Asymmetric Game Perfect Graphs and the Circular Coloring Game of Weighted Graphs

Dissertation zur Erlangung des Doktorgrades
an der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
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Oktober, 2011

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Acknowledgments

First, I would like to thank my supervisor Prof. Dr. Hans-Georg Carstens for giving me the opportunity to write this thesis. I am very grateful to him for his helpful advice, his motivating comments and permanent support throughout this period. I would like to express my sincere gratitude to Anush Khachatryan for so many fruitful discussions, for her patience and constant motivation. Further I would like to thank Prof. Dr. Michael Baake for taking the part of the first supervisor after Prof. Dr. Carstens' death[1]. Last but not least, I am very thankful to my parents who supported me all the time.

[1] †28.01.2012

Introduction

A *graph* G is defined to be a pair of vertices and edges. The vertices are denoted by V and the edges by $E \subseteq \binom{V}{2}$. Two vertices $x, y \in V$ are called *adjacent* if $(x, y) \in E$. One of the most interesting and important branches in the theory of graphs is the theory of graph coloring. A *vertex-coloring* of a graph $G = (V, E)$ is a map $c : V \to S$ such that $c(a) \neq c(b)$ whenever a and b are adjacent. The elements of the set $S = \{1, ..., k\}$ are called the available *colors*. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k such that G is colorable. Obviously the chromatic number is an element of $\mathbb{N}$. Graph coloring has found a lot of applications in daily life; it can be used for instance to optimize time schedules or assignment problems.

1991 H. L. Bodlaender raised the question of how many colors are required for a feasible coloring while considering the vertex coloring of a graph G as a game: two players Alice and Bob take turns coloring the vertices of a given graph G from a prespecified set of colors with Alice starting the game. Alice wins the game if the result is a feasible coloring and Bob wins the game if there exists at least one uncolored vertex and a feasible coloring is not possible any more. In [12] H. L. Bodlaender introduced the *coloring game* on a graph G and defined the *game chromatic number*, denoted by $\gamma(G)$, as the least integer k such that there exists a winning strategy for Alice for a given number of colors.

In our thesis we intend to generalize Boadlender's coloring game. In the respective chapters we introduce some relevant extensions and work out winning strategies for Alice for certain classes of graphs. In chapter 2 we extend the notion of game perfect graphs to the asymmetric coloring and introduce the asymmetric game perfect graphs. Further we investigate the two-person game considering circular colorings of weighted graphs in chapter 3. Chapter

4 is devoted to the circular game perfect graphs, an extension of the game perfect graphs.

Chapter 1 summarizes definitions and notations that will be used throughout the thesis.

In chapter 2 we turn our attention to game perfect graphs. The first who dealt with this theory was S. D. Andres in [14]. He defined a graph $G = (V, E)$ to be *game perfect* if and only if for every induced subgraph $H \subseteq G$ it holds $\gamma(H) = \omega(H)$, where $\omega(H) := \max\{ n \in \mathbb{N} \mid K_n \subseteq H\}$ is the clique number of G. In our thesis we introduce the notion of *asymmetric game perfect graphs*. The asymmetric coloring differs from the ordinary coloring in the fact that the players are allowed to color several vertices in a row. Asymmetric coloring was introduced by H. A. Kierstead in [15] and received huge interest since it changes the ordinary game dramatically and it opens room for new research. As already mentioned we introduce asymmetric game perfect graphs. In particular we consider the game for $n \in \mathbb{N}$ and $n \geq 2$, where Alice is allowed to color n vertices in a row and Bob one, the $(n, 1)$-coloring game. Contrary we also consider the game where Alice is allowed to color one vertex and Bob n vertices in a row, the $(1, n)$-coloring game. Precisely we determine the class of game perfect graphs for the $(n, 1)$-coloring game as well as for the $(1, n)$-coloring game for clique number 2. For this purpose we consider two variations of the ordinary asymmetric game, namely the A-game and the B-game. In the A-game Alice starts the game and has the right to miss a turn. In the B-game Bob has these rights. We determine the class of asymmetric g-game perfect graphs with clique number 2 for $g \in \{A, B\}$ for the $(n, 1)$-coloring game and the $(1, n)$-coloring game. In particular we show that only bipartite graphs can be g-game perfect and prove what conditions a bipartite graph must meet in order to be g-game perfect. Using these results we also determine the class of game perfect graphs with clique number 2 for the (n, m)-coloring game where $n, m \in \mathbb{N}$ and $n, m \geq 2$.

In chapter 3 we introduce the *circular two-person game on weighted graphs*. Roughly speaking, this new game emphasizes a combination of Bodlaender's two-person game and the circular coloring of weighted graphs defined by W.

A. Deuber and X. Zhu in [4] 1996, see definition 3.0.3. A graph $G^w = (V, E, w)$ is called *weighted* if there exists a mapping $w : V \to \mathbb{R}^+$ which assigns to each vertex $x \in V$ a vertex-weight $w(x) > 0$. In contrast to Boadlender's game we consider graphs with vertex-weights. Furthermore, the vertices are not assigned to natural numbers but to arcs on a given circle C^r in the Euclidean space $\mathbb{R}^2$ where $r \in \mathbb{R}^+$ is the circumference. For $(x, y) \in E$ the arcs, denoted by $f_w(x)$ and $f_w(y)$, are not allowed to overlap in a feasible coloring. Further, for each vertex the length of the assigned arc equals its weight. Thus, the new game provides a generalization of the regular two-person game. We define the *circular game chromatic number of weighted graphs*, denoted by $\gamma_c(G^w)$, as the least circumference r of C^r such that Alice has a winning strategy, where G^w is a weighted graph. First, some basic properties of the new parameter are investigated. In particular, we relate the circular game chromatic number of weighted graphs with the circular game chromatic number of ordinary graphs, denoted by $\gamma_c(G)$, and give conditions when $\gamma_c(G) \leq \gamma_c(G^w)$ and $\gamma_c(G) \geq \gamma_c(G^w)$ hold.

Afterwards, we turn our attention to the circular two-person game concerning relevant classes of weighted graphs. For all values of w we give upper and lower bounds of the new parameter concerning the respective classes of graphs. Moreover, we consider some relevant distributions of the vertex-weights and work out winning strategies for Alice. First, we investigate $\gamma_c(K_n^w)$, where K_n^w is a weighted complete graph on n vertices, see definition 1.1.8. We work out a winning strategy for Alice, while restricting the weights by the mapping $w : V \to \{k, l\}$, for $k, l \in \mathbb{R}^+$ and $k \neq l$. In particular, Alice's winning strategy has to be adapted to the frequencies that k and l appear in the complete graph. We proceed with the circular game chromatic number of weighted complete multipartite graphs, see definition 3.3.1. By applying techniques of W. Lin and X. Zhu in [18], we determine $\gamma_c(G^w)$ for $w : V \to \{k\}$, $k \in \mathbb{R}^+$. As in case of weighted complete graphs, we consider the weight function $w : V \to \{k, l\}$ and show how Alice's winning strategy varies depending on the frequencies that k and l appear in the complete multipartite graph. In addition, we investigate the circular game chromatic number of weighted cycles, see definition 1.1.11. While denoting a cycle C_n by its sequence of vertices $C_n = x_0...x_{n-1}x_0$, we consider the weight function $w : V \to \{k_0, ..., k_{m-1}\}$ for $\{k_0, ..., k_{m-1}\} \in \mathbb{R}$ and $m \bmod n = 0$, such that the vertex-weights are alternatingly assigned in ascending order. Further, we give an upper bound of the circular game chromatic number for the class of weighted trees. We work out an optimal strategy for Alice and show that the leaves of a tree can be left out of consideration. Next we give an upper bound for the circular game chromatic number of weighted planar graphs. For this purpose, we refer to H. A. Kierstead's activation strategy introduced in [2] 2000, which he applied for giving an upper bound for $\gamma(G)$ where G is a planar graph without vertex-weights. He estimated the maximum number of colored neighbors of an uncolored vertex v during the game. Using this strategy, we also manage to determine Bob's worst case scenario for the circular two-person game on weighted graphs and hence to give an upper bound for the circular game chromatic number of weighted planar graphs. Finally, we extend the notion of the cartesian product of ordinary graphs to the cartesian product of weighted graphs and investigate the circular game chromatic number of $K_2^w \times P_n^u$ and $K_2^w \times C_n^u$, where K_2 is the complete graph on 2 vertices, P_n is a path on n vertices and C_n is a cycle on n vertices.

The aim of chapter 4 is to introduce the *circular game-perfect graphs*. For this purpose we refer to X. Zhu, who defined in [20] that G is a circular perfect graph if $\chi_c(H) = \omega_c(H)$ for every induced subgraph $H \subseteq G$, where $\omega_c(H) := \max\{\frac{k}{d} : K_k^d \subseteq H\}$ is the *circular clique number of* H and K_d^k is a rational complete graph (introduced by A. Vince in [10]). Due to the fact that Alice and Bob are competitive, Zhu's method breaks down because the circular game chromatic number of the graph K_k^d does not equal $\frac{k}{d}$. Thus, we need to modify Zhu's definition of circular perfect graphs. For this purpose we consider the greatest K_k^d contained in G with $\omega_c(K_k^d) = \omega_c(G)$, denoted by $\Theta(G)$, and require for a circular game-perfect graph $\gamma_c(\Theta(G)) = \gamma_c(G)$. Furthermore, we give two criterions for a non circular-game nice graph G with circular clique number $2,5$ and $\Theta(G) = K_5^2$. However, this chapter shall indicate some initial results of circular game-perfect graphs and motivate the reader for further research.

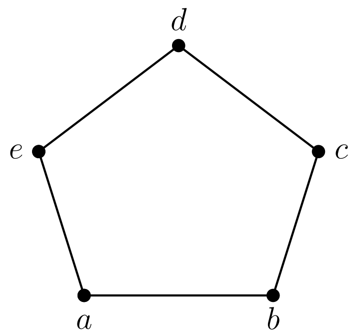
Chapter 1

Preliminaries

In this first chapter we want to give some basic graph theoretic definitions that will be used throughout this thesis. Although the terminology is not standard, we will use notations that are commonly used in the graph theoretic community.

Definition 1.1.1. A *graph* $G = (V, E)$ is an ordered pair of the sets V and E such that E is a subset of the set $\binom{V}{2}$ of unordered pairs of V. V denotes the set of *vertices* and E denotes the set of *edges*.

Example: The figure below shows a graph with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{(a, b), (b, c), (c, d), (d, e), (e, a)\}$.



Definition 1.1.2. Let $G = (V, E)$ be a graph. Two vertices $a, b \in V$ are called *adjacent* if (a, b) is an edge of G, i.e., $(a, b) \in E$. b is called a *neighbor* of a and vice versa. Otherwise a and b are *independent*. A vertex $a \in V$ is called *incident* with an edge $e \in E$ if $a \in e$. The *degree* of a is the amount of the

neighbors of a and is denoted by $d(a)$. The *maximum degree* of G is denoted by $\Delta(G) := \max\{d(a) \mid a \in V\}$.

For the graph in the example above we have $d(x) = 2$ for all $x \in V$ and hence $\Delta(G) = 2$.

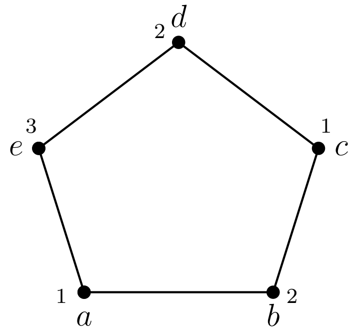
Definition 1.1.3. Let $G = (V, E)$ be a graph. A set $M \subset E$ of independent edges is called a *matching*. A *perfect matching* is a matching M such that every vertex of G is adjacent to an edge of M.
For $X, Y \subset V$ we say that M is a *matching from X to Y* if M saturates X and $Y \setminus X$ contains a cover of M.

Definition 1.1.4. A graph $G = (V, E)$ is called *connected* if for every partition of its vertex set into two nonempty sets X and Y there is an edge with one end in X and one end in Y; otherwise the graph is disconnected.

Definition 1.1.5. A graph $G' = (V', E')$ is called a *subgraph* of $G = (V, E)$, and G a *supergraph* of G', if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. If $G' \subseteq G$ and G' contains all edges $(a, b) \in E$ with $a, b \in V'$, then G' is an *induced* subgraph of G.

Definition 1.1.6. A *vertex-coloring* of a graph $G = (V, E)$ is a map $c : V \to S$ such that $c(a) \neq c(b)$ whenever a and b are adjacent. The elements of the set $S = \{1, ..., k\}$ are called the available *colors*. The *chromatic number*, $\chi(G)$, of a graph G is the minimum k such that G is colorable. If $\chi(G) = k$, G is said to be *k-chromatic*.

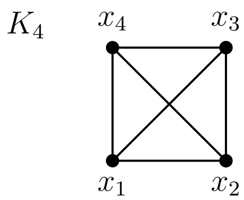
Example: Below we give a coloring for the graph of our first example, where the numbers indicate the assigned colors:



Remark 1.1.7. We will consider undirected, finite graphs, which means that $(a, b) = (b, a)$ and the set V is finite. Furthermore loops are not allowed, that is, for $a \in V$ we always have $(a, a) \notin E$.

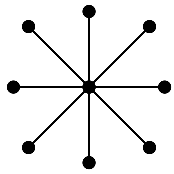
Definition 1.1.8. A graph $G = (V, E)$ is called *complete* if all vertices of G are pairwise adjacent. A complete graph on $n \in \mathbb{N}$ vertices is denoted by K_n. The *clique number* $\omega(G)$ of G is defined by $\omega(G) := \max\{n \in \mathbb{N} \mid K_n \subseteq G\}$.

Example:



Definition 1.1.9. A graph $G = (V, E)$ is called a *star* if there exists one and only one vertex $x \in V$ which is adjacent to all the other vertices and no other edges exist.

Example: A star with 8 neighbors also called *leaves*.



Definition 1.1.10. A graph $G = (V, E)$ is called *bipartite* if $V(G)$ admits a par-

tition into 2 classes such that vertices in the same partition are not adjacent. A bipartite graph in which every two vertices from different partition classes are adjacent is called *complete*. We denote a complete bipartite graph with independent sets M and N by $K_{m,n}$ where $|M| = m$ and $|N| = n$.

Definition 1.1.11. A graph $G = (V, E)$ is called a *path* if it is of the form

$$V = \{x_1, x_2, ..., x_n\} \quad E = \{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n)\}.$$

The number of the edges of a path is called its *length*. We denote a path from x_1 to x_n by $P_n = x_1...x_n$.

A graph $G = (V, E)$ is called a *cycle* if it is of the form

$$V = (x_1, x_2, ..., x_n) \quad E = \{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, x_1)\}.$$

The number of the edges of a cycle is called its *length*. We denote a cycle on n vertices by $C_n = x_1...x_n$.

Definition 1.1.12. For a graph $G = (V, E)$ and two vertices $x, y \in V$ we define the *distance* $d(x, y)$ of x and y by the length of a shortest path between x and y in G. If such a path does not exist we set $d(x, y) = \infty$. The greatest distance between any two vertices of G is called *diameter* of G and is denoted by $diam(G)$.

Chapter 2

Asymmetric Game Perfect Graphs

In this chapter we will bring together the notions of game perfectness and the asymmetric coloring game. First, we give the definitions of these two concepts and afterwards we combine them into one new definition, the asymmetric game perfectness. Our aim is to determine the class of asymmetric game perfect graphs with clique number 2. In sections 2.2 - 2.6 we wish to investigate the $A_{(1,n)}-$, $B_{(1,n)}-$, $B_{(n,1)}-$ as well as the $A_{(n,1)}-$game perfect graphs with clique number 2. Further, in sections 2.7, 2.8 and 2.9 we discuss the ordinary asymmetric game for the cases (a, b) for $a, b \in \mathbb{N}$ and $a, b \geq 2$.

The Two-Person Game on Graphs and the Game Chromatic Number

H. L. Bodlaender introduced in [12] the *two-person game on graphs* and the *game chromatic number* as follows:

Let $G = (V, E)$ be a graph and C be a set of given colors. Two players Alice and Bob take turns alternatingly with Alice moving first. Each move consists of choosing an uncolored vertex $v \in V$ and assigning one color from C such that adjacent vertices get distinct colors. Alice wins the game if all vertices of G are colored with the prespecified set of colors C; otherwise Bob wins, that is, neither Alice nor Bob can execute a feasible move.

The *game chromatic number* $\gamma(G)$ of G is the smallest amount of given colors such that there is a winning strategy for Alice with the given set of colors.

Since Alice and Bob are competitive obviously

$$\chi(G) \leq \gamma(G) \leq \Delta(G) + 1$$

holds for all graphs G.

Example: For more overview consider the vertex coloring of the graph $G = (V, E)$ below with vertices $a, b, c, d \in V$.

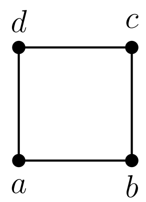


FIGURE: A cycle of length 4.

Obviously 2 colors suffice to color G in the ordinary game. However, the game chromatic number of G is 3. Without loss of generality Alice starts the game by coloring vertex a. The worst case occurs if Bob assigns to the opposing vertex, in this case c, another color. This move forces Alice to use a third color, since b and d are adjacent to a and c.

Game Perfect Graphs

S. D. Andres introduced in [14] the notion of game perfect graphs as follows:

A graph $G = (V, E)$ is called *game nice* if $\omega(G) = \gamma(G)$ holds where $\omega(G)$ is the clique number of G. Moreover, G is called *game perfect* if every induced subgraph $H \subseteq G$ is game nice.

S. D. Andres managed to determine the class of game perfect graphs with clique number 2. For this purpose he considered two variations of Bodlaender's graph coloring game which differ from the ordinary coloring game in the following way: In the *A-coloring game* Alice has the privilege to decide to start the game and to miss a turn. Where as, in the *B-coloring game* the same rules hold for Bob. For $p \in \{A, B\}$ the *p-game chromatic number*, denoted by $\gamma_p(G)$, is the smallest amount of given colors such that there is a winning strategy for Alice for the p-coloring game. Obviously it holds

$$\gamma_A(G) \leq \gamma(G) \leq \gamma_B(G).$$

For $p \in \{A, B\}$ a graph G is called *p-game nice* if $\gamma_p(G) = \omega(G)$. Moreover G is called *p-game perfect* if every induced subgraph $H \subseteq G$ is p-game nice.

Example: Consider the A-coloring game on a cycle of length 4 and assume that Alice decides to miss her first turn. If 2 colors are given, then after Bob's move Alice just has to color a neighbor of the colored vertex and she wins the game with 2 colors.

Asymmetric Coloring

We intend to consider a variation of the ordinary game namely the *asymmetric coloring game*. The first who dealt with this concept was H. A. Kierstead in [15]. He considered the case that each player has to color several vertices in a row instead of one vertex. He called this game *(a, b)-coloring game* where $a, b \in \mathbb{N}$ and Alice has to color a vertices while Bob has to color b vertices as long as there are uncolored vertices left and a feasible coloring is possible. The *asymmetric game chromatic number* $\gamma(G; a, b)$ is defined to be the smallest integer k of given colors such that Alice has a winning strategy for the (a, b)-coloring game.

2.1 Asymmetric Game Perfect Graphs

In this section we combine the notion of the asymmetric game and game perfectness and introduce the asymmetric game perfect graphs. Moreover we generalize the concept of the p-game perfect graphs with the asymmetric game.

Definition 2.1.1. Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{N}$. G is called *(a, b)-game nice* if $\omega(G) = \gamma(G; a, b)$. Moreover G is called *(a, b)-game perfect* if every induced subgraph H of G is (a, b)-game nice.

The aim of this chapter is to construct (a, b)-game perfect graphs with clique number 2 for the cases $(a, b) \in \{(1, n), (n, 1)\}$ where $n \in \mathbb{N}$ and $n \geq 2$. Using these results we also conclude the class of (n, m)-game perfect graphs with clique number 2 for $n, m \in \mathbb{N}$ and $n, m \geq 2$.

For this purpose we introduce the asymmetric versions of the A-coloring game, respectively, B-coloring game.

Let $p \in \{A, B\}$ and $a, b \in \mathbb{N}$. The *$p_{(a,b)}$-coloring game* is defined to be the (a, b)-coloring game with the rules of the p-coloring game, i.e., Alice or Bob starts the game and is allowed to miss a complete turn. Note that if one player decides to play, then the player has to color a, respectively, b vertices and is not allowed to break up as long as there are uncolored vertices left or a feasible coloring is possible.

The *asymmetric $p_{(a,b)}$-game chromatic number*, $\gamma_p(G; a, b)$, of a graph G is the smallest amount k of given colors, such that there is a winning strategy for Alice for the $p_{(a,b)}$-coloring game.

For $a, b \in \mathbb{N}$, it is easily seen that it holds:

$$\gamma_A(G; a, b) \leq \gamma(G; a, b) \leq \gamma_B(G; a, b).$$

As it is our purpose to characterize the asymmetric $p_{(a,b)}$-game perfect graphs, we introduce the following definitions of *$p_{(a,b)}$-nice graphs* as well as of *$p_{(a,b)}$-game perfect*.

Definition 2.1.2. Let $a, b \in \mathbb{N}$. For $p \in \{A, B\}$ a graph G is called *$p_{(a,b)}$-game nice* if $\omega(G) = \gamma_p(G; a, b)$. Moreover G is called *$p_{(a,b)}$-game perfect* if every induced subgraph $H \subseteq G$ is $p_{(a,b)}$-game nice.

2.2 $\mathbf{A_{(1,n)}}$-Game Perfect Graphs with Clique Number 2

For the purpose of determining the $A_{(1,n)}$-game perfect graphs with $\omega(G) = 2$, we first consider the case $n = 2$ which implies that Alice colors two vertices in a row each time she decides to take turn. However, Bob colors once. Afterwards, in corollary 2.2.5 we conclude the general case.

Proposition 2.2.1. *Let $G = (V, E)$ be a connected graph with $\omega(G) = 2$. Then G is $A_{(1,2)}$-game nice, if and only if G is a star or the K_2.*

Obviously a graph G is a star or the K_2 if and only if G does not contain an induced C_4 and an induced P_4. We will divide the proof into a sequence of the lemmas 2.2.2 and 2.2.3.

Lemma 2.2.2. *Let $G = (V, E)$ be a graph with $\omega(G) = 2$. Then $G = (V, E)$ is not $A_{(1,2)}$-game nice, if (i) or (ii) hold:*

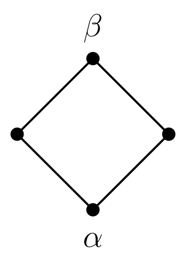
- *(i) C_4 is an induced subgraph of G,*
- *(ii) P_4 is an induced subgraph of G.*

Proof. Let C_4 with $V(C_4) = \{v_0, v_1, v_2, v_3\}$ or P_4 with $V(P_4) = \{u_0, u_1, u_2, u_3\}$ be induced subgraphs of G and assume that 2 colors $\{\alpha, \beta\}$ are given. Then Bob can apply the following strategy in order to win the game:

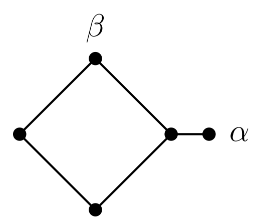
Since the $A_{(a,b)}$-coloring game is played on G, Alice has the opportunity to start the game or to miss the first turn.

(i) Assume that Alice decides to play in her first turn.

- If she colors a vertex $v_i \in V(C_4)$, then Bob replies by coloring vertex $v_{i+2 \bmod 4} \in V(C_4)$ with a color other than the one used by Alice. Further he colors any arbitrary vertex of G.
- If Alice colors any neighbor $w \in N(v_i)$ for $v_i \in V(C_4)$, then Bob colors vertex $v_j \in V(C_4)$ with $d(w, v_j) = 2$ with a color other than the one used by Alice. This is possible since each vertex on C_4 is uncolored. Further he colors any arbitrary vertex of G.



The cycle C_4



The cycle $C_4 \cup \{w\}$

In case that Alice misses her first turn or she does not color a vertex of C_4 or a neighbor of C_4, Bob colors any two vertices $\{v_i, v_{i+2 \bmod 4}\} \in V(C_4)$ with the colors α and β, respectively. Since $b = 2$, he has to color twice in a row. Again a feasible coloring of G is not possible anymore, since there is no color left for $\{v_{i+1 \bmod 4}, v_{i+3 \bmod 4}\} \in V(C_4)$.

(ii) Assume that Alice decides to play in her first turn.

- If she colors any vertex $x \in V(P_4)$, then Bob replies by coloring the vertex $y \in V(P_4)$ with $d(x, y) = 2$ with a color other than the one used by Alice. Further he colors any arbitrary vertex of G.

- If she colors any neighbor $w \in N(u_i)$ for $u_i \in V(P_4)$, then Bob colors vertex u_{i-1} or u_{i+1} (at least one exist) on P_4 with a color other than the one used by Alice and any arbitrary vertex of G.

The path P_4 The path $P_4 \cup \{w\}$

In case that Alice misses her first turn or she does not color a vertex of P_4 or a neighbor of P_4, Bob colors any two vertices $\{x, y\} \in V(P_4)$ with $d(x, y) = 2$ with two different colors α and β, respectively. Since $b = 2$, Bob has to color two vertices in a row. Obviously the common neighbor of x and y cannot be colored with the given set of colors.

Thus we can draw the conclusion that G is not $A_{(1,2)}$-game nice. □

Lemma 2.2.3. *Let $G = (V, E)$ be a connected graph with $\omega(G) = 2$. If G is not $A_{(1,2)}$-game nice, then C_4 or P_4 are induced subgraphs of G.*

Proof. Assume that P_4 and C_4 are not induced subgraphs of G and $\omega(G) = 2$. Then, obviously G is a star or the K_2. Alice winning strategy with two colors is to color the center of the star. Hence a star or a K_2 is $A_{(1,2)}$-game nice and the proof is complete.

□

Remark 2.2.4. The assumption that G must be connected in the lemma above is crucial. Suppose G is a graph which does not contain an induced P_4 or an induced C_4. Then G could be a forest of stars. By the time G consists of at least 2 stars, Bob can win the $A_{(1,2)}$-coloring game with 2 colors, since Alice is able to color only once and hence only one center of a star. Hence at least one star remains uncolored and Bob just has to color two leaves with 2 distinct colors in an uncolored star to win the game.

It is easy to see that proposition 2.2.1 also holds by the same arguments if we consider the $A_{(1,n)}$-coloring game for $n \in \mathbb{N}$ and $n \geq 3$. Hence we conclude:

Corollary 2.2.5. *Let $G = (V, E)$ be a connected graph with $\omega(G) = 2$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then G is $A_{(1,n)}$-game nice, if and only if G is a star or the K_2.*

2.3 $\mathrm{B}_{(1,\mathrm{n})}$-Game Perfect Graphs with Clique Number 2

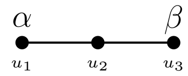
We consider first the case $n = 2$ such that Bob colors two vertices in a row, if he decides to play.

Proposition 2.3.1. *Let $G = (V, E)$ be a graph with $\omega(G) = 2$. Then G is $B_{(1,2)}$-game nice if and only if it does not contain the induced path P_3; in particular G is a forest of K_2's.*

Again we divide the proof into two lemmas.

Lemma 2.3.2. *Let $G = (V, E)$ be a graph with $\omega(G) = 2$. If P_3 is an induced subgraph of G, then G is not $B_{(1,2)}$-game nice.*

Proof. Let P_3 with $V(P_3) = \{u_1, u_2, u_3\}$ be an induced path of G and assume two colors $\{\alpha, \beta\}$ are given. Since the $B_{(1,2)}$-coloring game is played on G, it is up to Bob to start the game or miss each turn. Assume Bob starts the game by coloring vertices u_1 and u_3 with the colors α and β, respectively.



The path P_3

Since vertex u_2 cannot be colored anymore, a feasible coloring of G with two colors is not possible. Thus $\omega(G) \neq \gamma_B(G;1,2)$ and G is not $B_{(1,2)}$-game nice. □

It is easily seen that a graph which does not contain a path on three vertices with clique number 2 is a forest of K_2's, since the distance between each two vertices in the same component is 1. Hence we are left with the task of giving a winning strategy for Alice with two colors if G is a forest of K_2's:

Lemma 2.3.3. *Let $G = (V, E)$ be a graph with $\omega(G) = 2$. If P_3 is not an induced subgraph of G, then G is $B_{(1,2)}$-game nice.*

Proof. Since G consists of K_2's, each vertex has exactly one neighbor. It is obvious that a feasible coloring of G with two colors is possible. Hence it holds $\gamma_B(G;1,2) = \omega(G) = 2$. □

Proposition 2.3.1 also holds by the same arguments if we consider the $B_{(1,n)}$-coloring game for $n \in \mathbb{N}$ and $n \geq 3$. Hence we conclude:

Corollary 2.3.4. *Let $G = (V, E)$ be a graph with $\omega(G) = 2$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then G is $B_{(1,n)}$-game nice if and only if it does not contain the induced path P_3; in particular G is a forest of K_2's.*

2.4 $\mathbf{B_{(n,1)}}$-Game Perfect Graphs with Clique Number 2

We prove that every $B_{(2,1)}$-game nice graph is a bipartite graph and work out properties for bipartite graphs that are $B_{(2,1)}$-game nice. Afterwards we concentrate on the general case which is $n > 2$ and determine bipartite graphs that are $B_{(n,1)}$-game nice.

Lemma 2.4.1. *Let $G = (V, E)$ be a graph with clique number $\omega(G) = 2$. Then G is not $B_{(2,1)}$-game nice, if it contains an induced cycle C_n with $n = 2m + 1$ for $m \in \mathbb{N}$.*

Proof. Assume two colors are given. Since the chromatic number of a cycle C_n with odd number of vertices is 3 and $\chi(G) \leq \gamma(G) \leq \gamma_B(G)$ always holds, a feasible coloring of G is not possible. □

It is known that a graph which does not contain an odd cycle is bipartite. Hence with lemma 2.4.1 we can conclude the following:

Corollary 2.4.2. *Let $G = (V, E)$ be a $B_{(2,1)}$-game nice graph with $\omega(G) = 2$. Then G is a bipartite graph.*

Throughout the section it suffices to restrict our attention to bipartite graphs. In particular we will refer to a bipartite graph by $G_{m,l}$ with two independent sets M and L, where $|M| = m$ and $|L| = l$. Obviously $\omega(G) = 2$ always holds if G is bipartite.

