$$

Then by (A5) and [Tr1, Theorem 4.4], we deduce (2.3.1) for every $u,v\in\mathcal{F}$.

By (A5) $F_k=\{x\in\mathbb{R}^d,|x|\leq k\},k\in\mathbb{N}$ is an $\mathcal{E}$-nest. By [Tr2, Theorem 3.6], for $u_i(x)=x_i$, we have the Fukushima decomposition for $A^{[u_i]}:=u_i(X)-u_i(X_0)$, and let $M^{(i)}\in\dot{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$ be the associated local martingale additive functional. Here $\dot{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$ means that there exists $(M^k)_{k\in\mathbb{N}}\subset\dot{\mathcal{M}}$ such that for any k

$$M_t=M_t^k\quad\forall t\leq\sigma_{F_k^c},$$

where $\sigma_{F_k^c}=\inf\{t>0|X_t\in F_k^c\}$.

We define the stochastic integral $f\cdot M^{(i)}\in\dot{\mathcal{M}}$ for $f\in L^2(\mathbb{R}^d;\mu_{\langle M^{(i)}\rangle})$ as in [FOT94, p243], and for $L\in\dot{\mathcal{M}}$ we have

$$\langle f\cdot M^{(i)},L\rangle=f\cdot\langle M^{(i)},L\rangle,$$

where $f\cdot\langle M^{(i)},L\rangle_t=\int_0^t f(X_s)d\langle M^{(i)},L\rangle_s$.

Theorem 2.3.4 Suppose (A5) holds. Let $u\in C_0^1(\mathbb{R}^d)$, where $C_0^1(\mathbb{R}^d)$ denotes the continuous function with compact support and continuous first order derivative. Then for q.e. x under P^x,

$$M^{[u]} = \sum_{i=1}^{d} u_{x_i} \cdot M^{(i)}.$$

Proof By [Tr2, Theorem 3.6] we obtain that for q.e. x under P^x, for $t \geq 0$

$$\begin{aligned}\langle M^{[u]} - \sum_{i=1}^{d} u_{x_i} \cdot M^{(i)} \rangle_t =& \sum_{i,j=1}^{n} \int_0^t u_{x_i}(X_s) u_{x_j}(X_s) d\langle M^{(i)}, M^{(j)} \rangle_s \\ & - 2 \sum_{i,j=1}^{n} \int_0^t u_{x_i}(X_s) u_{x_j}(X_s) d\langle M^{(i)}, M^{(j)} \rangle_s \\ & + \sum_{i,j=1}^{n} \int_0^t u_{x_i}(X_s) u_{x_j}(X_s) d\langle M^{(i)}, M^{(j)} \rangle_s \\ =& 0.\end{aligned}$$

Then the assertion follows. □

Then by [Tr2, Lemma 2.4, Lemma 1.18], we have for q.e. x under P^x,

$$\langle M^{(i)}, M^{(j)} \rangle_t = 2 \int_0^t \sum_{l=1}^{k} \sigma_l^i(X_s) \sigma_l^j(X_s) ds. \tag{2.3.2}$$

Lemma 2.3.5 Suppose (A5) holds. Let $\mathcal{C}_1$ be a uniformly dense subset of $C_0(\mathbb{R}^d)$. Here $C_0(\mathbb{R}^d)$ denotes the continuous function with compact support. Then the family $\{f \cdot M^{[u]} : f \in \mathcal{C}_1, u \in C_0^\infty(\mathbb{R}^d)\}$ of stochastic integrals is dense in $(\dot{\mathcal{M}}, e)$.

Proof Suppose that an MAF $M \in \dot{\mathcal{M}}$ is e-orthogonal to the above family, namely, $\int_X f d\mu_{\langle M, M^{[u]} \rangle} = 0, \forall f \in \mathcal{C}_1, u \in C_0^\infty(\mathbb{R}^d)$. This identity extends to all $u \in \mathcal{F}$ by [Tr1, (13)] and (A5). Hence,

$$\langle M, M^{[u]} \rangle = 0 \qquad \forall u \in \mathcal{F}.$$

In particular, this holds for $u = G_\alpha g, \alpha > 0, \forall g \in C_0(\mathbb{R}^d)$. By [FOT94, Theorem A.3.20] we deduce that $M = 0$. □

Theorem 2.3.6 Suppose (A5) holds. Then the space $\dot{\mathcal{M}}$ can be represented by stochastic integrals based on $M^{(i)} = M^{[x_i]}, 1 \leq i \leq d$:

$$\dot{\mathcal{M}} = \{\sum_{i=1}^{d} f_i \cdot M^{(i)} : \sum_{i,j=1}^{d} \sum_{l=1}^{k} \int_{\mathbb{R}^d} (f_i f_j \sigma_i^l \sigma_j^l)(x) m(dx) < \infty\}, \tag{2.3.3}$$

and

$$e(\sum_{i=1}^{d} f_i \cdot M^{(i)}) = \sum_{i,j=1}^{d}\sum_{l=1}^{k}\int_{\mathbb{R}^d}(f_i f_j \sigma_i^l \sigma_j^l)(x)m(dx).$$

Proof The space of the right hand side of (2.3.3) is dense in $(\dot{\mathcal{M}}, e)$, since it contains the set

$$\{f \cdot M^{[u]} = \sum_{i=1}^{d}(fu_{x_i}) \cdot M^{(i)}; f \in C_0(\mathbb{R}^d), u \in C_0^1(\mathbb{R}^d)\},$$

which is dense in $(\dot{\mathcal{M}}, e)$ by Lemma 2.3.5. Hence, it is enough to show that the right hand side of (2.3.3) is closed in $(\dot{\mathcal{M}}, e)$.

Suppose that $\lim_{n\to\infty} e(M_n - M) = 0$, where

$$M_n = \sum_{i=1}^{d} f_i^{(n)} \cdot M^{(i)}, \sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij} f_i^{(n)} f_j^{(n)} dm < \infty, M \in \dot{\mathcal{M}}.$$

Set $f^n := (f_1^n, ..., f_d^n)$. Since

$$e(M_n - M_m) = \int_{\mathbb{R}^d} |(f^n - f^m)\sigma|^2 dm,$$

we deduce that $f^n\sigma$ converges in $L^2(\mathbb{R}^d, \mathbb{R}^k; m)$ to some function $h \in L^2(\mathbb{R}^d, \mathbb{R}^k; m)$ Let $f = h\tau$, where τ is the matrix that we have introduced at the beginning of Section 2.1 and $M' = \sum_{i=1}^{d} f_i \cdot M^{(i)}$, then

$$\begin{aligned} e(M_n - M') &= \int_D |(f^n - f)\sigma|^2 dm \\ &= \int_D |f^n\sigma - h|^2 dm. \end{aligned}$$

which converges to zero as $n \to \infty$. Therefore, we have $M = M'$ and

$$e(M) = \sum_{i,j=1}^{d}\sum_{l=1}^{k}\int_{\mathbb{R}^d}(f_i f_j \sigma_i^l \sigma_j^l)(x)m(dx) < \infty.$$

□

As a consequence, X satisfies *Fukushima representation theorem* mentioned before. To prove main results in this section we need the following lemma which is proved by Meyer in [M67] (see also [QY10, Lemma 2.2]).

Lemma 2.3.7 Let $K(\mathbb{R}^d) \subset \mathcal{B}_b(\mathbb{R}^d)$ be an algebra, which generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, and $\mathcal{C}_0$ be all $\xi = \xi_1 \cdots \xi_n$, $n \in \mathbb{N}$, $\xi_j = \int_0^\infty e^{-\alpha_j t} f_j(X_t) dt$, where

$\alpha_j \in \mathbb{Q}^+$, $f_j \in K(\mathbb{R}^d), j = 1, ..., n$. Then the completion of the σ-algebra generated by $\mathcal{C}_0$ is $\mathcal{F}_\infty$.

Moreover, by an analogous method to the proof of [QY10, Theorem 3.1] we have the martingale representation theorem for X.

Theorem 2.3.8 Suppose (A5) holds. Then there exists some $\mathcal{E}$-exceptional set $\mathcal{N}$ such that the following representation result holds: For every bounded $\mathcal{F}_\infty$-measurable random variable ξ, there exists an predictable process $(\phi_1, ..., \phi_d) : [0, \infty) \times \Omega \to \mathbb{R}^d$, such that for each probability measure ν, supported by $\mathbb{R}^d \setminus \mathcal{N}$, one has

$$\xi = E^\nu(\xi|\mathcal{F}_0) + \sum_{i=0}^{d} \int_0^\infty \phi_s^i dM_s^{(i)} \qquad P^\nu - a.s.,$$

and

$$E^\nu \int_0^\infty |\phi_s \sigma(X_s)|^2 ds \le \frac{1}{2} E^\nu \xi^2.$$

If another predictable process $\phi' = (\phi'_1, ..., \phi'_d)$ satisfies the same relations under a certain measure P^ν, then one has $\phi'_t \sigma(X_t) = \phi_t \sigma(X_t), dt \times dP^\nu - a.s.$.

Proof Suppose that $\mathcal{N}$ is some fixed exceptional set. By $\mathcal{K}$ we denote the class of bounded random variables for which the statement holds outside this set. First we prove that if $(\xi_n) \subset \mathcal{K}$ is a uniformly bounded increasing sequence and $\xi = \lim_{n\to\infty} \xi_n$ then $\xi \in \mathcal{K}$.

Indeed, since ξ and ξ_0 are bounded, $E^x|\xi_n - \xi|^2 \to 0$. Denoting by ϕ^n the process which represents ξ^n, we obtain

$$E^x \int_0^\infty |(\phi_s^p - \phi_s^n)\sigma(X_s)|^2 ds \le E^x(\xi_p - \xi_n)^2 \to 0, \text{ as } n, p \to \infty.$$

Now we want to pass to the limit with ϕ^n pointwise, so that the limit be predictable. In order to obtain a sequence of representable variables that converges rapidly enough under all measures $P^x, x \in \mathcal{N}^c$, we are going to construct them as follows. For each $l = 0, 1, ...$ set $n_l(x) = \inf\{n|E^x(\xi - \xi_n)^2 < \frac{1}{2^l}\}, \bar{\xi}_l = \xi_{n_l(x)}$. The process which represents ξ^l is simply obtained by the formula $\bar{\phi}^l = \phi^{n_l(X_0)}$. With this sequence we may pass to the limit and define $\psi_s = \limsup_{l\to\infty} \bar{\phi}_s^l \sigma(X_s)$ (where limsup is taken on each coordinate) and $\phi_s = \psi_s \tau(X_s)$ where τ is the matrix that we have introduced in the beginning of Section 2.1. Then we obatin

$$E^x \int_0^\infty |(\bar{\phi}_s^l - \phi_s)\sigma(X_s)|^2 ds \to 0, \text{ as } l \to \infty.$$

By this we obtain $\xi \in \mathcal{K}$.

Let $K(\mathbb{R}^d) \subset \mathcal{B}_b(\mathbb{R}^d)$ be a countable set which is closed under multiplication, generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and $R_\alpha(K(\mathbb{R}^d)) \subset K(\mathbb{R}^d)$ for each $\alpha \in \mathbb{Q}^+$. Such $K(\mathbb{R}^d)$ can be constructed as follows. Choose $N_0 \subset \mathcal{B}_b(\mathbb{R}^d)$ to be a countable set which generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. For $l \geq 1$ define $N_{l+1} = \{g_1...g_k, R_\alpha f g_1...g_k, f, g_i \in N_l, k \in \mathbb{N} \cup \{0\}, \alpha \in Q^+\}$ and $K(\mathbb{R}^d) := \cup_{l=0}^\infty N_l$.

Let $\mathcal{C}_0$ be all $\xi = \xi_1 \cdots \xi_n$, $n \in \mathbb{N}$, $\xi_j = \int_0^\infty e^{-\alpha_j t} f_j(X_t) dt$, where $\alpha_j \in \mathbb{Q}^+$, $f_j \in K(\mathbb{R}^d), j = 1, ..., n$. By Lemma 2.3.7 the completion of the σ-algebra generated by $\mathcal{C}_0$ is $\mathcal{F}_\infty$. By the first part of our proof a monotone class argument reduces the proof to the representation of a random variable in $\mathcal{C}_0$.

Let $\xi \in \mathcal{C}_0$. By Markov property of the process X, we obtain the following result (see e.g. [QY10, Theorem 3.1])

$$N_t = E^x(\xi|\mathcal{F}_t) = \sum_m Z_t^m,$$

where the sum is a finite one, and for each m, $Z^m = Z_t$ has the following form

$$Z_t = V_t u(X_t),$$

(the superscript m will be dropped if no confusion may arise), where $V_t = \prod_{i=1}^{k'} \int_0^t e^{-\beta_i s} g_i(X_s) ds$ and $u(x) = R_{\beta_1 + ... + \beta_k}(h_1(R_{\beta_2+...+\beta_k} h_2 ... (R_{\beta_k} h_k)...)$ for $\beta_i \in \mathbb{Q}^+, g_i, h_i \in K(\mathbb{R}^d)$. We have $u \in K(\mathbb{R}^d)$. Hence, by the Fukushima decomposition and the Fukushima representation we obtain

$$u(X_t) - u(X_0) = M_t^{[u]} + A_t^{[u]} = \sum_{j=1}^d \int_0^t G_s^j dM_s^{(j)} + A_t^{[u]} \qquad P^x - a.s.. \tag{2.3.4}$$

for some predictable processes G^j. Then by Itô's formula, we obtain

$$Z_t = Z_0 + \int_0^t u(X_s) dV_s + \int_0^t V_s dA_t^{[u]} + \sum_{j=1}^d \int_0^t V_s \cdot G_s^j dM_s^{(j)} \quad P^x - a.s..$$

Hence the martingale part of Z_t is $\sum_{j=1}^d \int_0^t V_s \cdot G_s^j dM_s^{(j)}$. We deduce that

$$N_t = \sum_{i=1}^d \int_0^t \sum_m V_s^m \cdot G_s^{m,i} dM_s^{(i)} \qquad P^x - a.s..$$

As a result, the representation holds for $\xi \in \mathcal{C}_0$. As (2.3.4) holds for every x outside a set of zero capacity. Then we take the exceptional set $\mathcal{N}$ in the assertion to be the union of all these exceptional sets corresponding to $u \in K(\mathbb{R}^d)$. $\square$

One may represent separately the positive and the negative parts and then we have the following corollary.

Corollary 2.3.9 Suppose (A5) holds. Let $\mathcal{N}$ be the set obtained in the preceding theorem. Then for any $\mathcal{F}_\infty$-measurable nonnegative random variable $\xi \geq 0$ there exists a predictable process $\phi = (\phi_1, ..., \phi_d) : [0, \infty) \times \Omega \to \mathbb{R}^d$ such that the following holds

$$\xi = E^x(\xi|\mathcal{F}_0) + \sum_{i=0}^{d} \int_0^\infty \phi_s^i dM_s^{(i)} \qquad P^x - a.s.,$$

and

$$E^x \int_0^\infty |\phi_s \sigma(X_s)|^2 ds \leq \frac{1}{2} E^x \xi^2,$$

for each point $x \in \mathcal{N}^c$ such that $E^x\xi < \infty$.

If another predictable process $\phi' = (\phi'_1, ..., \phi'_d)$ satisfies the same relations under a certain measure P^x, then one has $\phi'_t \sigma(X_t) = \phi_t \sigma(X_t), dt \times dP^x - a.s.$

2.3.2 Representation under P^m

In the following, we use the notation $\int_0^t \psi(s, X_s).dM_s := \sum_{i=1}^d \int_0^t \psi_i(s, X_s) dM_s^{(i)}$.

Lemma 2.3.10 Suppose (A1)-(A5) hold. If $u \in \mathcal{D}(L)$ and $\psi \in \tilde{\nabla} u$, then

$$u(X_t) - u(X_0) = \int_0^t \psi(X_s).dM_s + \int_0^t Lu(X_s)ds \qquad P^m - a.s..$$

Proof The assertion follows by the Fukushima decomposition, (2.3.1), (2.3.2) and Theorem 2.3.6. □

The aim of the rest of this section is to extend this representation to time dependent function $u(t, x)$.

Lemma 2.3.11 Suppose (A1)-(A5) hold. Let $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be such that

(i) $\forall s, u_s \in \mathcal{D}(L)$ and $s \to Lu_s$ is continuous in L^2.

(ii) $u \in C^1([0, T]; L^2)$.

Then clearly $u \in \mathcal{C}_T$. Moreover, for any $\psi \in \tilde{\nabla} u$ and any $s, t > 0$ such that $s + t < T$, the following relation holds P^m-a.s.

$$u(s + t, X_t) - u(s, X_0) = \int_0^t \psi(s + r, X_r).dM_r + \int_0^t (\partial_s + L)u(s + r, X_r)dr.$$

Proof We prove the above relation with $s = 0$, the general case being similar. Let

$0 = t_0 < t_1 < ... < t_p = t$ be a partition of the interval $[0, t]$ and write

$$u(t, X_t) - u(0, X_0) = \sum_{n=0}^{p-1} (u(t_{n+1}, X_{t_{n+1}}) - u(t_n, X_{t_n})).$$

Then, on account of the preceding lemma, each term of the sum is expressed as

$$\begin{aligned} & u(t_{n+1}, X_{t_{n+1}}) - u(t_n, X_{t_n}) \\ =& u(t_{n+1}, X_{t_{n+1}}) - u(t_{n+1}, X_{t_n}) + u(t_{n+1}, X_{t_n}) - u(t_n, X_{t_n}) \\ =& \int_{t_n}^{t_{n+1}} \psi^{n+1}(X_s).dM_s + \int_{t_n}^{t_{n+1}} Lu_{t_{n+1}}(X_s)ds + \int_{t_n}^{t_{n+1}} \partial_s u_s(X_{t_n})ds, \end{aligned}$$

where $\psi^{n+1} = (\psi_1^{n+1}, ..., \psi_d^{n+1}) \in \tilde{\nabla} u_{t_{n+1}}$ and the last integral is obtained by using the Leibnitz-Newton formula for the L^2-valued function $s \to u_s$. Below we estimate in L^2 the differences between each term in the last expression and the similar terms corresponding to the formula we have to prove. Here we use $mP_t \leq m$ i.e. $\int P_t f dm \leq \int f dm$ for $f \in \mathcal{B}^+$. This holds since $\hat{P}_t$ is sub-Markovian. Then we have

$$\begin{aligned} & E^m(\int_{t_n}^{t_{n+1}} \psi^{n+1}(X_s).dM_s - \int_{t_n}^{t_{n+1}} \psi(s, X_s).dM_s)^2 \\ =& E^m \int_{t_n}^{t_{n+1}} |(\psi^{n+1}(X_s) - \psi(s, X_s))\sigma(X_s)|^2 ds \\ \leq& \int_{t_n}^{t_{n+1}} \mathcal{E}^a(u_{t_{n+1}} - u_s)ds. \end{aligned}$$

Since $s \to Lu_s$ is continuous in L^2, it follows that $s \to u_s$ is continuous w.r.t. $\mathcal{E}_1^a$-norm. Hence the difference appearing in the last integral $\mathcal{E}^a(u_{t_{n+1}} - u_s)$ is uniformly small, provided the partition is fine enough. From this one deduces that

$$\sum_{n=0}^{p-1} \int_{t_n}^{t_{n+1}} \psi^{n+1}(X_s).dM_s \to \int_0^t \psi(s + r, X_r).dM_r.$$

The next difference is estimated by using Minkowski's inequality

$$\begin{aligned} & (E^m(\sum_{n=0}^{p-1} \int_{t_n}^{t_{n+1}} (Lu_{t_{n+1}} - Lu_s)(X_s)ds)^2)^{1/2} \\ \leq& \sum_{n=0}^{p-1} \int_{t_n}^{t_{n+1}} (E^m(Lu_{t_{n+1}} - Lu_s)^2(X_s))^{1/2} ds \\ \leq& \sum_{n=0}^{p-1} \int_{t_n}^{t_{n+1}} \|Lu_{t_{n+1}} - Lu_s\|_2 ds, \end{aligned}$$

so that it is similarly expressed as in integral of a uniformly small quantity.

For the last difference we write

$$
\begin{aligned}
&(E^m(\sum_{n=0}^{p-1}\int_{t_n}^{t_{n+1}}(\partial_s u_s(X_{t_n})-\partial_s u_s(X_s))ds)^2)^{1/2}\\
\leq&\sum_{n=0}^{p-1}\int_{t_n}^{t_{n+1}}(E^m(\partial_s u_s(X_{t_n})-\partial_s u_s(X_s))^2)^{1/2}ds\\
=&\sum_{n=0}^{p-1}\int_{t_n}^{t_{n+1}}(E^m(\partial_s u_s(X_{t_n})^2+P_{s-t_n}(\partial_s u_s)^2(X_{t_n})-2\partial_s u_s(X_{t_n})(P_{s-t_n}\partial_s u)(X_{t_n})))^{1/2}ds\\
=&\sum_{n=0}^{p-1}\int_{t_n}^{t_{n+1}}(E^m((\partial_s u_s(X_{t_n})-(P_{s-t_n}\partial_s u_s)(X_{t_n}))^2\\
&+(P_{s-t_n}(\partial_s u_s)^2(X_{t_n})-((P_{s-t_n}\partial_s u_s)(X_{t_n}))^2)))^{1/2}ds\\
\leq&\sum_{n=0}^{p-1}(\int_{t_n}^{t_{n+1}}\int(\partial_s u_s-P_{s-t_n}\partial_s u_s)^2+P_{s-t_n}(\partial_s u_s)^2-(P_{s-t_n}\partial_s u_s)^2dm))^{1/2}ds.
\end{aligned}
$$

From the hypotheses it follows that this will tend also to zero if the partition is fine enough. Hence the assertions follow. □

Theorem 2.3.12 Suppose (A1)-(A5) hold. Let $f \in L^1([0,T];L^2)$ and $\phi \in L^2(\mathbb{R}^d)$ and define

$$u_t := P_{T-t}\phi + \int_t^T P_{s-t}f_s ds.$$

Then for each $\psi \in \tilde{\nabla}u$ and for each $s \in [0,T]$, the following relation holds P^m-a.s.

$$u(s+t, X_t) - u(s, X_0) = \int_0^t \psi(s+r, X_r).dM_r - \int_0^t f(s+r, X_r)dr.$$

In particular, if u is a generalized solution of PDE (2.2.1), for each $t \in [s,T]$ the following BSDE holds P^m-a.s.

$$u(t, X_{t-s}) = \phi(X_{T-s}) + \int_t^T f(r, X_{r-s}, u(r, X_{r-s}), D_\sigma u(r, X_{r-s}))dr - \int_{t-s}^{T-s} \psi(s+r, X_r).dM_r.$$

Proof Assume first that ϕ and f satisfy the conditions in Proposition 2.1.6 (ii). Then we have u satisfies the conditions in Lemma 2.3.11. Then by Lemma 2.3.11, the assertion follows. For the general case we choose u^n associated (f^n, ϕ^n) as in

Proposition 2.1.9. Then we obtain that if $n \to \infty$, $\|u^n - u\|_T \to 0$. For u^n we have

$$(2.3.5) \quad u^n(s+t, X_t) - u^n(s, X_0) = \int_0^t \psi^n(s+r, X_r).dM_r - \int_0^t f^n(s+r, X_r)dr.$$

As

$$\begin{aligned} &E^m|\int_0^t (\psi^n(s+r, X_r) - \psi^p(s+r, X_r).dM_r|^2 \\ \leq &E^m \int_0^t |(\psi^n(s+r, X_r) - \psi^p(s+r, X_r))\sigma(X_r)|^2 dr \\ \leq &\int_0^t \mathcal{E}^a(u^n_{s+r} - u^p_{s+r})dr, \end{aligned}$$

letting $n \to \infty$ in (2.3.5) we obtain the assertion. □

2.4 BSDE's and Generalized Solutions

The set $\mathcal{N}$ obtained in Theorem 2.3.8 will be fixed throughout this section. By Theorem 2.3.8 we can solve BSDE's under all measures P^x, $x \in \mathcal{N}^c$, at the same time. We will treat systems of l equations, $l \in \mathbb{N}$, associated to $\mathbb{R}^l$-valued functions $f : [0,T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \mapsto \mathbb{R}^l$. These functions are assumed to depend on the past in general and it turns out that a good theory is developed assuming that they are predictable. This means that we consider the map $(s, \omega) \mapsto f(s, \omega, \cdot, \cdot)$ as a predictable process with respect to the canonical filtration of our process $(\mathcal{F}_t)$.

Lemma 2.4.1 Suppose (A5) holds. Let ξ be an $\mathcal{F}_T$-measurable random variable and $f : [0,T] \times \Omega \mapsto \mathbb{R}$ an $(\mathcal{F}_t)_{t\geq 0}$-predictable process. Let A be the set of all points $x \in \mathcal{N}^c$ for which the following integrability condition holds

$$E^x(|\xi| + \int_0^T |f(s,\omega)|ds)^2 < \infty.$$

Then there exists a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}$, $Z : [0,T) \times \Omega \mapsto \mathbb{R}^d$, such that under all measures P^x, $x \in A$, they have the following properties:

(i) Y is continuous,

(ii) Z satisfy the integrability condition

$$\int_0^T |Z_t \sigma(X_t)|^2 dt < \infty, \qquad P^x - a.s.,$$

(iii) The local martingale $\int_0^t Z_s.dM_s$, obtained by integrating Z against the coordinate martingales, is a uniformly integrable martingale,

(iv) they satisfy the equation

$$Y_t = \xi + \int_t^T f(s,\omega)ds - \int_t^T Z_s.dM_s, \qquad P^x - a.s., 0 \leq t \leq T.$$

If another pair (Y_t', Z_t') of predictable processes satisfies the above conditions (i),(ii),(iii),(iv), under a certain measure P^ν with the initial distribution ν supported by A, then one has $Y. = Y.', P^\nu - a.s.$ and $Z_t\sigma(X_t) = Z_t'\sigma(X_t), dt \times P^\nu - a.s.$.

Proof The representation of the positive and negative parts of the random variable $\xi + \int_0^T f_s ds$ give us the predictable process Z such that

$$\xi + \int_0^T f_s ds = E^{X_0}(\xi + \int_0^T f_s ds) + \int_0^T Z_s.dM_s.$$

Then we get the process Y by the formula

$$Y_t = E^{X_0}(\xi + \int_0^T f_s ds) + \int_0^t Z_s.dM_s - \int_0^t f_s ds.$$

□

Definition 2.4.2 Let ξ be an $\mathbb{R}^l$-valued, $\mathcal{F}_T$-measurable, random variable and $f : [0,T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \mapsto \mathbb{R}^l$ a measurable $\mathbb{R}^l$-valued function such that $(s,\omega) \mapsto f(s,\omega,\cdot,\cdot)$ as a process is predictable. Let $p > 1$ and ν be a probability measure supported by $\mathcal{N}^c$ such that $E^\nu|\xi|^p < \infty$. We say that a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}^l$, $Z : [0,T) \times \Omega \mapsto \mathbb{R}^l \otimes \mathbb{R}^d$ is a solution of the BSDE (2.4.1) in $L^p(P^\nu)$ with data (ξ, f) provided that Y is continuous under P^ν and satisfies both the integrability conditions

$$\int_0^T |f(t,\cdot,Y_t,Z_t\sigma(X_t))|dt < \infty, \qquad P^\nu - a.s.,$$

$$E^\nu(\int_0^T |Z_t\sigma(X_t)|^2 dt)^{p/2} < \infty,$$

and the following equation, with $0 \leq t \leq T$,

$$Y_t = \xi + \int_t^T f(s,\omega,Y_s,Z_s\sigma(X_s))ds - \int_t^T Z_s.dM_s, \qquad P^\nu - a.s.. \tag{2.4.1}$$

Let $f : [0,T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \mapsto \mathbb{R}^l$ be a measurable $\mathbb{R}^l$-valued function such that $(s,\omega) \mapsto f(s,\omega,\cdot,\cdot)$ is predictable and satisfies the following conditions:

(Ω1) (*Lipschitz condition in* z) There exists a constant $C > 0$ such that for all t, ω, y, z, z'

$$|f(t,\omega,y,z) - f(t,\omega,y,z')| \leq C|z - z'|.$$

(Ω2) (*Monotonicity condition in* y) There exists a function $\mu_t \in L^1([0,T],\mathbb{R})$ such that for all ω, y, y', z,

$$\langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z)\rangle \leq \mu_t|y - y'|^2,$$

and $\alpha_t := \int_0^t \mu_s ds < \infty$.

(Ω3) (*Continuity condition in* y) For t, ω and z fixed, the map

$$y \mapsto f(t,\omega,y,z),$$

is continuous.

We need the following notation

$$f^0(t,\omega) := f(t,\omega,0,0), \qquad f'(t,\omega,y) := f(t,\omega,y,0) - f(t,\omega,0,0),$$

$$f^{',r}(t,\omega) := \sup_{|y|\leq r} |f'(t,\omega,y)|.$$

Let ξ be an $\mathbb{R}^l$-valued, $\mathcal{F}_T$-measurable, random variable and, for each $p > 0$ denote by A_p the set of all points $x \in \mathcal{N}^c$ for which the following integrability conditions hold,

$$E^x \int_0^T f_t^{',r} dt < \infty, \qquad \forall r \geq 0, \tag{2.4.2}$$

$$E^x(|\xi|^p + (\int_0^T |f^0(s,\omega)|ds)^p) < \infty.$$

Denote by A_∞ the set of points $x \in \mathcal{N}^c$ for which (2.4.2) holds and with the property that $|\xi|, |f^0| \in L^\infty(P^x)$.

Proposition 2.4.3 Under the conditions (A5), (Ω1), (Ω2), (Ω3), there exists a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}^l, Z : [0,T) \times \Omega \mapsto \mathbb{R}^l \otimes \mathbb{R}^d$ that forms a solution of the BSDE (2.4.1) in $L^p(P^x)$ with data (ξ, f) for each point $x \in A_p$. Moreover, the following estimate holds with some constant K that depends only on C, μ and T,

$$E^x(\sup_{t\in[0,T]} |Y_t|^p + (\int_0^T |Z_t\sigma(X_t)|^2 dt)^{p/2}) \leq KE^x(|\xi|^p + (\int_0^T |f^0(s,\omega)|ds)^p), \qquad x \in A_p.$$

If $x \in A_\infty$, then $\sup_{t\in[0,T]} |Y_t| \in L^\infty(P^x)$.

If (Y'_t, Z'_t) is another solution in $L^p(P^x)$, for some point $x \in A_p$, then one has $Y_t = Y'_t$ and $Z_t\sigma(X_t) = Z'_t\sigma(X_t), dt \times P^x - a.s.$.

The proof is based on more or less standard methods. Therefore, we include it not here, but in the appendix below.

We shall now look at the connection between the solutions of BSDE's introduced in this section and PDE's studied in Section 2.2. In order to do this we have to consider BSDE's over time intervals like $[s,T]$, with $0 \le s \le T$. Since the present approach is based on the theory of Markov processes, which is a time homogeneous theory, we have to discuss solutions over the interval $[s,T]$, while the process and the coordinate martingales are indexed by a parameter in the interval $[0, T-s]$.

Let us give a formal definition for the natural notion of solution over a time interval $[s,T]$. Let ξ be an $\mathcal{F}_{T-s}$-measurable, $\mathbb{R}^l$-valued, random variable and $f : [s,T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l$ an $\mathbb{R}^l$-valued, measurable map such that $(f(s+l, \omega, \cdot, \cdot))_{l\in[0,T-s]}$ is predictable with respect to $(\mathcal{F}_l)_{l\in[0,T-s]}$. Let ν be a probability measure supported by $\mathcal{N}^c$ such that $E^\nu|\xi|^p < \infty$. We say a pair $(Y_t, Z_t)_{s\le t\le T}$ of processes $Y : [s,T] \times \Omega \to \mathbb{R}^l, Z : [s,T] \times \Omega \to \mathbb{R}^l \otimes \mathbb{R}^d$ is a solution in $L^p(P^\nu)$ of the BSDE (2.4.3) over the interval $[s,T]$ with data (ξ, f), provided that they have the property that reindexed as $(Y_{s+l}, Z_{s+l})_{l\in[0,T-s]}$ these processes are $(\mathcal{F}_l)_{l\in[0,T-s]}$-predictable, Y is continuous and together they satisfy the integrability conditions

$$\int_s^T |f(t, \cdot, Y_t, Z_t\sigma(X_{t-s}))|dt < \infty, \qquad P^\nu - a.s..$$

$$E^\nu(\int_s^T |Z_t\sigma(X_{t-s})|^2 dt)^{p/2} < \infty.$$

and the following equation under P^ν,

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r\sigma(X_{r-s}))dr - \int_{t-s}^{T-s} Z_{s+l}.dM_l, \qquad s \le t \le T. \tag{2.4.3}$$

The next result gives a probabilistic interpretation of Theorem 2.2.8. Let us assume that $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l$ is the measurable function appearing in the basic equation (2.2.1). Let $\phi : \mathbb{R}^d \to \mathbb{R}^l$ be measurable and for each $p > 1$, denote by A_p the set of points $(s,x) \in [0,T) \times \mathcal{N}^c$ with the following properties

$$E^x \int_s^T f'^{,r}(t, X_{t-s})dt < \infty, \qquad \forall r \ge 0. \tag{2.4.4}$$

$$E^x(|\phi|^p(X_{T-s})+(\int_s^T|f^0(t,X_{t-s})|ds)^p)<\infty.$$

Set $A=\cup_{p>1}A_p, A_{p,s}=\{x\in\mathcal{N}^c,(s,x)\in A_p\}$, and $A_s=\cup_{p>1}A_{p,s}, s\in[0,T)$. By the same arguments as in [BPS05, Theorem 5.4], we have the following results. In particular, we can reconstruct solutions to PDE (2.2.1) using Proposition 2.4.3.

Theorem 2.4.4 Assume that (A5) holds and f satisfies conditions (H1),(H2),(H3). Then there exist nearly Borel measurable functions $(u,\psi), u:A\to\mathbb{R}^l, \psi:A\to\mathbb{R}^l\otimes\mathbb{R}^d$, such that, for each $s\in[0,T)$ and each $x\in A_{p,s}$, the pair $(u(t,X_{t-s}),\psi(t,X_{t-s}))_{s\le t\le T}$ solves the BSDE (2.4.3) in $L^p(P^x)$ with data $(\phi(X_{T-s}),f(t,X_{t-s},y,z))$ over the interval $[s,T]$.

In particular, the functions u,ψ satisfy the following estimates, for $(s,x)\in A_p$,

$$E^x(\sup_{t\in[s,T]}|u(t,X_{t-s})|^p+(\int_s^T|\psi\sigma(t,X_{t-s})|^2dt)^{p/2})\le KE^x(|\phi(X_{T-s})|^p+(\int_s^T|f^0(t,X_{t-s})|dt)^p).$$

Moreover, suppose (A1)-(A4) hold, and the conditions in Theorem 2.2.11 hold when $m(dx)=dx$. If f and ϕ satisfy the conditions (H4) and (H5) then the complement of $A_{2.s}$ is m-negligible (i.e. $m(A^c_{2,s})=0$) for each $s\in[0,T)$, the class of $u1_{A_2}$ is an element of $\hat{F}^l$ which is a generalized solution of PDE (2.2.1), $\psi\sigma$ represents a version of $D_\sigma u$ and the following relations hold for each $(s,x)\in A$ and $1\le i\le l$,

$$u^i(s,x)=E^x(\phi^i(X_{T-s}))+\int_s^TE^xf^i(t,X_{t-s},u(t,X_{t-s}),D_\sigma u(t,X_{t-s}))dt. \tag{2.4.5}$$

Proof We will assume that ϕ and f^0 are bounded; the general case is then obtained by approximation. Then the sets $A_p, p>0$, are all equal. We construct the functions (u,ψ) on A as follows. For $s\in[0,T)$, denote by $(Y^s_t,Z^s_t)_{s\le t\le T}$ the solution in Proposition 2.4.3, of the BSDE (2.4.3) over the interval $[s,T]$, in $L^2(P^x)$, $x\in A_s$ with data $(\phi(X_{T-s}(\omega)),f(t,X_{t-s}(\omega),y,z))$. Since $X_r\in A_{s+r}$, P^x-a.s., by the uniqueness part of that proposition one deduces that

$$Y^{s+r}_t\circ\theta_r=Y^s_t,\qquad t\in[s+r,T),P^x-a.s.,$$

$$(Z^{s+r}_t\sigma(X_{t-s-r}))\circ\theta_r=Z^s_t\sigma(X_{t-s}),\qquad dt\times P^x-a.s.$$

for each fixed $r\in[0,T-s)$ and all measures $P^x, x\in A_s$. In particular, if we define

$$u(s,x):=E^x(Y^s_s),$$

we will have, for any $x \in A_s$,

$$u(t, X_{t-s}) = E^{X_{t-s}}(Y_t^t) = E^x(Y_t^t \circ \theta_{t-s}|\mathcal{F}_{t-s}) = Y_t^s \qquad P^x - a.s..$$

Set $W_l(s,\omega) := Z_{l+s}^s \sigma(X_l)(\omega)$, for $(s,\omega) \in [0,T) \times \Omega$ and $l \in [0, T-s)$. One has $W_l(r+s, \theta_r(\omega)) = W_{l+r}(s,\omega), dl \times P^x - a.s.$ In terms of the time-space Markov process $\hat{X}$ (see e.g. [BPS05, Section 4.2]), we have $W_l(\hat{\theta}_r(s,\omega)) = W_{l+r}(s,\omega)$. Therefore, $t \to U_{j,t}^i(s,\omega) = \int_0^{t\wedge T} W_{j,l}^i(s,\omega)dl$, with $1 \leq i \leq l$ and $1 \leq j \leq k$, represents an additive functional for the time-space process $\hat{X}$. By [Sh88, Theorem 66.2] we deduce that there exists a nearly Borel measurable function $\tilde{\psi}_j^i : [0,T) \times \mathbb{R}^d \to R$, such that $\tilde{\psi}_j^i(t, X_{t-s}(\omega)) = W_{j,t-s}^i(s,\omega), dt \times P^x$-a.s. for each $x \in A_s$. Define

$$\psi := \tilde{\psi}\tau.$$

Then we have $Z_t^s \sigma(X_{t-s}) = \psi\sigma(t, X_{t-s})dt \times P^x - a.s., \forall x \in A_s$. Now we have

$$u^i(s,x) = E^x(\phi^i(X_{T-s})) + \int_s^T E^x f^i(t, X_{t-s}, u(t, X_{t-s}), \psi\sigma(t, X_{t-s}))dt,$$

since ϕ and f^0 are bounded. In particular, we have that $t \to u(t, X_{t-s})$ is continuous P^x-a.s. for each $x \in A_s$, because u may be written as the difference of two $\hat{X}$-excessive functions with regular potential part (cf. [BG68]). This implies $u(\cdot, X_{\cdot-s}) = Y_\cdot^s$. $u(\cdot, X_{\cdot-s}), \psi(\cdot, X_{\cdot-s})$ solves the BSDE (2.4.3) in $L^p(P^x)$ over the time interval $[s,T]$. By Theorem 2.3.12, we have that u is a generalized solution of (2.2.1) and that $\psi\sigma$ represents a version of $D_\sigma u$. $\square$

Remark 2.4.5 In the above theorem, we need the analytic results, i.e. the existence of a generalized solution of nonlinear equation (2.2.1), to obtain the above results. In the following example, we drop the conditions (A1)-(A4), in particular, we don't need $|b\sigma| \in L^2(\mathbb{R}^d; m)$ and use the results that the existence of the solution of BSDE (2.4.3) to obtain the existence of a generalized solution of nonlinear equation (2.2.1), which is not covered by our analytic results in Section 2.2.

Example 2.4.6 Consider $d \geq 2$, $A = (a^{ij})$ a Borel-measurable mapping on $\mathbb{R}^d$ with values in the non-negative symmetric matrices on $\mathbb{R}^d$, and let $b = (b^i) : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel-measurable vector field. Consider the operator

$$L_{A,b}\psi = a^{ij}\partial_i\partial_j\psi + b^i\partial_i\psi, \qquad \forall\psi \in C_0^\infty(\mathbb{R}^d),$$

where we use the standard summation rule for repeated indices. By $H^{1,p}(\mathbb{R}^d, dx)$ we denote the standard Sobolev space of functions on $\mathbb{R}^d$ whose first order derivatives are in $L^p(\mathbb{R}^d, dx)$. Assume that for $p > d$

(C1)$a^{ij} = a^{ji} \in H^{1,p}_{\text{loc}}(\mathbb{R}^d, dx), 1 \leq i, j \leq d$.

(C2)$b^i \in L^p_{\text{loc}}(\mathbb{R}^d, dx)$.

(C3) for all V relatively compact in $\mathbb{R}^d$ there exist $\nu_V > 0$ such that

$$\nu_V^{-1}|h|^2 \leq \langle ha, h\rangle \leq \nu_V |h|^2 \text{ for all } h \in \mathbb{R}^d, x \in V$$

Here by $H^{1,p}_{loc}(\mathbb{R}^d, dx)$ we denote the class of all functions f on $\mathbb{R}^d$ such that $f\chi \in H^{1,p}(\mathbb{R}^d, dx)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. And $L^p_{\text{loc}}(\mathbb{R}^d, dx)$ denotes the class of all functions f on $\mathbb{R}^d$ such that $f\chi \in L^p(\mathbb{R}^d)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. Assume that there exists $V \in C^2(\mathbb{R}^d)$ ("Lyapunov function") such that

$$\lim_{|x|\to\infty} V(x) = +\infty, \qquad \lim_{|x|\to\infty} L_{A,b}V(x) = -\infty.$$

Examples of V can be found in [BRS00] and the reference therein.

Then by [BRS00, Theorem 2.2] there exists a probability measure μ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} L_{A,b}\psi d\mu = 0 \qquad \forall \psi \in C_0^\infty(\mathbb{R}^d).$$

Then by [BRS00, Theorem 2.1] we have $d\mu$ is absolutely continuous w.r.t. dx and that the density admits a representation φ^2, where $\varphi^2 \in H^{1,p}_{loc}(\mathbb{R}^d, dx)$. The closure of

$$\mathcal{E}^0(u, v) = \frac{1}{2}\int \langle \nabla ua, \nabla v\rangle d\mu; \ u, v \in C_0^\infty(\mathbb{R}^d),$$

on $L^2(\mathbb{R}^d, \mu)$ is a Dirichlet form.

Set $b^0 = (b_1^0, ..., b_d^0)$, where $b_i^0 := \sum_{j=1}^d (\partial_j a_{ij} + 2a^{ij}\partial_j\varphi/\varphi), i = 1, ..., d$, and $\beta := b - b^0$. Then, $\beta \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d, \mu)$. By [St1, Proposition 1.10 and Proposition 2.4] $(L, C_0^\infty(\mathbb{R}^d))$ is L^1-unique. By the proof of [St1, Proposition 2.4] we conclude that for $u \in b\mathcal{F}$ there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{E}(u_n - u, u_n - u) \to 0, n \to \infty$.

Consider the bilinear form

$$\mathcal{E}(u, v) = \frac{1}{2}\int \langle \nabla ua, \nabla v\rangle d\mu - \int \langle \frac{1}{2}\beta, \nabla u\rangle v d\mu \ u, v \in C_0^\infty(\mathbb{R}^d).$$

Then by the computation in [Tr2, Section 4d] we have that conditions (A5) hold for the bilinear form $\mathcal{E}$. Then we can use the first part of Theorem 2.4.4 to obtain the following results.

Theorem 2.4.7 Consider the bilinear form obtained in Example 2.4.6. If f satisfies conditions (H1),(H2),(H3). Then there exist nearly Borel measurable functions (u,ψ), $u : A \to \mathbb{R}^l$, $\psi : A \to \mathbb{R}^l \otimes \mathbb{R}^d$, such that, for each $s \in [0,T)$ and each $x \in A_{p,s}$, the pair $(u(t, X_{t-s}), \psi(t, X_{t-s}))_{s \le t \le T}$ solves the BSDE (2.4.3) in $L^p(P^x)$ with data $(\phi(X_{T-s}), f(t, X_{t-s}, y, z))$ over the interval $[s,T]$.

In particular, the functions u, ψ satisfy the following estimates, for $(s,x) \in A_p$,

$$E^x(\sup_{t\in[s,T]} |u(t,X_{t-s})|^p + (\int_s^T |\psi\sigma(t,X_{t-s})|^2 dt)^{p/2}) \le KE^x(|\phi(X_{T-s})|^p + (\int_s^T |f^0(t,X_{t-s})|dt)^p).$$

Moreover, suppose f and ϕ satisfy the conditions (H4) and (H5) then the complement of $A_{2.s}$ is μ-negligible (i.e. $\mu(A_{2,s}^c) = 0$) for each $s \in [0,T)$, the class of $u1_{A_2}$ is an element of $\hat{F}^l$ which is the unique generalized solution of (2.2.1), $\psi\sigma$ represents a version of $D_\sigma u$ and the following relations hold for each $(s,x) \in A$ and $1 \le i \le l$,

$$u^i(s,x) = E^x(\phi^i(X_{T-s})) + \int_s^T E^x f^i(t, X_{t-s}, u(t,X_{t-s}), D_\sigma u(t,X_{t-s}))dt.$$

Proof By [St1, Lemma 3.1] we have that for $u \in D(L_{A,b})$, $u \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(u,u) \le -\int Luud\mu$. Hence the first part of proof in Proposition 2.1.9 hold in this case i.e. the mild solution is equivalent to the generalized solution and (2.1.7), (2.1.8) hold. Hence, the uniqueness of the solution (2.2.1) follows by the same arguments as the uniqueness proof of Theorem 2.2.8. Moreover, the results in Theorem 2.3.12 hold. By the same arguments as in the proof of Theorem 2.4.4 the assertion follows. □

2.5 Further Examples

The following two examples discuss the case where PDE satisfies some boundary conditions.

Example 2.5.1 . Let $D \subset \mathbb{R}^d$ be a bounded domain. We choose $m(dx) = 1_D(x)dx$. If $\mathcal{E}$ is a sectorial Dirichlet form, it is associated to a reflecting diffusion X in the state space $\overline{D}$. Then by Theorem 2.2.8 there exists a solution to the non-linear parabolic equation

$$\begin{aligned} (\partial_t + L)u + f(t,x,u,D_\sigma u) &= 0, \qquad 0 \le t \le T, \\ u_T(x) &= \phi(x), \qquad x \in \mathbb{R}^d, \\ \frac{\partial u(t,\cdot)}{\partial \nu}|_{\partial D} &= 0, \qquad t > 0, \end{aligned}$$

where $\frac{\partial}{\partial \nu}$ denotes the normal derivative. Then Theorem 2.4.4 provides a probabilistic interpretation for this equation.

Example 2.5.2 . Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the cone condition. We choose $m(dx) = 1_D(x)dx$ and replace $C_0^\infty(\mathbb{R}^d)$ by $C_0^\infty(D)$. Then the results in Theorem 2.2.8 apply and there exists a solution $u_1 \in F = H_0^1(D)$ to the following non-linear parabolic equation:

$$\begin{aligned}(\partial_t + L)u + f(t, x, u, D_\sigma u) = 0, &\qquad 0 \le t \le T,\\ u_T(x) = \phi(x), &\qquad x \in \mathbb{R}^d.\end{aligned}$$

Assume $\mathcal{E}$ satisfies the weak sector condition. Let X^0 denote the diffusion associated with $\mathcal{E}^R$, where $\mathcal{E}^R$ denotes the Dirichlet form which has the same form as $\mathcal{E}$ with the reference measure $m(dx)$ replaced by dx. Then define

$$X_t := \begin{cases} X_t^0, & \text{if } t < \tau,\\ \Delta & \text{otherwise,}\end{cases}$$

where $\tau = \inf\{t \ge 0, X_t^0 \in D^c \cup \Delta\}$. Assume (A5) holds for X^0. We use Theorem 2.4.4 for X^0 with the data $(\phi(X_{T-s}^0)1_{\{T-s<\tau\}}, 1_{[0,\tau+s]}(r)f(r, X_{r-s}^0, Y_r, Z_r\sigma(X_{r-s}^0))$. Then there exist nearly Borel measurable functions $(u, \psi), u : A \to \mathbb{R}^l, \psi : A \to \mathbb{R}^l \otimes \mathbb{R}^d$, such that, for each $s \in [0, T)$ and each $x \in A_{p,s}$, the pair $(u(t, X_{t-s}^0), \psi(t, X_{t-s}^0))_{s\le t\le T}$ solves the BSDE

$$Y_t = \phi(X_{T-s}^0)1_{\{T-s<\tau\}} + \int_{t\wedge(\tau+s)}^{T\wedge(\tau+s)} f(r, X_{r-s}^0, Y_r, Z_r\sigma(X_{r-s}^0))dr - \int_{t-s}^{T-s} Z_{s+l}.dM_l, \qquad s \le t \le T.$$

Then by [Pa99, Proposition 2.6] we have

$$Y_t = 0, Z_t = 0 \text{ when } t \in [\tau + s, T],$$

and the pair $(u(t, X_{t-s}), \psi(t, X_{t-s}))_{s\le t\le T}$ solves the BSDE

$$Y_t = \phi(X_{T-s}) + \int_t^T f(r, X_{r-s}, Y_r, Z_r\sigma(X_{r-s}))dr - \int_{t-s}^{T-s} Z_{s+l}.dM_l, \quad s \le t \le T.$$

In particular, the functions u, ψ satisfy the following estimates, for $(s, x) \in A_p$,

$$E^x(\sup_{t\in[s,T]} |u(t, X_{t-s})|^p + (\int_s^T |\psi\sigma(t, X_{t-s})|^2 dt)^{p/2}) \le KE^x(|\phi(X_{T-s})|^p + (\int_s^T |f^0(t, X_{t-s})|dt)^p).$$

The class of $u1_{A_2}$ is an element in $\hat{F}^l$ which is an m-version of u_1, $\psi\sigma$ represents a

version of $D_\sigma u$ and the following relations hold for each $(s,x)\in A$ and $1\le i\le l$,
(2.5.1)

$$u^i(s,x) = E^x(\phi^i(X_{T-s})) + \int_s^T E^x f^i(t, X_{t-s}, u(t, X_{t-s}), \psi(t, X_{t-s})\sigma(X_{t-s}))dt.$$

2.6 Appendix

2.6.1 Appendix A. Proof of Theorem 2.2.8

[Uniqueness]

Let u_1 and u_2 be two solutions of equation (2.2.1). By using (2.1.7) for the difference u_1-u_2 we get

$$\begin{aligned}
&\|u_{1,t}-u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_{1,s}-u_{2,s})ds \\
\le& 2\int_t^T (f(s,\cdot,u_{1,s},D_\sigma u_{1,s}) - f(s,\cdot,u_{2,s},D_\sigma u_{2,s}), u_{1,s}-u_{2,s})ds + 2\alpha\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds \\
\le& 2\int_t^T C(|D_\sigma u_{1,s}-D_\sigma u_{2,s}|, |u_{1,s}-u_{2,s}|)ds + 2\alpha\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds \\
\le& (\frac{C^2}{c_1}+c_2+2\alpha)\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_{1,s}-u_{2,s})ds.
\end{aligned}$$

By Gronwall's lemma it follows that

$$\|u_{1,t}-u_{2,t}\|_2^2 = 0,$$

hence $u_1=u_2$.

