Franz-Peter Liebel

Computation of Causal Networks

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Preface

If a system is both large and complex it may be difficult to find out how it works. An expert system is a technique to solve a multifacetted problem. Probably the most important aspect of a medical expert system is its application to the development of new possibilities with the specific objective of optimizing diagnostic purposes. This book is dedicated to this topic. Its value is that it goes much further and deeper than a mere statement of principle, however valuable that might be. Indeed it analyses in detail how to construct a medical expert system with the help of causal networks.

This book will therefore hold interest for graduate students, researchers, and practicians in the fields of applied mathematics, computer sciences, medicine, and expert systems theory.

February 28, 2002

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Foreword

There are only a few medical expert systems which finally could reach practical application. As an example, there exists a quite useful expert system to decide between the alternatives "thrombosis" and "haemorrhage" in case of stroke.
In contrary to the restricted general applicability of existing expert systems, there are inspiring chances opened by this book. The causal network, forming the basis of operation, is not limited - shortening or expansion are allowed, as well as the connection to further causal networks. Furthermore, once the expert system is realized, the user is in a position to add or remove symptoms in order to practise a real dialogue between man and machine to decide the question "what, if".
It is a tempting idea to employ such a universal and powerful expert system which promises to reduce the daily strain caused by clumsy diagnoses. An early realization, even if only of modest size as a first step before gradual extension, would be more than desirable.

January, 2002

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COMPUTATION OF CAUSAL NETWORKS

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Dr. F.-P. Liebel

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Summary

We introduce a procedure to compute the probabilities of all events which could be causes of a given set of symptoms. We establish a causal network holding events with unknown probabilities as well as deterministic events. For any event with unknown probability a defining equation is constructed. The set of defining equations then forms a system of n non-linear equations, where n is the number of unknown probabilities.

These non-linear equations contain conditional probabilities of sometimes great complexity, making it unavoidable to realize a decomposition into factors. This is accomplished with the help of certain assumptions, which, in the medical field, impose no serious restrictions. The factorization finally yields conditional probabilities, conditioned on just a single event. Consequently, all sampling to obtain numerical values of such probabilities can be carried out easily.

Introduction

The expert system presented here deals with the task of computing the probabilities of all events which could be the cause of a given set of symptoms, concerning biological or technical systems of high structural complexity.
Regarding a living organism or a technical system of a certain size, we may define a number of measuring points which represent the current state of operation at those points. Each state of operation is normally kept within a prescribed interval by means of the systemic feedback control system. If there is a state of operation constantly situated outside of the associated interval, it represents a disturbance which resists the feedback control system. A disturbance is caused by influences inside or outside the system, and as an irregularity it may be itself the cause of subsequent irregularities.
If there are irregular states of operation within the system, we can use this information to compute the probabilities of the preceding events, i.e. the probabilities of the assumed causes of such irregularities.

As an introduction to the problems that we are concerned with, we discuss the situation by using a simple example. Let $\{F_1,\dots, F_5\}$ be a set of events representing effects in a given causal network. We are trying to determine the probability of all net nodes that possibly are the causes of events in $\{F_1,\dots, F_5\}$.
A solution is found immediately: If the symbol H denotes one of the assumed causes of $F_1,\dots, F_5$, we write $p(H \mid F_1 \dots F_5)$ and obtain the probability of the common appearance of H and the logic product $(F_1 \dots F_5)$.

However, it is hopeless to attempt to obtain the numerical values of large conditional probabilities by means of statistical methods. In addition, there is no chance to find an easy and immediate way to decompose $p(H \mid F_1 \dots F_5)$ into factors, since $F_1,\dots, F_5$ have common causes and are consequently (by definition) <u>not</u> <u>independent</u>.

Moreover, we have to consider the mutual influence of two causes, say H and K_1, if there exists a common effect of the two causes. In that case the ap-probability of H (to be defined below) depends on the ap-probability of K_1 and, on the other hand, the ap-probability of K_1 depends on the ap-probability of H.

Note:

We distinguish between the following probabilities concerning the arbitrarily chosen event H:

p(H)	a-priori probability of H.
$p(H \mid F_1 \ldots F_5)$	a-posteriori probability of H, conditioned upon a selection of events.
$p(H \mid H')$	ap-probability of H, representing the special case of an a-posteriori probability of H which is *conditioned upon all events within the considered causal network.* This probability will be defined precisely in Section 1.

Furthermore, it is evident that it is not enough to consider only the effects in order to compute the probability of net nodes within a causal arrangement of some size. We keep sight of the objective to compute the probabilities of causes like H and K_1 by considering all influences and every information attainable, i.e. by making use of all knowledge within reach (see Section 1, p.18).

It can be seen that it is indeed not easy to compute the probabilities of events which are assumed to be causes of events in $\{F_1, \ldots, F_5\}$. There are a number of questions:

a) Is it possible to decompose a conditional probability like $p(H \mid F_1 \ldots F_5)$ into factors? Which assumptions are needed in order to obtain a decomposition?
b) Which method is suitable for developing a calculation procedure that considers the mutual influence between the ap-probabilities of events? Is this procedure still applicable even when the causal network contains numerous events whose ap-probabilities depend on each other?

c) Which events in the "causal neighbourhood" of an event H have to be considered in order to calculate the ap-probability of H? What has to be done if these events have an unknown probability?

These questions pose difficult problems which certainly cannot be solved easily and immediately. Nevertheless, some of the solutions presented later on will be clear as well as extraordinarily compact. It will turn out that

1. the interpolation formula of DUDA (Eq.3.3) and the L-Theorem (Eq.4.3) are special cases of the General Interpolation Theorem (Eq.4.1),
2. the A→L-Theorem (Eq.7.6) and the A→L-Corollary 1 (Eq.8.1) are remarkable with respect to their compactness,
3. the problem of mutual influence can be tackled by forming a defining equation for every unknown ap-probability and solving the resulting system of non-linear equations.

The solution of the task may be divided into the following four areas:

I.

All knowledge relevant to the problem is transformed into a causal network. With this structure we are able to label the events which have to be taken into account in order to compute an unknown ap-probability.

To carry out this plan we have to introduce new mathematical concepts to standard probability theory, namely:

- The concept of '-events (see Section 1, p. 20), where p(H | H') is used to denote the ap-probability of an arbitrary event H.
- The concept of separated events (see Section 4, p. 53).

II.

The handling of '-events creates the need for interpolation formulas. Therefore we add the following new theorems to standard probability theory,

- the General Interpolation Theorem,
- the L-Theorem, and
- the Linear Interpolation Theorem.

III.

In the course of the computations we are confronted with conditional probabilities of sometimes large size. Therefore, we are in need of procedures which allow to reduce the complexity of these probabilities. This is the main reason to introduce assumptions which finally lead to the desired factorization theorems. To standard probability theory we therefore add

- the A→L-Theorem,
- the A→L-Corollary 1, and
- the factorization theorems which allow the decomposition of complex conditional probabilities into factors.

The development of these theorems is based on two new concepts:

- The concept of establishing events like (A→L), meaning "A creates L" and defining p(A→L | A), as well as
- the concept of self-reliant causes.

IV.

We aim to express every unknown ap-probability within the considered causal network as a function of the remaining unknown ap-probabilities. Thus, we obtain a non-linear system of equations in the unknown ap-probabilities. The problem of mutual influence is thus solved, and the iterative solution of the equation system yields the result.

In order to start the construction of the expert system, we have to assemble a causal network as a model of reality. Using net nodes, causal connections and inhibitors we are able to model the irregular transitions of the system considered.

Looking at already existing diagnostic expert systems, we find that some of these expert systems do not utilize causal networks, e.g. some of the diagnostic expert systems using the Gaussian least squares method. Thus, an important aid to file and represent the collected knowledge is missed, as well as the opportunity to reveal stochastic independence.

As a particular advantage of the causal network to be introduced in Section 1 below, we would like to emphasize the use of modular network pieces and, as a result of that, the possibility of a gradual refinement on extension. A further advantage of the causal network and of the procedure as a whole arises from its universal range of use, since the net and the computing procedure are applicable to different fields of knowledge without the need to change any rules or principles.

The property of having a universal range of use is indispensable with regard to the medical field, since we have to form groups with respect to age, sex, race, climate, genetic disposition and working place, each requiring a separate causal network. However, even the forming of numerous groups and hence the use of numerous causal networks is no problem for the diagnostic expert system we are going to develop, since the mathematical basis and the procedure itself possess a universal validity.

The expert system "Computation of causal networks", and especially the usage of it as a diagnostic expert system in the medical field, is the subject of two European patent requests.
The two requests for grant of a European patent carry the patent application numbers 91104386.7-2201 / 0504457 and 99105884.3-2201 / 1026616, European Patent Office.
(The request 99105884.3 uses the priority of a preceding request which has the patent application number 99102275.7.)

1. L-Net

In case of working with biological systems the task is to establish a list of all events which are assumed to be causes of a given set of symptoms, and then to sort these causes according to their probability.
In other words: for a given set of effects we first have to determine all hypothetical causes belonging to each single effect. Then we complete the causal arrangement of causes and effects by adding events which influence the probabilities of the registered hypothetical causes. Finally, we compute the ap-probabilities of all '-events by using the methods of probability theory.

It is appropriate to insert the effects, having ($p = 0$) or ($p = 1$), and the associated causes, having the unknown probability ($0 < p < 1$), into a causal network and to supplement this causal network by events being dependent of the inserted causes. The specific causal network which has been set up in this way to represent the pathological processes which might occur in plant, animal or man carries the name L-Net.
(In case of working on technical systems there are no changes concerning the causal network and the computing procedure.)

Definition (Net nodes)

D1.1 *The net nodes represent events which are equivalent to the irregular physiological states of the system under consideration.*

D1.2 *An irregular physiological state is a measurement or a parameter whose value is situated outside a prescribed interval.*

D1.3 *In particular, regular physiological states are not nodes of the L-Net.*

D1.4 *A node of the L-Net carrying a negation represents the nonexistence of this event.*

D1.5 *Each net node has an arbitrary number of causal arrows leading towards the node or leading away from it.*

Definition (Causal connections)

D1.6 Let L be an arbitrary event and A an arbitrarily chosen element within the set of L-generating causes. Then the causal process "A creates L" is interpreted as an event as well. The event "A creates L" has the graph

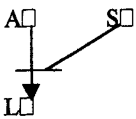
and the mathematical symbol A→L.

D1.7 A causal connection A→L from A to L as shown in the diagram above, means:
An event [A creates L] exists having the probability $0 < p < 1$.
A→L is called a transition.

Definition (Inhibitors)

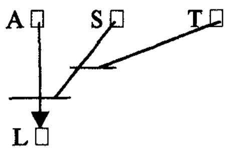
D1.8 An arbitrary event S which can directly influence the transition from the event A to the event L is called an inhibitor of A→L.
Directly means that there is no other known operating state between S and the inhibiting mechanism.

D1.9 The inhibitor S of A→L is shown in the following diagram:



D1.10 Provided A exists, the transition A→L has probability $p = 1$ if there are no events which influence A→L.
(The inhibiting event does not necessarily have to be known. If a diagram shows a causal arrow, e.g. A→L, then in general there are inhibitors acting upon that transition – otherwise we would not have to distinguish between A and L.)

D1.11 If S denotes an inhibitor of an arbitrary event A→L, it is allowed that the inhibiting mechanism caused by S is inhibited itself by another event T. This is shown by the following diagram:



S inhibits A→L, the event T counteracts this inhibition. Both types of events, i.e. the events with an immediate inhibiting effect on A→L as well as the events with an inhibiting effect on inhibiting mechanisms are collectively called "events with inhibiting activity", or short "inhibitors".

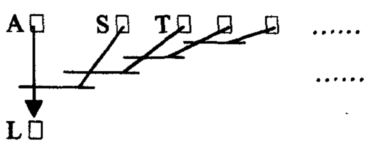
D1.12 Inhibitors represent <u>*regular*</u> <u>*or*</u> <u>*irregular*</u> *physiological states of the system in question.*

(Note: The events at the end of transitions are called net nodes; they represent irregular physiological states.

Inhibitors are not connected by means of transitions and they are not net nodes; they represent regular or irregular physiological states.)

<u>Comment on the definition of inhibitors</u>

a) It is possible to continue the inhibition of inhibiting mechanisms on and on. Therefore, it is allowed to extend the diagram of D1.11 as follows:



However, it became apparent that it is sufficient to name any influence upon a transition an inhibiting influence, no matter whether it concerns a single inhibition or the inhibition of an inhibition.

b) The A→L-Corollary 1 (Eq.8.1) does not need further information about inhibiting mechanisms than the summary product D (synonymous: logic product, product of events, compound of events), a product which is composed of A→L-influencing events, and which is brought into the computation as a whole.

c) All events exerting influences upon transitions are treated as inhibitors, even if they exert in fact an "accelerating" influence. Inhibitors of all types are considered to be present, regardless whether the inhibiting mode of operation is known or not. This method models the fact that an existing event A will transfer its probability $p = 1$ to the event A→L if there are no A→L-influencing events which decrease the probability to $p < 1$ in a joint action.

d) It turned out that accelerators are not essential. If we look at an arbitrary transition, we find that an increase in its probability, equivalent to the effect of an accelerator, can be regarded as a result of the inhibition of known or unknown inhibiting mechanisms. This organizational step helps to improve the simplicity of the causal network.

The assembly of the L-Net starts by stating hypothesis H. Hypothesis H represents an event which is assumed to be the most probable cause of the elements within a given set of symptoms. We start with H and keep in mind that it is our intention to list all potential causes and sort them according to their probability.

The search for the most probable cause at the beginning of the project is achieved by using a selection criterion (see Section 7, Eq.7.6, "A→L-Theorem"). We choose a leading symptom L , i.e. the most important symptom, and obtain H by using the following criterion.

The equation

$$\frac{p(L \mid H) - p(L \mid \overline{H})}{p(\overline{L} \mid \overline{H})} \geq \frac{p(L \mid A) - p(L \mid \overline{A})}{p(\overline{L} \mid \overline{A})}$$

shall hold for all events A, where A denotes an arbitrary element within the set of events which - in addition to H - are possible causes of L.

Assembling the network, we place the starting hypothesis H at the level of hypotheses and the events caused by H at the level of effects.
In a graphical representation the following symbols are used:

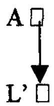
A □ event A having known probability p = 1.

$\overline{A}$ □ event A having known probability p = 0.

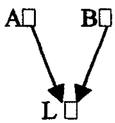
A'□ event A having known or unknown probability 0 < p < 1.

A is a cause of L; A and L both have known probability p = 1.

A is a cause of L; A has known probability p = 1,
L has known probability p = 0.



A is a cause of L; L has known or unknown probability 0 < p < 1.



A and B are causes of L. The statement is:
(A creates L) $\vee$ (B creates L).

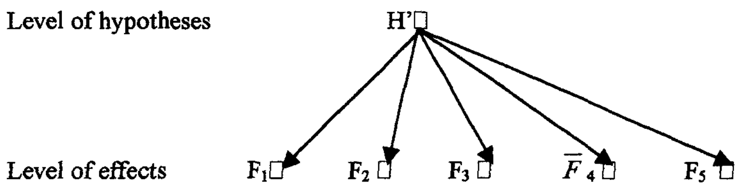
<u>Note:</u>

- If a diagram does not show any information about the stochastic dependence of two events, we are allowed to assume stochastic independence until further findings show the opposite.

- If a diagram does not show any information about the inhibition of an arbitrary transition A→L, we are <u>not</u> allowed to assume the absence of inhibitors. On the contrary: The transition A→L is influenced in any case by inhibiting mechanisms.

  But we may assume that the inhibitors of A→L act solely upon A→L, and that *these inhibitors do not depend on the inhibitors that belong to an arbitrary* transition B→L. If this statement does not hold, i.e. if there is information about the stochastic dependence of two inhibitors acting on different causal creation processes, we have to complete the diagram correspondingly.

In order to demonstrate the characteristics of the (so far unfamiliar) ap-probabilities, we establish a simple example:



Ill.1.1:
Example of a causal network to determine the ap-probability of H by means of the effects F_1 through F_5.

It is allowed to remove arbitrary elements from the set of effects, i.e. we are not forced to consider every event caused by H. Therefore, the events at the level of effects shown in Ill.1.1 are chosen arbitrarily, and it is permitted that there exist more consequences of H which we just ignored.

Consequently, we always determine an ap-probability by means of exactly those events which have been chosen to be part of the causal network.

If W(H) denotes a product of events (synonymous: compound of events, logic product) which contains the elements situated at the level of effects in Ill.1.1, we form the expression p(H | W(H)) and obtain the probability of H conditioned upon the events in W(H). We get:

$$p(H \mid W(H)) = p(H \mid F_1 F_2 F_3 \overline{F_4} F_5) \qquad (1.1)$$

Since p(H | W(H)) considers every event within the causal network in question, Eq.1.1 provides the ap-probability of H, i.e. the probability of the existence of the event H, determined by utilizing all knowledge and all information extractable from the causal network. Of course Eq.1.1 has practical use only if we are able to determine the numerical value of $p(H \mid F_1 F_2 F_3 \overline{F_4} F_5)$.

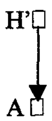
The set-up of such numerical values creates difficulties. We need a representative sample selected from a population which has the properties F_1 to F_5. Within this sample we count all events which possess the additional property H. However, it is a problem to extract representative samples if the population under considera-tion is characterized by more than two properties.

But if the condition upon the desired probability consists of only one event, and if, in addition, this probability is of the form *p(effect | cause)*, it is comparatively easy to obtain the appropriate sample. The development of probabilities condi-tioned upon a single event is an aim of the following sections.

The set-up of Eq.1.1 is based on the principle that an event Y is added to W(H) if H and Y are stochastically dependent, conditioned upon the rest of the elements in W(H). Since Ill.1.1 will be supplemented with further elements that belong to the L-Net, the possibility to get new candidates into W(H) is opened up. The neces-sary decision of whether H and an arbitrary event Y are stochastically dependent, conditioned upon the rest of the elements in W(H), can be made by using the knowledge contained in the causal network. For this purpose, it is necessary to discuss the equivalence of stochastic dependencies and causal structures following next.

Structure 1.1

A is caused by H



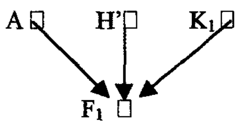
Structure 1.2

A is a cause of H

(Structure 1.1 $\vee$ 1.2 is valid) $\Rightarrow$ (H and A are stochastically dependent).

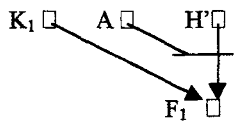
Structure 1.3

A and H are rivaling causes of F_1



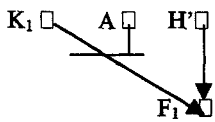
Structure 1.4

A inhibits $H \to F_1$



Structure 1.5

A inhibits $K_1 \to F_1$

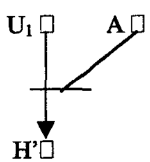


(Structure 1.3 $\vee$ 1.4 $\vee$ 1.5 is valid) $\Rightarrow$ (H and A are dependent, conditioned upon F_1)

$\Rightarrow$ $[p(H \mid F_1 A) \neq p(H \mid F_1 \bar{A})]$.

Structure 1.6

A inhibits $U_1 \to H$



(Structure 1.6 is valid) $\Rightarrow$ (H and A are stochastically dependent, conditioned upon U_1)

$\Rightarrow$ $[p(H \mid U_1 A) \neq p(H \mid U_1 \bar{A})]$.

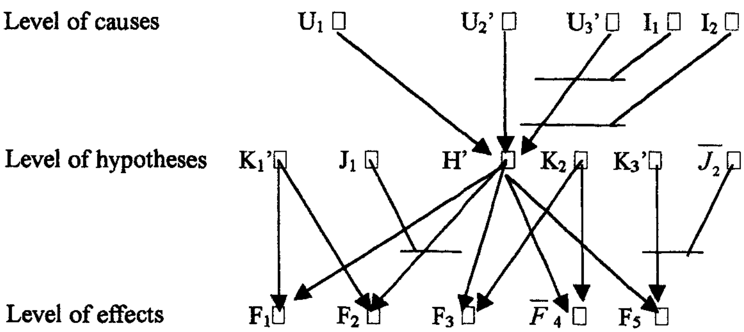
The nodes A and H shown in Structures 1.1 through 1.6 are dependent, and therefore A is a candidate for W(H). An extension of W(H) does not require that we have to accept the complete list of all events which influence the ap-probability of H. Type and number of elements used as members of W(H) can be chosen arbitrarily.

Later on we will have to restrict the unconstrained choice of W(H). In order to achieve conditional independence of the events caused by H, e.g. the conditional independence of $F_1,\ldots, F_5$, we will have to insert into W(H) every single cause of events contained in $\{F_1,\ldots, F_5\}$. But this constraint is alleviated by the fact that we may still reduce the number of effects and thus the number of causes belonging to it.

At present we intend to construct a causal network containing all events which possess, due to Structures 1.1 through 1.6, a conditioned or an unconditioned dependence of H. So we complete the L-Net in Ill.1.1 by adding

- causes of H,
- causes of events which are descendants of H,
- inhibitors of transitions directed to H,
- inhibitors of transitions directed to descendants of H.

Installing these events, we get an exemplary continuation of Ill.1.1 as follows:



Ill.1.2:
Example of a causal network to determine the ap-probability of H using events which possess a conditioned or an unconditioned dependence on H.

The following symbols are defined by using the node H as a reference:

URS(H) set of direct causes of H (German: Ursachen[1]).

FOL(H) set of direct effects of H (German: Folgen[1]).

DIFF(H) set of net nodes distinctive from H, being direct causes of elements belonging to FOL(H).
(DIFF(H) includes the events which have to be considered in the course of differential diagnostic procedures.)

INH(Z) set of net nodes which inhibit transitions leading to an arbitrary net node Z.

WERT(H) set of all net nodes influencing the ap-probability of H (screening neighbourhood of H; German: Wertungsumgebung[1]).

Assembling of L-Nets

The first net node inserted is H', whose ap-probability is assumed to be near 1. In order to construct causal connections to H, we have the events in URS(H), FOL(H), DIFF(H), INH(H) and $\bigcup_Z INH(Z)$*, Z* $\in$ *FOL(H) at our disposal.*

Any newly inserted event Y' that has unknown ap-probability may receive causal connections from arbitrary elements contained in URS(Y), FOL(Y), DIFF(Y), INH(Y) and $\bigcup_Z INH(Z)$*, Z* $\in$ *FOL(Y).*

Assembling terminates as a consequence of the assumption that additionally inserted events - with growing distances to H - will have only little influence on the ap-probability of H. The '-events which possess only '-events within their screening neighbourhood, i.e. within the sets mentioned above, will be removed from the causal network.

Assembling of L-Nets is subject to further constraints which result from assumptions to be introduced in Section 6.

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[1] *Three German words are used: Ursache ↔ cause,*
Folge ↔ effect,
Wertungsumgebung ↔ screening neighbourhood.

Definition (Screening neighbourhood)

Consider an L-Net and a net node H' having an unknown probability of existence. We define the screening neighbourhood WERT(H) as follows:

$$WERT(H) := URS(H) \cup FOL(H) \cup DIFF(H) \cup INH(H) \cup \bigcup_{Z \in FOL(H)} INH(Z). \quad (1.2)$$

(The reason for defining WERT(H) as shown in Eq.1.2 will be given in a comment at the end of this Section and in the course of Section 4.)

We transfer the elements belonging to the set WERT(H) into a product (synonymous: logic product, compound of events) which we denote by W(H). The elements of W(H) are negated, non-negated or apostrophized depending on their probabilities ($p = 0$) or ($p = 1$) or ($0 < p < 1$), respectively. Using W(H) we define:

Definition ('-event)

An arbitrarily chosen net node H' having known or unknown probability $0 < p < 1$ is called '-event (pronounced: prime event).
Let W(H) be a logic product containing all events of WERT(H), and let $p(H \mid W(H))$ be the ap-probability of H. We define:
$p(H \mid H') := p(H \mid W(H))$; consequently $H' := W(H)$.

a) *$H' := W(H)$ is the meaning of H' if the considered probability shows the event H in front of the conditioning line.*
b) *In all other cases, in particular if H' is contained in probabilities not having H in front of the conditioning line, the symbol H' has the following meaning: H' denotes an event having an ap-probability $0 < p(H \mid H') < 1$ which might be known or which can be computed by means of the L-Net computation system.*

   In particular, H' cannot be replaced by W(H).

Comment on the definition of '-events

Ill.1.2 may serve as an example to demonstrate the meaning of '-events. If W(H) contains all elements of WERT(H), we obtain:

$$p(H \mid H') := p(H \mid W(H))$$

$$= p(H \mid U_1U_2'U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2}) \qquad (1.3)$$

In order to point out an inadmissible action, we transform Eq.1.3 by inserting $W(U_2)$ in the place of U_2'. Because of $WERT(U_2) := \{H', U_1, U_3', I_1, I_2\}$ we have $W(U_2) = (H'\ U_1\ U_3'\ I_1\ I_2)$ and get the following transformation:

$$p(H \mid H')$$

$$=_{(\text{identical with Eq.1.3})}\ p(H \mid U_1U_2'U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2})$$

$$=_{(U_2'\text{ replaced})}\ p(H \mid U_1(H'U_1U_3'I_1I_2)U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2})$$

$$=_{(\text{duplicates removed})}\ p(H \mid U_1H'U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2})$$

$$=_{(H'\text{ replaced})}$$

$$p(H \mid U_1(U_1U_2'U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2})U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2})$$

$$=_{(\text{duplicates removed})}\ p(H \mid (U_1U_2'U_3'F_1F_2F_3\overline{F_4}F_5K_1'K_2K_3'I_1I_2J_1\overline{J_2}))$$

$$= p(H \mid H'). \qquad (1.4)$$

Eq.1.4 shows immediately that the replacement of U_2' by $W(U_2)$, which again contains H', does not make sense. The reason for U_2' being a member of W(H) and for H' being a member of $W(U_2)$ is given by the mutual influence of the ap-probabilities of H and U_2 on each other.

The problem of mutual influence, unsolved so far, will find a solution resulting from the development of the L-Net computation system.

<u>Comment on the definition of WERT(H)</u>

The L-Net represents reality. Events outside of the net do not exist, with the exception that we permit the presence of unknown inhibitors acting upon the causal generating processes. We are allowed to insert or to remove as many events as we want. But once the net is finished and the extraction of the desired equations has been started, we have to consider every single element within the L-Net. Yet we are authorized to ignore an element and to make no use of it in W(H) if this element is *separated* from H'.

In anticipation of Section 4 we outline the meaning of the separation property. If A $\notin$ W(H) denotes an arbitrary event, and if A is not connected to H' except via paths across elements in W(H), then A is called separated from H' by W(H). For this, the elements in W(H) may be negated, non-negated or apostrophized. The definition of separated events and a discussion of the consequences is given in Section 4.

Conclusion

Consider an arbitrary L-Net and an arbitrarily chosen net node H' with unknown ap-probability. If W(H) contains all elements which belong to WERT(H), we get for any event A $\notin$ W(H):

[H' is separated from A by W(H)] $\Rightarrow$ [p(H | W(H)) = p(H | W(H) A)]. (1.5)

(Please note: We do not demand p(H | W(H)) = p(H | W(H) A) = p(H | W(H) $\overline{A}$), i.e. the independence of A and H, conditioned on W(H).)

At the beginning of this section we established the rule to take an event A into W(H) if A and H show stochastic dependence in case of condition W(H). Clearly, we are in a position to remove A from W(H) again whenever A and H turn out to be stochastically independent if conditioned upon W(H). But even if only the "weak" separation property holds, i.e. if A is separated from H' in case of condition W(H), it is justified to remove A from W(H).

(The separation property is "weak" compared to the "strong" independence property, since independent events are always separated, but separated events are not independent in general. For further information see Section 4.)

2. *Interpolation functions for a single '-event*

A look at Eq.1.3 reveals that we need to consider the influence of the event U_2' upon $p(H \mid H')$ and, simultaneously, the influence of H' upon $p(U_2 \mid U_2')$. The problem of such mutual influences has not found a satisfying solution so far. We will eliminate the difficulty by installing all unknown existence probabilities in a system of equations.

In order to execute this plan we need a tool to express an unknown ap-probability as a function of the remaining unknown ap-probabilities. We demonstrate the facts by using the following example:

With *the exception of events which influence the transition* A→H, *no other* elements exist in addition to A' and H'. Both have an unknown ap-probability. We intend to express $p(H \mid H')$ as a function of $p(A \mid A')$.

We obtain:

$$p(H \mid H') = p(H \mid W(H)) = p(H \mid A'). \tag{2.1}$$

$$p(A \mid A') = p(A \mid W(A)) = p(A \mid H'). \tag{2.2}$$

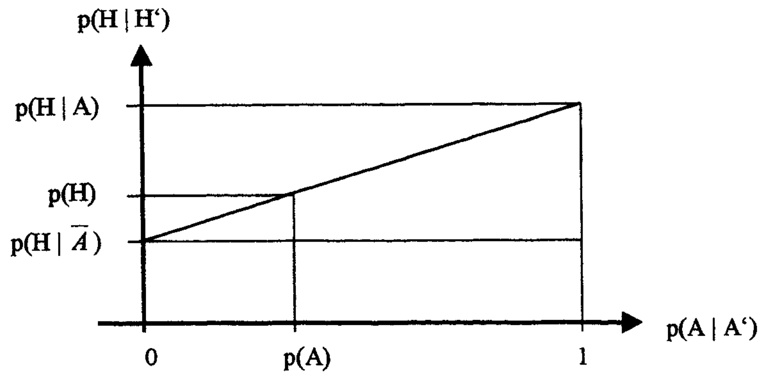
We impose the following requirements:

If $p(A \mid A') = 1$ then $p(H \mid H') = p(H \mid A)$.