For the purpose of characterizing bipartite graphs that are $B_{(2,1)}$-game nice, we need to introduce the following:

Definition 2.4.3. Let $G_{m,l}$ be a bipartite graph. A *central vertex* $v \in Q$ for $Q \in \{M, L\}$ is a vertex which is adjacent to every vertex from the independent set $\{M, L\} \setminus \{Q\}$, such that $d(v) = |\{M, L\} \setminus \{Q\}|$. We call a vertex $u \in Q$ for $Q \in \{M, L\}$ *semicentral* if $d(u) = |\{M, L\} \setminus \{Q\}| - 1$.

Example:

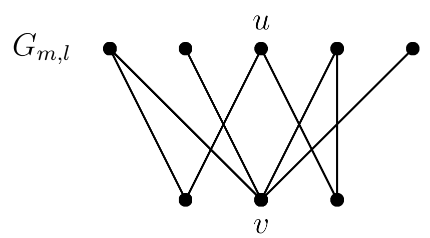


FIGURE: A bipartite graph $G_{m,l}$ with $|M| = 5$ and $|L| = 3$, where u and v are semicentral vertices.

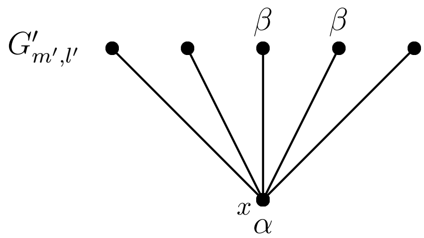
Lemma 2.4.4. *Let $G_{m,l}$ be a bipartite graph with $diam(G'_{m',l'}) \leq 2$ for each component $G'_{m',l'}$ of $G_{m,l}$. Then $G_{m,l}$ is $B_{(2,1)}$-game nice.*

Proof. Since the distance between two vertices is at most 2 each component of $G_{m,l}$ can be considered as a complete bipartite graph. We will give a winning strategy with two colors for Alice. Since the $B_{(2,1)}$ game is being played Bob is allowed to start the game or to miss a turn.

Assume Bob misses his first turn. Then Alice colors in an arbitrary component $G'_{m',l'}$ of $G_{m,l}$ two vertices $v \in M'$ and $u \in L'$ with α and β, respectively. Then the coloring of $G'_{m',l'}$ is fixed since it is a complete bipartite graph.

Assume Bob decides to play his first turn and assume that he colors a vertex $x \in G'_{m',l'}$ with α. Let x be a central vertex and without loss of generality let $x \in M'$, so that $M' = \{x\}$ and $m' = 1$.

- If $G'_{m',l'} \notin \{K_1, K_2\}$, then $l' \geq 2$. Alice's winning strategy is to color any two vertices from L' with β. This fixes the coloring of the component since the only vertex which is adjacent to the uncolored vertices in $G'_{m,l}$ is x. This implies that they must be colored with β.



- If $G'_{m',l'} \in \{K_1, K_2\}$, then Alice proceeds as in case where Bob missed his first turn.

Assume x is not central. Then $G'_{m',l'} \notin \{K_1, K_2\}$. Moreover this implies that there exists at least one vertex $z \in G'_{m',l'}$, such that $(x, z) \notin E(G'_{m',l'})$. Since $diam(G'_{m',l'}) \leq diam(G_{m,l}) \leq 2$, x and z are on a path $P = xyz$. Then Alice colors vertex y with β and the coloring is fixed since every vertex of $G'_{m',l'}$ has distance 1 from x or y. Otherwise if there exists a vertex h with $d(x, h) = 2$ and $d(y, h) = 2$, then there are two paths $P_1 = xrh$ and $P_2 = yth$. If $r = t$, then we have a K_3 which contradicts the assumption. If $r \neq t$ we have an odd cycle of the form $C_5 = xrhty$ which also contradicts the assumption. □

Lemma 2.4.5. *Let $G_{m,l}$ be a bipartite graph with $diam(G'_{m',l'}) \geq 6$ for some component $G'_{m',l'}$ of $G_{m,l}$. Then $G_{m,l}$ is not $B_{(2,1)}$-game nice.*

Proof. Assume that two colors α and β are given. We will give a winning strategy for Bob such that a proper coloring on $G_{m,l}$ is not possible. Consider the path $P = x_1x_2x_3x_4x_5x_6x_7$ of the component $G'_{m',l'}$, where $d(x_1, x_7) = 6$.

Bob misses the first turn until one of the following cases occurs.

Case 1. Assume Alice colors any vertex $x_i \in P$ with α and a second vertex $y \neq x_i$ by random.

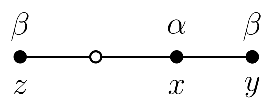
Assume $y \notin N(x_{i+2} \cup x_{i-2})$. If $i \in \{1, 2\}$, then Bob assigns to vertex x_{i+2} the color β and wins the game. If $i \in \{3, 4, 5\}$, then Bob assigns to vertex x_{i+2} or x_{i-2} the color β and wins the game. If $i \in \{6, 7\}$, then Bob assigns to vertex x_{i-2} the color β and wins the game.

Assume $y \in N(P)$ and y is colored with α, then Bob proceeds as above.

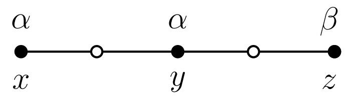
Assume $y \in N(P)$ and y is colored with β. If $i \in \{1, 2\}$ and $y \in N(x_{i+2})$, then Bob assigns to vertex x_{i+3} the color α and wins the game. In case $y \notin N(x_{i+2})$, Bob wins the game by coloring x_{i+2} with β. If $i \in \{3, 4, 5\}$, then Bob assigns to vertex x_{i+2} or x_{i-2} the color β and wins the game. If $i \in \{6, 7\}$ and $y \in N(x_{i-2})$, then Bob assigns to vertex x_{i-3} the color α and wins the game. In case $y \notin N(x_{i-2})$, Bob wins the game by coloring x_{i-2} with β.

Case 2. Assume Alice colors vertices $x, y \in P$. Consider the following case differentiation:

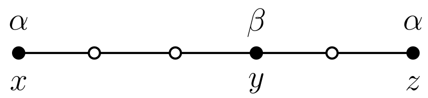
- Let $d(x, y) = 1$. Then there exists a vertex $z \in P$ with either $d(x, z) = 2$ and $d(y, z) = 3$ or $d(y, z) = 2$ and $d(x, z) = 3$. Without loss of generality assume that $d(x, z) = 2$ and $d(y, z) = 3$. Since $(x, y) \in E(G_{m,l})$, they only can be assigned different colors. Suppose x is colored with α and y with β. Then Bob assigns z the color β. Then there is no available color left for the vertex between z and x. See the following figure:



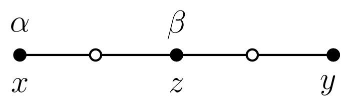
- Let $d(x, y) = 2$. Then there exists a vertex $z \in P$ with either $d(x, z) = 4$ and $d(y, z) = 2$ or $d(y, z) = 4$ and $d(x, z) = 2$. Without loss of generality assume that there exists a $z \in P$ such that $d(x, z) = 4$ and $d(y, z) = 2$ holds. Since $d(x, y) = 2$, obviously x and y has been colored with the same color, say α. The worst case occurs if Bob assigns z the color β, which implies that a feasible coloring of P is not possible anymore since the vertex between y and z cannot be colored anymore.



- Let $d(x, y) = 3$. Then there exists a $z \in P$, such that either $d(x, z) = 5$ and $d(y, z) = 2$ or $d(y, z) = 5$ and $d(x, z) = 2$. Assume there exists a $z \in P$, such that $d(x, z) = 5$ and $d(y, z) = 2$. Since $d(x, y) = 3$, Alice has colored x and y with different colors, otherwise, if she colored x and y with the same color, then there won't be an available color for one of the two uncolored vertices between them. Assume x is and y has been colored with α and β, respectively. Then Bob assigns vertex z the color α. Hence a feasible coloring is not possible anymore, since the vertex between y and z cannot be colored with α or β.



- Let $d(x, y) \geq 4$. Then there exists a $z \in P$, such that $d(x, z) = d(y, z) = 2$. Without loss of generality assume that x has the color α and y has been colored by random. Then Bob replies by coloring z with β. Hence the vertex between x and z cannot be colored anymore, since it is adjacent to x as well as to z. Thus Bob wins.



The cases that Alice colors 2 neighbors of P or 1 neighbor of P and an arbitrary vertex work with analogue arguments as case 1. $\square$

Next we investigate if the existence of semicentral or central vertices results in a $B_{(2,1)}$-game nice graph.

Lemma 2.4.6. *Let $G_{m,l}$ be a bipartite graph. If $G_{m,l}$ contains two semicentral vertices, $u \in M$ and $v \in L$ with $(u, v) \notin E(G_{m,l})$, then $G_{m,l}$ is $B_{(2,1)}$-game nice.*

Proof. We prove a winning strategy for Alice with two colors α and β for Alice. Since the $B_{(2,1)}$ game is being considered, Bob has the opportunity to start the game or to miss a turn.

Case 1: Bob decides to play:

- *Case 1.a:* Assume he colors an arbitrary vertex x different from u and v with the color α. Without loss of generality let $x \in L$ which implies that $(x, v) \notin E(G_{m,l})$. Then Alice replies by assigning vertex v the color α and vertex u the color β. Since v and u are semicentral, they are adjacent to each uncolored vertex in M and L, respectively. This implies that vertices from the independent set M can only be colored with the color β and vertices from L with α. Hence the coloring of $G_{m,l}$ is fixed by α and β.

- *Case 1.b:* Assume he colors a semicentral vertex. Without loss of generality assume he colors $v \in L$ with α. Then Alice replies by coloring vertex $u \in M$ with β. The second vertex she colors by random. Again this fixes the coloring of $G_{m,l}$, since each uncolored vertex from L and M is adjacent to u and v, respectively.

Case 2: Suppose Bob misses his first turn. Then Alice colors vertices u and v with different colors and the coloring is fixed. □

Corollary 2.4.7. *Let $G_{m,l}$ be a bipartite graph. If $G_{m,l}$ contains two central vertices $u \in M$ and $v \in L$, then $G_{m,l}$ is $B_{(2,1)}$-game nice.* □

Let us now consider the $B_{(n,1)}$-coloring game for $n \in \mathbb{N}$ and $n \geq 3$. Thus, we need to introduce the notion of n-central vertices.

Definition 2.4.8. Let $G_{m,l}$ be a bipartite graph. A set of n vertices $\{x_1, ..., x_n\} \in V(G_{m,l})$ where $n \in \mathbb{N}$ and $n \geq 3$ is called *n-central* if every vertex $y \in V(G_{m,l}) \setminus \{x_1, ..., x_n\}$ is adjacent to at least one vertex from $\{x_1, ..., x_n\}$.

Example:

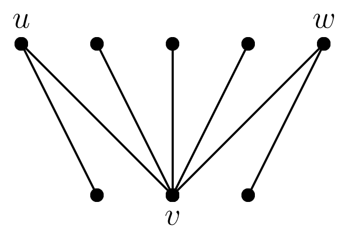


FIGURE: Vertices u, v and w are 3-central

Lemma 2.4.9. *Let $G_{m,l}$ be a bipartite graph and $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. If $G_{m,l}$ contains q vertices that are q-central, then $G_{m,l}$ is $B_{(n,1)}$-game nice.*

Proof. Let $\{x_1, ..., x_q\}$ be the q-central vertices of $G_{m,l}$ and let $\{x_1, ..., x_i\} \in M$ and $\{x_{i+1}, ..., x_q\} \in L$ with $i \in \{1, ..., q\}$.

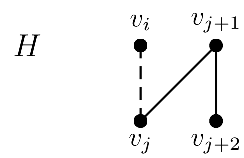
- If Bob colors a vertex $y \in \{x_1, ..., x_q\}$, then Alice checks to which independent set y belongs. In her turn Alice colors the remaining q-central vertices. The q-central vertices that are in the same independent set as y get the same color as y and the remaining get the second color. Obviously this fixes the coloring and Alice wins the game.

  If Bob colors a vertex $y \notin \{x_1, ..., x_q\}$, then Alice colors in her move the q-central vertices with the same conditions as above.

- If Bob decides to miss his first turn, then Alice colors the q-central vertices by coloring $\{x_1, ..., x_i\}$ with 1 and $\{x_{i+1}, ..., x_q\}$ with 2. □

Next we turn our attention again to the $B_{(2,1)}$-coloring game and give another property for bipartite graphs that are $B_{(2,1)}$-game nice.

Definition 2.4.10. Let $G_{m,l}$ be a bipartite graph. We call two vertices $v_i \in M$ and $v_j \in L$ *expandable*, if there is a path $P = v_j v_{j+1} v_{j+2}$, such that $(v_i, v_{j+2}) \notin E(G_{m,l})$. v_i and v_j are allowed to be adjacent vertices. In particular the induced subgraph $H \in G_{m,l}$ on $\{v_i, v_j, v_{j+1}, v_{j+2}\}$ is of the following form:



Lemma 2.4.11. *Let $G_{m,l}$ be a connected bipartite graph with $m, l \geq 3$. Then $G_{m,l}$ is not $B_{(2,1)}$-game nice if every pair of vertices from distinct independent sets is expandable.*

Proof. Assume that two colors α and β are given. We will give a winning strategy for Bob such that a feasible coloring of $G_{m,l}$ is not possible. Since the $B_{(2,1)}$ game is played, Bob has to start the game and is allowed to miss a turn. Assume he missed his first turn and Alice has colored two arbitrary vertices x and y in her first turn.

- Assume x and y are from the same independent set, say M. Since $m, l \geq 3$, there exists at least one uncolored vertex $z \in M$. Since the graph is connected, there exists a path $P_{zx} = zv_1u_1v_2u_2...v_{n-1}u_{n-1}v_nx$ where $v_1, ..., v_n \in L$ and $u_1, ..., u_{n-1} \in M$. Bob wins the game by coloring u_{n-1} with the color distinct from the color of x. In case $u_{n-1} = y$, Bob colors u_{n-2} with the color distinct from the color of x.

- Without loss of generality assume $x \in M$ and $y \in L$. Because of the assumption there exists either a path $P_y = yy'y''$ with $\{(x, y'), (x, y'')\} \notin E(G_{m,l})$ or a path $P_x = xx'x''$ with $\{(y, x'), (y, x'')\} \notin E(G_{m,l})$. Without loss of generality suppose there exists the path P_y and Alice has colored y with α. Then Bob replies by coloring vertex y'' with β. If Alice colored vertex x with β, then Bob could still color y'', since x and y'' are not adjacent. Thus we can conclude that there is no available color left for the vertex y', since $\{(y, y'), (y', y'')\} \in E(G_{m,l})$. □

Next we expand the above result for the $B_{(n,1)}$-coloring game for $n \geq 3$.

Definition 2.4.12. Let $G_{m,l}$ be a bipartite graph and $n \in \mathbb{N}$ with $n \geq 3$. We call n vertices $\{x_1, ..., x_n\}$ *n-expandable*, if there exists an $i \in \{1, ..., n\}$ such that $P = x_iy_1y_2$ is a path with $(x_j, y_2) \notin E(G_{m,l})$ for all $j \in \{1, ..., n\}$.

Lemma 2.4.13. *Let $G_{m,l}$ be a connected bipartite graph with $m, l \geq 3$ and let $n \in \mathbb{N}$ with $n \geq 3$. Then $G_{m,l}$ is not $B_{(n,1)}$-game nice if every set of n vertices is n-expandable.*

Proof. Bob misses his first turn and then by the same arguments as in 2.4.11 he wins the game after Alice has colored n vertices. □

Further we determine the class of $B_{(2,1)}$-game perfect graphs with clique number 2.

Lemma 2.4.14. *Let $G_{m,l}$ be a connected bipartite graph with $m, l \geq 3$. If $G_{m,l}$ is $B_{(2,1)}$-game nice, then $G_{m,l}$ contains either two semicentral vertices $x \in M$ and $y \in L$ with $(x, y) \notin E(G_{m,l})$ or two central vertices $u \in M$ and $v \in L$.*

Proof. Assume the assertion of the lemma is false. Then one of the following is satisfied:

(i) There exist in one independent set semicentral vertices and in the other there is no semicentral or central vertex.

(ii) There exist neither in M nor in L any semicentral vertices or central vertices.

(iii) If there exist two semicentral vertices $v \in M$ and $u \in L$, then (v, u) is an edge.

(iv) There exist only in one independent set a central vertex and in the other there is no semicentral or central vertex.

(v) There exist in M central vertices and in L semicentral vertices.

We will show that in each case each pair of vertices from distinct sets is expandable. Since the $B_{(2,1)}$ game is considered, Bob starts the game and is allowed to miss each turn. Assume that Bob misses the first turn and Alice colors vertices x and y. In the same manner as in the proof of lemma 2.4.11 we can show that it does not make sense for Alice if x and y are from the same independent set. Hence without loss of generality let $x \in M$ and $y \in L$.

Suppose (i) holds. Since y is not semicentral, there is at least one vertex $z \in M$, such that $(y, z) \notin E(G_{m,l})$. Since $G_{m,l}$ is connected, there is at least one path P_{xz} from x to z of even length with $P_{xz} = xa_1a_2...a_{k-1}z$. If $(y, a_2) \notin E(G_{m,l})$, then x and y are expandable. Otherwise if $(y, a_2) \in E(G_{m,l})$, then it has to be checked whether x and a_3 are adjacent. If $(x, a_3) \notin E(G_{m,l})$, then x and y are expandable. Otherwise it has to be checked whether y and a_4 are adjacent. In this manner we can proceed until we reach the following configuration, where $(x, a_{k-1}) \in E(G_{m,l})$. Because of the assumption that $(y, z) \notin E(G_{m,l})$, x and y are expandable.

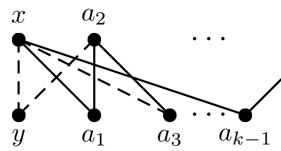


FIGURE: x and y are expandable since $(y, x) \notin E(G_{m,l})$.

The basic idea of the proof above is that for each pair x, y of vertices from distinct independent sets there exists a vertex z which is not adjacent to both. Since the graph is connected there exists a path from z to either x or y and so we proved that x and y are expandable. Hence for the following cases we have just to show that for each pair of vertices from distinct independent sets there exists a vertex which is not adjacent to both.

Suppose (ii) holds. This case is trivial.

Suppose (iii) holds. Because of (ii), it remains to consider the two semicentral vertices $v \in M$ and $u \in L$ that are adjacent. Obviously, since $(v, u) \in E$ there exists a vertex $z \in V$ which is not adjacent to both. Thus v and u are expandable.

Suppose (iv) holds. Because of (ii), it remains to consider the central vertex. Without loss of generality let $x \in M$ be a central vertex. Consider the pair (x, y) where $y \in L$. Since y is not a central vertex, there exists a vertex $z \in M$ which is not adjacent to y and since our graph is bipartite not to x.

Suppose (v) holds. Because of (ii), it remains to consider the central and the semicentral vertex. Let $x \in M$ be a central vertex and let $y \in L$ be a semicentral vertex and consider the pair (x, y). Since y is semicentral there exists a vertex $z \in M$ which is not adjacent to y and since our graph is bipartite not to x. Hence x and y are expandable. □

Finally we can draw the following conclusion.

Proposition 2.4.15. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m,\ |L| = l$ and $m, l \geq 3$. Then $G_{m,l}$ is $B_{(2,1)}$-game nice if and only if one of the properties P_1 and P_2 is satisfied.*

- P_1 : *$G_{m,l}$ contains two semicentral vertices $v \in M$ and $u \in L$ with $(v, u) \notin E(G_{m,l})$.*
- P_2 : *$G_{m,l}$ contains two central vertices $x \in M$ and $y \in L$.*

Proof. " $\Leftarrow$ " Suppose P_1 or P_2 is satisfied. Then by lemma 2.4.6 and corollary 2.4.7 $G_{m,l}$ is $B_{(2,1)}$-game nice.

" $\Rightarrow$ " Suppose $G_{m,l}$ is $B_{(2,1)}$-game nice. Then by lemma 2.4.14 the assertion is proved. □

Corollary 2.4.16. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m,\ |L| = l$ and $m, l \geq 3$. Then $G_{m,l}$ is $B_{(2,1)}$-game perfect if and only if for each induced subgraph $H \subseteq G_{m,l}$ one of the properties P_1 and P_2 from proposition 2.4.15 is satisfied.*

Now we can easily determine also the class of $B_{(n,1)}$-game perfect graphs with clique number 2.

Lemma 2.4.17. *Let $G_{m,l}$ be a connected bipartite graph with $m, l \geq 3$ and let $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. If $G_{m,l}$ is $B_{(n,1)}$-game nice, then $G_{m,l}$ contains q vertices that are q-central.*

Proof. Assume $G_{m,l}$ is a bipartite graph which does not contain q vertices that are q-central. Hence for every set of q vertices $\{x_1, ..., x_q\}$ there exists a vertex y which is not adjacent to $\{x_1, ..., x_q\}$. Hence by similar arguments as in 2.4.14 every set of q vertices is q-expandable which guarantees Bob's victory. □

Finally we conclude the following:

Proposition 2.4.18. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m$, $|L| = l$ and $m, l \geq 3$. Let $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. Then $G_{m,l}$ is $B_{(n,1)}$-game nice if and only if there exist a set of q vertices that are q-central.*

Proof. See lemmas 2.4.9, 2.4.13 and 2.4.17. □

Corollary 2.4.19. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m$, $|L| = l$ and $m, l \geq 3$. Let $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. Then $G_{m,l}$ is $B_{(n,1)}$-game perfect if and only if for each induced subgraph $H \subseteq G_{m,l}$ there exist a set of q vertices that are q-central.* □

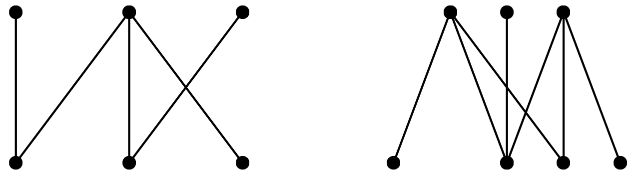
2.5 Construction of $\mathbf{B}_{(\mathbf{n},1)}$-Nice Graphs with Clique Number 2

The aim of this section is to construct $B_{(2,1)}$-game nice and $B_{(n,1)}$-game nice graphs for $n \geq 3$. In particular we will construct $B_{(2,1)}$-game nice graphs from graphs that are not $B_{(2,1)}$-game nice and do not contain an odd cycle. Thus we will take only bipartite graphs into our consideration. Consider the following operation:

Central-Operation: Let $G_{m,l}$ be a bipartite graph with independent sets $|M| = m$ and $|L| = l$.

- Add edges or vertices so that in M and in L there is a central vertex.
- Add vertices or edges so that in M and in L there are, respectively, semi-central vertices that are not adjacent to each other.

Obviously, if we apply the central-operation on a bipartite graph, the result will be a $B_{(2,1)}$-game nice graph. Consider the following two bipartite graphs:



Clearly the graphs above are not $B_{(2,1)}$-game nice. If we apply the central-operation, they become $B_{(2,1)}$-game nice. Hence, because of the central-operation, there cannot exist a bipartite forbidden configuration (note that forbidden configuration means that if the configuration is an induced subgraph, then the supergraph is not $B_{(2,1)}$-game nice).

Consider the following operation:

n-Central-Operation: Let $G_{m,l}$ be a bipartite graph with independent sets $M = |m|$ and $|L| = l$. Let $n, q \geq 3$ and $q \leq n$. Add to $G_{m,l}$ vertices or edges so that there exist q vertices that are q-central.

By lemma 2.4.9, the n-central-operation makes every bipartite graph $B_{(n,1)}$-nice.

2.6 $\mathbf{A_{(n,1)}}$-Game Perfect Graphs with Clique Number 2

As we determined the $B_{(n,1)}$-game nice graphs, we proceed with analyzing the $A_{(n,1)}$-game nice as well as $A_{(n,1)}$-game perfect graphs. Thus we assume that Alice is allowed to decide wether to play or to miss a turn. First we work under the assumption that $n = 2$ and afterwards we consider the general case which is $n > 2$.

According to lemma 2.4.1 we can draw for the $A_{(2,1)}$-coloring game the following conclusion.

Corollary 2.6.1. *Let $G = (V, E)$ be a $A_{(2,1)}$-game nice graph with $\omega(G) = 2$. Then*

G is a bipartite graph.

Hence we will refer to the bipartite graph $G_{m,l}$ with two independent sets M and L, where $|M| = m$ and $|L| = l$. Obviously $\omega(G) = 2$ always holds if G is bipartite.

Remark 2.6.2. Let $G = (V, E)$ be a bipartite graph with at least 3 vertices. Since the $A_{(2,1)}$-coloring game is played, Alice starts the game and is allowed to miss a turn. Assume she misses her first turn and Bob colors an arbitrary vertex x with the color α. Then Alice will play in her second turn at the latest, otherwise Bob could color a vertex y with $d(x, y) = 2$ with the color β. This implies that a feasible coloring of the graph is not possible anymore.

In the following we will analyze bipartite graphs with diameter greater or equal 7. For this purpose we will consider an induced path $P_8 = x_1x_2...x_8$ of length 7. In particular, if we deal with the neighborhood of P_8, then we only admit the case that $|N(z) \cap P_8| = 1$ for $z \in N(x_i)$ and $i \in \{1, ..., 8\}$. Otherwise if z is adjacent to another vertex x_j of P_8, then $d(x_i, x_j) = 2$ must hold. Otherwise, if $d(x_i, x_j) = k$, where k is odd, then $G'_{m',l'}$ would contain an odd cycle, which would contradict the assumption that $G'_{m',l'}$ is bipartite. In case that $k > 2$ and k even, then $d(x_i, x_j) < 7$, since there would exist the path $x_1...x_izx_j...x_8$, which contains at least one vertex less than P_8. However, if we consider the case $d(x_i, x_j) = 2$, then z would be on another path $P'_8 = x_1...x_izx_j...x_8$ of length 7. Hence we will ignore this case.

Lemma 2.6.3. *Let $G_{m,l}$ be a bipartite graph with $diam(G'_{m',l'}) \geq 7$ for some component $G'_{m',l'} \subseteq G_{m,l}$, $m' \leq m$ and $l' \leq l$. Then $G_{m,l}$ is not $A_{(2,1)}$-game nice.*

Proof. Assume two colors $\{\alpha, \beta\}$ are given. Since $diam(G'_{m',l'}) \geq 7$, there exists an induced path $P_8 = x_1x_2x_2x_3x_4x_5x_6x_7x_8$. Assume Alice misses the first move. Then Bob colors vertex x_3, say with α. Then one of the vertices Alice has to color is from $\{x_1, x_2\}$, otherwise Bob could attack x_2 by coloring x_1 with β. The second vertex she goes for is x_i, $i \in \{4, ..., 8\}$. If $i \in \{4, 5, 6\}$, then Bob replies by coloring vertex x_{i+2} with distinct color than the color of x_i. Otherwise if $i \in \{7, 8\}$, he will color x_{i-2} with distinct color than the color of x_i. Thus there won't be an available color left for x_{i+1} or x_{i-1}, respectively.

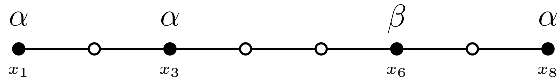


FIGURE: The coloring of P_8 if Alice decides to color x_1 and x_6.

Assume Alice colors in her first move vertices u and v. We will make a case differentiation in which components u, v are:
Case 1: Assume $u, v \in P_8$.

- Let $d(u, v) = 1$ and without loss of generality assume that $u = x_j$ and $v = x_{j+1}$ for $j \in \{1, ..., 7\}$. Then either $d(x_1, u) \geq 3$ or $d(v, x_8) \geq 3$. Let $d(x_1, u) = 3$, then there exists a path $x_1x_2x_3u$. Moreover let u be colored with α. Then Bob takes turn by coloring vertex x_2 with β, such that there is no available color left for x_3. The game proceeds in the same manner, if we assume that $d(v, x_8) = 3$.

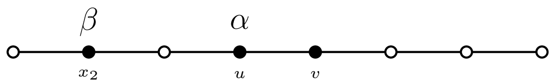


FIGURE: The coloring of P_8 if $d(x_1, u) = 3$.

- Let $d(u, v) = 2$ and without loss of generality assume that $u = x_j$ and $v = x_{j+2}$ for $j \in \{1, ..., 6\}$. Then either $d(x_1, u) \geq 3$ or $d(v, x_8) \geq 3$. Thus we can refer to the case $d(u, v) = 1$.

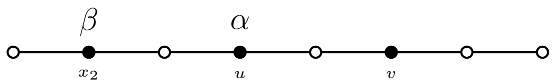


FIGURE: The coloring of P_8 if $d(x_1, u) = 3$.

- Let $d(u, v) = 3$, and without loss of generality assume that $u = x_j$ and $v = x_{j+3}$ for $j \in \{1, ..., 5\}$. Then either $d(x_1, u) \geq 2$ or $d(u, x_8) \geq 2$. Assume it holds $d(x_1, u) = 2$. Thus there exists the path x_1x_2u. Again we can argue similar as above.

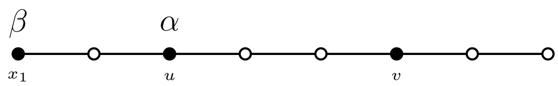


FIGURE: The coloring of P_8 if $d(x_1, u) = 2$.

- Let $d(u,v) \geq 4$. Then there is a path $uy_1y_2...y_iv$, where $i \in \{3,4,5,6\}$. Without loss of generality assume that Alice has colored vertex u with α. Then Bob replies by coloring y_2 with β, such that y_1 cannot be colored with a feasible color anymore.

