SDE in infinite dimensions: the reflection problem and the stochastic quasi-geostrophic equation

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Abstract

In this thesis we are concerned with the following two problems.

1. The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple.

We introduce a definition of BV functions in a Gelfand triple which is an extension of the definition of BV functions in [ADP10] by using Dirichlet form theory with an underlying Gaussian measure as reference measure. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Hilbert space H. We prove the existence and uniqueness of a strong solution of this problem when Γ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$.

We then generalize the above to the case where the Gaussian measure is replaced by a differentiable measure. Again we work in a Gelfand triple and use Dirichlet form theory. By this definition, we can consider the stochastic reflected quantization problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Banach space E. We prove the existence of a martingale solution of this problem when Γ is a regular convex set.

2. The stochastic quasi-geostrophic equation.

We study the 2d stochastic quasi-geostrophic equation in $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and multiplicative noise. We prove the existence of weak solutions with regular additive noise and the existence of martingale solutions with multiplicative noise and pathwise uniqueness under some condition in the general case, i.e. for all $\alpha \in (0,1)$. In the subcritical case $\alpha > 1/2$, we prove existence and uniqueness of (probabilistically) strong solutions and construct a Markov family of solutions. The large deviations principle in the subcritical case with multiplicative noise is also obtained.

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Chapter 0

Introduction

This thesis is devoted to stochastic differential equations in infinite dimensions. The Itô stochastic differential equations were introduced by Itô in the 1940s. Later the theory of stochastic differential equations became one of the most fruitful areas in the theory of stochastic processes. Since 1960s, motivated by a need to describe random phenomena from physics, chemistry, biology and so on, the theory of stochastic partial differential equations (SPDE) has made much progress. Stochastic partial differential equations can describe processes taking values in function spaces with random influence. Basic theoretical questions on existence and uniqueness of solutions have been considered under different conditions (cf. [DZ92], [PR97]). In this thesis, we will consider the existence and uniqueness of two problems: reflection problem and the stochastic quasi-geostrophic equation.

0.1 Reflection problem

In the first part of the thesis, we consider the following stochastic differential inclusion in the Hilbert space H:

$$\begin{cases} dX(t) + (AX(t) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x \in \Gamma, \end{cases} \tag{1.1}$$

if Γ is regular. Here $A : D(A) \subset H \to H$ is a self-adjoint strictly positive definite operator. $N_\Gamma(x)$ is the normal cone to Γ at x and $W(t)$ is a cylindrical Wiener process in H. The precise meaning of the above inclusion will be defined in Section 2.4.2. The solution to (1.1) is called reflected Ornslein-Uhlenbek (OU for short)-process.

(1.1) was first studied (strongly solved) in [NP92], when $H = L^2(0, 1)$, A is the Laplace operator with Dirichlet or Neumann boundary conditions and Γ is the convex set of all nonnegative functions of $L^2(0,1)$; see also [Za02]. In [BDL09] the authors study the situation when Γ is a regular convex set with nonempty interior. They get precise information about the corresponding Kolmogorov operator, but did not construct a strong solution to (1.1). It seems difficult to solve this problem by using general methods in SPDE theory.

In order to solve this problem, we introduce BV functions in a Gelfand triple, which is an extension of BV functions in a Hilbert space defined in [ADP10]. Let us recall that a function u is called a BV functions in $\mathbb{R}^n$ if and only if one of the following is satisfied:

i). there exist real finite measures $\mu_j, j = 1, ..., n$ on $\mathbb{R}^n$ such that:

$$\int_{\mathbb{R}^n} u D_j \phi dx = -\int_{\mathbb{R}^n} \phi d\mu_j, \forall \phi \in C_c(\mathbb{R}^n),$$

ii).

$$V(u) := \sup\{\int_{\mathbb{R}^n} u div\phi dx : \phi \in [C_c(\mathbb{R}^n)]^n, \|\phi\|_\infty \leq 1\} < \infty.$$

The equivalence of these two conditions can be proved by using Riesz representation theorem. But in infinite dimensions, since lack of local compactness, we cannot prove this equivalence directly. Fortunately, M. Fukushima proved a version of the Riesz-Markov representation theorem in infinite dimensions by using the quasi-regularity of the Dirichlet form (see [MR92]). Then M. Fukushima in [Fu00] gave a definition of BV functions in abstract Wiener spaces based upon Dirichlet form theory, and later extended by M. Fukushima and M. Hino in [FH01]. Here we introduce BV functions in a Gelfand triple, which can be used to solve the stochastic reflection problem.

Consider the Dirichlet form

$$\mathcal{E}^\rho(u, v) = \frac{1}{2} \int_H \langle Du, Dv \rangle \rho(z) \mu(dz)$$

(where μ is a Gaussian measure in H and ρ is a BV function) and its associated process. By using BV functions, we obtain a Skorohod-type representation for the associated process, if $\rho = I_\Gamma$ and Γ is a convex set.

In (1.1), we consider a convex set Γ. If Γ is a regular convex set, we show that I_Γ is a BV-function and thus obtain existence and uniqueness results for (1.1). By a modification of [Fu00] and using [BDL10], we obtain the existence of an (in the probabilistic sense) weak solution to (1.1). Then, we prove pathwise uniqueness. Thus, by a version of the Yamada-Watanabe Theorem (see [Ku07]), we deduce that (1.1) has a unique strong solution. We also consider the case when $\Gamma = K_\alpha$, where

$K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$, and prove our result about Skorohod-type representation and that I_{K_α} is a BV function in a Gelfand triple, if $\alpha > 0$.

The solution of the reflection problem is based on an integration by parts formula. The connection to BV functions is given in Theorem 2.2.1 below , which is a key result of this thesis. It asserts that the integration by parts formula for $\rho \cdot \mu$ gives a characterization of BV functions ρ, in the case where μ is a Gaussian measure. This is an extension of the characterization of BV functions in finite dimension. But an integration by parts fomula is in fact enough for the reflection problem. This we show in Section 2.5, exploiting the beautiful integration by parts formula for $K_\alpha, \alpha \geq 0$, proved in [Za02], which in case $\alpha = 0$, i.e, $K_0 = \{f \in L^2(0,1) : f \geq 0\}$, is with respect to a non-Gaussian measure, namely a Bessel bridge. Theorem 2.2.1 applies to prove that I_{K_α} is a BV function, but only if $\alpha > 0$.

Then we analogously define BV functions replacing the Gaussian measure with a differentiable measure in a Gelfand triple. Differentiable measures form a general class which contains more examples besides Gaussian measures (see [Bo10]). The definition of differentiable measure, namely to have integration by parts in sufficiently many directions, is essential for the definition of BV functions. We consider the Dirichlet form

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\sum_{k=1}^{\infty} \int_E \frac{\partial u}{\partial e_k}\frac{\partial v}{\partial e_k}\rho d\mu,$$

(where E is a Banach space with a Hilbert space $H \subset E$ continuously and densely, e_j is an orthonormal basis in H, μ is a differentiable measure in E and ρ is a BV function) and its associated process. Using BV functions, we obtain a Skorohod-type representation for the associated process, if $\rho = I_\Gamma$ and Γ is a convex set.

As a consequence of these results, we can consider the following stochastic differential inclusion in the Banach space E:

$$\begin{cases} dX(t) + (AX(t) + :p(X): + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{1.2}$$

if Γ is regular. Here $A : D(A) \subset H \to H$ is a self-adjoint operator. $N_\Gamma(x)$ is the normal cone to Γ at x and $W(t)$ is a cylindrical Wiener process in H. The solution to (1.2) is called reflected stochastic quantization process. We would like to stress that our results apply to models from 2D-quantum field theory ("$P(\phi)_2$-models") both in finite and infinite volume. The latter is much more difficult than the first.

The stochastic quantization problem with space dimension 2(without reflection term) was studied in [AR89] ("infinite and finite volume"), [AR91]("infinite and finite volume"), [RZ92]("finite volume"), [LR98]("finite volume") by using Dirichlet form theory. And Da Prato and Debussche in [DD03] proved the existence and uniqueness of a strong solution of this problem, but only in the finite volume case. By using BV functions, we obtain martingale solutions to the reflected stochastic quantization problem in finite and infinite volume.

0.2 Stochastic quasi-geostrophic equation

In the second part of this thesis, we are concerned with the following two dimensional (2D) stochastic quasi-geostrophic equation in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\frac{\partial\theta(t,\xi)}{\partial t} = -u(t,\xi)\cdot\nabla\theta(t,\xi) - \kappa(-\triangle)^\alpha\theta(t,\xi) + (G(\theta)\eta)(t,\xi), \tag{1.3}$$

with initial condition

$$\theta(0,\xi) = \theta_0(\xi),$$

where $\theta(t,\xi)$ is a real-valued function of $\xi \in \mathbb{T}^2$ and $t \geq 0$, $0 < \alpha < 1, \kappa > 0$ are real numbers. u is determined by θ through a stream function ψ via the following relations:

$$u = (u_1, u_2) = (-R_2\theta, R_1\theta) = R^\perp\theta. \tag{1.4}$$

Here R_j is the j-th periodic Riesz transform and $\eta(t,\xi)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below. The case $\alpha = \frac{1}{2}$ is called the critical case, the case $\alpha > \frac{1}{2}$ sub-critical and the case $\alpha < \frac{1}{2}$ super-critical.

This equation is an important model in geophysical fluid dynamics. The case $\alpha = 1/2$ exhibits similar features (singularities) as the 3D Navier-Stokes equations and can therefore serve as a model case for the latter. In the deterministic case this equation has been intensively investigated because of both its mathematical importance and its background in geophysical fluid dynamics (see for instance [CV06], [Re95], [CW99], [Ju03], [Ju04], [KNV07] and the references therein). In the deterministic case, the global existence of weak solutions has been obtained in [Re95] and one most remarkable result in [CV06] gives the existence of a classical solution for $\alpha = 1/2$. In [KNV07] another very important result is proved, namely that solutions for $\alpha = 1/2$ with periodic C^∞ data remain C^∞ for all times.

0.2.1 Existence and uniqueness of the solution

In this thesis we study the 2D stochastic quasi-geostrophic equation on $\mathbb{T}^2$ for general parameter $\alpha \in (0, 1)$ and for both additive as well as multiplicative noise.

For $\alpha \in (0, 1)$: We prove the existence of weak solutions in the sense of Definition 4.2.1 (ii) with additive noise (Theorem 4.2.4). We also prove the existence of martingale solutions for multiplicative noise under two different assumptions on G (see (G.1) and (G.2) in Section 4): under (G.1) we use Galerkin approximations and the compactness method in [FG95] (Theorem 4.3.2) and under (G.2) we use Aldous's criterion (Theorem 4.3.5). In order to prove the existence of (probabilistically strong) solutions in subsequent sections, we need L^p norm estimates for solutions, which are obtained by using the L^p-Itô formula proved in [Kr10]. But these L^p-norm estimates we cannot prove by Galerkin approximation, instead we use another approximation (Theorem 4.3.3). Pathwise uniqueness is obtained under some extra condition on the solution (Theorem 4.4.6). But, in general, we cannot prove a solution satisfies this condition, except for very special cases (see Remark 4.4.7).

For $\alpha > 1/2$: We obtain pathwise uniqueness (Theorem 4.4.1) and therefore get a (probabilistically strong) solution (Theorem 4.4.4) by the Yamada-Watanabe Theorem. In particular, it follows that the laws of the solutions form a Markov process.

For $\alpha = 1/2$: Using a result from the deterministic case in [KN09] and [CV06], we also prove that there exists a unique solution of the 2D stochastic quasi-geostrophic equation in the critical case driven by real linear multiplicative noise (Remark 4.4.7).

0.2.2 Large deviation principle

The large deviation theory concerns the asymptotic behavior of a family of random variables $\{\theta_\varepsilon\}$. It asserts that for some tail event A, $P(\theta_\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \to 0$. It also gives the exact rate of convergence (rate function)(cf. [DZ92, Chapter 12]). The large deviation principle was first established by Varadhan in [Va66]. Varadhan also studied the small time asymptotic of finite dimensional diffusion processes in [Va67]. Since then, important results about the large deviation principle have been established. For results on the large deviation principle for the stochastic differential equations in finite dimensional case we refer to [FW84]. For extensions to infinite dimensional diffusions or SPDE, we refer the reader to [DZ92, Li09, XZ09] and the references therein.

Here we will study the large deviation principle for the stochastic quasi-geostrophic equation for small multiplicative noise (Section 4.5) and the small time large deviations for this equation (Section 4.6) in the subcritical case. The large deviation principle for small multiplicative noise (Theorem 4.5.9) asserts that the probability of the deviation of the solution of stochastic quasi-geostrophic equation from the solution of the deterministic quasi-geostrophic equation converges exponentially fast. We use stochastic control and the weak convergence approach from [BD00]. The main difficulty lies in dealing with the nonlinear term since the solution to the stochastic quasi-geostrophic equation is not as regular as in the 2D Navier-Stokes case. To estimate the nonlinear term, we use Galerkin approximations and using the method in [GK96] we prove that these approximations converge in probability to the solution. The small time large deviation principle (Theorem 4.6.2) describes the behavior of θ when the time is very small. We will use the approach from [XZ09]. However, since the solution is not as regular as in for 2D Navier-Stokes equation, we cannot deal with the nonlinear term as in the 2D Navier-Stokes case. Instead, we establish the small time large deviation principle on a larger space.

Chapter 1

Preliminaries

In this chapter, we collect some definitions and results of stochastic analysis as preliminaries for the following chapters. All the content in chapter was included in [MR92]. We omit the proofs of the theorem and refer the readers to [MR92] for more details. In the first part, we recall the definition of quasi regular Dirichlet form and the important result of quasi regular Dirichlet form corresponding to a strong Markov process. In the second part, we recall some definitions and result in stochastic calculus associated with Dirichlet form.

1.1 Some basic concepts for Dirichlet forms

Let us recall the definition of Dirichlet form from [MR92]. Let E be a Hausdorff topological space and assume that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $C(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} := L^2(E, m)$ is a separable (real) Hilbert space. Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive closed form on $\mathcal{H}$, i.e. $D(\mathcal{E})$ is a dense linear subspace of $\mathcal{H}$, and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a positive definite bilinear map, $D(\mathcal{E})$ is a Hilbert space with inner product $\tilde{\mathcal{E}}_1(u, v) := \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u)) + (u, v)_{\mathcal{H}}$, and $\mathcal{E}$ satisfies the *weak sector condition*

$$|\mathcal{E}_1(u, v)| \leq K\mathcal{E}_1(u, u)^{1/2}\mathcal{E}_1(v, v)^{1/2},$$

$u, v \in D(\mathcal{E})$, with sector constant K. We will always denote the corresponding norm by $\|\cdot\|_{\tilde{\mathcal{E}}_1}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain that $\mathcal{E} \to \mathcal{H} \cong \mathcal{H}' \to \mathcal{E}'$ densely and continuously.

Definition 1.1 A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, m)$ is called *a Dirichlet* *form*, if for all $u \in D(\mathcal{E})$, one has that

$$u^+ \wedge 1 \in D(\mathcal{E}), \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0 \text{ and } \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$$

In infinite dimensional spaces, to construct a strong Markov process is sometimes difficult. However, the theory of quasi-regular Dirichlet form, which was introduced by Z. Ma and M. Rockner, provides an useful method to construct a strong Markov process in infinite dimensional spaces. This is an important development in the theory of Dirichlet form and will be used in the chapter 2 and chapter 3. Let's recall the definition of the quasi-regular Dirichlet form here. For this reason we introduce some useful notations.

Definition 1.2 (i) An increasing sequence $(F_k)_{k\geq 1}$ of closed subset of E is called an $\mathcal{E}$-nest, if $\cup D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ (w.r.t. $\|\cdot\|_{\tilde{\mathcal{E}}_1}$).

(ii) A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E \backslash F_k$.

(iii) A property of points in E holds $\mathcal{E}$-quasi-everywhere($\mathcal{E} - q.e.$) if the property holds outside some $\mathcal{E}$-exceptional set.

(iv) A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.) if there exists an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$, such that $\cup_{k\geq 1} F_k \subset E\backslash N$ and $f|_{F_k}$ is continuous for all k.

Definition 1.3 The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called *quasi-regular* if:

(i) There exists an $\mathcal{E}$-nest consisting of compact sets.

(ii) There exists a dense subset of $D(\mathcal{E})$ (w.r.t. $\|\cdot\|_{\tilde{\mathcal{E}}_1}$) whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exist $u_n \in \mathcal{F}, n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E\backslash N$.

Now we can formulate the existence theorem.

Theorem 1.4 Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular Dirichlet form on $L^2(E, m)$. Then there exists a pair $(M, \hat{M})$ of m-tight special standard process which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

Moreover, we have the following characterization of diffusion process which will be used in chapter 2 and chapter 3.

Definition 1.5 The quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is said to have the local property if:

$$\mathcal{E}(u, v) = 0, \text{ for all } u, v \in D(\mathcal{E}) \text{ with } supp[u] \cap supp[v] = \varnothing.$$

Theorem 1.6 A quasi-regular Dirichlet form possesses the local property if and only if it is associated with a pair of diffusions $(\mathcal{E}, D(\mathcal{E}))$.

1.2 Stochastic calculus associated with Dirichlet forms

In this section we assume that the Markov process $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P^x)$ is properly associated with the quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Now we introduce some definitions which will be relevant for our further investigations.

Definition 1.7 A family $(A_t)_{t\geq 0}$ of extended real valued functions on Ω is called an *additive functional* (abbreviated AF) of X if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a defining set $\Lambda \in \mathcal{F}_\infty$ and an $\mathcal{E}$-exceptional set $N \subset E$, such that $P^z[\Lambda] = 1$ for all $z \in E\backslash N, \theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0, |A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

An AF is called a *continuous additive functional* (abbreviated CAF) if $t \to A_t(\omega)$ is continuous on $[0, \infty)$ and a *positive continuous additive functional* (abbreviated PCAF) if $A_t(\omega) \geq 0$ for all $t \geq 0, \omega \in \Lambda$.

Definition 1.8 A positive measure μ on $(E, \mathcal{B}(E))$ is called *smooth* (w.r.t. $(\mathcal{E}, D(\mathcal{E}))$) if $\mu(N) = 0$ whenever $N \in \mathcal{B}(E)$ is $\mathcal{E}$-exceptional and there exists an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ of compact subsets of E such that

$$\mu(F_k) < \infty \text{ for all } k \in \mathbb{N}.$$

Theorem 1.9 There is a one to one correspondence between smooth measures μ of $(\mathcal{E}, D(\mathcal{E}))$ and PCAF's (A_t) of M which is specified by

$$\lim_{t\to 0} E_m[\frac{1}{t}\int_0^t f(X_s)dA_s] = \int f d\mu, \forall f \in \mathcal{B}^+(E).$$

For an additive functional A we define its energy

$$e(A) := \lim_{t\to 0} E_m[\frac{1}{t}A_t^2],$$

if this limit exists in $[0, \infty]$. Define

$$\mathcal{M} := \{M | M \text{ is a finite additive functional, } E^z[M_t^2] < \infty, E^z[M_t] = 0 \\ \text{for } \mathcal{E} - q.e. z \in E \text{ and all } t \geq 0\}.$$

$M \in \mathcal{M}$ is called a *martingale additive functional*(MAF). Furthermore, define

$$\dot{\mathcal{M}} = \{M \in \mathcal{M} | e(M) < \infty\}.$$

The elements of $\dot{\mathcal{M}}$ are called martingale additive functional's (MAF) of finite energy.

Define

$$\mathcal{N}_c := \{N | N \text{ is a finite continuous additive functional, } e(N) = 0, E^z[|N_t|] < \infty \\ \text{for } \mathcal{E} - q.e. z \in E \text{ and all } t \geq 0\}.$$

Now we recall the well-known Fukushima decomposition :

Theorem 1.8 If $u \in D(\mathcal{E})$, then there exists a unique $M^{[u]} \in \dot{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that

$$u(X) - u(X_0) = M^{[u]} + N^{[u]}.$$

Chapter 2

Reflection problem and BV functions in a Gelfand triple

In this chapter, we introduce a definition of BV functions in a Gelfand triple by using Dirichlet form theory. By this definition, we consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Hilbert space H. We prove the existence and uniqueness of a strong solution of this problem when Γ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$. The result in this chapter have been included in [RZZ11].

2.1 The Dirichlet form and the associated distorted OU-process

In this section, we consider a special kind of Dirichlet form and its associated distorted OU-process. Let H be a real separable Hilbert space (with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $|\cdot|$). We denote its Borel σ-algebra by $\mathcal{B}(H)$. Assume that:

Hypothesis 2.1.1 $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x \rangle \geq \delta |x|^2 \ \forall x \in D(A)$ for some $\delta > 0$ and A^{-1} is of trace class.

Since A^{-1} is trace class, there exists an orthonormal basis $\{e_j\}$ in H consisting of eigen-functions for A with corresponding eigenvalues $\alpha_j \in \mathbb{R}, j \in \mathbb{N}$, that is,

$$Ae_j = \alpha_j e_j, j \in \mathbb{N}.$$

Then $\alpha_j \geq \delta$ for all $j \in \mathbb{N}$.

Below $D\varphi : H \to H$ denotes the Fréchet-derivative of a function $\varphi : H \to \mathbb{R}$. By $C_b^1(H)$ we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For $K \subset H$, the space $C_b^1(K)$ is defined as the space of restrictions of all functions in $C_b^1(H)$ to the subset K. μ will denote the Gaussian measure in H with mean 0 and covariance operator

$$Q := \frac{1}{2}A^{-1}.$$

Since A is strictly positive, μ is nondegenerate and has full topological support. Let $L^p(H,\mu), p \in [1,\infty]$, denote the corresponding real L^p-spaces equipped with the usual norms $\|\cdot\|_p$. We set

$$\lambda_j := \frac{1}{2\alpha_j}\ \forall j \in \mathbb{N},$$

so that

$$Qe_j = \lambda_j e_j\ \forall j \in \mathbb{N}.$$

For $\rho \in L^1_+(H,\mu)$ we consider

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz), u, v \in C_b^1(F),$$

where $F := Supp[\rho\cdot\mu]$ and $L^1_+(H,\mu)$ denotes the set of all non-negative elements in $L^1(H,\mu)$. Let $QR(H)$ be the set of all functions $\rho \in L^1_+(H,\mu)$ such that $(\mathcal{E}^\rho, C_b^1(F))$ is closable on $L^2(F,\rho\cdot\mu)$. Its closure is denoted by $(\mathcal{E}^\rho,\mathcal{F}^\rho)$. We denote by $\mathcal{F}^\rho_e$ the extended Dirichlet space of $(\mathcal{E}^\rho,\mathcal{F}^\rho)$, that is, $u \in \mathcal{F}^\rho_e$ if and only if $|u| < \infty$ $\rho\cdot\mu - a.e.$ and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\rho$ such that $\mathcal{E}^\rho(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u$ $\rho\cdot\mu - a.e.$ as $n \to \infty$.

Theorem 2.1.2 Let $\rho \in QR(H)$. Then $(\mathcal{E}^\rho,\mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(F;\rho\cdot\mu)$ in the sense of Definition 1.3.

Proof The assertion follows from the main result in [RS92]. □

By virtue of Theorem 2.1.2 and Theorem 1.4, there exists a diffusion process $M^\rho = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$ on F associated with the Dirichlet form $(\mathcal{E}^\rho,\mathcal{F}^\rho)$. M^ρ will be called distorted OU-process on F. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1,1) = 0$, M^ρ is recurrent and conservative. We denote by $\mathbf{A}^\rho_+$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^ρ, and define $\mathbf{A}^\rho := \mathbf{A}^\rho_+ - \mathbf{A}^\rho_+$. For $A \in \mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}^\rho_0 := \{A \in \mathbf{A}^\rho | E_{\rho\cdot\mu}(\{A\}_t) < \infty \forall t > 0\}$. Each element in $\mathbf{A}^\rho_+$ has a corresponding positive $\mathcal{E}^\rho$-smooth measure on F by the Revuz correspondence. The set of all such measures will be denoted by S^ρ_+. Accordingly, $A_t \in \mathbf{A}^\rho$ corresponds to a $\nu \in S^\rho := S^\rho_+ - S^\rho_+$, the set of all $\mathcal{E}^\rho$-smooth signed measure in the sense that $A_t = A^1_t - A^2_t$ for $A^k_t \in \mathbf{A}^\rho_+, k = 1,2$ whose Revuz measures are $\nu^k, k = 1,2$ and $\nu = \nu^1 - \nu^2$ is the Hahn-Jordan decomposition of ν . The element of $\mathbf{A}^\rho$ corresponding to $\nu \in S^\rho$ will be denoted byA^ν.

Note that for each $l \in H$ the function $u(z) = \langle l, z\rangle$ belongs to the extended Dirichlet space $\mathcal{F}^\rho_e$ and

$$\mathcal{E}^\rho(l(\cdot), v) = \frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z) d\mu(z) \ \forall v \in C^1_b(F). \tag{2.1.1}$$

On the other hand, the AF $\langle l, X_t - X_0\rangle$ of M^ρ admits a unique decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy (Fukushima decomposition). More precisely, for every $l \in H$,

$$\langle l, X_t - X_0\rangle = M^l_t + N^l_t \ \forall t \geq 0 \ P_z - a.s. \tag{2.1.2}$$

for $\mathcal{E}^\rho$-q.e. $z \in F$.

Now for $\rho \in L^1(H, \mu)$ and $l \in H$, we say that $\rho \in BV_l(H)$ if there exists a constant $C_l > 0$,

$$|\int \langle l, Dv(z)\rangle \rho(z) d\mu(z)| \leq C_l \parallel v \parallel_\infty \ \ \forall v \in C^1_b(F). \tag{2.1.3}$$

By the same argument as in [FH01, Theorem 2.1], we obtain the following:

Theorem 2.1.3 Let $\rho \in L^1_+$ and $l \in H$.

(1) The following two conditions are equivalent:

(i)$\rho \in BV_l(H)$

(ii) There exists a (unique) signed measure ν_l on F of finite total variation such that

$$\frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z) d\mu(z) = -\int_F v(z)\nu_l(dz) \ \forall v \in C^1_b(F). \tag{2.1.4}$$

In this case, ν_l necessarily belongs to $S^{\rho+1}$.

Suppose further that $\rho \in QR(H)$. Then the following condition is also equivalent to the above:

(iii)$N^l \in \mathbf{A}^\rho_0$

In this case, $\nu_l \in S^\rho$, and $N^l = A^{\nu_l}$

(2) M^l is a martingale AF with quadratic variation process

$$\langle M^l\rangle_t = t|l|^2, t \geq 0. \tag{2.1.5}$$

Remark 2.1.4 Recall that the Riesz representation theorem of positive linear functionals on continuous functions by measures is not applicable to obtain Theorem 2.1.3, $(i) \Rightarrow (ii)$, because of the lack of local compactness. However, the quasi-regularity of the Dirichlet form provides a means to circumvent this difficulty.

In the rest of this section, we shall introduce a special class of $\rho \in QR(H)$, which will be used in Section 2.3 below.

A non-negative measurable function $h(s)$ on $\mathbb{R}^1$ is said to possess the *Hamza property* if $h(s) = 0\ ds - a.e.$ on the closed set $\mathbb{R}^1 \setminus R(h)$ where

$$R(h) = \{s \in \mathbb{R}^1 : \int_{s-\varepsilon}^{s+\varepsilon} \frac{1}{h(r)} dr < \infty \text{ for some } \varepsilon > 0\}.$$

We say that a function $\rho \in L^1_+(H, \mu)$ satisfies *the ray Hamza condition in direction* $l \in H$ ($\rho \in \mathbf{H}_l$ in notation) if there exists a non-negative function $\tilde{\rho}_l$ such that

$$\tilde{\rho}_l = \rho\ \mu - a.e. \text{ and } \tilde{\rho}_l(z + sl) \text{ has the Hamza property in } s \in \mathbb{R}^1 \text{ for each } z \in H.$$

We set $\mathbf{H} := \cap_k \mathbf{H}_{e_k}$, where e_k is as in Hypothesis 2.1.1. A function in the family $\mathbf{H}$ is simply said to satisfy the *ray Hamza condition.* By [AR90] $\mathbf{H} \subset QR(H)$, and thus we always have $\rho + 1 \in QR(H)$, since clearly $\rho + 1 \in \mathbf{H}$.

Next we will present some explicit description of the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ for $\rho \in \mathbf{H}$. For $e_j \in H$ as in Hypothesis 2.1.1, we set $H_{e_j} = \{se_j : s \in \mathbb{R}^1\}$. We then have the direct sum decomposition $H = H_{e_j} \oplus E_{e_j}$ given by

$$z = se_j + x, s = \langle e_j, z \rangle .$$

Let π_j be the projection onto the space E_{e_j} and μ_{e_j} be the image measure of μ under $\pi_j : H \to E_{e_j}$ i.e $\mu_{e_j} = \mu \circ \pi_j^{-1}$. Then we see that for any $F \in L^1(H, \mu)$

$$\int_H F(z)\mu(dz) = \int_{E_{e_j}} \int_{\mathbb{R}^1} F(se_j + x)p_j(s) ds \mu_{e_j}(dx), \tag{2.1.6}$$

where $p_j(s) = (1/\sqrt{2\pi\lambda_j})e^{-s^2/2\lambda_j}$. Thus by [AR90, Theorem3.10] for all $u, v \in D(\mathcal{E}^\rho)$,

$$\mathcal{E}^\rho(u, v) = \sum_{j=1}^\infty \mathcal{E}^{\rho, e_j}(u, v), \tag{2.1.7}$$

where

$$\mathcal{E}^{\rho,e_j}(u,v)=\frac{1}{2}\int_{E_{e_j}}\int_{R(\rho(\cdot e_j+x))}\frac{d\tilde{u}_j(se_j+x)}{ds}\times\frac{d\tilde{v}_j(se_j+x)}{ds}\rho(se_j+x)p_j(s)ds\mu_{e_j}(dx), \tag{2.1.8}$$

and $u,\tilde{u}_j$ satisfy $\tilde{u}_j=u\ \rho\mu-a.e$ and $\tilde{u}_j(se_j+x)$ is absolutely continuous in s on $R(\rho(\cdot\, e_j+x))$ for each $x\in E_{e_j}$. v and $\tilde{v}_j$ are related in the same way.

2.2 BV functions in a Gelfand triple and distorted OU-processes in F

We introduce BV functions in a Gelfand triple in this section, by which we can get the Skorohod type representation for the OU- process.

As in [FH01], we introduce some function spaces on H. Let

$$A_{1/2}(x):=\int_0^x(\log(1+s))^{1/2}ds, x\geq 0,$$

and let ψ be its complementary function, namely,

$$\psi(y):=\int_0^y(A'_{1/2})^{-1}(t)dt=\int_0^y(\exp(t^2)-1)dt.$$

Define

$$L(\log L)^{1/2}(H,\mu):=\{f:H\to\mathbb{R}|f\text{ Borel measurable, }A_{1/2}(|f|)\in L^1(H,\mu)\},$$

$$L^\psi(H,\mu):=\{g:H\to\mathbb{R}|g\text{ Borel measurable, }\psi(c|g|)\in L^1(H,\mu)\text{ for some }c>0\}.$$

From the general theory of Orlicz spaces (cf. [RR91]), we have the following properties.

(i) $L(\log L)^{1/2}$ and L^ψ are Banach spaces under the norms

$$\|f\|_{L(\log L)^{1/2}}=\inf\{\alpha>0|\int_H A_{1/2}(|f|/\alpha)d\mu\leq 1\},$$

$$\|g\|_{L^\psi}=\inf\{\alpha>0|\int_H\psi(|g|/\alpha)d\mu\leq 1\}.$$

(ii) For $f\in L(\log L)^{1/2}$ and $g\in L^\psi$, we have

$$\|fg\|_1\leq 2\|f\|_{L(\log L)^{1/2}}\|g\|_{L^\psi}. \tag{2.2.1}$$

(iii) Since μ is Gaussian, the function $x \mapsto \langle x, l\rangle$ belongs to L^{ψ}.

Let $c_j, j \in \mathbb{N}$, be a sequence in $[1, \infty)$. Define

$$H_1 := \{x \in H | \sum_{j=1}^{\infty} \langle x, e_j\rangle^2 c_j^2 < \infty\},$$

equipped with the inner product

$$\langle x, y\rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j\rangle \langle y, e_j\rangle.$$

Then clearly $(H_1, \langle, \rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H(\equiv H^*) \subset H_1^*.$$

It follows that

$${}_{H_1}\langle z, v\rangle_{H_1^*} = \langle z, v\rangle_H \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple. Furthermore, $\{\frac{e_j}{c_j}\}$ and $\{c_j e_j\}$ are orthonormal bases of H_1 and H_1^*, respectively.

We also introduce a family of H-valued functions on H by

$$(C_b^1)_{D(A)\cap H_1} := \{G : G(z) = \sum_{j=1}^{m} g_j(z) l^j, z \in H, g_j \in C_b^1(H), l^j \in D(A) \cap H_1\}.$$

Denote by D^* the adjoint of $D : C_b^1(H) \subset L^2(H, \mu) \to L^2(H, \mu; H)$. That is

$$Dom(D^*) := \{G \in L^2(H, \mu; H) | C_b^1 \ni u \mapsto \int \langle G, Du\rangle d\mu \text{ is continuous with respect to } L^2(H, \mu)\}.$$

Obviously, $(C_b^1)_{D(A)\cap H_1} \subset Dom(D^*)$. Then

$$\int_H D^*G(z) f(z)\mu(dz) = \int_H \langle G(z), Df(z)\rangle \mu(dz) \ \forall G \in (C_b^1)_{D(A)\cap H_1}, f \in C_b^1(H). \tag{2.2.2}$$

For $\rho \in L(\log L)^{1/2}(H, \mu)$, we set

$$V(\rho) := \sup_{G \in (C_b^1)_{D(A)\cap H_1}, \|G\|_{H_1} \leq 1} \int_H D^*G(z)\rho(z)\mu(dz).$$

A function ρ on H is called a BV function in the Gelfand triple (H_1, H, H_1^*)($\rho \in BV(H, H_1)$ in notation), if $\rho \in L(\log L)^{1/2}(H, \mu)$ and $V(\rho)$ is finite. When $H_1 =$

$H = H_1^*$, this coincides with the definition of BV functions defined in [ADP10] and clearly $BV(H,H) \subset BV(H,H_1)$. We can prove the following theorem by a modification of the proof of [Fu00, Theorem 3.1].

Remark 2.2.0 The introduction of BV functions in a Gelfand triple is natural and originates from standard ideas when working with infinite dimensional state spaces. The intersection of $BV_l(H)$, when l runs through $D(A) \cap H_1$, describes functions which are "componentwise of bounded variation" in the sense that their weak partial derivatives are measures. In contrast to finite dimensions this does not give rise to vector-valued measures representing their total weak derivatives or gradients. Therefore, one introduces an appropriate "tangent space" H_1^* to H, in which these total derivatives can be represented as a H_1^*-valued measure. This approach substantially extends the applicability of the theory of BV functions on Hilbert spaces. We document this by including the well-studied case of linear SPDE with reflection, more precisely, the randomly vibrating Gaussian string, forced to stay above a level $\alpha \geq 0$, (see [NP92], [Za02]), which (in the case of $\alpha > 0$) is then just a special case of our general approach.

Theorem 2.2.1 (i) $BV(H,H_1) \subset \bigcap_{l\in D(A)\cap H_1} BV_l(H)$.

(ii) Suppose $\rho \in BV(H,H_1) \cap L^1_+(H,\mu)$, then there exist a positive finite measure $\|d\rho\|$ on H and a Borel-measurable map $\sigma_\rho : H \to H_1^*$ such that $\|\sigma_\rho(z)\|_{H_1^*} = 1$ $\|d\rho\| - a.e$, $\|d\rho\|(H) = V(\rho)$,

$$\int_H D^*G(z)\rho(z)\mu(dz) = \int_H {}_{H_1}\langle G(z), \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) \ \forall G \in (C_b^1)_{D(A)\cap H_1} \quad (2.2.3)$$

and $\|d\rho\| \in S^{\rho+1}$.