If $p(A \mid A') = 0$ then $p(H \mid H') = p(H \mid \overline{A})$.

If $p(A \mid A') = p(A)$ then $p(H \mid H') = p(H)$.

In order to satisfy the three requirements, we interpolate between the extreme values $p(H \mid A)$ and $p(H \mid \overline{A})$ according to the value of $p(A \mid A')$. This leads to the following illustration:



Ill.2.1:

Interpolation between the extreme values p(H | H') = p(H | A) and p(H | H') = p(H | $\bar{A}$) according to the value of p(A | A'), supposing p(H | A) > p(H | $\bar{A}$).

Ill.2.1 yields:

$$p(H \mid H') = p(H \mid \bar{A}) + [p(H \mid A) - p(H \mid \bar{A})]p(A \mid A')$$
$$= p(H \mid A)p(A \mid A') + p(H \mid \bar{A})[1 - p(A \mid A')]$$
$$= p(H \mid A)p(A \mid A') + p(H \mid \bar{A})p(\bar{A} \mid A'). \qquad (2.3)$$

Please note:

Ill.2.1 allows the choice of either p(A |A') = p(A) or p(H | H') = p(H) in order to avoid inconsistent probabilities as a consequence of overspecification. However, this fact poses no problem, since we compute all unknown ap-probabilities simultaneously through one system of equations.

Because of p(H | H') = p(H | A'), Eq.2.3 yields an interpolation of p(H | A'):

$$p(H \mid A') = p(H \mid A)p(A \mid A') + p(H \mid \bar{A})p(\bar{A} \mid A'). \qquad (2.4)$$

__Warning:__ Despite the equal sign, the functions we established are interpolation formulas which in general do not reach equality.

Now we are in a position to specify the relationship between the unknown variables p(H | H') and p(A | A') as follows:

$$p(H \mid H') = p(H \mid A')$$
$$= p(H \mid A)p(A \mid A') + p(H \mid \overline{A})p(\overline{A} \mid A'). \qquad (2.5)$$
$$p(A \mid A') = p(A \mid H')$$
$$= p(A \mid H)p(H \mid H') + p(A \mid \overline{H})p(\overline{H} \mid H'). \qquad (2.6)$$

Eq.2.5 and Eq.2.6 represent the desired equation system in the variables p(H | H') and p(A | A').

The clear set-up of this system of equations instantly reveals that p(H | H') = p(H) and p(A | A') = p(A) represent a solution.

If we solve the system of equations by using the insertion method, we will obtain no other result. We demonstrate this computation in full length as an example.

Notation:

$y := p(H \mid H'),$ $\quad a := p(H \mid A),$

$x := p(A \mid A'),$ $\quad b := p(H \mid \overline{A}),$

$c := p(A \mid H),$

$d := p(A \mid \overline{H}).$

The use of this notation transforms Eqs.2.5 and 2.6 into the customary form of an equation system in the variables x and y:

$$y = ax + b(1-x). \qquad (2.5a)$$
$$x = cy + d(1-y). \qquad (2.6a)$$

The insertion method yields:

$$y = \frac{(a-b)d+b}{1-(a-b)(c-d)}. \qquad (2.5b)$$

$$x = \frac{(c-d)b+d}{1-(a-b)(c-d)}. \qquad (2.6b)$$

Now we have to show that $\frac{(a-b)d+b}{1-(a-b)(c-d)}$ = p(H) holds. Elementary computations yield:

$$(a-b) = p(H \mid A) - p(H \mid \overline{A}) = \frac{p(H)p(\overline{A}) - p(H\overline{A})}{p(A)p(\overline{A})}.$$

$$(c-d) = p(A \mid H) - p(A \mid \overline{H}) = \frac{p(A)p(\overline{H}) - p(A\overline{H})}{p(H)p(\overline{H})}.$$

$$(a-b)\,d+b = \frac{p(A \mid \overline{H})p(H) + p(H \mid \overline{A})p(A) - p(H \mid \overline{A})p(A \mid \overline{H})}{p(A)}.$$

$$1-(a-b)\,(c-d) = \frac{p(A \mid \overline{H})p(H) + p(H \mid \overline{A})p(A) - p(H \mid \overline{A})p(A \mid \overline{H})}{p(A)p(H)}.$$

$\Rightarrow$

$$\frac{(a-b)d+b}{1-(a-b)(c-d)} = p(H).$$

Analogously, we obtain $\frac{(c-d)b+d}{1-(a-b)(c-d)} = p(A)$.

The representation above also proves the following conjecture:

If a causal connection between events H' and A' is known, and there is no further information concerning existence or nonexistence of events within the screening neighborhood of H and A, then the ap-probabilities of H and A are not more exact than the a-priori-probabilities.

3. Properties of interpolation functions

We will rewrite the interpolation function Eq.2.4 and replace condition (A') by $(E_1..E_hE'_{h+1})$ in order to standardize the statements concerning interpolation functions. The symbol H denotes the event whose ap-probability needs to be determined. The symbols in $\{E_1,..., E_h\}$ indicate events with probabilities ($p = 0$) or ($p = 1$), i.e. negated or non-negated events, respectively. In order to simplify the equations we do not specify negated events. The apostrophized symbol E'_{h+1} represents an event with unknown ap-probability $0 < p(E_{h+1} \mid E'_{h+1}) < 1$.

Moreover, $V_{E'_{h+1}}$ denotes a product of events [synonymous: compound of events, logic product], obtained from the product $(E_1..E_hE'_{h+1})$ by removing E'_{h+1} - the event that acts as an index in $V_{E'_{h+1}}$ - as well as other arbitrarily chosen elements. The empty product $V_{E'_{h+1}} = \varnothing$ is permitted.

We demand that the interpolation functions which will be applied to the probability $p(H \mid E_1..E_hE'_{h+1})$ satisfy the following three interpolation points:

1st interpolation point:

$p(H \mid E_1..E_hE'_{h+1}) = p(H \mid E_1..E_h\overline{E}_{h+1})$ in case of $p(E_{h+1} \mid E'_{h+1}) = 0$.

2nd interpolation point:

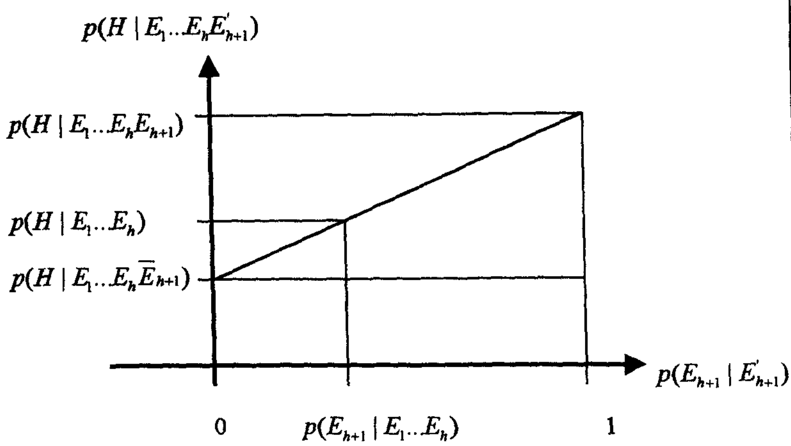
$p(H \mid E_1..E_hE'_{h+1}) = p(H \mid E_1..E_hE_{h+1})$ in case of $p(E_{h+1} \mid E'_{h+1}) = 1$.

3rd interpolation point:

$p(H \mid E_1..E_hE'_{h+1}) = p(H \mid E_1..E_h)$ in case of $p(E_{h+1} \mid E'_{h+1}) = p(E_{h+1} \mid V_{E'_{h+1}})$

with arbitrarily chosen $V_{E'_{h+1}}$.

Thus, Ill.2.1 is transformed into the following diagram:



Ill.3.1:

Graph belonging to the linear interpolation of $p(H \mid E_1..E_h E'_{h+1})$

in case $p(H \mid E_1...E_h E_{h+1}) > p(H \mid E_1...E_h \overline{E}_{h+1})$.

<u>Lemma</u>

<u>Interpolation function for a single '-event</u>

Concerning $p(H \mid E_1..E_k E'_{h+1})$, *the following notation is used:*

- E_1, ... , E_h *are negated or non-negated events (arbitrarily chosen but fixed),*
- E'_{h+1} *has probability* $0 < p(E_{h+1} \mid E'_{h+1}) < 1$,
- $V_{E'_{h+1}}$ *is an arbitrarily chosen part of the product* $(E_1..E_h)$, $V_{E'_{h+1}} = \varnothing$ *is permitted.*

Then the following equations satisfy the three interpolation points specified above:

$$p(H \mid E_1..E_hE'_{h+1})$$

$$= \frac{p(HE_1..E_hE_{h+1})\frac{p(E_{h+1} \mid E'_{h+1})}{p(E_{h+1} \mid V_{E'_{h+1}})} + p(HE_1..E_h\overline{E}_{h+1})\frac{p(\overline{E}_{h+1} \mid E'_{h+1})}{p(\overline{E}_{h+1} \mid V_{E'_{h+1}})}}{p(E_1..E_hE_{h+1})\frac{p(E_{h+1} \mid E'_{h+1})}{p(E_{h+1} \mid V_{E'_{h+1}})} + p(E_1..E_h\overline{E}_{h+1})\frac{p(\overline{E}_{h+1} \mid E'_{h+1})}{p(\overline{E}_{h+1} \mid V_{E'_{h+1}})}} \qquad (3.1)$$

Proof: Elementary.

Lemma

Linear interpolation function for a single '-event

Choosing the maximal $V_{E'_{h+1}}$ *, i.e.* $V_{E'_{h+1}} := (E_1..E_h)$, *Eq.3.1 yields a linear interpolation function which reads as follows:*

$$p(H \mid E_1..E_hE'_{h+1})$$

$$= p(H \mid E_1..E_hE_{h+1})p(E_{h+1} \mid E'_{h+1}) + p(H \mid E_1..E_h\overline{E}_{h+1})p(\overline{E}_{h+1} \mid E'_{h+1}). \qquad (3.2)$$

(Eq.3.2 is equivalent to the graph shown in Ill.3.1.)

Remarks:

In case of $\{E_1,..., E_h\} = \varnothing$ we consequently have $V_{E'_{h+1}} = \varnothing$, and Eq.3.1 becomes the interpolation formula of DUDA. [(See: DUDA, R.O., P.E. HART, N.J. NILSSON: Subjective Bayesian methods for rule-based inference systems. In: WEBER, B.L., N.J. NILSSON (eds.): Readings in artificial intelligence. Tioga Publ., Palo Alto Calif. (1981) p.192 - 199).]

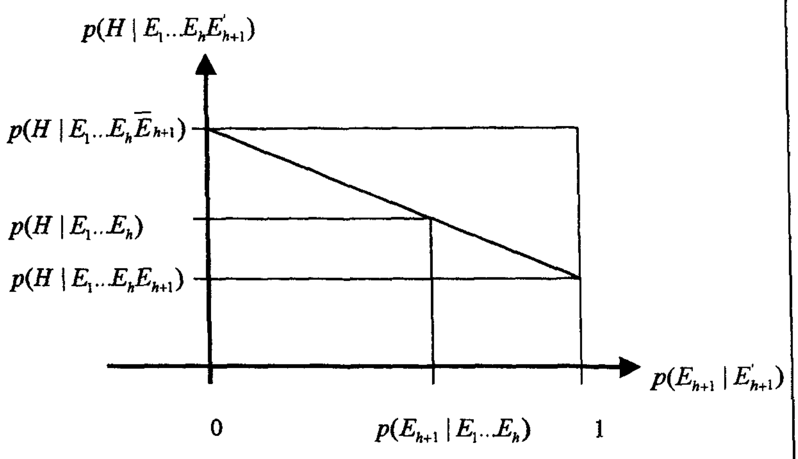
Written in the notation of DUDA, the following statement is given:

$$P(H \mid E') = P(H \mid E)\, P(E \mid E') + P(H \mid \overline{E}\,)\, P(\overline{E} \mid E'). \qquad (3.3)$$

Eq.3.3 is a first step to utilize "the observations E' relevant to E". Ideas going beyond this formula, or discoveries of other authors concerning the use of '-events, did not turn up so far.

In order to establish criteria which determine the choice of the logic product $V_{E'_{h+1}}$, we analyze the properties of the functions contained in Eq.3.1 by means of the graphs belonging to it.

First, we have to add another diagram to Ill.3.1, showing the linear interpolation function in the $p(H \mid E_1 \ldots E_h E_{h+1}) < p(H \mid E_1 \ldots E_h \overline{E}_{h+1})$ case. We obtain:



Ill.3.2:

Graph belonging to the linear interpolation of $p(H \mid E_1 \ldots E_h E'_{h+1})$ in case of $p(H \mid E_1 \ldots E_h E_{h+1}) < p(H \mid E_1 \ldots E_h \overline{E}_{h+1})$.

Looking at Eq.3.1 we find that the choice of the product $V_{E'_{h+1}}$ determines the graph of the corresponding interpolation function. In order to get a clear view of the facts, we vary $V_{E'_{h+1}}$ step by step from $\varnothing$ to the maximum size ($E_1 \ldots E_h$).

This leads to the following assignments:

$$p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1}). \tag{3.4}$$

$$p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1} \mid E_h). \tag{3.5}$$

$$p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1} \mid E_h E_{h-1}). \tag{3.6}$$

$$p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1} \mid E_h E_{h-1} E_{h-2}). \tag{3.7}$$

$$\vdots \qquad\qquad \vdots$$

$$p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1} \mid E_h .. E_1). \tag{3.8}$$

We abbreviate the variables contained in Eq.3.1 by the following symbols:

$p(H \mid E_1..E_h E'_{h+1}) =:$ y, and $p(E_{h+1} \mid E'_{h+1})$ =: x.

Eq.3.1 contains unconditioned probabilities. If we choose the values of $p(E_{h+1} \mid V_{E'_{h+1}})$ according to equations (3.4) through (3.8) and insert them into Eq.3.1, these unconditioned probabilities will be transformed into the corresponding conditional probabilities. We use the following abbreviations:

Choosing Eq.3.4: $a_0 := p(HE_1..E_h \mid E_{h+1})$.

$b_0 := p(HE_1..E_h \mid \overline{E}_{h+1})$.

$c_0 := p(E_1..E_h \mid E_{h+1})$.

$d_0 := p(E_1..E_h \mid \overline{E}_{h+1})$.

Choosing Eq.3.5: $a_1 := p(HE_1..E_{h-1} \mid E_h E_{h+1})$.

$b_1 := p(HE_1..E_{h-1} \mid E_h \overline{E}_{h+1})$.

$c_1 := p(E_1..E_{h-1} \mid E_h E_{h+1})$.

$d_1 := p(E_1..E_{h-1} \mid E_h \overline{E}_{h+1})$.

Choosing Eq.3.6: $a_2 := p(HE_1..E_{h-2} \mid E_{h-1} E_h E_{h+1})$.

$b_2 := p(HE_1..E_{h-2} \mid E_{h-1} E_h \overline{E}_{h+1})$.

$c_2 := p(E_1..E_{h-2} \mid E_{h-1} E_h E_{h+1})$.

$d_2 := p(E_1..E_{h-2} \mid E_{h-1} E_h \overline{E}_{h+1})$.

Choosing Eq.3.7: $a_3 := p(HE_1..E_{h-3} \mid E_{h-2}E_{h-1}E_hE_{h+1})$.

$b_3 := p(HE_1..E_{h-3} \mid E_{h-2}E_{h-1}E_h\overline{E}_{h+1})$.

$c_3 := p(E_1..E_{h-3} \mid E_{h-2}E_{h-1}E_hE_{h+1})$.

$d_3 := p(E_1..E_{h-3} \mid E_{h-2}E_{h-1}E_h\overline{E}_{h+1})$.

⋮ ⋮

Choosing Eq.3.8: $a_h := p(H \mid E_1..E_{h+1})$.

$b_h := p(H \mid E_1..\overline{E}_{h+1})$.

$c_h := 1$.

$d_h := 1$.

Utilizing this notations, we are able to transform Eq.3.1 into Eq.3.9, no matter which $V'_{E_{h+1}}$ we selected, and consequently no matter which indexing of a, b, c, d is currently valid. We obtain:

$$y = \frac{ax + b(1-x)}{cx + d(1-x)} \tag{3.9}$$

$$= \frac{ax + b - bx}{cx + d - dx}. \tag{3.10}$$

We proceed by showing the graphical appearance of the family of curves represented by Eq.3.10. To begin, we differentiate Eq.3.10:

$$y'(x) = \frac{ad - bc}{(cx + d - dx)^2}. \tag{3.11}$$

$$y''(x) = (-2)\frac{(ad - bc)(c - d)}{(cx + d - dx)^3}. \tag{3.12}$$

Using Eq.3.11, we obtain that $y'(x) \neq 0$ for all x, whenever $(ad - bc) \neq 0$. Therefore, y(x) shows no extremum and no point of inflexion within the open interval]0 , 1[. We obtain:

$$\left.\begin{aligned} y'_{(x=0)} &= \frac{ad-bc}{d^2}\,. \\ y'_{(x=1)} &= \frac{ad-bc}{c^2}\,. \end{aligned}\right\rangle \quad \text{These values characterize the gradient at the endpoints.}$$

We need to discuss cases I through IV, i.e. (a d – b c) > 0 and < 0, together with (c – d) > 0 and < 0. The choice of these terms determines the arrangement of abscissa and ordinate values.

The criterion to arrange the abscissa values is the following:

Proposition: (c > d) $\Rightarrow$ $p(E_{h+1} \mid E_1..E_h) > p(E_{h+1} \mid V_{E'_{h+1}})$.

Proof:

Using Eq.3.4 as an example, i.e. choosing $p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1})$, we demonstrate that $c_0 > d_0$ causes $p(E_{h+1} \mid E_1..E_h) > p(E_{h+1})$, which is the arrangement shown in Ill.3.3. Let

$$c_0 > d_0\,.$$

$$\Rightarrow \qquad p(E_1..E_h \mid E_{h+1}) > p(E_1..E_h \mid \overline{E}_{h+1})\,.$$

$$\Rightarrow \qquad \frac{p(E_1..E_h E_{h+1})}{p(E_{h+1})} > \frac{p(E_1..E_h \overline{E}_{h+1})}{p(\overline{E}_{h+1})}\,.$$

$$\Rightarrow \qquad p(E_{h+1} \mid E_1..E_h)p(\overline{E}_{h+1}) > p(\overline{E}_{h+1} \mid E_1..E_h)p(E_{h+1})\,.$$

$$\Rightarrow \quad p(E_{h+1} \mid E_1..E_h) - p(E_{h+1})p(E_{h+1} \mid E_1..E_h) > p(E_{h+1}) - p(E_{h+1})p(E_{h+1} \mid E_1..E_h)$$

$$\Rightarrow \qquad p(E_{h+1} \mid E_1..E_h) > p(E_{h+1})\,. \qquad \square$$

Analogous proofs yield in case of any arbitrarily chosen $V_{E'_{h+1}}$:

$$(c > d) \Rightarrow p(E_{h+1} \mid E_1..E_h) > p(E_{h+1} \mid V_{E'_{h+1}})\,. \tag{3.13}$$

$$(c < d) \Rightarrow p(E_{h+1} \mid E_1..E_h) < p(E_{h+1} \mid V_{E'_{h+1}})\,. \tag{3.14}$$

Ills.3.3 through 3.6 reflect the statements of (3.13) and (3.14) using $V_{E'_{h+1}} := \varnothing$.

The criterion to arrange the ordinate values is the following:

<u>Proposition:</u> (a d - b c) > 0 $\Rightarrow$ $p(H \mid E_1..E_h E_{h+1}) > p(H \mid E_1..E_h \overline{E}_{h+1})$.

We use two examples (1. and 2.) to demonstrate this proposition.

1.

$(a_0 d_0 - b_0 c_0)$

$$= p(HE_1..E_h \mid E_{h+1})p(E_1..E_h \mid \overline{E}_{h+1}) - p(HE_1..E_h \mid \overline{E}_{h+1})p(E_1..E_h \mid E_{h+1})$$

$$= \frac{p(H \mid E_1..E_{h+1})p(E_1..E_{h+1})}{p(E_{h+1})} p(E_1..E_h \mid \overline{E}_{h+1})$$

$$- \frac{p(H \mid E_1..E_h \overline{E}_{h+1})p(E_1..E_h \overline{E}_{h+1})}{p(\overline{E}_{h+1})} p(E_1..E_h \mid E_{h+1})$$

$$= p(E_1..E_h \mid E_{h+1})p(E_1..E_h \mid \overline{E}_{h+1})[p(H \mid E_1..E_h E_{h+1}) - p(H \mid E_1..E_h \overline{E}_{h+1})]$$

$$= c_0 d_0 [p(H \mid E_1..E_h E_{h+1}) - p(H \mid E_1..E_h \overline{E}_{h+1})].$$

Conclusion: If $(a_0 d_0 - b_0 c_0) > 0$ then $p(H \mid E_1..E_h E_{h+1}) > p(H \mid E_1..E_h \overline{E}_{h+1})$. □

2.

$(a_1 d_1 - b_1 c_1)$

$$= p(HE_1..E_{h-1} \mid E_h E_{h+1})p(E_1..E_{h-1} \mid E_h \overline{E}_{h+1})$$

$$- p(HE_1..E_{h-1} \mid E_h \overline{E}_{h+1})p(E_1..E_{h-1} \mid E_h E_{h+1})$$

$$= \frac{p(H \mid E_1..E_{h+1})p(E_1..E_{h+1})}{p(E_h E_{h+1})} p(E_1..E_{h-1} \mid E_h \overline{E}_{h+1})$$

$$- \frac{p(H \mid E_1..E_h \overline{E}_{h+1})p(E_1..E_h \overline{E}_{h+1})}{p(E_h \overline{E}_{h+1})} p(E_1..E_{h-1} \mid E_h E_{h+1})$$

$$= p(E_1..E_{h-1} \mid E_h E_{h+1})p(E_1..E_{h-1} \mid E_h \overline{E}_{h+1})$$

$$\cdot [p(H \mid E_1..E_h E_{h+1}) - p(H \mid E_1..E_h \overline{E}_{h+1})]$$

$$= c_1 d_1 [p(H \mid E_1..E_h E_{h+1}) - p(H \mid E_1..E_h \overline{E}_{h+1})].$$

Conclusion: If $(a_1 d_1 - b_1 c_1) > 0$ then $p(H \mid E_1..E_h E_{h+1}) > p(H \mid E_1..E_h \overline{E}_{h+1})$. □

The two examples (1. and 2. on page 34) yield equation (3.15) which will be of good use:

$$\frac{(a_0 d_0 - b_0 c_0)}{(a_1 d_1 - b_1 c_1)} = \frac{c_0 d_0}{c_1 d_1} \quad . \tag{3.15}$$

Case I:

$(ad - bc) > 0$ and c > d.

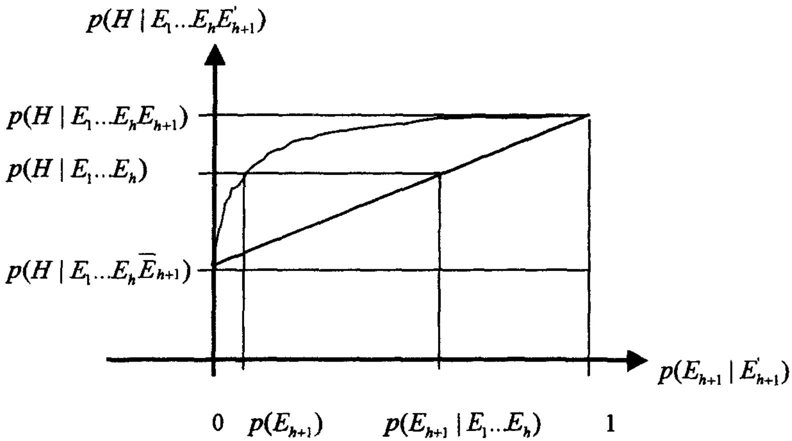
It follows that:

1) $y'_{(x=0)} > 0$ and $y'_{(x=1)} > 0$.

2) $y'_{(x=0)} > y'_{(x=1)}$.

3) $y''(x) < 0$ for all x, since $(c-d)x + d > 0$ for all x.

Consequently, the appropriate curve is situated above the linear interpolation shown in Ill.3.1. (The curved line in the following illustration reflects the assignment $p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1})$, i.e. it represents the simple case of Eq.3.4).



Ill.3.3:

Graph belonging to Eq.3.1 in case of $(ad - bc) > 0$ and c > d (Case I).

<u>Case II:</u>

$(ad - bc) > 0$ and $c < d$.

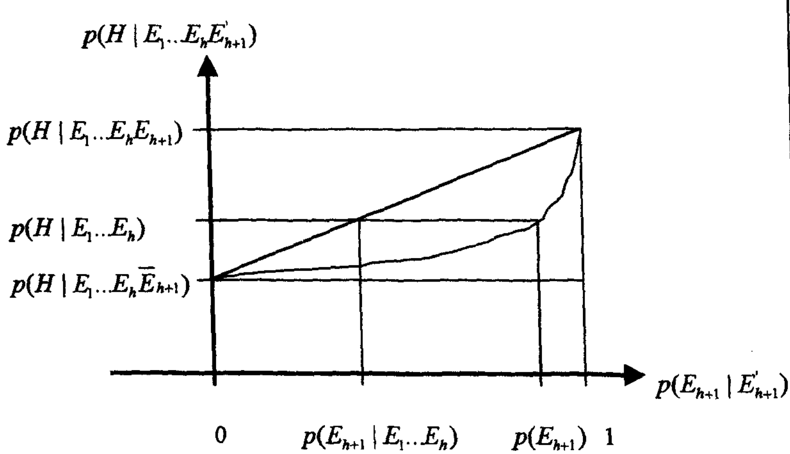
It follows that:

1) $y'_{(x=0)} > 0$ and $y'_{(x=1)} > 0$.

2) $y'_{(x=0)} < y'_{(x=1)}$.

3) $y''(x) > 0$ for all x.

Consequently, the appropriate curve is situated <u>below</u> the linear interpolation shown in <u>Ill.3.1.</u> (The curved line in the following illustration reflects the assignment $p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1})$, i.e. it represents the simple case of Eq.3.4).



Ill.3.4:

Graph belonging to Eq.3.1 in case of $(ad - bc) > 0$ and $c < d$ (Case II).

Case III:

$(ad - bc) < 0$ and c < d.

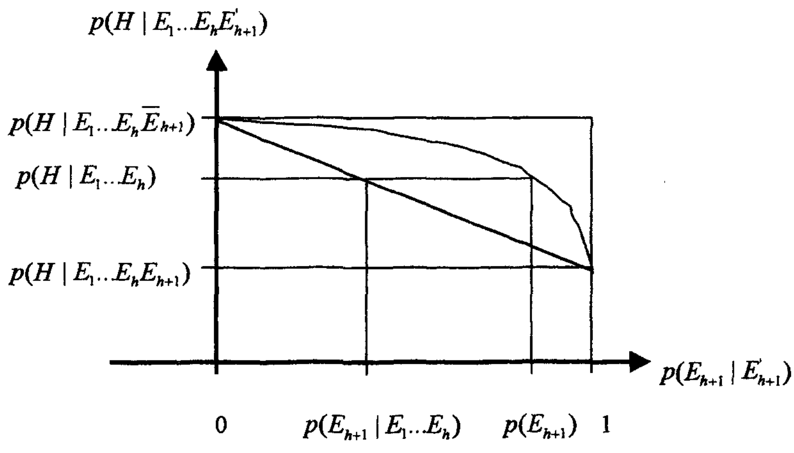
It follows that:

1) $y'_{(x=0)} < 0$ and $y'_{(x=1)} < 0$.

2) $y'_{(x=0)} > y'_{(x=1)}$.

3) $y''(x) < 0$ for all x.

Consequently, the appropriate curve is situated above the linear interpolation shown in Ill.3.2. (The curved line in the following illustration reflects the assignment $p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1})$, i.e. it represents the simple case of Eq.3.4).



Ill.3.5:

Graph belonging to Eq.3.1 in case of $(ad - bc) < 0$ and c < d (Case III).

<u>Case IV:</u>

$(ad-bc)<0$ and c > d.

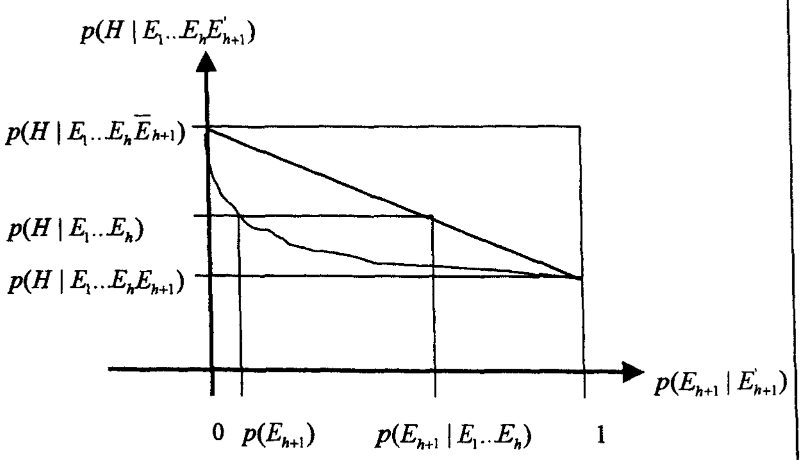
It follows that:

1) $y'_{(x=0)}<0$ and $y'_{(x=1)}<0$.

