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to my dad

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Contents

Chapter 1

Introduction

General Introduction

General equilibrium theory and cooperative game theory are different models in economic theory describing perfectly competitive or cooperative behavior in economic environments. This dissertation analyzes several aspects of the relation between these different approaches.

General equilibrium theory describes the behavior of agents on perfectly competitive markets. One considers some basic physical realities as the primitives of the model. These can be preferences of the agents, consumption sets, production sets and endowment vectors, but can also include several other aspects like time, uncertainty, financial assets and much more. In a next step one analyzes the equilibrium values of all variables of interest within a closed and interdependent system. Hereby, it is in particular interesting to consider equilibrium prices. The most prominent solution concept for Arrow-Debreu-Economies is the Walrasian or competitive equilibrium. It reflects the idea that given the prices agents act as price takers and maximize their utility subject to their budget constraints. Furthermore, firms also act as price takers and choose profit maximizing production plans. A tuple of price vectors, consumption plans of consumers and production plans of producers is called a competitive equilibrium if the prices are such that all markets clear, given the utility maximization of the agents and profit maximization of the firms. One typically considers questions like existence, uniqueness, stability of equilibria but also the analysis of efficiency or social desirability of competitive equilibria is of particular interest.

On the other hand, there is the theory on cooperative or coalitional games. This theory can be used to describe many different kinds of interactive decision situations where agents can form coalitions and make binding agreements about their cooperation. The aim here is to study what groups can reach rather than what individuals do. Hereby, one considers details of the behavior of the agents as a black box and just analyzes which outcomes are achievable for which coalition. Often, one even suppresses physical outcomes and considers only the allocations of utility that are feasible for the coalitions. The set of players together with the coalitional function is called a cooperative or coalitional game. The coalitional function describes for each coalition the utility allocations that are achievable for the coalition. Hereby, one often assumes that these outcomes are independent of the behavior of the agents outside the coalition. A solution concept for cooperative games assigns to each cooperative game a set of outcomes in terms of utility vectors. Such solution concepts can be single valued but may also be set valued. There exist several solution concepts for cooperative games like the core, inner core, Nash bargaining solution, Shapley value, kernel, nucleolus and many others. These solution concepts capture ideas like efficiency, stability, fairness, justice, equity or others and try to predict or suggest possible agreements. One of the most prominent solution concepts is the core where for elements of the core two conditions have to hold. First, they have to be contained in the set of utility allocations achievable for the grand coalition and, second, it must be impossible for any coalition to make each member of this coalition better off than in the core utility allocation.[1]

There exist mainly two approaches to study the relation between general equilibrium theory and cooperative game theory. The first approach is to analyze

[1] There exists a slightly different definition of the core. This alternative definition requires that core utility allocations have to be achievable for the grand coalition and, furthermore, that there is no coalition that can assign to each member of the coalition a utility that is at least as high as in the core utility allocation and to one member of the coalition a utility that is higher than in the core utility allocation.

an economy via concepts borrowed from game theory. Starting with a market Shapley (1955) considers markets as cooperative games with two kinds of players, seller and buyer. He introduces in this context the general notion of an 'abstract market game'. This is a cooperative game with certain conditions on the characteristic function. Shubik (1959) extends the ideas of Edgeworth (from 1881) and studies 'Edgeworth market games'. In particular he shows that if the number of players of both sides in an Edgeworth market game is the same, then the set of imputations coincides with the contract curve of Edgeworth. Furthermore, he considers non-emptiness conditions for the core of this class of games. Debreu and Scarf (1963) show that under certain assumptions a competitive allocation is in the core. Aumann (1964) investigates, based among others on the oceanic games from Milnor and Shapley (1978)[2], economies with a continuum of traders and obtains that in this case the core equals the set of equilibrium allocations. De Clippel and Minelli (2005) even show that competitive equilibrium allocations are under mild conditions not only in the core, but even in the inner core, a refinement of the core. The core convergence theorem by Debreu and Scarf (1963) shows that the core shrinks to the set of competitive equilibrium allocations if an economy becomes very large in a specific way.

The second approach is to consider cooperative games themselves as economies or markets and goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. In Shapley and Shubik (1975) they show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists

[2]The reference Milnor and Shapley (1978) is based on the Rand research memoranda from the early 1960's.

at least one market that has this payoff vector as its unique competitive payoff vector.

Cooperative games with non-transferable utility (NTU) can be considered as a generalization of TU games, where the transfer of the utility within a coalition does not take place at a fixed rate. After Shapley and Shubik (1969), Billera and Bixby (1974) investigated the NTU case and obtained similar results for compactly convexly generated NTU games. Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced NTU game, that is compactly convexly generated, is a market game. The inner core is a refinement of the core for NTU games. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to the transfer rates given by this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. The notion of the inner core was first described by Shapley and Shubik (1975) and formalized by Shapley (1984). Sufficient conditions for the non-emptiness of the inner core are studied in Qin (1994), Inoue (2010a) and in Bonnisseau and Iehlé (2007). Furthermore, de Clippel and Minelli (2005) give indirectly conditions for the non-emptiness of the inner core via economies. A recent contribution of Bonnisseau and Iehlé (2011) discusses necessary and sufficient conditions for the non-emptiness of the inner core. Hereby, they make use of the notion of payoff depend balancedness developed in Bonnisseau and Iehlé (2007). Qin (1994) analyzes the relation of the inner core with strictly inhibitive sets and de Clippel (2002) gives an axiomatization of the inner core.

It turns out that the inner core is a suitable concept in the context of NTU market games. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector. These results indicate the loss of information when going from markets to market games, since different markets can represent the same market game.

There exist other approaches to market games where different kinds of production are considered. In their work about TU-games Sun et al. (2008) consider economies with coalitional production. Motivated by Sun et al. (2008) and the approach of Billera and Bixby (1974), Inoue (2010b) considers the NTU-case. He proves that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy such that its set of competitive payoff vectors coincide with the inner core of the balanced cover of the original NTU game. Bejan and Gomez (2010) consider not necessarily balanced TU games. They show that the aspiration core of a TU game coincides with the set of competitive wages of two different types of direct production economies including coalitional production. Garratt and Qin (2000a) and Bejan and Gomez (2011) consider market games with time constraints or even time and location constraints. Bejan and Gomez (2011) show that in economies with time and location constraints and without free disposal every TU game is a TU market game in the sense that the game can be generated by an economy of this special type.

Trockel (1996, 2000) introduces an alternative approach and interprets in an NTU-context bargaining games directly as Arrow-Debreu or as coalition production economies. He shows that the unique equilibrium of such an economy coincides with the asymmetric Nash bargaining solution of the underlying game where the weights of the bargaining solution correspond to the shares in production. One difference to other literature is that he uses a stylized models with outputs in the production without requiring inputs.

Another contribution discussing the relation of bargaining solutions with competitive equilibria is an article by Sertel and Yildiz (2003). They consider pure exchange economies and study bargaining games that are induced by these economies. They prove "that there are distinct exchange economies whose Walrasian equilibrium welfare payoffs disagree but which define the same bargaining problem and should have hence determined the same bargaining solution and its payoffs." So, they show that in general there cannot be a bargaining solution that always yields the same payoffs as competitive equilibria.

But the results of Sertel and Yildiz (2003) just show the impossibility of a Walrasian bargaining solution in a very general setup. Under more restrictive conditions it is possible to give a bargaining solution that yields the same payoffs as Walrasian equilibria. John (2005) considers economies with linear utility functions and proportionally divided endowments. In this situation a certain asymmetric Nash bargaining solution yields exactly the competitive equilibrium allocations. Moulin (2003) mentions only in passing that there should exist a version of the results of John (2005) in the context of homogeneous utility functions.

Ervig and Haake (2005) also compare economies and bargaining games. They show that in their model the payoffs of competitive equilibria coincide with payoffs resulting from asymmetric versions of the Perles-Maschler bargaining solution. The main reason for their different result is that they restrict consumer demand by the total endowments of the economy.

Chipman and Moore (1979) discuss the relation of individual demand, aggregate demand and social welfare functions. They consider in particular the question whether the market demand function can be seen as the demand function of some representative consumer.

Contents of this dissertation

This dissertation studies several aspects of the relation of economies and cooperative games. Hereby, the focus is on the relation of solution concepts of the different fields. More precisely, we discuss the relation of competitive equilibria with solution concepts for cooperative games like core, inner core or asymmetric Nash bargaining solutions. We consider games and study which solutions appear as equilibria in economies representing these games. On the other hand we analyze when competitive equilibria of economies and cooperative solutions applied to induced games yield the same allocations.

The dissertation consists of four chapters. Hereby, the first three chapters are in joint work with Sonja Brangewitz (EBIM, Bielefeld University). We worked together on these chapters with equal overall contributions of both of us. These chapters are also a part of her dissertation.

Competitive Outcomes and the Core of TU Market Games

In the second chapter we investigate the relationship between certain subsets of the core for TU market games and competitive payoff vectors of certain markets linked to that game. Given a TU market we consider a certain market depending on a given compact, convex subset of the core. We prove that this market represents the game and further has the given set as the set of payoffs of competitive equilibria. This can be considered as the case in between the two extreme cases of Shapley and Shubik (1975). They remark already that their result can be extended to any closed convex subset of the core, but they omit the details of the proof which we present here. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases. Furthermore, it is interesting to see this result in contrast to the NTU version presented in the third chapter. While in chapter 2 convex, closed subsets of the core are considered it turns out that in the context of NTU market games the appropriate approach is to study compact subsets of the inner core. Also the techniques used in those chapters differ substantially.

More precisely, we denote with $N = \{1, 2..., n\}$ the set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Thus, a coalition is a non-empty subset of players. A *cooperative game with transferable utility (TU)* is given by the pair (N, v) where N is the player set and $v : \mathcal{N} \to \mathbb{R}$ is the *characteristic* or *coalitional function*.[3] One well known solution concept for TU games is the core. The *core* $C(v)$ of a TU game (N, v) is the set of payoff vectors where the value $v(N)$, the grand coalition N can achieve, is distributed and no coalition can improve upon,

$$C(v) = \{x \in \mathbb{R}^n | \, x(N) = v(N), \, x(S) \geq v(S) \quad \forall \, S \in \mathcal{N}\}.$$

We consider the relation of games to a certain class of pure exchange-economies

[3] Shapley and Shubik (1969) define the characteristic function as well for the empty set with $v(\emptyset) = 0$. Others, for example Billera and Bixby (1974), exclude the empty set from this definition.

called markets.

Definition 1 (market)**.** Let $N = \{1, 2..., n\}$ be the set of agents. A *market* is given by $\mathcal{E} = (X^i, \omega^i, u^i)_{i\in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $\omega^i \in X^i$ is the initial endowment vector,
- $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Having introduced the notion of markets one can define when games are related to markets and analyze which games are related to markets. We follow the notion of Shapley and Shubik (1975) and define a TU market game in the following way:

Definition 2 (TU market game)**.** A TU game (N, v) that is representable by a market is a *TU market game*. This means that there exists a market $\mathcal{E}$ such that $(N, v_{\mathcal{E}}) = (N, v)$ with

$$v_{\mathcal{E}}(S) = \max_{x^S\in F(S)} \sum_{i\in S} u^i(x^i) \quad \text{for all } S \in \mathcal{N}.$$

We analyze which kind of equilibria markets have that represent a game. Hereby, we use a notion of equilibrium that suppresses the explicit use of a numeraire commodity. In an extended model this commodity could be used to make transfers of utility between different agents possible.

Definition 3 (competitive solution)**.** A *competitive solution* is an ordered pair $(p^*, (x^{*i})_{i\in N})$, where p^* is an arbitrary n-vector of prices and x^{*N} is a feasible N-allocation, such that

$$u^i(x^{*i}) - p^* \cdot x^{*i} = \max_{x^i\in\mathbb{R}^l_+} [u^i(x^i) - p \cdot x^i] \quad \text{for all } i \in N.$$

The following notion describes payoffs of equilibria.

Definition 4 (competitive payoff vector)**.** A vector α^* is a *competitive payoff vector* if it arises from a competitive solution $(p^*, (x^{*i})_{i\in N})$ such that

$$\alpha^{*i} = u^i(x^{*i}) - p^* \cdot (x^{*i} - \omega^i).$$

A competitive payoff vector describes the payoffs of competitive equilibria in the complete model incorporating a numeraire commodity. The definition goes back to Shapley and Shubik (1975) and is more precisely discussed in chapter 2. In so called direct markets the competitive payoff vectors appear also as the set of equilibrium price vectors.

The main result of this chapter is the following theorem.

Theorem 1. *Let (N, v) be a TU market game and let A be a closed, convex subset of the core. Then there exists a market such that this market represents the game (N, v) and such that the set of competitive payoff vectors of this market is the set A.*

Competitive Outcomes and the Inner Core of NTU Market Games

In the third chapter we consider the classical approach using NTU market games. Hereby, it is well known that a market game can be represented by several markets. A natural question that arises in this context is which competitive equilibria those economies have. In particular, it is not clear which utility payoffs these equilibria generate and how they are related to the game. We investigate the case in between the two extreme cases of Qin (1993), where on the one hand there exists a market that has the complete inner core as set of its competitive payoff vector and on the other hand for any given inner core point there is a market that has this point as its unique competitive payoff vector. We extend the results of Qin (1993) to compact subsets of the inner core: Given an NTU market game we construct a market depending on a given compact subset of the inner core. This market represents the game and further has the given set as the set of payoff vectors of competitive equilibria. Hereby, we can not chose arbitrary compact subsets of the inner core but only subsets satisfying a condition called strict positive separability. This condition mainly requires that any point contained in the compact set can be strictly separated from the set of allocations of utility available for the grand coalition. As this condition is relatively mild our result shows that mainly any compact subset of the inner core of a given game can appear as the set of payoff vectors of an economy representing that game. The result is interesting itself as it gives new insights into the structural relation of market games and markets, but it could also be useful in the context of market foundations of compact valued solutions for cooperative games. The result is an NTU version of the results presented in chapter 2.

In our work we follow the notion of Billera and Bixby (1974) or Qin (1993). An *NTU (non-transferable utility) game* is a pair (N, V), that consists of a player set $N = \{1, ..., n\}$ and a coalitional function V. The coalitional function defines for every coalition a set of utility allocations this coalition can reach, regardless of what players outside this coalition do. Hence, the coalitional function V is defined as a mapping from the set of coalitions, $\mathcal{N}$, to the set of non-empty subsets of $\mathbb{R}^n$, such that for every coalition $S \in \mathcal{N}$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S)$ is non-empty and $V(S)$ is S-comprehensive, meaning $V(S) \supseteq V(S) - \mathbb{R}^S_+$. Hereby, $\mathbb{R}^S$ is defined as $\mathbb{R}^S = \{x \in \mathbb{R}^n | x_i = 0 \text{ if } i \notin S\} \subseteq \mathbb{R}^n$. The *core* $C(V)$ of an NTU game (N, V) is defined as the set of utility allocations that are achievable by the grand coalition N such that no coalition S can improve upon this allocation. Thus, in the NTU context the core is defined as

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i\}.$$

It turns out that in the context of NTU market games a refinement of the core, the inner core, is a suitable concept. To introduce it we need the following notion.

Definition 5 (λ-transfer game)**.** Let (N, V) be a compactly generated NTU game and let $\lambda \in \mathbb{R}^N_+$. Define the λ-*transfer game* of (N, V) by (N, V_λ) with

$$V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$$

where $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$.

The idea of the λ-transfer game is that we allow for transfers of utility within a coalition according to the transfer rates given by the vector λ. Now we can use this notion to introduce the inner core.

Definition 6 (inner core, Shubik (1984))**.** The *inner core* $IC(V)$ of a compactly generated NTU game (N, V) is

$$IC(V) = \{u \in V(N) | \exists\, \lambda \in \Delta \ \text{ such that } \ u \in C(V_\lambda)\}$$

where $C(V_\lambda)$ denotes the core of the λ-transfer game of (N, V).

We analyze the relation of games to economies. Therefore, we consider a particular class of economies called markets. Deviating from the definition in the TU case we consider economies with production where each agent owns his own firm. [4]

Definition 7 (market)**.** A *market* is given by $\mathcal{E} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,
- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

In a market we can describe which allocations are feasible for coalitions.

An S-allocation is a tuple $\left(x^i\right)_{i \in S}$ such that $x^i \in X^i$ for each $i \in S$. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \ \text{ for all } \ i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

[4]This type of economies was considered in Hurwicz (1960), Rader (1964), Billera (1974), Qin (1993), Qin and Shubik (2009), among others

A feasible S-allocation is an allocation that is feasible for a coalition S if the members of the coalition use their joint endowments and produce with their firms.

Having introduced this notion we can analyze which games are related to markets. An NTU game is called an NTU market game if there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible S-allocations in the market or that give less utility than some feasible S-allocation. The main result of this chapter is the following theorem.

Theorem 2. *Let (N, V) be an NTU market game and let A be a compact subset of the inner core of (N, V). Suppose that the game together with the set A satisfy the condition of strict positive separability. Then there exists a market such that*

a) this market represents the game (N, V) and

b) the set of competitive payoff vectors of this market is the set A.

Inner Core, asymmetric Nash and competitive payoffs

In the fourth chapter we discuss the relation of the inner core with the set of asymmetric Nash bargaining solutions for bargaining games. We show that the set of asymmetric Nash bargaining solutions for different strictly positive weights coincides with the inner core, if all points in the underlying bargaining set are strictly positive. Furthermore, we prove that every bargaining game is a market game. By using the results of Qin (1993) we conclude that for every possible vector of weights of the asymmetric Nash bargaining solution there exists an economy that has this asymmetric Nash bargaining solution as its unique competitive payoff vector. We relate the articles of Trockel (1996, 2005) with the ideas of Qin (1993). Our result can be seen as a market foundation of the asymmetric Nash bargaining solution in analogy to the results on non-cooperative foundations of cooperative games.

More precisely, we introduce the following notion of a comprehensive bargaining game.

Definition 8 (NTU bargaining game)**.** Define an *NTU bargaining game*(N, V) with the generating set B using the player set N and the coalitional function

$$V : \mathcal{N} \longrightarrow \mathcal{P}\left(\mathbb{R}^n\right)$$

defined by

$$\begin{aligned} V\left(\{i\}\right) &: = \{b \in \mathbb{R}^n | b_i \leq 0, b_j = 0, \forall j \neq i\} = \{0\} - \mathbb{R}_+^{\{i\}}, \\ V\left(S\right) &: = \{0\} - \mathbb{R}_+^S \quad \text{for all S with } 1 < |S| < n, \\ V\left(N\right) &: = \left\{b \in \mathbb{R}^n | \exists\, b' \in B : b \leq b'\right\} = B - \mathbb{R}_+^n. \end{aligned}$$

We apply the well known concept of an asymmetric Nash bargaining solution to this bargaining game.

Definition 9 (asymmetric Nash bargaining solution)**.** The *asymmetric Nash bargaining solution with a vector of weights* $\theta = (\theta_1, ..., \theta_n) \in \Delta_{++}^n$, for short θ-asymmetric, for a n-person NTU bargaining game (N, V) with disagreement point 0 is defined as the maximizer of the θ-asymmetric Nash product given by $\prod_{i=1}^n u_i^{\theta_i}$ over the set $V(N)$.[5]

If $\nu = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ the ν-asymmetric Nash bargaining solution is called the (symmetric) Nash bargaining solution. We obtain the following result about the relation of asymmetric Nash bargaining solutions with the inner core.

Proposition 1. *Let* (N, V) *be a* n*-person NTU bargaining game with disagreement point* 0 *and generating set* $B \subseteq \mathbb{R}_{++}^n$*.*

- *Suppose we have given a vector of weights* $\theta = (\theta_1, .., \theta_n) \in \Delta_{++}^n$*. Then the* θ*-asymmetric Nash bargaining solution,* a^θ*, is in the inner core of* (N, V)*.*
- *For any given inner core point* a^θ *we can find an appropriate vector of weights* $\theta = (\theta_1, .., \theta_n) \in \Delta_{++}^n$ *such that* a^θ *is the maximizer of the* θ*-asymmetric Nash product* $\prod_{i=1}^n u_i^{\theta_i}$*.*

[5]For bargaining games with a general threat point $d \in \mathbb{R}^n$ the θ-asymmetric Nash product is given by $\prod_{i=1}^n (u_i - d_i)^{\theta_i}$.

Combining this proposition with the results of Qin (1993) we obtain the main result of the chapter.

Proposition 2. *Given a n-person NTU bargaining game (N, V) (with disagreement point 0 and generating set from $\mathbb{R}^n_+$) and a vector of weights $\theta \in \Delta^n_{++}$, there is market that represents (N, V) and where additionally the unique competitive payoff vector of this market coincides with the θ-asymmetric Nash bargaining solution a^θ of the NTU bargaining game (N, V).*

Asymmetric Nash bargaining solutions and perfect competition

Chapter 5 builds on an unpublished mimeo by Reinhard John. The idea of this paper is to study the compatibility of competitive equilibria with concepts of bargaining theory and in particular with asymmetric Nash bargaining solutions. We consider a pure exchange economy and study this economy on the one hand with means of general equilibrium theory and on the other hand with means of cooperative bargaining theory. It turns out that sets of competitive equilibrium allocations and of allocations resulting from an asymmetric Nash bargaining solution coincide as long as one restricts attention to economies where agents have homogeneous (of degree 1) utility functions and where the initial endowments are proportionally distributed. We study what happens when these assumptions are relaxed or changed. Our result also holds if the agents have utility functions that are homogeneous of the same degree k with $0 < k \leq 1$. Moreover, we analyze the robustness of the result. Modifying the utility functions via certain monotone transformation of utility leads to a breakdown of the implications of the results. Furthermore, the unusual choice of the status quo point is analyzed in detail.

More precisely, we consider economies with n consumers $i = 1, ..., n$ and m commodities $j = 1, ..., l$. An economy is a tuple $\left(\left((X^i, u^i)_{i=1}^n\right), e\right)$. $X^i = \mathbb{R}^l_+$ is the consumption set of consumer i. Each consumer is described by a utility function

$$u^i : X^i \longrightarrow \mathbb{R} \tag{1.1}$$

which is weakly increasing, locally nonsatiated, concave, continuous and homogeneous of degree 1. Applying the concept of the Walrasian equilibrium in the context of a proportional division of the endowments leads to the following definition.

Definition 10. An allocation $\bar{x} \in A$ is called a Walras allocation with respect to (the ownership or income distribution) α if there exists a price vector $p = (p_j)_{j=1}^{l} \in \mathbb{R}_+^l \setminus \{0\}$ such that

$$\begin{aligned} \textit{For } i = 1, ..., n: \quad & \bar{x}^i \textit{ maximizes } u^i\left(x^i\right) \textit{ subject to} \\ & x_j^i \geq 0 \textit{ for all } i = 1, ..., n, j = 1, ..., l, \\ & p \cdot x^i \leq p \cdot (\alpha_i e) \textit{ for all } i = 1, ..., n \\ \sum_{i=1}^{n} \bar{x}^i \leq e \textit{ and } p \cdot & \left(\sum_{i=1}^{n} \bar{x}^i - e \right) = 0. \end{aligned}$$

In order to analyze this situation from the viewpoint of cooperative game theory we apply an asymmetric version of the Nash bargaining solution. That leads to the following definition.

Definition 11. A feasible allocation $\bar{x} \in A$ is called a Nash allocation with respect to α if it maximizes $\tilde{U}^{\alpha}(x) = \prod_{i=1}^{n} \left(u^i\left(x^i\right)\right)^{\alpha_i}$ on the set of all feasible allocation, i.e. if $\bar{x}$ is a solution to

$$\begin{aligned} \max \tilde{U}^{\alpha}(x) \quad & \textit{subject to} \\ & x_j^i \geq 0 \textit{ for all } i = 1, ..., n \textit{ and } j = 1, ..., l \\ & \sum_{i=1}^{n} x_j^i - e_j \leq 0 \textit{ for all } j = 1, ..., l \end{aligned}$$

The following proposition is the main result of chapter 5.

Proposition 3. *An allocation $\bar{x} = \left(\bar{x}^i\right)_{i=1}^{n}$ is a Nash allocation with respect to α if and only if it is a Walras allocation with respect to α.*

Short Overview

The main chapters of this thesis, each of which self contained in notation, are based on four articles. Chapters 2 and 3 consider extensions of the results of Shapley and Shubik (1975) and Qin (1993) to subsets of the core respectively the inner core. Chapter 2 considers the case of TU market games while in Chapter 3 the NTU case is analyzed. Chapter 4 investigates the relation of asymmetric Nash bargaining solutions with the inner core in the context of bargaining games. We conclude that asymmetric Nash bargaining solutions are related to certain markets. The fifth Chapter considers the relation of asymmetric Nash bargaining solutions and competitive equilibria but now starting with economies and looking at induced bargaining games.

Chapters 2 and 3 stress the loss of information when going from markets to games. They illustrate that it is impossible to reconstruct payoffs of equilibria of economies if one just has information about the possible allocations of utility, i.e. about the coalitional function. In contrast to this result (and also in contrast to the results of Sertel and Yildiz (2003)) chapter 5 illustrates that the conclusion is not correct if one restricts attention to economies with homogeneous utility functions (see chapter 5 for the detailed conditions). But this result is not very robust. Already small deviations from the assumptions lead to a breakdown of the results. So, only under very restrictive conditions it is possible to consider an asymmetric Nash bargaining solution as the bargaining solution describing payoffs of competitive equilibria.

Chapter 3 and Chapter 4 are also directly related. Chapter 4 makes direct use of the results of Qin (1993) and the results of Chapter 3 which show that essentially anything within the inner core can appear as the set of payoffs of competitive equilibria of some market. Therefore, in particular the set of utility payoffs given by an asymmetric Nash bargaining solution appear as the set of payoffs of competitive equilibria of some market. This market can be chosen as the market constructed by Qin (1993) or as the market given in Chapter 3.

Chapters 4 and 5 both discuss the relation of asymmetric Nash bargaining solutions with competitive equilibria of certain economies. Nevertheless, both approaches differ substantially. In chapter 4 we start with an arbitrary NTU bargaining game and pick a certain market representing this bargaining game. In this suitably chosen market utility payoff vectors of competitive equilibria and those allocations of utility, that an asymmetric Nash bargaining solution yields, coincide. In contrast to that, in chapter 5 we start with a given economy with certain properties. We show that the competitive equilibrium payoff vectors of this economy coincide with the vector of utilities given by an asymmetric Nash bargaining solution of the induced bargaining game.

To this point we have given a brief outline of the general context and developments which lead to this work. Since the questions and topics treated in the following chapters differ, a more detailed scientific placement of this work will be discussed in each chapter separately. This includes separate introductions and conclusions.

Introduction en Français

Introduction générale

La théorie de l'équilibre général et la théorie des jeux coopératifs sont deux modèles différents de l'économie théorique permettant de décrire les comportements compétitifs ou coopératifs. Cette thèse analyse différents aspects des relations entre ces deux théories.

La théorie de l'équilibre général décrit le comportement des agents sur des marchés parfaitement compétitifs. On considère des données physiques comme des éléments primitifs du modèle. Cela peut être les préférences des agents, les ensembles de consommation, les ensembles de production et les vecteurs de dotations initiales, mais aussi plusieurs autres aspects comme le temps, l'incertain, les actifs financiers et bien d'autres. Dans une étape ultérieure, on analyse les valeurs de toutes les variables endogènes à l'équilibre à l'intérieur d'un système complet d'équations interdépendantes. En conséquence, il est particulièrement intéressant de considérer les prix d'équilibre. Le concept de solution dominant pour les économies à la Arrow-Debreu est l'équilibre de Walras ou compétitif. Il traduit l'idée qu'étant donné les prix, les agents agissent comme s'ils n'avaient pas d'influence sur ceux-ci et maximisent leurs fonctions d'utilité sous leur contrainte budgétaire. De plus, les entreprises agissent aussi en prenant les prix comme une donnée et choisissent des productions qui maximisent le profit. La donnée d'un vecteur de prix, de consommations pour les consommateurs et de productions pour les producteurs est appelé un équilibre compétitif si les prix sont tels que l'offre est égal à la demande sur tous les marchés. Dans ce contexte, on étudie typiquement les questions d'existence, d'unicité, de stabilité des équilibres mais l'analyse de l'efficacité ou de l'efficience sociale est aussi particulièrement intéressante.

La théorie des jeux coopératifs ou des jeux avec coalitions est une autre approche des interactions entre agents économiques. Cette théorie peut être utilisée pour décrire de nombreux types d'interactions où les agents peuvent former des coalitions et faire des accords contraignants sur leur coopération. L'objectif ici est d'étudier quels groupes peuvent être atteints plutôt que ce que les individus vont faire. Ainsi, les détails sur le comportement des agents est traité comme une boite noire. On analyse seulement quelles conséquences sont réalisables par quelles coalitions en prenant en compte les préférences des agents sur ces conséquences. Souvent, on ne considère même pas les conséquences physiques mais seulement les niveaux d'utilité réalisables par les coalitions. L'ensemble des joueurs avec les fonctions de coalition est appelé un jeu coopératif. Les fonctions de coalition décrivent les niveaux d'utilité réalisables pour chaque coalition. On suppose souvent que ces niveaux d'utilités sont indépendant du comportement des agents hors de la coalition. Un concept de solution pour les jeux coopératifs associe à chaque jeu un ensemble de conséquences exprimées en terme de niveau d'utilité. Les ensembles de solutions peuvent être des singletons ou multivoques. Il existe plusieurs concepts de solution pour les jeux coopératifs comme le coeur, le coeur interne, la solution de marchandage à la Nash, la valeur de Shapley, le noyau, le nucleolus et bien d'autres. Ces concepts traduisent des notions d'efficacité, de stabilité, d'équité, de justice ou d'autres et tentent de prédire ou de suggérer des accords possibles. L'un des principaux concepts est le coeur pour lequel deux conditions doivent être satisfaites. Premièrement, les niveaux d'utilité dans le coeur doivent être réalisables par la grande coalition regroupant tous les joueurs et, deuxièmement, il ne doit pas être possible pour une coalition d'assurer à tous ses membres un niveau d'utilité strictement supérieur à celui offert par l'élément du coeur considéré.[6]

Il existe principalement deux approches pour étudier les liens entre théorie

[6] Il existe une version un peu différente pour le coeur. Celle-ci requiert que les niveaux d'utilité doivent être réalisables par la grande coalition et que, de plus, aucune coalition peut garantir à chacun de ses membres un niveau d'utilité supérieur ou égal à celui de l'élément proposé et strictement plus grand pour au moins un de ses membres.

de l'équilibre général et théorie des jeux coopératifs. La première approche est d'analyser une économie avec les concepts empruntés à la théorie des jeux. Shapley (1955) considère les marchés comme des jeux coopératifs avec deux types de joueurs, les vendeurs et les acheteurs. Il introduit dans ce contexte la notion générale d'un "jeu de marché abstrait". C'est un jeu coopératif avec certaines conditions sur les fonctions caractéristiques. Shubik (1959) étend les idées d'Edgeworth (de 1881) et étudie les "jeux de marché d'Edgeworth". En particulier, il montre que si le nombre d'agents des deux côtés dans un jeu de marché d'Edgeworth est le même, alors l'ensemble des conséquences coïncide avec les courbes des contrats d'Edgeworth. De plus, il considère des conditions suffisantes de non vacuité pour le coeur de cette classe de jeux. Debreu and Scarf (1963) montre que sous certaines hypothèses une allocation associée à un équilibre compétitif appartient au coeur. En partant des jeux océaniques introduit entre autres par Milnor and Shapley (1978)[7], Aumann (1964) examine les économies avec un continuum d'agents et obtient que le coeur coïncide alors avec l'ensemble des allocations d'équilibre. De Clippel and Minelli (2005) montre même que sous des conditions assez faibles, les allocations d'équilibre appartiennent non seulement au coeur mais aussi au coeur interne, qui est un raffinement du coeur. Le théorème de convergence vers le coeur de Debreu and Scarf (1963) montre que le coeur se rétrécit vers l'ensemble des allocations d'équilibre si l'économie devient de plus en plus grande dans un sens bien précis.

La deuxième approche, datant de Shapley and Shubik (1969), est de considérer les jeux coopératifs eux-mêmes comme des économies ou marchés. Dans Shapley and Shubik (1969), les auteurs considère les jeux de marchés avec utilité transférable (TU). Ce sont des jeux coopératifs TU qui sont dans un certain sens reliés aux économies ou marchés. Plus précisément, un marché représente un jeu si l'ensemble des niveaux d'utilité qu'une coalition peut atteindre dans le marché coïncide avec l'ensemble des niveaux d'utilité pour cette coalition donné par la fonction de coalition. S'il existe un marché qui représente le jeu, alors le jeu est appelé un jeu de marché. Shapley and Shubik (1969) montre l'identité entre la

[7]La référence Milnor and Shapley (1978) est basée sur le Rand research memoranda de début des années 60.

classe des jeux TU totalement équilibré avec la classe des jeux de marché TU. Dans Shapley and Shubik (1975), il montre qu'en partant d'un jeu de marché TU, tous les vecteurs de paiements dans le coeur du jeu sont des niveaux d'utilité d'équilibre d'un certain marché appelé marché direct, et que pour tout point dans le coeur, il existe au moins un marché tel que ce paiement est l'unique niveau d'utilité d'équilibre de ce marché.

Les jeux coopératifs à utilité non-transférable (NTU) peuvent être vu comme une généralisation des jeux à utilité transférable, où les transferts à l'intérieur d'une coalition ne peuvent pas être fait à un taux d'échange constant. Après Shapley and Shubik (1969), Billera and Bixby (1974) a étudié les jeux NTU et a obtenu des résultats similaires pour les jeux NTU générés par des convexes compacts. De manière analogue au résultat de Shapley and Shubik (1969), ils montrent que tous les jeux NTU totalement équilibrés et générés par des convexes compacts sont des jeux de marché.

Le coeur interne est un raffinement du coeur pour les jeux NTU. Un paiement appartient au coeur interne s'il existe un vecteur de taux de transfert tel que, étant donné ce vecteur, aucune coalition ne peut améliorer le paiement même si les niveaux d'utilités pourraient être transférés en suivant les taux donnés par le vecteur. Donc, un élément dans le coeur interne est un élément du coeur d'un jeux associé ou les ensembles d'utilités réalisables sont des demi-espaces définis par le vecteur de taux de transfert. La notion de coeur interne a été décrite pour la première fois par Shapley and Shubik (1975) et formalisé dans Shapley (1984). Des conditions suffisantes pour le non-vacuité du coeur interne ont été proposées dans Qin (1994), Inoue (2010a) et dans Bonnisseau and Iehlé (2007). De plus, de Clippel and Minelli (2005) donne des conditions indirectes pour la non-vacuité du coeur interne pour les jeux dérivés d'une économie. Une contribution récente de Bonnisseau and Iehlé (2011) discute des conditions nécessaires et suffisantes pour la non-vacuité du coeur interne. Qin (1994) analyse la relation entre le coeur interne avec les ensembles strictement inhibitifs et de Clippel (2002) donne une axiomatisation du coeur interne.

Il apparaît que le coeur interne est un concept approprié dans le contexte des jeux de marché NTU. Vérifiant une conjecture de Shapley and Shubik (1975), Qin

(1993) montre que le coeur interne d'un jeu de marché coïncide avec l'ensemble des vecteurs de paiements compétitifs du marché induit par le jeu. De plus, il montre que pour tout jeu de marché NTU et pour n'importe quel élément dans le coeur interne, il existe un marché qui représente le jeu et qui a pour unique paiement compétitif l'élément choisi dans le coeur interne. Ces résultats montrent la perte d'information lorsqu'on va des marchés aux jeux de marché, parce que différents marchés peuvent représenter le même jeu.