[Existence] The existence will be proved in four steps.

Step 1: Suppose there exists $r\in\mathbb{R}$ such that

$$r \ge 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f^{',1}\|_\infty),$$

where K is the constant appearing in Lemma 2.2.7 (2.2.5), and f is uniformly bounded on the set

$$A_r = [0,T]\times\mathbb{R}^d\times\{|y|\le r\}\times\mathbb{R}^l\otimes\mathbb{R}^k.$$

Define

$$M := \sup\{|f(t,x,y,z)| : (t,x,y,z)\in A_r\} < \infty.$$

Next we regularize f with respect to the variable y by convolution

$$f_n(t,x,y,z) = n^l \int_{R^l} f(t,x,y',z)\varphi(n(y-y'))dy',$$

where φ is a smooth nonnegative function with support contained in the ball $\{|y| \leq 1\}$ such that $\int \varphi = 1$. Then $f = \lim_{n\to\infty} f_n$ and for each n, $\partial_{y_i} f_n$ are uniformly bounded on A_{r-1}. Set

$$h_n(t,x,y,z) := f_n(t,x,\frac{r-1}{|y| \vee (r-1)}y,z).$$

Then each h_n satisfies the Lipschitz condition with respect to both y and z. Thus by Proposition 2.2.4 each h_n determines a solution $u_n \in \hat{F}^l$ of (2.2.1) with data (ϕ, h_n). By the same arguments as in [S09, Theorem 4.19], we have that h_n satisfies conditions (H1) and (H2') with the same constants ($C > 0$ and $\mu = 0$). As m is a finite measure and $f^{',1} \in L^\infty([0,T] \times \mathbb{R}^d)$, we have $f^{',1} \in L^2([0,T]; L^2)$. Since

$$\begin{aligned}|h_n(t,x,0,0) =& |f_n(t,x,0,0)| \\ \leq& n^l \int_{R^l} |f(t,x,y') - f^0(t,x) + f^0(t,x)||\varphi(n(-y'))|dy' \\ \leq& |f^0(t,x)| + f^{',1}(t,x),\end{aligned}$$

one deduces from Lemma 2.2.7 that $\|u_n\|_\infty \leq r-1$ and $\|u_n\|_T \leq K_T$. Since $h_n = f_n$ on A_{r-1}, it follows that u_n satisfies (2.2.1) with data (ϕ, f_n).

Now for $b > 0$, set

$$d_{n,b}(t,x) := \sup_{|y|\leq r-1, |z|\leq b} |f(t,x,y,z) - f_n(t,x,y,z)|.$$

Obviously one has $|d_{n,b}| \leq 2M$. Moreover, on account of the y-continuity and of the uniform z-continuity, one sees that for fixed t, x, b, the family of functions

$$\{f(t,x,\cdot,z) | |z| \leq b\},$$

is equicontinuous and then compact in $\mathcal{C}(\{|y| \leq r-1\})$. Since the convolution operators approach the identity uniformly on such a compact set, we get

$$\lim_{n\to\infty} d_{n,b}(t,x) = 0,$$

which implies $\lim_{n\to\infty} d_{n,b}(t,x) = 0$ in $L^2(dt \times m)$ because of our assumption that

$m(\mathbb{R}^d) < \infty$. Moreover, for $u \in \hat{F}^l, |u| \leq r-1$

$$\begin{aligned}|f(u, D_\sigma u) - f_n(u, D_\sigma u)| \leq & 1_{\{|D_\sigma u| \leq b\}} d_{n,b} + 2M 1_{\{|D_\sigma u| > b\}} \\ \leq & d_{n,b} + \frac{2M}{b} |D_\sigma u|.\end{aligned}$$

Next we will show that $(u_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_T$-Cauchy sequence. By (2.1.7) for the difference $u_l - u_n$, we have

$$\begin{aligned}
&\|u_{l,t} - u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s})ds \\
\leq & 2\int_t^T (f_l(s,\cdot,u_{l,s},D_\sigma u_{l,s}) - f_n(s,\cdot,u_{n,s},D_\sigma u_{n,s}), u_{l,s} - u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq & 2\int_t^T (|f_l(s,\cdot,u_{l,s},D_\sigma u_{l,s}) - f(s,\cdot,u_{l,s},D_\sigma u_{l,s})|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T (|f_n(s,\cdot,u_{n,s},D_\sigma u_{n,s}) - f(s,\cdot,u_{n,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T (|f(s,\cdot,u_{l,s},D_\sigma u_{l,s}) - f(s,\cdot,u_{l,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T (f(s,\cdot,u_{l,s},D_\sigma u_{n,s}) - f(s,\cdot,u_{n,s},D_\sigma u_{n,s}), u_{l,s} - u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq & 2\int_t^T (|f_l(s,\cdot,u_{l,s},D_\sigma u_{l,s}) - f(s,\cdot,u_{l,s},D_\sigma u_{l,s})|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T (|f_n(s,\cdot,u_{n,s},D_\sigma u_{n,s}) - f(s,\cdot,u_{n,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T C(|D_\sigma u_{l,s} - D_\sigma u_{n,s}|, |u_{l,s} - u_{n,s}|)ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq & 2\int_t^T (d_{l,b}(s,\cdot) + d_{n,b}(s,\cdot), |u_{l,s} - u_{n,s}|)ds + 2\int_t^T \frac{2M}{b}(|D_\sigma u_{l,s}| + |D_\sigma u_{n,s}|, |u_{l,s} - u_{n,s}|)ds \\
& + 2\int_t^T C(|D_\sigma u_{l,s} - D_\sigma u_{n,s}|, |u_{l,s} - u_{n,s}|)ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq & \int_t^T \|d_{l,b}(s,\cdot)\|_2^2 ds + \int_t^T \|d_{n,b}(s,\cdot)\|_2^2 ds + \frac{1}{b^2}\int_t^T (\|D_\sigma u_{l,s}\|_2^2 + \|D_\sigma u_{n,s}\|_2^2)ds \\
& + (1 + 4M^2 + \frac{C^2}{c_1} + 2\alpha + c_2)\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s})ds.
\end{aligned}$$

Since $\|u_n\|_T \leq K_T$, we have

$$\int_0^T \|D_\sigma u_{l,s}\|_2^2 ds \leq \frac{K_T}{c_1},$$

where the K_T is independent of l and b. Thus, for b, l, n large enough, for arbitrary $\varepsilon > 0$ we get

$$\|u_{l,t} - u_{n,t}\|_2^2 + \int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s})ds \leq \varepsilon + \tilde{K}\int_t^T \|u_{l,t} - u_{n,t}\|_2^2 ds,$$

where $\tilde{K}$ depends on C, M, μ, α. It is easy to see that Gronwall's lemma implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\hat{F}$. Define $u := \lim_{n\to\infty} u_n$ and take a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. We have

$$f(\cdot,\cdot,u_{n_k},D_\sigma u) \to f(\cdot,\cdot,u,D_\sigma u) \text{ in } L^2(dt\times m).$$

Since $\|u_{n_k} - u\|_T \to 0$, we obtain

$$\|D_\sigma u - D_\sigma u_{n_k}\|_{L^2(dt\times m)} \to 0.$$

Then by (H1), it follows that

$$\begin{aligned}&\lim_{k\to\infty}\|f(\cdot,\cdot,u_{n_k},D_\sigma u) - f(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k})\|_{L^2(dt\times m)}\\ \leq&\lim_{k\to\infty} C\|D_\sigma u - D_\sigma u_{n_k}\|_{L^2(dt\times m)}\\ =&0.\end{aligned}$$

We also have

$$\begin{aligned}&\|f(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k}) - f_{n_k}(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k})\|_{L^2(dt\times m)}\\ \leq&\|d_{n_k,b}\|_{L^2(dt\times m)} + \frac{2M}{b}\|D_\sigma u_{n_k}\|_{L^2(dt\times m)}.\end{aligned}$$

Letting $k\to\infty$ and then $b\to\infty$ the above equality converges to zero. Finally, we conclude

$$\begin{aligned}&\lim_{k\to\infty}\|f_{n_k}(u_{n_k},D_\sigma u_{n_k}) - f(u,D_\sigma u)\|_{L^2(dt\times m)}\\ \leq&\lim_{k\to\infty}\|f_{n_k}(u_{n_k},D_\sigma u_{n_k}) - f(u_{n_k},D_\sigma u_{n_k})\|_{L^2(dt\times m)}\\ &+\lim_{k\to\infty}\|f(u_{n_k},D_\sigma u_{n_k}) - f(u_{n_k},D_\sigma u)\|_{L^2(dt\times m)}\\ &+\lim_{k\to\infty}\|f(u_{n_k},D_\sigma u) - f(u,D_\sigma u)\|_{L^2(dt\times m)}\\ =&0.\end{aligned}$$

By passing to the limit in the mild equation associated to u_{n_k} with data (ϕ, f_{n_k}), it follows that u is the solution associated to (ϕ, f).

Step 2: In this step we will prove the assertion under the assumption that there exists some constant r such that $f^{',r}$ is uniformly bounded and

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f^{',1}\|_\infty),$$

where K is the constant appearing in Lemma 2.2.7 (2.2.5). Define

$$f_n(t,x,y,z) := f(t,x,y,\frac{n}{|z| \vee n} z).$$

$f_n \leq Cn + \|f^{',r}\|_\infty + \|f^0\|_\infty$ on A_r. Each of the functions f_n satisfies the same conditions as f and by Step 1, there exists a solution u_n associated to the data (ϕ, f_n). One has $\|u_n\|_\infty \leq r-1$, $\|u_n\|_T \leq K_T$. Conditions (H1) and (H2) yield

$$\begin{aligned}
&|(f_l(u_l, D_\sigma u_l) - f_n(u_n, D_\sigma u_n), u_l - u_n)| \\
\leq &C(|D_\sigma u_l - D_\sigma u_n|, |u_l - u_n|) + |(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|.
\end{aligned}$$

Since $f_n(t,x,y,z)1_{|z|\leq n} = f(t,x,y,z)1_{|z|\leq n}$, and for $n \leq l$, $|f_l - f_n|1_{|z|\geq n} \leq 2C|z|1_{|z|\geq n}$, we have

$$|(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)| \leq |(2C|D_\sigma u_n|1_{\{|D_\sigma u_n|\geq n\}}, |u_l - u_n|)|.$$

Then,

$$\begin{aligned}
&\|u_{l,t} - u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s})ds \\
\leq &2\int_t^T (f_l(u_{l,s}, D_\sigma u_{l,s}) - f_n(u_{n,s}, D_\sigma u_{n,s}), u_{l,s} - u_{n,s})ds + 2\alpha \int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq &2\int_t^T C(|D_\sigma u_l - D_\sigma u_n|, |u_l - u_n|)ds + 2\int_t^T |(2C|D_\sigma u_n|1_{\{|D_\sigma u_n|\geq n\}}, |u_l - u_n|)|ds \\
&+ 2\alpha \int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq &(\frac{C^2}{c_1} + 2\alpha + c_2)\int_t^T \|u_l - u_n\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_l - u_n)ds \\
&+ 8C(r-1)\int_t^T \int |D_\sigma u_n| 1_{\{|D_\sigma u_n|\geq n\}} dm ds
\end{aligned}$$

$$\begin{aligned}
\leq &(\frac{C^2}{c_1} + 2\alpha + c_2)\int_t^T \|u_l - u_n\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_l - u_n)ds \\
&+ 8C(r-1)(\int_t^T \|1_{\{|D_\sigma u_n|\geq n\}}\|_2^2 ds)^{\frac{1}{2}}(\int_t^T \|D_\sigma u_n\|_2^2 ds)^{\frac{1}{2}}.
\end{aligned}$$

As $\|u_n\|_T^2 \leq K_T$, we have $\int_0^T \|D_\sigma u_n\|_2^2 ds \leq \frac{K_T}{c_1}$. Hence,

$$n^2 \int_t^T \|1_{\{|D_\sigma u_n| \geq n\}}\|_2^2 ds \leq \int_t^T \|D_\sigma u_n 1_{\{|D_\sigma u_n| \geq n\}}\|_2^2 ds \leq \frac{K_T}{c_1}.$$

Therefore, for n big enough

$$\|u_{l,t} - u_{n,t}\|_2^2 + \int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s}) ds \leq (\frac{C^2}{c_1} + 2\alpha + c_2) \int_t^T \|u_l - u_n\|_2^2 ds + \varepsilon.$$

By Gronwalls' lemma it follows that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}^l$. Hence, $u := \lim_{n\to\infty} u_n$ is well defined. We can find a subsequence such that $(u_{n_k}, D_\sigma u_{n_k}) \to (u, D_\sigma u)$ a.e. and conclude

$$|f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f(u, D_\sigma u)| \leq C|\frac{n_k}{|D_\sigma u_{n_k}| \vee n_k} D_\sigma u_{n_k} - D_\sigma u| + |f(u_{n_k}, D_\sigma u) - f(u, D_\sigma u)| \to 0.$$

Since

$$\begin{aligned}
&|f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f(u, D_\sigma u)| \\
\leq &|f(u,0) - f(u, D_\sigma u)| + |f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f_{n_k}(u_{n_k}, 0)| + |f_{n_k}(u_{n_k}, 0) - f^0| + |f^0 - f(u,0)| \\
\leq &C(|D_\sigma u| + |D_\sigma u_{n_k}|) + 2f^{',r},
\end{aligned}$$

we have

$$f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) \to f(u, D_\sigma u) \text{ in } L^1([0,T], L^2).$$

We conclude that u is a solution of (2.2.1) associated to the data (ϕ, f).

Step 3: Now we only suppose that $f^{',1}$ is bounded. Hence, we can choose a constant r such that

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f^{',1}\|_\infty),$$

where K is the constant appearing in Lemma 2.2.7 (2.2.5). Let us define

$$f_n := \frac{n}{f^{',r} \vee n}(f - f^0) + f^0.$$

Easily we see that the f_n have the same properties as f. Since $f_n(t,x,y,z) = f(t,x,y,z)$ for $f^{',r} \leq n$, we have

$$\lim_{n\to\infty} f_n = f.$$

We introduce the following notation:

$$f_n^{',r}(t,x) := \sup_{|y|\le r} |f_n'(t,x,y)|, \text{ and } f_n'(t,x,y) := f_n(t,x,y,0) - f^0(t,x).$$

By the same arguments as in [S09, Theorem 4.19] we have

$$|f_n^{',r}| \le n \wedge |f^{',r}|.$$

Hence, by Step 2 we obtain that there exists a solution u_n associated to the data (ϕ, f^n) such that $\|u_n\|_\infty \le r-1, \|u_n\|_T \le M$, where M is a constant. For $n \le l$, we obtain

$$|f_l - f_n| \le (C|z| + |f'|)|\frac{l}{f^{',r} \vee l} - \frac{n}{f^{',r} \vee n}| \le (C|z| + |f'|)1_{\{f^{',r}>n\}}.$$

Hence

$$\int_t^T |(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|ds \le 2(r-1)\int_t^T \int_{\{f^{',r}>n\}} (C|D_\sigma u_n| + f^{',r})dmds.$$

We obtain as in the preceding steps:

$$\begin{aligned}
&\|u_{l,t} - u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}^{a,\hat{b}}(u_{l,s} - u_{n,s})ds \\
\le& 2\int_t^T (f_l(u_{l,s}, D_\sigma u_{l,s}) - f_n(u_{n,s}, D_\sigma u_{n,s}), u_{l,s} - u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\le& 2\int_t^T C(|D_\sigma u_l - D_\sigma u_n|, |u_l - u_n|)ds + 2\int_t^T |(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|ds \\
&+ 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\le& (\frac{C^2}{c_1} + 2\alpha + c_2)\int_t^T \|u_l - u_n\|_2^2 ds + \int_t^T \mathcal{E}^{a,\hat{b}}(u_l - u_n)ds \\
&+ 4(r-1)\int_t^T \int_{\{f^{',r}>n\}} (C|D_\sigma u_n| + f^{',r})dmds.
\end{aligned}$$

As

$$\lim_{n\to\infty}\int_t^T \int_{\{f^{',r}>n\}} f^{',r} dmds = 0,$$

and

$$\int_t^T \int_{\{f^{',r}>n\}} |D_\sigma u_n| dmdt \le \|1_{\{f^{',r}>n\}}\|_{L^2(dt\times m)} \|D_\sigma u_n\|_{L^2(dt\times m)} \to 0,$$

we have as above that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}^l$. Hence, $u := \lim_{n\to\infty} u_n$

exists in $\hat{F}^l$. We can find a subsequence such that $(u_{n_k}, D_\sigma u_{n_k}) \to (u, D_\sigma u)$ a.e. and we have that

$$\begin{aligned}
&|f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f(u, D_\sigma u)| \\
\leq& 1_{\{f'^{,r} \leq n_k\}} |f(u, D_\sigma u) - f(u_{n_k}, D_\sigma u_{n_k})| + 1_{\{f'^{,r} > n_k\}} [|f(u, D_\sigma u) - f^0| + |f(u, D_\sigma u) \\
&- f(u_{n_k}, D_\sigma u_{n_k})|] \\
\leq& |f(u, D_\sigma u) - f(u_{n_k}, D_\sigma u_{n_k})| + 1_{\{f'^{,r} > n_k\}} |f(u, D_\sigma u) - f^0| \\
\leq& |f(u_{n_k}, D_\sigma u) - f(u_{n_k}, D_\sigma u_{n_k})| + |f(u_{n_k}, D_\sigma u) - f(u, D_\sigma u)| + 1_{\{f'^{,r} > n_k\}} |f(u, D_\sigma u) - f^0|.
\end{aligned}$$

As in the above proof we have

$$f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) \to f(u, D_\sigma u),$$

in $L^1([0,T], L^2)$. We conclude that u is a solution of (2.2.1) associated to the data (ϕ, f).

Step 4: Now we prove the theorem without additional conditions. Define

$$f_n := \frac{n}{f'^{,1} \vee n}(f - f^0) + f^0.$$

Since $f_n(t,x,y,z) = f(t,x,y,z)$ for $f'^{,1} \leq n$, we have

$$\lim_{n \to \infty} f_n = f.$$

Introduce the following notation:

$$f_n'^{,1}(t,x) := \sup_{|y| \leq 1} |f_n'(t,x,y)| \text{ and } f_n'(t,x,y) := f_n(t,x,y,0) - f^0(t,x).$$

As in Step 3 we have

$$|f_n'^{,1}| \leq n \wedge |f'^{,1}|.$$

Since $f_n'^{,1}$ is uniformly bounded, we can apply Step 3. Then we get a solution u_n for the data (ϕ, f_n). The convergence of u_n can be shown analogously to Step 3. $\square$

2.6.2 Appendix B. Proof of Proposition 2.4.3

Let $M_x^p(\mathbb{R}^l)$ denote the set of (equivalence classes of)predictable processes $\{\phi_t\}_{t \in [0,T]}$ with values in $\mathbb{R}^l$ such that

$$\|\phi\|_{M_x^p} := (E^x[(\int_0^T |\phi_r|^2 dr)^{p/2}])^{1/p} < \infty.$$

$M^p_{\sigma,x}(\mathbb{R}^l \otimes \mathbb{R}^d)$ denotes the set of (equivalence classes of) predictable processes $\{\phi_t\}_{t\in[0,T]}$ with values in $\mathbb{R}^l \otimes \mathbb{R}^d$ such that

$$\|\phi\|_{M^p_{\sigma,x}} := (E^x[(\int_0^T |\phi_r\sigma(X_r)|^2 dr)^{p/2}])^{1/p} < \infty.$$

Fix $x \in A_p$.

We note that (Y, Z) solves the BSDE (2.4.1) with data (ξ, f) iff

$$(\bar{Y}_t, \bar{Z}_t) := (e^{\alpha_t} Y_t, e^{\alpha_t} Z_t),$$

solve the BSDE (2.4.1) with data $(e^{\alpha_T}\xi, f')$, where

$$f'(t, y, z) := e^{\alpha_t} f(t, e^{-\alpha_t} y, e^{-\alpha_t} z) - \mu_t y.$$

Therefore, we may replace $(\Omega 2)$ by

$$\langle y - y', f(t, \omega, y, z) - f(t, \omega, y', z)\rangle \le 0, \text{ for all } t, x, y, y', z.$$

Step 1 Assume that f is Lipschitz continuous with respect to both y and z. Define a mapping Φ from $B^2_x := M^2_x(\mathbb{R}^l) \times M^2_{\sigma,x}(\mathbb{R}^l \otimes \mathbb{R}^d)$ into itself as follows. Given $(U, V) \in B^2_x$, we can set $\Phi(U, V) := (Y, Z)$, where (Y, Z) is the solution of the BSDE (2.4.1) associated with data $(\xi, f(U, V\sigma(X)))$ given by Lemma 2.4.1. Then by Itô's formula and BDG inequality we get

$$E^x[\sup_{t\in[0,T]} |Y_t|^2] < \infty.$$

Let $(U, V), (U', V') \in B^2_x$, $(Y, Z) = \Phi(U, V)$, $(Y', Z') = \Phi(U', V')$, $(\bar{U}, \bar{V}) = (U - U', V - V')$, $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. It follows from Itô's formula that for each $\gamma \in \mathbb{R}$,

$$\begin{aligned} &e^{\gamma t} E^x|\bar{Y}_t|^2 + E^x \int_t^T e^{\gamma s}(\gamma|\bar{Y}_s|^2 + |\bar{Z}_s\sigma(X_s)|^2) ds \\ \le& 2K E^x \int_t^T e^{\gamma s}|\bar{Y}_s|(|\bar{U}_s| + |\bar{V}_s\sigma(X_s)|) ds \\ \le& 4K^2 E^x \int_t^T e^{\gamma s}|\bar{Y}_s|^2 + \frac{1}{2} E^x \int_t^T e^{\gamma s}(|\bar{U}_s|^2 + |\bar{V}_s\sigma(X_s)|^2) ds, \end{aligned}$$

where K is the Lipschitz constant of f. We choose $\gamma = 1 + 4K^2$. Then

$$E^x \int_0^T e^{\gamma s}(|\bar{Y}_s|^2 + |\bar{Z}_s\sigma(X_s)|^2) ds \le \frac{1}{2} E^x \int_0^T e^{\gamma s}(|\bar{U}_s|^2 + |\bar{V}_s\sigma(X_s)|^2) ds,$$

from which it follows that Φ is a strict contraction on B_x^2 equipped with the norm:

$$|||(Y,Z)|||_\gamma^x = (E^x \int_0^T e^{\gamma t}(|Y_t|^2 + |Z_t\sigma(X_t)|^2)dt)^{1/2}.$$

Define a sequence (Y^n, Z^n) by $(Y^{n+1}, Z^{n+1}) := \Phi(Y^n, Z^n)$. We have for $\gamma = 1 + 4K^2$

$$\begin{aligned}&E^x \int_0^T e^{\gamma s}(|Y_s^n - Y_s^{n+1}|^2 + |(Z_s^n - Z_s^{n+1})\sigma(X_s)|^2)ds \\ \le&\frac{1}{2^n} E^x \int_0^T e^{\gamma s}(|Y_s^0 - Y_s^1|^2 + |(Z_s^0 - Z_s^1)\sigma(X_s)|^2)ds.\end{aligned}$$

Then we have the a.s. pointwise convergence of $(Y_s^n, Z_s^n\sigma(X_s))$ under each measure P^x, $x \in A^2$. Denote the limit by $(Y_s, Z_s\sigma(X_s))$. Then this is the fixed point of Φ under the norm $|||(Y,Z)|||_\gamma^x$. So we have (Y_s, Z_s) is the solution of BSDE (2.4.1).

Step 2 We assume that f, ξ are bounded.

We need the following proposition.

Proposition B.1 Assume condition (A5). Given $V \in \cap_x M_{\sigma,x}^2(\mathbb{R}^l \otimes \mathbb{R}^d)$, there exists a unique pair of predictable processes $(Y_t, Z_t) \in M_x^2 \times M_{\sigma,x}^2(\mathbb{R}^l \otimes \mathbb{R}^d), \forall x \in \mathcal{N}^c$ satisfying under all P^x, $x \in \mathcal{N}^c$

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dM_s, \qquad 0 \le t \le T.$$

Using Proposition B.1, we can construct a mapping Φ from B_x^2 into itself as follows. For any $(U,V) \in B_x^2$, $(Y,Z) = \Phi(U,V)$ is the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dM_s, \qquad 0 \le t \le T.$$

Then as in **Step 1**, we have

$$\begin{aligned}&e^{\gamma t}E^x|\bar{Y}_t|^2 + E^x \int_t^T e^{\gamma s}(\gamma|\bar{Y}_s|^2 + |\bar{Z}_s\sigma(X_s)|^2)ds \\ =&2E^x \int_t^T e^{\gamma s}\langle \bar{Y}_s, f(Y_s, V_s\sigma(X_s)) - f(Y_s', V_s'\sigma(X_s))\rangle ds \\ \le&2KE^x \int_t^T e^{\gamma s}|\bar{Y}_s| \times |\bar{V}_s\sigma(X_s)|ds \\ \le&E^x \int_t^T e^{\gamma s}(2K^2|\bar{Y}_s|^2 + \frac{1}{2}|\bar{V}_s\sigma(X_s)|^2)ds.\end{aligned}$$

Then by the same arguments as in **Step 1**, we obtain the assertion of Proposition 2.4.3 if f, ξ are bounded.

Proof of Proposition B.1 We write $f(s,y)$ for $f(s,y,V_s)$.

By C we denote the constant satifying $|\xi|^2 + \sup_t |f(t,0)|^2 \le C$ a.s.. Define

$$f^n(t,y) := (\rho_n * f(t,\cdot))(y),$$

where $\rho_n : \mathbb{R}^l \mapsto \mathbb{R}^+$ is a sequence of smooth functions with compact support satisfying $\int \rho_n(z)dz = 1$, which approximate the Dirac measure at 0. Then each f^n is locally Lipschitz in y, uniformly with respect to s and ω.

Define for each $m \in \mathbb{N}$,

$$f^{n,m}(t,y) := f^n(t, \frac{\inf(m,|y|)}{|y|}y).$$

Then $f^{n,m}$ is globally Lipschitz and bounded, uniformly w.r.t. (t,ω). As in **Step 1**, we have a unique pair $(Y_t^{n,m}, Z_t^{n,m}) \in M_x^2 \times M_{\sigma,x}^2(\mathbb{R}^l \otimes \mathbb{R}^d)$ such that

$$Y_t^{n,m} = \xi + \int_t^T f^{n,m}(s, Y_s^{n,m})ds - \int_t^T Z_s^{n,m} dM_s, \qquad 0 \le t \le T.$$

By Itô's formula we have

$$|Y_t^{n,m}|^2 \le e^T C, \qquad 0 \le t \le T.$$

Consequently, for $m^2 > e^T C$, $(Y_t^{n,m}, Z_t^{n,m})$ does not depend on m. Therefore, we denote it by (Y_t^n, Z_t^n). Then by the same arguments as [BDHPS03, Proposition 3.2] we have

$$E^x(\sup_{0\le t\le T} |Y_t^k - Y_t^l|^2) + E^x(\int_0^T |(Z_t^k - Z_t^l)\sigma(X_t)|^2 dt) \le K E^x[\int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt].$$

By a similar argument as in the proof of Theorem 2.2.8, we obtain for fixed ω,

$$\sup_{k>l} \int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt \to 0, \text{ as } l \to \infty.$$

Then we have

$$\sup_{k>l} E^x \int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt \le E^x \sup_{k>l} \int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt \to 0, \qquad l \to \infty.$$

and we can obtain a sequence of representable variables that converges rapidly enough under all measures $P^x, x \in \mathcal{N}^c$. For each $l = 0, 1, ...$ set

$$n_l(x) = \inf\{n > n_{l-1}(x); \sup_{k \geq n} E^x[\int_0^T |f^k(t, Y_t^k) - f^n(t, Y_t^k)|^2 dt] < \frac{1}{2^l}\},$$

$$\bar{Y}^l = Y^{n_l(X_0)}, \bar{Z}^l = Z^{n_l(X_0)}.$$

With this sequence one may pass to the limit and define $Z'_s = \limsup_{l\to\infty} \bar{Z}^l_s \sigma(X_s)$ and $Z_s = Z'_s \tau(X_s)$. Then we obtain the claimed results. □

So far we have proved the assertion when ξ, f are bounded. Then by the same arguments as in [BDHPS03, Theorem 4.2], one proves the general case. □

Chapter 3

BSDE and generalized Dirichlet form: Infinite dimensional case

In this chapter we extend results in the previous chapter to infinite dimensional case. In Section 3.1 we give some basic assumptions on the operator L and prove some basic relations for linear equation. In Section 3.2, we use analytic methods to solve PDE (1.4). In Section 3.3, we prove the martingale representation theorem for the process associated with the operator L. By this we obtain the existence and uniqueness of the solution of BSDE (1.6) in Section 3.4. The relation between PDE and BSDE is also established in this section. Examples are given in Section 3.5. In Section 3.6, we use our results to a control problem for an application. The main results of this chapter have already been submitted for publication, see [Zhu b].

3.1 Preliminaries

Let E be a separable real Banach space and $(H, \langle\cdot,\cdot\rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously and ${}_{E'}\langle\cdot,\cdot\rangle_E = \langle\cdot,\cdot\rangle_H$ on $E' \times H$. Define the linear space of finitely based smooth functions on E by

$$\mathcal{F}C_b^\infty := \{f(l_1, ..., l_m) | m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, ..., l_m \in E'\}.$$

Here $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions with all partial derivatives bounded. For $u \in \mathcal{F}C_b^\infty$ and $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z + sk)|_{s=0}, z \in E,$$

be the Gâteaux derivative of u in direction k. It follows that for $u = f(l_1, ..., l_m) \in \mathcal{F}C_b^\infty$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1(z), ..., l_m(z))\langle l_i, k\rangle_H, z \in E.$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z).$$

Let μ be a finite positive measure on $(E, \mathcal{B}(E))$. By $L_{sym}(H)$ we denote the linear space of all symmetric and bounded linear operators on H equipped with usual operator norm $\|\cdot\|_{L^\infty(H)}$. Let $A : E \mapsto L_{sym}(H)$ be measurable such that $\langle A(z)h, h\rangle_H \geq 0$ for all $z \in E, h \in H$ and let $b : E \to H$ be $\mathcal{B}(E)/\mathcal{B}(H)$-measurable. Suppose the Pseudo inverse A^{-1} of A is measurable.

We denote by $|\cdot|_H$ the H-norm and set $\|u(z)\|_2^2 := \int |u(z)|^2 d\mu(z)$ for $u \in L^2(E, \mu)$. We also denote $(u, v)_{L^2(E,\mu)}$ by (u, v) for $u, v \in L^2(E, \mu)$. For $p \geq 1$, let $L^p(\mu)$, $L^p(\mu; H)$ denote $L^p(E, \mu)$, $L^p(E, \mu; H)$ respectively. If W is a function space, we will use bW to denote set of all the bounded functions in W.

Furthermore, we introduce the bilinear form

(3.1.1)
$$\mathcal{E}(u, v) := \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H d\mu(z) + \int \langle A(z)b(z), \nabla u(z)\rangle_H v(z) d\mu(z), u, v \in \mathcal{F}C_b^\infty.$$

We introduce the following conditions,

(A1) $\langle A(\cdot)k, k\rangle \in L^1(\mu)$ for $k \in H$ and the bilinear form

$$\mathcal{E}^A(u, v) = \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H d\mu(z); u, v \in \mathcal{F}C_b^\infty,$$

is closable on $L^2(E; \mu)$.

The closure of $\mathcal{F}C_b^\infty$ with respect to $\mathcal{E}_1^A := \mathcal{E}^A + \langle \cdot, \cdot\rangle_H$ is denoted by F. Then $(\mathcal{E}^A, F)$ is a well-defined symmetric Dirichlet form on $L^2(E, \mu)$. Set $\mathcal{E}_1^A(u) := \mathcal{E}_1^A(u, u), u \in F$.

(A2) Let $A^{1/2}b \in L^2(E; H, \mu)$, i.e. $\int |A^{1/2}b|_H^2 d\mu < \infty$, and there exists $\alpha \geq 0$ such that

(3.1.2)
$$\int \langle Ab, \nabla u^2\rangle_H d\mu \geq -\alpha\|u\|_2^2, \qquad u \in \mathcal{F}C_b^\infty, \forall u \geq 0.$$

Obviously, $\mathcal{E}$ from (3.1.1) immediately extends to all $u \in F, v \in bF$.

(A3) There exists a positivity preserving C_0-semigroup P_t on $L^2(E;\mu)$ such that for any $t \in [0,T], \exists C_T > 0$ such that

$$\|P_t f\|_\infty \leq C_T \|f\|_\infty,$$

and such that its L^2-generator $(L, \mathcal{D}(L))$ has the following properties: $b\mathcal{D}(L) \subset bF$ and for any $u \in bF$ there exists uniformly bounded $u_n \in \mathcal{D}(L)$ such that $\mathcal{E}_1^A(u_n - u) \to 0$ as $n \to \infty$ and that it is associated with the bilinear form $\mathcal{E}$ in (3.1.1) in the sense that $\mathcal{E}(u,v) = -(Lu, v)$ for $u, v \in b\mathcal{D}(L)$.

To obtain a semigroup P_t satisfying the above conditions, we can use generalized Dirichlet forms as introduced in Section 1.1.

Remark 3.1.1 (i) Some general criteria imposing conditions on A and μ in order that $\mathcal{E}^A$ be closable are e.g. given in [MR92, Chap II, Section 2] and [AR90].

(ii)In our case, due to our general conditions on b and f, we can't find a suitable Gelfand triple $V \subset H \subset V^*$ with V being a reflexive Banach space to apply the monotonicity method as in [Ba10] or [PR07].

(iii) We can construct a semigroup P_t satisfying (A3) by the theory of generalized Dirichlet forms. More precisely, if there exists a constant $\hat{c} \geq 0$ such that $\mathcal{E}_{\hat{c}}(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + \hat{c}(\cdot,\cdot)$ is a generalized Dirichlet form on a Hausdorff space E_1 with domain $\mathcal{F} \times \mathcal{V}$ in one of the following three senses:

(a)$(E_1, \mathcal{B}(E_1), m) = (E, \mathcal{B}(E), \mu)$,

$(\mathcal{A}, \mathcal{V}) = (\mathcal{E}^A, F)$,

$-\langle \Lambda u, v\rangle - \hat{c}(u,v) = \int \langle A(z)b(z), \nabla u(z)\rangle_H v(z) d\mu(z)$ for $u, v \in \mathcal{F}C_b^\infty$;

(b)$(E_1, \mathcal{B}(E_1), m) = (E, \mathcal{B}(E), \mu)$,

$\mathcal{A} \equiv 0$ and $\mathcal{V} = L^2(E,\mu)$,

$-\langle \Lambda u, v\rangle = \mathcal{E}_{\hat{c}}(u,v)$ for $u, v \in D$, where $D \subset \mathcal{F}C_b^\infty$ densely w.r.t. $\mathcal{E}_1^A$-norm and $D \subset \mathcal{D}(L)$;

(c) $\mathcal{E}_{\hat{c}} = \mathcal{A}$, $\Lambda \equiv 0$ (In this case $(\mathcal{E}_{\hat{c}}, \mathcal{V})$ is a sectorial Dirichlet form in the sense of [MR92]);

then there exists a sub-Markovian C_0-semigroup of contraction $P_t^{\hat{c}}$ associated with the generalized Dirichlet form $\mathcal{E}_{\hat{c}}$. Then $P_t := e^{\hat{c}t} P_t^{\hat{c}}$ satisfies (A3) and we have

$$\mathcal{D}(L) \subset \mathcal{F} \subset F.$$

(iv)The semigroup can also be constructed by other methods. (see e.g. [DR02], [BDR09]).

(v) By (A3) we have that $\mathcal{E}$ is positivity preserving, i.e.

$$\mathcal{E}(u, u^+) \geq 0 \ \forall u \in \mathcal{D}(L),$$

which can be obtained by the same arguments as in [St2, I Proposition 4.4]. By (3.1.2) and (A3), we have for $u \in b\mathcal{D}(L), u \geq 0$

$$\int Lud\mu = -\mathcal{E}(u, 1) = -\int \langle Ab, \nabla u\rangle_H d\mu = -\int \langle Ab, \nabla(u+\varepsilon)\rangle_H d\mu \leq -\alpha \int (u+\varepsilon)d\mu.$$

Letting $\varepsilon \to 0$, we have $\int Lud\mu \leq -\alpha \int ud\mu$. $(P_t)_{t\in[0,T]}$ is a C_0-semigroup on $L^1(E;\mu)$.

(vi) All the conditions are satisfied by the bilinear form considered in [St1, Section 4] and the operator in [D04, Chapter II,III,IV] (see Section 3.5 below).

(vii) The notion of quasi-regularity for generalized Dirichlet forms analogously to [MR92] has been introduced in [St2]. By this and a technical assumption an associated m-tight special standard process can be constructed. We will use stochastic calculus associated with this process to conclude our probabilistic results (see Section 3.3 below).

Let us recall the notations $\hat{F}, \mathcal{C}_T, \|\cdot\|_T$ associated with $\mathcal{E}^A$ from [BPS05]: $\mathcal{C}_T := C^1((0,T); L^2) \cap L^2(0,T;F)$, which turns out to be the appropriate space of test functions, i.e.

$$\begin{aligned}\mathcal{C}_T = \{\varphi : [0,T] \times E \to \mathbb{R} | \varphi_t \in F \text{ for almost each } t, \int_0^T \mathcal{E}^A(\varphi_t, \varphi_t)dt < \infty,\\ t \to \varphi_t \text{ is differentiable in } L^2 \text{and } t \to \partial_t\varphi_t \text{ is } L^2-\text{continuous on } [0,T]\}.\end{aligned}$$

We also set $\mathcal{C}_{[a,b]} := C^1([a,b]; L^2) \cap L^2([a,b]; F)$. For $\varphi \in \mathcal{C}_T$, we define

$$\|\varphi\|_T := (\sup_{t\leq T} \|\varphi_t\|_2^2 + \int_0^T \mathcal{E}^A(\varphi_t)dt)^{1/2}.$$

$\hat{F}$ is the completion of $\mathcal{C}_T$ with respect to $\|\cdot\|_T$. By [BPS05], $\hat{F} = C([0,T]; L^2) \cap L^2(0,T;F)$, and for every $u \in \hat{F}$ there exists a sequence $u^n \in \mathcal{F}C_b^{\infty,T}, n \in \mathbb{N}$, such that $\int_0^T \mathcal{E}_1^A(u_t - u_t^n)dt \to 0$, where

$$\mathcal{F}C_b^{\infty,T} := \{f(t, l_1, ..., l_m) | m \in \mathbb{N}, f \in C_b^\infty([0,T] \times \mathbb{R}^m), l_1, ..., l_m \in E'\}.$$

We also introduce the following space

$$W^{1,2}([0,T];L^2(E)) = \{u \in L^2([0,T];L^2); \partial_t u \in L^2([0,T];L^2)\},$$

where $\partial_t u$ is the derivative of u in the weak sense (see e.g. [Ba10]).

3.1.1 Linear Equations

We consider the linear equation

$$(3.1.3) \qquad \begin{aligned} (\partial_t + L)u + f = 0, &\qquad 0 \le t \le T \\ u_T(x) = \phi(x), &\qquad x \in E \end{aligned}$$

where $f \in L^1([0,T];L^2(E,\mu)), \phi \in L^2(E,\mu)$.

By [BPS05] we set $D_{A^{1/2}}\varphi := A^{1/2}\nabla\varphi$ for any $\varphi \in \mathcal{F}C_b^\infty$, define $V_0 = \{D_{A^{1/2}}\varphi : \varphi \in \mathcal{F}C_b^\infty\}$, and let V be the closure of V_0 in $L^2(E;H,\mu)$. Then we have the following results.

Proposition 3.1.2 Assume (A1) holds.

(i) For every $u \in F$ there is a unique element of V, which we denote by $D_{A^{1/2}}u$ such that

$$\mathcal{E}^A(u,\varphi) = \int \langle D_{A^{1/2}}u(x), D_{A^{1/2}}\varphi(x)\rangle_H \mu(dx), \qquad \forall \varphi \in \mathcal{F}C_b^\infty.$$

One has $A^{1/2}A^{-1/2}D_{A^{1/2}}u(x) = D_{A^{1/2}}u(x)$. Moreover, the above formula extends to $u, v \in F$,

$$\mathcal{E}^A(u,v) = \int \langle D_{A^{1/2}}u(x), D_{A^{1/2}}v(x)\rangle_H \mu(dx).$$

(ii) Furthermore, if $u \in \hat{F}$, there exists a measurable function $\phi : [0,T] \times E \mapsto H$ such that $|A^{1/2}\phi|_H \in L^2([0,T] \times E)$ and $D_{A^{1/2}}u_t = A^{1/2}\phi_t$ for almost all $t \in [0,T]$.

(iii)Let $u^n, u \in \hat{F}$ be such that $u^n \to u$ in $L^2((0,T) \times E)$ and $(D_{A^{1/2}}u^n)_n$ is a Cauchy-sequence in $L^2([0,T]\times E;H)$. Then $D_{A^{1/2}}u^n \to D_{A^{1/2}}u$ in $L^2((0,T)\times E;H)$, i.e. $D_{A^{1/2}}$ is closable as an operator from $\hat{F}$ into $L^2((0,T) \times E;H)$.

Proof (i) Since $\mathcal{E}^A$ is closable on $L^2(E;\mu)$, the assertion follows.

(ii)If $u \in \hat{F}$, we have $u^n \in \mathcal{F}C_b^{\infty,T}$, $n \in \mathbb{N}$, such that $\int_0^T \mathcal{E}_1^A(u^n - u)dt \to 0$. Hence, we define $\varphi := \lim_{n\to\infty} D_{A^{1/2}}u^n$ in $L^2((0,T) \times E;H)$ and $\phi := A^{-1/2}\varphi$. Since $A^{1/2}A^{-1/2}A^{1/2} = A^{1/2}$ and $\|A^{-1/2}A^{1/2}\| \le 1$, $D_{A^{1/2}}u^n = A^{1/2}A^{-1/2}A^{1/2}\nabla u^n \to A^{1/2}A^{-1/2}\varphi = A^{1/2}\phi$ in $L^2([0,T] \times E;H)$. Passing to a subsequence we may find a set $\Lambda \subset [0,T]$ such that $[0,T] \setminus \Lambda$ is negligible and for every $t \in \Lambda$, $\mathcal{E}_1^A(u_t^n - u_t) \to 0$

and $\|D_{A^{1/2}}u_t^n - A^{1/2}\phi_t\|_{L^2(E;H)} \to 0$. Then we have

$$\mathcal{E}^A(u_t,\varphi) = \int \langle A^{1/2}\phi_t(x), D_{A^{1/2}}\varphi(x)\rangle_H \mu(dx).$$

(iii) Let $v = \lim_n D_{A^{1/2}}u^n$. Passing to a subsequence we assume for almost every $t \in [0,T]$, $\|v_t - D_{A^{1/2}}u_t^n\|_{L^2(E;H)} \to 0$. We take $\varphi \in \mathcal{D}(L^A)$ where L^A is the generator associated to $(\mathcal{E}^A, F)$. Then

$$\begin{aligned}\int \langle v_t, D_{A^{1/2}}\varphi\rangle_H d\mu &= \lim_{n\to\infty}\int \langle D_{A^{1/2}}u_t^n, D_{A^{1/2}}\varphi\rangle_H d\mu = \lim_n \mathcal{E}^A(u_t^n,\varphi) = -\lim_n (u_t^n, L^A\varphi) \\ &= -\,(u_t, L^A\varphi) = \mathcal{E}^A(u_t,\varphi) = \int \langle D_{A^{1/2}}u_t, D_{A^{1/2}}\varphi\rangle_H d\mu.\end{aligned}$$

It follows that $v_t = D_{A^{1/2}}u_t$. □

For $u \in F, v \in bF$ we denote

$$\mathcal{E}(u,v) := \int \langle D_{A^{1/2}}u(x), D_{A^{1/2}}v(x)\rangle_H \mu(dx) + \int \langle A^{1/2}b, D_{A^{1/2}}u\rangle_H v\mu(dx).$$

Notation By $\tilde{\nabla}u$ we denote the set of all measurable functions $\phi: E \to H$, such that $A^{1/2}\phi = D_{A^{1/2}}u$ as elements of $L^2(\mu;H)$.

3.1.2 Solution of the Linear Equation

Definition 3.1.3 [*strong solution*] A function $u \in \hat{F} \cap L^1((0,T);\mathcal{D}(L))$ is called a strong solution of equation (3.1.3) with data (ϕ, f), if $t \mapsto u_t = u(t,\cdot)$ is L^2-differentiable on $[0,T], \partial_t u_t \in L^1((0,T);L^2)$ and the equalities in (3.1.3) hold in $L^2(\mu)$.

Definition 3.1.4 [*generalized solution*] A function $u \in \hat{F}$ is called a generalized solution of equation (3.1.3), if there exists a sequence of $\{u^n\}$ consisting of strong solutions with data (ϕ^n, f^n) such that

$$\|u^n - u\|_T \to 0, \|\phi^n - \phi\|_2 \to 0, \lim_{n\to\infty} f^n = f \text{ in } L^1([0,T];L^2(\mu)).$$

By (A3) and Remark 3.1.1 (v), for $0 \le t \le T$, P_t, as C_0-semigroup on $L^1(E;\mu)$, can be restricted to a semigroup on $L^p(E;\mu)$ for all $p \in [1,\infty)$ by the Riesz-Thorin Interpolation Theorem and the restricted semigroup (denoted again by P_t for simplicity) is strongly continuous on $L^p(E;\mu)$.

Proposition 3.1.5 Assume that (A1)-(A3) hold.

(i) Let $f \in C^1([0,T];L^p)$ for $p \in [1,\infty)$. Then

$$w_t := \int_t^T P_{s-t} f_s ds \in C^1([0,T];L^p),$$

and

$$\partial_t w_t(x) = -P_{T-t} f_T(x) + \int_t^T P_{s-t} \partial_s f_s(x) ds.$$

(ii) Assume that $\phi \in \mathcal{D}(L)$, $f \in C^1([0,T];L^2)$ and for each $t \in [0,T]$, $f_t \in \mathcal{D}(L)$. Define

$$u_t := P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Then u is a strong solution of (3.1.5) and, moreover, $u \in C^1([0,T];L^2)$.

Proof See the proof of [BPS05, Proposition 2.6]. □

Proposition 3.1.6 Suppose that conditions (A1)-(A3) hold. If u is a strong solution for (3.1.3), it is a mild solution for (3.1.3) i.e.

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Proof For fixed t, $\varphi \in \mathcal{D}(\hat{L})$

$$(u_T, \hat{P}_{T-t}\varphi) - (u_t, \varphi) = \int_t^T (-Lu_s - f_s, \hat{P}_{s-t}\varphi) ds + \int_t^T (u_s, \hat{L}\hat{P}_{s-t}\varphi) ds,$$

where $\hat{L}, \hat{P}_t$ denote the adjoints on $L^2(E,\mu)$ of L and P_t respectively. As u is a strong solution, we deduce that

$$(u_t, \varphi) = (P_{T-t}\phi + \int_t^T P_{s-t} f_s ds, \varphi).$$

Since $\mathcal{D}(\hat{L})$ is dense in L^2, we have the result. □

Proposition 3.1.7 Suppose that conditions (A1)-(A3) hold, $f \in L^1([0,T];L^2)$ and $\phi \in L^2$. Then the equation (3.1.3) has a unique generalized solution $u \in \hat{F}$

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds. \tag{3.1.4}$$

The solution satisfies the three relations:

(3.1.5)
$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s)ds \leq 2\int_t^T (f_s, u_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s\|_2^2 ds, \qquad 0 \leq t \leq T,$$

(3.1.6)
$$\|u\|_T^2 \leq M_T(\|\phi\|_2^2 + (\int_0^T \|f_t\|_2 dt)^2),$$

(3.1.7)
$$\int_0^T ((u_t, \partial_t\varphi_t) + \mathcal{E}^A(u_t, \varphi_t) + \int \langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H \varphi_t d\mu)dt = \int_0^T (f_t, \varphi_t)dt + (\phi, \varphi_T) - (u_0, \varphi_0),$$

for any $\varphi \in b\mathcal{C}_T$, where M_T is a constant depending on T. (3.1.7) can be extended easily to $\varphi \in bW^{1,2}([0,T]; L^2) \cap L^2([0,T]; F)$.

Moreover, if $u \in \hat{F}$ is bounded and satisfies (3.1.7) for any $\varphi \in b\mathcal{C}_T$ with bounded (f, ϕ), then u is a weak solution given by (3.1.4).