2) $y'_{(x=0)}<y'_{(x=1)}$.

3) $y''(x)>0$ for all x.

Consequently, the appropriate curve is situated <u>below</u> the linear interpolation shown in <u>Ill.3.2.</u> (The curved line in the following illustration reflects the assignment $p(E_{h+1}\,|\,V_{E'_{h+1}}):=p(E_{h+1})$, i.e. it represents the simple case of Eq.3.4).



Ill.3.6:

Graph belonging to Eq.3.1 in case of $(ad-bc)<0$ and $c>d$ (Case IV).

We still need to determine if two curves, derived from Eq.3.1 by using two different but arbitrarily chosen $V_{E'_{h+1}}$, intersect. For this, we utilize the abbreviations a_0, b_0, c_0, d_0 and a_1, b_1, c_1, d_1, which define two curves of Eq.3.1. We define a difference function D(x) as follows:

$$D(x) = \frac{a_0 x + b_0 - b_0 x}{c_0 x + d_0 - d_0 x} - \frac{a_1 x + b_1 - b_1 x}{c_1 x + d_1 - d_1 x}.$$

Differentiating D(x) and setting D'(x) = 0 yields:

$$D'(x) = \frac{a_0 d_0 - b_0 c_0}{(c_0 x + d_0 - d_0 x)^2} - \frac{a_1 d_1 - b_1 c_1}{(c_1 x + d_1 - d_1 x)^2}.$$

$$\sqrt{\frac{a_0 d_0 - b_0 c_0}{a_1 d_1 - b_1 c_1}} = \frac{(c_0 - d_0)x + d_0}{(c_1 - d_1)x + d_1}. \qquad (3.16)$$

UsingEq.3.15, we obtain:

$$\frac{\sqrt{c_0 d_0}}{\sqrt{c_1 d_1}} = \frac{(c_0 - d_0)x + d_0}{(c_1 - d_1)x + d_1}.$$

$$\Rightarrow \quad \sqrt{c_0 d_0}\,[(c_1 - d_1)x + d_1] = \sqrt{c_1 d_1}\,[(c_0 - d_0)x + d_0].$$

$$\Rightarrow \quad x[(c_1 - d_1)\sqrt{c_0 d_0} - (c_0 d_0)\sqrt{c_1 d_1}] = d_0\sqrt{c_1 d_1} - d_1\sqrt{c_0 d_0}.$$

$$\Rightarrow \quad x = \frac{d_0\sqrt{c_1 d_1} - d_1\sqrt{c_0 d_0}}{c_1\sqrt{c_0 d_0} - d_1\sqrt{c_0 d_0} - c_0\sqrt{c_1 d_1} + d_0\sqrt{c_1 d_1}}.$$

$$\Rightarrow \quad x = \frac{1}{1 + \dfrac{c_1\sqrt{c_0 d_0} - c_0\sqrt{c_1 d_1}}{d_0\sqrt{c_1 d_1} - d_1\sqrt{c_0 d_0}}}.$$

$$\Rightarrow \quad x = \frac{1}{1+Q},$$ using the assignment

$$Q := \frac{c_1\sqrt{c_0 d_0} - c_0\sqrt{c_1 d_1}}{d_0\sqrt{c_1 d_1} - d_1\sqrt{c_0 d_0}}.$$

Proposition: $Q \geq 0$.

Proof (by contradiction):

Assume Q < 0

$\Rightarrow$ [numerator(Q) > 0 and denominator(Q) < 0] or

[numerator(Q) < 0 and denominator(Q) > 0].

If [numerator(Q) > 0 and denominator(Q) < 0]

$$\Rightarrow \quad c_0\sqrt{c_1 d_1} < c_1\sqrt{c_0 d_0} \quad \text{and} \quad d_1\sqrt{c_0 d_0} > d_0\sqrt{c_1 d_1}$$

$$\Rightarrow \quad \left(\frac{c_0}{c_1}\right)^2 < \frac{d_0}{d_1}\frac{c_0}{c_1} \quad \text{and} \quad \frac{c_0}{c_1}\frac{d_0}{d_1} > \left(\frac{d_0}{d_1}\right)^2$$

$$\Rightarrow \quad \frac{c_0}{c_1} < \frac{d_0}{d_1} \quad \text{and} \quad \frac{c_0}{c_1} > \frac{d_0}{d_1}$$

$\Rightarrow$ contradiction.

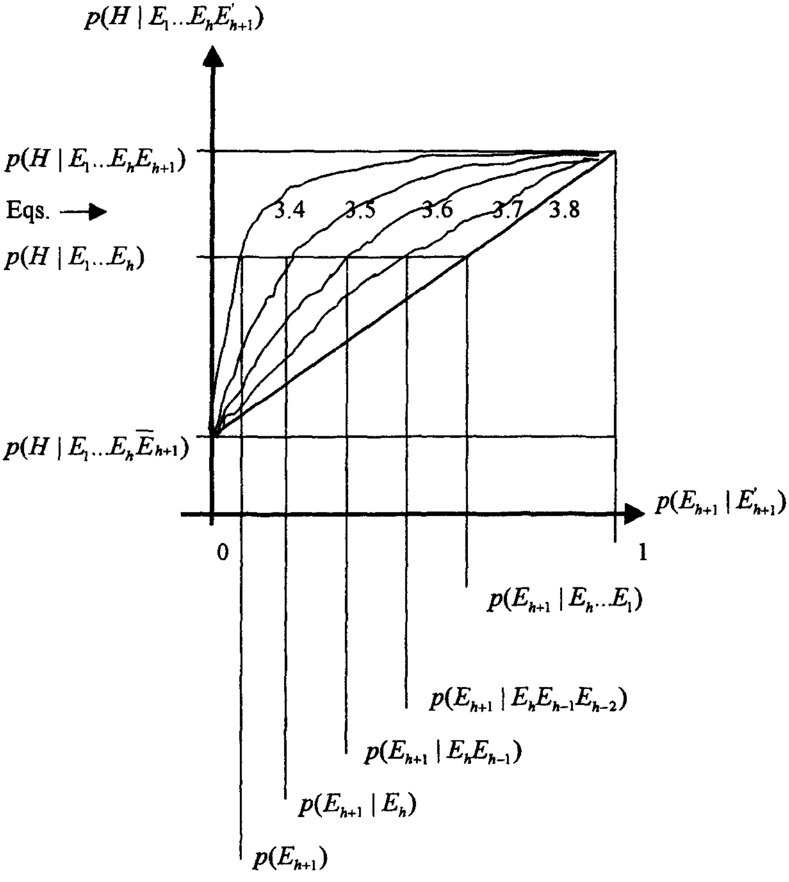
(If [numerator(Q) < 0 and denominator(Q) > 0], an analogous procedure is carried out.) □

We proved that from D'(x) = 0 follows $0 < x \leq 1$. The difference function D(x) therefore has at most one extremum within the open interval]0 , 1[. We conclude that any two interpolation functions belonging to cases I through IV, do not have common points other than at x = 0 and x = 1.

In order to give a summary representation of the (h+1) interpolation functions derived from Eq.3.1 by using the (h+1) different values of $p(E_{h+1} \mid V_{E'_{h+1}})$ due to Eq.3.4 through Eq.3.8, we choose the graph of Ill.3.3 and its parameters. This means we take Case I as an example and choose the following sequence:

$$p(E_{h+1}) < p(E_{h+1} \mid E_h) < p(E_{h+1} \mid E_h E_{h-1}) < \ldots < p(E_{h+1} \mid E_h \ldots E_1)\,. \qquad (*)$$

Using (*) we obtain the following summary:



Ill.3.7:

Interpolation functions derived from Eq.3.1 (shown according to Case I).

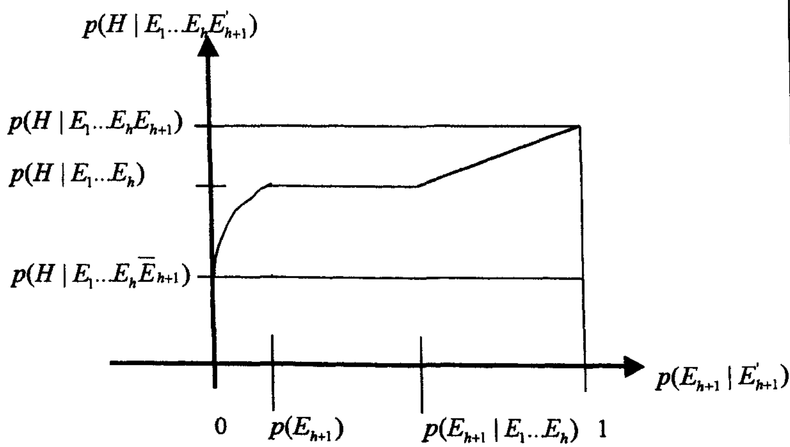
(h + 1) interpolation functions are created if (h + 1) different values of $p(E_{h+1} \mid V_{E'_{h+1}})$ are used due to Eq.3.4,..., Eq.3.8.

Supplement

We point out that the use of discontinuous interpolation functions has its justification in case of some special situations.
If we want to have an update of the probability p(H | E1... Eh) as a consequence of the event E'_{h+1} , we may utilize Eq.3.1 and choose $p(E_{h+1} \mid V_{E'_{h+1}})$ as follows:

$$p(E_{h+1} \mid V_{E'_{h+1}}) := \begin{cases} p(E_{h+1} \mid \varnothing)\ , \text{ if } p(E_{h+1} \mid E'_{h+1}) < p(E_{h+1}), \\ p(E_{h+1} \mid E_1..E_h), \text{ if } p(E_{h+1} \mid E'_{h+1}) > p(E_{h+1} \mid E_1..E_h), \\ p(E_{h+1} \mid E'_{h+1}), \text{ otherwise}. \end{cases} \quad (3.17)$$

The result of inserting (3.17) into Eq.3.1 is illustrated below, following the model of Ill.3.3.



Ill.3.8:

Graph of Eq.3.1 in case of $p(E_{h+1} \mid V_{E'_{h+1}})$, chosen according to (3.17).

The discontinuous interpolation function described in Ill.3.8 has been defined arbitrarily.

It follows the concept that we do not accept changes of $p(H \mid E_1 \ldots E_h)$ if both existence and nonexistence of E_{h+1} are far from being certain. Therefore we do not have a change of the ordinate values if $p(E_{h+1} \mid E'_{h+1})$ is situated between $p(E_{h+1})$ and $p(E_{h+1} \mid E_1 .. E_h)$.

If the presence of E_{h+1} is almost certain and if this fact meets our expectations, we allow a slight increase of $p(H \mid E_1 \ldots E_h)$ caused by E'_{h+1}. This explains the linear and consequently moderate changes of the ordinate values at the right-hand side of the curve shown in Ill.3.8.

But if the absence of E_{h+1} is almost certain and if this fact is surprising, we want to realize a strong response, i.e. we demand a quite considerable decrease of $p(H \mid E_1 \ldots E_h)$ as a consequence of E'_{h+1}.

In practice, however, the procedures used to update the probability $p(H \mid E_1 \ldots E_h)$ as a consequence of E'_{h+1} will preferably utilize $p(E_{h+1} \mid V_{E'_{h+1}}) := p(E_{h+1} \mid E_1 .. E_h)$ for simplicity.

Parts of the following Sections 4, 6 and 10 have been published in:

LIEBEL, F.-P.: Diagnostik mit Hilfe nicht-linearer Gleichungssysteme. BiBoS, Forschungszentrum Bielefeld-Bochum-Stochastik, Nr. 755 / 1 / 97, University of Bielefeld (1997).

4. Interpolation functions for more than one '-event

We need interpolation formulas in order to remove the prime from events contained in probabilities like $p(H \mid E_1..E_h E'_{h+1}..E'_k)$. The events $E_1,\ldots, E_h$ have probability ($p = 0$) or ($p = 1$), i.e. they are either negated or non-negated, whereas the events E_i', $i = h+1,.., k$, have probabilities $0 < p(E_i \mid E_i') < 1$ (known or unknown).

We will use expressions like $p(E_i \mid V_{E_i'})$, $i = h+1,\ldots, k$, where the symbol $V_{E_i'}$ denotes a logic product of events , derived from $(E_1..E_h E'_{h+1}..E'_i..E'_k)$ by removing E_i' - the event that forms the index in $V_{E_i'}$ - as well as other arbitrarily chosen elements. The empty product $V_{E_i'} = \varnothing$ is permitted.

We demand that interpolation functions suitable to compute $p(H \mid E_1.E_h E'_{h+1}..E'_k)$ satisfy the following three interpolation points, $i := h+1,\ldots, k$:

1 st interpolation point:

$p(H \mid E_1..E_h E'_{h+1}..E'_i..E'_k) = p(H \mid E_1..E_h E'_{h+1}..\bar{E}_i..E'_k)$ in case of $p(E_i \mid E_i') = 0$.

2 nd interpolation point:

$p(H \mid E_1..E_h E'_{h+1}..E'_i..E'_k) = p(H \mid E_1..E_h E'_{h+1}..E_i..E'_k)$ in case of $p(E_i \mid E_i') = 1$.

3 rd interpolation point:

$p(H \mid E_1..E_h E'_{h+1}..E'_i..E'_k) = p(H \mid E_1..E_h E'_{h+1}..E'_{i-1} E'_{i+1}..E'_k)$

in case of $p(E_i \mid E_i') = p(E_i \mid V_{E_i'})$ with arbitrarily chosen $V_{E_i'}$.

(The presence of the third interpolation point stems from the fact that no updating occurs with respect to E_i' if $p(E_i \mid E_i')$ reaches the value $p(E_i \mid V_{E_i'})$.)

Theorem

General Interpolation Theorem

For $p(H \mid E_1..E_h E'_{h+1}..E'_k)$*, consider the following symbols:*

- $E_1,\ldots, E_h$*, are negated or non-negated (arbitrarily chosen but fixed).*
- $E'_{h+1},\ldots,E'_k$*, have probabilities* $0 < p(E_i \mid E'_i) < 1$*,* $i = h+1,\ldots, k$*.*
- *For* $q_i \in \{0,1\}$*,* $i = h+1,\ldots, k$*,* $E_i^{(q_i)} := \overline{E}_i$ *if* $q_i = 0$ *and* $E_i^{(q_i)} := E_i$ *if* $q_i =$ *1.*
- $V_{E'_i}$ *is an arbitrarily chosen part of the product* $(E_1..E_h E'_{h+1}..E'_{i-1}E'_{i+1}..E'_k)$*,* $i = h+1,\ldots, k$*. The empty product* $V_{E'_i} = \varnothing$ *is permitted.*

Then the following equations satisfy the three interpolation points specified above:

$$p(H \mid E_1..E_h E'_{h+1}..E'_k) = \frac{\sum\limits_{q_{h+1},\ldots,q_k=0,1} p(HE_1..E_h E_{h+1}^{(q_{h+1})}..E_k^{(q_k)}) \prod\limits_{i=h+1}^{k} \frac{p(E_i^{(q_i)} \mid E'_i)}{p(E_i^{(q_i)} \mid V_{E'_i})}}{\sum\limits_{q_{h+1},\ldots,q_k=0,1} p(E_1..E_h E_{h+1}^{(q_{h+1})}..E_k^{(q_k)}) \prod\limits_{i=h+1}^{k} \frac{p(E_i^{(q_i)} \mid E'_i)}{p(E_i^{(q_i)} \mid V_{E'_i})}},$$

$E_i^{(0)} = \overline{E_i}$ *and* $E_i^{(1)} = E_i$*,* $i = h+1,\ldots, k$*.* *(4.1)*

Proof: Elementary.

The application of Eq.4.1 is an updating procedure.
As a result of the 3rd interpolation point we will have no update of p(H | H') with respect to E'_i, if $p(E_i \mid E'_i) = p(E_i \mid V_{E'_i})$. Therefore, the choice of $V_{E'_i}$ defines the "point of no alteration". We give three examples:

a) The update of *p(H | H')* = $p(H \mid E_1..E_h E'_{h+1}..E'_{i-1}E'_{i+1}..E'_k)$ as a consequence of E'_i may be undesired if the value of $p(E_i \mid E'_i)$ is determined exactly by those events which have just been used to calculate the value of *p(H | H')*, i.e. if we have $p(E_i \mid E'_i) = p(E_i \mid E_1..E_h E'_{h+1}..E'_{i-1}E'_{i+1}..E'_k)$.
Then we choose $V_{E'_i} := (E_1..E_h E'_{h+1}..E'_{i-1}E'_{i+1}..E'_k)$, the maximum $V_{E'_i}$.

b) In practice, $V_{E_i'} := (E_1..E_h)$ is used because this product avoids '-events.

c) $V_{E_i'} := \varnothing$ simplifies matters.

The handling of Eq.4.1 and the appearance of an interpolation concerning two uncertain events is shown by means of the example $p(U_2 \mid U_1 I_1 I_2 H' U_3')$. Here, $V_{H'}$ and $V_{U_3'}$ are chosen to have their maximum sizes as described in a) above, i.e. $V_{H'} := (U_1 I_1 I_2 U_3')$ and $V_{U_3'} := (U_1 I_1 I_2 H')$. Then Eq.4.1 yields:

$$p(U_2 \mid U_1 I_1 I_2 H' U_3') \tag{4.2}$$

$$= \frac{p(U_2U_1I_1I_2HU_3)\dfrac{p(H \mid H')}{p(H \mid U_1I_1I_2U_3')}\dfrac{p(U_3 \mid U_3')}{p(U_3 \mid U_1I_1I_2H')} + }{p(U_1I_1I_2HU_3)\dfrac{p(H \mid H')}{p(H \mid U_1I_1I_2U_3')}\dfrac{p(U_3 \mid U_3')}{p(U_3 \mid U_1I_1I_2H')} + } \cdots\cdots$$

$$\cdots\cdots \frac{p(U_2U_1I_1I_2H\overline{U}_3)\dfrac{p(H \mid H')}{p(H \mid U_1I_1I_2U_3')}\dfrac{p(\overline{U}_3 \mid U_3')}{p(\overline{U}_3 \mid U_1I_1I_2H')} + }{p(U_1I_1I_2H\overline{U}_3)\dfrac{p(H \mid H')}{p(H \mid U_1I_1I_2U_3')}\dfrac{p(\overline{U}_3 \mid U_3')}{p(\overline{U}_3 \mid U_1I_1I_2H')} + } \cdots\cdots$$

$$\cdots\cdots \frac{p(U_2U_1I_1I_2\overline{H}U_3)\dfrac{p(\overline{H} \mid H')}{p(\overline{H} \mid U_1I_1I_2U_3')}\dfrac{p(U_3 \mid U_3')}{p(U_3 \mid U_1I_1I_2H')} + }{p(U_1I_1I_2\overline{H}U_3)\dfrac{p(\overline{H} \mid H')}{p(\overline{H} \mid U_1I_1I_2U_3')}\dfrac{p(U_3 \mid U_3')}{p(U_3 \mid U_1I_1I_2H')} + } \cdots\cdots$$

$$\cdots\cdots \frac{p(U_2U_1I_1I_2\overline{H} \cdot \overline{U}_3)\dfrac{p(\overline{H} \mid H')}{p(\overline{H} \mid U_1I_1I_2U_3')}\dfrac{p(\overline{U}_3 \mid U_3')}{p(\overline{U}_3 \mid U_1I_1I_2H')}}{p(U_1I_1I_2\overline{H} \cdot \overline{U}_3)\dfrac{p(\overline{H} \mid H')}{p(\overline{H} \mid U_1I_1I_2U_3')}\dfrac{p(\overline{U}_3 \mid U_3')}{p(\overline{U}_3 \mid U_1I_1I_2H')}}.$$

(In the last line we write $\overline{H} \cdot \overline{U}_3$ in order to avoid confusion with $\overline{HU_3}$.)

It is obvious that Eq.4.2 meets the interpolation points specified above.

If we choose $V_{E_i'} := \varnothing$, i = h+1,..., k, the General Interpolation Theorem becomes the well-known L-Theorem.

We give the complete L-Theorem since the L-Theorem as stated in Eq.4.3 below and the historical L-Theorem have different wording even though the statements are identical.

L-Theorem

Special case of the General Interpolation Theorem using $V_{E_i'} := \varnothing$.

For $p(H \mid E_1 .. E_h E_{h+1}' .. E_k')$*, consider the following symbols:*

- $E_1, \ldots, E_h$ *are negated or non-negated (arbitrarily chosen but fixed).*
- $E_{h+1}', \ldots, E_k'$ *have probabilities* $0 < p(E_i \mid E_i') < 1$, $i = h+1, \ldots, k$.
- *For* $q_i \in \{0,1\}$, $i = h+1, \ldots, k$, $E_i^{(q_i)} := \overline{E}_i$ *if* $q_i = 0$ *and* $E_i^{(q_i)} := E_i$ *if* $q_i = 1$.
- $V_{E_i'} = \varnothing$ *for all* $i = h+1, \ldots, k$.

Then the following equations satisfy the three interpolation points specified above:

$$p(H \mid E_1 .. E_h E_{h+1}' .. E_k')$$

$$= \frac{\sum\limits_{q_{h+1},\ldots,q_k=0,1} p(HE_1 .. E_h E_{h+1}^{(q_{h+1})} .. E_k^{(q_k)}) \prod\limits_{i=h+1}^{k} \frac{p(E_i^{(q_i)} \mid E_i')}{p(E_i^{(q_i)})}}{\sum\limits_{q_{h+1},\ldots,q_k=0,1} p(E_1 .. E_h E_{h+1}^{(q_{h+1})} .. E_k^{(q_k)}) \prod\limits_{i=h+1}^{k} \frac{p(E_i^{(q_i)} \mid E_i')}{p(E_i^{(q_i)})}}, \qquad (4.3)$$

$E_i^{(0)} = \overline{E_i}$ *and* $E_i^{(1)} = E_i$, $i = h+1, \ldots, k$.

Moreover, Eq.4.1 may turn into a linear interpolation which can be handled much easier, and which therefore is the desired ideal.

We develop the Linear Interpolation Theorem which follows next.

Theorem

Linear Interpolation Theorem

If we choose $V_{E_i'} := (E_1..E_h)$, $i = h+1,...,k$, *Eq.4.1 provides a linear interpolation formula if, in addition, a) or b) holds:*

a) $(E_{h+2}'..E_k') = \varnothing$.

b) $\{E_{h+1}, ..., E_k\}$ *are independent, conditioned on* $(E_1 ... E_h)$.

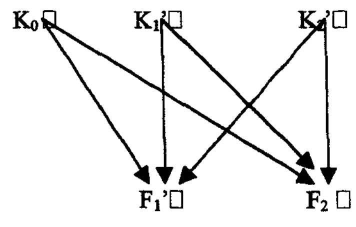
Then Eq.4.1 yields:

$$p(H \mid E_1..E_h E_{h+1}'..E_k') = \sum_{q_{h+1},...,q_k=0,1} p(H \mid E_1..E_h E_{h+1}^{(q_{h+1})}..E_k^{(q_k)}) \prod_{i=h+1}^{k} p(E_i^{(q_i)} \mid E_i'),$$

$$E_i^{(0)} = \overline{E_i} \text{ and } E_i^{(1)} = E_i,\ i = h+1,...,k. \qquad (4.4)$$

Proof: Elementary.

We demonstrate the effect of the Linear Interpolation Theorem by means of an example. This opens the possibility to introduce the concept of separated events and to explain the definition of the set WERT.



Ill.4.1:

Example to show the necessity of Requirement b) belonging to Eq.4.4.

We demand the interpolation of the probability $p(F_1 \mid F_2 K_0 K_1' K_2')$.

In order to show the necessity of Requirement b) mentioned above, we apply as a first step the General Interpolation Theorem (Eq.4.1) to $p(F_1 \mid F_2 K_0 K_1' K_2')$:

$p(F_1 \mid F_2 K_0 K_1^{'} K_2^{'})$ (4.4-1)

$$= _{(\text{Eq.4.1})} \frac{p(F_1 F_2 K_0 K_1 K_2) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid F_2 K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid F_2 K_0)} + \ldots}{p(F_2 K_0 K_1 K_2) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid F_2 K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid F_2 K_0)} + \ldots} \ldots$$

$$\ldots \frac{p(F_1 F_2 K_0 K_1 \overline{K_2}) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid F_2 K_0)} \frac{p(\overline{K_2} \mid K_2^{'})}{p(\overline{K_2} \mid F_2 K_0)} + \ldots}{p(F_2 K_0 K_1 \overline{K_2}) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid F_2 K_0)} \frac{p(\overline{K_2} \mid K_2^{'})}{p(\overline{K_2} \mid F_2 K_0)} + \ldots} \ldots$$

$$\ldots \frac{p(F_1 F_2 K_0 \overline{K_1} K_2) \frac{p(\overline{K_1} \mid K_1^{'})}{p(\overline{K_1} \mid F_2 K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid F_2 K_0)} + \ldots}{p(F_2 K_0 \overline{K_1} K_2) \frac{p(\overline{K_1} \mid K_1^{'})}{p(\overline{K_1} \mid F_2 K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid F_2 K_0)} + \ldots} \ldots$$

$$\ldots \frac{p(F_1 F_2 K_0 \overline{K_1} \overline{K_2}) \frac{p(\overline{K_1} \mid K_1^{'})}{p(\overline{K_1} \mid F_2 K_0)} \frac{p(\overline{K_2} \mid K_2^{'})}{p(\overline{K_2} \mid F_2 K_0)}}{p(F_2 K_0 \overline{K_1} \overline{K_2}) \frac{p(\overline{K_1} \mid K_1^{'})}{p(\overline{K_1} \mid F_2 K_0)} \frac{p(\overline{K_2} \mid K_2^{'})}{p(\overline{K_2} \mid F_2 K_0)}}.$$

Eq.4.4-1 does not reach linear interpolation due to Eq.4.4, since

$p(K_1 \mid F_2 K_0) p(K_2 \mid F_2 K_0) \neq p(K_1 K_2 \mid F_2 K_0)$,

i.e. Requirement b), needed to reach a linear interpolation, is violated.

Now we again refer to the configuration of Ill.4.1, but execute a negation upon F_2. Thus:

$p(F_1 \mid \overline{F_2} K_0 K_1^{'} K_2^{'})$ (4.4-2)

$$= _{(\text{Eq.4.1})} \frac{p(F_1 \overline{F_2} K_0 K_1 K_2) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid \overline{F_2} K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid \overline{F_2} K_0)} + \ldots etc.}{p(\overline{F_2} K_0 K_1 K_2) \frac{p(K_1 \mid K_1^{'})}{p(K_1 \mid \overline{F_2} K_0)} \frac{p(K_2 \mid K_2^{'})}{p(K_2 \mid \overline{F_2} K_0)} + \ldots etc.} \ldots$$

Since $p(K_1 \mid \overline{F_2} K_0) p(K_2 \mid \overline{F_2} K_0) = p(K_1 K_2 \mid \overline{F_2} K_0)$ [according to Eq.8.6 / A→L-Corollary 2], we are able to proceed as follows:

$$p(F_1 \mid \overline{F}_2 K_0 K_1' K_2') = \frac{p(F_1 \overline{F}_2 K_0 K_1 K_2)\dfrac{p(K_1 \mid K_1')}{p(K_1 K_2 \mid \overline{F}_2 K_0)}\dfrac{p(K_2 \mid K_2')}{1} + \ldots etc.}{p(\overline{F}_2 K_0 K_1 K_2)\dfrac{p(K_1 \mid K_1')}{p(K_1 K_2 \mid \overline{F}_2 K_0)}\dfrac{p(K_2 \mid K_2')}{1} + \ldots etc.} \ldots$$

$$= \frac{p(F_1 \mid \overline{F}_2 K_0 K_1 K_2)\dfrac{p(K_1 \mid K_1')}{1}\dfrac{p(K_2 \mid K_2')}{1} + \ldots etc.}{\dfrac{p(K_1 \mid K_1')}{1}\dfrac{p(K_2 \mid K_2')}{1} + \ldots etc.} \ldots$$

$$= p(F_1 \mid \overline{F}_2 K_0 K_1 K_2) p(K_1 \mid K_1') p(K_2 \mid K_2') + \ldots etc.,$$

i.e. linear interpolation.

Since F_1 and F_2 are independent, conditioned upon (K_0 K_1 K_2), we obtain:

$$p(F_1 \mid \overline{F}_2 K_0 K_1' K_2') = p(F_1 \mid K_0 K_1 K_2) p(K_1 \mid K_1') p(K_2 \mid K_2') + \ldots etc.$$

$$= p(F_1 \mid K_0 K_1' K_2'). \qquad (4.4\text{-}3)$$

Therefore, the Linear Interpolation Theorem gives Eq.4.4-3, but the Linear Interpolation Theorem does <u>not</u> give $p(F_1 \mid F_2 K_0 K_1' K_2') = p(F_1 \mid K_0 K_1' K_2')$, although the definition of WERT(F_1) provides this equation, i.e.

$$p(F_1 \mid L\text{-}NET) = p(F_1 \mid F_1')$$

$$= p(F_1 \mid F_2 K_0 K_1' K_2')$$

$$=_{(\text{definition of WERT})} p(F_1 \mid K_0 K_1' K_2').$$

This alleged inconsistency will be cleared up when we establish the defining equations which belong to the '-events of Ill.4.1. This gives the following equation system:

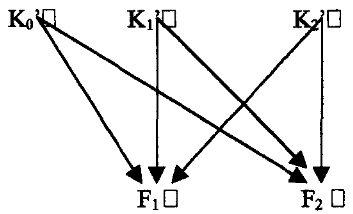
$$p(K_1 \mid K_1') = p(K_1 \mid F_1' F_2 K_0 K_2'). \qquad (4.4\text{-}4)$$

$$p(K_2 \mid K_2') = p(K_2 \mid F_1' F_2 K_0 K_1'). \qquad (4.4\text{-}5)$$

$$p(F_1 \mid F_1') = p(F_1 \mid \cancel{F_2} K_0 K_1' K_2') = p(F_1 \mid K_0 K_1' K_2'). \qquad (4.4\text{-}6)$$

(The removal of F_2 will be discussed in detail below.)