Il existe d'autres approches des jeux de marché où différents types de production sont introduites. Dans leur travail sur les jeux TU, Sun et al. (2008) considèrent des économies avec des productions par coalition. Inspiré par Sun et al. (2008) et l'approche de Billera and Bixby (1974), Inoue (2010b) étudie le cas des jeux NTU. Il montre que chaque jeu NTU généré par des ensembles compacts peut être représenté par une économie avec production par coalition. De plus, il prouve qu'il existe une économie avec production par coalition telle que son ensemble de paiements compétitifs coïncide avec le coeur interne de l'extension équilibrée du jeu NTU original. Bejan and Gomez (2010) considèrent des jeux TU non nécessairement équilibrés. Ils montrent que le coeur espéré d'un jeu TU coïncide avec l'ensemble des revenus compétitifs de deux différents types d'économies directes avec production dont des productions par coalition. Garratt and Qin (2000a) et Bejan and Gomez (2011) considèrent des jeux de marché avec des contraintes de temps ou des contraintes de temps et de localisation. Bejan and Gomez (2011) montre que dans les économies avec contraintes en temps et en localisation et sans libre disposition, chaque jeu TU est un jeu de marché TU dans le sens où le jeu peut être généré par une économie de ce type particulier.

Trockel (1996, 2000) introduit une approche alternative et interprète un jeu de marchandage dans un contexte NTU directement comme une économie à la Arrow-Debreu ou comme une économie avec des productions par coalition. Il montre que l'unique équilibre de cette économie correspond à la solution de marchandage de Nash asymétrique du jeu sous-jacent où les poids de la solution de marchandage sont les parts dans la production. Une différence avec le reste de la littérature est qu'il utilise un modèle où la production a des outputs mais pas d'inputs.

Une autre contribution au sujet des relations entre solutions de marchandage et équilibres compétitifs est l'article de Sertel and Yildiz (2003). Ils considèrent des économies d'échange pures et étudient le jeu de marchandage induit par ces économies. Ils montrent qu'il existe des économies d'échange différentes dont les niveaux d'utilité à l'équilibre de Walras sont différents mais qui induisent le même jeu de marchandage et qui donc devrait avoir la même solution de marchandage et donc les mêmes paiements. Ainsi, ils mettent en évidence que, en général, il n'existe pas de solution de marchandage qui donne toujours le même paiement que l'équilibre compétitif.

Mais les résultats de Sertel and Yildiz (2003) montre juste l'impossibilité d'une solution de marchandage walrasienne dans un contexte très général. Sous des hypothèses plus restrictives, il est possible de donner une solution de marchandage qui conduit aux mêmes paiements que l'équilibre walrasien. John (2005) considère des économies avec des fonctions d'utilité linéaires et des dotations initiales proportionnelles. Dans cette situation, une solution de marchandage de Nash asymétrique particulière conduit exactement à l'allocation de l'équilibre compétitif. Moulin (2003) mentionne seulement qu'il devrait exister une version des résultats de John (2005) pour des fonctions d'utilité homogène.

Ervig and Haake (2005) comparent aussi des économies avec des jeux de marchandage. Ils montrent que dans leur modèle, les niveaux d'utilité à l'équilibre compétitif coïncident avec ceux de la solution de marchandage asymétrique de Perles-Maschler. La principale raison qui justifie la différence de leur résultat est qu'il restreignent la demande des consommateurs par la dotation totale de l'économie.

Chipman and Moore (1979) discutent la relation entre demande individuelle, demande agrégée et fonction de bien-être sociale. Ils considèrent en particulier la question de savoir si la fonction de demande du marché peut être interprétée comme une fonction de demande d'un consommateur représentatif.

Contributions de la thèse

Cette thèse étudie plusieurs aspects des relations entre économies et jeux coopératifs. Dans la suite, nous nous concentrons sur les relations entre les concepts de solution dans les différents domaines. Plus précisément, nous discutons les relations entre équilibre compétitif d'une part et les concepts de solution des jeux coopératifs d'autre part comme le coeur interne ou les allocations de marchandage de Nash asymétrique. Nous partons des jeux et nous étudions quelles solutions correspondent à des équilibres des économies qui représentent ces jeux. Nous étudions également quels couples de solutions conduisent aux mêmes niveaux d'utilité.

La thèse est en quatre partie. Les trois premières sont un travail conjoint avec Sonja Brangewitz (EBIM, Université de Bielefeld).

Allocations compétitives et le coeur des jeux de marchés TU

Dans le second chapitre, nous analysons les relations entre certains sous-ensembles du coeur des jeux de marchés TU et les vecteurs de paiements compétitifs de certains marchés reliés à ces jeux. Etant donné un marché TU, nous construisons un marché dépendant d'un sous-ensemble donné convexe et compact du coeur. Le marché représente le jeu et de plus, l'ensemble des paiements compétitifs est égal au sous-ensemble donné a priori. Ce résultat peut être vu comme un résultat intermédiaire entre les deux cas extrêmes de Shapley and Shubik (1975). Les auteurs avaient déjà remarqués que leur résultat pouvait être étendu à n'importe quel ensemble fermé et convexe du coeur, mais ils n'avaient pas donné les détails de la démonstration qui est exposé ici. Ce cas plus général est en particulier intéressant car les deux théorèmes de Shapley and Shubik (1975) sont des cas particuliers. De plus, il est utile quand on le compare avec le résultat pour les jeux NTU présenté dans le troisième chapitre. Alors que dans le chapitre 2, l'ensemble de départ doit être convexe et fermé, il apparaît que pour les jeux de marchés NTU, la bonne approche est d'étudier les ensembles compacts du coeur interne. Notons aussi que les techniques utilisées dans les deux chapitres sont substantiellement différentes.

Soit $N = \{1, 2..., n\}$ l'ensemble des joueurs. L'ensemble de toutes les coalitions non vides est donné par $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Une coalition est donc un sous-ensemble non vide de l'ensemble des joueurs. Un *jeu coopératif avec utilité transférable (TU)* est la donnée d'une paire (N, v) où N est l'ensemble des joueurs et $v : \mathcal{N} \to \mathbb{R}$ est la *fonction caractéristique* ou *de coalition*.[8]

Un concept de solution bien connu pour les jeux TU est le coeur. Le *coeur* $C(v)$ du jeu TU (N, v) est l'ensemble des vecteurs de paiements ou la valeur $v(N)$ de la coalition de tous les joueurs est distribuée et aucune coalition ne peut améliorer cette distribution.

$$C(v) = \{x \in \mathbb{R}^n | \, x(N) = v(N), \, x(S) \geq v(S) \quad \forall \, S \in \mathcal{N}\}.$$

Nous considérons les relations entre les jeux avec une certaine classe d'économies d'échange pures appelées marchés.

Définition 1 (marché)**.** *Soit $N = \{1, 2..., n\}$ l'ensemble des agents. Un marché est donné par $\mathcal{E} = (X^i, \omega^i, u^i)_{i \in N}$ où pour chaque agent $i \in N$*

- *$X^i \subseteq \mathbb{R}^\ell_+$ est un l'ensemble de consomation qui est non vide, fermé et convexei et $\ell \geq 1$, $\ell \in \mathbb{N}$ est le nombre de biens,*
- *$\omega^i \in X^i$ est le vecteur de dotations initiales,*
- *$u^i : X^i \to \mathbb{R}$ est la fonction d'utilité continue et concave.*

Ayant introduit la notion de marché, nous pouvons définir comment les jeux sont reliés aux marchés et quels jeux sont reliés aux marchés. Nous suivons l'approche de Shapley and Shubik (1975) et définissons un jeu de marché TU de la façon suivante:

Définition 2 (jeu de marché TU)**.** *Un jeu TU (N, v) qui est représentable par un marché est un jeu de marché TU. Cela signifie qu'il existe un marché $\mathcal{E}$ tel*

[8] Shapley and Shubik (1969) définit la fonction caractéristique également pour l'ensemble vide par $v(\emptyset) = 0$. D'autres auteurs, par exemple Billera and Bixby (1974), excluent l'ensemble vide de l'ensemble des coalitions.

que $(N, v_{\mathcal{E}}) = (N, v)$ *avec*

$$v_{\mathcal{E}}(S) = \max_{x^S \in F(S)} \sum_{i \in S} u^i(x^i) \qquad \textit{pour tout } S \in \mathcal{N}.$$

Nous utilisons maintenant une définition d'équilibre sans expliciter le bien numéraire qui est utilisé pour faire les transferts d'utilité entre les agents.

Définition 3 (solution compétitive)**.** *Une solution compétitive est une paire* $(p^*, (x^{*i})_{i \in N})$*, où* p^* *est un vecteur de prix de dimension* n *et* x^{*N} *est une allocation réalisable pour la coalition de tous les agents vérifiant*

$$u^i(x^{*i}) - p^* \cdot x^{*i} = \max_{x^i \in \mathbb{R}^{\ell}_{+}} [u^i(x^i) - p \cdot x^i] \qquad \textit{pour tout } i \in N.$$

La définition suivante décrit les paiements d'équilibre.

Définition 4 (vecteur de paiements compétitifs)**.** *Un vecteur* α^* *est un vecteur de paiements compétitifs s'il est définit à partir d'une solution compétitive*$(p^*, (x^{*i})_{i \in N})$ *par*

$$\alpha^{*i} = u^i(x^{*i}) - p^* \cdot (x^{*i} - \omega^i).$$

Un vecteur de paiements compétitifs décrit les paiements à l'équilibre compétitif en intégrant le bien numéraire. La définition est introduite dans Shapley and Shubik (1975) et elle est plus précisément commentée dans le chapitre 2. Dans le marché particulier appelé marché direct, le vecteur de paiements est aussi le vecteur des prix d'équilibre.

Le principal résultat du chapitre 2 est le résultat suivant.

Théorème 1. *Soit* (N, v) *un jeu de marché TU et soit* A *un sous-ensemble convexe fermé du coeur. Alors, il existe un marché qui représente le jeu* (N, v) *et tel que l'ensemble des vecteurs de paiements compétitifs de ce marché est égal à l'ensemble* A.

Paiementss compétitifs et coeur interne des jeux de marché NTU

Dans le troisième chapitre, nous considérons les jeux de marchés NTU. Il est bien connu qu'un jeu de marché peut être représenté par plusieurs marchés. Une question naturelle dans ce contexte est d'étudier quels sont les équilibres compétitifs de ces économies, en particulier, quels sont les niveaux d'utilité générés par ces équilibres et comment ils sont reliés au jeu. Nous examinons le cas intermédiaire entre les deux cas extrêmes de Qin (1993), où d'un côté il existe un marché dont les vecteurs de paiements compétitifs sont tous les éléments du coeur interne et d'un autre côté, pour n'importe quel élément du coeur interne, il existe un marché tel que cet élément est l'unique vecteur de paiements compétitifs. Nous étendons les résultats de Qin (1993) aux sous-ensembles compacts du coeur interne: étant donné un jeu de marché NTU, nous construisons un marché dépendant du sous-ensemble donné. Ce marché représente le jeux et, de plus, a l'ensemble donné comme ensemble de vecteurs de paiements compétitifs. A vrai dire, nous ne pouvons pas choisir un sous-ensemble compact arbitraire du coeur interne mais seulement ceux satisfaisant une condition appelée séparation positive stricte. Cette condition exige principalement que tous les éléments du sous-ensemble compact peuvent être strictement séparés de l'ensemble des allocations en utilité réalisable par la coalition de tous les agents. Comme cette condition est relativement faible, notre résultat montre que presque tous les sous-ensembles compacts du coeur interne d'un jeu donné peuvent être l'ensemble des vecteurs de paiement compétitifs d'une économie représentant le jeu.

Le résultat est intéressant en lui-même car il donne de nouvelles intuitions sur la relation structurelle des jeux de marché et des marchés, mais il peut aussi être utile dans le contexte de l'analyse des fondations des concepts de solution pour les jeux coopératifs par les marchés. Le résultat est une version NTU de ceux présentés dans le chapitre 2.

Dans notre travail, nous adoptons les notations de Billera and Bixby (1974) ou Qin (1993). Un *jeu à utilité non transférable (NTU)* est une paire (N, V) où $N = \{1, ..., n\}$ est l'ensemble des agents et V une fonction de coalition. La fonction de coalition définit pour chaque coalition un ensemble d'allocations en utilité que la coalition peut réaliser, sans tenir compte des agents en dehors de la coalition. Donc, la fonction de coalition V est définie comme une application de l'ensemble des coalitions, $\mathcal{N}$, dans l'ensemble des sous-ensembles non vides de $\mathbb{R}^N$, tel que pour toute coalition $S \in \mathcal{N}$, $V(S) \subseteq \mathbb{R}^S$, $V(S)$ est non vide et S-comprehensif, c'est-à-dire que $V(S) \supseteq V(S) - \mathbb{R}^S_+$. Le *coeur* $C(V)$ du jeu NTU (N, V) est défini comme l'ensemble des allocations en utilité qui est réalisable par la coalition de tous les agent N et telles qu'aucune coalition S ne peut améliorer cette allocation. Donc, dans le contexte NTU, le coeur est défini par:

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ tel que } u'_i \leq u_i\}.$$

Il apparaît que dans le contexte des jeux de marché, un raffinement du coeur, le coeur interne, est un concept bien adapté. Pour le définir, nous avons besoin de la notion suivante.

Définition 5 (jeu avec λ-transfert)**.** *Soit (N, V) un jeu NTU généré par des compacts et soit $\lambda \in \mathbb{R}^N_+$. Le jeu avec λ-transfert de (N, V) est le jeu (N, V_λ) où*

$$V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$$

et $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$.

L'idée du jeu avec λ-transfert est que nous permettons des transferts d'utilité à l'intérieur des coalitions suivant les taux de transferts donnés par le vecteur λ. Maintenant, nous pouvons utiliser cette notion pour définir le coeur interne.

Définition 6 (coeur interne, Shubik (1984))**.** *Le coeur interne $IC(V)$ d'un jeu NTU généré par des compacts (N, V) est*

$$IC(V) = \{u \in V(N) | \exists \, \lambda \in \Delta \text{ \textit{tel que} } u \in C(V_\lambda)\}$$

où $C(V_\lambda)$ est le coeur du jeu avec λ-transfert associé à (N, V).

Nous analysons les relations entre jeux et économie. En conséquence, nous considérons un classe particulière d'économies appelée marchés. Nous considérons des économies avec production où les agents possèdent leur propre entreprise[9] ce qui diffère du cas TU.

Définition 7 (marché)**.** *Un marché est la donnée de* $\mathcal{E} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ *où pour chaque agent* $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ *est un ensemble de consommation non vide, fermé et convexe et* $\ell \geq 1$, $\ell \in \mathbb{N}$ *est le nombre de biens;*

- $Y^i \subseteq \mathbb{R}^\ell$ *est l'ensemble de production de l'agent* i*, qui est un sous-ensemble convexe et fermé vérifiant* $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$*;*

- $\omega^i \in X^i - Y^i$ *est le vecteur de dotations initiales;*

- *et* $u^i : X^i \to \mathbb{R}$ *est la fonction d'utilité continue et concave.*

Dans un marché, nous pouvons décrire les allocations réalisables de chaque coalition.

Une S-allocation est un élément $\left(x^i\right)_{i \in S}$ tel que $x^i \in X^i$ pour tout $i \in S$. L'ensemble des *S-allocations réalisables* est défini par

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \text{ pour tout } i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

Une S-allocation réalisable est une allocation réalisable par la coalition S si les membres de cette coalition utilisent conjointement leurs dotations initiales et leurs capacités de production.

Ayant introduit cette notion, nous pouvons analyser quels sont les jeux reliés aux marchés. Un jeu NTU est appelé un jeu de marché NTU s'il existe un marché tel que l'ensemble $V(S)$ des allocations en utilité réalisable par une coalition S donné par la fonction de coalition est égal à l'enveloppe compréhensive de l'ensemble des allocations en utilité qui est généré par les S-allocations réalisables du marché. Le résultat principal de ce chapitre est le résultat suivant.

[9] Ce type d'économie est considéré dans Hurwicz (1960), Rader (1964), Billera (1974), Qin (1993), Qin and Shubik (2009), parmi d'autres.

Théorème 2. *Soit (N, V) un jeu de marché NTU et soit A un sous-ensemble compact du coeur interne de (N, V). Supposons que le jeu et l'ensemble A vérifient la condition de stricte séparabilité positive. Alors, il existe un marché tel que*

a) ce marché représente le jeu (N, V) et

b) l'ensemble des vecteurs de paiements compétitifs de ce marché est égal à l'ensemble A.

Coeur interne, allocation de Nash asymétrique et paiements compétitifs

Dans le quatrième chapitre, nous étudions la relation du coeur interne avec l'ensemble des solutions de marchandage de Nash asymétriques pour les jeux de marchandage. Nous montrons que l'ensemble des solutions de marchandage de Nash asymétriques pour différents poids strictement positifs est égal au coeur interne si tous les éléments du jeu de marchandage sous-jacent sont strictement positifs. De plus, nous démontrons que chaque jeu de marchandage est un jeu de marché. En utilisant les résultats de Qin (1993), nous pouvons en conclure que pour chaque solution de marchandage de Nash asymétrique, il existe une économie qui a pour unique vecteur de paiement compétitif cette solution. Nous faisons ainsi un lien entre les articles de Trockel (1996, 2005) avec les idées de Qin (1993). Notre résultat peut être vu comme une fondation par les marchés de la solution de marchandage de Nash asymétrique comparable aux résultats sur la fondation non-coopérative des jeux coopératifs.

Plus précisément, nous introduisons la notion suivante d'un jeu de marchandage compréhensif.

Définition 8 (jeu de marchandage NTU)**.** *Nous définissons un jeu de marchandage NTU (N, V) avec un ensemble générateur B, un ensemble de joueurs N et une fonction de coalition*

$$V : \mathcal{N} \longrightarrow \mathcal{P}(\mathbb{R}^n)$$

définie par:

$$
\begin{aligned}
V(\{i\}) &:= \{b \in \mathbb{R}^n | b_i \leq 0, b_j = 0, \forall j \neq i\} = \{0\} - \mathbb{R}_+^{\{i\}}, \\
V(S) &:= \{0\} - \mathbb{R}_+^S \quad \textit{pour tout } S \textit{ avec } 1 < |S| < n, \\
V(N) &:= \left\{b \in \mathbb{R}^n | \exists\, b' \in B : b \leq b'\right\} = B - \mathbb{R}_+^n.
\end{aligned}
$$

Nous appliquons le concept bien connu de solution de marchandage de Nash asymétrique à ce jeu de marchandage.

Définition 9 (solution de marchandage de Nash asymétrique)**.** *La solution de marchandage de Nash asymétrique avec le vecteur de poids $\theta = (\theta_1, ..., \theta_n) \in \Delta_{++}^n$, en bref θ-asymétrique, pour un jeu de marchandage NTU avec n joueurs (N, V) et point de désaccord* 0 *est définie comme la solution de la maximisation du produit de Nash θ-asymétrique donnée par $\prod_{i=1}^n u_i^{\theta_i}$ sur l'ensemble $V(N)$*[10]*.*

Si $\nu = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ la solution de marchandage de Nash ν-asymétrique est appelée la solution de marchandage de Nash (symétrique). Nous obtenons le résultat suivant sur les relations entre coeur interne et solutions de marchandage de Nash asymétriques.

Proposition 4. *Soit (N, V) un jeu de marchandage NTU avec n joueurs ayant pour point de désaccord* 0 *et engendré par l'ensemble $B \subseteq \mathbb{R}_{++}^n$.*

- *Supposons que nous avons un vecteur de poids $\theta = (\theta_1, .., \theta_n) \in \Delta_{++}^n$. Alors la solution de marchandage de Nash θ-asymétrique, a^θ, appartient au coeur interne de (N, V).*

- *Pour n'importe quel élément a du coeur interne, nous pouvons trouver un vecteur de poids approprié $\theta = (\theta_1, .., \theta_n) \in \Delta_{++}^n$ tel que a est la solution de la maximisation du produit de Nash θ-asymétrique $\prod_{i=1}^n u_i^{\theta_i}$.*

En combinant ce résultat avec ceux de Qin (1993), nous obtenons le résultat principal de ce chapitre.

[10] Pour les jeux de marchandage avec un point de désaccord général $d \in \mathbb{R}^n$ le produit de Nash θ-asymétrique est donné par $\prod_{i=1}^n (u_i - d_i)^{\theta_i}$.

Proposition 5. *Etant donné un jeu de marchandage NTU à n joueurs (N, V) (avec point de désaccord 0 et engendré par un sous-ensemble de $\mathbb{R}^n_+$) et un vecteur de poids $\theta \in \Delta^n_{++}$, il existe un marché qui représente (N, V) et, de plus, l'unique vecteur de paiement compétitif de ce marché est la solution de marchandage de Nash θ-asymétrique a^θ du jeu de marchandage NTU (N, V).*

Quatrième article

Le chapitre 5 est basé sur une note non publiée de Reinhard John. L'idée du papier est d'étudier la compatibilité des équilibres compétitifs avec les concepts de la théorie du marchandage et en particulier, avec la solution de marchandage de Nash asymétrique. Nous considérons une économie d'échange pure et nous l'étudions à la fois d'un point de vue de la théorie de l'équilibre général et d'un point de vue de la théorie des jeux coopératifs. Il apparaît que les ensembles des allocations d'équilibre compétitives et les allocations obtenues par la solution de marchandage de Nash asymétrique coïncident lorsqu'on se restreint aux économies où les agents ont des préférences homogènes de degré 1 et lorsque les dotations initiales sont distribuées proportionnellement à la dotation initiale totale. Nous étudions également ce qu'il advient lorsque les hypothèses sont affaiblies ou modifiées. Notre résultat est encore vrai lorsque les agents ont des préférences qui sont homogènes pour le même degré k avec $0 \leq k \leq 1$. Nous analysons la robustesse du résultat. La modification des fonctions d'utilité par des transformations monotones ne permet plus d'obtenir ce résultat. De plus, le choix inhabituel du point de status quo est étudié en détail.

Plus précisément, nous considérons des économies avec n consommateurs $i = 1, ..., n$ et ℓ biens $j = 1, ..., \ell$. Une économie est un $n$uplet $\left(\left((X^i, u^i)_{i=1}^n\right), e\right)$. $X^i = \mathbb{R}^\ell_+$ est l'ensemble de consommation du consommateur i. Les préférences de chaque consommateur sont décrites par une fonction d'utilité

$$u^i : X^i \longrightarrow \mathbb{R} \tag{1.2}$$

qui est faiblement croissante, localement non saturée, concave, continue et homogène de degré 1. Le concept d'équilibre de Walras dans ce contexte avec une répartition proportionnelle des dotations initiales conduit à la définition suivante:

Définition 10. *Une allocation $\bar{x} \in A$ est appelée une allocation de Walras par rapport à la distribution des ressources α s'il existe un vecteur de prix $p = (p_j)_{j=1}^{\ell} \in \mathbb{R}_+^{\ell} \setminus \{0\}$ tel que*

$$\begin{aligned} \textit{Pour } i = 1, ..., n: \quad & \bar{x}^i \textit{ maximise } u^i\left(x^i\right) \textit{ sous les contraintes} \\ & x_j^i \geq 0 \textit{ pour tout } j = 1, ..., \ell, \\ & p \cdot x^i \leq p \cdot (\alpha_i e) \\ \sum_{i=1}^n \bar{x}^i \leq e \textit{ et } p \cdot & \left(\sum_{i=1}^n \bar{x}^i - e \right) = 0. \end{aligned}$$

Pour analyser la même situation d'un point de vue des jeux coopératif, nous utilisons la solution de marchandage asymétrique de Nash. Ceci nous donne la définition suivante:

Définition 11. *Une allocation réalisable $\bar{x} \in A$ est appelée une allocation de Nash par rapport à α si elle maximise $\tilde{U}^{\alpha}(x) = \prod_{i=1}^{n} \left(u^i\left(x^i\right)\right)^{\alpha_i}$ sur l'ensemble des allocations réalisables, c'est-à-dire si $\bar{x}$ est une solution de*

$$\begin{aligned} \max \tilde{U}^{\alpha}(x) \quad & \textit{sous les contraintes} \\ & x_j^i \geq 0 \textit{ pour tout } i = 1, ..., n \textit{ et } j = 1, ..., \ell \\ & \sum_{i=1}^n x_j^i - e_j \leq 0 \textit{ pour tout } j = 1, ..., \ell \end{aligned}$$

La proposition suivante constitue le résultat principal du chapitre 5.

Proposition 6. *Une allocation $\bar{x} = \left(\bar{x}^i\right)_{i=1}^n$ est une allocation de Nash par rapport à α si et seulement si c'est une allocation de Walras par rapport à α.*

Une rapide synthèse

Les principaux chapitres de cette thèse, chacun étant auto-suffisant pour les notations, sont basés sur quatre articles. Les chapitres 2 et 3 sont des extensions des résultats Shapley and Shubik (1975) et Qin (1993) à des sous-ensembles du coeur ou du coeur interne. Le chapitre 2 est consacré aux jeux de marché à utilité transférable alors que le chapitre 3 analyse les jeux à utilité non transférable. Le chapitre 4 étudie les relations entre la solution de marchandage de Nash asymétrique avec le coeur interne dans le contexte des jeux de marchandage. Nous pouvons en conclure que les solutions de marchandage de Nash asymétriques sont reliées à certains marchés. Le cinquième chapitre considère les relations entre les solutions de marchandage de Nash asymétriques et les équilibres compétitifs mais en partant maintenant des économies et en regardant les jeux induits.

Les chapitres 2 et 3 mettent en évidence le manque d'information lorsqu'on part des marchés vers les jeux. Ils montrent qu'il est impossible de déterminer les paiements associés à l'équilibre dans les économies si on a seulement l'information au sujet des allocations réalisables en utilité, c'est-à-dire, les fonctions coalitionnelles. Au contraire (et aussi au contraire des résultats de Sertel and Yildiz (2003)), le chapitre 5 montre que la conclusion n'est pas juste si on se restreint aux économies avec des préférences homogènes (voir le chapitre 5 pour les conditions précises). Mais ce résultat n'est pas très robuste. De petites déviations par rapport aux hypothèses rendent invalides le résultat. Ainsi, seulement sous des conditions restrictives, il est possible de considérer la solution de marchandage de Nash asymétrique comme la solution de marchandage décrivant les paiements des équilibres compétitifs.

Les chapitres 3 et 4 sont directement reliés. Le chapitre 4 utilise les résultats de Qin (1993) et les résultats du chapitre 3, qui montrent qu'essentiellement, n'importe quel ensemble à l'intérieur du coeur interne peut être représenté comme l'ensemble de paiements des équilibres compétitifs d'un marché. Donc, en particulier l'ensemble des paiements associés à une solution de marchandage de Nash asymétrique peut être représenté comme l'ensemble des paiements compétitifs d'un marché. Ce marché peut être construit en suivant soit Qin (1993) soit le chapitre 3.

Les chapitres 4 et 5 sont tous les deux consacrés aux relations entre solutions de marchandage de Nash asymétriques et équilibres de certaines économies. Cependant, les deux approches sont substantiellement différentes. Dans le chapitre 4, on part d'une jeu de marchandage NTU et on choisit un marché qui représente ce jeu. Pour ce marché soigneusement choisi, les vecteurs de paiements en utilité des équilibres compétitifs et ceux associés à une solution de marchandage de Nash asymétrique coïncident. Au contraire, dans le chapitre 5, on part d'un économie donnée avec certaines propriétés. On montre que le vecteur de paiement des équilibres compétitifs de cette économie coïncide avec celui donné par la solution de marchandage de Nash asymétrique du jeu de marchandage associé.

A ce stade, nous avons donné une brève présentation du contexte général et des développements qui conduisent à ce travail. Comme les questions et les sujets traités dans les chapitres ci-après diffèrent, un positionnement scientifique plus détaillé est donné dans chaque chapitre avec des introductions et des conclusions propres. Nous avons essayé d'utiliser des notations constantes dans les quatre chapitres. Nous avons réussi presque partout mais à certains moments des variantes sont nécessaires.

Chapter 2

Competitive Outcomes and the Core of TU Market Games

2.1 Abstract

We investigate the relationship between certain subsets of the core for TU market games and competitive payoff vectors of certain markets linked to that game. This can be considered as the case in between the two extreme cases of Shapley and Shubik (1975). They remark already that their result can be extended to any closed convex subset of the core, but they omit the details of the proof which we present here. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

2.2 Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. Furthermore, Shapley and Shubik (1975) show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector. Moreover, they claim that an analogous result holds also for closed convex subsets of the core. Shapley and Shubik (1975) give a hint how this can be shown but they omit the details of the proof. By following this remark of Shapley and Shubik (1975) we give a detailed proof how their two main results can be extended to any closed convex subset of the core. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

Similarly to the approach of Shapley and Shubik (1969, 1975), Inoue (2010b) uses coalition production economies as in Sun et al. (2008) instead of markets. Inoue (2010b) shows that every TU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy whose set of competitive payoff vectors coincides with the core of the balanced cover of the original TU game.

A different extension of Shapley and Shubik (1969, 1975) is Garratt and Qin (2000b). They consider time-constrained market games, where the agents are supposed to supply one unit of time to the market. Their main result is that a TU game is a time-constrained market game if and only if it is superadditive. This result of Garratt and Qin (2000b) was again extended by Bejan and Gomez

(2011) introducing additionally location and free disposal constraints. They show that in this sense the entire class of TU games can be considered as market games.

For NTU market games Brangewitz and Gamp (2011a) extend the NTU analogue to Shapley and Shubik (1975), namely Qin (1993), to closed subsets of the inner core. Hereby, the techniques used to show the results in the TU and the NTU case are notably different.

2.3 TU market games

In this section we state the main definitions and results on TU market games. The following introduction for TU market games is mainly based on Shapley and Shubik (1969) and Shapley and Shubik (1975).

Let $N = \{1, 2..., n\}$ be a set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Thus, a coalition is a non-empty subset of players. A *cooperative game with transferable utility (TU)* is given by the pair (N, v) where N is the player set and $v : \mathcal{N} \to \mathbb{R}$ is the *characteristic* or *coalitional function.*[1] A *subgame* (T, v_T) of a TU game (N, v) is a subset of players $T \in \mathcal{N}$ and the characteristic function v_T with $v_T(S) = v(S)$ for $S \subseteq T$, $S \neq \emptyset$. A *payoff vector* for a TU game (N, v) is a vector $x \in \mathbb{R}^n$. The payoff of a coalition $S \in \mathcal{N}$ is given by $x(S) = \sum_{i \in S} x_i$. The *core* $C(v)$ of a TU game (N, v) is the set of payoff vectors where the value $v(N)$, the grand coalition N can achieve, is distributed and no coalition can improve upon,

$$C(v) = \{x \in \mathbb{R}^n |\, x(N) = v(N),\, x(S) \geq v(S) \quad \forall\, S \in \mathcal{N}\}.$$

Given a set of players $N = \{1, 2..., n\}$, a family $\mathcal{B} \subseteq \mathcal{N}$ is a *balanced family* if there exist weights $\{\gamma_S\}_{S \in \mathcal{B}}$, with $\gamma_S \geq 0$, such that for all $i \in N$ we have

$$\sum_{S \in \mathcal{B},\, S \ni i} \gamma_S = 1.$$

[1]Shapley and Shubik (1969) define the characteristic function as well for the empty set with $v(\emptyset) = 0$. Others, for example Billera and Bixby (1974), exclude the empty set from this definition.

The weights γ_S do not depend on the individual players but on the coalition $S \in \mathcal{N}$. The above condition can be as well written as

$$\sum_{S \in \mathcal{N}} \gamma_S e^S = e^N$$

where $e^S \in \mathbb{R}^n$ is the vector with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Let the set of weights be denoted by $\Gamma(e^N)$. The balancing weights can be interpreted as the intensity with which player i participates in a coalition or the fraction of time he spends to be in this coalition.

A TU game (N, v) is *balanced* if for every balanced family $\mathcal{B}$ with weights $\{\gamma_S\}_{S \in \mathcal{B}}$ we have

$$\sum_{S \in \mathcal{B}} \gamma_S v(S) \leq v(N).$$

A TU game (N, v) is *totally balanced* if all its subgames are balanced. The *totally balanced cover* of a TU game (N, v) is the smallest TU game $(N, \bar{v})$ that is totally balanced and contains the game (N, v).

Shapley and Shubik (1969) recall the following result of Shapley (1965):

Theorem 3 (Shapley and Shubik (1969))**.** *A game has a non-empty core if and only if it is balanced.*

In oder to define a TU market game we first need to introduce the notion of a market. For the TU case it is sufficient to consider markets without production.

Definition 12 (market)**.** Let $N = \{1, 2..., n\}$ be the set of agents (or players). A *market* is given by $\mathcal{E} = (X^i, \omega^i, u^i)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}_+^\ell$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $\omega^i \in X^i$ is the initial endowment vector,
- $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in the case with non-transferable utility (NTU) usually markets with production are considered, see for example Billera and Bixby (1974) or Qin (1993).

Let $S \in \mathcal{N}$ be a coalition. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \;\middle|\; x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} x^i = \sum_{i \in S} \omega^i \right\}.$$

Elements of $F(S)$ are often denoted for short by x^S. The feasible S-allocations are those allocations the coalition S can achieve by redistributing their initial endowments within the coalition.

Now we define a TU market game in the following way:

Definition 13 (TU market game)**.** A TU game (N, v) that is representable by a market is a *TU market game*. This means there exists a market $\mathcal{E}$ such that $(N, v_{\mathcal{E}}) = (N, v)$ with

$$v_{\mathcal{E}}(S) = \max_{x^S \in F(S)} \sum_{i \in S} u^i(x^i) \quad \text{for all } S \in \mathcal{N}.$$

For a TU market game there exists a market such that the value a coalition S can reach according to the coalitional function coincides with the joint utility that is generated by feasible S-allocations in the market.

Given a TU game we can generate a market from this game in a “natural” way. Shapley and Shubik (1969) call this market a direct market.

Definition 14 (direct market)**.** A TU game (N, v) generates a *direct market* $\mathcal{D}_v = (X^i, \omega^i, u^i)_{i \in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+$,
- the initial endowment $\omega^i = e^{\{i\}}$ with $e^{\{i\}}_i = 1$ and $e^{\{i\}}_j = 0$ for $j \neq i$,
- the utility function $u^i(x) = \max \left\{ \sum_{S \in \mathcal{N}} \gamma_S v(S) \;\middle|\; \gamma_S \geq 0 \,\forall\, S \in \mathcal{N}, \sum_{S \in \mathcal{N}} \gamma_S e^S = x \right\}$.

The utility function $u^i(\cdot)$ of the direct market $\mathcal{D}_v$ is identical for every individual $i \in N$ and is homogeneous of degree 1, concave and continuous. Note that in a direct market every consumer owns initially his own (private) good or interpreted differently every player "is" himself a good. Using the direct market $\mathcal{D}_v$, Shapley and Shubik (1969) obtain the following characterization of TU market games.

Theorem 4 (Shapley and Shubik (1969))**.** *A game is a market game if and only if it is totally balanced.*

This means that in order to consider TU market games it is sufficient to consider just those TU games that are totally balanced. To obtain the above result Shapley and Shubik (1969) start by looking at an arbitrary TU game and its direct market. Hereafter, they consider the TU game of the direct market and show that it is equal to the totally balanced cover of the TU game they started with.

In a second paper Shapley and Shubik (1975) investigate the relationship between competitive payoffs, that arise from a competitive solution in the market, and the core of TU market games.