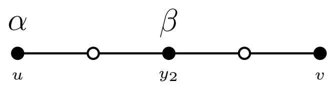
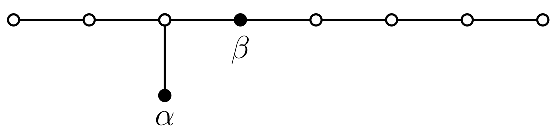


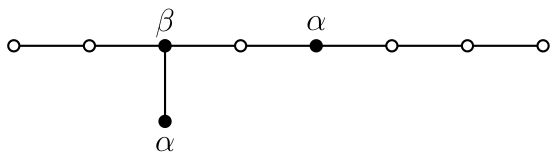
FIGURE: The coloring of P_8 if $d(u,v) = 4$.

Case 2: Let u or v be adjacent to P_8. Without loss of generality let $u \in N(x_i)$ for $i \in \{1,...,8\}$. In particular, assume that u has been colored with α.

- Assume $v \notin \{P_8, N(P_8)\}$, then Bob attacks x_i by coloring x_{i-1} or x_{i+1} with β, where obviously at least one of these vertices exists. Thus Bob wins.

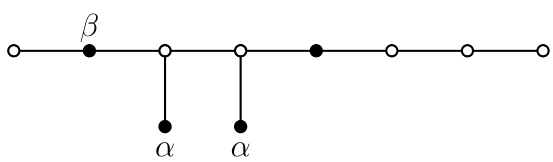


- Assume $v \in P_8$. Let $v = x_i$, then v has the color β, since $(u,v) \in E(G'_{m',l'})$. Bob wins by coloring either x_{i-2} or x_{i+2} with α, which implies that there is no available color left for either x_{i-1} or x_{i+1}. Note, that since P_8 is of length 7, at least one of the vertices $\{x_{i-2}, x_{i+2}\}$ exists.



If $v \neq x_i$, then it is easily seen that Bob would win the game.

- Assume $v \in N(P_8)$ and let $v \in N(x_j)$ for $i \neq j$. Then either x_i or x_j can be attacked by Bob. The details are left to the reader.



Case 3: Let $u, v \notin \{P_8, N(P_8)\}$. Then Bob wins the game since we can proceed analogously to the proof of the case where Alice misses her first turn. □

The proof of the following lemma runs by the same method as in lemma 2.4.6, where the $B_{(2,1)}$-coloring game was played and Bob decided to play his first turn which equals to the case that Alice misses her first turn.

Lemma 2.6.4. *Let $G_{m,l}$ be a bipartite graph. If $G_{m,l}$ contains two vertices $x \in M$ and $y \in L$ that are either semicentral with $(x, y) \notin E(G_{m,l})$ or central, then G is $A_{(2,1)}$-game nice.* □

Remark 2.6.5. Let $G_{m,m}$ be a bipartite graph with $m = 3$. Then $G_{m,m}$ is $A_{(2,1)}$-game nice.

Proof. Alice decides to miss her first turn. This implies Bob is forced to start the game. Assume Bob colors an arbitrary vertex. Alice decides to color the other two vertices in this independent set with the color already used by Bob. Then the coloring is fixed and Alice wins the game. □

Using lemma 2.4.9 we can easily conclude:

Lemma 2.6.6. *Let $G_{m,l}$ be a bipartite graph and let $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. If $G_{m,l}$ contains a set of q vertices that are q-central, then $G_{m,l}$ is $A_{(n,1)}$-game nice.* □

Our further procedure is to determine the class of $A_{(2,1)}$-game nice graphs and to make use of this result for determining also the class of $A_{(n,1)}$-game nice graphs for $n \geq 3$.

Definition 2.6.7. Let $G_{m,l} = (V, E)$ be a connected bipartite graph. $G_{m,l}$ is called *coverable* if for each vertex x there exist two vertices y, z such that $\{x, y, z\}$ are 3-central vertices of $G_{m,l}$.

Proposition 2.6.8. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m$ and $|L| = l$, with $m \geq 4$ and $l \geq 3$. Further assume $G_{m,l}$ neither contains two semicentral vertices $x \in M$ and $y \in L$ with $(x, y) \notin E(G_{m,l})$*

nor two central vertices $u \in M$ and $v \in L$. Then $G_{m,l}$ is $A_{(2,1)}$-game nice if and only if it is coverable.

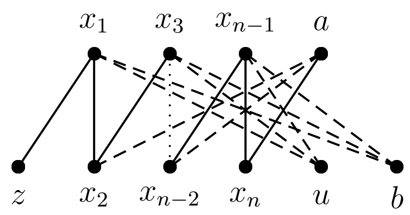
Proof. " $\Leftarrow$ ": Suppose two colors α and β are given. We will proof a winning strategy for Alice with two colors. Since the $A_{(2,1)}$-coloring game is played Alice has the right to start the game and to miss each turn. Assume she decides to miss the first turn and Bob colors an arbitrary vertex u with α. Since the graph is coverable there exist two vertices u' and u'', such that u, u', u'' are 3-central vertices of the graph. Alice replies by coloring u' and u'' according to the following rule: Vertices in the same independent set as u get the same color as u and vertices in the other independent set get different color than u. Clearly this implies that the coloring of $G_{m,l}$ is fixed, since u, u', u'' are 3-central vertices of $G_{m,l}$ and $G_{m,l}$ is bipartite. If Alice decides to miss also her second turn, she clearly loses the game.

" $\Rightarrow$ ": Assume that $G_{m,l}$ is $A_{(2,1)}$-game nice and not coverable.

- Suppose Alice starts the game. Since $G_{m,l}$ contains neither two semicentral vertices $x \in M$ and $y \in L$ with $(x, y) \notin E(G_{m,l})$ nor two central vertices $x \in M$ and $y \in L$, after Alice's turn there exist at least one vertex which is not adjacent to x and y. Hence using similar arguments as in 2.4.14, Alice loses the game.

- Suppose Alice decides to miss the first turn. Since $G_{m,l}$ is not coverable, there exists a vertex u such that there are no two other vertices u' and u'' so that $\{u, u', u''\}$ are 3-central vertices of $G_{m,l}$. Bob decides to color u. Without loss of generality assume he uses color α. No matter if Alice decides to color in her second turn or to miss it, there will be one vertex which is adjacent only to uncolored vertices. Without loss of generality let a, b be the vertices colored by Alice and let z be a vertex which is adjacent only to uncolored vertices. Obviously either $a, b \in L$ or $a, b \in M$ or $a \in L$ and $b \in M$ holds.

  Consider the case $a \in L$ and $b \in M$ first and assume $z \in M$. The proof below will not take into account how the vertices are colored. Since the graph is connected, there exists a path $P_{zu} = zx_1, x_2, ..., x_n, a$ from z to a. Bob does now the following: If x_{n-1} is not adjacent to u and to b, he colors x_{n-1} with a color different from the color of vertex a and hence he wins the game. If x_{n-1} is adjacent either to u or to b, Bob checks if x_{n-2} is adjacent to a. If x_{n-2} is not adjacent to a, Bob colors x_{n-2} with a color different from u if $(x_{n-1}, u) \in E$ or different from b if $(x_{n-1}, b) \in E$ (in case $(x_{n-1}, u) \in E$ and $(x_{n-1}, b) \in E$ he takes a color different from u or to b). If x_{n-2} is adjacent to a, Bob checks if x_{n-3} is adjacent to either u or b and goes on with the same method as above. This process will end at the latest if x_1 is adjacent either to u or to b. In this case Bob colors z with a color different from u if $(x_1, u) \in E$ or different from b if $(x_1, b) \in E$ (in case $(x_1, u) \in E$ and $(x_1, b) \in E$ he takes a color different from u or to b) and hence he wins the game. This is possible since z is not adjacent to a. Consider the figure below:



The case $a, z \in L$ and $b \in M$ works similar.

Now consider the case $a, b \in L$. Here we work with the path from z to u and the rest of the proof works similar as above.

Clearly if $a, b \in M$, Alice loses the game, since in M there is at least one uncolored vertex. □

Corollary 2.6.9. *Let $G_{m,l}$ be a connected bipartite graph where $m \geq 4$ and $l \geq 3$. $G_{m,l}$ is $A_{(2,1)}$-game perfect if and only if for each subgraph $H \subseteq G_{m,l}$ one of the following holds:*

(i) H contains in each independent set at least one central vertex.

(ii) H contains in each independent set at least one semicentral vertex such that there exist two semicentral vertices from different independent sets that are not adjacent.

(iii) H is coverable. □

Now we turn our attention to the $A_{(n,1)}$-coloring game.

Definition 2.6.10. Let $G_{m,l}$ be a connected bipartite graph. Further let $n \in \mathbb{N}$ with $n \geq 3$. $G_{m,l}$ is called *n-coverable* if for each vertex v_1 there exist $n-1$ vertices $v_2, ..., v_n$ such that $\{v_1, ..., v_n\}$ are n-central vertices of $G_{m,l}$.

Proposition 2.6.11. *Let $G_{m,l}$ be a connected bipartite graph with two independent sets $\{M, L\}$, where $|M| = m$ and $|L| = l$, with $m \geq 4$ and $l \geq 3$. Further let $n, q \in \mathbb{N}$ with $n, q \geq 3$ where $q \leq n$ and let $G_{m,l}$ does not contain q-central vertices. Then $G_{m,l}$ is $A_{(n,1)}$-game nice if and only if it is $(n+1)$-coverable.*

Proof. " $\Leftarrow$ " Alice misses her first turn and forces Bob to start the game. Assume Bob colors vertex x. Since $G_{m,l}$ is $(n+1)$-coverable, Alice is able to fix the coloring by the same method as in proof 2.6.8. If she misses her turn, she obviously loses the game.

" $\Rightarrow$ " Assume the graph $G_{m,l}$ is not $(n+1)$-coverable. Then Bob colors a vertex x for which there does not exist n vertices that are $(n+1)$-central vertices of $G_{m,l}$. Then by similar arguments as in 2.6.8, Bob wins the game. □

Corollary 2.6.12. *Let $G_{m,l}$ be a connected bipartite graph where $m \geq 4$ and $l \geq 3$ and let $n, q \in \mathbb{N}$ with $n, q \geq 3$ and $q \leq n$. $G_{m,l}$ is $A_{(n,1)}$-game perfect if and only if for each subgraph $H \subseteq G_{m,l}$ one of the following holds:*

(i) H contains q-central vertices.

(ii) H is $(n+1)$-coverable. □

Remark 2.6.13. Because of 2.6.8 and 2.6.11, the central operation and n-central operation from section 2.5, respectively, also produces $A_{(2,1)}$-game nice graphs and $A_{(n,1)}$-game nice graphs for $n \geq 3$.

2.7 $(\mathbf{n}, \mathbf{1})$-Game Perfect Graphs with Clique Number 2

In this section we turn our attention to the ordinary asymmetric game and will use the previous results to determine $(n, 1)$-game perfect graphs and $(1, n)$-game perfect graphs with clique number 2 for $n \in \mathbb{N}$ and $n \geq 2$. That is, we consider the asymmetric coloring game where Alice starts the game and no player is allowed to miss a turn.

As we did in the previous sections, we need to consider only bipartite graphs. We call a bipartite graph $G_{m,l}$ with independent sets M and L *special* if $G_{m,l}$ has at least one central vertex in M and in L,respectively, or if $G_{m,l}$ has at least two semicentral vertices v, w with $v \in M$ and $w \in L$ and $(v, w) \notin E(G_{m,l})$.

Let us first investigate $(2, 1)$-game nice graphs.

Proposition 2.7.1. *Let $G = (V, E)$ be a graph with clique number 2. Then G is $(2, 1)$-game nice if and only if it holds either*

(i) G consists of a special graph and an arbitrary number of edges.

(ii) G consists of special graphs $\{G_1, ..., G_j\}$ so that $\sum_{i=1}^{j} |V(G_i)| \mod 3 = 2$ and at most one coverable graph, or

(iii) G consists of special graphs $\{G_1, ..., G_j\}$ so that $\sum_{i=1}^{j} |V(G_i)| \mod 3 = 2$ and coverable graphs $\{G_{j+1}, ..., G_m\}$ so that $|V(G_k)| \mod 3 = 0$ for all $k \in \{j + 1, ..., m\}$.

Proof. Suppose H is a special graph. Alice wins the $(2, 1)$-coloring game on H by coloring two central vertices in different independent sets or by coloring two semicentral vertices in different independent sets that are not connected by an edge. Thus Alice also wins the game if H consists of more than one special graph, by coloring in each step the central, respectively, the semicentral vertices of each special graph.

According to 2.6.8 Alice has a winning strategy for a coverable graph if Bob starts the game, otherwise she loses the game if two colors are given.

The conditions in the assumptions guarantee that Alice achieves to color all special graphs and that Bob starts coloring a coverable graph.

Otherwise if a special graph is not given or if Alice starts coloring in a coverable graph, she loses the game when 2 colors are given by 2.6.8. □

Next we want to determine the class of $(n, 1)$-game perfect graphs for $n \geq 3$. Let $q \in \mathbb{N}$ with $q \leq n$. We call a bipartite graph $G_{m,l}$ *n-special* if it has a set of q vertices that are q-central.

Proposition 2.7.2. *Let $G = (V, E)$ be a graph with clique number 2 and let $n \in \mathbb{N}$ with $n \geq 3$. Then G is $(n, 1)$-game nice if and only if it holds either*

(i) G consists of a special or an n-special graph and an arbitrary number of edges.

(ii) G consists of an arbitrary number of special graphs and n-special graphs $\{G_1, ..., G_j\}$ so that $\sum_{i=1}^{j} |V(G_i)| \mod n+1 = n$ and at most one $(n+1)$-coverable graph, or

(iii) G consists of an arbitrary number of special graphs and n-special graphs $\{G_1, ..., G_j\}$ so that $\sum_{i=1}^{j} |V(G_i)| \mod n+1 = n$ and $(n+1)$-coverable graphs $\{G_{j+1}, ..., G_m\}$ so that $|V(G_k)| \mod n+1 = 0$ for $k \in \{j+1, ..., m\}$.

Proof. By lemma 2.4.9, Alice has a winning strategy for an n-special graph and by lemma 2.6.11, Alice has a winning strategy if Bob starts the game in an $(n+1)$-coverable graph. Obviously by the assumptions, Alice is able to start coloring in a n-special graph and she easily achieves that Bob will always be the first who colors in an $(n+1)$-coverable graph.

Otherwise a p-special graph for $p \in \mathbb{N}$ and $p > n$ is given or Alice colors in an $(n+1)$-coverable graph first and loses the game if two colors are given by 2.6.11. □

2.8 $(1, n)$-Game Perfect Graphs with Clique Number 2

In this section we will determine all $(1, n)$-game perfect graphs with clique number 2 and $n \geq 2$ using the results from 2.2.5 and 2.3.4.

Proposition 2.8.1. *Let $G = (V, E)$ be a graph with clique number 2. Then G is $(1, 2)$-game nice if and only if G consists of an arbitrary number of edges and at most one star.*

Proof. As we already showed, more than one star is not allowed since Bob could win the game after his first move. Thus only one star is allowed and clearly an arbitrary number of edges since the asymmetric game chromatic number of an edge is 2. Alice's winning strategy is to color the center of the star and the rest of the coloring is trivial. □

Finally we determine also the class of $(1, n)$-game perfect graphs with clique number 2 for $n \geq 3$.

Proposition 2.8.2. *Let $G = (V, E)$ be a graph with clique number 2 and $n \geq 3$. Then G is $(1, n)$-game nice if and only if G consists of an arbitrary number of edges and at most one star.*

Proof. The proof runs as in 2.8.1. □

2.9 (n, m)-Game Perfect Graphs with Clique Number 2 and $n, m \geq 2$

In the last section of this chapter we want to determine the class of (n, m)-game perfect graphs with clique number 2 and $n, m \geq 2$. By the result so far we can easily conclude:

Corollary 2.9.1. *Let $G = (V, E)$ be a graph with clique number 2 and let $n, m \in \mathbb{N}$ with $n, m \geq 2$. Then G is (n, m)-game nice if and only if G consists of at most one* *special graph or at most one n-special graph and an arbitrary number of edges.*

Proof. Clearly if G consists of at most one special graph or at most one n-special graph and an arbitrary number of edges, then the graph has game chromatic number equal to 2. If G is not of that form, then after Alice's move there exists at least one vertex x whose neighbors are all uncolored. Hence Bob wins the game if 2 colors are given. $\square$

Open Problems: In this chapter we introduced the notion of asymmetric game perfect graphs and determined the asymmetric game perfect graphs with clique number 2. Further open questions are to determine asymmetric game perfect graphs with clique number greater or equal to 3. Moreover the aim could be to find an analogue to the well-known strong perfect graph theorem which states that a graph is perfect if and only if it is a berge graph (see [19]).

Chapter 3

The Circular Two-Person Game on Weighted Graphs

In this chapter we intend to generalize the theory of the circular coloring by bringing together the notion of the circular coloring of weighted graphs and the two-person game on graphs. After introducing the circular two-person game for weighted graphs and the circular game chromatic number of weighted graphs, denoted by $\gamma_c(G^w)$, we work out some basic properties of the new parameter. In particular $\gamma_c(G^w)$ is related to $\gamma_c(G)$ for $w(v_i) \geq 1$ and $w(v_i) \leq 1$ for all $v_i \in V(G^w)$. However, we look more closely at the more general case $w(v_i) > 0$ and show that the relation between $\gamma_c(G^w)$ and $\gamma_c(G)$ is dependent on the distribution of the vertex-weights. Afterwards we analyze the new parameter for the class of complete graphs, complete multipartite graphs, bipartite graphs, cycles as well as trees and planar graphs. In addition we restrict our attention to certain distributions of vertex-weights.

For the purpose of introducing the circular two-person game on weighted graphs we need to summarize some relevant material on circular coloring in terms of weighted graphs.

Weighted Graphs

Definition 3.0.2. A graph $G^w = (V, E, w)$ is called *weighted* if there exists a mapping $w : V \to \mathbb{R}^+$ which assigns to each vertex $x \in V(G^w)$ a *vertex-weight*

$w(x) > 0$.

For a weighted graph $G^w = (V, E, w)$ with $V(G^w) = \{v_1, ..., v_n\}$ and $w(G^w) = \{w_{v_1}, ..., w_{v_n}\}$, we define

$$w_{\max}(G^w) := \max\{w_{v_i} \mid 1 \leq i \leq n\}$$

as the *maximum weight of* $V(G^w)$ and

$$w_{\min}(G^w) := \min\{w_{v_i} \mid 1 \leq i \leq n\}$$

as the *minimum weight of* $V(G^w)$.

Note that there exists also the notion of *edge-weights*, where a mapping $A : V \times V \to \mathbb{R}^+ \cup \{0\}$ assigns to each pair of vertices (x, y) an edge-weight $A(x, y) := a_{x,y} \geq 0$. Since we deal with graphs without edge-weights, we set $a_{x,y} = 1$ if $(x, y) \in E(G)$ and $a_{x,y} = 0$, if $(x, y) \notin E(V)$.

Further we give the definition of circular coloring of weighted graphs introduced by W. A. Deuber and X. Zhu in [4]

Circular Coloring of Weighted Graphs

Definition 3.0.3. Let $G^w = (V, E, w)$ be a graph with an nonnegative weight function $w : V \to [0, \infty)$. For $r \in \mathbb{R}^+$ let C^r be a circle with circumference r. An *r-circular coloring* of G is a mapping f_w which assigns to each vertex of G an open arc of C^r such that

(i) if $(x, y) \in E(G^w)$ then $f_w(x) \cap f_w(y) = \emptyset$ and

(ii) for all vertices $x \in V(G^w)$ the length of the arc $f_w(x)$ is at least $w(x)$.

The *circular chromatic number* $\chi_c(G^w)$ is defined as

$$\chi_c(G^w) = \inf\{r : \text{there is an } r\text{-circular coloring of } G^w\}.$$

In particular they prove that $\chi_c(G^w)$ is rational if $w(x_i)$ are rational for all $x_i \in V(G^w)$.

Circular coloring of weighted graphs is a generalization of circular coloring of non-weighted graphs introduced by X. Zhu in [9]. The difference is that in circular coloring of non-weighted graphs the vertices are assigned to open unit length arcs since they are not weighted; in particular if we set $w : V \to \{1\}$ in definition 3.0.3 we have the definition of the ordinary circular chromatic number.

Circular coloring of non-weighted graphs were introduced as an equivalent definition of (k, d)-coloring by A. Vince in [10] and it is a generalization of the regular coloring since the outcome is not necessary an element of $\mathbb{N}$ but $\mathbb{Q}$. In particular for the circular chromatic number denoted by χ_c and the ordinary chromatic number χ it holds the following relation for a graph G:

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

For simplicity of notation we write circular coloring instead of r-circular coloring if it is clear from the context. Throughout the thesis we denote an arc on C^r by $b_{(x_1,x_2)}$, characterized by its initial-point x_1 and end-point x_2 on C^r in the clockwise direction, where $x_1, x_2 \in \mathbb{R}^+$. We define its *length* by

$$l(b_{(x_1,x_2)}) = \begin{cases} x_2 - x_1, & \text{if } x_2 \geq x_1, \\ r - (x_1 - x_2), & \text{else.} \end{cases}$$

Let $b_{(x_1,x_2)}$ and $b_{(y_1,y_2)}$ be two arcs on C^r. We define the *distance between* $b_{(x_1,x_2)}$ and $b_{(y_1,y_2)}$ by

$$d(b_{(x_1,x_2)}, b_{(y_1,y_2)}) := \min\{l(b_{(x_2,y_1)}), l(b_{(y_2,x_1)})\}.$$

3.1 The Circular Two-Person Game on Weighted Graphs

In this section we introduce the combination of circular coloring of weighted graphs and the two person game.

Definition 3.1.1. Let $G^w = (V, E, w)$ be a weighted graph and C^r a circle with a given circumference $r \in \mathbb{R}^+$. Two players Alice and Bob take turns alternatingly with Alice moving first. Each move consists of assigning any uncolored vertex $v \in V$ to an open arc $f_w(v)$ on C^r for $r \in \mathbb{R}^+$ such that

(i) if $(x, y) \in E(G^w)$, then $f_w(x) \cap f_w(y) = \emptyset$ and

(ii) $l(f_w(x)) = w(x)$ for all vertices $x \in V(G^w)$.

Alice wins the game if there is a feasible circular coloring of G^w on the given C^r. Otherwise Bob wins. The *circular game chromatic number* $\gamma_c(G^w)$ of $G^w = (V, E, w)$ is the least circumference r of C^r for which there exists a winning strategy for Alice.

Remark 3.1.2. For $w(v) = 1$ for all $v \in V(G)$, one gets the circular game chromatic number $\gamma_c(G)$ of a graph G introduced by W. Lin and X. Zhu in [18].

Remark 3.1.3. We restricted the assumption $w(x) \geq l(f_w(x))$ for $x \in V(G^w)$ of the circular weighted coloring (see 3.0.3) by setting $l(f_w(x)) = w(x)$, since in case of two competing players, Bob would assign $l(f_w(x)) = r$ for a vertex $x \in V(G^w)$ if possible, such that a proper coloring of G^w won't be guaranteed anymore.

It is easily seen that

$$\chi_c(G^w) \leq \gamma_c(G^w).$$

The equality holds if Bob cooperates. However, we obtain for the trivial upper bound

$$\gamma_c(G^w) \leq \max_{x \in V(G^w)} \Big\{ w(x) + \sum_{y \in N(x)} w(y) + \big(d(x) - 1\big) w(x) \Big\}$$

because a coloring of an arbitrary vertex x is ensured, independent of how the corresponding arcs of all neighbors of x have been assigned to C^r.

Example: Consider the following graph $G^w = (V, E, w)$ with vertex set $V(G^w) = \{x_1, x_2, x_3\}$ and vertex-weights $w(x_1) = 3, w(x_2) = 3$ and $w(x_3) = 1$. Without loss of generality assume that Alice has assigned vertex x_1 to $b_{(0,3)}$ in her first move and it is Bob's turn. If he assigns vertex x_3 to the arc $b_{(6-\varepsilon,7-\varepsilon)}$ for an $\varepsilon > 0$, then vertex x_2 cannot be assigned between $f_w(x_1)$ and $f_w(x_3)$. Otherwise the corresponding arc of x_2 would overlap either with $f_w(x_1)$ or $f_w(x_3)$ and a proper coloring of G^w would not be possible because x_2 is adjacent to x_1 and x_3. However, a circle with circumference 10 is the trivial upper bound for the given distribution of the vertex-weights and guarantees Alice's victory.

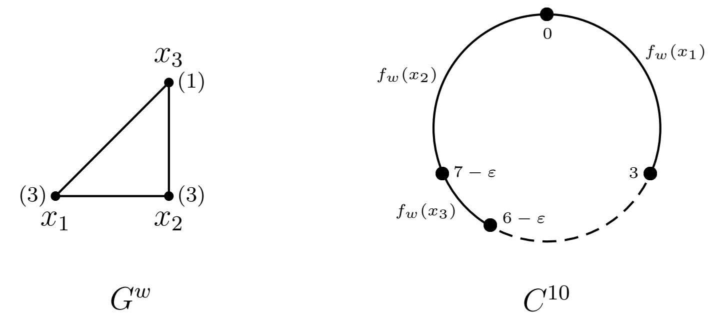


FIGURE: The coloring of G^w on C^{10},
where the numbers in the brackets indicate the vertex-weights.

We proceed with a general result about the game chromatic number of weighted graphs which demonstrates an elementary property of this parameter.

Proposition 3.1.4. *Let $G^w = (V, E, w)$ be a weighted graph and $G' = (V', E')$ the corresponding graph without vertex-weights with $V = V'$ and $E = E'$. Then the following holds:*

(i) If $w(v_i) \geq 1$ for all $v_i \in V(G^w)$, then $\gamma_c(G') \leq \gamma_c(G^w)$.

(ii) If $w(v_i) \leq 1$ for all $v_i \in V(G^w)$, then $\gamma_c(G') \geq \gamma_c(G^w)$.

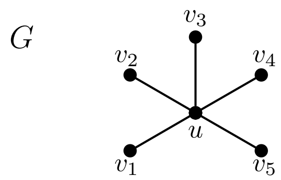
Proof. Assume that a circle with circumference r is given. (i) Assume the vertex-weights are greater or equal than 1. We proof that each winning strategy on G^w is a winning strategy on G'. Assume there exists a winning strategy σ for Alice, such that the mapping f_σ assigns each vertex of G^w to an open arc on C^r, if Alice plays strategy σ. Further let $g_{f_\sigma} : V \to \mathbb{R}^+$ be the mapping which assigns every vertex its initial-point on C^r. Suppose that $x \in V(G^w)$

and $x' \in V(G')$ is the corresponding vertex in G'. Then each time Alice takes turn she plays strategy σ, such that she places vertex x' to the unit length arc with the initial-point $g_{f_\sigma}(x)$. Since $l(f_\sigma(x)) \geq 1$, corresponding arcs of adjacent vertices still not overlap. Thus, strategy σ guarantees a proper coloring of G'.

(ii) Assume the vertex-weights of G^w are less or equal than 1 and there exists a strategy σ', such that Alice wins the game if she plays σ' on G'. Then the proof follows by the same method as above. □

One may conjecture that $\gamma_c(G^w)$ can be related to $\gamma_c(G)$ if we drop the assumption $w(v_i) \geq 1$ or $w(v_i) \leq 1$ for all $v_i \in V(G^w)$, respectively, but consider the general case $w(v_i) > 0$. The following consideration demonstrates that a general result cannot be received.

Let $G = (V, E)$ be of the following form:



Vertex u is called the *center* and vertices $\{v_1, ..., v_5\}$ the *leaves* of G, where each leave is adjacent to the center and has degree 1. Such a graph is called a *star*. It is easily seen that $\gamma_c(G) = 2$, if all vertex-weights equal 1. Assume the circle C^2 is given. Then Alice's winning strategy is to assign u to C^2 in her first turn. Without loss of generality assume she goes for the unit length arc $b_{(0,1)}$. Since the remaining uncolored vertices are independent, Bob is indifferent which vertex to color when he takes turn. Assume he colors v_1. Since $(u, v_1) \in E(G)$, he can only place the corresponding arc of v_1 on $b_{(1,2)}$. Since $(v_1, v_j) \notin E(G)$ for all $j \in \{2, ..., 5\}$, there is a proper coloring of the remaining uncolored vertices on C^2, since the corresponding arcs can overlap, moreover they can be placed on each other.

We now turn to the case that the vertices are assigned vertex-weights and let $k := \max\{w(v_j) \mid j \in \{1, ..., 5\}\}$. Assume, due to Alice's winning strategy above, she colors first vertex u by assigning the corresponding arc to $b_{(0,w(u))}$.

- Let $w(u) + k \leq 2$ and suppose C^2 is given. Then by the assumption and because of $w(v_j) \leq k$, $f_w(v_j)$ can be placed on C^2 for all $j \in \{1, ..., 5\}$.

- Let $w(u)+k > 2$ and assume that C^2 is given. Let $v' \in \{v_1, ..., v_5\}$ be a vertex with vertex-weight k. Then by the assumption $f_w(u)$ and $f_w(v')$ must overlap, which is a contradiction since (u, v') is an edge. This implies that a circle with circumference $r = 2$ does not suffice.

Thus, we can conclude that

$$\gamma_c(G^w) \begin{cases} \leq \gamma_c(G), & \text{if } w(u) + k \leq 2, \\ > \gamma_c(G), & \text{else.} \end{cases}$$

3.2 The Circular Game Chromatic Number of Weighted Complete Graphs

It is our purpose to study the circular two-person game on weighted complete graphs for some relevant distributions of vertex-weights; we analyze $\gamma_c(G^w)$, whereas we restrict the vertex-weights by the mapping $w : V \to \{k\}$ for $k \in \mathbb{R}^+$, which provides a natural characterization of the case that the graph has no vertex-weights. We make use of this result and determine the upper and lower bounds of $\gamma_c(G^w)$ for all values of w. Finally we restrict our attention to the case that $w : V \to \{k, l\}$ for $k, l \in \mathbb{R}^+$. For the respective distribution of the vertex-weights we give a strategy for Alice and figure out the worst case strategy Bob could apply.