We compare Ill.4.1 and its set of equations with the following example:



Ill.4.2:

Example derived from Ill.4.1 by moving the prime from F_1 to K_0.

We demand the computation of $p(K_0 \mid K_0')$, $p(K_1 \mid K_1')$ and $p(K_2 \mid K_2')$.

The set of equations to compute the L-Net shown in Ill.4.2 is as follows:

$$p(K_1 \mid K_1') = p(K_1 \mid F_1F_2K_0'K_2'). \qquad (4.4\text{-}7)$$

$$p(K_2 \mid K_2') = p(K_2 \mid F_1F_2K_0'K_1'). \qquad (4.4\text{-}8)$$

$$p(K_0 \mid K_0') = p(K_0 \mid F_1F_2K_1'K_2') \qquad (4.4\text{-}9)$$

We transform Eq.4.4-9 in the same manner as Eq.9.1 in Section 9. It follows:

$$p(K_0 \mid K_0') = \frac{p(K_0F_1F_2K_1'K_2')}{p(K_0F_1F_2K_1'K_2') + p(\overline{K_0}F_1F_2K_1'K_2')}$$

$$= \frac{1}{1 + \dfrac{p(\overline{K_0}F_1F_2K_1'K_2')}{p(K_0F_1F_2K_1'K_2')}}$$

$$= \frac{1}{1 + \dfrac{p(F_1F_2 \mid \overline{K_0}K_1'K_2')p(\overline{K_0} \mid K_1'K_2')}{p(F_1F_2 \mid K_0K_1'K_2')p(K_0 \mid K_1'K_2')}}$$

$$= \frac{1}{1 + \dfrac{p(F_1 \mid F_2\overline{K_0}K_1'K_2')p(F_2 \mid \overline{K_0}K_1'K_2')p(\overline{K_0} \mid K_1'K_2')}{p(F_1 \mid F_2K_0K_1'K_2')p(F_2 \mid K_0K_1'K_2')p(K_0 \mid K_1'K_2')}}. \qquad (4.4\text{-}9^*)$$

We see that Eq.4.4-9* contains $p(F_1 \mid F_2K_0K_1'K_2')$, which also is part of Eq.4.4-6. The problem at hand consists of the necessity to name the properties which allow to remove F_2 from $p(F_1 \mid F_2K_0K_1'K_2')$. Such properties exist in case of Eq.4.4-6 but not in case of Eq.4.4-9*. The reasons for that are the following:

I.)

If $p(F_1 \mid F_2K_0K_1'K_2')$ belongs to Eq.4.4-9* the ap-probability values of K_1' and K_2' <u>are</u> <u>not</u> <u>given</u> since

- the probability of F_1 depends on the ap-probability values of K_1' and K_2', and
- the ap-probability values of K_1' and K_2' depend on the probability of F_1.

Since $p(F_1 \mid F_2K_0K_1'K_2')$ is <u>not</u> <u>a</u> <u>variable</u> of an L-Net computation system, the mutual dependence is not settled. Therefore, F_1 and F_2 are not independent in case of condition $(K_0K_1'K_2')$.

II.)

The situation differs if we look at Eq.4.4-6.

<u>Premise</u>

$p(K_1 \mid K_1')$, $p(K_2 \mid K_2')$ and $p(F_1 \mid F_1')$ are part of an L-Net computation system.

<u>Proposition</u>

$$p(F_1 \mid F_1') = p(F_1 \mid \cancel{F_2}K_0K_1'K_2') = p(F_1 \mid K_0K_1'K_2'). \qquad \textit{(identical to 4.4-6)}$$

<u>Explanation</u>

If $p(F_1 \mid F_2K_0K_1'K_2')$ belongs to Eq.4.4-6 then $p(F_1 \mid F_2K_0K_1'K_2') =: p(F_1 \mid F_1')$ is a variable of an L-Net computation system. The variables $p(K_1 \mid K_1')$, $p(K_2 \mid K_2')$ and $p(F_1 \mid F_1')$ are understood to have definite values which are determined by the L-Net. If we now remove F_2 from $p(F_1 \mid F_2K_0K_1'K_2')$, <u>without removing it from the L-Net</u>, there will occur no change of the probability $p(F_1 \mid F_2K_0K_1'K_2')$. This holds since F_2 influences F_1 only via paths across K_1' and K_2', and the ap-probabilities $p(K_1 \mid K_1')$ and $p(K_2 \mid K_2')$ remain unchanged if the L-Net remains unchanged. Removing F_2 from $p(F_1 \mid F_2 K_0 K_1' K_2')$ without consequences is equivalent to the <u>separation</u> of F_1' from F_2 by means of $(K_0 K_1' K_2')$.

Because of the proposition above, we were able to establish Eq.1.2 and define the set WERT(H) in the way we did [e.g. without additional events which might belong to the sets FOL(U_1), ..., FOL(U_3)].

We now use the example of Ill.4.1 to define separated events in a less formal way.

<u>Definition (Separated events)</u>

Consider the structure of Ill.4.1.

An arbitrary event F_1' is separated from an arbitrary event F_2 by means of an arbitrary conditioning product, containing negated, non-negated or apostrophized events, e.g. $(K_0 K_1' K_2')$, if

- *the ap-probabilities of the participating '-events, i.e. $p(F_1 \mid F_1')$, $p(K_1 \mid K_1')$ and $p(K_2 \mid K_2')$, are variables of an L-Net computation system, and*
- *<u>all</u> connections between F_1' and F_2 lead across the elements included in the separating product, i.e. across $\{K_0, K_1', K_2'\}$.*

Then $p(F_1 \mid F_1') = p(F_1 \mid \not{F_2} K_0 K_1' K_2') = p(F_1 \mid K_0 K_1' K_2')$.

<u>Importance of separated events</u>

The logic product W(H), no matter if the events in W(H) are negated, non-negated or apostrophized, separates the event H' from any event A $\notin$ W(H) if the causal connections between H' and A consist only of paths across events included in W(H). So p(H | W(H)) can be kept small since the integration of all elements belonging to WERT(H) into the product W(H) produces the effect that we may ignore all net nodes "outside".

Finally, we show the handling of Eq.4.4 and, in particular, that the three requested interpolation points are satisfied. As an example, we choose $p(H \mid U_1 I_1 I_2 U_2' U_3')$. If U_2 and U_3 are independent, conditioned upon (U_1 I_1 I_2), Requirement b) of the Linear Interpolation Theorem is satisfied. It follows:

$$
\begin{aligned}
p(H \mid U_1 I_1 I_2 U_2' U_3') =_{(Eq.4.4)} \; & p(H \mid U_1 I_1 I_2 U_2 U_3)\; p(U_2 \mid U_2') p(U_3 \mid U_3') \\
& + p(H \mid U_1 I_1 I_2 U_2 \overline{U_3})\; p(U_2 \mid U_2') p(\overline{U_3} \mid U_3') \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} U_3)\; p(\overline{U_2} \mid U_2') p(U_3 \mid U_3') \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} \cdot \overline{U_3})\; p(\overline{U_2} \mid U_2') p(\overline{U_3} \mid U_3') .
\end{aligned}
\qquad (4.5)
$$

It is obvious that Eq.4.5 satisfies the 1st and the 2nd interpolation point.

The 3rd interpolation point is defined by using $p(E_i \mid E_i') = p(E_i \mid V_{E_i'})$ with an arbitrary $V_{E_i'}$. Since the Linear Interpolation Theorem requires $V_{E_i'}$:= (E_1...E_h), we choose $V_{U_2'} = V_{U_3'} := (U_1 I_1 I_2)$. Hence,

$$
\begin{aligned}
p(H \mid U_1 I_1 I_2 U_2' U_3') = \; & p(H \mid U_1 I_1 I_2 U_2 U_3)\; p(U_2 \mid U_1 I_1 I_2) p(U_3 \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 U_2 \overline{U_3})\; p(U_2 \mid U_1 I_1 I_2) p(\overline{U_3} \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} U_3)\; p(\overline{U_2} \mid U_1 I_1 I_2) p(U_3 \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} \cdot \overline{U_3})\; p(\overline{U_2} \mid U_1 I_1 I_2) p(\overline{U_3} \mid U_1 I_1 I_2) \\
= \; & p(H \mid U_1 I_1 I_2 U_2 U_3)\; p(U_2 U_3 \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 U_2 \overline{U_3})\; p(U_2 \overline{U_3} \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} U_3)\; p(\overline{U_2} U_3 \mid U_1 I_1 I_2) \\
& + p(H \mid U_1 I_1 I_2 \overline{U_2} \cdot \overline{U_3})\; p(\overline{U_2} \cdot \overline{U_3} \mid U_1 I_1 I_2) \\
= \; & p(H U_2 U_3 \mid U_1 I_1 I_2) + p(H U_2 \overline{U_3} \mid U_1 I_1 I_2) \\
& + p(H \overline{U_2} U_3 \mid U_1 I_1 I_2) + p(H \overline{U_2} \cdot \overline{U_3} \mid U_1 I_1 I_2) \\
= \; & p(H \mid U_1 I_1 I_2),
\end{aligned}
$$

i.e. the 3rd interpolation point is satisfied.

Excursion

We are going to discuss the properties needed to turn the Linear Interpolation Function (Eq.4.4) into an equation. This problem is presented in a simplified manner using Ill.1.2 and $p(H \mid U_1\, I_1\, I_2\, U_2'\, U_3')$ as an example:

$$p(H \mid U_1 I_1 I_2 U_2' U_3') = _{\text{(extension)}} \frac{\sum_{q_2,q_3=0,1} p(HU_1 I_1 I_2 U_2' U_3' U_2^{(q_2)} U_3^{(q_3)})}{\sum_{q_2,q_3=0,1} p(U_1 I_1 I_2 U_2' U_3' U_2^{(q_2)} U_3^{(q_3)})}$$

$$= \frac{p(H \mid U_1 I_1 I_2 U_2' U_3' U_2 U_3)\, p(U_2 U_3 \mid U_1 I_1 I_2 U_2' U_3') + etc.....}{p(U_2 U_3 \mid U_1 I_1 I_2 U_2' U_3') + etc.....} .....$$

(4.6)

In order to transform Eq.4.6 into the form of the Linear Interpolation Theorem, we need special properties which are defined by Eq.4.7 and Eq.4.8 (U_2 and U_3 are arbitrarily negated or non-negated):

$p(H \mid U_1\, I_1\, I_2\, U_2'\, U_3'\, U_2\, U_3) \;=\; p(H \mid U_1\, I_1\, I_2\, U_2\, U_3).$

(4.7)

$p(U_2\, U_3 \mid U_1\, I_1\, I_2\, U_2'\, U_3') \;=\; p(U_2 \mid U_2')\, p(U_3 \mid U_3').$ *(4.8)*

Considering now the configuration of L-Nets and the concept of separated events we may conclude:

1.

Eq.4.7 holds if $W(U_2)$ and $W(U_3)$ include only events which are connected to H' solely via U_2 or U_3, respectively, or which are already elements of the conditioning product $(U_1\, I_1\, I_2\, U_2\, U_3)$.

2.

Eq.4.8 holds if U_2 and U_3 are independent when conditioned upon $(U_1 I_1 I_2 U_2' U_3')$, i.e. if $(U_1 I_1 I_2 U_2' U_3')$ includes no events which are able to destroy the demanded independence of U_2 and U_3; in particular, the independence of two events is lost if there is a common effect of these two events having p = 1.

5. Causality and dependence

The arrangement of events in a causal network offers the opportunity to detect stochastic dependencies. Furthermore, it opens a way to name the conditions which are needed to turn dependent events into conditionally independent events.

A) Dependent successors

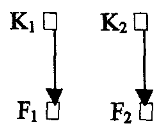
First, we shortly repeat the note in front of Ill.1.1.

Note:

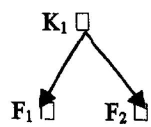
- If a diagram does not show any information about the stochastic dependence of two net nodes or two inhibitors, we are allowed to assume stochastic independence until further statements are declared.
- If a diagram does not show any information about the inhibition of an arbitrary transition A→L, we are not allowed to assume the absence of inhibitors. On the contrary: The transition A→L is influenced in any case by inhibiting events.

Structure 5.1:

Let K_1, K_2 be dependent.



Special case $K_1 \equiv K_2$.



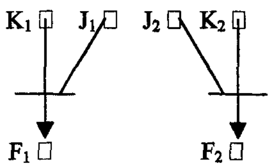
(Structure 5.1 is valid) $\Rightarrow$ *(F_1 and F_2 are stochastically dependent).*

Structure 5.2:

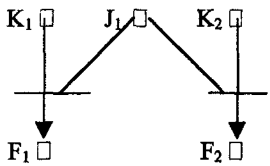
Let

K_1, K_2 be independent, and

J_1, J_2 be dependent.



Special case $J_1 \equiv J_2$:



(Structure 5.2 is valid) $\Rightarrow$ *(F_1 and F_2 are stochastically dependent).*

Structure 5.3:

Let

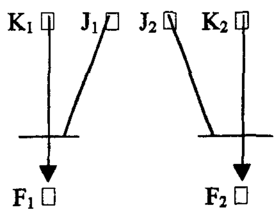
K_1, K_2 be independent,

J_1, J_2 be independent, and

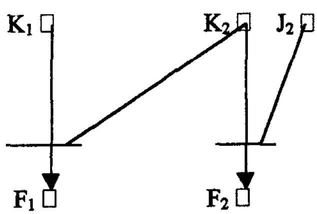
J_1, K_2 be dependent

or

J_2, K_1 be dependent.



Special case, e.g. $J_1 \equiv K_2$:



(Structure 5.3 is valid) $\Rightarrow$ *(F_1 and F_2 are stochastically dependent).*

Sufficient condition for dependent events F_1, F_2

(Structure 5.1 ∨ 5.2 ∨ 5.3 is valid) ⇒ (F_1, F_2 are stochastically dependent).

(C.5.1)

In general, the converse of (C.5.1) does not hold. But we may state:

Necessary condition for dependent events F_1, F_2

If the left-hand side of (C.5.1) specifies a complete list of networks which produce two dependent successors F_1 and F_2, the conclusion may be reversed:

(The successors F_1, F_2 are dependent) ⇒ (Structure 5.1 ∨ 5.2 ∨ 5.3 is valid).

(C.5.2)

B) Eliminating the dependence of successors by conditioning

Stochastically dependent events that are descendant nodes in a causal network may reach conditional independence by means of specific conditions. These conditions will be made explicit in the case of the successors F_1 and F_2 contained in Structure 5.1 through 5.3.

Eliminating the dependence between F_1, F_2

If Structure 5.1 ∨ 5.2 ∨ 5.3 is valid

⇒

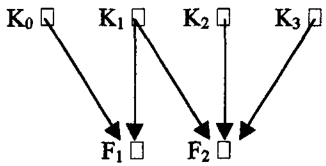
F_1, F_2 are independent in case of a condition which contains one of the events $\{X_k, Y_k\}$ for all $k = 1,\ldots,4$, according to the following enumeration:

1) *Either $X_1 \in URS(F_1)$ or $Y_1 \in URS(F_2)$ if X_1 and Y_1 are dependent.*
2) *Either $X_2 \in INH(F_1)$ or $Y_2 \in INH(F_2)$ if X_2 and Y_2 are dependent.*
3) *Either $X_3 \in INH(F_1)$ or $Y_3 \in URS(F_2)$ if X_3 and Y_3 are dependent.*
4) *Either $X_4 \in URS(F_1)$ or $Y_4 \in INH(F_2)$ if X_4 and Y_4 are dependent.*

For each option 1) through 4) we give an example.

<u>Examples:</u> Eliminating the dependence of F_1, F_2.

<u>ad 1):</u>

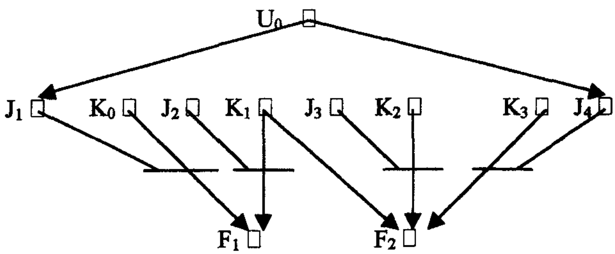


URS(F_1) = { K_0, K_1 },
↙↗
URS(F_2) = { K_1, K_2, K_3 }.

⟩ F_1, F_2 are independent if conditioned upon K_1.

(↙↗ means: stochastically dependent.)

<u>ad 2):</u>

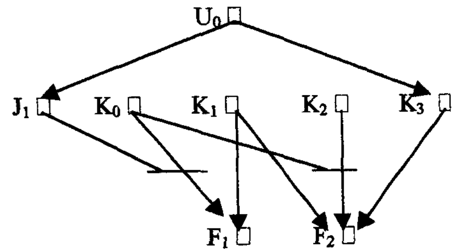


URS(F_1) = { K_0, K_1 },
↙↗
URS(F_2) = { K_1, K_2, K_3 }.

INH(F_1) = { J_1, J_2 },
↖↘
INH(F_2) = { J_3, J_4 }.

⟩ F_1, F_2 are independent if conditioned upon (K_1 J_1) or (K_1 J_4).

ad 3), 4):



URS(F_1) = { K_0, K_1 },

URS(F_2) = { K_1, K_2, K_3 }.

URS(F_1) = { K_0, K_1 },

INH(F_2) = { K_0 }.

INH(F_1) = { J_1 },

URS(F_2) = { K_1, K_2, K_3 }.

F_1, F_2 are independent if conditioned upon (K_1 K_0 J_1) or (K_1 K_0 K_3).

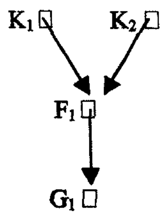
C) Dependent parent nodes as a result of specific conditions

Stochastically independent events that are causes may become dependent in case of specific conditions.

Structure 5.4:

Let

K_1, K_2 be independent.

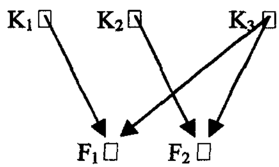


(Structure 5.4 is valid) $\Rightarrow$ *(K_1 and K_2 are stochastically dependent when conditioned upon F_1 or G_1.*

Struktur 5.5:

Let

K_1, K_2 be independent.



(Structure 5.5 is valid) $\Rightarrow$ *(K_1 and K_2 are stochastically dependent when conditioned upon (F_1 F_2)).*

D) Eliminating the conditional dependence of parent nodes by changing the condition

The events K_1 and K_2 contained in Structure 5.4 lose the property of being independent if they are conditioned upon F_1, but they may become independent again when conditioned upon $\overline{F_1}$. This requires that certain assumptions which are named in A→L-Corollary 2 (see Section 8, Eq.8.6) are satisfied.

The events K_1 and K_2 contained in Structure 5.5, which lose independence as a result of the condition (F_1 F_2), may become independent again in case of an extended condition.

(Structure 5.5 is valid) $\Rightarrow$ *(K_1 and K_2 are independent, conditioned upon the logic product (F_1 F_2 K_3)).*

Importance attached to independent events

Ill.1.2 shows that the events F_i , i = 1,..., 5 are stochastically dependent since these level-of-effects events have common causes at the H/K-level. It is therefore definitely wrong to assume that the F_i are independent.

But if we look at two arbitrarily chosen events F_1, $F_2 \in \{F_i \mid i = 1,..., 5\}$, and introduce a condition (K_1 K_2 K_3 J H) according to the following requirements,

- J is the logic product of all known inhibitors which act upon the transitions leading to F_1 and F_2 ,
- (K_1 K_2 K_3 H) includes all causes of F_1 and F_2,

we may have a chance to reach

$$p(F_1 F_2 \mid K_1 K_2 K_3 J H) = p(F_1 \mid K_1 K_2 K_3 J H)\ p(F_2 \mid K_1 K_2 K_3 J H). \qquad (5.3)$$

This equation holds if, in addition, we impose the restriction that unknown elements belonging to INH(F_1) are independent of the unknown elements in INH(F_2) (see Section 6, Assumption IIc).

To sum up, it can be said that the F_i , situated at the level of effects, reach stochastic independence if we use a conditioning product which contains all causes of the F_i, $i = 1,..., 5$, and if, in addition, the events in INH(F_{i_0}) are independent of the events in INH(F_{i_1}), for all $i_0 \neq i_1$.

Although we are forced to admit the stochastic dependence of the F_i as a consequence of their common causes located at the H/K-level, there is no such general restriction concerning the independence of two elements A and B, $A \in$ INH(F_{i_0}) and $B \in$ INH(F_{i_1}). If there is information that events in INH(F_{i_0}) depend on some events in INH(F_{i_1}), we take at least one event of the detected paired dependence and place it into the product J.

The equivalence of causality and independence - in conjunction with the necessity of using independent events - leads to the assumptions stated in Section 6.

6. Assumptions

L-Nets are structured according to the following assumptions in order to facilitate the intended computations. The causal net, i.e. the web of transitions which start and end at irregular physiological states, must necessarily allow these assumptions. We discuss the situation in detail and introduce a number of symbols which again are defined using the event H of III.1.2 as a reference.

U product of elements from the set URS(H); $U := (U_1\ U_2'\ U_3')$.

F product of elements from the set FOL(H); $F := (F_1\ F_2\ F_3\ \overline{F_4}\ F_5)$.

K product of elements from the set DIFF(H); $K := (K_1'\ K_2\ K_3')$.

I product of elements from the set INH(H); $I := (I_1\ I_2)$.

J product of elements from the sets $INH(F_1),\ldots, INH(F_5)$; $J := (J_1\ \overline{J_2}\ )$.

Assumptions I, IIa, IIb, IIc, and III, as well as the complementary Assumptions IIa, IIb*, IIc* apply equivalently to every non-inhibiting net node which has an unknown ap-probability.*

Assumption I

The events of F, I, J possess the probability $(p = 0)$ or $(p = 1)$.

Comment on Assumption I

The problem at hand is to determine the probabilities of hypothetical causes, i.e. the probabilities of events belonging to the H/K-level, to the U-level or to levels situated "further up" (see Fig.1.2). The computation of these values is done by using the probabilities of inhibiting events, of F-level-events or of events "further down", all of which should not be uncertain in order to achieve precise results.

Assumption IIa

$URS(F_{i_0})$ includes all causes of F_{i_0}, for any $F_{i_0} \in F$.

Comment on Assumption IIa

Medical diagnoses suffer from possible incorrectness unless every cause of each single symptom is taken into consideration, and this applies to computer aided procedures as well as to pencil and paper methods. But the main purpose of Assumption IIa is the fact that elements in F can reach conditional independence only if, among other requirements, the conditioning product contains each and every cause.

Assumption IIb

The events in URS(F_{i_0}) are self-reliant causes of F_{i_0}, for any $F_{i_0} \in F$.

Comment on Assumption IIb

Self-reliance as a property of causes is going to be defined precisely when introducing the A→L-Theorem below. To get the idea: Self-reliant causes of an arbitrarily chosen net node F_1 show the characteristics that

- the events in URS(F_1) are independent, and that
- an arbitrary F_1-generating process is not influenced by other causes or by the inhibitors of another F_1-generating process.

For the case of medical diagnoses we may assume that Assumption IIb is satisfied if the causes of F_1 do not show a close relationship.

Assumption IIc

For any $\{F_{i_0}, F_{i_1}\} \subset \{F_i \mid i \in N\}$, $\{K_1, K_2\} \subset$ URS(F_{i_0}), $K_3 \in$ URS(F_{i_1}) and all events X, Y, Z we have:

- *$X \in$ URS(F_{i_0}) is independent of $Y \in$ URS(F_{i_0}) and $Z \in$ URS(F_{i_1}).*
- *$X \in$ INH($K_1 \rightarrow F_{i_0}$) is independent of $Y \in$ INH($K_2 \rightarrow F_{i_0}$) and $Z \in$ INH($K_3 \rightarrow F_{i_1}$).*
- *$X \in$ URS(F_{i_0}) is independent of $Y \in$ INH(F_{i_0}) and $Z \in$ INH(F_{i_1}).*

Comment on Assumption IIc

1. In order to obtain Eq.9.1, independence of events in (K J H) is required.
2. Eq.9.2 also needs independence of events in (K J H).

3. Eq.9.3 requires conditional independence of the $F_i \in F$. This is satisfied if the conditioning product contains all causes of the F_i, and if additionally the elements in INH(F_{i_0}) are independent of the elements in INH(F_{i_1}), for any $F_{i_0}, F_{i_1} \in F$.

Assumption III

$p(U\,F\,K\,I\,J \mid H) = p(U\,I \mid H)\; p(F\,K\,J \mid H)$, event H negated or non-negated.

Comment on Assumption III

Assumption III mathematically expresses the separation of events in (U I) from events in (F K J) brought about by means of the net node H. Therefore, the nodes belonging to (U I) and the nodes lying "beneath" the (U I)-level are connected only via paths across the node H.

Assumption III, which is used to obtain Eq.9.1, causes no problems even though there are '-events located in front of the conditioning lines.

In anticipation of Section 7 below, we take a closer look at the statement of Assumption IIb in order to determine the parts which are covered by Assumption IIc.

Proposition

Let $F_{i_0} \in F$ and K_1, $K_2 \in$ URS(F_{i_0}) be arbitrary events.

If [elements in URS(F_{i_0}) are independent]

$\wedge$ [elements in INH($K_1 \rightarrow F_{i_0}$) are independent of the elements in INH($K_2 \rightarrow F_{i_0}$)]

$\wedge$ [elements in URS(F_{i_0}) are independent of the elements in INH(F_{i_0})]

$\Rightarrow$

the elements in URS(F_{i_0}) are self-reliant causes of F_{i_0}, i.e. any element belonging to URS(F_{i_0}) meets the Assumptions 1/A→L through 4/A→L (Eq.7.6).

The proposition leads to the following corollary:

Assumption IIc $\Rightarrow$ *Assumption IIc/single* $F_{i_0} \in F$ $\Rightarrow$ *Assumption IIb.*

We are able to handle probabilities like p(F | K J H) [see Eq.9.1] because we introduced the necessary assumptions. But we still need similar assumptions to be able to transform p(H | U I).

Assumptions IIa, IIb and IIc have been established to arrange the events at the F-level and the H/K-level. We now "move up to the next floor" and declare the H/K-level to be the new level of effects, and the U-level above to be the new level of causes. This action yields the new set of assumptions which we name IIa*, IIb* and IIc* and which read as follows:

<u>Complementary assumptions</u>

<u>Assumption IIa</u>*

The set URS(H) includes all causes of the event H.

<u>Assumption IIb</u>*

The events in URS(H) are self-reliant causes of H.

<u>Assumption IIc</u>*

Let U_1, $U_2 \in URS(H)$ *be arbitrary events.*

- U_1, U_2 *are independent.*
- *The elements in* $INH(U_1 \rightarrow H)$ *are independent of the elements in* $INH(U_2 \rightarrow H)$.
- *The elements in URS(H) are independent of the elements in INH(H).*

In Section 9 we will introduce a procedure to decompose conditional probabilities with respect to inhibiting events contained in the conditioning product. Such a decomposition into factors needs the following assumption.

<u>Assumption concerning inhibitors</u>

Let $K_1 \in URS(F_{i_0})$ *be an arbitrary event and* $I_1, \ldots, I_r$ *the inhibitors of* $K_1 \rightarrow F_{i_0}$.

If K_1 *exists,* $I_1, \ldots, I_r$ *are assumed to be self-reliant causes of the event* $\overline{(K_1 \rightarrow F_{i_0})}$.

Structure and appearance of the causal network depend upon the assumptions which we had to establish. Because of the equivalence of causality and independence, assumptions which demand stochastic independence or the separation of events have great impact:

IIc Independence of two arbitrarily chosen elements belonging to one or two arbitrary sets included in the set of sets $\{URS(F_i), INH(F_i) \mid F_i \in F\}$.

IIc* Independence of two arbitrarily chosen elements belonging to one or two arbitrary sets included in $\{URS(H), INH(H)\}$.

III Separation of the elements in (U I) from the elements in (F K J) by means of H.

This seems to impose tremendous independence demands. But as a matter of fact there is almost only one essential requirement, namely - short and inexact - the independence of causes and inhibitors situated at the level of hypotheses.

a) There are no fundamental reasons against the assumption of independent events included in an arbitrary set URS(F_{i_0}), $F_{i_0} \in F$, because these events generally do not have common predecessors.

b) If the elements contained in URS(F_{i_0}) are not closely related (since they do not have the same ancestors), we may assume the same for the elements in INH(F_{i_0}).

c) Inhibitors represent regular or irregular physiological states while causes belong to irregular physiological states only. Therefore, the assumption that causes are independent of inhibitors may not be too far from reality.

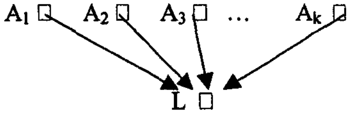
Altogether there is justified hope to satisfy Assumption IIc.

Prospect

In the course of Section 9 we utilize the assumptions in order to obtain Eq.9.1. Since Eq.9.1 includes conditional probabilities of sometimes extended size, we need a decomposition into factors. This requires the A→L-Theorem and the A→L-Corollaries which are presented in the Sections 7 and 8.