Definition 15 (competitive solution)**.** A *competitive solution* is an ordered pair $(p^*, (x^{*i})_{i \in N})$, where p^* is an arbitrary n-vector of prices and x^{*N} is a feasible N-allocation, such that

$$u^i(x^{*i}) - p^* \cdot x^{*i} = \max_{x^i \in \mathbb{R}^l_+} [u^i(x^i) - p \cdot x^i] \quad \text{for all } i \in N.$$

We are in a setting with transferable utility. Thus, there is implicitly the additional commodity money, that makes the transfer of utility possible. Suppose ξ_0^i are the initial money holdings of agent i. Then his "true" maximization problem is

$$\max_{x^i \in \mathbb{R}^l_+} [u^i(x^i) + \xi_0^i - p \cdot \left(x^i - \omega^i\right)].$$

Since the solution of the maximization problem is independent of the initial money holdings and the initial endowment, it is equivalent to solve the in the definition above stated maximization problem.

Definition 16 (competitive payoff vector)**.** A vector α^* is a *competitive payoff vector* if it arises from a competitive solution $(p^*, (x^{*i})_{i\in N})$ such that

$$\alpha^{*i} = u^i(x^{*i}) - p^* \cdot (x^{*i} - \omega^i).$$

Shapley and Shubik (1975) show the following two relationships between the core and competitive payoff vectors.

Theorem 5 (1, Shapley and Shubik (1975))**.** *Every payoff vector in the core of a TU market game is competitive in the direct market of that game.*

Theorem 6 (2, Shapley and Shubik (1975))**.** *Among the markets that generate a given totally balanced TU game, there exists a market having any given core point as its unique competitive payoff vector.*

These two theorems represent the two extreme cases where on the one hand the whole core equals the set of competitive payoff vectors of the direct market and one the other hand a given core point is the unique competitive payoff vector of a certain other market. The main ideas to prove the above two theorems are the following: For the first result Shapley and Shubik (1975) use the direct market to show that its competitive payoff vectors coincide with the core of the TU market game. To prove the second theorem they introduce a second game with a modified coalitional function for the grand coalition N. Afterwards they look at the direct market of the original game with a modified utility function depending on a given core point. Finally they show that this market represents the original TU game and has a given core point as its unique competitive payoff vector.

2.4 Results on TU market games

Shapley and Shubik (1975) already remark that for TU market games a extension of their proof for their second theorem leads to the following result.

Theorem 7. *Let (N, v) be a totally balanced TU game and let A be a closed, convex subset of the core. Then there exists a market such that this market rep-*

resents the game (N, v) and such that the set of competitive payoff vectors of this market is the set A.

Shapley and Shubik (1975) omit the details of the proof. We elaborate on them here. They remark that it is enough to change the definition of the utility function.

In the following we first define the according market and show afterwards in two steps that this market satisfies the properties we require.

Let (N, v) be a totally balanced TU game with $N = \{1, ..., n\}$ the set of players and the coalitional function v. Let $\mathcal{D}_v$ be its direct market as defined before. For $d \in \mathbb{R}_{++}$ define the TU game (N, v_d) by

$$v_d(S) = v(S) \quad \text{for all } S \subset N$$

and

$$v_d(N) = v(N) + d.$$

Since $d > 0$ the game (N, v_d) is totally balanced. Analogously let $\mathcal{D}_{v_d}$ be the direct market of (N, v_d). Let $(u_d^i)_{i \in N}$ denote the utility functions of $\mathcal{D}_{v_d}$, i.e.

$$u_d^i(x) = \max\left\{ \sum_{S \in \mathcal{N}} \gamma_S v_d(S) \middle| \gamma_S \geq 0 \, \forall \, S \in \mathcal{N}, \sum_{S \in \mathcal{N}} \gamma_S e^S = x \right\}.$$

As the utility functions u_d^i in the direct market $\mathcal{D}_{v_d}$ are identical for every individual $i \in N$, we write for short u_d.

Let A be a any non-empty closed convex subset of the core. For $\alpha \in A$ let $u_{d,\alpha}$ be defined as

$$u_{d,\alpha}(x) = \min(u_d(x), \alpha \cdot x).$$

Then define the function $u_{d,A}$ by

$$u_{d,A}(x) = \min_{\alpha \in A} u_{d,\alpha}(x).$$

Since $u_{d,A}$ is continuous and concave we can define a market by

$$\mathcal{E}_{v_d} = \left(\mathbb{R}^n_+, e^{\{i\}}, u^i_{d,A}\right)_{i\in N}.$$

with $u^i_{d,A} = u_{d,A}$ for all $i \in N$. It is easy to see that $u_{d,A}$ is homogeneous of degree 1.

Next, we show first that the market game of this market is (N, v) and second that the set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ is exactly the set A.

Proposition 7. *The market $\mathcal{E}_{v_d}$ represents the game (N, v).*

Proof. Recall that for the market $\mathcal{E}_{v_d}$ the set

$$F(S) = \left\{ x^S \in \mathbb{R}^{n\cdot S}_+ | \sum_{i\in S} x^i = \sum_{i\in S} e^{\{i\}} \right\}$$

is the set of feasible allocations for a coalition $S \in \mathcal{N}$.

Looking at the market game generated by the market $\mathcal{E}_{v_d}$ we obtain

$$\begin{aligned}
v_{\mathcal{E}_{v_d}}(S) &= \max_{x^S\in F(S)} \sum_{i\in S} u^i_{d,A}(x^i) \\
&= |S| \max_{x^S\in F(S)} \sum_{i\in S} \frac{1}{|S|} u_{d,A}(x^i) \\
&\overset{(1)}{=} |S| \max_{x^S\in F(S)} u_{d,A}\left(\frac{e^S}{|S|}\right) \\
&= |S| u_{d,A}\left(\frac{e^S}{|S|}\right) \\
&\overset{(2)}{=} u_{d,A}(e^S) \\
&= \min_{\alpha\in A} u_{d,\alpha}(e^S) \\
&= \min_{\alpha\in A}(\min(u_d(e^S), \alpha\cdot e^S)) \\
&\overset{(3)}{=} \min_{\alpha\in A}(\min(v_d(S), \alpha\cdot e^S)) \\
&= \min_{\alpha\in A}(v_d(S), \alpha\cdot e^S) \\
&\overset{(4)}{=} v(S)
\end{aligned}$$

The detailed arguments are the following:

(1) First observe that $\sum_{i\in S}\frac{1}{|S|}u_{d,A}(x^i) \leq u_{d,A}\left(\sum_{i\in S}\frac{x^i}{|S|}\right) = u_{d,A}\left(\frac{e^S}{|S|}\right)$ from the concavity of $u_{d,A}$ and the market clearing condition. We take the maximum on both sides over the feasible S-allocations $F(S)$ and we observe that $\bar{x}^i = \frac{1}{|S|}e^S$ for all $i \in S$ is a feasible S-allocation. Therefore, we obtain that setting $\left(\bar{x}^i\right)_{i\in S}$ maximizes the expression on the left side and hence we get equality.

(2) The equality follows from the homogeneity of degree 1 of $u_{d,A}$.

(3) Using the totally balancedness of the game (N, v_d) we obtain

$$u_d(e^S) = \max\left\{\sum_{T\in\mathcal{N}}\gamma_T v_d(T)\middle|(\gamma_T)\geq 0, \sum_{T\in\mathcal{N}}\gamma_T e^T = e^S\right\} = v_d(S).$$

(4) For $S \subset N$ this minimum is equal to $v(S)$, since α is in the core of the TU game (N, v) and therefore $\alpha \cdot e^S \geq v(S) = v_d(S)$. For $S = N$ the minimum is equal to $\alpha' \cdot e^N$ for some $\alpha' \in A$ and since α' is in the core of (N, v) we have $\alpha' \cdot e^N = v(N)$. As $d > 0$ we have $v(N) < v_d(N)$.

Thus $v_{\mathcal{E}_{v_d}} = v$ and hence the market $\mathcal{E}_{v_d}$ generates the game (N, v). □

Proposition 8. *The set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ are coincides with the set A.*

Proof. The proof is divided into five parts:

1. First, suppose $\left((x^{*i})_{i\in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then competitive payoffs are of the form $\left(p^* \cdot e^{\{i\}}\right)_{i\in N}$.

   From the definition of a competitive solution it follows that $(x^{*i})_{i\in N}$ clears the markets,
   $$\sum_{i=1}^{n} x^{*i} = \sum_{i=1}^{n} e^{\{i\}} = e^N$$
   and maximizes for each trader i his trading profit given by
   $$u_{d,A}(x^i) - p \cdot x^i.$$

Moreover, we have from the existence of a maximum and the fact that the trading profit as a function of the consumption bundle is homogeneous of degree 1 that

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} = 0.$$

Looking at the competitive payoffs of competitive solutions we observe

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

2. Second, suppose $\left((x^{*i})_{i\in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is as well a competitive solution in the market $\mathcal{E}_{v_d}$. In addition the competitive payoffs coincide.

   From the fact that the trading profit equals zero we obtain

$$\begin{aligned} u_{d,A}\left(\frac{1}{n}e^N\right) - p^* \cdot \frac{1}{n}e^N &= u_{d,A}\left(\frac{1}{n}\sum_{i=1}^n x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^n x^{*i} \\ &\overset{(1)}{=} \frac{1}{n}\sum_{i=1}^n u_{d,A}\left(x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^n x^{*i} \\ &= \frac{1}{n}\left(\sum_{i=1}^n u_{d,A}\left(x^{*i}\right) - p^* \cdot \sum_{i=1}^n x^{*i}\right) \\ &= \frac{1}{n}\left(\sum_{i=1}^n \left(u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i}\right)\right) \\ &= 0. \end{aligned}$$

   The detailed argument is the following:

   (1) Using the concavity of $u_{d,A}$ gives us "$\geq$" and from maximality of x^{*i} we obtain the equality.

   As already seen in 1., looking at the competitive payoffs of these competitive solutions we observe

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = u_{d,A}\left(\frac{1}{N}e^N\right) - p^* \cdot \left(\frac{1}{N}e^N\right) + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

To summarize these results mean that looking for competitive solutions and their competitive payoffs we can focus on possible equilibrium prices of the allocation $\left(\frac{1}{N}e^N\right)_{i\in N}$. Then those competitive solutions give us all possible competitive payoffs.

3. Third, as in the proof of Proposition 7, equality (3)

$$u_d\left(\frac{1}{N}e^N\right) = \frac{1}{N}v_d(N) > \frac{1}{N}v(N) = u_{d,A}\left(\frac{1}{N}e^N\right)$$

and furthermore

$$u_{d,A}\left(\frac{1}{N}e^N\right) = \alpha' \cdot \left(\frac{1}{N}e^N\right)$$

for all $\alpha' \in A$. Because of the continuity of $u_d(\cdot)$ it follows for all $\alpha' \in A$ that $u_d(x) > \alpha' \cdot x$ for x in a small neighborhood of $\frac{1}{N}e^N$. Thus, in a neighborhood of $\frac{1}{N}e^N$, $u_{d,A}(x) = \min_{\alpha'\in A}(\alpha' \cdot x)$.

4. Forth, it remains to check for which prices p^* the pair $\left(\left(\frac{1}{N}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. In a first step we show that each $p^* \in A$ can be chosen as an equilibrium price vector, in a second step we show that any $p^* \notin A$ cannot be an equilibrium price vector. For the second step it is enough to concentrate on $p^* \in C(v) \setminus A$ as we have seen in 1. that the equilibrium price vector determines the competitive payoff vector, which are necessarily in the core.

   Step 1: Suppose $p^* \in A$. Then for all $x^i \in \mathbb{R}^n_+$ we have

$$\min_{\alpha'\in A}\left(\alpha' \cdot x^i\right) - p^* \cdot x^i \le p^* \cdot x^i - p^* \cdot x^i = 0$$

   and furthermore

$$\min_{\alpha'\in A}\left(\alpha' \cdot \left(\frac{1}{N}e^N\right)\right) - p^* \cdot \left(\frac{1}{N}e^N\right) = 0.$$

   Hence, $x^i = \frac{1}{N}e^N$ maximizes the trading profit of agent i. Furthermore, the markets clear, as $\sum_{i\in N} \frac{1}{N}e^N = e^N$.

   So, the pair $\left(\left(\frac{1}{N}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution.

Step 2: Suppose $p^* \in C(v) \setminus A$. Recall that the set A is compact and convex. Hence, we can apply the separating hyperplane theorem[2] and obtain that there exists $\bar{x} \in \mathbb{R}^n_+$ such that for all $\alpha \in A$

$$\alpha \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

Therefore we conclude that

$$\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

Now, for sufficiently small $\varepsilon > 0$ we have that $\frac{1}{N}e^N + \varepsilon\bar{x}$ is in a neighborhood of $\frac{1}{N}e^N$ where we have $u_{d,A}(x) = \min_{\alpha' \in A}(\alpha' \cdot x)$. But

$$\min_{\alpha' \in A}\left(\alpha' \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right)\right) - p^* \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right) = \varepsilon\left(\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x}\right) > 0.$$

This implies that $\frac{1}{N}e^N$ does not maximize agent i's trading profit for $p^* \notin A$.

5. To summarize the line of argument:

If $\left((x^{*i})_{i\in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then by 2. we have that $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. By 4. we show that $p^* \in A$ and by 1. we know that its competitive payoff vector is equal to p^*. On the other hand if $p^* \in A$ then by 4. we have that $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. The competitive payoff vector is equal to p^*.

□

2.5 Concluding Remarks

Shapley and Shubik (1975) investigate the relationship between competitive payoffs of markets that represent a cooperative game and their relation to solution concepts for cooperative games. We presented the details of the proof of Shapley and Shubik (1975), that extends their two main results to closed, convex subsets

[2] See for example Mas-Colell et al. (1995, Theorem M.G.2, p.948).

of the core. This shows also the two theorems of Shapley and Shubik (1975). In a further contribution (Brangewitz and Gamp, 2011a) we establish an analogue result for NTU market games.

Chapter 3

Competitive Outcomes and the Inner Core of NTU market games

3.1 Abstract

We consider the inner core as a solution concept for cooperative games with non-transferable utility (NTU) and its relationship to competitive equilibria of markets that are induced by an NTU game. We investigate the relationship between certain subsets of the inner core for NTU market games and competitive payoff vectors of markets linked to the NTU market game. This can be considered as the case in between the two extreme cases of Qin (1993). We extend the results of Qin (1993) to a large class of closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of its inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely and thus we obtain a class of markets with the desired property.

3.2 Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. Furthermore, Shapley and Shubik (1975) show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector.

Cooperative games with non-transferable utility (NTU) can be considered as a generalization of TU games, where the transfer of the utility within a coalition does not take place at a fixed rate. In this paper we consider NTU market games. After Shapley and Shubik (1969), Billera and Bixby (1974) investigated the NTU case and obtained similar results for compactly convexly generated NTU games. Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced NTU game, that is compactly convexly generated, is a market game. The inner core is a refinement of the core for NTU games. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector.

Similarly to the approach of Billera and Bixby (1974), Inoue (2010b) uses coalition production economies as in Sun et al. (2008) instead of markets. Inoue (2010b) shows that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy whose set of competitive payoff vectors coincides with the inner core of the balanced cover of the original NTU game.

Here we consider the classical approach using markets. We investigate the case in between the two extreme cases of Qin (1993), where on the one hand there exists a market that has the complete inner core as its set of competitive payoff vectors and on the other hand there is a market that has a given inner core point as its unique competitive payoff vector. We extend the results of Qin (1993) to closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of the inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely. Several parameters in our construction can be chosen in different ways. Thus, we obtain a class of markets with the desired property.

Shapley and Shubik (1975) remark that in the TU case their result can be extended to any closed and convex subset of the core. Whether a similar result analogously to the one of Shapley and Shubik (1975) holds for NTU market games, was up to now not clear. Our result shows, that in the NTU case it is even possible to focus on closed, typically non-convex, subsets of the inner core.

The inner is one solution concept for NTU games. Extending the results of Qin (1993) to closed subsets of the inner core means in particular to show such a result for all solution concepts selecting closed subsets of the inner core.

3.3 NTU market games

Let $N = \{1, ..., n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ be a set of players. Let $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ be the set of coalitions. Define for a coalition $S \in \mathcal{N}$ the following sets $\mathbb{R}^S = \{x \in \mathbb{R}^n | x_i = 0 \text{ if } i \notin S\} \subseteq \mathbb{R}^n$, $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S | x_i \geq 0 \text{ for all } i \in S\} \subseteq \mathbb{R}^n_+$,

$\mathbb{R}^S_{++} = \{x \in \mathbb{R}^S | x_i > 0 \text{ for all } i \in S\} \subseteq \mathbb{R}^n_{++}$. For a vector $a \in \mathbb{R}^n$ and a coalition $S \in \mathcal{N}$ let a^S denote the vector, where for $i \in S$ we have $a^S_i = a_i$ and $a^S_j = 0$ for $j \notin S$. Moreover, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ denote the inner product by $a \cdot b = \sum_{i=1}^n a_i b_i$ and the Hadamard product by $a \circ b = (a_1 b_1, ..., a_n b_n)$.

An *NTU (non-transferable utility) game* is a pair (N, V), that consists of a player set $N = \{1, ..., n\}$ and a coalitional function V, which defines for every coalition the utility allocations this coalition can reach, regardless of what the other players outside this coalition do. Hence, define the coalitional function V from the set of coalitions, $\mathcal{N}$, to the set of non-empty subsets of $\mathbb{R}^n$, such that for every coalition $S \in \mathcal{N}$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S)$ is non-empty and $V(S)$ is S-comprehensive, meaning $V(S) \supseteq V(S) - \mathbb{R}^S_+$.

The literature on NTU market games, as for example Billera and Bixby (1974) and Qin (1993), considers NTU games that are compactly and convexly generated. An NTU game (N, V) is *compactly (convexly) generated* if for all coalitions $S \in \mathcal{N}$ there exists a compact (convex) set $C^S \subseteq \mathbb{R}^S$ such that the coalitional function has the form $V(S) = C^S - \mathbb{R}^S_+$.

Given a player set $N = \{1, ..., n\}$ the set of balancing weights is defined by $\Gamma(e^N) = \left\{(\gamma_S)_{S \subseteq N} | \gamma_S \geq 0 \ \forall \ S \subseteq N, \sum_{S \subseteq N} \gamma_S e^S = e^N\right\}$. The balancing weights can be interpreted in the following way: Every player i has one unit of time that he can split over all the coalitions, he is a member of, with the constraint that a coalition has to agree on a common weight. Thereby, each player has to spend all his time. The weight γ_S can be seen as well as the intensity with which each player participates in the coalition $S \in \mathcal{N}$. In particular, if we have a partition of the player set into a coalition S and its complement $N \setminus S$ a balancing weight can be defined by $\gamma_S = \gamma_{N \setminus S} = 1$ and $\gamma_T = 0$ for all other coalitions T except for S and $N \setminus S$. An NTU game (N, V) is *balanced* if for all balancing weights $\gamma \in \Gamma(e^N)$ we have $\sum_{S \subseteq N} \gamma_S V(S) \subseteq V(N)$. Moreover, an NTU game (N, V) is *totally balanced* if it is balanced in all subgames. This means for all coalitions $T \in \mathcal{N}$ and for all balancing weights $\gamma \in \Gamma(e^T) = \left\{(\gamma_S)_{S \subseteq T} | \gamma_S \geq 0 \ \forall \ S \subseteq T, \sum_{S \subseteq T} \gamma_S e^S = e^T\right\}$ we have $\sum_{S \subseteq T} \gamma_S V(S) \subseteq V(T)$.

In order to define an NTU market game we first consider the notion of a market which is less general than the notion of an economy according to for example Arrow and Debreu (1954). In a market the number of consumers coincides with the number of producers. Each consumer has his own private production set. In contrast to the usual notion of an economy a market is assumed to have concave and not just quasi concave utility functions.

Definition 17 (market)**.** A *market* is given by $\mathcal{E} = \left\{ \left(X^i, Y^i, \omega^i, u^i \right)_{i \in N} \right\}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,
- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

As pointed out before in a market each consumer is assumed have his own private production set. This assumption is not as restrictive as it appears to be. A given private ownership economy can be transformed into an economy with the same number of consumers and producers without changing the set of competitive equilibria or possible utility allocations, see for example Qin and Shubik (2009, section 4).

In the following, we often consider markets where $X^i \subseteq \mathbb{R}^{kn}_+$ with $k, n \in \mathbb{N}$. Then, consumption vectors are usually written as $x^i = \left(x^{(1)i}, ..., x^{(k)i} \right) \in X^i$ where $x^{(m)i} \in \mathbb{R}^n_+$ for $m = 1, ..., k$. In a sense, we divide the kn consumption goods in k consecutive groups of n goods. The vector $x^{(m)i}$ is the m^{th} group of n consumption goods of the consumption vector x^i. We use an analogous notation for the production goods and price vectors.

Given a market we define which allocations are considered as feasible for some coalition $S \in \mathcal{N}$. An S-allocation is a tuple $\left(x^i \right)_{i \in S}$ such that $x^i \in X^i$ for each

$i \in S$. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

Hence, an S-allocation is feasible if there exist for all $i \in S$ production plans $y^i \in Y^i$ such that $\sum_{i \in S}(x^i - \omega^i) = \sum_{i \in S} y^i$. We refer to a feasible S-allocation in the following together with suitable production plans as a feasible S-allocation $\left(x^i\right)_{i \in S}$ with $\left(y^i\right)_{i \in S}$.

In the definition of feasibility it is implicitly assumed that by forming a coalition the available production plans are the sum of the individually available production plans. This approach is different from the idea to use coalition production economies, where every coalition has already in the definition of the economy its own production possibility set. Nevertheless, a market can be transformed into a coalition production economy by defining the production possibility set of a coalition as the sum of the individual production possibility sets.

Given the notion of a market and of feasible allocations for coalitions $S \in \mathcal{N}$ we define an NTU market game in the following way:

Definition 18 (NTU market game)**.** An NTU game (N, V) that is representable by a market is an *NTU market game*. This means there exists a market $\mathcal{E}$ such that $(N, V_{\mathcal{E}}) = (N, V)$ with

$$V_{\mathcal{E}}(S) = \left\{ u \in \mathbb{R}^S | \, \exists \, (x^i)_{i \in S} \in F(S), u_i \leq u^i(x^i), \, \forall \, i \in S \right\}.$$

For an NTU market game there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible S-allocations in the market or that give less utility than some feasible S-allocation.

One of the main results on NTU market games in Billera and Bixby (1974) is the following:

Theorem 8 (2.1, Billera and Bixby (1974))**.** *An NTU game (N, V) is an NTU market game if and only if it is totally balanced and compactly convexly generated.*

Hence, in order to study NTU market games, it is sufficient to look at those NTU games that are totally balanced and compactly convexly generated.

For the succeeding analysis, it will be useful to shift a given NTU game in the following way (compare Billera and Bixby (1973b, Proposition 2.2)): Given a vector $c \in \mathbb{R}^n$ define the coalitional function $(V+c)$ via $(V+c)(S) = V(S) + \sum_{i \in S} c_i$. To represent a shifted game by a market we have to shift the utility function of agent i by c_i. Hence, the shifted game with coalitional function $(V+c)$ is again a market game. Furthermore, shifting the utility functions of the agents does not change the set of competitive equilibria. Having this idea of shifting in mind we will focus in some proofs on games where for every coalition $S \in \mathcal{N}$ we have $C^S \subseteq \mathbb{R}^S_{++}$.

To prove the above result Billera and Bixby (1974) introduce the notion of an induced market that arises from a compactly convexly generated NTU game.

Definition 19 (induced market)**.** Let (N, V) be a compactly convexly generated NTU game. The *induced market* of the game (N, V) is defined by

$$\mathcal{E}_V = \left\{ (X^i, Y^i, u^i, \omega^i)_{i \in N} \right\}$$

with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{2n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S \in \mathcal{N}} \left(C^S \times \{-e^S\} \right) \right] \subseteq \mathbb{R}^{2n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i(x^i) = x_i^{(1)i}$.

It can easily be seen that this is a market according to the previous definition. Note that in an induced market we have input and output goods. Initially every consumer owns one unit of his personal input good that can only be used for the production process. By using his input good the consumer can get utility just from his personal output good. The consumption and production set are the same for every player. Just the utility functions and the initial endowments are dependent on the player.

The individual production sets in an induced market are convex cones and identical for all agents. In this situation taking the sum over production sets of some agents leads to the same production set. Setting $Y = \sum_{i \in N} Y^i$ the condition for feasibility of S-allocations reduces to $\sum_{i \in S}(x^i - \omega^i) \in Y$. Furthermore, for convex-cone technologies the competitive equilibrium profits are equal to 0. This means that in equilibrium we do not have to specify shares of the production as it usually done in private ownership economies.
Thus, as long as the individual production sets are convex cones and identical for all agents, we could alternatively consider a model for the production where we have only one production set for all agents and possible coalitions without specifying the shares. This model could be used instead of the production setup in the definition of a market.

In the definition of the induced market it is assumed that every individual has already the production possibilities, that become available if coalitions form, included in his personal production set. This means he already knows everything that can be produced in the different coalitions, even if he does not possess the necessary input commodities himself. Starting with an NTU game the utility allocations a coalition can reach in the derived induced market are not described by defining production sets individually for every coalition but by using input and output commodities. A utility allocation, that is reachable in the NTU game by a coalition S, is reachable in the induced market by the same coalition if the individuals pool their initial endowments using “one general” production possibility set. Utility allocations that require the cooperation of individuals outside the coalition S are technologically possible but can actually not be produced as the input commodities of these individuals are needed. In contrast to this interpretation in coalition production economies every coalition has its own production set.

The main proof of the above theorem from Billera and Bixby (1974) relies on Billera (1974). In a similar manner as Shapley and Shubik (1969), he starts with an NTU game, (N, V), and looks at the induced market of that game, $\mathcal{E}_V$, and afterwards at the NTU game that is induced by the induced market, $V_{\mathcal{E}_V}$. He shows that this game coincides with the totally balanced cover of the game (N, V).

The next step is to investigate the existing literature on and to study the relationship between solution concepts in cooperative game theory, as the inner core, and those in general equilibrium theory, as the notion of a competitive equilibrium. Analogously to the TU case of Shapley and Shubik (1975), Qin (1993) shows that the inner core of an NTU market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector. Before we extend the results of Qin (1993) we recall the basic definitions and state his main results. We start with the definition of the inner core and the notion of competitive payoff vectors in the context of NTU market games. Afterwards, we state the main results of Qin (1993) and comment on the ideas he uses to prove them.

In order to define the inner core we first consider a game that is related to a compactly generated NTU game, called the λ-transfer game. Fix a transfer rate vector $\lambda \in \mathbb{R}^n_+$. Define $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$ as the maximal sum of weighted utilities that coalition S can achieve given the transfer rate vector λ. The *λ-transfer game*, denoted as (N, V_λ), of (N, V) is defined by taking the same player set N and the coalitional function $V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$. Qin (1994, p.433) gives the following interpretation of the λ-transfer game: "The idea of the λ-transfer game may be captured by thinking of each player as representing a different country. The utilities are measured in different currencies, and the ratios λ_i/λ_j are the exchange rates between the currencies of i and j." As for the λ-transfer game only proportions matter we can assume without loss of generality that λ is normalized, i.e. $\lambda \in \Delta = \left\{\lambda \in \mathbb{R}^n_+ | \sum_{i=1}^n \lambda_i = 1\right\}$. Define the positive unit simplex by $\Delta_{++} = \left\{ \lambda \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n \lambda_i = 1 \right\}$.

The inner core is a refinement of the core. The *core* $C(V)$ of an NTU game (N, V) is defined as the set of utility allocations that are achievable by the grand coalition N such that no coalition S can improve upon this allocation. Thus,

$$C(V) = \left\{ u \in V(N) \middle| \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i \right\}.$$

A utility allocation is in the *inner core* $IC(V)$ of a compactly generated game (N, V) if it is achievable by the grand coalition N and if additionally there exists a transfer rate vector $\lambda \in \Delta$ such that this utility allocation is in the core of the λ-transfer game. More precisely:

Definition 20 (inner core)**.** The *inner core* of a compactly generated NTU game (N, V) is given by

$$IC(V) = \{u \in V(N) | \, \exists \, \lambda \in \Delta \text{ such that } u \in C(V_\lambda)\}.$$

Qin (1993, Remark 1, p. 337) remarks that if the NTU game is compactly convexly generated the vectors of supporting weights for a utility vector in the inner core must all be strictly positive. This can be seen by the following argument: If for one player $i \in N$ λ_i is equal to 0, then the core of the λ-transfer game is empty, because player i can improve upon any $u \in V_\lambda(N)$ by forming the singleton coalition $\{i\}$.

Qin (1994) considers sufficient conditions for the inner core to be non-empty. In particular he shows that a compactly generated NTU game (N, V), where $V(N)$ is convex, has a non-empty inner core if it is balanced with slack, meaning that for balancing weights $(\gamma_S)_{S \subseteq N}$ with $\gamma_N = 0$ we have $\sum\limits_{S \subset N} \gamma_S V(S) \subset \operatorname{int}_{\mathbb{R}^n} V(N)$ where $\operatorname{int}_{\mathbb{R}^n} V(N)$ is the interior of $V(N)$ relative to $\mathbb{R}^n$. Other contributions related to the non-emptiness of the inner core can be found for example in Iehlé (2004), Bonnisseau and Iehlé (2007) or Inoue (2010a).

We now define a competitive equilibrium for a market $\mathcal{E}$.

Definition 21 (competitive equilibrium)**.** A *competitive equilibrium* for a market

$\mathcal{E}$ is a tuple

$$\big((\hat{x}^i)_{i\in N}, (\hat{y}^i)_{i\in N}, \hat{p}\big) \in \mathbb{R}_+^{\ell n} \times \mathbb{R}_+^{\ell n} \times \mathbb{R}_+^{\ell}$$

such that

(i) $\sum_{i\in N} \hat{x}^i = \sum_{i\in N}(\hat{y}^i + \omega^i)$ (market clearing),

(ii) for all $i \in N$, $\hat{y}^i$ solves $\max_{y^i\in Y^i} \hat{p}\cdot y^i$ (profit maximization),

(iii) and for all $i \in N$, $\hat{x}^i$ is maximal with respect to the utility function u^i in the budget set $\{x^i \in X^i | \hat{p}\cdot x^i \leq \hat{p}\cdot(\omega^i + \hat{y}^i)\}$ (utility maximization).

Given a competitive equilibrium its *competitive payoff vector* is defined as $\left(u^i\left(\hat{x}^i\right)\right)_{i\in N}$.

Qin (1993) investigates the relationship between the inner core of an NTU market game and the set of competitive payoff vectors of a market that represents this game. He establishes, following a conjecture of Shapley and Shubik (1975), the two theorems below analogously to the TU-case of Shapley and Shubik (1975).

Theorem 9 (1, Qin (1993))**.** *The inner core of an NTU market game coincides with the set of competitive payoff vectors of the induced market by that game.*

Theorem 10 (3, Qin (1993))**.** *For every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector.*

To show his first result Qin (1993) uses the notion of the induced market of a compactly convexly generated NTU game as it was already used by Billera and Bixby (1974). It turns out that the set of competitive equilibrium payoff vectors of the induced market coincides with the inner core. For his second result Qin (1993) fixes an inner core point, denoted by u^* [1], and chooses one transfer rate vector $\lambda^*_{u^*}$ from an associated λ-transfer game. He modifies the given NTU game by applying a suitable strictly monotonic transformation on the utility allocations a coalition can reach. In this modified game the given inner core point u^* can be

[1] Qin (1993) considers only NTU games where for all coalitions $S \in \mathcal{N}$ the generating sets satisfy $C^S \subseteq \mathbb{R}^S_+$ and $C^S \cap \mathbb{R}^S_{++} \neq \emptyset$ and hence has $u^* \gg 0$.

strictly separated from the set of utility allocations the grand coalition can reach (excluding u^*). Denote the modified game by $(N, \bar{V})$ and the convex compact sets generating this game by $(\bar{C}^S)_{S\in\mathcal{N}}$. A market to prove Theorem 3 of Qin (1993) can be defined as follows:

Define for all coalitions $S \in \mathcal{N}$

$$\begin{aligned} A_S^1 &= \left\{ \left(u^S, -e^S, -e^S, -e^S, 0\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^2 &= \left\{ \left(u^S, 0, -e^S, 0, -e^S\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^3 &= \left\{ \left(u^S, 0, 0, -e^S, -e^S\right) | u^S \in \bar{C}^S \right\} \subseteq \mathbb{R}^{5n}. \end{aligned}$$

Let $\mathcal{E}_{\bar{V},u^*} = \left\{ \left(X^i, Y^i, \omega^i, u^i\right)_{i\in N} \right\}$ be the market with for every individual $i \in N$

- the consumption set $X^i = X = \mathbb{R}^n_+ \times \{(0,0,0)\} \times \mathbb{R}^n_+ \subseteq \mathbb{R}^{5n}_+$,
- the production set $Y^i = Y = convexcone \left[\bigcup_{S\subseteq N} \left(A_S^1 \cup A_S^2 \cup A_S^3\right) \right] \subseteq \mathbb{R}^{5n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}\right) \in \mathbb{R}^{5n}_+$,
- the utility function $u^i(x^i) = \min \left\{ x_i^{(1)i}, \frac{(\lambda^*_{u^*} \circ u^*) \cdot x^{(5)i}}{\lambda^*_{u^* i}} \right\}$ with $x^i = (x^{(1)i}, 0, 0, 0, x^{(5)i}) \in X^i$ and $x_k^{(1)i}$ is the k^{th} entry of $x^{(1)i}$.

Note that, similarly to the induced market, all individuals have the same consumption sets and the same production sets. The individuals differ in their initial endowment vectors and their utility functions. Qin (1993) introduces the sets A_S^1, A_S^2, A_S^3 in order to be able to show that the equilibrium price vector for the 5^{th} group of n goods, $\hat{p}^{(5)}$, is strictly positive. The i^{th} consumer obtains utility from the i^{th} component of the vector of the 1^{st} group of n goods and from all the 5^{th} n goods. The dependence of the utility function on all components of the 5^{th} group of n goods is crucial to show the positiveness of $\hat{p}^{(5)}$. To prove his result Qin (1993) shows that the market $\mathcal{E}_{\bar{V},u^*}$ represents the modified game and that the given inner core point is the unique competitive payoff vector of this economy. By applying the inverse strictly monotonic transformation to the utility functions he obtains his result.

In order to extend the results of Qin (1993) to a large class of closed subsets of the inner core we make use of the fact that for compactly convexly generated NTU games competitive payoff vectors need necessarily to be in the inner core. To see this we use a modified version of Proposition 1 from de Clippel and Minelli (2005).

Let $N = \{1, ..., n\}$ be the set of agents and $\{1, .., \ell\}$ be the set of commodities. Let $X^i \subseteq \mathbb{R}_+^\ell$ be a convex set containing 0, the consumption set of agent i. Each individual has a continuous, concave, (weakly) increasing and locally non-satiated utility function $u^i : \mathbb{R}_+^\ell \to \mathbb{R}$ and an initial endowment vector $\omega^i \in \mathbb{R}_+^\ell \setminus \{0\}$. Let $Y^i \subseteq \mathbb{R}^\ell$ be a non-empty and closed convex cone, the production set of agent i's firm.

Lemma 1. *Let* $\left(\left(\hat{x}^i\right)_{i \in N}, \left(\hat{y}^i\right)_{i \in N}, \hat{p}\right)$ *be a competitive equilibrium such that* $\hat{p} \cdot \omega^i > 0$ *for all individuals* $i \in N$. *Then* $\left(u^i\left(\hat{x}^i\right)\right)_{i \in N}$ *is in the inner core of the game induced by the economy.*

The proof of Lemma 1 can be found in Appendix 3.6.1.