Clearly, by the definition 1.1.8 for every two vertices $u, v \in V(K_n^w)$, $f_w(u)$ and $f_w(v)$ cannot overlap.

Proposition 3.2.1. *Let $K_n^w = (V, E, w)$ be a weighted complete graph with $w : V \to \{k\}$, for $k \in \mathbb{R}^+$. Then it holds:*

$$\gamma_c(K_n^w) \leq \begin{cases} k \cdot \left(\lceil \frac{n}{2} \rceil + 2 \cdot \lfloor \frac{n}{2} \rfloor\right), & \textit{if } n \textit{ odd}, \\ k \cdot \left(\frac{3n}{2} - 1\right), & \textit{if } n \textit{ even}. \end{cases}$$

Proof. The proof is divided into two steps. First we give an strategy for Alice and afterwards we determine the best case strategies for Bob. We shall calculate how many vertices each of the players assign to C^r for n odd and even during the game, such that due to the respective strategies of the players, $\gamma_c(K_n^w)$ is determined.

Let $V(K_n^w) = \{v_1, ..., v_n\}$. Since K_n^w is complete and $w(v_i) = k$ for every $i \in \{1, ..., n\}$, the order of the vertices assigned by both players throughout the game is not decisive, such that the choice of the vertices to color can be at random.

Alice's strategy: Initially Alice assigns an arbitrary vertex to an arbitrary arc on C^r. Let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order where $B := \emptyset$ at the beginning. Each time a vertex is colored Alice updates B. Throughout the game she proceeds as follows. Let v_i be an uncolored vertex.

- If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k$, then she colors v_i with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
- If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k$ for every $1 \leq j \leq m$, then she looses the game by the assumption that all vertices are adjacent to each other.

Bob's strategy: Again let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order and $v_l \in V(K_n^w)$ be an uncolored vertex.

- If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$, then he colors v_l with an arc of distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
- If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2k$ for every $1 \leq j \leq m$, then the number of the vertices that can be colored is fixed by the structure of the graph.

If he colored v_l with a distance d' with $d' < k - \varepsilon$ to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$, he would waste d' of C^r, but simultaneously safe $d - d'$. If he colored v_l with a distance d'' with $k < d'' < k + k - \varepsilon$ to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$, another uncolored vertex could be colored between them, such that again he would waste d' of C^r, but simultaneously safe $d - d'$ of C^r. We can conclude that the required circumference r of C^r for a feasible coloring of K_n^w increases by k each time Bob takes turn. One could conjecture that r increases by $k - \varepsilon$ for an $\varepsilon > 0$ instead of k.

But then Bob could place his arcs with distance $k - \frac{\varepsilon}{n}$ such that an increase of r by $k - \varepsilon$ won't suffice Alice for achieving a feasible coloring.

Suppose n is odd. Since Alice starts as well as finishes the game, if possible, she colors $\lceil \frac{n}{2} \rceil$ and Bob $\lfloor \frac{n}{2} \rfloor$ vertices, respectively, whereas due to his strategy destroys $k \cdot \lfloor \frac{n}{2} \rfloor$ additional arcs. Thus, the following conclusion can be drawn:

$$\gamma_c(K_n^w) \leq k \cdot \left\lceil \frac{n}{2} \right\rceil + 2k \cdot \left\lfloor \frac{n}{2} \right\rfloor = k \cdot \left(\left\lceil \frac{n}{2} \right\rceil + 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor \right),$$

for n odd.

Suppose n is even. Since Alice starts but Bob finishes the game, if possible, both will color $\frac{n}{2}$ vertices. We shall determine how often Bob achieves to destroy arcs of length k. For this purpose we can assume without loss of generality that both players color in cyclic order and restrict our attention to Bob's last turn. Then we can conclude by the strategies of both players that at that time $\frac{n}{2}$ and $\frac{n}{2} - 1$ vertices have been colored by Alice and Bob, respectively, where $k \cdot (\frac{n}{2} - 1)$ additional space on C^k have been destroyed by Bob.

It is sufficient to admit an arc of length k for coloring the last vertex and hence Bob won't be able to destroy another arc on C^r. Thus, a circle with circumference $k \cdot (n-1) + k \cdot \left(\frac{n}{2} - 1\right) + k$ suffices in order to achieve a proper coloring on C^r, such that we can conclude that

$$\gamma_c(K_n^w) \leq k \cdot \left(\frac{3n}{2} - 1 \right),$$

for n even. □

Corollary 3.2.2. *Let $K_n^w = (V, E, w)$ be a weighted complete graph with $V(K_n^w) =$*

$\{v_1, ..., v_n\}$. *Then*

$$\sum_{i=1}^{n} w(v_i) \leq \gamma_w(K_n^w) \leq \begin{cases} w_{\max}(V) \cdot \left(\lceil \frac{n}{2} \rceil + 2 \cdot \lfloor \frac{n}{2} \rfloor\right), & \text{if } n \text{ odd,} \\ w_{\max}(V) \cdot \left(\frac{3n}{2} - 1\right), & \text{if } n \text{ even.} \end{cases}$$

Proof. For determining the upper bound, we can refer to the proposition 3.2.1, since the distribution which increases $\gamma_w(K_n^w)$ at most is $w : V \to \{w_{\max}(V)\}$. For the lower bound It suffices to consider $\chi_c(K_n^w)$, which obviously equals $\sum_{i=1}^{n} w(v_i)$. □

Another relevant distribution of the vertex-weights on complete graphs is indicated in the following proposition. We restrict the mapping w by $w : V \to \{k, l\}$ for $k, l \in \mathbb{R}^+$, and determine the circular game chromatic number for $K_n^w = (V, E, w)$. For this purpose we consider the cases n even and n odd, where the proof of case n odd follows by the same method as in case n even. In particular we work on two different strategies Bob could apply and determine which one demonstrates the worst case strategy from Alice's viewpoint.

Let us introduce the following temporary notations: From now on we will consider the partition P of $V(K_n^w)$, where $P = \big\{\{V(K_n^{w,k})\}, \{V(K_n^{w,l})\}\big\}$, with $V(K_n^{w,k}) := \{v_i \mid w(v_i) = k\}$ and $V(K_n^{w,l}) := \{v_j \mid w(v_j) = l\}$. Further let $|V(K_n^{w,k})| = p$ and $|V(K_n^{w,l})| = q$. Moreover, we call a vertex $v \in V(K_n^{w,k})$ *k-vertex* and a vertex $u \in V(K_n^{w,l})$ *l-vertex*.

Proposition 3.2.3. *Let* $K_n^w = (V, E, w)$ *be a weighted complete graph with* $w : V \to \{k, l\}$ *where* $k > l$ *for* $k, l \in \mathbb{R}^+$ *and* $p, q \geq 2$.

(i) Let n *be even. Then the following hold:*

(a) For $p < q$

$$\gamma_c(K_n^w) \leq \max\left\{k \cdot p + l \cdot \left(\frac{3q + p}{2} - 1\right),\ k \cdot (2p - 1) + l \cdot (p - 1)\right\}.$$

(b) For $p \geq q$

$$\gamma_c(K_n^w) \leq \begin{cases} \max\Big\{k \cdot \big(\frac{3p+q}{2} - 1\big) + l \cdot \lceil \frac{q}{2} \rceil, \\ k \cdot \frac{3p-q}{2} + l \cdot (2q-1)\Big\}, & \textit{for } p-q \textit{ even,} \\ \max\Big\{k \cdot \big(\frac{3p+q}{2} - 1\big) + l \cdot \lceil \frac{q}{2} \rceil, \\ k \cdot \big(p + \lfloor \frac{p-q}{2} \rfloor + 1\big) + l \cdot (2q-1)\Big\}, & \textit{for } p-q \textit{ odd.} \end{cases}$$

(ii) Let n *be odd. Then the following hold:*

(a) For $p < q$

$$\gamma_c(K_n^w) \leq \max\Big\{k \cdot p + l \cdot \Big(q + \Big\lfloor \frac{n}{2} \Big\rfloor\Big),\ k \cdot (2p-1) + l \cdot (p-1)\Big\}.$$

(b) For $p \geq q$

$$\gamma_c(K_n^w) \leq \begin{cases} \max\Big\{k \cdot \big(p + \lfloor \frac{n}{2} \rfloor\big) + l \cdot \lceil \frac{q}{2} \rceil, \\ k \cdot \frac{3p-q}{2} + l \cdot (2q-1)\Big\}, & \textit{for } p-q \textit{ even,} \\ \max\Big\{k \cdot \big(p + \lfloor \frac{n}{2} \rfloor\big) + l \cdot \lceil \frac{q}{2} \rceil, \\ k \cdot \big(p + \lfloor \frac{p-q}{2} \rfloor + 1\big) + l \cdot (2q-1)\Big\}, & \textit{for } p-q \textit{ odd.} \end{cases}$$

Proof. Without loss of generality let $V(K_n^{w,k}) = \{v_1, ..., v_p\}$ and $V(K_n^{w,l}) = \{v_{p+1}, v_{p+2}, ..., v_n\}$. Let $U_k \subseteq V(K_n^{w,k})$ and $U_l \subseteq V(K_n^{w,l})$ be subsets that contain all uncolored k-vertices and l-vertices, respectively. Initially $U_j = V(K_n^{w,j})$ for $j \in \{k, l\}$.

(i) Suppose n is even. The basic idea of the proof is the following: After giving a strategy for Alice we work out two strategies for the respective cases (a) and (b). These turn out to be the two possible worst cases by the structure of the graph. Afterwards we determine which one increases a required circumference r of C^r at most.

(a) Let $p < q$.

Alice's strategy: Let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order where $B := \emptyset$ at the beginning. Each time a vertex is colored Alice updates B, U_k and U_l. Initially Alice colors an arbitrary vertex from U_k with an arbitrary arc on C^r. Throughout the game she proceeds as follows.

- Assume $U_k \neq \emptyset$ with $v_i \in U_k$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$, then she colors v_i with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
  If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2k$ for all $1 \leq j \leq m$, then the number of the k-vertices that can be colored is fixed.
- Assume $U_k = \emptyset$ and $U_l \neq \emptyset$ with $v_i \in U_l$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$, then she colors v_i with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
  If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2l$, then the number of the l-vertices that can be colored is fixed.

As already mentioned we proceed with strategies of Bob, denoted by σ_1 and σ_2. Let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order.

Strategy σ_1

- If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$, then he colors an arbitrary vertex from K_n^w with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
- If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2l$ for every $1 \leq j \leq m$, then the number of the vertices that can be colored is fixed by the structure of the graph.

Assume Bob applies strategy σ_1. By the assumption n is even, Bob colors the last uncolored vertex, if possible. Assume v_n is the last uncolored vertex. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) = w(v_n)$, then he has to color v_n with an arc either next to $b_{j-1 \bmod m}$ or next to $b_{j \bmod m}$. Thus, he cannot destroy another arc of length $l - \varepsilon$. This implies that he colors $\frac{n}{2} - 1$ times with arcs of distance $l - \varepsilon$ to placed arcs. Then the required circumference increases by l each time Bob takes turn besides his last turn. Thus, the circumference of the given circle has to be at least

$$k \cdot p + l \cdot q + l \cdot \left(\frac{n}{2} - 1\right) =$$

$$k \cdot p + l \cdot \left(\frac{3q + p}{2} - 1\right). \tag{3.1}$$

Strategy σ_2

- Assume $U_k \neq \emptyset$ and $U_l \neq \emptyset$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k + l$, then he assigns the corresponding arc of an arbitrary vertex from $(K_n^{w,l})$ with distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of the k-vertices that can be colored is fixed. For the remaining uncolored l-vertices see the next case.

- Assume $U_k = \emptyset$ and $U_l \neq \emptyset$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$, then he assigns the corresponding arc of an arbitrary vertex from $(K_n^{w,l})$ with distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the coloring of the remaining uncolored vertices is fixed.

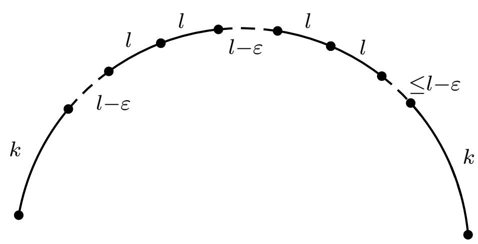
Assume Bob applies strategy σ_2. Then by the assumption $p < q$ and Alice's strategy Alice colors all k-vertices and Bob only l-vertices throughout the game. By the time the last k-vertex is colored, Bob will have colored $(p - 1)$ l-vertices with arcs of distance $k - \varepsilon$ to placed arcs. Since $p < q$ there are still uncolored l-vertices left. If we assume that the remaining uncolored l-vertices can be assigned into the $k - \varepsilon$ gaps produced by Bob, then the circumference of the given circle has to be at least

$$k \cdot p + l \cdot (p - 1) + k \cdot (p - 1) =$$

$$k \cdot (2p - 1) + l \cdot (p - 1). \tag{3.2}$$

We are left with the task of determining Bob's optimal strategy. Assume Bob plays strategy σ_2 and that a circle with circumference $k \cdot (2p-1) + l \cdot (p-1)$ is not sufficient for a proper coloring of all uncolored l-vertices. By the respective strategies of the players the remaining l-vertices are being colored with arcs in the $k - \varepsilon$ gaps as follows: Each time Bob takes turn he colors an l-vertex with an arc of a distance of $l - \varepsilon$ to an already placed one. Alice colors with arcs next to each other. Thus, the last gap is less or equal to $l - \varepsilon$. In case of equality Bob is indifferent to apply σ_1 or σ_2. Otherwise he should play strategy σ_1, since then he is able to destroy more free space on the circle.

The following figure demonstrates a feasible coloring of l-vertices in such a $k - \varepsilon$ gap:



To find out which strategy is the best for Bob, we need to compare the outcomes of both strategies. Hence, if the inequality

$$\underbrace{k \cdot p + l \cdot \left(\frac{3q+p}{2} - 1\right)}_{(3.1)} < \underbrace{k \cdot (2p-1) + l \cdot (p-1) \Leftrightarrow}_{(3.2)}$$

$$l \cdot \left(\frac{3q+p}{2} - 1\right) - l \cdot (p-1) < k \cdot (2p-1) - k \cdot p \Leftrightarrow$$

$$l \cdot \left(\frac{3q+p}{2} - 1 - p + 1\right) < k \cdot (2p - 1 - p) \Leftrightarrow$$

$$l \cdot \frac{3q-p}{2} < k \cdot (p-1) \Leftrightarrow$$

$$l \cdot \frac{3q-p}{2(p-1)} < k$$

holds, Bob prefers to play strategy σ_2. Thus, we can draw the following conclusion:

- If $\frac{3q-p}{2(p-1)} \cdot l < k$ holds, Bob plays strategy σ_2.
- If $\frac{3q-p}{2(p-1)} \cdot l > k$ holds, Bob plays strategy σ_1.

- If $\frac{3q-p}{2(p-1)} \cdot l = k$ holds, Bob is indifferent which strategy to play.

(b) Let $p \geq q$ and $p - q$ be even.

Alice's strategy: Let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order where $B := \emptyset$ at the beginning. Each time a vertex is colored Alice updates B, U_k and U_l. Initially she colors vertex $v_i \in U_k$ with an arbitrary arc. Throughout the game she proceeds as follows:

- Assume $U_k \neq \emptyset$ and $U_l \neq \emptyset$.

  (1) If there exist two arcs $b_{j-1 \bmod m} = b_{(x_1,x_2)}$ and $b_{j \bmod m} = b_{(y_1,y_2)}$ with $l \leq d(b_{j-1 \bmod m}, b_{j \bmod m}) \leq k-\varepsilon$ for $1 \leq j \leq m$ and an $\varepsilon > 0$ and with $l(b_{(x_2,y_1)}) = d(b_{j-1 \bmod m}, b_{j \bmod m})$, she colors a vertex $v_i \in U_l$ with an arc next to $b_{j-1 \bmod m}$. If $l(b_{(x_2,y_1)}) > d(b_{j-1 \bmod m}, b_{j \bmod m})$, she colors a vertex $v_i \in U_l$ with an arc next to $b_{j \bmod m}$. Otherwise she proceeds with step (2).

  (2) If there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$ for $1 \leq j \leq m$, then she colors a vertex $v_i \in U_k$ with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise for some $1 \leq j \leq m$ it holds $k \leq d(b_{j-1 \bmod m}, b_{j \bmod m}) \leq 2k-\varepsilon$ and for some j' it holds $d(b_{j'-1 \bmod m}, b_{j' \bmod m}) \leq k - \varepsilon$ for an $\varepsilon > 0$. Then the number of the remaining k-vertices that can be colored is fixed.

- Assume $U_k = \emptyset$ and $U_l \neq \emptyset$. If there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$ for $1 \leq j \leq m$, then she colors a vertex $v_i \in U_l$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of the remaining l-vertices that can be colored is fixed.

- Assume $U_k \neq \emptyset$ and $U_l = \emptyset$. Then Alice proceeds as in step (2).

According to Alice's winning strategy there are two possible worst case strategies σ_1' and σ_2' Bob could apply. Let $B := \{b_0, .., b_{m-1}\}$ for $m - 1 \leq n - 1$ be the set of all assigned arcs on C^r in cyclic order.

Strategy σ_1'

- Assume $|U^l(K_n^w)| \geq 2$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$, then he colors a vertex $v_i \in U_l$ with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of vertices that can be colored is fixed.

- Assume $|U^l(K_n^w)| = 1$ with $v_i \in U_l$ and $U^k(K_n^w) \neq \emptyset$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k + l$, then he colors v_i with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$ with distance $k - \varepsilon$ for an $\varepsilon > 0$. Otherwise he assigns v_i to an arbitrary arc on C^r.

- Assume $U^l(K_n^w) = \emptyset$ and $U^k(K_n^w) \neq \emptyset$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$, then he colors a vertex $v_i \in U_k$ with an arc of distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of the k-vertices that can be colored is fixed.

Suppose Bob plays strategy σ_1'. Then by the assumption $p \geq q$ and Alice's strategy Bob colors $q-1$ l-weighted vertices with arcs of distance $l-\varepsilon$ and one l-vertex with an arc of distance $k - \varepsilon$. As soon as $U_l = \emptyset$ he proceeds to color with arcs of distance $k - \varepsilon$. Since $p - q$ is even he destroys another $k \cdot \left(\frac{p-q}{2} - 1\right)$ free space on C^r. Hence, a circumference

$$r = k \cdot p + l \cdot q + l \cdot (q-1) + k + k \cdot \left(\frac{p-q}{2} - 1\right) =$$

$$k \cdot \frac{3p - q}{2} + l \cdot (2q - 1) \tag{3.3}$$

suffice for guaranteeing a proper coloring of K_n^w.

Strategy σ_2'

- Assume $U_l \neq \emptyset$ and that there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k + l$. Then Bob colors a vertex $v_i \in U_l$ with an arc with distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.

- Assume $U_l \neq \emptyset$ and $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k + l$ for every $1 \leq j \leq m$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$, then he colors a vertex $v_i \in U_l$ with an arc with distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of vertices that can be colored is fixed.

- Assume $U_l = \emptyset$ and $U_k \neq \emptyset$. If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$, then he colors a vertex $v_i \in U_k$ with an arc with distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Otherwise the number of k-vertices that can be colored is fixed.

Suppose Bob applies strategy σ_2'. Then by the assumption $p \geq q$ and by Alice's strategy both players color the l-vertices alternately until $U_l = \emptyset$ after Alice has colored a k-vertex in her first turn. Thus, she colors $\lfloor \frac{q}{2} \rfloor$ and Bob $\lceil \frac{q}{2} \rceil$ l-vertices, respectively. In particular Bob colors with an arc of distance of $k - \varepsilon$ throughout the game and Alice colors the l-vertices with arcs in these gaps. Thus, $k \cdot \left(\frac{n}{2} - 1\right) - l \cdot \lfloor \frac{q}{2} \rfloor$ free space is being destroyed by Bob on C^r. Taking into account that the k-vertices have to be colored with p arcs and considering that Bob colors $\lceil \frac{q}{2} \rceil$ l-vertices, a circle with circumference

$$r = k \cdot p + k \cdot \left(\frac{n}{2} - 1\right) + l \cdot \left\lceil \frac{q}{2} \right\rceil =$$

$$k \cdot \left(\frac{3p + q}{2} - 1\right) + l \cdot \left\lceil \frac{q}{2} \right\rceil \tag{3.4}$$

ensures Alice's victory.

Further we want to give a relation of k and l in order to figure out which strategy to prefer. If

$$\underbrace{k \cdot \frac{3p - q}{2} + l \cdot (2q - 1)}_{(3.3)} < \underbrace{k \cdot \left(\frac{3p + q}{2} - 1\right) + l \cdot \left\lceil \frac{q}{2} \right\rceil}_{(3.4)} \Leftrightarrow$$

$$l \cdot (2q - 1) - l \cdot \left\lceil \frac{q}{2} \right\rceil < k \cdot \left(\frac{3p + q}{2} - 1\right) - k \cdot \frac{3p - q}{2} \Leftrightarrow$$

$$l \cdot \left(2q - \left\lceil \frac{q}{2} \right\rceil - 1\right) < k \cdot \left(\frac{3p + q}{2} - 1 - \frac{3p - q}{2}\right) \Leftrightarrow$$

$$l \cdot \left(2q - \left\lceil \frac{q}{2} \right\rceil - 1\right) < k \cdot (q - 1) \Leftrightarrow$$

$$l \cdot \frac{2q - \left\lceil \frac{q}{2} \right\rceil - 1}{q - 1} < k$$

holds, Bob prefers to play strategy σ_2'. If

$$l \cdot \frac{2q - \left\lceil \frac{q}{2} \right\rceil - 1}{q - 1} > k$$

holds, he decides to play strategy σ_1'. In case of equality he is indifferent which strategy to play.

Let $p - q$ be odd. The proof runs almost analogue as in case $p - q$ even. The only difference occurs in strategy σ_1'. Instead of $l \cdot (q-1) + k + k \cdot \left(\frac{p-q}{2} - 1\right)$ Bob destroys $l \cdot (q-1) + k + \left\lfloor \frac{p-q}{2} \right\rfloor$ free space on C^r, since $p - q$ is odd.

(ii) Suppose n is odd. Since the proof works very similar as in case (i), we leave the details to the reader. □

3.3 The Circular Game Chromatic Number of Weighted Complete Multipartite Graphs

The aim of this section is to analyze the circular game chromatic number for the class of complete multipartite graphs. First, we determine $\gamma_c(K^w_{s_1,\ldots,s_n})$ for $w(v_i) = k$ for all $v_i \in V(K^w_{s_1,\ldots,s_n})$. This result is needed for the purpose of determining the circular game chromatic number of weighted complete multipartite graphs for all values of w. Therefore, we apply techniques of W. Lin and X. Zhu who worked out in [18] the circular game chromatic number of complete graphs without vertex-weights. In addition we give an upper bound of $\gamma_c(K^w_{s_1,\ldots,s_n})$ for $w : V \to \{k, l\}$ for $k, l \in \mathbb{R}^+$ such that $w(u) = w(v)$ for all $(u, v) \notin E(K^w_{s_1,\ldots,s_n})$.

Definition 3.3.1. For an integer $r \geq 2$ a graph $G = (V, E)$ is called *r-partite* or *multipartite* if $V(G)$ admits a partition into r classes such that vertices in the same partition class must not be adjacent. A multipartite graph in which every two vertices from different partition classes are adjacent is called *complete*.

Let $\{S_1, ..., S_r\}$ be the set of the partition classes, then a complete multipartite graph is denoted by $K_{s_1,...,s_r}$, where $s_i = |S_i|$ for all $i \in \{1, ..., r\}$. Particularly, we call the S_i *independent sets*. For $m = 2$ we call the graph K_{s_1,s_2} *complete bipartite*.

Further we call an independent set S_i *colored* if and only if every vertex from S_i is colored. Additionally, we call S_i *uncolored* if and only if every vertex from S_i is not colored.

Proposition 3.3.2. *Let* $(K^w_{s_1,...,s_n}) = (V, E, w)$ *be a weighted complete multipartite graph with* $s_i \geq 4$ *for* $i \in \{1, ..., n\}$ *and* $w : V \to \{k\}$ *for* $k \in \mathbb{R}$. *Then for* $l \in \mathbb{N} \setminus \{0\}$

$$\gamma_c(K^w_{s_1,...,s_n}) = \begin{cases} k(8l+1), & \textit{if} \;\; n = 3l+1, \\ k(8l+4), & \textit{if} \;\; n = 3l+2, \\ k(8l+7), & \textit{if} \;\; n = 3l+3. \end{cases}$$

Proof. Case $n = 3l + 1$: We prove that if the given circle C^r has circumference at least $k(8l + 1)$, then Alice has a winning strategy. First we make some considerations. Since Alice starts the game, she will color in an independent set which contains only uncolored vertices. Obviously it does not make sense for Bob to color also in an independent set which contains only uncolored vertices because throughout the game he is able to color in an independent set which contains at least one colored vertex and so to "destroy" more space on the circle C^r. Contrary, Alice's aim is to color in independent sets that contain only uncolored vertices, since once every independent set contains at least one colored vertex, the coloring is fixed and Bob is not able to "destroy" more space on the circle C^r. Hence, we give below a winning strategy for Alice which will be used as long as there exist independent sets with uncolored vertices.

For simplicity we use an equivalent version of circular coloring for the rest of the proof: Let $G^w = (V, E, w)$ be a weighted graph and let C^r be a circle with circumference r. Let a, b be two points on the circle C^r. We denote by $d(a, b)$ the minimal distance between a and b on the circle C^r. A mapping $f_w : V(G^w) \to C^r$ which assigns every vertex of G^w to a point on C^r is called an *r-circular coloring* if for every edge $(x, y) \in E(G^w)$, we have $d(f_w(x), f_w(y)) \geq$

$\max\{w(x), w(y)\}$.

Hence, for the rest of the proof, the players Alice and Bob won't choose arcs to color the vertices but points which satisfies the rules of the definition above.

Alice's strategy: Initially Alice colors an arbitrary vertex.

Suppose $i \in \mathbb{N} \setminus \{0\}$ vertices have already been colored with $b_1, ..., b_i$ being the points and suppose it is Alice's turn. Assume the points $b_1, ..., b_i$ occur in the clockwise direction on C^r and that there does not exist any duplicates. If there exist $j, j' \in \{1, ..., i\}$ with $j \neq j'$ and $b_j = b_{j'}$, then we drop the duplicates from the list $b_1, ..., b_i$ so that at every point there is only one vertex assigned. Since Alice colors every time in an uncolored independent set, the point which she will choose is not allowed to be one of the already placed points.

- If there are two points with $d(b_j, b_{j+1}) \geq 3k$ for $1 \leq j \leq i-1$, then Alice colors a vertex from an uncolored independent set with the point $b_j + 2k$ where $b_j + 2k$ lies between b_j and b_{j+1} and $d(b_j, b_j + 2k) = 2k$.
- If $d(b_j, b_{j+1}) < 3k$ for every $1 \leq j \leq i-1$, Alice uses an arbitrary free point for the rest of the game.

By the strategy above the procedure of the game is the following independent of which strategy Bob applies:

Suppose it is Alice's t-th move and suppose that $b_1, ..., b_{2t-2}$ are the already placed points in the clockwise direction on C^r. Assume in the t-th move there are two points with $d(b_j, b_{j+1}) \geq 3k$ for $1 \leq j \leq i-1$ but in her $(t+1)$-th turn, each distance $d(b_j, b_{j+1})$ of the already placed points (that are $b_1, ..., b_{2t}$ now) has length less than $3k$.

After Alice finished her t-th move, $2t-1$ points have been placed by both players. Let $b_1, ..., b_{2t-1}$ denote those points in the clockwise direction on C^r. These $2t-1$ points divide C^r into $2t-1$ intervals, that we denote by $\{(b_1, b_2), (b_2, b_3), ..., (b_{2t-1}, b_1)\}$ in the clockwise direction. We may assume that in his next move Bob chooses a point from the interval (b_{2t-1}, b_1). By definition of t, after Bob's move, each interval of C^r has length less than $3k$. This implies that after Alice t-th move and before Bob's t-th move the interval (b_{2t-1}, b_1) has length at least $2k$ and less than $6k$.

If every interval has length less that $3k$, the later moves of the game are trivial for Alice and Bob. Alice will use a point from the remaining intervals to color a vertex from an independent set which contains only uncolored vertices and Bob will "destroy" the remaining intervals by coloring vertices from independent sets that contain colored vertices. If p points have been placed to reach the configuration that each interval has length less than $3k$ and q of the intervals have length at least $2k$, then Alice wins the game if and only if $\lceil \frac{p+q}{2} \rceil \geq n$.

Depending of the length of the interval (b_{2t-1}, b_1), which is produced after Alice's t-th move, we divide the remaining proof into some cases.

- First consider the case that the interval (b_{2t-1}, b_1) has length less than $4k$. According to Alice's strategy every time she colors a vertex, she produces an interval of length $2k$. Hence, Alice created $t-1$ intervals of length $2k$. During the game Bob may use some of these intervals to color his vertices. Assume he did so s times. Then without loss of generality $2s$ of the intervals $\{(b_1, b_2), ..., (b_{2t-1}, b_1)\}$ have length k and the remaining $t-1-s$ by Alice produced intervals have length $2k$. The other $t-1-s$ by Bob produced intervals have lengths at least k but less than $3k$. Assume q of these $t-1-s$ intervals have length at least $2k$ and $t-1-s-q$ of them have length less than $2k$. Thus, after Alice colored t vertices and by the assumption that (b_{2t-1}, b_1) has length at least $2k$ and less than $4k$, in the remaining moves $\lfloor \frac{t-1-s+q+1}{2} \rfloor = \lfloor \frac{t-s+q}{2} \rfloor$ more points can be chosen by Alice. Thus, if

$$t + \frac{t-s+q}{2} \geq n \Leftrightarrow$$
$$3t - s + q \geq 2n,$$

  then at least n points can be chosen by Alice and hence the coloring of the complete multipartite graph is fixed and Alice wins the game. Further we have to prove that indeed $3t - s + q \geq 2n$ holds.