7. A→L-Theorem

We want to obtain statements about the probability of a transition directed from an existing parent event towards a possible subsequent event.
We use the arrangement of events in the L-Net and look at an arbitrarily chosen net node L and its set of causes URS(L). The point of interest is the probability which may be assigned to a generating process ending at the event L and starting at an arbitrary element of URS(L).
If there are k causes $A_1,\ldots, A_k$ of a given net node L, i.e. if the L-Net



represents the current situation, the expression $p(L \mid A_1\overline{A}_2\ldots\overline{A}_k)$ primarily gives the probability of event L in case of condition $(A_1\overline{A}_2\ldots\overline{A}_k)$. Since there is no other cause of L in addition to A_1, the expression $p(L \mid A_1\overline{A}_2\ldots\overline{A}_k)$ simultaneously gives the probability of the event "A_1 creates L" in case of the condition $(A_1\overline{A}_2\ldots\overline{A}_k)$.

The event [A_1 creates L] has the meaning [the system state A_1 leads to the system state L], or [the system state L develops from the system state A_1], or short [there is a transition from A_1 to L]. In this context we consider the cause A_1, the generated event L and the causal generating process "A_1 creates L" as events which occur together.
Correspondingly, the expression $p(L \mid A_1A_2\overline{A}_3\ldots\overline{A}_k)$ represents the conditional probability of the event [(A_1 creates L) ∨ (A_2 creates L)]. In case of condition $(A_1A_2\overline{A}_3\ldots\overline{A}_k)$, the causal process [$A_1$ creates L] may be inhibited or accelerated by A_2 and the causal process [A_2 creates L] may be inhibited or accelerated by A_1. Then no method of standard probability theory is suited to decide which fraction of the generating process belongs to A_1 and which fraction belongs to A_2. In this case, we will not be able to assign a numerical value to the conditional probability

p[(A_1 creates L)| $A_1 A_2 \overline{A_3} ... \overline{A_k}$]. All that can be done is to unite A_1 and A_2 and determine p[(A_1 A_2) creates L | $A_1 A_2 \overline{A_3} ... \overline{A_k}$].

Summary: If L denotes an arbitrary event and A an arbitrary element taken from the set URS(L), we regard the causal process [A creates L] as an event as well and denote it by A→L. The objective is to develop formulas to compute p(A→L | A).

<u>Note</u>

In the past we used the symbol A.L instead of A→L.

A.L and A→L are synonymous symbols.

Because of the definition of A→L the following statements hold:

1. If A is an L-generating event then $\overline{A}$ is not, i.e. $\overline{A}$→L = ∅.
2. If (A→L) then A and L must exist.
3. If $\overline{A}$ or $\overline{L}$ then ($\overline{A \to L}$).

Consequently, we obtain the following equations:

p(A→L) = p(A (A→L)) = p(A (A→L) L). (7.1)

p(A→L | $\overline{A}$) = 0. (7.2)

p(A→L | A $\overline{L}$) = 0. (7.3)

If the symbol A does not represent a single event but a logic product of events, the definition of [A creates L] is extended as follows:

<u>*Definition*</u> *((A_1...A_k) creates L)*

In case of A := (A_1 ... A_k) and A→L := (A_1 ... A_k)→L we define:

p[(A_1 ... A_k)→L | A_1 ... A_k] = p[(A_1→L) ∨... ∨(A_k→L) | A_1 ... A_k],

$p[\overline{(A_1 ... A_k) \to L} \mid A_1 ... A_k] = p[(\overline{A_1 \to L}) \wedge ... \wedge (\overline{A_k \to L}) \mid A_1 ... A_k]$.

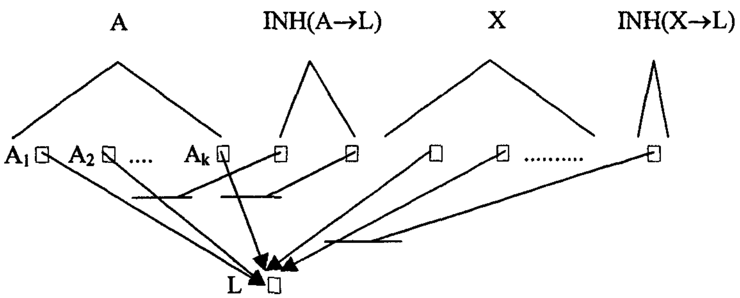
Applying this definition, we obtain Eqs.7.4 and 7.5, which correspond to Eqs.7.2

and 7.3 above.

$$p(\,(A_1 \ldots A_k)\rightarrow L \mid \bar{A}_1 \ldots . \bar{A}_k\,) \quad = 0. \qquad (7.4)$$

$$p(\,(A_1 \ldots A_k)\rightarrow L \mid A_1 \ldots A_k\,\bar{L}\,) = 0, \qquad (7.5)$$

In order to state the A→L-Theorem we use symbols which are introduced next:



Ill.7.1

View of the symbols used to establish the A→L–Theorem.

A arbitrarily chosen, but fixed product of non-negated elements belonging to URS(L), $A := (A_1 \ldots A_k)$.

X arbitrarily chosen product of negated or non-negated events belonging to URS(L) \ {elements in A}.

A→L transition "A creates L". In case of $A := (A_1 \ldots A_k)$, the symbol A→L means: "A_1 creates L" ∨ ... ∨ "A_k creates L".

INH(A→L) set of events which inhibit $A_1\rightarrow L, \ldots, A_k\rightarrow L$, or which increase or decrease the inhibitions acting upon $A_1\rightarrow L, \ldots, A_k\rightarrow L$.

INH(L) set of events which inhibit the transitions leading to L, or which increase or decrease the inhibitions acting upon the transitions leading to L.

D arbitrary product of negated or non-negated elements belonging to INH(A→L).

We demand that an arbitrary A and all X satisfy the following assumptions:

Assumption 1/A→L:

$p((A \to L)(X \to L) \mid A\,X) = p(A \to L \mid A\,X)\, p(X \to L \mid A\,X).$

Assumption 2/A→L:

$p(A \to L \mid A\,X) = p(A \to L \mid A).$

Assumption 3/A→L:

$p(X \to L \mid A\,X) = p(X \to L \mid \overline{A}_1 \ldots \overline{A}_k X).$

Assumption 4/A→L:

$p(X \mid A) = p(X \mid \overline{A}_1 \ldots \overline{A}_k).$

Definition (self-reliant cause)

Event A is called a self-reliant cause of event L, if A satisfies Assumptions 1/A→L through 4/A→L for any X.

Self-reliant causes are the subject of Assumption IIb which is included in Assumption IIc (see Section 6):

Assumption IIc $\Rightarrow$ *Assumption IIc/single* $F_{i_0} \in F$ $\Rightarrow$ *Assumption IIb.*

A→L-Theorem

Let $A := (A_1 \ldots A_k)$ *be a self-reliant cause of the event L. Then*

$$p(A \to L \mid A) = \frac{p(L \mid A) - p(L \mid \overline{A}_1 \ldots \overline{A}_k)}{p(\overline{L} \mid \overline{A}_1 \ldots \overline{A}_k)}. \tag{7.6}$$

$$p(A \to L \mid A\,L) = \frac{p(L \mid A) - p(L \mid \overline{A}_1 \ldots \overline{A}_k)}{p(L \mid A) \cdot p(\overline{L} \mid \overline{A}_1 \ldots \overline{A}_k)}. \tag{7.7}$$

Proof:

1.

From Assumptions 1/A→L and 3/A→L we obtain for any X:

$$p(X \to L \mid AX) = p(X \to L \mid \bar{A}_1 \ldots \bar{A}_k X).$$

$$p(X \to L \mid AX(\overline{A \to L})) = p(X \to L \mid A\,X).$$

⇒

$$p(X \to L \mid AX(\overline{A \to L})) = p(X \to L \mid \bar{A}_1 \ldots \bar{A}_k X).$$

⇒

$$\frac{p((X \to L)AX(\overline{A \to L}))}{p(AX(\overline{A \to L}))} = \frac{p((X \to L)\bar{A}_1 \ldots \bar{A}_k X)}{p(\bar{A}_1 \ldots \bar{A}_k X)}.$$

⇒

$$\frac{p(X(X \to L) \mid A(\overline{A \to L}))}{p(X \mid A(\overline{A \to L}))} = \frac{p(X(X \to L) \mid \bar{A}_1 \ldots \bar{A}_k)}{p(X \mid \bar{A}_1 \ldots \bar{A}_k)}. \qquad (*)$$

Assumption 2/A→L yields:

$$p(\overline{A \to L} \mid AX) = p(\overline{A \to L} \mid A).$$

⇒

$$\frac{p((\overline{A \to L})AX)}{p(AX)} = \frac{p((\overline{A \to L})A)}{p(A)}.$$

⇒

$$p(X \mid A(\overline{A \to L})) = p(X \mid A).$$

⇒ (because of Assumption 4/A→L)

$$p(X \mid A(\overline{A \to L})) = p(X \mid \bar{A}_1 \ldots \bar{A}_k). \qquad (**)$$

From Eqs.(*) and (**) follows for any X:

$$p(X(X \to L) \mid A(\overline{A \to L})) = p(X(X \to L) \mid \bar{A}_1 \ldots \bar{A}_k). \qquad (***)$$

⇒

$$p(L \mid A(\overline{A \to L})) = p(L \mid \bar{A}_1 \ldots \bar{A}_k).$$

The last line follows, since

- Eq.(***) holds for any $\hat{X}$,
- only (X) is able to generate L, and
- the probability of (X(X→L)) conditioned upon $(A(\overline{A \to L}))$ is equal to the probability of (X(X→L)) conditioned upon $(\overline{A}_1 \ldots \overline{A}_k)$.

2.

Without any further assumptions, only using extensions, transformations and the statement of part 1., we proceed as follows:

$$p(LA) = p(LA(A \to L)) + p(LA(\overline{A \to L}))$$

$\Rightarrow$

$$p(LA) = p(L \mid A(A \to L))\, p(A(A \to L)) + p(L \mid A(\overline{A \to L}))\, p(A(\overline{A \to L}))$$

$\Rightarrow$

[because of $p(L \mid A(A \to L)) = 1$ *and* $p(L \mid A(\overline{A \to L})) = p(L \mid \overline{A}_1 \ldots \overline{A}_k)$ *]*

$$p(LA) = p(A(A \to L)) + p(L \mid \overline{A}_1 \ldots \overline{A}_k)\, p(A(\overline{A \to L})).$$

$\Rightarrow$

$$p(LA) =$$

$$p(A \to L \mid A)p(A) + p(L \mid \overline{A}_1 \ldots \overline{A}_k)p(A) - p(L \mid \overline{A}_1 \ldots \overline{A}_k)p(A \to L \mid A)p(A).$$

$\Rightarrow$

$$p(LA) = p(A \to L \mid A)\ [\ p(A) - p(A)p(L \mid \overline{A}_1 \ldots \overline{A}_k)\ ] + p(A)p(L \mid \overline{A}_1 \ldots \overline{A}_k).$$

$\Rightarrow$

$$p(A \to L \mid A) = \frac{p(LA) - p(A)p(L \mid \overline{A}_1 \ldots \overline{A}_k)}{p(A) - p(A)p(L \mid \overline{A}_1 \ldots \overline{A}_k)}.$$

$\Rightarrow$

$$p(A \to L \mid A) = \frac{p(L \mid A) - p(L \mid \overline{A}_1 \ldots \overline{A}_k)}{p(\overline{L} \mid \overline{A}_1 \ldots \overline{A}_k)}.$$

Eq.7.7, the second statement of the A→L-Theorem, holds, since

$$p(A \to L \mid AL) = \frac{p((A \to L)AL)}{p(AL)}$$

$$= \frac{p((A \to L)A)}{p(AL)}$$

$$= \frac{p(A \to L \mid A)}{p(L \mid A)} .$$ □

<u>Complementary remark to Eq.7.6:</u>

<u>Proposition:</u> $0 \le p(A \to L \mid A) \le 1$.

<u>Proof</u> (by contradiction):

Assumption 1: $p(A \to L \mid A) > 1$.

⇒

$$\frac{p(L \mid A) - p(L \mid \overline{A}_1 ... \overline{A}_k)}{1 - p(L \mid \overline{A}_1 ... \overline{A}_k)} > 1.$$

⇒

$$p(L \mid A) - p(L \mid \overline{A}_1 ... \overline{A}_k) > 1 - p(L \mid \overline{A}_1 ... \overline{A}_k).$$

⇒

$p(L \mid A) > 1$.

⇒

Contradiction.

Assumption 2: $p(A \to L \mid A) < 0$.

⇒

$$\frac{p(L \mid A) - p(L \mid \overline{A}_1 ... \overline{A}_k)}{1 - p(L \mid \overline{A}_1 ... \overline{A}_k)} < 0.$$

⇒

$p(L \mid A) < p(L \mid \overline{A}_1 ... \overline{A}_k)$.

⇒

Contradiction. □

Deduction from the A→L-Theorem

If we know about the existence of two events A and L, the A→L-Theorem draws no distinction between [A is the cause of L] and conversely [L is the cause of A]. In other words: In case of two existing events A and L, i.e. if the conditioning logic product (A L) is valid, the event A→L has the same probability as the event L→A. This is the subject of the following rule.

In analogy to the Multiplication Rule we write the L-Rule as follows:

Multiplication Rule: $p(A \mid L)\, p(L) = p(L \mid A)\, p(A).$

L-Rule: $p(A \to L \mid A\, L) = p(L \to A \mid L\, A).$

In a different form:

Multiplication Rule: $p(A\, L) = p(A \mid L)\, p(L) = p(L \mid A)\, p(A).$

L-Rule: $\dfrac{p(AL) - p(A)p(L)}{p(AL)\,p(\overline{A} \cdot \overline{L})} = p(A \to L \mid A\, L) = p(L \to A \mid L\, A).$

Proof of the L-Rule:

$$p(A \to L \mid AL) = \frac{p(L \mid A) - p(L \mid \overline{A})}{p(L \mid A) \cdot p(\overline{L} \mid \overline{A})}$$

$$= \frac{p(LA)p(\overline{A}) - p(L\overline{A})p(A)}{p(LA)\,p(\overline{L} \cdot \overline{A})}$$

$$= \frac{p(LA) - p(LA)p(A) - p(L\overline{A})p(A)}{p(LA)\,p(\overline{L} \cdot \overline{A})}$$

$$= \frac{p(LA) - p(A)[p(LA) + p(L\overline{A})]}{p(LA)\,p(\overline{L} \cdot \overline{A})}$$

$$= \frac{p(LA) - p(A)p(L)}{p(LA)\,p(\overline{L} \cdot \overline{A})}.$$

(Continued on next page)

$$= \frac{p(LA) - p(A)p(L)}{p(LA)p(\overline{L} \cdot \overline{A})}$$

$$= \frac{p(AL) - p(L)p(A)}{p(AL)p(\overline{A} \cdot \overline{L})}$$

$$= \frac{p(AL) - p(L)[p(LA) + p(\overline{L}A)]}{p(AL)p(\overline{A} \cdot \overline{L})}$$

$$= \frac{p(AL) - p(AL)p(L) - p(A\overline{L})p(L)}{p(AL)p(\overline{A} \cdot \overline{L})}$$

$$= \frac{p(AL)p(\overline{L}) - p(A\overline{L})p(L)}{p(AL)p(\overline{A} \cdot \overline{L})}$$

$$= \frac{p(A \mid L) - p(A \mid \overline{L})}{p(A \mid L)p(\overline{A} \mid \overline{L})}$$

$$= p(L \to A \mid LA) .$$ □

The following conjecture (regarded to be almost certainly true) has found its formal proof as a result of the L-Rule:

It is impossible to determine the <u>direction</u> of a causal connection between two existing events by means of stochastic evaluations.

In the course of the expert system's development, which started in 1985, it has become a custom to use the letter L as a characterization. Originally, the symbol L indicated the "<u>l</u>eading symptom" (German: <u>L</u>eitsymptom).

8. A→L-Corollaries

The A→L-Theorem depends on Assumptions 1/A→L through 4/A→L which are contained in Assumption IIc. However, it may happen that the independence of the events in URS(L) and in INH(L), or the independence of the events in URS(L) of the events in INH(L), cannot be reached in any case. We then make use of the possibility to reduce the independence demands by defining subsets of events which are allowed to contain dependent events.

We use the following notation:

A arbitrarily chosen but fixed logic product consisting of non-negated elements in URS(L), $A := (A_1 \ldots A_k)$.

B logic product consisting of arbitrary non-negated elements in URS(L) \ {elements in A}.

C logic product consisting of negated elements in URS(L) \ {elements in (A B)}.

D arbitrary logic product consisting of negated or non-negated elements in INH(A→L).

$\hat{X}$ arbitrarily chosen logic product consisting of negated or non-negated events in URS(L) \ {elements in (A B C)}.

Assumptions 1/Cor.1 through 4/Cor.1 needed for A→L-Corollary 1

For an arbitrarily chosen event $A := (A_1 \ldots A_k)$ and any $\hat{X}$ we demand:

1/Cor.1: $p((A \to L)((B\hat{X}) \to L) \mid ABCD\hat{X})$

$$= p(A \to L) \mid ABCD\hat{X})p((B\hat{X}) \to L \mid ABCD\hat{X}).$$

2/Cor.1: $p(A \to L \mid ABCD\hat{X}) = p(A \to L \mid ABCD).$

3/Cor.1: $p((B\hat{X}) \to L \mid ABCD\hat{X}) = p((B\hat{X}) \to L \mid \bar{A}_1 \ldots \bar{A}_k BCD\hat{X}).$

4/Cor.1: $p(\hat{X} \mid ABCD) = p(\hat{X} \mid \bar{A}_1 \ldots \bar{A}_k BCD).$

A comparison with Assumption IIc reveals the difference that now

- only events in URS(L) \ {elements in (B C)} have to be independent and not all events in URS(L), and
- only events in INH(L) \ [INH(C→L) ∪ {elements in D}] are demanded to be independent and not all events in INH(L).

Hence, we establish the following two steps to handle dependent events:

1. If URS(L) contains two elements being stochastically dependent, we assign one of them to B or C.
2. If INH(A→L) contains an element which depends on an element in INH((B $\hat{X}$)→L), it is sufficient (according to the statements in Section 5) to insert this INH(A→L)-element into D in order to preserve the independence of A→L and (B $\hat{X}$)→L.

But we have to accept the drawback that the formulas (belonging to the A→L - Corollary 1) will become large for extensive logic products B, C, D.

<u>*Proposition*</u>

Let K_1, K_2 ∈ URS(L) be arbitrary events.

If [elements in URS(L) \ {elements in (B C)} are independent]

∧ [elements in INH(K_1→L) \ (INH(C→L) ∪{elements in D}) are independent of the elements in INH(K_2→L) \ (INH(C→L) ∪{elements in D})]

∧ [elements in URS(L) \ {elements in (B C) } are independent of the elements in INH(L) \ (INH(C→L) ∪{elements in D})]

⇒

any element in URS(L) satisfies Assumptions 1/Cor.1 through 4/Cor.1.

<u>*Definition (conditional self-reliance)*</u>

If an event A ∈ URS(L) satisfies Assumptions 1/Cor.1 through 4/Cor.1., it is called a self-reliant cause of L, conditioned upon (B C D).

A→L-Corollary 1

Let A be a self-reliant cause of L, conditioned upon (B C D).

Then

$$p(A \to L \mid ABCD) = \frac{p(L \mid ABCD) - p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(\bar{L} \mid \bar{A}_1 ... \bar{A}_k BCD)}. \quad (8.1)$$

$$p(A \to L \mid ABCDL) = \frac{p(L \mid ABCD) - p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(L \mid ABCD) \cdot p(\bar{L} \mid \bar{A}_1 ... \bar{A}_k BCD)}. \quad (8.2)$$

Proof:

1.

From 1/Cor.1 and 3/Cor.1 it follows that for any $\hat{X}$:

$$\left.\begin{array}{l} p((B\hat{X}) \to L \mid ABCD\hat{X}) = p((B\hat{X}) \to L \mid \bar{A}_1 ... \bar{A}_k BCD\hat{X}). \\ p((B\hat{X}) \to L \mid ABCD\hat{X}(\overline{A \to L})) = p((B\hat{X}) \to L \mid ABCD\hat{X}). \end{array}\right\rangle$$

$\Rightarrow$

$$p((B\hat{X}) \to L \mid ABCD\hat{X}(\overline{A \to L})) = p((B\hat{X}) \to L \mid \bar{A}_1 ... \bar{A}_k BCD\hat{X}).$$

$\Rightarrow$

$$\frac{p([(B\hat{X}) \to L]ABCD\hat{X}(\overline{A \to L}))}{p(ABCD\hat{X}(\overline{A \to L}))} = \frac{p([(B\hat{X}) \to L]\bar{A}_1 ... \bar{A}_k BCD\hat{X})}{p(\bar{A}_1 ... \bar{A}_k BCD\hat{X})}.$$

$\Rightarrow$

$$\frac{p(\hat{X}[(B\hat{X}) \to L] \mid ABCD(\overline{A \to L}))}{p(\hat{X} \mid ABCD(\overline{A \to L}))} = \frac{p(\hat{X}[(B\hat{X}) \to L] \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(\hat{X} \mid \bar{A}_1 ... \bar{A}_k BCD)}. \quad (*)$$

2/Cor.1 yields for any $\hat{X}$:

$$p(\overline{A \to L} \mid ABCD\hat{X}) = p(\overline{A \to L} \mid ABCD).$$

$\Rightarrow$

$$\frac{p((\overline{A.L})ABCD\hat{X})}{p(ABCD\hat{X})} = \frac{p((\overline{A.L})ABCD)}{p(ABCD)}.$$

$\Rightarrow$

$p(\hat{X} \mid ABCD(\overline{A \to L})) = p(\hat{X} \mid ABCD).$

$\Rightarrow$ *(because of 4/Cor.1)*

$p(\hat{X} \mid ABCD(\overline{A \to L})) = p(\hat{X} \mid \overline{A}_1 ... \overline{A}_k BCD).$ (**)

From Eq.(*) and Eq.(**) we now obtain for any $\hat{X}$ *:*

$p(\hat{X}[(B\hat{X}) \to L] \mid ABCD(\overline{A \to L})) = p(\hat{X}[(B\hat{X}) \to L] \mid \overline{A}_1 ... \overline{A}_k BCD).$ (***)

$\Rightarrow$

$p(L \mid ABCD(\overline{A \to L})) = p(L \mid \overline{A}_1 ... \overline{A}_k BCD).$

The last line follows, since

- Eq.(***) holds for any $\hat{X}$,
- only (B $\hat{X}$) is able to generate L, and
- the probability of $(\hat{X}(B\hat{X}) \to L)$ conditioned upon $(ABCD(\overline{A \to L}))$ is equal to the probability of $(\hat{X}(B\hat{X}) \to L)$ conditioned upon $(\overline{A}_1 ... \overline{A}_k BCD)$.

2.

Without any further assumptions, only using extensions, transformations and the statement of part 1., we proceed as follows:

$$p(LABCD) = p(LABCD(A \to L)) + p(LABCD(\overline{A \to L})).$$

$\Rightarrow$

$$p(LABCD) = p(L \mid ABCD(A \to L))p(ABCD(A \to L)) + p(L \mid ABCD(\overline{A \to L}))p(ABCD(\overline{A \to L}))$$

$\Rightarrow$ (because of $p(L \mid ABCD(A \to L)) = 1$ and

$p(L \mid ABCD(\overline{A \to L})) = p(L \mid \overline{A}_1 ... \overline{A}_k BCD)$)

$$p(LABCD) = p(ABCD(A \to L)) + p(L \mid \overline{A}_1 ... \overline{A}_k BCD)p(ABCD(\overline{A \to L})).$$

$\Rightarrow$

$$p(LABCD) = p(A \to L \mid ABCD)p(ABCD) + p(L \mid \overline{A}_1 ... \overline{A}_k BCD)p(ABCD) - p(L \mid \overline{A}_1 ... \overline{A}_k BCD)p(A \to L \mid ABCD)p(ABCD).$$

$\Rightarrow$

$$p(LABCD) = p(A \to L \mid ABCD) \cdot [\ p(ABCD) - p(ABCD)p(L \mid \bar{A}_1 ... \bar{A}_k BCD)\ ] + p(ABCD)p(L \mid \bar{A}_1 ... \bar{A}_k BCD).$$

$\Rightarrow$

$$p(A \to L \mid ABCD) = \frac{p(LABCD) - p(ABCD)p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(ABCD) - p(ABCD)p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}.$$

$\Rightarrow$

$$p(A \to L \mid ABCD) = \frac{p(L \mid ABCD) - p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(\bar{L} \mid \bar{A}_1 ... \bar{A}_k BCD)}.$$

Eq.8.2, the second statement of the A→L-Corollary 1, holds, since:

$$p(A \to L \mid ABCDL) = \frac{p((A \to L)ABCDL)}{p(ABCDL)} = \frac{p((A \to L)ABCD)}{p(ABCDL)} = \frac{p(A \to L \mid ABCD)}{p(L \mid ABCD)}.$$

□

A→L-Corollary 1 allows to reduce the independence demands of the A→L-Theorem step by step.

If (A B C) contains all events in URS(L), i.e. if $\hat{X} := \varnothing$, Assumptions 2/Cor.1 and 4/Cor.1 expire. This is the subject of the following A→L-Corollary 1.1.

A→L-Corollary 1.1

Let (A B C) contain all events in URS(L). Let $A := (A_1 ... A_k)$ satisfy the following assumptions:

1/Cor.1.1: $p((A \to L)\ (B \to L) \mid A\ B\ C\ D) = p(A \to L \mid A\ B\ C\ D)\ p(B \to L \mid A\ B\ C\ D)$.

2/Cor.1.1: *expired.*

3/Cor.1.1: $p(B \to L \mid A\ B\ C\ D) = p(B \to L \mid \bar{A}_1 \ ...\ \bar{A}_k\ B\ C\ D)$.

4/Cor.1.1: *expired.*

Then

$$p(A \to L \mid ABCD) = \frac{p(L \mid ABCD) - p(L \mid \bar{A}_1 ... \bar{A}_k BCD)}{p(\bar{L} \mid \bar{A}_1 ... \bar{A}_k BCD)}. \qquad (8.3)$$

Proof:

The result follows from A→L-Corollary 1 for $\hat{X} := \varnothing$.

Complementary remark to Eq.8.3

Proposition:

If [elements in INH(A→L) are independent of the events in INH(B→L)]

∧ [events in A are independent of the events in INH(B→L)]

⇒

A satisfies 1/Cor.1.1 and 3/Cor.1.1.

A→L-Corollary 1 results in the following A→L-Corollary 1.2, if all causes of L are negated with the exception of event A.

A→L-Corollary 1.2

Let (A C) contain all events in URS(L).

Remark: All assumptions are expired.

Then

$$p(A \to L \mid ACD) = \frac{p(L \mid ACD) - p(L \mid \bar{A}_1 ... \bar{A}_k CD)}{p(\bar{L} \mid \bar{A}_1 ... \bar{A}_k CD)}$$

$$= p(L \mid ACD). \qquad (8.4)$$

Proof:

This is a direct consequence of A→L-Corollary 1, since $p(L \mid \bar{A}_1 ... \bar{A}_k CD) = 0$.

For some applications the following might be significant:

Let A := $(A_1 \dots A_k)$ and C := $(\bar{A}_u \dots \bar{A}_v)$ represent logic products which contain all known elements from URS(L), i.e. A includes all known causes of L having p = 1 while C includes all known causes of L having p = 0. The logic product B consists of all unknown causes of L.