3.4 An extension of the Results of Qin (1993)

In the above two theorems Qin (1993) considers on the one hand the whole inner core and on the other hand a single point in the inner core. In this section we extend the results of Qin (1993) by showing a similar result for closed subsets of the inner core. In the following we consider NTU market games and closed subsets of the inner core with certain properties. We want to ensure that for every point in a subset of the inner core, denoted by A, of a given NTU market game (N, V) we can find a normal vector such that this point is strictly separated from the set $V(N)$ without the point by the hyperplane using this normal vector. If we assume that the individual rational part of $V(N)$ is strictly convex, then this property is satisfied. Moreover, we want to assume that this set of normal vectors, where each normal vector corresponds to one point of the set A, is bounded below by a strictly positive vector. This means that the exchange rates, represented by the normal vectors, within the set A cannot be too extreme. We make the following definition:

Definition 22 (strict positive separability)**.** A pair $[(N, V), A]$ consisting of a compactly, convexly generated and totally balanced NTU game (N, V) and a closed subset A of its inner core satisfies *strict positive separability [SPS]* if the following condition holds:

> There exists an $\varepsilon > 0$ and a mapping $\lambda : A \to \Delta_{++}$, that associates to every point $x \in A$ a normal vector $\lambda(x) = \lambda^x$, such that
>
> – every point $x \in A$ can be strictly separated from the set $V(N) \setminus \{x\}$ using this normal vector λ^x, i.e.
>
> $$\lambda^x \cdot x > \lambda^x \cdot y \quad \text{for all } y \in V(N) \setminus \{x\},$$
>
> – for all $x \in A$ every coordinate of the normal vector λ^x is strictly greater than ε, i.e.
>
> $$\lambda_i^x > \varepsilon \quad \text{for all } i \in N.$$

For a pair $[(N, V), A]$ satisfying strict positive separability there might exist more than on mapping λ and more than one ε. In the following we always consider one fixed mapping λ together with one fixed ε satisfying the conditions. Whenever λ or ε appear we mean the ones we fixed knowing that we might have chosen different ones.

The assumption of strict positive separability is not as restrictive as it might appear. It is satisfied for example if the individual rational part of $V(N)$ is strictly convex and A is a closed subset of the interior of the inner core.

Note that from $\varepsilon < \lambda_i^x = \frac{\lambda_i^x}{1} \leq \frac{\lambda_i^x}{\lambda_j^x}$ it follows that

$$\varepsilon < \min_{i,j \in N} \frac{\lambda_i^x}{\lambda_j^x} \quad \text{for all } \lambda^x,\ x \in A.$$

Figure 3.1 illustrates the idea of strict positive separability with some examples. Assume that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 3.1.

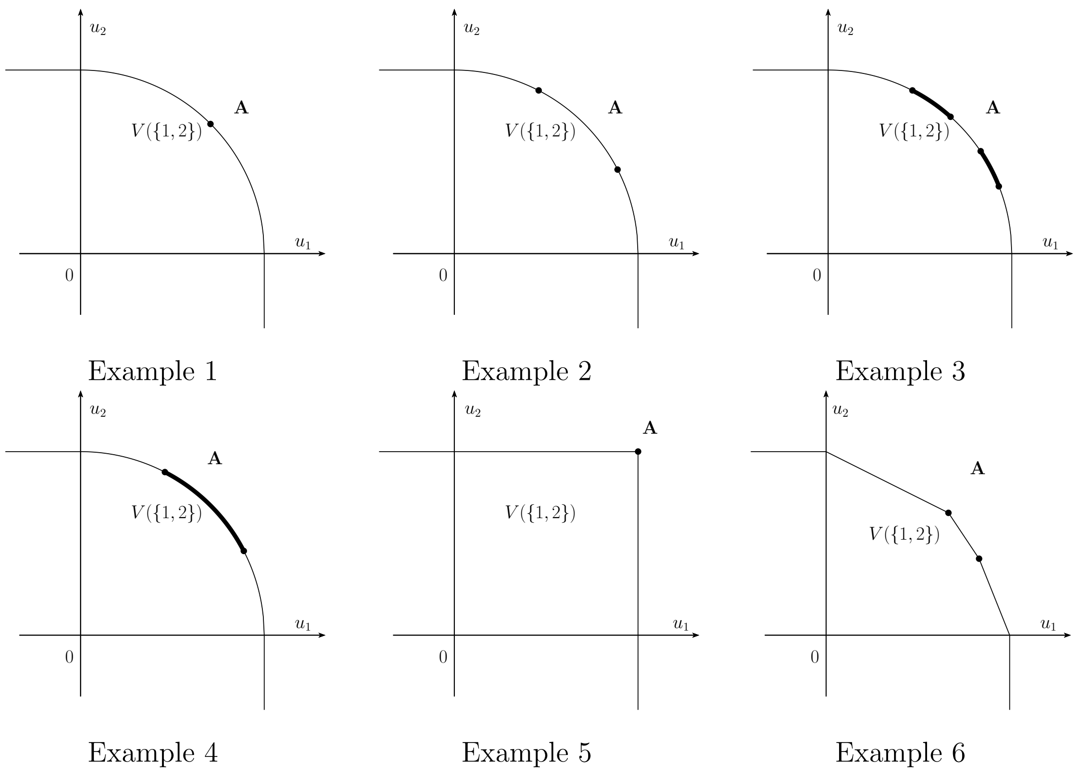


Figure 3.1: Examples where SPS is satisfied.

In Examples 1, 2, 3 and 4 the set $V(\{1,2\})$ is strictly convex. Here the inner core is given by all points on the efficient boundary without the two points on the axes. Thus, the NTU game together with every closed subset of its inner core satisfies SPS. This holds in particular for single points, finite sets, closed and connected sets or finite unions of closed sets.

Example 5 illustrates the case where the set $V(\{1,2\})$ is generated by a square and thus the inner core consists only of the corner point. In this case all the vectors in the strictly positive two-dimensional simplex support this inner core point. In order to establish SPS we just take one of these supporting vectors.

In Example 6 the set $V(\{1,2\})$ is generated by a polyhedron. The set A is a finite set, consisting of some corner points of the polyhedron. For each of these corner points there exists a strictly positive normal vector that strictly separates it from $V(\{1,2\})$ without this corner point. The NTU game (N,V) and this choice of the set A satisfy SPS.

Figure 3.2 shows some examples that do not satisfy strict positive separability. As before assume that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 3.2.

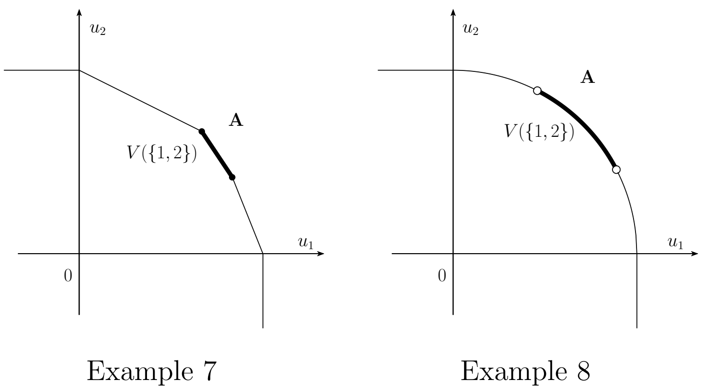


Figure 3.2: Examples where SPS is not satisfied.

In contrast to Example 6, in Example 7 the set A is chosen to be the line segment connecting two neighboring corner points of a polyhedron. Hence, all points in the set A have a common normal vector. Thus, each of this points cannot be strictly separated from the polyhedron without this point. Therefore, SPS is not satisfied. In Example 8 each point in the set A can be strictly separated from $V(\{1,2\})$ without the point. Nevertheless SPS is not satisfied, as the set A is not closed.

The properties, that we require at this point by considering only $[(N,V),A]$ satisfying SPS, are stronger than the properties, that we really need. For example it is sufficient if we can strictly separate each point in the boundary of A from

A without it. Nevertheless, we choose to consider $[(N, V), A]$ which satisfy SPS, because they allow for an easy interpretation. After the presentation of the main results we discuss the question, how this can be weakened such that cases as in Example 6 are included in our results.

Now we prove the following result:

Theorem 11. *Let $[(N, V), A]$ satisfy strict positive separability. Then there exists a market such that this market represents the game (N, V) and such that the set of competitive payoff vectors of this market is the set A.*

We show this result for NTU games where for every coalition $S \in \mathcal{N}$ we have $C^S \subseteq \mathbb{R}^S_{++}$. Due to the remark on page 57 this is not a restriction as we can shift an arbitrary given NTU game such that this condition is satisfied. After having applied our results we shift back the obtained economies such that they represent the original game. Hence, in the following if we consider an NTU game, we always assume for every coalition $S \in \mathcal{N}$ that we have $C^S \subseteq \mathbb{R}^S_{++}$.

Before beginning with the construction of a market satisfying the properties mentioned above, we introduce an auxiliary game and some notation.

Let $[(N, V), A]$ satisfy SPS. Let $(N, \tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap\limits_{a \in A} \{z \in \mathbb{R}^n | \lambda^a \cdot z \leq \lambda^a \cdot a\} & \text{if } S = N \end{cases}$$

where λ^a is as in the definition of SPS.

Note that to define the game $(N, \tilde{V})$ we use for every point of the set $a \in A$ just one normal vector that strictly separates this point from $V(N) \setminus \{a\}$. The games (N, V) and $(N, \tilde{V})$ are equal except for the grand coalition N. For the coalition N we extend the set $V(N)$ depending on the normal vectors of the set A. For illustration purposes figure 3.3 shows as an example for two players the sets $V(\{1, 2\})$ and $\tilde{V}(\{1, 2\})$.

To describe the relation between $(N, \tilde{V})$ and (N, V) we introduce the following

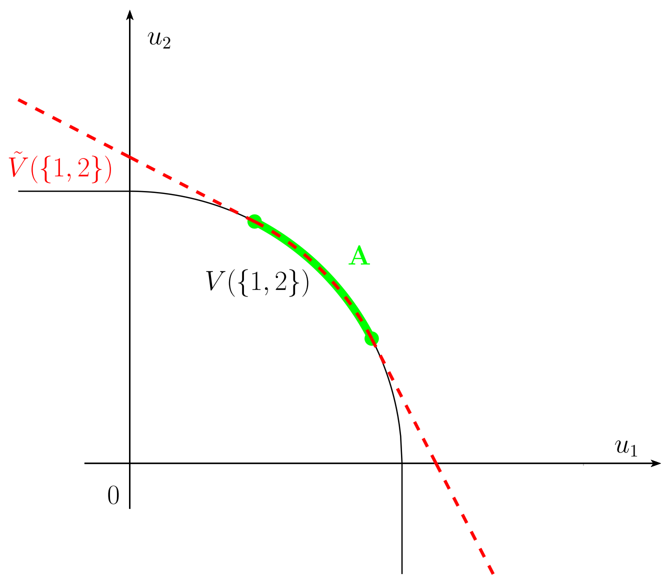


Figure 3.3: Example: The sets $V(\{1,2\})$ and $\tilde{V}(\{1,2\})$ for $N = \{1,2\}$.

notation: Let $z \in \tilde{V}(N)$ and

$$\bar{t}^z = \min\left\{t \in \mathbb{R}_+ | z - te^N \in V(N)\right\}.$$

Define

$$\tilde{C}^N = \left\{z \in \tilde{V}(N) \middle| \exists t \in \mathbb{R}_+ \text{ such that } z - te^N \in C^N\right\}.$$

Then we also have $\tilde{C}^N = \left\{z \in \tilde{V}(N) \middle| z - \bar{t}^z e^N \in C^N\right\}$.

The following remark is easy to verify:

Remark 1.

1. The game (N, V) is contained in the game $(N, \tilde{V})$. This means we have $V(S) \subseteq \tilde{V}(S)$ for all $S \subseteq N$.

2. The set $\tilde{C}^N$ is convex and furthermore, $C^N \subseteq \tilde{C}^N$.

3. The game $(N, \tilde{V})$ is a convexly generated and totally balanced NTU-game, but it is not compactly generated. In particular we have $\tilde{V}(N) \neq \tilde{C}^N - \mathbb{R}^n_+$.

4. SPS ensures in particular: If we take x in $V(N)$ outside from A, then x is in the interior of $\tilde{V}(N)$,

$$x \in V(N) \setminus A \Rightarrow x \in \text{int}\left(\tilde{V}(N)\right).$$

The second point of the remark can be seen as follows: Take $z_1, z_2 \in \tilde{C}^N$ and $\alpha \in [0,1]$. Then there exist t^{z_1} and t^{z_2} such that $z_1 - t^{z_1}e^N \in C^N$ and $z_2 - t^{z_2}e^N \in C^N$. As C^N is per assumption convex $\alpha\left(z_1 - t^{z_1}e^N\right) + (1-\alpha)\left(z_2 - t^{z_2}e^N\right) \in C^N$. As well the set $\tilde{V}(N)$, as an intersection of halfspaces, is convex and hence $\alpha z_1 + (1-\alpha)z_2 \in \tilde{V}(N)$. Thus taking $t^{\alpha z_1 + (1-\alpha)z_2} = \alpha t^{z_1} + (1-\alpha)t^{z_2}$ shows that $(\alpha z_1 + (1-\alpha)z_2) - t^{\alpha z_1 + (1-\alpha)z_2}e^N = \alpha\left(z_1 - t^{z_1}e^N\right) + (1-\alpha)\left(z_2 - t^{z_2}e^N\right) \in C^N$. Therefore, we have $\alpha z_1 + (1-\alpha)z_2 \in \tilde{C}^N$. Hence, $\tilde{C}^N$ is convex.

Definition 23. Define the mapping $P_A : \tilde{V}(N) \longrightarrow V(N)$ via

$$P_A(x) = x - \bar{t}^x e^N.$$

The following figure illustrates the mapping P_A for the example from figure 3.3.

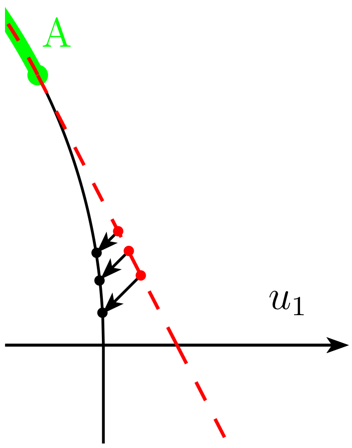


Figure 3.4: Illustration of the mapping P_A for the example from figure 3.3.

Note, that if $x \in V(N)$ then $\bar{t}^x = 0$ and $P_A(x) = x$.

Remark 2.

1. The mapping P_A is continuous and its image is $V(N)$.
2. The set $\tilde{C}^N$ can be written as

$$\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| P_A(z) \in C^N \right\} = P_A^{-1}\left(C^N\right),$$

thus we have $P_A\left(\tilde{C}^N\right) = C^N$.

3.4.1 The basic idea

First, we present an intermediate result, which is interesting in itself. For $[(N,V),A]$ satisfying SPS we construct a market such that this market represents the given game and such that the set of payoff vectors of competitive equilibria with strictly positive price vectors coincides with the given set A. In the last chapter we show, how we deal with the case, when the equilibrium price vectors are not necessarily strictly positive, using a more complicated market with a similar structure.

Definition 24. Let $[(N,V),A]$ satisfy SPS. Then the market $\mathcal{E}^0_{V,A}$ is defined by

$$\mathcal{E}^0_{V,A} = \left\{ \left(X^i, Y^i, u^i, \omega^i\right)_{i\in N} \right\}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{4n}$,
- the production set

$$Y^i = convexcone \left[\left(\bigcup_{S\in\mathcal{N}\setminus\{N\},\ c^S\in C^S} \left(c^S, -e^S, c^S, -e^S\right) \right) \right.$$
$$\left. \cup \left(\bigcup_{\tilde{c}^N\in\tilde{C}^N} \left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right) \right) \right] \subseteq \mathbb{R}^{4n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(\left(x^{(1)}, 0, x^{(3)}, 0\right)\right) = \min\left(x^{(1)}_i, x^{(3)}_i\right)$.

Note that this market has the same consumption and production set for every individual $i \in N$. The individuals differ in their initial endowment vectors and their utility functions. There are input and output commodities. The 2^{nd} group and the 4^{th} group of n commodities are the input commodities and every individual $i \in N$ owns one unit of his personal input commodity in the i^{th} component of the 2^{nd} and the 4^{th} group of n goods. The 1^{st} and the 3^{rd} group of n goods are the output commodities, from whose i^{th} component player $i \in N$ obtains utility. The construction of this market is based on the idea of the induced market in Billera and Bixby (1974) or Qin (1993).

We now need to establish first that the market $\mathcal{E}^0_{V,A}$ is indeed a market for the NTU market game (N, V).

Lemma 2. *The market $\mathcal{E}^0_{V,A}$ represents the game (N, V).*

The proof of Lemma 2 is inspired by Billera (1974).

Proof.

- As $V(S) = C^S - \mathbb{R}^S_+$ it is enough to show, that for all $S \in \mathcal{N}$ the payoff vectors in the set C^S can be achieved by coalition S in the market $\mathcal{E}^0_{V,A}$. Let $z \in C^S$. We show, that there exists a feasible S-allocation $\left(x^i\right)_{i \in S}$ with $\left(y^i\right)_{i \in S}$ such that $u^i\left(x^i\right) = z_i$ for all $i \in S$.
  Define for $i \in S$ the consumption plan
  $$x^i = \left(z^{\{i\}}, 0, z^{\{i\}}, 0\right)$$
  and let
  $$y^i = \frac{1}{|S|}\left(z, -e^S, z, -e^S\right)$$
  be the production plan for all $i \in S$. By the definition of the consumption sets we observe $x^i \in X^i$ for all $i \in S$. With regard to the production sets for $S \neq N$ we have immediately $y^i \in Y^i$ for all $i \in S$. For $S = N$ note that $z \in V(N) \subseteq \tilde{V}(N)$ and thus $P_A(z) = z$. Hence, we have $y^i \in Y^i$ for all $i \in N$. Observe that
  $$\sum_{i \in S}\left(x^i - \omega^i\right) = \sum_{i \in S} y^i.$$

Hence, $\left(x^i\right)_{i\in S}$ is a feasible S-allocation and

$$u^i\left(x^i\right) = z_i \quad \text{for all } i \in S.$$

- Let $\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0\right)_{i\in S}$ be a feasible S-allocation with $\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}\right)_{i\in S}$ in the market $\mathcal{E}^0_{V,A}$.
The feasibility implies

$$\left(\sum_{i\in S} \bar{x}^{(1)i}, -e^S, \sum_{i\in S} \bar{x}^{(3)i}, -e^S\right) = \sum_{i\in S}\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}\right).$$

Each production set is a convex cone of a union of convex sets. Hence, an arbitrary production plan can be written in the following way: Choose one suitable element from each of the convex sets and build a linear combination (with non-negative coefficients) of these elements. For the 1^{st} and the 2^{nd} group of n commodities we obtain, that there exist $\alpha^i_R \in \mathbb{R}_+$ for all $R \in \mathcal{N}$, $z^i_R \in C^R$ for all $R \in \mathcal{N} \setminus \{N\}$ and $\tilde{z}^i_N \in \tilde{C}^N$, such that

$$\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right) = \sum_{R\in\mathcal{N}\setminus\{N\}} \alpha^i_R\left(z^i_R, -e^R\right) + \alpha^i_N\left(P_A\left(\tilde{z}^i_N\right), -e^N\right).$$

As $P_A\left(\tilde{C}^N\right) = C^N$ there exists $z^i_N \in C^N$ such that $P_A\left(\tilde{z}^i_N\right) = z^i_N$ and hence we have

$$\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right) = \sum_{R\in\mathcal{N}} \alpha^i_R\left(z^i_R, -e^R\right).$$

As feasibility implies $\left(\sum\limits_{i\in S} \bar{x}^{(1)i}, -e^S\right) = \sum\limits_{i\in S}\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}\right)$, for the 2^{nd} group of n coordinates we have that

$$\begin{aligned} e^S &= \sum_{i\in S}\sum_{R\in\mathcal{N}} \alpha^i_R e^R \\ &= \sum_{R\in\mathcal{N}}\left(\sum_{i\in S}\alpha^i_R\right) e^R. \end{aligned}$$

Thus $\alpha^i_R > 0$ implies $R \subseteq S$ and if we define $\alpha(R) = \sum\limits_{i\in S}\alpha^i_R$, then $\left(\alpha(R)\right)_{R\subseteq S}$ is a balanced family for the coalition S. Looking at the 1^{st}

group of n coordinates we have

$$\begin{aligned}
\sum_{i\in S}\bar{x}^{(1)i} &= \sum_{R\subseteq S}\sum_{i\in S}\alpha_R^i z_R^i \\
&= \sum_{\{R\subseteq S|\alpha(R)>0\}}\alpha(R)\left(\frac{1}{\alpha(R)}\sum_{i\in S}\alpha_R^i z_R^i\right).
\end{aligned}$$

Since C^R is convex we have

$$\frac{1}{\alpha(R)}\sum_{i\in S}\alpha_R^i z_R \in C^R$$

and hence, using totally balancedness, $\sum\limits_{i\in S}\bar{x}^{(1)i}\in V(S)$.

>From the definition of the utility function we obtain $u^i\left(\bar{x}^{(1)i},0,\bar{x}^{(3)i},0\right)\le \bar{x}_i^{(1)i}$. Since $\left(\bar{x}_i^{(1)i}\right)_{i\in S}\le\sum\limits_{i\in S}\bar{x}^{(1)i}\in V(S)$ we have by the S-comprehensiveness of $V(S)$ that $\left(u^i\left(\bar{x}^{(1)i},0,\bar{x}^{(3)i},0\right)\right)_{i\in S}\in V(S)$.

□

We verify that the payoff vectors in the set A are indeed competitive payoff vectors of the market $\mathcal{E}^0_{V,A}$:

Proposition 9. *Every point in the set A is equilibrium payoff vector of the market $\mathcal{E}^0_{V,A}$.*

Proof. Let $a\in A$ and $\lambda^a\in\Delta$ be a normal vector such that a is in the core of the λ^a-transfer game. We know that λ^a is strictly positive (compare the remark on page 60). By the assumption that $C^N\subseteq\mathbb{R}^N_{++}$ we know that a is strictly positive. To prove the proposition, we show that the consumption and production plans

$$\left(\hat{x}^i\right)_{i\in N}=\left(\left(a^{\{i\}},0,a^{\{i\}},0\right)\right)_{i\in N}$$

and

$$\left(\hat{y}^i\right)_{i\in N}=\left(\left(\frac{1}{n}\left(a,-e^N,a,-e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = (\lambda^a, \lambda^a \circ a, \lambda^a, \lambda^a \circ a)$$

constitute a competitive equilibrium in the market $\mathcal{E}^0_{V,A}$.

First note that as $P_A(a) = a$ we have $\hat{y}^i \in Y^i$ for all $i \in N$. According to the remark above, the price system $\hat{p}$ is strictly positive. As we have a convex-cone-technology maximum profits are zero. We observe

$$\hat{p} \cdot \hat{y}^i = \frac{1}{n} \left(\lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N + \lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N \right) = 0.$$

Hence, the production plan $\hat{y}^i$ is profit maximizing.

As we have a min-type or Leontief utility function, it is optimal for each agent i to spend his budget in a way such that $\hat{x}_i^{(1)i} = \hat{x}_i^{(3)i}$ and that he does not consume anything of the other commodities. Furthermore, he has to spend all his budget, because the preferences are locally non-satiated and continuous. The budget constraint is satisfied with equality,

$$\hat{p} \cdot \hat{x}^i = \lambda^a \cdot \left(a^{\{i\}} + a^{\{i\}} \right) = (\lambda^a \circ a) \cdot \left(e^{\{i\}} + e^{\{i\}} \right) = \hat{p} \cdot \omega^i$$

and

$$\hat{x}^{(1)i} = a^{\{i\}} = \hat{x}^{(3)i}.$$

Hence, the consumption vector $\hat{x}^i$ is utility maximizing on the budget set of agent i.

Furthermore, the market clearing condition

$$\sum_{i \in N} \hat{x}^i = \sum_{i \in N} \omega^i + \sum_{i \in N} \hat{y}^i$$

is satisfied.

Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i \left(\hat{x}^i \right) \right)_{i \in N} = a.$$

□

Looking again at the competitive equilibrium price vectors in the proof of Proposition 9 note: For a competitive equilibrium with payoff vector $a \in A$ the equilibrium price vector for the 1^{st} (respectively 3^{rd}) group of n goods, the output goods, is the normal vector λ^a separating the point a from $V(N)$. The transfer rate vectors coincide with the equilibrium prices for the output goods of the market. The input goods are priced by $\lambda^a \circ a$. This is the transfer rate vector weighted by the according point of the set A. Interpreted differently: The input goods are first weighted by the point a of the set A and afterwards they are priced by the transfer rate vector λ^a. The relationship of the transfer rate vectors and the prices of competitive equilibria was observed in several publications discussing the relation between NTU games and economies. Examples are Shubik (1985), Shapley (1987), Trockel (1996) and Qin (1993). Shapley (1987, p. 192) states: "There is a strong analogy though no formal equivalence that we know of between the comparison weights that we must introduce in order to obtain a feasible transfer value and the prices in a competitive market." Here we obtain a formal equivalence for the prices of the output goods and an indirect link for the prices of the input goods. Trockel (1996) investigated this equivalence for NTU bargaining games and Qin (1993) obtained very similar equilibrium prices as we have here.

Next, we consider the utility allocations outside the set A. Using Lemma 1 it is sufficient to consider those vectors in the inner core.

Proposition 10. *Any payoff vector of a competitive equilibrium of the market $\mathcal{E}^0_{V,A}$ with a strictly positive equilibrium price vector is an element of the set A.*

Proof. Lemma 1 ensures that every competitive equilibrium payoff vector is in the inner core. Assume that there exists a competitive equilibrium $((x^i)_{i\in N}, (y^i)_{i\in N}, p)$ such that its payoff vector $(u^i(x^i))_{i\in N}$ is in the inner core but not in the set A and such that the equilibrium price vector is strictly positive, $p \gg 0$.

Then, there exists an element c^N in the inner core outside A such that $u^i(x^i) = c^N_i$ for all player $i = 1, ..., n$. Let $x^i = (x^{(1)i}, x^{(2)i}, x^{(3)i}, x^{(4)i})$. By the definition of the consumption set we know $x^{(2)i} = x^{(4)i} = 0$ and by the definition of the utility function we obtain $x^{(1)i}_i \geq c^N_i$ and $x^{(3)i}_i \geq c^N_i$ for all $i = 1, ..., n$.

<u>Claim 1:</u> >From the utility maximization and the strict positivity of the price vector it follows that we need to have

$$x_i^{(1)i} = c_i^N = x_i^{(3)i}.$$

The proof of Claim 1 can be found in Appendix 3.6.2.

We get by the market clearing condition: $y = \sum_{i \in N} \left(x^i - \omega^i\right) = \left(c^N, -e^N, c^N, -e^N\right)$. But the production plan $y = (c^N, -e^N, c^N, -e^N)$ is not profit maximizing.[2]

To see this notice the following: As c^N is in the inner core but outside the set A there exists a $\tilde{c}^N$ with $P_A\left(\tilde{c}^N\right) = c^N$ and $\tilde{c}^N \gg c^N$. Consider the production plan $\left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right)$. Looking at the profits and using the strict positivity of the price vector we observe

$$\begin{aligned} p \cdot y &= p^{(1)} \cdot c^N - p^{(2)} \cdot e^N + p^{(3)} \cdot c^N - p^{(4)} \cdot e^N \\ &< p^{(1)} \cdot c^N - p^{(2)} \cdot e^N + p^{(3)} \cdot \tilde{c}^N - p^{(4)} \cdot e^N \\ &= p^{(1)} \cdot P_A\left(\tilde{c}^N\right) - p^{(2)} \cdot e^N + p^{(3)} \cdot \tilde{c}^N - p^{(4)} \cdot e^N \\ &\leq 0. \end{aligned}$$

Thus, we have found a production plan that has strictly higher profits than y. This is a contradiction, since y needs to be profit maximizing.

It follows that with strictly positive price vectors the allocations outside the set A but in the inner core cannot be competitive equilibrium payoff vectors. □

Combining the two propositions above we obtain the following theorem:

Theorem 12. *Let $[(N, V), A]$ satisfy strict positive separability. The set of payoff vectors of competitive equilibria with a strictly positive equilibrium price vector of the market $\mathcal{E}^0_{V,A}$ coincides with the set A.*

[2] Since the individual production sets are convex cones, to check profit maximization it is sufficient to consider the joint production plans. We have $\sum_{i=1}^n Y^i = Y^j$ for any $j \in N$.

Positive equilibrium price vectors are required to obtain the above results

Up to now we always considered competitive equilibria with only strictly positive equilibrium price vectors. This was indeed necessary. If we also allow for price vectors that are not strictly positive, then we can construct a competitive equilibrium with competitive payoff vectors outside the given set A. To see this fix $a \notin A$ but in the inner core. Then there exists $\tilde{a} \in \tilde{C}^N$ such that $P_A(\tilde{a}) = a$ and $\tilde{a} \gg a$. Consider

$$\begin{aligned} \hat{x}^i &= \left((P_A(\tilde{a}))^{\{i\}}, 0, \tilde{a}^{\{i\}}, 0\right) = \left(a^{\{i\}}, 0, \tilde{a}^{\{i\}}, 0\right) \text{ for all } i \in N, \\ \hat{y}^i &= \left(\frac{1}{n}\left(P_A(\tilde{a}), -e^N, \tilde{a}, -e^N\right)\right) = \left(\frac{1}{n}\left(a, -e^N, \tilde{a}, -e^N\right)\right) \text{ for all } i \in N, \\ \hat{p} &= (\lambda^a, \lambda^a \circ a, 0, 0) \end{aligned}$$

where λ^a is one normal vector from a λ^a-transfer game and $(P_A(\tilde{a}))^{\{i\}}$ is the vector that has as its i^{th} coordinate the i^{th} coordinate of $P_A(\tilde{a})$ and zero coordinates otherwise. Analogously define $\tilde{a}^{\{i\}}$.

We show that $\left((\hat{x}^i)_{i\in N}, (\hat{y}^i)_{i\in N}, \hat{p}\right)$ constitutes a competitive equilibrium with the payoff vector $a \notin A$.

- First note that $u^i(\hat{x}^i) = \min\{a_i, \tilde{a}_i\} = a_i$, since we have $\tilde{a} \gg a$.
- For the profit maximization we obtain

  $$\hat{p} \cdot \hat{y}^i = \frac{1}{n}\left(\lambda^a \cdot a - (\lambda^a \circ a) \cdot e^N\right) = 0.$$

  Since the maximum profits are zero, $\hat{y}^i$ is profit maximizing.
- For the utility maximization we obtain that the budget constraint is satisfied with equality,

  $$\hat{p} \cdot \hat{x}^i = \lambda^a \cdot a^{\{i\}} = (\lambda^a \circ a) \cdot e^{\{i\}} = \hat{p} \cdot \omega^i,$$

  and furthermore individual i spends all his budget for the i^{th} commodity in the 1^{st} group of n goods. Since the prices are equal to zero for the 3^{rd} and 4^{th} group of n goods he can consume $\hat{x}_i^{(3)i} = \tilde{a}_i$ without using any of his budget. Thus, $\hat{x}^i$ is utility maximizing.

- Moreover, the market clearing condition is satisfied

$$\sum_{i \in N} \hat{x}^i = \sum_{i \in N} \omega^i + \sum_{i \in N} \hat{y}^i.$$

Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i\left(\hat{x}^i\right)\right)_{i \in N} = a \notin A.$$

3.4.2 The main results

In order to deal with the general case without assuming the strict positivity of price vectors, we modify the market from the previous section in an appropriate way. This modification allows us to show, that the prices of the 3^{rd} group of n commodities are strictly positive, $p^{(3)} \gg 0$. For the rest of this section let $[(N, V), A]$ satisfy SPS. To simplify the notation of the market, we introduce some sets before:

For the definition of the production sets define for all coalitions $S \in \mathcal{N} \setminus \{N\}$

$$\begin{aligned} A_S^1 &= \left\{\left(c^S, -e^S, c^S, -e^S, -e^S\right) | c^S \in C^S\right\}, \\ A_S^2 &= \left\{\left(c^S, 0, c^S, -e^S, 0\right) | c^S \in C^S\right\}, \\ A_S^3 &= \left\{\left(c^S, 0, c^S, 0, -e^S\right) | c^S \in C^S\right\} \end{aligned}$$

and for the grand coalition N define

$$\begin{aligned} A_N^1 &= \left\{\left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right) | \tilde{c}^N \in \tilde{C}^N\right\}, \\ A_N^2 &= \left\{\left(P_A\left(\tilde{c}^N\right), 0, \tilde{c}^N, -e^N, 0\right) | \tilde{c}^N \in \tilde{C}^N\right\}, \end{aligned}$$

$$A_N^3 = \left\{ \left(P_A \left(\tilde{c}^N \right), 0, \tilde{c}^N, 0, -e^N \right) | \tilde{c}^N \in \tilde{C}^N \right\}.$$

In order to obtain the result without the assumption of strictly positive price vectors, we modify the utility functions, the production and consumption sets. The utility functions do not depend anymore only on the two personal output commodities but also on the whole second group of output commodities. For that we add 'a little bit' of utility from the other players output goods. This 'little bit' is described by using the $\varepsilon > 0$ from the definition of SPS.

Definition 25 (induced A-market)**.** Let $[(N,V),A]$ satisfy strict positive separability. Let $\varepsilon > 0$ such that $\varepsilon < \min_{i,j \in N} \frac{\lambda_i^a}{\lambda_j^a}$ for all $a \in A$. The *induced A-market* of the game (N,V) and the set A is defined by

$$\mathcal{E}_{V,A,\varepsilon} = \{(X^i, Y^i, u^i, \omega^i)_{i \in N}\}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{0\} \times \mathbb{R}_+^n \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S \in \mathcal{N}} \left(A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}} \right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with

$$u^i \left(x^{(1)}, 0, x^{(3)}, 0, 0 \right) = \min \left(x_i^{(1)}, x_i^{(3)} + \varepsilon \sum_{j \neq i} x_j^{(3)} \right).$$

Note that this market is very similar to the market we defined in the previous section. We change the definition of the production and consumption sets slightly by introducing a further input commodity. Moreover, the utility functions here depend on all coordinates of the 3^{rd} group of n goods.

Having defined the induced A-market we prove the following theorem, which is the main result of this paper:

Theorem 13. *Let $[(N,V),A]$ satisfy strict positive separability. Then there exists a market such that this market represents the game (N,V) and such that the set of competitive payoff vectors of this market is the set A.*

To prove the above theorem we use the induced A-market $\mathcal{E}_{V,A,\varepsilon}$ as defined before. We divide the proof of this Theorem into 3 parts: First we show, that $\mathcal{E}_{V,A,\varepsilon}$ represents the game (N, V), in the second part we prove, that every vector in the set A is a competitive payoff vector, and in the third part we show that competitive payoff vectors always belong to the set A.

Lemma 3. *The induced A-market $\mathcal{E}_{V,A,\varepsilon}$ represents the game (N, V).*

The proof of Lemma 3 is inspired by Billera (1974).

Proof.