Furthermore, if $\rho \in QR(H)$, $\|d\rho\|$ is $\mathcal{E}^\rho$-smooth in the sense that it charges no set of zero $\mathcal{E}_1^\rho$-capacity. In particular, the domain of integration H on both sides of (2.2.3) can be replaced by F, the topological support of $\rho\mu$.

Also, σ_ρ and $\|d\rho\|$ are uniquely determined, that is, if there are σ'_ρ and $\|d\rho\|'$ satisfying relation (2.2.3), then $\|d\rho\| = \|d\rho\|'$ and $\sigma_\rho(z) = \sigma'_\rho(z)$ for $\|d\rho\| - a.e.z$

(iii) Conversely, if Eq.(2.2.3) holds for $\rho \in L(\log L)^{1/2}(H,\mu)$ and for some positive finite measure $\|d\rho\|$ and a map σ_ρ with the stated properties, then $\rho \in BV(H,H_1)$ and $V(\rho) = \|d\rho\|(H)$.

(iv) Let $W^{1,1}(H)$ be the domain of the closure of $(D, C_b^1(H))$ with norm

$$\|f\| := \int_H (|f(z)| + |Df(z)|)\mu(dz).$$

Then $W^{1,1}(H) \subset BV(H,H)$ and Eq.(2.2.3) is satisfied for each $\rho \in W^{1,1}(H)$. Furthermore,

$$\|d\rho\| = |D\rho| \cdot \mu, V(\rho) = \int_H |D\rho|\mu(dz), \sigma_\rho = \frac{1}{|D\rho|} D\rho I_{\{|D\rho|>0\}}.$$

Proof (i) Let $\rho \in BV(H, H_1)$ and $l \in D(A) \cap H_1$. Take $G \in (C_b^1)_{D(A)\cap H_1}$ of the type

$$G(z) = g(z)l, z \in H, g \in C_b^1(H). \tag{2.2.4}$$

By (2.2.2)

$$\begin{aligned}\int_H D^*G(z)f(z)\mu(dz) &= \int_H \langle G(z), Df(z)\rangle\mu(dz) \\ &= -\int_H \langle l, Dg(z)\rangle f(z)\mu(dz) + 2\int_H \langle Al, z\rangle g(z)f(z)\mu(dz) \ \forall f \in C_b^1(H);\end{aligned}$$

consequently,

$$D^*G(z) = -\langle l, Dg(z)\rangle + 2g(z)\langle Al, z\rangle. \tag{2.2.5}$$

Accordingly,

$$\int_H \langle l, Dg(z)\rangle\rho(z)\mu(dz) = -\int_H D^*G(z)\rho(z)\mu(dz) + 2\int_H \langle Al, z\rangle g(z)\rho(z)\mu(dz). \tag{2.2.6}$$

For any $g \in C_b^1(H)$, satisfying $\|g\|_\infty \le 1$, by (2.2.1) the right hand side is dominated by

$$V(\rho)\|l\|_{H_1} + 4\|\rho\|_{L(\log L)^{1/2}}\|\langle Al, \cdot\rangle\|_{L^\psi} < \infty,$$

hence, $\rho \in BV_l(H)$.

(ii) Suppose $\rho \in L_+^1(H, \mu) \bigcap BV(H, H_1)$. By (i) and Theorem 2.1.3 for each $l \in D(A) \cap H_1$, there exists a finite signed measure ν_l on H for which Eq.(2.1.4) holds. Define

$$D_l^A\rho(dz) := 2\nu_l(dz) + 2\langle Al, z\rangle\rho(z)\mu(dz).$$

In view of (2.2.6), for any G of type (2.2.4), we have

$$\int_H D^*G(z)\rho(z)\mu(dz) = \int_H g(z)D_l^A\rho(dz), \tag{2.2.7}$$

which in turn implies

$$V(D_l^A\rho)(H) = \sup_{g\in C_b^1(H), \|g\|_\infty \le 1} \int_H g(z)D_l^A\rho(dz) \le V(\rho)\|l\|_{H_1}, \tag{2.2.8}$$

where $V(D_l^A\rho)$ denotes the total variation measure of the signed measure $D_l^A\rho$.

For the orthonormal basis $\{\frac{e_j}{c_j}\}$ of H_1, we set

$$\gamma_\rho^A := \Sigma_{j=1}^\infty 2^{-j} V(D^A_{\frac{e_j}{c_j}}\rho),\ v_j(z) := \frac{dD^A_{\frac{e_j}{c_j}}\rho(z)}{d\gamma_\rho^A(z)}, z \in H, j \in \mathbb{N}. \tag{2.2.9}$$

γ_ρ^A is a positive finite measure with $\gamma_\rho^A(H) \le V(\rho)$ and v_j is Borel-measurable. Since $D^A_{\frac{e_j}{c_j}}\rho$ belongs to $S^{\rho+1}$, so does γ_ρ^A . Then for

$$G_n := \sum_{j=1}^n g_j \frac{e_j}{c_j} \in (C_b^1)_{D(A)\cap H_1}, n \in \mathbb{N}, \tag{2.2.10}$$

by (2.2.7) the following equation holds

$$\int_H D^* G_n(z)\rho(z)\mu(dz) = \sum_{j=1}^n \int_H g_j(z) v_j(z) \gamma_\rho^A(dz). \tag{2.2.11}$$

Since $|v_j(z)| \le 2^j\ \gamma_\rho^A$-a.e. and $C_b^1(H)$ is dense in $L^1(H, \gamma_\rho^A)$, we can find $v_{j,m} \in C_b^1(H)$ such that

$$\lim_{m\to\infty} v_{j,m} = v_j\ \gamma_\rho^A - a.e.,$$

Substituting

$$g_{j,m}(z) := \frac{v_{j,m}(z)}{\sqrt{\sum_{k=1}^n v_{k,m}(z)^2 + 1/m}}, \tag{2.2.12}$$

for $g_j(z)$ in (2.2.10) and (2.2.11) we get a bound

$$\sum_{j=1}^n \int_H g_{j,m}(z) v_j(z) \gamma_\rho^A(dz) \le V(\rho),$$

because $\|G_n(z)\|_{H_1}^2 = \sum_{j=1}^n g_{j,m}(z)^2 \le 1\ \forall z \in H$. By letting $m \to \infty$, we obtain

$$\int_H \sqrt{\sum_{j=1}^n v_j(z)^2} \gamma_\rho^A(dz) \le V(\rho)\ \forall n \in \mathbb{N}.$$

Now we define

$$\|d\rho\| := \sqrt{\sum_{j=1}^\infty v_j(z)^2} \gamma_\rho^A(dz) \tag{2.2.13}$$

and $\sigma_\rho : H \to H_1^*$ by

$$\sigma_\rho(z) = \begin{cases} \sum_{j=1}^{\infty} \frac{v_j(z)}{\sqrt{\sum_{k=1}^{\infty} v_k(z)^2}} \cdot c_j e_j, & \text{if } z \in \{\sum_{k=1}^{\infty} v_k(z)^2 > 0\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.14}$$

Then

$$\|d\rho\|(H) \leq V(\rho),\ \|\sigma_\rho(z)\|_{H_1^*} = 1\ \|d\rho\| - a.e., \tag{2.2.15}$$

$\|d\rho\|$ is $S^{\rho+1}$-smooth and σ_ρ is Borel-measurable. By (2.2.11) we see that the desired equation (2.2.3) holds for $G = G_n$ as in (2.2.10). It remains to prove (2.2.3) for any G of type (2.2.4), i.e. $G = g \cdot l, g \in C_b^1(H), l \in D(A) \cap H_1$. In view of (2.2.6), Eq.(2.2.3) then reads

$$-\int_H \langle l, Dg(z)\rangle \rho(z)\mu(dz) + 2\int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz) = \int_H g(z) {}_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz). \tag{2.2.16}$$

We set

$$k_n := \sum_{j=1}^{n} \langle l, e_j\rangle e_j = \sum_{j=1}^{n} \langle l, \frac{e_j}{c_j}\rangle_{H_1} \frac{e_j}{c_j}, G_n(z) := g(z)k_n.$$

Thus $k_n \to l$ in H_1 and $Ak_n \to Al$ in H as $n \to \infty$. But then also

$$\lim_{n\to\infty} \int_H \langle Dg, k_n\rangle \rho d\mu = \int_H \langle Dg, l\rangle \rho d\mu,$$

and

$$|\int_H g(z)\langle Ak_n, z\rangle \rho(z)\mu(dz) - \int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz)|$$
$$\leq 2\|g\|_\infty \|\rho\|_{L(\log L)^{1/2}} \|\langle Ak_n - Al, \cdot\rangle\|_{L^\psi}.$$

Furthermore,

$$\lim_{n\to\infty} \int_H g(z) {}_{H_1}\langle k_n, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) = \int_H g(z) {}_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz).$$

So letting $n \to \infty$ yields (2.2.16).

If $\rho \in QR(H)$, we can get the claimed result by the same arguments as above.

Uniqueness follows by the same argument as [FH01, Theorem 3.9]. Suppose that σ'_ρ and $\|d\rho\|'$ are another pair. Then,

$$\int_E {}_{H_1}\langle G(z), \gamma\rangle_{H_1^*} \xi(dz) = 0 \text{ for every } G \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1},$$

where $\xi = \|d\rho\| + \|d\rho\|'$ and $\gamma = \sigma_\rho \frac{d\|d\rho\|}{d\xi} - \sigma'_\rho \frac{d\|d\rho\|'}{d\xi}$. Taking a uniformly bounded sequence $G_n \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1}$ so that ${}_{H_1}\langle G_n(z), \gamma\rangle_{H_1^*} \to \|\gamma\|_{H_1^*}\xi$ a.e., we get $\gamma = 0\xi$ a.e. Therefore, $\|\sigma_\rho\|_{H_1^*}\frac{d\|d\rho\|}{d\xi} = \|\sigma'_\rho\|_{H_1^*}\frac{d\|d\rho\|'}{d\xi}$ ξ a.e. Since $\|\sigma_\rho\|_{H_1^*} = 1$ $\|d\rho\|$-a.e. $\|\sigma_\rho\|_{H_1^*}\frac{d\|d\rho\|}{d\xi} = \frac{d\|d\rho\|}{d\xi}$ ξ-a.e. Similarly, $\|\sigma'_\rho\|_{H_1^*}\frac{d\|d\rho\|'}{d\xi} = \frac{d\|d\rho\|'}{d\xi}$ ξ-a.e. Then $\frac{d\|d\rho\|}{d\xi} = \frac{d\|d\rho\|'}{d\xi}$ ξ-a.e. which implies $d\|d\rho\| = d\|d\rho\|'$. Also it follows that $\sigma_\rho = \sigma'_\rho$.

(iii) Suppose $\rho \in L(\log)^{1/2}(H,\mu)$ and that Eq.(2.2.3) holds for some positive finite measure $\|d\rho\|$ and some map σ_ρ with the properties stated in (ii). Then clearly

$$V(\rho) \le \|d\rho\|(H)$$

and hence $\rho \in BV(H, H_1)$. To obtain the converse inequality, set

$$\sigma_j(z) := \langle c_j e_j, \sigma_\rho(z)\rangle_{H_1^*} = {}_{H_1}\langle \frac{e_j}{c_j}, \sigma_\rho(z)\rangle_{H_1^*}, j \in \mathbb{N}.$$

Fix an arbitrary n. As in the proof of (ii) we can find functions

$$v_{j,m} \in C_b^1(H), \qquad \lim_{m\to\infty} v_{j,m}(z) = \sigma_j(z)\ \|d\rho\| - a.e.$$

Define $g_{j,m}(z)$ by (2.2.12). Substituting $G_{n,m}(z) := \sum_{j=1}^n g_{j,m}(z)\frac{e_j}{c_j}$ for $G(z)$ in (2.2.3) then yields

$$\sum_{j=1}^n \int_H g_{j,m}(z)\sigma_j(z)\|d\rho\|(dz) \le V(\rho).$$

By letting $m \to \infty$, we get

$$\int_H \sqrt{\sum_{j=1}^n \sigma_j(z)^2}\|d\rho\|(dz) \le V(\rho)\ \forall n \in \mathbb{N}.$$

We finally let $n \to \infty$ to obtain $\|d\rho\|(H) \le V(\rho)$.

(iv) Obviously the duality relation (2.2.2) extends to $\rho \in W^{1,1}(H)$ replacing $f \in C_b^1(H)$. By defining $\|d\rho\|$ and $\sigma_\rho(z)$ in the stated way, the extended relation (2.2.2) is exactly (2.2.3). □

Theorem 2.2.2 Let $\rho \in QR(H) \cap BV(H, H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 2.2.1(ii). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P_z\text{−a.s..} \tag{2.2.17}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

In particular, if $\rho \in BV(H, H)$, then $\forall z \in F \backslash S$, $l \in D(A) \cap H$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle + \frac{1}{2}\int_0^t \langle l, \sigma_\rho(X_s)\rangle dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P_z\text{−a.s..}$$

Proof Let $\{e_j\}$ be the orthonormal basis of H introduced above. Define for all $k \in \mathbb{N}$

$$W_k^z(t) := \langle e_k, X_t - z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} + \int_0^t \langle Ae_k, X_s\rangle ds. \tag{2.2.18}$$

By (2.1.1) and (2.2.16) we get for all $k \in \mathbb{N}$

$$\mathcal{E}^\rho(e_k(\cdot), g) = \int_H g(z)\langle Ae_k, z\rangle \rho(z)\mu(dz) - \frac{1}{2}\int_H g(z)_{H_1}\langle e_k, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) \ \forall g \in C_b^1(H).$$

By Theorem 2.1.3 it follows that for all $k \in \mathbb{N}$

$$N_t^{e_k} = \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Ae_k, X_s\rangle ds. \tag{2.2.19}$$

Here we get from (2.2.18), (2.2.19) and the uniqueness of decomposition (2.1.2) that for $\mathcal{E}^\rho$-q.e. $z \in F$,

$$W_k^z(t) = M_t^{e_k} \ \forall t \geq 0 \ P_z\text{−a.s.,}$$

where the $\mathcal{E}^\rho$-exceptional set and the zero measure set does not depend on e_k. Indeed, we can choose the capacity zero set $S = \cup_{j=1}^\infty S_j$, where S_j is the $\mathcal{E}^\rho$-exceptional set for e_j, and for $z \in F \backslash S$, we can use the same method to get a zero measure set independent of e_k. By Dirichlet form theory we get $\langle M^{e_i}, M^{e_j}\rangle_t = t\delta_{ij}$. So for $z \in F \setminus S$, W_k^z is an $\mathcal{M}_t$-Wiener process under P_z. Thus, with W^z being an $\mathcal{M}_t$-cylindrical Wiener process given by $W^z(t) = (W_k^z(t)e_k)_{k\in\mathbb{N}}$, (2.2.17) is satisfied for $P_z - a.e.$, where $z \in F \setminus S$. □

2.3 Reflected OU-processes

In this section we consider the situation where $\rho = I_\Gamma \in BV(H, H_1)$, where $\Gamma \subset H$ and

$$I_\Gamma(x) = \begin{cases} 1, & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \Gamma^c. \end{cases}$$

Denote the corresponding objects σ_ρ, $\|dI_\Gamma\|$ in Theorem 2.2.1(ii) by $-\mathbf{n}_\Gamma$, $\|\partial\Gamma\|$ respectively. Then formula (2.2.3) reads

$$\int_\Gamma D^*G(z)\mu(dz) = -\int_F {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*} \|\partial\Gamma\|(dz)\ \forall G \in (C_b^1)_{D(A)\cap H_1},$$

where the domain of integration F on the right hand side is the topological support of $I_\Gamma \cdot \mu$. F is contained in $\bar{\Gamma}$, but we shall show that the domain of integration on the right hand side can be restricted to $\partial\Gamma$. We need to use the associated distorted OU-process M^{I_Γ} on F, which will be called reflected OU-process on Γ.

First we consider a μ-measurable set $\Gamma \subset H$ satisfying

$$I_\Gamma \in BV(H, H_1) \cap \mathbf{H}. \tag{2.3.1}$$

Remark 2.3.1 We emphasize that if Γ is a convex closed set in H, then obviously $I_\Gamma \in \mathbf{H}$. Indeed, for each $z, l \in H$ the set $\{s \in \mathbb{R} | z + sl \in \Gamma\}$ is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_\Gamma \in QR(H)$.

By a modification of [Fu00, Theorem 4.2], we can prove the following theorem.

Theorem 2.3.2 Let $\Gamma \subset H$ be μ-measurable satisfying condition (2.3.1). Then the support of $\|\partial\Gamma\|$ is contained in the boundary $\partial\Gamma$ of Γ, and the following generalized Gauss formula holds:

$$\int_\Gamma D^*G(z)\mu(dz) = -\int_{\partial\Gamma} {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*} \|\partial\Gamma\|(dz)\ \forall G \in (C_b^1)_{D(A)\cap H_1}. \tag{2.3.2}$$

Proof For any G of type (2.2.4) we have from (2.1.1), (2.2.5) and (2.2.7) that

$$\mathcal{E}^{I_\Gamma}(l(\cdot), g) - \int_\Gamma g(z)\langle Al, z\rangle \mu(dz) = -\frac{1}{2}\int_F g(z) D_l^A I_\Gamma(dz). \tag{2.3.3}$$

Since the finite signed measure $D_l^A I_\Gamma$ charges no set of zero $\mathcal{E}_1^{I_\Gamma}$-capacity, Eq.(2.3.3) readily extends to any $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $g \in \mathcal{F}_b^{I_\Gamma} := \mathcal{F}^{I_\Gamma} \cap L^\infty(\Gamma, \mu)$.

Denote by Γ^0 the interior of Γ. Then $\Gamma^0 \subset F \subset \bar{\Gamma}$. In view of the construction of the measure $\|dI_\Gamma\|$ in Theorem 2.2.1, it suffices to show that for $\frac{e_j}{c_j} \in D(A) \cap H_1$

$$V(D^A_{\frac{e_j}{c_j}} I_\Gamma)(\Gamma^0) = 0.$$

By linearity and since positive constants interchange with sup, it suffices to show that,

$$V(D^A_{e_j} I_\Gamma)(\Gamma^0) = 0. \tag{2.3.4}$$

Take an arbitrary $\varepsilon > 0$ and set

$$U := \{z \in H : d(z, H\backslash\Gamma^0) > \varepsilon\}, V := \{z \in H : d(z, H\backslash\Gamma^0) \geq \varepsilon\},$$

where d is the metric distance of the Hilbert space H. Then $\bar{U} \subset V$ and V is a closed set contained in the open set Γ^0. We define a function h by

$$h(z) := 1 - E_z(e^{-\tau_V}), z \in F, \tag{2.3.5}$$

where τ_V denotes the first exit time of M^{I_Γ} from the set V. The nonnegative function h is in the space $\mathcal{F}_b^{I_\Gamma}$ and furthermore it is $\mathcal{E}^{I_\Gamma}$-quasicontinuous because it is M^{I_Γ} finely continuous.

Moreover,

$$h(z) > 0 \ \forall z \in U, \qquad h(z) = 0 \ \forall z \in F\backslash V. \tag{2.3.6}$$

Set

$$\nu_j(dz) := h(z) D^A_{e_j} I_\Gamma(dz) \tag{2.3.7}$$

and

$$I_g^j := \mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) - \int_\Gamma g(z)h(z)\langle Ae_j, z\rangle \mu(dz). \tag{2.3.8}$$

Then Eq.(2.3.3) with the $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $gh \in \mathcal{F}_b^{I_\Gamma}$ replacing g implies

$$I_g^j = -\frac{1}{2}\int_F g(z)\nu_j(dz).$$

In order to prove (2.3.4), it is enough to show that $I_g^j = 0$ for any function $g(z)$ of the type

$$g(z) = f(\langle e_j, z\rangle, \langle l_2, z\rangle, ..., \langle l_m, z\rangle); l_2, ..., l_m \in H, f \in C_0^1(R^m), \tag{2.3.9}$$

for we have then $\nu_j = 0$.

On account of (2.1.8) we have the expression

$$\mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) = \mathcal{E}^{I_\Gamma, e_j}(e_j(\cdot), gh) = \frac{1}{2}\int_{E_{e_j}}\int_{R_x} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s) ds \mu_{e_j}(dx), \tag{2.3.10}$$

where $R_x = R(I_\Gamma(\cdot e_j + x)), F_x := \{s : se_j + x \in F\}$ for $x \in E_{e_j}$ and $\tilde{h}$ is a $I_\Gamma \cdot \mu$-version of h appearing in the description of (2.1.8). For $x \in E_{e_j}$ set

$$V_x := \{s : se_j + x \in V\}, \Gamma_x^0 := \{s : se_j + x \in \Gamma^0\}.$$

We then have the inclusion $V_x \subset \Gamma_x^0 \subset R_x \cap F_x$. By (2.3.6), $h(se_j + x) = 0$ for any $x \in E_{e_j}$ and for any $s \in R_x \setminus V_x$. On the other hand, there exists a Borel set $N \subset E_{e_j}$ with $\mu_{e_j}(N) = 0$ such that for each $x \in E_{e_j} \backslash N$,

$$h(se_j + x) = \tilde{h}(se_j + x)\ ds - a.e.$$

Here we set $h \equiv 0$ on $H \backslash F$. Since $\tilde{h}(\cdot e_j + x)$ is absolutely continuous in s, we can conclude that

$$\tilde{h}(se_j + x) = 0\ \forall x \in E_{e_j} \backslash N,\ \forall s \in R_x \backslash V_x.$$

Fix $x \in E_{e_j} \backslash N$ and let I be any connected component of the one dimensional open set R_x. Furthermore, for any function g of type (2.3.9) we denote the support of $g(\cdot e_j + x)$ by K_x (which is a compact set) and choose a bounded open interval J containing K_x. Then $I \cap V_x \cap K_x$ is a closed set contained in the bounded open interval $I \cap J$ and

$$g\tilde{h}(se_j + x) = 0\ \forall s \in (I \cap J) \backslash (I \cap V_x \cap K_x).$$

Therefore, an integration by part gives

$$\int_{I \cap J} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s) ds = \int_{I \cap J} \frac{1}{\lambda_j}(g\tilde{h})(se_j + x) s p_j(s) ds.$$

Combining this with (2.3.8) and (2.3.10), we arrive at

$$I_g^j = \int_{E_{e_j}}\int_{R_x} \frac{1}{2\lambda_j}(g\tilde{h})(se_j + x) s p_j(s) ds \mu_{e_j}(dx) - \int_H g(z) h(z) \langle Ae_j, z\rangle I_\Gamma(z) \mu(dz) = 0.$$

□

Now we state Theorem 2.2.2 for $\rho = I_\Gamma$.

Theorem 2.3.3 Suppose $\Gamma \subset H$ is a μ-measurable set satisfying condition (2.3.1). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s) \rangle_{H_1^*} dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s \rangle ds \ P_z\text{–a.s.}. \tag{2.3.11}$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in F \backslash S$

$$I_{\partial\Gamma}(X_s) dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \ P_z - a.s.. \tag{2.3.12}$$

In particular, if $\rho \in BV(H, H)$, then $\forall z \in F \backslash S, l \in D(A) \cap H$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t \langle l, \mathbf{n}_\Gamma(X_s) \rangle dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s \rangle ds \ \forall t \geq 0 \ P_z - a.s..$$

Proof All assertions except for (2.3.12) follow from Theorem 2.2.2 for $\rho := I_\Gamma$. (2.3.12) follows by Theorem 2.3.2 and [FOT94, Theorem 5.1.3]. □

2.4 Stochastic reflection problem on a regular convex set

In this section, we get the existence and uniqueness of the solution for (1.1) if Γ is a regular convex set. We also extend these results to the non-symmetric case. Now we consider Γ satisfying [BDT09] Hypothesis 1.1 (ii) with $K := \Gamma$, that is:

Hypothesis 2.4.1 There exists a convex C^∞ function $g : H \to \mathbb{R}$ with $g(0) = 0, g'(0) = 0$, and $D^2 g$ strictly positive definite, that is,$\langle D^2 g(x) h, h \rangle \geq \gamma |h|^2 \ \forall h \in H$ for some $\gamma > 0$, such that

$$\Gamma = \{x \in H : g(x) \leq 1\}, \partial\Gamma = \{x \in H : g(x) = 1\}$$

Moreover, we also suppose that $D^2 g$ is bounded on Γ and $|Q^{1/2} Dg|^{-1} \in \cap_{p>1} L^p(H, \mu)$.

Remark 2.4.2 By [BDT09, Lemma 1.2], Γ is convex and closed and there exists some constant $\delta > 0$ such that $|Dg(x)| \leq \delta \ \forall x \in \Gamma$.

2.4.1 Reflected OU processes on regular convex sets

Under Hypothesis 2.4.1, by [BDT10, Lemma A.1] we can prove that $I_\Gamma \in BV(H,H) \cap QR(H)$:

Theorem 2.4.3 Assume that Hypothesis 2.4.1 holds. Then $I_\Gamma \in BV(H,H) \cap QR(H)$.

Proof We first note that trivially by Remark 2.3.1 we have that $I_\Gamma \in QR(H)$. Let

$$\rho_\varepsilon(x) := \exp(-\frac{(g(x)-1)^2}{\varepsilon} 1_{\{g\geq 1\}}), x \in H.$$

Thus,

$$\lim_{\varepsilon\to 0} \rho_\varepsilon = I_\Gamma.$$

Moreover,

$$D\rho_\varepsilon = -\frac{2}{\varepsilon}\rho_\varepsilon 1_{\{g\geq 1\}} Dg(g-1)\ \mu - a.e..$$

By [BDT10, Lemma A.1] we have for $\varphi \in C_b^1(H)$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_H \varphi(x) 1_{\{g(x)\geq 1\}}(g(x)-1)\langle Dg(x), z\rangle \rho_\varepsilon(x)\mu(dx) = \frac{1}{2}\int_{\partial\Gamma} \varphi(y)\langle n(y), z\rangle \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dy),$$

where $n := Dg/|Dg|$ is the exterior normal to $\partial\Gamma$ at y and $\mu_{\partial\Gamma}$ is the surface measure on $\partial\Gamma$ induced by μ (cf. [BDT09], [BDT10], [Ma97]), whereas by (2.2.2) for any $\varphi \in C_b^1(H)$ and $z \in D(A)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_H \varphi(x) 1_{\{g(x)\geq 1\}}(g(x)-1)\langle Dg(x), z\rangle \rho_\varepsilon(x)\mu(dx) \\
=& -\lim_{\varepsilon\to 0} \frac{1}{2}\int_H \langle D\rho_\varepsilon(x), \varphi(x) z\rangle \mu(dx) \\
=& -\frac{1}{2}\lim_{\varepsilon\to 0}\int_H \rho_\varepsilon(x) D^*(\varphi z)(x)\mu(dx) \\
=& -\frac{1}{2}\int_H 1_\Gamma(x) D^*(\varphi z)(x)\mu(dx).
\end{aligned}$$

Thus,

$$\int_H 1_\Gamma(x) D^*(\varphi z)(x)\mu(dx) = -\int_{\partial\Gamma} \varphi(x)\langle n(x), z\rangle \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dx)\ \forall z \in D(A), \varphi \in C_b^1. \tag{2.4.1}$$

By the proof of [BDT10, Lemma A.1], we get that g is a non-degenerate map. So we can use the co-area formula (see [Ma97, Theorem 6.3.1, Ch. V] or [BDT10, (A.4)]):

$$\int_H f\mu(dx) = \int_0^\infty [\int_{g=r} f(y)\frac{1}{|Q^{1/2}Dg(y)|}\mu_{\Sigma_r}(dy)]dr.$$

By [Ma97, Theorem 6.2, Ch. V] the surface measure is defined for all $r \geq 0$, moreover [Ma97, Theorem 1.1, Corollary 6.3.2, Ch. V] imply that $r \mapsto \mu_{\Sigma_r}$ is continuous in the topology induced by $D_r^p(H)$ for some $p \in (1,\infty), r \in (0,\infty)$(cf [Ma97]) on the measures on $(H, \mathcal{B}(H))$. Take $f \equiv 1$ in the co-area formula, then by the continuity property of the surface measure with respect to r we have that $\frac{1}{|Q^{1/2}Dg(y)|}\mu_{\Sigma_r}(dy)$ is a finite measure supported in $\{g = r\}$. By Remark 2.4.2 and since $\mu_{\partial\Gamma} = \mu_{\Sigma_1}$, we have that $\frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}$ is a finite measure. And hence by Theorem 2.2.1 (iii), we get $I_\Gamma \in BV(H,H)$.

□

Thus by Theorem 2.3.3 we immediately get the following.

Theorem 2.4.4 Assume Hypothesis 2.4.1. Then there exists an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P_z - a.e.$$

where $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ and

$$\|\partial\Gamma\|(dy) = \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dy),$$

where $\mu_{\partial\Gamma}$ is the surface measure induced by μ (c.f [BDT09], [BDT10], [Ma97]).

Remark 2.4.5 It can be shown that for $x \in \partial\Gamma$, $\mathbf{n}_\Gamma(x) = \frac{Dg}{|Dg|}$ is the exterior normal to Γ, i.e the unique element in H of unit length such that

$$\langle \mathbf{n}_\Gamma(x), y - x\rangle \leq 0 \ \forall y \in \Gamma.$$

2.4.2 Existence and uniqueness of solutions

Let $\Gamma \subset H$ and our linear operator A satisfy Hypothesis 2.4.1 and Hypothesis 2.1.1, respectively. Consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{2.4.2}$$

where $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, i.e.

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \leq 0 \ \forall y \in \Gamma\}.$$

Definition 2.4.6 A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (2.4.2) if the following conditions hold.

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ $P - a.s.$;

(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s.$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle dL_s - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P - a.s.$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

Remark 2.4.7 By Remark 2.4.5 we know that $\mathbf{n}_\Gamma(x) \in N_\Gamma(x)$ for all $x \in \partial\Gamma$. Hence by Definition 2.4.6 (ii) it follows that Definition 2.4.6 is appropriate to define a solution for the multi-valued equation (2.4.2).

We denote the semigroup with the infinitesimal generator $-A$ by $S(t)$, $t \geq 0$.

Definition 2.4.8 A pair of continuous $H \times \mathbb{R}$ valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$ is called a mild solution of (2.4.2) if

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ $P - a.s.$;

(ii) L is an increasing process with the property

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s.$$

and

$$X_t = S(t)x + \int_0^t S(t-s)dW_s - \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s \ \forall t \in [0, T] \ P - a.s.$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ. In particular, the appearing integrals have to be well defined.

Lemma 2.4.9 The process given by

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s$$

is P-a.s. continuous and adapted to $\mathcal{F}_t, t \in [0,T]$. This especially implies that it is predictable.

Proof As $|S(t-s)\mathbf{n}_\Gamma(X_s)| \leq M_T|\mathbf{n}_\Gamma(X_s)|, s \in [0,T]$, the integrals $\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, t \in [0,T]$, are well defined. For $0 \leq s \leq t \leq T$,

$$\begin{aligned}
&|\int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u - \int_0^t S(t-u)\mathbf{n}_\Gamma(X_u)dL_u| \\
\leq &|\int_0^s [S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)dL_u| + |\int_s^t S(t-u)\mathbf{n}_\Gamma(X_u)dL_u| \\
\leq &\int_0^s |[S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)|dL_u + \int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|dL_u,
\end{aligned}$$

where the first summand converges to zero as $s \uparrow t$ or $t \downarrow s$, because

$$|1_{[0,s)}(u)[S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)| \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

For the second summand we have

$$\int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|dL_u \leq M_T(L_t - L_s) \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

By the same arguments as in [Ro10, Lemma 5.1.9] we conclude that the integral is adapted to $\mathcal{F}_t, t \in [0,T]$. □

Theorem 2.4.10 $(X(t), L_t), t \in [0,T]$, is a solution of (2.4.2) if and only if it is a mild solution.

Proof ($\Rightarrow$) First, we prove that for arbitrary $\zeta \in C^1([0,T], D(A))$ the following equation holds:

$$\langle X_t, \zeta_t\rangle = \langle x, \zeta_0\rangle + \int_0^t \langle \zeta_s, dW_s\rangle - \int_0^t \langle \mathbf{n}_\Gamma(X_s), \zeta_s\rangle dL_s + \int_0^t \langle X_s, -A\zeta_s + \zeta_s'\rangle ds \ \forall t \geq 0 \ P-a.s.. \tag{2.4.3}$$

If $\zeta_s = \eta f_s$ for $f \in C^1([0,T])$ and $\eta \in D(A)$, by Itô's formula we have the above relation for such ζ. Then by [Ro10, Lemma G.0.10] and the same arguments as the proof of Proposition G.0.11 we obtain the above formula for all $\zeta \in C^1([0,T], D(A))$. As in [Ro10, Proposition G.0.11], for the resolvent $R_n := (n+A)^{-1} : H \to D(A)$

and $t \in [0,T]$ choosing $\zeta_s := S(t-s)nR_n\eta, \eta \in H$, we deduce from (2.4.3) that

$$\begin{aligned}\langle X_t, nR_n\eta\rangle =&\langle x, S(t)nR_n\eta\rangle + \int_0^t \langle S(t-s)nR_n\eta, dW_s\rangle - \int_0^t \langle \mathbf{n}_\Gamma(X_s), S(t-s)nR_n\eta\rangle dL_s\\ &+ \int_0^t \langle X_s, AS(t-s)nR_n\eta\rangle + \langle X_s, -AS(t-s)nR_n\eta\rangle ds\\ =&\langle S(t)x + \int_0^t S(t-s)dW_s + \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, nR_n\eta\rangle \ \forall t \in [0,T] \ P-a.s..\end{aligned}$$

Letting $n \to \infty$, we conclude that $(X(t), L_t), t \in [0,T]$, is a mild solution.

($\Leftarrow$) By Lemma 2.4.9 and [Ro10, Theorem 5.1.3], we have

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s \qquad \text{and} \qquad \int_0^t S(t-s)dW_s, t \in [0,T],$$

have predictable versions. And we use the same notation for the predictable versions of the respective processes. As (X_t, L_t) is a mild solution, for all $\eta \in D(A)$ we get

$$\begin{aligned}\int_0^t \langle X_s, A\eta\rangle ds = &\int_0^t \langle S(s)x, A\eta\rangle ds - \int_0^t \langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u, A\eta\rangle ds\\ &+ \int_0^t \langle \int_0^s S(s-u)dW_u, A\eta\rangle ds \ \forall t \in [0,T] \ P-a.s..\end{aligned}$$

The assertion that $(X(t), L_t), t \in [0,T]$, is a solution of (2.4.2) now follows as in the proof of [Ro10, Proposition G.0.9] because

$$\begin{aligned}\int_0^t \langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u, A\eta\rangle ds =& \int_0^t \int_0^s \langle \mathbf{n}_\Gamma(X_u), -\frac{d}{ds}S(s-u)\eta\rangle dL_u ds\\ =& -\langle \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, \eta\rangle + \langle \int_0^t \mathbf{n}_\Gamma(X_s)dL_s, \eta\rangle.\end{aligned}$$

□

Below, we prove (2.4.2) has a unique solution in the sense of Definition 2.4.6.

Theorem 2.4.11 Let $\Gamma \subset H$ satisfy Hypothesis 2.4.1. Then the stochastic inclusion (2.4.2) admits at most one solution in the sense of Definition 2.4.6.