Proof Define u by (3.1.4). First assume that ϕ, f are bounded and satisfy the conditions of Proposition 3.1.5 (ii). Then, since u is bounded and by Proposition 3.1.5 we know that u is a strong solution of (3.1.3), hence it obviously satisfies (3.1.7). Furthermore, $u \in C^1([0,T]; L^2)$. Hence, actually $u \in b\mathcal{C}_T$ and consequently, for $t_0 \in [0,T]$

$$\int_{t_0}^T ((u_t, \partial_t u_t) + \mathcal{E}^A(u_t, u_t) + \int \langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H u_t d\mu)dt = \int_{t_0}^T (f_t, u_t)dt + (\phi, u_T) - (u_{t_0}, u_{t_0}).$$

By (3.1.2) we have $\int \langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H u_t d\mu \geq -\alpha\|u_t\|_2^2$ then we have

(3.1.8)
$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s)ds \leq 2\int_t^T (f_s, u_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s\|_2^2 ds, \qquad 0 \leq t \leq T.$$

As

$$\begin{aligned}\int_t^T (f_s, u_s)ds &= \int_t^T ((f_s, P_{T-s}\phi) + (f_s, \int_s^T P_{r-s}f_r dr))ds \\ &\leq \int_t^T \|f_s\|_2\|P_{T-s}\phi\|_2 ds + \int_t^T \|f_s\|_2\|\int_s^T P_{r-s}f_r dr\|_2 ds \\ &\leq M_0 e^{T-t}(\|\phi\|_2 \int_t^T \|f_s\|_2 ds + \int_t^T (\|f_s\|_2 \int_s^T \|f_r\|_2 dr)ds),\end{aligned}$$

and
$$\int_t^T \|u_s\|_2^2 ds \leq M_{T-t}(\|\phi\|_2^2 + (\int_0^T \|f_t\|_2 dt)^2),$$

we obtain

$$\|u_t\|_2^2 + \int_t^T \mathcal{E}^A(u_s)ds \leq M_{T-t}(\|\phi\|_2^2 + (\int_0^T \|f_t\|_2 dt)^2).$$

Hence, it follows that

$$\|u\|_T^2 \leq M_T(\|\phi\|_2^2 + (\int_0^T \|f_t\|_2 dt)^2). \tag{3.1.9}$$

Here the constant M_{T-t} may change from line to line, but it is independent of f, ϕ. Next we will prove the result for general data ϕ and f. Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of functions in $bC^1([0,T];L^2(\mu))$ such that $f_t \in \mathcal{D}(L)$ for a.e. $t \in [0,T]$ and $\int_0^T \|f_t^n - f_t\|_2 dt \to 0$. (Such a sequence exists, since $\{\alpha_t g(x); \alpha_t \in C_0^\infty[0,T], g \in b\mathcal{D}(L)\}$ is dense in $L^1([0,T];L^2)$). Take functions $(\phi^n)_{n\in\mathbb{N}} \subset b\mathcal{D}(L)$ such that $\phi^n \to \phi$ in L^2. Let u^n denote the solution given by (3.1.4) with $f = f^n, \phi = \phi^n$.

By linearity, $u^n - u^m$ is associated with $(\phi^n - \phi^m, f^n - f^m)$. Since (3.1.9) implies that

$$\|u^n - u^m\|_T^2 \leq M_T(\|\phi^n - \phi^m\|_2^2 + (\int_0^T \|f_t^n - f_t^m\|_2 dt)^2),$$

we deduce that $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}$. Then $u = \lim_{n\to\infty} u^n$ in $\|\cdot\|_T$ is a generalized solution of (3.1.3) and we have

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Next we prove (3.1.5) (3.1.6) (3.1.7) for u. For $\varphi \in b\mathcal{C}_T$, we have

(3.1.10)

$$\int_0^T ((u_t^n, \partial_t\varphi_t) + \mathcal{E}^A(u_t^n, \varphi_t) + \int \langle A^{1/2}b, D_{A^{1/2}}u_t^n\rangle_H \varphi_t d\mu)dt = \int_0^T (f_t^n, \varphi_t)dt + (\phi^n, \varphi_T) - (u_0^n, \varphi_0).$$

Since we have

$$|\int_0^T \mathcal{E}^A(u_t^n - u_t, \varphi_t)dt| \leq (\int_0^T \mathcal{E}^A(u_t^n - u_t)dt)^{\frac{1}{2}}(\int_0^T \mathcal{E}^A(\varphi_t)dt)^{\frac{1}{2}} \to 0, \text{ as } n \to \infty,$$

and

$$\begin{aligned}
&|\int_0^T \int \langle A^{1/2}b, D_{A^{1/2}}(u_t^n - u_t)\rangle_H \varphi_t d\mu dt| \\
\leq& \|\varphi\|_\infty (\int_0^T \int |A^{1/2}b|_H^2 d\mu dt)^{\frac{1}{2}} (\int_0^T \int |D_{A^{1/2}}(u_t^n - u_t)|_H^2 d\mu dt)^{\frac{1}{2}} \\
=& \|\varphi\|_\infty (\int_0^T \int |A^{1/2}b|_H^2 d\mu dt)^{\frac{1}{2}} (\int_0^T \mathcal{E}^A(u_t^n - u_t)dt)^{\frac{1}{2}} \\
\to& 0, \text{ as } n \to \infty,
\end{aligned}$$

we deduce that

$$\int_0^T ((u_t,\partial_t\varphi_t)+\mathcal{E}^A(u_t,\varphi_t)+\int\langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H\varphi_t d\mu)dt = \int_0^T (f_t,\varphi_t)dt+(\phi,\varphi_T)-(u_0,\varphi_0),$$

for any $\varphi\in b\mathcal{C}_T$.

The relations (3.1.5) (3.1.6) hold for the approximating functions:

$$\|u_t^n\|_2^2+2\int_t^T \mathcal{E}^A(u_s^n)ds \le 2\int_t^T (f_s^n,u_s^n)ds+\|\phi^n\|_2^2+2\alpha\int_t^T \|u_s^n\|_2^2 ds, \qquad 0\le t\le T.$$

$$\|u^n\|_T^2 \le M_T(\|\phi^n\|_2^2+(\int_0^T \|f_t^n\|_2 dt)^2).$$

Since $\|u_t^n\|_T\to\|u_t\|_T$, $n\to\infty$, we conclude

$$\lim_{n\to\infty}\int_0^T \mathcal{E}^A(u_t^n)dt = \int_0^T \mathcal{E}^A(u_t)dt.$$

It is easy to see that $\lim_{n\to\infty}\int_t^T (f_s^n,u_s^n)ds = \int_t^T (f_s,u_s)ds$. Then by passing to the limit, $n\to\infty$ in the above relations, we get (3.1.5) and (3.1.6) for u.

[Uniqueness] Let $v\in\hat{F}$ be another generalized solution of (3.1.3) and let $(v^n)_{n\in\mathbb{N}}$, $(\tilde{\phi}^n)_{n\in\mathbb{N}}$, $(\tilde{f}^n)_{n\in\mathbb{N}}$ be the corresponding approximating sequences in the definition of the generalized solution. By Proposition 3.1.6

$$\sup_{t\in[0,T]}\|u_t^n-v_t^n\|_2^2 \le M_T(\|\phi^n-\tilde{\phi}^n\|_2^2+(\int_0^T \|f_t^n-\tilde{f}_t^n\|_2 dt)^2).$$

Letting $n\to\infty$, this implies $u=v$.

For the last result we have $\forall t_0\in[0,T],\varphi\in b\mathcal{C}_T$

(3.1.11)

$$\int_{t_0}^T ((u_t,\partial_t\varphi_t)+\mathcal{E}^A(u_t,\varphi_t)+\int\langle A^{1/2}b, D_{A^{1/2}}u_t\rangle\varphi_t dm)dt = \int_{t_0}^T (f_t,\varphi_t)dt+(\phi,\varphi_T)-(u_{t_0},\varphi_{t_0}).$$

For $t\ge\frac{1}{n}$, define

$$u_t^n := n\int_0^{\frac{1}{n}} u_{t-s}ds, \qquad f_t^n := n\int_0^{\frac{1}{n}} f_{t-s}ds, \qquad \phi^n := n\int_0^{\frac{1}{n}} u_{T-s}ds.$$

By a similar argument as the proof of Proposition 2.1.9, u^n also fulfills (3.1.11) with f^n,ϕ^n. For the mild solution v associated with f,ϕ, the above relation also holds with v^n replacing u^n. Hence we have

$$\int_{t_0}^T (((u-v)_t^n, \partial_t\varphi_t)+\mathcal{E}^A((u-v)_t^n, \varphi_t)+\int\langle A^{1/2}b, D_{A^{1/2}}(u-v)_t^n\rangle\varphi_t dm)dt = -((u-v)_{t_0}^n, \varphi_{t_0}).$$

Since $(u-v)_t^n \in b\mathcal{C}_{[\frac{1}{n},T]}$, the above equation holds with $(u-v)_t^n$ as a test function. So we have

$$\|(u-v)_{t_0}^n\|_2^2 + 2\int_{t_0}^T \mathcal{E}^A((u-v)_t^n, (u-v)_t^n)dt \leq 2\alpha\int_{t_0}^T \|(u-v)_t^n\|_2^2 dt.$$

By Gronwall's lemma it follows that

$$\|(u-v)_{t_0}^n\|_2^2 = 0.$$

Letting $n \to \infty$, we have $\|u_{t_0}-v_{t_0}\|_2 = 0$. Then letting $t_0 \to 0$, we have $\|u_0-v_0\| = 0$. Therefore, $u_t = P_{T-t}\phi + \int_t^T P_{s-t}f_s ds$ is a generalized solution for (3.1.3). $\square$

3.1.3 Basic Relations for the Linear Equation

In this section we assume that (A1)-(A3) hold.

Lemma 3.1.8 If u is a bounded generalized solution of equation (3.1.3) with some function $\phi \geq 0, \phi \in L^2 \cap L^\infty$, then u^+ satisfies the following relation with $0 \leq t_1 < t_2 \leq T$

$$\|u_{t_1}^+\|_2^2 \leq 2\int_{t_1}^{t_2} (f_s, u_s^+)ds + \|u_{t_2}^+\|_2^2.$$

Proof Choose the approximation sequence u^n for u as in the existence proof of Proposition 3.1.7. Denote its related data by f^n, ϕ^n .

We have the following equations:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u_t^n - u_t\|_2 = 0, \qquad \lim_{n\to\infty}\int_0^T \mathcal{E}^A(u_t^n - u_t)dt = 0,$$

$$\lim_{n\to\infty}\int_0^T \|f_t^n - f_t\|_2 dt = 0, \qquad \lim_{n\to\infty}\|\phi^n - \phi\|_2 = 0.$$

Suppose that the following holds

$$\|(u_{t_1}^n)^+\|_2^2 \leq 2\int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)ds + \|(u_{t_2}^n)^+\|_2^2, \tag{3.1.12}$$

where $0 \leq t_1 \leq t_2 \leq T$. Since $\|u^n\|_2$ are uniformly bounded, we have

$$\lim_{n\to\infty}\int_{t_1}^{t_2}(f_s^n,(u_s^n)^+)ds=\int_{t_1}^{t_2}(f_s,u_s^+)ds.$$

By letting $n \to \infty$ in equation (3.1.12) we get for $0 \leq t_1 \leq t_2 \leq T$,

$$\|u_{t_1}^+\|_2^2 \leq 2\int_{t_1}^{t_2}(f_s,u_s^+)ds+\|u_{t_2}^+\|_2^2.$$

Therefore, the problem is reduced to the case where u belongs to $b\mathcal{C}_T$; in the remainder we assume $u \in b\mathcal{C}_T$. (3.1.7), written with $u^+ \in bW^{1,2}([0,T];L^2) \cap L^2([0,T];F)$ as test functions, takes the form

$$\begin{aligned}&\int_{t_1}^{t_2}(u_t,\partial_t(u_t^+))dt+\int_{t_1}^{t_2}\mathcal{E}^A(u_t,u_t^+)dt+\int_{t_1}^{t_2}\int\langle A^{1/2}b,D_{A^{1/2}}u_t\rangle u_t^+d\mu dt\\=&\int_{t_1}^{t_2}(f_t,u_t^+)dt+(u_{t_2},u_{t_2}^+))-(u_{t_1},u_{t_1}^+)).\end{aligned}\tag{3.1.13}$$

By [Ba10, Theorem 1.19] we obtain

$$\int_{t_1}^{t_2}(u_t,\partial_t(u_t^+))dt=\frac{1}{2}(\|u_{t_2}^+\|_2^2-\|u_{t_1}^+\|_2^2).$$

Then

$$\begin{aligned}&\|u_{t_1}^+\|_2^2+2\int_{t_1}^{t_2}\mathcal{E}^A(u_t,u_t^+)dt+2\int_{t_1}^{t_2}\int\langle A^{1/2}b,D_{A^{1/2}}u_t\rangle_H u_t^+d\mu dt\\=&2\int_{t_1}^{t_2}(f_t,u_t^+)dt+\|u_{t_2}^+\|_2^2.\end{aligned}\tag{3.1.14}$$

Next we prove for $u \in bF$

$$\mathcal{E}(u,u^+)\geq 0.\tag{3.1.15}$$

We have the above relation for $u \in \mathcal{D}(L)$. For $u \in bF$, by (A3) we choose a uniformly bounded sequence $\{u_n\} \subset \mathcal{D}(L)$ such that $\mathcal{E}_1^A(u_n-u) \to 0$, $n \to \infty$. Then we have

$$\begin{aligned}&|\int\langle A^{1/2}b,D_{A^{1/2}}u\rangle_H u^+d\mu-\int\langle A^{1/2}b,D_{A^{1/2}}u_n\rangle_H u_n^+d\mu|\\\leq&|\int\langle A^{1/2}b,D_{A^{1/2}}u_n-D_{A^{1/2}}u\rangle_H u_n^+d\mu|+|\int\langle A^{1/2}b,D_{A^{1/2}}u\rangle_H(u_n^+-u^+)d\mu|\end{aligned}$$

$$\leq M(\int |D_{A^{1/2}}u_n - D_{A^{1/2}}u|_H^2 d\mu)^{\frac{1}{2}} + |\int \langle A^{1/2}b, D_{A^{1/2}}u\rangle_H (u_n^+ - u^+)d\mu|$$
$$\to 0, \text{ as } n \to \infty.$$

Since $\mathcal{E}^A(u^+) \leq \mathcal{E}^A(u)$, $\sup_n \mathcal{E}^A(u_n^+) \leq \sup_n \mathcal{E}^A(u_n) < \infty$, we also have

$$\begin{aligned}
&|\mathcal{E}^A(u_n, (u_n)^+) - \mathcal{E}^A(u, u^+)| \\
\leq &|\mathcal{E}_1^A(u_n - u, (u_n)^+) + \mathcal{E}_1^A(u, (u_n)^+ - u^+)| \\
&+ |(u_n - u, (u_n)^+)| + |(u, (u_n)^+ - u^+)| \\
\leq &(\mathcal{E}_1^A(u_n - u))^{\frac{1}{2}}(\mathcal{E}_1^0((u_n)^+))^{\frac{1}{2}} + |\mathcal{E}_1^A(u, (u_n)^+ - u^+)| \\
&+ (\|(u_n)^+\|_2\|u_n - u\|_2 + \|(u_n)^+ - u^+\|_2\|u\|_2) \\
\to &0, \text{ as } n \to \infty.
\end{aligned}$$

As a result we obtain (3.1.15) for bounded $u \in F$. So we have

$$\|u_{t_1}^+\|_2^2 \leq 2\int_{t_1}^{t_2} (f_t, u_t^+)dt + \|u_{t_2}^+\|_2^2.$$

□

To extend the class of solutions we are working with, to allow f to belong to $L^1(dt \times d\mu)$, we need the following proposition. It is a modified version of the above lemma.

Lemma 3.1.9 Let $u \in \hat{F}$ be bounded and $f \in L^1(dt \times d\mu)$, be such that the weak relation (3.1.7) is satisfied with test functions in $b\mathcal{C}_T$ and some function $\phi \geq 0$, $\phi \in L^2 \cap L^\infty$. Then u^+ satisfies the following relation for $0 \leq t_1 < t_2 \leq T$

$$\|u_{t_1}^+\|_2^2 \leq 2\int_{t_1}^{t_2} (f_s, u_s^+)ds + \|u_{t_2}^+\|_2^2.$$

Proof First note that we prove analogously to the proof of Lemma 3.1.8 that for each $u \in b\mathcal{C}_T$ satisfying the weak relation (3.1.7) with data (ϕ, f) over the interval $[t_1, t_2]$, where $\varepsilon \leq t_1 \leq t_2 \leq T$ for $\varepsilon > 0$, the following holds:

$$\|u_{t_1}^+\|_2^2 \leq 2\int_{t_1}^{t_2} (f_t, u_t^+)dt + \|u_{t_2}^+\|_2^2.$$

For $u \in \hat{F}$ we take approximating functions u^n with data (ϕ^n, f^n) as in the last proof of Proposition 3.1.7. Then u^n satisfies the weak relation (3.1.7) for the data ϕ^n, f^n with test functions in $b\mathcal{C}_T$ over the interval $[\varepsilon, t_2]$ and $\frac{1}{n} \leq \varepsilon \leq t_2 \leq T$. Note that

$$\lim_{n\to\infty}\int_{\varepsilon}^{T}\|f_t^n - f_t\|_1 dt = 0.$$

We have

$$\|(u_{t_1}^n)^+\|_2^2 \le 2\int_{t_1}^{t_2}(f_t^n,(u_t^n)^+)dt + \|(u_{t_2}^n)^+\|_2^2,$$

where $\varepsilon \le t_1 \le t_2 \le T$ for $\varepsilon > 0$. The convergence of all terms, which do not depend on f, follows by the same arguments as the proof of Lemma 3.1.8. Since u is bounded, it is easy to see that u^n is uniformly bounded. Then we have

$$\begin{aligned}
&\lim_{n\to\infty}|\int_{t_1}^{t_2}(f_s^n,(u_s^n)^+)ds - \int_{t_1}^{t_2}(f_s,u_s^+)ds|\\
\le& M\lim_{n\to\infty}\int_{t_1}^{t_2}\|f_s^n - f_s\|_1 ds + \lim_{n\to\infty}\int_{t_1}^{t_2}(f_s,(u_s^n)^+ - u_s^+)ds\\
=&0.
\end{aligned}$$

Finally, we obtain that

$$\|u_{t_1}^+\|_2^2 \le 2\int_{t_1}^{t_2}(f_t,u_t^+)dt + \|u_{t_2}^+\|_2^2,$$

where $\varepsilon \le t_1 \le t_2 \le T$ for $\varepsilon > 0$. Letting $\varepsilon \to 0$ the results follows. □

The next proposition is a modification of [BPS05, Proposition 2.9]. It represents a version of the maximum principle.

Proposition 3.1.10 Let $u \in \hat{F}$ be bounded and $f \in L^1(dt\times d\mu), f \ge 0$, be such that the weak relation (3.1.7) is satisfied with test functions in $b\mathcal{C}_T$ and some function $\phi \ge 0$, $\phi \in L^2 \cap L^\infty$. Then $u \ge 0$ and it is represented by the following relation:

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t}f_s ds,$$

where we use P_t is a C_0-semigroup on $L^1(E;\mu)$ to make $P_{s-t}f_s$ meaningful.

Proof Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of bounded functions on $[0,T]\times E$ such that

$$0 \le f^n \le f^{n+1} \le f, \qquad \lim_{n\to\infty} f^n = f.$$

Since f^n is bounded, we have $f^n \in L^1([0,T];L^2)$. Next we define

$$u_t^n = P_{T-t}\phi + \int_t^T P_{s-t}f_s^n ds.$$

Then $u^n \in \hat{F}$ is the unique generalized solution for the data (ϕ, f^n) by Proposition 3.1.7. Clearly $0 \le u^n \le u^{n+1}$ for $n \in \mathbb{N}$. Define $y := u^n - u$ and $\tilde{f} := f^n - f$. Then $\tilde{f} \le 0$ and y satisfies the weak relation (3.1.7) for the data $(0, \tilde{f})$. Therefore by Lemma 3.1.9, we have for $t_1 \in [0,T]$

$$\|y_{t_1}^+\|_2^2 \le 2\int_{t_1}^T (\tilde{f}_s, y_s^+)ds \le 0.$$

We conclude that $\|y_{t_1}^+\|_2^2 = 0$. Therefore, $u \ge u^n \ge 0$ for $n \in \mathbb{N}$. Set $v := \lim_{n\to\infty} u^n$. By relation (3.1.5) for u^n and f^n we have

$$\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s^n)ds \le 2\int_t^T (f_s^n, u_s^n)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s^n\|_2 ds,$$

which implies that

$$\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s^n)ds \le 2M\int_t^T\int |f_s^n| d\mu ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s^n\|_2^2 ds.$$

By Gronwall's lemma we have $\sup_n \sup_{t\in[0,T]} \|u_t^n\|_2^2 \le$const. We obtain that $\lim_{n\to\infty} \|u_t^n - v_t\|_2^2 = 0$ and

$$\lim_{n\to\infty} |\int_t^T \int (f_s^n u_s^n - f_s v_s) d\mu ds| = 0.$$

By [MR92, Lemma 2.12] we have

$$\int_t^T \mathcal{E}^A(v_s)ds \le \int_t^T \liminf_{n\to\infty} \mathcal{E}^A(u_s^n)ds \le \liminf_{n\to\infty} \int_t^T \mathcal{E}^A(u_s^n)ds.$$

Finally, we get for $t \in [0,T]$

$$\begin{aligned}
\|v_t\|_2^2 + 2\int_t^T \mathcal{E}^A(v_s)ds &\le \lim_{n\to\infty} \|u_t^n\|_2^2 + 2\liminf_{n\to\infty}\int_t^T \mathcal{E}^A(u_s^n)ds \\
&\le \lim_{n\to\infty} (2\int_t^T (f_s^n, u_s^n)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s^n\|_2 ds) \\
&= 2\int_t^T (f_s, v_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|v_s\|_2 ds.
\end{aligned}$$

Since the right side of this inequality is finite and $t \mapsto v_t$ is L^2-continuous, it follows that $v \in \hat{F}$.

Now we show that v satisfies the weak relation (3.1.7) for the data (ϕ, f). As

$\varphi^n(t) := \|u_t^n - v_t\|_2$ is continuous and decreasing to 0, we conclude by Dini's theorem

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - v_t\|_2 = 0,$$

and therefore

$$\lim_{n\to\infty} \int_0^T \|u_t^n - v_t\|_2^2 = 0.$$

Furthermore, there exists $K \in \mathbb{R}_+$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$|\int_0^T \mathcal{E}^A(u_s^{n_k})ds| \le K \qquad \forall k \in \mathbb{N}.$$

In particular,

$$\int_0^T \int |D_{A^{1/2}} u_s^{n_k}|_H^2 d\mu ds \le K \qquad \forall k \in \mathbb{N}.$$

We obtain

$$\lim_{k\to\infty} \int_0^T \mathcal{E}^A(u_s^{n_k}, \varphi_s)ds = \int_0^T \mathcal{E}^A(v_s, \varphi_s)ds,$$

and

$$\lim_{k\to\infty} \int_0^T \int \langle A^{1/2}b, D_{A^{1/2}} u_s^{n_k}\rangle_H \varphi_s d\mu ds = \int_0^T \int \langle A^{1/2}b, D_{A^{1/2}} v_s\rangle_H \varphi_s d\mu ds,$$

which implies (3.1.7) for v associated to (ϕ, f). Clearly $u - v$ satisfies (3.1.7) with data $(0,0)$ for $\varphi \in b\mathcal{C}_T$. By Proposition 3.1.7 we have $u - v = 0$. Since

$$v_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds,$$

the assertion follows. □

Corollary 3.1.11 Let $u \in \hat{F}$ be bounded and $f \in L^1(dt \times d\mu)$ be such that the weak relation (3.1.7) is satisfied with test functions in $b\mathcal{C}_T$ and some function $\phi \in L^2 \cap L^\infty$. Assume there exists $g \in bL^1(dt \times d\mu)$ such that $f \le g$. Then u has the following representation:

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Proof Define $f^n := (f \vee (-n)) \wedge g, n \in \mathbb{N}$. Then $(f^n)_{n\in\mathbb{N}}$ is a sequence of bounded functions such that $f^n \downarrow f$ and $f^n \le g$ then by the same arguments as the proof of Proposition 3.1.10, the assertion follows. □

The following proposition is a modification of [BPS05, Proposition 2.10] . It is essential for the analytical treatment of the non-linear equation (1.4) which is done in the next section.

Proposition 3.1.12 Let $u = (u^1, ..., u^l)$ be a vector valued function where each component is a generalized solution of the linear equation (3.1.3) associated to data (f^i, ϕ^i), which are assumed to be bounded and to satisfy the condition in Proposition 3.1.5 (ii) for $i = 1, ..., l$. Let ϕ, f denote the vectors $\phi = (\phi^1, ..., \phi^l), f = (f^1, ..., f^l)$ and $D_{A^{1/2}}u$ denote the matrix whose rows consist of $D_{A^{1/2}}u^i$. Then the following relations hold μ-almost everywhere:

$$|u_t|^2 + 2\int_t^T P_{s-t}(|D_{A^{1/2}}u_s|_H^2)ds = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle ds, \tag{3.1.16}$$

and

$$|u_t| \leq P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle ds. \tag{3.1.17}$$

Here we write $\hat{x} = x/|x|$, for $x \in \mathbb{R}^l$, $x \neq 0$ and $\hat{x} = 0$, if $x = 0$.

Proof By Proposition 3.1.5 (ii) we have $u \in b\mathcal{C}_T$.

First we assume $l = 1$. If we check that u^2 satisfies (3.1.7) with data $(2uf - 2|D_{A^{1/2}}u|_H^2, \phi^2)$ for $\varphi \in b\mathcal{C}_T$, then (3.1.16) will follow by Corollary 3.1.11. We have the following relations:

$$\int_0^T (u_t^2, \partial_t\varphi_t)dt = 2\int_0^T (u_t, \partial_t(u_t\varphi_t))dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T),$$

$$\mathcal{E}^A(u_t^2, \varphi_t) = 2\mathcal{E}^A(u_t, u_t\varphi_t) - 2(|D_{A^{1/2}}u_t|_H^2, \varphi_t),$$

and

$$\int \langle A^{1/2}b, D_{A^{1/2}}(u_t^2)\rangle_H \varphi_t d\mu = 2\int \langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H u_t\varphi_t d\mu.$$

Since u is a generalized solution of (3.1.3), we have

$$\int_0^T (u_t, \partial_t(u_t\varphi_t))dt - (u_T, u_T\varphi_T) + (u_0, u_0\varphi_0) - \int_0^T (f_t, u_t\varphi_t)dt$$
$$= -\int_0^T \mathcal{E}^A(u_t, u_t\varphi_t)dt - \int_0^T \langle A^{1/2}b, D_{A^{1/2}}u_t\rangle_H u_t\varphi_t d\mu dt.$$

By the above relation, we obtain
(3.1.18)
$$\int_0^T (u_t^2, \partial_t\varphi_t)dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T) + \int_0^T (\mathcal{E}^A(u_t^2, \varphi_t) + \langle A^{1/2}b, D_{A^{1/2}}(u_t^2)\rangle_H \varphi_t d\mu)dt$$
$$= 2\int_0^T (f_t u_t, \varphi_t)dt - \int_0^T 2(|D_{A^{1/2}}u_t|_H^2, \varphi_t)dt.$$

Hence, by Corollary 3.1.11, (3.1.16) holds in the case $l = 1$. To deduce this relation in the case $l > 1$ it suffices to add the relations corresponding to the components $|u_t^i|^2, i = 1, ..., l$. For (3.1.17), let us define for $\varepsilon > 0$, $h_\varepsilon(t) := \sqrt{t+\varepsilon} - \sqrt{\varepsilon}$ for $t \geq 0$. We have the following relations by integration by parts:

$$\mathcal{E}^A(h_\varepsilon(|u|^2), \varphi) = \mathcal{E}^A(|u|^2, h_\varepsilon'(|u|^2)\varphi) - (h_\varepsilon''(|u|^2)|D_{A^{1/2}}(|u|^2)|_H^2, \varphi),$$

$$\int_0^T (h_\varepsilon(|u_t|^2), \partial_t\varphi_t)dt = \int_0^T (|u_t|^2, \partial_t(\varphi_t h_\varepsilon'(|u_t|^2)))dt - (|u_T|^2, \varphi_T h_\varepsilon'(|u_T|^2))$$
$$+ (|u_0|^2, \varphi_0 h_\varepsilon'(|u_0|^2)) + (h_\varepsilon(|u_T|^2), \varphi_T) - (h_\varepsilon(|u_0|^2), \varphi_0).$$

If we choose $\varphi h_\varepsilon'(|u|^2)$ as a test function in (3.1.18), we have

$$\int_0^T (|u_t|^2, \partial_t(\varphi_t h_\varepsilon'(|u_t|^2)))dt + (|u_0|^2, \varphi_0 h_\varepsilon'(|u_0|^2)) - (|u_T|^2, \varphi_T h_\varepsilon'(|u_T|^2))$$
$$+ \int_0^T (\mathcal{E}^A(|u_t|^2, \varphi_t h_\varepsilon'(|u|_t^2)) + \int \langle A^{1/2}b, D_{A^{1/2}}(|u_t|^2)\rangle_H \varphi_t h_\varepsilon'(|u_t|^2)d\mu)dt$$
$$= 2\int_0^T (\langle f_t, u_t\rangle, \varphi_t h_\varepsilon'(|u_t|^2))dt - \int_0^T 2(|D_{A^{1/2}}u_t|_H^2, \varphi_t h_\varepsilon'(|u_t|^2))dt.$$

By the above relations we have

$$\int_0^T (h_\varepsilon(|u_t|^2), \partial_t\varphi_t)dt - (h_\varepsilon(|u_T|^2), \varphi_T) + (h_\varepsilon(|u_0|^2), \varphi_0)$$
$$+ \int_0^T (\mathcal{E}^A(h_\varepsilon(|u_t|^2), \varphi_t) + \int \langle A^{1/2}b, D_{A^{1/2}}(h_\varepsilon(|u_t|^2))\rangle_H \varphi_t d\mu)dt$$
$$= -\int_0^T (h_\varepsilon''(|u_t|^2)|D_{A^{1/2}}(|u_t|^2)|_H^2, \varphi_t)dt + 2\int_0^T (\langle f_t, u_t\rangle h_\varepsilon'(|u_t|^2), \varphi_t)dt$$
$$- \int_0^T 2(h_\varepsilon'(|u_t|^2)|D_{A^{1/2}}u_t|_H^2, \varphi_t)dt.$$

As

$$|D_{A^{1/2}}(|u|^2)|_H^2 = 4|\sum_i u^i D_{A^{1/2}}u^i|_H^2 \leq 4(\sum_i |u^i|\cdot|D_{A^{1/2}}u^i|_H)^2 \leq 4(\sum_i (u^i)^2)(\sum_i |D_{A^{1/2}}u^i|_H^2),$$

we deduce

$$
\begin{aligned}
&2\langle f,u\rangle h_\varepsilon'(|u|^2)-2h_\varepsilon'(|u|^2)|D_{A^{1/2}}u|_H^2-h_\varepsilon''(|u|^2)|D_{A^{1/2}}(|u|^2)|_H^2\\
=&\frac{\langle f,u\rangle-|D_{A^{1/2}}u|_H^2}{(|u|^2+\varepsilon)^{\frac12}}+\frac{|\sum_i u^iD_{A^{1/2}}u^i|_H^2}{(|u|^2+\varepsilon)^{\frac32}}\\
\le&\frac{\langle f,u\rangle}{(|u|^2+\varepsilon)^{\frac12}}-\frac{\varepsilon|D_{A^{1/2}}u|_H^2+|u|^2|D_{A^{1/2}}u|_H^2-\sum_i(u^i)^2\sum_i\langle D_{A^{1/2}}u^i,D_{A^{1/2}}u^i\rangle_H}{(|u|^2+\varepsilon)^{\frac32}}\\
\le&\frac{\langle f,u\rangle}{(|u|^2+\varepsilon)^{\frac12}}.
\end{aligned}
$$

By Proposition 3.1.10 we deduce that

$$h_\varepsilon(|u_t|^2)\le P_{T-t}h_\varepsilon(|\phi|^2)+\int_t^T P_{s-t}\frac{\langle f_s,u_s\rangle}{(|u_s|^2+\varepsilon)^{\frac12}}ds.$$

Letting $\varepsilon\to0$, the assertion follows. □

The next corollary is a version of the above proposition for general data. Here we use P_t is a C_0-semigroup on L^1.

Corollary 3.1.13 Let $u=(u^1,...,u^l)$ be a vector valued function where each component is a generalized solution of the linear equation (3.1.3) associated to data $f^i\in L^1([0,T];L^2),\phi^i\in L^2$ for $i=1,...,l$. Let ϕ,f denote the vectors $\phi=(\phi^1,...,\phi^l),f=(f^1,...,f^l)$ and $D_{A^{1/2}}u$ denote the matrix whose rows consist of $D_{A^{1/2}}u^i$. Then the following relations hold μ-almost everywhere

$$|u_t|^2+2\int_t^T P_{s-t}(|D_{A^{1/2}}u_s|_H^2)ds=P_{T-t}|\phi|^2+2\int_t^T P_{s-t}\langle u_s,f_s\rangle ds, \tag{3.1.19}$$

and

$$|u_t|\le P_{T-t}|\phi|+\int_t^T P_{s-t}\langle\hat u_s,f_s\rangle ds. \tag{3.1.20}$$

Proof Analogously to the proof of Proposition 3.1.12 it is enough to verify (3.1.19) for $l=1$. For $\phi\in L^2,f\in L^1([0,T],L^2)$, take ϕ_n,f_n as in Proposition 3.1.7. Then we have

(a). $u_{n,t}:=P_{T-t}\phi_n+\int_t^T P_{s-t}f_{n,s}ds$ is a generalized solution ,

(b). $\lim_{n\to\infty}\int_t^T\|f_{n,s}-f_s\|_2ds=0$,

(c). $\lim_{n\to\infty}\|\phi_n-\phi\|_2=0$,

(d). $\lim_{n\to\infty}\|u_n-u\|_T=0$.

By Proposition 3.1.12 we have

$$(3.1.21)\quad |u_{n,t}|^2+2\int_t^T P_{s-t}(|D_{A^{1/2}}u_{n,s}|_H^2)ds=P_{T-t}|\phi_n|^2+2\int_t^T P_{s-t}\langle u_{n,s},f_{n,s}\rangle ds.$$

By (b) and (d) we obtain

$$\begin{aligned}
&\|\int_t^T P_{s-t}((u_{n,s},f_{n,s})-(u_s,f_s))ds\|_1\\
\leq &C\int_t^T(\|u_{n,s}\|_2\|f_{n,s}-f_s\|_2+\|f_s\|_2\|u_{n,s}-u_s\|_2)ds\\
\leq &C(\sup_{s\in[0,T]}\|u_{n,s}\|_2\int_t^T\|f_{n,s}-f_s\|_2ds+\sup_{s\in[0,T]}\|u_{n,s}-u_s\|\int_t^T\|f_s\|_2ds)\\
\to &0,\ \text{as } n\to\infty.
\end{aligned}$$

Here we used P_t is a C_0-semigroup on $L^1(E;\mu)$. By (d) we obtain

$$\begin{aligned}
&\int_t^T\||D_{A^{1/2}}u_{n,s}|_H^2-|D_{A^{1/2}}u_s|_H^2\|_1ds\\
\leq&((\int_t^T\||D_{A^{1/2}}u_{n,s}|_H\|_2^2ds)^{\frac12}+(\int_t^T\||D_{A^{1/2}}u_s|_H\|_2^2ds)^{\frac12})(\int_t^T\||D_{A^{1/2}}u_{n,s}-D_{A^{1/2}}u_s|_H\|_2^2ds)^{\frac12}\\
=&((\int_t^T\mathcal{E}^A(u_{n,s})ds)^{\frac12}+(\int_t^T\mathcal{E}^A(u_s)ds)^{\frac12})(\int_t^T\mathcal{E}^A(u_{n,s}-u_s)ds)^{\frac12}\\
\to&0,\ \text{as } n\to\infty,
\end{aligned}$$

and obtain

$$\lim_{n\to\infty}\int_t^T P_{s-t}(|D_{A^{1/2}}u_{n,s}|_H^2)ds=\int_t^T P_{s-t}|D_{A^{1/2}}u_s|_H^2ds.$$

Passing to the limit in equation (3.1.21) we obtain (3.1.19). (3.1.20) follows by the same method. $\square$

Lemma 3.1.14 If $f,g\in L^1([0,T];L^2)$ and $\phi\in L^2$, then the following relations hold μ-a.e.:

$$(3.1.22)\qquad \int_t^T P_{s-t}(f_sP_{T-s}\phi)ds\leq\frac12 P_{T-t}\phi^2+\int_t^T\int_s^T P_{s-t}(f_sP_{r-s}f_r)drds.$$

Proof Define

$$h_t=P_{T-t}\phi,\qquad v_t=\int_t^T P_{s-t}f_sds.$$

By (3.1.19) we deduce

$$h_t^2 + 2\int_t^T P_{s-t}|D_{A^{1/2}}h_s|_H^2 ds = P_{T-t}\phi^2,$$

$$v_t^2 + 2\int_t^T P_{s-t}|D_{A^{1/2}}v_s|_H^2 ds = 2\int_t^T P_{s-t}(f_s\int_s^T P_{r-s}f_r dr)ds,$$

and

$$h_t v_t + 2\int_t^T P_{s-t}\langle D_{A^{1/2}}h_s, D_{A^{1/2}}v_s\rangle_H ds = \int_t^T P_{s-t}(f_s P_{T-s}\phi)ds.$$

So, we have

$$\begin{aligned}\int_t^T P_{s-t}(f_s P_{T-s}\phi)ds &= h_t v_t + 2\int_t^T P_{s-t}\langle D_{A^{1/2}}h_s, D_{A^{1/2}}v_s\rangle_H ds\\ &\leq \frac{1}{2}(h_t^2 + v_t^2) + \int_t^T P_{s-t}(|D_{A^{1/2}}h_s|_H^2 + |D_{A^{1/2}}v_s|_H^2)ds\\ &= \frac{1}{2}P_{T-t}\phi^2 + \int_t^T\int_s^T P_{s-t}(f_s P_{r-s}f_r)drds.\end{aligned}$$

□

3.2 The Non-linear Equation

In the case of non-linear equations, we are going to treat systems of equations, with the unknown functions and their first-order derivatives mixed in the non-linear term of the equation. The non-linear term is a given measurable function $f : [0,T] \times E \times \mathbb{R}^l \times H^l \to \mathbb{R}^l$, $l \in \mathbb{N}$. We are going to treat the following system of equations.

$$(\partial_t + L)u + f(\cdot,\cdot,u,D_{A^{1/2}}u) = 0, \qquad u_T = \phi. \tag{3.2.1}$$

The function ϕ is assumed to be in $L^2(E, d\mu; \mathbb{R}^l)$.

Definition 3.2.1 [*Generalized solution of the nonlinear equation*] A generalized solution of equation (3.2.1) is a system $u = (u^1, u^2, ..., u^l)$ of l elements in $\hat{F}$, which has the property that each function $f^i(\cdot,\cdot,u,D_{A^{1/2}}u)$ belongs to $L^1([0,T]; L^2(\mu))$ and such that there is a sequence $\{u_n\}$ which consists of strong solutions to (3.2.1) with data (ϕ_n, f_n) such that

$$\|u_n - u\|_T \to 0, \|\phi_n - \phi\|_2 \to 0,$$

and

$$\lim_{n\to\infty} f_n(\cdot,\cdot,u_n,D_{A^{1/2}}u_n) = f(\cdot,\cdot,u,D_{A^{1/2}}u) \text{ in } L^1([0,T];L^2(\mu)).$$

Definition 3.2.2 [*Mild solution*] A mild solution of equation (3.2.1) is a system $u=(u^1,u^2,...,u^l)$ of l elements in $\hat{F}$, which has the property that each function $f^i(\cdot,\cdot,u,D_{A^{1/2}}u)$ belongs to $L^1([0,T];L^2(\mu))$ and such that for every $i\in\{1,...,l\}$, the following equation holds

$$u^i(t,x) = P_{T-t}\phi^i(x) + \int_t^T P_{s-t}f^i(s,\cdot,u_s,D_{A^{1/2}}u_s)(x)ds, \mu - a.e.. \tag{3.2.2}$$

Lemma 3.2.3 u is a generalized solution of the nonlinear equation (3.2.1) if and only if it is a mild solution of equation (3.2.1).

Proof The assertion follows by Proposition 3.1.7. □

We will use the following notation:

$$|u|_H := \sum |u^i|_H, u\in L^2(E;H^l,d\mu),$$

$$\|\phi\|_2^2 := \sum_{i=1}^l \|\phi^i\|_2^2, \phi\in L^2(E,d\mu;\mathbb{R}^l),$$

$$\mathcal{E}(u,v) := \sum_{i=1}^l \mathcal{E}(u^i,v^i),\ \mathcal{E}^A(u,v) := \sum_{i=1}^l \mathcal{E}^A(u^i,v^i), u,v\in F^l,$$

$$\|u\|_T^2 := \sup_{t\le T}\|u_t\|_2^2 + \int_0^T \mathcal{E}^A(u_t)dt, u\in\hat{F}^l.$$

3.2.1 The Case of Lipschitz Conditions

In this subsection we consider a measurable function $f:[0,T]\times E\times\mathbb{R}^l\times H^l\to\mathbb{R}^l$ such that

$$|f(t,x,y,z)-f(t,x,y',z')| \le C(|y-y'|+|z-z'|_H), \tag{3.2.3}$$

with t,x,y,y',z,z' arbitrary and C a constant independent of t,x. We set $f^0(t,x) := f(t,x,0,0)$.

Proposition 3.2.4 Suppose that conditions (A1)-(A3) hold and f satisfies condition (3.2.3), $f^0\in L^2([0,T]\times E,dt\times d\mu;\mathbb{R}^l)$ and $\phi\in L^2(E;\mathbb{R}^l)$. Then the equation

(3.2.1) admits a unique solution $u \in \hat{F}^l$ and it satisfies the following estimate

$$\|u\|_T^2 \le e^{T(1+2C+C^2+2\alpha)}(\|\phi\|_2^2 + \|f^0\|^2_{L^2([0,T]\times E)}).$$

Proof If $u \in \hat{F}^l$, then by relation (3.2.3) we have

$$\begin{aligned} |f(\cdot,\cdot,u,D_{A^{1/2}}u)| &\le |f(\cdot,\cdot,u,D_{A^{1/2}}u) - f(\cdot,\cdot,0,0)| + |f(\cdot,\cdot,0,0)| \\ &\le C(|u| + |D_{A^{1/2}}u|_H) + |f^0|. \end{aligned}$$

As $f^0 \in L^2([0,T]\times E, dt\times d\mu;\mathbb{R}^l)$ and $|D_{A^{1/2}}u|_H$ is an element of $L^2([0,T]\times E)$, we get $f(\cdot,\cdot,u,D_{A^{1/2}}u) \in L^2([0,T]\times E;\mathbb{R}^l)$.

Now we define the operator $A : \hat{F}^l \to \hat{F}^l$ by

$$(Au)^i(t,x) = P_{T-t}\phi^i(x) + \int_t^T P_{s-t}f^i(s,\cdot,u_s,D_{A^{1/2}}u_s)(x)ds, \qquad i = 1,...,l.$$

Then Proposition 3.1.7 implies that $Au \in \hat{F}^l$. In the following we write $f^i_{u,s} := f^i(s,\cdot,u_s,D_{A^{1/2}}u_s)$. Since $(Au)^i_t - (Av)^i_t = \int_t^T P_{s-t}(f^i_{u,s} - f^i_{v,s})ds$ is the mild solution with data $(f^i_u - f^i_v, 0)$, by the same argument as the proof of Proposition 3.1.7 we have

$$\begin{aligned} \|\int_t^T P_{s-t}(f^i_{u,s} - f^i_{v,s})ds\|^2_{[t,T]} &\le M_T(\int_t^T \|f_{u,s} - f_{v,s}\|_2 ds)^2 \\ &\le M_T(T-t)\int_t^T \|f_{u,s} - f_{v,s}\|_2^2 ds \\ &\le M_T(T-t)\int_t^T (\|u_s - v_s\|_2^2 + \||D_{A^{1/2}}u_s - D_{A^{1/2}}v_s|_H\|_2^2)ds \\ &\le M_T(T-t)\|u-v\|^2_{[t,T]}, \end{aligned}$$

where M_T may change from line to line. Here

$$\|u\|_{[T_a,T_b]} := (\sup_{t\in[T_a,T_b]} \|u_t\|_2^2 + \int_{T_a}^{T_b} \mathcal{E}^A(u_t)dt)^{\frac{1}{2}},$$

where $0 \le T_a \le T_b \le T$. Fix T_1 sufficiently small such that $\gamma := M_T(T - T_1) < 1$. Then we have :

$$\|Au - Av\|^2_{[T_1,T]} < \gamma\|u-v\|^2_{[T_1,T]}.$$

Then there exists a unique $u_1 \in \hat{F}_{[T_1,T]}$ such that $Au_1 = u_1$ where $\hat{F}_{[T_a,T_b]} := C([T_a,T_b];L^2)\cap L^2((T_a,T_b);F)$ for $T_a \in [0,T]$ and $T_b \in [T_a,T]$.

By the same method as above, we define the operator $A^1 : \hat{F}^l \to \hat{F}^l$ by

$$(A^1u)^i(t,x) = P_{T_1-t}u_1^i(T_1,x) + \int_t^{T_1} P_{s-t}f^i(s,\cdot,u_s,D_{A^{1/2}}u_s)(x)ds, \qquad i=1,...,l.$$

Then we have

$$\|A^1u - A^1v\|^2_{[t,T_1]} \le M_T(T_1-t)\|u-v\|^2_{[t,T_1]}.$$

Now we choose $T_2 < T_1$ such that $M_T(T_1-T_2) < 1$. Then we have that there exists a unique $u_2 \in \hat{F}_{[T_2,T_1]}$ such that $A^1u_2 = u_2$. Define $u := u_1 1_{[T_1,T]} + u_2 1_{[T_2,T_1)}$. By a similar argument as in the proof of Proposition 2.2.4, u is a solution on $[T_2,T]$. Therefore, we construct a solution over the interval $[T_2,T]$. Clearly there exists $n \in \mathbb{N}$ such that $T < n(T-T_1)$. Hence, the construction is done after n steps.

In order to obtain the estimate in the statement, we write

$$\begin{aligned}
&|\int_t^T (f_{u,s},u_s)ds| \\
\le &\int_t^T |(f_s^0,u_s)|ds + C\int_t^T \|u_s\|_2^2 ds + C\int_t^T \||D_{A^{1/2}}u_s|_H\|_2\|u_s\|_2 ds \\
\le &\frac{1}{2}\int_t^T \|f_s^0\|_2^2 ds + (\frac{1}{2} + C + \frac{1}{2}C^2)\int_t^T \|u_s\|_2^2 ds + \frac{1}{2}\int_t^T \mathcal{E}^A(u_s)ds.
\end{aligned}$$

By relation (3.1.5) of Proposition 3.1.7 it follows that

$$\begin{aligned}
&\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s)ds \le 2\int_t^T (f_{u,s},u_s)ds + \|\phi\|_2^2 + 2\alpha\int_t^T \|u_s\|_2^2 ds \\
&\le \|\phi\|_2^2 + \int_t^T \|f_s^0\|_2^2 ds + (1+2C+C^2+2\alpha)\int_t^T \|u_s\|_2^2 ds + \int_t^T \mathcal{E}^A(u_s)ds.
\end{aligned}$$

Now by Gronwall's lemma the desired estimate follows.

[Uniqueness] Let u_1 and u_2 be two solutions of equation (3.2.1). By using (3.1.5) for the difference $u_1 - u_2$ we get

$$\begin{aligned}
&\|u_{1,t} - u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}^A(u_{1,s}-u_{2,s})ds \\
\le &2\int_t^T (f(s,\cdot,u_{1,s},D_{A^{1/2}}u_{1,s}) - f(s,\cdot,u_{2,s},D_{A^{1/2}}u_{2,s}), u_{1,s}-u_{2,s})ds + 2\alpha\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds \\
\le &2\int_t^T C(|D_{A^{1/2}}u_{1,s} - D_{A^{1/2}}u_{2,s}|, |u_{1,s}-u_{2,s}|)ds + (2\alpha + C)\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds \\
\le &(2\alpha + C^2 + C)\int_t^T \|u_{1,s}-u_{2,s}\|_2^2 ds + \int_t^T \mathcal{E}^A(u_{1,s}-u_{2,s})ds.
\end{aligned}$$

By Gronwall's lemma it follows that

$$\|u_{1,t} - u_{2,t}\|_2^2 = 0,$$

hence $u_1 = u_2$. □

3.2.2 The Case of Monotonicity Conditions

Let $f : [0,T] \times E \times \mathbb{R}^l \times H^l \to \mathbb{R}^l$ be a measurable function and $\phi \in L^2(E, \mu; \mathbb{R}^l)$ be the final condition of (3.2.1). In this subsection we impose the following conditions:

(H1) [*Lipschitz condition in z*] There exists a fixed constant $C > 0$ such that for t, x, y, z, z' arbitrary

$$|f(t,x,y,z) - f(t,x,y,z')| \leq C|z - z'|_H.$$

(H2) [*Monotonicity condition in y*] For x, y, y', z arbitrary, there exists a function $\mu \in L^1([0,T]; \mathbb{R})$ such that

$$\langle y - y', f(t,x,y,z) - f(t,x,y',z) \rangle \leq \mu_t |y - y'|^2.$$

We set $\alpha_t := \int_0^t \mu_s ds$.

(H3) [*Continuity condition in y*] For t, x and z fixed, the map

$$\mathbb{R}^l \ni y \mapsto f(t,x,y,z)$$

is continuous.

We need the following notation

$$f^0(t,x) := f(t,x,0,0), \qquad f'(t,x,y) := f(t,x,y,0) - f(t,x,0,0),$$

$$f'^{,r}(t,x) := \sup_{|y| \leq r} |f'(t,x,y)|.$$

(H4) For each $r > 0$, $f'^{,r} \in L^1([0,T]; L^2)$.