- As $V(S) = C^S - \mathbb{R}^S_+$ it is enough to show, that the payoffs in the set C^S can be achieved by coalition S in the market $\mathcal{E}_{V,A,\varepsilon}$. Let $z \in C^S$. We show, that there exists a feasible S-allocation $\left(x^i\right)_{i\in S}$ with $\left(y^i\right)_{i\in S}$ such that $u^i\left(x^i\right) = z_i$ for all $i \in S$.
  Define for $i \in S$ the consumption plan
  $$x^i = \left(z^{\{i\}}, 0, z^{\{i\}}, 0, 0\right)$$
  and let
  $$y^i = \frac{1}{|S|}\left(z, -e^S, z, -e^S, -e^S\right)$$
  be the production plan for all $i \in S$. By the definition of the consumption sets we observe $x^i \in X^i$ for all $i \in S$. With regard to the production sets for $S \neq N$ we have immediately $y^i \in Y^i$ for all $i \in S$. For $S = N$ note that $z \in V(N) \subseteq \tilde{V}(N)$ and thus $P_A(z) = z$. Hence, we have $y^i \in Y^i$ for all $i \in N$. Observe that
  $$\sum_{i\in S}\left(x^i - \omega^i\right) = \sum_{i\in S} y^i.$$
  Hence, $\left(x^i\right)_{i\in S}$ is a feasible S-allocation and
  $$u^i\left(x^i\right) = z_i \quad \text{for all } i \in S.$$
- Let $\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right)_{i\in S}$ be a feasible S-allocation with $\left(\bar{y}^{(1)i}, \bar{y}^{(2)i}, \bar{y}^{(3)i}, \bar{y}^{(4)i}, \bar{y}^{(5)i}\right)_{i\in S}$ in the market $\mathcal{E}_{V,A,\varepsilon}$.

The feasibility implies

$$\left(\sum_{i\in S}\bar{x}^{(1)i},-e^S,\sum_{i\in S}\bar{x}^{(3)i},-e^S,-e^S\right)=\sum_{i\in S}\left(\bar{y}^{(1)i},\bar{y}^{(2)i},\bar{y}^{(3)i},\bar{y}^{(4)i},\bar{y}^{(5)i}\right).$$

Each production set is a convex cone of a union of convex sets. Hence, an arbitrary production plan can be written in the following way: Choose one suitable element from each of the convex sets and build a linear combination (with non-negative coefficients) of these elements. For the 1^{st} and the 2^{nd} group of n commodities we obtain, that there exist $\alpha_R^i\in\mathbb{R}_+$ for all $R\in\mathcal{N}$, $z_R^i\in C^R$ for all $R\in\mathcal{N}\setminus\{N\}$ and $\tilde{z}_N^i\in\tilde{C}^N$, such that

$$\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)=\sum_{R\in\mathcal{N}\setminus\{N\}}\alpha_R^i\left(z_R^i,-e^R\right)+\alpha_N^i\left(P_A\left(\tilde{z}_N^i\right),-e^N\right).$$

As $P_A\left(\tilde{C}^N\right)=C^N$ there exists $z_N^i\in C^N$ such that $P_A\left(\tilde{z}_N^i\right)=z_N^i$ and hence we have

$$\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)=\sum_{R\in\mathcal{N}}\alpha_R^i\left(z_R^i,-e^R\right).$$

As feasibility implies $\left(\sum\limits_{i\in S}\bar{x}^{(1)i},-e^S\right)=\sum\limits_{i\in S}\left(\bar{y}^{(1)i},\bar{y}^{(2)i}\right)$, for the 2^{nd} group of n coordinates we have that

$$\begin{aligned}e^S&=\sum_{i\in S}\sum_{R\in\mathcal{N}}\alpha_R^i e^R\\&=\sum_{R\in\mathcal{N}}\left(\sum_{i\in S}\alpha_R^i\right)e^R.\end{aligned}$$

Thus $\alpha_R^i>0$ implies $R\subseteq S$ and if we define $\alpha(R)=\sum\limits_{i\in S}\alpha_R^i$, then $(\alpha(R))_{R\subseteq S}$ is a balanced family for the coalition S. Looking at the 1^{st} group of n coordinates we have

$$\sum_{i\in S}\bar{x}^{(1)i}=\sum_{R\subseteq S}\sum_{i\in S}\alpha_R^i z_R^i$$

$$= \sum_{\{R \subseteq S | \alpha(R) > 0\}} \alpha(R) \left(\frac{1}{\alpha(R)} \sum_{i \in S} \alpha_R^i z_R^i \right).$$

Since C^R is convex we have

$$\frac{1}{\alpha(R)} \sum_{i \in S} \alpha_R^i z_R \in C^R$$

and hence, using totally balancedness, $\sum_{i \in S} \bar{x}^{(1)i} \in V(S)$.

>From the definition of the utility function we obtain $u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right) \leq \bar{x}_i^{(1)i}$. Since $\left(\bar{x}_i^{(1)i}\right)_{i \in S} \leq \sum_{i \in S} \bar{x}^{(1)i} \in V(S)$ we have by the S-comprehensiveness of $V(S)$ that $\left(u^i\left(\bar{x}^{(1)i}, 0, \bar{x}^{(3)i}, 0, 0\right)\right)_{i \in S} \in V(S)$.

□

Proposition 11. *Every point in A is an equilibrium payoff vector of the market $\mathcal{E}_{V,A,\varepsilon}$.*

Proof. The above proposition holds by an argument similar to the one used in the proof of Proposition 9. Let $a \in A$ and $\lambda^a \in \Delta$ an associated normal vector. We know that λ^a is strictly positive (compare the remark on page 60). Note that the consumption and production plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(a^{\{i\}}, 0, a^{\{i\}}, 0, 0\right)\right)_{i \in N}$$

and

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n}\left(a, -e^N, a, -e^N, -e^N\right)\right)\right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^a, \frac{2}{3}\left(\lambda^a \circ a\right), \lambda^a, \frac{2}{3}\left(\lambda^a \circ a\right), \frac{2}{3}\left(\lambda^a \circ a\right)\right)$$

constitute a competitive equilibrium in the market $\mathcal{E}_{V,A,\varepsilon}$. The equilibrium price vector is strictly positive since a and λ^a are strictly positive.

As we have a convex-cone-technology maximum profits are zero. We observe

$$\hat{p} \cdot \hat{y}^i = \frac{1}{n}\left(\lambda^a \cdot a - \frac{2}{3}\left(\lambda^a \circ a\right) \cdot e^N + \lambda^a \cdot a - \frac{2}{3}\left(\lambda^a \circ a\right) \cdot e^N - \frac{2}{3}\left(\lambda^a \circ a\right) \cdot e^N\right) = 0.$$

Hence, the production plan $\hat{y}^i$ is profit maximizing.

Next we show that the consumption vector x^i is utility maximizing on the budget set of agent i.

- First notice that the budget constraint is satisfied with equality,

$$\hat{p} \cdot \hat{x}^i = \lambda^a \cdot \left(a^{\{i\}} + a^{\{i\}}\right) = \frac{2}{3} \left(\lambda^a \circ a\right) \cdot \left(e^{\{i\}} + e^{\{i\}} + e^{\{i\}}\right) = \hat{p} \cdot \omega^i.$$

- Second the consumption vector of agent i satisfies

$$\hat{x}_i^{(1)i} = \hat{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \hat{x}_j^{(3)i}.$$

  This means agent i consumes in a way such that he receives the "same amount of utility" from the 1^{st} group of n goods and the 3^{rd} group of n goods. For an agent with a min-type or Leontief utility function it is a necessary condition for utility maximization to consume in such a way (as long as we have strictly positive prices). This can be seen by similar arguments like in the proof of Claim 1.

- Third, it remains to check that $\hat{x}^i$ is indeed utility maximizing for agent i on his budget set. Hereby, the crucial point to see is, that agent i only consumes his personal output goods, and not the output goods of the other agents. In particular, this means for the 3^{rd} group of n commodities $\hat{x}_j^{(3)i} = 0$ for $j \neq i$.

  First look at the consumption of the 3^{rd} group of n goods when half of the wealth, $\lambda^a \cdot a^{\{i\}}$, is used for these goods.

  If agent i spends the wealth only for his personal output commodity, he consumes $\hat{x}^{(3)i} = a^{\{i\}}$. Then we have $\hat{p}^{(3)} \cdot \hat{x}^{(3)i} = \lambda^a \cdot a^{\{i\}}$. Suppose now agent i changes his consumption plan for the 3^{rd} group of n commodities to a plan $\tilde{x}^{(3)i}$, where he consumes as well one of the other agents output goods, meaning $\tilde{x}_j^{(3)i} > 0$ for one $j \neq i$. To do this agent i needs to decrease the consumption in his personal output good and hence $\hat{x}_i^{(3)i} > \tilde{x}_i^{(3)i}$. Set $\delta = \hat{x}_i^{(3)i} - \tilde{x}_i^{(3)i}$. Then this δ he consumes less gives him an available budget of $\lambda_i^a \delta$, that he can now use to spend for the other agents commodity j. If agent i now spends $\lambda_i^a \delta$ for good j, he can purchase $\frac{\lambda_i^a}{\lambda_j^a}\delta$ units of good j which gives him an additional level of "utility" in good j of the 3^{rd} group of n goods.

Look at

$$\begin{aligned}
&\hat{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \hat{x}_j^{(3)i} - \left(\tilde{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \tilde{x}_j^{(3)i} \right) \\
&= \hat{x}_i^{(3)i} - \left(\hat{x}_i^{(3)i} - \delta + \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \cdot \delta \right) \\
&= \delta - \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \cdot \delta \\
&= \delta \left(1 - \varepsilon \frac{\lambda_i^a}{\lambda_j^a} \right).
\end{aligned}$$

The above expression is positive since $\varepsilon < \frac{\lambda_j^a}{\lambda_i^a}$ for all $i, j \in N$ and hence $\varepsilon \frac{\lambda_i^a}{\lambda_j^a} < \frac{\lambda_j^a}{\lambda_i^a} \frac{\lambda_i^a}{\lambda_j^a} = 1$. Thus we have

$$\hat{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \hat{x}_j^{(3)i} > \tilde{x}_i^{(3)i} + \varepsilon \sum_{j \neq i} \tilde{x}_j^{(3)i}.$$

The potential loss of utility from consuming less of his personal output commodity is higher than the potential gain from consuming agent j's output commodity given a fixed wealth.

A similar argument also holds true, when agent i changes the consumption in a way such that he consumes output goods of several other agents.

Thus agent i cannot increase his utility by changing his consumption plan for the 3^{rd} group of n commodities from $\hat{x}^{(3)i}$ to $\tilde{x}^{(3)i}$ and consuming output commodities of the other agents $j \neq i$ instead of his own output commodities.

Now it is easy to see, that spending half of the total wealth for each of the two groups of output commodities leads to the same amount of utility in both arguments of the min-type utility function and is hence utility maximizing.

Furthermore, the market clearing condition

$$\sum_{i\in N}\hat{x}^i=\sum_{i\in N}\omega^i+\sum_{i\in N}\hat{y}^i$$

is satisfied.
Thus, we have found a competitive equilibrium with equilibrium payoff vector

$$\left(u^i\left(\hat{x}^i\right)\right)_{i\in N}=a.$$

□

In the above proof the competitive equilibrium price vectors are linked to the transfer rate vectors of points in the set A similarly as in the proof of Proposition 9. The output goods are directly priced by the transfer rate vectors and the input goods are priced by the transfer rate vectors weighted by the according point of the set A (multiplied by $\frac{2}{3}$).

It remains to show, that vectors not belonging to the set A cannot be competitive payoff vectors. The crucial point is to show, that $p^{(3)}$ is strictly positive.

Lemma 4. *Let $((x^i)_{i\in N},(y^i)_{i\in N},p)$ be any competitive equilibrium for the induced A-market. Then $p^{(3)}$ is strictly positive.*

Proof. Let $((x^i)_{i\in N},(y^i)_{i\in N},p)$ be a competitive equilibrium for the induced A-market. By the market clearing condition we have

$$\sum_{i\in N}x^i=\sum_{i\in N}y^i+\left(0,e^N,0,e^N,e^N\right)$$

and by profit maximization $p\cdot y^i=0$ for all $i\in N$. By the definition of the production set for each $i\in N$ there exist $\gamma_S^{i1},\ \gamma_S^{i2},\ \gamma_S^{i3}\geq 0$ for all $S\in\mathcal{N}$, $u_S^{i1},\ u_S^{i2},\ u_S^{i3}\in C^S$ for all $S\in\mathcal{N}\setminus\{N\}$ and $\tilde{u}_N^{i1},\ \tilde{u}_N^{i2},\ \tilde{u}_N^{i3}\in\tilde{C}^N$ such that

$$\begin{aligned}y^i=&\sum_{S\in\mathcal{N}\setminus\{N\}}\left(\sum_{j=1}^{3}\gamma_S^{ij}u_S^{ij},\ -\gamma_S^{i1}e^S,\ \sum_{j=1}^{3}\gamma_S^{ij}u_S^{ij},\ -\left(\gamma_S^{i1}+\gamma_S^{i2}\right)e^S,\ -\left(\gamma_S^{i1}+\gamma_S^{i3}\right)e^S\right)\\&+\left(\sum_{j=1}^{3}\gamma_N^{ij}P_A\left(\tilde{u}_N^{ij}\right),\ -\gamma_N^{i1}e^N,\ \sum_{j=1}^{3}\gamma_N^{ij}\tilde{u}_N^{ij},\ -\left(\gamma_N^{i1}+\gamma_N^{i2}\right)e^N,\ -\left(\gamma_N^{i1}+\gamma_N^{i3}\right)e^N\right).\end{aligned}$$

As $P_A\left(\tilde{C}^N\right) = C^N$ there exist $u_N^{ij} \in C^N$ such that $P_A\left(\tilde{u}_N^{ij}\right) = u_N^{ij}$ for $j = 1, 2, 3$. Thus, we have for all $i \in N$

$$y^i = \sum_{S \in \mathcal{N} \setminus \{N\}} \left(\sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\gamma_S^{i1} e^S,\ \sum_{j=1}^{3} \gamma_S^{ij} u_S^{ij},\ -\left(\gamma_S^{i1} + \gamma_S^{i2}\right) e^S,\ -\left(\gamma_S^{i1} + \gamma_S^{i3}\right) e^S \right)$$
$$+ \left(\sum_{j=1}^{3} \gamma_N^{ij} u_N^{ij},\ -\gamma_N^{i1} e^N,\ \sum_{j=1}^{3} \gamma_N^{ij} \tilde{u}_N^{ij},\ -\left(\gamma_N^{i1} + \gamma_N^{i2}\right) e^N,\ -\left(\gamma_N^{i1} + \gamma_N^{i3}\right) e^N \right).$$

By the definition of the consumption set we need to have $x^{(2)i} = x^{(4)i} = x^{(5)i} = 0$ for all $i \in N$. Hence, for all $i \in N$, we obtain, using the market clearing condition and the definition of the production sets, for all coalitions $S \in \mathcal{N}$

$$\sum_{T \subseteq N} \gamma_T^{i1} e^T = e^S,$$
$$\sum_{T \subseteq N} \left(\gamma_T^{i1} + \gamma_T^{i2}\right) e^T = e^S,$$
$$\sum_{T \subseteq N} \left(\gamma_T^{i1} + \gamma_T^{i3}\right) e^T = e^S.$$

It follows that $\gamma_S^{i2} = \gamma_S^{i3} = 0$ for all $i \in N$ and for all $S \in \mathcal{N}$ and that for some $i \in N$ and some $S \in \mathcal{N}$ we have $\gamma_S^{i1} > 0$.
Suppose now, that $p_i^{(3)} = 0$ for at least one $i \in N$. We show, that this leads to a contradiction.

First observe: If $p_i^{(3)} = 0$ for one $i \in N$, then $p_k^{(3)} = 0$ for all $k \in N$.

To see this suppose $p_k^{(3)} > 0$ for some $k \in N$. For every individual $j \in N$ the consumption bundle x^j maximizes his utility function over his budget set $\{\hat{x}^j \in X^j | p \cdot \hat{x}^j \leq p \cdot \omega^j\}$. This implies, if $p_i^{(3)} = 0$ that agent j does not consume any good that has a positive price. If he did so, this would decrease his available budget whereas he can reach the same utility from consuming good i that is for free. Precisely $p_i^{(3)} = 0$ implies $x_k^{(3)j} = 0$ for all $j \in N$ and for all $k \in N$ such that $k \neq i$ and $p_k^{(3)} > 0$.
However, the market clearing condition and the definition of the production

set require

$$\sum_{j \in N} x^{(3)j} = \sum_{S \in \mathcal{N} \setminus \{N\}} \gamma_S^{i1} u_S^{i1} + \gamma_N^{i1} \tilde{u}_N^{i1} \gg 0,$$

since $u_S^{i1} \in C^S \subseteq \mathbb{R}_{++}^S$ and $\tilde{u}_N^{i1} \geq u_N^{i1} \in C^N \subseteq \mathbb{R}_{++}^N$. Hence, we obtain a contradiction and thus $p^{(3)} = 0$.

Since $u^j(\check{x}^j) > u^j(\bar{x}^j)$ whenever $\check{x}_j^{(1)j} > \bar{x}_j^{(1)j}$ and $\check{x}^{(3)j} > \bar{x}^{(3)j}$, it follows from $p^{(3)} = 0$ that $p_j^{(1)}$ must be positive. This holds for all $j \in N$, thus $p^{(1)} \gg 0$. Since $C^S \subseteq \mathbb{R}_{++}^S$, it follows that $p^{(1)} \cdot u_S^{i1} > 0$. Since the maximal profits are equal to zero because of the convex-cone-technology, it must be true that

$$p^{(1)} \cdot u_S^{i1} - p^{(2)} \cdot e^S - p^{(4)} \cdot e^S - p^{(5)} \cdot e^S = 0. \tag{$\star$}$$

For any $j \in N$ choose $u \in C^{\{j\}} \cap \mathbb{R}_{++}^{\{j\}}$ and $\gamma > 0$. Then

$$\left(\gamma u, 0, \gamma u, -\gamma e^{\{j\}}, 0\right) \in Y^j$$

and

$$p \cdot \left(\gamma u, 0, \gamma u, -\gamma e^{\{j\}}, 0\right) = \gamma \left(p_j^{(1)} u - p_j^{(4)}\right).$$

Since $p^{(1)} \gg 0$, $p_j^{(4)}$ must be positive, because otherwise this would contradict the fact, that maximal profits are 0. Thus, $p^{(4)} \gg 0$. Similarly $p^{(5)} \gg 0$. Therefore, from the equation ($\star$) above we obtain using $-p^{(5)} \cdot e^S < 0$ and $-p^{(2)} \cdot e^S \leq 0$

$$p^{(1)} \cdot u_S^{i1} - p^{(4)} \cdot e^S > 0.$$

Hence, we have

$$p \cdot \left(u_S^{i1}, 0, u_S^{i1}, -e^S, 0\right) = p^{(1)} \cdot u_S^{i1} + p^{(3)} \cdot u_S^{i1} - p^{(4)} \cdot e^S = p^{(1)} \cdot u_S^{i1} - p^{(4)} \cdot e^S > 0.$$

But $\left(u_S^{i1}, 0, u_S^{i1}, -e^S, 0\right) \in Y^i$ as it is of the form as points in the set A_S^2. This is a contradiction to the fact, that the maximal profits are zero. Thus $p^{(3)} \gg 0$. □

We use this result to show the remaining Proposition that completes the proof of the theorem:

Proposition 12. *Any payoff vector of a competitive equilibrium of the market $\mathcal{E}_{V,A,\varepsilon}$ is an element of the set A.*

Proof. Suppose there exists a competitive equilibrium $\left(\left(x^i\right)_{i\in N}\left(y^i\right)_{i\in N}, p\right)$, such that $\left(u^i\left(x^i\right)\right)_{i\in N} = c^N$ with $c^N \notin A$.

>From Lemma 1 we know that c^N is in the inner core.

That Lemma 1 is applicable can be seen as follows: We know that $p \cdot \omega^i > 0$. Otherwise agent i would have a budget of 0 and we needed to have $p_i^{(2)} = p_i^{(4)} = p_i^{(5)} = 0$. This would mean that the production plan $\left(c^{\{i\}}, -e^{\{i\}}, c^{\{i\}}, -e^{\{i\}}, -e^{\{i\}}\right)$ with $c^{\{i\}} \in C^{\{i\}}$ has strictly positive profits. This would be a contradiction. Thus, for all individuals $i \in N$ we have $p \cdot \omega^i > 0$.

By Lemma 4 we know $p^{(3)} \gg 0$. Furthermore we know

$$y = \sum_{i\in N} y^i = \left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right)$$

for some $\tilde{c}^N \in \tilde{C}^N$ satisfying $P_A\left(\tilde{c}^N\right) = c^N$ as any other production would contradict the market clearing condition in the 1^{st} group of n coordinates. From the profit maximization we know that $\tilde{c}^N$ has to be chosen on the boundary of $\tilde{C}(N)$ and hence, since $c^N \notin A$, we have $\tilde{c}^N \gg c^N$. By the market clearing condition (for the 3^{rd} group of n coordinates) we have

$$\sum_{i\in N} x^{(3)i} = \tilde{c}^N. \qquad (\star\star)$$

Furthermore, by utility maximization we obtain

$$c_i^N = x_i^{(3)i} + \varepsilon \sum_{j\neq i} x_j^{(3)i}. \qquad (\star\star\star)$$

As $c^N \ll \tilde{c}^N$, equation $(\star\star\star)$ implies, that we have $x_i^{(3)i} < \tilde{c}_i^N$ for all $i \in N$.

Hence, for every $i \in N$ we have $\sum_{j\neq i} x_i^{(3)j} > 0$. Thus, for every $i \in N$ there exists $j \neq i$ satisfying $x_i^{(3)j} > 0$. Define a mapping $M : N \longrightarrow N$ in the following way: Every $i \in N$ is mapped to one $j \neq i$ satisfying $x_i^{(3)j} > 0$. Then, we can find

$k \in N$ and $t \in \mathbb{N}$ such that $M^t(k) = k$.

We use these results to show some constraints on the equilibrium prices: As $x_k^{(3)M(k)} > 0$, the utility maximization of agent $M(k)$ implies, that we have $p_k^{(3)} \leq \varepsilon p_{M(k)}^{(3)}$. Otherwise, agent $M(k)$ would not consume good k, but instead more of good $M(k)$. In the same way, we can show similar equations for other prices and obtain

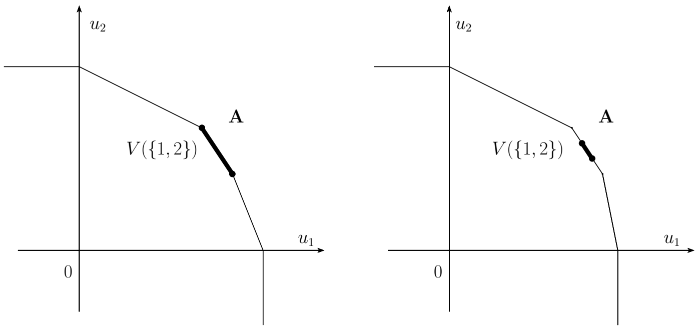
$$p_k^{(3)} \leq \varepsilon p_{M(k)}^{(3)} \leq \varepsilon^2 p_{M^2(k)}^{(3)} \leq ... \leq \varepsilon^t p_{M^t(k)}^{(3)} = \varepsilon^t p_k^{(3)}.$$

But $\varepsilon^t < 1$. This is a contradiction.

□

As already mentioned before, assuming SPS is more restrictive than actually needed. Requiring the strict separation property for all points in the set A can be weakened to requiring it only for the boundary points of the set A. In fact, we need for the construction of the auxiliary game $(N, \tilde{V})$ that outside the set A the efficient boundary is strictly enlarged. This means the property that if we take $x \in V(N) \setminus A$, then x being in the interior of $\tilde{V}(N)$ is the crucial property to eliminate equilibria with a payoff vector outside the set A. Using this weaker assumption allows a choice of the set A as in Example 7. An example, where even this weaker version of the strict positive separability property is violated, and where our approach cannot be applied can be found in Figure 3.5. Assume as before that we have always two players and that the coalitional function is given by $V(\{1\}) = V(\{2\}) = \{0\} - \mathbb{R}_+$ and $V(\{1,2\})$ is given as indicated in Figure 3.5.

In contrast to Example 7, in Example 8 the set A is chosen in such a way that it is a closed interval of a line segment connecting two neighboring corner points, but not the whole line segment. Because of the polyhedral structure none of the points in the set A can be strictly separated from the set $V(\{1,2\})$ without the point.



Example 7 Example 9

Figure 3.5: Examples where SPS is not satisfied.

Another important aspect of our result is the fact that the induced A-market is not determined uniquely. We have some freedom in different aspects of our construction and obtain a whole class of markets, that can be used to prove our main theorem:

- First, to define the induced A-market we use the auxiliary NTU game $(N, \tilde{V})$ where we enlarge the given NTU game (N, V). For this enlargement we use for every inner core point one of its normal vectors. This normal vector is not always unique.

- Second, for the auxiliary game $(N, \tilde{V})$ we define the mapping P_A which can be chosen in different ways. The important property is that for the points outside the given subset of the inner core, A, we have $P_A(z) \gg z$ for all $z \in IC(A) \setminus A$. Moreover, for points in the given set A we require $P_A(z) = z$ for all $z \in A$.

- Third, we add to the utility function of the induced A-market an ε-term, that needs to be between certain bounds and hence is not determined uniquely. Moreover, we can choose different ε for different players.

3.5 Concluding Remarks

In this paper we have continued the work of Shapley and Shubik (1975) and Qin (1993) to investigate competitive payoff vectors of markets that represent a cooperative game and their relation to solution concepts for cooperative games.

We extend the results of Qin (1993) to a large class of closed subsets of the inner core: Given an NTU market game we construct the induced A-market depending on a given closed subset of its inner core. This market represents the game and further has the given set as the set of payoff vectors of competitive equilibria. More precisely, inspired by the construction of the induced market of Billera and Bixby (1974) and by the markets that Qin (1993) uses to prove his two main results, we define a market in an appropriate way to generalize the results of Qin (1993) to a large class of closed subsets of the inner core. It turns out that this market is not determined uniquely and thus we obtain a whole class of markets that has the given closed subset of the inner core as the set of payoff vectors of competitive equilibria.

In the literature it was already known that one game can be represented by several markets, see Billera and Bixby (1974) or Qin (1993). Our work confirms that going from NTU games to markets some structural information is added that is not present in the NTU game. To a given NTU market game we can associate a huge class of markets that represents the NTU game. In particular, by choosing the structure, that we add, we can control the set of payoffs of competitive equilibria.

Another point of view on our results is to analyze situations where we start with given markets and consider the induced games. Looking at competitive equilibria and how they appear in the game, we observe that almost everything is possible. Depending on the specific market the set of competitive equilibrium payoff vectors might fill up the whole inner core or be almost any closed subset, in particular any single point. Hence, our result demonstrates that we can not expect to observe more game theoretic properties of competitive equilibria than knowing that competitive payoffs are in the inner core. Only by imposing additional structural assumptions on the markets, for example restricting the class of utility functions, we may observe additional game theoretic properties.

We establish a link between closed subsets of the inner core and competitive payoffs of certain economies. Extending the results of Qin (1993) to closed subsets of the inner core means in particular to establish a link for all solution concepts selecting closed subsets of the inner core. Therefore, our results can be seen as a market foundation of game theoretic solution concepts that select closed subsets of the inner core. For the particular class of bargaining games a more precise presentation of the idea of a market foundation can be found in Trockel (1996, 2005) and Brangewitz and Gamp (2011b).

The result presented here includes the result of Qin (1993) for a single point in the inner core. This holds also in a very general setup by using monotone transformations of utilities in the same way as it was done in Qin (1993). Nevertheless, if we consider closed subsets of the inner core that contain more than a single point, the idea to transform the utilities seems not to work. Due to this fact we assume some separation properties on the game and the given closed subset of its inner core.

Furthermore, by investigating the NTU case we realized that a simple generalization of the approach of Shapley and Shubik (1975) in the framework of Qin (1993) does not work and we need to stay closer to the results on NTU games. More precisely, changing the utility function in the market, that Qin (1993) uses to prove his second result, in analogy to the TU case of Shapley and Shubik (1975) to

$$u^i(x^i) = \min\left\{x_i^{(1)i}, \min_{u^* \in A}\left\{\frac{\left(\lambda^{u^*} \circ u^*\right) \cdot x^{(5)i}}{\lambda_i^{u^*}}\right\}\right\}$$

does not lead to markets with the desired properties.

Having our result in mind there remains the open question if we can further weaken our assumptions such that the results can be proved for more general cases. Another interesting related line of research is to continue to look at the class of games that are linked to coalition production economies as analyzed by Inoue (2010b). Given a balanced NTU game Inoue (2010b) defines a coalition production economy such that this economy represents the game and has moreover the whole inner core as the set of competitive equilibrium payoff vectors. It remains an open question if one can find analogously to Qin (1993) and to this work a coalition production economy such that one inner core point or a certain subset of the inner core are competitive equilibrium payoff vectors in this coalition production economy. Moreover, it is interesting to compare the set of competitive equilibrium allocations of different market representations of a given NTU market game. Does there exist a general and more simple method to obtain desired competitive payoffs? Can we characterize a class of NTU games where this is possible? What happens if we restrict our attention for example to bargaining games?

3.6 Appendix

3.6.1 Proof of Lemma 1

For the proof of Lemma 1 we follow the idea of de Clippel and Minelli (2005).

Proof. Let $\left(\hat{x}^i\right)_{i\in N}$ and $\left(\hat{y}^i\right)_{i\in N}$ be a competitive equilibrium allocation at a price $\hat{p} \in \mathbb{R}^\ell_+ \setminus \{0\}$. For each individual $i \in N$ define the set

$$C^i = \left\{(u,m) \in \mathbb{R}^2 | \exists z^i \in X^i : u \leq u^i\left(z^i\right) - u^i\left(\hat{x}^i\right), m \leq \hat{p} \cdot \left(\omega^i + \hat{y}^i - z^i\right)\right\}.$$

By the concavity of u^i, this set is convex. On the other hand, $C^i \cap \mathbb{R}^2_{++} = \emptyset$, as $\hat{x}^i$ is optimal for individual i in his budget set.

> Suppose $(u,m) \in C^i$ and $(u,m) \gg 0$, then there exists $z^i \in X^i$ with $u(\hat{x}^i) < u^(z^i)$ and $\hat{p} \cdot z^i < \hat{p} \cdot (\omega^i + \hat{y}^i)$ which means z^i gives individual i a higher utility as $\hat{x}^i$ and is affordable under the price system $\hat{p}$. This is in contradiction to the optimality of $\hat{x}^i$.

By the separating hyperplane theorem there exists a non-zero, non-negative vector $\left(\alpha^i, \beta^i\right) \in \mathbb{R}^2_+$ such that we can separate 0 from C^i and obtain

$$\alpha^i u^i\left(\hat{x}^i\right) \geq \alpha^i u^i\left(z^i\right) - \beta^i \hat{p} \cdot \left(z^i - \omega^i - \hat{y}^i\right)$$

for all $z^i \in X^i$.
As $\hat{p} \cdot \omega^i > 0$, it follows from the above inequality that we have $\alpha^i > 0$.

> To see this suppose $\alpha^i = 0$ ($\beta^i > 0$). Then, as in equilibrium $\hat{p} \cdot \hat{y}^i = 0$, we obtain from the above inequality
>
> $$0 \leq \hat{p} \cdot \left(z^i - \omega^i - \hat{y}^i\right) \quad \text{for all } z^i \in X^i,$$
>
> which is not true, as $0 \in X^i$ and $\hat{p} \cdot \hat{y}^i = 0$. Thus $\alpha^i > 0$.

We can assume $\alpha^i = 1$ without the loss of generality. Moreover, monotonicity and locally non-satiation of the utility function imply that $\beta^i > 0$. Let $\lambda^i = \frac{1}{\beta^i}$.

Summing up over all $i \in S$ we obtain

$$\sum_{i\in S} \lambda^i u^i \left(\hat{x}^i\right) \geq \sum_{i\in S} \lambda^i u^i \left(z^i\right) - \hat{p} \cdot \sum_{i\in S} \left(z^i - \omega^i - \hat{y}^i\right)$$

for all $S \subseteq N$ and for all $z^i \in \mathbb{R}^\ell_+$ with $i \in S$.

If a coalition S could λ-improve on x with $\left(\bar{x}^i\right)_{i\in S}$ (with the production plan $\bar{y}^i \in Y^i$), then the previous inequality would be violated, because we have, due to feasibility,

$$\sum_{i\in S} \left(\bar{x}^i - \omega^i - \bar{y}^i\right) \leq 0$$

and thus we obtain a contradiction by

$$\begin{aligned}
\sum_{i\in S} \lambda^i u^i \left(\bar{x}^i\right) &> \sum_{i\in S} \lambda^i u^i \left(\hat{x}^i\right) \\
&\geq \sum_{i\in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i\in S} \left(\bar{x}^i - \omega^i - \hat{y}^i\right) \\
&\geq \sum_{i\in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i\in S} \left(\bar{x}^i - \omega^i\right) \\
&\geq \sum_{i\in S} \lambda^i u^i \left(\bar{x}^i\right) - \hat{p} \cdot \sum_{i\in S} \bar{y}^i \\
&\geq \sum_{i\in S} \lambda^i u^i \left(\bar{x}^i\right).
\end{aligned}$$

□

3.6.2 Proof of Claim 1

Proof. We show

$$x_i^{(1)i} = x_i^{(3)i}$$

by contradiction. Then it immediately follows from $u^i(x^i) = c_i^N$ that

$$x_i^{(1)i} = x_i^{(3)i} = c_i^N.$$

Suppose $x_i^{(3)i} > x_i^{(1)i}$. This cannot be utility maximizing in the presence of strictly positive prices. If player i consumes a little bit less of the i^{th} good of the 3^{rd} group of n goods and invests the - not anymore used - additional budget in the i^{th} good of the 1^{st} group of n goods, then he can strictly increase his utility. Precisely, from the assumption $u^i(x^i) = c_i^N$ and $x_i^{(3)i} > x_i^{(1)i}$ it follows that $x_i^{(1)i} = c_i^N$. For δ sufficiently small, i.e. $0 < \delta < x_i^{(3)i} - x_i^{(1)i}$, player i can increase his utility by consuming δ less of the i^{th} good of the 3^{rd} group of n goods and increasing the consumption in the i^{th} good of the 1^{st} group of n goods by $\frac{p_i^{(3)}}{p_i^{(1)}}\delta$. To consume $\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta e^{\{i\}}, 0, x^{(3)i} - \delta e^{\{i\}}, 0\right)$ is still budget feasible for player i, because

$$p^{(1)}\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta e^{\{i\}}\right) + p^{(3)}\left(x^{(3)i} - \delta e^{\{i\}}\right) = p^{(1)}x^{(1)i} + p^{(3)}x^{(3)i} \leq p \cdot \omega^i.$$

Hereby, the last inequality follows from the budget feasibility of x^i. Moreover, the utility of consumer i strictly increases, since

$$u^i\left(x^{(1)i} + \frac{p_i^{(3)}}{p_i^{(1)}}\delta, 0, x^{(3)i} - \delta, 0\right) > x_i^{(1)i} = u^i\left(x^{(1)i}, 0, x^{(3)i}, 0\right)$$

by the choice of δ. This is a contradiction to the assumption that x^i is utility maximizing. Hence, we have $x_i^{(3)i} \leq x_i^{(1)i}$.

By exchanging the roles of $x_i^{(1)i}$ and $x_i^{(3)i}$ we can analogously show $x_i^{(3)i} \geq x_i^{(1)i}$. Therefore, we have $x_i^{(3)i} = x_i^{(1)i}$. □

Chapter 4

Inner Core, Asymmetric Nash Bargaining Solutions and Competitive Payoffs

4.1 Abstract

We investigate the relationship between the inner core and asymmetric Nash bargaining solutions for n-person bargaining games with complete information. We show that the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core if all points in the underlying bargaining set are strictly positive. Furthermore, we prove that every bargaining game is a market game. By using the results of Qin (1993) we conclude that for every possible vector of weights of the asymmetric Nash bargaining solution there exists an economy that has this asymmetric Nash bargaining solution as its unique competitive payoff vector. We relate the literature of Trockel (1996, 2005) with the ideas of Qin (1993). Our result can be seen as a market foundation for every asymmetric Nash bargaining solution in analogy to the results on non-cooperative foundations of cooperative games.