  The sum of the intervals $\{(b_1, b_2), ..., (b_{2t-1}, b_1)\}$ is equal to r. Because the interval (b_{2t-1}, b_1) has length less than $4k$ we can conclude that

$$k(2s + 2(t-1-s) + 3q + 2(t-1-s-q) + 4) = k(4t + q - 2s) > r.$$

Since $n = 3l+1$ and because of $n \geq 4$ clearly while $s = 0$ we can conclude $q \geq 2$ and while $s \geq 1$ we can conclude $q \geq 1$. Thus, we have

$$k(4t - 2s + q) > k(8l + 1) \Rightarrow$$

$$4t - 2s + q \geq 8l + 2 \Rightarrow$$
$$3t - \frac{3}{2}s + \frac{3}{4}q \geq 6l + \frac{3}{2} \Rightarrow$$
$$3t - s + q - 1 \geq 6l + 1 \Rightarrow$$
$$3t - s + q \geq 6l + 2.$$

- Now we consider the case that the interval (b_{2t-1}, b_1) has length at least $4k$ but less than $5k$. Then two more points can be chosen in the interval (b_{2t-1}, b_1). Hence, after Alice has finished her t-th move, in the remaining moves $\lfloor \frac{t-1-s+q+2}{2} \rfloor = \lfloor \frac{t-s+q+1}{2} \rfloor$ more points can be chosen by Alice. Thus, if
$$t + \frac{t - s + q + 1}{2} \geq n \Rightarrow$$
$$3t - s + q + 1 \geq 2n$$

holds, Alice wins the game. Again the sum of the intervals $\{(b_1, b_2), ..., (b_{2t-1}, b_1)\}$ is equal to r. Hence,

$$k(2s + 2(t - 1 - s) + 3q + 2(t - s - q - 1) + 5) = k(4t - 2s + q + 1) > r.$$

Since $n = 3l + 1$, we have

$$k(4t - 2s + q + 1) > k(8l + 1) \Rightarrow$$

$$4t - 2s + q + 1 \geq 8l + 2 \Rightarrow$$
$$3t - \frac{3}{2}s + \frac{3}{4}q + \frac{3}{4} \geq 6l + \frac{3}{2} \Rightarrow$$
$$3t - s + q \geq 6l + 1 \Rightarrow$$

$$3t - s + q + 1 \geq 6l + 2.$$

- Now we consider the case that the interval (b_{2t-1}, b_1) has length at least $5k$. Then three more points can be chosen in the interval (b_{2t-1}, b_1). Hence, after Alice has finished her t-th move, in the remaining moves $\lfloor\frac{t-1-s+q+3}{2}\rfloor = \lfloor\frac{t-s+q+2}{2}\rfloor$ more points can be chosen by Alice. Thus, if

$$t + \frac{t-s+q+2}{2} \geq n \Rightarrow$$

$$3t - s + q + 2 \geq 2n$$

holds, Alice wins the game. Again the sum of the intervals $\{(b_1, b_2), ..., (b_{2t-1}, b_1)\}$ is equal to r. Hence,

$$k(2s + 2(t-1-s) + 3q + 2(t-s-q-1) + 6) = k(4t - 2s + q + 2) > r.$$

Since $n = 3l + 1$, we have

$$k(4t - 2s + q + 2) > k(8l + 1) \Rightarrow$$

$$4t - 2s + q + 2 \geq 8l + 2 \Rightarrow$$

$$3t - \frac{3}{2}s + \frac{3}{4}q + \frac{3}{2} \geq 6l + \frac{3}{2} \Rightarrow$$

$$3t - s + q + 1 \geq 6l + 1 \Rightarrow$$

$$3t - s + q + 2 \geq 6l + 2.$$

Next we prove that $\gamma_w(K_{s_1,...,s_n}) \geq k(8l + 1)$. Assume a circle C^r with $r = k(8l + 1) - \varepsilon$ for $\varepsilon > 0$ is given. Further we give a winning strategy for Bob.

Bob's strategy: Suppose it is Bob's turn and i points $b_1, ..., b_i$ have already been placed. Moreover, assume that the points occur in the clockwise direction. These points divide C^r into i intervals $\{(b_1, b_2), ..., (b_i, b_1)\}$.

- If there is an interval, say (b_j, b_{j+1}) for $1 \leq j \leq i-1$, of length at least $3k$, then Bob chooses the point $b_j + 2k - \frac{\varepsilon}{n}$ and he colors a vertex from an independent set which already contains at least one colored vertex.

- Otherwise, if every interval has length less than $3k$, the number of vertices that can be colored is fixed.

Now assume it is Bob's t-th move and he can still find an interval of length at least $3k$, but in his $(t+1)$-th move, each interval has length less than $3k$. Then after his t-th move, the circle C^r is divided into $2t$ intervals $\{(b_1, b_2), ..., (b_{2t}, b_1)\}$. We may assume that in Alice's next move she chooses a point from the interval (b_{2t}, b_1). By the definition of t, after Alice's $(t+1)$-th move, each of the intervals has length less than $3k$. Thus, the interval (b_{2t}, b_1) has length at least $2k$ and less than $6k$.

If p points are chosen to reach a configuration in which each interval has length less than $3k$ and q of the intervals have length at least $2k$, then Bob wins the game if and only if $\lceil \frac{p+q}{2} \rceil < n$.

According to the length of the interval (b_{2t}, b_1), we divide the remaining proof into three cases.

- First consider the case that the interval (b_{2t}, b_1) has length less than $3k$. Assume that q of the intervals $\{(b_1, b_2), ..., (b_{2t}, b_1)\}$ have length at least $2k$. Obviously if $t + \lceil \frac{q}{2} \rceil < n$, Bob wins the game.

  The sum of the lengths of all the intervals $\{(b_1, b_2), ..., (b_{2t}, b_1)\}$ is equal to $k(8l + 1) - \varepsilon$. Thus,

  $$k\Big(t\big(2 - \frac{\varepsilon}{n}\big) + 2q + (t - q)\Big) = k\big(3t + q - \frac{t\varepsilon}{n}\big) < r = k(8l + 1) - \varepsilon.$$

  Hence, we can conclude

  $$3t + q < 8l + 1. \tag{3.5}$$

  If $t \leq 2l$, then because $q \leq t$, we have

  $$2t + q + 1 \leq 3t + 1 \leq 6l + 1 < 2n.$$

  Hence,

  $$t + \frac{q}{2} + \frac{1}{2} < n \Rightarrow t + \left\lceil \frac{q}{2} \right\rceil < n.$$

  If $t \geq 2l + 1$, because of 3.5 we have

  $$2t + q + 1 < 8l + 2 - t \leq 8l + 2 - 2l - 1 = 6l + 1 \leq 2n,$$

  and hence $t + \lceil \frac{q}{2} \rceil < n$.

- Assume that the interval (b_{2t}, b_1) has length at least $3k$ and less than $4k$. Now two more points can be chosen from the interval (b_{2t}, b_1). Let q be the number of intervals which has length at least $2k$ among $\{(b_1, b_2), ..., (b_{2t-1}, b_{2t})\}$. To prove that Bob wins the game we have to show that $t + \lceil \frac{q+2}{2} \rceil < n$ holds.

  Again the sum of the lengths of all the intervals $\{(b_1, b_2), ..., (b_{2t}, b_1)\}$ is equal to $r = k(8l+1) - \varepsilon$. We can conclude

  $$k\Big(t\big(2 - \frac{\varepsilon}{n}\big) + 2q + (t-1-q) + 3\Big) = k\big(3t + q + 2 - \frac{t\varepsilon}{n}\big) < r = k(8l+1) - \varepsilon.$$

  Thus,

  $$3t + q + 2 < 8l + 1. \tag{3.6}$$

  If $t \leq 2(l-1)$, then because $q \leq t-1$, we have

  $$2t + q + 3 \leq 3t + 2 \leq 6l - 4 < 2n.$$

  Hence,

  $$t + \frac{q+2}{2} + \frac{1}{2} < n \Rightarrow t + \left\lceil \frac{q+2}{2} \right\rceil < n.$$

  If $t \geq 2l$, because of 3.6 we have

  $$2t + q + 3 < 8l + 2 - t \leq 8l + 2 - 2l = 6l + 2 = 2n,$$

  and hence $t + \lceil \frac{q+2}{2} \rceil < n$.

- Assume that the interval (b_{2t}, b_1) has length at least $4k$. In this case three more points can be chosen from the interval (b_{2t}, b_1). Let q be the number of intervals which has length al least $2k$ between $\{(b_1, b_2), ..., (b_{2t-1}, b_{2t})\}$. Bob wins the game if $t + \lceil \frac{q+3}{2} \rceil < n$ holds.

  Again the sum of the lengths of all the intervals $\{(b_1, b_2), ..., (b_{2t}, b_1)\}$ is equal to $r = k(8l+1) - \varepsilon$. We can conclude

  $$k\Big(t\big(2 - \frac{\varepsilon}{n}\big) + 2q + (t-1-q) + 4\Big) = k\big(3t + q + 3 - \frac{t\varepsilon}{n}\big) < r = k(8l+1) - \varepsilon.$$

  Thus,

  $$3t + q + 3 < 8l + 1. \tag{3.7}$$

If $t \leq 2(l-1)$, then because $q \leq t-1$, we have

$$2t + q + 4 \leq 3t + 3 \leq 6l - 3 < 2n.$$

Hence,

$$t + \frac{q+3}{2} + \frac{1}{2} < n \Rightarrow t + \left\lceil \frac{q+3}{2} \right\rceil < n.$$

If $t \geq 2l$, because of 3.7 we have

$$2t + q + 4 < 8l + 2 - t \leq 8l + 2 - 2l = 6l + 2 = 2n,$$

and hence $t + \lceil \frac{q+3}{2} \rceil < n$.

Case $n = 3l + 2$: Works with a similar calculation as case $n = 3l + 1$.

Case $n = 3l + 3$: Works with a similar calculation as case $n = 3l + 1$. □

Corollary 3.3.3. *Let $K^w_{s_1,\dots,s_n} = (V, E, w)$ be a weighted complete multipartite graph, with $s_i \geq 4$ for all $i \in \{1, \dots, n\}$ and $w : V \to \mathbb{R}^+$. Then for $l \in \mathbb{N} \setminus \{0\}$*

$$\sum_{i=1}^{n} w_{\max}(S_i) \leq \gamma_w(K^w_{s_1,\dots,s_n}) \leq \begin{cases} w_{\max}(K^w_{s_1,\dots,s_n}) \cdot (8l+1), & \textit{if } \; n = 3l+1 \\ w_{\max}(K^w_{s_1,\dots,s_n}) \cdot (8l+4), & \textit{if } \; n = 3l+2 \\ w_{\max}(K^w_{s_1,\dots,s_n}) \cdot (8l+7), & \textit{if } \; n = 3l+3. \end{cases}$$

Proof. By the structure of the graph, clearly the lower bound turns out to be the sum of the maximum vertex-weight of each independent set.

For determining the upper bound we consider the distribution of the vertex-weights $w : V \to \{w_{\max}(K^w_{s_1,\dots s_n})\}$, which increases $\gamma_c(K^w_{s_1,\dots,s_n})$ at most. Hence, we can refer to proposition 3.3.2. □

In the following we work under the assumption that vertices from same independent sets are being assigned either to a vertex-weight k or l, whereas $k > l$ for $k, l \in \mathbb{R}^+$. For this purpose we introduce some temporary notations:

Consider the vertex-set $V(K^w_{s_1,\dots,s_n})$ as a disjoint union $V = V^k \dot{\cup} V^l$, where

$$V^k(K^w_{s_1,\dots,s_n}) := \Big\{ S_i \mid w(v) = k, v \in S_i, i \in \{1, \dots, n\} \Big\}$$

and

$$V^l(K^w_{s_1,\dots,s_n}) := \Big\{ S_j \mid w(v) = l, v \in S_j, j \in \{1, \dots, n\} \Big\}$$

with $p := |V^k|$ and $q := |V^l|$. Beyond let $u^k := \tilde{k} - 2p$, where $\tilde{k}$ is the number of all vertices with vertex-weight k. For an independent set S_i for $i \in \{1, ..., n\}$ we denote the vertices by $\{v_{i_1}, ..., v_{i_{s_i}}\}$.

We give a strategy of Alice independent of the structure of the graph and work out a worst case strategy of Bob for the respective cases $2q > p$, $2q < p$ and $2q = p$. In particular we restrict our attention to the two possible worst case strategies σ_1 and σ_2 Bob could apply, that differ among others in the following way: If Bob applies σ_1 he colors a vertex with an arc on C^r of distance $k - \varepsilon$, for an $\varepsilon > 0$, as long as there is an independent set $S_i \in V^k$ which contains only uncolored vertices. If he applies σ_2 he colors a vertex with an arc of distance $l - \varepsilon$, for an $\varepsilon > 0$, each time he takes turn.

Proposition 3.3.4. *Let $K^w_{s_1,...,s_n} = (V, E, w)$ be a weighted complete multipartite graph with $s_i \geq 4$ for all $i \in \{1, ..., n\}$ and let $w : V \to \{k, l\}$ for $k, l \in \mathbb{R}^+$ and $k > l$ such that $w(u) = w(v)$ for all $u, v \in S_i$ and for all $i \in \{1, ..., n\}$. Further let $k \leq 2l$, $q > 1$ and $p > 2$. Then the following holds:*

(i) Let $2q > p$. Then

$$\gamma_c(K^w_{s_1,...,s_n}) \leq \begin{cases} k(2p + q - 1) + l(p + 2q - 1), & if \quad u^k \geq q - 1, \\ k(2p + u^k) + l(p + 3q - u^k - 2), & if \quad u^k < q - 1. \end{cases}$$

(ii) Let $2q < p$. Then

$$\gamma_c(K^w_{s_1,...,s_n}) \leq \max\{k(3p - 2),\ \ k(2p + q - 1) + l(p + 2q - 1)\}.$$

(iii) Let $2q = p$. Then

$$\gamma_c(K^w_{s_1,...,s_n}) \leq k(2p + q - 1) + l(p + 2q - 1).$$

Proof. The proof is divided into three steps. After introducing Alice's strategy we give the two possible worst case strategies σ_1 and σ_2 Bob can apply. Finally due to the strategies of the players and the structure of the graph $\gamma_c(K^w_{s_1,...,s_n})$ is determined.

Without loss of generality assume that $\{S_1, ..., S_p\} \in V^k$ and $\{S_{p+1}, ..., S_n\} \in V^l$. Further let $C \subseteq V$ be the set of all colored vertices during the game.

Alice's Strategy: Initially Alice colors v_{1_1} with an arbitrary arc on C^r and updates C. Assume there are uncolored independent sets and let $B := \{b_0, ..., b_{m-1}\}$ be the set of all assigned arcs on C^r in cyclic order. Further let i_0 be the least index from $\{1, ..., n\}$ such that S_{i_0} is an uncolored independent set.

- If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2w(v_{i_{0_1}})$, then she colors $v_{i_{0_1}}$ with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$. Afterwards she updates C and B.
- If $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2w(v_{i_{0_1}})$ for all $1 \leq j \leq m$, then the number of the remaining uncolored vertices with vertex-weight $w(v_{i_0})$ that can be colored is fixed.

If there does not exist any uncolored independent set, then the coloring of the graph is fixed because vertices from the same independent set can be colored with the same arc.

Let us introduce the possible worst case strategies of Bob, denoted by σ_1 and σ_2, taking Alice's strategy into account. In particular after a vertex has been colored, Bob updates C independent of who has colored it.

Strategy σ_1: Let $B := \{b_0, ..., b_{m-1}\}$ be the set of all assigned arcs on C^r in cyclic order.

(1) Assume there exists an uncolored independent set S_i for $i \in \{1, ..., p\}$. Since Alice initially colors vertex $v_{1_1} \in S_1$, there exists an $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \cap C \neq \emptyset$ and S_{i_0} contains an uncolored vertex. Let $v_{i_{0_h}} \in S_{i_0}$ for $h \in \{1, ..., s_{i_0}\}$ be uncolored.

  - Assume there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$. Then Bob colors $v_{i_{0_h}}$ with an arc of distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
  - Assume that $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2k$ for $1 \leq j \leq m$. Then the number of the k-vertices that can be colored is fixed.

(2) Assume $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$ and there exists an i_0 for $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \not\subset C$. Let $v_{i_{0_h}} \in S_{i_0}$ for $h \in \{1, ..., s_{i_0}\}$ be uncolored.

- Assume there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k + l$. Then Bob colors $v_{i_{0_h}}$ with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
- Assume that $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k + l$ for $1 \leq j \leq m$.

  If there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ such that $k < d(b_{j-1 \bmod m}, b_{j \bmod m}) < k + l$, then he colors $v_{i_{0_h}}$ with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.

  Otherwise $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k$ for $1 \leq j \leq m$. By the assumption there exists an i_0 for $i_0 \in \{1, ..., p\}$ such that S_{i_0} contains an uncolored vertex u. Let v be a colored vertex from S_{i_0}. Without loss of generality assume that v is colored with the arc $b_{j \bmod m}$. Then Bob colors vertex u with an arc next to $b_{j-1 \bmod m}$ which obviously overlaps with $b_{j \bmod m}$ because $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k$.

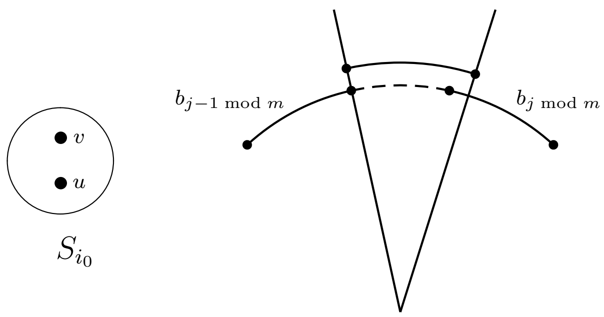


FIGURE: v is colored with $b_{j \bmod m}$. Since $(u, v) \in E(K^w_{s_1,...,s_n})$, u can be colored with an arc that is allowed to overlap with $b_{j \bmod m}$.

(3) Assume that $S_i \subset C$ for all $i \in \{1, ..., p\}$ and that there exists an uncolored independent set S_j for $j \in \{p + 1, ..., n\}$. Then by Alice's strategy there exists an $i_0 \in \{p + 1, ..., n\}$ such that $S_{i_0} \cap C \neq \emptyset$ and S_{i_0} contains an uncolored vertex. Let $v_{i_{0_h}} \in S_{i_0}$ for $h \in \{1, ..., s_{i_0}\}$ be uncolored.

- Assume there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2l$. Then Bob colors $v_{i_{0_h}}$ with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.

- Assume that $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2l$ for $1 \leq j \leq m$. Then the number of the l-vertices that can be colored is fixed.

Strategy σ_2: Let $B := \{b_0, ..., b_{m-1}\}$ be the set of all assigned arcs on C^r in cyclic order. Suppose there exists an uncolored independent set S_i for $i \in \{1, ..., n\}$.

(1) Assume there exists an uncolored independent set S_i for $i \in \{1, ..., p\}$. Since Alice initially colors vertex $v_{1_1} \in S_1$, there exists an $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \cap C \neq \emptyset$ and S_{i_0} contains an uncolored vertex. Let $v_{i_{0_h}} \in S_{i_0}$ for $h \in \{1, ..., s_{i_0}\}$ be uncolored.

  - Assume there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq 2k$. Then Bob colors $v_{i_{0_h}}$ with an arc of distance $k - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
  - Assume that $d(b_{j-1 \bmod m}, b_{j \bmod m}) < 2k$ for $1 \leq j \leq m$. Then the number of the k-vertices that can be colored is fixed.

(2) Assume $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$ and there exists an i_0 for $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \not\subset C$. Let $v_{i_{0_h}} \in S_{i_0}$ for $h \in \{1, ..., s_{i_0}\}$ be uncolored.

  - Assume that there exist two arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $d(b_{j-1 \bmod m}, b_{j \bmod m}) \geq k + l$. Then Bob colors $v_{i_{0_h}}$ with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
  - Assume that $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k + l$ for $1 \leq j \leq m$.
    If there exist arcs $b_{j-1 \bmod m}$ and $b_{j \bmod m}$ for $1 \leq j \leq m$ with $k < d(b_{j-1 \bmod m}, b_{j \bmod m}) < k + l$, he colors $v_{i_{0_h}}$ with an arc next to $b_{j-1 \bmod m}$ or $b_{j \bmod m}$.
    Otherwise $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k$ for $1 \leq j \leq m$. By the assumption there exists an i_0 for $i_0 \in \{1, ..., p\}$ such that S_{i_0} contains an uncolored vertex u. Let v be a colored vertex from S_{i_0}. Without loss of generality assume that v is colored with the arc $b_{j \bmod m}$. Then Bob colors vertex u with an arc next to $b_{j-1 \bmod m}$ which obviously overlaps with $b_{j \bmod m}$ because $d(b_{j-1 \bmod m}, b_{j \bmod m}) < k$.

(3) Assume that $S_i \subset C$ for all $i \in \{1, ..., p\}$ and that there exists an uncolored independent set S_j for $j \in \{p+1, ..., n\}$. Then by Alice's strategy there is an i_0 for $i_0 \in \{p+1, ..., n\}$ such that S_{i_0} contains an uncolored vertex. Let $v_{i_{0_h}} \in S_{i_0}$ be uncolored for $h \in \{1, ..., s_{i_0}\}$.

- Assume that there exist two arcs $b_{j-1 \text{ mod } m}$ and $b_{j \text{ mod } m}$ for $1 \le j \le m$ with $d(b_{j-1 \text{ mod } m}, b_{j \text{ mod } m}) \ge 2l$, then Bob colors $v_{i_{0_h}}$ with an arc of distance $l - \varepsilon$ for an $\varepsilon > 0$ next to $b_{j-1 \text{ mod } m}$ or $b_{j \text{ mod } m}$.
- Assume If $d(b_{j-1 \text{ mod } m}, b_{j \text{ mod } m}) < 2l$ for $1 \le j \le m$. Then the number of the l-vertices that can be colored is fixed.

As soon as $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., n\}$, the coloring of the entire graph is fixed because vertices from the same independent set can be colored with arcs that are allowed to overlap.

It is of our interest to determine the circumference r of C^r depending on which strategy Bob goes for. For this purpose we consider the intersection of Alice's strategy with σ_1 and σ_2. However, independent of which strategy Bob goes for, we can conclude the following: at the time when $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$ and $S_j \cap C = \emptyset$ for all $j \in \{p+1, ..., n\}$ and Alice is the next one to color, the amount of the colored vertices is $2q - 1$ until $S_j \cap C \neq \emptyset$ for all $j \in \{p+1, ..., n\}$. In particular Alice will color for q times, while Bob will color for $q - 1$ times.

Assume that Bob applies strategy σ_1.

Suppose it is Bob's turn and it holds $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$ and $S_j \cap C = \emptyset$ for all $j \in \{p+1, ..., n\}$. Without loss of generality assume that Alice colored vertex v_{p_1} at last. Moreover, we can follow that Alice has assigned for p times and Bob for $p - 1$ times vertices from V^k, while Bob created a $(k - \varepsilon)$-gap each time he took turn. Thus, we can conclude that

$$r \ge kp + 2k \cdot (p - 1).$$

$$r \ge k(3p - 2). \tag{3.8}$$

For determining the further increasing of r we are left with the task of considering the cases $2q-1>p-1,\ 2q-1<p-1$ and $2q-1=p-1$.

Let $2q-1>p-1$.

This implies that the number of the $(k-\varepsilon)$-gaps is less than the number of the colorings until $S_i\cap C\neq\emptyset$ for all $i\in\{p+1,...,n\}$. Thus, a circle with circumference $k(3p-2)$ does not suffice for assigning all arcs of vertices from V^l inside these gaps. Then by step (2) r increases at least by $k+l$ and by (3.8) it is easily seen that it holds

$$r\geq k(3p-1)+l. \tag{3.9}$$

It remains to consider the coloring of V^l taking into account that it is Alice's turn now. Let $\{b_0,...,b_{2p-1}\}$ be the set of all assigned arcs in cyclic order. By the strategies they proceed to color with arcs between the already assigned arcs b_{2p-1} and b_0 for $2q-1-(p-1)=2q-p$ times until they color into the $(k-\varepsilon)$-gaps.

- Let p be even. Then both assign $\frac{2q-p}{2}=q-\frac{p}{2}$ vertices to C^r, respectively. In particular Alice starts and Bob finishes coloring between the already assigned arcs b_{2p-1} and b_0, whereas Alice colors $q-\frac{p}{2}$ l-vertices.

  If $u^k\geq q-\frac{p}{2}$, then Bob achieves to color only vertices from V^k until $S_n\cap C\neq\emptyset$. Thus, Bob has colored $k\cdot(q-\frac{p}{2}-1)$ k-vertices while destroying $(q-\frac{p}{2}-1)$ arcs as soon as Alice colored the last vertex with an arc between b_{2p-1} and b_0. The last arc which Bob destroys suffices to be of length l. Thus, by (3.9) it follows that

  $$r=k(3p-1)+l+l\Big(q-\frac{p}{2}\Big)+k\Big(q-\frac{p}{2}-1\Big)+l\Big(q-\frac{p}{2}-1\Big)+l,$$

  such that a circle with circumference of

  $$r=k\Big(\frac{5p+2q}{2}-2\Big)+l(2q-p+1) \tag{3.10}$$

  suffices in order to get a proper coloring of the graph.

  If $u^k<q-\frac{p}{2}$, then Bob manages to place only u^k arcs of vertices from V^k. The remaining $q-\frac{p}{2}-u^k$ vertices he colors are vertices from V^l. Due to

step (3) each time he takes turn he produces an $(l-\varepsilon)$-gap, such that the total amount of the produced $(l-\varepsilon)$-gaps is $q-\frac{p}{2}-1$. Hence, because of (3.9) we can conclude that for the circumference r of the circle C^r it holds:

$$r = k(3p-1)+l+l\Big(q-\frac{p}{2}\Big)+ku^k+l\Big(q-\frac{p}{2}-u^k-1\Big)+l\Big(q-\frac{p}{2}-1\Big)+l$$

$$r = k(3p+u^k-1)+l\Big(3(q-\frac{p}{2})-u^k\Big). \tag{3.11}$$

- Let p be odd. Since Alice starts coloring, she colors $\lceil\frac{2q-p}{2}\rceil = \lceil q-\frac{p}{2}\rceil$ and Bob colors $\lfloor\frac{2q-p}{2}\rfloor = q-\frac{p}{2}-\frac{1}{2}$ vertices with arcs between b_{2p-1} and b_0.

  If $u^k \geq q-\frac{p}{2}-\frac{1}{2}$, then Bob assigns another $\lfloor\frac{2q-p}{2}\rfloor$ vertices with vertex-weight k on C^r, whereas he destroys additional $\lfloor\frac{2q-p}{2}\rfloor$ arcs of length $l-\varepsilon$. Thus, the circumference increases by $k\lfloor q-\frac{p}{2}\rfloor+l\lfloor q-\frac{p}{2}\rfloor$, such that by (3.9) a feasible coloring is guaranteed if

$$r = k(3p-1)+l+l\Big\lceil q-\frac{p}{2}\Big\rceil+k\Big\lfloor q-\frac{p}{2}\Big\rfloor+l\Big\lfloor q-\frac{p}{2}\Big\rfloor =$$

$$k\cdot\frac{5p+2q-3}{2}+l(2q-p+1). \tag{3.12}$$

  If $u^k < q-\frac{p}{2}-\frac{1}{2}$, then Bob manages to colors only u^k vertices from V^k. The remaining $\lfloor\frac{2q-p}{2}\rfloor-u^k$ vertices he colors have vertex-weight l. By his strategy each time he takes turn he produces an $(l-\varepsilon)$-gap, such that the total amount of the produced $(l-\varepsilon)$-gaps is $\lfloor\frac{2q-p}{2}\rfloor$. Thus, because of (3.9) it can be drawn the conclusion that it holds:

$$r = k(3p-1)+l+l\Big\lceil\frac{2q-p}{2}\Big\rceil+ku^k+l\Big(\Big\lfloor\frac{2q-p}{2}\Big\rfloor-u^k\Big)+l\Big\lfloor\frac{2q-p}{2}\Big\rfloor =$$

$$k(3p+u^k-1)+l\Big(3(q-\frac{p}{2})-u^k+\frac{1}{2}\Big). \tag{3.13}$$

Let $2q < p$.

Since $2q-1 < p-1$, the number of the $(k-\varepsilon)$-gaps suffices for placing the corresponding arcs of all vertices with vertex-weight l inside these gaps. Then a circle with circumference $r = kp+2k(p-1)$ suffices in order to achieve a feasible coloring of the graph. Thus, it holds

$$r = kp+2k(p-1) = k(3p-2). \tag{3.14}$$

Let $2q = p$.

Since $2q - 1 = p - 1$, one could assume that a circle with circumference $r = k(3p - 2)$ suffices for achieving a proper coloring of the graph. But since from the time when Alice has colored vertex v_{p_1}, both players colors for q times, respectively. This implies that the vertices of one independent set from V^l cannot be colored because $|V^l| = q$. Hence, r increases by l. It follows that a circle with circumference

$$r = k(3p - 2) + l \tag{3.15}$$

suffices.

Assume Bob applies strategy σ_2.

Then according to strategy σ_2 we can conclude that $n - 1$ $(l - \varepsilon)$-gaps arise since the game is over as soon as Alice has colored a vertex in S_n. Further lq additionally arcs are being added, which implies that because of the structure of the graph Alice cannot place an arc inside the $(l-\varepsilon)$-gaps since $w(v_{i_j}) > l-\varepsilon$ for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., s_i\}$. Finally it has to be considered that r increases by at least kp, because each time Bob colors he assigns vertices from V^k, as long as there are vertices with vertex-weight k left. Thus,

$$r \geq kp + kp + l(n - 1) + lq =$$

$$2kp + l(p + 2q - 1). \tag{3.16}$$

- If $u^k \geq q - 1$, then because of (3.16) we can conclude that

$$r = 2kp + l(p + 2q - 1) + k(q - 1) =$$

$$k(2p + q - 1) + l(p + 2q - 1). \tag{3.17}$$

- If $u^k < q - 1$, then Bob manages to assign u^k vertices with vertex-weight k as well as $q - 1 - u^k$ vertices with vertex-weight l. Hence, because of (3.16) we can conclude that

$$r = 2kp + l(p + 2q - 1) + ku^k + l(q - 1 - u^k) =$$

$$k(2p + u^k) + l(p + 3q - u^k - 2). \tag{3.18}$$

We are left with the task of determining the circular game chromatic number of $K^w_{s_1,\dots,s_n}$. Hence, we have to figure out, which of both strategies Bob should apply, if he considers the structure of the graph:

(i) Let $2q > p$ and suppose that p is even. The proof falls naturally into three parts, namely we consider the cases $u^k \geq q-1, u^k < q - \frac{p}{2}$ as well as $q - \frac{p}{2} \leq u^k < q-1$.