(H5) $\|\phi\|_\infty < \infty, \|f^0\|_\infty < \infty$.

As $\mu(E) < \infty$ we have $|\phi| \in L^2, |f^0| \in L^2([0,T]; L^2)$. The conditions (H1), (H4), and (H5) imply that if $u \in \hat{F}$ is bounded, then $|f(u, D_{A^{1/2}} u)| \in L^1([0,T]; L^2)$. Under the above conditions, even if E is equal to a Hilbert space, it seems impossible to apply general monotonicity methods to the map $\mathcal{V} \ni u \mapsto f(t, \cdot, u(\cdot), D_{A^{1/2}} u) \in \mathcal{V}'$

because of lack of a suitable reflexive Banach space $\mathcal{V}$ such that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. Therefore, also here we proceed developing a hands-on approach to prove existence and uniqueness of solutions for equation (3.2.1) as done in [BPS05], [S09] and in particular, Chapter 2. Then by the same arguments as the proof of Lemma 2.2.5, the following lemma follows:

Lemma 3.2.5 In (H2) without loss of generality we assume that $\mu_t \equiv 0$.

Lemma 3.2.6 Suppose that conditions (A1)-(A3), (H1) and the following weaker form of condition (H2) (with $\mu_t \equiv 0$) hold,

$$(H2')\langle y, f'(t,x,y)\rangle \leq 0,$$

for all t, x, y. If u is a solution of (3.2.1), then there exists a constant K which depends on C, T, α such that

$$\|u\|_T^2 \leq K(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt).$$

Proof Since u is a generalized solution of (3.2.1), we have by Proposition 3.1.7

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s)ds \leq 2\int_t^T (f_s, u_s)ds + \|u_T\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds.$$

Conditions (H1) and (H2') yield

$$\begin{aligned}
\langle f_s(u_s, D_{A^{1/2}}u_s), u_s\rangle =& \langle f_s(u_s, D_{A^{1/2}}u_s) - f_s(u_s, 0) + f_s'(u_s) + f_s^0, u_s\rangle \\
\leq& |f_s(u_s, D_{A^{1/2}}u_s) - f_s(u_s,0)||u_s| + \langle f_s'(u_s), u_s\rangle + |f_s^0||u_s| \\
\leq& (C|D_{A^{1/2}}u_s|_H + |f_s^0|)|u_s|.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\|u_t\|_2^2 + 2\int_t^T \mathcal{E}^A(u_s)ds \\
\leq& 2\int_t^T \int (C|D_{A^{1/2}}u_s|_H + |f_s^0|)|u_s| d\mu ds + \|u_T\|_2^2 + 2\alpha \int_t^T \|u_s\|_2^2 ds \quad . \\
\leq& \int_t^T \mathcal{E}^A(u_s)ds + (C^2 + 1 + 2\alpha)\int_t^T \|u_s\|_2^2 ds + \int_t^T \|f_s^0\|_2^2 ds + \|u_T\|_2^2.
\end{aligned}$$

Then Gronwall's lemma yields

$$\|u\|_T^2 \leq K(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt).$$

□

Lemma 3.2.7 Assume that the conditions (A1)-(A3), (H1) and (H2') hold. If u is a generalized solution of (3.2.1) , then there exists a constant K, which depends on C, α and T such that

$$\|u\|_\infty \le K(\|\phi\|_\infty + \|f^0\|_\infty). \tag{3.2.4}$$

Proof By Corollary 3.1.13, we have

(3.2.5)

$$|u_t|^2 + 2\int_t^T P_{s-t}(|D_{A^{1/2}}u_s|_H^2)ds = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s(u_s, D_{A^{1/2}}u_s)\rangle ds.$$

Follow the same arguments as the proof of Lemma 3.2.6 we deduce

$$\langle f_s(u_s, D_{A^{1/2}}u_s), u_s\rangle \le (C|D_{A^{1/2}}u_s|_H + |f_s^0|)|u_s|.$$

By Corollary 3.1.13 (3.1.20) we get

$$|u_s| \le P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_{A^{1/2}}u_r|_H + |f_r^0|)dr.$$

Then we have

$$\begin{aligned}&\int_t^T P_{s-t}\langle f_s(u_s, D_{A^{1/2}}u_s), u_s\rangle ds\\ \le &\int_t^T P_{s-t}[(P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_{A^{1/2}}u_r|_H + |f_r^0|)dr)(C|D_{A^{1/2}}u_s|_H + |f_s^0|)]ds.\end{aligned}$$

So by (3.2.5) and Lemma 3.1.14 we have

$$\begin{aligned}&|u_t|^2 + 2\int_t^T P_{s-t}(|D_{A^{1/2}}u_s|^2)ds\\ \le &P_{T-t}|\phi|^2 + 2(\int_t^T P_{s-t}[(P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_{A^{1/2}}u_r| + |f_r^0|)dr)(C|D_{A^{1/2}}u_s| + |f_s^0|)]ds)\\ \le &3P_{T-t}|\phi|^2 + 2C^2\int_t^T\int_s^T P_{s-t}(|D_{A^{1/2}}u_s|P_{r-s}|D_{A^{1/2}}u_r|)drds + 2\int_t^T\int_s^T P_{s-t}(|f_s^0|P_{r-s}|f_r^0|)drds\\ &+ 2\int_t^T\int_s^T P_{s-t}[P_{r-s}(C|D_{A^{1/2}}u_r| + |f_r^0|)(C|D_{A^{1/2}}u_s| + |f_s^0|)]drds.\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\int_t^T \int_s^T P_{s-t}[P_{r-s}(C|D_{A^{1/2}}u_r| + |f_r^0|)(C|D_{A^{1/2}}u_s| + |f_s^0|)]drds \\
\leq &\frac{1}{2}\int_t^T \int_s^T [P_{s-t}(C|D_{A^{1/2}}u_s| + |f_s^0|)^2] + P_{s-t}[(P_{r-s}(C|D_{A^{1/2}}u_r| + |f_r^0|))^2]drds \\
\leq &\int_t^T \int_s^T C^2 P_{s-t}|D_{A^{1/2}}u_s|^2 + P_{s-t}|f_s^0|^2 + \frac{1}{2}P_{r-t}(C|D_{A^{1/2}}u_r| + |f_r^0|)^2 drds \\
\leq &2C^2(T-t)\int_t^T P_{s-t}|D_{A^{1/2}}u_s|^2 ds + 2(T-t)\int_t^T P_{s-t}|f_s^0|^2 ds.
\end{aligned}
$$

By Schwartz's inequality one has

$$
\begin{aligned}
&\int_t^T \int_s^T P_{s-t}(|D_{A^{1/2}}u_s|_H P_{r-s}|D_{A^{1/2}}u_r|_H)drds \\
\leq &\int_t^T \int_s^T \frac{1}{2}(P_{s-t}|D_{A^{1/2}}u_s|_H^2)drds + \int_t^T \int_s^T \frac{1}{2}(P_{r-t}|D_{A^{1/2}}u_r|_H^2)drds \\
\leq &(T-t)\int_t^T P_{s-t}|D_{A^{1/2}}u_s|_H^2 ds.
\end{aligned}
$$

Hence we conclude

$$
\begin{aligned}
&|u_t|^2 + 2\int_t^T P_{s-t}(|D_{A^{1/2}}u_s|_H^2)ds \\
\leq &2P_{T-t}|\phi|^2 + 6C^2(T-t)\int_t^T P_{s-t}|D_{A^{1/2}}u_s|_H^2 ds + 6(T-t)\int_t^T \int_s^T P_{s-t}(|f_s^0|^2)drds.
\end{aligned}
$$

Hence, we deduce by iteration the estimate over the interval $[0,T]$. We obtain from the above estimate:

$$
\begin{aligned}
|u_t|^2 \leq &\sup_{t\in[0,T]}\sup_{x\in E}\tilde{K}(P_{T-t}|\phi|^2 + (T-t)\int_t^T P_{s-t}|f_s^0|^2 ds) \\
\leq &\sup_{t\in[0,T]}\tilde{K}(\|\phi^2\|_\infty + T^2\|f^0\|_\infty^2) \\
\leq &K^2(\|\phi\|_\infty^2 + \|f^0\|_\infty^2),
\end{aligned}
$$

which implies (3.2.4). □

The proof here is different from the finite dimensional case, i.e. the proof of Theorem 2.2.8, since a unit ball in H is not compact. And it is inspired from the probabilistic approach to prove the existence of the solution of the BSDE of [BDHPS03].

Theorem 3.2.8 Suppose the conditions (A1)-(A3), (H1)-(H5) hold. Then there exists a unique generalized solution of equation (3.2.1), and it satisfies the following estimates with constants K_1 and K_2 independent of u, ϕ, f

$$\|u\|_T^2 \leq K_1(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2 dt),$$

and

$$\|u\|_\infty \leq K_2(\|\phi\|_\infty + \|f^0\|_\infty).$$

Proof [Uniqueness] Let u_1 and u_2 be two solutions of equation (3.2.1). By using (3.1.5) for the difference $u_1 - u_2$ we get

$$\begin{aligned}
&\|u_{1,t} - u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}^A(u_{1,s} - u_{2,s})ds \\
\leq& 2\int_t^T (f(s,\cdot,u_{1,s}, D_{A^{1/2}}u_{1,s}) - f(s,\cdot,u_{2,s}, D_{A^{1/2}}u_{2,s}), u_{1,s} - u_{2,s})ds + 2\alpha\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds \\
\leq& 2\int_t^T C(|D_{A^{1/2}}u_{1,s} - D_{A^{1/2}}u_{2,s}|_H, |u_{1,s} - u_{2,s}|)ds + 2\alpha\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds \\
\leq& (C^2 + 2\alpha)\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds + \int_t^T \mathcal{E}^A(u_{1,s} - u_{2,s})ds.
\end{aligned}$$

By Gronwall's lemma it follows that

$$\|u_{1,t} - u_{2,t}\|_2^2 = 0,$$

hence $u_1 = u_2$.

[Existence] The existence will be proved in two steps.

Step 1. Suppose f is bounded. We define

$$M := \sup |f(t,x,y,z)|.$$

We need the following proposition.

Proposition 3.2.9 If f satisfies the condition in **Step 1**, then for $v \in \hat{F}^l$, there exists a unique generalized solution $u \in \hat{F}^l$ for the equation

$$(\partial_t + L)u + f(\cdot,\cdot,u,D_{A^{1/2}}v) = 0, \qquad u_T = \phi.$$

Following the same arguments as in Lemma 3.2.5, we assume that $2C^2+2\alpha+\mu_t \leq 0$.

For each $v \in \hat{F}^l$, we define $Av = u$ where u is the unique generalized solution obtained by Proposition 3.2.9. Let $v_1, v_2 \in \hat{F}^l$. By applying (3.1.5) to the difference $u_1 - u_2$ we obtain

$$
\begin{aligned}
&\|u_{1,t} - u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}^A(u_{1,s} - u_{2,s})ds \\
\leq& 2\int_t^T (f(s,\cdot,u_{1,s},D_{A^{1/2}}v_{1,s}) - f(s,\cdot,u_{2,s},D_{A^{1/2}}v_{2,s}), u_{1,s} - u_{2,s})ds + 2\alpha\int_t^T \|u_{1,s} - u_{2,s}\|_2^2 ds \\
\leq& 2\int_t^T C(|D_{A^{1/2}}v_{1,s} - D_{A^{1/2}}v_{2,s}|_H, |u_{1,s} - u_{2,s}|)ds + \int_t^T (2\alpha + \mu_s)\|u_{1,s} - u_{2,s}\|_2^2 ds \\
\leq& \int_t^T (2C^2 + 2\alpha + \mu_s)\|u_{1,s} - u_{2,s}\|_2^2 ds + \frac{1}{2}\int_t^T \mathcal{E}^A(v_{1,s} - v_{2,s})ds \\
\leq& \frac{1}{2}\int_t^T \mathcal{E}^A(v_{1,s} - v_{2,s})ds.
\end{aligned}
$$

Consequently we have $\|Av_1 - Av_2\|_T \leq \frac{1}{2}\|v_1 - v_2\|_T$. Then the fixed point u of A is the solution for (3.2.1).

Proof of Proposition 3.2.9 We write $f(t,x,y) = f(t,x,y,D_{A^{1/2}}v)$ and uniqueness follows as above. Now we prove the existence of the solution.

We regularize f with respect to the variable y by convolution:

$$f_n(t,x,y,z) = n^l \int_{\mathbb{R}^l} f(t,x,y')\varphi(n(y-y'))dy'$$

where φ is a smooth nonnegative function with support contained in the ball $\{|y| \leq 1\}$ such that $\int \varphi = 1$. Then $f = \lim_{n\to\infty} f_n$ and for each n, $\partial_{y_i} f_n$ are uniformly bounded. Then each f_n satisfies a Lipschitz condition with respect to both y and z. Thus by Proposition 3.2.4 each f_n determines a solution $u_n \in \hat{F}^l$ of (3.2.1) with data (ϕ, f_n). By the same arguments as in [S09, Theorem 4.19], we have that each f_n satisfies conditions (H1) and (H2') with $C = 0$ and $\mu = 0$. Since

$$
\begin{aligned}
|f_n(t,x,0,0)| \leq& n^l \int_{|y'|\leq \frac{1}{n}} |f(t,x,y')||\varphi(n(-y'))|dy' \\
\leq& M,
\end{aligned}
$$

one deduces from Lemma 3.2.7 that $\|u_n\|_\infty \leq K$ and $\|u_n\|_T \leq K_T$.

Since the convolution operators approximate the identity uniformly on compact sets, we get for fixed t, x,

$$\lim_{n\to\infty} d'_{n,K}(t,x) := \sup_{|y|\leq K} |f(t,x,y) - f_n(t,x,y)| = 0.$$

Next we will show that $(u_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_T$-Cauchy sequence. By (3.1.5) for the difference $u_l - u_n$, we have

$$
\begin{aligned}
&\|u_{l,t}-u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}^A(u_{l,s}-u_{n,s})ds \\
\leq & 2\int_t^T (f_l(s,\cdot,u_{l,s}) - f_n(s,\cdot,u_{n,s}), u_{l,s}-u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s}-u_{n,s}\|_2^2 ds \\
\leq & 2\int_t^T (|f_l(s,\cdot,u_{l,s}) - f(s,\cdot,u_{l,s})|, |u_{l,s}-u_{n,s}|)ds \\
& + 2\int_t^T (|f_n(s,\cdot,u_{n,s}) - f(s,\cdot,u_{n,s})|, |u_{l,s}-u_{n,s}|)ds \\
& + 2\int_t^T (f(s,\cdot,u_{l,s}) - f(s,\cdot,u_{n,s}), u_{l,s}-u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s}-u_{n,s}\|_2^2 ds \\
\leq & 2\int_t^T (d'_{l,K}(s,\cdot) + d'_{n,K}(s,\cdot), |u_{l,s}-u_{n,s}|)ds + 2\alpha\int_t^T \|u_{l,s}-u_{n,s}\|_2^2 ds \\
\leq & \int_t^T \|d'_{l,K}(s,\cdot)\|_2^2 ds + \int_t^T \|d'_{n,K}(s,\cdot)\|_2^2 ds + (2+2\alpha)\int_t^T \|u_{l,s}-u_{n,s}\|_2^2 ds,
\end{aligned}
$$

and that $\lim_{n\to\infty}\int_t^T \|d'_{n,r}(s,\cdot)\|_2^2 ds = 0$. Thus, for l, n large enough, we get for an arbitrary $\varepsilon > 0$

$$
\|u_{l,t}-u_{n,t}\|_2^2 + \int_t^T \mathcal{E}^A(u_{l,s}-u_{n,s})ds \leq \varepsilon + \tilde{K}\int_t^T \|u_{l,t}-u_{n,t}\|_2^2 ds,
$$

where $\tilde{K}$ depends on C, M, μ, α. It is easy to see that Gronwall's lemma then implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\hat{F}$. Define $u := \lim_{n\to\infty} u_n$ and take a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. We have

$$
f(\cdot,\cdot,u_{n_k}) \to f(\cdot,\cdot,u) \text{ in } L^2(dt\times d\mu).
$$

Since $\|u_{n_k} - u\|_T \to 0$, we obtain

$$
\||D_{A^{1/2}}u - D_{A^{1/2}}u_{n_k}|_H\|_{L^2(dt\times d\mu)} \to 0.
$$

We conclude that

$$
\begin{aligned}
&\lim_{k\to\infty}\|f_{n_k}(u_{n_k}) - f(u)\|_{L^2(dt\times d\mu)} \\
\leq & \lim_{k\to\infty}\|f_{n_k}(u_{n_k}) - f(u_{n_k})\|_{L^2(dt\times d\mu)} + \lim_{k\to\infty}\|f(u_{n_k}) - f(u)\|_{L^2(dt\times d\mu)} \\
\leq & \lim_{k\to\infty}\|d'_{n_k,r}\|_{L^2(dt\times d\mu)} + \lim_{k\to\infty}\|f(u_{n_k}) - f(u)\|_{L^2(dt\times d\mu)} \\
= & 0.
\end{aligned}
$$

By passing to the limit in the mild equation associated to u_{n_k} with data (ϕ, f_{n_k}), it follows that u is the solution associated to $(\phi, f(u, D_{A^{1/2}}v))$. $\square$

Step 2. Now we consider the general case. Let r be a positive real number such that

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty),$$

where K is the constant appearing in Lemma 3.2.7 (3.2.4). Let θ_r be a smooth function such that $0 \leq \theta_r \leq 1, \theta_r(y) = 1$ for $|y| \leq r$ and $\theta_r(y) = 0$ if $|y| \geq r+1$. For each $n \in \mathbb{N}$, we set $q_n(z) := z\frac{n}{|z|_H \vee n}$ and

$$h_n(t,x,y,z) := \theta_r(y)(f(t,x,y,q_n(z)) - f_t^0)\frac{n}{f^{\prime,r+1} \vee n} + f_t^0.$$

We have

$$\begin{aligned}
|h_n(t,x,y,z)| &\leq |f(t,x,y,q_n(z)) - f(t,x,y,0) + f(t,x,y,0) - f_t^0|1_{\{|y| \leq r+1\}}\frac{n}{f^{\prime,r+1} \vee n} + f_t^0 \\
&\leq C|q_n(z)|_H + \frac{nf^{\prime,r+1}}{f^{\prime,r+1} \vee n} + f_t^0 \\
&\leq (1+C)n + f_t^0.
\end{aligned}$$

We easily show that h_n satisfies (H1) and (H3). So, we only need to prove (H2). For $y, y' \in \mathbb{R}^l$, if $|y| > r+1$, $|y'| > r+1$, the inequality is trivially satisfied and thus we concentrate on the case $|y'| \leq r+1$. We have

$$\begin{aligned}
&\langle y - y', h_n(t,x,y,z) - h_n(t,x,y',z)\rangle \\
=&\theta_r(y)\frac{n}{f^{\prime,r+1} \vee n}\langle y - y', f(t,x,y,q_n(z)) - f(t,x,y',q_n(z))\rangle \\
&+ \frac{n}{f^{\prime,r+1} \vee n}(\theta_r(y) - \theta_r(y'))\langle y - y', f(t,x,y',q_n(z)) - f_t^0\rangle.
\end{aligned}$$

The first term of the right hand side of the previous equality is negative. For the second term, since θ_r is $C(r)$-Lipschitz, we obtain

$$\begin{aligned}
(\theta_r(y) - \theta_r(y'))\langle y - y', f(t,x,y',q_n(z)) - f_t^0\rangle \leq& C(r)|y - y'|^2|f(t,x,y',q_n(z)) - f_t^0| \\
\leq& C(r)(Cn + f'_{r+1}(t))|y - y'|^2,
\end{aligned}$$

and thus

$$\frac{n}{f^{\prime,r+1} \vee n}(\theta_r(y) - \theta_r(y'))\langle y - y', f(t,x,y',q_n(z)) - f_t^0\rangle \leq C(r)(C+1)n|y - y'|^2.$$

Then each h_n satisfies the assumptions in **Step 1**, and denote u_n is the generalized solution of (3.2.1) with data (h_n, ϕ). We have

$$\langle y, h_n'(t,x,y)\rangle = \langle y, h_n(t,x,y,0) - h_n(t,x,0,0)\rangle = \langle y, f(t,x,y,0) - f_t^0\rangle \frac{n\theta_r(y)}{f'^{,r+1} \vee n} \leq 0.$$

Hence, Lemma 3.2.7 implies that $\|u_n\|_\infty \leq r-1$, $\|u_n\|_T \leq K_T$. So, u_n is a solution with data (f_n, ϕ), where $f_n(t,x,y,z) = (f(t,x,y,q_n(z)) - f_t^0)\frac{n}{f'^{,r+1} \vee n} + f_t^0$. For this function (H2) is satisfied with $\mu_t = 0$. Conditions (H1) and (H2) yield

$$\begin{aligned}
&|(f_l(u_l, D_{A^{1/2}}u_l) - f_n(u_n, D_{A^{1/2}}u_n), u_l - u_n)| \\
\leq &C(|D_{A^{1/2}}u_l - D_{A^{1/2}}u_n|_H, |u_l - u_n|) + |(f_l(u_n, D_{A^{1/2}}u_n) - f_n(u_n, D_{A^{1/2}}u_n), u_l - u_n)|.
\end{aligned}$$

For $n \leq l$, we have

$$\begin{aligned}
|f_l(u_n, D_{A^{1/2}}u_n) - f_n(u_n, D_{A^{1/2}}u_n)| \leq &2C|D_{A^{1/2}}u_n|_H 1_{\{|D_{A^{1/2}}u_n|_H \geq n\}} \\
&+ 2C|D_{A^{1/2}}u_n|_H 1_{\{f'^{,r+1} > n\}} + 2f'^{,r+1} 1_{\{f'^{,r+1} > n\}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|u_{l,t} - u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}^A(u_{l,s} - u_{n,s})ds \\
\leq &2\int_t^T (f_l(u_{l,s}, D_{A^{1/2}}u_{l,s}) - f_n(u_{n,s}, D_{A^{1/2}}u_{n,s}), u_{l,s} - u_{n,s})ds + 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq &2\int_t^T C(|D_{A^{1/2}}u_l - D_{A^{1/2}}u_n|_H, |u_l - u_n|)ds + 2\int_t^T (2C|D_{A^{1/2}}u_n|_H 1_{\{|D_{A^{1/2}}u_n|_H \geq n\}}, |u_l - u_n|)ds \\
&+ 2\int_t^T (2C|D_{A^{1/2}}u_n|_H 1_{\{f'^{,r+1} > n\}}, |u_l - u_n|)ds + 2\int_t^T (2f'^{,r+1} 1_{\{f'^{,r+1} > n\}}, |u_l - u_n|)ds \\
&+ 2\alpha\int_t^T \|u_{l,s} - u_{n,s}\|_2^2 ds \\
\leq &(C^2 + 2\alpha)\int_t^T \|u_l - u_n\|_2^2 ds + \int_t^T \mathcal{E}^A(u_l - u_n)ds \\
&+ 8C(r-1)\int_t^T \int |D_{A^{1/2}}u_n|_H 1_{\{|D_{A^{1/2}}u_n|_H \geq n\}} d\mu ds \\
&+ 8C(r-1)\int_t^T |D_{A^{1/2}}u_n|_H 1_{\{f'^{,r+1} > n\}} d\mu ds + 8C(r-1)\int_t^T \int f'^{,r+1} 1_{\{f'^{,r+1} > n\}} d\mu ds.
\end{aligned}$$

Since $\|u_n\|_T^2 \leq K_T$, we have $\int_0^T \||D_{A^{1/2}}u_n|_H\|_2^2 ds \leq K_T$. Hence,

$$n^2 \int_t^T \|1_{\{|D_{A^{1/2}}u_n|_H \geq n\}}\|_2^2 ds \leq \int_t^T \||D_{A^{1/2}}u_n 1_{\{|D_{A^{1/2}}u_n|_H \geq n\}}|_H\|_2^2 ds \leq K_T.$$

As
$$\lim_{n\to\infty}\int_t^T\int_{\{f^{',r}>n\}} f^{',r}d\mu ds=0,$$
and
$$\int_t^T\int_{\{f^{',r}>n\}}|D_{A^{1/2}}u_n|_H d\mu dt\leq \|1_{\{f^{',r}>n\}}\|_{L^2(dt\times d\mu)}\||D_{A^{1/2}}u_n|_H\|_{L^2(dt\times d\mu)}\to 0,$$
for n big enough we obtain
$$\|u_{l,t}-u_{n,t}\|_2^2+\int_t^T\mathcal{E}^A(u_{l,s}-u_{n,s})ds\leq (C^2+2\alpha)\int_t^T\|u_l-u_n\|_2^2ds+\varepsilon.$$

By Gronwall's lemma it is easy to see that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}^l$. Hence, $u:=\lim_{n\to\infty}u_n$ is well defined. We find a subsequence such that $(u_{n_k},D_{A^{1/2}}u_{n_k})\to(u,D_{A^{1/2}}u)$ a.e.
$$f(u_{n_k},D_{A^{1/2}}u)\to f(u,D_{A^{1/2}}u),$$
and conclude that
$$\begin{aligned}
&|f_{n_k}(u_{n_k},D_{A^{1/2}}u_{n_k})-f(u,D_{A^{1/2}}u)|\\
\leq& 1_{\{f^{',r}\leq n_k\}}|f(u,D_{A^{1/2}}u)-f(u_{n_k},q_{n_k}(D_{A^{1/2}}u_{n_k}))|\\
&+1_{\{f^{',r}>n_k\}}[|f(u,D_{A^{1/2}}u)-f^0|+|f(u,D_{A^{1/2}}u)-f(u_{n_k},q_{n_k}(D_{A^{1/2}}u_{n_k}))|]\\
\leq& |f(u,D_{A^{1/2}}u)-f(u_{n_k},q_{n_k}(D_{A^{1/2}}u_{n_k}))|+1_{\{f^{',r}>n_k\}}|f(u,D_{A^{1/2}}u)-f^0|\\
\leq& |f(u_{n_k},D_{A^{1/2}}u)-f(u_{n_k},q_{n_k}(D_{A^{1/2}}u_{n_k}))|+|f(u_{n_k},D_{A^{1/2}}u)-f(u,D_{A^{1/2}}u)|\\
&+1_{\{f^{',r}>n_k\}}|f(u,D_{A^{1/2}}u)-f^0|\\
\to& 0 \text{ a.e..}
\end{aligned}$$

Since
$$\begin{aligned}
&|f_{n_k}(u_{n_k},D_{A^{1/2}}u_{n_k})-f(u,D_{A^{1/2}}u)|\\
\leq& |f(u,0)-f(u,D_{A^{1/2}}u)|+|f_{n_k}(u_{n_k},D_{A^{1/2}}u_{n_k})-f_{n_k}(u_{n_k},0)|\\
&+|f_{n_k}(u_{n_k},0)-f^0|+|f^0-f(u,0)|\\
\leq& C(|D_{A^{1/2}}u|_H+|D_{A^{1/2}}u_{n_k}|_H)+2f^{',r},
\end{aligned}$$
we obtain
$$f_{n_k}(u_{n_k},D_{A^{1/2}}u_{n_k})\to f(u,D_{A^{1/2}}u)$$
in $L^1([0,T],L^2)$. We conclude that u is a generalized solution of (3.2.1) associated to the data (ϕ,f).

$\square$

3.3 Martingale representation for the processes

Some of the basic results on backward equations rely on the following well-known representation theorem (see, e.g., [FT02]). The Wiener process in a Hilbert space has the *martingale representation property*: any martingale with respect to the filtration generated by the Wiener process can be expressed as an Itô integral against the Wiener process. Now we extend this martingale representation theorem for the process associated with the operator L.

3.3.1 Representation under P^x

In order to obtain the results for the probabilistic part, we need that $\mathcal{E}$ is a quasi-regular generalized Dirichlet form (Definition 1.3) in the sense of Remark 3.1.1 (iii) with $\hat{c} \equiv 0$. There is a Markov process $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P^x)$ which is properly associated in the resolvent sense with $\mathcal{E}$, i.e. $R_\alpha f := E^x \int_0^\infty e^{-\alpha t} f(X_t)dt$ is $\mathcal{E}$-quasi-continuous m-version of the resolvent G_α of $\mathcal{E}$ for $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap L^2(E;\mu)$. The coform $\hat{\mathcal{E}}$ introduced in Section 1.1 is a generalized Dirichlet form with the associated resolvent $(\hat{G}_\alpha)_{\alpha>0}$ and there exists an μ-tight special standard process properly associated in the resolvent sense with $\hat{\mathcal{E}}$. We always assume that $(\mathcal{F}_t)_{t\geq 0}$ is the (universally completed) natural filtration of X_t. From now on, we obtain all the results under the above assumptions.

As mentioned in Remark 3.1.1 (vii), such a process can be constructed by quasi-regularity ([St2, IV. 1. Definition 1.7]) and a structural condition ([St2, IV. 2. D3] on the domain $\mathcal{F}$ of the generalized Dirichlet form.

We use the spaces $\mathcal{M}, \dot{\mathcal{M}}$ which are introduced in Section 1.2.

Define for $k \in E$,

$$\mathcal{E}_k(u,v) := \int \frac{\partial u}{\partial k}\frac{\partial v}{\partial k} d\mu, \qquad u, v \in \mathcal{F}C_b^\infty.$$

$k \in E$ is called μ-admissible if $(\mathcal{E}_k, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$.

We consider the following conditions:

(A4) There exist constants $c, C_1 > 0$ such that $cId_H \leq A(z) \leq C_1 Id_H$ for all $z \in E$. There exists a countable dense subset $\{e_k\}$ of E', which is an orthonormal basis of H, consisting of μ-admissible elements in E, and $u_k(\cdot) :=_{E'} \langle e_k, \cdot\rangle_E \in \mathcal{F}$.

(A4') There exists a countable dense subset $\{e_k\}$ of E', which is an orthonormal basis of H, consisting of μ-admissible elements in E, and $u_k(\cdot) =_{E'} \langle e_k, \cdot\rangle_E \in \mathcal{F}$. Furthermore, $A(z)e_k = \lambda_k(z)e_k$ for some non-negative Borel measurable functions

λ_k.

Remark 3.3.1 Condition (A4) can be replaced by condition (A4') and all results below can be proved by the same arguments. For simplicity, we only give the proof under condition (A4).

By the existence of $\{e_k\}$, (A1) follows from [MR92, Proposition 3.8]. Set

$$\mathcal{F}C_b^\infty(\{e_k\}) := \{f({}_{E'}\langle e_1, \cdot\rangle_E, ..., {}_{E'}\langle e_m, \cdot\rangle_E)|m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m)\}.$$

(A5) The process X associated with $\mathcal{E}$ above is a continuous conservative Hunt process in the state space $E \cup \{\partial\}$, $\alpha\hat{G}_\alpha$ is sub-Markovian and strongly continuous on $\mathcal{V}$, and $\hat{\mathcal{E}}$ is quasi-regular. Furthermore, $\mathcal{F}C_b^\infty(\{e_k\}) \subset \mathcal{F}$ and for $u \in \mathcal{F}$, there exists a sequence $\{u_n\} \subset \mathcal{F}C_b^\infty(\{e_k\})$ such that $\mathcal{E}(u_n - u) \to 0, n \to \infty$.

If $\mathcal{E}$ satisfies (A2) and (A4), the bilinear form can be written as

$$\mathcal{E}(u,v) := \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H d\mu(z) + \int \langle A(z)b(z), \nabla u(z)\rangle_H v(z) d\mu(z), u \in F, v \in bF.$$

Again we set $D_{A^{1/2}}u := A^{1/2}\nabla u$.

For an initial distribution $\mu \in \mathcal{P}(E)$ (where $\mathcal{P}(E)$ denotes all the probabilities on E,) we will prove that the *Fukushima reprensentation property* mentioned in [QY10] holds for X, i.e. there is an algebra $K(E) \subset \mathcal{B}_b(E)$ which generates the Borel σ-algebra $\mathcal{B}(E)$ and is invariant under R^α for $\alpha > 0$, and there are countable continuous martingales $M^i, i \in \mathbb{N}$, over $(\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P^\mu)$ such that for any potential $u = R^\alpha f$ where $\alpha > 0$ and $f \in K(E)$, the martingale part $M^{[u]}$ of the semimartingale $u(X_t) - u(X_0)$ has a martingale representation in terms of M^i, that is, there are predictable processes $F_i, i \in \mathbb{N}$ on $(\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu)$ such that

$$M_t^{[u]} = \sum_{j=1}^{\infty} \int_0^t F_s^j dM_s^j \qquad P^\mu - a.e..$$

By Theorem 1.7, if $\hat{G}_\alpha$ is sub-Markovian and strongly continuous on $\mathcal{V}$, the Fukushima's decomposition holds for $u \in \mathcal{F}$. In this case we set $M^k := M^{[u_k]}$, with $u_k(\cdot) := \langle e_k, \cdot\rangle_H$. These martingales are called *coordinate martingales.*

Let us first calculate the energy measure related to $\langle M^{[u]}\rangle, u \in \mathcal{F}C_b^\infty$. By [Tr2, formula (23)], for bounded $g \in L^2(E,\mu)$, we have

$$\begin{aligned}
&\int \hat{G}_\gamma g d\mu_{\langle M^{[u]}\rangle}\\
=&\lim_{\alpha\to\infty}\alpha(U^{\alpha+\gamma}_{\langle M^{[u]}\rangle}1,\hat{G}_\gamma g)\\
=&\lim_{\alpha\to\infty}\lim_{t\to\infty}E_{\hat{G}_\gamma g\cdot\mu}(\alpha e^{-(\gamma+\alpha)t}\langle M^{[u]}\rangle_t)+\lim_{\alpha\to\infty}E_{\hat{G}_\gamma g\cdot\mu}(\int_0^\infty \langle M^{[u]}\rangle_t\alpha(\gamma+\alpha)e^{-(\gamma+\alpha)t}dt)\\
=&\lim_{\alpha\to\infty}\lim_{t\to\infty}\alpha\langle\mu_{\langle M^{[u]}\rangle},e^{-(\gamma+\alpha)t}\int_0^t\hat{P}_s\hat{G}_\gamma gds\rangle\\
&+\lim_{\alpha\to\infty}\alpha(\gamma+\alpha)(\int_0^\infty e^{-(\gamma+\alpha)t}E_{\hat{G}_\gamma g\cdot\mu}((u(X_t)-u(X_0)-N_t^{[u]})^2)dt)\\
=&\lim_{\alpha\to\infty}\alpha(\gamma+\alpha)(\int_0^\infty e^{-(\gamma+\alpha)t}E_{\hat{G}_\gamma g\cdot\mu}((u(X_t)-u(X_0))^2)dt)\\
=&\lim_{\alpha\to\infty}2\alpha(u-\alpha G_\alpha u,u\hat{G}_\gamma g)-\alpha(u^2,\hat{G}_\gamma g-\alpha\hat{G}_\alpha\hat{G}_\gamma g)\\
=&2(-Lu,u\hat{G}_\gamma g)-(-Lu^2,\hat{G}_\gamma g)\\
=&2\mathcal{E}(u,u\hat{G}_\gamma g)-\mathcal{E}(u^2,\hat{G}_\gamma g)\\
=&2\mathcal{E}^A(u,u\hat{G}_\gamma g)-\mathcal{E}^A(u^2,\hat{G}_\gamma g)+2\int\langle Ab,\nabla u\rangle_H u\hat{G}_\gamma g\mu(dx)\\
&-\int\langle Ab,\nabla(u^2)\rangle_H\hat{G}_\gamma g\mu(dx)\\
=&2\mathcal{E}^A(u,u\hat{G}_\gamma g)-\mathcal{E}^A(u^2,\hat{G}_\gamma g)\\
=&2\int\langle A\nabla u,\nabla(u\hat{G}_\gamma g)\rangle_H d\mu-\int\langle A\nabla(u^2),\nabla(\hat{G}_\gamma g)\rangle_H d\mu\\
=&2\int\langle A\nabla u,\nabla u\rangle_H\hat{G}_\gamma g d\mu.
\end{aligned}$$

Then by [Tr2, Theorem 2.5], we have

$$\mu_{\langle M^{[u]}\rangle}=2\langle A\nabla u,\nabla u\rangle_H\cdot d\mu.$$

By [Tr2, Proposition 2.19], for $u\in\mathcal{F}C_b^\infty$ and $u=f({}_{E'}\langle e_1,\cdot\rangle_E,...,{}_{E'}\langle e_m,\cdot\rangle_E)$, we have

$$M_t^{[u]}=\sum_{i=1}^n\int_0^t\langle\nabla u(X_s),e_i\rangle_H dM_s^i.$$

Then by the same arguments as in [F90, Theorem 3.1] (see also proof of Theorem 2.3.6 and Lemma 3.3.2), we have that under P^x for quasi every point x (where the exceptional set depends on u, v) , and every $u, v \in \mathcal{F}$,

$$M_t^{[u]} = \sum_{i=1}^{\infty} \int_0^t \langle \nabla u(X_s), e_i \rangle_H dM_s^i, \tag{3.3.1}$$

where $\sum_{i=1}^{\infty} \int_0^t \langle \nabla u(X_s), e_i \rangle_H dM_s^i = \lim_{n\to\infty} \sum_{i=1}^{n} \int_0^t \langle \nabla u(X_s), e_i \rangle_H dM_s^i$ in $(\dot{\mathcal{M}}, e)$ and we have

$$\langle M^{[u]}, M^{[v]} \rangle_t = 2 \int_0^t \langle A(X_s) \nabla u(X_s), \nabla v(X_s) \rangle_H ds. \tag{3.3.2}$$

In particular,

$$\langle M^i, M^j \rangle_t = 2 \int_0^t a_{ij}(X_s) ds, \tag{3.3.3}$$

where $a_{ij}(z) := \langle A(z) e_i, e_j \rangle_H$.

Lemma 3.3.2 Suppose (A4)(A5) hold. For $u \in \mathcal{F}$ and V_t is a continuous adapted process, with $|V_t| \leq M, \forall t, \omega$, we have for q.e. $x \in E$,

$$\int_0^t V_s dM_s^{[u]} = \sum_{i=1}^{\infty} \int_0^t V_s \langle \nabla u(X_s), e_i \rangle_H dM_s^i \qquad P^x - a.s.. \tag{3.3.4}$$

If (A4'), (A5) hold, then for some $\psi \in \tilde{\nabla} u$, we have for q.e. $x \in E$,

$$\int_0^t V_s dM_s^{[u]} = \sum_{i=1}^{\infty} \int_0^t V_s \langle \psi(X_s), e_i \rangle_H dM_s^i \qquad P^x - a.s..$$

Proof By [Tr2, Remark 2.2], for $\nu \in \hat{S}_{00}$ and $B^n := \sum_{i=1}^{n} \int_0^t V_s \langle \nabla u(X_s), e_i \rangle_H dM_s^i$ we have

$$\begin{aligned}
E^\nu (B^{n+m} - B^n)^2 =& E^\nu \Big(\sum_{i=n+1}^{n+m} \int_0^t V_s \langle \nabla u(X_s), e_i \rangle_H dM_s^i \Big)^2 \\
=& E^\nu \Big(\sum_{i,j=n+1}^{n+m} \int_0^t V_s^2 a_{ij}(X_s) \langle \nabla u(X_s), e_i \rangle_H \langle \nabla u(X_s), e_j \rangle_H ds \Big) \\
\leq& C_1 M^2 E^\nu \Big(\sum_{i=n+1}^{n+m} \int_0^t \langle \nabla u(X_s), e_i \rangle_H^2 ds \Big) \\
\leq& C_1 M^2 e^t |\hat{U}_1 \nu|_\infty \sup_t \frac{1}{t} E^\mu \Big(\sum_{i=n+1}^{n+m} \int_0^t \langle \nabla u(X_s), e_i \rangle_H^2 ds \Big)
\end{aligned}$$

$$=C_1M^2e^t|\hat{U}_1\nu|_\infty\sum_{i=n+1}^{n+m}\int\langle\nabla u(z),e_i\rangle_H^2\mu(dz)\to 0,\ \text{as } n,m\to\infty,$$

where $\hat{S}_{00},\hat{U}_1\nu$ are introduced in Section 1.2. Then we define $\sum_{i=1}^\infty\int_0^t V_s\langle\nabla u(X_s),e_i\rangle_H dM_s^i:=\lim_{n\to\infty}B^n$ in $(\dot{\mathcal{M}},e)$. Furthermore, we have

$$\begin{aligned}
&E^\nu(\int_0^t V_s dM_s^{[u]}-\sum_{i=1}^n\int_0^t V_s\langle\nabla u(X_s),e_i\rangle_H dM_s^i)^2\\
\le& E^\nu\int_0^t\sum_{i,j=n+1}^\infty V_s^2 a_{ij}(X_s)\langle\nabla u(X_s),e_i\rangle_H\langle\nabla u(X_s),e_j\rangle_H ds\\
\le& M^2C_1e^t|\hat{U}_1\nu|_\infty\sup_t\frac{1}{t}E^\mu\int_0^t\sum_{i=n+1}^\infty\langle\nabla u(X_s),e_i\rangle_H^2 ds\\
\le& M^2C_1e^t|\hat{U}_1\nu|_\infty\int_0^t\sum_{i=n+1}^\infty\langle\nabla u(z),e_i\rangle_H^2\mu(dz)\to 0,\ \text{as } n\to\infty.
\end{aligned}$$

So, we have

$$\int_0^t V_s dM_s^{[u]}=\sum_{i=1}^\infty\int_0^t V_s\langle\nabla u(X_s),e_i\rangle_H dM_s^i\qquad P^\nu-a.s..$$

Then by [Tr2, Theorem 2.5] and Theorem 1.4, the assertions follow. □

Moreover, by a modification of the proof of [QY10, Theorem 3.1] , we have the martingale representation theorem for X which is similar to [BPS05].

Theorem 3.3.3 Suppose that (A4) or (A4') and (A5) hold. Then there exists some exceptional set $\mathcal{N}$ such that the following representation result holds: For every bounded $\mathcal{F}_\infty$-measurable random variable ξ, there exists predictable processes $\phi:[0,\infty)\times\Omega\to H$, such that for each probability measure ν, supported by $E\setminus\mathcal{N}$, one has

$$\xi=E^\nu(\xi|\mathcal{F}_0)+\sum_{i=0}^\infty\int_0^\infty\langle\phi_s,e_i\rangle_H dM_s^i\qquad P^\nu-a.e.,$$

where $M^i=M^{[u_i]}$ with $u_i:={}_{E'}\langle e_i,\cdot\rangle_E, i\in\mathbb{N}$ are the coordinate martingales, and

$$E^\nu\int_0^\infty\langle A(X_s)\phi_s,\phi_s\rangle_H ds\le\frac{1}{2}E^\nu\xi^2.$$

If another predictable process ϕ' satisfies the same relations under a certain measure P^ν, then one has $A^{1/2}(X_t)\phi'_t=A^{1/2}(X_t)\phi_t, dt\times dP^\nu-a.e.$

Proof Suppose that $\mathcal{N}$ is some fixed exceptional set. By $\mathcal{K}$ we denote the class of bounded random variables for which the statement holds outside this set. We claim that if $(\xi_n) \subset \mathcal{K}$ is a uniformly bounded increasing sequence and $\xi = \lim_{n\to\infty} \xi_n$, then $\xi \in \mathcal{K}$. Indeed, we have $E^x|\xi_n - \xi|^2 \to 0$. Let ϕ^n denote the process which represents ξ_n. Then

$$E^x \int_0^\infty |\phi_s^n - \phi_s^p|_H^2 ds \leq \frac{1}{c} E^x \int_0^\infty \langle A(X_s)(\phi_s^n - \phi_s^p), \phi_s^n - \phi_s^p \rangle_H ds \leq \frac{1}{2c} E^x |\xi_p - \xi_n|^2.$$

Now we want to pass to the limit with ϕ^n pointwise, so that the limit become predictable. For each $l = 0, 1, ...$ set

$$n_l(x) := \inf\{n | E^x(\xi - \xi_n)^2 < \frac{1}{2^l}\}$$

$$\bar{\xi}_l := \xi_{n_l(X_0)}.$$

Then one has $\bar{\xi}_l = \xi_{n_l(x)}$ on the set $\{X_0 = x\}$, and $E^x(\xi - \bar{\xi}_l)^2 < \frac{1}{2^l}$ for any $x \in \mathcal{N}^c$. The process which represents $\bar{\xi}_l$ is simply obtained by the formula $\bar{\phi}^l = \phi^{n_l(X_0)}$. Then define $\phi_s = \lim_{l\to\infty} \bar{\phi}_s^l$ in H. By the same arguments as the proof of Lemma 3.3.1, we obtain

$$\begin{aligned} E^x(\sum_{i=1}^\infty \int_0^\infty \langle \phi_s - \bar{\phi}_s^l, e_i \rangle_H dM_s^i)^2 &= \lim_{k\to\infty} E^x(\sum_{i=1}^k \int_0^\infty \langle \phi_s - \bar{\phi}_s^l, e_i \rangle_H dM_s^i)^2 \\ &= \lim_{k\to\infty} E^x(\sum_{i,j=1}^k \int_0^\infty a_{ij}(X_s) \langle \phi_s - \bar{\phi}_s^l, e_i \rangle_H \langle \phi_s - \bar{\phi}_s^l, e_j \rangle_H ds) \\ &\leq C_1 E^x \int_0^\infty |\phi_s - \bar{\phi}_s^l|_H^2 ds \to 0, \text{ as } l \to \infty. \end{aligned}$$

Therefore, we have $\xi \in \mathcal{K}$.

Let $K(E) \subset \mathcal{B}_b(E)$ be a countable set which is closed under multiplication, generates the Borel σ-algebra $\mathcal{B}(E)$ and $R^\alpha(K(E)) \subset K(E)$ for $\alpha \in \mathbb{Q}^+$. Such $K(E)$ can be constructed as follows. We choose a countable set $N_0 \subset b\mathcal{B}(E)$ which generates the Borel σ-algebra $\mathcal{B}(E)$. Since E as a separable Banach space is strongly Lindelöf, such a set N_0 can easily be constructed (see [MR92, Section 3.3]). For $l \geq 1$ we define $N_{l+1} = \{g_1 \cdot ... \cdot g_k, U^\alpha g_1 \cdot g_2 \cdot ... \cdot g_k, g_i \in N_l, k \in \mathbb{N} \cup \{0\}, \alpha \in \mathbb{Q}^+\}$ and $K(E) := \cup_{l=0}^\infty N_l$ (c.f. [FOT94, Lemma 7.1.1]).

Let $\mathcal{C}_0$ be all $\xi = \xi_1 \cdots \xi_n$ for some $n \in \mathbb{N}$, $\xi_j = \int_0^\infty e^{-\alpha_j t} f_j(X_t) dt$, where $\alpha_j \in \mathbb{Q}^+$, $f_j \in K(E), j = 1, ..., n$. Since the results in Lemma 2.3.7 also hold in this case, we see that the universal completion of the σ-algebra generated by $\mathcal{C}_0$ is $\mathcal{F}_\infty$. By the first claim, a monotone class argument reduces the proof to the representation of a random variable in $\mathcal{C}_0$.

Let $\xi \in \mathcal{C}_0$. By Markov property of the process (see e.g. [QY10, Theorem 3.1]) , we have

$$N_t = E^x(\xi|\mathcal{F}_t) = \sum_m Z_t^m$$

where the sum is finite , and for each m, $Z^m = Z_t$ has the following form

$$Z_t = V_t u(X_t)$$

(the superscript m will be dropped if no confusion may arise), where $V_t = \prod_{i=1}^{k'} \int_0^t e^{-\beta_i s} g_i(X_s) ds$ and $u(x) = U^{\beta_1+\ldots+\beta_k}(h_1(U^{\beta_2+\ldots+\beta_k} h_2 \ldots (U^{\beta_k} h_k)\ldots)$ for $\beta_i \in \mathbb{Q}^+, g_i, h_i \in K(E)$. Obviously, $u \in K(E)$. Hence, by Fukushima's decomposition and Fukushima's representation property we have

$$\text{(3.3.5)}\quad u(X_t) - u(X_0) = M_t^{[u]} + A_t^{[u]} = \sum_{j=1}^{\infty} \int_0^t \langle \nabla u(X_s), e_j \rangle_H dM_s^j + A_t^{[u]} \qquad P^x - a.s..$$

Then by Itô's formula and Lemma 3.3.2, we have

$$\begin{aligned} Z_t &= Z_0 + \int_0^t u(X_s) dV_s + \int_0^t V_s dA_s^{[u]} + \int_0^t V_s dM_s^{[u]} \\ &= Z_0 + \int_0^t u(X_s) dV_s + \int_0^t V_s dA_s^{[u]} + \sum_{i=1}^{\infty} \int_0^t V_s \langle \nabla u(X_s), e_i \rangle_H dM_s^i. \end{aligned}$$

Hence,

$$N_t = \sum_{i=1}^{\infty} \int_0^t V_s \langle \nabla u(X_s), e_i \rangle_H dM_s^i \qquad P^x - a.s..$$

Define $\phi_s = V_s \nabla u(X_s)$, then the representation holds for $\xi \in \mathcal{C}_0$. As (3.3.4) and (3.3.5) hold for every x outside of an exceptional set of null capacity, the exceptional set $\mathcal{N}$ in the statement will be the union of all these exceptional sets corresponding to $u \in K(E)$ and the exceptional sets related to V in Lemma 3.3.2. $\square$

If in the preceding theorem, ξ is nonnegative, we drop the boundedness assumption.