4.2 Introduction

The inner core and asymmetric Nash bargaining solutions represent solution concepts for cooperative games. The inner core is defined for cooperative games whereas asymmetric Nash bargaining solutions are usually only applied to a subclass of cooperative games, namely bargaining games. A recent contribution of Compte and Jehiel (2010) generalizes the symmetric Nash bargaining solution to other cooperative games (with transferable utility). In this paper we consider the relationship between the inner core and asymmetric Nash bargaining solutions for bargaining games. Moreover, as an application of these results we show how asymmetric Nash bargaining solutions can be justified in a general equilibrium framework as a competitive payoff vector of a certain economy.

In the first section we give a literature overview to motivate our ideas. In the second section we recall the definitions of the inner core, a bargaining game and asymmetric Nash bargaining solutions. Afterwards, we investigate for bargaining games the relationship between the inner core and the set of asymmetric Nash bargaining solutions. Finally, we apply these results to market games and obtain by this a market foundation of asymmetric Nash bargaining solutions.

4.3 Motivation and Background

The inner core is a refinement of the core for cooperative games with non-transferable utility (NTU). For cooperative games with transferable utility (TU) the inner core coincides with the core. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector.

The Nash bargaining solution for bargaining games, a special class of cooperative games, where just the singleton and the grand coalition are allowed to form, goes back to Nash (1950, 1953). The (symmetric) Nash bargaining solution is defined as the maximizer of the product of the utilities over the individual rational bargaining set or as the unique solution that satisfies the following axioms: Invariance to affine linear Transformations, Pareto Optimality, Symmetry and Independence of Irrelevant Alternatives. If the bargaining power of the players is different an asymmetric Nash bargaining solution can be defined as the maximizer of an accordingly weighted Nash product. Concerning the axiomatization this means that the Symmetry axiom is replaced by an appropriate Asymmetry axiom, see Roth (1979). In addition to the axiomatic approach the literature studies non-cooperative foundations to justify cooperative solutions like the (asymmetric) Nash bargaining solution. The idea is to find an appropriate non-cooperative game whose equilibrium outcomes coincide with a given cooperative solution (see for example Bergin and Duggan (1999), Trockel (2000)). Here, we study the foundation of the asymmetric Nash bargaining solution by having this solution as a payoff vector of a competitive equilibrium in a certain economy.

There are different approaches to consider the relationship between cooperative games and economies or markets. On the one hand for example Shapley (1955), Shubik (1959) Debreu and Scarf (1963) and Aumann (1964) consider economies as games. On the other hand there is the approach to start with a cooperative game and to consider related economies as it was introduced by Shapley and Shubik (1969, 1975).

Starting with a market Shapley (1955) considers markets as cooperative games with two kinds of players, seller and buyer. He introduces in this context the general notion of an 'abstract market game'. This is a cooperative game with certain conditions on the characteristic function. Shubik (1959) extends the ideas of Edgeworth (from 1881) and studies 'Edgeworth market games'. In particular he shows that if the number of players of both sides in an Edgeworth market game is the same, then the set of imputations coincides with the contract curve of Edgeworth. Furthermore, he considers non-emptiness conditions for the core of this class of games. Debreu and Scarf (1963) show that under certain assumptions a competitive allocation is in the core. Aumann (1964) investigates, based among others on the oceanic games from Milnor and Shapley (1978)[1], economies with a continuum of traders and obtains that in this case the core equals the set of equilibrium allocations.

Starting with a cooperative game Shapley and Shubik (1969) look at these problems from a different viewpoint and study which class of cooperative games can be represented by a market. A market represents a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. Shapley and Shubik (1969) call any game that can be represented by a market a 'market game'. In the TU-case it turns out that every totally balanced TU game is a market game. Furthermore, Shapley and Shubik (1975) start with a TU game and show that every payoff vector in the core of that game is competitive in a certain market, the direct market. The direct market has a nice structure: Besides a numeraire commodity there are as many goods as players and initially every player owns one unit of 'his personal commodity'. Moreover, Shapley and Shubik (1975) show that for a given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector.

The idea of market games was applied to NTU games by Billera and Bixby (1974). Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced game, that is compactly convexly generated, is an NTU market game. Qin (1993) compares the inner core of NTU market games with the competitive payoff vectors of markets that represent this game. He shows that for a given NTU market game there exists a market such that the set of equilibrium payoff vectors coincides with the inner core of the game. In a second result, he shows that given an inner core point there exists a market, which represents the

[1]The reference Milnor and Shapley (1978) is based on the Rand research memoranda from the early 1960's.

game and has this given inner core point as its unique competitive equilibrium payoff. Brangewitz and Gamp (2011a) extend the results of Qin (1993) to a large class of closed subsets of the inner core.

Apart from this literature Trockel (1996, 2005) considers bargaining games directly as Arrow-Debreu or coalition production economies. One difference to other literature is that he allows to obtain output in the production without requiring input. In contrast to Shapley and Shubik (1969, 1975), Trockel (1996, 2005) considers NTU games rather than TU games. Motivated by the approach of Sun et al. (2008) and the approach of Billera and Bixby (1974), Inoue (2010b) uses coalition production economies instead of markets. Inoue (2010b) shows that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy such that its set of competitive payoff vectors coincides with the inner core of the balanced cover of the original NTU game.

Here, we show that we can apply the main results of Qin (1993) to a special class of NTU games, namely bargaining games. By that we obtain a market foundation of the asymmetric Nash bargaining solution. In contrast to Trockel (1996, 2005) we do not use Arrow-Debreu or coalition production economies directly but we consider bargaining games as market games by using the economies of Qin (1993). By this we relate the approach of Trockel (1996, 2005) on the one hand with the ideas of Qin (1993) on the other hand. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions in analogy to the results on non-cooperative foundations of cooperative games (see Trockel (2000), Bergin and Duggan (1999)).

4.4 Inner Core and Asymmetric Nash Bargaining Solution

4.4.1 NTU Games and the Inner Core

Let $N = \{1, ..., n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ be the set of players. Let $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ be the set of non-empty coalitions and $\mathcal{P}(\mathbb{R}^n) = \{A | A \subseteq \mathbb{R}^n\}$ be the set of all subsets of $\mathbb{R}^n$. Define $\mathbb{R}^S_+ = \left\{x \in \mathbb{R}^n_+ | x_i = 0, \forall i \notin S\right\}$.

Definition 26 (NTU game)**.** An *NTU game* is a pair (N, V), where the coalitional function is defined as

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

such that for all non-empty coalitions $S \subseteq N$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$ and $V(S)$ is S-comprehensive.

Definition 27 (compactly (convexly) generated)**.** An NTU game (N, V) is *compactly (convexly) generated* if for all $S \in \mathcal{N}$ there exists a compact (convex) $C^S \subseteq \mathbb{R}^S$ such that the coalitional function can be written as $V(S) = C^S - \mathbb{R}^S_+$.

In order to define the inner core we first consider a game that is related to a compactly generated NTU game. Given a compactly generated NTU game we define for a given transfer rate vector $\lambda \in \mathbb{R}^N_+$ the λ-transfer game.

Definition 28 (λ-transfer game)**.** Let (N, V) be a compactly generated NTU game and let $\lambda \in \mathbb{R}^N_+$. Define the λ*-transfer game* of (N, V) by (N, V_λ) with

$$V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$$

where $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$.

Qin (1994, p.433) gives the following interpretation of the λ-transfer game: "The idea of the λ-transfer game may be captured by thinking of each player as representing a different country. The utilities are measured in different currencies, and the ratios λ_i/λ_j are the exchange rates between the currencies of i and j." As for the λ-transfer game only proportions matter we can assume without loss of generality that λ is normalized, i.e. $\lambda \in \Delta^n = \left\{\lambda \in \mathbb{R}^n_+ \,|\, \sum_{i=1}^n \lambda_i = 1\right\}$. Define the positive unit simplex by $\Delta^n_{++} = \left\{\lambda \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n \lambda_i = 1\right\}$.

The inner core is a refinement of the core. The *core* $C(V)$ of an NTU game (N, V) is defined as those utility allocations that are achievable by the grand coalition N such that no coalition S can improve upon this allocation. Thus,

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i\}.$$

Definition 29 (inner core, Shubik (1984))**.** The *inner core* $IC(V)$ of a compactly generated NTU game (N, V) is

$$IC(V) = \{u \in V(N) | \exists \lambda \in \Delta \text{ such that } u \in C(V_\lambda)\}$$

where $C(V_\lambda)$ denotes the core of the λ-transfer game of (N, V).

This means a vector u is in the inner core if and only if u is affordable by the grand coalition N and if u is in the core of an appropriately chosen λ-transfer game. If a utility allocation u is in the inner core, then u is as well in the core.

For compactly convexly generated NTU games we have the following remark:

Remark 3 (Qin (1993), Remark 1, p. 337)**.** The vectors of supporting weights for a utility vector in the inner core must all be strictly positive.

4.4.2 Asymmetric Nash Bargaining Solutions

We consider a special class of NTU games, where only the singleton or the grand coalition can form, namely NTU bargaining games. Two-person bargaining games with complete information and the (symmetric) Nash bargaining solution were originally defined by Nash (1950).

Alternatively to the notion based on Nash (1950)[2] we adapt the notation and interpret bargaining games here as a special class of NTU games where only the

[2]Following the idea of Nash (1950) a *n-person bargaining game* with complete information is defined as a pair (B, d) with the following properties:

1. $B \subseteq \mathbb{R}^n$,
2. B is convex and compact,

grand coalition can profit from cooperation. Smaller coalitions are theoretically possible but there are no incentives to form them as everybody obtains the same utility as being in a singleton coalition. Starting from the definition of a bargaining game based on Nash (1950) we define an NTU bargaining game. Let $B \subseteq \mathbb{R}^n$ be a compact, convex set and assume that there exists at least one $b \in B$ with $b \gg 0$. For normalization purposes we assume here that the disagreement outcome is 0 and that $B \subseteq \mathbb{R}^n_+$. Nevertheless the results presented here can easily be generalized to the case that the disagreement point is not equal to 0.

Definition 30 (NTU bargaining game)**.** Define an *NTU bargaining game*[3] (N, V) with the generating set B using the player set N and the coalitional function

$$V : \mathcal{N} \longrightarrow \mathcal{P}\left(\mathbb{R}^n\right)$$

defined by

$$\begin{aligned} V\left(\{i\}\right) &:= \{b \in \mathbb{R}^n | b_i \leq 0, b_j = 0, \forall j \neq i\} = \{0\} - \mathbb{R}^{\{i\}}_+, \\ V\left(S\right) &:= \{0\} - \mathbb{R}^S_+ \quad \text{for all S with } 1 < |S| < n, \\ V\left(N\right) &:= \left\{b \in \mathbb{R}^n | \exists\, b' \in B : b \leq b'\right\} = B - \mathbb{R}^n_+. \end{aligned}$$

The definition of an NTU bargaining game reflects the idea that smaller coalitions than the grand coalition do not gain from cooperation. They cannot reach higher utility levels as the singleton coalitions for all its members simultaneously. Only in the grand coalition every individual can be made better off. In the further analysis we use the above comprehensive version of an n-person NTU bargaining game.

One solution concept for bargaining games with complete information is that

3. $d \in B$ and there exists at least one element $b \in B$ such that $d \ll a$.

($d \ll b$ if and only if $d_i < b_i$ for all $i = 1, ..., n$. This means that there is a utility allocation in B that gives every player a strictly higher utility than the disagreement point.)
B is called the feasible or decision set and d is called the status quo, conflict or disagreement point.

[3]Billera and Bixby (1973a, Section 4) modeled bargaining games in the same way.

of an asymmetric Nash bargaining solution. To define this solution we take as the set of possible vectors of weights or bargaining powers the strictly positive n-dimensional unit simplex Δ^n_{++}.

Definition 31 (asymmetric Nash bargaining solution)**.** The *asymmetric Nash bargaining solution with a vector of weights* $\theta = (\theta_1, ..., \theta_n) \in \Delta^n_{++}$, for short θ-asymmetric, for a n-person NTU bargaining game (N, V) with disagreement point 0 is defined as the maximizer of the θ-asymmetric Nash product given by $\prod_{i=1}^n u_i^{\theta_i}$ over the set $V(N)$.[4]

Hereby, we consider the symmetric Nash bargaining solution as one particular asymmetric Nash bargaining solution, namely the one with the vector of weights $\theta = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$. Hence, the correct interpretation of "asymmetric" in this sense is "not necessarily symmetric".

As the NTU bargaining game (N, V) is compactly convexly generated, the set $V(N)$ is closed and convex and hence the maximizer above exists. Note that the assumption that the vectors of weights are from Δ^n_{++} instead of $\mathbb{R}^n_{++}$ can be made without loss of generality.

The asymmetric Nash bargaining solution is a well-known solution concept for bargaining games. Similarly to the symmetric Nash bargaining solution the asymmetric Nash bargaining solution satisfies the axioms Invariance to affine linear Transformations, Pareto Optimality and Independence of Irrelevant Alternatives. As for example shown in Roth (1979, p.20), these axioms together with an appropriate asymmetry assumption fixing the vector of weights characterize an asymmetric Nash bargaining solution. Dropping only the Symmetry axiom without making an appropriate asymmetry assumption is not sufficient to characterize the set of asymmetric Nash bargaining solutions. Peters (1992, p.17–25) shows that one needs to consider so called "bargaining solutions corresponding to weighted hierarchies" which include as a special case the asymmetric Nash bargaining solutions.

[4]For bargaining games with a general threat point $d \in \mathbb{R}^n$ the θ-asymmetric Nash product is given by $\prod_{i=1}^n (u_i - d_i)^{\theta_i}$.

4.4.3 Inner Core versus Asymmetric Nash Bargaining Solutions

Having introduced the concept of the inner core and the asymmetric Nash bargaining solution, we investigate the relationship of these concepts for NTU bargaining games. As in NTU bargaining games only the grand coalition can profit from cooperation, looking at the inner core only transfer possibilities within the grand coalition need to be considered. Hereby, it turns out that there is a close connection between the inner core and asymmetric Nash bargaining solutions:

Proposition 13. *Let (N, V) be a n-person NTU bargaining game with disagreement point 0 and generating set $B \subseteq \mathbb{R}^n_{++}$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the θ-asymmetric Nash bargaining solution, a^θ, is in the inner core of (N, V).*
- *For any given inner core point a^θ we can find an appropriate vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$ such that a^θ is the maximizer of the θ-asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$.*

Proof.

"$\Rightarrow$" Suppose a^θ is the θ-asymmetric Nash bargaining solution. To prove that a^θ is in the inner core of (N, V), we need to show that a^θ is in the core of the NTU bargaining game (N, V) and that there exists a transfer rate vector $\lambda^\theta \in \Delta^n_+$ such that a^θ is in the core of the λ^θ-transfer game (N, V_{λ^θ}).

a^θ is the maximizer of the θ-asymmetric Nash product

$$\prod_{i=1}^n u_i^{\theta_i}$$

over $V(N)$. Since there exists at least one $u \gg 0$ in $V(N)$ the θ-asymmetric Nash product is strictly positive and thus a^θ is as well the maximizer of the logarithm of the θ-asymmetric Nash product

$$g(u) = \sum_{i=1}^n \theta_i log(u_i).$$

Since a^θ is the maximizer of the θ-asymmetric Nash product, a^θ is Pareto optimal. Thus, there is no coalition S that can improve upon a^θ. Remember that we are considering bargaining games. Thus in particular no singleton coalition can improve upon a^θ. We conclude that a^θ has to be in the core of the bargaining game (N, V).

Next, we show that a^θ is as well in the core of an appropriately chosen λ-transfer game. The gradient of the function $g(u)$ at a^θ is given by $\frac{\partial g}{\partial x}(a^\theta) = \left(\frac{\theta_1}{a_1^\theta}, ..., \frac{\theta_n}{a_n^\theta}\right)$. We show now, that we have

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \leq \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta$$

for all $x \in V(N)$.[5] To see this, let $x \in V(N)$ and $t \in [0,1]$ and define $x^t = tx + (1-t)a^\theta$. Observe that $x^t \in V(N)$ since $V(N)$ is convex. Now we get using the maximality of a^θ and by applying Taylor's Theorem that

$$0 \geq g(x^t) - g(a^\theta) = (x^t - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}\left(|x^t - a^\theta|^2\right) = t(x - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}(t^2).$$

This means that we have

$$\frac{\partial g}{\partial x}(a^\theta)(x - a^\theta) \leq 0$$

and hence

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \leq \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta.$$

Taking the normalized gradient, defining

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, ..., \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right)$$

and observing that $\lambda^\theta \gg 0$ we obtain that a^θ is in the core of the λ^θ-transfer game (N, V_{λ^θ}).

"$\Leftarrow$" If $a \in \mathbb{R}_+^n$ is some given vector in the inner core of (N, V), then a is in the core of (N, V) and there exists a transfer rate vector $\lambda \in \Delta_+^n$ such that

[5] Compare for the idea of this argument Rosenmüller (2000, p. 549).

a is in the core of the λ-transfer game (N, V_λ). Since a is in the core of the λ-transfer game and the NTU bargaining game (N, V) is compactly generated, we know that λ needs to be strictly positive in all coordinates. Otherwise at least one coalition could still improve upon a. We have $a \gg 0$, because a is in the inner core. If we now take the vector of weights of the asymmetric Nash bargaining solution equal to

$$\theta = (\theta_1, .., \theta_n) = \left(\frac{\lambda_1 a_1}{\sum_{i=1}^n \lambda_i a_i}, ..., \frac{\lambda_n a_n}{\sum_{i=1}^n \lambda_i a_i} \right)$$

then a is the maximizer of the asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$ over $V(N)$. Hereby, similar arguments as in the first step can be used to show that this is the appropriate choice of θ. Hence, a is the asymmetric Nash bargaining solution with weights θ of the bargaining game (N, V).

□

One direction of Proposition 13 can be generalized to the case where the generating set is a subset of $\mathbb{R}^n_+$ but not a subset of $\mathbb{R}^n_{++}$. The set of asymmetric Nash bargaining solutions is always contained in the inner core, but the inner core might be strictly larger then the set of asymmetric Nash bargaining solutions. This can be seen in the following two-player example with disagreement point $(0, 0)$:

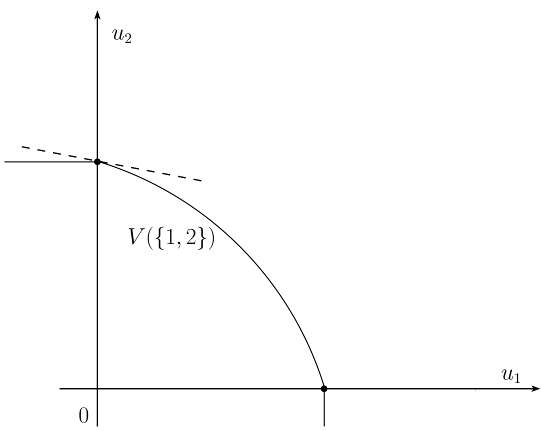


Figure 4.1: Example.

The two points on the axis are in this example in the inner core, as there exits a strictly positive transfer rate vector λ, such that they are in the core of the λ-transfer game. But they cannot result from an asymmetric Nash bargaining solution as any of these solutions chooses only points that are strictly larger than the disagreement point in all coordinates. Thus, the inner core is in this example strictly larger than the set of asymmetric Nash bargaining solutions.

Hence, in general for underlying bargaining sets from $\mathbb{R}^n_+$ and not necessarily from $\mathbb{R}^n_{++}$ Proposition 13 reduces to the following statement:

Proposition 14. *Let (N, V) be a n-person NTU bargaining game with disagreement point 0 and underlying bargaining set from $\mathbb{R}^n_+$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the asymmetric Nash bargaining solution a^θ for θ is in the inner core of (N, V).*

4.5 Application to Market Games

4.5.1 Market Games

In this section we use the result from the preceding section to investigate the relationship between asymmetric Nash bargaining solutions and competitive payoffs of a market that represents the n-person NTU bargaining game. We start by showing that every NTU bargaining game is a market game. Afterwards, we apply the results of Qin (1993) and Brangewitz and Gamp (2011a) to our results from the previous section.

Definition 32 (market)**.** A *market* is given by $\mathcal{E} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$ is the number of commodities,

- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,

- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in a market the number of consumers coincides with the number of producers. Each consumer has his own private production set. This assumption is not as restrictive as it appears to be. A given private ownership economy can be transformed into an economy with the same number of consumers and producers without changing the set of competitive equilibria or possible utility allocations, see for example Qin and Shubik (2009, section 4). Differently from the usual notion of an economy a market is assumed to have concave and not just quasi-concave utility functions.

Let $S \in \mathcal{N}$ be a coalition. The feasible S-allocations are those allocations that the coalition S can achieve by redistributing their initial endowments and by using the production possibilities within the coalition.

Definition 33 (feasible S-allocation)**.** The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i\in S} \middle| x^i \in X^i \text{ for all } i \in S, \sum_{i\in S}(x^i - \omega^i) \in \sum_{i\in S} Y^i \right\}.$$

Hence, an S-allocation is feasible if there exist for all $i \in S$ production plans $y^i \in Y^i$ such that $\sum_{i\in S}(x^i - \omega^i) = \sum_{i\in S} y^i$.

In the definition of feasibility it is implicitly assumed that by forming a coalition the available production plans are the sum of the individually available production plans. This approach is different from the idea to use coalition production economies, where every coalition has already in the definition of the economy its own production possibility set. Nevertheless, a market can be “formally” transformed into a coalition production economy by defining the production possibility set of a coalition as the sum of the individual production possibility sets.

Definition 34 (NTU market game)**.** An NTU game that is representable by a market is a *NTU market game*, this means there exists a market $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i\in N}$ such that $(N, V_{\mathcal{E}}) = (N, V)$ with

$$V_{\mathcal{E}}(S) = \{u \in \mathbb{R}^S | \, \exists \, (x^i)_{i\in S} \in F(S), u_i \leq u^i(x^i), \, \forall \, i \in S\}.$$

For an NTU market game there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible S-allocations in the market or that give less utility than some feasible S-allocation.

In order to show that every NTU bargaining game is a market game we use the following result from Billera and Bixby (1974):

Theorem 14 (2.1, Billera and Bixby (1974))**.** *An NTU game is an NTU market game if and only if it is totally balanced and compactly convexly generated.*

Proposition 15. *Every bargaining game (N, V) is a market game.*[6]

Proof. We show that every bargaining game is totally balanced. Suppose we have an n-person NTU bargaining game. For totally balancedness we need to check that for every coalition $T \subseteq N$ and for all balancing weights

$$\gamma \in \Gamma(e^T) = \left\{ (\gamma_S)_{S \subseteq T} \in \mathbb{R}_+ | \sum_{S \subseteq T} \gamma_S e^S = e^T \right\}$$

we have

$$\sum_{S \subseteq T} \gamma_S V(S) \subseteq V(T).$$

Since the worth each coalition $S \subsetneq N$ can achieve is $V(S) = \{0\} - \mathbb{R}_+$ and since the grand coalition N can achieve $V(N) = B - \mathbb{R}^n_+$ with at least one element $b \in B$ with $b \gg 0$, we have for all $S \subseteq N$ that $V(S) \subseteq V(N)$ holds. Since for all $S \subseteq N$ we have for the balancing weights $0 \leq \gamma_S \leq 1$ and $\sum_{S \subseteq T} \gamma_S e^S = e^T$ the balancedness condition is satisfied. Thus, the bargaining game is totally balanced and hence a market game. □

We now define a competitive equilibrium for a market $\mathcal{E}$.

Definition 35 (competitive equilibrium)**.** A *competitive equilibrium* for a market $\mathcal{E}$ is a tuple

$$\left((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p}\right) \in \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell}_+$$

[6]This result was already observed by Billera and Bixby (1973a, Theorem 4.1). In their proof they define a market representation of a bargaining game with $m \leq n^2$ commodities and nondecreasing utility functions.

such that

(i) $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} (\hat{y}^i + \omega^i)$ (market clearing),

(ii) for all $i \in N$, $\hat{y}^i$ solves $\max_{y^i \in Y^i} \hat{p} \cdot y^i$ (profit maximization),

(iii) and for all $i \in N$, $\hat{x}^i$ is maximal with respect to the utility function u^i in the budget set $\{x^i \in X^i | \hat{p} \cdot x^i \leq \hat{p} \cdot (\omega^i + \hat{y}^i)\}$ (utility maximization).

Given a competitive equilibrium $\left((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p}\right)$ its *competitive payoff vector* is defined as $\left(u^i\left(\hat{x}^i\right)\right)_{i \in N}$.

Qin (1993) investigates the relationship between the inner core of an NTU market game and the set of competitive payoff vectors of a market that represents this game. He establishes, following a conjecture of Shapley and Shubik (1975), the two theorems below analogously to the TU-case of Shapley and Shubik (1975).

Theorem 15 (3, Qin (1993))**.** *For every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector.*

Theorem 16 (1, Qin (1993))**.** *For every NTU market game, there is a market that represents the game and further has the whole inner core as its competitive payoff vectors.*[7]

4.5.2 Results

Now we apply Theorem 3 of Qin (1993) to prove the existence of an economy corresponding to some vector of weights $\theta \in \Delta^n_{++}$, such that the unique competitive payoff vector of this economy coincides with the θ-asymmetric Nash bargaining solution of the n-person NTU bargaining game.

Proposition 16. *Given a n-person NTU bargaining game (N, V) (with disagreement point 0 and generating set from $\mathbb{R}^n_+$) and a vector of weights $\theta \in \Delta^n_{++}$, there is market that represents (N, V) and where additionally the unique competitive payoff vector of this market coincides with the θ-asymmetric Nash bargaining solution a^θ of the NTU bargaining game (N, V).*

[7] A market that satisfies this property is the so called "induced market" introduced by Billera and Bixby (1974). Its definition can be found in Qin (1993).

Proof. (N, V) is a market game by Proposition 15. Moreover, Proposition 13 (or Proposition 14 respectively) shows, that the θ-asymmetric Nash bargaining solution a^θ is an element of the inner core. Thus, we can apply Theorem 3 from Qin (1993). □

The market behind Proposition 16 can be taken from Qin (1993) or Brangewitz and Gamp (2011a) taking necessary monotone transformations of the original game as done in Qin (1993) into consideration. A version of these markets for NTU bargaining games can be found in Appendix 4.7.1 and 4.7.3.

An Alternative Market for Proposition 16

The two markets from Qin (1993) or Brangewitz and Gamp (2011a) have a quite complicated structure. In the following we give a simpler version a market, where strictly positive prices are required. This market is a modification from Brangewitz and Gamp (2011a).

Given a n-person NTU bargaining game (N, V) and a vector of weights $\theta \in \Delta_{++}$. Let a^θ be the θ-asymmetric bargaining solution. From Proposition 13 (or Proposition 14 respectively) we know that the corresponding λ^θ-transfer game is (N, V_{λ^θ})

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Figure 4.2 illustrates as an example for $N = \{1, 2\}$ the sets $V(\{1, 2\})$ and $V_{\lambda^\theta}(\{1, 2\})$ for an NTU bargaining game with disagreement point $(0, 0)$.

Let $z \in V_{\lambda^\theta}(N)$ and $\bar{t}^z = \min \left\{ t \in \mathbb{R}_+ | z - te^N \in V(N) \right\}$. Define the mapping P_θ by $P_\theta : V_{\lambda^\theta}(N) \longrightarrow V(N)$ via $P_\theta(z) = z - \bar{t}^z e^N$. Figure 4.3 illustrates for the same example as in Figure 4.2 the mapping P_θ.

The market for the NTU bargaining game (N, V) and vector of weights θ, denoted by $\mathcal{E}_{V,\theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}_+^n \times \mathbb{R}_+^n \times \{0\} \subseteq \mathbb{R}^{3n}$,

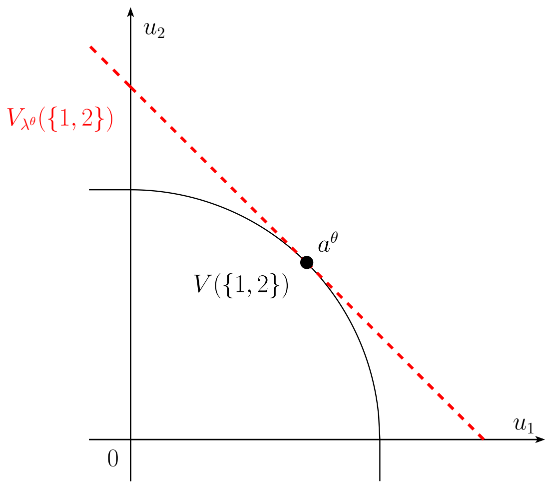


Figure 4.2: Illustration of the sets $V(\{1,2\})$ and $V_{\lambda^\theta}(\{1,2\})$.

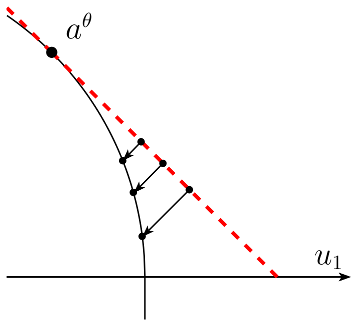


Figure 4.3: Illustration of the mapping P_θ.

- the production set

$$Y^i = convexcone\left[\left(\bigcup_{S\in\mathcal{N}\setminus\{N\}}\left\{\left(0,0,-e^S\right)\right\}\right)\right.$$
$$\left.\bigcup\left(\bigcup_{c\in\left(V_{\lambda^\theta}(N)\cap\mathbb{R}^n_+\right)}\left\{\left(P_\theta(c),c,-e^N\right)\right\}\right)\right]\subseteq\mathbb{R}^{3n},$$

- the initial endowment vector $\omega^i = \left(0,0,e^{\{i\}}\right)$,

- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(2)i}\right)$ where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(2)i}$ and $x_j^{(2)i}$.

It can be shown using the arguments of Brangewitz and Gamp (2011a) that the market $\mathcal{E}_{V,\theta}$ represents the NTU bargaining game (N, V) and has as its unique competitive equilibrium payoff vector (assuming strictly positive equilibrium price vectors) the θ-asymmetric Nash bargaining solution a^θ. For the method of proof and the details we refer to Brangewitz and Gamp (2011a). Here, we only state how the competitive equilibria of the market $\mathcal{E}_{V,\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, \left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta, 2\ \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\theta}$.

Considering NTU bargaining games as NTU market games there is a market such that the same sets of utility allocations are reachable in the game and the market. Proposition 16 shows that in the class of markets representing a given NTU bargaining game there is a market that has a given asymmetric Nash bargaining solution (with a fixed vector of weights) as its unique competitive payoff vector. We establish a link between utility allocations coming from asymmetric Nash bargaining in NTU bargaining games and payoffs arising from competitive equilibria in certain markets. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions. Instead of considering non-cooperative games to give foundations of cooperative solutions, we link cooperative behavior described by asymmetric Nash bargaining with competitive behavior in markets.

In addition a similar interpretation holds true for the whole inner core and certain of its subsets. Combining Proposition 13 with Theorem 1 of Qin (1993) we obtain:

Proposition 17. *Let* (N,V) *be a* n*-person NTU bargaining game with disagreement point* 0 *and generating set from* $\mathbb{R}^n_{++}$*. Then there is market that represents* (N,V) *and where additionally the set of asymmetric Nash solutions of* (N,V) *coincides with the set of competitive payoff vectors of the market.*

Proof. (N,V) is a market game by Proposition 15 and the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core of (N,V) by Proposition 13. Thus, we can apply Theorem 1 of Qin (1993). □

The two results of Qin (1993) we use above represent two extreme cases. On the one hand he uses the whole inner core and on the other hand he uses only one single point from the inner core. Brangewitz and Gamp (2011a) show how the results of Qin (1993) can be extended to a large class of closed subsets of the inner core. Using their results we obtain:

Proposition 18. *Given a* n*-person NTU bargaining game* (N,V) *(with disagreement point* 0 *and generating set from* $\mathbb{R}^n_+$*) and a closed set* $\Theta \subset \Delta^n_{++}$ *of strictly positive vectors of weights. Moreover, assume that every* θ*-asymmetric Nash bargaining solution* a^θ *with vector of weights* $\theta \in \Theta$ *can be strictly separated from the set* $V(N) \setminus \{a^\theta\}$*.*[8] *Then there is market that represents the NTU bargaining game* (N,V) *and the set of competitive payoff vectors of this market coincides with the set of* θ*-asymmetric Nash bargaining solutions with vectors of weights* $\theta \in \Theta$*,* $\{a^\theta | \theta \in \Theta\}$*, of the NTU bargaining game* (N,V)*.*

Proof. (N,V) is a market game by Proposition 15. Moreover, Proposition 13 (or Proposition 14 respectively) shows, that the θ-asymmetric Nash bargaining

[8] More details concerning this assumptions and how they might be weakened can be found in Brangewitz and Gamp (2011a).

solution with a vector of weights $\theta \in \Delta^n_{++}$ is an element of the inner core. Furthermore, note that the set of vectors of weights Θ is assumed to be closed. If we take now as a parameter the vectors of bargaining weights θ and consider the function that associates to every vector of weights θ the θ-asymmetric Nash bargaining solution a^θ, we observe that this function is continuous in θ.[9] Moreover, as continuous functions map compact sets into compact sets, we know that if we take a closed set of vectors of weights Θ that the set of θ-asymmetric Nash bargaining solutions $\{a^\theta | \theta \in \Theta\}$ is closed. Therefore, the assumptions in Brangewitz and Gamp (2011a) are satisfied and their result can be applied. $\square$

Proposition 17 can be regarded as the other extreme case in contrast to the result in Proposition 16. Knowing that competitive payoff vectors are under weak assumptions always in the inner core (compare de Clippel and Minelli (2005), Brangewitz and Gamp (2011a)), in the class of markets representing a game the market behind Proposition 17 is the market with the largest set of possible competitive payoff vectors.

Proposition 18 has the following interpretation: If the vector of weights or interpreted differently the bargaining power is not exactly known, then as an approximation using Proposition 18 we obtain the coincidence of the set of asymmetric Nash bargaining solutions with a closed subset of weight vectors and the set of competitive payoff vectors of a certain market.

4.6 Concluding Remarks

The results above show that asymmetric Nash bargaining solutions as solution concepts for bargaining games are linked via the inner core to competitive payoff vectors of certain markets. Thus, our result can be seen as a market foundation of the asymmetric Nash bargaining solutions. This result holds for bargaining games in general as any asymmetric Nash bargaining solution is always in the inner core (Proposition 14). The idea of a market foundation parallels the one that is used in implementation theory. Here, rather than giving a non-cooperative

[9] To see this we use Theorem 2.4 of Fiacco and Ishizuka (1990) applied to maximization problems.

foundation for solutions of cooperative games, we provide a market foundation. Our result may be seen as an existence result.

Another interesting related line of research, that we do not follow here, is to consider the recent definition of Compte and Jehiel (2010) of the coalitional Nash bargaining solution. They consider cooperative games with transferable utility (TU) and define the coalitional Nash bargaining solution as the point in the core that maximizes the Nash product (with equal weights). Thus, using Theorem 2 of Shapley and Shubik (1975) for TU market games, where they define for any given core point a market that has this point as its unique competitive payoff vector, gives a market foundation as well for the symmetric coalitional Nash bargaining solution by choosing the symmetric coalitional Nash bargaining solution as this given core point. It seems interesting to study how this idea can be generalized for asymmetric coalitional Nash bargaining solutions or for (asymmetric) coalitional Nash bargaining solutions for NTU games.