- Assume that $u^k \geq q-1$. Then

$$\gamma_c(K^w_{s_1,\dots,s_n}) \leq k(2p+q-1) + l(p+2q-1),$$

since

$$\overbrace{k(2p+q-1)+l(p+2q-1)}^{(3.17)} > \overbrace{k\Big(\frac{5p+2q}{2}-2\Big)+l(2q-p+1)}^{(3.10)}.$$

Suppose the inequality is false, such that

$$k(2p+q-1)+l(p+2q-1) \leq k\Big(\frac{5p+2q}{2}-2\Big)+l(2q-p+1) \Leftrightarrow$$

$$k(2p+q-1-\frac{5p+2q}{2}+2) \leq l(2q-p+1-p-2q+1) \Leftrightarrow$$

$$k\Big(\frac{2+4p+2q-5p-2q}{2}\Big) \leq l(2-2p) \Leftrightarrow$$

$$k\cdot\frac{2-p}{2} \leq 2l(1-p)\Big| : \quad \underbrace{\frac{2-p}{2}}_{<0 \text{ since } p>2} \quad \Leftrightarrow$$

$$k \geq 2l\cdot\frac{2(1-p)}{2-p}.$$

But this contradicts the assumption $k \leq 2l$, because $\frac{1-p}{2-p} > 1$.

- Assume that $u^k < q - \frac{p}{2}$. Then

$$\gamma_c(K^w_{s_1,\dots,s_n}) \leq k(2p+u^k) + l(p+3q-u^k-2),$$

since

$$\overbrace{k(2p+u^k)+l(p+3q-u^k-2)}^{(3.18)} > \overbrace{k(3p+u^k-1)+l\big(3(q-\frac{p}{2})-u^k\big)}^{(3.11)}.$$

Suppose the inequality is false, such that

$$k(2p+u^k)+l(p+3q-u^k-2) \leq k(3p+u^k-1)+l\big(3(q-\frac{p}{2})-u^k\big) \Leftrightarrow$$

$$k(2p+u^k-3p-u^k+1) \leq l\big(3(q-\frac{p}{2})-u^k-p-3q+u^k+2\big) \Leftrightarrow$$

$$k(1-p) \leq l\Big(2-\frac{5p}{2}\Big)\Big| : \qquad \underbrace{(1-p)}_{<0 \text{ since } p>2} \qquad \Leftrightarrow$$

$$k \geq 2l \cdot \frac{2-\frac{5p}{2}}{2(1-p)}$$

But it is easily seen that $\frac{2-\frac{5p}{2}}{2(1-p)} > 1$. Otherwise $\frac{2-\frac{5p}{2}}{2(1-p)} \leq 1$ contradicts the assumption $k \leq 2l$. Assume

$$\frac{2-\frac{5p}{2}}{2(1-p)} \leq 1\Big| \cdot \qquad \underbrace{2(1-p)}_{<0 \text{ since } p>2} \qquad \Leftrightarrow$$

$$2-\frac{5p}{2} \geq 2-2p \Leftrightarrow$$

$$-\frac{5p}{2} \geq -2p \Leftrightarrow$$

$$\frac{5p}{2} \leq 2p \Leftrightarrow$$

$$5p \leq 4p \ \lightning$$

- Assume that $q-\frac{p}{2} \leq u^k < q-1$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq k(2p+u^k)+l(p+3q-u^k-2),$$

since

$$\overbrace{k(2p+u^k)+l(p+3q-u^k-2)}^{(3.18)} > \overbrace{k\Big(\frac{5p+2q}{2}-2\Big)+l(2q-p+1)}^{(3.10)}.$$

Suppose the inequality is false, such that

$$k(2p+u^k)+l(p+3q-u^k-2) \leq k\Big(\frac{5p+2q}{2}-2\Big)+l(2q-p+1) \Leftrightarrow$$

$$l(p+3q-u^k-2-2q+p-1) \leq k\Big(\frac{5p+2q}{2}-2-2p-u^k\Big) \Leftrightarrow$$

$$l(2p+q-u^k-3) \leq k \cdot \frac{p+2q-2u^k-4}{2} \Leftrightarrow$$

$$2l \cdot \frac{2p+q-u^k-3}{p+2q-2u^k-4} \leq k \tag{3.19}$$

$p+2q-2u^k-4 > 0$, because $u^k < q-1$ and $p+2q-2(q-1)-4 = p-2 > 0$. Thus, the equivalence does not change.

It holds $\frac{2p+q-u^k-3}{p+2q-2u^k-4} > 1$, because of the following calculation: Assume the inequality is false, then

$$\frac{2p+q-u^k-3}{p+2q-2u^k-4} \leq 1 \Leftrightarrow$$

$$2p+q-u^k-3 \leq p+2q-2u^k-4 \Leftrightarrow$$

$$p-q+u^k+1 \leq 0.$$

Since by the assumption $q-\frac{p}{2} \leq u^k < q-1$, it holds $p-q+u^k+1 > 0$ which is a contradiction.

Thus, by (3.19) we can conclude that the inequality

$$2l \cdot \frac{2p+q-u^k-3}{p+2q-2u^k-4} \leq k$$

is true which is a contradiction to the assumption $k \leq 2l$.

Suppose p is odd. As in case p even, the proof falls into three cases, namely we work on the cases $u^k \geq q-1$, $u^k < q-\frac{q}{2}-\frac{1}{2}$ and $q-\frac{q}{2}-\frac{1}{2} \leq u^k < q-1$.

- Assume that $u^k \geq q-1$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq k(2p+q-1)+l(p+2q-1),$$

since

$$\overbrace{k(2p+q-1)+l(p+2q-1)}^{(3.17)} > \overbrace{k \cdot \frac{5p+2q-3}{2}+l(2q-p+1)}^{(3.12)}.$$

Suppose the inequality is false, such that

$$k(2p+q-1)+l(p+2q-1) \leq k\cdot\frac{5p+2q-3}{2}+l(2q-p+1) \Leftrightarrow$$

$$l(p+2q-1-2q+p-1) \leq k\Big(\frac{5p+2q-3}{2}-2p-q+1\Big) \Leftrightarrow$$

$$l(2p-2) \leq k\cdot\frac{5p+2q-3-4p-2q+2}{2} \Leftrightarrow$$

$$2l(p-1) \leq k\cdot\frac{p-1}{2} \Leftrightarrow$$

$$4l \leq k.$$

But this contradicts the assumption $k \leq 2l$.

- Assume that $u^k < q-\frac{p}{2}-\frac{1}{2}$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq k(2p+u^k)+l(p+3q-u^k-2),$$

since

$$\overbrace{k(2p+u^k)+l(p+3q-u^k-2)}^{(3.18)} > \overbrace{k(3p+u^k-1)+l\Big(3(q-\frac{p}{2})-u^k+\frac{1}{2}\Big)}^{(3.13)}.$$

Suppose the inequality is false, such that

$$k(2p+u^k)+l(p+3q-u^k-2) \leq k(3p+u^k-1)+l\Big(3(q-\frac{p}{2})-u^k+\frac{1}{2}\Big) \Leftrightarrow$$

$$l\big(p+3q-u^k-2-3q+\frac{3p}{2}+u^k-\frac{1}{2}\big) \leq k(3p+u^k-1-2p-u^k) \Leftrightarrow$$

$$\frac{5}{2}l(p-1) \leq k(p-1) \Leftrightarrow$$

$$\frac{5}{2}\,l \leq k.$$

But since $k \leq 2l$, the inequality is false.

- Assume that $q-\frac{p}{2}-\frac{1}{2} \leq u^k < q-1$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq k(2p+u^k)+l(p+3q-u^k-2),$$

since

$$\overbrace{k(2p+u^k)+l(p+3q-u^k-2)}^{(3.18)} > \overbrace{k\cdot\frac{5p+2q-3}{2}+l(2q-p+1)}^{(3.12)}.$$

Suppose the inequality is false, such that

$$\begin{aligned} k(2p+u^k)+l(p+3q-u^k-2) &\leq k\cdot\frac{5p+2q-3}{2}+l(2q-p+1) \Leftrightarrow \\ l(p+3q-u^k-2-2q+p-1) &\leq k\Big(\frac{5p+2q-3}{2}-2p-u^k\Big) \Leftrightarrow \\ l(2p+q-u^k-3) &\leq k\cdot\frac{5p+2q-3-4p-2u^k}{2} \Leftrightarrow \\ l(2p+q-u^k-3) &\leq k\cdot\frac{p+2q-2u^k-3}{2} \Leftrightarrow \\ 2l\cdot\frac{2p+q-u^k-3}{p+2q-2u^k-3} &\leq k. \end{aligned} \tag{3.20}$$

The equivalence did not change, since $p+2q-2u^k-3>0$ because of the following: Since $u^k<q-1$, it suffices to prove $p+2q-2u^k-3>0$ for $u^k=q-1$. Then

$$p+2q-2u^k-3 = p+2q-2(q-1)-3 = p+2q-2q+2-3 = p-1,$$

which is true because of the assumption $p>2$. In particular it holds $\frac{2p+q-u^k-3}{p+2q-2u^k-3}>1$ because of the following calculation: Let

$$\begin{aligned} \frac{2p+q-u^k-3}{p+2q-2u^k-3} &\leq 1 \Leftrightarrow \\ 2p+q-u^k-3 &\leq p+2q-2u^k-3 \Leftrightarrow \\ 0 &\leq q-p-u^k. \end{aligned}$$

Since $u^k<q-1$, it suffices to prove that $0\leq q-p-u^k$ for $u^k=q-1$, since $q-p-u^k>q-p-(q-1)$. Then

$$0 \leq q-p-(q-1) = q-p-q+1 = 1-p$$

which is a contradiction, since $p > 2$.

Thus, because of (3.20) we can conclude that the inequality

$$2l \cdot \frac{2p + q - u^k - 3}{p + 2q - 2u^k - 3} \leq k$$

is true, which contradicts the assumption $k \leq 2l$.

Finally we can conclude that in each of the three cases Bob decides to play strategy σ_2.

(ii) Let $2q < p$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq \max\{\overbrace{k(3p-2)}^{(3.14)}, \overbrace{k(2p+q-1) + l(p+2q-1)}^{(3.17)}\}.$$

Let us consider in which case Bob goes for σ_1 and σ_2, respectively. For this purpose we consider the following inequality:

$$k(3p-2) \leq k(2p+q-1) + l(p+2q-1) \Leftrightarrow$$

$$k(3p-2-2p-q+1) \leq l(p+2q-1) \Leftrightarrow$$

$$k(p-q-1) \leq l(p+2q-1) \Leftrightarrow$$

$$k \leq l \cdot \frac{p+2q-1}{p-q-1}.$$

Thus, by the assumption $l < k \leq 2l$ the following conclusions can be drawn:

- If $k < l \cdot \frac{p+2q-1}{p-q-1}$, then Bob should apply strategy σ_2.
- If $k > l \cdot \frac{p+2q-1}{p-q-1}$, then Bob should apply strategy σ_1.
- If $k = l \cdot \frac{p+2q-1}{p-q-1}$, then Bob is indifferent, which strategy he should play.

(iii) Let $2q = p$. Then

$$\gamma_c(K^w_{s_1,\ldots,s_n}) \leq k(2p+q-1) + l(p+2q-1),$$

since

$$\overbrace{k(2p+q-1) + l(p+2q-1)}^{(3.17)} > \overbrace{k(3p-2) + l}^{(3.15)}.$$

Assume the inequality is false, then

$$k(2p+q-1)+l(p+2q-1) \leq k(3p-2)+l \Leftrightarrow$$

$$k(2p+q-1-3p+2) \leq l(1-p-2q+1) \Leftrightarrow$$

$$k(q-p+1) \leq l(2-p-2q) \Leftrightarrow$$

$$k \geq l \cdot \frac{2-p-2q}{q-p+1} \quad \Big| \quad \text{since } q < p.$$

But this contradicts the assumption $k \leq 2l$, since $\frac{2-p-2q}{q-p+1} > 2$, because of the following: Assume that

$$\frac{2-p-2q}{q-p+1} \leq 2 \Leftrightarrow$$

$$2-p-2q \geq 2(q-p+1) \Leftrightarrow$$

$$2-p-2q \geq 2q-2p+2 \Leftrightarrow$$

$$p \geq 4q \ \lightning$$

Hence, in this case Bob should apply *Strategy* σ_2. □

Remark 3.3.5. One may ask whether it makes sense for Bob to color a vertex with an arc of distance different than $k-\varepsilon$ and $l-\varepsilon$ for an $\varepsilon > 0$. Let r' be the required circumference for a proper coloring of the graph. Assume Bob colors an arbitrary vertex with an arc b_{i+1} of distance x to an assigned arc b_i.

Let $x \geq k$ and assume that there exists an independent set $S_{i_0} \in V^k$ for which it holds $S_{i_0} \cap C = \emptyset$. Then a vertex from S_{i_0} can be colored with an arc between b_i and b_{i+1}.

- If $x-k < k-\varepsilon$, then r' increases only by $x-k$ instead of $k-\varepsilon$.
- If $x-k > k-\varepsilon$, then Bob did not destroy any free space on $C^{r'}$ in this turn.

Let $k-\varepsilon > x \geq l$ and assume that there exists an independent set $S_{i_0} \in V^k$ for which it holds $S_{i_0} \cap C = \emptyset$. Then r' increases only by x instead of $k-\varepsilon$. Assume there exists an independent set $S_{j_0} \in V^l$ where $S_{j_0} \cap C = \emptyset$, then an arbitrary vertex from S_{j_0} can be colored with an arc between b_i and b_{i+1}.

- If $x - l < l - \varepsilon$, then r' increases by $x - l$ instead of $l - \varepsilon$.
- If $x - l > l - \varepsilon$, then Bob did not destroy any free space on $C^{r'}$ in this turn.

Let $x < l - \varepsilon$. Then r' increases by x instead of $l - \varepsilon$.

3.4 The Circular Game Chromatic Number of Weighted Cycles

In this section we turn our attention to the circular game chromatic number of weighted cycles. We give an upper bound for the entire class of weighted cycles as well as consider some certain distributions of the weights on a cycle. In particular we analyze the so called alternating-weighted cycles.

Let us introduce the following notion: Let $G^w = (V, E, w)$ be a graph. Let $y \in V(G^w)$ and $N(y) = \{x_1, ..., x_m\}$ be the set of all neighbors of y. Suppose vertex x_1 is already colored and $\{y, x_2, ..., x_m\}$ are uncolored yet. Then we say that there is an *attack* on y, if x_i for an $i \in \{2, ..., m\}$ is colored on C^r, such that $0 < d(f_w(x_1), f_w(x_i)) < w(y)$. In case that the given colors are from $\mathbb{N}$ we say that there is an *attack* on y, if x_2 is colored with a different color than x_1.

Example:

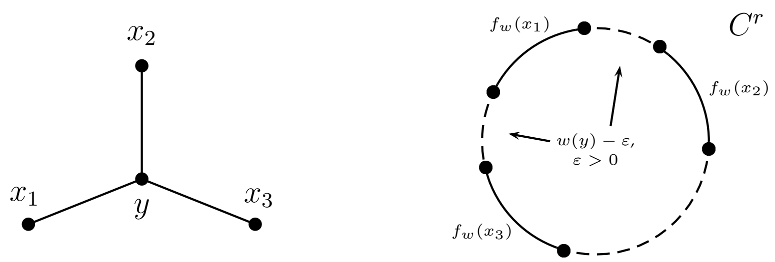


FIGURE: Vertex y is attacked twice because $(x_i, y) \in E(G^w)$ for $i \in \{1, 2, 3\}$ and $0 < d(f_w(x_1), f_w(x_2)) < w(y)$ and $0 < d(f_w(x_1), f_w(x_3)) < w(y)$ on C^r.

In the following proposition we work on the upper bound of the circular game chromatic number on the class of weighted cycles independent of the given distribution of the vertex-weights. In particular we utilize the simple structure of a cycle, that is, each vertex has degree 2. Thus, it can be attacked once at most.

Proposition 3.4.1. *Let $C_n^w = (V, E, w)$ be a weighted cycle be a cycle on n vertices. Then*

$$\gamma_c(C_n^w) \leq 4 \cdot w_{\max}(C_n^w).$$

Proof. Without loss of generality assume that each vertex from $V(C_n^w)$ achieves the maximum weight $k := w_{\max}(C_n^w)$. Obviously this increases the required circumference r of C^r at most and hence we receive an upper bound of the game chromatic number of the class of weighted cycles. Assume that a circle C^r with circumference $r = 4k$ is given.

Without loss of generality assume that initially Alice colors vertex x_0. Since $w(x_i) = k$ for all $i \in \{0, ..., n-1\}$, the worst case occurs if Bob attacks either x_{n-1} or x_1 by coloring x_{n-2} or x_2 with an arc of distance $k - \varepsilon$ to $f_w(x_0)$. Assume that by symmetry he attacks x_1 such that $d(f_w(x_0), f_w(x_2)) = k - \varepsilon$. Then x_1 is attacked since $(x_0, x_1), (x_1, x_2) \in E(C_n^w)$ and the corresponding arc of x_1 cannot be placed between $f_w(x_0)$ and $f_w(x_2)$. However, there is still enough space for $f_w(x_1)$ by the assumption $r = 4k$ and $d(f_w(x_2), f_w(x_0)) = k - \varepsilon$. Hence, the trivial upper bound is required for Alice's victory. □

Not previously discussed is the case that the vertices of C^w are assigned to a certain distribution of weights $\{w(x_0), w(x_1), ..., w(x_{n-1})\}$. We intend to analyze the circular game chromatic number of the class of the so called alternating-weighted cycles, while referring to a cycle by the natural sequence of its vertices $x_0, x_1, ..., x_{n-1}$.

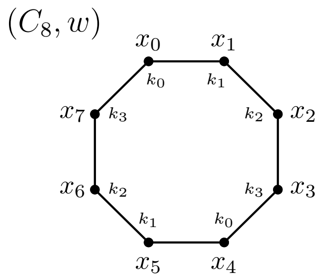
Definition 3.4.2. Let m be a positive integer with $m \leq n$ and $C_n^w = (V, E, w)$ be a cycle with $|V| = n$ whereas $n \bmod m = 0$. We call $C_n^w = (V, E, w)$ $(k_0, ..., k_{m-1})$-*alternating-weighted* iff $w : V \to \{k_0, ..., k_{m-1}\}$ with $k_i = w(x_{i+m\times j})$, where $i + m \times j \leq n - 1$ for $j \in \mathbb{N}_0$. and $i \in \{0, ..., m-1\}$.

Example: Let $C_n^w = (V, E, w)$ be $(k_0, ..., k_{m-1})$-alternating-weighted with

$n = 8$ and $m = 4$. Then

$$w(x_0) = w(x_4) = k_0, \;\; w(x_1) = w(x_5) = k_1,$$

$$w(x_2) = w(x_6) = k_2, \;\; w(x_3) = w(x_7) = k_3.$$



Proposition 3.4.3. *Let $C_n^w = (V, E, w)$ be a (k_0, k_1)-alternating-weighted cycle with $k_0 \neq k_1$. Then $\gamma_c(C_n^w) = 2(k_0 + k_1)$.*

Proof. Assume that a circle C^r with circumference $r = 2(k_0 + k_1)$ is given. Without loss of generality assume that in her first turn Alice colored vertex x_0 and it is Bob's turn. Because $w(x_{n-1}) = w(x_1)$, we can refer to Bob's worst case strategy as in proposition 3.4.1. Suppose Bob attacks x_1 by coloring x_2 such that $d(f_w(x_0), f_w(x_2)) = k_1 - \varepsilon$. Since $l(f_w(x_0)) = l(f_w(x_2)) = k_0$, vertex x_1 can still be colored with an arc between $f_w(x_2)$ and $f_w(x_0)$. Thus, a circle with circumference

$$\begin{aligned} r &= 2 \cdot w(x_0) + 2 \cdot w(x_1) \\ &= 2k_0 + 2k_1 \end{aligned}$$

suffices in order to achieve a proper coloring of $\{x_0, x_1, x_2\}$. The coloring of the remaining uncolored vertices is trivial. Thus, Alice wins the game. □

For generalizing proposition 3.4.3 we introduce the following notion: Let M be a finite set with $M \subset \mathbb{N}$. Then we define $\max_2\{M\}$ to be the second highest integer of the set M.

Proposition 3.4.4. *Let $C_n^w = (V, E, w)$ be a $(k_0, \ldots, k_{m-1})$-alternating-weighted cycle with $k_0 < k_1 < \ldots < k_{m-1}$ for $m \in \mathbb{N}$ and $m \geq 3$.*

(i) For $\big(|V| - |\{x \in V \ ; \ w(x) \notin \{k_0, k_{m-3}, k_{m-2}, k_{m-1}\}\}|\big) \bmod 2 = 0$ *we have*

$$\gamma_c(C_n^w) = \max_2\big\{k_{m-2}+2k_{m-1}+k_0,\ k_{m-3}+2k_{m-2}+k_{m-1},\ k_{m-4}+2k_{m-3}+k_{m-2}\big\}.$$

(ii) For $\big(|V| - |\{x \in V \ ; \ w(x) \notin \{k_0, k_{m-3}, k_{m-2}, k_{m-1}\}\}|\big) \bmod 2 = 1$ *we have*

$$\gamma_c(C_n^w) = \max\{k_{m-4} + 2k_{m-3} + k_{m-2},\ k_{m-1} + k_{m-2}\}.$$

Proof. (i) Let C_n^w be a $(k_0, \ldots, k_{m-1})$-alternating-weighted cycle and let $\big(|V| - |\{x \in V \ ; \ w(x) \notin \{k_0, k_{m-3}, k_{m-2}, k_{m-1}\}\}|\big) \bmod 2 = 0$. Consider the following strategies of Bob:

Strategy 1: Bob attacks a vertex with weight k_{m-1}. Then a circle with circumference $k_{m-2} + 2k_{m-1} + k_0$ is required in order to guarantee Alice's victory, since vertices with weight k_{m-1} are adjacent to vertices with weights k_{m-2} and k_0.

Strategy 2: Bob attacks a vertex with weight k_{m-2}. Then a circle with circumference $k_{m-3} + 2k_{m-2} + k_{m-1}$ is required, since vertices with weight k_{m-2} are adjacent to vertices with weights k_{m-3} and k_{m-1}.

An attack of a vertex with weight $k_0, ..., k_{m-3}$ would destroy less space on the circle since $k_0 < k_1 < ... < k_{m-2} < k_{m-1}$. Thus, by the assumption $k_0 < k_1 < \ldots < k_{m-1}$ either strategy 1 or strategy 2 turns out to be the worst case.

Alice's strategy

Let $\hat{w} \in \{k_{m-2}, k_{m-1}\}$ be the weight which results to the worst case for the given cycle C_n^w. Initially she colors a vertex with weight $\hat{w}$. Then she proceeds as follows: Assume Bob has colored vertex $x_{i \bmod n}$ for $i \in \{0, ..., n-1\}$ in his last turn.

- If $w(x_{i-1 \bmod n}) = \hat{w}$, then she colors $x_{i-1 \bmod n}$ (the same for $x_{i+1 \bmod n}$).
- If $w(x_{i-1 \bmod n}) \neq \hat{w}$ and if $w(x_{i+1 \bmod n}) \neq \hat{w}$, then she colors an arbitrary vertex with weight $\hat{w}$, if possible. Otherwise she colors an arbitrary vertex.

- If $w(x_{i \bmod n}) = \hat{w}$, then she colors an arbitrary vertex with weight $\hat{w}$, if possible. Otherwise she colors an arbitrary vertex.
- If each vertex with weight $\hat{w}$ is colored, she colors an arbitrary vertex.

If Alice applies the strategy above, she is able to avoid each attack on vertices with weight $\hat{w}$, since it holds $m \geq 3$. Hence, Bob cannot attack two such vertices simultaneously. However, if she tried to avoid the attacks on vertices that create the second worst case, Bob would be able to attack vertices with weight $\hat{w}$, since these vertices are connected by an edge. Hence, this strategy is the best possible and we can conclude for the circular game chromatic number the following:

- If the worst case is that Bob attacks vertices with weight k_{m-1}, Alice will avoid these attacks and hence Bob will attack vertices with weight k_{m-2}.
- Otherwise Bob will attack vertices with weight either k_{m-3} or k_{m-1}. In case $k_{m-4}+2k_{m-3}+k_{m-2} > k_{m-2}+2k_{m-1}+k_0$ he will attack vertices with weight k_{m-3}. In case $k_{m-4}+2k_{m-3}+k_{m-2} < k_{m-2}+2k_{m-1}+k_0$ he will attack vertices with weight k_{m-1}. In case of equality he is indifferent.

Finally we conclude that

$$\gamma_c(C_n^w) = \max_2\big\{k_{m-2}+2k_{m-1}+k_0,\ 2k_{m-2}+k_{m-3}+k_{m-1},\ 2k_{m-4}+k_{m-3}+k_{m-2}\big\}.$$

(ii) Next consider the case $\big(|V| - |\{x \in V\ ;\ w(x) \notin \{k_0, k_{m-3}, k_{m-2}, k_{m-1}\}\}|\big) \bmod 2 = 1$. Then obviously Alice is able to avoid attacks on vertices with weight k_{m-1} and k_{m-2} by coloring vertices with weight $k_1, k_2, ..., k_{m-4}$. By the assumption $\big(|V| - |\{x \in V\ ;\ w(x) \notin \{k_0, k_{m-3}, k_{m-2}, k_{m-1}\}\}|\big) \bmod 2 = 1$, Alice can force Bob to color always in a sequence of vertices with weights $k_{m-3}, k_{m-2}, k_{m-1}, k_0$ first and hence Bob can attack only vertices with weight k_{m-3} twice.

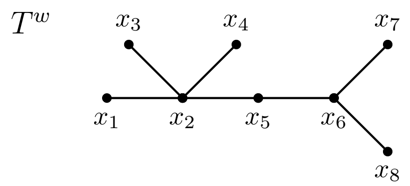
□

3.5 The Circular Game Chromatic Number of Weighted Trees

A graph which does not contain any cycles is called a *forest*, denoted by $F = (V, E)$. A connected forest is called a *tree*, denoted by $T = (V, E)$.

Let $T^w = (V, E, w)$ be a weighted tree. We call a vertex with degree 1 a *leaf* and denote the set of all leaves by $\overline{V}(T)$. Further we call a vertex with degree greater than 1 an *interior vertex* and denote the set of all interior-vertices by $\mathring{V}(T)$.

Example: Consider the following tree $T^w = (V, E, w)$ with $w(x_1) = w(x_2) = 3$ and $w(x_i) = 2$ for all $i = \{3, ..., 8\}$. Then $\overline{V}(T) = \{x_1, x_3, x_4, x_7, x_8\}$ and $\mathring{V}(T) = \{x_2, x_5, x_7\}$ where $\overline{w}_{\max}(T^w) = \mathring{w}_{\max}(T^w) = 3$.



In this section we intend to determine the circular game chromatic number of weighted trees. For our purpose we make use of an algorithm which is due to U. Faigle, U. Kern, H. A. Kierstead, and W. T. Trotter in [1]. They introduced a winning strategy for Alice such that 4 colors are sufficient for coloring a tree if the regular game is played, where the colors are natural numbers. For the sake of completeness we give below this algorithm:

At the beginning of the game Alice colors an arbitrary vertex $r \in V(T)$ which will be called *root* for the rest of the game. During the game Alice maintains a subtree T_0 of T that contains all the vertices colored so far. Alice initializes $T_0 = \{r\}$.

Suppose Bob has just colored vertex v. Let P be the directed path from r to v in T and let u be the last vertex P has in common with T_0. Then Alice does the following:

(1) Update $T_0 := T \cup P$.

(2) If u is uncolored, assign a feasible color to u.

(3) If u is colored and T_0 contains an uncolored vertex $v \in T_0$, assign a feasible color to v.

(4) If all vertices in T_0 are colored, color any vertex v adjacent to T_0 and update $T_0 := T_0 \cup \{v\}$.

The next proposition gives an upper bound for the circular game chromatic number for weighted trees. Note that $N(x)$ is the set of all neighbors of x for $x \in V$.

Proposition 3.5.1. *Let $T^w = (V, E, w)$ be a weighted tree. Then the following holds*

$$\gamma_c(T^w) \leq \max_{y \in \mathring{V}(T)} \{m \cdot w(y) + \max_{z_i \in N(y)} \sum_{i=1}^{m} w(z_i)\},$$

whereas

- $m = 3$ *if* $|N(y)| \geq 3$ *and*
- $m = 2$ *if* $|N(y)| = 2$

for $y \in \mathring{V}(T^w)$.