Proof Let (u, L^1) and (v, L^2) be two solutions of (2.4.2), and let $\{e_k\}_{k\in N}$ be the eigenbasis of A from above. We then have

$$\langle e_k, u(t)-v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s)-v(s)\rangle ds + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle dL_s^2 = 0$$

Setting $\phi_k(t) := \langle e_k, u(t) - v(t)\rangle$, we obtain

$$\begin{aligned}
\phi_k^2(t) =& 2\int_0^t \phi_k(s) d\phi_k(s) \\
=& -2\Big(\int_0^t \langle \alpha_k e_k, u(s)-v(s)\rangle\langle e_k, u(s)-v(s)\rangle ds + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^1 \\
& - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^2\Big) \\
\le& -2\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^1 + 2\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^2.
\end{aligned} \tag{2.4.4}$$

By dominated convergence theorem for all $t \ge 0$ we have $P-a.s$:

$$\begin{aligned}
&\sum_{k\le N}\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^1 \\
&\to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s)-v(s)\rangle dL_s^1 \text{ as } N\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k\le N}\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s)-v(s)\rangle dL_s^2 \\
&\to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s)-v(s)\rangle dL_s^2 \text{ as } N\to\infty.
\end{aligned}$$

Summing over $k \le N$ in (2.4.4) and letting $N \to \infty$ yield that for all $t \ge 0$ $P-a.s$

$$|u(t)-v(t)|^2 \le 2\int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s)-u(s)\rangle dL_s^1 + 2\int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s)-v(s)\rangle dL_s^2$$

By Remark 2.4.5 it follows that

$$|u(t)-v(t)|^2 \le 0,$$

which implies

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t).$$

□

Combining Theorem 2.4.4 and 2.4.11 with the Yamada-Watanabe Theorem, we now obtain the following:

Theorem 2.4.12 If Γ satisfies Hypothesis 2.4.1, then there exists a Borel set $M \subset H$ with
$I_\Gamma \cdot \mu(M) = \mu(\Gamma)$ such that for every $x \in M$, (2.4.2) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 2.4.6 and $P(X_0 = x) = 1$. Moreover $X(t) \in M$ for all $t \geq 0$ P-a.s.

Proof By Theorem 2.4.4 and Theorem 2.4.11, one sees that [Ku07, Theorem 3.14] a) is satisfied for the solution (X, L). So, the assertion follows from [Ku07, Theorem 3.14] b). $\square$

Remark 2.4.13 Following the same arguments as in the proof of [RSZ08, Theorem 2.1], we can give an alternative proof of Theorem 2.4.12 for a stronger notion of strong solutions (see 2.6 Appendix). Also, because of Theorem 2.4.10, by a modification of [On04, Theorem 12.1], we can prove the Yamada Watanabe Theorem for the mild solution in Definition 2.4.8, and then also a corresponding version of Theorem 2.4.12 for mild solutions for (2.4.2). This will be contained in forthcoming work.

2.4.3 The non-symmetric case

In this section, we extend our results to the non-symmetric case. For $\Gamma \subset H$ satisfying Hypothesis 2.4.1, we consider the non-symmetric Dirichlet form,

$$\mathcal{E}^\Gamma(u, v) = \int_\Gamma (\frac{1}{2}\langle Du(z), Dv(z)\rangle + \langle B(z), Du(z)\rangle v(z))\mu(dz), u, v \in C_b^1(\Gamma),$$

where B is a map from Γ to H such that

$$B \in L^\infty(\Gamma \to H, \mu), \int_\Gamma \langle B, Du\rangle d\mu \geq 0 \text{ for all } u \in C_b^1(\Gamma), u \geq 0. \qquad (2.4.5)$$

Then $(\mathcal{E}, C_b^1(\Gamma))$ is a densely defined bilinear form on $L^2(\Gamma; \mu)$ which is positive definite, since for all $u \in C_b^1(\Gamma)$

$$\mathcal{E}^\Gamma(u, u) = \int_\Gamma \frac{1}{2}(\langle Du(z), Du(z)\rangle + \langle B(z), Du^2(z)\rangle(z))\mu(dz) \geq 0.$$

Furthermore, by the same argument as [MR92, II.3.e] we have $(\mathcal{E}, C_b^1(\Gamma))$ is closable on $L^2(\Gamma, \mu)$ and its closure $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a Dirichlet form on $L^2(\Gamma, \mu)$. We denote the extended Dirichlet space of $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ by $\mathcal{F}_e^\Gamma$: Recall that $u \in \mathcal{F}_e^\Gamma$ if and only if $|u| < \infty$ $I_\Gamma \cdot \mu - a.e.$ and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\Gamma$ such that

$\mathcal{E}^\Gamma(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u\ I_\Gamma \cdot \mu - a.e.$ as $n \to \infty$. This Dirichlet form satisfies the weak sector condition

$$|\mathcal{E}_1^\Gamma(u, v)| \leq K \mathcal{E}_1^\Gamma(u, u)^{1/2} \mathcal{E}_1^\Gamma(v, v)^{1/2}.$$

Furthermore, we have:

Theorem 2.4.14 Suppose $\Gamma \subset H$ satisfies Hypothesis 2.4.1. Then $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a quasi-regular local Dirichlet form on $L^2(\Gamma; \mu)$.

Proof The assertion follows by [MR92 IV,4b] and [RS92]. □

By virtue of Theorem 2.4.14 and [MR92], there exists a diffusion process $M^\Gamma = (X_t, P_z)$ on Γ associated with the Dirichlet form $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$. Since constant functions are in $\mathcal{F}^\Gamma$ and $\mathcal{E}^\Gamma(1, 1) = 0$, M^Γ is recurrent and conservative. We denote by $\mathbf{A}_+^\Gamma$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^Γ, and define $\mathbf{A}^\Gamma = \mathbf{A}_+^\Gamma - \mathbf{A}_+^\Gamma$. For $A \in \mathbf{A}^\Gamma$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}_0^\Gamma = \{A \in \mathbf{A}^\Gamma | E_{I_\Gamma \cdot \mu}(\{A\}_t) < \infty\ \forall t > 0\}$. Each element in $\mathbf{A}_+^\Gamma$ has a corresponding positive $\mathcal{E}^\Gamma$-smooth measure on Γ by the Revuz correspondence. The totality of such measures will be denoted by S_+^Γ. Accordingly, $\mathbf{A}^\Gamma$ corresponds to $S^\Gamma = S_+^\Gamma - S_+^\Gamma$, the set of all $\mathcal{E}^\Gamma$-smooth signed measure in the sense that $A_t = A_t^1 - A_t^2$ for $A_t^k \in \mathbf{A}_+^\rho, k = 1, 2$ whose Revuz measures are $\nu^k, k = 1, 2$ and $\nu = \nu^1 - \nu^2$ is the Hahn-Jordan decomposition of ν. The element of $\mathbf{A}$ corresponding to $\nu \in S$ will be denoted byA^ν.

Note that for each $l \in H$ the function $u(z) = \langle l, z \rangle$ belongs to the extended Dirichlet space $\mathcal{F}_e^\Gamma$ and

$$\mathcal{E}^\Gamma(l(\cdot), v) = \int_\Gamma (\frac{1}{2} \langle l, Dv(z) \rangle + \langle B(z), l \rangle v(z)) \mu(dz)\ \forall v \in C_b^1(\Gamma). \tag{2.4.6}$$

On the other hand, the AF $\langle l, X_t - X_0 \rangle$ of M^Γ admits a decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l \in H$

$$\langle l, X_t - X_0 \rangle = M_t^l + N_t^l\ \forall t \geq 0\ P_z - a.s. \tag{2.4.7}$$

for $\mathcal{E}^\rho$-q.e. $z \in \Gamma$.

Then we have the following:

Theorem 2.4.15 Suppose $\Gamma \subset H$ satisfies Hypothesis 2.4.1.

(1) The next three conditions are equivalent:

(i)$N^l \in A_0$.

(ii)$|\mathcal{E}^\Gamma(l(\cdot), v)| \leq C \|v\|_\infty\ \forall v \in C_b^1(\Gamma)$.

(iii) There exists a finite (unique) signed measure ν_l on Γ such that

$$\mathcal{E}^\Gamma(l(\cdot), v) = -\int_\Gamma v(z)\nu_l(dz)\ \forall v \in C_b^1(\Gamma). \tag{2.4.8}$$

In this case, ν_l is automatically smooth, and

$$N^l = A^{\nu_l}.$$

(2) M^l is a martingale AF with quadratic variation process

$$\langle M^l\rangle_t = t|l|^2, t \geq 0. \tag{2.4.9}$$

Proof (1) By [Os88, Theorem 5.2.7] and the same arguments as in [Fu99], we can extend Theorem 6.2 in [Fu99] to our nonsymmetric case to prove the assertions.

(2)Since

$$\mathcal{E}^\Gamma(u, v) = \int_\Gamma (\frac{1}{2}\langle Du(z), Dv(z)\rangle + \langle B(z), Du(z)\rangle v(z))\mu(dz), u, v \in \mathcal{F}^\Gamma,$$

by [Os88 Theorem 5.1.5] for $u \in C_b^1(\Gamma)$, $f \in \mathcal{F}^\Gamma$ bounded we have

$$\begin{aligned}\int \tilde{f}(x)\mu_{\langle M^{[u]}\rangle}(dx) =& 2\mathcal{E}^\Gamma(u, uf) - \mathcal{E}^\Gamma(u^2, f)\\ =& 2\int_\Gamma (\frac{1}{2}\langle Du(z), D(u\tilde{f})(z)\rangle + \langle B(z), Du(z)\rangle u(z)\tilde{f}(z))\mu(dz)\\ &- \int_\Gamma (\frac{1}{2}\langle D(u(z)^2), D\tilde{f}(z)\rangle + \langle B(z), D(u^2)(z)\rangle \tilde{f}(z))\mu(dz)\\ =& \int_\Gamma \langle Du(z), Du(z)\rangle \tilde{f}(z)\mu(dz).\end{aligned}$$

Here $\tilde{f}$ denotes the $\mathcal{E}^\Gamma$-quasi-continuous version of f, $\mu_{\langle M^{[u]}\rangle}$ is the Reuvz measure for $\langle M^{[u]}\rangle$ and $M^{[u]}$ is the martingale additive functional in the Fukushima decomposition for $u(X_t)$. Hence we have

$$\mu_{\langle M^{[u]}\rangle}(dz) = I_\Gamma\langle Du(z), Du(z)\rangle \cdot \mu(dz).$$

By [Os88, (5.1.3)] we also have

$$e(\langle M^l\rangle) = e(M^l) = \int_\Gamma \frac{1}{2}\langle l, l\rangle \mu(dz)$$

where $e(M^l)$ is the energy of M^l. Then (2.4.9) easily follows. □

By Theorem 2.2.1 we can now prove the following:

Theorem 2.4.16 Suppose $\Gamma \subset H$ satisfies Hypothesis 2.4.1. Then there is an $\mathcal{E}^\Gamma$-exceptional set $S \subset \Gamma$ such that $\forall z \in \Gamma \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated OU-process M^Γ on Γ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s) \rangle_{H_1^*} dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s \rangle ds - \int_0^t \langle l, B(X_s) \rangle ds \; P_z\text{–a.s.} \tag{2.4.11}$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in \Gamma \backslash S$

$$I_{\partial\Gamma}(X_s) dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \; P_z - a.s.. \tag{2.4.12}$$

Here $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ, and

$$\|\partial\Gamma\|(dy) = \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|} \mu_{\partial\Gamma}(dy),$$

where $\mu_{\partial\Gamma}$ the surface measure induced by μ.

Proof By (2.4.6) and (2.2.16) we have

$$\mathcal{E}^\Gamma(l(\cdot), v) = \int_\Gamma \frac{1}{2} \langle l, Dv(z) \rangle + \langle B(z), l \rangle v(z) \mu(dz)$$

$$= \int_\Gamma \langle B(z), l \rangle v(z) \mu(dz) + \int_\Gamma v(z) \langle Al, z \rangle \mu(dz) + \frac{1}{2} \int_{\partial\Gamma} v(z) \langle l, \mathbf{n}_\Gamma(z) \rangle \|\partial\Gamma\|(dz).$$

Thus, by Theorem 2.4.15

$$N_t^l = -\langle Al, \int_0^t X_s(\omega) ds \rangle - \langle l, \int_0^t B(X_s(\omega)) ds \rangle - \frac{1}{2} \langle l, \int_0^t \mathbf{n}_\Gamma(X_s(\omega)) dL_s^{\|\partial\Gamma\|}(\omega) \rangle.$$

By Theorem 2.4.15 and the same method as in Theorem 2.2.2 one then proves the first assertion, and the last assertion follows by Theorem 2.4.3 and 2.4.4. □

Let $\Gamma \subset H$ and our linear operator A satisfy Hypothesis 2.4.1 and Hypothesis 2.1.1, respectively. As in Section 2.4.2 we shall now prove the existence and uniqueness of a solution of the following stochastic differential inclusion on the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + B(X(t)) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{2.4.13}$$

where B satisfies condition (2.4.5), $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, i.e.

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \leq 0 \ \forall y \in \Gamma\}.$$

Definition 2.4.17 A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (2.4.13) if the following conditions hold.

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ P-a.s;

(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s,$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle dL_s - \int_0^t \langle l, B(X_s)\rangle ds - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P-a.s.,$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

Below we prove (2.4.13) has a unique solution in the sense of Definition 2.4.17.

Theorem 2.4.18 Let $\Gamma \subset H$ satisfy Hypothesis 2.4.1 and B satisfy the monotonicity condition

$$\langle B(u) - B(v), u - v\rangle \geq -\alpha|u - v|^2 \tag{2.4.14}$$

for all $u, v \in \Gamma$, for some $\alpha \in [0, \infty)$ independent of u, v. The stochastic inclusion (2.4.13) admits at most one solution in the sense of Definition 2.4.17.

Proof Let (u, L^1) and (v, L^2) be two solutions of (2.4.13), and let $\{e_k\}_{k\in N}$ be the eigenbasis of A from above. We then have

$$\langle e_k, u(t) - v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle ds + \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle ds$$

$$+ \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle dL_s^2 = 0.$$

Setting $\phi_k(t) := \langle e_k, u(t) - v(t)\rangle$, and we have

$$\begin{aligned}
\phi_k^2(t) =& 2\int_0^t \phi_k(s) d\phi_k(s) \\
=& -2\Big(\int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle \langle e_k, u(s) - v(s)\rangle ds \\
&+ \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds \\
&\quad + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2\Big) \\
\leq& -2\int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds \\
&\quad -2\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1 + 2\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2.
\end{aligned} \tag{2.4.15}$$

By the same argument as Theorem 2.4.11, we have the following $P - a.s$:

$$\sum_{k\leq N} \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds$$

$$\to \int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \text{ as } N \to \infty,$$

$$\sum_{k\leq N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s) - v(s)\rangle dL_s^1 \text{ as } N \to \infty,$$

and

$$\sum_{k\leq N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2 \text{ as } N \to \infty.$$

Summing over $k \leq N$ in (2.4.15) and letting $N \to \infty$ yield that for all $t \geq 0, P - a.s$

$$\begin{aligned}
|u(t) - v(t)|^2 &+ 2\int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \\
&\leq 2\int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s) - u(s)\rangle dL_s^1 + 2\int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2.
\end{aligned}$$

By Remark 2.4.4 it follows that

$$|u(t) - v(t)|^2 + 2\int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \leq 0.$$

By (2.4.14) and Gronwall's Lemma it follows that

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t).$$

□

Combining Theorem 2.4.16 and 2.4.18 with the Yamada-Watanabe Theorem, we obtain the following:

Theorem 2.4.19 If Γ satisfies Hypothesis 2.4.1 and B in (2.4.13) satisfies (2.4.14), then there exists a Borel set $M \subset H$ with $I_\Gamma \cdot \mu(M) = \mu(\Gamma)$ such that for every $x \in M$, (2.4.13) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 2.4.17 and $P(X_0 = x) = 1$. Moreover, $X(t) \in M$ for all $t \geq 0$ P-a.s.

Proof The proof is completely analogous to that of Theorem 2.4.12. □

2.5 Reflected OU-processeses on a class of convex sets

Below for a topological space X we denote its Borel σ-algebra by $\mathcal{B}(X)$. In this section, we consider the case where $H := L^2(0,1), \rho = I_{K_\alpha}$, where $K_\alpha := \{f \in H | f \geq -\alpha\}, \alpha \geq 0$, and $A = -\frac{1}{2}\frac{d^2}{dr^2}$ with Dirichlet boundary conditions on (0,1). So in this case $e_j = \sqrt{2}\sin(j\pi r),\ j \in \mathbb{N}$, is the corresponding eigenbases. We recall that (cf [Za02]) we have $\mu(C_0([0,1])) = 1$. In [Za02], L.Zambotti proved the following integration by parts formulae in this situation:

For $\alpha > 0$,

$$\int_{K_\alpha} \langle l, D\varphi\rangle d\mu = -\int_{K_\alpha} \varphi(x)\langle x, l''\rangle \mu(dx) - \int_0^1 drl(r)\int \varphi(x)\sigma_\alpha(r, dx),\ \forall l \in D(A), \varphi \in C_b^1(H),$$

for $\alpha = 0$,

$$\int_{K_0} \langle l, D\varphi\rangle d\nu = -\int_{K_0} \varphi(x)\langle x, l''\rangle \nu(dx) - \int_0^1 drl(r) \int \varphi(x)\sigma_0(r, dx), \ \forall l \in D(A), \varphi \in C_b^1(H), \tag{2.5.1}$$

where ν is the law of the Bessel Bridge of dimension 3 over $[0, 1]$ which is zero at 0 and 1, $\sigma_\alpha(r, dx) = \sigma_\alpha(r)\mu_\alpha(r, dx)$, and for $\alpha > 0$, σ_α is a positive bounded function, and for $\alpha = 0$, $\sigma_0(r) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}}$, where $\mu_\alpha(r, dx), \alpha \geq 0$, are probability kernels from $(H, \mathcal{B}(H))$ to $([0, 1], \mathcal{B}([0, 1]))$.

Remark 2.5.1 Since each l in $D(A)$ has a second derivative in L^2, its first derivative is bounded, hence l goes faster than linear to zero at any point where l is zero, in particular at the boundary points $r = 0$ and $r = 1$. Hence the second integral in the right hand side of the above equality is well-defined.

We know by (2.2.5) that for all $l \in D(A)$

$$D^*(\varphi(\cdot)l) = -\langle l, D\varphi\rangle - \varphi\langle l'', \cdot\rangle.$$

Hence for $\alpha > 0$,

$$\int_{K_\alpha} D^*(\varphi(\cdot)l) d\mu = \int_0^1 l(r) \int \varphi(x)\sigma_\alpha(r, dx)dr \ \forall l \in D(A), \varphi \in C_b^1(H). \tag{2.5.2}$$

Now take

$$c_j := \begin{cases} (j\pi)^{\frac{1}{2}+\varepsilon}, & \text{if } \alpha > 0 \\ (j\pi)^\beta, & \text{if } \alpha = 0, \end{cases} \tag{2.5.3}$$

where $\varepsilon \in (0, \frac{3}{2}]$ and $\beta \in (\frac{3}{2}, 2]$ respectively, and define

$$H_1 := \{x \in H | \sum_{j=1}^\infty \langle x, e_j\rangle^2 c_j^2 < \infty\},$$

equipped with the inner product

$$\langle x, y\rangle_{H_1} := \sum_{j=1}^\infty c_j^2 \langle x, e_j\rangle\langle y, e_j\rangle.$$

We note that $D(A) \subset H_1$ continuously for all $\alpha \geq 0$, since $\varepsilon \leq \frac{3}{2}, \beta \leq 2$. Furthermore, $(H_1, \langle , \rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H(\equiv H^*) \subset H_1^*.$$

It follows that

$$_{H_1}\langle z, v\rangle_{H_1^*} = \langle z, v\rangle_H \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple.

The following is the main result of this section.

Theorem 2.5.2 If $\alpha > 0$, then $I_{K_\alpha} \in BV(H, H_1) \cap \mathbf{H}$.

Proof First for σ_α as in (2.5.2) we show that for each $B \in \mathcal{B}(H)$ the function $r \mapsto \sigma_\alpha(r, B)$ is in H_1^* and that the map $B \mapsto \sigma_\alpha(\cdot, B)$ is in fact an H_1^*-valued measure of bounded variation, i.e

$$\sup\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\cup}_{n=1}^{\infty} B_n\} < \infty,$$

that is,

$$\sup\{\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_\alpha(r, B_n)\sin(j\pi r)dr)^2)^{1/2} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\cup}_{n=1}^{\infty} B_n\} < \infty,$$

where $\dot{\cup}_{n=1}^{\infty} B_n$ means disjoint union.

For $\alpha > 0$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_\alpha(r, B_n)\sin(j\pi r)dr)^2)^{1/2}\\
\leq&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_\alpha(r, B_n)dr)^2)^{1/2}\\
\leq&C\sum_{n=1}^{\infty}\int_0^1 \sigma_\alpha(r, B_n)dr\\
=&C\int_0^1 \sigma_\alpha(r)dr < \infty.
\end{aligned}$$

Thus σ_α in (2.5.2) is of bounded variation as an H_1^*-valued measure. Hence by the theory of vector-valued measures (cf [AMMP10, Section 2.1]), there is a unit vector field $n_\alpha : H \to H_1^*$, such that $\sigma_\alpha = n_\alpha\|\sigma_\alpha\|$, where $\|\sigma_\alpha\|(B) := \sup\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, B = \dot{\cup}_{n=1}^{\infty} B_n\}$ is a nonnegative measure, which is finite by the above proof. So (2.5.2) becomes

$$\int_{K_\alpha} D^*(\varphi(\cdot)l)d\mu = \int {}_{H_1}\langle \varphi(x)l, n_\alpha(x)\rangle_{H_1^*}\|\sigma_\alpha\|(dx) \ \forall l \in D(A), \varphi \in C_b^1(H),$$

which by linearity extends to all $G \in (C_b^1)_{D(A)\cap H_1}$. Thus by Theorem 2.2.1(iii), we get that $I_{K_\alpha} \in BV(H, H_1)$. $I_{K_\alpha} \in \mathbf{H}$ follows by Remark 2.3.1. $\square$

Remark 2.5.3 It has been proved by Guan Qingyang that I_{K_α} is not in $BV(H, H)$. If we take $H = H_1 = H_1^*$ and define $B_n^i := \{x \in H : \inf_{t\in[\frac{i-1}{n},\frac{i}{n}]} x(t) = -\alpha\}$ for $i = 1, ..., n$, then $\sigma_\alpha(r, B_n^i) = I_{[\frac{i-1}{n},\frac{i}{n}]}(r)$. Thus $\|\sigma_\alpha\|(H) \geq \sqrt{n}$. Letting $n \to \infty$, we have $\|\sigma_\alpha\|(H) \to \infty$. So I_{K_α} is not in $BV(H, H)$.

Theorem 2.5.4 For $\alpha = 0$, then there exist a positive finite measure $\|\sigma_0\|$ on H and a Borel-measurable map $n_0 : H \to H_1^*$ such that $\|n_0(z)\|_{H_1^*} = 1$ $\|\sigma_0\| - a.e$, and for any $l \in D(A), \varphi \in C_b^1(H)$

$$-\int_{K_0} \langle l, D\varphi\rangle d\nu - \int_{K_0} \varphi(x)\langle x, l''\rangle \nu(dx) = \int {}_{H_1}\langle \varphi(x)l, n_0(x)\rangle_{H_1^*} \|\sigma_0\|(dx). \quad (2.5.4)$$

Proof For $\alpha = 0$ using that $|\sin(j\pi r)| \leq 2j\pi r(1-r)$ $\forall r \in [0, 1]$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_0(r, B_n)\sin(j\pi r)dr)^2)^{1/2}\\
\leq&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_0(r, B_n)2j\pi r(1-r)dr)^2)^{1/2}\\
\leq&C\sum_{n=1}^{\infty}\int_0^1 \sigma_0(r, B_n)r(1-r)dr\\
=&C\int_0^1 \sigma_0(r)r(1-r)dr < \infty
\end{aligned}$$

Thus σ_0 in (2.5.1) is of bounded variation as an H_1^*-valued measure. Hence by the theory of vector-valued measures (cf [AMMP10, Section 2.1]), there is a unit vector field $n_0 : H \to H_1^*$, such that $\sigma_0 = n_0\|\sigma_\alpha\|$, where $\|\sigma_0\|(B) := \sup\{\sum_{n=1}^{\infty} \|\sigma_0(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, B = \dot\cup_{n=1}^{\infty} B_n\}$ is a nonnegative measure, which is finite by the above proof. So the result follows by (2.5.1). $\square$

Since here $\mu(K_0) = 0$, we have to change the reference measure of the Dirichlet form. Consider

$$\mathcal{E}^{K_0}(u, v) = \frac{1}{2}\int_{K_0} \langle Du, Dv\rangle d\nu, u, v \in C_b^1(K_0).$$

Since $I_{K_0} \in \mathbf{H}$ by Remark 2.3.1, the closure of $(\mathcal{E}^{I_{K_0}}, C_b^1(K_0))$ is also a quasi-regular local Dirichlet form on $L^2(F; \rho\cdot\nu)$ in the sense of [MR92, IV Definition 3.1]. As before, there exists a diffusion process $M^{I_{K_0}} = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$ on F associated with this Dirichlet form. $M^{I_{K_0}}$ will also be called distorted OU-process on K_0. As before, $M^{I_{K_0}}$ is recurrent and conservative. As before, we also have the associated PCAF and the Revuz correspondence.

Combining these two cases: for $\alpha > 0$ by Theorem 2.2.2 and for $\alpha = 0$ by the same argument as Theorem 2.2.2, since we have (2.5.4), we have the following theorem.

Theorem 2.5.5 Let $\rho := I_{K_\alpha}, \alpha \geq 0$ and consider the measure $\|\sigma_\alpha\|$ and n_α appearing in Theorem 2.5.2 and Theorem 2.5.4. Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l \in D(A)$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s \rangle + \frac{1}{2} \int_0^t {}_{H_1}\langle l, n_\alpha(X_s) \rangle_{H_1^*} dL_s^{\|\sigma_\alpha\|} - \int_0^t \langle Al, X_s \rangle ds \ P_z - a.e. \tag{2.5.5}$$

Here $L_t^{\|\sigma_\alpha\|}$ is the real valued PCAF associated with $\|\sigma_\alpha\|$ by the Revuz correspondence with respect to M^ρ, satisfying

$$I_{\{X_s + \alpha \neq 0\}} dL_s^{\|\sigma_\alpha\|} = 0, \tag{2.5.6}$$

and for $l \in H_1$ with $l(r) \geq 0$ we have

$$\int_0^t {}_{H_1}\langle l, n_\alpha(X_s) \rangle_{H_1^*} dL_s^{\|\sigma_\alpha\|} \geq 0. \tag{2.5.7}$$

Furthermore, for all $z \in F$

$$P_z[X_t \in C_0[0,1] \text{ for a.e. } t \in [0, \infty)] = 1. \tag{2.5.8}$$

Proof For $\alpha > 0$, the first part of the assertion follows by Theorem 2.2.2 and the uniqueness part of Theorem 2.2.1 (ii). For $\alpha = 0$, the assertion follows by the same argument as in Theorem 2.2.2. (2.5.6) and (2.5.7) follow by the property of σ_α in [Za02]. By [Pa67, p.135 Theorem 2.4], we have $C_0[0,1]$ is a Borel subset of $L^2[0,1]$. By [FOT94, (5.1.13)], we have

$$E_{\rho\mu}[\int_{k-1}^k 1_{F \backslash C_0[0,1]}(X_s) ds] = \rho\mu(F \setminus C_0[0,1]) = 0 \ \forall k \in \mathbb{N},$$

hence

$$E_{\rho\mu}[\int_0^\infty 1_{F \backslash C_0[0,1]}(X_s) ds] = 0.$$

Since $E_x[\int_0^\infty 1_{F \backslash C_0[0,1]}(X_s) ds]$ is a 0-excessive function in $x \in K_\alpha$, it is finely continuous with respect to the process X. Then for $\mathcal{E}^\rho -$ q.e. $z \in F$,

$$E_z[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds] = 0,$$

thus, for $\mathcal{E}^\rho -$ q.e. $z \in F$,

$$P_z[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds = 0] = 1.$$

As a consequence, we have that $\Lambda_0 := \{X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)\}$ is measurable and for $\mathcal{E}^\rho -$ q.e. $z \in F$

$$P_z(\Lambda_0) = 1.$$

As $\Lambda_0 = \cap_{t\in\mathbb{Q},t>0}\theta_t^{-1}\Lambda_0$ and since by [ASZ09] we have that the semigroup associated with X_t is strong Feller, by the Markov property as in [DR02, Lemma 7.1], we obtain that for any $z \in F, t \in \mathbb{Q}, t > 0$,

$$P_z(\theta_t^{-1}\Lambda_0) = 1.$$

Hence for any $z \in F$ we have

$$P_z[X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)] = 1.$$

□

Remark 2.5.6 We emphasize that in the present situation it was proved in [NR92, Theorem 1.3] that for all initial conditions $x \in H$, there exists a unique strong solution to (1.1). By [Za02] the solution in [NP92] is associated to our Dirichlet form, hence satisfies (2.5.5) by Theorem 2.5.5. Hence it follows that the solution in [NP92, Theorem 1.3] is solution to an infinite-dimensional Skorohod problem.

2.6 Appendix

Another proof of Yamada-Watanabe theorem for Theorem 2.4.12's use We follow the same arguments as the proof of [RSZ08]:

We use the following spaces

$$\mathbb{B}_H := \{\omega \in C(\mathbb{R}^+; H), \int_0^T |\omega(t)|dt < \infty \text{ for all } T \in [0,\infty)\}$$

equipped with the metric

$$\rho(\omega_1,\omega_2) := \sum_{k=1}^{\infty} 2^{-k}[(\int_0^k |\omega_1(t)-\omega_2(t)|dt + \sup_{t\in[0,k]} |\omega_1(t)-\omega_2(t)|)\wedge 1].$$

Obviously, $(\mathbb{B}_H,\rho)$ is a complete separable metric space. Let $\mathcal{B}_t(\mathbb{B}_H)$ denote the σ-algebra generated by all maps $\pi_s:\mathbb{B}_H\to H, s\in[0,t]$ where $\pi_s(\omega):=\omega(s),\omega\in\mathbb{B}_H$.

$$\mathbb{B}_{\mathbb{R}} := \{\omega\in C(\mathbb{R}^+;\mathbb{R}),\omega(0)=0,\int_0^T |\omega(t)|dt<\infty \text{ for all } T\in[0,\infty)\}$$

equipped with the metric

$$\rho(\omega_1,\omega_2) := \sum_{k=1}^{\infty} 2^{-k}[(\int_0^k |\omega_1(t)-\omega_2(t)|dt + \sup_{t\in[0,k]} |\omega_1(t)-\omega_2(t)|)\wedge 1].$$

Obviously, $(\mathbb{B}_{\mathbb{R}},\rho)$ is a complete separable metric space. Let $\mathcal{B}_t(\mathbb{B}_{\mathbb{R}})$ denote the σ-algebra generated by all maps $\pi_s:\mathbb{B}_{\mathbb{R}}\to\mathbb{R}, s\in[0,t]$ where $\pi_s(\omega):=\omega(s),\omega\in\mathbb{B}_{\mathbb{R}}$.

By Theorem 2.4.3, we can choose a measurable set M, such that $I_\Gamma\cdot\mu(M)=1$ and for every $z\in M$, there exists a process W satisfying under P^z, W_t is a $\mathcal{M}_t$ cylindrical Wiener process, and any $l\in D(A)$

$$\langle l, X_t(\omega)-X_0(\omega)\rangle = \langle l, W_t(\omega)-\frac{1}{2}\int_0^t \mathbf{n}_\Gamma(X_s(\omega))dL_s^{\|\partial\Gamma\|}(\omega)\rangle - \langle Al, \int_0^t X_s(\omega)ds\rangle, P^z\text{−}a.e.$$

$L^{\|\partial\Gamma\|}$ is an increasing process which enjoy the property

$$\int_0^t I_{\partial\Gamma}(X_s(\omega))dL_s^{\|\partial\Gamma\|}(\omega) = L_t^{\|\partial\Gamma\|}(\omega), t\geq 0$$

and $X(t)\in M$ for all $t\geq 0$ P^z-a.s. We can choose one-to-one Hilbert-Schmidt operator J from H into another Hilbert space $(U,\langle,\rangle_U)$ and $\bar{W}(t):=\sum_{j=1}^{\infty}\beta_k(t)Je_k$, where $\beta_k(t)$ is independent $\mathcal{M}_t$-Brownian motions. Set

$$\mathbb{W}_0 := \{\omega\in C(\mathbb{R}^+;U),\omega(0)=0\}$$

equipped with the supremum norm and Borel σ-algebra $\mathcal{B}(\mathbb{W}_0)$. Let $\mathcal{B}_t(\mathbb{W}_0)$ be the σ-algebra generated by all maps $\pi_s:\mathbb{W}_0\to U, s\in[0,t]$ where $\pi_s(\omega):=\omega(s)$.

For fixed probability measure ν on $(H,\mathcal{B}(H)$ with $\nu(M)=1$, define $P^\nu := \int P^x\nu(dx)$ and a probability measure $\tilde{P}^\nu$ on $(H\times\mathbb{B}_H\times\mathbb{B}_{\mathbb{R}}\times\mathbb{W}_0, \mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes$

$\mathcal{B}(\mathbb{B}_\mathbb{R}) \otimes \mathcal{B}(\mathbb{W}_0))$, by

$$\tilde{P}^\nu := P^\nu \circ (X(0), X, L^{\|\partial\Gamma\|}, \bar{W})^{-1}$$

and P^J denotes the distribution of $\bar{W}_t$ on $(\mathbb{W}_0, \mathcal{B}(\mathbb{W}_0))$.

[Step 1] There exists a family $K_\nu((x,\omega), d\omega_1^1, d\omega_1^2), x \in H, \omega \in \mathbb{W}_0$, of probability measures on $(\mathbb{B}_H \times \mathbb{B}_\mathbb{R}, \mathcal{B}(\mathbb{B}_H) \otimes \mathcal{B}(\mathbb{B}_\mathbb{R}))$ having the following properties:

(i) For every $A \in \mathcal{B}(\mathbb{B}_H) \otimes \mathcal{B}(\mathbb{B}_\mathbb{R})$ the map

$$H \times \mathbb{W}_0 \ni (x,\omega) \mapsto K_\nu((x,\omega), A)$$

is $\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0)$-measurable.

(ii) For every $\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{B}_H) \otimes \mathcal{B}(\mathbb{B}_\mathbb{R}) \otimes \mathcal{B}(\mathbb{W}_0)$-measurable map $f : H \times \mathbb{B}_H \times \mathbb{B}_\mathbb{R} \times \mathbb{W}_0 \to [0,\infty)$ we have

$$\begin{aligned} &\int_{H\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{W}_0} f(x,\omega_1^1,\omega_1^2,\omega)\tilde{P}^\nu z(dx, d\omega_1^1, d\omega_1^2, d\omega) \\ &= \int_{\mathbb{W}_0}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} f(z,\omega_1^1,\omega_1^2,\omega)K_\nu((z,\omega), d\omega_1^1, d\omega_1^2)P^J(d\omega). \end{aligned}$$

(iii) If $t \in [0,\infty)$ and $f : \mathbb{B}_H \times \mathbb{B}_\mathbb{R} \to [0,\infty)$ is $\mathcal{B}_t(\mathbb{B}_H) \otimes \mathcal{B}_t(\mathbb{B}_\mathbb{R})$-measurable, then

$$H \times \mathbb{W}_0 \ni (x,\omega) \mapsto \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} f(\omega_1^1,\omega_1^2)K_\nu((x,\omega), d\omega_1^1, d\omega_1^2)$$

is $\overline{\mathcal{B}(H) \otimes \mathcal{B}_t(\mathbb{W}_0)}^{\nu\otimes P^J}$-measurable, where $\overline{\mathcal{B}(H) \otimes \mathcal{B}_t(\mathbb{W}_0)}^{\nu\otimes P^J}$ denotes the completion with respect to $\nu \otimes P^J$ in $\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0)$.

Let $\Pi : H \times \mathbb{B}_H \times \mathbb{B}_\mathbb{R} \times \mathbb{W}_0 \to H \times \mathbb{W}_0$ be the canonical projection. Since $X(0)$ is $\mathcal{F}_0$-measurable, hence P^ν-independent of W, it follows that

$$\tilde{P}^\nu \circ \Pi^{-1} = P^\nu \circ (X(0), \bar{W})^{-1} = \nu \otimes P^J.$$

Hence by the existence result on regular conditional distributions, the existence of the family $K_\nu((x,\omega), d\omega_1^1, d\omega_1^2), x \in H, \omega \in \mathbb{W}_0$ satisfying (i) and (ii) follows.