Corollary 3.3.4 Suppose that (A4) or (A4') and (A5) hold. Let $\mathcal{N}$ be the set obtained in Theorem 3.3.3. Then for any $\mathcal{F}_\infty$-measurable nonnegative random variable $\xi \geq 0$ there exists a predictable process $\phi : [0, \infty) \times \Omega \to H$ such that

$$\xi = E^x(\xi|\mathcal{F}_0) + \sum_{i=0}^{\infty} \int_0^{\infty} \langle \phi_s, e_i \rangle_H dM_s^i \qquad P^x - a.e.,$$

where $M^i, i \in \mathbb{N}$, are as in Theorem 3.3.3, and

$$E^x \int_0^\infty \langle A(X_s)\phi_s, \phi_s\rangle_H ds \leq \frac{1}{2} E^x \xi^2,$$

for each point $x \in \mathcal{N}^c$ such that $E^x \xi < \infty$.

If another predictable process ϕ' satisfies the same relations under a certain measure P^x, then one has $A^{1/2}(X_t)\phi'_t = A^{1/2}(X_t)\phi_t, dt \times dP^x - a.e.$

3.3.2 Representation under P^μ

As usual we set $\int_0^t \psi_s.dM_s = \sum_{i=0}^\infty \int_0^t \langle \psi_s, e_i\rangle_H dM_s^i$.

Lemma 3.3.5 Suppose that (A1)-(A3), (A5) and (A4) or (A4') hold. If $u \in \mathcal{D}(L)$, $\psi \in \tilde{\nabla} u$, then

$$u(X_t) - u(X_0) = \int_0^t \psi(X_s).dM_s + \int_0^t Lu(X_s)ds \qquad P^\mu - a.s..$$

Proof Corollary 3.3.4 and (3.3.1) imply the assertion. □

Then by the same arguments as the proof of Lemma 2.3.6 we extend this representation to time dependent functions $u(t,x)$.

Lemma 3.3.6 Suppose that (A1)-(A3), (A5) and (A4) or (A4') hold. Let $u : [0,T] \times E \to \mathbb{R}$ be such that

(i) $\forall s, u_s \in \mathcal{D}(L)$ and $s \to Lu_s$ is continuous in L^2.

(ii) $u \in C^1([0,T]; L^2)$.

Then clearly $u \in \mathcal{C}_T$, and, moreover, for any $\psi \in \tilde{\nabla} u$ and any $s,t > 0$ such that $s + t < T$, the following relation holds P^μ-a.s.

$$u(s+t, X_t) - u(s, X_0) = \int_0^t \psi_{s+r}(X_r).dM_r + \int_0^t (\partial_s + L)u_{s+r}(X_r)dr.$$

Theorem 3.3.7 Supppose that (A1)-(A3), (A5) and (A4) or (A4') hold. Let $f \in L^1([0,T]; L^2)$ and $\phi \in L^2(E)$ and define

$$u_t := P_{T-t}\phi + \int_t^T P_{s-t} f_s ds.$$

Then for each $\psi \in \tilde{\nabla} u$ and for each $s \in [0,T]$, the following relation holds P^μ-a.s.

$$u(s+t, X_t) - u(s, X_0) = \int_0^t \psi(s+r, X_r).dM_r - \int_0^t f(s+r, X_r)dr.$$

Furthermore, if u is a generalized solution of PDE (3.2.1), for each $t \in [s,T]$ the following BSDE holds P^μ-a.s.

$$\begin{aligned} u(t, X_{t-s}) =& \phi(X_{T-s}) + \int_t^T f(r, X_{r-s}, u(r, X_{r-s}), A^{1/2}\nabla u(r, X_{r-s}))dr \\ &- \int_{t-s}^{T-s} \psi(s+r, X_r).dM_r. \end{aligned}$$

Proof First assume that ϕ and f satisfy the conditions in Proposition 3.1.5 (ii). Then we have that u satisfies the conditions in Lemma 3.3.6 and by Lemma 3.3.6, the assertion follows. For the general case we choose u^n associated with (f^n, ϕ^n) as in Proposition 3.1.7. Then $\|u^n - u\|_T \to 0$ as $n \to \infty$. For u^n we have

$$u^n(s+t, X_t) - u^n(s, X_0) = \int_0^t \nabla u^n_{s+r}(X_r).dM_r - \int_0^t f^n(s+r, X_r)dr. \tag{3.3.6}$$

As

$$\begin{aligned} &E^\mu \Big| \int_0^t (\nabla u^n_{s+r}(X_r) - \nabla u^p_{s+r}(X_r)).dM_r\Big|^2 \\ \leq& E^\mu \int_0^t \langle A(X_r)(\nabla u^n_{s+r}(X_r) - \nabla u^p_{s+r}(X_r)), \nabla u^n_{s+r}(X_r) - \nabla u^p_{s+r}(X_r)\rangle_H dr \\ \leq& \int_0^t \mathcal{E}^A(u^n_{s+r} - u^p_{s+r})dr, \end{aligned}$$

letting $n \to \infty$ in (3.3.6), we obtain the assertions. □

3.4 BSDE's and Weak Solutions

The set $\mathcal{N}$ obtained in Theorem 3.3.3 will be fixed throughout this section. By Theorem 3.3.3, we solve the BSDE under all measures P^x, $x \in \mathcal{N}^c$, at the same time. We will treat systems of l equations, $l \in \mathbb{N}$, associated to $\mathbb{R}^l$-valued functions $f : [0,T] \times \Omega \times \mathbb{R}^l \times H^l \mapsto \mathbb{R}^l$, assumed to be predictable. This means that we consider the map $(s,\omega) \mapsto f(s,\omega,\cdot,\cdot)$ as a process which is predictable with respect to the canonical filtration of our process $(\mathcal{F}_t)$.

Lemma 3.4.1 Suppose that (A4) or (A4') and (A5) hold. Let ξ be an $\mathcal{F}_T$-

measurable random variable and $f : [0,T] \times \Omega \mapsto \mathbb{R}$ an $(\mathcal{F}_t)_{t\geq 0}$-predictable process. Let D be the set of all points $x \in \mathcal{N}^c$ for which the following integrability condition holds

$$E^x(|\xi| + \int_0^T |f(s,\omega)|ds)^2 < \infty.$$

Then there exists a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}$, $Z : [0,T) \times \Omega \mapsto H$, such that under all measures P^x, $x \in D$, they have the following properties:

(i) Y is continuous;

(ii) Z satisfies the integrability condition

$$\int_0^T |A^{1/2}(X_t)Z_t|_H^2 dt < \infty, \qquad P^x - a.s.;$$

(iii) the local martingale obtained integrating Z against the coordinate martingales, i.e. $\int_0^t Z_s.dM_s$, is a uniformly integrable martingale;

(iv)

$$Y_t = \xi + \int_t^T f(s,\omega)ds - \int_t^T Z_s.dM_s, \qquad P^x - a.s., 0 \leq t \leq T.$$

If another pair (Y_t', Z_t') of predictable processes satisfies the above conditions (i),(ii),(iii),(iv), under a certain measure P^ν with the initial distribution ν supported by D, then one has $Y. = Y.'$, $P^\nu - a.s.$ and $A^{1/2}(X_t)Z_t = A^{1/2}(X_t)Z_t'$, $dt \times P^\nu - a.s.$.

Proof The representations of the positive and negative parts of the random variable $\xi + \int_0^T f_s ds$ give us a predictable process Z such that

$$\xi + \int_0^T f_s ds = E^{X_0}(\xi + \int_0^T f_s ds) + \int_0^T Z_s.dM_s.$$

Then we obtain the desired process Y by the formula

$$Y_t = E^{X_0}(\xi + \int_0^T f_s ds) + \int_0^T Z_s.dM_s - \int_0^t f_s ds.$$

□

Definition 3.4.2 Let ξ be an $\mathbb{R}^l$-valued, $\mathcal{F}_T$-measurable, random variable and $f : [0,T] \times \Omega \times \mathbb{R}^l \times H^l \mapsto \mathbb{R}^l$ a measurable $\mathbb{R}^l$-valued function such that $(s,\omega) \mapsto f(s,\omega,\cdot,\cdot)$ is a predictable process. Let $p > 1$ and ν be a probability measure supported by $\mathcal{N}^c$ such that $E^\nu|\xi|^p < \infty$. We say that a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}^l$, $Z : [0,T) \times \Omega \mapsto H^l$ is a solution of the BSDE in $L^p(P^\nu)$ with data (ξ, f) provided Y is continuous under P^ν and it satisfies both the integrability conditions

$$\int_0^T |f(t,\cdot,Y_t,A^{1/2}(X_t)Z_t)|dt < \infty, \qquad P^\nu - a.s.,$$

and

$$E^\nu(\int_0^T |A^{1/2}(X_t)Z_t|_H^2 dt)^{p/2} < \infty,$$

and the following equation holds

(3.4.1)
$$Y_t = \xi + \int_t^T f(s,\omega,Y_s,A^{1/2}(X_s)Z_s)ds - \int_t^T Z_s.dM_s, \qquad P^\nu - a.s., 0 \le t \le T.$$

Let $f : [0,T] \times \Omega \times \mathbb{R}^l \times H^l \mapsto \mathbb{R}^l$ be a measurable $\mathbb{R}^l$-valued function such that $(s,\omega) \mapsto f(s,\omega,\cdot,\cdot)$ is predictable and it satisfies the following conditions:

(Ω1) [Lipschitz condition in z] There exists a constant $C > 0$ such that for all t,ω,y,z,z'

$$|f(t,\omega,y,z) - f(t,\omega,y,z')| \le C|z - z'|_H.$$

(Ω2) [Monotonicity condition in y] For ω, y, y', z arbitrary, there exists a function $\mu_t \in L^1([0,T],\mathbb{R})$ such that

$$\langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z)\rangle \le \mu_t|y - y'|^2,$$

and $\alpha_t := \int_0^t \mu_s ds$.

(Ω3) [Continuity condition in y] For t,ω and z fixed, the map

$$y \mapsto f(t,\omega,y,z)$$

is continuous.

We use the following notations:

$$f^0(t,\omega) := f(t,\omega,0,0), \qquad f'(t,\omega,y) := f(t,\omega,y,0) - f(t,\omega,0,0),$$

$$f^{',r}(t,\omega) := \sup_{|y|\le r} |f'(t,\omega,y)|.$$

Let ξ be an $\mathbb{R}^l$-valued, $\mathcal{F}_T$-measurable, random variable and, for each $p > 0$ let A_p denote the set of all points $x \in \mathcal{N}^c$ such that

$$E^x \int_0^T f_t^{',r} dt < \infty, \qquad \forall r \ge 0, \tag{3.4.2}$$

and

$$E^x(|\xi|^p + (\int_0^T |f^0(s,\omega)|ds)^p) < \infty.$$

Let A_∞ denote the set of points $x \in \mathcal{N}^c$ for which (3.4.2) holds and with the property that $|\xi|, |f^0| \in L^\infty(P^x)$.

The method to prove the following proposition is standard, and it is included in the Appendix.

Proposition 3.4.3 Assume that (A4) or (A4') and (A5) holds. Under conditions $(\Omega 1), (\Omega 2), (\Omega 3)$ there exists a pair $(Y_t, Z_t)_{0\leq t\leq T}$ of predictable processes $Y : [0,T) \times \Omega \mapsto \mathbb{R}^l, Z : [0,T) \times \Omega \mapsto H^l$ that forms a solution of the BSDE (3.4.1) in $L^p(P^x)$ with data (ξ, f) for each point $x \in A_p$. Moreover, the following estimate holds with some constant K that depends only on c, C, μ and T,

$$E^x(\sup_{t\in[0,T]} |Y_t|^p + (\int_0^T |A^{1/2}(X_t)Z_t|_H^2 dt)^{p/2}) \leq K E^x(|\xi|^p + (\int_0^T |f^0(s,\omega)|ds)^p), \qquad x \in A_p.$$

If $x \in A_\infty$, then $\sup_{t\in[0,T]} |Y_t| \in L^\infty(P^x)$.

If (Y_t', Z_t') is another solution in $L^p(P^x)$ for some point $x \in A_p$, then one has $Y_t = Y_t'$ and $A^{1/2}(X_t)Z_t = A^{1/2}(X_t)Z_t', dt \times P^x - a.s.$.

We shall now look at the connection between the solutions of BSDE's introduced in this section and the PDE's studied in Section 3.2. In order to do this we have to consider BSDE's over time intervals $[s,T]$, with $0 \leq s \leq T$ as done in Section 2.4.

Let us give a formal definition for the natural notion of solution over a time interval $[s,T]$. Let ξ be an $\mathcal{F}_{T-s}$-measurable, $\mathbb{R}^l$-valued, random variable and $f : [s,T] \times \Omega \times \mathbb{R}^l \times H^l \to \mathbb{R}^l$ an $\mathbb{R}^l$-valued, measurable map such that $(f(s+l,\omega,\cdot,\cdot))_{l\in[0,T-s]}$ is predictable with respect to $(\mathcal{F}_l)_{l\in[0,T-s]}$. Let ν be a probability measure supported by $\mathcal{N}^c$ such that $E^\nu|\xi|^p < \infty$. We say a pair $(Y_t, Z_t)_{s\leq t\leq T}$ of processes $Y : [s,T] \times \Omega \to \mathbb{R}^l, Z : [s,T] \times \Omega \to H^l$ is a solution in $L^p(P^\nu)$ of the BSDE over the interval $[s,T]$ with data (ξ, f), provided they have the property that reindexed as $(Y_{s+l}, Z_{s+l})_{l\in[0,T-s]}$ these processes are $(\mathcal{F}_l)_{l\in[0,T-s]}$-predictable, Y is continuous and together they satisfy the integrability conditions

$$\int_s^T |f(t,\cdot,Y_t, A^{1/2}(X_{t-s})Z_t)|dt < \infty, \qquad P^\nu - a.s.,$$

and

$$E^\nu(\int_s^T |A^{1/2}(X_{t-s})Z_t|_H^2 dt)^{p/2} < \infty,$$

and the following equation under P^ν holds

$$(3.4.3) \quad Y_t = \xi + \int_t^T f(r, Y_r, A^{1/2}(X_{r-s})Z_r)dr - \int_{t-s}^{T-s} Z_{s+l}.dM_l, \qquad s \le t \le T.$$

The next result gives the probabilistic interpretation of Theorem 3.2.8. Let us assume that $f : [0,T] \times E \times \mathbb{R}^l \times H^l \to \mathbb{R}^l$ is the measurable function appearing in the basic equation (3.2.1).

Let $\phi : E \to \mathbb{R}^l$ be measurable and for each $p > 1$, let A_p denote the set of all points $(s,x) \in [0,T) \times \mathcal{N}^c$ such that

$$E^x \int_s^T f^{',r}(t, X_{t-s})dt < \infty, \qquad \forall r \ge 0,$$

and

$$E^x(|\phi|^p(X_{T-s}) + (\int_s^T |f^0(t, X_{t-s})|ds)^p) < \infty.$$

Set $D := \cup_{p>1} A_p$, $A_{p,s} := \{x \in \mathcal{N}^c, (s,x) \in A_p\}$, and $A_s := \cup_{p>1} A_{p,s}$, $s \in [0,T)$.

By the same arguments as Theorem 2.4.4, we obtain the following results.

Theorem 3.4.4 Assume that (A4) or (A4') and (A5) holds and that the function f satisfies conditions (H1),(H2),(H3). Then there exist nearly Borel measurable functions $(u,\psi), u : D \to \mathbb{R}^l, \psi : D \to H^l$, such that, for each $s \in [0,T)$ and each $x \in A_{p,s}$, the pair $(u(t, X_{t-s}), \psi(t, X_{t-s}))_{s \le t \le T}$ solves BSDE (3.4.3) in $L^p(P^x)$ with data $(\phi(X_{T-s}), f(t, X_{t-s}, y, z))$ over the interval $[s,T]$.

In particular, the functions u, ψ satisfy the following estimate, for $(s,x) \in A_p$,

$$\begin{aligned} &E^x(\sup_{t \in [s,T]} |u(t, X_{t-s})|^p + (\int_s^T |A^{1/2}\psi(t, X_{t-s})|^2 dt)^{p/2}) \\ \le &KE^x(|\phi(X_{T-s})|^p + (\int_s^T |f^0(t, X_{t-s})|dt)^p). \end{aligned}$$

Moreover, if (A1)-(A3) hold and f, ϕ satisfy conditions (H4) and (H5), then the complement of $A_{2.s}$ is μ-negligible (i.e. $\mu(A^c_{2,s}) = 0$) for each $s \in [0,T)$, the class of $u1_{A_2}$ is an element of $\hat{F}^l$ which is a generalized solution of (3.2.1), ψ represents a version of ∇u and the following relation holds for each $(s,x) \in D$ and $1 \le i \le l$,

$$(3.4.4) \quad u^i(s,x) = E^x(\phi^i(X_{T-s})) + \int_s^T E^x f^i(t, X_{t-s}, u(t, X_{t-s}), A^{1/2}(X_{t-s})\psi(t, X_{t-s}))dt.$$

3.5 Examples

In this section, we give some examples satisfying our assumptions (A1)-(A5).

Example 3.5.1 (Ornstein-Uhlenbeck semigroup) Given two separable Hilbert spaces H and U, consider the stochastic differential equation

$$dX(t) = A_1X(t)dt + BdW(t), \qquad X(0) = x \in H, \tag{3.5.1}$$

where $A_1 : D(A_1) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA_1}, $B : U \to H$ is a bounded linear operator, and W is a cylindrical Wiener process in U. Assume

(i) $\|e^{tA_1}\| \leq Me^{\omega t}$ for $\omega < 0$, $M \geq 0$, and all $t \geq 0$.

(ii) For any $t > 0$ the linear operator Q_t, defined as

$$Q_t x = \int_0^t e^{sA_1} C e^{sA_1^*} x ds, \;\; x \in H, t \geq 0,$$

where $C = BB^*$, is of trace class.

(iii) $Ce^{tA_1^*} = e^{tA_1}C$.

μ denotes the Gaussian measure in H with mean 0 and covariance operator Q_∞. Then the bilinear form

$$\mathcal{E}(u, u) := \frac{1}{2}\int_H |C^{1/2}\nabla u|^2 d\mu, \;\; u \in \mathcal{F}C_b^\infty,$$

is closable. The closure of $\mathcal{F}C_b^\infty$ with respect to $\mathcal{E}_1$ is denoted by F. $(\mathcal{E}, F)$ is a generalized Dirichlet form in the sense of Remark 3.1.1 (iii) with $(E_1, \mathcal{B}(E_1), m) = (H, \mathcal{B}(H), \mu)$, $(\mathcal{A}, \mathcal{V}) = (\mathcal{E}, F)$ and $\Lambda = 0$. In particular, it is a symmetric Dirichlet form associated with the O-U process given by (3.5.1) and satisfies conditions (A1)-(A5) (see [D04, ChapterII]).

Example 3.5.2 Let H be a real separable Hilbert space (with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $|\cdot|$) and μ a finite positive measure on H. We denote its Borel σ-algebra by $\mathcal{B}(H)$. For $\rho \in L^1_+(H, \mu)$ we consider the following bilinear form

$$\mathcal{E}^\rho(u, v) = \frac{1}{2}\int_H \langle \nabla u, \nabla v \rangle \rho(z)\mu(dz), u, v \in \mathcal{F}C_b^\infty,$$

where $L^1_+(H, \mu)$ denotes the set of all non-negative elements in $L^1(H, \mu)$. There are many examples for ρ such that $\mathcal{E}^\rho$ is closable. For example if $\rho d\mu$ is a "Log-Concave" measure in the sense of [ASZ09], and more examples can be found in [MR92]. The closure of $\mathcal{F}C_b^\infty$ with respect to $\mathcal{E}_1$ is denoted by F. $(\mathcal{E}, F)$ is a generalized Dirichlet form in the sense of Remark 3.1.1 (iii) with $(E_1, \mathcal{B}(E_1), m) = (H, \mathcal{B}(H), \mu)$, $(\mathcal{A}, \mathcal{V}) = (\mathcal{E}, F)$ and $\Lambda = 0$. In particular, it is a symmetric Dirichlet form and satisfies (A1)-(A5). Assume that: $A_1 : D(A_1) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle A_1 x, x\rangle \geq \delta|x|^2$ $\forall x \in D(A_1)$ for some $\delta > 0$ and A_1^{-1} is of trace class. μ will denote the Gaussian measure in H with mean 0 and covariance operator

$$Q := \frac{1}{2}A_1^{-1}.$$

We are concerned with the following two cases.

1. Choose $\rho = \frac{e^{-2U(x)}}{\int_H e^{-2U(y)}dy}$ for a Borel map $U : H \to (-\infty, +\infty]$ with $\int_H e^{-2U(y)}dy \in (0, \infty)$. Under some regular condition for U, the process associated with $\mathcal{E}^\rho$ is the solution of the following SPDE

$$dX(t) = (A_1 X(t) + \nabla U(X(t))dt + dW(t), \qquad X(0) = x \in H.$$

2. $\rho = 1_{\{|x|_H \leq 1\}}$. This case has been studied in [ASZ09], [RZZ] and it is associated with a reflected O-U process ([RZZ]). The Kolomogorov equation associated with $\mathcal{E}$ has been studied in [BDT09] and the solution corresponds to the Kolomogorov equation with Neumman boundary condition.

Example 3.5.3 Consider the same situation in Example 3.5.1 and assume that, in addition we are given a nonlinear function $F : H \to H$ such that there exists $K > 0$, $|F(x) - F(y)|_H \leq K|x - y|, x, y \in H$ and $\langle F(x) - F(y), x - y\rangle \leq 0, x, y \in H$. A_1 is an operator which satisfies the condition in Example 3.5.2 and $A_1^{-1+\delta}$ is trace-class for some $\delta \in (0, \frac{1}{2})$. We are concerned with the stochastic differential equation

$$dX(t) = (A_1 X(t) + F(X(t))dt + BdW(t), \qquad X(0) = x \in H. \tag{3.5.2}$$

The Kolomogrov operator associated with (3.5.2) is given by

$$K_0\varphi = \frac{1}{2}Tr[CD^2\varphi] + \langle x, A_1^* D\varphi\rangle + \langle F(x), D\varphi\rangle, \varphi \in \mathcal{E}_{A_1}(H),$$

where $\mathcal{E}_{A_1}(H) :=$ linear span $\{\varphi_h(x) = e^{i\langle h,x\rangle} : h \in D(A_1^*)\}$. Assume the semigroup e^{tA_1} is analytic. Then by [DZ02, Theorem 11.2.21] there exists a unique invariant measure ν for K_0 i.e.

$$\int K_0\varphi d\nu = 0, \text{ for all } \varphi \in \mathcal{E}_{A_1}(H),$$

and ν is absolutely continuous with respect to μ from Example 3.5.1 and for $\rho = \frac{d\nu}{d\mu}$

we have that $\rho \in W^{1,2}(H,\mu)$ and $D\log\rho \in W^{1,2}(H,\nu;H)$.

Then by [Tr2, Section 4.2], we know that the bilinear form on $L^2(H;\nu)$ associated with K_0 is a generalized Dirichlet form in the sense of Remark 3.1.1 (iii) with $(E_1, \mathcal{B}(E_1), m) = (H, \mathcal{B}(H), \nu)$, $(\mathcal{A}, \mathcal{V}) = (0, L^2(H,\nu))$ and $\Lambda = K_0$. It satisfies conditions (A1)-(A5). There are even more general conditions on F and A_1 which can be found in [DRW09, Theorem 5.2] such that conditions (A1)-(A5) hold.

The following example is given in [Tr2, Section 4.2].

Example 3.5.4 Assume that E is a separable real Hilbert space with inner product $\|\cdot\|_E^{1/2}$ and $H \subset E$ densely by a Hilbert-Schmidt map. Let $B : E \to E$ be a Borel measurable vector field satisfying the following conditions:

(B.1) $\lim_{\|z\|_E\to\infty}\langle B(z), z\rangle = -\infty$.

(B.2) ${}_{E'}\langle l, B\rangle_E : E \to \mathbb{R}$ is weakly continuous for all $l \in E'$.

(B.3) There exist $C_1, C_2, d \in (0,\infty)$, such that $\|B(z)\|_E \le C_1 + C_2\|z\|_E^d$.

Then by [BRS00, Theorem 5.2] there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that ${}_{E'}\langle l, B\rangle_E \in L^2(E;\mu)$ for all $l \in E'$ and such that

$$\int \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle \nabla u, B\rangle_E d\mu = 0 \text{ for all } u \in \mathcal{F}C_b^\infty,$$

where Δ_H is the Gross-Laplacian, i.e., $\Delta_H u = \sum_{i,j=1}^m \frac{\partial f}{\partial x_i \partial x_j}(l_1(z), .., l_m(z))\langle l_i, l_j\rangle_H$ for $u = f(l_1, ..., l_m) \in \mathcal{F}C_b^\infty$. Assume $B(z) = -z + v(z), v : E \to H$. For the bilinear form associated with $Lu = \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle \nabla u, B\rangle_E, u \in \mathcal{F}C_b^\infty$ on $L^2(E,\mu)$ is a generalized Dirichlet form in the sense of Remark 3.1.1 (iii) with $(E_1, \mathcal{B}(E_1), m) = (H, \mathcal{B}(H), \nu)$, $(\mathcal{A}, \mathcal{V}) = (0, L^2(H,\nu))$ and $\Lambda = L$. It satisfies conditions (A1)-(A5).

3.6 A control problem

In this section, we assume conditions (A1)-(A5) and $\|A^{1/2}(\cdot)\|_{L^\infty(H)}, \|A^{-1/2}(\cdot)\|_{L^\infty(H)} \in L^\infty(E,\mu)$ and consider the case $l = 1$.

Proposition 3.6.1 Let (β,γ) be an $(\mathbb{R}, H)$-valued predictable process, $\varphi \in \cap_{x\in\mathcal{N}^c} M_x^2(\mathbb{R})$ and $\xi \in \cap_{x\in\mathcal{N}^c} L^2(P^x)$. β is assumed to be bounded from above, $|\gamma|_H$ is bounded and $\xi \in \mathcal{F}_T$. Then the linear BSDE

$$-dY_t = [\varphi_t + Y_t\beta_t + \langle A^{1/2}(X_t)Z_t, \gamma_t\rangle_H]dt - Z_t.dM_t, \qquad Y_T = \xi,$$

has a solution (Y, Z) in $M_x^2(\mathbb{R}) \times M_x^2(H), \forall x \in \mathcal{N}^c$, where $M_x^2(\mathbb{R})$ and $M_x^2(H)$ are defined in the appendix. Moreover, Y_t is given by the closed formula

$$Y_t = E^x[\xi\Gamma_T^t + \int_t^T \Gamma_s^t \varphi_s ds|\mathcal{F}_t] \ P^x - a.s., \forall x \in \mathcal{N}^c,$$

where Γ_s^t is the adjoint process defined for $s \geq t$ by the forward linear SDE

$$d\Gamma_s^t = \Gamma_s^t[\beta_s ds + A^{-1/2}(X_s)\gamma_s.dM_s], \qquad \Gamma_t^t = 1.$$

In particular, if ξ and φ are nonnegative, the process Y is nonnegative. If, in addition, $Y_0 = 0$, then, for any t, $Y_t = 0$ a.s., $\xi = 0$ a.s., and $\varphi_t = 0 \ dP^x \otimes dt$ a.s..

Proof By the same arguments as in [BPS02, Lemma 7.1] the assertion follows. □

Theorem 3.6.2 (Comparison Theorem). Let $x \in A_2$. Let (f^1, ξ^1) and (f^2, ξ^2) be two standard parameters of BSDE (3.4.1), where f^1, f^2 satisfy conditions $(\Omega 1)-(\Omega 3)$ and

$$f^i(s, \omega, y, z) := h_i(s, \omega) + \langle A^{1/2}(X_s)c_i(s, \omega), z\rangle_H, \text{ for } i = 1, 2,$$

where $(h_i, c_i), i = 1, 2$ are bounded $(\mathbb{R}, H)$-valued predictable processes. Let (Y^1, Z^1) and (Y^2, Z^2) be the associated square-integrable solutions. We suppose that

(a) $\xi^1 \geq \xi^2 \ P^x$-a.s.

(b)$\delta_2 f_t = f^1(t, Y_t^2, A^{1/2}(X_t)Z_t^2) - f^2(t, Y_t^2, A^{1/2}(X_t)Z_t^2) \geq 0$, $dP^x \otimes dt$-a.s.

Then we have that P^x-almost surely: $Y_t^1 \geq Y_t^2$, for all $t \geq 0$.

Proof The pair $(\delta Y, \delta Z)$ is the solution of the following linear BSDE:

$$-d\delta Y_t = [\langle \Delta_z f(t), A^{1/2}(X_t)\delta Z_t\rangle_H + \delta_2 f_t]dt - \delta Z_t.dM_t, \qquad \delta Y_T = \xi^1 - \xi^2,$$

$$\langle \Delta_z f(t), e_i\rangle_H := \langle A^{1/2}(X_s)c_1, e_i\rangle_H,$$

if $\langle A^{1/2}(X_t)(Z_t^1 - Z_t^2), e_i\rangle_H$ is not equal to 0, and $\langle \Delta_z f(t), e_i\rangle_H := 0$, otherwise. Then by the same arguments as in [BPS02, Theorem 7.2] and using Proposition 3.6.1, the assertion follows. □

Now we consider a control problem associated to the Markov process X. An admissible control is a process $\theta(t, \omega)$ which is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and which takes values in a compact subset K of some metric space. We denote by Θ the class of admissible controls.

A bounded measurable function $c : [0, T] \times E \times K \to H$ is given and we suppose that it is continuous with respect of the last variable. For a given admissible control θ we define $N_t^\theta = \int_0^t c_s(X_s, \theta_s).dM_s, \Gamma_t^\theta = \exp(N_t^\theta - \frac{1}{2}\langle N^\theta\rangle_t)$, and $P^{\theta,x} = \Gamma^\theta.P^x$. The payoff function of the control problem is defined as

$$J^\theta(x) = E^{\theta,x}[\phi(X_T) + \int_0^T h(s, X_s, \theta_s)ds],$$

where ϕ and h are bounded measurable functions and h is continuous in θ. One wants to minimize the payoff function, that is to calculate the value function

$$J^*(x) = \inf_{\theta\in\Theta} J^\theta(x),$$

and to find an optimal control θ^*, that is an admissible control such that $J^*(x) = J^{\theta^*}(x)$.

In what follows we restrict our analysis to points $x \in \mathcal{N}^c$. We calculate $J^\theta(x)$ by solving the BSDE

$$Y_t^\theta = \phi(X_T) + \int_t^T g_s(X_s, A^{1/2}(X_s)Z_s^\theta, \theta_s)ds - \int_t^T Z_s^\theta.dM_s, \tag{3.6.1}$$

where $g : [0, T] \times E \times H \times K \to \mathbb{R}$ is the Hamiltonian defined by

$$g(s, x, z, \theta) = h(s, x, \theta) + 2\langle A^{1/2}(x)c(s, x, \theta), z\rangle_H.$$

Then by Itô's formula and the same arguments as in [BPS02, Section 7] we have

$$J^\theta(x) = Y_0^{\theta,x},$$

where $Y_0^{\theta,x}$ is the initial value of the solution of the preceding equation (3.6.1) under P^x.

In order to calculate the value function and to produce the optimal control we have to solve the following BSDE

$$Y_t^* = \phi(X_T) + \int_t^T g^*(s, X_s, A^{1/2}(X_s)Z_s^*)ds - \int_t^T Z_s^*.dM_s, \tag{3.6.2}$$

where $g^*(s, x, z) = \inf_{\theta\in K} g(s, x, z, \theta)$.

It is easy to check that $z \to g^*(s, x, z)$ is Lipschitz continuous, so that there exists a unique solution (Y^*, Z^*) of equation (3.6.2). Then we know the initial value of the solution is a constant: $Y_0^{*,x} = E^x Y_0^*$.

Since g is continuous as a function of θ and K is a compact set, the infimum is attained at a point θ^* and one may choose a measurable function $(s, x, z) \to \theta^*(s, x, z)$ which realizes the infimum. We construct the optimal control in the following way: $\theta_s^* := \theta^*(s, \omega) = \theta^*(s, X_s(\omega), Z_s^*(\omega))$.

Corollary 3.6.3 Under the above hypotheses $J^*(x) = Y_0^{*,x}$ and θ^* is an optimal control.

Proof Using Theorem 3.6.2 and by the same arguments as in [BPS02, Corollary 7.3], the assertion follows. □

We now may interpret the solution of the Hamilton Jacobi Bellman equation as the value function of the above control problem. The HJB equation is

$$(\partial_t + L)u + g^*(t, x, D_{A^{1/2}}u) = 0, u(T, x) = \phi.$$

We have proved that this equation admits a unique solution in the sense of mild equations and it satisfies $u(0, x) = Y_0^{*,x} = J^*(x)$.

3.7 Appendix

For $p \geq 1$, let $M_x^p(\mathbb{R}^l)$ denote the set of all (equivalent classes of) predictable processes $\{\phi_t\}_{t\in[0,T]}$ with values in $\mathbb{R}^l$ such that

$$\|\phi\|_{M_x^p} = (E^x[(\int_0^T |\phi_r|^2 dr)^{p/2}])^{1/p} < \infty.$$

Let $M_x^p(H^l)$ denote the set of all (equivalent classes of) predictable processes $\{\phi_t\}_{t\in[0,T]}$ with values in H^l such that

$$\|\phi\|_{M_x^p} = (E^x[(\int_0^T |\phi_r|_H^2 dr)^{p/2}])^{1/p} < \infty.$$

$\mathcal{S}_x^p(\mathbb{R}^l)$ denotes the set of all $\mathbb{R}^l$-valued, adapted and càdlàg processes $\{\phi_t\}_{t\in[0,T]}$ such that

$$\|\phi\|_{\mathcal{S}_x^p(\mathbb{R}^l)} = E^x[\sup_t |X_t|^p]^{1/p} < \infty.$$

Let ν be a measure supported by $\mathcal{N}^c$.

Lemma A.1 Let $\{K_t\}_{t\in[0,T]}$ and $\{H_t\}_{t\in[0,T]}$ be two progressively measurable processes with values in $\mathbb{R}^l$ and H^l respectively, such that P^ν-a.s.

$$\int_0^T (|K_t| + |A^{1/2}(X_t)H_t|_H^2)dt < \infty.$$

We consider the $\mathbb{R}^l$-valued semimartingale $\{Y_t\}_{t\in[0,T]}$ defined by

$$Y_t = Y_0 + \int_0^t K_s ds + \int_0^t H_s.dM_s.$$

Then, for any $p \geq 1$, we have

$$\begin{aligned}|Y_t|^p - 1_{p=1}L_t =&|Y_0|^p + p\int_0^t |Y_s|^{p-1}\langle \hat{Y}_s, K_s\rangle ds + p\int_0^t |Y_s|^{p-1}\langle \hat{Y}_s, H_s.dM_s\rangle \\ &+ \frac{p}{2}\int_0^t |Y_s|^{p-2}1_{Y_s\neq 0}\{(2-p)(|A^{1/2}(X_s)H_s|_H^2 \\ &- \sum_{j,k=1}^{l}\langle A(X_s)H_s^j, H_s^k\rangle_H Y_s^j Y_s^k/|Y_s|^2) + (p-1)|A^{1/2}(X_s)H_s|_H^2\}ds,\end{aligned}$$

where $\{L_t\}_{t\in[0,T]}$ is a continuous, increasing process with $L_0 = 0$, which increases only on the boundary of the random set $\{t \in [0,T], Y_t = 0\}$.

Proof We consider the function $u_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1/2}$. We have

$$\nabla u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)x, \qquad D^2 u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)I + p(p-2)u_\varepsilon^{p-4}(x)(x\otimes x).$$

Then Itô's formula leads to the equality,

$$\begin{aligned}u_\varepsilon^p(Y_t) =&u_\varepsilon^p(Y_0) + p\int_0^t u_\varepsilon^{p-2}(Y_s)\langle Y_s, K_s\rangle ds + p\int_0^t u_\varepsilon^{p-2}(Y_s)\langle Y_s, H_s.dM_s\rangle \\ &+ \frac{1}{2}\sum_{j,k}\int_0^t D_{jk}^2 u_\varepsilon^p(Y_s)\langle A(X_s)H_s^j, H_s^k\rangle_H ds.\end{aligned}$$

It remains to pass to the limit when $\varepsilon \to 0$ in this identity. We have

$$\int_0^t u_\varepsilon^{p-2}(Y_s)\langle Y_s, K_s\rangle ds \to \int_0^t |Y_s|^{p-1}\langle \hat{Y}_s, K_s\rangle ds.$$

and uniformly on $[0,T]$ in P^ν-probability,

$$\int_0^t u_\varepsilon^{p-2}(Y_s)\langle Y_s, H_s.dM_s\rangle \to \int_0^t |Y_s|^{p-1}\langle \hat{Y}_s, H_s.dM_s\rangle.$$

This convergence of stochastic integrals follows from the following convergence

$$\int_0^T |Y_r|^2 1_{Y_r\neq 0}|A^{1/2}(X_s)H_r|_H^2(|Y_r|^{p-2} - u_\varepsilon^{p-2}(Y_r))^2 dr \to 0,$$

which is clear from the dominated convergence theorem.

We have

$$\begin{aligned}
&\sum_{j,k} D^2_{jk} u^p_\varepsilon(Y_s)\langle A(X_s)H^j_s, H^k_s\rangle_H \\
=&p(2-p)(|Y_s|u_\varepsilon^{-1}(Y_s))^{4-p}|Y_s|^{p-2}1_{Y_s\neq 0}(|A^{1/2}(X_s)H_s|^2_H \\
&-\sum_{j,k=1}^{l}\langle A(X_s)H^j_s, H^k_s\rangle_H Y^j_s Y^k_s/|Y_s|^2) \\
&+p(p-1)(|Y_s|u_\varepsilon^{-1}(Y_s))^{4-p}|Y_s|^{p-2}1_{Y_s\neq 0}|A^{1/2}(X_s)H_s|^2_H + C^\varepsilon_s(p),
\end{aligned}$$

where $C^\varepsilon_s(p) = p\varepsilon^2|A^{1/2}(X_s)H_s|^2_H u_\varepsilon^{p-4}(Y_s)$. Furthermore, one has

$$|A^{1/2}(X_s)H_s|^2_H \geq \sum_{j,k=1}^{l}\langle A(X_s)H^j_s, H^k_s\rangle_H Y^j_s Y^k_s/|Y_s|^2.$$

Moreover,

$$\frac{|Y_s|}{u_\varepsilon(Y_s)} \uparrow 1_{Y_s\neq 0},$$

as $\varepsilon \to 0$. Hence by monotone convergence, as $\varepsilon \to 0$,

$$\begin{aligned}
&\int_0^t (|Y_s|u_\varepsilon^{-1}(Y_s))^{4-p}|Y_s|^{p-2}1_{Y_s\neq 0}\{(2-p)(|A^{1/2}(X_s)H_s|^2_H \\
&-\sum_{j,k=1}^{l}\langle A(X_s)H^j_s, H^k_s\rangle_H Y^j_s Y^k_s/|Y_s|^2) + (p-1)|A^{1/2}(X_s)H_s|^2_H\}ds
\end{aligned}$$

converges to

$$\begin{aligned}
&\int_0^t |Y_s|^{p-2}1_{Y_s\neq 0}\{(2-p)(|A^{1/2}(X_s)H_s|^2_H - \sum_{j,k=1}^{l}\langle A(X_s)H^j_s, H^k_s\rangle_H Y^j_s Y^k_s/|Y_s|^2) \\
&+(p-1)|A^{1/2}(X_s)H_s|^2_H\}ds,
\end{aligned}$$

P^ν-a.s. for all $0 \leq t \leq T$. By the same arguments as in [BDHPS03, Lemma 2.2], we have $\{L^\varepsilon_t(p) := \int_0^t C^\varepsilon_s(p)ds\}_{t\in[0,T]}$ converges, as $\varepsilon \to 0$, to a continuous increasing process $\{L_t(p)\}_{t\in[0,T]}$, and $L_t(p) \equiv 0$ for $p \neq 1$. Furthermore, $L_t(1)$ increases only on the boundary of the random set $\{t \in [0,T], Y_t = 0\}$. Now the assertions follows. □

Corollary A.2 If (Y, Z) is a solution of the BSDE, $p \geq 1$, $c(p) = p[(p-1)\wedge 1]/2$ and $0 \leq t \leq u \leq T$, then

$$|Y_t|^p + c(p)\int_t^u |Y_s|^{p-2}1_{Y_s\neq 0}|A^{1/2}(X_s)Z_s|^2_H ds$$

$$
\begin{aligned}
&\leq |Y_u|^p + p\int_t^u |Y_s|^{p-1}\langle \hat{Y}_s, f(s, Y_s, A^{1/2}(X_s)Z_s)\rangle ds \\
&- p\int_t^u |Y_s|^{p-1}\langle \hat{Y}_s, Z_s.dM_s\rangle.
\end{aligned}
$$

Now we state some estimates concerning solutions to the BSDE. In the following we assume that $p > 1$ and make use of the following assumption

$$
\text{(A)}: \qquad \forall (t, y, z) \in [0, T] \times \mathbb{R}^l \otimes H^l, \qquad \langle \hat{y}, f(t, y, z)\rangle \leq f_t + \mu|y| + C|z|, \; P^\nu - a.s.,
$$

where $\mu \in \mathbb{R}, C \geq 0$ and $\{f_t\}_{t\in[0,T]}$ is a non-negative progressively measurable process. Let us set $F := \int_0^T f_r dr$.

Lemma A.3 Suppose assumption (A) holds and that for some $p > 0$, F^p is integrable. Let (Y, Z) be a solution to the BSDE. If $Y \in \mathcal{S}_\nu^p$, then $Z \in \mathcal{M}_\nu^p$ and there exists a constant C_p depending only on p such that for every $a \geq \mu + C^2$,

$$
E^\nu[(\int_0^T e^{2at}|A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2}] \leq C_p E^\nu[\sup_t e^{apt}|Y_t|^p + (\int_0^T e^{ar} f_r dr)^p].
$$

Proof We note that (Y, Z) solves the BSDE with data (ξ, f) iff

$$
(\bar{Y}_t, \bar{Z}_t) := (e^{at}Y_t, e^{at}Z_t)
$$

solves the BSDE with data $(e^{aT}\xi, f')$, where

$$
f'(t, y, z) := e^{at} f(t, e^{-at}y, e^{-at}z) - ay.
$$

We restrict ourselves to the case that $a = 0$ and $\mu + C^2 \leq 0$. For each integer $n \geq 1$, let us introduce the stopping time

$$
\tau_n := \inf\{t \in [0, T], \int_0^t |A^{1/2}(X_r)Z_r|_H^2 dr \geq n\} \wedge T.
$$

By Itô's formula we get

$$
|Y_0|^2 + \int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr = |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} \langle Y_r, f(r, Y_r, Z_r)\rangle dr - 2\int_0^{\tau_n} \langle Y_r, Z_r.dM_r\rangle.
$$

By (A) and since $\mu + C^2 \leq 0$, we have

$$
2\langle y, f(r, y, z)\rangle \leq 2|y|f_r + 2\mu|y|^2 + 2C^2|y|^2 + |z|^2/2 \leq 2|y|f_r + |z|^2/2.
$$

Hence we deduce that

$$\frac{1}{2}\int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr \leq Y_*^2 + 2Y_* \int_0^T f_r dr + 2|\int_0^{\tau_n} \langle Y_r, Z_r.dM_r\rangle|,$$

where Y_* denotes $\sup_{t\in[0,T]} |Y_t|$, and thus

$$(\int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2} \leq c_p(Y_*^p + (\int_0^T f_r dr)^p + |\int_0^{\tau_n} \langle Y_r, Z_r.dM_r\rangle|^{p/2}).$$

By the BDG inequality, we get

$$\begin{aligned} c_p E^\nu[|\int_0^{\tau_n} \langle Y_r, Z_r.dM_r\rangle|^{p/2}] \leq & d_p E^\nu[(\int_0^{\tau_n} |Y_r|^2 |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/4}] \\ \leq & d_p E^\nu[|Y_*|^{p/2}(\int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/4}] \\ \leq & \frac{d_p^2}{2} E^\nu[Y_*^p] + \frac{1}{2} E^\nu[(\int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2}]. \end{aligned}$$

Furthermore, we have

$$E^\nu[(\int_0^{\tau_n} |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2}] \leq C_p E^\nu[Y_*^p + (\int_0^T f_r dr)^p].$$

Letting $n \to \infty$ we obtain

$$E^\nu[(\int_0^T |A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2}] \leq C_p E^\nu[Y_*^p + (\int_0^T f_r dr)^p].$$

□

Proposition A.4 Suppose that assumption (A) holds and that for some $p > 1$, F belongs to L^p. Let (Y, Z) be a solution to the BSDE where Y belongs to $\mathcal{S}_\nu^p$. Then there exists a constant C_p, depending only on p, such that for every $a \geq \mu + C^2/[1 \wedge (p-1)]$,

$$E^\nu[\sup_t e^{apt}|Y_t|^p + (\int_0^T e^{2at}|A^{1/2}(X_r)Z_r|_H^2 dr)^{p/2}] \leq C_p E^\nu[e^{apT}|\xi|^p + (\int_0^T e^{ar} f_r dr)^p].$$

Proof We restrict ourselves the proof to the case $a = 0$ and $\mu + C^2/[1\wedge(p-1)] \leq 0$; By Corollary A.2 and $\langle \hat{y}, f(r, y, z)\rangle \leq f_r + \mu|y| + C|z|$, we have

$$|Y_t|^p + c(p)\int_t^T |Y_r|^{p-2} 1_{Y_s\neq 0} |A^{1/2}(X_r)Z_r|_H^2 dr$$

$$\leq |\xi|^p + p\int_t^T |Y_r|^{p-1} f_r + \mu |Y_r|^p + C|Y_r|^{p-1}|A^{1/2}(X_r)Z_r|_H dr - p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, Z_r.dM_r\rangle.$$

Now by the same arguments as in [BDHPS03, Proposition 3.2] the assertion follows.
□

Proof of Proposition 3.4.3 We consider P^x for $x \in A_p$.

We note that (Y, Z) solves the BSDE with data (ξ, f) iff

$$(\bar{Y}_t, \bar{Z}_t) := (e^{\alpha_t} Y_t, e^{\alpha_t} Z_t)$$

solves the BSDE with data $(e^{\alpha_T}\xi, f')$, where

$$f'(t, y, z) := e^{\alpha_t} f(t, e^{-\alpha_t} y, e^{-\alpha_t} z) - \mu_t y.$$

We replace $(\Omega 2)$ by the condition that for t, ω, y, y', z arbitrary,

$$\langle y - y', f(t, \omega, y, z) - f(t, \omega, y', z)\rangle \leq 0.$$

Step 1. We assume that f is Lipschitz continuous with respect to both y and z, and that f, ξ are bounded. We define a mapping Φ from $B_x^2 := M_x^2(\mathbb{R}^l) \times M_x^2(H^l)$ into itself as follows. Given $(U, V) \in B_x^2$, $\Phi(U, V) := (Y, Z)$, where (Y, Z) is the solution of the BSDE (3.4.1) associated with the data $(\xi, f(U, A^{1/2}(X)V))$ by Lemma 3.4.1. As

$$\langle \int_0^{\cdot} \phi_s.dM_s\rangle_t = \int_0^t \langle A(X_s)\phi_s, \phi_s\rangle_H ds,$$

by Itô's formula and the BDG inequality, we have

$$E^x[\sup_{t\in[0,T]} |Y_t|^2] < \infty.$$

Let $(U, V), (U', V') \in B_x^2$, $(Y, Z) = \Phi(U, V)$, $(Y', Z') = \Phi(U', V')$, $(\bar{U}, \bar{V}) = (U - U', V - V')$, $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. It follows from Itô's formula that for each $\gamma \in \mathbb{R}$,

$$\begin{aligned}
&e^{\gamma t} E^x |\bar{Y}_t|^2 + E^x \int_t^T e^{\gamma s}(\gamma |\bar{Y}_s|^2 + |A^{1/2}(X_s)\bar{Z}_s|_H^2) ds \\
\leq &2K E^x \int_t^T e^{\gamma s} |\bar{Y}_s|(|\bar{U}_s| + |A^{1/2}(X_s)\bar{V}_s|_H) ds \\
\leq &4K^2 E^x \int_t^T e^{\gamma s}|\bar{Y}_s|^2 + \frac{1}{2} E^x \int_t^T e^{\gamma s}(|\bar{U}_s|^2 + |A^{1/2}(X_s)\bar{V}_s|_H^2) ds,
\end{aligned}$$

where K is the Lipschitz constant of f. We choose $\gamma = 1 + 4K^2$. Then

$$E^x \int_0^T e^{\gamma s}(|\bar{Y}_s|^2 + |A^{1/2}(X_s)\bar{Z}_s|_H^2)ds \leq \frac{1}{2}E^x \int_0^T e^{\gamma s}(|\bar{U}_s|^2 + |A^{1/2}(X_s)\bar{V}_s|_H^2)ds,$$

from which it follows that Φ is a strict contraction on B_x^2 equipped with the norm:

$$|||(Y,Z)|||_\gamma^x = (E^x \int_0^T e^{\gamma t}(|Y_t|^2 + |A^{1/2}(X_t)Z_t|_H^2)dt)^{1/2}.$$

We define a sequence (Y^n, Z^n) by $(Y^{n+1}, Z^{n+1}) = \Phi(Y^n, Z^n)$. For $\gamma = 1 + 4K^2$, we have

$$\begin{aligned} &E^x \int_0^T e^{\gamma s}(|Y_s^n - Y_s^{n+1}|^2 + |A^{1/2}(X_s)(Z_s^n - Z_s^{n+1})|_H^2)ds \\ \leq &\frac{1}{2^n}E^x \int_0^T e^{\gamma s}(|Y_s^0 - Y_s^1|^2 + |A^{1/2}(X_s)(Z_s^0 - Z_s^1)|_H^2)ds. \end{aligned}$$

Then for any $x \in A^2$ we have the a.e. pointwise convergence of (Y_s^n, Z_s^n) under P^x. We denote the limit by (Y_s, Z_s). Then it is the fixed point of Φ under the norm $|||(Y,Z)|||_\gamma^x$. So we have (Y_s, Z_s) is the solution of the BSDE.

Step 2. We assume f, ξ are bounded.

We need the following proposition.