Proof. Let $T^w = (V, E, w)$ be a weighted tree. Due to [1] Bob achieves to attack a vertex twice at most such that throughout the game the neighbors of any uncolored vertex $x_0 \in \mathring{V}(T^w)$ are colored with at most three distinct open arcs on C^r of distances $w(x_0) - \varepsilon$ between each other, provided that $|N(x_0)| \geq 2$. Then a circumference of

$$r = m \cdot w(x_0) + \max_{x_{0_i} \in N(x_0)} \sum_{i=1}^{m} w(x_{0_i})$$

suffices in order to achieve a proper coloring of x_0. Thus, we can conclude that a circle with circumference

$$\max_{y \in \mathring{V}(T^w)} \{m \cdot w(y) + \max_{z_i \in N(y)} \sum_{i=1}^{m} w(z_i)\}$$

guarantees a coloring of T^w. □

Remark 3.5.2. One will ask why we restricted our attention only to interior-vertices. However, it is worth pointing out that a circumference $r = \max_{u \in \overline{V}(T^w)} \{w_u + w_v\}$, where $(u, v) \in E(T^w)$ of C^r does suffice for achieving a feasible coloring of the leaves and their respective neighbors because of the following consideration: Let $x \in \overline{V}(T^w)$. Since $d(u) = 1, u$ cannot be attacked by Bob. Thus, we need only to consider the maximum sum of the weights of connected vertices, such that one vertex is a leaf. Thus, one may conjecture that

$$\gamma_c(T^w) \leq$$

$$\max \Big\{ \underbrace{\max_{u \in \overline{V}(T^w)} \{w(u) + w\big(N(u)\big)\}}_{(i)}, \underbrace{\max_{y \in \mathring{V}(T^w)} \{m \cdot w(y) + \max_{z_i \in N(y)} \sum_{i=1}^{m} w(z_i)\}}_{(ii)} \Big\},$$

for

- $m = 3$ if $|N(y)| \geq 3$ and
- $m = 2$ if $|N(y)| = 2$.

However, according to Alice's strategy it holds $(i) < (ii)$ because of the following illustration:

Let $(x_k, x_{k+1}) \in E(T^w)$ with $x_k \in \mathring{V}(T^w)$ and $x_{k+1} \in \overline{V}(T^w)$ and assume that

$$\max_{u \in \overline{V}(T^w)} \{w(u) + w\big(N(u)\big)\} = w(x_k) + w(x_{k+1}).$$

Since $x_k \in \mathring{V}(T^w)$, $|N(x_k)| \geq 2$. Then a circle with circumference $r = w(x_k) + w(x_{k+1})$ does not suffice for achieving a proper coloring because a circle with circumference

$$m \cdot w(x_k) + \max_{z_i \in N(x_k)} \sum_{i=1}^{m} w(z_i) > w(x_k) + w(x_{k+1})$$

is required for guaranteeing the coloring of x_k.

3.6 The Circular Game Chromatic Number of Weighted Planar Graphs

The aim of this section is to determine the circular marking game number of weighted planar graphs which turns out to be an upper bound of the circular game chromatic number. X. Zhu showed in [3] that the ordinary game chromatic number is bounded by the marking game number, $col_g(G)$, and proved that $col_g(G) \leq 19$ for all planar graphs. In [2] H. A. Kierstead gave a slight improvement and showed that $col_g(G) \leq 18$ for all planar graphs using his well known activation strategy, which turned out to be a strong tool for giving upper bounds for the ordinary game chromatic number for a lot of graph classes. In particular, he introduced a new parameter $r(G)$, called *rank*, which bounds the marking game number of every graph. For our purpose we extend X. Zhu's marking game to the circular marking game on weighted graphs and make use of results and techniques by H. A. Kierstead, while adapting the definitions for the case of circular coloring of weighted graphs.

3.6.1 The Circular Marking Game on Weighted Graphs

For the rest of this section we make use of the following well known notations.

Definition 3.6.1. Let $G = (V, E)$ be a graph and let $\Pi(G)$ be the set of all linear orderings on the vertices of G. For a linear ordering L we say $x > y$ if y appears before x in L for $x, y \in V(G)$. Further for a linear ordering $L \in \Pi(G)$ we define the *orientation* G_L of G with respect to L by $E_L = \{(v, u) \mid (v, u) \in E(G) \text{ and } v > u \text{ in } L\}$.

Moreover, for $u \in V$ let $V_L^+(u) := \{v \in V \mid v < u\}$ and $V_L^-(u) := \{v \in V \mid v > u\}$; in particular let $V^+[u] := V^+(u) \cup \{u\}$ and $V^-[u] := V^-(u) \cup \{u\}$. For a vertex $u \in V(G)$ we denote the *outneighborhood* of u in G_L by $N^+_{G_L}(u)$ and the *inneighborhood* of u in G_L by $N^-_{G_L}(u)$. Let $N^+_{G_L}[u] := N^+_{G_L}(u) \cup \{u\}$ and $N^-_{G_L}[u] := N^-_{G_L}(u) \cup \{u\}$.

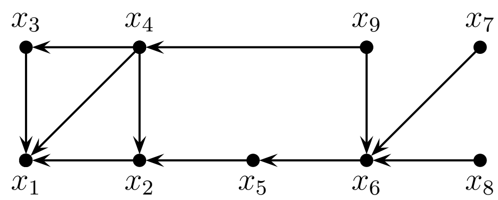
The various degrees of u are denoted by $d^+_{G_L}(u) = |N^+_{G_L}(u)|$ and $d^-_{G_L}(u) = |N^-_{G_L}(u)|$. The *maximum outdegree* of G_L is denoted by $\Delta^+_{G_L}$ and the *maximum* *indegree* by $\Delta^{-}_{G_L}$.

For our purposes we introduce the notion of the circular weighted outneighborhood which is the required circumference for coloring a vertex and it's outneighborhood.

Definition 3.6.2. Let $G^w = (V, E, w)$ be a weighted graph and G^w_L an orientation on G^w. We denote the *circular weighted outneighborhood* of u by

$$\varphi_{G^w_L}(u) := \begin{cases} \sum\limits_{x \in N^+_{G_L}(u)} w(x) + d^+_{G_L}(u) \cdot w(u), & \text{if } N^+_{G_L}(u) \neq \emptyset, \\ w(u), & \text{else.} \end{cases}$$

Example:



Let $w(x_1, ..., x_4) = 2$ and $w(x_5, ..., x_9) = 3$. Then $\varphi_{G^w_L}(x_9) = w(x_4) + w(x_6) + \lceil (2-1)(w(x_9) - \varepsilon) \rceil + w(x_9) = 2 + 3 + 3 + 3 = 11$ and $\varphi_{G^w_L}(x_1) = 2$ because $N_{G^+_L}(x_1) = \emptyset$.

The Circular Marking Game on Weighted Graphs

Let $G^w = (V, E, w)$ be a weighted graph and let t be a given integer. The *circular marking game* is played on G^w by Alice and Bob with Alice acting first. In each move the players take turns marking vertices from the shrinking set $U \in V$ of unmarked vertices; obviously initially $U = V$. This results in a linear ordering $L \in \Pi(G)$ of the vertices of G^w with $x < y$ if and only if x is marked before y for $x, y \in L$. The *cw-score* of the game is equal to $\Phi_{G^w_L} := \max\limits_{u \in V} \varphi_{G^w_L}(u)$. Alice wins if the cw-score is at most the given integer t; otherwise Bob wins.

The *circular marking game number of a weighted graph* G^w, denoted by $col^g_c(G^w)$, is the least integer t such that Alice has a winning strategy for the circular marking game, that is $\Phi_{G^w_L} \leq t$.

Lemma 3.6.3. *Let $G^w = (V, E, w)$ be a weighted graph. Then*

$$\gamma_c(G^w) \leq col_c^g(G^w).$$

Proof. Suppose $col_c^g(G^w) = t$ and let σ be the optimal strategy for Alice for the circular marking game. Further assume that a circle C^r is given with circumference $r = t$. Obviously Alice wins the coloring game if she follows σ and colors the vertices by First-Fit, that are to be marked. □

The Activation Strategy

As already mentioned we refer to H. A. Kierstead's algorithm for our purpose. For the sake of completeness we give below the activation strategy: Let L be a linear ordering from the set $\Pi(G)$ of all linear orderings on V and U the set of all unmarked vertices.

Strategy $S(L)$. Let $A \subseteq V$ be the set of *active* vertices; in particular $A = \emptyset$ at the beginning of the game. Each time a vertex $u \in V$ is being activated Alice updates $A := A \cup \{u\}$. At the beginning Alice activates the least vertex from L and marks it. Assume at his last turn Bob marked vertex b. Then Alice proceeds as follows:

(i) $x := b$;

(ii) while $x \notin A$ do $A := A \cup \{x\}$; $s(x) := \min_L \left\{N^+[x] \cap (U \cup \{b\})\right\}$; $x := s(x)$ od;

(iii) if $x \neq b$ then mark x
else $y := \min_L U$; if $y \notin A$, then $A := A \cup \{y\}$ fi; mark y fi;

The Circular Rank of a Weighted Graph

H. A. Kierstead also introduced a parameter called the *rank* of a graph G which restricts the marking game number. For the case of the circular marking game on weighted graphs we extend the definition of the rank and introduce the *circular rank of weighted graphs*. For the sake of completeness we give the definition of the rank introduced by H. A. Kierstead in [2]. For this purpose he made use of the so called *matching number*.

For $u \in V$ the *matching number* $m(u, L, G)$ of u with respect to L is defined as the size of the largest set $M(u, L, G) \subset N^-[u]$ such that there exists a partition $\{X, Y\}$ of $M(u, L, G)$ and there exist matchings N_1 from $X \subset N^-[u]$ to $V^+(u)$ and N_2 from $Y \subset N^-(u)$ to $V^+[u]$. The *rank* $r(L, G)$ of L with respect to G and *rank* $r(G)$ of G are defined as follows:

$$r(u, L, G) := d^+_{G_L}(u) + m(u, L, G)$$

$$r(L, G) := \max_{u \in V} r(u, L, G)$$

$$r(G) := \min_{L \in \Pi(G)} r(L, G)$$

Definition 3.6.4 (Circular Rank of Weighted Graphs). Let $G^w = (V, E, w)$ be a weighted graph and $L \in \Pi(G)$ be a linear ordering of V. The *circular rank of* G^w with respect to L is defined as follows:

$$r_c(u, L, G^w) :=$$

$$\sum_{x \in N^+_{G_L}(u)} w(x) + \sum_{y \in M(u,L,G)} w(y) + (d^+_{G_L}(u) + m(u, L, G) - 1) \cdot w(u)$$

$$r_c(L, G^w) := \max_{u \in V} \left\{ w(u) + r_c(u, L, G^w) \right\}$$

$$r_c(G^w) := \min_{L \in \Pi(G)} r_c(L, G^w)$$

The following proposition shows that the circular rank of a weighted graph is an upper bound of the circular marking game number of a weighted graph.

Proposition 3.6.5. *For any weighted graph $G^w = (V, E, w)$ and any linear ordering $L \in \Pi(G)$ if Alice uses the strategy $S(L)$ for the circular marking game on G^w, then the cw-score will be at most $r_c(L, G^w)$. In particular, $col^g_c(G^w) \leq r_c(G^w)$.*

Proof. Let $L \in \Pi(G)$ and suppose that Alice decides to play the activation strategy for the circular marking game. Further let $L' \in \Pi(G)$ be the linear ordering which we obtain if the game ends, i.e., the order the vertices were marked.

Obviously it holds

$$|N(u) \cap A| \leq d^+_{G_L}(u) + |N^-(u) \cap A|.$$

H. A. Kierstead proved in [2] that at any time of the game any unmarked vertex u is adjacent to at most $d^+_{G_L}(u) + m(u, L, G)$ active vertices, since every vertex marked by Bob becomes active by Alice and every vertex marked by Alice is already active. That is $|N(u) \cap A| \leq d^+_{G_L}(u) + m(u, L, G)$.

Thus we can conclude that:

$$\varphi_{G^w_{L'}}(u) \leq \sum_{x \in N(u) \cap A} w(x) + (|N(u) \cap A| - 1) \cdot w(u) + w(u) \leq$$

$$\sum_{x \in N^+_{G_L}(u)} w(x) + \sum_{y \in N^-_{G_L}(u) \cap A} w(y) +$$

$$\left(d^+_{G_L}(u) + |N^-_{G_L}(u) \cap A| - 1\right) \cdot w(u) + w(u) \leq$$

$$\sum_{x \in N^+_{G_L}(u)} w(x) + \sum_{y \in M(u,L,G)} w(y) +$$

$$\left(d^+_{G_L}(u) + m(u, L, G) - 1\right) \cdot w(u) + w(u) =$$

$$r_c(u, L, G^w) + w(u).$$

Hence,

$$\varphi_{G^w_{L'}}(u) \leq r_c(u, L, G^w) + w(u)$$

and finally we have

$$\Phi_{G^w_{L'}} \leq r_c(L, G^w).$$

□

The proposition above shows that the circular marking game number for weighted graphs is bounded by the circular rank of weighted graphs. Hence, in order to determine the circular game chromatic number of a class of weighted graphs one may determine the circular marking game number by determining the circular rank of this class.

The Circular Marking Game Number of Weighted Planar Graphs

In this section we use proposition 3.6.5 for giving an upper bound for the circular marking game number of weighted planar graphs. First we give a definition of planar graphs.

Definition 3.6.6. A graph $G = (V, E)$ is called *planar*, if it can be drawn in the plane such that the edges do not intersect.

Corollary 3.6.7. *Let $G^w = (V, E, w)$ be a weighted planar graph. Then*

$$col_c^g(G^w) \leq$$

$$\min_{L\in\Pi(G)} \max_{u\in V} \left\{ \sum_{x\in N^+_{G_L}(u)} w(x) + \sum_{y\in M(u,L,G)} w(y) + 17w(u) \right\}.$$

Proof. Let $G^w = (V, E, w)$ be a weighted planar graph. H. A. Kierstead proved in [2] that at any time during the regular coloring game for an unmarked vertex $u \in V$ we have

$$d^+_{G_L}(u) + m(u, L, G) \leq 17.$$

Using this result and proposition 3.6.5, we can conclude that for an unmarked vertex $u \in V$

$$r_c(u, L, G^w) \leq \sum_{x\in N^+_{G_L}(u)} w(x) + \sum_{y\in M(u,L,G)} w(y) + 16w(u)$$

holds. Hence,

$$r_c(L, G^w) \leq \max_{u\in V} \left\{ \sum_{x\in N^+_{G_L}(u)} w(x) + \sum_{y\in M(u,L,G)} w(y) + 17w(u) \right\}$$

and

$$r_c(G^w) \leq$$

$$\min_{L\in\Pi(G)} \max_{u\in V} \left\{ \sum_{x\in N^+_{G_L}(u)} w(x) + \sum_{y\in M(u,L,G)} w(y) + 17w(u) \right\}$$

Finally because of proposition 3.6.5 the assumption is proven. □

3.7 The Circular Game Chromatic Number of Weighted Cartesian Product Graphs

The aim of this section is to introduce a definition of the cartesian product of weighted graphs and to analyze the circular game chromatic number of special families of weighted cartesian product graphs. In particular, we consider the cartesian product of the weighted complete graph K_2^w and H^u, where H^u is a weighted path P_n^u or a weighted cycle C_n^u on n vertices for $n \in \mathbb{N}$, respectively. For this purpose we refer to [7], where T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, I. Peterin worked on the game chromatic number of K_2 and H, where H is a P_n or a C_n for graphs without vertex-weights.

Definition 3.7.1. Let $G^w = (V_G, E_G, w)$ and $H^u = (V_H, E_H, u)$ be two weighted graphs with their respective weight functions $w : V_G \to \mathbb{R}^+$ and $u : V_H \to \mathbb{R}^+$. The *cartesian product* $G^w \times H^u$ of G^w and H^u is a graph with vertex-set $V\big(G^w \times H^u\big) := V_G \times V_H$, such that two vertices $(u_1 v_1) \in V\big(G^w \times H^u\big)$ and $(u_2 v_2) \in V\big(G^w \times H^u\big)$ (we denote vertices without a coma to avoid irritation) are adjacent if and only if either $u_1 = u_2$ and $(v_1, v_2) \in E_H$ or $v_1 = v_2$ and $(u_1, u_2) \in E_G$. The vertex-weights of $G^w \times H^u$ are defined by the weight function $w \times u : V_G \times V_H \to \mathbb{R}^+$ with $(w \times u)(xy) = w(x) \cdot u(y)$ where $(xy) \in V_G \times V_H$. Particularly, G^w and H^u are called *factor graphs* of $G^w \times H^u$.

Let $G^w \times H^u$ be the cartesian product of the graphs G^w and H^u. For a vertex $x \in V_H$ the subgraph G_x^w of $G^w \times H^u$, induced by $\{(yx) : y \in V_G\}$ is called a *G^w-fiber*.

Our definition agrees with the definition of cartesian product graphs without vertex-weights, given in [7], if $w(x) = u(y) = 1$ for all $x \in V_G$ and $y \in V_H$. For simplicity of notation we write $G \times H$ instead of $G^w \times H^u$ if the weight functions are clear from the context.

First, we work on the cartesian product of $K_2^w \times P_n^u$, where n is a positive integer. Let us consider $K_2^w \times P_n^u$ for all n:

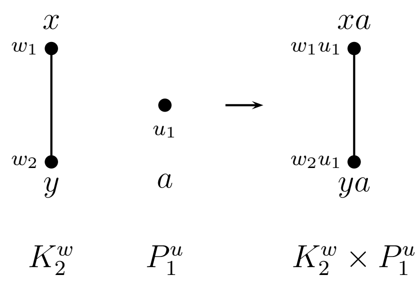


FIGURE: $K_2^w \times P_1^u$ is an edge with $V(K_2^w \times P_1^u) = \{(xa), (ya)\}$ and $E(K_2^w \times P_1^u) = \{(xa, ya)\}$.

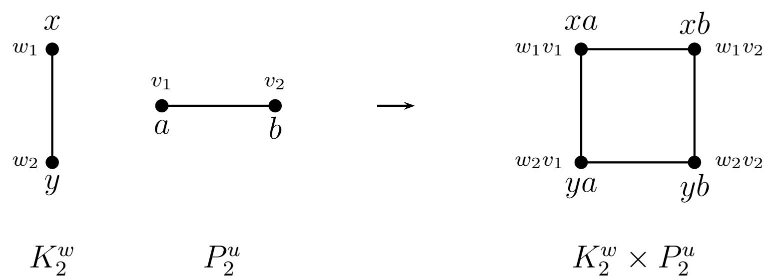


FIGURE: $K_2^w \times P_2^u$ is a cycle of length 4, where $V(K_2^w \times P_2^u) = \{(xa), (xb), (ya), (yb)\}$ and $E(K_2^w \times P_2^u) = \{(xa, xb), (xb, yb), (yb, ya), (ya, xa)\}$.

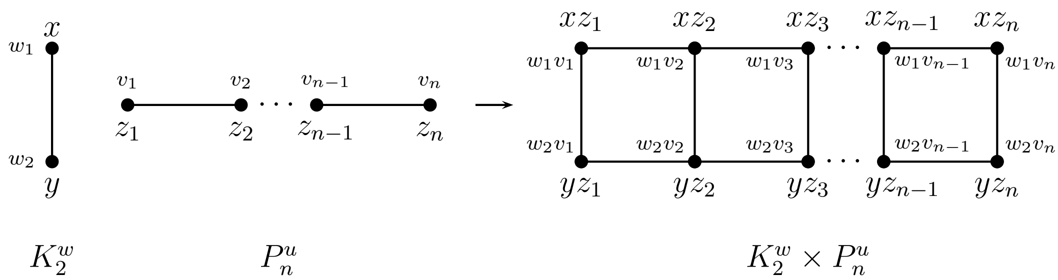


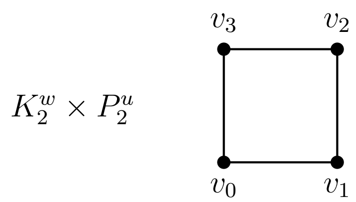
FIGURE: $K_2^w \times P_n^u$ is a ladder with vertex-set $V(K_2^w \times P_n^u) = \{(xz_1), (xz_2), ..., (xz_n), (yz_1), (yz_2), ..., (yz_n)\}$ and edge-set $E(K_2^w \times P_n^u) = \{(xz_i, xz_{i+1}), ((yz_i, yz_{i+1}) : 1 \leq i \leq n-1\} \cup \{xz_i, yz_i : 1 \leq i \leq n\}$.

Proposition 3.7.2. *Let $K_2^w = (V_{K_2}, E_{K_2}, w)$ be the weighted complete graph on 2 vertices and $P_n^u = (V_{P_n}, E_{P_n}, u)$ be a weighted path for $n \in \mathbb{N}$. Then*

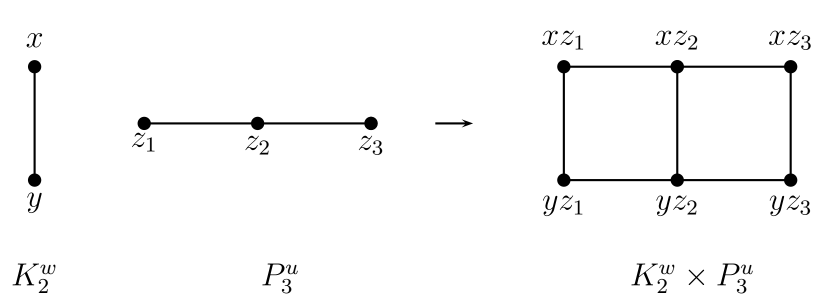
$$\gamma_c\big(K_2^w \times P_n^u\big) \begin{cases} = \sum\limits_{x \in V(K_2 \times P_n)} (w \times u)(x), & \textit{for } n = 1, \\ \leq 4 \max\limits_{x \in V(K_2 \times P_n)} \{(w \times u)(x)\}, & \textit{for } n = \{2,3\}, \\ \geq 5 \min\limits_{x \in V(K_2 \times P_n)} \{(w \times u)(x)\}, & \textit{for } n \geq 4, \\ \leq 6 \max\limits_{x \in V(K_2 \times P_n)} \{(w \times u)(x)\}, & \textit{for } n \geq 4. \end{cases}$$

Proof. For $n = 1$ the cartesian product of the weighted factor graphs K_2^w and P_1^u is again the K_2^w. Obviously the circular game chromatic number of K_2^w is the sum of the vertex-weights.

Let $n = 2$. Then the graph $K_2^w \times P_2^u$ is a cycle on 4 vertices. Let $V(K_2 \times P_2) = \{v_0, v_1, v_2, v_3\}$ with $E(K_2 \times P_2) = \{(v_0, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_0)\}$ and vertex-weights $\{(w \times u)(v_0), (w \times u)(v_1), (w \times u)(v_2), (w \times u)(v_3)\}$. Let $(w \times u)(v_i) = k$ for all $i \in \{0, ..., 3\}$ for $k \in \mathbb{R}^+$. Then we can refer to the upper bound of the circular game chromatic number of weighted cycles, see proposition 3.4.1.



Let $n = 3$. Then $K_2^w \times P_3^u$ is a *ladder* with $V(K_2 \times P_3) = \{xz_1, xz_2, xz_3, yz_1, yz_2, yz_3\}$ and $E(K_2 \times P_3) = \{(xz_1, xz_2), (xz_2, xz_3), (yz_1, yz_2), (yz_2, yz_3), (xz_1, yz_1), (xz_2, yz_2), (xz_3, yz_3)\}$, see the figure below:



For determining the upper bound we need to consider the case $(w \times u)(v) = k$, for all $v \in V(K_2 \times P_n)$ and $k \in \mathbb{R}^+$, which increases $\gamma_c\big(K_2^w \times P_n^u\big)$ at most.

Assume a circle C^r with circumference $r = 4k$ is given. By symmetry there are two possibilities for Alice's first move: she colors a vertex with degree either 2 which is a vertex from $\{xz_1, xz_3, yz_1, yz_3\}$ or 3 which is a vertex from $\{xz_2, yz_2\}$. Suppose that she decides to colors a vertex with degree 3, say xz_2. Then the worst case occurs if Bob attacks vertex yz_2 by coloring yz_1 or yz_3 with an arc of distance $k - \varepsilon$ to $f_{(w\times u)}(xz_2)$. Without loss of generality assume that he decides to color yz_1. If Alice proceeds to color yz_2 with an arc next to $f_{(w\times u)}(yz_1)$ of distance 0, then the remaining uncolored vertices have degree 2, such that they can be attacked once at most. Thus, after her second move the coloring of the entire graph is fixed.

We are left with the task of considering the case that Alice starts coloring a vertex with degree 2. Without loss of generality assume that she colors vertex xz_1 to an arbitrary open arc on C^r. Then we have to consider 4 cases for Bob's strategy.

- Assume Bob colors a vertex with degree 3. Then after Alice's second move the coloring of the graph is fixed since the remaining uncolored vertices have degree 2, if Alice colors the second vertex with degree 3.
- If he colors vertex xz_3, then Alice replies by coloring yz_2 with $f_{(w\times u)}(xz_1)$. Again the coloring is fixed after Alice second move, since each uncolored vertex have only colored neighbors which implies that it cannot be attacked.
- If he colors vertex yz_1, then Alice colors a vertex with degree 3. She colors either yz_2 with $f_{(w\times u)}(xz_1)$ or xz_2 with $f_{(w\times u)}(yz_1)$. In both cases the remaining vertices can be attacked once at most.
- If he colors vertex yz_3, then Alice colors vertex xz_2 with $f_{(w\times u)}(yz_3)$ and clearly the remaining vertices can be attacked once at most.

Thus, in every case a vertex is being attacked once at most which indicates that a circle with circumference $4 \max_{x\in V(K_2\times P_n)} \{(w \times u)(x)\}$ guarantees Alice's victory.

Let $n \geq 4$. We consider the two fibers $xz_1, xz_2, ..., xz_n$ and $yz_1, yz_2, ..., yz_n$ of P_n^u, where we denote them by $v_1, v_2, ..., v_n$ and $v_1', v_2', ..., v_n'$, with $xz_i = v_i$ and $yz_i = v_i'$ for $i \in \{1, ..., n\}$. Moreover, assume that $(w \times u)(v_i) = (w \times u)(v_i') = k$, for $i \in \{1, ..., n\}$ and $k \in \mathbb{R}^+$. The graph $K_2^w \times P_n^u$ contains 4 vertices with degree 2, that are $U = \{v_1, v_n, v_1', v_n'\}$, and $2n-4$ vertices with degree 3, that are $W = \{v_2, v_3, ..., v_{n-1}, v_2', v_3', ..., v_{n-1}'\}$.

By the symmetry of the graph there are two possible moves for Alice's first turn.

(a) Suppose a circle with circumference $r = 5k$ is given and Alice starts the game by assigning vertex $v_i \in W$ to C^{5k}. The worst case occurs if Bob colors vertex $v_{i+1}' \in W$ with an arc of distance $k - \varepsilon$ to $f_{(w \times u)}(v_i)$. In case that $v_{i+1}' \in U$, Bob chooses vertex v_{i-1}' which is from W, if $v_{i+1}' \in U$ by the assumption $n \geq 4$. For the rest of the proof we assume that $v_{i+1}' \in W$. Obviously after Alice's second move either the tuple $(v_{i+1}, v_{i+2}, v_{i+2}', v_{i+3})$ or the tuple $(v_i', v_{i-1}', v_{i-1}, v_{i-2}')$ remains uncolored. Assume the vertices $v_{i+1}, v_{i+2}, v_{i+2}'$ and v_{i+3} are still uncolored and it is Bob's turn. Then Bob replies by coloring vertex v_{i+2} with an arc of distance $k - \varepsilon$ to $f_{(w \times u)}(v_{i+1}')$ such that $f_{(w \times u)}(v_{i+2}) \cap f_{(w \times u)}(v_i) = \emptyset$. Thus, vertex v_{i+1} is being attacked twice, where each time Bob destroys $k - \varepsilon$ space on the circle, while the three neighbors $\{v_i, v_{i+2}, v_{i+1}'\}$ of v_{i+1} have been placed to distinct k-unit length arcs. Since there is no space for vertex v_{i+1} on C^{5k} left a proper coloring of the graph is not possible.

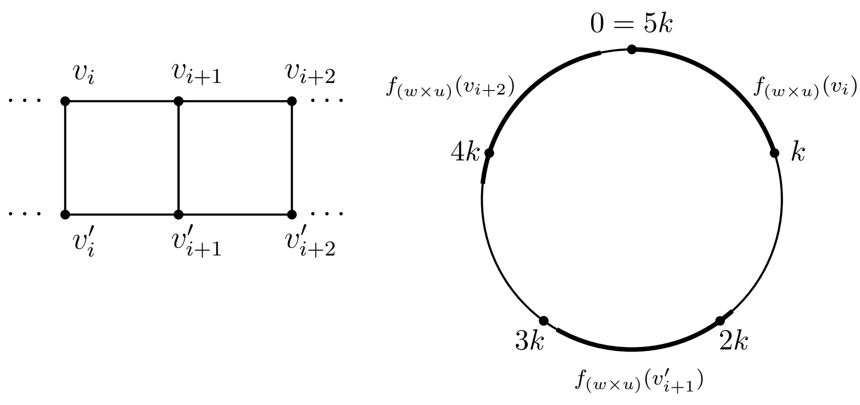


FIGURE: The coloring on C^{5k}, if Alice started with vertex $v_i \in W$, that is $d(v_i) = 3$.

(b) The proof of the following is motivated by [7].

Suppose $r = 5k - \varepsilon$ for an $\varepsilon > 0$ and Alice starts the game by coloring vertex $v_1 \in U$. Assume that Bob colors vertex v_3' with $f_{(w\times u)}(v_1)$, such that. Now let us consider the path $P = v_1'v_2'v_2v_3$. Since the number of the vertices from $V(K_2 \times P_n) - (\{v_1, v_3'\} + P)$ is even, Bob forces Alice to be the first who has to color a vertex from P. Since $(v_1, v_2), (v_1, v_1'), (v_3', v_3), (v_3', v_2') \in E(K_2 \times P_n)$, the remaining space on $C^{5k-\varepsilon}$, where vertices v_1', v_2', v_2, v_3 can be assigned is $4k - \varepsilon$. But since the circular game chromatic number of a weighted path with all vertex-weights equal to k is $4k$ and $\{f_{(w\times u)}(v_1'), f_{(w\times u)}(v_2'), f_{(w\times u)}(v_2), f_{(w\times u)}(v_3)\}$ cannot overlap with $f_{(w\times u)}(v_1)$, a circle with circumference $5k - \varepsilon$ does not suffice for achieving a proper coloring of the graph.