To prove (iii) it suffices to show that for $t \in [0,\infty)$ and for all $A_0 \in \mathcal{B}(H), A_1 \in \mathcal{B}_t(\mathbb{B}_H) \otimes \mathcal{B}_t(\mathbb{B}_\mathbb{R}), A \in \mathcal{B}_t(\mathbb{W}_0)$ and

$A' := \{\pi_{r_1} - \pi_t \in B_1, \pi_{r_2} - \pi_{r_1} \in B_2, ..., \pi_{r_k} - \pi_{r_{k-1}} \in B_k\}, t \le r_1 < ... < r_k, B_1, ..., B_k \in \mathcal{B}(H)$

$$\int_{A_0}\int_{\mathbb{W}_0} 1_{A\cap A'}(\omega)K_\nu((x,\omega),A_1)P^J(d\omega)\nu(dx)$$
$$=\int_{A_0}\int_{\mathbb{W}_0} 1_{A\cap A'}(\omega)E_{\nu\otimes P^J}(K_\nu(\cdot,A_1)|\mathcal{B}(H)\otimes\mathcal{B}_t(\mathbb{W}_0))P^J(d\omega)\nu(dx)$$

since the system of all $A\cap A', A\in\mathcal{B}_t(\mathbb{W}_0), A'$ as above generates $\mathcal{B}(\mathbb{W}_0)$. By (ii) above, the left-hand side of above relation is equal to

$$\begin{aligned}
&\int_{H\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{W}_0} 1_{A_0}(x)1_{A\cap A'}(\omega)1_{A_1}(d\omega_1^1,d\omega_1^2)\tilde{P}^\nu(dx,d\omega_1^1,d\omega_1^2,d\omega)\\
=&\int_\Omega 1_{A_0}(X(0))1_{A_1}(X,L^{\|\partial\Gamma\|})1_A(\bar{W})1_{A'}(\bar{W})dP^\nu\\
=&\int_\Omega 1_{A'}(\bar{W})dP^z\cdot\int_\Omega 1_{A_0}(X(0))1_{A_1}(X,L^{\|\partial\Gamma\|})1_A(\bar{W})dP^\nu\\
=&P^J(A')\int_{H\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{W}_0} 1_{A_0}(x)1_A(\omega)1_{A_1}(d\omega_1^1,d\omega_1^2)\tilde{P}^\nu(dx,d\omega_1^1,d\omega_1^2,d\omega)\\
=&P^J(A')\int_{A_0}\int_A K_\nu((x,\omega),A_1)P^J(d\omega)\nu(dx)\\
=&P^J(A')\int_{A_0}\int_A E_{\nu\otimes P^J}(K_\nu(\cdot,A_1)|\mathcal{B}(H)\otimes\mathcal{B}_t(\mathbb{W}_0))((x,\omega))P^J(d\omega)\nu(dx)\\
=&\int_{A_0}\int_{\mathbb{W}_0} 1_{A\cap A'}(\omega)E_{\nu\otimes P^J}(K_\nu(\cdot,A_1)|\mathcal{B}(H)\otimes\mathcal{B}_t(\mathbb{W}_0))((x,\omega))P^J(d\omega)\nu(dx)
\end{aligned}$$

[step 2] For $z\in H$ define a measure Q^z on

$$(H\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{W}_0,\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{W}_0))$$

by

$$\begin{aligned}
Q^z(A):=\int_H\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}}\int_{\mathbb{W}_0} &1_A(x,\omega_1^1,\omega_1^2,\omega_2^1,\omega_2^2,\omega)\\
&K_\nu((x,\omega),d\omega_1^1,d\omega_1^2)K_\nu((x,\omega),d\omega_2^1,d\omega_2^2)P^J(d\omega)\delta_z(dx)
\end{aligned}$$

Define the stochastic basis

$$\tilde{\Omega}:=H\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{W}_0$$

$$\tilde{\mathcal{F}}^z:=\overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{W}_0)}^{Q_z}$$

$\tilde{\mathcal{F}}_t^z:=\cap_{\varepsilon>0}\sigma(\mathcal{B}(H)\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_H)\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_H)\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{W}_0),\mathcal{N}_z)$,

where $\mathcal{N}_z:=\{N\in\tilde{\mathcal{F}}^z,Q^z(N)=0\}$ and define maps

$$\Pi_0:\tilde{\Omega}\to H,(x,\omega_1^1,\omega_1^2,\omega_2^1,\omega_2^2,\omega)\mapsto x,$$

$$\Pi_i^j : \tilde{\Omega} \to \mathbb{B}_H \text{ or } \mathbb{B}_{\mathbb{R}}, (x, \omega_1^1, \omega_1^2, \omega_2^1, \omega_2^2, \omega) \mapsto \omega_i^j, i, j = 1, 2,$$

$$\Pi_3 : \tilde{\Omega} \to \mathbb{W}_0, (x, \omega_1^1, \omega_1^2, \omega_2^1, \omega_2^2, \omega) \mapsto \omega \in \mathbb{W}_0.$$

Then, obviously,

$$Q^z \circ \Pi_0^{-1} = \delta_z$$

and

$$Q^z \circ \Pi_3^{-1} = P^J$$

By definition Π_3 is $(\tilde{\mathcal{F}}_t^z)$-adapted. Furthermore, for $0 \le s < t, y \in H$, and $A_0, \tilde{A}_0 \in \mathcal{B}(H), A_i \in \mathcal{B}_s(\mathbb{B}_H) \otimes \mathcal{B}_s(\mathbb{B}_{\mathbb{R}}), i = 1, 2, A_3 \in \mathcal{B}_s(\mathbb{W}_0)$,

$$\begin{aligned}
&\int_{\tilde{A}_0} E_{Q^z}(\exp(i\langle y, \Pi_3(t) - \Pi_3(s)\rangle)1_{A_0 \times A_1 \times A_2 \times A_3})\nu(dz) \\
=& \int_{\tilde{A}_0} \int_{\mathbb{W}_0} \exp(i\langle y, \omega(t) - \omega(s)\rangle)1_{A_0}(x)1_{A_3}(\omega)K_\nu((x,\omega), A_1)K_\nu((x,\omega), A_2)P^J(d\omega)\nu(dz) \\
=& \int_{\tilde{A}_0} \int_{\mathbb{W}_0} \exp(i\langle y, \omega(t) - \omega(s)\rangle)P^J(d\omega)Q^z(A_0 \times A_1 \times A_2 \times A_3)\nu(dz)
\end{aligned}$$

Now by a monotone class argument, we have that Π_3 is an $(\tilde{\mathcal{F}}_t^z)$-Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}^z, Q^z)$.

Then we can conclude that there exists $N_0 \in \mathcal{B}(H)$ with $\nu(N_0) = 0$ and for all $x \in N_0^c$, Π_3 is an $(\tilde{\mathcal{F}}_t^z)$-Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}^z, Q^z)$.

[Step 3] There exists $N_1 \in \mathcal{B}(H), N_0 \subset N_1$, with $\nu(N_1) = 0$ such that for $x \in N_1^c$, $(\Pi_1^1, \Pi_1^2, J^{-1}\Pi_3)$ and $(\Pi_2^1, \Pi_2^2, J^{-1}\Pi_3)$ with stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}^z, Q^z, (\tilde{\mathcal{F}}_t^z))$ satisfy (i)(ii)(iii) in Definition 2.4.4 for $X = \Pi_i^1, L = \Pi_i^2, W = J^{-1}\Pi_3, i = 1, 2$ such that

$$\Pi_1^1(0) = \Pi_2^1(0) = z \qquad Q^z - a.e.,$$

therefore, $\Pi_1^1 = \Pi_2^1, \Pi_1^2 = \Pi_2^2$ $Q^z - a.e.$.

For this we need to consider the set $A_i \in \tilde{\mathcal{F}}^z$ defined by

$$\begin{aligned}
A_i :=& \{\langle l, \Pi_i^1(t) - \Pi_0\rangle = \langle l, J^{-1}\Pi_3(t) - \frac{1}{2}\int_0^t \mathbf{n}_\Gamma(\Pi_i^1(s))d\Pi_i^2(t)\rangle - \langle Al, \int_0^t \Pi_i^1(t)ds\rangle, l \in D(A)\} \\
&\cap \{\Pi_i^2(t) \text{ is an increasing process }, \int_0^t I_{\partial\Gamma}(\Pi_i^1(s))d\Pi_i^2(s) = \Pi_i^2(t), t \ge 0\}
\end{aligned}$$

Define $A \in \mathcal{B}(H) \otimes \mathcal{B}(\mathbb{B}_H) \otimes \mathcal{B}(\mathbb{B}_{\mathbb{R}}) \otimes \mathcal{B}(\mathbb{W}_0)$ analogously with Π_i^1, Π_i^2 replaced by the canonical projection from $H \times \mathbb{B}_H \times \mathbb{B}_{\mathbb{R}} \times \mathbb{W}_0$ onto the second and the third and Π_0, Π_3 by the canonical projection onto the first and the forth coordinate respectively. Then by step 1 (ii), for $i = 1, 2$

$$\begin{aligned}&\int_H \int_{\mathbb{W}_0} \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} 1_{A_i}(x,\omega_1^1,\omega_1^2,\omega_2^1,\omega_2^2,\omega)\\ &K_\nu((x,\omega),d\omega_1^1,d\omega_1^2)K_\nu((x,\omega),d\omega_2^1,d\omega_2^2)P^J(d\omega)\nu(dx)\\ =&\tilde{P}^\nu(A) = P^\nu(\{(X(0),X,L^{\|\partial\Gamma\|},\bar{W})\in A\}) = 1\end{aligned}$$

Then we have for μ-a.e. $z \in H \cap M$

$$Q^z(A_i) = Q^z(A_{i,z}) = 1$$

where for $i = 1, 2$

$$\begin{aligned}A_{i,z} :=&\{\langle l, \Pi_i^1(t) - z\rangle = \langle l, J^{-1}\Pi_3(t) - \frac{1}{2}\int_0^t \mathbf{n}_\Gamma(\Pi_i^1(s))d\Pi_i^2(t)\rangle - \langle Al, \int_0^t \Pi_i^1(t)ds\rangle, l \in D(A)\}\\ &\cap \{\Pi_i^2(t) \text{ is an increasing process }, \int_0^t I_{\partial\Gamma}(\Pi_i^1(s))d\Pi_i^2(s) = \Pi_i^2(t), t\geq 0\}\end{aligned}$$

and have the results for [step 3].

[Step 4] There exists a $\overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0)}^{\nu\otimes P^J}/\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})$-measurable map

$$F_\nu : H\times\mathbb{W}_0 \to \mathcal{B}(H)\times\mathbb{B}_\mathbb{R}$$

such that

$$K_\nu((z,\omega),\cdot) = \delta_{F_\nu(z,\omega)}$$

for $\nu\otimes P^J$-a.e. $(x,\omega)\in H\times\mathbb{W}_0$. Furthermore, F_ν is $\overline{\mathcal{B}(H)\otimes\mathcal{B}_t(\mathbb{W}_0)}^{\nu\otimes P^J}/\mathcal{B}_t(\mathbb{B}_H)\otimes \mathcal{B}_t(\mathbb{B}_\mathbb{R})$-measurable. Then we have

$$(X, L^{\|\partial\Gamma\|}) = F_\nu(z,\bar{W}) \qquad P^z - a.e.$$

As for all $z\in N_1^c$,

$$\begin{aligned}1 =& Q^z(\{\Pi_1^1 = \Pi_2^1, \Pi_2^1 = \Pi_2^2\})\\ =&\int_{\mathbb{W}_0}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} 1_D(\omega_1^1,\omega_1^2,\omega_2^1,\omega_2^2)K_\nu((x,\omega),d\omega_1^1,d\omega_1^2)K_\nu((x,\omega),d\omega_2^1,d\omega_2^2)P^J(d\omega)\end{aligned}$$

where $D := \{(\omega_1^1,\omega_1^2,\omega_1^1,\omega_1^2)\in\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\times\mathbb{B}_H\times\mathbb{B}_\mathbb{R} | (\omega_1^1,\omega_1^2)\in\mathbb{B}_H\times\mathbb{B}_\mathbb{R}\}$. Hence by [Ro10] Lemma 2.2, there exists $N\in\mathcal{B}(H)\otimes\mathcal{B}(W_0)$ such that $\nu\otimes P^J(N) = 0$ and for all $(x,\omega)\in N^c$ there exists $F_\nu(z,\omega)\in\mathbb{B}_H\times\mathbb{B}_\mathbb{R}$ such that

$$K_\nu((z,\omega),d\omega_1^1,d\omega_1^2) = \delta_{F_\nu(z,\omega)}(d\omega_1^1,d\omega_1^2).$$

Set $F_\nu(x,\omega) := 0$, if $(x,\omega) \in N$. Then F_ν is $\overline{\mathcal{B}(H)\otimes\mathcal{B}_t(\mathbb{W}_0)}^{\delta_z\otimes P^J}/\mathcal{B}_t(\mathbb{B}_H)\otimes\mathcal{B}_t(\mathbb{B}_\mathbb{R})$-measurable.

As

$$
\begin{aligned}
&P^\nu(\{(X, L^{\|\partial\Gamma\|}) = F_\nu(X(0),\bar{W})\})\\
=&\int_H\int_{\mathbb{W}_0}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} 1_{\{(\omega_1^1,\omega_1^2)=F_z(x,\omega)\}}(x,\omega_1^1,\omega_1^2,\omega)\delta_{F_\nu(x,\omega)}(d\omega_1^1,d\omega_1^2)P^J(d\omega)\nu(dx)\\
=&1
\end{aligned}
$$

We have $(X, L^{\|\partial\Gamma\|}) = F_\nu(X(0),\bar{W})$ P^ν-a.e..

[step 5] Let W' be another standard Wiener process on a stochastic basis $(\Omega',\mathcal{F}',P',(\mathcal{F}'_t))$ and $\xi:\Omega'\to H$ an $\mathcal{F}'_0/\mathcal{B}(H)$-measurable map and $\nu := P'\circ\xi^{-1}$. Set $(X',L') := F_\nu(z,\bar{W}')$ then (X',L',W') is a (weak) solution of (2.4.2) with $X'(0)=\xi$ P'-a.s.

By the measurability properties of F_ν it follows that X' is adapted. We have

$$
\begin{aligned}
P'(\{X'(0)=\xi\}) =& P'(\{(\xi,0)=F_\nu(z,\bar{W}')(0)\})\\
=&\nu\otimes P^J(\{(x,\omega)\in H\times\mathbb{W}_0 | (x,0)=F_\nu(x,\omega)(0)\})\\
=&P^\nu(\{(X(0),0)=F_\nu(X(0),\bar{W})(0)\})=1
\end{aligned}
$$

To see that (X',L') is a solution of (2.4.2), we consider the set $A\in\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{W}_0)$ defined in the step 3. We have to show that

$$
P'(\{(X'(0),X',L',\bar{W}')\in A\})=1
$$

We have

$$
\begin{aligned}
&\int 1_A(X'(0),F_\nu(X'(0),\bar{W}'),\bar{W}')dP'\\
=&\int_H\int_{\mathbb{W}_0} 1_A(x,F_\nu(x,\omega),\omega)P^J(d\omega)\nu(dx)\\
=&\int_H\int_{\mathbb{W}_0}\int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} 1_A(x,\omega_1^1,\omega_1^2,\omega)\delta_{F_\nu(x,\omega)}(d\omega_1^1,d\omega_1^2)P^J(d\omega)\nu(dx)\\
=&\int 1_A(x,\omega_1^1,\omega_1^2,\omega)\tilde{P}^\nu(dx,d\omega_1^1,d\omega_1^2,d\omega)\\
=&P^\nu(\{(X(0),X,L^{\|\partial\Gamma\|},\bar{W})\in A\})=1
\end{aligned}
$$

Then we have the results.

[Step 6] Define $F(x,\omega) := F_{\delta_x}(x,\omega)$ for $x\in M,\omega\in\mathbb{W}_0$ and $F(x,\omega):=0$ for $x\in H\backslash M,\omega\in\mathbb{W}_0$. Then ν be a probability measure on $(H,\mathcal{B}(H))$ with $\nu(M)=1$

and F_ν as constructed in [Step 4]. Then for ν-a.e. $x \in H$

$$F(x,\cdot) = F_\nu(x,\cdot) \qquad P^J - a.e..$$

Furthermore, $F(x,\cdot)$ is $\overline{\mathcal{B}_t(\mathbb{W}_0)}^{P^J}/\mathcal{B}_t(\mathbb{B}_H)\otimes\mathcal{B}_t(\mathbb{B}_\mathbb{R})$-measurable for all $x \in H, t \in [0,\infty)$, where $\overline{\mathcal{B}_t(\mathbb{W}_0)}^{P^J}$ denotes the completion of $\mathcal{B}_t(\mathbb{W}_0)$ with respect to P^W in $\mathcal{B}(\mathbb{W}_0)$.

For fixed $z \in M$,

$$\bar{\Omega} := H \times \mathbb{B}_H \times \mathbb{B}_\mathbb{R} \times \mathbb{W}_0$$
$$\bar{\mathcal{F}}^z := \overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{W}_0)}^{Q_z}$$

Define a measure $\bar{Q}^z$ on $(\bar{\Omega},\bar{\mathcal{F}})$ by

$$\bar{Q}^z(A) := \int_H \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} \int_{\mathbb{W}_0} 1_A(x,\omega^1,\omega^2,\omega) K_\nu((x,\omega),d\omega^1,d\omega^2)P^J(d\omega)\delta_z(dx)$$

$$\bar{\mathcal{F}}^z_t := \cap_{\varepsilon>0}\sigma(\mathcal{B}(H)\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_H)\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}_{t+\varepsilon}(\mathbb{W}_0),\bar{\mathcal{N}}_z),$$

where $\bar{\mathcal{N}}_z := \{N, \bar{Q}^z(N)=0\}$ and define maps

$$\Pi_0 : \tilde{\Omega} \to H, (x,\omega^1,\omega^2,\omega) \mapsto x,$$

$$\Pi^j : \tilde{\Omega} \to \mathbb{B}_H \text{ or } \mathbb{B}_\mathbb{R}, (x,\omega^1,\omega^2,\omega) \mapsto \omega^j, j=1,2,$$
$$\Pi_3 : \tilde{\Omega} \to \mathbb{W}_0, (x,\omega^1,\omega^2,\omega) \mapsto \omega \in \mathbb{W}_0.$$

As in [Step 3] one shows that (Π^1,Π^2,Π^3) on $(\bar{\Omega},\bar{\mathcal{F}}^z,\bar{Q}^z,\bar{\mathcal{F}}^z_t)$ is a (weak) solution to (2.4.2) with $\Pi^1(0)=z$ $\bar{Q}^z$-a.e.. And by [Step 5], $(F_{\delta_z}(z,\Pi_3),\Pi_3)$ on the stochastic basis $(\bar{\Omega},\bar{\mathcal{F}}^z,\bar{Q}^z,\bar{\mathcal{F}}^z_t)$ is a (weak) solution to (2.4.2) with $F_{\delta_z}(z,\Pi_3)(0)=z$. Hence, it follows that $F_{\delta_z}(z,\Pi_3) = (\Pi^1,\Pi^2)$ $\bar{Q}^z-$a.s..

For $A \in \mathcal{B}(H)\otimes\mathcal{B}(\mathbb{B}_H)\otimes\mathcal{B}(\mathbb{B}_\mathbb{R})\otimes\mathcal{B}(\mathbb{W}_0)$

$$\begin{aligned}
&\int_M \int_{\mathbb{W}_0} \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} 1_A(x,\omega^1,\omega^2,\omega)\delta_{F_\nu(x,\omega)}(d\omega^1,d\omega^2)P^J(d\omega)\nu(dx)\\
&= \int_M \bar{Q}_x(A)\nu(dx)\\
&= \int_M \int_{\bar{\Omega}} 1_A(\Pi_0, F_{\delta_x}(x,\Pi_3),\Pi_3)d\bar{Q}^x\nu(dx)\\
&= \int_M \int_{\mathbb{B}_H\times\mathbb{B}_\mathbb{R}} \int_{\mathbb{W}_0} 1_A(x,\omega^1,\omega^2,\omega)\delta_{F_{\delta_x}(x,\omega)}(d\omega^1,d\omega^2)P^J(d\omega)\nu(dx),
\end{aligned}$$

which implies the assertion. The last results follow by the same arguments as in [PR07, Lemma E.1.16]. $\square$

Chapter 3

BV functions for differentiable measure

The main motivation of this chapter is to give a definition of BV functions which can take BV functions in a Gelfand triple and BV functions in abstract Wiener space as examples. So we introduce a definition of BV functions for differentiable measure in a Gelfand triple by using Dirichlet form theory. We also give examples of BV functions which cannot be BV functions in a Gelfand triple or BV functions in abstract Wiener space. As an application, we consider the reflected stochastic quantization problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Hilbert space H. We prove the existence of a martingale solution of this problem when Γ is a regular convex set.

3.1 The Dirichlet form and the associated distorted process

In this section, we consider a special kind of Dirichlet form and its associated distorted process. Let E be a Banach space, and H be a real separable Hilbert space (with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $|\cdot|$) continuously and densely embedded in E. We denote its Borel σ-algebra by $\mathcal{B}(H)$. Here identifying H with its dual we obtain the continuous and dense embeddings

$$E^* \subset H(\equiv H^*) \subset E.$$

It follows that

$${}_{E^*}\langle z, v \rangle_E = \langle z, v \rangle_H \forall z \in E^*, v \in H.$$

Let $L^p(E,\mu), p\in[1,\infty]$, denote the corresponding real L^p-spaces equipped with the usual norms $\|\cdot\|_p$ and let $L^p_+(E,\mu)$ denote the set of all non-negative elements in $L^p(E,\mu)$. Assume that:

Hypothesis 3.1.1 $Q: H\to H$ is a strictly positive linear bounded operator and there exists an orthonormal basis $\{e_j\}$ in H consisting of eigen-functions for Q with corresponding eigenvalues $\lambda_j\in\mathbb{R}_+, j\in\mathbb{N}$, that is,

$$Qe_j=\lambda_j e_j, j\in\mathbb{N}.$$

Also, $\{e_j\}\subset E^*$.

Let

$$\mathcal{F}C_b^1=\{u: u(z)=f({}_{E^*}\langle l_1,z\rangle_E, {}_{E^*}\langle l_2,z\rangle_E,...,{}_{E^*}\langle l_m,z\rangle_E), z\in E, l_1,l_2,...,l_m\in E^*, f\in C_b^1(\mathbb{R}^m)\}.$$

Define for $u\in\mathcal{F}C_b^1$ and $l\in H$,

$$\frac{\partial u}{\partial l}(z):=\frac{d}{ds}u(z+sl)|_{s=0}, z\in E,$$

that is,

$$\frac{\partial u}{\partial l}=\sum_{j=1}^m \partial_j f({}_{E^*}\langle l_1,z\rangle_E, {}_{E^*}\langle l_2,z\rangle_E,...,{}_{E^*}\langle l_m,z\rangle_E)\langle l_j,l\rangle.$$

Denote by Du the H-derivative of $u\in\mathcal{F}C_b^1$, namely, it is a map from E to H such that

$$\langle Du,l\rangle=\frac{\partial u}{\partial l}.$$

Let μ be a finite positive Radon measure on E has the following property: if a function $\varphi\in\mathcal{F}C_b^1$ is equal to zero μ-almost everywhere, then $\frac{\partial\varphi}{\partial l}=0$ μ-almost everywhere. In particular, this holds for a measure with a complete support. Now we introduce the following definition from [Bo10].

Definition 3.1.2 A measure μ on E is called differentiable along a vector v in the sense of Fomin if there exists a signed measure $d_v\mu$ of bounded variation such that for any $\varphi\in\mathcal{F}C_b^1$ the following equality holds:

$$\int\frac{\partial\varphi(x)}{\partial v}\mu(dx)=-\int\varphi(x)d_v\mu(dx),$$

and $d_v\mu$ is absolutely continuous with respect to μ.

If v is a fixed vector, then the density of $d_v\mu$ with respect to μ will be denoted by β_v. We denote by $H(\mu)$ the space

$$\{h \in E : \mu \text{ is differentiable along } h \text{ and } \|\beta_h\|_2 < \infty\},$$

endowed with the norm $\|h\|_{H(\mu)} = \|\beta_h\|_2$. The space $H(\mu)$ is a Hilbert space continuous embedded in E.

Let μ be a differentiable measure on E, and in the following we assume $Q^{1/2}(H) \subset H(\mu)$.

For $\rho \in L^1_+(E, \mu)$ we consider

$$\mathcal{E}^\rho(u, v) = \frac{1}{2} \sum_{k=1}^{\infty} \int_E \frac{\partial u}{\partial e_k} \frac{\partial v}{\partial e_k} \rho d\mu, u, v \in \mathcal{F}C_b^1,$$

where $F := Supp[\rho \cdot \mu]$. Let $QR(E)$ be the set of all functions $\rho \in L^1_+(E, \mu)$ such that $(\mathcal{E}^\rho, \mathcal{F}C_b^1)$ is closable on $L^2(F, \rho\cdot \mu)$. Its closure is denoted by $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. We denote by $\mathcal{F}_e^\rho$ the extended Dirichlet space of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$, that is, $u \in \mathcal{F}_e^\rho$ if and only if $|u| < \infty \ \rho \cdot \mu - a.e.$ and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\rho$ such that $\mathcal{E}^\rho(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u \ \ \rho \cdot \mu - a.e.$ as $n \to \infty$.

Theorem 3.1.3 Let $\rho \in QR(E)$. Then $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(F; \rho \cdot \mu)$ in the sense of Definition 1.3.

Proof The assertion follows from the main result in [RS92]. □

By virtue of Theorem 3.1.3 and Theorem 1.4, there exists a diffusion process $M^\rho = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, \ P_z)$ on F associated with the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. M^ρ will be called distorted process on F. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1, 1) = 0$, M^ρ is recurrent and conservative. We denote by $\mathbf{A}_+^\rho$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^ρ, and define $\mathbf{A}^\rho := \mathbf{A}_+^\rho - \mathbf{A}_+^\rho$. For $A \in \mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}_0^\rho := \{A \in \mathbf{A}^\rho | E_{\rho\cdot\mu}(\{A\}_t) < \infty \forall t > 0\}$. Each element in $\mathbf{A}_+^\rho$ has a corresponding positive $\mathcal{E}^\rho$-smooth measure on F by the Revuz correspondence. The set of all such measures will be denoted by S_+^ρ. Accordingly, $A_t \in \mathbf{A}^\rho$ corresponds to a $\nu \in S^\rho := S_+^\rho - S_+^\rho$, the set of all $\mathcal{E}^\rho$-smooth signed measure in the sense that $A_t = A_t^1 - A_t^2$ for $A_t^k \in \mathbf{A}_+^\rho, k = 1, 2$ whose Revuz measures are $\nu^k, k = 1, 2$ and $\nu = \nu^1 - \nu^2$ is the Hahn-Jordan decomposition of ν . The element of $\mathbf{A}^\rho$ corresponding to $\nu \in S^\rho$ will be denoted byA^ν.

Note that for each $l \in E^*$ the function $u(z) = {}_{E^*}\langle l, z\rangle_E$ belongs to the extended

Dirichlet space $\mathcal{F}^\rho_e$ and

$$\mathcal{E}^\rho(l(\cdot), v) = \frac{1}{2}\int \frac{\partial v(z)}{\partial l}\rho(z)d\mu(z)\ \forall v \in \mathcal{F}C^1_b. \tag{3.1.1}$$

On the other hand, the AF ${}_{E^*}\langle l, X_t - X_0\rangle_E$ of M^ρ admits a unique decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l \in E^*$,

$$_{E^*}\langle l, X_t - X_0\rangle_E = M^l_t + N^l_t\ \forall t \geq 0\ P_z - a.s. \tag{3.1.2}$$

for $\mathcal{E}^\rho$-q.e. $z \in F$.

Now for $\rho \in L^1(E, \mu)$ and $l \in E^*$, we say that $\rho \in BV_l(E)$ if there exists a constant $C_l > 0$,

$$|\int_E \frac{\partial v(z)}{\partial l}\rho(z)d\mu(z)| \leq C_l \parallel v \parallel_\infty\ \ \forall v \in \mathcal{F}C^1_b. \tag{3.1.3}$$

By the same argument as in [FH01, Theorem 2.1], we obtain the following:

Theorem 3.1.4 Let $\rho \in L^1_+$ and $l \in E^*$.

(1) The following two conditions are equivalent:

(i)$\rho \in BV_l(E)$

(ii) There exists a (unique) signed measure ν_l on F of finite total variation such that

$$\frac{1}{2}\int \frac{\partial v(z)}{\partial l}\rho(z)d\mu(z) = -\int_F v(z)\nu_l(dz)\ \forall v \in \mathcal{F}C^1_b. \tag{3.1.4}$$

In this case, ν_l necessarily belongs to $S^{\rho+1}$.

Suppose further that $\rho \in QR(E)$. Then the following condition is also equivalent to the above:

(iii)$N^l \in \mathbf{A}^\rho_0$

In this case, $\nu_l \in S^\rho$, and $N^l = A^{\nu_l}$

(2) M^l is a martingale AF with quadratic variation process

$$\langle M^l\rangle_t = t|l|^2, t \geq 0. \tag{3.1.5}$$

3.2 BV functions and distorted processes in F

We introduce BV functions in this section, by which we can get the Skorohod type representation for the process.

Let $c_j, j \in \mathbb{N}$, be a sequence in $[1, \infty)$. Define

$$H_1 := \{x \in H | \sum_{j=1}^{\infty} \langle x, e_j \rangle^2 c_j^2 < \infty\},$$

equipped with the inner product

$$\langle x, y \rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j \rangle \langle y, e_j \rangle.$$

Then clearly $(H_1, \langle, \rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H(\equiv H^*) \subset H_1^*.$$

It follows that

$${}_{H_1}\langle z, v \rangle_{H_1^*} = \langle z, v \rangle_H \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple. Furthermore, $\{\frac{e_j}{c_j}\}$ and $\{c_j e_j\}$ are orthonormal bases of H_1 and H_1^*, respectively.

We also introduce a family of H-valued functions on E by

$$(\mathcal{F}C_b^1)_{Q^{1/2}(H) \cap H_1} := \{G : G(z) = \sum_{j=1}^{m} g_j(z) l^j, z \in E, g_j \in \mathcal{F}C_b^1, l^j \in Q^{1/2}(H) \cap H_1\}.$$

Denote by D^* the adjoint of $D : \mathcal{F}C_b^1 \subset L^2(E, \mu) \to L^2(E, \mu; H)$. That is

$$Dom(D^*) := \{G \in L^2(E, \mu; H) | C_b^1 \ni u \mapsto \int_E \langle G, Du \rangle d\mu \text{ is continuous with respect to } L^2(E, \mu)\}.$$

Obviously, $(\mathcal{F}C_b^1)_{Q^{1/2}(H) \cap H_1} \subset Dom(D^*)$. Then

$$\int_E D^* G(z) f(z) \mu(dz) = \int_E \langle G(z), Df(z) \rangle \mu(dz) \ \forall G \in (\mathcal{F}C_b^1)_{Q^{1/2}(H) \cap H_1}, f \in \mathcal{F}C_b^1. \tag{3.2.1}$$

For $\rho \in L^2(E,\mu)$, we set

$$V(\rho) := \sup_{G\in(\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1},\|G\|_{H_1}\le 1} \int_E D^*G(z)\rho(z)\mu(dz). \tag{3.2.2}$$

A function ρ on E is called an BV function in the Gelfand triple $(H_1, H, H_1^*)(\rho \in BV(H, H_1)$ in notation), if $\rho \in L^2(E,\mu)$ and $V(\rho)$ is finite. We can prove the following theorem by a modification of the proof of Theorem 2.2.1 in chapter 2.

Theorem 3.2.1 (i) $BV(H,H_1) \subset \bigcap_{l\in Q^{1/2}(H)\cap H_1\cap E^*} BV_l(E)$.

(ii) Suppose $\rho \in BV(H,H_1)\cap L_+^2(E,\mu)$, then there exist a positive finite measure $\|d\rho\|$ on E and a Borel-measurable map $\sigma_\rho : E \to H_1^*$ such that $\|\sigma_\rho(z)\|_{H_1^*} = 1\ \|d\rho\| - a.e, \|d\rho\|(E) = V(\rho)$,

$$\int_E D^*G(z)\rho(z)\mu(dz) = \int_E {}_{H_1}\langle G(z), \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz)\ \forall G \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1} \tag{3.2.3}$$

and $\|d\rho\| \in S^{\rho+1}$.

Furthermore, if $\rho \in QR(E)$, $\|d\rho\|$ is $\mathcal{E}^\rho$-smooth in the sense that it charges no set of zero $\mathcal{E}_1^\rho$-capacity. In particular, the domain of integration E on both sides of (3.2.3) can be replaced by F, the topological support of $\rho\mu$.

Also, σ_ρ and $\|d\rho\|$ are uniquely determined, that is, if there are σ'_ρ and $\|d\rho\|'$ satisfying relation (3.2.3), then $\|d\rho\| = \|d\rho\|'$ and $\sigma_\rho(z) = \sigma'_\rho(z)$ for $\|d\rho\| - a.e.z$

(iii) Conversely, if Eq.(3.2.3) holds for $\rho \in L^2(E,\mu)$ and for some positive finite measure $\|d\rho\|$ and a map σ_ρ with the stated properties, then $\rho \in BV(H,H_1)$ and $V(\rho) = \|d\rho\|(E)$.

(iv) Let $W^{1,2}(E)$ be the domain of the closure of $(D, \mathcal{F}C_b^1)$ with norm

$$\|f\|^2 := \int_E (|f(z)|^2 + |Df(z)|^2)\mu(dz).$$

Then $W^{1,2}(E) \subset BV(H,H)$ and Eq.(3.2.3) is satisfied for each $\rho \in W^{1,2}(E)$. Furthermore,

$$\|d\rho\| = |D\rho|\cdot\mu, V(\rho) = \int_E |D\rho|\mu(dz), \sigma_\rho = \frac{1}{|D\rho|}D\rho I_{\{|D\rho|>0\}}.$$

Proof (i) Let $\rho \in BV(H,H_1)$. Take $G \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1}$ of the type

$$G(z) = g(z)l, z \in E, g \in \mathcal{F}C_b^1, l \in Q^{1/2}(H)\cap H_1. \tag{3.2.4}$$

By (3.2.1)

$$\int_E D^*G(z)f(z)\mu(dz) = \int_E \langle G(z), Df(z)\rangle \mu(dz)$$
$$= -\int_E \langle l, Dg(z)\rangle f(z)\mu(dz) - \int_E \beta_l(z)g(z)f(z)\mu(dz)\ \forall f \in \mathcal{F}C_b^1;$$

consequently,

$$D^*G(z) = -\langle l, Dg(z)\rangle - g(z)\beta_l(z). \tag{3.2.5}$$

Accordingly,

$$\int_E \langle l, Dg(z)\rangle \rho(z)\mu(dz) = -\int_E D^*G(z)\rho(z)\mu(dz) - \int_E \beta_l(z)g(z)\rho(z)\mu(dz). \tag{3.2.6}$$

For any $g \in \mathcal{F}C_b^1$, satisfying $\|g\|_\infty \leq 1$, by (3.2.2) the right hand side is dominated by

$$V(\rho)\|l\|_{H_1} + \|\rho\|_2\|\beta_l\|_2 < \infty,$$

hence, $\rho \in BV_l(H)$.