Proposition A.5 Suppose f, ξ are bounded. Given $V_t \in \cap_x M_x^2(H^l)$, there exists a unique pair of predictable processes $(Y_t, Z_t) \in M_x^2 \times M_x^2(H^l), \forall x \in \mathcal{N}^c$, satisfying for all P^x, $x \in \mathcal{N}^c$

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dM_s, \qquad 0 \leq t \leq T.$$

Using Proposition A.5, we construct a mapping Φ from B_x^2 into itself as follows. For any $(U,V) \in B_x^2$, $(Y,Z) = \Phi(U,V)$ is the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dM_s, \qquad 0 \leq t \leq T.$$

Then as in **Step 1**, we have

$$e^{\gamma t}E^x|\bar{Y}_t|^2 + E^x \int_t^T e^{\gamma s}(\gamma|\bar{Y}_s|^2 + |A^{1/2}(X_s)\bar{Z}_s|_H^2)ds$$

$$
\begin{aligned}
&=2E^x\int_t^T e^{\gamma s}\langle \bar{Y}_s, f(Y_s, A^{1/2}(X_s)V_s) - f(Y_s', A^{1/2}(X_s)V_s')\rangle ds\\
&\leq 2KE^x\int_t^T e^{\gamma s}|\bar{Y}_s| \times |A^{1/2}(X_s)\bar{V}_s|_H ds\\
&\leq E^x\int_t^T e^{\gamma s}(2K^2|\bar{Y}_s|^2 + \frac{1}{2}|A^{1/2}(X_s)\bar{V}_s|_H^2)ds.
\end{aligned}
$$

Then by the same argument as in Step 1, the assertion of Proposition 3.4.3 follows, if f, ξ are bounded.

Proof of Proposition A.5 We shall write $f(s,y)$ for $f(s,y,V_s)$ and $|\xi|^2 + \sup_t |f(t,0)|^2 \leq C$ a.s.. We define

$$f^n(t,y) := (\rho_n * f(t,\cdot))(y),$$

where $\rho_n : \mathbb{R}^l \mapsto \mathbb{R}^+, n \in \mathbb{N}$ is a sequence of smooth functions with compact support which approximates the Dirac measure at 0, satisfying $\int \rho_n(z)dz = 1$. Then f^n is locally Lipschitz in y, uniformly with respect to s and ω.

Define for each $m \in \mathbb{N}$

$$f^{n,m}(t,y) := f^n(t, \frac{\inf(m,|y|)}{|y|}y).$$

Then $f^{n,m}$ is globally Lipschitz and bounded, uniformly w.r.t. (t,ω). As in Step 1, we have a unique pair $(Y_t^{n,m}, Z_t^{n,m}) \in M_x^2(\mathbb{R}^l) \times M_x^2(H^l)$ such that

$$Y_t^{n,m} = \xi + \int_t^T f^{n,m}(s, Y_s^{n,m})ds - \int_t^T Z_s^{n,m}.dM_s, \qquad 0 \leq t \leq T.$$

By Itô's formula, we have

$$|Y_t^{n,m}|^2 \leq e^T C, \qquad 0 \leq t \leq T.$$

Consequently, for $m^2 > e^T C$, $(Y_t^{n,m}, Z_t^{n,m})$ does not depend on m. Therefore, we denote it by (Y_t^n, Z_t^n). $(Y^l - Y^k, Z^l - Z^k)$ is the solution of BSDE associated with $(f^l(t, y + Y_t^k) - f^k(Y_t^k), 0)$. Hence by Proposition A.4, we have

$$E^x(\sup_{0\leq t\leq T}|Y_t^k - Y_t^l|^2) + E^x(\int_0^T |A^{1/2}(X_t)(Z_t^k - Z_t^l)|_H^2 dt) \leq KE^x[\int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt].$$

For fixed (t,ω) we have,

$$\sup_{k>l}\int_0^T |f^k(t,Y_t^k) - f^l(t,Y_t^k)|^2 dt \to 0, \text{ as } l \to \infty.$$

Then we obtain

$$\sup_{k>l} E^x \int_0^T |f^k(t,Y_t^k)-f^l(t,Y_t^k)|^2 dt \leq E^x \sup_{k>l} \int_0^T |f^k(t,Y_t^k)-f^l(t,Y_t^k)|^2 dt \to 0, \text{ as } l \to \infty.$$

Therefore, we obtain a sequence of representable variables that converges rapidly enough under all measures $P^x, x \in \mathcal{N}^c$. For each $l = 0, 1, ...$ set

$$n_l(x) = \inf\{n > n_{l-1}(x); \sup_{k\geq n} E^x[\int_0^T |f^k(t,Y_t^k) - f^n(t,Y_t^k)|^2 dt] < \frac{1}{2^l}\}.$$

$$\bar{Y}^l = Y^{n_l(X_0)}, \bar{Z}^l = Z^{n_l(X_0)}.$$

With this sequence one may pass to the limit and define $Z_s := \limsup_{l\to\infty} \bar{Z}_s^l$ and the assertion follows.. □

Now we continue the proof of Proposition 3.4.3.

Step 3. Now we assume that ξ and $\sup_t |f_t^0|$ are bounded random variables. Let r be a positive real number such that

$$\sqrt{e^(1 + C^2)T}(\|\xi\|_\infty + T\|f^0\|_\infty) < r.$$

Let θ_r be a smooth function such that $0 \leq \theta_r \leq 1, \theta_r(y) = 1$ for $|y| \leq r$ and $\theta_r(y) = 0$, if $|y| \geq r+1$. For $n \in \mathbb{N}$, we set $q_n(z) := z\frac{n}{|z|_H \vee n}$ and

$$h^n(t,y,z) := \theta_r(y)(f(t,y,q_n(z)) - f_t^0)\frac{n}{f^{\prime,r+1} \vee n} + f_t^0.$$

By [BDHPS03, Theorem 4.2], we have that each h_n satisfies $(\Omega 2)$ with a positive constant and (ξ, h^n) are bounded. The BSDE associated to (ξ, h^n) has a unique solution (Y^n, Z^n) in the space $\mathcal{S}_x^2 \times \mathcal{M}_x^2(H^l)$. By the same arguments as in [Pa99, Proposition 2.4] we have $\|Y^n\|_\infty < r$. By Proposition A.4, $\|Z^n\|_{\mathcal{M}_x^2(H^l)} \leq r'$. Hence (Y^n, Z^n) is a solution to the BSDE associated to (ξ, f_n) where

$$f^n(t,y,z) := (f(t,y,q_n(z)) - f_t^0)\frac{n}{f^{\prime,r+1} \vee n} + f_t^0.$$

Since $(Y^{n+i} - Y^n, Z^{n+i} - Z^n)$ is the solution of BSDE associated with $(f^{n+i}(t, y + Y_t^n, z + Z_t^n) - f^n(Y_t^n, Z_t^n), 0)$ and

$$\begin{aligned}
&\langle y, f^{n+i}(t, y+Y_t^n, z+Z_t^n) - f^n(t, Y_t^n, Z_t^n)\rangle \\
=&\langle y, f^{n+i}(t, y+Y_t^n, z+Z_t^n) - f^{n+i}(t, Y_t^n, Z_t^n)\rangle + \langle y, f^{n+i}(t, Y_t^n, Z_t^n) - f^n(t, Y_t^n, Z_t^n)\rangle \\
\leq& C|y||z| + |y||f^{n+i}(t, Y_t^n, Z_t^n) - f^n(t, Y_t^n, Z_t^n)|,
\end{aligned}$$

by Proposition A.4 we have

$$
\begin{aligned}
&E^x(\sup_{0\le t\le T}|Y_t^{n+i}-Y_t^n|^2)+E^x(\int_0^T|A^{1/2}(X_t)(Z_t^{n+i}-Z_t^n)|_H^2dt)\\
\le& KE^x[\int_0^T|f^{n+i}(t,Y_t^n,Z_t^n)-f^n(t,Y_t^n,Z_t^n)|^2dt].
\end{aligned}
$$

Since $\|Y^n\|_\infty\le r$, we have

$$
|f^{n+i}(t,Y_t^n,Z_t^n)-f^n(t,Y_t^n,Z_t^n)|\le 2C|Z_t^n|_H1_{|Z_t^n|_H>n}+2C|Z_t^n|_H1_{f^{\prime,r+1}>n}+2f^{\prime,r+1}(t)1_{f^{\prime,r+1}>n},
$$

and the above formula converges to 0, uniformly in i as $n\to\infty$. Follow the same arguments as in the proof of Proposition A.5, the assertion follows in this case.

Step 4. Consider the general case. For each $n\in\mathbb{N}$, let us define

$$
\xi^n:=q_n(\xi),\qquad f^n(t,y,z):=f(t,y,z)-f_t^0+q_n(f_t^0).
$$

For each pair (ξ^n,f^n), the BSDE has a unique solution (Y^n,Z^n) in L^2 by Step 3. By Proposition A.4, we have

$$
\begin{aligned}
&E^x(\sup_{0\le t\le T}|Y_t^{n+i}-Y_t^n|^p)+E^x(\int_0^T|A^{1/2}(X_t)(Z_t^{n+i}-Z_t^n)|_H^2dt)^{p/2}\\
\le& K_1E^x[|\xi^{n+i}-\xi^n|^p+(\int_0^T|q_{n+i}(f_t^0)-q_n(f_t^0)|dt)^p].
\end{aligned}
$$

The right hand side of the last inequality clearly tends to 0, as $n\to\infty$, uniformly in i and the assertion follows. $\square$

Chapter 4

Stochastic quasi-geostrophic equation

In this chapter, we study the 2D stochastic quasi-geostrophic equation in $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and multiplicative noise. We prove it is uniquely ergodic provided the noise is non-degenerate for $\alpha > \frac{2}{3}$. In this case, the convergence to the (unique) invariant measure is exponentially fast. In the general case, we prove the existence of Markov selections. In Section 4.1, we introduce some notations and preliminaries for quasi-geostrophic equation. In Section 4.2, we prove the existence of Markov selections for the solution of the stochastic quasi-geostrophic equation with $\alpha \in (0,1)$. In Section 4.3, we prove the Markov semigroup associated with the solution of the stochastic quasi-geostrophic equation is strong Feller and irreducible if the noise is non-degenerate. Furthermore, it is strongly mixing. In Section 4.4 we prove that the convergence to the (unique) invariant measure is exponentially fast. In Section 4.5, we prove the above results if the noise is mildly degenerate. The main results of this chapter have already been submitted for publication, see [RZZ12].

4.1 Notations and Preliminaries

We consider the usual abstract form of equations (1.7)-(1.9). In the following, we will restrict ourselves to flows which have zero average on the torus, i.e.

$$\int_{\mathbb{T}^2} \theta d\xi = 0.$$

Thus (1.9) can be restated as

$$u = (-\frac{\partial\psi}{\partial\xi_2}, \frac{\partial\psi}{\partial\xi_1}) \text{ and } (-\triangle)^{1/2}\psi = -\theta.$$

Set $H = \{f \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f d\xi = 0\}$ and let $|\cdot|$ and $\langle .,. \rangle$ denote the norm and inner product in H respectively. On the periodic domain $\mathbb{T}^2$, $\{\sin(k\xi)|k \in \mathbb{Z}^2_+\} \cup \{\cos(k\xi)|k \in \mathbb{Z}^2_-\}$ form an eigenbasis of $-\triangle$. Here $\mathbb{Z}^2_+ = \{(k_1,k_2) \in \mathbb{Z}^2|k_2 > 0\} \cup \{(k_1,0) \in \mathbb{Z}^2|k_1 > 0\}, \mathbb{Z}^2_- = \{(k_1,k_2) \in \mathbb{Z}^2| -k \in \mathbb{Z}^2_+\}, x \in \mathbb{T}^2$, and the corresponding eigenvalues are $|k|^2$. Define

$$\|f\|^2_{H^s} = \sum_k |k|^{2s}\langle f, e_k\rangle^2$$

and let H^s denote the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. Set $\Lambda = (-\triangle)^{1/2}$. Then

$$\|f\|_{H^s} = |\Lambda^s f|.$$

By the singular integral theory of Calderón and Zygmund (cf. [St70, Chapter 3]), for any $p \in (1,\infty)$, there is a constant $C = C(p)$, such that

$$\|u\|_{L^p} \le C(p)\|\theta\|_{L^p}. \tag{4.1.1}$$

Fix $\alpha \in (0,1)$ and define the linear operator $A : D(A) = H^{2\alpha}(\mathbb{T}^2) \subset H \to H$ as $Au := \kappa(-\triangle)^\alpha u$. The operator A is positive definite and selfadjoint with the same eigenbasis as that of $-\triangle$ mentioned above. Denote the eigenvalues of A by $0 < \lambda_1 \le \lambda_2 \le \cdots$, and renumber the above eigenbasis correspondingly as e_1, e_2,.... We also set $\|u\| := |A^{1/2}u|$, then $\|\theta\|^2 \ge \lambda_1|\theta|^2$.

First we recall the following important product estimates (cf. [Re95, Lemma A.4]):

Lemma 4.1.1 Suppose that $s > 0$ and $p \in (1,\infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|g\|_{H^{s,p_2}} + \|g\|_{L^{p_3}}\|f\|_{H^{s,p_4}}), \tag{4.1.2}$$

with $p_i \in (1,\infty), i = 1,...,4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We shall use as well the following useful Sobolev inequality (cf [St70, Chapter

V]):

Lemma 4.1.2 Suppose that $q > 1, p \in [q, \infty)$ and

$$\frac{1}{p} + \frac{\sigma}{2} = \frac{1}{q}.$$

Suppose that $\Lambda^\sigma f \in L^q$, then $f \in L^p$ and there is a constant $C \geq 0$ such that

$$\|f\|_{L^p} \leq C\|\Lambda^\sigma f\|_{L^q}.$$

We consider the abstract stochastic evolution equation in place of Eqs (1.7)-(1.9),

$$\begin{cases} d\theta(t) + A\theta(t)dt + u(t) \cdot \nabla\theta(t)dt = G(\theta(t))dW(t), \\ \theta(0) = \theta_0, \end{cases} \tag{4.1.3}$$

where u satisfies (1.9) and $W(t)$ is a cylindrical Wiener process in a separable Hilbert space K defined on a probability space $(\Omega, \mathcal{F}, P)$. Here G is a mapping from H^α to $L_2(K, H)$, where $L_2(K, H)$ denote all the Hilbert-Schmidt operator from K to H.

Consider the following conditions:

(G.1) (i) $|G(\theta)|^2_{L_2(K,H)} \leq \lambda_0|\theta|^2 + \rho, \theta \in H^\alpha$, for some positive real numbers λ_0 and ρ.

(ii) If $y, y_n \in H^\alpha$ such that $y_n \to y$ in H, then $\lim_{n\to\infty} \|G(y_n)^*(v) - G(y)^*(v)\|_K = 0$ for all $v \in C^\infty(\mathbb{T}^2)$.

Remark 4.1.3 Note that, because $divu = 0$ for regular functions θ and v, we have

$$\langle u(s) \cdot \nabla(\theta(s) + \psi), \theta(s) + \psi\rangle = 0,$$

so

$$\langle u(s) \cdot \nabla\theta(s), \psi\rangle = -\langle u(s) \cdot \nabla\psi, \theta(s)\rangle.$$

4.2 Markov selections in the general case

In this section, we will use [GRZ09, Theorem 4.7] to get an almost sure Markov family $(P_x)_{x\in L^2}$ for Eq. (4.1.3). Here we use the same notations as [GRZ09]. Below we choose

$$H = \mathbb{Y} = L^2(\mathbb{T}^2)$$

and

$$\mathbb{X} = (H^{2+2\alpha})^*, \qquad \mathbb{X}^* = H^{2+2\alpha}.$$

Then $\mathbb{X}$ is a Hilbert space and $\mathbb{X}^* \subset \mathbb{Y}$ compactly. Let $\mathcal{E} = \{e_i, i \in \mathbb{N}\}$ be the orthonormal basis of H introduced in Section 4.1. We define the operator $\mathcal{A}$ as follows: for $\theta \in C^\infty(\mathbb{T}^2)$

$$\mathcal{A}(\theta) := -\kappa(-\Delta)^\alpha\theta - u \cdot \nabla\theta,$$

where u satisfies (1.9). Then by Lemma 4.2.3 below, $\mathcal{A}$ can be extended to an operator $\mathcal{A} : H \to \mathbb{X}$. For θ not in H define $\mathcal{A}(\theta) := \infty$.

Set

$$\Omega := C([0,\infty); \mathbb{X}),$$

and let $\mathcal{B}$ denote the σ-field of Borel sets of Ω and let $\mathcal{P}(\Omega)$ denote the set of all probability measures on $(\Omega, \mathcal{B})$. Define the canonical process $\xi : \Omega \to \mathbb{X}$ as

$$\xi_t(\omega) = \omega(t).$$

For each t, $\mathcal{B}_t = \sigma(\xi_s : 0 \le s \le t)$. Given $P \in \mathcal{P}(\Omega)$ and $t > 0$, let $P(\cdot|\mathcal{B}_t)(\omega)$ denote a regular conditional probability distribution of P given $\mathcal{B}_t$. In particular, $P(\cdot|\mathcal{B}_t)(\omega) \in \mathcal{P}(\Omega)$ for every $\omega \in \Omega$ and for any bounded $\mathcal{B}$-measurable function f on Ω

$$E^P[f|\mathcal{B}_t] = \int_\Omega f(y)P(dy|\mathcal{B}_t), \quad P-a.s.,$$

and there exists a P-null set $N \in \mathcal{B}_t$ such that for every ω not in N

$$P(\cdot|\mathcal{B}_t)(\omega)_{|\mathcal{B}_t} = \delta_\omega (= \text{ Dirac measure at } \omega),$$

hence

$$P(\{y : y(s) = \omega(s), s \in [0,t]\}|\mathcal{B}_t)(\omega) = 1.$$

In particular, we can consider $P(\cdot|\mathcal{B}_t)(\omega)$ as a measure on $(\Omega^t, \mathcal{B}^t)$, i.e.,

$$P(\cdot|\mathcal{B}_t)(\omega) \in \mathcal{P}(\Omega^t),$$

where $\Omega^t := C([t,\infty); \mathbb{X})$ and $\mathcal{B}^t := \sigma(\xi_s : s \ge t)$.

We say $P \in \mathcal{P}(\Omega)$ is concentrated on the paths with values in H, if there exists $A \in \mathcal{B}$ with $P(A) = 1$ such that $A \subset \{\omega \in \Omega : \xi_t(\omega) \in H, \forall t \ge 0\}$. The set of such measures is denoted by $\mathcal{P}_H(\Omega)$. The shift operator $\Phi_t : \Omega \to \Omega^t$ is defined by

$$\Phi_t(\omega)(s) = \omega(s-t), \ s \ge t.$$

Following [GRZ09, Definitions 2.5], we introduce the following notions.

Definition 4.2.1 A family $(P_x)_{x\in H}$ of probability measures in $\mathcal{P}_H(\Omega)$, is called an *almost sure Markov family* if for any $A \in \mathcal{B}$, $x \mapsto P_x(A)$ is $\mathcal{B}(H)/\mathcal{B}([0,1])$-measurable, and for each $x \in H$ there exists a Lebesgue null set $T_{P_x} \subset (0,\infty)$ such that for all t not in T_{P_x} and P_x-almost all $\omega \in \Omega$

$$P_x(\cdot|\mathcal{B}_t)(\omega) = P_{\omega(t)} \circ \Phi_t^{-1}.$$

We now introduce the following notion of a martingale solution to Eq. (4.1.3) and write $\xi(t)$ instead of ξ_t.

Definition 4.2.2 Let $x_0 \in H$. A probability measure $P \in \mathcal{P}(\Omega)$ is called a martingale solution of Eq. (4.1.3) with initial value x_0, if:

(M1) $P(\xi(0) = x_0) = 1$ and for any $n \in \mathbb{N}$

$$P\{\xi \in \Omega : \int_0^n \|\mathcal{A}(\xi(s))\|_{\mathbb{X}} ds + \int_0^n \|G(\xi(s))\|^2_{L_2(K;H)} ds < +\infty\} = 1;$$

(M2) for every $l \in \mathcal{E}$, the process

$$M_l(t,\xi) :=_{\mathbb{X}} \langle \xi(t), l\rangle_{\mathbb{X}^*} - \int_0^t {}_{\mathbb{X}}\langle \mathcal{A}(\xi(s)), l\rangle_{\mathbb{X}^*} ds$$

is a continuous square-integrable $\mathcal{F}_t$-martingale under P, whose quadratic variation process is given by

$$\langle M_l\rangle(t,\xi) := \int_0^t \|G^*(\xi(s))(l)\|_K^2 ds,$$

where the asterisk denotes the adjoint operator of $G(\xi(s))$;

(M3) for any $p \in \mathbb{N}$, there exist a continuous positive real function $t \mapsto C_{t,p}$ (only depending on p and $\mathcal{A}, G$), a lower semi-continuous positive real functional $\mathcal{N}_p : \mathbb{Y} \to [0,\infty]$, and a Lebesgue null set $T_P \subset (0,\infty)$ such that for all $0 \le s \in [0,\infty)\backslash T_P$ and for all $t \ge s$

$$E^P[\sup_{r\in[s,t]} |\xi(r)|^{2p} + \int_s^t \mathcal{N}_p(\xi(r))dr|\mathcal{B}_s] \le C_{t-s}(|\xi(s)|^{2p} + 1).$$

First, we prove the following lemma.

Lemma 4.2.3 For any $\theta_1, \theta_2 \in C^\infty(\mathbb{T}^2)$,

$$\|(-\Delta)^\alpha\theta_1 - (-\Delta)^\alpha\theta_2\|_{\mathbb{X}} \le C_1|\theta_1 - \theta_2|,$$

$$\|u_1\cdot\nabla\theta_1-u_2\cdot\nabla\theta_2\|_{\mathbb{X}}\leq C_2(|\theta_1|+|\theta_2|)|\theta_1-\theta_2|,$$

for constants C_1, C_2. In particular, the operator $\mathcal{A}: C^\infty(\mathbb{T}^2)\to\mathbb{X}$ extends to an operator $\mathcal{A}: H\to\mathbb{X}$ by continuity.

Proof We only prove the second assertion, the first can be proved analogously. By the Sobolev embedding theorem we have

$$\begin{aligned}
&\|u_1\cdot\nabla\theta_1-u_2\cdot\nabla\theta_2\|_{\mathbb{X}}\\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\leq 1}|\langle u_1\cdot\nabla\theta_1-u_2\cdot\nabla\theta_2,w\rangle|\\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\leq 1}|\langle u_1\cdot\nabla w,\theta_1\rangle-\langle u_2\cdot\nabla w,\theta_2\rangle|\\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\leq 1}|\langle (u_1-u_2)\cdot\nabla w,\theta_1\rangle+\langle u_2\cdot\nabla w,\theta_1-\theta_2\rangle|\\
\leq& C[\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\leq 1}\|\nabla w\|_{C(\mathbb{T}^2)}](|u_1-u_2|\cdot|\theta_1|+|\theta_1-\theta_2|\cdot|u_2|)\\
\leq& C(|\theta_1|+|\theta_2|)|\theta_1-\theta_2|.
\end{aligned}$$

In the last inequality we use (4.1.1) and the constant C changes from line to line. □

In order to use [GRZ09, Theorem 4.7], we define the functional $\mathcal{N}_1$ on $\mathbb{Y}$ as follows:

$$\mathcal{N}_1(\theta):=\begin{cases}|\Lambda^\alpha\theta|^2, & \text{if } \theta\in H^\alpha,\\ +\infty, & \text{otherwise}.\end{cases}$$

It is obvious that $\mathcal{N}_1\in\mathfrak{U}^2$, defined in [GRZ09, Section 4]. We recall that a lower semicontinuous function $\mathcal{N}:\mathbb{Y}\to[0,\infty]$ belongs to $\mathfrak{U}^2$ if $\mathcal{N}(x)=0$ implies $x=0$, $\mathcal{N}(cy)\leq c^2\mathcal{N}(y),\forall c\geq 0, y\in\mathbb{Y}$ and $\{y\in\mathbb{Y}:\mathcal{N}(y)\leq 1\}$ is relatively compact in $\mathbb{Y}$.

Theorem 4.2.4 Let $\alpha\in(0,1)$ and assume G satisfies (G.1). Then for each $x_0\in H$, there exists a martingale solution $P\in\mathcal{P}(\Omega)$ starting from x_0 to Eq. (4.1.3) in the sense of Definition 4.2.2.

Proof We only need to check (C1)-(C3) in [GRZ09, Section 4] for the above $\mathcal{A}$ and G. For the reader's convenience, we give them as follows:

(C1)(Demi-Continuity) For any $x\in\mathbb{X}^*$, if y_n strongly converges to y in $\mathbb{Y}$, then

$$\lim_{n\to\infty}{}_{\mathbb{X}}\langle\mathcal{A}(y_n),x\rangle_{\mathbb{X}^*}={}_{\mathbb{X}}\langle\mathcal{A}(y),x\rangle_{\mathbb{X}^*},$$

and

$$\lim_{n\to\infty}\|G^*(y_n)(x)-G^*(y)(x)\|_K=0.$$

(C2)(Coercivity Condition) There exist $\lambda_1 \geq 0$ and $N_1 \in \mathfrak{U}^2$ such that for all $x \in \mathbb{X}^*$

$$_{\mathbb{X}}\langle \mathcal{A}(x), x\rangle_{\mathbb{X}^*} \leq -\mathcal{N}_1(x) + \lambda_1(1 + |x|^2).$$

(C3)(Growth Condition) There exist $\lambda_2, \lambda_3, \lambda_4 > 0$ and $\gamma' \geq \gamma > 1$ such that for all $x \in \mathbb{Y}$

$$\|A(x)\|_{\mathbb{X}}^{\gamma} \leq \lambda_2 \mathcal{N}_1(x) + \lambda_3(1 + |x|^{\gamma'}),$$

$$\|G(x)\|_{L_2(K;H)}^2 \leq \lambda_4(1 + |x|^2),$$

where $\mathcal{N}_1$ is as in (C2).

(C1) holds since Lemma 4.2.3 implies demi-continuity of $\mathcal{A}$ and G.

(C2) follows, because noting that for $\theta \in \mathbb{X}^*$

$$\langle u \cdot \nabla\theta, \theta\rangle = 0,$$

we have

$$\langle \mathcal{A}(\theta), \theta\rangle = -\mathcal{N}_1(\theta).$$

Also (C3) is clear since by Lemma 4.2.3

$$\|\mathcal{A}(\theta)\|_{\mathbb{X}} \leq C|\theta|^2$$

and

$$\|G(\theta)\|_{L_2(K;H)} \leq C(|\theta| + 1).$$

□

The set of all such martingale solutions with initial value x_0 is denoted by $\mathcal{C}(x_0)$. Using [GRZ09, Theorem 4.7], we now obtain the following:

Theorem 4.2.5 Let $\alpha \in (0,1)$. Assume G satisfies (G.1). Then there exists an almost sure Markov family $(P_{x_0})_{x_0 \in H}$ for Eq. (4.1.3) and $P_{x_0} \in \mathcal{C}(x_0)$ for each $x_0 \in H$.

4.3 Ergodicity for $\alpha > \frac{2}{3}$

In this section, we assume that $\alpha > \frac{2}{3}$, $K = H$, and that G satisfies:

Assumption 4.3.1 There are an isomophism Q_0 of H and a number $s \geq 1$ such that $G = A^{-\frac{s+\alpha}{2\alpha}} Q_0^{1/2}$, and furthermore, G satisfies

$$\int (\sum_j |Ge_j|^2)^{p/2} d\xi \le C,$$

for some fixed $p \in ((\alpha - \frac{1}{2})^{-1}, \infty)$, (which is e.g. always the case if $Q_0 = I$).

For $x := \theta_0 \in L^p$, let P_x denote the law of the corresponding solution θ to (4.1.3). Then by [RZZ12, Theorems 5.4 and 5.5] the measures $P_x, x \in L^p$, form a Markov process. Let $(P_t)_{t\ge 0}$ be the associated transition semi-group on $\mathcal{B}_b(H)$, defined as

$$P_t(\varphi)(x) := E_x[\varphi(\xi_t)], \qquad x \in L^p, \varphi \in \mathcal{B}_b(H), \tag{4.3.1}$$

where E_x denotes expectation under P_x.

4.3.1 The strong Feller property for $\alpha > \frac{2}{3}$

In this subsection we prove that its transition semigroup has the strong Feller property under appropriate conditions.

Remark 4.3.2 (i) Since in our case $\alpha < 1$, the linear part $(-\Delta)^\alpha$ in (1.7) is less regularizing.As $G = A^{-\frac{s+\alpha}{2\alpha}} Q_0^{1/2}$, we get the trajectories z of the associated O-U process to be in $C([0,\infty), H^{s+2\alpha-1-\varepsilon})$ for every $\varepsilon > 0$ (c.f. [DZ92, Theorem 5.16], [DO06, Proposition 3.1]). However, in order to prove the weak-strong uniqueness principle (see (4.3.2) and Theorem 4.3.4 below) and the strong Feller property of the semigroup associated with the solution of the cutoff equation (see Proposition 4.3.5 below), we need $z \in C([0,\infty), H^{s+1-\alpha+\sigma_1})$ for some $\sigma_1 > 0$. Therefore, we need $s + 2\alpha - 1 > s + 1 - \alpha$, i.e. $\alpha > \frac{2}{3}$. The situation of the 3D-Navier-Stokes equation is different. While in our case the needed regularity of z is higher than the regularity of our solution space $C((0,\infty), H^s)$ for the cutoff equation (4.3.2), for the 3-D Navier-Stokes equation the needed regularity of z is the same as for the solution of the cutoff equation.

(ii) Since $\alpha < 1$, we can't use the same type of estimate as in [FR08] (c.f. [FR08, Lemma D.2]) to obtain our results. We use Lemma 4.1.1 and choose suitable parameters (s, σ_1, σ_2) such that the approach in [FR08] can be modified to apply here (see (4.3.6)-(4.3.10), (4.3.13) and so on).

(iii) It seems difficult to use the Kolmogorov equation method as in [DD03], [DO06] or a coupling approach as in [O07] in our situation. In fact, to get a uniform H^s-norm estimate for the solutions of the Galerkin approximations of the equation (1.1) for some $s > 0$, the regularity, needed for the trajectories of the associated Ornstein-Uhlenbeck (O-U) process z is higher than H^s, which is entirely different from the situation of the 3-D Navier-Stokes equation. According to the method in [DD03], DO06] and [O07], we should use the solutions' $H^{s+\alpha}$-norm to control the $H^{s+\alpha}$-norm of the derivative of the solutions as required for the Bismut-Elworthy-Li formula. In particular, the associated O-U process z should be also in $H^{s+\alpha}$. However, under Assumption 4.3.1 for the noise, our O-U process z is only in $L^2([0,T],H^{s+2\alpha-1})$. As a result, for their method to apply here, we need even $\alpha \geq 1$.

Fix $s \geq 1$ as in Assumption 4.3.1 and set $\mathcal{W} := H^s$ and $|x|_{\mathcal{W}} := \|x\|_{H^s}$.

Now we state the main result of this section.

Theorem 4.3.3 Under Assumption 4.3.1, $(P_t)_{t\geq 0}$ is $\mathcal{W}$-strong Feller, i.e. for every $t > 0$ and $\psi \in \mathcal{B}_b(H)$, $P_t\psi \in C_b(\mathcal{W})$.

We shall use [FR08, Theorem 5.4], which is an abstract result to prove the strong Feller property. In order to use [FR08, Theorem 5.4], we follow the idea of [FR08, Theorem 5.11] to construct $P_x^{(R)}$. We introduce an equation which differs from the original one by a cut-off only, so that with large probability they have the same trajectories on a small random time interval (see (4.3.3) below). We consider the equation

$$d\theta(t) + A\theta(t)dt + \chi_R(|\theta|^2_{\mathcal{W}})u(t)\cdot\nabla\theta(t)dt = GdW(t), \tag{4.3.2}$$

where $\chi_R : \mathbb{R} \to [0,1]$ is of class C^∞ such that $\chi_R(|\theta|) = 1$ if $|\theta| \leq R$, $\chi_R(|\theta|) = 0$ if $|\theta| > R+1$ and with its first derivative bounded by 1. Then, if we can prove the following Theorem 4.3.4 and Proposition 4.3.5, Theorem 4.3.3 follows.

Theorem 4.3.4 (Weak-strong uniqueness) Suppose Assumption 4.3.1 holds. Then for every $x \in \mathcal{W}$, Eq. (4.3.2) has a unique martingale solution $P_x^{(R)}$, with

$$P_x^{(R)}[C([0,\infty);\mathcal{W})] = 1.$$

Let $\tau_R : \Omega \to [0,\infty]$ be defined as

$$\tau_R(\omega) = \inf\{t \geq 0 : |\omega(t)|^2_{\mathcal{W}} \geq R\},$$

and $\tau_R(\omega) = \infty$ if this set is empty. If $x \in \mathcal{W}$ and $|x|^2_{\mathcal{W}} < R$, then

$$\lim_{\varepsilon\to 0} P_{x+h}^{(R)}[\tau_R \geq \varepsilon] = 1, \text{ uniformly in } h \in \mathcal{W}, |h|_{\mathcal{W}} < 1. \tag{4.3.3}$$

Moreover,

$$E^{P_x^{(R)}}[\varphi(\xi_t)1_{[\tau_R\geq t]}] = E^{P_x}[\varphi(\xi_t)1_{[\tau_R\geq t]}], \tag{4.3.4}$$

for every $t \geq 0$ and $\varphi \in \mathcal{B}_b(H)$, where P_x is the martingale solution of (4.1.3).

Proof Let z denote the solution to

$$dz(t) + Az(t)dt = GdW(t),$$

with initial data $z(0) = 0$ and let $v_x^{(R)}$ be the solution to the auxiliary problem

$$\frac{dv^{(R)}(t)}{dt} + Av^{(R)}(t) + u^{(R)}(t) \cdot \nabla(v^{(R)}(t) + z(t))\chi_R(|v^{(R)} + z|^2_{\mathcal{W}}) = 0, \tag{4.3.5}$$

with $v^{(R)}(0) = x$. Here $u^{(R)}(t) = u_{v^{(R)}}(t) + u_z(t)$, $u_{v^{(R)}}$ and u_z satisfy (1.9) with θ replaced by $v^{(R)}$ and z, respectively. Moreover, define $\theta^{(R)} := v^{(R)} + z$, which is a martingale solution to equation (4.3.2). We denote its law on Ω by $P_x^{(R)}$. By Assumption 4.3.1 the trajectories of the noise belong to

$$\Omega^* := \bigcap_{\beta\in(0,\frac{1}{2}),\kappa\in[0,\frac{s+\alpha}{2\alpha}-\frac{1}{2\alpha})} C^{\beta}([0,\infty); D(A^{\kappa})),$$

with probability one. Hence, the analyticity of the semigroup generated by A implies that for each $\omega \in \Omega^*$, $z(\omega) \in C([0,\infty), D(\Lambda^{s+2\alpha-1-\varepsilon}))$ for every $\varepsilon > 0$.

Now, for $\omega \in \Omega^*$ we prove that Eq. (4.3.5) with $z(\omega)$ replacing z has a unique global weak solution in the space $C([0,\infty); \mathcal{W})$. First, we obtain the following a-priori estimate for suitable $\sigma_1, \sigma_2 > 0$ with $\sigma_2 \leq s, \sigma_2 + \sigma_1 = 1, s + \sigma_1 - \alpha + 1 < s + 2\alpha - 1 < s + \alpha$, where we used that $\alpha > \frac{2}{3}$ since $0 < \sigma_1 < 3\alpha - 2$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\Lambda^s v^{(R)}|^2 + \kappa|\Lambda^{s+\alpha}v^{(R)}|^2 \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s-\alpha+1}R(u^{(R)}\theta^{(R)})| \cdot |\Lambda^{s+\alpha}v^{(R)}| \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s-\alpha+1+\sigma_1}\theta^{(R)}||\Lambda^{\sigma_2}\theta^{(R)}| \cdot |\Lambda^{s+\alpha}v^{(R)}| \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^{s-\alpha+1+\sigma_1}v^{(R)}| + |\Lambda^{s-\alpha+1+\sigma_1}z|) \cdot |\Lambda^{s+\alpha}v^{(R)}| \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(C|\Lambda^s v^{(R)}|^{1-r}|\Lambda^{s+\alpha}v^{(R)}|^r + |\Lambda^{s-\alpha+1+\sigma_1}z|) \cdot |\Lambda^{s+\alpha}v^{(R)}| \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^s v^{(R)}|^2 + |\Lambda^{s-\alpha+1+\sigma_1}z|^2) + \frac{\kappa}{2}|\Lambda^{s+\alpha}v^{(R)}|^2 \\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(C(R) + |\Lambda^{s-\alpha+1+\sigma_1}z|^2) + \frac{\kappa}{2}|\Lambda^{s+\alpha}v^{(R)}|^2,
\end{aligned} \tag{4.3.6}$$

where $r := \frac{1-\alpha+\sigma_1}{\alpha}$. Here in the second inequality we used Lemmas 4.1.1 and 4.1.2, and in the fourth inequality we used the Gagliardo-Nirenberg inequality and in the fifth inequality we used Young's inequality. Let P_n be the orthogonal projection in H onto the space spanned by $e_1, ...e_n$. Consider the ordinary differential equation

$$\frac{dv_n(t)}{dt} + Av_n(t) + P_n(u_n(t) \cdot \nabla(v_n(t) + z(t))\chi_R(|v_n + z|^2_{\mathcal{W}})) = 0,$$

with initial condition

$$v_n(0) = P_n v_0.$$

Here u_n satisfies (1.9) with θ replaced by $v_n + z$. Denote the solution of the following approximate equation by v_n. We obtain that the sequence v_n is bounded in $L^\infty(0, T; H)$ and in $L^2(0, T; H^\alpha)$. It is obvious that there exists an element $v^{(R)} \in L^\infty(0, T; H) \cap L^2(0, T; H^\alpha)$ and a sub-sequence v'_m such that

$$v'_m \to v^{(R)} \text{ in } L^2(0, T; H^{s+\alpha}) \text{ weakly, and in } L^\infty(0, T; H^s) \text{ weak-star, as } m' \to \infty.$$

In order to prove the strong convergence in $L^2(0, T; H^s)$, we need to use [FG95, Theorem 2.1]. So we just need to prove that $\|v_n\|_{W^{\gamma,2}(0,T,H^{-3})}$ is bounded for some $1/2 < \gamma < 1$, which can be obtained by estimated each term of the approximate equation. Then by compact embedding, we have $v'_m \to v^{(R)}$ in $L^2(0, T; H^s) \cap C([0, T]; H^{-\beta})$ strongly for some $\beta > 3$. Note that v_n also satisfies

$$\langle v_n(t), \psi\rangle + \int_0^t \langle A^{1/2}v_n(s), A^{1/2}\psi\rangle ds - \int_0^t \chi_R(|v_n+z|^2_{\mathcal{W}})\langle u_n(s)\cdot\nabla\psi, v_n(s)+z(s)\rangle ds = \langle P_n v_0, \psi\rangle,$$

for all $t \in [0, T]$ and all $\psi \in C^1(\mathbb{T}^2)$. Then taking the limit in above equation, we obtain that (4.3.5) has a weak solution in $L^\infty([0, T], \mathcal{W})$.

[Continuity] For each $\omega \in \Omega^*$, σ_1 and σ_2 as above, since $s-\alpha+1+\sigma_1 < s+2\alpha-1$, we have $z \in C([0, \infty); D(\Lambda^{s-\alpha+1+\sigma_1}))$. For $s > 3 - 3\alpha$, $s_0 = s - \alpha$, multiplying the equations (4.3.5) by $\frac{d}{dt}\Lambda^{2s_0}v^{(R)}$, we obtain

(4.3.7)
$$\begin{aligned}
\frac{\kappa}{2}\frac{d}{dt}|\Lambda^{s_0+\alpha}v^{(R)}|^2 + |\Lambda^{s_0}\dot{v}^{(R)}|^2 \leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s_0+1}R(u^{(R)}\theta^{(R)})| \cdot |\Lambda^{s_0}\dot{v}^{(R)}| \\
\leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s_0+1+\sigma_1}\theta^{(R)}||\Lambda^{\sigma_2}\theta^{(R)}| \cdot |\Lambda^{s_0}\dot{v}^{(R)}| \\
\leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^{s+\alpha}v^{(R)}|^2 + |\Lambda^{s_0+\alpha}v^{(R)}|^2 + |\Lambda^{s_0+1+\sigma_1}z|^2) \\
& + \frac{1}{2}|\Lambda^{s_0}\dot{v}^{(R)}|^2.
\end{aligned}$$

Here in the second inequality we used Lemmas 4.1.1 and 4.1.2, and in the third inequality we used the Gagliardo-Nirenberg inequality and Young's inequality.

As $\int_0^T |\Lambda^{s+\alpha}v^{(R)}(t_1)|^2 dt_1$ can be dominated by the same arguments as (4.3.6), we get an a-priori estimate for the time derivative $\frac{d}{dt}v^{(R)}$ in $L^2(0,T;H^{s_0})$. Then by [Te84], we obtain $v^{(R)}\in C([0,T],\mathcal{W})$.

[Uniqueness] Let θ_1,θ_2 be two solutions of Eq. (4.3.5) in $C([0,\infty);\mathcal{W})$ and set $w:=\theta_1-\theta_2$ and $u_w:=u_1-u_2$. Then by a similar argument as in the proof of [RZZ12, Theorem 5.1], we have for small $\varepsilon_0>0$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Lambda^{s_0}w|^2+\kappa|\Lambda^{s_0+\alpha}w|^2=&-(\chi_R(|\theta_1|^2_{\mathcal{W}})-\chi_R(|\theta_2|^2_{\mathcal{W}}))\langle\Lambda^{s_0+\varepsilon_0-\alpha}(u_1\cdot\nabla\theta_1),\Lambda^{s_0+\alpha-\varepsilon_0}w\rangle\\&-\chi_R(|\theta_2|^2_{\mathcal{W}})\langle\Lambda^{s_0-\alpha}(u_1\cdot\nabla w+u_w\cdot\nabla\theta_2),\Lambda^{s_0+\alpha}w\rangle\\=&I+II+III.\end{aligned}$$

As

$$|\chi_R(|\theta_1|^2_{\mathcal{W}})-\chi_R(|\theta_2|^2_{\mathcal{W}})|\leq C(R)|w|_{\mathcal{W}}[1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}})+1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})],$$

we have for σ_1,σ_2 as above,
(4.3.8)

$$\begin{aligned}I\leq&C[1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}})+1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})]|w|_{\mathcal{W}}\cdot|\Lambda^{s_0-\alpha+\varepsilon_0+1+\sigma_1}\theta_1||\Lambda^{\sigma_2}\theta_1|\cdot|\Lambda^{s_0+\alpha-\varepsilon_0}w|\\\leq&C(R,|\theta_1|_{\mathcal{W}},|\theta_2|_{\mathcal{W}})|w|_{\mathcal{W}}|\Lambda^{s_0+\alpha-\varepsilon_0}w|\\\leq&C(R,|\theta_1|_{\mathcal{W}},|\theta_2|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}},\end{aligned}$$

where $s_0+\alpha=s$. Here in the first inequality we used Lemmas 4.1.1 and 4.1.2, and in the third inequality we used the Gagliardo-Nirenberg inequality and Young's inequality. In a similar way, we obtain

$$II\leq C(R,|\theta_1|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}},$$

and

$$III\leq C(R,|\theta_2|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}}.$$

Then we obtain

$$\frac{1}{2}\frac{d}{dt}|\Lambda^{s_0}w|^2+\kappa|\Lambda^{s_0+\alpha}w|^2\leq C(R,\sup_{t\in[0,T]}|\theta_1(t)|_{\mathcal{W}},\sup_{t\in[0,T]}|\theta_2(t)|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{3\kappa}{4}|w|^2_{\mathcal{W}}.$$

By Gronwall's lemma we have $|\Lambda^{s_0}w|=0$, which implies $w=0$.

So Eq. (4.3.5) has a unique global weak solution in the space $C([0,\infty);\mathcal{W})$.

Next, we prove (4.3.3). In order to do so, it is sufficient to show that $P_x^{(R)}[\tau_R<\varepsilon]\leq C(\varepsilon,R)$ with $C(\varepsilon,R)\downarrow 0$ as $\varepsilon\downarrow 0$, for all $x\in\mathcal{W}$, with $|x|^2_{\mathcal{W}}\leq\frac{R}{8}$. So, fix $\varepsilon>0$ small enough, let $\Theta_{\varepsilon,R}:=\sup_{t\in[0,\varepsilon]}|\Lambda^{s-\alpha+1+\sigma_1}z(t)|$ and assume that $\Theta^2_{\varepsilon,R}\leq\frac{R}{8}$. Setting $\varphi(t):=|v^{(R)}|^2_{\mathcal{W}}+\Theta^2_{\varepsilon,R}$, by (4.3.6) we get $\dot\varphi\leq C(R)$. This implies, together with the bounds on x and $\Theta_{\varepsilon,R}$, that

$$|\theta^{(R)}(t)|^2_{\mathcal{W}} \leq 2(|v^{(R)}(t)|^2_{\mathcal{W}} + |z(t)|^2_{\mathcal{W}}) \leq R,$$

for ε small enough. In particular, since this holds for all $t \leq \varepsilon$, it follows that $\tau_R \geq \varepsilon$. Hence

$$P_x^{(R)}[\tau_R < \varepsilon] \leq P_x^{(R)}[\sup_{t\in[0,\varepsilon]} |\Lambda^{s+1+\sigma_1-\alpha} z(t)|^2 > \frac{R}{8}].$$

Letting $\varepsilon \downarrow 0$, we have $P_x^{(R)}[\tau_R < \varepsilon] \to 0$, and the claim is proved, since the probability above is independent of x.

Finally, the same arguments as in the proof of [RZZ12 Theorem 5.1] imply that

$$\theta_x(t \wedge \tau_R(\theta_x^{(R)})) = \theta_x^{(R)}(t \wedge \tau_R(\theta_x^{(R)})) \quad \forall t, P_x - a.s..$$

Moreover, since θ is H-valued weakly continuous, we obtain $\tau_R(\theta_x^{(R)}) = \tau_R(\theta)$. $\square$

In order to apply [FR08, Theorem 5.4], we now only need the following result.

Proposition 4.3.5 For every $R > 0$, the transition semi-group $(P_t^{(R)})_{t\geq 0}$ associated to Eq. (4.3.2) is $\mathcal{W}$-strong Feller.

Proof We shall provide formal estimates, that can, however, be made rigorous through Galerkin approximations. Let $(\Sigma, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, $(W_t)_{t\geq 0}$ a cylindrical Wiener process on H and, for every $x \in \mathcal{W}$, let $\theta_x^{(R)}$ be the solution to Eq. (4.3.2). By the Bismut, Elworthy and Li formula,

$$D_y(P_t^{(R)}\psi)(x) = \frac{1}{t} E^{\mathbb{P}}[\psi(\theta_x^{(R)}(t)) \int_0^t \langle G^{-1} D_y \theta_x^{(R)}(s), dW(s)\rangle],$$

where $D_y(P_t^{(R)}\psi)$ denotes $\langle D(P_t^{(R)}\psi), y\rangle$ for $y \in H$, and thus, for $\|\psi\|_\infty \leq 1$, by the B-D-G inequality

$$|(P_t^{(R)}\psi)(x_0 + h) - (P_t^{(R)}\psi)(x_0)| \leq \frac{C}{t} \sup_{\eta\in[0,1]} E^{\mathbb{P}}[(\int_0^t |G^{-1} D_h \theta^{(R)}_{x_0+\eta h}(s)|^2 ds)^{1/2}].$$

The proposition is proved once we prove that the right-hand side of the above inequality converges to 0 as $|h|_{\mathcal{W}} \to 0$.

Fix $x \in \mathcal{W}$, $y \in H$ and write $\theta = \theta_x^{(R)}, D\theta = D_y\theta, Du = D_y u$. The term $D\theta$ solves the following equation

$$\frac{d}{dt} D\theta + \kappa\Lambda^{2\alpha}(D\theta) = -[\chi_R(|\theta|^2_{\mathcal{W}})[Du \cdot \nabla\theta + u \cdot \nabla D\theta] + 2\chi_R'(|\theta|^2_{\mathcal{W}})\langle\theta, D\theta\rangle_{\mathcal{W}} u \cdot \nabla\theta].$$

Multiplying the above equation with $\Lambda^{2s}D\theta$ and taking the inner product in L^2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\Lambda^s D\theta|^2+\kappa|\Lambda^{s+\alpha}(D\theta)|^2\\ =&-\langle[\chi_R(|\theta|^2_{\mathcal{W}})[Du\cdot\nabla\theta+u\cdot\nabla D\theta]+2\chi'_R(|\theta|^2_{\mathcal{W}})\langle\theta,D\theta\rangle_{\mathcal{W}}u\cdot\nabla\theta],\Lambda^{2s}D\theta\rangle.\end{aligned}$$

For the first term on the left hand side, we have for $|\theta|^2_{\mathcal{W}}\le R$

(4.3.9)

$$\begin{aligned}&|\langle Du\cdot\nabla\theta,\Lambda^{2s}D\theta\rangle|\\ =&|\langle\Lambda^{s-\alpha}(Du\cdot\nabla\theta),\Lambda^{s+\alpha}D\theta\rangle|\\ \le&C|\Lambda^{s-\alpha+1+\sigma_1}\theta|\cdot|\Lambda^{\sigma_2}D\theta|\cdot|\Lambda^{s+\alpha}D\theta|+C|\Lambda^{s-\alpha+1+\sigma_1}D\theta|\cdot|\Lambda^{\sigma_2}\theta|\cdot|\Lambda^{s+\alpha}D\theta|\\ \le&\varepsilon|\Lambda^{s+\alpha}D\theta|^2+C(C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)|\Lambda^s D\theta|^2,\end{aligned}$$

for σ_1,σ_2 as above, where we used Lemmas 4.1.1, 4.1.2 in the first inequality as well as the Gagliardo-Nirenberg inequality and Young's inequality in the second inequality.