Our approach parallels the one in Trockel (1996, 2005). Trockel (1996) is based on a direct interpretation of a n-person bargaining game as an Arrow-Debreu economy with production and private ownership, a so called bargaining economy. He shows that, given a bargaining economy, the consumption vector of the unique stable Walrasian equilibrium coincides with the asymmetric Nash bargaining solution with the vector of weights corresponding to the shares in the production of the bargaining economy. The main difference between our result and his is that Trockel (1996) did not consider markets in the sense of Billera and Bixby (1974) or Qin (1993) and thus his bargaining economies do not constitute the kind of market representation as defined in Billera and Bixby (1974) or Qin (1993). Similarly Trockel (2005) uses coalition production economies to establish a core equivalence of the Nash bargaining solution. By using the markets of Qin (1993) we obtained a market foundation of the asymmetric Nash bargaining solution. This can be seen as a link between the literature on market games (as in Billera and Bixby (1974), Qin (1993)) and the ideas of Trockel (1996, 2005).

4.7 Appendix

4.7.1 The Market behind Proposition 16 from Qin (1993)

Qin (1993) considers NTU games in general and does not restrict his attention to NTU bargaining games. The market behind Proposition 16 from Qin (1993) has a simpler structure if we restrict our attention to NTU bargaining games. The difference lies in the description of the private production sets.

To show his result Qin (1993) modifies the given NTU game by applying a strictly monotonic transformation to the utility functions. This allows him to assume that the given inner core point can be strictly separated in the modified NTU game. Qin (1993) shows that this market represents the modified game and that the given inner core point is the unique competitive payoff vector of this economy. By applying the inverse strictly monotonic transformation to the utility functions he obtains his result. As we do not want to restrict our attention to bargaining games with strictly convex generating sets, a similar transformation need to be applied to the NTU bargaining game to use the market defined below.

The transformed bargaining game is denoted by $(N, \bar{V})$ with generating set $\bar{C}^N$. Define for the grand coalition N the following sets

$$
\begin{aligned}
A_N^1 &= \left\{ \left(u^N, -e^N, -e^N, -e^N, 0\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\
A_N^2 &= \left\{ \left(u^N, 0, -e^N, 0, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\
A_N^3 &= \left\{ \left(u^N, 0, 0, -e^N, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n},
\end{aligned}
$$

and for the remaining coalitions

$$
\begin{aligned}
A_S^1 &= \left\{ \left(0, -e^S, -e^S, -e^S, 0\right) \right\} \subseteq \mathbb{R}^{5n}, \\
A_S^2 &= \left\{ \left(0, 0, -e^S, 0, -e^S\right) \right\} \subseteq \mathbb{R}^{5n}, \\
A_S^3 &= \left\{ \left(0, 0, 0, -e^S, -e^S\right) \right\} \subseteq \mathbb{R}^{5n},
\end{aligned}
$$

Let $\theta \in \Theta$ be a given vector of weights and a^θ the θ-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Let $\mathcal{E}_{\bar{V},\theta} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ be the market with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{(0,0,0)\} \times \mathbb{R}_+^n \subseteq \mathbb{R}_+^{5n}$,
- the production set $Y^i = convexcone \left[\bigcup_{S \subseteq N} \left(A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$,
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}\right) \in \mathbb{R}_+^{5n}$,
- the utility function $u^i(x^i) = \min \left\{ x_i^{(1)i}, \frac{\sum_{j=1}^n \lambda_j^\theta a_j^\theta x_j^{(5)i}}{\lambda_i} \right\}$ where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(5)i}$ and $x_j^{(5)i}$.

Qin (1993) shows that the market $\mathcal{E}_{\bar{V},\theta}$ represents the modified NTU bargaining game $(N, \bar{V})$ and has as its unique competitive payoff vector a^θ, a given inner core point. For the method of proof and the details we refer to Qin (1993). Here, we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_{\bar{V},\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, 0, 0, e^{\{i\}} \right) \right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n} \left(a^\theta, -e^N, -e^N, -e^N, 0 \right) \right) \right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{1}{3} \left(\lambda^\theta \circ a^\theta \right), \frac{1}{3} \left(\lambda^\theta \circ a^\theta \right), \frac{1}{3} \left(\lambda^\theta \circ a^\theta \right), \lambda^\theta \circ a^\theta \right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute the unique competitive equilibrium in the market $\mathcal{E}_{\bar{V},\theta}$.

4.7.2 The Market behind Proposition 17 from Qin (1993)

Similarly to Proposition 16 the market behind Proposition 17 from Qin (1993), called the induced market of an NTU market game, simplifies for NTU bargaining games to:

Definition 36 (induced market)**.** Let (N,V) be NTU bargaining game. The *induced market* of the game (N,V) is defined by $\mathcal{E}_V = (X^i, Y^i, u^i, \omega^i)_{i\in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{2n}$,

- the production set

$$Y^i = convexcone\left[\bigcup_{S\in\mathcal{N}\setminus N} \{(0,-e^S)\} \cup \left(C^N \times \{-e^N\}\right)\right] \subseteq \mathbb{R}^{2n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}\right)$,

- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i(x^i) = x_i^{(1)i}$
  where $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate.

Qin (1993) shows that the market $\mathcal{E}_V$ represents the NTU bargaining game (N,V) and has as its set of competitive payoff vectors the whole inner core. For the method of proof and the details we refer to Qin (1993). Here, we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_V$ look like:

Let $\theta \in \Theta$ be a given vector of weights and a^θ the θ-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}\right).$$

The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_V$.

4.7.3 The Market behind Proposition 18

Similarly to Proposition 16 and Proposition 17 the market behind Proposition 18 from Brangewitz and Gamp (2011a), called the induced A-market of an NTU market game, can be simplified for NTU bargaining games (under the assumptions of Proposition 18). For $\theta \in \Theta$ define

$$\lambda^{\theta} = \left(\frac{\frac{\theta_1}{a_1^{\theta}}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^{\theta}}}, \ldots, \frac{\frac{\theta_n}{a_n^{\theta}}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^{\theta}}} \right).$$

Let $(N, \tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap\limits_{\theta \in \Theta} \left\{ u \in \mathbb{R}^n | \lambda^{\theta} \cdot u \leq \lambda^{\theta} \cdot a^{\theta} \right\} & \text{if } S = N \end{cases}$$

where a^{θ} denotes the θ-asymmetric Nash bargaining solution.

Define the mapping $P_{\Theta} : \tilde{V}(N) \longrightarrow V(N)$ via

$$P_{\Theta}(x) = x - \bar{t}^{x} e^{N}.$$

Define

$$\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| \exists t \in \mathbb{R}_+ \text{ such that } z - t e^N \in C^N \right\}.$$

Then we also have $\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| z - \bar{t}^{z} e^N \in C^N \right\}$.

For the definition of the production sets define for all coalitions $S \in \mathcal{N} \setminus \{N\}$

$$\begin{aligned} A_S^1 &= \left\{ \left(0, -e^S, 0, -e^S, -e^S\right) \right\}, \\ A_S^2 &= \left\{ \left(0, 0, 0, -e^S, 0\right) \right\}, \\ A_S^3 &= \left\{ \left(0, 0, 0, 0, -e^S\right) \right\} \end{aligned}$$

and for the grand coalition N define

$$A_N^1 = \left\{ \left(P_{\Theta}\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N \right) | \tilde{c}^N \in \tilde{C}^N \right\},$$

$$A_N^2 = \left\{ \left(P_\Theta\left(\tilde{c}^N\right), 0, \tilde{c}^N, -e^N, 0 \right) | \tilde{c}^N \in \tilde{C}^N \right\},$$
$$A_N^3 = \left\{ \left(P_\Theta\left(\tilde{c}^N\right), 0, \tilde{c}^N, 0, -e^N \right) | \tilde{c}^N \in \tilde{C}^N \right\}.$$

The market $\mathcal{E}_{V,\Theta}$ using the closed set of weights Θ of the NTU bargaining game is defined by

$$\mathcal{E}_{V,\Theta} = (X^i, Y^i, u^i, \omega^i)_{i \in N}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{0\} \times \mathbb{R}_+^n \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- the production set $Y^i = \mathit{convexcone} \left[\bigcup_{S \in \mathcal{N}} \left(A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with

$$u^i\left(x^i\right) = \min \left(x_i^{(1)i}, x_i^{(3)i} + \varepsilon \sum_{j \neq i} x_j^{(3)i} \right)$$

  where ε is chosen such that $\varepsilon < \lambda_i^\theta = \frac{\lambda_i^\theta}{1} \leq \frac{\lambda_i^\theta}{\lambda_j^\theta}$ for all $\theta \in \Theta$ and $x^{(1)i}$ denotes the first group of n goods of x^i and $x_j^{(1)i}$ its j^{th} coordinate; similarly $x^{(3)i}$ and $x_j^{(3)i}$.

Using Brangewitz and Gamp (2011a) it can be shown that the market $\mathcal{E}_{V,\Theta}$ represents the NTU bargaining game (N, V) and its set of competitive equilibrium payoff vectors coincides with the set $\{a^\theta | \theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011a).

The competitive equilibria of the market $\mathcal{E}_{V,\Theta}$ are of the following form: Let $\theta \in \Theta$ be the vector of weights and a^θ the θ-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, \left(a^\theta\right)^{\{i\}}, 0, 0 \right) \right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, a^\theta, -e^N, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right)\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\Theta}$.

In addition to the market $\mathcal{E}_{V,\Theta}$ Brangewitz and Gamp (2011a) define a market where the set of payoff vectors of competitive equilibria with a strictly positive equilibrium price vectors coincides with the set $\{a^\theta | \theta \in \Theta\}$. This market, denoted by $\mathcal{E}^0_{V,\Theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{4n}$,
- the production set

$$\begin{aligned} Y^i = convexcone &\left[\left(\bigcup_{S\in\mathcal{N}\setminus\{N\}} \left\{\left(0, -e^S, 0, -e^S\right)\right\}\right)\right. \\ &\left.\cup \left(\bigcup_{\tilde{c}^N \in \tilde{C}^N} \left(P_\Theta\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right)\right)\right] \subseteq \mathbb{R}^{4n}, \end{aligned}$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(3)i}\right)$.

Similarly as for the market presented before, it can be shown using Brangewitz and Gamp (2011a) that the market $\mathcal{E}^0_{V,\Theta}$ represents the NTU bargaining game (N, V) and its set of competitive equilibrium payoff vectors with strictly positive prices coincides with the set $\{a^\theta | \theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011a). Here, we only state how the competitive equilibria of the market $\mathcal{E}^0_{V,\theta}$ look like:

Let $\theta \in \Theta$ be the vector of weights and a^θ the θ-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, \left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, a^\theta, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta \circ a^\theta, \lambda^\theta, \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}^0_{V,\Theta}$.

Chapter 5

Asymmetric Nash Bargaining Solutions and perfect Competition

5.1 Introduction

The idea of this paper is to study the compatibility of competitive equilibria with concepts of bargaining theory and in particular with the asymmetric Nash bargaining solution. We consider a pure exchange economy and study this economy on the one hand with means of general equilibrium theory and on the other hand with means of cooperative bargaining theory. It turns out that sets of competitive equilibrium allocations and of allocations resulting from an asymmetric Nash bargaining solution coincide as long as one restricts attention to economies where agents have homogeneous (of degree 1) utility functions and where the initial endowments are proportionally distributed. We also study what happens when these assumptions are relaxed or changed. Our result holds as well if the agents have utility functions that are homogeneous of the same degree k with $0 < k \leq 1$.

It is well known that in general there does not exist a cooperative solution that always yields the same allocations as the competitive equilibrium. Sertel and Yildiz (2003) show in the context of pure exchange economies interpreted as bargaining games "that there are distinct exchange economies whose Walrasian equilibrium welfare payoffs disagree but which define the same bargaining problem and should have hence determined the same bargaining solution and its payoffs." In the economies, that they consider, agents have utility functions that are not homogeneous. Furthermore, the endowments are not proportionally distributed. Therefore, the results of Sertel and Yildiz (2003) just show the impossibility of a Walrasian bargaining solution in a very general setup. Under more restrictive conditions it is possible to give a bargaining solution that yields the same allocation as Walrasian equilibria. John (2005) considers economies with linear utility functions and proportionally divided endowments. In this situation a certain asymmetric Nash bargaining solution yields exactly competitive equilibrium allocations. Moulin (2003) mentions only in passing that there should exist a version of the result of John (2005) in the context of homogeneous utility functions. Our work offers such a more general version of the results of John (2005) but also demonstrates the limitations of the approach. The result is not robust and already with other utility representations of the same preferences the implications of the result do not hold anymore.

Comparing economies and games it is well known that payoffs of utility allocations generated by competitive equilibria are in the core of the game induced by the economy. De Clippel and Minelli (2005) even show that payoffs of competitive equilibrium allocations are under mild conditions not only in the core, but even in the inner core, a refinement of the core. But this result can be regarded as being sharp. For example Qin (1993) shows for markets, a certain class of economies with production, that every single point in the inner core of a so called market game can be the payoff of an equilibrium of some economy inducing this game. So, payoffs of utility allocations generated by competitive equilibria can be mainly anything within the inner core. This conclusion is no more correct under the restrictive assumption of homogeneous utility functions. Our result illustrates that, given these assumptions, payoffs of competitive equilibria are not only some subset of the inner core but are always a certain point in the inner core - the point that the asymmetric Nash bargaining solution chooses.

There is another branch of literature studying the relation of competitive equilibria and the (asymmetric) Nash bargaining solution. Trockel (1996, 2000) introduces an alternative approach and interprets in an NTU-context bargaining games directly as Arrow-Debreu or as coalition production economies. He shows that the unique equilibrium of such an economy coincides with the asymmetric Nash bargaining solution of the underlying game where the weights of the bargaining solution correspond to the shares in production. One difference to other literature is that he uses a stylized models with outputs in the production without requiring inputs. Also Brangewitz and Gamp (2011b) study the relation of competitive equilibria and the asymmetric Nash bargaining solution. They show that given a bargaining game there exists a market that represents this game and the utility allocations given by competitive equilibria coincide with those utility allocations given by the Nash bargaining solution. Comparing Trockel (1996, 2000) and Brangewitz and Gamp (2011a) with the results of this work the main difference is that Trockel (1996, 2000) as well as Brangewitz and Gamp (2011a) start with a cooperative game and look at certain induced economies. The competitive equilibrium allocations in those economies coincide with the allocations generated by an asymmetric Nash bargaining solution. Here we start with an economy and compare this result with an induced bargaining game.

Ervig and Haake (2005) also compare economies and bargaining games. They show that in their model the payoffs of competitive equilibria coincide with payoffs resulting from asymmetric versions of the Perles-Maschler bargaining solution. The main reason for their different result is that they restrict consumer demand by the total endowments of the economy.

One aim of this work is to clarify the relation of the articles of Sertel and Yildiz (2003) and John (2005) with results of Chipman and Moore (1979) and Polterovich (1975). Chipman and Moore (1979) discuss the relation of individual demand, aggregate demand and social welfare functions. They consider in particular the question whether the market demand function can be seen as the demand function of some representative consumer. Hereby, they also consider homothetic preferences and homogeneous utility functions. The main difference to our result is that they do not take the viewpoint of cooperative game theory and do not mention the relation to the asymmetric Nash bargaining solution. As they consider the preferences as the data of the model - and not the utility functions like it is typically done in cooperative game theory - they do not study problems arising from different utility representations of the same preferences or from the unusual choice of the status quo point. Polterovich (1975) also considers pure exchange economies. He introduces a concept that maps sets of utility functions on feasible allocations. So, his concept is similar to a social choice rule. It turns out that this concept is related to competitive equilibria and to the Nash bargaining solution. Furthermore, he proves a result about aggregate demand that is very close to the result of Chipman and Moore (1979).

5.2 Basic definitions

5.2.1 Economies

We consider economies with n consumers $i = 1, ..., n$ and m commodities $j = 1, ..., l$ where we also use the notation $I = \{1, ..., n\}$ and $J = \{1, ..., l\}$. An economy is a tuple $\left(\left((X^i, u^i)_{i=1}^n\right), e\right)$. $X^i = \mathbb{R}_+^l$ is the consumption set of consumer i. Each consumer is described by a utility function

$$u^i : X^i \longrightarrow \mathbb{R} \tag{5.1}$$

which is weakly increasing, locally nonsatiated, concave, continuous and homogeneous of degree 1.

The total endowments of an economy are given by the commodity vector $e = (e_j)_{j=1}^l \in \mathbb{R}_{++}^l$. Denote with E the set of all economies satisfying these properties.

Denote for $a, b \in \mathbb{R}^l$ with $a \cdot b$ the scalar product of a and b, i.e. $a \cdot b = \sum_{i=1}^l a_i b_i$. An allocation $x = (x^i)_{i=1}^n \in \bigtimes_{i=1}^n X^i$ is feasible if it satisfies the inequality $\sum_{i=1}^n x^i \leq e$. Denote with $A \subset \bigtimes_{i=1}^n X^i$ the set of feasible allocations.

To study competitive equilibria of these economies we will have to specify which agent is endowed with which amount of goods. We will focus on the case that each agent is endowed with some fraction of the total endowments meaning that for each $i \in I$ there exists some $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ such that the endowments of agent i are given by $e^i = \alpha_i e$. Now we can look at the allocations that we obtain by applying the concept of the competitive equilibrium to this situation. We focus on allocations given by competitive equilibria. Therefore, we use the following definition:

Definition 37. An allocation $\bar{x} = \left(\bar{x}^i\right)_{i=1}^n \in A$ is called a Walras allocation with respect to (the ownership or income distribution) α if there exists a price vector $p = (p_j)_{j=1}^l \in \mathbb{R}_+^l \setminus \{0\}$ such that

$$\begin{aligned}
&i) \quad \textit{For } i = 1, ..., n: \quad \bar{x}^i \textit{ maximizes } u^i\left(x^i\right) \textit{ subject to} \\
&\qquad\qquad\qquad\qquad\quad x_j^i \geq 0 \textit{ for all } i = 1, ..., n, j = 1, ..., l, \\
&\qquad\qquad\qquad\qquad\quad p \cdot x^i \leq p \cdot (\alpha_i e) \textit{ for all } i = 1, ..., n \\
&ii) \quad \sum_{i=1}^n \bar{x}^i \leq e \textit{ and } p \cdot \left(\sum_{i=1}^n \bar{x}^i - e\right) = 0.
\end{aligned}$$

5.2.2 Economies as bargaining games

Another approach to model the situation above is to use cooperative game theory and in particular bargaining theory. One first analyzes which allocations of utility are feasible in these economies and uses this to define an induced bargaining game. In a second step one can apply one of the solution concepts described in the literature about bargaining games. We start with defining the relevant concepts:

Definition 38. An n-player bargaining game is a pair $V = (U, d)$ with the following properties:

1. $U \subset \mathbb{R}^n,\ d \in U$,
2. U is convex and closed,
3. $U_d = \{x \in U \big| x \geq d\}$ is bounded,
4. U is comprehensive, i.e. $x \in U$ and $z \leq x$ implies $z \in U$.

Hereby, the status quo point $d \in \mathbb{R}^n$ is describing the utilities of the agents if they do not agree to cooperate. If all agents agree to cooperate they are able to achieve any of the distributions of utilities described by the set U.
A bargaining solution some class of bargaining games $\mathcal{U}^0$ is a mapping φ that maps every bargaining game from $\mathcal{U}^0$ to $\mathbb{R}^n$ and that satisfies

1. φ is feasible, i.e. $\varphi(U, d) \in U$,
2. φ is individually rational, i.e. $\varphi(U, d) \geq d$,
3. φ is Pareto efficient, i.e. $\varphi(U, d)$ is Pareto efficient in U.

The (symmetric) Nash bargaining solution is defined as the maximizer of the product of the utilities over the individual rational bargaining set or as the unique solution that satisfies the following axioms: Invariance to positive affine linear Transformations, Pareto Optimality, Symmetry and Independence of Irrelevant Alternatives. An asymmetric version of the Nash bargaining solution can be defined as the maximizer of an accordingly weighted Nash product.

Definition 39. Let $\alpha = (\alpha_i)_{i=1}^n$ be a vector of weights, i.e. $\alpha_i > 0$ for $i = 1, ..., n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Then the α-asymmetric Nash bargaining solution is the bargaining solution that maps a bargaining game (U, d) to the (unique) maximizer of the the function

$$U^{\alpha}(u) = \prod_{i=1}^{n} (u_i - d_i)^{\alpha_i} \tag{5.2}$$

over the set U.

Concerning the axiomatization this means that the Symmetry axiom is replaced by an appropriate Asymmetry axiom, see for example Roth (2008, Theorem 3). Then, the α-asymmetric Nash bargaining solution is characterized through the axioms Covariance with affine linear transformations of utility, Independence of Irrelevant Alternatives and Individual rationality in addition to an condition describing the solution on symmetric hyperplane games. Hereby, we observe for the case $\alpha = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ the well known (symmetric) Nash bargaining solution as a special case of the asymmetric bargaining solution. Hence, the expression "asymmetric Nash bargaining solution" should be understood as a specific not necessarily symmetric version of the Nash bargaining solution.

Now, we can consider bargaining games induced by pure exchange economies. Define a mapping U from the set of economies E to $\mathcal{P}\left(\mathbb{R}^n_+\right)$. Given an economy $f = \left((u_i,)_{i=1}^n, e\right)$ the set

$$U(f) := \left\{u \in \mathbb{R}^n \middle| \exists x \in A : u_i \leq u^i(x^i) \text{ } for \text{ } all \text{ } i \in I\right\} \subseteq \mathbb{R}^n$$

consists of all feasible utility vectors for this economy. Now we can define a bargaining game induced by an economy:

Definition 40 (induced bargaining game)**.** The bargaining game induced by an economy $f \in E$ is the pair $(U(f), 0)$.

By the properties of f it is obvious that $U(f)$ is compactly, convexly generated and comprehensive and furthermore $0 \in U(f)$. Hence, $(U(f), 0)$ has the usual properties required for a bargaining problem with status-quo point $0 \in \mathbb{R}^n_+$.

Now, we can apply the α-asymmetric Nash bargaining solution to this bargaining game. To be able to compare the utility allocations given by the α-asymmetric Nash bargaining solution with Walras allocations with respect to α we translate utility allocations to allocations of goods leading to the utilities. Thus, define the mapping $\tilde{U}^{\alpha}$ from the set of allocations to the reals defined by

$$\tilde{U}^{\alpha}(x) = \prod_{i=1}^{n} \left(u^i\left(x^i\right)\right)^{\alpha_i}.$$

Definition 41. A feasible allocation $\bar{x} \in A$ is called a Nash allocation with respect to α if it maximizes $\tilde{U}^{\alpha}$ on the set of all feasible allocations, i.e. if $\bar{x}$ is a solution to

$$\begin{aligned} \max \tilde{U}^{\alpha}(x) \quad & \textit{subject to} \\ & x_j^i \geq 0 \textit{ for all } i = 1, ..., n \textit{ and } j = 1, ..., l \\ & \sum_{i=1}^{m} x_j^i - e_j \leq 0 \textit{ for all } j = 1, ..., l \end{aligned}$$

Now, the following Lemma shows that this definition is the "correct" definition.

Lemma 5. *A feasible allocation $x = \left(x^i\right)_{i=1}^n$ is a Nash allocation with respect to α if and only if the vector of utilities $\left(u^i\left(x^i\right)\right)_{i=1}^n$ is the α-Nash bargaining solution of the bargaining game $(U(f), 0)$.*

Proof. Lemma 5 follows directly from the definitions. □

5.2.3 A generalization of Chipman and Moore (1979)

The results of Chipman and Moore (1979) will play a crucial role in the proof of the main result. We present a modified and more general version of their result. Hereby, we denote with $G^i\left(p, \omega^i\right)$ the Walrasian demand correspondence of agent i at price $p \in \mathbb{R}_+^l$ if he is endowed with the commodity bundle $\omega^i \in \mathbb{R}_{++}^l$; denote with $G(p, \omega)$ the set of maximizers of the function $\tilde{U}^{\alpha}$ on the set $\left\{x \in \bigtimes_{i=1}^{n} X^i \middle| \sum_{i=1}^{n} p \cdot x^i \leq p \cdot \omega\right\}$.

Theorem 17. *If there exists $k > 0$ such that for each agent i his utility function u^i is homogeneous of degree k and weakly increasing and if the income shares are fixed to α ($\alpha_i \geq 0\ \forall i, \sum \alpha_i = 1$), then for any endowment vector $\omega \in \mathbb{R}^l_+$ we have*

$$G(p,\omega) = \sum_{i=1}^{n} G^i(p,\alpha_i\omega).$$

Proof. For the case $k = 1$ this Theorem is just a reformulation of Theorem 4.2 of Chipman and Moore (1979) and hence follows directly from their result. While Chipman and Moore (1979) denote their result in terms of Marshallian demand we use the notation in terms of Walrasian demand here. This does not make a crucial difference as for any price $p \in \mathbb{R}^l_+$ we have that agent i owns the fraction α_i of the total wealth.

For the case $k \neq 1$ the proof of Chipman and Moore (1979) can be adapted by using the following two results.

First, if a utility function u^i is homogeneous of degree k (and satisfies the other assumptions) then the corresponding indirect utility function is of the form

$$V_i(p, p \cdot (\alpha_i\omega)) = \frac{(p \cdot (\alpha_i\omega))^k}{\gamma_i(p)}$$

for a suitable function $\gamma_i : \mathbb{R}^l_+ \longrightarrow \mathbb{R}_+$. This follows directly from the fact that for homothetic preferences the income elasticity of the demand is equal to 1.

Second, one considers the maximization of the product

$$\prod_{i=1}^{n} \left(\frac{(d_i(p \cdot \omega))^k}{\gamma_i(p)} \right)^{\alpha_i} = \frac{(p \cdot \omega)^k}{\prod_{i=1}^{n} \gamma_i(p)^{\alpha_i}} \prod_{i=1}^{n} d_i^{\alpha_i k}$$

as a function of $(d_1, ..., d_n)$ with respect to the constraint $d_i \geq 0, \sum d_i = 1$. Following the proof of Chipman and Moore (1979) the crucial step is to show that setting $d_i = \alpha_i$ for all i maximizes this product.

Obviously, any solution satisfies $d_i > 0$. Removing constant factors and applying the monotone transformations $\sqrt[k]{\ }$ and ln to the target function we obtain the equivalent maximization problem

$$\max \sum_{i=1}^{n} \alpha_i \ln (d_i) \tag{5.3}$$

$$\text{w.r.t. } \sum d_i = 1. \tag{5.4}$$

Using the Lagrange approach we get that for a maximizer of this problem $\hat{d}_1, ..., \hat{d}_n$ there exists $\lambda \geq 0$ such that for all i

$$\frac{\alpha_i}{\hat{d}_i} = \lambda$$

and

$$\sum \hat{d}_i = 1.$$

It is easy to see that $\lambda = 1$ and $\hat{d}_i = \alpha_i$ is the unique solution to this system of equations.
Hence, $\hat{d}_i = \alpha_i$ is the unique solution for this maximization problem.

Together with Theorems 3.8 and 3.9 from Chipman and Moore (1979) this completes the proof. □

This result shows that under the given assumptions the function $\tilde{U}^{\alpha}$ can be regarded as the utility function of a representative consumer. While Chipman and Moore (1979) denote their result in terms of Marshallian demand we use the notation in terms of Walrasian demand here. Furthermore, compared with the version of Chipman and Moore (1979) this version is more general as it includes the case that utility functions are homogeneous of degree k. Hereby, one should be aware of the fact that the utility functions of all agents have to be homogeneous of the same degree.

5.3 Results

5.3.1 The main results

Proposition 19. *An allocation $\bar{x} = \left(\bar{x}^i\right)_{i=1}^{m}$ is a Nash allocation with respect to α if and only if it is a Walras allocation with respect to α.*

Proof. 1. Assume that $\bar{x}$ is a Walras allocation with respect to α. By the definition of a Walras allocation there exists a price vector $p \in \mathbb{R}^l_+ \setminus \{0\}$ such that $(p, \bar{x})$ is an α-Walrasian equilibrium. By definition this means that for each i the vector $\bar{x}^i$ maximizes $u^i\left(x^i\right)$ subject to $x^i \geq 0$ and $p \cdot x^i \leq p \cdot (\alpha_i e)$. By Theorem 17 this implies that $\bar{x}$ is a maximizer of the function

$$\tilde{U}^\alpha(x) = \prod_{i=1}^n u^i\left(x^i\right)^{\alpha_i}$$

on the set

$$B^p = \left\{ x \in \bigtimes_{i=1}^n X^i | \sum_{i=1}^n p \cdot x^i \leq p \cdot e \right\}.$$

Notice that

$$A \subseteq B^p$$

and that

$$\bar{x} \in A.$$

Hence, $\bar{x}$ also maximizes $\tilde{U}^\alpha(x)$ on the set A. So, $\bar{x}$ is a Nash allocation with respect to α.

2. Let $\bar{x} = (\bar{x})_{i=1}^n$ be a Nash allocation with respect to α. By the definition of Nash allocations the allocation $\bar{x}$ maximizes $\tilde{U}^\alpha(x) = \prod_{i=1}^n u^i\left(x^i\right)^{\alpha_i}$ on the set A. Notice that the function U^α is quasiconcave and that for each $i \in I$ the utility function u^i is concave.

   Therefore the set

$$T = \left\{ z \in \mathbb{R}^l_+ | \exists x^i \in X^i,\ i = 1, ..., n,\ \textit{such that} \sum x^i = z,\ \tilde{U}^\alpha\left(x^1, ..., x^n\right) > \tilde{U}^\alpha\left(\bar{x}\right) \right\}$$

   is convex. Furthermore, we observe that $e \notin T$. By the separating Hyperplane Theorem there exists an vector $p \in \mathbb{R}^l$ with $p \neq 0$ such that $p \cdot y > p \cdot e)$ for all $y \in T$.

   As the utility functions are weakly increasing and locally nonsatiated and as the function U^α is strictly increasing , we observe that for all $\underline{x} \in A$ with $\underline{x} > \bar{x}$ we have $\tilde{U}^\alpha(\underline{x}) > \tilde{U}^\alpha(\bar{x})$. It follows that $p \geq 0$ holds.

   Thus, for any allocation $\hat{x} = (\hat{x})_{i=1}^n$ with $\tilde{U}^\alpha\left(\hat{x}\right) > \tilde{U}^\alpha\left(\bar{x}\right)$ we have that

$p \cdot \left(\sum \hat{x}^i\right) > p \cdot e$. Furthermore, we observe $p \cdot \left(\sum_{i=1}^{n} \bar{x}^i\right) \leq p \cdot e$. Therefore, $\bar{x}$ maximizes $\tilde{U}^{\alpha}(x)$ on the set B^p. Applying Theorem 17 it follows that for all i the vector $\bar{x}^i$ maximizes u^i on $\left\{x^i \in \mathbb{R}^l_+ \middle| p \cdot x^i \leq p \cdot (\alpha_i e)\right\}$. Hence, the tuple $\left(p, \left(\bar{x}^i\right)_{i=1}^n\right)$ is an α-Walrasian equilibrium and the allocation $\bar{x}$ is a Walras allocation with respect to α.

$\square$

There are some immediate consequences of this result that can be used to derive properties of competitive equilibria under the given assumptions:

- Fix some vector of weights α. It is well known that the α-asymmetric Nash bargaining solution yields a single point. Hence, each agent's utility is constant on the set of of Nash allocations with respect to α. As Nash allocations coincide with Walras allocations the same holds true for Walras allocations with respect to α.

- As the utility functions are concave and as each agent's utility is constant on the set of Nash allocations, it is easy to see that the set of Nash allocations with respect to α is convex. The same holds true for the set of Walras allocations with respect to α.

- It is easy to see that an allocation $\bar{x}$ can be at the same time Nash allocation with respect to α^1 and α^2 for different vectors of weights $\alpha^1,\ \alpha^2$. Then, $\bar{x}$ is also a Walras allocations with respect to α^1 and α^2. This implies that $\bar{x}$ is an equilibrium allocation for different prices $p^1 \in \mathbb{R}^l_+$ and $p^2 \in \mathbb{R}^l_+$. Thus, it can happen that an allocation is an equilibrium allocation for different prices (and different endowments).

- In this context it is individual rational to apply the α-asymmetric Nash bargaining solution with status quo point 0. This follows from the fact that Nash allocations coincide with Walras allocation together with the fact that Walras allocations are individual rational. This point is not a priori clear as individual rationality has to be seen in relation to the utility of the endowments vectors. Individual rationality as given in the axiomatization of the α-asymmetric Nash bargaining solution with status quo point 0 just shows that the each agent receives at least 0.

The implications of Proposition 19 also hold if all the utility functions are homogeneous of the same degree k with $0 < k < 1$ instead of homogeneous of degree 1. Hereby, one should be aware of the fact if a function is homogeneous of degree $k > 1$ then this function can not be concave.[1]

Corollary 1. *Suppose for some* $0 < k < 1$ *the utility functions of all agents* $i \in I$ *are homogeneous of degree* k *and that in addition all assumptions (except homogeneity of degree* 1*) from Section 2 hold. Then, an allocation* $\bar{x} = \left(\bar{x}^i\right)_{i=1}^m$ *is a Nash allocation with respect to* α *if and only if it is a Walras allocation with respect to* α*.*

Proof. This Proposition can be shown in the same way like Proposition 19. Hereby, it is important to see that the homogeneity of degree 1 enters in the proof of Proposition 19 only indirectly via Theorem 17. But Theorem 17 is also valid for utility functions that are homogeneous of degree k with $0 < k < 1$.

□

To prove Corollary 1 it is important that all the utility functions are homogeneous of the same degree. If the utility functions are homogeneous of different degrees the implication of the result does not have to hold. This point will be discussed more precisely in subsection 5.5.3.

5.3.2 Non-proportional endowments

If the initial endowments are not proportionally distributed there is the problem that it is not clear for which α the α-symmetric Nash bargaining solution could be applied.

If the initial endowments are $\omega = \left(\omega^i\right)_{i=1}^n$ one can compute an equilibrium price

[1] To see this suppose that a function f is homogeneous of degree $k > 0$ and concave and suppose that for some x we have $f(x) > 0$. Then we have $f(x) = f\left(\frac{1}{2}0 + \frac{1}{2}2x\right) \geq \frac{1}{2}f(0) + \frac{1}{2}f(2x) = 0 + \frac{1}{2}f(2x) = 2^{k-1}f(x) \Rightarrow 1 \geq 2^{k-1}$. If $k > 1$ this inequality is not satisfied.

vector $p(\omega)$. As seen before, this price vector does not have to be unique. Now, one can compute the vector of weights $\alpha(p(\omega),\omega) \in \mathbb{R}^n_+$ given by

$$\alpha_i(p(\omega),\omega) := \left\{ t \in \mathbb{R}_+ \middle| p(\omega) \cdot \left(te^N\right) = p \cdot \omega^i \right\}$$

for $i \in N$. Using this definition, it is easy to see that Walras allocations (coming from the vector of initial endowments ω) are also Nash allocations with respect to $\alpha(p(\omega),\omega)$. Hereby, one can not just look on the data of the model and knows for which α the α-bargaining solution is related to competitive equilibria (given ω). One first has to compute $\alpha(p(\omega),\omega)$ via the equilibrium prices and then is able to construct α. Moreover, we observe that the vector $\alpha(p(\omega),\omega)$ depends on the choice of the equilibrium price vector and the equilibrium price vector does not have to be unique. In particular, consider the situation that ω is already a Nash allocation with respect to some α. Then, it can happen that ω is an equilibrium allocations for different prices which leads to different choices of α.
On the other hand consider that some allocation $\bar{x}$ is a Nash allocation for some α. Then, by Proposition 19 there exists a price $p \in \mathbb{R}^l_+$ such that $(p,\bar{x})$ is an α-Walrasian equilibrium. Then, $\bar{x}$ is obviously an equilibrium allocation for all $\left(\omega^i\right)_{i=1}^n$ satisfying $p \cdot \omega^i = p \cdot (\alpha_i e)$ for $i = 1, ..., n$. Moreover, one should recall that $\bar{x}$ can also be a Nash allocation with respect to some $\alpha' \neq \alpha$. In this way, one could find even more vectors of endowments such that $\bar{x}$ is a Walras allocation with respect to these endowments. Again, we observe the problem that the relation of α and ω is only indirect via the equilibrium prices.