(ii) Suppose $\rho \in L^1_+(E,\mu)\bigcap BV(H,H_1)$. By (i) and Theorem 3.1.4 for each $l \in Q^{1/2}(H) \cap H_1 \cap E^*$, there exists a finite signed measure ν_l on E for which Eq.(3.1.4) holds. Define

$$D_l^A\rho(dz) := 2\nu_l(dz) - \beta_l(z)\rho(z)\mu(dz).$$

In view of (3.2.6), for any G of type (3.2.4), we have

$$\int_E D^*G(z)\rho(z)\mu(dz) = \int_E g(z)D_l^A\rho(dz), \tag{3.2.7}$$

which in turn implies

$$V(D_l^A\rho)(E) = \sup_{g\in\mathcal{F}C_b^1, \|g\|_\infty\leq 1} \int_E g(z)D_l^A\rho(dz) \leq V(\rho)\|l\|_{H_1}, \tag{3.2.8}$$

where $V(D_l^A\rho)$ denotes the total variation measure of the signed measure $D_l^A\rho$.

For the orthonormal basis $\{\frac{e_j}{c_j}\}$ of H_1, we set

$$\gamma_\rho^A := \Sigma_{j=1}^\infty 2^{-j} V(D^A_{\frac{e_j}{c_j}}\rho),\ v_j(z) := \frac{dD^A_{\frac{e_j}{c_j}}\rho(z)}{d\gamma_\rho^A(z)}, z \in E, j \in \mathbb{N}. \tag{3.2.9}$$

γ_ρ^A is a positive finite measure with $\gamma_\rho^A(E) \leq V(\rho)$ and v_j is Borel-measurable. Since

$D^A_{\frac{e_j}{c_j}}\rho$ belongs to $S^{\rho+1}$, so does γ^A_ρ . Then for

$$G_n := \sum_{j=1}^n g_j \frac{e_j}{c_j} \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)\cap H_1}, n \in \mathbb{N}, \tag{3.2.10}$$

by (3.2.7) the following equation holds

$$\int_E D^* G_n(z)\rho(z)\mu(dz) = \sum_{j=1}^n \int_E g_j(z) v_j(z) \gamma^A_\rho(dz). \tag{3.2.11}$$

Since $|v_j(z)| \le 2^j$ γ^A_ρ-a.e. and $\mathcal{F}C_b^1$ is dense in $L^1(E, \gamma^A_\rho)$, we can find $v_{j,m} \in \mathcal{F}C_b^1$ such that

$$\lim_{m\to\infty} v_{j,m} = v_j \ \gamma^A_\rho - a.e..$$

Substituting

$$g_{j,m}(z) := \frac{v_{j,m}(z)}{\sqrt{\sum_{k=1}^n v_{k,m}(z)^2 + 1/m}}, \tag{3.2.12}$$

for $g_j(z)$ in (3.2.10) and (3.2.11) we get a bound

$$\sum_{j=1}^n \int_E g_{j,m}(z) v_j(z) \gamma^A_\rho(dz) \le V(\rho),$$

because $\|G_n(z)\|^2_{H_1} = \sum_{j=1}^n g_{j,m}(z)^2 \le 1 \ \forall z \in E$. By letting $m \to \infty$, we obtain

$$\int_E \sqrt{\sum_{j=1}^n v_j(z)^2} \gamma^A_\rho(dz) \le V(\rho) \ \forall n \in \mathbb{N}.$$

Now we define

$$\|d\rho\| := \sqrt{\sum_{j=1}^\infty v_j(z)^2} \gamma^A_\rho(dz) \tag{3.2.13}$$

and $\sigma_\rho : E \to H_1^*$ by

$$\sigma_\rho(z) = \begin{cases} \sum_{j=1}^\infty \frac{v_j(z)}{\sqrt{\sum_{k=1}^\infty v_k(z)^2}} \cdot c_j e_j, & \text{if } z \in \{\sum_{k=1}^\infty v_k(z)^2 > 0\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.14}$$

Then

$$\|d\rho\|(E) \le V(\rho), \ \|\sigma_\rho(z)\|_{H_1^*} = 1 \ \|d\rho\| - a.e., \tag{3.2.15}$$

$\|d\rho\|$ is $S^{\rho+1}$-smooth and σ_ρ is Borel-measurable. By (3.2.11) we see that the desired equation (3.2.3) holds for $G = G_n$ as in (3.2.10). It remains to prove (3.2.3) for any

G of type (3.2.4), i.e. $G = g \cdot l, g \in \mathcal{F}C_b^1, l \in Q^{1/2}(H) \cap H_1$. In view of (3.2.6), Eq.(3.2.3) then reads

$$-\int_E \langle l, Dg(z)\rangle\rho(z)\mu(dz) - \int_E g(z)\beta_l(z)\rho(z)\mu(dz) = \int_E g(z)_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz). \tag{3.2.16}$$

We set

$$k_n := \sum_{j=1}^n \langle l, e_j\rangle e_j = \sum_{j=1}^n \langle l, \frac{e_j}{c_j}\rangle_{H_1}\frac{e_j}{c_j} = \sum_{j=1}^n \langle l, \lambda_j^{1/2}e_j\rangle_{Q^{1/2}(H)}\lambda_j^{1/2}e_j, G_n(z) := g(z)k_n.$$

Thus $k_n \to l$ in H_1 and $k_n \to l$ in $Q^{1/2}(H)$ as $n \to \infty$. So $\|\beta_{k_n} - \beta_l\|_2 \to 0$. But then also

$$\lim_{n\to\infty}\int_E \langle Dg, k_n\rangle\rho d\mu = \int_E \langle Dg, l\rangle\rho d\mu,$$

and

$$\begin{aligned}|\int_E g(z)\beta_{k_n}(z)\rho(z)\mu(dz) - \int_E g(z)\beta_l(z)\rho(z)\mu(dz)|\\ \leq \|g\|_\infty\|\rho\|_2\|\beta_{k_n} - \beta_l\|_2.\end{aligned}$$

Furthermore,

$$\lim_{n\to\infty}\int_E g(z)_{H_1}\langle k_n, \sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz) = \int_E g(z)_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz).$$

So letting $n \to \infty$ yields (3.2.16).

If $\rho \in QR(E)$, we can get the claimed result by the same arguments as above.

Uniqueness follows by the same method as Theorem 2.2.1.

(iii) Suppose $\rho \in L^2(E, \mu)$ and that Eq.(3.2.3) holds for some positive finite measure $\|d\rho\|$ and some map σ_ρ with the properties stated in (ii). Then clearly

$$V(\rho) \leq \|d\rho\|(E)$$

and hence $\rho \in BV(H, H_1)$. To obtain the converse inequality, set

$$\sigma_j(z) := \langle c_j e_j, \sigma_\rho(z)\rangle_{H_1^*} =_{H_1} \langle \frac{e_j}{c_j}, \sigma_\rho(z)\rangle_{H_1^*}, j \in \mathbb{N}.$$

Fix an arbitrary n. As in the proof of (ii) we can find functions

$$v_{j,m} \in \mathcal{F}C_b^1, \qquad \lim_{m\to\infty} v_{j,m}(z) = \sigma_j(z)\ \|d\rho\| - a.e.$$

Define $g_{j,m}(z)$ by (3.2.12). Substituting $G_{n,m}(z) := \sum_{j=1}^n g_{j,m}(z)\frac{e_j}{c_j}$ for $G(z)$ in (3.2.3) then yields

$$\sum_{j=1}^{n}\int_E g_{j,m}(z)\sigma_j(z)\|d\rho\|(dz)\le V(\rho).$$

By letting $m\to\infty$, we get

$$\int_E\sqrt{\sum_{j=1}^{n}\sigma_j(z)^2}\|d\rho\|(dz)\le V(\rho)\ \forall n\in\mathbb{N}.$$

We finally let $n\to\infty$ to obtain $\|d\rho\|(E)\le V(\rho)$.

(iv) Obviously the duality relation (3.2.1) extends to $\rho\in W^{1,2}(E)$ replacing $f\in\mathcal{F}C_b^1$. By defining $\|d\rho\|$ and $\sigma_\rho(z)$ in the stated way, the extended relation (3.2.1) is exactly (3.2.3). $\square$

Now we give the following examples of BV functions by using the result in [Pu98]. Now let f satisfies the conditions in [Pu98, Section 4], i.e. there exists a $Cap_{1,12}$-quasi-continuous function $f\in H^{2,12}(E)$ such that $|QDf|^{-1}\in L^{12}(E,\mu)$, $D^*(QDf)\in L^2(E,\mu)$, and

$$U=f^{-1}((-\infty,0)).$$

Here $H^{2,12}(E)$ is the completion of the space $\mathcal{F}C_b^\infty$ with respect to the norm

$$\|\varphi\|_{2,12}^{12}=\int(\varphi^2(x)+\sum_k\lambda_k(\frac{\partial\varphi}{\partial e_k})^2+\sum_{k,h}\lambda_k\lambda_h(\frac{\partial}{\partial e_k}\frac{\partial\varphi}{\partial e_h})^2)^6\mu(dx).$$

$Cap_{1,12}$ is defined by the following:

$$Cap_{1,12}(U)=\inf\{\|\varphi\|_{1,12}:\varphi\ge 0,\varphi\ge 1\mu\text{ almost everywhere on }U\}\quad\text{for open set}U,$$

$$Cap_{1,12}(A)=\inf\{Cap_{1,12}(U):U\text{ is open},U\supset A\}\quad\text{for arbitary set}A,$$

where

$$\|\varphi\|_{1,12}^{12}=\int(\varphi^2(x)+\sum_k\lambda_k(\frac{\partial\varphi}{\partial e_k})^2)^6d\mu.$$

The set $\Sigma=f^{-1}(0)$ will be called the surface of U, denoted by ∂U. By [Pu98, Section 3], we have the finite measure ν on Σ (see [Pu98, Section 3] for the construction of ν). Here we take $Q^{1/2}H$ as the H used in [Pu98]. Then by [Pu98 , Theorem 4.1], we have the following theorem.

Theorem 3.2.2 Assume there exists a $Cap_{1,12}$-quasi-continuous function $f\in H^{2,12}(E)$ such that $|QDf|^{-1}\in L^{12}(E,\mu)$, $D^*(QDf)\in L^2(E,\mu)$, then I_U is an BV function with $H_1^* = Q^{-1/2}H, H_1 = Q^{1/2}H$, and

$$\int_U D^*G(z))\mu(dz) = -\int_\Sigma {}_{H_1}\langle G(z), n_U(z)\rangle_{H_1^*} \|d\rho\|(dz)\ \forall G \in (\mathcal{F}C_b^1)_{Q^{1/2}(H)},$$

where $n_U(z) = Df(z)/\|Df(z)\|_{H_1^*}$ and $\|d\rho\|(dz) = \|Df(z)\|_{H_1^*}\nu(dz)$ is a finite measure on Σ. Moreover, if $\|Df(z)\|_{H_1^*}$ is finite on Σ for some $H_1^* \subset Q^{-1/2}H$, then $I_U \in BV(H, H_1)$.

Theorem 3.2.3 Let $\rho \in QR(E) \cap BV(H, H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 3.2.1(ii). Then for any smooth measure γ under P_γ there exists an $\mathcal{M}_t$- cylindrical Wiener process W, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in H_1 \cap E^* \cap Q^{1/2}(H)$

$${}_{E^*}\langle l, X_t - X_0\rangle_E = \int_0^t \langle l, dW_s^z\rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} + \frac{1}{2}\int_0^t \beta_l(X_s)ds\ \forall t \geq 0\ P_\gamma\text{−a.s.}. \tag{3.2.17}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

Proof Let $\{e_j\}$ be the orthonormal basis of H introduced above. Define for all $k \in \mathbb{N}$

$$W_k(t) := {}_{E^*}\langle e_k, X_t - z\rangle_E - \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} - \frac{1}{2}\int_0^t \beta_{e_k}(X_s)ds. \tag{3.2.18}$$

By (3.1.1) and (3.2.16) we get for all $k \in \mathbb{N}$

$$\mathcal{E}^\rho(e_k(\cdot), g) = -\frac{1}{2}\int_E g(z)\beta_{e_k}(z)\rho(z)\mu(dz) - \frac{1}{2}\int_E g(z)_{H_1}\langle e_k, \sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz)\ \forall g \in \mathcal{F}C_b^1.$$

By Theorem 3.1.4 it follows that for all $k \in \mathbb{N}$

$$N_t^{e_k} = \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} + \frac{1}{2}\int_0^t \beta_{e_k}(X_s)ds. \tag{3.2.19}$$

Here we get from (3.2.18), (3.2.19) and the uniqueness of decomposition (3.1.2) that,

$$W_k(t) = M_t^{e_k}\ \forall t \geq 0\ P_\gamma\text{−a.s.}.$$

By Dirichlet form theory we get $\langle M^{e_i}, M^{e_j}\rangle_t = t\delta_{ij}$. So W_k is an $\mathcal{M}_t$-Wiener process under P_γ. Thus, with W being an $\mathcal{M}_t$- cylindrical Wiener process given by $W(t) = (W_k(t)e_k)_{k\in\mathbb{N}}$, (3.2.17) is satisfied for $P_\gamma - a.e.$. □

3.3 Examples

Now we want to give examples which cannot be covered by the BV functions in abstract Wiener space and BV functions in a Gelfand triple and by using it, we can consider the reflected stochastic quantization equation.

3.3.1 Reflected stochastic quantization equations with finite volume

In this section we apply our BV functions theory to the stochastic quantization of $(\mathcal{P}(\phi)_2-)$ field theory in finite volume. The stochastic quantization problem was studied in [AR89], [AR91], [RZ92] and [LR98] by using the Dirichlet form theory and get the existence and uniqueness of the martingale problem. And Da Prato and Debussche in [DD03] proved the existence and uniqueness of a strong solution of this problem. We consider the reflected problem in this case. Let $H = L^2([0, 2\pi]^2)$, and denote the complete orthonormal system by $\{e_k = \frac{1}{2\pi} e^{i\langle k,\cdot\rangle}\}_{k\in\mathbb{Z}^2}$. Define for $\alpha \in \mathbb{R}^+$,

$$H^\alpha := \{u \in H : \sum_k |k|^{2\alpha} \langle u, e_k\rangle^2 < \infty\},$$

and for $\alpha \in \mathbb{R}^-$, define H^α be the dual of $H^{-\alpha}$. Set $E = H^{-s}, E^* = H^s$ for some $s > 0$. Also set $\mu_0 = N(0, (-\triangle + 1)^{-1}) := N(0, C)$, where $\triangle$ is Laplace operator on $[0, 2\pi]^2$ with Dirichlet boundary condition. Then μ_0 is a measure supported on E.

Let us introduce the renormalized power. Set $W_z(x) = \langle x, C^{-1/2} z\rangle$, for $z \in C^{1/2}(H)$. We have for any $N \in \mathbb{N}$,

$$x_N(\xi) := \sum_{|k|\le N} \langle x, e_k\rangle e_k(\xi) = \rho_N W_{\eta_N(\xi)}(x) \qquad \text{for } x\mu - a.e. \text{ in } H,$$

where

$$\rho_N = \frac{1}{2\pi}[\sum_{|k|\le N} \frac{1}{1+|k|^2}]^{1/2}$$

and

$$\eta_N(\xi) = \frac{1}{\rho_N} \sum_{|k|\le N} \frac{\overline{e_k(\xi)}}{\sqrt{1+|k|^2}} e_k.$$

Now for any $n \in \mathbb{N}$, we set

$$: x_N^n : (\xi) = \rho_N^n H_n(W_{\eta_N(\xi)}(x)) \qquad \text{for } x\mu - a.e. \text{ in } H.$$

Here H_n are the Hermite polynomials, i.e. $H_n(t) = \sum_{m=0}^{[n/2]} (-1)^m C_{nm} t^{n-2m}$ with

$C_{nm} = n!/[(n-2m)!2^m m!]$. By [GJ86, Section 8.5], there exists an element $: x^n : (h)$ such that the sequence $\langle : x_N^n :, h\rangle \to : x^n : (h)$ in $L^2(E, \mu_0), n \to \infty$ for $h \in H$. By [DD03, Lemma 3.2], $: x_N^n :$ is bounded in $L^2(E, \mu_0, H^r(G))$ for any $r < 0$. Thus $h \in H^{-r} \to : x^n : (h) \in L^2(E, \mu_0)$ is continuous. So by [AR91, Proposition 6.9], there exists a $\mathcal{B}(H^{-2+r})/\mathcal{B}(H^{-2+r})$ measurable map $: x^n : H^{-2+r} \to H^{-2+r}$ such that $: x^n : (h) =_{H^{-2+r}} \langle : x^n :, h\rangle_{H^{2-r}}$. Finally, we set $: P(x) := \sum_{n=0}^{2N} a_n : x^n :$. Now we assume that $a_n \in \mathbb{R}$ and $a_{2N} > 0$.

Let

$$\mu = \frac{\exp\left(-\int_{[0,2\pi]^2} : P(x) : dx\right)}{\int \exp\left(-\int_{[0,2\pi]^2} : P(x) : dx\right) d\mu_0}\mu_0.$$

Now set $Qe_k = \frac{1}{|k|^{4+2s}} e_k$, so $Q^{1/2}(H) = H^{2+s}$. Then by [GlJ86, (9.1.32)] we have the following:

Theorem 3.3.1 $Q^{1/2}(H) \subset H(\mu)$. Moreover for each $l \in Q^{1/2}H$, we have

$$\beta_l(x) =_{H^{-s-2}} \langle -\sum_{n=1}^{2N} n a_n : x^{n-1} :, l\rangle_{H^{2+s}} +_{H^s} \langle \Delta l - l, x\rangle_{H^{-s}}.$$

Now fix $k \in \mathbb{N}, a \in \mathbb{R}$ and take $U = \{x \in H^{-s} :_{H^{-s}} \langle x, e_k\rangle_{H^s} \leq a\}, \rho = I_U$. Then by Theorem 3.2.2, ρ is an BV function with $H = H_1 = H_1^*$. Since U is a convex closed set, then $\rho \in QR(E)$. Thus we can apply Theorem 3.2.3 directly and by a modification get the following:

Theorem 3.3.2 There is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in H^{2+s}$

$$\begin{aligned} {}_{E^*}\langle l, X_t - X_0\rangle_E = & \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t \langle l, n_U(X_s)\rangle dL_s^{\|d\rho\|} \\ & + \frac{1}{2}\int_0^t {}_{H^{-s-2}}\langle -\sum_{n=1}^{2N} n a_n : X_r^{n-1} :, l\rangle_{H^{2+s}} +_{H^s} \langle \Delta l - l, X_r\rangle_{H^{-s}} dr \ \forall t \geq 0 \ P_z\text{−a.s.}. \end{aligned}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$I_{\partial U}(X_s) dL_s^{\|d\rho\|} = dL_s^{\|d\rho\|} \ P - a.s.$$

and $n_U(z) = e_k$.

We can also take $U = \{x \in E : \|x\|_E \leq 1\}, \rho = I_U$. Then by Theorem 3.2.2, ρ is an BV function with $H_1 = H_1^* = H$. Since U is a convex closed set, then

$\rho \in QR(E)$. Thus as Theorem 3.3.2, we get the following:

Theorem 3.3.3 There is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in H^{2+s}$

$$\begin{aligned}{}_{E^*}\langle l, X_t - X_0\rangle_E = &\int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t \langle l, n_U(X_s)\rangle dL_s^{\|d\rho\|} \\ &+\frac{1}{2}\int_0^t {}_{H^{-2-s}}\langle -\sum_{n=1}^{2N} na_n : X_r^{n-1} :, l\rangle_{H^{2+s}} + {}_{H^s}\langle \Delta l - l, X_r\rangle_{H^{-s}} dr \ \forall t \geq 0 \ P_z\text{−a.s..}\end{aligned}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$I_{\partial U}(X_s)dL_s^{\|d\rho\|} = dL_s^{\|d\rho\|} \ P - a.s.$$

and $n_U(x) = \frac{(-\triangle)^{-s}x}{|(-\triangle)^{-s}x|}$.

Now we want to construct an example which is an BV functions in Gelfand triple with $H \neq H_1$. Set $z_n = \sum_{k=1}^n \frac{1}{|k|^s}e_k$. Then it is obvious that ${}_{E^*}\langle z_n, \cdot\rangle_E$ as a function on E converges to some function in $H^{2,12}(E,\mu)$. We will denote this function by $z(x)$. By [RS92, Lemma 2.4], $z(x)$ has a $Cap_{1,12}$-quasi-continuous versions. It is easy to check the conditions in Theorem 3.2.2 are satisfied. We take $U = \{x \in E : z(x) \leq a\}$ for some $a \in \mathbb{R}$ such that $\mu(U) > 0$, and $\rho = I_U$. Then by Theorem 3.2.2, ρ is an BV function with $H_1 = H^1, H_1^* = H^{-1}$. Since $z(x + se_k)$ is continuous in s, we have$\rho \in QR(E)$. Thus as Theorem 3.3.2, we get the following:

Theorem 3.3.4 There is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in H^{2+s}$

$$\begin{aligned}{}_{E^*}\langle l, X_t - X_0\rangle_E = &\int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t {}_{H^1}\langle l, n_U(X_s)\rangle_{H^{-1}} dL_s^{\|d\rho\|} \\ &+\frac{1}{2}\int_0^t {}_{H^{-2-s}}\langle -\sum_{n=1}^{2N} na_n : X_r^{n-1} :, l\rangle_{H^{2+s}} + {}_{H^s}\langle \Delta l - l, X_r\rangle_{H^{-s}} dr \ \forall t \geq 0 \ P_z\text{−a.s..}\end{aligned}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$I_{\partial U}(X_s)dL_s^{\|d\rho\|} = dL_s^{\|d\rho\|} \ P - a.s.$$

and $n_U(x) = \frac{\sum_k e_k/|k|^s}{\|\sum_k e_k/|k|^s\|_{H^{-1}}}$.

Remark 3.3.5 From above three theorems, we get martingale solutions to the reflected stochastic quantization equations. By the same argument as above, we can also obtain martingale solution to the stochastic reflected OU equations with space dimension 2.

3.3.2 Reflected stochastic quantization equations with infinite volume

In this section, we consider the reflected stochastic quantization equations with infinite volume. Let $\mathcal{S}'(\mathbb{R}^2)$ be the space of tempered Schwartz distributions on $\mathbb{R}^2$ and $\mathcal{S}(\mathbb{R}^2)$ the associated test function space equipped with the usual topology. Let μ_0 be the mean zero Gaussian measure on $(\mathcal{S}'(\mathbb{R}^2), \mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ with covariance

$$\int {}_{\mathcal{S}}\langle k_1, z\rangle_{\mathcal{S}'}\,{}_{\mathcal{S}}\langle k_2, z\rangle_{\mathcal{S}'}\mu_0(dz) = \int\int(-\triangle+1)^{-1}(x-y)k_1(x)k_2(y)dxdy =: \langle k_1, k_2\rangle_{H_2},$$

where $(-\triangle+1)^{-1})$ denotes the Green function of the operator $(-\triangle+1)$ on $\mathbb{R}^2$. Now for $n\in\mathbb{N}$, let $\mathcal{S}_{-n}$ denote the Hilbert subspace of $\mathcal{S}'(\mathbb{R}^2)$ which is the dual of $\mathcal{S}_n$ defined as the completion of $\mathcal{S}$ w.r.t the norm

$$\|k\|_n := [\sum_{|m|\leq n}\int_{\mathbb{R}^2}(1+|x|^2)^n|(\frac{\partial^{m_1}}{\partial x_1^{m_1}}, \frac{\partial^{m_2}}{\partial x_2^{m_2}})k(x)|^2dx]^{1/2}.$$

For $h\in H_2$ we define $X_h\in L^2(\mathcal{S}',\mu_0)$ by $X_h := \lim_{n\to\infty} {}_{\mathcal{S}}\langle k_n, \cdot\rangle_{\mathcal{S}'}$ in $L^2(\mathcal{S}',\mu_0)$ where k_n is any sequence in $\mathcal{S}$ such that $k_n\to h$ in H_2. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}',\mu_0) = \bigoplus_{n\geq 0}\mathcal{H}_n.$$

For $h\in L^2(\mathbb{R}^2, dx)$ and $n\in\mathbb{N}$, define $: z^n : (h)$ to be the unique element in $\mathcal{H}_n$ such that

$$\int : z^n : (h) : \prod_{j=1}^n X_{k_j} :_n d\nu = n!\int_{\mathbb{R}^2}\prod_{j=1}^n(\int_{\mathbb{R}^2}(-\triangle+1)^{-1}(x-y_j)k_j(y_j)dy_j)h(x)dx$$

where $k_1, ..., k_n\in\mathcal{S}(\mathbb{R}^2)$ and $::_n$ means orthogonal projection onto $\mathcal{H}_n$ (see [S74, V.1] for existence of $: z^n : (h)$).

From now on we fix $N\in\mathbb{N}, a_n\in\mathbb{R}, 0\leq n\leq 2N$, and define for $h\in L^2(\mathbb{R}^2, dx)$

$$: P(z) : (h) := \sum_{n=0}^{2N} a_n : z^n : (h) \text{ with } a_{2N} > 0.$$

We have that $\exp(-:P(z):(h))\in L^p(\mathcal{S}',\nu)$ for all $p\in[1,\infty)$ if $h\geq 0$ (cf. [AR91, Section 7]), hence the following probability measures (called space-time cut-off quantum fields) are well-defined for $\Lambda\in\mathcal{B}(\mathbb{R}^2),\Lambda$ bounded,

$$\mu_\Lambda:=\frac{\exp\left(-:P(z):(1_\Lambda)\right)}{\int\exp\left(-:P(z):(1_\Lambda)\right)d\mu_0}\mu_0.$$

It has been proven that the weak limit

$$\lim_{\Lambda\to\mathbb{R}^2}\mu_\Lambda=:\mu$$

exists as a probability measure on $(\mathcal{S}'(\mathbb{R}^2),\mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ (see [AR91, Section 7]). In particular, $\mu(\mathcal{S}_{-n})=1$ for some $n\in\mathbb{N}$. Thus we take $E=\mathcal{S}_{-n},H=L^2(\mathbb{R}^2,dx)$ for some n big enough. Since the embedding $H\subset E$ is Hilbert-Schmidt(cf. [H80, A.3]), by [AR89, Proposition 3.9], there exists an orthonormal basis e_n of H and l_n such that l_ne_n is an orthonormal basis of E. Now take $Q^{1/2}H=\mathcal{S}_n$. Then by [AR91, Theorem 7.11], $Q^{1/2}(H)\subset H(\mu)$ and for each $l\in\mathcal{S}$,

$$\beta_l(z):=-\sum_{n=1}^{2N}na_n:z^{n-1}:(l)-{}_{\mathcal{S}_n}\langle(-\triangle+1)l,z\rangle_{\mathcal{S}_{-n}},z\in\mathcal{S}_{-n}.$$

Then by the same argument as last section, we also obtain the following two theorems.

Now fix $k\in\mathbb{N},a\in\mathbb{R}$ and take $U=\{x\in\mathcal{S}_{-n}:{}_{\mathcal{S}_{-n}}\langle x,e_k\rangle_{\mathcal{S}_n}\leq a\},\rho=I_U$. Then by Theorem 3.2.2, ρ is an BV function with $H=H_1=H_1^*$. Since U is a convex closed set, then $\rho\in QR(E)$. Thus we can apply Theorem 3.2.3 directly and by a modification get the following:

Theorem 3.3.6 There is an $\mathcal{E}^\rho$-exceptional set $S\subset F$ such that $\forall z\in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l\in\mathcal{S}$

$$\begin{aligned}{}_{E^*}\langle l,X_t-X_0\rangle_E=&\int_0^t\langle l,dW_s^z\rangle-\frac{1}{2}\int_0^t\langle l,n_U(X_s)\rangle dL_s^{\|d\rho\|}\\&+\frac{1}{2}\int_0^t-\sum_{n=1}^{2N}na_n:X_r^{n-1}:(l)+_{\mathcal{S}_n}\langle\Delta l-l,X_r\rangle_{\mathcal{S}_{-n}}dr\ \forall t\geq 0\ P_z\text{-a.s..}\end{aligned}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$I_{\partial U}(X_s)dL_s^{\|d\rho\|}=dL_s^{\|d\rho\|}\ P-a.s.$$

and $n_U(z)=e_k$.

We can also take $U = \{x \in E : \|x\|_E \leq 1\}, \rho = I_U$. Then by Theorem 3.2.2, ρ is an BV function with $H_1 = H_1^* = H$. Since U is a convex closed set, then $\rho \in QR(E)$. Thus as Theorem 3.3.2, we get the following:

Theorem 3.3.7 There is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in \mathcal{S}$

$$
{}_{E^*}\langle l, X_t - X_0\rangle_E = \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t {}_{E^*}\langle l, n_U(X_s)\rangle_E dL_s^{\|d\rho\|}
$$

$$
+\frac{1}{2}\int_0^t -\sum_{n=1}^{2N} na_n : X_r^{n-1} : (l) +_{\mathcal{S}_n} \langle \Delta l - l, X_r\rangle_{\mathcal{S}_{-n}} dr \ \forall t \geq 0 \ P_z\text{−a.s..}
$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$
I_{\partial U}(X_s) dL_s^{\|d\rho\|} = dL_s^{\|d\rho\|} \ P - a.s.
$$

and $n_U(x) = \frac{\sum_k \frac{1}{l_k^2}\langle x, e_k\rangle e_k}{|\sum_k \frac{1}{l_k^2}\langle x, e_k\rangle e_k|}$.

3.3.3 Other examples

Consider $\mu = \varphi^2\mu_0$. Here μ_0 is the Gaussian measure in H with mean 0 and covariance operator$Q := \frac{1}{2}A^{-1}$., where A satisfies Hypothesis 2.1.1 in last chapter, i.e.$A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x\rangle \geq \delta|x|^2 \ \forall x \in D(A)$ for some $\delta > 0$ and A^{-1} is of trace class. Assume

$$
\int_H |D\varphi|^2 d\mu_0 < \infty, \varphi > 0. \tag{3.3.1}
$$

Then by Young's inequality we can deduce

$$
\varphi(x) \cdot \langle e_k, x\rangle \in L^2(H, \mu_0), \forall k \in \mathbb{N}.
$$

By [MR92, II.3.d], for $l \in D(A)$, we have

$$
\beta_l(z) = -2\langle Al, z\rangle + 2\langle l, \frac{D\varphi(z)}{\varphi(z)}\rangle.
$$

We can use Theorem 3.2.3 get the following result.

Theorem 3.3.8 Let $\rho \in QR(H) \cap BV(H, H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 3.2.1(ii). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that

$\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle + \frac{1}{2} \int_0^t {}_{H_1}\langle l, \sigma_\rho(X_s) \rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s \rangle ds + \int_0^t \langle l, \frac{D\varphi(X_s)}{\varphi(X_s)} \rangle ds \ P_z\text{-a.s.}.$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

Assume f satisfies the same conditions as in Theorem 3.2.2 and

$$U = f^{-1}((-\infty, 0)).$$

Theorem 3.3.9 Let $I_U \in QR(H)$ satisfying the same conditions as in Theorem 3.2.2 and $|Df|$ is finite on ∂U, then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted process M^ρ on F satisfy the following: for $l \in D(A)$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t \langle l, n_U(X_s) \rangle dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s \rangle ds + \int_0^t \langle l, \frac{D\varphi(X_s)}{\varphi(X_s)} \rangle ds \ P_z\text{-a.s.}.$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence satisfying

$$I_{\partial U}(X_s) dL_s^{\|d\rho\|} = dL_s^{\|d\rho\|} \ P - a.s.$$

and $n_U(z) = DF(z)/|DF(z)|$ is the normal to Σ.

Now consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) - \frac{D\varphi(X_t)}{\varphi(X_t)} + N_U(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{3.3.2}$$

where $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_U(x)$ is the normal cone to U at x, i.e.

$$N_U(x) = \{z \in H : \langle z, y - x \rangle \leq 0 \ \forall y \in U\}.$$

Definition 3.3.10 A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (3.3.2) if the following conditions hold.

(i) $X(t) \in U$ for all $t \in [0,T]$ $P-a.s.$;

(ii) L is an increasing process with the property that

$$I_{\partial U}(X_s)dL_s = dL_s \ P-a.s.$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_U(X_s)dL_s\rangle - \int_0^t \langle Al, X_s\rangle ds + \int_0^t \langle l, \frac{D\varphi(X_s)}{\varphi(X_s)}\rangle ds \ \forall t \geq 0 \ P-a.s.$$

where $\mathbf{n}_U$ is the exterior normal to U.

Then by a modification of Theorem 2.4.11, we can get the pathwise uniqueness.

Theorem 3.3.11 Assume $U \subset H$ satisfies the same conditions as in Theorem 3.3.9, and $\log\varphi$ is a concave function. Then the stochastic inclusion (3.3.2) admits at most one solution in the sense of Definition 3.3.9.

Combining Theorem 3.3.9 and 3.3.11 with the Yamada-Watanabe Theorem, we now obtain the following:

Theorem 3.3.12 Assume U satisfies the same conditions as in Theorem 3.3.9, and $\log\varphi$ is a concave function. Then there exists a Borel set $M \subset H$ with $I_U \cdot \mu(M) = \mu(U)$ such that for every $x \in M$, (3.3.2) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 3.3.10 and $P(X_0 = x) = 1$. Moreover $X(t) \in M$ for all $t \geq 0$ P-a.s.

As an example, we can take $f = \langle x, x\rangle - 1, \varphi(x) = e^{-|x|^4}$ and $\log\varphi$ is a concave function. Then we can check that all the conditions in Theorem 3.3.12 is satisfied, and by using Theorem 3.3.12, we get there exists a unique probabilistically strong solution in the sense of Definition 3.3.10 for the following problem:

$$\begin{cases} dX(t) + (AX(t) + |X_t|^2 X_t + N_U(X(t)))dt \ni dW(t), \\ X(0) = x. \end{cases}$$

Chapter 4

The stochastic quasi-geostrophic equation

In this chapter we study the 2d stochastic quasi-geostrophic equation in $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and multiplicative noise. We prove the existence of weak solutions with regular additive noise and the existence of martingale solutions with multiplicative noise and pathwise uniqueness under some condition in the general case, i.e. for all $\alpha \in (0,1)$. In the subcritical case $\alpha > 1/2$, we prove existence and uniqueness of (probabilistically) strong solutions and construct a Markov family of solutions. The large deviations principle in the subcritical case for multiplicative noise has also been established. Part of result in this chapter has been included in [RZZ12].

4.1 Notations and preliminaries

We consider the usual abstract form of equations (1.3)-(1.4). In the following, we will restrict ourselves to flows which have zero average on the torus, i.e.

$$\int_{\mathbb{T}^2} \theta d\xi = 0.$$

Thus (1.4) can be restated as

$$u = (-\frac{\partial \psi}{\partial \xi_2}, \frac{\partial \psi}{\partial \xi_1}) \text{ and } (-\triangle)^{1/2}\psi = -\theta.$$

Set $H = \{f \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f dx = 0\}$ and let $|\cdot|$ and $\langle .,. \rangle$ denote the norm and inner product in H respectively. On the periodic domain $\mathbb{T}^2$, $\{\sin(k\xi)|k \in \mathbb{Z}^2_+\} \cup \{\cos(k\xi)|k \in \mathbb{Z}^2_-\}$ form an eigenbasis of $-\triangle$. Here $\mathbb{Z}^2_+ = \{(k_1, k_2) \in \mathbb{Z}^2 | k_2 >$

$0\} \cup \{(k_1, 0) \in \mathbb{Z}^2 | k_1 > 0\}, \mathbb{Z}^2_- = \{(k_1, k_2) \in \mathbb{Z}^2 | -k \in \mathbb{Z}^2_+\}, x \in \mathbb{T}^2$, and the corresponding eigenvalues are $|k|^2$. Define

$$\|f\|^2_{H^s} = \sum_k |k|^{2s} \langle f, e_k \rangle^2$$

and let H^s denote the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. Set $\Lambda = (-\triangle)^{1/2}$. Then

$$\|f\|_{H^s} = |\Lambda^s f|.$$

By the singular integral theory of Calderón and Zygmund (cf [St70, Chapter 3]), for any $p \in (1, \infty)$, there is a constant $C = C(p)$, such that

$$\|u\|_{L^p} \leq C(p) \|\theta\|_{L^p}. \tag{4.1.1}$$

Fix $\alpha \in (0, 1)$ and define the linear operator $A : D(A) = H^{2\alpha}(\mathbb{T}^2) \subset H \to H$ as $Au := \kappa(-\triangle)^\alpha u$. The operator A is positive definite and selfadjoint with the same eigenbasis as that of $-\triangle$ mentioned above. Denote the eigenvalues of A by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and renumber the above eigenbasis correspondingly as $e_1, e_2, \ldots$. We also set $\|u\| := |A^{1/2} u|$, then $\|\theta\|^2 \geq \lambda_1 |\theta|^2$.