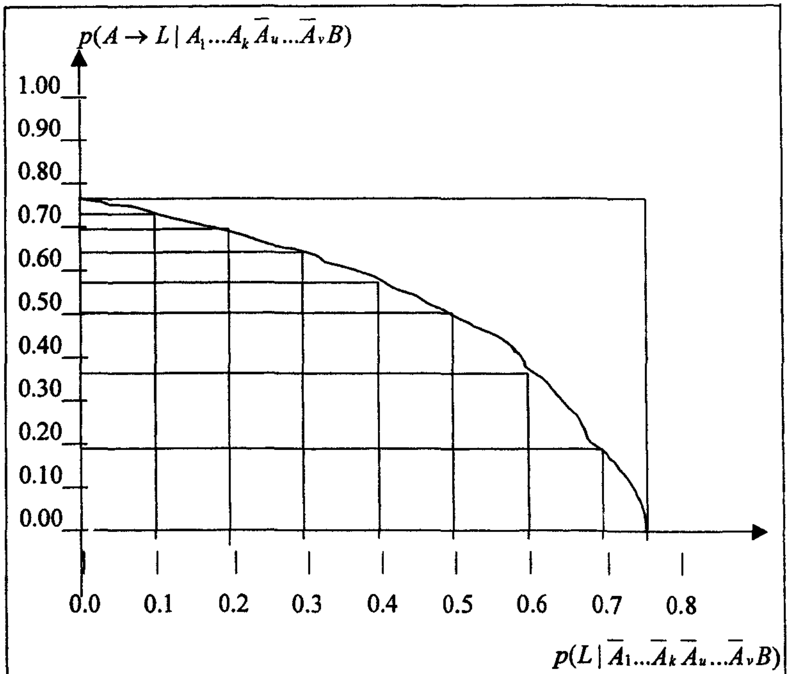
Without loss of generality we set D := ∅.
If there are no causes of L outside of (A C), i.e. if B = ∅, we have
$p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B) = 0$.
If we assume B ≠ ∅ and thus have $p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B) > 0$, we want to express the probability of "A creates L" in terms of $p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)$. If we assign an arbitrary value, e.g. 0.75, to $p(L \mid A_1 \dots A_k \bar{A}_u \dots \bar{A}_v B)$, we obtain from Eq.8.3

$$p(A \to L \mid A_1 \dots A_k \bar{A}_u \dots \bar{A}_v B) = \frac{0.75 - p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)}{1 - p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)}, \tag{8.5}$$

which leads to the following table of values:

$p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)$	$\dfrac{0.75 - p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)}{1 - p(L \mid \bar{A}_1 \dots \bar{A}_k \bar{A}_u \dots \bar{A}_v B)}$	$p(A \to L \mid A_1 .. A_k \bar{A}_u .. \bar{A}_v B)$
0.00	0.75 : 1.00	0.7500
0.01	0.74 : 0.99	0.7475
0.02	0.73 : 0.98	0.7449
0.03	0.72 : 0.97	0.7423
0.04	0.71 : 0.96	0.7396
0.05	0.70 : 0.95	0.7368
0.06	0.69 : 0.94	0.7340
0.07	0.68 : 0.93	0.7312
0.08	0.67 : 0.92	0.7283
0.09	0.66 : 0.91	0.7253
0.10	0.65 : 0.90	0.7222

$p(L \mid \bar{A}_1...\bar{A}_k\bar{A}_u...\bar{A}_vB)$	$\frac{0.75 - p(L \mid \bar{A}_1...\bar{A}_k\bar{A}_u...\bar{A}_vB)}{1 - p(L \mid \bar{A}_1...\bar{A}_k\bar{A}_u...\bar{A}_vB)}$	$p(A \to L \mid A_1..A_k\bar{A}_u..\bar{A}_vB)$
0.1	0.65 : 0.9	0.72
0.2	0.55 : 0.8	0.69
0.3	0.45 : 0.7	0.64
0.4	0.35 : 0.6	0.58
0.5	0.25 : 0.5	0.50
0.6	0.15 : 0.4	0.38
0.7	0.05 : 0.3	0.17
0.75	0.00 : 0.25	0



Ill.8.1

Graph of Eq.8.5.

Obviously, the probability of [A creates L] decreases if unknown causes increase.

The A→L-Theorem and its Corollaries can also be used to answer a question which emerged in Section 5.

<u>Conjecture</u>

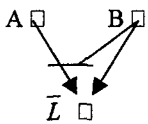
Let independent events A and B be causes of an event L. Then

$p(AB \mid \bar{L}) = p(A \mid \bar{L}) p(B \mid \bar{L})$.

This statement, although it seems to be next to certain, is <u>not</u> true in general.

The reason is that the inhibitors and their dependencies are not defined. If the independence of A and B is the only information about the structuring of the causal network, it is of course not forbidden that the inhibitors belonging to A→L are dependent of the inhibitors belonging to B→L, or that for instance the event B influences A→L.

<u>Example:</u>



The events A and B are independent, but $p(AB \mid \bar{L}) \neq p(A \mid \bar{L}) p(B \mid \bar{L})$. This can be seen immediately by considering the fact that event L with p = 0 causes the probability of A to decrease if the probability of B decreases.

The following corollary states the properties which are needed to maintain the independence of A and B in case of the condition $\bar{L}$.

<u>A→L-Corollary 2</u>

Let events A and B be arbitrary causes of an event L. If A and B are self-reliant causes of L, then

$$p(AB \mid \bar{L}) = p(A \mid \bar{L}) p(B \mid \bar{L}). \qquad (8.6)$$

Proof:

$$p(\overline{(AB)\to L} \mid AB) = p(\overline{(A\to L)\vee(B\to L)} \mid AB)$$

$$= p(\overline{(A\to L)}\,\overline{(B\to L)} \mid AB)$$

= *(because of Assumption 1/A→L)*

$$p(\overline{A\to L} \mid AB)\,p(\overline{B\to L} \mid AB)$$

= *(because of Assumptions 2/A→L and 3/A→L)*

$$p(\overline{A\to L} \mid A)\,p(\overline{B\to L} \mid B). \qquad (*)$$

Applying the A→L-Theorem, Eq(*) becomes:

$$\frac{p(\overline{L} \mid AB)}{p(\overline{L} \mid \overline{A}\cdot\overline{B})} = \frac{p(\overline{L} \mid A)}{p(\overline{L} \mid \overline{A})}\frac{p(\overline{L} \mid B)}{p(\overline{L} \mid \overline{B})}.$$

⇒ *(because A and B are independent)*

$$\frac{p(\overline{L}AB)}{p(\overline{L}\cdot\overline{A}\cdot\overline{B})} = \frac{p(\overline{L}A)}{p(\overline{L}\cdot\overline{A})}\frac{p(\overline{L}B)}{p(\overline{L}\cdot\overline{B})}.$$

⇒

$$\frac{p(AB \mid \overline{L})}{p(\overline{A}\cdot\overline{B} \mid \overline{L})} = \frac{p(A \mid \overline{L})}{p(\overline{A} \mid \overline{L})}\frac{p(B \mid \overline{L})}{p(\overline{B} \mid \overline{L})}.$$

⇒

$$p(AB \mid \overline{L})\,[1 - p(A \mid \overline{L}) - p(B \mid \overline{L}) + p(A \mid \overline{L})p(B \mid \overline{L})]$$

$$= p(A \mid \overline{L})p(B \mid \overline{L})\,[1 - p(A \mid \overline{L}) - p(B \mid \overline{L}) + p(AB \mid \overline{L})]. \qquad (**)$$

⇒

$$p(AB \mid \overline{L})\,[1 - p(A \mid \overline{L}) - p(B \mid \overline{L})]$$

$$= p(A \mid \overline{L})p(B \mid \overline{L})\,[1 - p(A \mid \overline{L}) - p(B \mid \overline{L})].$$

⇒

$$p(AB \mid \overline{L}) = p(A \mid \overline{L})p(B \mid \overline{L}).$$

Eq.(**) holds, since

$$
\begin{aligned}
p(\bar{A}\cdot\bar{B} \mid \bar{L}) &= 1 - p(\overline{\bar{A}\cdot\bar{B}} \mid \bar{L}) \\
&= 1 - p(A \vee B \mid \bar{L}) \\
&= 1 - [p(A \mid \bar{L}) + p(B \mid \bar{L}) - p(AB \mid \bar{L})] \\
&= 1 - p(A \mid \bar{L}) - p(B \mid \bar{L}) + p(AB \mid \bar{L}).
\end{aligned}
\qquad \square
$$

The independence of self-reliant events is not lost if conditioned on $\bar{L}$. If the events A and B are only independent but not self-reliant, Eq.8.6 does not hold.

A→L-Corollary 2 is used in the course of Section 9 below.
If I_1 and I_2 denote two inhibitors which act upon an arbitrary transition $K_1 \to F_1$, and if I_1 and I_2 are self-reliant causes of $(\overline{K_1 \to F_1})$ for an existing event K_1, A→L-Corollary 2 yields:

$$p(I_1 I_2 \mid \overline{\overline{K_1 \to F_1}}) = p(I_1 \mid \overline{\overline{K_1 \to F_1}})\, p(I_2 \mid \overline{\overline{K_1 \to F_1}}).$$

A→L-Theorem and A→L-Corollary 1 have been published in 1991. See:

LIEBEL, F.-P.: Wahrscheinlichkeit der Entstehung eines Folgezustands aus einer vorhandenen Ursache. In: Österreichische Zeitschrift für Statistik und Informatik (ZSI), 21. Jg. (1991), Heft 3 – 4.

9. Computation of ap-probabilities

In order to compute the ap-probability p(H | H') we make use of the assumptions and transform $p(H \mid H') = p(H \mid UFKIJ)$, which is identical to Eq.1.3, as follows:

$$
\begin{aligned}
p(H \mid UFKIJ) &= \frac{p(HUFKIJ)}{p(HUFKIJ) + p(\overline{H}UFKIJ)} \\
&= \frac{1}{1 + \dfrac{p(UFKIJ \mid \overline{H})p(\overline{H})}{p(UFKIJ \mid H)p(H)}} \\
&=_{\text{(Assumpt.III)}} \frac{1}{1 + \dfrac{p(FKJ \mid \overline{H})p(UI \mid \overline{H})p(\overline{H})}{p(FKJ \mid H)p(UI \mid H)p(H)}} \\
&= \frac{1}{1 + \dfrac{p(FKJ\overline{H})p(UI \mid \overline{H})}{p(FKJH)p(UI \mid H)}} \\
&= \frac{1}{1 + \dfrac{p(F \mid KJ\overline{H})p(KJ\overline{H})p(UI \mid \overline{H})}{p(F \mid KJH)p(KJH)p(UI \mid H)}} \\
&= \frac{1}{1 + \dfrac{p(F \mid KJ\overline{H})p(\overline{H} \mid KJ)p(UI \mid \overline{H})}{p(F \mid KJH)p(H \mid KJ)p(UI \mid H)}} \\
&=_{\text{(Assumpt.IIc)}} \frac{1}{1 + \dfrac{p(F \mid KJ\overline{H})p(\overline{H})p(UI \mid \overline{H})}{p(F \mid KJH)p(H)p(UI \mid H)}} \\
&= \frac{1}{1 + \dfrac{p(F \mid KJ\overline{H})p(UI\overline{H})}{p(F \mid KJH)p(UIH)}} \\
&= \frac{1}{1 + \dfrac{p(F \mid KJ\overline{H})p(\overline{H} \mid UI)}{p(F \mid KJH)p(H \mid UI)}} \,. \qquad (9.1)
\end{aligned}
$$

The next step will be the computation of the conditioned probabilities p(F | K J H) and p(H | U I) which appear in Eq.9.1. Since these probabilities normally contain apostrophized events within their conditioning products, we have to apply an interpolation procedure. (Early in Section 6 we defined K and U to represent the products $K := (K_1^{'}K_2K_3^{'})$ and $U := (U_1U_2^{'}U_3^{'})$.)

We first consider *p(F | K J H)*. According to Assumption IIc, the events in the logic product *(K J H)*, i.e. K_1, K_2, K_3, J_1, J_2, *and H*, are independent. Since K_1 and K_3 are independent in case of the condition $(K_2\, J\, H)$, we are able to use the Linear Interpolation Theorem to obtain:

$$
\begin{aligned}
&p(F_1F_2...F_5 \mid K_1^{'}K_2K_3^{'}JH) \\
= \; & p(F_1F_2...F_5 \mid K_1K_2K_3JH)p(K_1 \mid K_1^{'})p(K_3 \mid K_3^{'}) \qquad (*) \\
& + p(F_1F_2...F_5 \mid K_1K_2\overline{K_3}JH)p(K_1 \mid K_1^{'})p(\overline{K_3} \mid K_3^{'}) \\
& + p(F_1F_2...F_5 \mid \overline{K_1}K_2K_3JH)p(\overline{K_1} \mid K_1^{'})p(K_3 \mid K_3^{'}) \\
& + p(F_1F_2...F_5 \mid \overline{K_1}K_2\overline{K_3}JH)p(\overline{K_1} \mid K_1^{'})p(\overline{K_3} \mid K_3^{'}).
\end{aligned}
$$

In order to establish Eq.(*) we demonstrate another method which does not utilize the Linear Interpolation Theorem. However, the independence of the events in (K J H) is also needed.

We execute the following simple transformation by using the chain rule:

$$
\begin{aligned}
p(F \mid KJH) = \; & p(F_1...F_5 \mid K\,J\,H) \\
= \; & p(F_1 \mid F_2...F_5\,K\,J\,H) \qquad (**) \\
& \cdot p(F_2 \mid F_3...F_5\,K\,J\,H) \\
& \cdot p(F_3 \mid \overline{F}_4\, F_5\,K\,J\,H) \\
& \cdot p(\overline{F}_4 \mid F_5\,K\,J\,H) \\
& \cdot p(F_5 \mid K\,J\,H).
\end{aligned}
$$

We now apply the L-Theorem (Eq.4.3) five times to the right-hand side of Eq.(**) and obtain:

$$p(F_1F_2...F_5 \mid K_1^{'}K_2K_3^{'}JH)$$

$$= \frac{p(F_1F_2...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(F_2...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$\cdot \frac{p(F_2F_3...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(F_3...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$\cdot \frac{p(F_3\overline{F_4}...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(\overline{F_4}...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$\cdot \frac{p(\overline{F_4}F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$\cdot \frac{p(F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$= \frac{p(F_1F_2...F_5K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(K_1K_2K_3JH)\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

$$= \frac{p(F_1F_2...F_5K_1K_2K_3JH)\cdot\frac{1}{p(K_2JH)}\cdot\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc.}{p(K_1K_2K_3JH)\cdot\frac{1}{p(K_2JH)}\cdot\frac{p(K_1 \mid K_1^{'})}{p(K_1)}\frac{p(K_3 \mid K_3^{'})}{p(K_3)} + ...etc}$$

= *(because the events in (K J H) are independent)*

$$p(F_1F_2...F_5 \mid K_1K_2K_3JH)p(K_1 \mid K_1^{'})p(K_3 \mid K_3^{'}) + ...etc.$$

□

Note:

We are allowed to apply the Linear Interpolation Theorem to the left-hand side of Eq.(**) because Requirement b) of the Linear Interpolation Theorem is satisfied. But we are not allowed to do the same to all factors on the right-hand side of Eq.(**) since only the last one meets Requirement b).

The difficulties concerning the computation of the probability p(F | K J H) do not differ from the questions which emerged in the course of establishing Eq.4.4-9. There we discussed

$$p(F_1 \mid F_2 K_0 K_1^{'} K_2^{'}) \neq p(F_1 \mid K_0 K_1^{'} K_2^{'}),$$

which is of the same form as

$$p(F_1 \mid F_2 ... F_5 K_1^{'} K_2 K_3^{'} JH) \neq p(F_1 \mid K_1^{'} K_2 K_3^{'} JH).$$

The last mentioned inequality holds since $p(F_1 \mid F_2 ... F_5 K_1^{'} K_2 K_3^{'} JH)$ is not part of an L-Net computation system (see Section 4, Eq.4.4-9.) and thus F_1 is not separated from $(F_2 ... F_5)$ by means of $(K_1^{'} K_2 K_3^{'} JH)$.

This leads to the following conclusion:

- *It is not permitted to decompose* $p(F_1 F_2 ... F_5 \mid K_1^{'} K_2 K_3^{'} JH)$ *into the factors* $p(F_1 \mid K_1^{'} K_2 K_3^{'} JH), ..., p(F_5 \mid K_1^{'} K_2 K_3^{'} JH)$.
- *It is necessary to apply the Linear Interpolation Theorem (or another interpolation procedure) to* $p(F_1 F_2 ... F_5 \mid K_1^{'} K_2 K_3^{'} JH)$ *first and only then execute the decomposition*

$$p(F_1 F_2 ... F_5 \mid K_1 K_2 K_3 JH) = p(F_1 \mid K_1 K_2 K_3 JH) \cdot ... \cdot p(F_5 \mid K_1 K_2 K_3 JH).$$

This conclusion is expressed in Eqs.9.2 and 9.3 below.

The Linear Interpolation Theorem, or the L-Theorem as shown above, yields for n '-events $K_1^{'}, ..., K_n^{'}$:

$$p(F \mid K_1^{'} .. K_n^{'} JH) = \sum_{q_1, ..., q_n = 0,1} p(F \mid K_1^{(q_1)} ... K_n^{(q_n)} JH) p(K_1^{(q_1)} \mid K_1^{'}) \cdot ... \cdot p(K_n^{(q_n)} \mid K_n^{'}) . \quad (9.2)$$

Eq.9.2 includes the probabilities $p(F \mid K_1^{(q_1)}...K_n^{(q_n)}JH)$ which do not contain apostrophized events. Since the conditioning product contains all causes of events belonging to F [see Assumption IIa], and since the elements in INH(F_{i_0}) are independent of the elements in INH(F_{i_1}) for any $F_{i_0}, F_{i_1} \in$ F [see Assumption IIc], we obtain:

$$p(F \mid K_1^{(q_1)}...K_n^{(q_n)}JH) = \prod_{F_i \in F} p(F_i \mid K_1^{(q_1)}...K_n^{(q_n)}JH). \qquad (9.3)$$

Eq.9.3 contains the probabilities $p(F_i \mid K_1...K_n\ J\ H)$, $F_i \in F$. In order to decompose probabilities of this form into factors we develop the theorems T-9.4 and T.9.12 below.

Eq.9.4 of Theorem T-9.4 will produce conditional probabilities which contain only one cause with p = 1 in their conditional product. The inhibitors remain unchanged. Factorization Theorem T-9.12 will then provide the tools to factor with respect to the inhibitors.

Theorem (T-9.4) *(Factorization with respect to causes)*

Let {H, K_1, ..., K_n} denote the set of all causes of an arbitrarily chosen event F_1.
Let all elements in {H, K_1, ..., K_n} be self-reliant causes of F_1.
Let I_1, ... , I_r be the inhibitors of $K_1 \rightarrow F_1$, and J_1, ... , J_s the inhibitors of $K_2 \rightarrow F_1$.
Then

$$\begin{aligned} p(\overline{F}_1 \mid HK_1...K_nI_1...I_rJ_1...J_s) = \ & p(\overline{F}_1 \mid H\overline{K}_1\overline{K}_2...\overline{K}_n) \qquad (9.4) \\ & \cdot p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1...I_r) \\ & \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_nJ_1...J_s) \\ & \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1\overline{K}_2K_3\overline{K}_4...\overline{K}_n) \\ & \vdots \\ & \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_{n-1}K_n). \end{aligned}$$

In particular, we have:

$$p(\overline{F}_1 \mid HK_1...K_n)$$
$$= p(\overline{F}_1 \mid H\overline{K}_1...\overline{K}_n)p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot...\cdot p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n). \qquad (9.5)$$

Remark:

If the conditioning product on the left-hand side of Eq.9.4 or 9.5 contains a negated event, e.g. $(H\overline{K}_1K_2...K_n)$, *then terms containing solely negated causes* $[p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1\overline{K}_2...\overline{K}_nI_1...I_r)$ *or* $p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}\overline{K}_n)]$ *are replaced by 1 in Eq.9.4 and 9.5, respectively.* *(9.6)*

<u>*Proof:*</u>

$$p(F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$$
$$= p[(H \to F_1)\vee(K_1 \to F_1)\vee...\vee(K_n \to F_1) \mid HK_1...K_nI_1...I_rJ_1...J_s] \qquad (9.7)$$
$$= 1 - p[\overline{(H \to F_1)}\wedge\overline{(K_1 \to F_1)}\wedge...\wedge\overline{(K_n \to F_1)} \mid HK_1...K_nI_1...I_rJ_1...J_s]$$
$$= 1 - [1 - p(H \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)]$$
$$\cdot[1 - p(K_1 \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)]$$
$$\vdots$$
$$\cdot[1 - p(K_n \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)]$$

= *(because of the self-reliance property)*

$$1 - [1 - p(H \to F_1 \mid H\overline{K}_1...\overline{K}_n)] \qquad (9.7a)$$
$$\cdot[1 - p(K_1 \to F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1...I_r)]$$
$$\cdot[1 - p(K_2 \to F_1 \mid \overline{H}\cdot\overline{K}_1K_2\overline{K}_3...\overline{K}_nJ_1...J_s)]$$
$$\cdot[1 - p(K_3 \to F_1 \mid \overline{H}\cdot\overline{K}_1\overline{K}_2K_3\overline{K}_4...\overline{K}_n)]$$
$$\vdots$$
$$\cdot[1 - p(K_n \to F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n)]$$

= *(applying the A→L-Theorem or A→L-Corollary 1)* *(continued on next page)*

$$= 1 - [\,1 - \frac{p(F_1 \mid H\overline{K}_1...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}\,]$$

$$\cdot [\,1 - \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n I_1...I_r) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n I_1...I_r)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n I_1...I_r)}\,]$$

$$\cdot [\,1 - \frac{p(F_1 \mid \overline{H}\cdot\overline{K}_1 K_2\overline{K}_3...\overline{K}_n J_1...J_s) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n J_1...J_s)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n J_1...J_s)}\,]$$

$$\cdot [\,1 - \frac{p(F_1 \mid \overline{H}\cdot\overline{K}_1\overline{K}_2 K_3\overline{K}_4...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}\,]$$

$$\vdots$$

$$\cdot [\,1 - \frac{p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}\,].$$

The rest of the proof is shown using the last factor:

$$1 - \frac{p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$= \frac{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n) + p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$= \frac{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$= \frac{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}\,.$$

Hence, $p(F_1 \mid HK_1...K_n I_1...I_r J_1...J_s)$

$$= 1 - \frac{p(\overline{F}_1 \mid H\overline{K}_1...\overline{K}_n)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)} \cdot \frac{p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n I_1...I_r)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$\cdot \frac{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1 K_2\overline{K}_3...\overline{K}_n J_1...J_s)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)} \cdot \frac{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1\overline{K}_2 K_3\overline{K}_4...\overline{K}_n)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$\cdot \,.....\, \cdot \frac{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n)}{p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}\,,$$

and the result follows, since $p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n) = 1$.

Eq.9.5, the second statement of Theorem T-9.4, represents a special case which has been explicitly mentioned for its easily remembered form. Statement (9.6) follows from Eq.9.7 if we consider the definition of A→L in Section 7, which demands $\overline{A} \to L = \varnothing$. □

Eq.9.5 turns out to be extraordinarily convenient because it allows to decompose the probability $p(\overline{F}_1 \mid H K_1 \ldots K_n)$ right away, just by writing down the factors. Moreover, we notice that the factors of the product

$$p(\overline{F}_1 \mid H\overline{K}_1\overline{K}_2\ldots\overline{K}_n)p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2\ldots\overline{K}_n)\cdot\ldots\ldots\cdot p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1\ldots\overline{K}_{n-1}K_n)$$

possess a remarkable symmetry. Each factor contains exactly one F_1-generating cause which has p = 1, while all other causes of F_1 have p = 0. This greatly simplifies the statistic sampling needed to determine the numerical values of such probabilities.

"Simplification of statistic sampling" means that a comparatively large population can be used. In the case of e.g. $p(\overline{F}_1 \mid H\overline{K}_1\ldots\overline{K}_n)$, the only required property of the population is H; none of $K_1,\ldots,K_n$ is requireded. Using this population we count all occurrences having property F_1 and obtain the relative frequency $h(F_1 \mid H\overline{K}_1\ldots\overline{K}_n)$ which allows the approximation

$$p(\overline{F}_1 \mid H\overline{K}_1\ldots\overline{K}_n) \approx 1 - h(F_1 \mid H\overline{K}_1\ldots\overline{K}_n).$$

If we look back at Eq.9.4 we detect the probabilities $p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2\ldots\overline{K}_nI_1\ldots I_r)$ and $p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1K_2\overline{K}_3\ldots\overline{K}_nJ_1\ldots J_s)$. It is possible to factorize once again, this time with respect to the inhibitors.

We keep the notation used to state Theorem T-9.4 above. In order to decompose a probability such as $p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2\ldots\overline{K}_nI_1\ldots I_r)$ into factors with respect to the inhibitors we need the following assumption which has already been stated at the end of Section 6.

Assumption concerning inhibitors

If K_1 exists, the inhibitors $I_1,\dots, I_r$ are assumed to be self-reliant causes of the event $\overline{(K_1 \to F_1)}$.

Comment

Considering the events $I_1,\dots, I_r$, which are assumed to be self-reliant causes of $\overline{(K_1 \to F_1)}$ in case of an existing event K_1, we may use A→L-Corollary 2 without any change. The notation of A→L-Corollary 2 corresponds with the notation of the assumption above as follows:

	A→L-Corollary 2	Assumption concerning inhibitors
Generated event	$\overline{L}$	$\overline{(K_1 \to F_1)}$
Set of causes	{A, B}	$\{ I_1, \dots, I_r \}$.

$I_1, \dots, I_r$ are stochastically independent given K_1.

According to A→L-Corollary 2, the independence of two events I_1 and I_2 is not lost in case of a conditioning element which is commonly generated but actually not existent. In the present case, the independence of I_1 and I_2 is not lost by the commonly generated event $\overline{(K_1 \to F_1)}$ in its negated form, i.e. $\overline{\overline{(K_1 \to F_1)}}$, which means $(K_1 \to F_1)$.

Therefore, the events $I_1,\dots, I_r$, which are independent if conditioned upon (K_1), do not loose the independence property when the condition (K_1) is increased to $(K_1 (K_1 \to F_1))$.

We use the independence of $I_1, \dots, I_r$, conditioned upon $(K_1 (K_1 \to F_1))$, in order to establish the following theorem (Theorem T-9.9), which is needed to develop Theorem T-9.12.

Theorem (T-9.9) *(Factorization with respect to inhibitors)*

Let $\{H, K_1, \ldots, K_n\}$ denote the set of all causes of an arbitrarily chosen event F_1. Let all elements in $\{H, K_1, \ldots, K_n\}$ be self-reliant causes of F_1, and let $I_1, \ldots, I_r$ be the inhibitors of $K_1 \to F_1$.

Assume that $I_1, \ldots, I_r$ are self-reliant causes of $\overline{(K_1 \to F_1)}$ if K_1 exists.

Then

$$p(K_1 \to F_1 \mid K_1 I_1 \ldots I_r) =$$

$$p(K_1 \to F_1 \mid K_1) \cdot \frac{p(K_1 \to F_1 \mid K_1 I_1)}{p(K_1 \to F_1 \mid K_1)} \cdot \ldots\ldots \cdot \frac{p(K_1 \to F_1 \mid K_1 I_r)}{p(K_1 \to F_1 \mid K_1)} . \qquad (9.8)$$

If we define factors $i_1, \ldots, i_r$ to represent the quotients

$$i_1 := \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)}, \ \ldots, \ i_r := \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_r)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)},$$

we obtain the easily remembered form:

$$p(K_1 \to F_1 \mid K_1 I_1 \ldots I_r) \ = \ p(K_1 \to F_1 \mid K_1) \cdot i_1 \cdot \ldots \cdot i_r . \qquad (9.9)$$

Proof:

$$p(K_1 \to F_1 \mid K_1 I_1 \ldots I_r)$$

$$= \frac{p[(K_1 \to F_1) K_1 I_1 \ldots I_r]}{p(K_1 I_1 \ldots I_r)}$$

$$= \frac{p[I_1 \ldots I_r \mid K_1 (K_1 \to F_1)] \cdot p[K_1 (K_1 \to F_1)]}{p(K_1) p(I_1) \cdot \ldots \cdot p(I_r)}$$

= *(because of the independence of $I_1, \ldots, I_r$ given $[K_1(K_1 \to F_1)]$)*

$$\frac{p[I_1 \mid K_1 (K_1 \to F_1)]}{p(I_1)} \cdot \ldots\ldots \cdot \frac{p[I_r \mid K_1 (K_1 \to F_1)]}{p(I_r)} \cdot p(K_1 \to F_1 \mid K_1)$$

$$= \frac{p[(K_1 \to F_1) K_1 I_1]}{p[(K_1 \to F_1) K_1] p(I_1)} \cdot \ldots\ldots \cdot \frac{p[(K_1 \to F_1) K_1 I_r]}{p[(K_1 \to F_1) K_1] p(I_r)} \cdot p(K_1 \to F_1 \mid K_1)$$

$$= \frac{p(K_1 \to F_1 \mid K_1 I_1)}{p(K_1 \to F_1 \mid K_1)} \cdot \ldots\ldots \cdot \frac{p(K_1 \to F_1 \mid K_1 I_r)}{p(K_1 \to F_1 \mid K_1)} \cdot p(K_1 \to F_1 \mid K_1) .$$

This proves Eq.9.8.

We show the validity of Eq.9.9 by transforming the first quotient of Eq.9.8.

$$\frac{p(K_1 \to F_1 \mid K_1 I_1)}{p(K_1 \to F_1 \mid K_1)}$$

$$= \textit{(because of self-reliance)} \frac{p(K_1 \to F_1 \mid K_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}{p(K_1 \to F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}$$

$$= \textit{(applying the A}\to\textit{L-Theorem)} \quad \frac{p(F_1 \mid K_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n) - p(F_1 \mid \overline{K}_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}{p(\overline{F}_1 \mid \overline{K}_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}$$

$$\cdot \frac{p(\overline{F}_1 \mid \overline{K}_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n) - p(F_1 \mid \overline{K}_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}$$

$$= \textit{(because of } p(F_1 \mid \overline{K}_1 I_1) = p(F_1 \mid \overline{K}_1)\textit{)}$$

$$\frac{p(F_1 \mid K_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1 ... \overline{K}_n)}{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1 ... \overline{K}_n)} \qquad (9.10)$$

$$= \textit{(because of } p(F_1 \mid \overline{H} \cdot \overline{K}_1 ... \overline{K}_n) = 0\textit{)}$$

$$\frac{p(F_1 \mid K_1 I_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 ... \overline{K}_n)}. \qquad (9.11)$$

□

We finally combine Theorems T-9.4 and T-9.9 in order to establish Theorem T-9.12 below. This theorem is a tool suited to factor even large conditional probabilities of the form $p(F_1 \mid H\ K_1\ ...\ K_n\ I_1\ ...\ I_r\ J_1\ ...\ J_s)$ right away, just by writing down the factors. Each of these factors contains exactly one cause with p = 1, and at most a single inhibitor belonging to that existing cause. Since the handling of Eq.9.12 needs no further calculations it is very well suited to automate the corres-ponding procedure.

(The factors contained in Eq.9.12 are also utilized as a summary of the probabilities in use.)