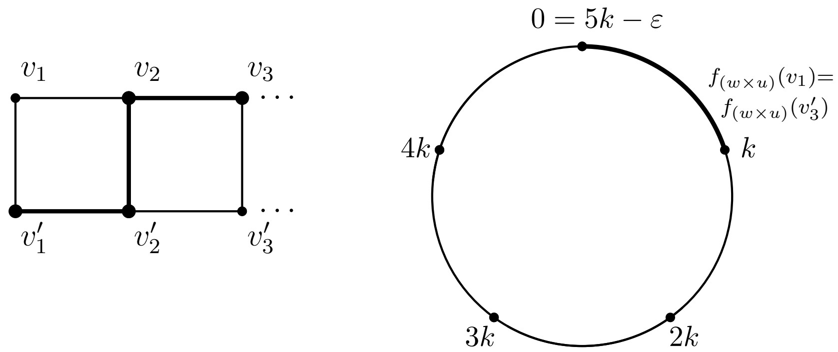


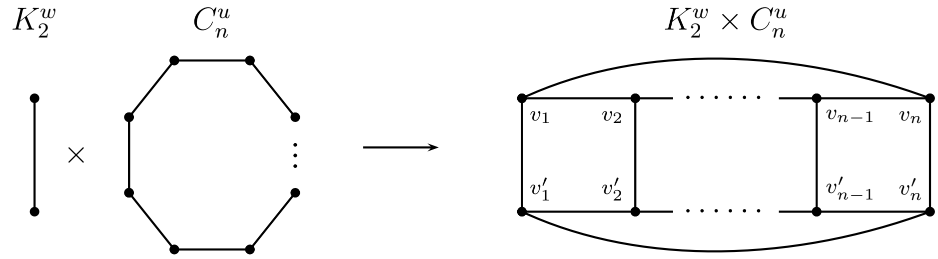
FIGURE: The coloring on $C^{5k-\varepsilon}$, if Alice started with vertex $v_1 \in U$, that is $d(v_1) = 2$. The thick line stands for the path $P = \{v_1', v_2', v_2, v_3\}$.

Finally considering case (a) as the worst case scenario (since Bob achieves 2 attacks) and (b) as the best case we can conclude that

$$5 \min_{x \in V(K_2 \times P_n)} \{(w \times u)(x)\} \leq \gamma_c\big(K_2^w \times P_n^u\big) \leq 6 \max_{x \in V(K_2 \times P_n)} \{(w \times u)(x)\},$$

where we refer to he trivial upper bound, since the degree of an arbitrary vertex is at most 3 and hence it can be attacked twice at most. $\square$

We proceed to determine the circular game chromatic number for the cartesian product of the factor graphs K_2^w and C_n^u, where $C_n^u = (V, E, u)$ is a weighted cycle on n vertices. Again we denote the vertices of the two fibers of C_n^u with $v_1, ..., v_n$ and $v'_1, ..., v'_n$, respectively. It is easily seen that the graph $K_2^w \times C_n^u$ is a 3-regular graph. The following figure demonstrates the cartesian product of K_2^w and the cycle C_n^u on n vertices:

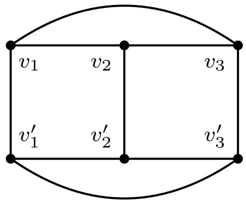


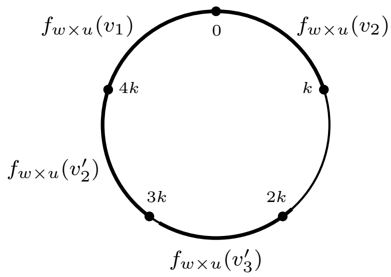
Proposition 3.7.3. *Let $K_2^w = (V_{K_2}, E_{K_2}, w)$ be a weighted complete graph on 2 vertices and $C_n^u = (V_{C_n}, E_{C_n}, u)$ be a weighted cycle for an integer $n \geq 3$. Then*

$$\gamma_c\big(K_2^w \times C_n^u\big) \begin{cases} = 5 \max\limits_{x \in V(K_2 \times C_n)} \{(w \times u)(x)\}, & \textit{for } n = 3, \\ \leq 6 \max\limits_{x \in V(K_2 \times C_n)} \{(w \times u)(x)\}, & \textit{for } n \geq 4. \end{cases}$$

Proof. Again we assume that $(w \times u)(x) = k$ for all $x \in V(K_2 \times C_n)$ with $k \in \mathbb{R}^+$ which increases the circular game chromatic number at most.

Let $n = 3$ and assume that a circle with circumference $5k$ is given. By symmetry Alice starts the game by coloring vertex v_2 with the arc $b_{(0,k)}$. Clearly the worst case occurs if Bob colors vertex v'_3 with an arc of distance of $k - \varepsilon$ to $f_{(w \times u)}(v_2)$. In this manner he attacks vertices v_3 and v'_2. If Alice responds by coloring v'_2 (or v_3) in her next move, Bob can attack one of them by coloring v_1 (or v'_1) with an arc of distance smaller than k to the arcs $f_{(w \times u)}(v_2)$ or $f_{(w \times u)}(v'_3)$. Hence, she has to go for coloring vertices v_1 or v'_1. By symmetry she is indifferent which one to choose. Assume she colors vertex v_1 with the arc $b_{(4k,5k)}$, such that v_3 can be colored with an arc between $3k - \varepsilon$ and $4k$. Moreover, vertex v'_2 cannot be attacked by Bob twice, since $(v_1, v'_1) \in E(K_2 \times C_3)$ and $(v_1, v'_2) \notin E(K_2^w \times C_3^u)$ so that v'_2 can be colored with the arc $f_{(w \times u)}(v_1)$. Thus, a proper coloring on C^{5k} is guaranteed. The following figure demonstrates a proper coloring on the graph $K_2^w \times C_3^u$:





If we consider a circle with circumference $r = 5k - \varepsilon$, Bob could place the arc $f_{w\times u}(v_3')$ on $b_{(2k-\delta, 3k-\delta)}$, for a $\delta < \varepsilon$, such that there won't be enough space for the corresponding arc of v_3.

Let $n = 4$. Then we can consider the graph $K_2^w \times C_4^u$ as the weighted complete bipartite graph $K_{4,4}^{w\times u}$ without a perfect matching M, such that $V := \{v_1, v_2', v_3, v_4'\}$ and $V' := \{v_1', v_2, v_3', v_4\}$ are the two independent sets of $V(K_{4,4} - M)$. Let us introduce the temporary notations $x_1 := v_1,\ x_2 := v_2',\ x_3 := v_3,\ x_4 := v_4'$ for the vertices of V and $x_1' := v_3',\ x_2' := v_4,\ x_3' := v_1',\ x_4' := v_2$ for the vertices of V', where $M = \{(x_1, x_1'), (x_2, x_2'), (x_3, x_3'), (x_4, x_4')\}$. See the following figure:

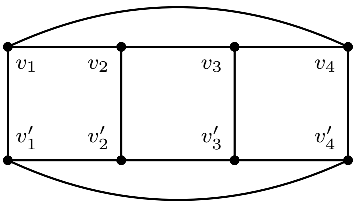


FIGURE: $K_2^w \times C_4^u$

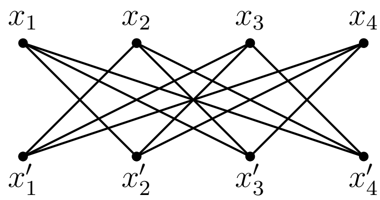


FIGURE: $K_{4,4}^{w\times u}$

Assume that a circle with circumference $r = 6k$ is given. By symmetry Alice is indifferent which vertex to color first, hence assume that she colors x_1. By symmetry we need to consider three different cases for Bob's first move: He colors a vertex from V, from $V' - \{x_1'\}$ or x_1'.

Case 1: Assume that Bob colors vertex $x_2 \in V$ with an arbitrary arc. Then Alice colors x_1' so that $f_{(w\times u)}(x_1) \cap f_{(w\times u)}(x_1') = \emptyset$. By the assumption $(x_1', x_i) \in E(K_{4,4})$ for $i \in \{2, 3, 4\}$ the corresponding arcs of the remaining uncolored vertices from V cannot overlap with $f_{(w\times u)}(x_1')$, such that the corresponding arcs of the vertices $\{x_2', x_3', x_4'\}$ can be placed on $f_{(w\times u)}(x_1')$. And vice versa the corresponding arcs of the vertices x_3 and x_4 can be placed on $f_{(w\times u)}(x_1)$ because by the assumption $(x_1, x_i') \in E(K_{4,4})$ for $i \in \{2,3,4\}$ the corresponding arcs of the remaining uncolored vertices from V' cannot overlap with $f_{(w\times u)}(x_1)$. Thus, the coloring of the entire graph is fixed after Alice's second move and a circle C^r with circumference $6k$ suffices for Alice's victory.

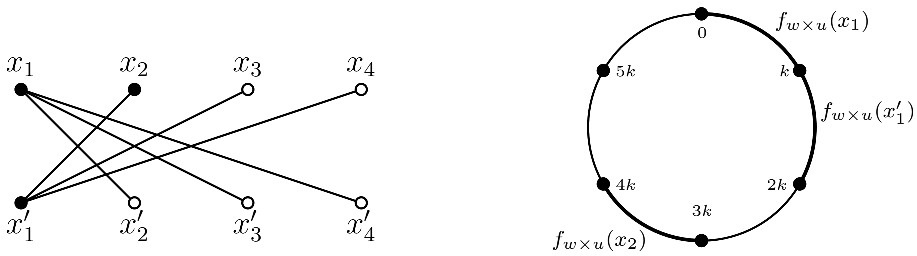


FIGURE: Since $f_{(w\times u)}(x_1) \cap f_{(w\times u)}(x_1') = \emptyset$ and $\{(x_1, x_i'), (x_1', x_i)\} \in E(K_{4,4})$ for $i \in \{2,3,4\}$ the coloring of the remaining uncolored vertices is fixed on C^{6k}.

Case 2: Assume that Bob colors vertex $x_2' \in V' - \{x_1'\}$. Obviously $f_{(w\times u)}(x_1) \cap f_{(w\times u)}(x_2') = \emptyset$ because $(x_1, x_2') \in E(K_{4,4} - M)$. Then after Alice's second move the coloring of the graph is fixed if she colors x_2 such that $f_{(w\times u)}(x_1) = f_{(w\times u)}(x_2)$. The proof is analogue as in case 1.

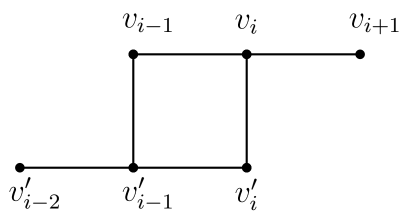
Case 3: Assume that Bob colors vertex x_1'. Let $f_{(w\times u)}(x_1) \cap f_{(w\times u)}(x_1') = \emptyset$. Then after Alice's second move the coloring of $K_{4,4}^{w\times u} - M$ is fixed if Alice colors x_2 with the arc $f_{(w\times u)}(x_1)$. Again we can refer to case 1. Let $f_{(w\times u)}(x_1) \cap f_{(w\times u)}(x_1') \neq \emptyset$ and without loss of generality assume that the arcs $f_{(w\times u)}(x_1)$ and $f_{(w\times u)}(x_1')$ are placed in cyclic order. Then Alice colors an arbitrary vertex from V', say x_2', with an arc of distance k to $f_{(w\times u)}(x_1')$ in the clockwise direction.

(i) Assume Bob colors a vertex from $\{x_3', x_4'\}$, say x_3'. Then after Alice's second move the coloring is fixed if she colors x_3 with an arc either between $f_{(w\times u)}(x_1')$ and $f_{(w\times u)}(x_2')$ or next to $f_{(w\times u)}(x_2')$ in the clockwise direction. In each case she wins the game since Bob is forced to color either $\{x_2, x_4\}$ with $f_{(w\times u)}(x_3)$ or x_4' with $f_{(w\times u)}(x_3')$.

(ii) Assume that Bob colors a vertex from $\{x_3, x_4\}$. Then in the same manner as in case (i) the coloring is fixed after Alice's second move.

(iii) Assume that Bob colors vertex x_2. If $f_{(w\times u)}(x'_2) \cap f_{(w\times u)}(x_2) = \emptyset$, we can refer to the case (i). Thus, let $f_{(w\times u)}(x'_2) \cap f_{(w\times u)}(x_2) \neq \emptyset$ and assume that $d(f_{(w\times u)}(x_1), f_{(w\times u)}(x_2)) \geq k$. Then Alice replies by coloring a vertex from $\{x_3, x_4\}$ with an arc between $f_{(w\times u)}(x'_1)$ and $f_{(w\times u)}(x'_2)$. Again Alice wins since $f_{(w\times u)}(x'_3)$ and $f_{(w\times u)}(x'_4)$ as well as $f_{(w\times u)}(x_3)$ and $f_{(w\times u)}(x_4)$ can overlap. Suppose $d(f_{(w\times u)}(x_1), f_{(w\times u)}(x_2)) < k$ which implies that no other arc can be placed between. Then Alice colors vertex x_3 with an arc of distance k to $f_{(w\times u)}(x_2)$ in the clockwise direction. Then x'_3, x_4 and x'_4 are the remaining uncolored vertices. Since $(x'_3, x_4) \in E(K_{4,4})$ but $\{(x'_3, x'_4), (x'_4, x_4)\} \notin E(K_{4,4})$, it is easily seen that the coloring is fixed now and a circle with circumference $6k$ suffices in order to achieve a proper coloring.

Let $n \geq 5$ and consider the induced subgraph on the vertices $\{v_{i-1}, v_i, v_{i+1}, v'_{i-2}, v'_{i-1}, v'_i\}$.



Without loss of generality assume Alice colors vertex v_{i-1} and Bob responds by coloring v'_i with an arc of distance $k - \varepsilon$ to $f_{(w\times u)}(v_{i-1})$. After Alice's next move v_i or v'_{i-1} remain uncolored. Since the graph is 3-regular, Bob will be able to attack one of them twice by coloring v_{i+1} or v'_{i-2}. Hence, the circumference of the given circle has to be $6k$ for achieving a proper coloring. For $n \geq 5$ Bob is able to find an induced graph of the above form no matter which vertex Alice colors in her first move. Thus, Bob is able to produce two attacks on one vertex which also is the worst case since our graph is 3-regular. □

Chapter 4

Circular Game-Perfect Graphs

We introduced the notion of asymmetric game perfectness in chapter 1. The objective of this chapter is to combine the notion of game perfectness and circular coloring. The result will be the *circular game-perfect graphs*. For this purpose we modify a definition of X. Zhu, who introduced the circular perfect graphs in [20]. He made use of the so called rational complete graphs $\mathcal{K}_k^d$, which goes back to the work of A. Vince in [10].

Definition 4.0.4 (Rational Complete Graphs). Let $r = \frac{k}{d} \geq 2$ be a rational number. We denote by K_k^d the graph with vertex set $\{0, 1, 2, ..., k-1\}$, and (i, j) is an edge if and only if $d \leq |i - j| \leq k - d$. K_k^d is called *rational complete*.

Example:

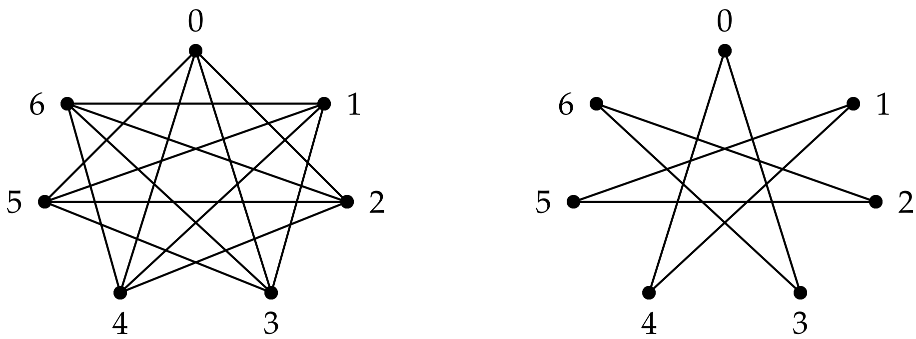


FIGURE: A K_7^2 with $r = 3,5$ and a K_7^3 with $r = 2,5$

Observe that if $d = 1$, then K_k^1 is the complete graph K_k. In particular A.

Vince proved that $\chi_c(K_k^d) = \frac{k}{d}$.

So far, X. Zhu introduced the circular perfect graphs as follows: A graph G is called circular perfect if $\chi_c(H) = \omega_c(H)$ for every induced subgraph $H \subseteq G$, where $\omega_c(H) := \max\{\frac{k}{d} : K_k^d \subseteq H\}$ is the *circular clique number of* H.

The main difficulty in working out a definition for circular game-perfect graphs is that the circular game chromatic number of K_k^d does not equal to the circular clique number of it because Alice and Bob are competitive. In the following we work out a definition of circular game-perfect graphs.

4.1 Circular Game-Perfect Graphs

Let $G = (V, E)$ be a graph. We classify the K_k^d s by their respective circular clique number and denote by $\Omega_m(G)$ the set of all K_k^d s with circular clique number $m \in \mathbb{Q}$ that are subgraphs of G.

Definition 4.1.1. Let $G = (V, E)$ be a graph with $\omega_c(G) = m$. We call the greatest $K_k^d \in \Omega_m(G)$ with respect to k *maximal rational complete subgraph* of G and denote it by $\Theta(G)$.

The choice of $\Theta(G)$ is due to the following consideration: In [10] and [21] it was shown that if $\frac{k'}{d'} = \frac{k}{d}$ with $k > k'$, then the graph $K_{k'}^{d'}$ is an induced subgraph of the graph K_k^d. Thus, for all $K_k^d \in \Omega_m(G)$ we can conclude that K_k^d is an induced subgraph of $\Theta(G)$ and therefore $\gamma_c(\Theta(G)) \geq \gamma_c(K_k^d)$.

Definition 4.1.2. A graph $G = (V, E)$ is *circular game-perfect* if for every induced subgraph $H \subseteq G$ with $\omega_c(H) > 2$ it holds $\gamma_c(H) = \gamma_c\big(\Theta(H)\big)$; in particular G is *circular game-nice* if it holds $\gamma_c(G) = \gamma_c\big(\Theta(G)\big)$.

Obviously a graph G is circular game-perfect if every induced subgraph $H \subseteq G$ is circular game-nice.

Remark 4.1.3. Consider the graph $G = (V, E)$ below and let $\mathcal{H}$ be the set of all induced subgraphs of G. Obviously $\omega(H) \in \{1, 2, 3\}$ for every $H \in \mathcal{H}$, since every induced subgraph of G is either a vertex, a path or the K_3 with $V(K_3) = \{a, b, c\}$. If $H = \{v\}$, then $\omega(v) = \chi(v) = 1$. If H is a path, then $\omega(H) = \chi(H) = 2$. Otherwise if $K_3 \subseteq H$, then $\omega(H) = \chi(H) = 3$. Thus, G is perfect.

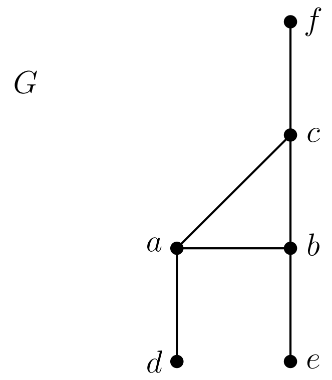


FIGURE: G is perfect but not circular game-perfect.

If we ignore the assumption $\omega_c(H) > 2$ in the definition of circular game-perfectness, then the examination of circular game-perfect graphs won't be noteworthy because of the following fact: Consider the induced subgraph $P \in \mathcal{H}$ with $P = dacf$. Since $K_4^2 = \Theta(P)$ we need to consider the circular game chromatic number of an edge which is 2. But since P is a path, $\gamma_c(P) = 4$. Thus, it holds that $\gamma_c(P) \neq \gamma_c(K_4^2)$. We could draw the conclusion that a graph G is not circular game-perfect if it contains the P_4, which would be a huge restriction so that the class of circular game-perfect graphs becomes irrelevant.

A solution of the problem described above could be to take the K_k^d s with circular clique number 2 and $k > 2$ out of consideration. The advantage would be that the graph above would be circular game-perfect and hence an examination of circular game-perfect graphs with clique number greater that 2 would be possible.

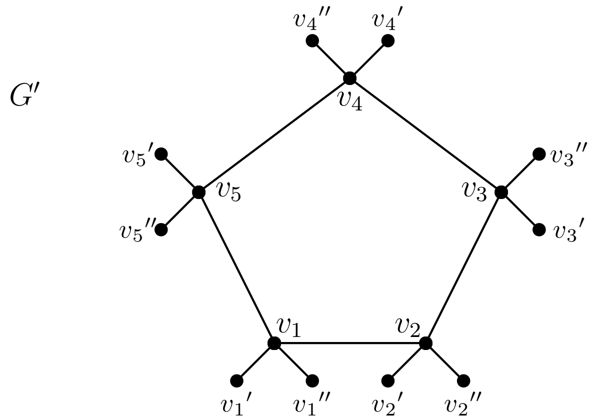
4.2 Circular Game-Perfect Graphs with $\Theta(G) = K_5^2$

This section deals with two conditions for non circular-game nice graphs G with circular clique number $2,5$ and $\Theta(G) = K_5^2$. Note that the K_5^2 is the cycle

C_5. Thus, the circular game chromatic number of K_5^2 is 4 for the case that each vertex is assigned to the vertex-weight 1, see 3.4.1. Hence, it is of our interest to find those graphs with $\omega_c = 2,5$, $\Theta = K_5^2$ and $\gamma_c = 4$. For the remainder of the section let C^r be a circle with circumference $r \geq 2$ and let $f(v)$ be the placed arc of the vertex v on C^r.

Proposition 4.2.1. *Let $G = (V, E)$ be a graph with $\omega_c(G) = 2,5$ and $\Theta(G) = K_5^2$. Then $G = (V, E)$ is not circular game-nice if (i) or (ii) holds:*

(i) *The graph G is of the form G'.*



(ii) *For every vertex $v \in V(G)$ there exists at least one vertex $v' \neq v$, such that property $\mathcal{P}$ is satisfied:*

$\mathcal{P}$: $d(v, v') = 2$ and $|N(v) \cap N(v')| = 2$ where $d(u) \geq 3$ and $d(u') \geq 3$ for $u, u' \in N(v) \cap N(v')$. Moreover, for $x \in N(u') \setminus \{v, v'\}$ it holds $(u, x) \notin E(G)$ and $y \in N(u) \setminus \{v, v'\}$ it holds $(u', y) \notin E(G)$.

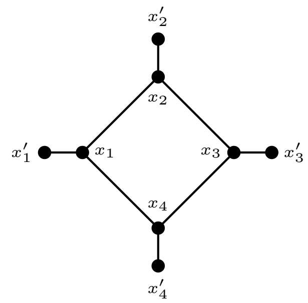
Proof. Suppose $G = (V, E)$ is a graph with $\omega_c(G) = 2,5$, $\Theta(G) = K_5^2$. We work out a winning strategy for Bob, such that independent of which strategy Alice applies she will lose the game if a circle C^r with circumference $r = 4$ is given. Suppose, contrary to our claim, that a circle C^4 is sufficient. In particular Bob's strategy will be to attack an uncolored vertex v twice such that v cannot be assigned to C^4 anymore which implies that a feasible coloring of the graph fails.

(i) Consider the graph G'. By symmetry of G' there are only two possible starting points for Alice. Either she colors a vertex from $\{v_1, ..., v_5\}$ or in $\{v'_1, ..., v'_5, v''_1, ..., v''_5\}$.

- Suppose initially Alice colors a vertex from $\{v_1, ..., v_5\}$. Without loss of generality assume that she colors vertex v_1 with the arc $b_{(0,1)}$. Then by symmetry of G' Bob replies by coloring v'_5 with an arc of distance $1 - \varepsilon$ to $b_{(0,1)}$. Without loss of generality assume that Bob assigns the vertices on C^4 in the clockwise direction, which implies that he colors v'_5 with the arc $b_{(2-\varepsilon,3-\varepsilon)}$. Thus, vertex v_5 is attacked once. In her next move Alice has to color v_5; otherwise she loses the game: If she colors vertex v''_5, Bob will attack v_5 for the second time by coloring vertex v_4. If she colors vertex v_4, then Bob replies with coloring vertex v''_5. In particular she is only able to use an arc between $3 - \varepsilon$ to 4 for v_5. Bob then colors vertex v'_4 with an arc of distance $1 - \varepsilon$ to $f(v_5)$. Thus, vertex v_4 is attacked once and Alice is forced to color v_4; otherwise she loses. In this manner Bob continues until v_3 is colored by Alice. Then clearly it holds $f(v_1) \cap f(v_3) \in [0, 3 - \varepsilon]$. Hence, Bob wins the game by coloring vertex v'_2 such that for v_2 there is no feasible arc left.

- Suppose Alice starts the game by coloring a vertex in $\{v'_1, ..., v'_5, v''_1, ..., v''_5\}$. Without loss of generality assume that she colors v'_1 with the arc $b_{(0,1)}$. Then Bob colors vertex v''_1 with the arc $b_{(2-\varepsilon,3-\varepsilon)}$ so that Alice is forced to color v_1 in her next move. Otherwise in his next turn Bob would attack v_1 for the second time by coloring either v_2 or v_5 with an arc of distance $1-\varepsilon$ to $f(v''{}_1)$. From this point we refer to the case above for Bob's winning strategy.

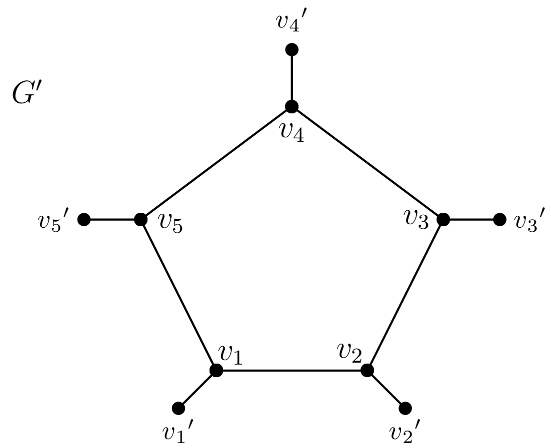
(ii) Assume that G satisfies property $\mathcal{P}$. Then G contains a subgraph H of the following form:



By symmetry assume that Alice colors vertex x_1 with the arc $b_{(0,1)}$. Then Bob replies by coloring vertex x_3 with the arc $b_{(2-\varepsilon,3-\varepsilon)}$. This implies that vertices $\{x_2, x_4\}$ are being attacked once. After Alice's second move one of the vertices $\{x_2, x_4\}$ will stay uncolored. Suppose x_2 is uncolored and it is Bob's turn. Then he colors vertex x_2' with an arc between $3-\varepsilon$ and 4. This leads to the second attack of x_2.

Since $x_1, x_2', x_3 \in N(x_2)$, by Bob's strategy x_2 cannot be assigned to C_4, otherwise the corresponding arc of x_2 would overlap with $f(x_1)$, $f(x_2')$ or $f(x_3)$. Thus, a feasible coloring of G fails. In particular x_2' is allowed to be colored with an arbitrary arc because $(x_2', x_4) \notin E(G)$ by the property $\mathcal{P}$ and
$\{(x_1, x_2'), (x_3, x_2')\} \notin E(G)$ by the assumption $\omega_c(G) = 2, 5$. □

Remark 4.2.2. The graph G' in 4.2.1 (i) becomes game-nice, if we slightly modify it, such that it is of the following form:

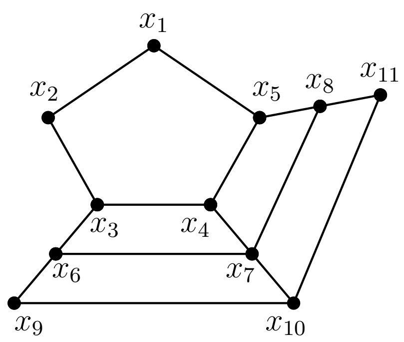


If we assume that a circle C^r with circumference $r = 4$ is given, then Alice is able to fix the coloring of the graph after her second or third turn.

The details of the proof are left to the reader.

The following example demonstrates that property $\mathcal{P}$ must not hold for every vertex of the graph G such that G is not circular game-nice:

Example: Consider the graph G.



Vertices $\{x_1, x_2, x_{10}\}$ do not satisfy $\mathcal{P}$. However, independent of which strategy Alice applies, Bob can force her to color one of the vertices $\{x_3, x_4, ..., x_{11}\}$ first, since the number of vertices on C_5 with degree 2 is even. If Alice colors one of the vertices $\{x_3, x_4, ..., x_9, x_{11}\}$ Bob wins the game with the same strategy as in the proof of proposition 4.2.1 (ii). If Alice colors x_{10}, then Bob replies by coloring x_6 with an arc of distance $1 - \varepsilon$ to $f(x_{10})$. Then Alice is forced to color x_7 in order not to lose the game and Bob afterwards colors x_5 with an arc of distance $1 - \varepsilon$ to $f(x_7)$. Hence, Bob attacks x_4 and x_8 once and Alice is able to save only one of them in her next turn, so that either x_4 or x_8 will be attacked twice by Bob. Thus, a C^4 is not sufficient to guarantee Alice's victory.

Open Problems

In proposition 4.2.1 we gave two properties, such that a graph G with $\omega_c(G) = 2,5$ and $\Theta(G) = K_5^2$ is not circular game-nice if they hold. One may ask for the set of all forbidden properties that are not allowed to hold for a circular game-nice graph G with $\omega_c(G) = 2,5$ and $\Theta(G) = K_5^2$, such that we can determine the class of such circular game-nice graphs. Moreover, it is interesting to investigate circular game-perfect graphs with clique number greater than $2,5$.

Eventually we introduced in this chapter a sensible definition of circular game-perfectness and pointed out briefly a topic in which further research can be made. The aim could be to give an analogue of the strong perfect graph theorem which holds for the ordinary chromatic number and states that a graph is perfect if and only if it is a berge graph. 2006, the authors Maria Chudnovsky, Paul Seymour, Neil Robertson and Robin Thomas published a 178-page paper (see [19]) where they proved this claim.

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