First we recall the following important product estimates (cf. [Re95, Lemma A.4]):

Lemma 4.1.1 Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|g\|_{H^{s,p_2}} + \|g\|_{L^{p_3}} \|f\|_{H^{s,p_4}}), \tag{4.1.2}$$

with $p_i \in (1, \infty), i = 1, \ldots, 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We shall use as well the following useful Sobolev inequality (cf. [St70, Chapter V]):

Lemma 4.1.2 Suppose that $q > 1, p \in [q, \infty)$ and

$$\frac{1}{p} + \frac{\sigma}{2} = \frac{1}{q}.$$

Suppose that $\Lambda^\sigma f \in L^q$, then $f \in L^p$ and there is a constant $C \geq 0$ such that

$$\|f\|_{L^p} \leq C \|\Lambda^\sigma f\|_{L^q}.$$

The following compact embedding results will be used later.

Lemma 4.1.3 ([FG95, Theorem 2.1]) Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B. Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Let X be the space

$$X = L^p(0, T; B_0) \cap W^{\alpha,p}(0, T; B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0, T; B)$ is compact.

Lemma 4.1.4 ([FG95, Theorem 2.2]) If $B_1 \subset \tilde{B}$ are two Banach spaces with compact embedding, and the real numbers $\alpha \in (0, 1), p > 1$ satisfy $\alpha p > 1$, then the space $W^{\alpha,p}(0, T; B_1)$ is compactly embedded into $C([0, T]; \tilde{B})$. Similarly, if the Banach spaces $B_1, ..., B_n$ are compactly embedded into $\tilde{B}$ and the real numbers $\alpha_1, ..., \alpha_n \in (0, 1), p_1, ...p_n > 1$ satisfy $\alpha_i p_i > 1, \forall i = 1, ..., n$, then the space

$$W^{\alpha_1,p_1}(0, T; B_1) + ... + W^{\alpha_n,p_n}(0, T; B_n)$$

is compactly embedded into $C([0, T]; \tilde{B})$.

4.2 Existence of solutions for additive noise

In this section, we consider the abstract stochastic evolution equation in place of Eqs (1.3)-(1.4),

$$\begin{cases} d\theta(t) + A\theta(t)dt + u(t) \cdot \nabla\theta(t)dt = G(\theta(t))dW(t), \\ \theta(0) = \theta_0 \in H, \end{cases} \tag{4.2.1}$$

where u satisfies (1.4) and $W(t)$ is a cylindrical Wiener process in a separable Hilbert space K defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. Here G is a measurable mapping from H^α to $L_2(K, H)$.

Definition 4.2.1 (i) We say that there exists a (probabilistically) strong solution to (4.2.1) over the time interval $[0, T]$ if for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists an $\mathcal{F}_t$-adapted process $\theta : [0, T] \times \Omega \to H$ such that for $P - a.s.\ \omega \in \Omega$

$$\theta(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; H^\alpha) \cap C([0, T]; H_w)$$

and P-a.s.

$$\langle\theta(t),\varphi\rangle+\int_0^t\langle A^{1/2}\theta(s),A^{1/2}\varphi\rangle ds-\int_0^t\langle u(s)\cdot\nabla\varphi,\theta(s)\rangle ds=\langle\theta_0,\varphi\rangle+\langle\int_0^t G(\theta(s))dW(s),\varphi\rangle,$$

for all $t\in[0,T]$ and all $\varphi\in C^1(\mathbb{T}^2)$, (assuming also that all integrals in the equation are defined). Here $C([0,T];H_w)$ denotes the space of H-valued weakly continuous functions on $[0,T]$.

(ii)If θ is not an $\mathcal{F}_t$-adapted process, then for additive noise the equation is still defined. In this case we call θ a (probabilistically) weak solution.

Remark 4.2.2 Note that, because $divu=0$ for regular functions θ and v, we have

$$\langle u(s)\cdot\nabla(\theta(s)+\psi),\theta(s)+\psi\rangle=0,$$

so

$$\langle u(s)\cdot\nabla\theta(s),\psi\rangle=-\langle u(s)\cdot\nabla\psi,\theta(s)\rangle.$$

Thus the integral equation in Definition 4.2.1 corresponds to equation (4.2.1).

Assumption 4.2.3 Assume that G does not depend on θ and $\mathrm{Tr}(\Lambda^{2(1+\sigma-\alpha)+\varepsilon}\mathrm{GG}^*)<\infty$ for some $\varepsilon>0$, where $\sigma:=(1-2\alpha)\vee 0$.

Consider the OU equation

$$dz(t)+Az(t)dt=GdW(t).$$

It is known that the process

$$z(t)=\int_0^t e^{-(t-s)A}GdW(s)$$

is a solution with continuous trajectories.

By [DZ92], under Assumption 4.2.3, $\sup_{0\leq t\leq T}\|\nabla z(t)\|_{L^q}<\infty\ P-a.s.$ with $q=(\frac{1}{\alpha}+\varepsilon)\vee 2$ for some $\varepsilon>0$.

Theorem 4.2.4 Let $\alpha\in(0,1)$ and suppose that Assumption 4.2.3 holds. Then for each initial condition $\theta_0\in H$, there exists a weak solution θ of equation (4.2.1) over $[0,T]$ with initial condition $\theta(0)=\theta_0$.

Proof By the classical change of variable $v(t)=\theta(t)-z(t)$ we obtain the differential equation

$$\frac{dv(t)}{dt}+Av(t)+u(t)\cdot\nabla(v(t)+z(t))=0. \tag{4.2.2}$$

For almost all given paths of the process $z(t)$ we study this equation as a deterministic evolution equation.

Let P_n be the orthogonal projection in H onto the linear space spanned by $e_1, ...e_n$. Consider the ordinary differential equation

$$\frac{dv_n(t)}{dt} + Av_n(t) + P_n(u_n(t) \cdot \nabla(v_n(t) + z(t))) = 0,$$

with initial condition

$$v_n(0) = P_n v_0.$$

Here u_n satisfies (1.4) with θ replaced by $v_n + z$.

Its solution satisfies

$$\frac{1}{2}\frac{d}{dt}|v_n|^2 + \|v_n\|^2 = \langle -u_n(t) \cdot \nabla(v_n(t) + z(t)), v_n(t)\rangle.$$

Here $\omega \in \Omega$ is fixed. For simplicity, in the following estimate, we set $v = v_n$ and $u(t) = u_v(t) + u_z(t)$, u_v and u_z satisfying (1.4) with θ replaced by v and z, respectively. So

$$\begin{aligned} |\langle -u(t) \cdot \nabla(v(t) + z(t)), v(t)\rangle| =& |\langle u_v(t) \cdot \nabla z(t), v(t)\rangle + \langle u_z(t) \cdot \nabla z(t), v(t)\rangle| \\ \leq& C\|\nabla z\|_{L^q}\|v\|_{L^p}^2 + C\|\nabla z\|_{L^q}\|z\|_{L^p}\|v\|_{L^p}. \end{aligned}$$

Here $\frac{1}{q} + \frac{2}{p} = 1$. Since

$$\|v\|_{L^p}^2 \leq C\|v\|_{H^{\alpha-\varepsilon_1}}^2 \leq C\|v\|^{2\beta}|v|^{2(1-\beta)},$$

where $\beta = \frac{\alpha-\varepsilon_1}{\alpha}$, by Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 \leq \varepsilon\|v\|^2 + C(\varepsilon)|v|^2 + C(\varepsilon)|v|^2\|\nabla z\|_{L^q}^{1/(1-\beta)} + C\|\nabla z\|_{L^q}^4.$$

Therefore, for all $t \in [0, T]$,

$$|v(t)|^2 \leq e^{\int_0^t C(1+\|\nabla z(s)\|_{L^q}^{1/(1-\beta)})ds}|v_0|^2 + C\int_0^t e^{\int_\tau^t C(1+\|\nabla z(s)\|_{L^q}^{1/(1-\beta)})ds}\|\nabla z(\tau)\|_{L^q}^4 d\tau, \tag{4.2.3}$$

and for $[r, t] \subset [0, T]$,

$$\int_r^t \|v\|^2 d\tau \leq |v(r)|^2 + C\int_r^t (|v|^2 + |v|^2\|\nabla z\|_{L^q}^{1/(1-\beta)} + \|\nabla z\|_{L^q}^4)d\tau. \tag{4.2.4}$$

Then by Assumption 4.2.3, all the terms in (4.2.3) and (4.2.4) containing z are uniformly bounded in t. Therefore, from (4.2.3) and (4.2.4) (which hold true for

v_n) we obtain that the sequence v_n is bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;H^\alpha)$. It is obvious that there exists an element $v \in L^\infty(0,T;H) \cap L^2(0,T;H^\alpha)$ and a sub-sequence v'_m such that

$$v'_m \to v \text{ in } L^2(0,T;H^\alpha) \text{ weakly, and in } L^\infty(0,T;H) \text{ weak-star, as } m \to \infty.$$

In order to prove the strong convergence in $L^2(0,T;H)$, we need to use Lemma 4.1.3. So we just need to prove that $\|v_n\|_{W^{\gamma,2}(0,T,H^{-3})}$ is bounded for some $1/2 < \gamma < 1$. Then by compact embedding, we have $v'_m \to v$ in $L^2(0,T;H) \cap C([0,T];H^{-\beta})$ strongly for some $\beta > 3$. Note that v_n also satisfies

$$\langle v_n(t), \psi\rangle + \int_0^t \langle A^{1/2}v_n(s), A^{1/2}\psi\rangle ds - \int_0^t \langle u_n(s)\cdot\nabla\psi, v_n(s)+z(s)\rangle ds = \langle P_n v_0, \psi\rangle, \tag{4.2.5}$$

for all $t \in [0,T]$ and all $\psi \in C^1(\mathbb{T}^2)$. Then taking the limit in (4.2.5), we obtain the result.

Now decompose v_n as

$$v_n(t) = P_n v_0 - \int_0^t A v_n(s) ds - \int_0^t P_n(u_n(s)\cdot\nabla(v_n(s)+z(s)))ds.$$

By (4.2.4) we obtain

$$\|\int_0^\cdot A v_n(s) ds\|_{W^{1,2}(0,T,H^{-\alpha})} \le C.$$

And by $H^2 \subset L^\infty$, we have for $\theta \in H^1, \psi \in H^3$,

$$|\langle u\cdot\nabla\theta, \psi\rangle| = |\langle u\cdot\nabla\psi, \theta\rangle| \le |\theta|^2\|\nabla\psi\|_\infty \le |\theta|^2\|\psi\|_{H^3}.$$

Then

$$\|P_n(u_n\cdot\nabla(v_n+z))\|_{L^2(0,T;H^{-3})} \le T^{1/2}\sup_{0\le s\le T}|v_n(s)+z(s)|^2 \le C,$$

whence

$$\|\int_0^\cdot P_n(u_n(s)\cdot\nabla(v_n(s)+z(s)))ds\|_{W^{1,2}(0,T,H^{-3})} \le C.$$

Clearly for a Banach space B, $W^{1,2}(0,T;B) \subset W^{\gamma,2}(0,T;B)$. So we have proved

$$\|v_n\|_{W^{\gamma,2}(0,T,H^{-3})} \le C.$$

Thus the assertion follows.

□

4.3 Martingale solutions in the general case

In this section, we consider multiplicative noise in the general case $\alpha \in (0,1)$. First we introduce the following definition of a martingale solution.

Definition 4.3.1 We say that there exists a martingale solution of the equation (4.2.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, a cylindrical Wiener process W on the space K and a progressively measurable process $\theta : [0,T]\times\Omega \to H$, such that for P-a.e. $\omega \in \Omega$,

$$\theta(\cdot,\omega) \in L^\infty(0,T;H) \cap L^2(0,T;H^\alpha) \cap C([0,T];H^{-\beta}),$$

where $\beta > 3$, and such that P-a.s.

$$\langle\theta(t),\phi\rangle+\int_0^t \langle A^{1/2}\theta(s), A^{1/2}\phi\rangle ds-\int_0^t \langle u(s)\cdot\nabla\phi, \theta(s)\rangle ds = \langle\theta_0,\phi\rangle+\langle\int_0^t G(\theta(s))dW(s),\phi\rangle, \tag{4.3.1}$$

for $t \in [0,T]$ and all $\phi \in C^1(\mathbb{T}^2)$.

Let $f_n, n \in \mathbb{N}$, be an ONB of K and consider the following two conditions:

(G.1)(i) $|G(\theta)|^2_{L_2(K,H)} \le \lambda_0|\theta|^2 + \rho, \theta \in H^\alpha$, for some positive real numbers λ_0 and ρ.

(ii) If $y, y_n \in H^\alpha$ such that $y_n \to y$ in H, then $\lim_{n\to\infty} \|G(y_n)^*(v)-G(y)^*(v)\|_K = 0$ for all $v \in C^\infty(\mathbb{T}^2)$.

(G.2)For $y \in K$

$$G(u)y = \sum_{k=1}^\infty (b_k\Lambda^\alpha u + c_k u)\langle y, f_k\rangle_K, u \in H^\alpha,$$

where $b_k, c_k \in C^\infty(\mathbb{T}^2)$ satisfying $\sum_k b_k^2(\xi) < 2\kappa, \sum_k c_k^2(\xi) < M, \xi \in \mathbb{T}^2$.

Theorem 4.3.2 Let $\alpha \in (0,1)$. Under Assumption (G.1), there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (4.2.1).

Proof [Step 1] Let P_n be the orthogonal projection in H onto the space spanned by $e_1, ...e_n$. Consider the Faedo-Galerkin approximation.

$$\begin{cases} d\theta_n(t) + A\theta_n(t)dt + P_n(u_n(t)\cdot\nabla\theta_n(t))dt = P_nG(\theta_n(t))dW(t), \\ \theta_n(0) = P_n\theta_0, \end{cases} \tag{4.3.2}$$

where u_n satisfy (1.4) with θ replaced by θ_n. Since all the coefficients are smooth in P_nH, this equation has a martingale solution $\theta_n \in L^2(\Omega; C([0,T]; P_nH))$.

Since we have

$$\langle u_n(t)\cdot\nabla\theta_n(t),\theta_n\rangle=0,$$

by Itô's formula, for all $p\geq 2$ we have

$$d|\theta_n(t)|^p+p|\theta_n(t)|^{p-2}\|\theta_n\|^2dt\leq p|\theta_n(t)|^{p-2}\langle G(\theta_n)dW(t),\theta_n\rangle+\frac{1}{2}p(p-1)|\theta_n|^{p-2}|P_nG(\theta_n)|^2_{L_2(K,H)}dt.$$

By classical arguments, we easily show that there exist positive constants $C_1(p),C_2$, for each $p\geq 2$, such that (cf [FG95, Appendix 1])

$$E(\sup_{0\leq s\leq T}|\theta_n(s)|^p)\leq C_1(p), \tag{4.3.3}$$

and

$$E\int_0^T\|\theta_n(s)\|^2ds\leq C_2. \tag{4.3.4}$$

[Step 2] Now decompose θ_n as

$$\theta_n(t)=P_n\theta_0-\int_0^t A\theta_n(s)ds-\int_0^t P_n(u_n(s)\cdot\nabla\theta_n(s))ds+\int_0^t P_nG(\theta_n(s))dW(s).$$

By (4.3.4) we obtain

$$E\|\int_0^t A\theta_n(s)ds\|_{W^{1,2}(0,T,H^{-\alpha})}\leq C.$$

And by $H^2\subset L^\infty$ we have for $\theta\in H^1, v\in H^3$

$$|\langle u\cdot\nabla\theta,v\rangle|=|\langle u\cdot\nabla v,\theta\rangle|\leq|\theta|^2\|\nabla v\|_\infty\leq|\theta|^2\|v\|_{H^3}.$$

Then

$$E\|P_n(u_n\cdot\nabla\theta_n)\|_{L^2(0,T;H^{-3})}\leq T^{1/2}E[\sup_{0\leq s\leq T}|\theta_n(s)|^2]\leq C,$$

whence

$$E\|\int_0^t P_n(u_n(s)\cdot\nabla\theta_n)ds\|_{W^{1,2}(0,T,H^{-3})}\leq C.$$

By [FG95, Lemma 2.1], Assumption (G.1), and (4.3.3), (4.3.4), we have

$$E\|\int_0^t P_nG(\theta_n(s))dW(s)\|_{W^{\gamma,2}(0,T;H)}\leq C.$$

Clearly, for a Banach space B, $W^{1,2}(0,T;B)\subset W^{\gamma,2}(0,T;B)$ for $0<\gamma<1$. So, we have proved

$$E\|\theta_n\|_{W^{\gamma,2}(0,T,H^{-3})}\leq C.$$

Recalling (4.3.4), this implies that the laws $\mathcal{L}(\theta_n), n \in \mathbb{N}$ are bounded in probability in

$$L^2(0,T;H^\alpha) \cap W^{\gamma,2}(0,T,H^{-3})$$

and thus are tight in $L^2(0,T;H)$ by Lemma 4.1.3.

Arguing similarly for the term $\int_0^t P_n G(\theta_n(s))dW(s)$, on the basis of the estimate (4.3.3), we apply Lemma 4.1.4 and have that the family $\mathcal{L}(\theta_n), n \in \mathbb{N}$, is tight in $C([0,T];H^{-\beta})$, for all given $\beta > 3$. Thus, we find a subsequence, still denoted by θ_n, such that $\mathcal{L}(\theta_n)$ converges weakly in

$$L^2(0,T;H) \cap C(0,T,H^{-\beta}).$$

By Skorohod's embedding theorem, there exist a stochastic basis $(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\in[0,T]}, P^1)$ and, on this basis, $L^2(0,T;H) \cap C(0,T,H^{-\beta})$-valued random variables $\theta^1, \theta_n^1, n \geq 1$, such that θ_n^1 has the same law as θ_n on $L^2(0,T;H) \cap C(0,T,H^{-\beta})$, and $\theta_n^1 \to \theta^1$ in $L^2(0,T;H) \cap C(0,T,H^{-\beta}), P^1$ -a.s. For θ_n^1 we also have (4.3.3) and (4.3.4). Hence it follows that

$$\theta^1(\cdot,\omega) \in L^2(0,T;H^\alpha) \cap L^\infty(0,T;H) \text{ for } P^1 - a.e\ \omega \in \Omega.$$

For each θ_n^1 we have that u_n^1 satisfies (1.4) with θ replaced by θ_n^1.

For each $n \geq 1$, define the process

$$M_n^1(t) := \theta_n^1(t) - P_n\theta_0^1 + \int_0^t A\theta_n^1(s)ds + \int_0^t P_n(u_n^1(s)\cdot\nabla\theta_n^1(s))ds.$$

In fact M_n^1 is a square integrable martingale with respect to the filtration

$$\{\mathcal{G}_n^1\}_t = \sigma\{\theta_n^1(s), s \leq t\}$$

with quadratic variation

$$\langle\langle M_n^1\rangle\rangle_t = \int_0^t P_n G(\theta_n^1(s))G(\theta_n^1(s))^* P_n ds.$$

For all $s \leq t \in [0,T]$, all bounded continuous functions on $L^2(0,s;H)$ or $C([0,s];H^{-\beta})$, and all v, z smooth, we have

$$E(\langle M_n^1(t) - M_n^1(s), v\rangle \phi(\theta_n^1|_{[0,s]})) = 0$$

and

$$E((\langle M_n^1(t),v\rangle\langle M_n^1(t),z\rangle-\langle M_n^1(s),v\rangle\langle M_n^1(s),z\rangle$$
$$-\int_s^t\langle G(\theta_n^1)^*P_nv,G(\theta_n^1)^*P_nz\rangle dr)\phi(\theta_n^1|_{[0,s]}))=0.$$

Take the limit in the above equation, we obtain that for all $s\leq t\in[0,T]$, all bounded continuous functions on $L^2(0,s;H)$ or $C([0,s];H^{-\beta})$, and all v,z smooth, we have

$$E(\langle M^1(t)-M^1(s),v\rangle\phi(\theta^1|_{[0,s]}))=0$$

and

$$E((\langle M^1(t),v\rangle\langle M^1(t),z\rangle-\langle M^1(s),v\rangle\langle M^1(s),z\rangle-\int_s^t\langle G(\theta^1)^*v,G(\theta^1)^*z\rangle dr)\phi(\theta^1|_{[0,s]}))=0,$$

where

$$M^1(t):=\theta^1(t)-\theta_0^1+\int_0^tA\theta^1(s)ds+\int_0^t(u^1(s)\cdot\nabla\theta^1(s))ds.$$

Thus the conclusion of the proof follows by a martingale representation theorem (cf. [DZ92]). □

In order to get an estimate for the L^p norm, we need to use another approximation.

Theorem 4.3.3 Let $\alpha\in(0,1)$. If $G\in L_2(K,H)$ satisfies (G.1) and also the following conditions: for all $\theta\in H^\alpha\cap L^p(\mathbb{T}^2)$,

$$\int(\sum_j|G(\theta)(f_j)|^2)^{p/2}d\xi\leq C(\int|\theta|^pd\xi+1),\forall t>0,\tag{4.3.5}$$

with $2<p<\infty$ for some constant $C:=C(p)>0$ and for all $\theta_1,\theta_2\in H^\alpha\cap L^p(\mathbb{T}^2)$,

$$\int(\sum_j|(G(\theta_1)-G(\theta_2))(f_j)|^2)^{p/2}d\xi\leq C\int|\theta_1-\theta_2|^pd\xi,\tag{4.3.6}$$

then there exists a martingale solution $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P,W,\theta)$ to (4.2.1). Moreover, if $\theta_0\in L^p(\mathbb{T}^2)$ with $p>2$, then

$$E\sup_{t\in[0,T]}\|\theta(t)\|_{L^p}<\infty.$$

Remark 4.3.4 Typical examples for G satisfying (4.3.5) have the following form:

for $\theta \in H^\alpha$

$$G(\theta)y = \sum_{k=1}^{\infty} b_k \langle y, f_k \rangle_K \theta, \qquad y \in K$$

where b_k are C^∞ functions on $\mathbb{T}^2$ satisfying $\sum_{k=1}^{\infty} b_k^2(\xi) \leq M$.

Proof [Step 1] We first establish the existence of L^p-bounded solutions of the linear equation:

$$d\theta(t) + A\theta(t)dt + w(t) \cdot \nabla\theta(t)dt = k_\delta * G(\theta)dW(t), \tag{4.3.7}$$

with a given coefficient function $w(t)$ which satisfies $divw(t) = 0$ and $\sup_{t\in[0,T]} \|w(t)\|_{C^3} \leq C$. Here $k_\delta * G(\theta)$ means for $y \in K$, $k_\delta * G(\theta)(y) = k_\delta * (G(\theta)(y))$, where k_δ is the periodic Poisson kernel in $\mathbb{T}^2$ given by $\widehat{k_\delta}(\zeta) = e^{-\delta|\zeta|}, \zeta \in \mathbb{Z}^2$. First, we consider G not depending on θ. Now take $z = \int_0^t e^{-(t-s)A} k_\delta * GdW(s), v = \theta - z$. We have

$$dv(t) + Av(t)dt + w(t) \cdot \nabla(v + z(t))dt = 0,$$

which is easily seen to have a solution $v \in C([0,T]; H) \cap L^2([0,T]; H^\alpha)$. We have for any $s > 0$,

$$\frac{d}{dt}|\Lambda^s v|^2 + 2|\Lambda^{s+\alpha} v|^2 \leq C(\|w\|_{C^3(\mathbb{T}^2)})|\Lambda^s v|^2 + |\Lambda^{s+\alpha} v|^2 + C(|\Lambda^{s+\alpha} z|).$$

By this estimate and a standard argument we prove that if $v(t_0) \in H^s$, then $v \in C([t_0,T], H^s) \cap L^2([t_0,T], H^{s+\alpha})$. Then we obtain $v \in C((0,T]; H^s)$ for any $3 > s > 0$. Thus we get the existence of L^p-bounded solutions for additive noise. Then consider the mapping $\Gamma : L^1(\Omega, L^\infty([0,T], L^p)) \to L^1(\Omega, L^\infty[0,T], L^p))$ defined by $\Gamma(\theta_1) = \theta$, where θ satisfies (4.3.7) with $G(\theta)$ replaced by $G(\theta_1)$. Thus, by considering the norm $[E\sup_{s\in[0,T]}(e^{-\beta s}\|\theta(s)\|_{L^p}^p)]^{1/p}$ for suitable $\beta \in (0,\infty)$ and a similar calculation as (4.3.9) below, we obtain Γ maps $L^1(\Omega, L^\infty[0,T], L^p))$ into itself and is a contraction. Thus, the equation $\theta_1 = \Gamma(\theta_1)$ has a unique solution. Hence (4.3.7) has a unique L^p bounded solution.

[Step 2] Now we construct an approximation of (4.2.1).

We pick a smooth $\phi \geq 0$, with supp$\phi \subset [1,2], \int_0^\infty \phi = 1$, and for $\delta > 0$ let

$$U_\delta[\theta](t) := \int_0^\infty \phi(\tau)(k_\delta * R^\perp\theta)(t - \delta\tau)d\tau,$$

where k_δ is the periodic Poisson Kernel in $\mathbb{T}^2$ given by $\widehat{k_\delta}(\zeta) = e^{-\delta|\zeta|}, \zeta \in \mathbb{Z}^2$, and we set $\theta(t) = 0, t < 0$. We take a zero sequence δ_n and consider the equation:

$$d\theta_n(t) + A\theta_n(t)dt + u_n(t) \cdot \nabla\theta_n(t)dt = k_{\delta_n} * G(\theta)dW(t), \tag{4.3.8}$$

with initial data $\theta_n(0)=\theta_0$ and $u_n=U_{\delta_n}[\theta_n]$. For a fixed n, this is a linear equation in θ_n on each subinterval $[t_k,t_{k+1}]$ with $t_k=k\delta_n$, since u_n is determined by the values of θ_n on the two previous subintervals. By [Step1], we obtain the existence of a solution to (4.3.8).

[Step 3] It is sufficient to show that θ_n converge to the solution of (4.2.1). This follows by similar arguments as in the proof of Theorem 4.3.2. Just as in Theorem 4.3.2, we only need to prove

$$E\|\theta_n\|_{W^{\gamma,2}(0,T,H^{-3})}\leq C.$$

Here we can't bound $|u_n|$ by $|\theta_n|$, pointwise in time. Instead, we have

$$\sup_{[0,t]}|u_n|\leq C\sup_{[0,t]}|\theta_n|.$$

Thus by a small modification of the proof of Theorem 4.3.2, we get the martingale solution $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P,W,\theta)$ to (4.2.1).

[Step 4] Now we prove the last statement. It is sufficient to prove that

$$E\sup_{t\in[0,T]}\|\theta_n(t)\|_{L^p}^p\leq C.$$

We write for simplicity $\theta(t)=\theta_n(t,\xi)$. By [Kr10, Lemma 5.1], we have

$$\begin{aligned}
\|\theta(t)\|_{L^p}^p=&\|\theta_0\|_{L^p}^p+\int_0^t[-p\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)(\Lambda^{2\alpha}\theta(s)+u(s)\cdot\nabla\theta(s))d\xi\\
&+\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)d\xi]ds\\
&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s)\\
\leq&\|\theta_0\|_{L^p}^p+\int_0^t\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)d\xi ds\\
&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s)\\
\leq&\|\theta_0\|_{L^p}^p+\int_0^t(\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^pd\xi+C(\varepsilon)\int(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds\\
&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s).
\end{aligned}$$

Then by the Burkholder-Davis-Gundy inequality, Minkowski's inequality and the same estimate as in the proof of (6.4) in [Kr10] and (4.3.1) we have

$$\begin{aligned}
E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p \leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds \\
& + pE(\int_0^t(\int_{\mathbb{T}^2}|\theta(s)|^{p-1}(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{1/2}d\xi)^2 ds)^{1/2} \\
\leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds \\
& + pE\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^{p-1}(\int_0^t(\int_{\mathbb{T}^2}(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2}d\xi)^{2/p}ds)^{1/2} \\
\leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds
\end{aligned}$$

$$\begin{aligned}
& + C(T)E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^{p-1}(\int_0^t(\int_{\mathbb{T}^2}(\sum_j |G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds)^{1/p} \\
\leq & E\|\theta_0\|_{L^p}^p + \varepsilon E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p + C_1 E\int_0^t \|\theta(s)\|_{L^p}^p ds + C_2 \qquad (4.3.9) \\
\leq & E\|\theta_0\|_{L^p}^p + \varepsilon E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p + C_1\int_0^t E\sup_{s\in[0,\sigma]}\|\theta(s)\|_{L^p}^p d\sigma + C_2.
\end{aligned}$$

By Gronwall's lemma, the assertion follows. □

Theorem 4.3.5 Let $\alpha\in(0,1)$. Under Assumption (G.2), there exists a martingale solution $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P,W,\theta)$ to (4.2.1).

Proof The proof is similar to the one for Theorem 4.3.2. The only difference is the proof of $\theta(\cdot,\omega)\in C([0,T];H^{-\beta})$. Here by Aldous' criterion it suffices to check that for all stopping times $\tau_n\leq T$ and $\delta_n\to 0$,

$$\lim_n E\|\theta_n(\tau_n+\delta_n)-\theta_n(\tau_n)\|_{H^{-\beta}}=0.$$

This can however be checked easily. □

4.4 Uniqueness of solutions

In this section, we will prove pathwise uniqueness for equation (4.2.1). First we prove uniqueness in the subcritical case.

Theorem 4.4.1 Assume $\alpha > \frac{1}{2}$. If G satisfies the following condition

$$\|\Lambda^{-1/2}(G(u) - G(v))\|^2_{L_2(K,H)} \le \beta|\Lambda^{-1/2}(u-v)|^2 + \beta_1|\Lambda^{\alpha-\frac{1}{2}}(u-v)|^2, \qquad (4.4.1)$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v, and $\beta_1 < 2\kappa$, then (4.2.1) admits at most one probabilistically strong solution in the sense of Definition 4.2.1 such that

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^q} < \infty, \qquad P-a.s.,$$

with $0 < 1/q < \alpha - \frac{1}{2}$, and

$$E\sup_{t\in[0.T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Remark If in Remark 4.3.4 $b_k = \mu_k e_k$ for $\mu_k \in \mathbb{R}$, then (4.4.1) is satisfied.

Proof Let θ_1, θ_2 be two solutions of (4.2.1), and let $\{e_k\}_{k\in\mathbb{N}}$ be the eigenbasis of A from above. Then their difference $\theta = \theta_1 - \theta_2$ satisfies

$$\begin{aligned}&\langle\psi, \theta(t)\rangle - \int_0^t \langle u\cdot\nabla\psi, \theta_1\rangle ds - \int_0^t \langle u_2\cdot\nabla\psi, \theta\rangle ds + \kappa\int_0^t \langle\theta, \Lambda^{2\alpha}\psi\rangle ds \\ =&\int_0^t \langle\psi, (G(\theta_1) - G(\theta_2))dW(s)\rangle.\end{aligned} \qquad (4.4.2)$$

Now set $\phi_k = \langle e_k, \theta(t)\rangle, \varphi_k = \langle\Lambda^{-1}e_k, \theta(t)\rangle$. Itô's formula and (4.4.2) yield

$$\begin{aligned}\phi_k\varphi_k =&\int_0^t \phi_k d\varphi_k + \int_0^t \varphi_k d\phi_k + \langle\varphi_k, \phi_k\rangle(t)\\ =&2\int_0^t \langle u\cdot\nabla e_k, \theta_1\rangle\langle\Lambda^{-1}\theta, e_k\rangle + \langle u_2\cdot\nabla e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle - \kappa\langle\Lambda^{2\alpha}e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds\\ &+2\int_0^t \langle\Lambda^{-1}\theta, e_k\rangle\langle e_k, (G(\theta_1) - G(\theta_2))dW(s)\rangle\\ &+\int_0^t \langle(G(\theta_1) - G(\theta_2))^*e_k, (G(\theta_1) - G(\theta_2))^*\Lambda^{-1}e_k\rangle ds.\end{aligned} \qquad (4.4.3)$$

The dominated theorem implies:

$$\sum_{k\le N}\int_0^t \langle u\cdot\nabla e_k, \theta_1\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds \to \int_0^t {}_{H^{-1}}\langle u\cdot\nabla\theta_1, \Lambda^{-1}\theta\rangle_{H^1} ds, N\to\infty,$$

$$\sum_{k\le N}\int_0^t \langle u_2\cdot\nabla e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds \to \int_0^t {}_{H^{-1}}\langle u_2\cdot\nabla\theta, \Lambda^{-1}\theta\rangle_{H^1} ds, N\to\infty,$$

and

$$\sum_{k\leq N}\int_0^t\langle\Lambda^{2\alpha}e_k,\theta\rangle\langle\Lambda^{-1}\theta,e_k\rangle ds\to\int_0^t\langle\theta,\Lambda^{2\alpha-1}\theta\rangle ds, N\to\infty.$$

Furthermore, since

$$\int_0^t|\Lambda^{-1/2}\theta|^2\|\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds$$

$$\leq C\sup_{s\leq t}|\theta(s)|^2\int_0^t\|\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds<\infty,$$

we obtain

$$\sum_{k\leq N}\int_0^t\langle\Lambda^{-1}\theta,e_k\rangle\langle e_k,(G(\theta_1)-G(\theta_2))dW(s)\rangle\to$$

$$M_t:=\int_0^t\langle\Lambda^{-1/2}\theta,\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))dW(s)\rangle, N\to\infty.$$

Finally, the following inequality holds:

$$\sum_{k\leq N}\int_0^t\langle(G(\theta_1)-G(\theta_2))^*e_k,(G(\theta_1)-G(\theta_2))^*\Lambda^{-1}e_k\rangle ds\leq\int_0^t\|\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds.$$

Thus, summing up over $k\leq N$ in (4.4.3) and letting $N\to\infty$ we obtain

$$\begin{aligned}&|\Lambda^{-1/2}\theta|^2+2\kappa\int_0^t|\Lambda^{\alpha-\frac12}\theta|^2ds\\ \leq&2M(t)+2\int_0^t {}_{H_{-1}}\langle u\cdot\nabla\theta_1,\Lambda^{-1}\theta\rangle_{H_1}+{}_{H_{-1}}\langle u_2\cdot\nabla\theta,\Lambda^{-1}\theta\rangle_{H_1}ds\\ &+\int_0^t\|\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds.\end{aligned}$$

By [Re95] we have

$${}_{H^{-1}}\langle u\cdot\nabla\theta_1,\Lambda^{-1}\theta\rangle_{H^1}=0,$$

and

$$\begin{aligned}|{}_{H_{-1}}\langle u_2\cdot\nabla\theta,\Lambda^{-1}\theta\rangle_{H_1}|\leq&\|u_2\|_{L^q}\|\theta\|_{L^p}\|\nabla\Lambda^{-1}\theta\|_{L^p}\leq C\|u_2\|_{L^q}\|\theta\|_{H^{1/q}}\|\nabla\Lambda^{-1}\theta\|_{H^{1/q}}\\ \leq&C\|\theta_2\|_{L^q}\|\Lambda^{-1}\theta\|^2_{H^{1+\frac1q}}\leq C\|\theta_2\|_{L^q}\|\Lambda^{-1}\theta\|^{2/N}_{H^{1/2}}\|\Lambda^{-1}\theta\|^{2(1-\frac1N)}_{H^{\frac12+\alpha}}\\ \leq&\varepsilon|\Lambda^{\alpha-\frac12}\theta|^2+C\|\theta_2\|^N_{L^q}|\Lambda^{-1/2}\theta|^2.\end{aligned}$$

Here $\frac1q+\frac2p=1$ for $0\leq 1/q<\alpha-1/2, N=\frac{\alpha}{\alpha-\frac12-\frac1q}$ and we use $H^{1/q}\hookrightarrow L^p$ continuously.