The second term can be estimated similarly. For the third term, by Lemmas 4.1.1, 4.1.2 we have

$$\begin{aligned}|\langle u\cdot\nabla\theta,\Lambda^{2s}D\theta\rangle|=&|\langle\Lambda^{s-\alpha}(u\cdot\nabla\theta),\Lambda^{s+\alpha}D\theta\rangle|\\ \le&C|\Lambda^{s-\alpha+1+\sigma_1}\theta||\Lambda^{\sigma_2}\theta|\cdot|\Lambda^{s+\alpha}D\theta|\\ \le&C(|\Lambda^{s+\alpha}v|+|\Lambda^{s-\alpha+1+\sigma_1}z|)|\Lambda^s\theta||\Lambda^{s+\alpha}D\theta|.\end{aligned}\tag{4.3.10}$$

Then we obtain

$$\frac{1}{2}\frac{d}{dt}|\Lambda^s D\theta|^2+\kappa|\Lambda^{s+\alpha}(D\theta)|^2\le\frac{\kappa}{2}|\Lambda^{s+\alpha}(D\theta)|^2+C(C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)|\Lambda^s D\theta|^2.$$

From Gronwall's inequality we finally obtain

$$\int_0^t|\Lambda^{s+\alpha}(D\theta(l))|^2dl\le\exp(C\int_0^t(C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2dl))|\Lambda^s h|^2.$$

By (4.3.6) we obtain

$$E\int_0^t|\Lambda^{s+\alpha}(D\theta(l))|^2dl\le\sum_{n=1}^\infty\exp(Ct(C(R)+cn^2))P(\sup_{(0,t)}|\Lambda^{s-\alpha+1+\sigma_1}z|>n)|\Lambda^s h|^2.$$

Because of Assumption 4.3.1 and since z is a Gaussian process, one deduces that there exist $\eta, C > 0$ such that

$$P[\sup_{l\in[0,t]} |\Lambda^{s-\alpha+1+\sigma_1} z(l)|^2 > R_0] \le Ce^{-\eta\frac{R_0^2}{t}},$$

(see e.g. [FR07, Proposition 15]). Then for $t_0^2 \le \frac{\eta}{cC}$, we obtain

$$E\int_0^{t_0} |\Lambda^{s+\alpha}(D\theta(s))|^2 ds \le c(t_0, R)|\Lambda^s h|^2,$$

which, as $G = Q_0^{-1/2}\Lambda^{s+\alpha}$, implies the assertion for t_0. For general t, by the semi-group property the assertion follows easily. □

4.3.2 A support theorem for $\alpha > 2/3$

A Borel probability measure μ on H is fully supported on $\mathcal{W}$ if $\mu(U) > 0$ for every non-empty open set $U \subset \mathcal{W}$. Set $\mathcal{W}_1 := D(\Lambda^{s-\alpha+1+\sigma_1})$, where σ_1 is the same as in the proof of Theorem 4.3.4 and we will use it below.

Lemma 4.3.6 (Approximate controllability) Let $R > 0$, $T > 0$. Let $x \in \mathcal{W}$ and $y \in \mathcal{W}$, with $Ay \in \mathcal{W}_1$, such that

$$|x|_{\mathcal{W}}^2 \le \frac{R}{2} \qquad |y|_{\mathcal{W}}^2 \le \frac{R}{2}.$$

Then there exist (a control function) $\omega \in \mathrm{Lip}([0,\mathrm{T}]; \mathcal{W}_1)$ and

$$\theta \in C([0,T]; \mathcal{W}) \cap L^2([0,T]; D(\Lambda^{s+\alpha})),$$

such that θ solves the equation

$$\theta(t) - x + \int_0^t A\theta(r) + \chi_R(|\theta|_{\mathcal{W}}^2)u(r)\cdot\nabla\theta(r)dr = \omega(t) \quad dt - a.e.t \in [0,T], \tag{4.3.11}$$

with $\theta(0) = x$ and $\theta(T) = y$, and

$$\sup_{t\in[0,T]} |\theta(t)|_{\mathcal{W}}^2 \le R. \tag{4.3.12}$$

Proof First consider $\omega = 0$. Then by an inequality similar to (4.3.6), we get

$$\frac{d}{dt}|\theta|_{\mathcal{W}}^2 + \kappa|\Lambda^\alpha\theta|_{\mathcal{W}}^2 \le C(R)|\theta|_{\mathcal{W}}^2.$$

Hence by Gronwall's lemma $\theta(t) \in D(\Lambda^{s+\alpha})$ for almost every $t \in [0,T]$ and, by solving again the equation with one of these regular points as initial condition, by Lemma 4.1.1 we have

$$\frac{d}{dt}|\Lambda^{\alpha+s}\theta|^2+\kappa|\Lambda^{2\alpha+s}\theta|^2_{\mathcal{W}}\le C|\Lambda^{2\alpha+s}\theta||\Lambda^{s+1+\sigma}\theta|\|\theta\|_{L^p}\le C(R)|\Lambda^{s+\alpha}\theta|^2+\frac{\kappa}{2}|\Lambda^{s+2\alpha}\theta|^2,$$

where $\sigma=\frac{2}{p}<2\alpha-1$ and where we used the L_p-estimate in the same way as in the proof of [Re95, Theorem 3.3]. Then we find a small $T_*\in(0,\frac{T}{2})$ such that $|\theta(t)|^2_{\mathcal{W}}\le R$ and $A\theta(T_*)\in\mathcal{W}_1$ for all $t\le T_*$. Define θ to be the solution above for $t\in[0,T_*]$ and extended by linear interpolation between y and $\theta(T_*)$ in $[T_*,T]$. Then obviously (4.3.12) follows.

Next, if we set

$$\eta:=\partial_t\theta+A\theta+\chi_R(|\theta|^2_{\mathcal{W}})u\cdot\nabla\theta,\quad T_*\le t\le T,$$

$\omega:=0$ for $t\le T_*$ and $\omega(t)=\int_{T_*}^t\eta_s ds$ for $t\in[T_*,T]$, we also have (4.3.11). It remains to prove that $\eta\in L^\infty(0,T;\mathcal{W}_1)$. For the first two terms of η this is obvious. For the non-linear term we have that

$$|u\cdot\nabla\theta|_{\mathcal{W}_1}\le C|\Lambda^{2\alpha}\theta|^2_{\mathcal{W}_1},$$

for any $\theta\in D(\Lambda^{s+\sigma_1+1+\alpha})$. □

Let $l\in(0,\frac{1}{2})$ and $p>1$ such that $l-\frac{1}{p}>0$. Under this assumption we see that for every $\alpha_1<\frac{s+\alpha-1}{2\alpha}$ the map

$$\omega\mapsto z(\cdot,\omega):W^{l,p}([0,T];D(A^{\alpha_1}))\to C([0,T];D(A^{\alpha_1+l-\frac{1}{p}-\varepsilon}))$$

is continuous, for all $\varepsilon>0$, where z is the solution to the Stokes problem

$$z(t)+\int_0^t Az(s)ds=\omega(t).$$

In particular, it is possible to find $\alpha_1\in(0,\frac{s+\alpha-1}{2\alpha})$, s and p such that the above map is continuous from $W^{l,p}([0,T];D(A^{\alpha_1}))$ to $C([0,T];D(\Lambda^{s-\alpha+1+\sigma_1}))$.

Lemma 4.3.7 (Continuity with respect to the control functions) Let l,p and α_1 be chosen as above, and let $\omega_n\to\omega$ in $W^{l,p}([0,T];D(A^{\alpha_1}))$. Let θ be the solution to equation (4.3.11) corresponding to ω and some initial condition x, and let

$$\tau=\inf\{t\ge 0:|\theta(t)|^2_{\mathcal{W}}\ge R\},$$

where as usual we set $\inf\emptyset=\infty$. For each $n\in\mathbb{N}$, define similarly θ_n and τ_n

corresponding to ω_n with the same initial condition x. If $\tau > T$, then $\tau_n > T$ for n large enough and

$$\theta_n \to \theta \text{ in } C([0,T];\mathcal{W}).$$

Proof Set $v_n := \theta_n - z_n$ for each $n \in \mathbb{N}$, and $v := \theta - z$, where z_n, z are the solutions to the Stokes problem corresponding to ω_n, ω respectively. Since $\omega_n \to \omega$ in $W^{l,p}([0,T]; D(A^{\alpha_1}))$, we can find a common lower bound for $(\tau_n)_{n\in\mathbb{N}}$ and τ. For every time smaller than this lower bound t_0, by (4.3.6) we have

$$\sup_{(0,t_0)} |\Lambda^s \theta_n|^2 \leq R, \qquad \sup_{(0,t_0)} |\Lambda^s \theta|^2 \leq R, \qquad \sup_{(0,t_0)} |\Lambda^{s-\alpha+1+\sigma_1} z_n| \leq C(R),$$

and

$$\sup_{(0,t_0)} |\Lambda^{s-\alpha+1+\sigma_1} z| \leq C(R), \qquad \int_0^{t_0} |\Lambda^{s+\alpha} v_n(l)|^2 dl \leq C(R), \qquad \int_0^{t_0} |\Lambda^{s+\alpha} v(l)|^2 dl \leq C(R),$$

where $C(R)$ is a constant depending only on R. Moreover, we obtain for $t \leq t_0$

$$\begin{aligned}
&\frac{d}{dt}|v - v_n|^2_{\mathcal{W}} + 2\kappa|\Lambda^\alpha (v_n - v)|^2_{\mathcal{W}} \\
=&\langle u_n \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle - \langle u \cdot \nabla\theta, \Lambda^{2s}(v - v_n)\rangle \\
=&[\langle (u_{v_n} - u_v) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle + \langle u \cdot \nabla(v_n - v), \Lambda^{2s}(v - v_n)\rangle \\
&+ \langle (u_{z_n} - u_z) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle + \langle u \cdot \nabla(z_n - z), \Lambda^{2s}(v - v_n)\rangle].
\end{aligned}$$

For the first term on the right hand side, by using Lemmas 4.1.1, 4.1.2 we have

$$\begin{aligned}
&|\langle (v_n - v) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle| \\
\leq& C|\Lambda^{s+\alpha}(v - v_n)||\Lambda^{s-\alpha+1+\sigma_1}(v - v_n)||\Lambda^{\sigma_2}\theta_n| \\
&+ C|\Lambda^{s+\alpha}(v - v_n)||\Lambda^{s-\alpha+1+\sigma_1}\theta_n||\Lambda^{\sigma_2}(v - v_n)| \\
\leq& \frac{\kappa}{4}|\Lambda^{s+\alpha}(v - v_n)|^2 + (C(R) + |\Lambda^s v_n|^2 + |\Lambda^{s+\alpha} v_n|^2)|\Lambda^s(v - v_n)|^2 \\
&+ c|\Lambda^{s-\alpha+1+\sigma_1} z_n|^2 |\Lambda^s (v - v_n)|^2.
\end{aligned} \tag{4.3.13}$$

The other term can be estimated similarly. Then we obtain

$$\begin{aligned}
&\frac{d}{dt}|v - v_n|^2_{\mathcal{W}} + 2\kappa|\Lambda^\alpha (v_n - v)|^2_{\mathcal{W}} \\
\leq& \kappa|\Lambda^\alpha (v_n - v)|^2_{\mathcal{W}} + C(C(R) + |\Lambda^\alpha v_n|^2_{\mathcal{W}} + |\Lambda^\alpha v|^2_{\mathcal{W}})(|v - v_n|^2_{\mathcal{W}} + |\Lambda^{s-\alpha+1+\sigma_1}(z - z_n)|^2).
\end{aligned}$$

Here σ_1, σ_2 are as above. Then by Gronwall's lemma

$$|v - v_n|^2_{\mathcal{W}} \leq \Theta_n \exp(C\int_0^t (C(R) + |\Lambda^\alpha v_n|^2_{\mathcal{W}} + |\Lambda^\alpha v|^2_{\mathcal{W}}) dl) \int_0^t (C(R) + |\Lambda^\alpha v_n|^2_{\mathcal{W}} + |\Lambda^\alpha v|^2_{\mathcal{W}}) dl,$$

where $\Theta_n = \sup_{[0,T]} |\Lambda^{s-\alpha+1+\sigma_1}(z - z_n)|$. We conclude $\theta_n \to \theta$ in $C([0,T];\mathcal{W})$. Now, since $\tau > T$, if $S = \sup_{t\in[0,T]} |\Lambda^s\theta(t)|^2$, then $S < R$ and we find $\delta > 0$ (depending only on R and S) and $n_0 \in \mathbb{N}$ such that $\Theta_n < \delta$ and $|v_n - v|^2_{\mathcal{W}} < \delta$ for all $n \geq n_0$, and so

$$|\theta_n(t)|_{\mathcal{W}} \leq |v_n(t) - v(t)|_{\mathcal{W}} + \Theta_n + |\theta(t)|_{\mathcal{W}} \leq 2\sqrt{\delta} + \sqrt{S} \leq \sqrt{R-\delta}.$$

Then $\tau_n > T$ for all $n \geq n_0$. $\square$

Theorem 4.3.8 Suppose Assumption 4.3.1 holds and for $x \in H$ let P_x be the distribution of the solution of (4.1.3) with initial value $\theta_0 = x$. Then for every $x \in \mathcal{W}$ and every $T > 0$, the image measure of P_x at time T is fully supported on $\mathcal{W}$.

Proof Fix $x \in \mathcal{W}$ and $T > 0$. We need to show that for every $y \in \mathcal{W}$ and $\varepsilon > 0$, $P_x[|\theta_T - y|_{\mathcal{W}} < \varepsilon] > 0$. Let $\bar{y} \in \mathcal{W} \cap D(A)$ such that $A\bar{y} \in \mathcal{W}_1$ and $|y - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}$. Choose $R > 0$ such that $3|x|^2_{\mathcal{W}} < R$ and $3|y|^2_{\mathcal{W}} < R$. Then by Theorem 4.3.4,

$$\begin{aligned} P_x[|\theta_T - y|_{\mathcal{W}} < \varepsilon] \geq & P_x[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}] \geq P_x[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T] \\ = & P_x^{(R)}[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T]. \end{aligned}$$

By Lemma 4.3.6, there is a control $\bar{\omega} \in W^{l,p}([0,T]; D(A^{\alpha_1}))$, with l, p and α_1 chosen as in Lemma 4.3.7, such that the solution $\bar{\theta}$ to the control problem (4.3.11) corresponding to $\bar{\omega}$ satisfies $\bar{\theta}(0) = x, \bar{\theta}(T) = \bar{y}$ and $|\bar{\theta}(t)|^2_{\mathcal{W}} \leq \frac{2}{3}R$. By Lemma 4.3.7, there exists $\delta > 0$ such that for all $\omega \in W^{l,p}([0,T]; D(A^{\alpha_1}))$ with $|\omega - \bar{\omega}|_{W^{l,p}([0,T];D(A^{\alpha_1}))} < \delta$, we have

$$|\theta(T,\omega) - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2} \text{ and } \sup_{t\in[0,T]} |\theta(t,\omega)|^2_{\mathcal{W}} < R,$$

where $\theta(\cdot,\omega)$ is the solution to the control problem (4.3.11) corresponding to ω and starting at x. Hence

$$P_x^{(R)}[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T] \geq P_x^{(R)}[|\eta - \bar{\omega}|_{W^{l,p}([0,T];D(A^{\alpha_1}))} < \delta],$$

where $\eta_t = \theta_t - x + \int_0^t (A\theta_s + \chi_R(|\theta_s|^2_{\mathcal{W}})u \cdot \nabla\theta_s)ds$, hence $\theta_T = \theta(T,\eta)$, and the right hand side of the inequality above is strictly positive since by Assumption 4.3.1 η is a Brownian motion in $D(A^{\alpha_1})$.

4.3.3 Existence of invariant measures for $\alpha > \frac{2}{3}$

In this subsection, we prove the existence of invariant measures. Let θ_n denote the solution of the usual Galerkin approximation

$$\begin{cases} d\theta_n(t) + A\theta_n(t)dt + P_n(u_n(t)\cdot\nabla\theta_n(t))dt = P_nG(\theta_n(t))dW(t), \\ \theta_n(0) = P_nx. \end{cases}$$

Lemma 4.3.9 Let $\alpha > \frac{2}{3}$. If $x \in H^1, n \in \mathbb{N}, t > 0$, then there exist $\delta_1 > 0$ and $\gamma_0 > 0$ such that

$$E[\int_0^t |A^{\delta_1}\theta_n|_{\mathcal{W}}^{2\gamma_0} dr] \leq C(1+t)(|x|^2+1),$$

where C is independent of x and R.

Proof We apply Itô's formula to the function $(1+|\Lambda^\delta\theta|^2)^{-p}$ for $\delta > 2-2\alpha$ and get

$$\begin{aligned} &\frac{1}{(1+|\Lambda^\delta\theta|^2)^p} - \frac{1}{(1+|\Lambda^\delta x|^2)^p} \\ =&2p\int_0^t \frac{|\Lambda^{\delta+\alpha}\theta|^2}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr + 2p\int_0^t \frac{\langle\Lambda^{\delta-\alpha}(u\cdot\nabla\theta), \Lambda^{\delta+\alpha}\theta\rangle}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr \\ &-2p\int_0^t \frac{\langle\Lambda^\delta\theta, \Lambda^\delta GdW_r\rangle}{(1+|\Lambda^\delta\theta|^2)^{p+1}} - p\int_0^t \frac{\mathrm{Tr}[\mathrm{GG}^*\Lambda^{2\delta}]}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr \\ &+2p(p+1)\int_0^t \frac{|\Lambda^\delta G\theta|^2}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr, \end{aligned}$$

where for simplicity we write $\theta = \theta_n$. Choosing σ_1', σ_2' with $\sigma_2' \leq \delta, \sigma_2' + \sigma_1' = 1, \delta + \sigma_1' - \alpha + 1 < \delta + \alpha$ the non-linear part is estimated as follows:

$$\begin{aligned} |\langle\Lambda^{\delta-\alpha}(u\cdot\nabla\theta), \Lambda^{\delta+\alpha}\theta\rangle| \leq& C|\Lambda^{\delta-\alpha+1+\sigma_1'}\theta|\cdot|\Lambda^{\sigma_2'}\theta||\Lambda^{\delta+\alpha}\theta| \\ \leq& C|\Lambda^\delta\theta|^m + |\Lambda^{\delta+\alpha}\theta|^2, \end{aligned}$$

with $m = \frac{2(3\alpha-1-\sigma_1')}{2\alpha-1-\sigma_1'}$.

Then for p big enough we obtain

$$E\int_0^t \frac{|\Lambda^{\delta+\alpha}\theta_n|^2}{(1+|\Lambda^\delta\theta_n|^2)^{p+1}}dr \leq C(1+t).$$

Since by Young's inequality

$$|\Lambda^{\delta+\alpha}\theta_n|^{2\gamma_p} \leq c[\frac{|\Lambda^{\delta+\alpha}\theta_n|^2}{(1+|\Lambda^\delta\theta_n|^2)^{p+1}} + 1 + |\Lambda^\delta\theta_n|^2],$$

for $\delta \le \alpha$ we obtain

$$E[\int_0^t |\Lambda^{\delta+\alpha}\theta_n|^{2\gamma_p} dr] \le C(1+t)(|x|^2+1). \tag{4.3.14}$$

If $\delta > \alpha$, we already know that some power of $|\Lambda^\delta \theta_n|$ is integrable with respect to $dt \otimes P$. Then one proceeds as in the previous case to obtain (4.3.14). We choose $\delta + \alpha > s$ and obtain the assertions. $\square$

Theorem 4.3.10 Let $\alpha > \frac{2}{3}$ and suppose Assumption 4.3.1 holds. Then there exists a unique invariant measure ν on $\mathcal{W}$ for the transition semigroup $(P_t)_{t\ge 0}$. Moreover:

(i) The invariant measure ν is ergodic.

(ii) The transition semigroup $(P_t)_{t\ge 0}$ is $\mathcal{W}$-strong Feller, irreducible, and therefore strongly mixing. Furthermore, $P_t(x,dy), t>0, x\in\mathcal{W}$, are mutually equivalent.

(iii) There are $\delta_1 > 0$ and $\gamma_0 > 0$ such that

$$\int |A^{\delta_1}x|_{\mathcal{W}}^{2\gamma_0} d\nu < \infty.$$

Proof Choose $x_0 \in H^1$ and define

$$\mu_t = \frac{1}{t}\int_0^t P_r^* \delta_{x_0} dr.$$

Since

$$\int |A^{\delta_1}x|_{\mathcal{W}}^{2\gamma_0} \mu_t(dx) = \frac{1}{t} E_{x_0}[\int_0^t |A^{\delta_1}\theta|_{\mathcal{W}}^{2\gamma_0} dr],$$

by Lemma 4.3.9 we obtain

$$\int |A^{\delta_1}x|_{\mathcal{W}}^{2\gamma_0} \mu_t(dx) \le C.$$

This implies that μ_t is tight on $\mathcal{W}$. The strong Feller property of P_t follows from Theorem 4.3.3. Hence, a limit point of μ_t is an invariant measure for $(P_t)_{t\ge 0}$. Therefore, by Doob's theorem, the strongly mixing property is a consequence of the irreducibility.

Remark 4.3.11 If we don't assume that G satisfies $\int (\sum_j |G(e_j)|^2)^{p/2} d\xi \le C$, for some fixed $p \in ((\alpha - \frac{1}{2})^{-1}, \infty)$, the solution of equation (4.1.3) may be not unique. Then we can also prove the above results for each Markov selection $P_x, x \in \mathcal{W}$, corresponding to (4.1.3) and the respective semigroup $(P_t)_{t\ge 0}$ by similar arguments as [R08].

Remark 4.3.12 (i) (Mildly degenerate noise) We can also consider the ergodicity of the equation driven by a mildly degenerate noise as in [EH01]. For this we have to use an extension of the Bismut-Elworthy-Li formula. We have the same problem as explained in Remark 4.3.2. So, we can just get the result for $\alpha > 2/3$.

(ii) (Degenerate noise) There are many papers considering 2D Navier-Stokes equation driven by degenerate noise. Contrary to the 2D Navier-Stokes equation, no Foias-Prodi type estimate is available for the quasi-geostrophic equation. It seems impossible to use a coupling approach as in [KS02], [BKL02], [M02] to prove ergodicity in the case where equation (4.1.3) is driven by a degenerate noise. It also seems difficult to use the method in [HM06] to prove ergodicity.

4.4 Exponential convergence for $\alpha > \frac{2}{3}$

First we introduce the same approximation as in the proof of [RZZ12, Theorem 4.3]: We pick a smooth $\phi \geq 0$, with supp$\phi \subset [1,2]$, $\int_0^\infty \phi = 1$, and for $\delta > 0$ let

$$U_\delta[\theta](t) := \int_0^\infty \phi(\tau)(k_\delta * R^\perp\theta)(t-\delta\tau)d\tau,$$

where k_δ is the periodic Poisson Kernel in $\mathbb{T}^2$ given by $\widehat{k_\delta}(\zeta) = e^{-\delta|\zeta|}, \zeta \in \mathbb{Z}^2$, and we set $\theta(t) = 0, t < 0$. We take a zero sequence δ_n and consider the equation:

$$d\theta_n(t) + A\theta_n(t)dt + u_n(t)\cdot\nabla\theta_n(t)dt = k_{\delta_n} * G(\theta)dW(t),$$

with initial data $\theta_n(0) = \theta_0$ and $u_n = U_{\delta_n}[\theta_n]$, where $k_\delta * G(\theta)$ means for $y \in K$, $k_\delta * G(\theta)(y) = k_\delta * (G(\theta)(y))$. For a fixed n, this is a linear equation in θ_n on each subinterval $[t_k, t_{k+1}]$ with $t_k = k\delta_n$, since u_n is determined by the values of θ_n on the two previous subintervals.

As will be seen below, we shall need uniform L^p-estimates, and a crucial ingredient to prove them is Krylov's L^p-Itô formula. In order to obtain a uniform estimate, the L^p-estimate known from the deterministic case (see e.g. [Re95]) is not strong enough for our purpose. Therefore, we need the following result, which is an improved version of the "positivity lemma" from [Re95, Lemma 3.2].

Lemma 4.4.1 (Improved positivity Lemma) For $\alpha \in (0,1)$, and $\theta \in L^p$ with $\Lambda^{2\alpha}\theta \in L^p$, for some $2 < p < \infty$,

$$\int |\theta|^{p-2}\theta(\Lambda^{2\alpha} - \frac{2\lambda_1}{p})\theta \geq 0.$$

Proof Denote the semigroup with respect to $-\Lambda^{2\alpha}+\frac{2\lambda_1}{p}$ and $-\Lambda^{2\alpha}$ in L^2 by P_t^0 and P_t^1, respectively. Then we have $P_t^0 f = e^{2t\lambda_1/p}P_t^1 f$. Since

$$\|P_t^1 f\|_{L^2}\leq e^{-\lambda_1 t}\|f\|_{L^2},$$

and

$$\|P_t^1 f\|_{L^\infty}\leq \|f\|_{L^\infty},$$

by the interpolation theorem, we have

$$\|P_t^1 f\|_{L^p}\leq e^{-2\lambda_1 t/p}\|f\|_{L^p}.$$

Thus,

$$\|P_t^0 f\|_{L^p}\leq \|f\|_{L^p}.$$

Then we have

$$\frac{d}{dt}\|P_t^0\theta\|_{L^p}^p=\int|P_t^0\theta|^{p-2}(P_t^0\theta)(P_t^0(-\Lambda^{2\alpha}+\frac{2\lambda_1}{p})\theta)dx\leq 0.$$

Letting $t\to 0$, we obtain our result. □

Proposition 4.4.2 Let $\alpha>\frac{1}{2}$. Let θ denote the solution of equation (4.1.3). Then for $2<p<\infty$

$$E\|\theta(t)\|_{L^p}^p\leq \|x\|_{L^p}^p e^{-\lambda_1 t}+\frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

Proof Using [Kr10, Lemma 5.1] for θ_n, we obtain

$$\begin{aligned}
\|\theta(t)\|_{L^p}^p=&\|\theta(s)\|_{L^p}^p+\int_s^t[-p\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)(\Lambda^{2\alpha}\theta(l)+u(l)\cdot\nabla\theta(l))d\xi dl\\
&+\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(l)|^{p-2}(\sum_j|k_{\delta_n}*G(e_j)|^2)d\xi]dl\\
&+p\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)k_{\delta_n}*Gd\xi dW(l)\\
\leq&\|\theta(s)\|_{L^p}^p-2\lambda_1\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^p d\xi dl\\
&+\int_s^t\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(l)|^{p-2}(\sum_j|k_{\delta_n}*G(e_j)|^2)d\xi dl\\
&+p\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)k_{\delta_n}*Gd\xi dW(l)
\end{aligned}$$

$$\leq \|\theta(s)\|_{L^p}^p - 2\lambda_1 \int_s^t \int_{\mathbb{T}^2} |\theta(l)|^p d\xi dl + \int_s^t (\varepsilon \int_{\mathbb{T}^2} |\theta(l)|^p d\xi + C(\varepsilon) \int (\sum_j |k_{\delta_n} * G(e_j)|^2)^{p/2} d\xi) dl$$
$$+ p \int_s^t \int_{\mathbb{T}^2} |\theta(l)|^{p-2}\theta(l) k_{\delta_n} * G d\xi dW(l),$$

where we used Lemma 4.4.1 to get the first inequality and our assumption on G to get the last inequality. Here for simplicity we write $\theta(t) = \theta_n(t,x)$. Taking expectation we obtain

$$E\|\theta_n(t)\|_{L^p}^p \leq E\|\theta_n(s)\|_{L^p}^p - E\lambda_1 \int_s^t \int_{\mathbb{T}^2} |\theta_n(l)|^p d\xi dl + C(\varepsilon,p)(t-s).$$

Then by Gronwall's lemma we have

$$E\|\theta_n(t)\|_{L^p}^p \leq \|\theta(0)\|_{L^p}^p e^{-\lambda_1 t} + \frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

Then taking the limit $n \to \infty$ in the above inequality we deduce

$$E\|\theta(t)\|_{L^p}^p \leq \|x\|_{L^p}^p e^{-\lambda_1 t} + \frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

□

Lemma 4.4.3 Let $\alpha > \frac{2}{3}$ and suppose Assumption 4.1.3 is satisfied with $s > 3-2\alpha$. Let θ denote the solution of (4.1.3) and take p as in Assumption 4.1.3. Then for every $R_0 \geq 1$, there exist values $T_1 = T_1(R_0)$ and $K_1 = K_1(R_0)$ such that if $|\theta_0| \leq R_0$, $\sup_{t\in[0,T_1]} \|\theta\|_{L^p}^p \leq R_0$, and $\sup_{t\in[0,T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 \leq R_0$ for some $0 < \delta < 3\alpha - 2 - \sigma_1$, then $|\Lambda^{s+\delta}\theta(T_1)|^2 \leq K_1$.

Proof By Itô's formula, we obtain that there exists $K_0 = K_0(R_0) > 0$ and for P-a.s. ω, $\exists\ t_0(\omega) > 0$ such that

$$|\Lambda^\alpha \theta(t_0)|^2 \leq K_0.$$

For any $r > 0$, by Lemmas 4.1.1, 4.1.2 we have the following a-priori estimate for $N = \frac{\alpha}{\alpha - \frac{1}{2} - \frac{1}{p}}$ and $\sigma = \frac{2}{p}$,

$$\begin{aligned}
\frac{d}{dt}|\Lambda^r v|^2 + |\Lambda^{r+\alpha} v|^2 &\leq |\langle u\cdot\nabla\theta, \Lambda^{2r} v\rangle| \\
&\leq C|\Lambda^{r+\alpha} v| \cdot |\Lambda^{r-\alpha+1+\sigma}\theta| \cdot \|\theta\|_{L^p} \qquad (4.4.1)\\
&\leq \frac{1}{4}|\Lambda^{r+\alpha} v|^2 + C\|\theta\|_{L^p}^N |\Lambda^r v|^2 + C|\Lambda^{r-\alpha+1+\sigma} z|^2 \cdot \|\theta\|_{L^q}^2.
\end{aligned}$$

We choose the approximation θ_n as at the beginning of this subsection with initial time $t = 0$ replaced by the initial time $t = t_0(\omega)$ and $\theta_n(t_0) = \theta(t_0)$. Set $z_n =$

$\int_{t_0}^t e^{-(t-s)A} k_{\delta_n} * GdW(s)$. Then we have the following L^p-norm estimate for $v_n := \theta_n - z_n$,

$$\frac{d}{dt}\|v_n\|_{L^p}^p \leq pC\|\nabla z_n\|_\infty(\|v_n\|_{L^p}^p + \|z_n\|_{L^p}\|v_n\|_{L^p}^{p-1}).$$

Thus we have

$$\frac{d}{dt}\|v_n\|_{L^p} \leq C\|\nabla z_n\|_\infty(\|v_n\|_{L^p} + \|z_n\|_{L^p}).$$

Then by Gronwall's lemma and since $s > 3 - 2\alpha$, we obtain the desired uniform L^p-norm estimates for v_n. Moreover, by (4.4.1) and Gronwall's lemma we obtain the uniform H^r-norm estimates for v_n. By a similar argument as in the proof of [RZZ12, Theorem 3.4] we have that v_n converges to some process $\tilde{v}$ in $L^2([t_0, T], H)$ such that $\tilde{v} + z$ is the solution of (4.1.3) in $[t_0, T]$. Then by the uniqueness proof in [RZZ12, Theorem 5.1] we have $\tilde{v} = v$ in $[t_0, \infty)$, which implies that $v \in L^\infty([t_0, \infty), H^r) \cap L^2_{\text{loc}}([t_0, \infty), H^{r+\alpha})$ P-a.s. ω. Therefore, (4.4.1) also holds for v with $t \in [t_0, \infty)$.

Then by (4.4.1) for $r = \alpha$, we obtain that there exist $K_1 = K_1(R_0) > 0$ and $t_1 = t_1(\omega) > t_0(\omega)$ such that $|\Lambda^{2\alpha} v(t_1)| \leq K_1$. Using (4.4.1) for $r = 2\alpha$ we obtain that there exists $T_0 = T_0(R_0)$ such that $|\Lambda^{2\alpha} v(T_0)| \leq K_1$. Then we proceed analogously and obtain that there exists $T_1 = T_1(R_0)$ such that $|\Lambda^{s+\delta} v(T_1)| \leq K_1$ for some $0 < \delta < 3\alpha - 2 - \sigma_1$. □

Lemma 4.4.4 Let $\alpha > \frac{2}{3}$ and suppose Assumption 4.1.3 holds with $s > 3 - 2\alpha$. Then for each $R \geq 1$ there are $T_1 > 0$ and a compact subset $K \subset \mathcal{W}$ such that

$$\inf_{\|x\|_{L^p} \leq R} P_{T_1}(x, K) > 0,$$

for p as in Assumption 4.1.3.

Proof Define $K := \{x : |\Lambda^{s+\delta} x|^2 \leq K_1(R_0)\}$, where $K_1(R_0), \delta$ comes from the previous lemma. By Lemma 4.4.3, for $R \leq R_0$ we have

$$\begin{aligned}\inf_{\|x\|_{L^p} \leq R} P_{T_1}(x, K) \geq \inf_{\|x\|_{L^p} \leq R} (1 - P_x[\sup_{t \in [0, T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 > R_0] \\ - P_x[\sup_{t \in [0, T_1]} \|\theta\|_{L^p}^p > R_0]),\end{aligned}$$

where we used Lemma 4.4.3 in the last step. Under Assumption 4.1.3, since z is a Gaussian process, one deduces that there exist $\eta, C > 0$ such that

$$P_x[\sup_{t \in [0, T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 > R_0] \leq Ce^{-\eta \frac{R_0^2}{T_1}},$$

(see e.g. [FR06, Proposition 15]). Also by [RZZ12, Theorem 4.3], we obtain

$$\sup_{\|x\|_{L^p}\le R} P_x[\sup_{t\in[0,T_1]} \|\theta\|^p_{L^p} > R_0] \le \sup_{\|x\|_{L^p}\le R} \frac{E_x[\sup_{t\in[0,T_1]} \|\theta\|^p_{L^p}]}{R_0} \le \frac{C(R)}{R_0}.$$

Choosing R_0 big enough, we prove the assertion. □

The exponential convergence now follows from Lemma 4.4.4 and an abstract result of [GM05, Theorem 3.1]. For p in Assumption 4.1.3, let $V : L^p \to \mathbb{R}$ be a measurable function and define $\|\phi\|_V := \sup_{x\in L^p} \frac{|\phi(x)|}{V(x)}$ and $\|\nu\|_V := \sup_{\|\phi\|_V\le 1}\langle \nu, \phi\rangle$ for a signed measure ν.

Theorem 4.4.5 Let $\alpha > \frac{2}{3}$. Suppose that Assumption 4.1.3 holds with $s > 3-2\alpha$ and let $V(x) := 1 + \|x\|^p_{L^p}$ for p as in Assumption 4.1.3. Then there exist $C_{\exp} > 0$ and $a > 0$ such that

$$\|P^*_t\delta_{x_0} - \mu\|_{TV} \le \|P^*_t\delta_{x_0} - \mu\|_V \le C_{\exp}(1 + \|x_0\|^p_{L^p})e^{-at},$$

for all $t > 0$ and $x_0 \in L^p$, where $\|\cdot\|_{TV}$ is the total variation distance on measures.

Proof By similar arguments as in the proof of Lemma 4.4.3 we obtain $P_t(x, \mathcal{W}) = 1$ for $x \in L^p$. By [GM05, Theorem 3.1], we need to verify the following four conditions,

1. the measures $(P_t(x, \cdot))_{t>0,x\in L^p}$ are equivalent,
2. $x \to P_t(x, \Gamma)$ is continuous in $\mathcal{W}$ for all $t > 0$ and all Borel sets $\Gamma \subset H$,
3. for each $R \ge 1$ there exist $T_1 > 0$ and a compact subset $K \subset \mathcal{W}$ such that
$$\inf_{\|x\|_{L^p}\le R} P_{T_1}(x, K) > 0,$$
4. there exist $k, b, c > 0$ such that for all $t \ge 0$,
$$E^{P_x}[\|\theta(t)\|^p_{L^p}] \le k\|x\|^p_{L^p}e^{-bt} + c.$$

Condition 1 can be verified by [GM05, Lemma 3.2] and since $P_t(x, \mathcal{W}) = 1$ for $x \in L^p$. The other conditions can be verified by Theorem 4.3.2, Lemma 4.4.4 and Proposition 4.4.2. □

Remark 4.4.6 (i) For $\alpha > \frac{3}{4}$ we can get a better result following a similar argument as in [R08]. Namely, there exist $C_{\exp} > 0$ and $a > 0$ such that

$$\|P^*_t\delta_{x_0} - \mu\|_{TV} \le \|P^*_t\delta_{x_0} - \mu\|_V \le C_{\exp}(1 + |x_0|^2)e^{-at},$$

for all $t > 0$ and $x_0 \in H$. Here P_t could be every Markov selection associated to the solution of equation (4.1.3).

(ii) The reason why $\alpha > \frac{3}{4}$ is needed, is as follows: As in Theorem 4.3.3, we can prove P_t is H^s-strong Feller with $s > 3-3\alpha$. And for a solution θ of equation (4.1.3) starting from $x \in H$, we can prove that it will enter H^α only under our condition on the noise. If the process θ enters H^s, we can prove that it satisfies the above four conditions. Hence, to obtain exponential convergence for every $x \in H$, we need the process starting from $x \in H$ to enter H^s. Hence we need $3-3\alpha < s < \alpha$, i.e. $\alpha > \frac{3}{4}$.

4.5 Ergodicity for $\alpha > 3/4$ driven by mildly degenerate noises

In this section, we assume that $\alpha > \frac{3}{4}$, $W(t)$ is a cylinder Wiener process on H and G is additive. We recall here that on the periodic domain $\mathbb{T}^2$, $\{\cos(kx)|k \in \mathbb{Z}^2_+\} \cup \{\sin(kx)|k \in \mathbb{Z}^2_-\}$ form an eigenbasis of $-\triangle$, which we denote by e_k in this section. Here $\mathbb{Z}^2_+ = \{(k_1,k_2) \in \mathbb{Z}^2|k_2 > 0\} \cup \{(k_1,0) \in \mathbb{Z}^2|k_1 > 0\}, \mathbb{Z}^2_- = \{(k_1,k_2) \in \mathbb{Z}^2|-k \in \mathbb{Z}^2_+\}, x \in \mathbb{T}^2$ and the corresponding eigenvalues are $|k|^2$. Define $\mathbb{Z}^2_* = \mathbb{Z}^2\backslash\{(0,0)\}$. Moreover, denote for any $N > 0$, $Z_L(N) = [-N,N]^2 \setminus (0,0)$ and $Z_H(N) = \mathbb{Z}^2_* \setminus Z_L(N)$.

4.5.1 The strong Feller property for $\alpha > 3/4$

Given $N \geq 1$, let $\pi_N : H \to H$ be the projection onto the subspace of H generated by all modes k such that $|k|_\infty := \max|k_i| \leq N$. Assume that $\alpha > \frac{3}{4}$ and G satisfies:

Assumption 4.5.1 The operator $G : H \to H$ is linear bounded and there are $\gamma > 1$ and an integer $N_0 \geq 1$ such that

[A1] (diagonality) G is diagonal on the basis $(e_k)_{k\in Z^2_*}$,

[A2] (finite degeneracy) $\pi_{N_0}G = 0$ and $\ker((Id-\pi_{N_0})G) = \{0\}$,

[A3] (regularity) $(Id-\pi_{N_0})A^{\frac{\gamma+\alpha}{2\alpha}}G$ is bounded invertible on $(Id-\pi_{N_0})H$.

Under Assumption 4.5.1, (G.1) is satisfied obviously and $QdW = \sum_{k\in Z_H(N_0)} e_k g_k dw_k$, where $(w_k)_{k\in Z_H(N_0)}$ is a sequence of independent 2D Brownian motions. Let $(P_x)_{x\in H}$ be any a.s. Markov process obtained in Theorem 4.2.5, and $(P_t)_{t\geq 0}$ be the associated transition semi-group on $\mathcal{B}_b(H)$, defined as

$$P_t(\varphi)(x) = E^x[\varphi(\xi_t)], \qquad x \in H, \varphi \in \mathcal{B}_b(H). \tag{4.5.1}$$

In this subsection we prove it has the strong Feller property under Assumption 4.5.1.

Choose s such that $\gamma < \gamma - \alpha + 1 < s < s - \alpha + 1 < \gamma + 2\alpha - 1$, and set $\mathcal{W} = D(\Lambda^s)$ and $|x|_{\mathcal{W}} = |\Lambda^s x|$. By this choice, we know that $\alpha > \frac{3}{4}$ is required.

Now we state the main result of this section.

Theorem 4.5.2 Under Assumption 4.5.1. Then $(P_t)_{t\geq 0}$ is $\mathcal{W}$-strong Feller, i.e. for every $t > 0$ and $\psi \in \mathcal{B}_b(H)$, $P_t\psi \in C_b(\mathcal{W})$.

In order to prove Theorem 4.5.2, we follow the approach of [EH01], [RX10] to construct P_x^ρ. We introduce an equation which differs from the original one by a cut-off only, so that with large probability they have the same trajectories on a small deterministic time interval. We consider the equation

$$d\theta^\rho(t) + A\theta^\rho(t)dt + \chi(\frac{|\theta^\rho|_{\mathcal{W}}}{3\rho})u^\rho(t)\cdot\nabla\theta^\rho(t)dt = G(\theta^\rho)dW(t), \tag{4.5.2}$$

where

$$G(\theta^\rho) = G + (1 - \chi(\frac{|\theta^\rho|_{\mathcal{W}}}{\rho}))\bar{G},$$

and $\bar{G}$ is non-degenerate operator on $\pi_{N_0}H$, and $\chi : \mathbb{R} \to [0, 1]$ of class C^∞ such that $\chi(r) = 1$ if $r \leq 1$, $\chi(r) = 0$ if $r \geq 2$ and with derivative bounded by 1. We could choose $\bar{G}$ is diagonal on the basis (e_k) i.e. $\bar{G}e_k = g_k e_k$, for $k \in Z_L(N_0)$. By the same arguments as in Theorem 4.3.4, we obtain the following results, where we used $s - \alpha + 1 < \gamma + 2\alpha - 1$.

Theorem 4.5.3 (Weak-strong uniqueness) Under Assumption 4.5.1. For every $x \in \mathcal{W}$, Eq. (4.5.2) has a unique martingale solution P_x^ρ, with

$$P_x^\rho[C([0,\infty); \mathcal{W})] = 1.$$

Let $\tau_\rho : \Omega \to [0, \infty]$ be defined as

$$\tau_\rho(\omega) = \inf\{t \geq 0 : |\omega(t)|_{\mathcal{W}} \geq \rho\},$$

and $\tau_\rho(\omega) = \infty$ if this set is empty. If $x \in \mathcal{W}$ and $|x|_{\mathcal{W}} < \rho$, then

$$\lim_{\varepsilon\to 0} P_{x+h}^\rho[\tau_\rho \geq \varepsilon] = 1, \text{ uniformly in } h \in \mathcal{W}, |h|_{\mathcal{W}} < 1. \tag{4.5.3}$$

Moreover, on $[0, \tau_\rho]$, the probability measure P_x^ρ coincides with any martingale solution P_x of the equation (4.1.3), namely

$$E^{P_x^\rho}[\varphi(\xi_t)1_{[\tau_\rho\geq t]}] = E^{P_x}[\varphi(\xi_t)1_{[\tau_\rho\geq t]}], \tag{4.5.4}$$

for every $t \geq 0$ and $\varphi \in \mathcal{B}_b(H)$.

4.5.2 Strong-Feller property of cutoff dynamics

In order to apply [FR08, Theorem 5.4], we need to prove the following result.

Theorem 4.5.4 There is $\rho_0 > 0$ such that for $\rho \geq \rho_0$, the transition semi-group $(P_t^\rho)_{t\geq 0}$ associated to Eq. (4.5.2) is $\mathcal{W}$-strong Feller.

Fix $N \geq N_0$ (whose value will be suitably chosen later). In this and the following subsection we shall denote with the superscript L the quantities projected onto the modes smaller than N and with the superscript H those projected onto the modes larger than N. We divide the equation (4.5.2) into the low and high frequency parts (dropping the ρ in θ^ρ for simplicity),

$$\begin{cases} d\theta^L + A\theta^L dt + \chi(\frac{|\theta|_{\mathcal{W}}}{3\rho})B_L(\theta,\theta)dt = G_L(\theta)dW_t^L, \\ d\theta^H + A\theta^H dt + \chi(\frac{|\theta|_{\mathcal{W}}}{3\rho})B_H(\theta,\theta)dt = G_H dW_t^H, \end{cases} \tag{4.5.5}$$

where $\theta^L = \pi_N\theta$, $\theta^H = (Id-\pi_N)u, W^L = \pi_N W$, $W^H = (Id-\pi_N)W$, $B_L = \pi_N B$, $B_H = (Id-\pi_N)B$ for $B(\theta,\theta) = u\cdot\nabla\theta$, $G_L(\theta) = G(\theta)\pi_N$ and $G_H = G(\theta)(Id-\pi_N)$.

With the above separation for the dynamics, it is natural to define the Frechet derivatives for their low and high frequency parts. We will always use the notations D_L and D_H to denote the derivatives with respect to H^L (resp. H^H) of a differentiable function defined on H. For instance, for any stochastic process $X(t,x)$ on H with $X(0,x) = x$, $D_H X^L(t,x) : H^H \to H^L$ is defined by

$$D_H X^L(t,x)h = D_h X^L(t,x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[X^L(t,x+\varepsilon h) - X^L(t,x)], h \in H^H.$$

Let $C_b^k(\mathcal{W})$ be the set of functions on $\mathcal{W}$ with bounded 0-th,...,k-th order derivatives. Similarly, for $\psi \in C_b^1(\mathcal{W})$, $D_L\psi(x)$ and $D_H\psi(x)$ can be defined.

To prove Theorem 4.5.4, we need to approximate (4.5.5) by the following more regular dynamics

$$\begin{cases} d\theta^{\delta,\rho} + A\theta^{\delta,\rho}dt + e^{-A_H\delta}\chi(\frac{|\theta^{\delta,\rho}|_{\mathcal{W}}}{3\rho})B(\theta^{\delta,\rho},\theta^{\delta,\rho})dt = G(\theta^{\delta,\rho})dW_t, \\ \theta^{\delta,\rho}(0) = x, \end{cases} \tag{4.5.6}$$

where $\delta > 0$ and $A_H = (Id-\pi_N)A$ (the existence and uniqueness of weak solution to equation (4.5.6) is standard).

Define two maps $\Phi_t(\cdot)$ and $\Phi_t^\delta(\cdot)$ from H to H by

$$\Phi_t(x) := \theta^\rho(t) \text{ and } \Phi_t^\delta(x) := \theta^{\delta,\rho}(t),$$

where $\theta^\rho(t), \theta^{\delta,\rho}(t)$ are the solutions to (4.5.5) and (4.5.6) respectively.

The key points for the proofs of Proposition 4.5.6 are the following two inequalities and Lemma 4.5.5. For given $\beta \geq 1$, there exist constants $C_1 > 0, C_2 > 0$ such that for every $\theta_1, \theta_2 \in D(\Lambda^{\beta+1+\sigma_1}), \sigma_1 + \sigma_2 = 1$,

$$|\Lambda^\beta B(\theta_1, \theta_2)| \leq C_1(|\Lambda^{\beta+1+\sigma_1}\theta_1||\Lambda^{\sigma_2}\theta_2| + |\Lambda^{\beta+1+\sigma_1}\theta_1||\Lambda^{\sigma_2}\theta_2|). \tag{4.5.7}$$

$$|\Lambda^\beta e^{-At} B(\theta_1, \theta_2)|_H \leq \frac{C_2}{t^{1-\frac{\varepsilon}{2\alpha}}}|\Lambda^\beta\theta_1|_H|\Lambda^\beta\theta_2|_H, \tag{4.5.8}$$

for some $0 < \varepsilon < 2\alpha - 1$. (4.5.7) and (4.5.8) can be obtained by Lemma 4.1.1.

Lemma 4.5.5 (c.f. [DZ92]) Let $G : H \to H$ be a linear bounded operator such that $\Lambda^{\gamma+\alpha}G$ is also bounded. Then for any $\varepsilon_1 < \frac{1}{2}(\gamma + 2\alpha - 1 - s)$ and $\beta_1 < \frac{1}{2}(\gamma + 2\alpha - 1 - s) - \varepsilon_1$, there exists $C(\varepsilon, \beta, p, T) > 0$ such that

$$E[\sup_{0\leq t\leq T} |\Lambda_1^\varepsilon \int_0^t e^{-A(t-l)} G dW_l|_{\mathcal{W}}^p] \leq C(\varepsilon_1, \beta_1, p, T)T^{\beta_1 p}.$$

By (4.5.7), (4.5.8) and Lemma 4.5.5 we obtain the following estimates.