Theorem (T-9.12) *(Combined Factorization Theorem)*

Let $\{H, K_1,\ldots, K_n\}$ denote the set of all causes of an arbitrarily chosen event F_1.
Let all elements in $\{H, K_1,\ldots, K_n\}$ be self-reliant causes of F_1.
Let $I_1,\ldots, I_r$ and $J_1,\ldots, J_s$ be the inhibitors of $K_1 \rightarrow F_1$ and $K_2 \rightarrow F_1$, respectively.
Assume that, if both K_1 and K_2 exist, $I_1,\ldots, I_r$ and $J_1,\ldots, J_s$ are self-reliant causes of $(\overline{K_1 \rightarrow F_1})$ and $(\overline{K_2 \rightarrow F_1})$, respectively.
Let

$$i_1 := \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)}, \ldots, i_r := \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_r)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)},$$

$$j_1 := \frac{p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n J_1)}{p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n)}, \ldots\ldots, j_s := \frac{p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n J_s)}{p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n)}.$$

Then

$$\begin{aligned} p(\overline{F}_1 \mid H K_1 \ldots K_n I_1 \ldots I_r J_1 \ldots J_s) = \; & p(\overline{F}_1 \mid H \overline{K}_1 \ldots \overline{K}_n) \\ & \cdot [1 - p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n) \cdot i_1 \ldots i_r] \\ & \cdot [1 - p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n) \cdot j_1 \ldots j_s] \\ & \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \overline{K}_2 K_3 \overline{K}_4 \ldots \overline{K}_n) \\ & \vdots \\ & \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_{n-1} K_n). \end{aligned} \qquad (9.12)$$

<u>Proof:</u> Application of Theorem T-9.9 to Eq.9.7a yields the result. □

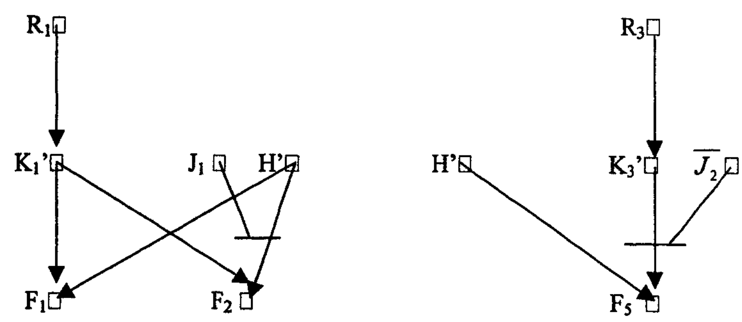
We again look back at Eq.9.1 which contains the probabilities p(F | K J H) and p(H | U I).
Since we were successful in transforming both types of probabilities into the form *p(effects | causes ∧ inhibitors)*, we are in a position to interpolate and decompose p(F | K J H) and p(H | U I) in the same way. The essential assumptions have already been stated in Section 6.

10. Computation of an L-Net

The L-Net of Ill.1.2, which has been established in order to determine the ap-probability of H, will be completed by a number of L-Net modules. These L-Net modules are created "around" arbitrarily chosen '-events and then connected to the already existing causal network.

This work-step is not mandatory, e.g. $p(K_1 \mid K_1^{'})$ may be determined without any further information than that of Ill.1.2. But the ap-probability of H, which is our main interest, is inexact if there were not enough efforts to obtain reliable ap-probabilities for the other events at the H/K-level. Furthermore, the events situated at the H/K-level have a very special importance since they "compete" with H. Therefore, it is almost unavoidable to extend an L-Net to a certain degree by further L-Net modules.

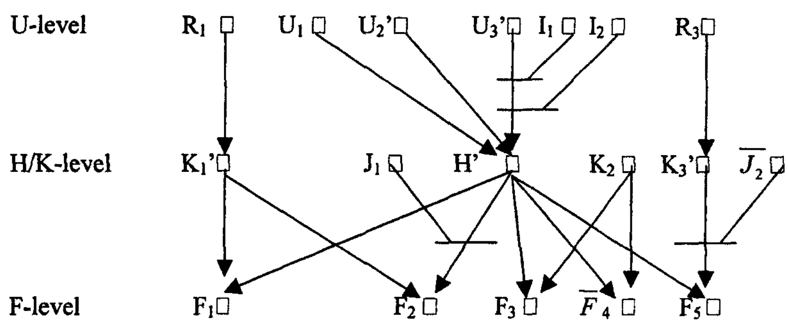
We continue the example of Ill.1.2 by arranging two L-Net modules around $K_1^{'}$ and $K_3^{'}$ as follows:



Ill.10.0:

Example of two L-Net modules in order to extend Ill.1.2. The L-Nets are created around events K_1' and K_3'.

The connection of the L-Net modules to the already existing causal network in Ill.1.2 results in:



Ill.10.1:

Example of an L-Net, completed in order to determine the ap-probabilities of the '-events at the H/K-level. (Derived from Ill.1.2.)

Step 1

The L-Net module around H comprises the sets which belong to WERT(H), i.e. the sets URS(H), FOL(H), DIFF(H), INH(H) and $\bigcup_{Z \in FOL(H)} INH(Z)$. The set DIFF(H) has to be present entirely in any case (Assumption IIa), while the set URS(H) has to be maximal only if needed (Assumption IIa*), e.g. when Eq.9.4 or Eq.9.12. are to be used. The remaining structure of the L-Net module follows the independence requirements listed in Section 6.

We are interested in the ap-probabilities of the '-events situated at the H/K-level.

Step 2

We refer to the example of Eq.1.3 in order to state the ap-probabilities of the events H', K_1', K_3', U_2', U_3'. This results in the following equation system.

$$
\begin{aligned}
p(H \mid H') &= p(H \mid W(H)) \qquad \text{(identical with Eq.1.3)} \\
&= p(H \mid U\,F\,K\,I\,J) \qquad\qquad (10.1) \\
&= p(H \mid U_1 U_2' U_3' F_1 F_2 F_3 \overline{F_4} F_5 K_1' K_2 K_3' I_1 I_2 J_1 \overline{J_2}).
\end{aligned}
$$

$$
\begin{aligned}
p(K_1 \mid K_1') &= p(K_1 \mid W(K_1)) \\
&= p(K_1 \mid F_1 F_2 H' J_1 R_1). \qquad (10.2)
\end{aligned}
$$

$$
\begin{aligned}
p(K_3 \mid K_3') &= p(K_3 \mid W(K_3)) \\
&= p(K_3 \mid F_5 H' \overline{J_2} R_3). \qquad (10.3)
\end{aligned}
$$

$$
\begin{aligned}
p(U_2 \mid U_2') &= p(U_2 \mid W(U_2)) \\
&= p(U_2 \mid H' U_1 U_3' I_1 I_2) \qquad (10.4)
\end{aligned}
$$

$$
\begin{aligned}
p(U_3 \mid U_3') &= p(U_3 \mid W(U_3)) \\
&= p(U_3 \mid H' U_1 U_2' I_1 I_2) \qquad (10.5)
\end{aligned}
$$

Advantages compared to other computation systems

The ap-probabilities of the '-events contained in a causal network have a mutual influence on each other which is of considerable impact. This property usually frustrates all efforts to compute probabilities of net nodes in the neighbourhood of nodes which have unknown probability as well. Here, we eliminate this difficulty by simultaneously installing all unknown ap-probabilities into one equation system.

The target now is to express each ap-probability in terms of the remaining ap-probabilities. However, such calculations pose no problems when using interpolation formulas.

Step 3

Application of the General Interpolation Theorem (Eq.4.1) or the Linear Interpolation Theorem (Eq.4.4) to the equation system consisting of Eq.10.1 through Eq.10.5 leads to the following equation system:

$$p(H \mid H') =_{(Eq.9.1)} \frac{1}{1+\dfrac{p(F_1...F_5 \mid K_1'K_2K_3'J_1\overline{J}_2\overline{H})p(\overline{H} \mid U_1U_2'U_3'I_1I_2)}{p(F_1...F_5 \mid K_1'K_2K_3'J_1\overline{J}_2H)p(H \mid U_1U_2'U_3'I_1I_2)}}. \quad (10.1a)$$

To compute Eq.10.1a we use Eq.9.2:

$$p(F_1...F_5 \mid K_1'K_2K_3'J_1\overline{J}_2H)$$
$$= \sum_{q_1,q_3=0,1} p(F_1...F_5 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H)p(K_1^{(q_1)} \mid K_1')p(K_3^{(q_3)} \mid K_3').$$

Eq.9.3 yields:

$$p(F_1...F_5 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = \prod_{i=1}^{5} p(F_i \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H).$$

Since the elements in (K J H) are self-reliant causes, we have:

$$p(F_1 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = p(F_1 \mid K_1^{(q_1)}H),$$
$$p(F_2 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = p(F_2 \mid K_1^{(q_1)}J_1H),$$
$$p(F_3 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = p(F_3 \mid K_2H),$$
$$p(\overline{F}_4 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = p(\overline{F}_4 \mid K_2H),$$
$$p(F_5 \mid K_1^{(q_1)}K_2K_3^{(q_3)}J_1\overline{J}_2H) = p(F_5 \mid K_3^{(q_3)}\overline{J}_2H).$$

Furthermore, we need

$p(H \mid U_1U_2'U_3'I_1I_2)$ = *identical to Eq.4.5.*

$$p(K_1 \mid K_1') =_{(Eq.9.1)} \frac{1}{1+\dfrac{p(F_1F_2 \mid H'J_1\overline{K}_1)p(\overline{K}_1 \mid R_1)}{p(F_1F_2 \mid H'J_1K_1)p(K_1 \mid R_1)}}. \quad (10.2a)$$

(Further transformations analogous to Eq.10.1a.)

$$p(K_3 \mid K_3') =_{(Eq.9.1)} \frac{1}{1+\dfrac{p(F_5 \mid H'\overline{J}_2\overline{K}_3)p(\overline{K}_3 \mid R_3)}{p(F_5 \mid H'\overline{J}_2K_3)p(K_3 \mid R_3)}}. \quad (10.3a)$$

$p(U_2 \mid U_2') =_{(Eq.4.1)}$ *identical to Eq.4.2.* *(10.4a)*

$p(U_3 \mid U_3') =_{(Eq.4.1)}$ *like Eq.10.4a, with U_2 and U_3 exchanged.* *(10.5a)*

Step 4

We choose $p(U_3 \mid U_3') := 1$ and introduce the following abbreviations:

x := $p(H \mid H')$,

y := $p(K_1 \mid K_1')$,

z := $p(K_3 \mid K_3')$,

u := $p(U_2 \mid U_2')$.

Eqs.10.1a through 10.5a contain '-events which will be removed by means of the Linear Interpolation Theorem (Eq.4.4) which transforms the equation system into the following form:

$$x = \frac{1}{1 + \frac{a_0 yz + a_1 y \cdot \bar{z} + a_2 \bar{y} \cdot z + a_3 \bar{y} \cdot \bar{z}}{b_0 yz + b_1 y \cdot \bar{z} + b_2 \bar{y} \cdot z + b_3 \bar{y} \cdot \bar{z}} \cdot \left(\frac{1}{c_0 u + c_1 \bar{u}} - 1 \right)}, \qquad (10.1b)$$

where

$a_0 := p(F_1 ... F_5 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H})$,

$a_1 := p(F_1 ... F_5 \mid K_1 K_2 \overline{K_3} J_1 \overline{J_2} \cdot \overline{H})$,

$a_2 := p(F_1 ... F_5 \mid \overline{K_1} K_2 K_3 J_1 \overline{J_2} \cdot \overline{H})$,

$a_3 := p(F_1 ... F_5 \mid \overline{K_1} K_2 \overline{K_3} J_1 \overline{J_2} \cdot \overline{H})$,

$b_0 := p(F_1 ... F_5 \mid K_1 K_2 K_3 J_1 \overline{J_2} H)$,

$b_1 := p(F_1 ... F_5 \mid K_1 K_2 \overline{K_3} J_1 \overline{J_2} H)$,

$b_2 := p(F_1 ... F_5 \mid \overline{K_1} K_2 K_3 J_1 \overline{J_2} H)$,

$b_3 := p(F_1 ... F_5 \mid \overline{K_1} K_2 \overline{K_3} J_1 \overline{J_2} H)$,

$c_0 := p(H \mid U_1 I_1 I_2 U_2 U_3)$,

$c_1 := p(H \mid U_1 I_1 I_2 \overline{U_2} U_3)$.

$$y = \frac{1}{1 + \frac{d_0 x + d_1 \bar{x}}{e_0 x + e_1 \bar{x}} \cdot \left(\frac{1}{f_0} - 1 \right)} \quad , \tag{10.2b}$$

where

$d_0 := p(F_1 F_2 \mid HJ_1 \overline{K_1})$,

$d_1 := p(F_1 F_2 \mid \overline{H} J_1 \overline{K_1})$,

$e_0 := p(F_1 F_2 \mid HJ_1 K_1)$,

$e_1 := p(F_1 F_2 \mid \overline{H} J_1 K_1)$,

$f_0 := p(K_1 \mid R_1)$.

$$z = \frac{1}{1 + \frac{g_0 x + g_1 \bar{x}}{h_0 x + h_1 \bar{x}} \cdot \left(\frac{1}{k_0} - 1 \right)} \tag{10.3b}$$

where

$g_0 := p(F_5 \mid H\overline{J_2} \cdot \overline{K_3})$,

$g_1 := p(F_5 \mid \overline{H} \cdot \overline{J_2} \cdot \overline{K_3})$,

$h_0 := p(F_5 \mid H\overline{J_2} K_3)$,

$h_1 := p(F_5 \mid \overline{H} \cdot \overline{J_2} K_3)$,

$k_0 := p(K_3 \mid R_3)$.

$$u = l_0 x + l_1 \bar{x} \tag{10.4b}$$

where

$l_0 := p(U_2 \mid U_1 I_1 I_2 H U_3)$,

$l_1 := p(U_2 \mid U_1 I_1 I_2 \overline{H} U_3)$.

$$v := p(U_3 \mid U_3') := 1. \tag{10.5b}$$

($p(U_3 \mid U_3') := 1$ has been arbitrarily chosen.)

Step5

We decompose into factors and assign numerical values.

$$
\begin{aligned}
a_0 := p(F_1 ... F_5 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \; &= \; p(F_1 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \\
&\quad \cdot p(F_2 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \\
&\quad \cdot p(F_3 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \\
&\quad \cdot p(\overline{F_4} \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \\
&\quad \cdot p(F_5 \mid K_1 K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) \\
&= \; p(F_1 \mid K_1 \overline{H}) && := 0.2 \\
&\quad \cdot p(F_2 \mid K_1 J_1 \overline{H}) && := 0.2 \\
&\quad \cdot p(F_3 \mid K_2 \overline{H}) && := 0.2 \\
&\quad \cdot p(\overline{F_4} \mid K_2 \overline{H}) && := 0.8 \\
&\quad \cdot p(F_5 \mid K_3 \overline{J_2} \cdot \overline{H}) && := 0.1 \\
&= \; 0.2 \cdot 0.2 \cdot 0.2 \cdot 0.8 \cdot 0.1 \\
&= \; 0.00064.
\end{aligned}
$$

$$a_1 := p(F_1 ... F_5 \mid K_1 K_2 \overline{K_3} J_1 \overline{J_2} \cdot \overline{H}) := 0.$$

$$a_2 := p(F_1 ... F_5 \mid \overline{K_1} K_2 K_3 J_1 \overline{J_2} \cdot \overline{H}) := 0.$$

$$a_3 := p(F_1 ... F_5 \mid \overline{K_1} K_2 \overline{K_3} J_1 \overline{J_2} \cdot \overline{H}) := 0.$$

$$
\begin{aligned}
b_0 := p(F_1 ... F_5 \mid K_1 K_2 K_3 J_1 \overline{J_2} H) \; &= \; p(F_1 \mid K_1 H) && := 0.5 \\
&\quad \cdot p(F_2 \mid K_1 J_1 H) && := 0.4 \\
&\quad \cdot p(F_3 \mid K_2 H) && := 0.5 \\
&\quad \cdot p(\overline{F_4} \mid K_2 H) && := 0.5 \\
&\quad \cdot p(F_5 \mid K_3 \overline{J_2} H) && := 0.5 \\
&= \; 0.5 \cdot 0.4 \cdot 0.5 \cdot 0.5 \cdot 0.5 \\
&= \; 0.025.
\end{aligned}
$$

$$
\begin{aligned}
b_1 := p(F_1 ... F_5 \mid K_1 K_2 \overline{K_3} J_1 \overline{J_2} H) &= p(F_1 \mid K_1 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(F_2 \mid K_1 J_1 H) && := 0.4 \text{ (as above)} \\
&\quad \cdot p(F_3 \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(\overline{F_4} \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(F_5 \mid \overline{K_3} \cdot \overline{J_2} H) && := 0.1 \\
&= 0.5 \cdot 0.4 \cdot 0.5 \cdot 0.5 \cdot 0.1 \\
&= 0.005.
\end{aligned}
$$

$$
\begin{aligned}
b_2 := p(F_1 ... F_5 \mid \overline{K_1} K_2 K_3 J_1 \overline{J_2} H) &= p(F_1 \mid \overline{K_1} H) && := 0.2 \\
&\quad \cdot p(F_2 \mid \overline{K_1} J_1 H) && := 0.2 \\
&\quad \cdot p(F_3 \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(\overline{F_4} \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(F_5 \mid K_3 \overline{J_2} H) && := 0.5 \text{ (as above)} \\
&= 0.2 \cdot 0.2 \cdot 0.5 \cdot 0.5 \cdot 0.5 \\
&= 0.005.
\end{aligned}
$$

$$
\begin{aligned}
b_3 := p(F_1 ... F_5 \mid \overline{K_1} K_2 \overline{K_3} J_1 \overline{J_2} H) &= p(F_1 \mid \overline{K_1} H) && := 0.2 \text{ (as above)} \\
&\quad \cdot p(F_2 \mid \overline{K_1} J_1 H) && := 0.2 \text{ (as above)} \\
&\quad \cdot p(F_3 \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(\overline{F_4} \mid K_2 H) && := 0.5 \text{ (as above)} \\
&\quad \cdot p(F_5 \mid \overline{K_3} \cdot \overline{J_2} H) && := 0.1 \text{ (as above)} \\
&= 0.2 \cdot 0.2 \cdot 0.5 \cdot 0.5 \cdot 0.1 \\
&= 0.001.
\end{aligned}
$$

$c_0 := p(H \mid U_1 I_1 I_2 U_2 U_3) := 0.8.$

$c_1 := p(H \mid U_1 I_1 I_2 \overline{U_2} U_3) := 0.4.$ *These assignments result in:*

$$
x = \frac{1}{1 + \frac{0.0064yz}{0.025yz + 0.005y \cdot \bar{z} + 0.005\bar{y} \cdot z + 0.001\bar{y} \cdot \bar{z}} \cdot \left(\frac{1}{0.8u + 0.4\bar{u}} - 1 \right)}. \qquad (10.1c)
$$

$d_0 := p(F_1F_2 \mid HJ_1\overline{K_1}) \;=\; p(F_1 \mid HJ_1\overline{K_1})$

$\quad\cdot\, p(F_2 \mid HJ_1\overline{K_1})$

$\quad= \; p(F_1 \mid H\overline{K_1}) \qquad := 0.2$ *(as above)*

$\quad\cdot\, p(F_2 \mid HJ_1\overline{K_1}) \qquad := 0.2$ *(as above)*

$\quad= 0.2 \cdot 0.2 = 0.04.$

$d_1 := p(F_1F_2 \mid \overline{H}J_1\overline{K_1}) \;:= 0.$

$e_0 := p(F_1F_2 \mid HJ_1K_1) \;=\; p(F_1 \mid HJ_1K_1)$

$\quad\cdot\, p(F_2 \mid HJ_1K_1)$

$\quad= \; p(F_1 \mid HK_1) \qquad := 0.5$ *(as above)*

$\quad\cdot\, p(F_2 \mid HJ_1K_1) \qquad := 0.4$ *(as above)*

$\quad= 0.5 \cdot 0.4 = 0.2.$

$e_1 := p(F_1F_2 \mid \overline{H}J_1K_1) \;=\; p(F_1 \mid \overline{H}K_1) \qquad := 0.2$ *(as above)*

$\quad\cdot\, p(F_2 \mid \overline{H}J_1K_1) \qquad := 0.2$ *(as above)*

$\quad= 0.2 \cdot 0.2 = 0.04.$

$f_0 := p(K_1 \mid R_1) := 0.6.$ *These assignments result in:*

$$y = \frac{1}{1 + \frac{0.04x}{0.2x + 0.04\overline{x}} \cdot \left(\frac{1}{0.6} - 1\right)}. \qquad (10.2c)$$

$g_0 := p(F_5 \mid H\overline{J_2} \cdot \overline{K_3}) \;:= 0.1$ *(as above).*

$g_1 := p(F_5 \mid \overline{H} \cdot \overline{J_2} \cdot \overline{K_3}) := 0.$

$h_0 := p(F_5 \mid H\overline{J_2}K_3) \;:= 0.5$ *(as above).*

$h_1 := p(F_5 \mid \overline{H} \cdot \overline{J_2}K_3) \;:= 0.1$*(as above).*

$k_0 := p(K_3 \mid R_3) \;:= 0.7.$ *These assignments result in:*

$$z = \frac{1}{1 + \frac{0.1x}{0.5x + 0.1\overline{x}} \cdot \left(\frac{1}{0.7} - 1\right)} \qquad (10.3c)$$

$l_0 := p(U_2 \mid U_1 I_1 I_2 H U_3) := 0.6$

$l_1 := p(U_2 \mid U_1 I_1 I_2 \overline{H} U_3) := 0.1.$

These assignments result in:

$$u = 0.6x + 0.1\overline{x} \tag{10.4c}$$

To summarize, we obtained the following equation system:

$$x = \frac{1}{1 + \frac{0.0064yz}{0.025yz + 0.005y\cdot\overline{z} + 0.005\overline{y}\cdot z + 0.001\overline{y}\cdot\overline{z}}\cdot\left(\frac{1}{0.8u + 0.4\overline{u}} - 1\right)}. \tag{10.1c}$$

$$y = \frac{1}{1 + \frac{0.04x}{0.2x + 0.04\overline{x}}\cdot\left(\frac{1}{0.6} - 1\right)}. \tag{10.2c}$$

$$z = \frac{1}{1 + \frac{0.1x}{0.5x + 0.1\overline{x}}\cdot\left(\frac{1}{0.7} - 1\right)}. \tag{10.3c}$$

$$u = 0.6x + 0.1\overline{x}. \tag{10.4c}$$

Step 6

In order to solve the equation system we employ the commercial computer algebra system "Maple 6", whose output is as follows:

eqn1:=x=1/(1+(((0.00064*y*z)/((0.025*y*z)+(0.005*y*(1-z))+(0.005*(1-y)*z)+(0.001*(1-y)*(1-z))))*((1/((0.8*u)+(0.4*(1-u))))-1)));

$$eqn1 := x = \frac{1}{1 + \frac{.00064\, y\, z\left(\frac{1}{.4\, u + .4} - 1\right)}{.025\, y\, z + .005\, y\,(1 - z) + .005\,(1 - y)\, z + .001\,(1 - y)(1 - z)}}$$

eqn2:=y=1/(1+(((0.04*x)/((0.20*x)+(0.04*(1-x))))*((1/0.6)-1)));

$$eqn2 := y = \frac{1}{1 + \frac{.02666666668\,x}{.16\,x + .04}}$$

eqn3:=z=1/(1+(((0.1*x)/((0.5*x)+(0.1*(1-x))))*((1/0.7)-1)));

$$eqn3 := z = \frac{1}{1 + \frac{.0428571429\,x}{.4\,x + .1}}$$

eqn4:=u=(0.6*x)+(0.1*(1-x));

$$eqn4 := u = .5\,x + .1$$

solve ({eqn1,eqn2,eqn3,eqn4},{x,y,z,u});

$\{u = -1.042618895, z = .8926149951, y = .8423611317, x = -2.285237791\}$,
$\{x = -.2446786040, y = -.1500744630, u = -.02233930198, z = -.2546833095\}$,
$\{z = -.4372489087, y = -.2431224964, x = -.2421083162, u = -.02105415808\}$,
$\{x = .9862353880, z = .9212550827, y = .8826419881, u = .5931176940\}$

Thus we obtain the final result:

$p(H \mid H') = 98.6\%$.

$p(K_1 \mid K_1') = 88.3\%$.

$p(K_3 \mid K_3') = 92.1\%$.

$p(U_2 \mid U_2') = 59.3\%$.

<u>Note</u>

If there are no hidden causes we have, according to Factorization Theorem T-9.4, the following decomposition into factors:

$p(F_1 \mid H K_1) = 1 - p(\overline{F}_1 \mid H\overline{K}_1)p(\overline{F}_1 \mid K_1\overline{H})$.

$p(F_2 \mid K_1 J_1 H) = 1 - p(\overline{F}_2 \mid HJ_1\overline{K}_1)p(\overline{F}_2 \mid K_1\overline{H})$.

$p(F_3 \mid K_2 H) = 1 - p(\overline{F}_3 \mid H\overline{K}_2)p(\overline{F}_3 \mid K_2\overline{H})$.

$$p(\overline{F}_4 \mid K_2 H) = 1 - p(F_4 \mid K_2 H)$$

$$= 1 - [1 - p(\overline{F}_4 \mid H\overline{K}_2)p(\overline{F}_4 \mid K_2\overline{H})]$$

$$= p(\overline{F}_4 \mid H\overline{K}_2)p(\overline{F}_4 \mid K_2\overline{H}).$$

$$p(F_5 \mid K_3\overline{J}_2 H) = 1 - p(\overline{F}_5 \mid H\overline{K}_3)p(\overline{F}_5 \mid K_3\overline{J}_2\overline{H}).$$

In order to keep the example easy to understand, we immediately assigned numerical values to the probabilities on the left-hand side.

However, in practice the factorization theorems are used to decompose extended conditional probabilities into factors, for which numerical values can easily be determined.

The Factorization Theorem T-9.12 has not been used since all probabilities contained less than two inhibitors.

Final remark

The prospect of letting everyone receive medical treatment in line with the latest findings is brought about by expert systems. The expert system presented here enables the utilization of every single symptom and the inclusion of the total number of hypothetical diagnoses, and it does not estimate but carries out computation processes which produce precise numerical values.

11. Appendix

Hidden causes

Theorems T-9.4 and T-9.12 use the assumption that the set {H, K_1,...,K_n} contains all causes of F_1 and that there are no unknown F_1-generating events. But if $p(F_1 \mid \overline{H} \cdot \overline{K}_1 ... \overline{K}_n) > 0$ and if we nonetheless apply T-9.4 or T-9.12, we will not obtain precise result.

At first, we consider Eq.9.4 and amend that formula to work in the case of hidden causes, i.e. if $p(F_1 \mid \overline{H} \cdot \overline{K}_1 ... \overline{K}_n) > 0$.

Theorem (T-11.1) *(Eq.9.4 for the case of hidden causes)*

Let {H, K_1, ..., K_n} denote the set of all known causes of an arbitrarily chosen event F_1. Let there be unknown causes of F_1, for which we write the summary designation K_y. Let all F_1-generating events be self-reliant causes of F_1.

Let I_1,..., I_r and J_1,..., J_s be the inhibitors of $K_1 \rightarrow F_1$ and $K_2 \rightarrow F_1$, respectively.

Then

$$p(\overline{F}_1 \mid HK_1...K_nI_1...I_rJ_1...J_s) = p(\overline{F}_1 \mid H\overline{K}_1...\overline{K}_n) \cdot p(\overline{F}_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1...I_r) \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_nJ_1...J_s) \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1\overline{K}_2K_3\overline{K}_4...\overline{K}_n) \cdots p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_{n-1}K_n) \cdot f, \quad (11.1)$$

where $f = \dfrac{1}{\left(p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)\right)^{m-1}}$, *m := number of elements in {H, K_1,...,K_n}.*

The value of $p(\overline{F}_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$, when $p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n) > 0$, is f-times the result obtained from Eq.9.4, $f > 1$.

Proof:

$$p(F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$$

$$= p[(H \to F_1) \vee (K_1 \to F_1) \vee .. \vee (K_n \to F_1) \vee (K_y \to F_1) \mid HK_1...K_nI_1..I_rJ_1..J_s]$$

$$= 1 - p[\overline{(H \to F_1)} \wedge .... \wedge \overline{(K_n \to F_1)} \wedge \overline{(K_y \to F_1)} \mid HK_1....K_nI_1...I_rJ_1...J_s]$$

$$= 1 - [\,1 - p(H \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)\,]$$

$$\cdot[\,1 - p(K_1 \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)\,]$$

$$\vdots$$

$$\cdot[\,1 - p(K_n \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)\,]$$

$$\cdot[\,1 - p(K_y \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)\,]. \qquad (*)$$

The last factor is transformed as follows:

$$1 - p(K_y \to F_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$$

$$= 1 - p(K_y \to F_1) \qquad \text{(since } K_y \text{ is a self-reliant cause of } F_1\text{)}$$

$$= 1 - p[(K_y \to F_1)(K_y)] \qquad \text{(because of Eq. 7.1)}$$

$$= 1 - p(K_y \to F_1 \mid K_y)p(K_y)$$

$$= 1 - p(K_y \to F_1 \mid K_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)p(K_y)$$

$$= 1 - \frac{p(F_1 \mid K_y\overline{H}\cdot\overline{K}_1...\overline{K}_n) - p(F_1 \mid \overline{K}_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{K}_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)}\, p(K_y) \qquad \text{(A}\to\text{L-Theorem)}$$

$$= 1 - p(F_1 \mid K_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)p(K_y).$$

Proposition: $p(F_1 \mid K_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)p(K_y) = p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)$.