Now by (4.4.1) we have

$$|\Lambda^{-1/2}\theta|^2 \leq M(t) + \int_0^t C\|\theta_2\|_{L^q}^N |\Lambda^{-1/2}\theta|^2 ds + \beta \int_0^t |\Lambda^{-1/2}(\theta_1 - \theta_2)|^2 ds.$$

Let

$$\tau_n^1 := \inf\{t > 0, \|\theta_2(t)\|_{L^q} > n\}.$$

Then by the weak continuity of θ_2, τ_n^1 are stopping times with respect to $\mathcal{F}_{t+}$, ($\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$)and $\|\theta_2(t \wedge \tau_n^1)\|_{L^q} \leq n$ for large n. Also let τ_n^2 be a localizing sequence of stopping times for M and $\tau_n := \tau_n^1 \wedge \tau_n^2$. Then, since $M(t \wedge \tau_n)$ is a martingale with respect to $\mathcal{F}_{t+}$, we get

$$\begin{aligned}E|\Lambda^{-1/2}\theta(t\wedge\tau_n)|^2 \leq & Cn^N E\int_0^{t\wedge\tau_n} |\Lambda^{-1/2}\theta|^2 ds + \beta E \int_0^{t\wedge\tau_n} |\Lambda^{-1/2}\theta|^2 ds \\ = & C(n)\int_0^t E|\Lambda^{-1/2}\theta(s\wedge\tau_n)|^2 ds + \beta\int_0^t E|\Lambda^{-1/2}\theta(s\wedge\tau_n)|^2 ds.\end{aligned}$$

By Gronwall's inequality, we get $|\Lambda^{-1/2}\theta(t \wedge \tau_n)|^2 = 0$ $P - a.s.$, and recalling that $\tau_n \to T$ as $n \to \infty$, we obtain that $\theta(t) = 0$ $P - a.s.$ for $t \leq T$, thus completing the proof. $\square$

From the proof we immediately obtain the following result.

Corollary 4.4.2 Assume $\alpha > \frac{1}{2}$. If there exists a probabilistically strong solution θ_2 in the sense of Definition 4.2.1 such that

$$\sup_{t\in[0,T]} \|\theta_2(t)\|_{L^q} < \infty, \qquad P - a.s.$$

for some q with $0 < 1/q < \alpha - \frac{1}{2}$ and G satisfies (4.4.1), then θ_2 is the only solution to (4.2.1) such that

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta_2(t)|^2 < \infty.$$

Thus, combining Theorem 4.4.1, Theorem 4.3.3 and the Yamada-Watanabe Theorem in [Ku07], we get the following existence and uniqueness result.

Theorem 4.4.3 Assume $\alpha > \frac{1}{2}$ and that G satisfies (4.4.1), (G.1) (4.3.5) and (4.3.6) for some p with $0 < 1/p < \alpha - \frac{1}{2}$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution θ of equation (4.2.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^p} < \infty, \qquad P - a.s,$$

and

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Combining Theorem 4.4.3 and Corollary 4.4.2, we obtain the following more general existence and uniqueness result.

Theorem 4.4.4 Assume $\alpha > \frac{1}{2}$ and that G satisfies (4.4.1), (G.1), (4.3.5) and (4.3.6) with $0 < 1/p < \alpha - \frac{1}{2}$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution θ of equation (4.2.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Moreover, the solution satisfies

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^p} < \infty, \qquad P - a.s..$$

Theorem 4.4.5 (Markov property) Assume $\alpha > \frac{1}{2}$ and that G satisfies (G.1),(4.4.1) and (4.3.5),(4.3.6) with $0 < 1/p < \alpha - \frac{1}{2}$. If $\theta_0 \in L^p$, then for every bounded, $\mathcal{B}(H)$-measurable $F : H \to \mathbb{R}$, and all $s, t \in [0, T]$, $s \le t$

$$E(F(\theta(t))|\mathcal{F}_s)(\omega) = E(F(\theta(t, s, \theta(s)(\omega)))) \text{ for } P - a.s.\omega \in \Omega.$$

Here $\theta(t, s, \theta(s)(\omega))$ denotes the solution to (4.2.1) starting from $\theta(s)$ at time s satisfying

$$E \sup_{t\in[s,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Proof By Theorem 4.4.4, we have $\theta(t) = \theta(t, s, \theta(s))$ P-a.s.. Then by the same arguments as in [PR07, Proposition 4.3.3] and the Yamada-Watanabe Theorem in [RSZ08], the assertion follows. □

Set

$$p_t(x, dy) := P \circ (\theta(t, x))^{-1}(dy), 0 \le t \le T, x \in H.$$

Here and in the following, we use $\theta(t, x)$ to denote a solution with initial value x. We set for $\mathcal{B}(H)$-measurable $F : H \to \mathbb{R}$, and $t \in [0, T], x \in H$

$$P_t F(x) := \int F(y) p_t(x, dy),$$

provided F is $p_t(x,dy)$-integrable. Then by Theorem 4.4.5, we have for $F: H \to \mathbb{R}$, bounded and $\mathcal{B}(H)$-measurable, $s,t \geq 0$,

$$P_s(P_tF)(x) = P_{s+t}F(x), x \in L^p \text{ with } 0 < 1/p < \alpha - \frac{1}{2}.$$

Theorem 4.4.6 Let $\alpha \in (0,1)$. If G satisfies the Lipschitz condition

$$\|G(u) - G(v)\|^2_{L_2(K,H)} \leq \beta|u-v|^2 + \beta_1\|u-v\|^2_{H_\alpha} \tag{4.4.4}$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v, and $\beta_1 < 2\kappa$, then (4.2.1) admits at most one solution in the sense of Definition 4.2.1 such that

$$E \sup_{t\in[0,T]} |\theta(t)|^4 < \infty$$

and

$$\int_0^T \|\Lambda^{1-\alpha+\varepsilon}\theta(t)\|^q_{L^p} dt < \infty, \frac{1}{p} + \frac{\alpha}{q} = \frac{\alpha+\varepsilon}{2} \qquad P-a.s.,$$

where $\varepsilon \in (0,\alpha]$ and $q < \infty$.

Proof By the same argument as in the proof of Theorem 4.4.1, we get (4.4.2). Set $\phi_k := \langle e_k, \theta(t)\rangle$. Then Itô's formula and (4.4.2) yield

$$\begin{aligned}\phi_k^2 =& 2\int_0^t \phi_k d\phi_k + [\phi_k](t)\\ =& 2\int_0^t \langle u\cdot\nabla e_k, \theta_1\rangle\langle\theta, e_k\rangle + \langle u_2\cdot\nabla e_k, \theta\rangle\langle\theta, e_k\rangle - \kappa\langle\Lambda^{2\alpha}e_k,\theta\rangle\langle\theta, e_k\rangle ds\\ &+2\int_0^t \langle\theta, e_k\rangle\langle e_k, (G(\theta_1) - G(\theta_2))dW(s)\rangle\\ &+\int_0^t \langle (G(\theta_1)-G(\theta_2))^*e_k, (G(\theta_1)-G(\theta_2))^*e_k\rangle ds.\end{aligned} \tag{4.4.5}$$

Since

$$\begin{aligned}|\langle u_2\cdot\nabla\theta,\varphi\rangle| \leq& \|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}}\|\Lambda^{\alpha-\varepsilon}(u_2\theta)\|_{L^{p_1'}}\\ \leq& C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}}(\|\theta_2\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta\|_{L^{q_2}} + \|\theta\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta_2\|_{L^{q_2}})\\ \leq& C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}}(|\theta_2| + |\theta_1|)^{2-\beta-\gamma}(|\Lambda^\alpha\theta_1| + |\Lambda^\alpha\theta_2|)^{\beta+\gamma}\\ \leq& C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|^{p_2'}_{L^{p_1}}(|\theta_2|^2 + |\theta_1|^2) + |\Lambda^\alpha\theta_2|^2 + |\Lambda^\alpha\theta_1|^2,\end{aligned}$$

the term $u_2\cdot\nabla\theta$ can be considered as an element in $(H^{1-\alpha+\varepsilon,p_1})'$. Here $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p_1'}$

and $\beta+\gamma=\frac{1}{\alpha}(\alpha-\varepsilon+\frac{2}{p_1}), p_2=2/(\beta+\gamma)$.

By a similar calculation for $\langle u\cdot\nabla\theta_1,\theta\rangle$, the dominated convergence theorem yields the following:

$$\sum_{k\le N}\int_0^t\langle u\cdot\nabla e_k,\theta_1\rangle\langle\theta,e_k\rangle ds\to\int_0^t{}_{(H^{1-\alpha+\varepsilon,p_1})'}\langle u\cdot\nabla\theta_1,\theta\rangle_{H^{1-\alpha+\varepsilon,p_1}}ds, N\to\infty,$$

$$\sum_{k\le N}\int_0^t\langle u_2\cdot\nabla e_k,\theta\rangle\langle\theta,e_k\rangle ds\to\int_0^t{}_{(H^{1-\alpha+\varepsilon,p_1})'}\langle u_2\cdot\nabla\theta,\theta\rangle_{H^{1-\alpha+\varepsilon,p_1}}ds, N\to\infty,$$

and

$$\sum_{k\le N}\int_0^t\langle\Lambda^{2\alpha}e_k,\theta\rangle\langle\theta,e_k\rangle ds\to\int_0^t\langle\theta,\Lambda^{2\alpha}\theta\rangle ds, N\to\infty.$$

Furthermore, since

$$\int_0^t|\theta|^2\|G(u)-G(v)\|^2_{L_2(K,H)}ds\le C\sup_{s\le t}|\theta(s)|^2\int_0^t\|G(u)-G(v)\|^2_{L_2(K,H)}ds<\infty,$$

we obtain

$$\sum_{k\le N}\int_0^t\langle\theta,e_k\rangle\langle e_k,(G(\theta_1)-G(\theta_2))dW(s)\rangle\to M_t:=\int_0^t\langle\theta,(G(\theta_1)-G(\theta_2))dW(s)\rangle, N\to\infty.$$

Finally, the following inequality holds:

$$\sum_{k\le N}\int_0^t\langle(G(\theta_1)-G(\theta_2))^*e_k,(G(\theta_1)-G(\theta_2))e_k\rangle ds\le\int_0^t\|G(\theta_1)-G(\theta_2)\|^2_{L_2(K,H)}ds.$$

Thus, summing up over $k\le N$ in (4.4.5) and letting $N\to\infty$ we obtain

$$\begin{aligned}|\theta(t)|^2+2\kappa\int_0^t|\Lambda^\alpha\theta|^2ds\le&2M(t)+2\int_0^t\langle u\cdot\nabla\theta_1,\theta\rangle+\langle u_2\cdot\nabla\theta,\theta\rangle ds\\&+\int_0^t\|(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds.\end{aligned}$$

We have

$$\langle u_2\cdot\nabla\theta,\theta\rangle=0,$$

and by a similar calculation as in the proof of [Ju05, Theorem 3.3], we have

$$\begin{aligned}|\langle u\cdot\nabla\theta_1,\theta\rangle|\le&\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}\|\Lambda^{\alpha-\varepsilon}(u\theta)\|_{L^{p_1'}}\le C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}\|\theta\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta\|_{L^{q_2}}\\\le&C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}|\theta|^{2-\beta-\gamma}|\Lambda^\alpha\theta|^{\beta+\gamma}\\\le&\varepsilon|\Lambda^\alpha\theta|^2+C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|^{p_2'}_{L^{p_1}}|\theta|^2.\end{aligned}$$

Here $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p_1'}$ and $\beta+\gamma=\frac{1}{\alpha}(\alpha-\varepsilon+\frac{2}{p_1}), p_2=2/(\beta+\gamma)$.

Now by (4.4.4) we have

$$|\theta(t)|^2\le M(t)+\int_0^t C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}^{p_2'}|\theta|^2ds+\beta\int_0^t|(\theta_1-\theta_2)|^2ds.$$

Define the stopping time

$$\tau_n:=\inf\{t>0,\int_0^t\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}^{p_2'}ds>n\}.$$

Applying Gronwall's lemma, we have

$$|\theta(t\wedge\tau_n)|^2\le|M(t\wedge\tau_n)|e^{\int_0^{t\wedge\tau_n}C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}^{p_2'}ds+\beta t}\le|M(t\wedge\tau_n)|e^{Cn+\beta t}.$$

Consequently,

$$\begin{aligned}E|\theta(t\wedge\tau_n)|^4\le&e^{2Cn+2\beta t}E\int_0^{t\wedge\tau_n}|\theta|^2\|G(\theta_1)-G(\theta_2)\|_{L_2(K,H)}^2ds\\\le&\beta^2e^{2Cn+2\beta t}\int_0^tE|\theta(s\wedge\tau_n)|^4ds.\end{aligned}$$

By Gronwall's lemma, we get $|\theta(t\wedge\tau_n)|^2=0\ P-a.s.$, and recalling that $\tau_n\to T$ as $n\to\infty$, we obtain that $\theta(t)=0\ P-a.s.$ for $t\le T$, thus completing the proof. □

Remark 4.4.7 For $\alpha=1/2$, consider

$$d\theta=[A\theta+u\cdot\nabla\theta)]dt+\sum_{j=1}^m b_j\theta\circ dw_j(t),\tag{4.4.6}$$

for $b_j\in\mathbb{R}$, and independent 1-dimensional Brownian motions w_j. Consider the process

$$\beta(t)=e^{-\sum_{j=1}^m b_jw_j(t)}.$$

Then, the process $v(t)$ defined by transformation

$$v(t)=\beta(t)\theta(t),$$

satisfies the equation (which depends on a random parameter)

$$\frac{dv}{dt}=Av+\beta^{-1}u_v\cdot\nabla v.\tag{4.4.7}$$

Then by the same argument as in the proof of [CC04, Theorem 3.1], we obtain the local existence and uniqueness of smooth solutions starting from H^1 periodic initial data. More precisely, for P-almost every $\omega \in \Omega$, there exists a time $t(\omega, |\Lambda\theta_0|)$, such that $v \in C((0,t), H^m)$ for any $m > 0$. On the other hand, by the same arguments as in [CV06, Section 2], we obtain for any $T > 0$, there exists $M(\omega, |\Lambda\theta_0|)$ such that

$$\|v(t,\cdot)\|_\infty \leq M \text{ for } t \in [0,T].$$

Then

$$\|\beta^{-1}u_v(t,\cdot)\|_{\text{BMO}} \leq M_1(\omega, |\Lambda\theta_0|, T) \text{ for } t \in [0,T].$$

Hence by [KN09, Theorem 1.1], we obtain that there exists $\gamma(\omega, |\Lambda\theta_0|, T) > 0$, such that

$$\|v(\cdot,t)\|_{C^\gamma(\mathbb{T}^2)} \leq C(\omega, |\Lambda\theta_0|, T).$$

Then by the same arguments as in the proofs of [CW07, Theorem 3.1] and [CV06, Theorem 10], we obtain

$$\|v(\cdot,t)\|_{C^1(\mathbb{T}^2)} \leq C_1(\omega, |\Lambda\theta_0|, T) \text{ for } t \in [0,T].$$

By this a-priori bound and the local existence, we obtain a global regular solution v for P-almost every $\omega \in \Omega$. Define

$$\theta(t,\xi) := \beta(t)^{-1}v(t,\xi).$$

Then we obtain a solution θ such that

$$\sup_{t\in[0,T]} \|\Lambda^{1-\alpha+\varepsilon}\theta\|_{L^p} < \infty, \frac{1}{p} \leq \frac{\alpha+\varepsilon}{2} \qquad P-a.s..$$

So, for this special linear multiplicative noise, we obtain a solution satisfying the condition in Theorem 4.4.6. Unfortunately, we don't get this result for more general noise and $\alpha = \frac{1}{2}$ since the results and the method in the deterministic case (e.g. [CV06], [KNV07], [KN09]) cannot be applied directly.

4.5 The large deviations result for small noise in the subcritical case

In this section, for $\alpha > 1/2$ we want to consider the large deviation principle for small noise stochastic quasi-geostrophic equation. Here we will use the weak convergence approach established by Budhiraja and Dupuis in [BD00]. Let us first recall some standard definitions and results from the large deviation theory. Let $\{X^\varepsilon\}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in some Polish space E.

Definition 4.5.1 (Rate function) A function $I : E \to [0, \infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in E : I(x) \le K\}$ is compact for each $K < \infty$.

Definition 4.5.2 (I). (Large deviation principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of E

$$-\inf_{x \in A^o} I(x) \le \liminf_{\varepsilon \to 0} \varepsilon^2 \log P(X^\varepsilon \in A) \le \limsup_{\varepsilon \to 0} \varepsilon^2 \log P(X^\varepsilon \in A) \le -\inf_{x \in \bar{A}} I(x),$$

where A^o and $\bar{A}$ are respectively the interior and the closure of A in E.

(II). (Laplace principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the Laplace principle with rate function I if for each bounded continuous real-valued function h defined on E

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log E\{\exp[-\frac{1}{\varepsilon^2} h(X^\varepsilon)]\} = \inf_{x \in E}\{h(x) + I(x)\}.$$

Suppose $W(t)$ is an cylindrical Wiener process on Hilbert space K (with the inner product $\langle \cdot, \cdot \rangle_0$, and norm $|\cdot|_0$) defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$,(i.e. the path of W take values in $C([0,T], H_1)$, where H_1 is another Hilbert space such that the embedding $K \subset H_1$ is Hilbert-Schmidt.) Let $\mathcal{A}$ denote the class of K valued $\{\mathcal{F}_t\}$-predictable processes ϕ which satisfy $\int_0^T |\phi(s)|_0^2 ds < \infty$ a.s. and $S_N := \{v \in L^2([0,T], K) : \int_0^T |v(s)|_0^2 ds \le N\}$. Define $\mathcal{A}_N := \{\phi \in \mathcal{A} : \phi(\omega) \in S_N, P - a.s\}$.

Suppose $g^\varepsilon : C([0,T], H_1) \to E$ is a measurable map and $X^\varepsilon = g^\varepsilon(W)$. We are interested in the large deviation principle for X^ε as $\varepsilon \to 0$. Consider the following Hypothesis:

Hypothesis 4.5.3 There exists a measurable map $g^0 : C([0,T], H_1) \to E$ such that the following hold:

1. Let $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$ for some $M < \infty$. If v^ε converge to v as S_M-valued random elements in distribution, then $g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s) ds)$ converge in distribution to $g^0(\int_0^\cdot v(s) ds)$.

2. For every $M < \infty$, the set $K_M = \{g^0(\int_0^\cdot v(s) ds) : v \in S_M\}$ is a compact subset of E.

The following theorem was proven in [BD00].

Theorem 4.5.4 If $\{g^\varepsilon\}$ satisfies Hypothesis 4.5.3, then $\{X^\varepsilon\}$ satisfies the Laplace principle (hence large deviation principle) on E with the good rate function I given by

$$I(f)=\inf_{\{v\in L^2([0,T],K):f=g^0(\int_0^\cdot v(s)ds)\}}\{\frac{1}{2}\int_0^T|v(s)|_0^2ds\}. \quad (4.5.1)$$

In this section, we consider the abstract stochastic evolution equation in place of Eqs (1.3)-(1.4),

$$\begin{cases} d\theta(t)+A\theta(t)dt+u(t)\cdot\nabla\theta(t)dt=G(\theta)dW(t),\\ \theta(0)=\theta_0, \end{cases} \quad (4.5.2)$$

where u satisfies (1.4).

Hypothesis 4.5.5 Assume G satisfies the following conditions:

i) $G:H\to L_2(K,H)$ is continuous and $|G(\theta)|^2_{L_2(K,H)}\le\lambda_0|\theta|^2+\rho,\theta\in H$, for some positive real numbers λ_0 and ρ.

ii)For some p with $0<1/p<\alpha-\frac{1}{2}$,

$$\int(\sum_j|G(\theta)(e_j)|^2)^{p/2}d\xi\le C(\int|\theta|^pd\xi+1), \quad (4.5.3)$$

and

$$\int(\sum_j|(G(\theta_1)-G(\theta_2))(e_j)|^2)^{p/2}d\xi\le C\int|\theta_1-\theta_2|^pd\xi. \quad (4.5.4)$$

iii)

$$\|\Lambda^{-1/2}(G(u)-G(v))\|^2_{L_2(K,H)}\le C|\Lambda^{-1/2}(u-v)|^2+\beta_1|\Lambda^{\alpha-\frac{1}{2}}(u-v)|^2, \quad (4.5.5)$$

for some $\beta_1<2\kappa$.

Under Hypothesis 4.5.5, by Theorem 4.4.4, for $\theta_0\in L^p$, there exists a pathwise unique strong solution of (4.5.2) in $L^\infty([0,T],H)\cap L^2([0,T],H^\alpha)\cap C([0,T],H^{-\beta})$, where $\beta>3$. The main difficulty lies in dealing with the nonlinear term since the solution to the stochastic quasi-geostrophic equation is not as regular as in the 2D Navier-Stokes case. To estimate the nonlinear term, we use Galerkin approximations and using the method in [GK96] we prove that these approximations converge in probability to the solution.

Lemma 4.5.6 ([GK96, Lemma 1.1]) Let Z_n be a sequence of random elements in a Polish space (E,ρ) equipped with the Borel σ-algebra. Then Z_n converges in probability to an E-valued random element if and only if for every pair of subsequences Z_l and Z_m there exists a subsequence $v_k:=(Z_{l(k)},Z_{m(k)})$ converging weakly to a random element v supported on the diagonal $\{(x,y)\in E\times E:x=y\}$.

Theorem 4.5.7 Assume Hypothesis 4.5.5, then θ_n converge to θ in probability.

Proof In Theorem 4.3.3, we proved that θ_n is tight in $L^2([0,T],H)\cap C([0,T],H^{-\beta})$. In order to use Lemma 4.5.6, we now take two subsequences θ_l,θ_m of θ_n. Then obviously (θ_l,θ_m,W) is tight in

$$L^2([0,T],H)\cap C([0,T],H^{-\beta})\times L^2([0,T],H)\cap C([0,T],H^{-\beta})\times C([0,T],H_1).$$

Then by Skorokhod's embedding theorem, there exists subsequences l_j,m_j, a probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$ carrying on $\hat{\theta}_{l_j},\bar{\theta}_{m_j},\hat{W}_j$ such that the distribution of $(\hat{\theta}_{l_j},\bar{\theta}_{m_j},\hat{W}_j)$ and $(\theta_{l_j},\theta_{m_j},W)$ coincide, and for P a.e.

$$\hat{\theta}_{l_j}\to\hat{\theta},\qquad \bar{\theta}_{l_j}\to\bar{\theta},\qquad \hat{W}_j\to\hat{W}$$

in the corresponding topology.

Thus $(\hat{\theta},\hat{W})$ and $(\bar{\theta},\hat{W})$ are the solutions of (4.5.2). Then by pathwise uniqueness, we have $\hat{\theta}=\bar{\theta}$. Thus the result follows from Lemma 4.5.6. □

Consider the stochastic quasi-geostrophic equation with multiplicative noise given by

$$d\theta^\varepsilon(t)+A\theta^\varepsilon(t)dt+u^\varepsilon(t)\cdot\nabla\theta^\varepsilon(t)dt=\sqrt{\varepsilon}G(\theta^\varepsilon)dW(t) \tag{4.5.6}$$

with $\theta^\varepsilon(0)=\theta_0\in L^p$. By Theorem 4.4.5, under Hypothesis 4.5.5, there exists a pathwise unique strong solution of (4.5.6) in $L^\infty([0,T],H)\cap L^2([0,T],H^\alpha)\cap C([0,T],H^{-\beta})$, for $\beta>3$. Therefore, there exists a Borel-measurable function $g^\varepsilon: C([0,T],H_1)\to L^\infty([0,T],H)\cap L^2([0,T],H^\alpha)\cap C([0,T],H^{-\beta})$ such that $\theta^\varepsilon(\cdot)=g^\varepsilon(W(\cdot))$ a.s..

Now the aim is to prove the large deviation principle for θ^ε. For this purpose we need to impose the following assumptions on G.

Hypothesis 4.5.8 Assume G satisfies the following conditions:

i) $G(\theta)$ is a bounded operator from K to H^δ with $\delta>r:=(2-2\alpha)\vee\alpha$ such that

$$\|G(\theta)\|_{L(K,H^{1-\alpha})}\le C(\|\theta\|_{H^1}+1),\qquad \|G(\theta)\|_{L(K,H^\delta)}\le C(\|\theta\|_{H^{1+\alpha}}+1). \tag{4.5.7}$$

ii)

$$\|G(u)-G(v)\|_{L_2(K,H)}\le C\|u-v\|_{H^\alpha}.$$

Remark (4.5.7) can also be changed to

$$\|G(\theta)\|_{L(K,H^\delta)} \le C(\|\theta\|_{H^{\delta+\alpha}} + 1).$$

Let θ_v be the solution of

$$d\theta_v(t) + A\theta_v(t)dt + u_v(t)\cdot\nabla\theta_v(t)dt = G(\theta_v)v(t)dt \tag{4.5.8}$$

with $\theta_v(0) = \theta_0$ and $v \in L^2([0,T],K)$. By Hypothesis 4.5.5 and 4.5.8, we obtain

$$\|G(\theta)v\|_{L^p} \le C|v|_0(\|\theta\|_{L^p} + 1),$$

$$\|G(\theta)v\|_{H^{1-\alpha}} \le C|v|_0(\|\theta\|_{H^1} + 1),$$
$$|\Lambda^{-1/2}(G(\theta_1) - G(\theta_2))v| \le C|v|_0|\Lambda^{-1/2}(\theta_1 - \theta_2)|.$$

By [Re95, Theorems 3.5, 3.7], we know that there exists a unique solution $\theta_v \in L^\infty([0,T],H^1) \cap L^2([0,T],H^{1+\alpha}) \cap C([0,T],H^{-\beta})$ for (4.5.8).

Define $g^0 : C([0,T],H_1) \to L^\infty([0,T],H) \cap L^2([0,T],H^\alpha) \cap C([0,T],H^{-\beta})$ by

$$g^0(h) = \begin{cases} \theta_v, \text{ if } h = \int_0^\cdot v(s)ds \text{ for some } v \in L^2([0,T],K), \\ 0, \text{ otherwise.} \end{cases}$$

The following result shows that g^ε satisfies Hypothesis 4.5.3 so that Theorem 4.5.4 is applicable to establish the large deviation principle for θ^ε.

Theorem 4.5.9 Suppose Hypothesis 4.5.5 and 4.5.8 hold, then $\{\theta^\varepsilon\}$ satisfies the Laplace principle (hence large deviation principle) on

$$L^\infty([0,T],H) \cap L^2([0,T],H^\alpha) \cap C([0,T],H^{-\beta})$$

with a good rate function given by (4.5.1).

Proof To prove the theorem, it suffices to verify the two conditions in Hypothesis 4.5.3.

[Step 1] First we show that the set $K_M = \{g^0(\int_0^\cdot v(s)ds) : v \in S_M\}$ is a compact subset of $L^\infty([0,T],H) \cap L^2([0,T],H^\alpha) \cap C([0,T],H^{-\beta})$. Let $\{\theta_n\}$ be a sequence in K_M where θ_n corresponds to the solution of (4.5.8) with $v_n \in S_M$ in place of v. By the weak compactness of S_M, there exists a subsequence of $\{v_n\}$ which converges to a limit v weakly in $L^2([0,T],K)$. Let $w_n = \theta_n - \theta_v$. It suffices to show that $w_n \to 0$ in $L^\infty([0,T],H) \cap L^2([0,T],H^\alpha) \cap C([0,T],H^{-\beta})$ as $n \to \infty$.

By [Re95] we have

$$\langle u_n \cdot \nabla w_n, w_n\rangle = 0.$$

By Lemma 4.1.2 and (4.1.1) we get

$$\begin{aligned}|\langle (u_n - u_v)\cdot \nabla\theta_v, w_n\rangle| \leq |\Lambda\theta_v||w_n|^2_{L^4} \leq C|\Lambda\theta_v||w_n|^{2(1-\frac{1}{2\alpha})}|\Lambda^\alpha w_n|^{\frac{1}{\alpha}} \\ \leq \frac{\kappa}{2}|\Lambda^\alpha w_n|^2 + C|\Lambda\theta_v|^{2\alpha/(2\alpha-1)}|w_n|^2.\end{aligned} \tag{4.5.9}$$

Thus

$$\begin{aligned}|w_n(t)|^2 + 2\kappa\int_0^t |\Lambda^\alpha w_n|^2 ds =& 2\int_0^t -\langle u_n\cdot\nabla\theta_n, w_n\rangle + \langle u_v\cdot\nabla\theta_v, w_n\rangle ds \\ &+\int_0^t \langle G(\theta_n)v_n(s) - G(\theta_v)v(s), w_n(s)\rangle ds \\ =& -2\int_0^t \langle (u_n - u_v)\cdot\nabla\theta_v, w_n\rangle ds \\ &+\int_0^t \langle (G(\theta_n) - G(\theta_v))v_n(s), w_n(s)\rangle ds \\ &+\int_0^t \langle G(\theta_v)(v_n(s) - v(s)), w_n(s)\rangle ds \\ \leq& \int_0^t \kappa|\Lambda^\alpha w_n|^2 + C(|\Lambda\theta_v|^{2\alpha/(2\alpha-1)} + |v_n|_0^2)|w_n|^2 \\ &+\langle G(\theta_v)(v_n(s) - v(s)), w_n(s)\rangle ds.\end{aligned}$$

Define

$$h_n(t) = \int_0^t G(\theta_v)(v_n(s) - v(s))ds.$$

Since $H^\delta \subset H^r$ is compact and $v_n \to v$ weakly in $L^2([0,T];K)$, by (4.5.7), it is easy to show that $h^n \to 0$ in $C([0,T],H^r)$(cf. [Li09, Lemma 3.2]) by using the Arzèla-Ascoli theorem(more precisely, this convergence may only hold for a subsequence, but it is enough for our use and we denote the convergent subsequence still by h_n). Also we have

$$\begin{aligned}&\int_0^t \langle G(\theta_v)(v_n(s) - v(s)), w_n(s)\rangle ds = \langle w_n(t), h_n(t)\rangle - \int_0^t \langle w_n'(s), h_n(s)\rangle ds \\ =&\langle w_n(t), h_n(t)\rangle + \int_0^t \langle Aw_n(s) + u_n\cdot\nabla\theta_n - u_v\cdot\nabla\theta_v, h_n(s)\rangle ds \\ &-\int_0^t \langle G(\theta_{v_n})v_n(s) - G(\theta_v)v(s)), h_n(s)\rangle ds \\ =:&I_1 + I_2 + I_3\end{aligned} \tag{4.5.10}$$

Note that

$$I_1 \le \varepsilon |w_n(t)|^2 + C|h_n(t)|^2;$$

$$I_3 \le C \sup_{s\in[0,T]} \|h_n(s)\|_{H^s}.$$

For $\varphi \in H^{2-2\alpha}$, by Lemma 4.1.1 and (4.1.1), we obtain

$$|\langle u_n\cdot\nabla\theta_n - u_v\cdot\nabla\theta_v, \varphi\rangle| \le C|\Lambda^{2\alpha-1}(u_n\theta_n - u_v\theta_v)||\Lambda^{2-2\alpha}\varphi| \le (|\Lambda^\alpha\theta_n|^2 + |\Lambda^\alpha\theta_v|^2)|\Lambda^{2-2\alpha}\varphi|,$$

hence

$$\|u_n\cdot\nabla\theta_n - u_v\cdot\nabla\theta_v\|_{H^{-(2-2\alpha)}} \le |\Lambda^\alpha\theta_n|^2 + |\Lambda^\alpha\theta_v|^2.$$

Therefore,

$$\begin{aligned} I_2 \le & \int_0^t (\|Aw_n(s)\|_{H^{-\alpha}} + \|u_n\cdot\nabla\theta_n - u_v\cdot\nabla\theta_v\|_{H^{-(2-2\alpha)}})\|h_n(s)\|_{H^r} ds \\ \le & C \sup_{s\in[0,T]} \|h_n(s)\|_{H^r} \int_0^t (\|w_n\|_{H^\alpha} + \|\theta_n\|^2_{H^\alpha} + \|\theta_v\|^2_{H^\alpha}) ds \\ \le & C \sup_{s\in[0,T]} \|h_n(s)\|_{H^r}. \end{aligned}$$

By the Gronwall lemma and (4.5.10) we have

$$|w_n(t)|^2 + \frac{\kappa}{2}\int_0^t |\Lambda^\alpha w_n|^2 ds \le C \sup_{s\in[0,T]} \|h_n(s)\|_{H^s} e^{C\int_0^t |\Lambda\theta_v|^{2\alpha/(2\alpha-1)} + |v_n|_0^2 ds}.$$

Since $\theta_v \in L^\infty([0,T], H^1) \cap L^2([0,T], H^{1+\alpha})$, we obtain

$$\sup_{t\in[0,T]} |w_n(t)|^2 + \frac{\kappa}{2}\int_0^T |\Lambda^\alpha w_n|^2 ds \to 0, \qquad n\to\infty.$$

[Step 2] Let v_ε converge to v as S_M-valued random elements in distribution. By the Girsanov Theorem $\theta_{v_\varepsilon} = g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds)$ solves the following equation

$$d\theta_{v_\varepsilon}(t) + A\theta_{v_\varepsilon}(t)dt + u_{v_\varepsilon}(t)\cdot\nabla\theta_{v_\varepsilon}(t)dt = G(\theta_{v_\varepsilon})v_\varepsilon(t)dt + \sqrt{\varepsilon}G(\theta_{v_\varepsilon})dW(t). \quad (4.5.11)$$

Since S_M is Polish space, by the Skorokhod theorem, we can construct processes $(\tilde{v}_\varepsilon, \tilde{v}, \tilde{W}_\varepsilon)$ such that the joint distribution of $(\tilde{v}_\varepsilon, \tilde{W}_\varepsilon)$ is the same as that of (v_ε, W), and the distribution of v coincides with that of $\tilde{v}$, and $\tilde{v}_\varepsilon \to \tilde{v}$ a.s. in the topology.

Set $w_\varepsilon(t) := \theta_{\tilde{v}_\varepsilon} - \theta_{\tilde{v}}$. It suffices to prove that $w_\varepsilon \to 0$ in probability in $L^\infty([0,T], H) \cap L^2([0,T], H^\alpha) \cap C([0,T], H^{-\beta})$.

Let $\theta^n_{\tilde{v}_\varepsilon}$ be the solution of the Galerkin approximations to (4.5.11). Then by

Theorem 4.5.7, we know that $\theta^n_{\tilde{v}_\varepsilon}$ converge in probability to $\theta_{\tilde{v}_\varepsilon}$ as elements in $L^2([0,T],H)$. Also let θ^n_v be the solution of Galerkin approximation to (4.5.8), then $\theta^n_{\tilde{v}} \to \theta_{\tilde{v}}$ $P-a.s.$ as element in $L^2([0,T],H)$.