Proposition 4.5.6 For every $T > 0$ and $p \geq 2$, there exist some $C_i = C_i(p, \rho, \gamma, \alpha) > 0, i = 1, 2$ such that

$$E[\sup_{0\leq t\leq T} |\Phi_t - \Phi_t^\delta|_{\mathcal{W}}^p] \leq C_1 e^{C_1 T}|e^{-A\delta} - Id|_{\mathcal{L}(\mathcal{W})}^p, \tag{4.5.9}$$

$$E[\sup_{0\leq t\leq T} |D\Phi_t - D\Phi_t^\delta|_{\mathcal{L}(\mathcal{W})}^p] \leq C_2 e^{C_2 T}|e^{-A\delta} - Id|_{\mathcal{L}(\mathcal{W})}^p. \tag{4.5.10}$$

For any $\psi \in C_b^1(\mathcal{W}), h \in \mathcal{W}$ and $t > 0$,

$$\lim_{\delta\to 0+} |D_h E[\psi(\Phi_t^\delta)] - D_h E[\psi(\Phi_t)]| = 0 \tag{4.5.11}$$

Proof Denote $\Psi_t = \Phi_t - \Phi_t^\delta$, we have

$$\begin{aligned}\Psi_t =& \int_0^t e^{-A(t-r)}[B(\Phi_r,\Phi_r)\chi(\frac{|\Phi_r|_{\mathcal{W}}}{3\rho}) - e^{-A\delta}B(\Phi_r^\delta,\Phi_r^\delta)\chi(\frac{|\Phi_r^\delta|_{\mathcal{W}}}{3\rho})]dr\\ &+\int_0^t e^{-A(t-r)}[G(\Phi_r)-G(\Phi_r^\rho)]dW_r\\ :=& \int_0^t I_1 dr + \int_0^t I_2 dW_r.\end{aligned}$$

By (4.5.7) and (4.5.8) we obtain

$$|I_1|_{\mathcal{W}} \leq C_1(t-r)^{-1+\frac{\varepsilon}{2\alpha}}|Id-e^{-A\delta}|_{\mathcal{L}(\mathcal{W})} + C_2(t-r)^{-1+\frac{\varepsilon}{2\alpha}}|\Psi_r|_{\mathcal{W}},$$

for ε in (4.5.8). By Lemma 4.5.5 we obtain

$$E[\sup_{0\leq t\leq T}|\int_0^t I_2 dW_r|^p] \leq C_3 T^{\beta_1 p}E[\sup_{0\leq t\leq T}|\Psi_t|_{\mathcal{W}}^p],$$

with $p\geq 2$ and β_1 in Lemma 4.5.5. By the above two estimates and induction argument (4.5.9) follows. (4.5.10) can be obtained by the same method and (4.5.11) follows by (4.5.9) and (4.5.10). □

Moreover, we obtain the following estimates by using (4.5.7), (4.5.8) and Lemma 4.5.5. We choose $0<\varepsilon_0<\frac{1}{2}(\gamma+2\alpha-1-s), 0<\varepsilon_1<\frac{2\alpha-1}{2\alpha}$ and define $\widetilde{\mathcal{W}} := D(\Lambda^{s+\varepsilon_0})$ and $|x|_{\widetilde{\mathcal{W}}} = |\Lambda^{s+\varepsilon_0}x|$.

Lemma 4.5.7 For any $T>0, p\geq 2$ and $\delta\geq 0$, there exist some $C_i = C_i(p,\rho,\gamma,\alpha) > 0, i=1,...,7$ such that

$$E[\sup_{0\leq t\leq T}|\Phi_t^\delta|_{\mathcal{W}}^p] \leq C_1 e^{C_1 T}|x|_{\mathcal{W}}^p, \tag{4.5.12}$$

$$E[\sup_{0\leq t\leq T}|\Phi_t^\delta|_{\widetilde{\mathcal{W}}}^p] \leq C_2 e^{C_2 T}|x|_{\widetilde{\mathcal{W}}}^p, \tag{4.5.13}$$

$$E[\sup_{0\leq t\leq T}|t^{\frac{\varepsilon_0}{2\alpha}}\Phi_t^\delta|_{\widetilde{\mathcal{W}}}^p] \leq C_3 e^{C_3 T}|x|_{\mathcal{W}}^p, \tag{4.5.14}$$

$$E[\sup_{0\leq t\leq T}|D_h\Phi_t^\delta|_{\mathcal{W}}^p] \leq C_4 e^{C_4 T}|h|_{\mathcal{W}}^p, \qquad h\in\mathcal{W}, \tag{4.5.15}$$

$$E[\int_0^t |\Lambda^{\alpha+\gamma}D_h\Phi_\tau^\delta|^2 d\tau] \leq C_5 e^{C_5 t}|h|_{\mathcal{W}}^2, \qquad h\in\mathcal{W}, \tag{4.5.16}$$

(4.5.17) $$E[\sup_{0\leq t\leq T}|D_{h^L}\Phi_t^{\delta,H}|_{\mathcal{W}}^p] \leq (T^{p\varepsilon_1}\vee T^{\varepsilon_0 p})C_6 e^{C_6 T}|h^L|_{\mathcal{W}}^p, \qquad h^L\in\mathcal{W}^L,$$

(4.5.18) $$E[\sup_{0\leq t\leq T}|D_{h^H}\Phi_t^{\delta,L}|_{\mathcal{W}}^p] \leq (T^{\varepsilon_1 p}\vee T^{\varepsilon_0 p})C_7 e^{C_7 T}|h^H|_{\mathcal{W}}^p, \qquad h\in\mathcal{W}^H.$$

Proof (4.5.12)-(4.5.14) can be proved by a similar argument as the proof of (4.5.9), so we omit them here. For (4.5.15), we have that for every $h\in\mathcal{W}$, $D_h\Phi_t$ satisfies the following equation

$$\begin{aligned}
D_h\Phi_t =&e^{-At}h + \int_0^t e^{-A(t-r)}(B(D_h\Phi_r,\Phi_r)+B(\Phi_r,D_h\Phi_r))\chi(\frac{|\Phi_r|_{\mathcal{W}}}{3\rho})\\
&+ e^{-A(t-r)}B(D_h\Phi_r,\Phi_r)\chi'(\frac{|\Phi_r|_{\mathcal{W}}}{3\rho})\frac{1}{3\rho}\frac{\langle D_h\Phi_r,\Phi_r\rangle_{\mathcal{W}}}{|\Phi_r|_{\mathcal{W}}}dr\\
&- \int_0^t e^{-A(t-r)}\chi'(\frac{|\Phi_r|_{\mathcal{W}}}{\rho})\frac{1}{\rho}\frac{\langle D_h\Phi_r,\Phi_r\rangle_{\mathcal{W}}}{|\Phi_r|_{\mathcal{W}}}G_L dW_s^L,
\end{aligned}$$

By (4.5.7), (4.5.8) and Lemma 4.5.5, (4.5.15) follows. Similarly we obtain (4.5.17) and (4.5.18).

Let us prove (4.5.16). By Itô's formula, we have for $\sigma_1+\sigma_2=1, 0<\sigma_1<(s-\gamma+\alpha-1)\wedge(2\alpha-1)$,

$$\begin{aligned}
&E|\Lambda^\gamma D_h\Phi_t|^2 + 2\int_0^t E|\Lambda^{\gamma+\alpha}D_h\Phi_l|^2 dl\\
\leq&|\Lambda^\gamma h|^2 + C\int_0^t E|\Lambda^s D_h\Phi_l|^2 dl + C\int_0^t E[|\Lambda^{\gamma+\alpha}D_h\Phi_l||\Lambda^{\gamma-\alpha}B(\Phi_l,D_h\Phi_l)|\chi(\frac{|\Phi_l|_{\mathcal{W}}}{3\rho})]dl\\
\leq&|\Lambda^\gamma h|^2 + C\int_0^t E|\Lambda^s D_h\Phi_l|^2 dl + C\int_0^t E[|\Lambda^{\gamma+\alpha}D_h\Phi_l||\Lambda^{\gamma-\alpha+1+\sigma_1}\Phi_l||\Lambda^{\sigma_2}D_h\Phi_l|\\
&+|\Lambda^{\gamma+\alpha}D_h\Phi_l||\Lambda^{\gamma-\alpha+1+\sigma_1}D_h\Phi_l||\Lambda^{\sigma_2}\Phi_l|)|\chi(\frac{|\Phi_l|_{\mathcal{W}}}{3\rho})|]dl.
\end{aligned}$$

Then by $\gamma-\alpha+1<s, \gamma-\alpha+1+\sigma_1<\gamma+\alpha$, we obtain

$$E|\Lambda^\gamma D_h\Phi_t|^2 + 2\int_0^t E|\Lambda^{\gamma+\alpha}D_h\Phi_l|^2 dl \leq |\Lambda^\gamma h|^2 + C\int_0^t E|\Lambda^\gamma D_h\Phi_l|^2 dl,$$

which implies (4.5.16) by Gronwall's lemma. □

Lemma 4.5.8 There exists some constant $p>1$ such that for every $x\in\widetilde{\mathcal{W}}$, $h\in\mathcal{W}^L$, $\psi\in C_b^1(H)$

$$|E[(D_L\psi)(\Phi_t^\delta(x))D_h\Phi_t^{\delta,L}(x)]| \leq \frac{Ce^{Ct}(1+|x|_{\widetilde{\mathcal{W}}})^p}{t^p}\|\psi\|_\infty|h|_{\mathcal{W}}.$$

The proof of Lemma 4.5.8 will be given in the next section. Now we could prove Theorem 4.5.4 by using Lemmas 4.5.7, 4.5.8. The proof is a modification of the proof of [RX10, Theorem 3.1].

Proof of Theorem 4.5.4 Set $S_t\psi(x) = E[\psi(\Phi_t^\delta)]$ for any $\psi \in C_b^2(\mathcal{W})$, we prove the theorem in the following two steps.

Step 1. Estimate $DS_t\psi(x)$ for all $x \in \widetilde{\mathcal{W}}$. By (4.5.16) we know $y_t^H = G_H^{-1}D_{h^H}\Phi_t^{\delta,H} \in H^H dt \times dP$-a.s.. Hence we could proceed as in the proof of Proposition 5.2 of [EH01] to get

$$D_{h^H}S_t\psi(x) = \frac{2}{t}E[\psi(\Phi_t^\delta)\int_{\frac{t}{4}}^{\frac{3t}{4}} \langle y_s^H, dW_s^H\rangle_H] + \frac{2}{t}E[\int_{\frac{t}{4}}^{\frac{3t}{4}} D_L S_{t-s}\psi(\Phi_s^\delta)D_{h^H}\Phi_s^{\delta,L}ds].$$

Hence, by B-D-G inequality and (4.5.16), we obtain

$$\begin{aligned}|D_{h^H}S_t\psi(x)| \leq &\frac{2}{t}\|\psi\|_\infty(\int_{\frac{t}{4}}^{\frac{3t}{4}} E|y_r^H|^2dr)^{1/2} + \frac{2}{t}\int_{\frac{t}{4}}^{\frac{3t}{4}} E[|D_L S_{t-r}\psi(\Phi_s^\delta)|_{\mathcal{W}'}|D_{h^H}\Phi_r^{\delta,L}|_{\mathcal{W}}]dr\\ \leq &\frac{2}{t}ce^{ct}\|\psi\|_\infty|h^H|_{\mathcal{W}} + \frac{2}{t}\int_{\frac{t}{4}}^{\frac{3t}{4}} E[|D_L S_{t-r}\psi(\Phi_s^\delta)|_{\mathcal{W}'}|D_{h^H}\Phi_r^{\delta,L}|_{\mathcal{W}}]dr.\end{aligned}$$

For the low frequency part, according to Lemma 4.5.8, we obtain

$$\begin{aligned}|D_{h^L}S_t\psi(x)| =&|E[D_L S_{\frac{t}{2}}\psi(\Phi_{t/2}^\delta)D_{h^L}\Phi_{t/2}^{\delta,L}]| + |E[D_H S_{\frac{t}{2}}\psi(\Phi_{t/2}^\delta)D_{h^L}\Phi_{t/2}^{\delta,H}]|\\ \leq &C_2e^{C_2t}(1+|x|_{\widetilde{\mathcal{W}}})^p t^{-p}\|\psi\|_\infty|h^L|_{\mathcal{W}} + E[|D_H S_{\frac{t}{2}}\psi(\Phi_{t/2}^\delta)|_{\mathcal{W}'}|D_{h^L}\Phi_{t/2}^{\delta,H}|_{\mathcal{W}}],\end{aligned}$$

where $p > 1$ is the constant in Lemma 4.5.8.

By this we obtain for $0 < t \leq T$ and T sufficiently small, (see e.g. [RX10, Theorem 3.1])

$$|DS_t\psi(x)|_{\mathcal{W}'} \leq \frac{C(1+|x|_{\widetilde{\mathcal{W}}}^p)^p}{t^p}\|\psi\|_\infty,$$

with $C = C(T, \rho, \gamma, \alpha)$.

Step 2. Strong Feller property of P_t^ρ. For any $h \in \mathcal{W}$ and $0 < t \leq T$, we have

$$\begin{aligned}|D_h S_{2t}\psi(x)|^2 =&|E[DS_t\psi(\Phi_t^\delta)D_h\Phi_t^\delta]|^2 \leq E[|DS_t\psi(\Phi_t^\delta)|_{\mathcal{W}'}^2]E[|D_h\Phi_t^\delta|_{\mathcal{W}}^2]\\ \leq &\frac{C}{t^{2p}}\|\psi\|_\infty^2 E[(1+|\Phi_t^\delta|_{\widetilde{\mathcal{W}}})^{2p}]|h|_{\mathcal{W}}^2 \leq \frac{C}{t^{2p+\varepsilon_0 p/\alpha}}\|\psi\|_\infty^2(1+|x|_{\mathcal{W}})^{2p}|h|_{\mathcal{W}}^2,\end{aligned}$$

where $C = C(T, \rho, \alpha, \gamma)$. Let $\delta \to 0$, we have

$$|D_h P_{2t}\psi(x)| \leq \frac{C}{t^{p+\varepsilon_0 p/(2\alpha)}}\|\psi\|_\infty(1+|x|_{\mathcal{W}})^p|h|_{\mathcal{W}}, 0 < t \leq T.$$

This implies that $(P_t^\rho)_{t\in(0,T]}$ is strong Feller. The extension of the strong Feler property to arbitrary $T > 0$ is standard.

4.5.3 Malliavin calculus

Proof of Lemma 4.5.8

In this subsection, we will only study the equation (4.5.6), following the idea of [N85], [EH01] . We will simply write $\Phi_t = \Phi_t^\delta$ throughout this subsection since all the estimates obtained are independent of δ.

Given $v \in L^2_{\text{loc}}(\mathbb{R}^+, H)$, the Malliavin derivative of Φ_t in direction v, denoted by $\mathcal{D}_v\Phi_t$, is defined by

$$\mathcal{D}_v\Phi_t = \lim_{\varepsilon\to 0} \frac{\Phi_t(W+\varepsilon V, x) - \Phi_t(W, x)}{\varepsilon},$$

where $V(t) = \int_0^t v(s)ds$. The direction v can be random and is adapted to the filtration generated by W. The Malliavin derivatives on the low and high frequency parts, denoted by $\mathcal{D}_v\Phi_t^L$ and $\mathcal{D}_v\Phi_t^H$, can be defined in a similar way. $\mathcal{D}_v\Phi_t^L$ and $\mathcal{D}_v\Phi_t^H$ satisfy the following two SPDEs respectively:

$$\begin{aligned}
&d\mathcal{D}_v\Phi^L + [A\mathcal{D}_v\Phi^L + \tilde{B}_L(\mathcal{D}_v\Phi^L, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + \tilde{B}_L(\mathcal{D}_v\Phi^H, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) \\
&+ B_L(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L + B_L(\Phi,\Phi)D_H(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^H]dt \\
=&[D_LG_L(\Phi)\mathcal{D}_v\Phi^L + D_HG_L(\Phi)\mathcal{D}_v\Phi^H]dW_t^L + G_L(\Phi)v^L dt,
\end{aligned} \tag{4.5.19}$$

(4.5.20)

$$\begin{aligned}
&d\mathcal{D}_v\Phi^H + [A\mathcal{D}_v\Phi^H + e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^L, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^H, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) \\
&+ e^{-A_H\delta}B_H(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L + e^{-A_H\delta}B_H(\Phi,\Phi)D_H(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^H]dt \\
=&G_Hv^H dt,
\end{aligned}$$

with $\mathcal{D}_v\Phi_0^L = 0$ and $\mathcal{D}_v\Phi_0^H = 0$, where $\tilde{B}(u,v) = B(u,v) + B(v,u)$. Moreover, we define a flow between s and t by $J_{s,t}(x)$, $s \le t$, which satisfies the following equation: for all $h \in H^L$

(4.5.21)

$$\begin{aligned}
&dJ_{s,t}h + [AJ_{s,t}h + \tilde{B}_L(J_{s,t}h, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + B_L(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))J_{s,t}h]dt \\
=&D_LG_L(\Phi)J_{s,t}hdW_t^L,
\end{aligned}$$

with $J_{s,s}(x) = Id \in \mathcal{L}(H^L, H^L)$. The inverse $J_{s,t}^{-1}(x)$ satisfies

(4.5.22)
$$dJ_{s,t}^{-1}h + J_{s,t}^{-1}[Ah + \tilde{B}_L(h,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + B_L(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))h - \mathrm{Tr}((\mathrm{D_L G_L(\Phi_t)})^2)\mathrm{h}]\mathrm{dt}$$
$$= - J_{s,t}^{-1}D_L G_L(\Phi)hdW_t^L.$$

Simply writing $J_t = J_{0,t}$, clearly $J_{s,t} = J_t J_s^{-1}$.

We follow the ideas in Section 6.1 of [EH01] to develop a Malliavin calculus for (4.5.6). One of the key points for this approach is to find an adapted process $v \in L^2_{\mathrm{loc}}(\mathbb{R}^+, H)$ so that

(4.5.23)
$$G_H v^H(t) = e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^L,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + e^{-A_H\delta}B_H(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L,$$

which implies that $\mathcal{D}_v\Phi_t^H = 0$ for all $t > 0$.

Proposition 4.5.9 There exists $v \in L^2_{\mathrm{loc}}(\mathbb{R}^+, H)$ satisfying (4.5.23), and

$$\mathcal{D}_v\Phi_t^L = J_t\int_0^t J_s^{-1}G_L(\Phi_s)v^L(s)ds \text{ and } \mathcal{D}_v\Phi_t^H = 0.$$

Proof We first claim that

(4.5.24)
$$e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^L,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + e^{-A_H\delta}B_H(\Phi,\Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L \in (D(\Lambda^{\gamma+\alpha}))^H.$$

Since $\mathcal{D}_v\Phi_t^L$ is finite dimensional, $\mathcal{D}_v\Phi_t^L \in \mathcal{W}$. The two terms on the left hand of (4.5.24) can all be bounded in the same way, for instance

$$\begin{aligned}|\Lambda^{\gamma+\alpha}e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^L,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho})| =& |\Lambda^{\gamma+\alpha}e^{-A_H\delta}\tilde{B}_H(\mathcal{D}_v\Phi^L,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho})| \\ \le& C_1\delta^{\frac{-\alpha-\gamma}{2\alpha}}|\mathcal{D}_v\Phi_t^L|_{\mathcal{W}}|\Phi_t|_{\mathcal{W}},\end{aligned}$$

and (4.5.24) follows immediately. Hence by Assumption [A3] for G, there exists at least one $v^H \in L^2_{\mathrm{loc}}(\mathbb{R}^+, H^H)$ satisfying (4.5.23). Thus equation (4.5.20) is a homogeneous linear equation and has a unique solution

$$\mathcal{D}_v\Phi_t^H = 0,$$

for all $t > 0$. Then equation (4.5.19) now reads as

$$\begin{aligned}&d\mathcal{D}_v\Phi^L + [A\mathcal{D}_v\Phi^L + \tilde{B}_L(\mathcal{D}_v\Phi^L, \Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + B_L(\Phi, \Phi)D_L(\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L]dt \\ =&D_L G_L(\Phi)\mathcal{D}_v\Phi^L dW_t^L + G_L(\Phi)v^L dt,\end{aligned}$$

with $\mathcal{D}_v\Phi_0^L = 0$. Applying the Itô's formula to the product $J^{-1}\mathcal{D}_v\Phi^L$ we see immediately that

$$\mathcal{D}_v\Phi_t^L = J_t\int_0^t J_s^{-1}G_L(\Phi_s)v^L(s)ds. \tag{4.5.25}$$

Let $N \geq N_0$ be the integer fixed and M is the dimension of $\pi_N H$. Consider $v_1, ..., v_L \in L^2_{\text{loc}}(\mathbb{R}^+; H)$ with each of them satisfying Proposition 4.5.9. Set

$$v = [v_1, ..., v_M],$$

we have

$$\mathcal{D}_v\Phi_t^H = 0, \qquad \mathcal{D}_v\Phi_t^L = J_t\int_0^t J_s^{-1}G_L(\Phi_s)v^L(s)ds.$$

Choose $v^L(s) = (J_s^{-1}G_L(\Phi_s))^*$ and define the *Malliavin matrix*

$$\mathcal{M}_t = \int_0^t J_s^{-1}G_L(\Phi_s)(J_s^{-1}G_L(\Phi_s))^* ds.$$

The following two lemmas are crucial for the proof of Lemma 4.5.8. The first one can be proved by a similar argument as Lemma 4.5.7 and [RX10, Lemma 4.2], so we omit it here.

Lemma 4.5.10 For any $T > 0$ and $p \geq 2$, there exist some $C_i = C_i(p, \rho, \gamma, \alpha) > 0 (i = 1, ..., 4)$ such that

$$E(\sup_{0\leq t\leq T}|J_t(x)h^L|_{\mathcal{W}}^p) \leq C_1 e^{C_1 T}|h^L|_{\mathcal{W}}^p, \tag{4.5.26}$$

$$E(\sup_{0\leq t\leq T}|J_t^{-1}(x)h^L|_{\mathcal{W}}^p) \leq C_2 e^{C_2 T}|h^L|_{\mathcal{W}}^p, \tag{4.5.27}$$

$$E(\sup_{0\leq t\leq T}|J_t^{-1}(x)h^L - h^L|_{\mathcal{W}}^p) \leq T^{p/2}C_3 e^{C_3 T}|h^L|_{\mathcal{W}}^p, \tag{4.5.28}$$

$$E(\sup_{0\le t\le T}|\Phi_t(x)-e^{-At}x|^p_{\mathcal{W}})\le (T^{\varepsilon_0 p}\vee T^{p\varepsilon_1})C_4e^{C_4T}, \tag{4.5.29}$$

Suppose that v_1, v_2 satisfy Proposition 4.5.9 and $p\ge 2$, then

$$E(\sup_{0\le t\le T}|\mathcal{D}_{v_1}\Phi^L_t(x)|^p_{\mathcal{W}})\le C_5e^{C_5T}E[\int_0^T|v^L_1(s)|^p_{\mathcal{W}}ds], \tag{4.5.30}$$

(4.5.31)

$$E(\sup_{0\le t\le T}|\mathcal{D}^2_{v_1v_2}\Phi^L_t(x)|^p_{\mathcal{W}})\le C_6e^{C_6T}(E[\int_0^T|v^L_1(s)|^{2p}_{\mathcal{W}}ds])^{1/2}(E[\int_0^T|v^L_2(s)|^{2p}_{\mathcal{W}}ds])^{1/2},$$

$$E(\sup_{0\le t\le T}|\mathcal{D}_{v_1}D_h\Phi^L_t(x)|^p_{\mathcal{W}})\le C_7e^{C_7T}|h|^p_{\mathcal{W}}(E[\int_0^T|v^L_1(s)|^{2p}_{\mathcal{W}}ds])^{1/2}, \tag{4.5.32}$$

with $h\in\mathcal{W}$ and $C_i=C_i(p,\rho,\gamma,\alpha)>0, i=5,6,7$.

Lemma 4.5.11 Suppose that Φ_t is the solution to equation (4.5.6) with initial data $x\in\widetilde{\mathcal{W}}$. Then $\mathcal{M}_t\in\mathcal{L}(\mathcal{W}^L,\mathcal{W}^L)$ is invertible almost surely. Denote $\lambda_{\min}(t)$ the smallest eigenvalue of $\mathcal{M}_t$. Then there exists some $q>1$ such that for every $p>0$ there is some $C=C(p,\rho,\alpha,\gamma)$ such that

$$P[|1/\lambda_{\min}(t)|\ge 1/\varepsilon^q]\le\frac{C\varepsilon^{\varepsilon_0 p}(1+|x|_{\widetilde{\mathcal{W}}})^p}{t^p}. \tag{4.5.33}$$

Now we are ready to prove Lemma 4.5.8, which is a modification from [EH01, Proposition 7.12].

Proof of Lemma 4.5.8 Let us consider for $i,k=1...M$,

$$\psi_{ik}(\Phi_t)=\psi(\Phi_t)\sum_{j=1}^M[(\mathcal{D}_v\Phi^L_t)^{-1}]_{ij}[D_L\Phi^L_t]_{jk}.$$

For $h\in\mathcal{W}^L$, we obtain

$$\begin{aligned}D_L\psi_{ik}(\Phi_t)\mathcal{D}_v\Phi^L_th=&D_L\psi(\Phi_t)\mathcal{D}_v\Phi^L_th\sum_{j=1}^M[(\mathcal{D}_v\Phi^L_t)^{-1}]_{ij}[D_L\Phi^L_t]_{jk}\\&+\psi(\Phi_t)\sum_{j=1}^M\mathcal{D}_{vh}\{[(\mathcal{D}_v\Phi^L_t)^{-1}]_{ij}[D_L\Phi^L_t]_{jk}\}.\end{aligned}$$

Set $h=h_i$, where h_i is the standard orthonormal basis of $\mathbb{R}^M$ and sum over i, we obtain

$$
\begin{aligned}
&E[D_L\psi(\Phi_t)\mathcal{D}_{h_k}\Phi_t^L h]\\
=&E[\sum_{i=1}^M \mathcal{D}_{vh_i}\psi_{ik}(\Phi_t)] - E[\psi(\Phi_t)\sum_{i,j=1}^M \mathcal{D}_{vh_i}\{[(\mathcal{D}_v\Phi_t^L)^{-1}]_{ij}[D_L\Phi_t^L]_{jk}\}].
\end{aligned}
$$

For the first term on the right hand of the above equality, by Bismut integration by parts formula(see e.g. [EH01, Proposition 6.1]) and $\mathcal{D}_v\Phi_t^L = J_t\mathcal{M}_t$,

$$
|E[\sum_{i=1}^M D_L\psi_{ik}(\Phi_t)\mathcal{D}_{vh_i}\Phi_t^L]|
$$

$$
\begin{aligned}
\leq &\sum_{i,j=1}^M |E[\psi(\Phi_t)[J_t^{-1}\mathcal{M}_t^{-1}]_{ij}[D_L\Phi_t^L]_{jk}\int_0^t \langle v^L h_i, dW_r\rangle]|\\
\leq &\|\psi\|_\infty \sum_{i,j=1}^M E[\frac{1}{\lambda_{\min}}|J_t^{-1}h_j||[D_{h_k}\Phi_t^L]_{jk}|\int_0^t \langle v^L h_i, dW_r\rangle|].
\end{aligned}
$$

Then by B-D-G inequality, (4.5.33), (4.5.27), we obtain

$$
|E[\sum_{i=1}^M \mathcal{D}_{vh_i}\psi_{ik}(\Phi_t)]| \leq \|\psi\|_\infty Ce^{Ct}(1+|x|_{\widetilde{\mathcal{W}}})^p t^{-p}.
$$

The other term can be estimated similarly, the assertion follows. □

4.5.4 Hörmander's systems

Let us consider the SPDE for θ^L in Stratanovich form as

(4.5.34)

$$
d\theta^L + A\theta^L dt + \chi(\frac{|\theta|_{\mathcal{W}}}{3\rho})B_L(\theta,\theta)dt - \frac{1}{2}\sum_{k\in Z_L(N_0)} Dg_k(\theta)e_k\cdot g_k(\theta)e_k dt = \sum_{k\in Z_L(N)} g_k(u)\circ dw_k(t)e_k,
$$

where $g_k(\theta) = (1-\chi(\frac{|\theta|_{\mathcal{W}}}{\rho}))g_k$ for $k \in Z_L(N_0)$ and $g_k(\theta) = g_k$ for $k \in Z_L(N)\backslash Z_L(N_0)$. For any $x \in \mathcal{W}$, it is clear that if $k \in Z_L(N_0)$

$$
Dg_k(x)e_k \cdot g_k(x)e_k = -\frac{1}{\rho}\chi'(\frac{|x|_{\mathcal{W}}}{\rho})(1-\chi(\frac{|x|_{\mathcal{W}}}{\rho}))g_k^2\frac{\langle x, e_k\rangle_{\mathcal{W}}}{|x|_{\mathcal{W}}}e_k.
$$

For any two Banach spaces E_1, E_2, we denote the set of all C^∞ functions $E_1 \to E_2$, which are polynomially bounded together with all their derivatives by $P(E_1, E_2)$. If

$K \in P(H, H^L)$ and $X \in P(H,H)$, define $[X,K]_L$ by

$$[X,K]_L(x) = DK(x)X(x) - D_L X^L(x)K(x), x \in H.$$

For instance, $[A,K]_L \in P(D(A), H^L)$ with $[A,K]_L(x) = DK(x)Ax - A_L K(x)$. Define

$$X^0(x) = Ax + \chi(\frac{|x|_{\mathcal{W}}}{\rho})e^{-\delta A_H}B(x,x) + \sum_{k\in Z_L(N_0)} \frac{1}{2\rho}\chi'(\frac{|x|_{\mathcal{W}}}{\rho})(1-\chi(\frac{|x|_{\mathcal{W}}}{\rho}))g_k^2\frac{\langle x, e_k\rangle_{\mathcal{W}}}{|x|_{\mathcal{W}}}e_k.$$

Definition 4.5.12 The Hörmander's system $\mathbf{K}$ for equation (4.5.34) is defined as follows: given any $y \in \mathcal{W}$, define

$$\begin{aligned}
\mathbf{K}_0(y) =& \{g_k(y)e_k : k \in Z_L(N)\},\\
\mathbf{K}_1(y) =& \{[X^0(y), g_k(y)e_k]_L : k \in Z_L(N)\}\\
\mathbf{K}_2(y) =& \{[g_k(y)e_k, K(y)]_L : K \in \mathbf{K}_1(y), k \in Z_L(N)\},
\end{aligned}$$

and $\mathbf{K}(y) = \mathbf{K}_0(y) \cup \mathbf{K}_1(y) \cup \mathbf{K}_2(y)$.

Proposition 4.5.13 There exist $\bar\rho > 0$ and $\bar N \geq N_0$ (which depend only on N_0 and G) such that if $\rho \geq \bar\rho$ and $N \geq \bar N$, then the following property holds : for every $x \in \mathcal{W}$ and $h \in H^L$ there exist $\sigma > 0$ and $R > 0$ such that

$$\inf_{\delta>0}\sup_{K\in\mathbf{K}}\inf_{|y-x|_{\mathcal{W}}\leq R} |\langle K(y), h\rangle_{\mathcal{W}}| \geq \sigma |h|_{\mathcal{W}}.$$

Proof The basic idea of the proof follows from [EH01, Theorem 7.8] and [RX10, Proposition 4.5]. It is sufficient to show that there is a (finite) set $\tilde{\mathbf{K}} \subset \mathbf{K}(y)$ such that span$\tilde{\mathbf{K}} = H^L$. We choose $R \leq \frac{1}{4}\rho$.

Case 1: $|x|_{\mathcal{W}} \geq R + 2\rho$. In this case $g_k(y) = g_k$, we can take $\tilde{\mathbf{K}} = \mathbf{K}_0$.

Case 2: $|x|_{\mathcal{W}} \leq \rho - R$. In this case $X^0(y) = Ay + e^{-\delta A_H}B(y,y)$. Since

$$\langle B(e^{il\cdot x}, e^{im\cdot x}), e^{ik\cdot x})\rangle = \begin{cases} -\frac{1}{|l|}(l^\perp \cdot m), & \text{if } l+m=k,\\ 0, & \text{if } l+m\neq k,\end{cases}$$

we could calculate $B(e_l, e_m)$ easily. For instance, for $l, m, l-m \in \mathbb{Z}^2_+$, we have

$$B(e_l, e_m) = -\frac{1}{2|l|}(l^\perp \cdot m)e_{l+m} + \frac{1}{2|l|}(l^\perp \cdot m)e_{l-m}.$$

For $l, -m, l+m \in \mathbb{Z}_+^2$, we have

$$B(e_l, e_m) = -\frac{1}{2|l|}(l^\perp \cdot m)e_{-l-m} + \frac{1}{2|l|}(l^\perp \cdot m)e_{m-l}.$$

We have for $l, m \in Z_L(N)\backslash Z_L(N_0)$, $l+m = k \in Z_L(N_0)$,

$$[g_l e_l, [B(y,y), g_m e_m]_L]_L = -\pi_N B(g_l e_l, g_m e_m) - \pi_N B(g_m e_m, g_l e_l).$$

Hence, we choose $N \geq N_0$ large enough so that for each $k \in Z_L(N_0)$ there are $l, m \in Z_L(N)\backslash Z_L(N_0)$ such that l and m are linearly independent and $k = l+m$ (or $k = l - m$). Then the vectors $[g_l e_l, [B(y,y), g_m e_m]_L]_L, g_l e_l$, where l, m run over $Z_L(N)\backslash Z_L(N_0)$ span H^L. Then we can take $\tilde{\mathbf{K}} = \mathbf{K}_0 \cup \mathbf{K}_2$.

Case 3: $\rho - R \leq |x|_{\mathcal{W}} \leq 2\rho + R$. Write $X^0(y) = X^{01}(y) + X^{02}(y)$ where $X^{01}(y) = Ay + e^{-\delta A_H}B(y,y)$ and $X^{02}(y) = \sum_{k \in Z_L(N_0)} \frac{1}{2\rho}\chi'(\frac{|x|_{\mathcal{W}}}{\rho})(1-\chi(\frac{|x|_{\mathcal{W}}}{\rho}))g_k^2 \frac{\langle x, e_k\rangle_{\mathcal{W}}}{|x|_{\mathcal{W}}} e_k$. By Case 2, we obtain $[g_l e_l, [X^{01}(y), g_m e_m]_L]_L, g_l e_l$ span the whole H^L. And it is easy to see $|[g_l e_l, [X^{01}(y), g_m e_m]_L]_L| \leq \frac{c}{\rho^3}$. So, for ρ large enough, $[g_l e_l, [X^0(y), g_m e_m]_L]_L$ span H^L. Take $\tilde{\mathbf{K}} = \mathbf{K}_0 \cup \mathbf{K}_2$. $\square$

4.5.5 Proof of Lemma 4.5.11

We follow the idea of the proof of [N85, Theorem 4.2] by using Proposition 4.5.13 and the following Norris's Lemma ([N85, Lemma 4.1]).

Lemma 4.5.14 (Norris' Lemma). Let $a, y \in \mathbb{R}$. Let β_t be a real-valued predictable process and γ_t and u_t be adapted H-valued processes. Let

$$a_t = a + \int_0^t \beta_s ds + \int_0^t \langle \gamma_s, dW_s\rangle, \qquad Y_t = y + \int_0^t a_s ds + \int_0^t \langle u_s dW_s\rangle,$$

Suppose that $T < t_0$ is a bounded stopping time such that for some constant $C < \infty$:

$$|\beta_t|, |\gamma_t|, |a_t|, |u_t| \leq C \text{ for all } t \leq T.$$

Then for any $r > 8$ and $\nu > \frac{r-8}{9}$ there is $C = C(T, q, \nu)$ such that

$$P[\int_0^T Y_t^2 dt < \varepsilon^r, \int_0^T (|a_t|^2 + |u_t|^2)dt \geq \varepsilon] < Ce^{-\frac{1}{\varepsilon^\nu}}.$$

Proof of Lemma 4.5.11 Denote $\mathcal{S}^L = \{\eta \in \mathcal{W}^L; |\eta|_{\mathcal{W}^L} = 1\}$. As for $\eta \in \mathcal{S}^L$,

$$\langle \mathcal{M}_t \eta, \eta \rangle_{\mathcal{W}} = \sum_{k \in Z_L(N)} \frac{1}{|k|^{2s}} \int_0^t |\langle J_s^{-1}(g_k(\Phi_s)e_k), \eta \rangle_{\mathcal{W}}|^2 ds,$$

(4.5.33) is equivalent to

(4.5.35)
$$P[\inf_{\eta \in \mathcal{S}^L} \sum_{k \in Z_L(N)} \frac{1}{|k|^{2s}} \int_0^t |\langle J_s^{-1}(g_k(\Phi_s)e_k), \eta \rangle_{\mathcal{W}}|^2 ds \le \varepsilon^q] \le \frac{C\varepsilon^{\frac{\varepsilon_0 p}{2\alpha}}(1 + |x|_{\widetilde{\mathcal{W}}})^p}{t^p},$$

for all $p > 0$, where $g_k(\theta) = (1 - \chi(\frac{|\theta|_{\mathcal{W}}}{\rho}))g_k$ for $k \in Z_L(N_0)$ and $g_k(\theta) = g_k$ for $k \in Z_L(N) \backslash Z_L(N_0)$.

Define a stopping time τ by

$$\tau = \inf\{s > 0 : |\Phi_s(x) - x|_{\mathcal{W}} > R, |J_s^{-1} - Id|_{\mathcal{L}(\mathcal{W})} > c\},$$

where $R > 0$ is the same as in Proposition 4.5.13 and $c > 0$ is sufficiently small. By (4.5.29) and the easy fact $|e^{-At}x - x|_{\mathcal{W}} \le Ct^{\frac{\varepsilon_0}{2\alpha}}|x|_{\widetilde{\mathcal{W}}}$, we have for any $p \ge 2$

$$\begin{aligned} E[| \sup_{0 \le t \le T} |\Phi_t - x|_{\mathcal{W}}^p] \le & E[\sup_{0 \le t \le T} |e^{-At}x - x|_{\mathcal{W}}^p + \sup_{0 \le t \le T} |\Phi_t(x) - e^{-At}x|_{\mathcal{W}}^p] \\ \le & C_1(1 + |x|_{\widetilde{\mathcal{W}}})^p (T^{\frac{p\varepsilon_0}{2\alpha}} \vee T^{p\varepsilon_1}). \end{aligned}$$

Combining the above inequality and (4.5.28), we have for $\varepsilon_0 < \varepsilon_1$

(4.5.36)
$$P(\tau \le \varepsilon) = C_1 \varepsilon^{\frac{p\varepsilon_0}{2\alpha}}(1 + |x|_{\widetilde{\mathcal{W}}})^p,$$

for all $p > 0$.

By the same arguments as [N85, p127], (4.5.35) holds as long as for any $\eta \in \mathcal{S}^L$, we have some neighborhood $\mathcal{N}(\eta)$ of η and some $k \in Z_L(N)$ so that

(4.5.37)
$$\sup_{\eta' \in \mathcal{N}(\eta)} P[\int_0^{t \wedge \tau} |\langle J_s^{-1}(g_k(\Phi_s)e_k), \eta \rangle_{\mathcal{W}}|^2 ds \le \varepsilon^q] \le \frac{C\varepsilon^{\frac{\varepsilon_0 p}{2\alpha}}(1 + |x|_{\widetilde{\mathcal{W}}})^p}{t^p}.$$

According to Definition 4.5.12 and Proposition 4.5.13, for any $\eta \in \mathcal{S}^L$, there exists a $K \in \mathbf{K}$ and a neighborhood $\mathcal{N}$ of η in $\mathcal{S}^L$ such that

$$\inf_{|y - x|_{\mathcal{W}} \le R} \inf_{|V - Id|_{\mathcal{L}(\mathcal{W})}} \inf_{\eta \in \mathcal{N}} |\langle VK(y), \eta \rangle_{\mathcal{W}}| \ge \frac{\sigma}{2}.$$

By this and (4.5.36) we deduce that for any $\eta \in \mathcal{S}^L$, we have some neighborhood

$\mathcal{N}(\eta)$ of η

$$\sup_{\eta' \in \mathcal{N}(\eta)} P[\int_0^{t\wedge\tau} |\langle J_s^{-1} K(\Phi_s), \eta\rangle_{\mathcal{W}}|^2 ds \le \varepsilon^q] \le P[\tau \wedge t < 2\varepsilon/\sigma] \le \frac{C\varepsilon^{\frac{\varepsilon_0 p}{2\alpha}}(1+|x|_{\widetilde{\mathcal{W}}})^p}{t^p}. \tag{4.5.38}$$

Now we prove that (4.5.38) implies (4.5.37). Without loss of generality, assume that $K \in \mathbf{K}_2$, so there exists some $g_k e_k$ and $g_l e_l$ such that

$$K_0(y) := g_k(y)e_k, K_1(y) := [X^0(y), g_k(y)e_k], K = K_2 := [g_l(y)e_l, K_1(y)].$$

Take $Y(t) = \langle J_t^{-1} K_1(\Phi_t), \eta\rangle, a(t) = \langle J_t^{-1}[X_0, K_1](\Phi_t), \eta\rangle$ and $u^i(t) = \langle J_t^{-1}[g_i e_i, K_1](\Phi_t), \eta\rangle$. Applying Lemma 4.5.14, we obtain

$$P(\int_0^{t\wedge\tau} |\langle J_s^{-1} K_1(\Phi_s), \eta\rangle|^2 \le \varepsilon^r, \int_0^{t\wedge\tau} |\langle J_s^{-1} K_2(\Phi_s), \eta\rangle|^2 \ge \varepsilon) \le Ce^{-\frac{1}{\varepsilon^\nu}}.$$

Hence, by (4.5.38) we obtain

$$P[\int_0^{t\wedge\tau} |\langle J_s^{-1} K_1(\Phi_s), \eta\rangle_{\mathcal{W}}|^2 ds \le \varepsilon^r] \le \frac{C\varepsilon^{\frac{\varepsilon_0 p}{2\alpha}}(1+|x|_{\widetilde{\mathcal{W}}})^p}{t^p}.$$

By a similar but simpler arguments we have (4.5.37). □

4.5.6 Controllability and support

In this subsection we prove the support theorem for the solution of the equation (4.3.1) by the control theory.

Proposition 4.5.15 Suppose that Assumption 4.5.1 holds. Let $(P_x)_{x\in H}$ be the Markov solution of equaion (4.3.1). For every $x \in \mathcal{W}$ and $T > 0$, and every $\mathcal{W}$-open set $U \subset \mathcal{W}$, $P_x(\xi_T \in U) > 0$.

To prove Proposition 4.5.15, by the proof of Theorem 4.3.8, we only need to prove the following control problem.

Lemma 4.5.16 Given any $T > 0, x, y \in \mathcal{W}$ and $\varepsilon > 0$, there exist ρ_0, $\theta \in C([0,T]; \mathcal{W})$ and $\omega \in L^\infty([0,T]; H)$ such that θ solves the following equation,

$$\partial_t \theta + A\theta + B(\theta, \theta) = G\omega, \tag{4.5.39}$$

with $\theta(0) = x$ and $|\theta(T) - y| \le \varepsilon$, and $\sup_{t\in[0,T]} |\theta(t)|_{\mathcal{W}} \le \rho_0$.

Proof Let $z \in D(\Lambda^{3\alpha+s})$ such that $|y - z|_{\mathcal{W}} \le \frac{\varepsilon}{2}$. It suffices to show that there exist

θ, ω satisfying the conditions of lemma and $|u(T)-z|_{\mathcal{W}} \le \frac{\varepsilon}{2}$. Decompose $\theta = \theta^H + \theta^L$, then equation (4.5.39) can be written as

$$d\theta^L + A\theta^L dt + B_L(\theta,\theta)dt = 0. \tag{4.5.40}$$

$$d\theta^H + A\theta^H dt + B_H(\theta,\theta)dt = G\omega. \tag{4.5.41}$$

We split $[0,T]$ into the pieces $[0,T_1], [T_1,T_2], [T_2,T_3]$ and $[T_3,T]$ with T_1, T_2, T_3 to be chosen along the proof.

Step 1: regularization of the initial condition. Set $\omega = 0$ on $[0,T_1]$. By the same arguments as Lemma 4.3.6, we can find a time T_1 such that $\sup_{[0,T_1]} |\theta(t)|_{\mathcal{W}} \le \rho_0$ and $\theta(T_1) \in D(\Lambda^{3\alpha+s})$.

Step 2: high modes led to zero. Choose a smooth function ψ on $[T_1,T_2]$ such that $0 \le \psi \le 1, \psi(T_1) = 1$ and $\psi(T_2) = 0$ and set $\theta^H(t) = \psi(t)\theta^H(T_1)$ for $t \in [T_1,T_2]$. As θ^L is finite dimensional, an estimate yields

$$\frac{d}{dt}|\theta^L|^2_{\mathcal{W}} + |\Lambda^\alpha \theta^L|^2_{\mathcal{W}} \le c(|\theta^L|^2_{\mathcal{W}} + |\theta^H|^2_{\mathcal{W}})^2,$$

and $|\theta(t)|^2_{\mathcal{W}} \le |\theta^L(t)|^2_{\mathcal{W}} + |\theta^H(T_1)|^2_{\mathcal{W}} \le \rho_0$ for $T_1 \le t \le T_2 := \frac{T}{2} \wedge (T_1 + (4c|x|^2_{\mathcal{W}})^{-1})$. Plug θ^L in (4.5.41), take

$$\omega(t) = \psi'(t)G^{-1}\theta^H(T_1) + \psi(t)G^{-1}A\theta^H(T_1) + G^{-1}B_H(\theta(t),\theta(t)).$$

As $\theta(T_1) \in D(\Lambda^{3\alpha+s})$, $|G^{-1}A\theta^H(T_1)| < \infty$ and $|G^{-1}B_H(\theta(t),\theta(t))| \le c|A\theta(t)|^2_{\mathcal{W}} \le C(|A\theta^H(T_1)|^2_{\mathcal{W}} + |\theta^L(t)|^2_{\mathcal{W}})$ for $t \in [T_1,T_2]$. Hence, $\omega \in L^\infty([T_1,T_2],H)$.

Step 3: low modes close to z. Let $\theta^L(t)$ be the linear interpolation between $\theta^L(T_2)$ and z^L for $t \in [T_2,T_3]$. Write $\theta(t) = \sum \theta_k(t)e_k$, then (4.5.40) is written as

$$\dot\theta_k + |k|^{2\alpha}\theta_k + B_k(\theta,\theta) = 0, k \in Z_L(N_0). \tag{4.5.42}$$

Let us choose a suitable θ^H to simplify the above $B_k(\theta,\theta)$. Consider the set $\{(l_k, m_k) : k \in Z_L(N_0)\}$ such that (a) $l_k + m_k = k$.

(b)$l_k \nparallel m_k$ for all $k \in Z_L(N_0)$.

(c) For every $k \in Z_L(N_0)$, $|l_k|, |m_k| \ge 2^{(2N_0+1)^2}$.

(d) If $k_1 \ne k_2$, then $|l_{k_1} \pm l_{k_2}|, |m_{k_1} \pm m_{k_2}|, |l_{k_1} \pm m_{k_2}|, |m_{k_1} \pm l_{k_2}| \ge 2^{(2N_0+1)^2}$.

Define

$$\theta^H(t) = \sum_{k\in Z_L(N_0)} (\theta_{l_k}(t)e_{l_k} + \theta_{m_k}e_{m_k}),$$

with $\theta_{l_k}(t), \theta_{m_k}$ to be determined below. By (c) and (d), it is easy to see

$$B_k(\theta^L, \theta^H) = B_k(\theta^H, \theta^L) = 0,$$

$$B_k(e_{l_{k_1}}, e_{l_{k_2}}) = B_k(e_{m_{k_1}}, e_{m_{k_2}}) = B_k(e_{l_{k_1}}, e_{m_{k_2}}) = B_k(e_{m_{k_1}}, e_{l_{k_2}}) = 0.$$

Then (4.5.40) is simplified to the following equation

$$\dot{\theta}_k + |k|^{2\alpha}\theta_k + B_k(\theta^L, \theta^L) + \tilde{B}_k(\theta_{l_k} e_{l_k}, \theta_{m_k} e_{m_k}) = 0, k \in Z_L(N_0). \tag{4.5.43}$$

One can easily find a solution θ^H for equation (4.5.43) which is smooth in t and by construction θ is finite dimensional. Hence $\theta(t)$ is smooth in space and time for $t \in [T_2, T_3]$ and $\sup|\theta(t)|_{\mathcal{W}}$ can be bounded by $|\theta^L(T_2)|_{\mathcal{W}}, z^L$ and $T_3 - T_2$. We set $\omega = G^{-1}[\dot{\theta}^H + A\theta^H + B_H(\theta, \theta)]$ and $\omega \in L^\infty([T_2, T_3], H)$.

Step 4: high modes close to z. In the interval $[T_3, T]$ we choose θ^H as the linear interpolation between $\theta^H(T_3)$ and z^H. Let θ^L be the solution to (4.5.40) on $[T_3, T]$ with the choice of θ^H given above. Since $\theta(T_3) \in D(\Lambda^{3\alpha+s})$ and $\theta^L(T_3) = z^L$, we know $\sup_{T_3 \le t \le T} |\theta^L(t) - z^L|_{\mathcal{W}} \le \frac{\varepsilon}{2}$ if $T - T_3$ is small enough. Then as in Step 1, we can find $\omega \in L^\infty([T_3, T], H)$ solving (4.5.41). It is clear that $\sup_{T_3 \le t \le T} |\theta(t)|_{\mathcal{W}} \le c|z|_{\mathcal{W}} + C|\theta(T_3)|_{\mathcal{W}}$. $\square$

We also obtain the following ergodic properties by the same arguments as Theorems 4.3.12 and 4.4.3.

Theorem 4.5.17 Assume Assumption 4.5.1. There exists a Markov process $\theta(\cdot, \nu)$ on a probability space $(\Omega, \mathcal{F}, P_\nu)$ which is a martingale stationary solution of the stochastic quasi-geostrophic equation (4.1.3). The law ν of $\theta(t, \nu)$ is the unique invariant measure on $\mathcal{W}$ of the transition semigroup $(P_t)_{t\ge 0}$. Moreover

(i) the invariant measure ν is ergodic,

(ii) the transition semigroup $(P_t)_{t\ge 0}$ is strong Feller, irreducible, and therefore strongly mixing.

(iii) there exist $C_{\exp} > 0$ and $a > 0$ such that

$$\|P_t^*\delta_{x_0} - \mu\|_{TV} \le \|P_t^*\delta_{x_0} - \mu\|_V \le C_{\exp}(1 + |x_0|^2)e^{-at},$$

for all $t > 0$ and $x_0 \in H$.

Remark 4.5.18 We can also prove approximate controllability of the solution of the stochastic quasi-geostrophic equation for $\alpha > 1/2$ driven by finite dimensional noise. Since the proof is similar to [S06], we don't give all the details of the proof. For more details, we refer to [S06].

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