Proof: *If F_1 exists then* $p(K_y \mid F_1\overline{H}\cdot\overline{K}_1...\overline{K}_n) = 1$.

$$\Rightarrow \quad p(K_y) = \frac{p(K_y)}{p(K_y \mid F_1\overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$= \frac{p(K_y \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{p(K_y \mid F_1\overline{H}\cdot\overline{K}_1...\overline{K}_n)} \qquad \text{(self-reliance)}$$

$$= \frac{p(K_y\overline{H}\cdot\overline{K}_1...\overline{K}_n)p(F_1\overline{H}\cdot\overline{K}_1...\overline{K}_n)}{p(K_yF_1\overline{H}\cdot\overline{K}_1...\overline{K}_n)p(\overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

(continued on next page)

$$= \frac{p(F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}{p(F_1 \mid K_y \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)},$$

i.e. the proposition. With this result we proceed from Eq.(*):

$$= 1 - [\, 1 - p(H \rightarrow F_1 \mid H\overline{K}_1 \ldots \overline{K}_n)\,]$$

$$\cdot [\, 1 - p(K_1 \rightarrow F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1 \ldots I_r)\,]$$

$$\cdot [\, 1 - p(K_2 \rightarrow F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n J_1 \ldots J_s)\,]$$

$$\cdot [\, 1 - p(K_3 \rightarrow F_1 \mid \overline{H} \cdot \overline{K}_1 \overline{K}_2 K_3 \overline{K}_4 \ldots \overline{K}_n)\,]$$

$$\vdots$$

$$\cdot [\, 1 - p(K_n \rightarrow F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_{n-1} K_n)\,] \cdot [1 - p(F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)].$$

$$= 1 - \frac{p(\overline{F}_1 \mid H\overline{K}_1 \ldots \overline{K}_n)}{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}$$ (see proof of Theorem T-9.4)

$$\cdot \frac{p(\overline{F}_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1 \ldots I_r)}{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}$$

$$\cdot \frac{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n J_1 \ldots J_s)}{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}$$

$$\cdot \frac{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \overline{K}_2 K_3 \overline{K}_4 \ldots \overline{K}_n)}{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}$$

$$\vdots$$

$$\cdot \frac{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_{n-1} K_n)}{p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)} \cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)$$

$$= 1 - p(\overline{F}_1 \mid H\overline{K}_1 \ldots \overline{K}_n)$$

$$\cdot p(\overline{F}_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1 \ldots I_r)$$

$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n J_1 \ldots J_s)$$

$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \overline{K}_2 K_3 \overline{K}_4 \ldots \overline{K}_n)$$

$$\vdots$$

$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_{n-1} K_n) \cdot \frac{1}{\left(p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)\right)^{m-1}}.$$

□

Theorem T-11.1 states that using Eq.11.1 instead of Eq.9.4 increases the value of $p(\overline{F}_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$ by the factor f.

As an example we choose $p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n) := 0.01$ and m := 11 which yields

f = $[p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)]^{-10} = 0.99^{-10}$ = 1.1057.

Therefore, if $p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n) := 0.01$, and if we use Eq.11.1 instead of Eq.9.4, the value of $p(\overline{F}_1 \mid HK_1...K_nI_1...I_rJ_1...J_s)$ increases by 10.6 %.

<u>Theorem (T-11.2)</u> *(Eq.9.12 for the case of hidden causes)*

Let $\{H, K_1, ..., K_n\}$ denote the set of all <u>known</u> causes of an arbitrarily chosen event F_1. Let there be <u>unknown</u> causes of F_1 which entail $p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n) > 0$.
Let all F_1-generating events be self-reliant causes of F_1.
Let $I_1,..., I_r$ and $J_1,..., J_s$ be the inhibitors of $K_1 \rightarrow F_1$ and $K_2 \rightarrow F_1$, respectively.
Assume that, if both K_1 and K_2 exist, $I_1,..., I_r$ and $J_1,..., J_s$ are self-reliant causes of $(\overline{K_1 \rightarrow F_1})$ and $(\overline{K_2 \rightarrow F_1})$, respectively.
Then

$$p(\overline{F}_1 \mid HK_1 ... K_nI_1 ... I_rJ_1 ... J_s) = p(\overline{F}_1 \mid H\overline{K}_1...\overline{K}_n) \qquad (11.2)$$
$$\cdot [1 - p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) \cdot i_1...i_r]^*$$
$$\cdot [1 - p(F_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_n) \cdot j_1...j_s]^*$$
$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1\overline{K}_2K_3\overline{K}_4...\overline{K}_n)$$
$$\vdots$$
$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_{n-1}K_n) \cdot f^*,$$

where

$$i_1 := \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}, ..., \quad i_r := \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_r)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)},$$

$$j_1 := \frac{p(F_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_nJ_1)}{p(F_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_n)}, ..., j_s := \frac{p(F_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_nJ_s)}{p(F_1 \mid \overline{H} \cdot \overline{K}_1K_2\overline{K}_3...\overline{K}_n)}.$$

(Continued on next page)

[1 - $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r$] can be obtained from*

[1 - $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r$] by means of the following replacements:

$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)$ replaced by $\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$,

$i_1 := \dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n I_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}$ replaced by $\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n I_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}$,

$i_2,...,i_r$ accordingly.

[1 - $p(F_1 \mid \overline{H}\cdot\overline{K}_1 K_2\overline{K}_3...\overline{K}_n)\cdot j_1...j_s$] is obtained through an analogous procedure.*

$$f^* = \frac{1}{\left(p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)\right)^{m^*-1}},$$

m^* := number of factors without square brackets contained in Eq.11.2.

<u>Proof</u>

The replacement of $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)$ follows from the A→L-Theorem,

the replacement of $i_1 := \dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n I_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}$ follows from Eq.9.10,

the factor f^* follows from Eq.11.1. □

Since

$$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) > \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)} \quad (11.2a)$$

and

$$\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n I_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)} > \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n I_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)} \quad (11.2b)$$

we have that [1 - $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r$] < [1 - $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r$]*. This holds since $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)$ and $i_1,..., i_r$ are replaced in each case with something smaller.

Of course Theorem T-11.2 is quite unwieldy. Therefore, we give the following approximations.

Theorem (T-11.3) *(Approximation of T-11.2)*

Consider the setting of T-11.2.

Then Eq.11.2 can be approximated thus:

$$p(\overline{F}_1 \mid H K_1 \ldots K_n I_1 \ldots I_r J_1 \ldots J_s) \approx p(\overline{F}_1 \mid H \overline{K}_1 \ldots \overline{K}_n) \tag{11.3}$$

$$\cdot [1 - p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n) \cdot i_1 \ldots i_r \cdot f_{K_1}]$$

$$\cdot [1 - p(F_1 \mid \overline{H} \cdot \overline{K}_1 K_2 \overline{K}_3 \ldots \overline{K}_n) \cdot j_1 \ldots j_s \cdot f_{K_2}]$$

$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \overline{K}_2 K_3 \overline{K}_4 \ldots \overline{K}_n)$$

$$\vdots$$

$$\cdot p(\overline{F}_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_{n-1} K_n) \cdot f^* .$$

The first expression in square brackets which in full length reads

$$[1 - \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)}{1} \frac{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 \ldots \overline{K}_n I_1)}{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 \ldots \overline{K}_n)} \ldots \frac{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 \ldots \overline{K}_n I_r)}{p(F_1 \mid K_1 \overline{H} \cdot \overline{K}_2 \ldots \overline{K}_n)} f_{K_1}]$$

contains the factor

$$f_{K_1} := \prod_{\lambda} \left(\frac{num(q_\lambda) - p(F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}{num(q_\lambda)} \right),$$

where $num(q_\lambda)$ *denotes the <u>num</u>erator of the <u>q</u>uotient at position* λ.

All other expressions in square brackets contain analogous factors.

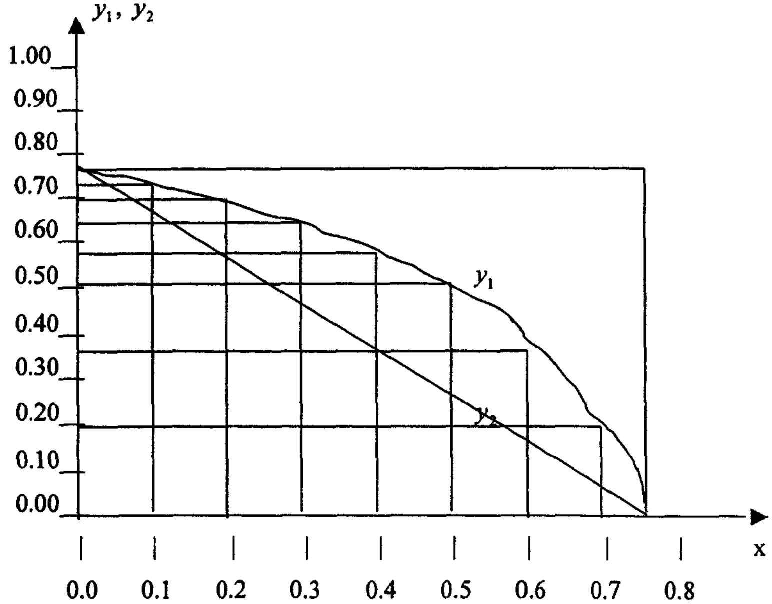
<u>*Proof:*</u>

$$[1 - \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)}{1} \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_1)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)} \cdots\cdots \frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n I_r)}{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)}]$$

contains $p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n)$, which in case of hidden causes has to be replaced with $\frac{p(F_1 \mid \overline{H} K_1 \overline{K}_2 \ldots \overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}{1 - p(F_1 \mid \overline{H} \cdot \overline{K}_1 \ldots \overline{K}_n)}$.

We set $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) := 0.75$, $p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n) := x$ and consider

$y_1 = \dfrac{0.75 - x}{1 - x}$ as well as $y_2 = 0.75 - x$. See also Ill.8.1.



Ill.11.1

Approximation of $y_1 = \dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$ *by*

$y_2 = p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)$.

According to (11.2a) and Ill.11.1 we have:

$$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) > \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}$$

$$> p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n).$$

Therefore,

$$[1 - \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}{1} \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)} ..... \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_r)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}]$$

increases if, in case of hidden causes, $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)$ is replaced with $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)$.

This replacement is equivalent to a multiplication of $p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)$ with the constant $\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}$ since:

$$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)$$

$$= p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}.$$

The multiplicative constant $\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}$ itself can be written in the form $\frac{num(q_1) - p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n)}{num(q_1)}$.

We now consider $\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H} \cdot \overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n) - p(F_1 \mid \overline{H} \cdot \overline{K}_1..\overline{K}_n)}$, which represents the next replacement. In order to give an example similar to the above we set

$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n) := 0.75,$

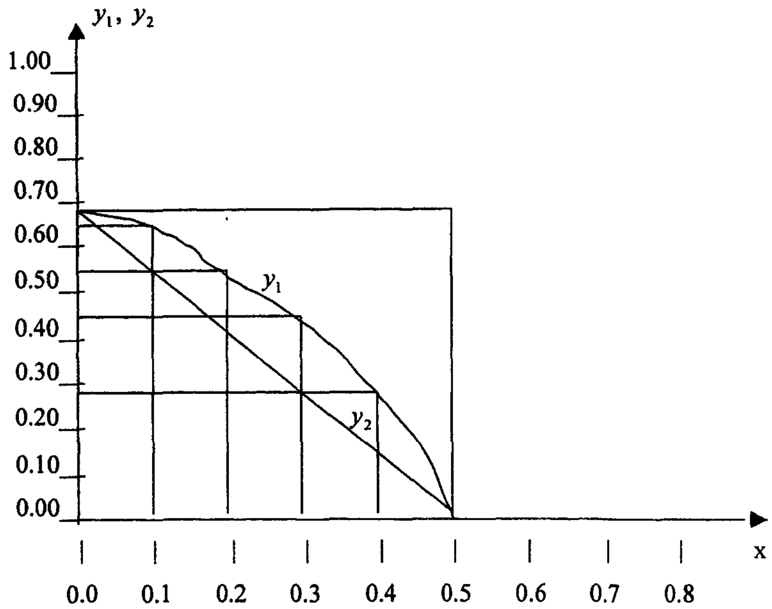
$p(F_1 \mid \overline{H} \cdot \overline{K}_1...\overline{K}_n) := x,$

$p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1) := 0.5$

and consider $y_1 = \frac{0.5 - x}{0.75 - x}$ as well as $y_2 = \frac{0.5 - x}{0.75}$.

Table of values:

x	0.00	0.10	0.20	0.30	0.40	0.50
y_1	0.67	0.62	0.55	0.44	0.29	0.00



Ill.11.2

Approximation of $y_1 = \dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}$ *by*

$$y_2 = \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}.$$

According to (11.2b) and Ill.11.2 we have:

$$\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)} > \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}$$

$$> \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}.$$

Therefore,

$$[1 - \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}{1} \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)} \ldots\ldots \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_r)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}]$$

increases if, in case of hidden causes, $\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}$ is replaced with

$$\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}.$$

This replacement is equivalent to a multiplication of $\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}$ with

the constant $\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}$ since:

$$\frac{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1..\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2..\overline{K}_n)}$$

$$= \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)}\ \frac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}.$$

$\dfrac{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_nI_1)}$ itself can be written in the form

$$\frac{num(q_2) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{num(q_2)}. \qquad \square$$

<u>**Theorem (T-11.4)**</u> *(Approximation of T-11.3)*

Consider the setting of T-11.2 and T-11.3.

Then the following approximation of Eq. 11.3 holds:

$$\begin{aligned} p(\overline{F}_1 \mid H K_1 \ldots K_n I_1 \ldots I_r J_1 \ldots J_s) \approx\ & p(\overline{F}_1 \mid H\overline{K}_1...\overline{K}_n) \qquad (11.4)\\ & \cdot [1 - p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r] \\ & \cdot [1 - p(F_1 \mid \overline{H}\cdot\overline{K}_1K_2\overline{K}_3...\overline{K}_n)\cdot j_1...j_s] \\ & \cdot p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1\overline{K}_2K_3\overline{K}_4...\overline{K}_n) \\ & \vdots \\ & \cdot p(\overline{F}_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_{n-1}K_n)\cdot\frac{f^*}{f_{K_1}\cdot f_{K_2}}. \end{aligned}$$

<u>Proof</u>

Eq.11.3 contains the expression $[\,1 - p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r \cdot f_{K_1}]$. We write this expression in the form [1 – a· f], using the following abbreviations:

$a := p(F_1 \mid \overline{H}K_1\overline{K}_2...\overline{K}_n)\cdot i_1...i_r$,

$f := f_{K_1}$.

We consider the functions $y_1 = 1 - a\cdot f$, $y_2 = \frac{1-a}{f}$, $y_3 = \frac{1-a}{f^{\frac{1}{2}}}$.

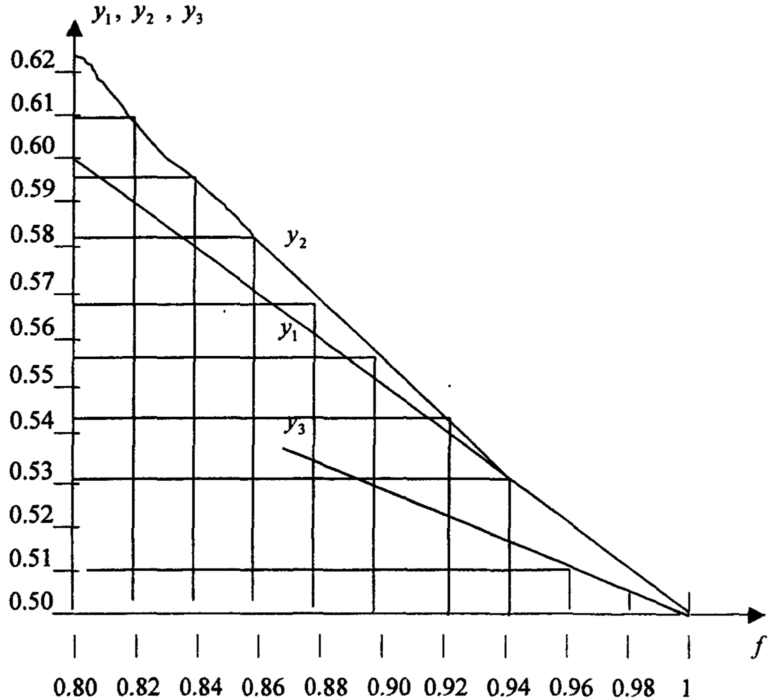
We set a := 0.5 which leads to the following table of values:

f	$f^{\frac{1}{2}}$	$y_1 = 1 - a\cdot f$	$y_2 = \frac{1-a}{f}$	$y_3 = \frac{1-a}{f^{\frac{1}{2}}}$
1	1	0.50	0.5000	0.5000
0.98	0.9899	0.51	0.5102	0.5051
0.96	0.9798	0.52	0.5208	0.5103
0.94	0.9695	0.53	0.5314	0.5157
0.92	0.9592	0.54	0.5435	0.5213
0.90	0.9487	0.55	0.5556	0.5270
0.88	0.9381	0.56	0.5682	0.5330
0.86	0.9274	0.57	0.5814	0.5391
0.84	0.9165	0.58	0.5952	0.5456
0.82	0.9055	0.59	0.6098	0.5522
0.80	0.8944	0.60	0.6250	0.5590

We will replace $[1 - a\cdot f]$ with $[\frac{1-a}{f}]$ and we will not use the option $[\frac{1-a}{f^{\frac{1}{2}}}]$.

See the illustration below.

We have f ≤ 1; f is close to 1 since $f_{K_1} := \prod_{\lambda}\left(\frac{num(q_\lambda) - p(F_1 \mid \overline{H}\cdot\overline{K}_1...\overline{K}_n)}{num(q_\lambda)}\right)$.



Ill.11.3

Approximation of $y_1 = 1 - a \cdot f$ by $y_2 = \dfrac{1-a}{f}$ or $y_3 = \dfrac{1-a}{f^{\frac{1}{2}}}$. a:= 0.5.

The replacement of $[1 - a \cdot f]$ with $[\dfrac{1-a}{f}]$ is one of several options. In general, the expressions $[\dfrac{1-a}{f^t}]$ are suitable to replace $[1 - a \cdot f]$. We chose t:= 1. □

Index of symbols

A	single event or logic product of k non-negated events; A := $(A_1...A_k)$.
$\overline{A}$	negation; event A has the probability p = 0.
A'	'-event; see definition in Section 1, page 20.
A→L	event "A creates L" ; in case of A:= $(A_1...A_k)$ we get $[(A_1 ... A_k)\rightarrow L]$ which means "A_1 creates L" ∨...∨ "A_k creates L".
A.L	synonymous to A→L; A.L is the symbol formerly used.
ap	ap-probability; special case of an a-posteriori-probability which is conditioned upon all events within the considered causal network.
B	logic product consisting of arbitrary non-negated elements in URS(L) \ {elements in A}.
C	logic product consisting of arbitrary negated elements in URS(L) \ {elements in (A B)}.
D	arbitrary logic product consisting of negated or non-negated elements in INH(A→L).
DIFF(H)	set of net nodes distinctive from H, being direct causes of elements in FOL(H). DIFF(H) includes events which have to be considered in the course of <u>diff</u>erential diagnostic procedures.
F	product of elements which belong to FOL(H); F:= $(F_1\ F_2\ F_3\ \overline{F_4}\ F_5)$.
F_i	events included in F.
FOL(H)	set of direct effects of H (German: <u>Fol</u>gen).
H	hypothesis; H represents the most probable cause of a given set of symptoms.
h	relative frequency.
I	product of elements belonging to the set INH(H); I:= $(I_1\ I_2)$.
INH(A→L)	set of events which <u>inh</u>ibit $A_1\rightarrow L$, ..., $A_k\rightarrow L$, or which increase or decrease inhibitions acting upon $A_1\rightarrow L$, ..., $A_k\rightarrow L$.

INH(L)	set of events which inhibit transitions leading to L, or which increase or decrease inhibiting mechanisms acting upon these transitions.
J	product of elements belonging to the sets INH(F_1),..., INH(F_5); J:= (J_1 $\overline{J_2}$).
K	product of elements belonging to the set DIFF(H); K:= (K_1' K_2 K_3').
$K_1 \rightarrow F_1$	event "K_1 creates F_1".
L	arbitrary event; L represents the "leading symptom" (German: Leit-symptom).
p(A | A')	ap-probability of the event A.
p(A→L|A)	probability of the transition A→L in case the event A exists.
q_i	variable of value 0 or 1.
R_1, R_3	events belonging to URS(K_1) or URS(K_3), respectively. (Representatives.)
U	product of elements belonging to the set URS(H); U:= (U_1 U_2' U_3').
URS(H)	set of direct causes of H (German: Ursachen).
$V_{E_i'}$	product of events, derived from the product $(E_1 ... E_h E_{h+1}' ... E_i' ... E_k')$ by removing E_i' as well as other arbitrarily chosen elements; the empty product $V_{E_i'} = \varnothing$ is permitted.
WERT(H)	set of all net nodes influencing the ap-probability of H (screening neighborhood of H; German: Wertungsumgebung).
W(H)	logic product consisting of the events in WERT(H).
X	arbitrarily chosen product of negated or non-negated events in URS(L) \ {elements in A}.
$\hat{X}$	arbitrarily chosen logic product consisting of negated or non-negated events in URS(L) \ {elements in (A B C)}.

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(In chronological order)

[DHN81] ***DUDA, R.O.; P.E. HART; N.J. NILSSON: Subjective Bayesian methods for rule-based inference systems. In: WEBER, B.L.; N.J. NILSSON (eds.): Readings in artificial intelligence. Tioga Publ., Palo Alto Calif. (1981), pp. 192 - 199.***

Some authors argue that the equation (ib. p. 194)

$$P(H \mid E') = P(H \mid E, E')P(E \mid E') + P(H \mid \overline{E}, E')P(\overline{E} \mid E')$$

reaches the form

$$P(H \mid E') = P(H \mid E)P(E \mid E') + P(H \mid \overline{E})P(\overline{E} \mid E') \qquad (*)$$

only if H and E' are independent given E.

This objection is justified. In addition, we demonstrated in Section 4, Eq.4.7, that we are allowed to write

$p(H \mid U_1\, I_1\, I_2\, U_2'\, U_3'\, U_2\, U_3) = p(H \mid U_1\, I_1\, I_2\, U_2\, U_3)$

only if H and (U_2' U_3') are independent given (U_1 I_1 I_2 U_2 U_3).

However, the criticism loses its justification if we do not demand equality but obviously use interpolation formulas. Now, we may refer to (*) and start off from there, all the more because we were able to prove that (*) is included in Eq.4.1, the General Interpolation Theorem.

[BS84] ***BUCHANAN, B.G.; E.H. SHORTLIFFE: Rule-based inference systems. Addison-Wesley, Massachusetts (1984).***

This book is one of the earliest publications concerning diagnostic expert systems. It shows quite clearly that it is almost unavoidable to try an extension of standard probability theory in order to get some more tools than only Bayes' Theorem. We have been inspired by that opinion, so we also tried to add some new theorems to probability theory.

Reference [Ada84] below points out that the authors have not been too successful in creating an extension of probability theory. In [Ada84, p.270], J. Barclay

Adams states: "The fact that in trying to create an alternative to probability theory or reasoning Shortliffe and Buchanan duplicated the use of standard theory..." Furthermore, he continues on page 271 that this "...demonstrates the difficulty of creating a useful and internally consistent system that is not isomorphic to a portion of probability theory".

[Ada84] ***ADAMS, J.B.: A probability model of medical reasoning and the MYCIN model. In [BS84, Chapter 12].***

The statements of J. Barclay Adams with respect to the MYCIN model are of an unveiling clearness. It is a pity that he did not give a single hint on how to improve that model.

[Pea86] ***PEARL, J.: Fusion, propagation and structuring in belief networks. Artificial Intelligence 29 (1986), pp. 241 – 287.***

The author considers a triplet x_1, x_2, x_3, where x_1 is connected to x_3 via x_2 (ib. p. 248). "The two links, connecting the pairs (x_1, x_2) and (x_2, x_3), can join at the midpoint, x_2, in one of three possible ways

(1) $x_1 \leftarrow x_2 \rightarrow x_3$,

(2) $x_1 \rightarrow x_2 \rightarrow x_3$ or $x_1 \leftarrow x_2 \leftarrow x_3$,

(3) $x_1 \rightarrow x_2 \leftarrow x_3$."

These configurations are discussed in view of the independence of x_1 and x_3, given x_2. This points out that it is necessary to pay attention to the equivalence of causal connections and stochastic dependence. (We devoted a special section to this subject (Section 5)).

On page 248, the author proceeds to separated events, thus giving another contribution to "a graphical criterion for testing conditional independence". However, the paper is not concerned with net nodes having unknown probability.

(We discuss events with known or unknown probability, and in the course of Section 4 we handle the separation mechanisms that apply to them.)

The author also mentions the "screening neighbourhood" (ib. p. 249). It consists of "direct parents, direct successors and all direct parents of the latter." In our

opinion this is not the complete set of neighbours. For example, a causal connection A→B can be influenced by an event C, where C is an inhibitor of A→B and not a member of the screening neighbourhood defined by the Pearl book.
Starting at page 250, the book contains a description of belief propagation in belief trees, but we did not follow the method given there (pp. 250 – 287 / Chapter 2 and 3), but favoured an algebraic solution based on non-linear equation systems.

[BS96] ***BAXT, W.G.; J. SKORA: Prospective validation of artificial neural network trained to identify acute myocardial infarction. Lancet 347 (1996), pp. 12 – 15.***
This paper describes one of the very few medical expert systems in actual use. It employs the Gaussian least squares method, which is completely different from the method we had in mind, and which we therefore did not adopt.

[Lie91] ***LIEBEL, F.-P.: Wahrscheinlichkeit der Entstehung eines Folgezustands aus einer vorhandenen Ursache. In: Österreichische Zeitschrift für Statistik und Informatik (ZSI), 21. Jg. (1991), Heft 3 – 4, S. 125 – 146.***
Original publication of the A→L-Theorem and A→L-Corollary 1.

[Lie99-Pat] ***LIEBEL, F.-P.: Verfahren zur Ermittlung von Wahrscheinlichkeiten pathophysiologischer Zustände als Zwischenergebnisse der Diagnosefindung. European Patent Office, Patent Application No. 99105884.3–2201/ 1026616, date of receipt March 24, 1999.***
The expert system "Computation of causal networks" is the subject of two European patent applications: 91104386.7 (1991) and 99105884.3 (1999).

[Lie99-Pre] ***LIEBEL, F.-P.: Berechnung kausaler Strukturen. Institut für Sportmedizin der Universität Bielefeld (Director: Prof. Dr. E. Zimmermann), BiBoS-Preprint 836 - 4 - 99 (1999), revised version (2000).***
The original version of this book, in German.

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Bielefeld, January 2002

Dr. F.-P. Liebel

Peter V. Zysno

Qualitative Verbundmessung

Die moderne Psychologie sucht die vielschichtigen Eigenschaften und Ressourcen des Menschen qualitativ und quantitativ durch empirische Verfahren zu erfassen und damit zugleich intersubjektiv kontrollierbarer zu machen. Im Kern geht es darum, eine geeignete Operationalisierung für das jeweilige Konstrukt zu finden, die dessen Charakteristika möglichst authentisch verkörpert. Die Formulierung dieses Zusammenhangs ist gleichermaßen kreative wie diffizile Arbeit am Fundament. Seine Solidität ist außerordentlich wichtig, weil es die Aussagekraft aufbauender wissenschaftlicher Sätze und den Realitätsgehalt praktischer Schlußfolgerungen trägt. Der grundlegenden Bedeutung entsprechend wurde im Laufe der Zeit eine recht stattliche Kollektion von Verfahren entwickelt, die aber ein eher unauffälliges Dasein im Methodenarchiv führen. Ihr Abschreckungspotential beruht vor allem auf der mathematischen Einkleidung und dem nicht unerheblichen Durchführungsaufwand. Als Gegenströmung erfreuen sich naive Zahlenetiketten und Einschätzungsskalen für jede Gelegenheit zunehmender Verbreitung. Die sind zwar einfach zu handhaben, genügen in ihrer Aussagekraft aber bestenfalls sehr bescheidenen Ansprüchen und sind mißbräuchlicher Verwendung wehrlos ausgesetzt.

Die qualitative Verbundmessung verfolgt einen neuen Weg, um die beiden Grundanliegen nach valider Eigenschaftserfassung und einfacher Durchführung zu bedienen. Sie stellt einen rational und empirisch überprüfbaren Bezug zwischen definiertem psychologischen Konstrukt, menschlichem Verhalten und zahlenmäßiger Repräsentation her. Sie legt besonderen Wert auf die Explikation des empirischen Relativs, die Wiederspiegelung der definierten Eigenschaft im konkreten Verhalten. Bestätigt das Geflecht der tatsächlichen Reaktionen die hypothetische Struktur, wird das Datengefüge in eine homomorphe Zahlenkonfiguration übertragen. Die Beziehung zwischen den individuellen Reaktionen und den Skalenaussagen bleibt stets transparent.

Für Urteilskonzepte und Bevorzugungsdaten, für klassifikatorische und kontinuierliche Merkmale stehen Prozeduren zur Verfügung, die überwiegend ohne große Schwierigkeiten praktisch umzusetzen sind. Die Resultate lassen sich aufgrund ihrer Datennähe eindeutig interpretieren. Beispiele aus der angewandten Psychologie illustrieren jedes Verfahren.

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