Set $w^n_\varepsilon(t) := \theta^n_{\tilde{v}_\varepsilon} - \theta^n_{\tilde{v}}$, then Itô's formula and (4.5.9) implies that

$$\begin{aligned}
|w^n_\varepsilon(t)|^2 + 2\kappa\int_0^t |\Lambda^\alpha w^n_\varepsilon|^2 ds =& 2\int_0^t -\langle u^n_{v_\varepsilon}\cdot\nabla\theta^n_{v_\varepsilon}, w^n_\varepsilon\rangle + \langle u^n_v\cdot\nabla\theta^n_v, w^n_\varepsilon\rangle ds\\
&+\int_0^t \langle P_nG(\theta^n_{v_\varepsilon}(s))v_\varepsilon(s) - P_nG(\theta^n_v(s))v(s), w^n_\varepsilon(s)\rangle ds\\
&+\sqrt{\varepsilon}\int_0^t \langle w^n_\varepsilon, P_nG(\theta^n_{v_\varepsilon})dW\rangle + \frac{\varepsilon}{2}\int_0^t \|P_nG(\theta^n_{v_\varepsilon})\|^2_{L_2(K,H)}ds\\
=&-2\int_0^t \langle u_{w^n_\varepsilon}\cdot\nabla\theta^n_v, w^n_\varepsilon\rangle ds\\
&+\int_0^t \langle (G(\theta^n_{v_\varepsilon}(s)) - G(\theta^n_v(s)))v_\varepsilon(s), w^n_\varepsilon(s)\rangle ds\\
&+\int_0^t \langle G(\theta^n_v(s))(v_\varepsilon(s)-v(s)), w^n_\varepsilon(s)\rangle ds\\
&+\sqrt{\varepsilon}\int_0^t \langle w^n_\varepsilon, G(\theta^n_{v_\varepsilon})dW\rangle + \frac{\varepsilon}{2}\int_0^t \|P_nG(\theta^n_{v_\varepsilon})\|^2_{L_2(K,H)}ds\\
\le&\int_0^t \kappa|\Lambda^\alpha w^n_\varepsilon|^2 + C(|\Lambda\theta^n_v|^{2\alpha/(2\alpha-1)} + |v_\varepsilon|_0^2)|w^n_\varepsilon|^2 ds\\
&+\int_0^t \langle G(\theta^n_v(s))(v_\varepsilon(s)-v(s)), w^n_\varepsilon(s)\rangle ds\\
&+\sqrt{\varepsilon}\int_0^t \langle w^n_\varepsilon, G(\theta^n_{v_\varepsilon})dW\rangle + \frac{\varepsilon}{2}\int_0^t \|P_nG(\theta^n_{v_\varepsilon})\|^2_{L_2(K,H)}ds.
\end{aligned}$$

Here we write $v_\varepsilon = \tilde{v}_\varepsilon$ for simplicity. Now let $n\to\infty$, we obtain

$$\begin{aligned}
&|w_\varepsilon(t)|^2 + \kappa\int_0^t |\Lambda^\alpha w_\varepsilon|^2 ds \le \int_0^t C(\sup_n |\Lambda\theta^n_v|^{2\alpha/(2\alpha-1)} + |v_\varepsilon|_0^2)|w_\varepsilon|^2 ds\\
&+\int_0^t \langle G(\theta_v(s))(v_\varepsilon(s)-v(s)), w_\varepsilon(s)\rangle ds + \sqrt{\varepsilon}\int_0^t \langle w_\varepsilon, G(\theta_{v_\varepsilon})dW\rangle\\
&+\frac{\varepsilon}{2}\int_0^t \|G(\theta_{v_\varepsilon})\|^2_{L_2(K,H)}ds.
\end{aligned} \tag{4.5.12}$$

Similarly we define

$$h_\varepsilon(t) = \int_0^t G(\theta_v(s))(v_\varepsilon(s)-v(s))ds.$$

Then $h_\varepsilon(t) \to 0$ in $C([0,T],H^r)$. By Itô's formula and the same arguments as

(4.5.10), we have

$$\int_0^t \langle G(\theta_v(s))(v_\varepsilon(s)-v(s)), w_\varepsilon(s)\rangle ds \leq \varepsilon|w_\varepsilon(t)|^2 + C(1+\int_0^t |\Lambda^\alpha \theta_{v_\varepsilon}|^2 ds) \sup_{s\in[0,T]} \|h_\varepsilon(s)\|_{H^s}$$
$$-\sqrt{\varepsilon}\int_0^t \langle h_\varepsilon, G(\theta_{v_\varepsilon})dW\rangle.$$

By the Burkhölder-Davis-Gundy inequality one has

$$\sqrt{\varepsilon}E\sup_{t\in[0,T]}|\int_0^t \langle w_\varepsilon - h_\varepsilon, G(\theta_{v_\varepsilon})dW\rangle| \leq 2\sqrt{\varepsilon}E(\int_0^T |w_\varepsilon-h_\varepsilon|^2 \|G(\theta_{v_\varepsilon})\|^2_{L_2(K,H)}ds)^{1/2}$$
$$\leq C\sqrt{\varepsilon}$$

Combining the above estimates with (4.5.12) and applying the Gronwall lemma we have

$$\sup_{s\in[0,t]}|w_\varepsilon(s)|^2 + \frac{\kappa}{2}\int_0^t |\Lambda^\alpha w_\varepsilon|^2 ds \leq \big(C(1+\int_0^t |\Lambda^\alpha \theta_{v_\varepsilon}|^2 ds)\sup_{s\in[0,T]}\|h_\varepsilon(s)\|_{H^s}$$
$$+\sqrt{\varepsilon}\sup_{t\in[0,T]}|\int_0^t \langle w_\varepsilon - h_\varepsilon, G(\theta_{v_\varepsilon})dW\rangle|$$
$$+\frac{\varepsilon}{2}\int_0^t \|G(\theta_{v_\varepsilon})\|^2_{L_2(K,H)}ds\big)e^{C\int_0^T \sup_n |\Lambda\theta_v^n|^{2\alpha/(2\alpha-1)}+|v_\varepsilon|_0^2 dr}.$$

Define

$$\tau_{N,\varepsilon} := T\wedge \inf\{t : \int_0^t |\Lambda^\alpha \theta_{v_\varepsilon}(s)|^2 ds > N\}.$$

Then we have

$$\sup_{t\in[0,\tau_{N,\varepsilon}]}|w_\varepsilon(t)|^2 + \frac{\kappa}{2}\int_0^{\tau_{N,\varepsilon}} |\Lambda^\alpha w_\varepsilon|^2 ds \to 0$$

in probability as $\varepsilon \to 0$.

Let N be fixed. It is easy to show that for a suitable constant C

$$\liminf_{\varepsilon\to 0} P(\tau_{N,\varepsilon} = T) \geq 1 - \frac{C}{N}.$$

Therefore,

$$\sup_{t\in[0,T]}|w_\varepsilon(t)|^2 + \frac{\kappa}{2}\int_0^T |\Lambda^\alpha w_\varepsilon|^2 ds \to 0$$

in probability as $\varepsilon \to 0$. □

4.6 The small time large deviations result in the subcritical case

In this section, we consider the small time large deviations result. The approach is similar to [XZ09]. We consider again the stochastic quasi-geostrophic equation (4.5.2) and G satisfies Hypothesis 4.5.5, then by Theorem 4.4.4, for $\theta_0 \in L^p$, there exists a pathwise unique strong solution of (4.5.2) in $L^\infty([0,T],H) \cap L^2([0,T],H^\alpha) \cap C([0,T],H^{-\beta})$, for $\beta > 3$.

Moreover, we consider the following conditions:

A.1) There exists a constant L such that $\|G(\theta)\|^2_{L_2(K,H^1)} \leq L(1+\|\theta\|^2_{H^1})$ for all $\theta \in H^1$.

A.2) There exists a constant L_1 such that $\|G(\theta)-G(\theta_1)\|^2_{L_2(K,H^1)} \leq L_1\|\theta-\theta_1\|^2_{H^1}$ for all $\theta, \theta_1 \in H^1$.

Let $\varepsilon > 0$, by the scaling property of the Brownian motion, it is easy to see that $\theta(\varepsilon t)$ coincides in law with the solution of the following equation:

$$d\theta^\varepsilon(t) + \varepsilon A\theta^\varepsilon(t)dt + \varepsilon u^\varepsilon(t)\cdot\nabla\theta^\varepsilon(t)dt = \sqrt{\varepsilon}G(\theta^\varepsilon)dW(t) \tag{4.6.1}$$

with $\theta^\varepsilon(0) = \theta_0$. Let μ^ε be the law of θ^ε on $L^\infty([0,T],H^{-1/2})$.

Remark 4.6.1 Since the solution is not as regular as in for 2D Navier-Stokes equation, we cannot deal with the nonlinear term as in the 2D Navier-Stokes case. So we cannot consider the problem on $L^\infty(0,T,H)$ as Xu and Zhang did in [XZ09]. Here we can only obtain the large deviation principle on $L^\infty([0,T],H^{-1/2})$.

Theorem 4.6.2 Suppose Hypothesis 4.5.3, A.1), A.2) holds, then for $\theta_0 \in L^p$, μ^ε satisfies a large deviation principle on $L^\infty([0,T],H^{-1/2})$ with the rate function I given by

$$I(f) = \inf_{\{v\in L^2([0,T],K):f=\theta_0+\int_0^t G(f(s))v(s)ds)\}} \{\frac{1}{2}\int_0^T |v(s)|_0^2 ds\}. \tag{4.6.2}$$

Proof Let v^ε be the solution of the stochastic equation

$$v^\varepsilon(t) = \theta_0 + \sqrt{\varepsilon}\int_0^t G(v^\varepsilon(s))dW(s),$$

and ν^ε be the law of v^ε on $L^\infty([0,T],H^{-1/2})$. Then by [Li09], we know that ν^ε satisfies a large deviation principle with the rate function I. Our main task is to show that two families of the probability measures μ^ε and ν^ε are exponentially equivalent, that is, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\varepsilon \log P(\sup_{0\le t\le T}|\Lambda^{-1/2}(\theta^\varepsilon(t)-v^\varepsilon(t))|^2>\delta)=-\infty. \tag{4.6.3}$$

Then Theorem 4.6.2 follows from [DZ93, Theorem 4.2.13].

Now we prove the following Lemmas.

Lemma 4.6.3

$$\lim_{M\to\infty}\sup_{0<\varepsilon\le 1}\varepsilon\log P(\sup_{0\le t\le T}\|\theta^\varepsilon(t)\|_{L^p}^p>M)=-\infty.$$

Proof Now consider the approximation $\theta^{\varepsilon,n}$ to θ^ε as in Theorem 4.3.3 and by [Kr10, Lemma 5.1], we have

$$\begin{aligned}
\|\theta(t)\|_{L^p}^p =&\|\theta_0\|_{L^p}^p+\varepsilon\int_0^t[-p\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)(\Lambda^{2\alpha}\theta(s)+u(s)\cdot\nabla\theta(s))dx\\
&+\frac{1}{2}p(p-1)\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|G(\theta(s))(e_j)|^2)dx]ds\\
&+p\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s)\\
&\le\|\theta_0\|_{L^p}^p+\int_0^t\frac{1}{2}p(p-1)\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|G(\theta(s))(e_j)|^2)dxds\\
&+p\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s)\\
&\le\|\theta_0\|_{L^p}^p+\varepsilon\int_0^t(\int_{\mathbb{T}^2}|\theta(s)|^pdx+C\int(\sum_j|G(\theta(s))(e_j)|^2)^{p/2}dx)ds\\
&+p\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s).
\end{aligned}$$

Here we write for simplicity $\theta(t)=\theta^{\varepsilon,n}(t,x)$.

Then by Hypothesis 4.5.5 (ii), we have

$$\begin{aligned}
\sup_{t\in[0,T]}\|\theta(t)\|_{L^p}^p\le&\|\theta_0\|_{L^p}^p+\varepsilon CT+C\varepsilon\int_0^T\sup_{t\in[0,s]}\|\theta(t)\|_{L^p}^pds\\
&+p\sqrt{\varepsilon}\sup_{0\le t\le T}|\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s)|.
\end{aligned}$$

Hence, for $q \geq 2$ we obtain

$$\begin{aligned}(E(\sup_{t\in[0,T]} \|\theta(t)\|_{L^p}^{pq}))^{1/q} \leq & \|\theta_0\|_{L^p}^p + \varepsilon CT + C\varepsilon(E(\int_0^T \sup_{t\in[0,s]} \|\theta(t)\|_{L^p}^p ds)^q)^{1/q} \\ & + p\sqrt{\varepsilon}(E \sup_{0\leq t\leq T} |\int_0^t \int_{\mathbb{T}^2} |\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s)|^q)^{1/q}.\end{aligned}$$

To estimate the stochastic integral term, we will use the following result from [BY82] and [Da76] that there exists a universal constant c such that for any $q \geq 2$ and for any continuous martingale M_t with $M_0 = 0$, one has

$$\|M_t^*\|_{L^q} \leq cq^{1/2}\|\langle M\rangle_t^{1/2}\|_{L^q}, \tag{4.6.4}$$

where $M_t^* = \sup_{0\leq s\leq t} |M_s|$.

Using this result and Minkowski's inequality we have

$$\begin{aligned}& p\sqrt{\varepsilon}(E \sup_{0\leq t\leq T} |\int_0^t \int_{\mathbb{T}^2} |\theta(s)|^{p-2}\theta(s)G(\theta(s))dxdW(s)|^q)^{1/q} \\ & \leq pc\sqrt{q\varepsilon}(E(\int_0^T (\int_{\mathbb{T}^2} |\theta(s)|^{p-1}(\sum_j |G(\theta(s))(e_j)|^2)^{1/2}dx)^2 ds)^{q/2})^{1/q} \\ & \leq pc\sqrt{q\varepsilon}(E(\sup_{s\in[0,T]} \|\theta(s)\|_{L^p}^{p-1}(\int_0^T (\int_{\mathbb{T}^2} (\sum_j |G(\theta(s))(e_j)|^2)^{p/2}dx)^{2/p} ds)^{1/2})^q)^{1/q} \\ & \leq pc\sqrt{q\varepsilon}(E(\sup_{s\in[0,T]} \|\theta(s)\|_{L^p}^{p-1}(\int_0^T (\int_{\mathbb{T}^2} (\sum_j |G(\theta(s))(e_j)|^2)^{p/2}dx) ds)^{1/p})^q)^{1/q} \\ & \leq \frac{1}{2}(E \sup_{s\in[0,T]} \|\theta(s)\|_{L^p}^{pq})^{1/q} + c(p)(q\varepsilon)^{p/2}(E(\int_0^T (\int_{\mathbb{T}^2} (\sum_j |G(\theta(s))(e_j)|^2)^{p/2}dx) ds)^q)^{1/q} \\ & \leq \frac{1}{2}(E \sup_{s\in[0,T]} \|\theta(s)\|_{L^p}^{pq})^{1/q} + c(p)(q\varepsilon)^{p/2}(\int_0^T 1 + (E\|\theta(s)\|_{L^p}^{pq})^{1/q} ds).\end{aligned}$$

Thus

$$\begin{aligned}(E(\sup_{t\in[0,T]} \|\theta(t)\|_{L^p}^{pq}))^{1/q} \leq & 2\|\theta_0\|_{L^p}^p + \varepsilon CT + C\varepsilon \int_0^T (E \sup_{t\in[0,s]} \|\theta(t)\|_{L^p}^{pq})^{1/q} ds \\ & + c(p)(q\varepsilon)^{p/2}(\int_0^T 1 + (E\|\theta(s)\|_{L^p}^{pq})^{1/q} ds).\end{aligned}$$

Applying Gronwall's lemma we obtain

$$(E(\sup_{t\in[0,T]} \|\theta(t)\|_{L^p}^{pq}))^{1/q} \leq (2\|\theta_0\|_{L^p}^p + \varepsilon CT + c(p)(q\varepsilon)^{p/2}T)\exp(C\varepsilon + c(p)(q\varepsilon)^{p/2}).$$

Let $n \to \infty$ we have

$$(E(\sup_{t\in[0,T]} \|\theta^\varepsilon(t)\|_{L^p}^{pq}))^{1/q} \le (2\|\theta_0\|_{L^p}^p + \varepsilon CT + c(p)(q\varepsilon)^{p/2}T)\exp(C\varepsilon + c(p)(q\varepsilon)^{p/2}).$$

Since

$$P(\sup_{0\le t\le T} \|\theta^\varepsilon(t)\|_{L^p}^p > M) \le M^{-q}E(\sup_{t\in[0,T]} \|\theta^\varepsilon(t)\|_{L^p}^{pq}),$$

let $q = 2/\varepsilon$ we get

$$\begin{aligned}\varepsilon \log P(\sup_{0\le t\le T} \|\theta^\varepsilon(t)\|_{L^p}^p > M) &\le -2\log M + 2\log(E(\sup_{t\in[0,T]} \|\theta^\varepsilon(t)\|_{L^p}^{pq}))^{1/q}\\ \le &-2\log M + 2\log(2\|\theta_0\|_{L^p}^p + \varepsilon CT + CT) + 2C\varepsilon + 2C,\end{aligned}$$

hence the proof is complete. □

Since H^1 is dense in H, there exists a sequence $\theta_0^n \subset H^1$ such that $\lim_n |\theta_0^n - \theta_0| = 0$. Let θ_n^ε be the solution of (4.6.2) with the initial value θ_0^n. From the proof of Lemma 4.6.3, it follows that

$$\lim_{M\to\infty} \sup_n \sup_{0<\varepsilon\le 1} \varepsilon \log P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t)\|_{L^p}^p > M) = -\infty. \tag{4.6.5}$$

Let v_n^ε be the solution of (4.6.3) with the initial value θ_0^n. We have the following result whose proof is very similar to (but simpler than) Lemma 4.6.3.

Lemma 4.6.4 For any $n \in \mathbb{Z}^+$,

$$\lim_{M\to\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P(\sup_{0\le t\le T} \|v_n^\varepsilon(t)\|_{H^1}^2 > M) = -\infty.$$

Lemma 4.6.5 For any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) = -\infty.$$

Proof For $M > 0$, we define the following stopping times with respect to $\mathcal{F}_t^+$,

$$\tau_{\varepsilon,M} = \inf\{t : \|\theta^\varepsilon(t)\|_{L^p}^p > M\}.$$

Clearly,

$$\begin{aligned}&P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{1/2}}^2 > \delta, \sup_{0\le t\le T} \|\theta^\varepsilon(t)\|_{L^p}^p \le M)\\ \le &P(\sup_{0\le t\le T\wedge\tau_{\varepsilon,M}} \|\theta_n^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta).\end{aligned} \tag{4.6.6}$$

Let k be a positive constant and $N=\frac{\alpha}{\alpha-\frac{1}{2}-\frac{1}{p}}$.

Applying Ito's formula to $e^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|\theta^\varepsilon(s)\|_{L^p}^N ds}|\Lambda^{-1/2}(\theta^\varepsilon(t\wedge\tau_{\varepsilon,M})-\theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}))|^2$, we get

$$\begin{aligned}
&e^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|\theta^\varepsilon(s)\|_{L^p}^N ds}|\Lambda^{-1/2}(\theta^\varepsilon(t\wedge\tau_{\varepsilon,M})-\theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}))|^2\\
&+2\varepsilon\kappa\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon(r)\|_{L^p}^N dr}|\Lambda^{\alpha-\frac{1}{2}}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2ds\\
=&|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^2-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{L^P}^N dr}\|\theta^\varepsilon\|_{L^P}^N|\Lambda^{-\frac{1}{2}}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2ds\\
&-2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{L^P}^N dr}\langle u^\varepsilon\cdot\nabla\theta^\varepsilon-u_n^\varepsilon\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(\theta^\varepsilon-\theta_n^\varepsilon)\rangle ds\\
&+2\sqrt{\varepsilon}\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{L^P}^N dr}\langle\Lambda^{-1/2}(\theta^\varepsilon-\theta_n^\varepsilon),\Lambda^{-1/2}(G(\theta^\varepsilon)-G(\theta_n^\varepsilon))dW(s)\rangle\\
&+\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{L^P}^N dr}\|\Lambda^{-1/2}(G(\theta^\varepsilon)-G(\theta_n^\varepsilon))\|_{L_2(K,H)}^2ds.
\end{aligned}$$

Notice that

$$\langle u^\varepsilon\cdot\nabla\theta^\varepsilon-u_n^\varepsilon\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(\theta^\varepsilon-\theta_n^\varepsilon)\rangle=\langle(u_n^\varepsilon-u^\varepsilon)\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\rangle+\langle u^\varepsilon\cdot\nabla(\theta_n^\varepsilon-\theta^\varepsilon),\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\rangle.$$

By [Re95], we have

$$\langle(u_n^\varepsilon-u^\varepsilon)\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\rangle=0, \tag{4.6.7}$$

and

$$\begin{aligned}
|\langle u^\varepsilon\cdot\nabla(\theta_n^\varepsilon-\theta^\varepsilon),\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\rangle|\le&\|u^\varepsilon\|_{L^p}\|\theta_n^\varepsilon-\theta^\varepsilon\|_{L^{p'}}\|\nabla\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\|_{L^{p'}}\\
\le&C\|u^\varepsilon\|_{L^p}\|(\theta_n^\varepsilon-\theta^\varepsilon)\|_{H^{1/p}}\|\nabla\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\|_{H^{1/p}}\\
\le&C\|u^\varepsilon\|_{L^p}\|\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\|_{H^{1+\frac{1}{p}}}^2\\
\le&C\|\theta^\varepsilon\|_{L^p}\|\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\|_{H^{1/2}}^{2/N}\|\Lambda^{-1}(\theta_n^\varepsilon-\theta^\varepsilon)\|_{H^{\frac{1}{2}+\alpha}}^{2(1-\frac{1}{N})}\\
\le&\kappa|\Lambda^{\alpha-\frac{1}{2}}(\theta_n^\varepsilon-\theta^\varepsilon)|^2+C\|\theta^\varepsilon\|_{L^p}^N|\Lambda^{-1/2}(\theta_n^\varepsilon-\theta^\varepsilon)|^2.
\end{aligned} \tag{4.6.8}$$

Here $\frac{1}{p}+\frac{2}{p'}=1$ for $0\le 1/p<\alpha-1/2$, and we use $H^{1/p}\hookrightarrow L^{p'}$. Therefore,

$$\begin{aligned}
&e^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|\theta^\varepsilon(s)\|_{L^p}^N ds}|\Lambda^{-1/2}(\theta^\varepsilon(t\wedge\tau_{\varepsilon,M})-\theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}))|^2\\
&+2\varepsilon\kappa\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon(r)\|_{L^p}^N dr}|\Lambda^{\alpha-\frac{1}{2}}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2ds\\
\le&|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^2-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{L^P}^N dr}\|\theta^\varepsilon\|_{L^P}^N|\Lambda^{-\frac{1}{2}}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2ds
\end{aligned}$$

$$+2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{LP}^N dr}\kappa|\Lambda^{\alpha-\frac{1}{2}}(\theta_n^\varepsilon-\theta^\varepsilon)|^2+C\|\theta^\varepsilon\|_{L^p}^N|\Lambda^{-1/2}(\theta_n^\varepsilon-\theta^\varepsilon)|^2ds$$
$$+2\sqrt{\varepsilon}\int_0^{t\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{LP}^N dr}\langle\Lambda^{-1/2}(\theta^\varepsilon-\theta_n^\varepsilon),\Lambda^{-1/2}(G(\theta^\varepsilon)-G(\theta_n^\varepsilon))dW(s)\rangle$$
$$+C\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon\|_{LP}^N dr}|\Lambda^{-1/2}(\theta^\varepsilon-\theta_n^\varepsilon)|^2ds.$$

Choosing $k>2C$ and using (4.6.4), we have

$$\begin{aligned}&(E[\sup_{0\le s\le t\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon(r)\|_{L^p}^N dr}|\Lambda^{-1/2}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}\\ \le&2|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^4\\ &+C(q\varepsilon+\varepsilon^2)\int_0^t(E[\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon(r)\|_{L^p}^N dr}|\Lambda^{-1/2}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}ds.\end{aligned}$$

Applying Gronwall's lemma, one obtains,

$$\begin{aligned}&(E[\sup_{0\le s\le T\wedge\tau_{\varepsilon,M}} e^{-k\varepsilon\int_0^s\|\theta^\varepsilon(r)\|_{L^p}^N dr}|\Lambda^{-1/2}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}\\ \le&2|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^4e^{C(q\varepsilon+\varepsilon^2)}.\end{aligned}$$

Hence,

$$\begin{aligned}&(E[\sup_{0\le s\le T\wedge\tau_{\varepsilon,M}}|\Lambda^{-1/2}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}\\ \le&2e^{2kM^{N/p}T}|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^4e^{C(q\varepsilon+\varepsilon^2)}.\end{aligned}$$

Fix M, and take $q=2/\varepsilon$ to get

$$\begin{aligned}&\sup_{0<\varepsilon\le1}\varepsilon\log P(\sup_{0\le t\le T}\|\theta_n^\varepsilon(t)-\theta^\varepsilon(t)\|_{H^{-1/2}}>\delta)\\ \le&\sup_{0<\varepsilon\le1}\varepsilon\log\frac{E[\sup_{0\le s\le T\wedge\tau_{\varepsilon,M}}|\Lambda^{-1/2}(\theta^\varepsilon(s)-\theta_n^\varepsilon(s))|^{2q}]}{\delta^q}\\ \le&2kM^{N/p}T+\log2|\Lambda^{-\frac{1}{2}}(\theta_0-\theta_0^n)|^4-2\log\delta+C\to-\infty,\ \text{as } n\to\infty.\end{aligned}\tag{4.6.9}$$

By Lemma 4.6.3, for any $R>0$, there exists a constant M such that for any $\varepsilon\in(0,1]$, the following inequality holds

$$P(\sup_{0\le t\le T}\|\theta^\varepsilon(t)\|_{L^p}^p>M)\le e^{-R/\varepsilon}.\tag{4.6.10}$$

For such M, by (4.6.6), and (4.6.9), there exists a constant N_0 such that for any $n\ge N_0$,

$$\sup_{0<\varepsilon\le1}\varepsilon\log P(\sup_{0\le t\le T}\|\theta_n^\varepsilon(t)-\theta^\varepsilon(t)\|_{H^{1/2}}>\delta,\ \sup_{0\le t\le T}\|\theta^\varepsilon(t)\|_{L^p}^p)\le-R.\tag{4.6.11}$$

Putting (4.6.10) and (4.6.11) together, one sees that there exists a positive integer N_0 such that for any $n \geq N_0, \varepsilon \in (0,1]$

$$P(\sup_{0\leq t\leq T} \|\theta_n^\varepsilon(t) - \theta^\varepsilon(t)\|^2_{H^{-1/2}} > \delta) \leq 2e^{-R/\varepsilon}.$$

Since R is arbitrary, the conculsion in the lemma follows. □

The next lemma can be proved similarly as Lemma 4.6.5.

Lemma 4.6.6 For any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P(\sup_{0\leq t\leq T} \|v_n^\varepsilon(t) - v^\varepsilon(t)\|^2_{H^{-1/2}} > \delta) = -\infty.$$

Lemma 4.6.7 For any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq T} |\Lambda^{-1/2}(\theta_n^\varepsilon(t) - v_n^\varepsilon(t))|^2 > \delta) = -\infty.$$

Proof For $M > 0$, we define the following stopping times:

$$\tau^n_{\varepsilon,M} = \inf\{t : \|v_n^\varepsilon(t)\|^2_{H^1} > M\}.$$

Then we have

$$\begin{aligned} &P(\sup_{0\leq t\leq T} \|\theta_n^\varepsilon(t) - v_n^\varepsilon(t)\|^2_{H^{-1/2}} > \delta, \sup_{0\leq t\leq T} \|v^\varepsilon(t)\|^2_{H^1} \leq M) \\ \leq &P(\sup_{0\leq t\leq T\wedge\tau^n_{\varepsilon,M}} \|\theta_n^\varepsilon(t) - v_n^\varepsilon(t)\|^2_{H^{-1/2}} > \delta). \end{aligned} \tag{4.6.12}$$

Applying Ito's formula to $|\Lambda^{-1/2}(v_n^\varepsilon(t\wedge\tau^n_{\varepsilon,M}) - \theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}))|^2$, we get

$$\begin{aligned} &|\Lambda^{-1/2}(v_n^\varepsilon(t\wedge\tau^n_{\varepsilon,M}) - \theta_n^\varepsilon(t\wedge\tau^n_{\varepsilon,M}))|^2 + 2\varepsilon\kappa\int_0^{t\wedge\tau^n_{\varepsilon,M}} |\Lambda^{\alpha-\frac{1}{2}}(v_n^\varepsilon(s) - \theta_n^\varepsilon(s))|^2 ds \\ =&2\varepsilon\int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle Av_n^\varepsilon(s), \Lambda^{-1}(v_n^\varepsilon(s) - \theta_n^\varepsilon(s))\rangle ds + 2\varepsilon\int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle u_n^\varepsilon\cdot\nabla\theta_n^\varepsilon, \Lambda^{-1}(v_n^\varepsilon - \theta_n^\varepsilon)\rangle ds \\ &+2\sqrt{\varepsilon}\int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle\Lambda^{-1/2}(v_n^\varepsilon - \theta_n^\varepsilon), \Lambda^{-1/2}(G(v_n^\varepsilon) - G(\theta_n^\varepsilon))dW(s)\rangle \\ &+\varepsilon\int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\Lambda^{-1/2}(G(v_n^\varepsilon) - G(\theta_n^\varepsilon))\|^2_{L_2(K,H)} ds. \end{aligned}$$

Notice that by the similar argument as (4.6.7) and (4.6.8), we have

$$\begin{aligned}|\langle u_n^\varepsilon\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(v_n^\varepsilon-\theta_n^\varepsilon)\rangle| =&|\langle(u_n^\varepsilon-u_{v_n}^\varepsilon)\cdot\nabla\theta_n^\varepsilon,\Lambda^{-1}(\theta_n^\varepsilon-v_n^\varepsilon)\rangle\\&+\langle u_{v_n}^\varepsilon\cdot\nabla(\theta_n^\varepsilon-v_n^\varepsilon),\Lambda^{-1}(\theta_n^\varepsilon-v_n^\varepsilon)\rangle\\&+\langle u_{v_n}^\varepsilon\cdot\nabla v_n^\varepsilon,\Lambda^{-1}(\theta_n^\varepsilon-v_n^\varepsilon)\rangle|\\ \le&\frac{\kappa}{2}|\Lambda^{\alpha-\frac12}(\theta_n^\varepsilon-v_n^\varepsilon)|^2\\&+C\|v_n^\varepsilon\|_{L^p}^N|\Lambda^{-1/2}(\theta_n^\varepsilon-v_n^\varepsilon)|^2+C\|v_n^\varepsilon\|_{L^4}^4.\end{aligned}$$

Thus

$$\begin{aligned}&|\Lambda^{-1/2}(v_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n)-\theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n))|^2+2\varepsilon\kappa\int_0^{t\wedge\tau_{\varepsilon,M}^n}|\Lambda^{\alpha-\frac12}(v_n^\varepsilon(s)-\theta_n^\varepsilon(s))|^2ds\\ \le&2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\frac{\kappa}{2}|\Lambda^{\alpha-\frac12}(v_n^\varepsilon(s)-\theta_n^\varepsilon(s))|^2+C|\Lambda^{\alpha-\frac12}v_n^\varepsilon|^2ds\\ &+2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\frac{\kappa}{2}|\Lambda^{\alpha-\frac12}(\theta_n^\varepsilon-v_n^\varepsilon)|^2+C\|v_n^\varepsilon\|_{L^p}^N|\Lambda^{-1/2}(\theta_n^\varepsilon-v_n^\varepsilon)|^2+C\|v_n^\varepsilon\|_{L^4}^4ds\\ &+2\sqrt{\varepsilon}\int_0^{t\wedge\tau_{\varepsilon,M}^n}\langle\Lambda^{-1/2}(v_n^\varepsilon-\theta_n^\varepsilon),\Lambda^{-1/2}(G(v_n^\varepsilon)-G(\theta_n^\varepsilon))dW(s)\rangle\\ &+\varepsilon C\int_0^{t\wedge\tau_{\varepsilon,M}^n}|\Lambda^{-1/2}(v_n^\varepsilon-\theta_n^\varepsilon)|^2ds.\end{aligned}$$

Using Gronwall's lemma, we obtain

$$\begin{aligned}&|\Lambda^{-1/2}(v_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n)-\theta_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n))|^2\\ \le&(2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}C|\Lambda^{\alpha-\frac12}v_n^\varepsilon|^2+C\|v_n^\varepsilon\|_{L^4}^4ds+\end{aligned}$$

$$2\sqrt{\varepsilon}|\int_0^{t\wedge\tau_{\varepsilon,M}^n}\langle\Lambda^{-1/2}(v_n^\varepsilon-\theta_n^\varepsilon),\Lambda^{-1/2}(G(v_n^\varepsilon)-G(\theta_n^\varepsilon))dW(s)\rangle|)e^{\varepsilon C\int_0^{t\wedge\tau_{\varepsilon,M}^n}\|v_n^\varepsilon\|_{L^p}^Nds+Ct\varepsilon}.$$

Using (4.6.4), we have

$$\begin{aligned}&(E[\sup_{0\le s\le t\wedge\tau_{\varepsilon,M}^n}|\Lambda^{-1/2}(v_n^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}\\ \le&Ce^{\varepsilon CM^{N/2}+Ct\varepsilon}(\varepsilon M+\varepsilon M^2+q\varepsilon\int_0^t(E[\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}}|\Lambda^{-1/2}(v_n^\varepsilon(r)-\theta_n^\varepsilon(r))|^2]^q)^{2/q}ds).\end{aligned}$$

Applying Gronwall's lemma, one obtains,

$$\begin{aligned}&(E[\sup_{0\le s\le T\wedge\tau_{\varepsilon,M}}|\Lambda^{-1/2}(v_n^\varepsilon(s)-\theta_n^\varepsilon(s))|^2]^q)^{2/q}\\ \le&Ce^{\varepsilon CM^{N/2}+Ct\varepsilon}(\varepsilon M+\varepsilon M^2)\exp CqT\varepsilon e^{\varepsilon CM^{N/2}+Ct\varepsilon}.\end{aligned}$$

Fix M, and take $q = 2/\varepsilon$ we have

$$
\begin{aligned}
&\varepsilon \log P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t) - v_n^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta)\\
\le&\varepsilon \log \frac{E[\sup_{0\le s\le T\wedge\tau_{\varepsilon,M}} |\Lambda^{-1/2}(v_n^\varepsilon(s) - \theta_n^\varepsilon(s))|^{2q}]}{\delta^q}\\
\le&\log C(\varepsilon M + \varepsilon M^2) - 2\log\delta + Ce^{\varepsilon CM^{N/2}+Ct\varepsilon} + \varepsilon CM^{N/2} + Ct\varepsilon \to -\infty, \text{ as } \varepsilon \to 0.
\end{aligned} \tag{4.6.13}
$$

Thus, there exists a ε_0 such that for any ε satisfying $0 < \varepsilon \le \varepsilon_0$,

$$
P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t) - v_n^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta, \sup_{0\le t\le T} \|v^\varepsilon(t)\|_{H^1}^2 \le M) \le e^{-R/\varepsilon}. \tag{4.6.14}
$$

By Lemma 4.6.4 and (4.6.14), one sees that there exists ε_0 such that for any ε satisfying $0 < \varepsilon \le \varepsilon_0$,

$$
P(\sup_{0\le t\le T} \|\theta_n^\varepsilon(t) - v_n^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) \le 2e^{-R/\varepsilon}.
$$

Since R is arbitrary, the conculsion in the lemma holds. $\square$

By Lemmas 4.6.5, 4.6.6, we have for any $R > 0$, there exists N_0 satisfying

$$
P(\sup_{0\le t\le T} \|\theta_{N_0}^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) \le e^{-R/\varepsilon}. \text{ for any } \varepsilon \in (0,1],
$$

and

$$
P(\sup_{0\le t\le T} \|v_{N_0}^\varepsilon(t) - v^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) \le e^{-R/\varepsilon}. \text{ for any } \varepsilon \in (0,1].
$$

By Lemma 4.6.7, for such N_0, there exists ε_0 such that for any ε satisfying $0 < \varepsilon \le \varepsilon_0$,

$$
P(\sup_{0\le t\le T} \|\theta_{N_0}^\varepsilon(t) - v_{N_0}^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) \le e^{-R/\varepsilon}.
$$

Thus, for any ε satisfying $0 < \varepsilon \le \varepsilon_0$,

$$
P(\sup_{0\le t\le T} \|\theta^\varepsilon(t) - v^\varepsilon(t)\|_{H^{-1/2}}^2 > \delta) \le 3e^{-R/\varepsilon}.
$$

Since R is arbitrary, we conclude that

$$
\lim_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\le t\le T} |\Lambda^{-1/2}(\theta^\varepsilon(t) - v^\varepsilon(t))|^2 > \delta) = -\infty.
$$

$\square$

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