To summarize, these result illustrate that it is mathematically possible to relate Walras allocation and Nash allocations even if the vector of initial endowments is not proportionally distributed. Nevertheless, there is a problem with the economic interpretation. The relation of ω and α is in both cases indirectly via equilibrium prices. Thus, the concepts of cooperative game theory and general equilibrium theory are mixed. We do no more analyze distinct concepts and hence a comparison of those is not that meaningful.

5.4 Discussion of the status quo point 0

The status quo point 0 in the previous section may be considered as a surprising choice. The usual interpretation of the status quo point is that this point describes the utilities of the agents in the case that cooperation fails to happen. In the situation of a pure exchange economy no cooperation could be understood as saying that no trade happens and each agents stays with his initial endowments. Hence, the first idea for the choice of a status quo point could be to choose it as the vector of utilities that agents obtain by consuming their own initial endowments.

We choose the status quo point $0 \in \mathbb{R}^n$ here. Hereby, the following example shows that implications of the result of the previous section only hold if one really chooses the status quo point 0. Choosing the utilities of the endowments of the agents as status quo point can (and typically does) lead to a failure of the results:

Example 1. Consider an economy with two agents $i = 1, 2$ and two commodities $j = 1, 2$. The utility function of agent 1 is given by

$$u^1(x_1^1, x_2^1) = x_1^1 + 2x_2^1$$

and the utility function of agent 2 is given by

$$u^2(x_1^2, x_2^2) = 2x_1^2 + x_2^2.$$

The endowments of the agents are given by

$$\omega^1 = \omega^2 = (1, 2)$$

what means that the set of total endowments is splitted according to $\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$.
Denote by U the allocations of utility that are feasible in this economy.
The unique Walrasian equilibrium price vector is $\bar{p} = (1, \frac{1}{2})$. This leads to the following demand in equilibrium: For agent 1:

$$x^{1*}\left(\bar{p}, \omega^1\right) = (0, 4),$$

for agent 2:

$$x^{2*}\left(\bar{p},\omega^2\right) = (2,0)$$

They constitute the unique Walras allocation with respect to $\left(\frac{1}{2},\frac{1}{2}\right)$.
The utility of the endowments for agent 1 is $u^1(\omega^1) = 5$ and for agent 2 is $u^2(\omega^2) = 4$.
The $\left(\frac{1}{2},\frac{1}{2}\right)$-Nash solution of the bargaining game $(U,(5;4))$ yields the allocation of utility $\left(\frac{13}{2},\frac{19}{4}\right)$. This corresponds to the allocation of goods $\hat{x} = \left(\left(0,\frac{13}{4}\right),\left(2,\frac{3}{4}\right)\right)$. This allocation can be regarded as a Nash allocation for the status quo point $(5;4)$, but this allocation is different from the Walras allocation $x^* = ((0,4)\,(2,0))$.

The $\left(\frac{1}{2},\frac{1}{2}\right)$-Nash solution of the bargaining game $(U,(0))$ is $u^* = (8,4)$. Hence, the Nash allocation with respect to $\left(\frac{1}{2},\frac{1}{2}\right)$ is $x^* = ((0,4),(2,0))$ and this coincides with the Walras allocation with respect to $\left(\frac{1}{2},\frac{1}{2}\right)$.

Figure 1 illustrates the situation of this example. On the axes the utility levels of the agents are drawn. The point $A = (8,4)$ is the Nash allocation with respect to $\left(\frac{1}{2},\frac{1}{2}\right)$ and at the same time Walras allocation with respect to $\left(\frac{1}{2},\frac{1}{2}\right)$. The point $B = (5,4)$ denotes the vector of utilities of the endowments. So it is easy to see that agent 2 can not improve through trade in the equilibrium compared to the utility he receives through his endowments. If the point B is chosen as the reference point the $\left(\frac{1}{2},\frac{1}{2}\right)$-symmetric Nash bargaining solution yields the point $C = \left(\frac{13}{2},\frac{19}{4}\right)$ where both agents have a higher utility than in point B.

There is a further reason why the status quo point 0 has to be chosen here. Remember that the utility functions of the agents are positively homogeneous. Hereby, the notation hides the fact that the utility functions are homogeneous in relation to the reference point $0 \in \mathbb{R}^l$, the consumption bundle that describes no consumption. Looking at the induced bargaining game this translates to the status quo $0 \in \mathbb{R}^n_+$ where payoffs in terms of utility are in relation to the utility of consuming nothing, i.e. $0 = \left(u^i\left(0\right)\right)_{i=1}^n$.

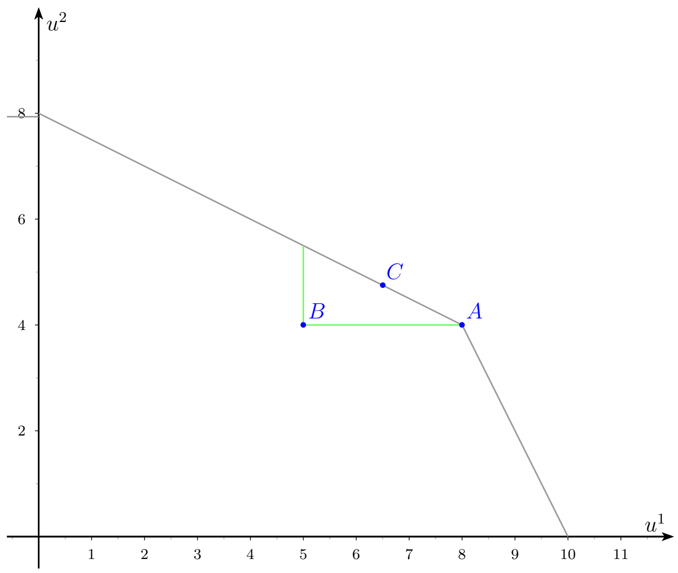


Figure 5.1: Example

5.5 Robustness of the result

5.5.1 Non-homogeneous utility functions

Looking at the results from section 5.3 one could think that these results are in conflict with the results of Sertel and Yildiz (2003) or the results of Qin (1993). While here competitive outcomes are captured by the asymmetric bargaining solution, the results of Qin (1993) and others show that competitive outcomes can be mainly anything within the inner core. The reason for these different results are the more restrictive assumptions (in particular homogeneity of the utility functions) in this work. The question arises whether there are less restrictive assumptions which still lead to the same result.

We begin with an example in the spirit of the results of Sertel and Yildiz (2003). In this example Nash allocations with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$ and Walras allocations with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$ do not coincide. Moreover, in this example the set of Walras allocations can not coincide with the set of allocations chosen by any (single-valued) bargaining solution. The example given in Sertel and Yildiz (2003) illustrates the same fact. In their example the utility functions are not homogeneous and the endowments are not proportionally distributed. In contrast, we give an example where the endowments are proportionally distributed but the utility functions are not homogeneous. Hence, our example illustrates that certain assumptions on the utility functions are necessary to show our result. The example is a modification of an economy given in Mas-Colell et al. (1995, Example 15.B.2).

Example 2. Consider a pure exchange economy with two consumers 1 and 2 and two goods where quantities of these goods are typically denoted by x and y. Let $\bar{z} = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$. Let $X^1 = X^2 = \mathbb{R}^2_+$ be the consumption set of the agents, let

$$u^1(x, y) = x - \frac{1}{8}y^{-8}$$

and

$$u^2(x, y) = (y - 2 + \bar{z}) - \frac{1}{8}(x + \bar{z} - 2)^{-8}$$

be the utility functions of the agents. The endowments of the agents are given by

$$\omega^1 = \omega^2 = (2, \bar{z}).$$

Hereby, each of the agents owns one half of the total endowments.

Looking for competitive equilibria one obtains that the price vectors $p_1 = (1, 1);\ p_2 = (1, 2);\ p_3 = (1, \frac{1}{2})$ are competitive equilibrium price vectors.
The Walrasian demands $x^i(p, \omega^i)$ of agent $i = 1, 2$ for the prices p_1 and p_2 are:

- $x^1(p_1, \omega^1) \approx (1,77; 1)$,
- $x^1(p_2, \omega^1) \approx (1,69; 0,93)$,
- $x^2(p_1, \omega^2) \approx (2,23; 0,54)$,
- $x^2(p_2, \omega^2) \approx (2,31; 0,62)$.

The corresponding indirect utility functions $v^i\left(p, \omega^i\right)$ take the following values in equilibrium:

- $v^1\left(p_1, \omega^1\right) \approx 1,64$,
- $v^1\left(p_2, \omega^1\right) \approx 1,46$,

- $v^2\left(p_1,\omega^2\right) \approx 1,64$,
- $v^2\left(p_2,\omega^2\right) \approx 1,77$.

We observe in this single economy different payoffs in terms of utility for different equilibria. If we model the situation with a bargaining situation and apply any bargaining solution we obtain a unique payoff in terms of utility. Hence, no bargaining solution can obtain all different equilibrium allocations at the same time.
To summarize this example shows the economy defined above is also an example in the spirit of Sertel and Yildiz (2003). In contrast to the example of Sertel and Yildiz (2003), the total endowments are proportionally distributed in our example.

The example illustrates that the implications of the results from section 5.3 do not hold if one allows for a very large class of utility functions. But for which class of utility functions is the result valid? Is it the class of homogeneous utility functions or is the result even valid for a larger class of utility functions? It is complicated to give a clear answer to these questions because homogeneity is used here as a local property in relation to the point $0 \in \mathbb{R}^l$. For example, if the utilities are not homogeneous in a subset of the consumption set which is not important for the maximization problems for Nash and Walras allocations the implications of the result remain valid. The question is whether structural deviations from homogeneity lead to a breakdown of the results. In subsection 5.5.2 and 5.5.3 we consider two types of deviations from homogeneity. We will see that although we modify the utility functions in both cases in a way, such that they still represent the same preferences, after these modifications Nash and Walras allocations do no more coincide. Thus, these results illustrate that the result is not robust with respect to these perturbations of the utility functions.

5.5.2 Shifts of utility functions

For the first modification we consider a situation with two agents and two commodities. For simplicity we assume that $e = (1,1)$. Given some $0 < b < 1$ we

can use the vector $\alpha = (b, 1-b)$ as vector of weights. Suppose that the utility functions of the agents u^1 and u^2 are strictly increasing, concave, homogeneous of degree 1 and continuously differentiable. Furthermore, suppose that for each $i \in I$ and for each $j \in J$ we have for all $x^i_{-j} \in \mathbb{R}^{l-1}_{++}$ that $\lim_{x^i_j \to 0} \frac{\partial u^i\left(x^i_j, x^i_{-j}\right)}{\partial x^i_j} = \infty$ holds. These conditions ensure interior solutions of the maximization problems that one considers to find Nash or Walras allocations. Furthermore, Nash and Walras allocation can be characterized via first order conditions. Given the results from Proposition 19 we know that Walras allocations with respect to α and Nash allocations with respect to α coincide.
We modify the situation by changing the utility function of agent 1 to $u^{1\varepsilon} = u^1 + \varepsilon$ for some $\varepsilon \neq 0$. We will refer to situation A as the situation where agent 1 has the utility function u^1 and to situation B as the situation where agent 1 has the utility function $u^{1\varepsilon}$.

Proposition 20. *In situation B Walras allocations with respect to α and Nash allocations with respect to α do not coincide.*

Proof. Suppose that $\bar{x}$ is a Walras allocation with respect to α in situation B. As adding ε to u^1 is a monotone transformation of utility we have that $\bar{x}$ is a Walras allocation with respect to α in situation A. By proposition 19 $\bar{x}$ is also a Nash allocation with respect to α in situation A. Now, suppose to the contrary that $\bar{x}$ is also a Nash allocation with respect to α in situation B.

First, consider situation A. Under the given conditions Nash allocations can be characterized by first order conditions. Plugging in the feasibility conditions we can rewrite the constrained maximization problem

$$\max \left[u^1\left(x^1_1, x^1_2\right)\right]^{\alpha} \left[u^2\left(x^2_1, x^2_2\right)\right]^{1-\alpha} \text{ with respect to } x^1 + x^2 \leq (1,1)$$

to the unconstrained maximization problem

$$\max \left[u^1\left(x^1_1, x^1_2\right)\right]^{\alpha} \left[u^2\left(1 - x^1_1, 1 - x^1_2\right)\right]^{1-\alpha}.$$

Taking the partial derivative with respect to x^1_1 and looking at the first order condition we obtain

$$
\begin{aligned}
&\alpha\left(u^1\left(\bar{x}_1^1,\bar{x}_2^1\right)\right)^{\alpha-1}\frac{\partial u^1}{\partial x_1^1}(\bar{x}_1^1,\bar{x}_2^1)u^2\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)^{1-\alpha}\\
&-(1-\alpha)\left(u^1(\bar{x}_1^1,\bar{x}_2^1)\right)^{\alpha}\left(u^2\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)\right)^{-\alpha}\frac{\partial u^2}{\partial x_1^2}\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)\\
&=0. \qquad (5.5)
\end{aligned}
$$

As u^2 is strictly increasing and hence $\frac{\partial u^2}{\partial x_1^2}>0$, equation (5.5) can be rewritten as

$$
u^1\left(\bar{x}_1^1,\bar{x}_2^1\right)=\frac{\alpha}{1-\alpha}u^2\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)\frac{\frac{\partial u^1}{\partial x_1^1}(x_1^1,x_2^1)}{\frac{\partial u^2}{\partial x_1^2}\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)}. \qquad (5.6)
$$

If one now considers situation B one obtains the following equation in the same way:

$$
u^1\left(\bar{x}_1^1,\bar{x}_2^1\right)+\varepsilon=\frac{\alpha}{1-\alpha}u^2\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)\frac{\frac{\partial u^{1\varepsilon}}{\partial x_1^1}(x_1^1,x_2^1)}{\frac{\partial u^2}{\partial x_1^2}\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)}. \qquad (5.7)
$$

But $\frac{\partial u^{1\varepsilon}}{\partial x_1^1}=\frac{\partial u^1}{\partial x_1^1}$. Hence, it follows that

$$
u^1\left(\bar{x}_1^1,\bar{x}_2^1\right)+\varepsilon=\frac{\alpha}{1-\alpha}u^2\left(1-\bar{x}_1^1,1-x_2^1\right)\frac{\frac{\partial u^1}{\partial x_1^1}(\bar{x}_1^1,\bar{x}_2^1)}{\frac{\partial u^2}{\partial x_1^2}\left(1-\bar{x}_1^1,1-\bar{x}_2^1\right)}. \qquad (5.8)
$$

Using equation (5.6) it follows

$$
\varepsilon=0.
$$

Hence, as $\varepsilon\neq 0$, we have a contradiction. Thus, $\bar{x}$ is not a Nash allocations with respect to α in situation B. □

5.5.3 Different degrees of homogeneity

Another way to modify the utility functions is to modify them in a way such that they are not homogeneous of the same degree. As already mentioned before the implications of Proposition 19 only hold if the utility functions of all agents are homogeneous of the same degree k. To see this we start with a situation where

the utility functions of all agents are homogeneous of degree 1. Then, for some agent j we modify the utility function of this agent by applying the monotone transformation $(.)^m$ for a sufficiently small $m > 0$. By applying this monotone transformation we do not change the preferences of the agent. Nevertheless, we change the degree of homogeneity of the function from homogeneity of degree 1 to homogeneity of degree m. Obviously, this transformation does not change the set of Walras allocations with respect to α. But we will see that this modification changes the set of Nash allocations with respect to α.
More general, we first consider the situation that we modify the utility function of agent j by applying the transformation $(.)^{m_j}$ with $m_j > 0$. This does not change the set of Walras allocations with respect to α but it changes the set of Nash allocations in the following way. Looking at the Nash product this transforms to

$$\prod_{i=1}^{n} \left(\left(u^i \left(x^i \right) \right)^{m_i} \right)^{\alpha_i}.$$

Maximizing this product is equivalent to maximizing the product

$$\prod_{i=1}^{n} \left(u^i \left(x^i \right) \right)^{\beta_i}$$

with $\beta_i = \frac{m_i \alpha_i}{\sum_{j=1}^{n} m_j \alpha_j}$ where we have $\sum_{i=1}^{n} \beta_i = 1$. Hence, this transformation leads to an application of the β-asymmetric Nash bargaining solution. Hereby, by choosing the transformations m_i for $i = 1, ..., n$ we can basically choose any vector of weights β. The following Proposition shows that using this transformation we can change the set of Nash allocations (with respect to α). Hereby, we use the modification of just one utility function.

Proposition 21. *Consider a situation where the utility functions of all agents $i \in I$ are weakly increasing, locally nonsatiated, concave, continuous and homogeneous of degree 1 and fix some vector of weights α. Suppose that for some Nash allocation with respect to α denoted by $\bar{x}$ there exists an agent $j \in I$ and a feasible allocation $\hat{x}$ with*

$$\prod_{i \neq j} \left(u^i \left(\hat{x}^i \right) \right)^{\alpha_i} > \prod_{i \neq j} \left(u^i \left(\bar{x}^i \right) \right)^{\alpha_i}.$$

Then, there exists an $m > 0$ with the following property: If the utility function of agent j is changed to $\tilde{u}^j = \left(u^j\right)^m$ then Walras allocations with respect to α and Nash allocations with respect to α do not coincide.

Proof. Suppose to the contrary that for any $0 < m \leq 1$ after the monotone transformation $(.)^m$ Nash allocations with respect to α and Walras allocations with respect to α coincide. We will refer to the situation before applying the monotone transformation as situation 1 and to the situation after the transformation as situation 2.
First notice that applying the monotone transformation $(.)^m$ does not change the set of Walras allocations with respect to α. We will show that for sufficiently small $m > 0$ this transformation changes the set of Nash allocations with respect to α.
As $\bar{x}$ maximizes also in situation 2 the α-asymmetric Nash product, we have

$$\left(u^j\left(\bar{x}^j\right)^m\right)^{\alpha_j} \prod_{i \neq j} \left(u^i\left(\bar{x}^i\right)\right)^{\alpha_i} \geq \left(u^j\left(\hat{x}^j\right)^m\right)^{\alpha_j} \prod_{i \neq j} \left(u^i\left(\hat{x}^i\right)\right)^{\alpha_i}.$$

This implies

$$\frac{\left(u^j\left(\bar{x}^j\right)^m\right)^{\alpha_j}}{\left(u^j\left(\hat{x}^j\right)^m\right)^{\alpha_j}} \geq \frac{\prod_{i \neq j} \left(u^i\left(\hat{x}^i\right)\right)^{\alpha_i}}{\prod_{i \neq j} \left(u^i\left(\bar{x}^i\right)\right)^{\alpha_i}}$$

and hence

$$\left(\frac{u^j\left(\bar{x}^j\right)}{u^j\left(\hat{x}^j\right)}\right)^m \geq \left(\prod_{i \neq j} \frac{\left(u^i\left(\hat{x}^i\right)\right)^{\alpha_i}}{\left(u^i\left(\bar{x}^i\right)\right)^{\alpha_i}}\right)^{\frac{1}{\alpha_j}}.$$

Notice, that the term on the right does not depend on l and is larger than 1. Furthermore, $\lim_{m \to 0} \left(\frac{u^j\left(\bar{x}^j\right)}{u^j\left(\hat{x}^j\right)}\right)^m = 1$. Hence, for sufficiently small $m > 0$ the inequality is not satisfied. This is a contradiction. Hence, there exists an $m > 0$ such that in situation 2 Nash and Walras allocations do not coincide. $\square$

If the Pareto surface of the induced bargaining game in situation 1 is smooth, then even for all $0 < m < 1$ the sets of Nash and Walras allocations do not coincide any more. If the Pareto surface of the induced bargaining game has a kink then it may happen that for some set of real numbers $m > 0$ we observe the same set of Nash allocations with respect to α.

Example 1 (continued)
In example 1 we have a kink at the Walras allocation. The vector of utilities $(8,4)$ is the Nash bargaining solution for all vectors of weights α with $\alpha_1 \in \left[\frac{1}{2}, \frac{4}{5}\right]$. If we apply the monotone transformation to the utility function of agent 2 for some m this is equivalent to using the vector of weights $\left(\frac{\frac{1}{2}}{\frac{1}{2}m+\frac{1}{2}}, \frac{\frac{1}{2}m}{\frac{1}{2}m+\frac{1}{2}}\right)$ in the asymmetric Nash bargaining solution. Hence, we observe that after applying the monotone transformation for any $m \in \left[\frac{1}{4}, 1\right]$ Nash allocations with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$ and Walras allocations with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$ still coincide. If $m < \frac{1}{4}$ then the set of Nash allocations changes and hence Walras and Nash allocations do not coincide.

There is also a positive implication of this analysis. Suppose that the utility functions of the agents are not homogeneous of the same degrees and the utility function of agent i is homogeneous of degree k_i. Then, after applying the concave, monotone transformation $(.)^{\frac{\min_{i \in I} k_i}{k_i}}$ to the utility function of agent i the utility functions of all agents are homogeneous of the same degree (and still concave). Hence, after this transformation Walras and Nash allocations coincide.

5.6 Relation to Polterovich (1975)

It is interesting to see our result as well as the result of Chipman and Moore (1979) in contrast to an article written by Polterovich (1975). This article is not very well known since it was originally published in Russian language (in 1973). We have only access to one printed version of a translation of it. In this article Polterovich considers economies where the number of agents, income shares, consumption sets and production are fixed and only the (cardinal) utility functions vary. Hereby, he uses an unusual kind of production describing the "total vector outputs of consumer goods available to society" without requiring inputs. To compare this result with ours we focus in the following on the case that only one vector of output goods is available to the society, namely the vector of total endowments e. Polterovich introduces a set valued concept that maps sets of utility functions $(u^1, ..., u^n)$ from a class of utility functions F to feasible allocations. Thus, the concept is similar to a social choice rule. The difference is that it is not defined on the preferences of the agents but on the cardinal utility functions of the agents. This so called "**solution of the distribution problem**" is characterized through 4 axioms. Axiom 1 describes invariance of the solution with respect to positive linear and affine transformations of the utility functions. Axiom 2 requires invariance of the solution with respect to changes of the utility functions in favor of the allocations that are already chosen before the change of the utility functions. This axiom can be regarded as a cardinal and weaker version of Maskin monotonicity. The condition would follow from Maskin monotonicity. Axiom 3 describes the solution on economies where the agents have identical, linear and monotonically increasing utility functions. It captures ideas like efficiency and proportionality of the utilities of incomes in those economies. The fourth axiom says that given a set of utility functions each agent's utility is constant on the set of all allocations the solution yields. The formal versions of the axioms are given in the appendix.

Polterovich (1975) proves in Theorem 6 that under certain conditions the solution of the distribution problem exists. Furthermore, if the solution exists it yields exactly the set of competitive equilibrium allocations. The conditions he requires include the case that the class of utility functions F is the set of utility functions which are positive and homogeneous. Thus, the solution of the distribution can be applied in the context of our work.

In a second part Polterovich (1975) considers a characterization of competitive equilibrium allocations as maximizers of a certain maximization problem.

Theorem 18 (Polterovich, 1975)**.** *Suppose that for each i the utility function u^i is positive, homogeneous of degree $k_i > 0$,nonnegative on $X^i = \mathbb{R}^m_+$ and that X^i contains a vector where u^i is strictly positive.*

(a) The tupel $\left(\left(x^i\right)_{i=1}^n, p^*\right)$ is a Walrasian equilibrium

if and only if

(b) the allocation $\left(x^i\right)_{i=1}^n$ *forms a solution of the problem*

$$\begin{aligned} \max \ & \sum_{i=1}^n \frac{\alpha_i}{k_i} \ln u^i\left(x^i\right) \\ \textit{with respect to } & \sum_{i=1}^n x^i \leq e, \\ & x^i \in X^i, \end{aligned} \tag{5.9}$$

and p^* *is a vector of Lagrange multipliers corresponding to the inequality of (5.9).*

Hereby, one should be aware that, given the conditions of the theorem, the function $\sum\limits_{i=1}^n \frac{\alpha_i}{k_i} \ln u^i\left(x^i\right)$ does not have to be concave or quasiconcave. In the proof he claims that for a maximzer x^* in (b) it follows that

$$\sum_{i=1}^n \frac{\alpha_i}{k_i} \ln u^i\left(x^{*i}\right) \geq \sum_{i=1}^n \frac{\alpha_i}{k_i} \ln u^i\left(x^i\right) + p^* \cdot \left(e - \sum_{i=1}^n x^i\right)$$

for all $x^i \in X^i$, $i = 1, ..., n$. Hence, the second part implicitly assumes the existence of Lagrange multipliers and (quasi)concavity properties of the target function. This should have been assumed in the conditions of the Theorem. Thus, in fact a maximzer in b) maximizes the target function with respect to $p^* \cdot \left(\sum x^i\right) \leq p^* \cdot e$, given a price vector p^*. Hence, this result is the analogue of the result of Chipman and Moore (1979) but only analyzing equilibrium allocations and not properties of the demand in general. On the other hand the result is more general in the sense that it includes the case that the utility functions are homogeneous of a degree not equal to 1. From a historical point of view this is interesting as Polterovich (1975) was published earlier than Chipman and Moore (1979).

In a short remark Polterovich (1975) mentions that, given the conditions of Theorem 18, assuming that the utility functions are homogeneous of the same degree $0 < k \leq 1$ and assuming that $\alpha = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$, then his solution of the distribution yields the same allocation like the symmetric Nash solution for status quo point 0 in an induced bargaining game. He does not elaborate on the details of the proof. In particular, he does not assume concavity of the utility functions but claims that in the induced bargaining game "all of Nash's assumptions are satisfied" where he refers to Nash (1950). But concavity would be necessary to show that the induced bargaining game is a bargaining game in the sense of Nash (1950). Without this assumption the bargaining set does not have to be convex. Moreover, his arguments and the application of Theorem 18 require that the function $\sum_{i=1}^{n} \frac{\alpha_i}{k_i} \ln u^i\left(x^i\right)$ is quasiconcave. This does not follow from the other assumptions and so the remark as presented in Polterovich (1975) is not correct.

Nevertheless, if one assumes concavity of the single utility functions u^i, then the function $\sum_{i=1}^{m} \frac{\alpha_i}{k_i} \ln u^i\left(x^i\right)$ is concave. Thus, if one adds this assumption the results become correct. Then, the (symmetric) Nash solution yields the same allocations as the solution of the distribution problem of Polterovich and, furthermore, the allocations given by the solution of the distribution problem and competitive equilibrium allocations coincide. Thus, indirectly Polterovich (1975) shows that Nash allocations and competitive equilibrium allocations coincide and obviously he was aware of that. He did not pay attention to this relation as he was more interested in the relation of the solution of the distribution problem with the Nash bargaining solution. The results of Polterovich were not observed in the western world as for example Shubik (1984) writes: "There is a strong analogy though no formal equivalence that we know of between the comparison weights that we must introduce in order to obtain a feasible transfer value and the prices in a competitive market." Shubik refers hereby to the λ-transfer value and must have know that in the context of bargaining games the λ-transfer value coincides with the Nash bargaining solution. We can state that Polterovich was the first who observed such a relation.

To summarize, the results of Polterovich (1975) can be used to show our result in a very similar way like it is done in Section 5.3. Furthermore, Polterovich (1975) mentions already a relation of the Nash bargaining solution with competitive equilibria. This result is not as general as ours as Polterovich only considers the symmetric Nash bargaining solution. Moreover, the conditions he requires are not sufficient to show the result.

5.7 Conclusion

This article analyzes the relation of asymmetric Nash bargaining solutions and competitive equilibria in the context of homogeneous utility functions and a proportional division of the endowments. It is shown that given some restrictive assumptions a certain asymmetric Nash bargaining solutions yields the same allocations as competitive equilibria. It is interesting to see this result in contrast to the results of Sertel and Yildiz (2003) who claim that there does not exist a Walrasian bargaining solution. Our result suggests that the Nash bargaining solution is in some sense a Walrasian bargaining solution.

We observe that the result requires an unusual choice of the status quo point. The result also strongly depends on the choice of the utility function representing the preferences. It is not robust with regard to monotone transformations of the utility functions. This fact highlights the differences between the cardinal utilities in cooperative game theory and the ordinal preferences in general equilibrium theory.

5.8 Appendix

5.8.1 The solution of the distribution problem

We recall the definition of Polterovich (1975) of his **solution of the distribution problem**. Hereby, we focus on the case that one vector of outputs is available to the society, namely the vector of the total endowments $e \in \mathbb{R}^l_+$. $X^i \subset \mathbb{R}^l_+$ denotes the consumption set of agent i. We consider vectors of utility functions of the agents $f = \left(f^1, ..., f^n\right)$ that are elements of some class of vectors of utility functions F.

Definition 42. The correspondence $D : F \longrightarrow \bigtimes_{i=1}^{n} X^i$ is called a solution of the distribution problem in class F, (and the allocation D(f) is called valid), if the following four conditions hold:

1. The mapping D is invariant with respect to positive linear transformations of utility function: if $f = \left(f^1, ..., f^n\right) \in F, f^i = \varphi^i$ for all $i \neq j,\ \varphi^j = af^j + b,\ a > 0,\ b \in \mathbb{R}$, then $D(f) = D(\varphi)$.

2. Under a change of preferences in favor of the valid allocation, it remains valid: if $f,\ \varphi \in F,\ f^i = \varphi^i$ for all $i \neq j$, $\varphi^j\left(x^j\right) \leq f^j\left(x^j\right)$ for any $x^j \in X^j$, $z = \left(z^1, ..., z^n\right) \in D(f)$ and $\varphi^j\left(z^j\right) = f^j\left(z^j\right)$, then $z \in D(\varphi)$.

3. Let all utility functions be identical, linear and monotonically increasing with respect to all their arguments: $f^i\left(x^i\right) = c \cdot x^i,\ i = 1, ..., n,\ c \in \mathbb{R}^l,\ c \geq 0$. If the feasible allocation $z = \left(z^1, ..., z^n\right)$

   (a) reaches a maximum utility under a given technology, i.e.

   $$\sum_{i=1}^{n} c \cdot x^i = c \cdot e$$

   , and if it

   (b) provides proportionality in the utility of income,i.e.

   $$\alpha_i c \cdot z^j = \alpha_j c \cdot z^i,\ i, j = 1, ..., n,$$

then it follows that z is a valid allocation, i.e. $z \in D(f)$.

4. For any $f \in F$ all valid allocations are equivalent: if $x, z \in D(f)$ it follows that $f^i\left(x^i\right) = f^i\left(z^i\right),\ i = 1, ..., n$.

Chapter 6

Concluding Remarks

Within the four chapters of this thesis we have studied several problems arising in the context of market games. Each chapter discusses its respective topic in detail and ends with a conclusion summarizing its results. Nevertheless for completeness we will briefly restate our achievements at this point:

First, we analyze a problem in the context of TU market games. We show that given a closed, convex subset of the core of a TU market game there exists a market which represent the game and furthermore has the given subset of the core as the set of its competitive payoff vectors. The main insight is that within the class of markets representing a TU market game the markets can have many different kinds of payoffs. Our work generalizes the results of Shapley and Shubik (1975) as it contains their results as a special case. Furthermore, it is interesting to see the techniques of the proofs in contrast to the techniques used in the NTU case.

In the third chapter of this work we analyze NTU marked games. We extend the results of Qin (1993) to a large class of closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of its inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. Our work confirms that going from NTU games to markets some structural information is added that is not present in the NTU game. To a given NTU market game we can associate a huge class of markets that represents the NTU game. In particular, by choosing the structure, that we add, we can control the set of payoffs of competitive equilibria.

In the fourth chapter we consider NTU bargaining games and prove that those games are NTU market games. Our results show that asymmetric Nash bargaining solutions as solution concepts for bargaining games are linked via the inner core to competitive payoff vectors of certain markets. Thus, our result can be seen as a market foundation of the asymmetric Nash bargaining solutions. This result holds for bargaining games in general as any asymmetric Nash bargaining solution is always in the inner core. The idea of a market foundation parallels the one that is used in implementation theory. Here, rather than giving a non-cooperative foundation for solutions of cooperative games, we provide a market foundation. Our result may be seen as an existence result.

In the last chapter we consider pure exchange economies and study these economies on the one hand with means of general equilibrium theory and on the other hand with means of cooperative bargaining theory. We prove that sets of competitive equilibrium allocations and of allocations resulting from an asymmetric Nash bargaining solution coincide as long as one restricts attention to economies where agents have homogeneous (of degree $k \leq 1$) utility functions and where the initial endowments are proportionally distributed. This result suggests that in the context of homogeneous utility functions the asymmetric Nash bargaining solution can be seen as a Walrasian bargaining solution.

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Short Curriculum Vitae

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Summary

This dissertation studies several aspects of the relation of general equilibrium theory and cooperative game theory. Hereby, the focus is on the relation of solution concepts of the different fields. We discuss the relation of competitive equilibria with solution concepts for cooperative games like core, inner core or asymmetric Nash bargaining solutions. We consider games and study which solutions appear as equilibria in economies representing these games. On the other hand we analyze when competitive equilibria of economies and cooperative solutions applied to induced games yield the same allocations.

The main chapters of this thesis, each of which self contained in notation, are based on four articles. Chapters 2 and 3 consider extensions of the results of Shapley and Shubik (1975) and Qin (1993) to subsets of the core respectively the inner core. Chapter 2 considers the case of TU market games while in Chapter 3 the NTU case is analyzed. Chapter 4 investigates the relation of asymmetric Nash bargaining solutions with the inner core in the context of bargaining games. We conclude that asymmetric Nash bargaining solutions are related to certain markets. The fifth Chapter considers the relation of asymmetric Nash bargaining solutions and competitive equilibria but now starting with economies and looking at induced bargaining games.

Keywords

Market Games, Coalitional Market Games, Competitive Payoffs, Core, Inner Core, Asymmetric Nash Bargaining Solutions.

Résumé ein Français

Cette thèse étudie plusieurs aspects des relations entre la théorie de l'équilibre général et la théorie des jeux coopératifs. Dans la suite, nous nous concentrons sur les relations entre les concepts de solution dans les différents domaines. Nous discutons les relations entre équilibre compétitif d'une part et les concepts de solution des jeux coopératifs d'autre part comme le coeur interne ou les allocations de marchandage de Nash asymétrique. Nous partons des jeux et nous étudions quelles solutions correspondent à des équilibres des économies qui représentent ces jeux. Nous étudions également quels couples de solutions conduisent aux mêmes niveaux d'utilité.

Les principaux chapitres de cette thèse, chacun étant auto-suffisant pour les notations, sont basés sur quatre articles. Les chapitres 2 et 3 sont des extensions des résultats Shapley and Shubik (1975) et Qin (1993) à des sous-ensembles du coeur ou du coeur interne. Le chapitre 2 est consacré aux jeux de marché à utilité transférable alors que le chapitre 3 analyse les jeux à utilité non transférable. Le chapitre 4 étudie les relations entre la solution de marchandage de Nash asymétrique avec le coeur interne dans le contexte des jeux de marchandage. Nous pouvons en conclure que les solutions de marchandage de Nash asymétriques sont reliées à certains marchés. Le cinquième chapitre considère les relations entre les solutions de marchandage de Nash asymétriques et les équilibres compétitifs mais en partant maintenant des économies et en regardant les jeux induits.

Mots clés

Jeux de Marchés, Jeux de Marchés Coopératifs, Paiements Compétitifs, Coeur, Coeur Interne, Solutions de Négociation Asymétriques de Nash.