AN ANALYSIS INTO STRUCTURES AND MECHANISMS OF 5TH GRADE CHILDREN'S MATHEMATICAL KNOWLEDGE

Ipke Wachsmuth, Universität Osnabrück

INTRODUCTION

From analyses of children's behavior when learning mathematics emerges the crucial role of the mental representation of their mathematical knowledge. Instead of being coherently organized, relevant knowledge structures which need to be coordinated for success with a broad range of mathematical situations appear to develop initially as isolated "packets" which are more or less restricted to the situational context in which they were acquired.

The postulate that knowledge is stored in memory in discrete and self-contained units has been raised by several authors and has been captured in notions like "frames" (Davis, 1980), "microworlds" (Lawler, 1981), "Subjektive Erfahrungsbereiche" (Bauersfeld, 1983), and others not mentioned here. Besides this postulate, we will take into account that organizational structures of the memory system are given by the kind of the connections established between such memory units. That is, we view the structures of the knowledge in memory to be constituted by at least the following two things:

(i) self-contained memory units ("packets of knowledge");
(ii) connections between these ("organizing path networks").

Both are the result of the individual's interaction with the outside world and of internal reflective processes, and both can be subject to pedagogical interventions. In such structures is constituted the personal knowledge and beliefs of an individual, including all misconceptions and inaccuracies.

Another postulate we submit to in this paper is that in learning takes place accretion, that is, everything committed to memory is stored permanently. Access structures may change but representations are not actually deleted from memory (e.g., Davis, 1980). This postulate becomes critical when memory units concern the same topic of knowledge, for it will support the fact that competing knowledge may come to coexist in memory.

Out of the context of a two-year clinical study on the acquisition of rational number concepts with 4th/5th grade children (see acknowledgement), interview material was chosen for an analysis of characteristical features of mental representation structures of children's knowledge, and of cognitive mechanisms acting on such representation structures. The piece of work presented here will focus on one child's performance on a complex problem solving task to

give a detailed example in support of such hypothetical mechanisms. Interviews with other subjects on the same task are available and will be used to back-up, and further expand on, these hypotheses in future work. Cognitive phenomena which may interfere with successful performance will be discussed, and questions will be posed on the nature of pedagogical activities taking account of such mechanisms.

CONTEXT

The "Gray levels study" was done as part of the assessments in the Rational Number Project after completing 30 weeks of experimental instruction, and was reported on earlier (Wachsmuth, Behr, and Post, 1983; a detailed report is in progress). In videotaped one-on-one clinical interviews, sixteen 5th-grade subjects were presented with a scale of 11 distinct gray levels increasing in darkness from 0% (white) to 100% (black) in stages of 10%, and with a set of twelve fraction cards: 0/20, 1/5, 2/7, 6/20, 2/5, 4/10, 6/15, 2/4, 4/8, 4/6, 6/9, 12/15. Interpreted as representing ink mixtures with a parts black ink in b parts solution, these cards were to be ordered and attached to the scale according to their "darkness". Requiring the coordinated application of a broad scale of relevant skills, this task was expected to elicit particular features of children's rational number knowledge.

One subject, called Terri in this paper, arrived at the following solution:

0	10	20	30	40	50	60	70	80	90	100%
$\frac{0}{20}$	$\frac{1}{5}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{4}{6}$	$\frac{4}{8}$	$\frac{4}{10}$	$\frac{6}{9}$	$\frac{6}{15}$	$\frac{6}{20}$	$\frac{12}{15}$

Terri (age 11;6;24) was a low-achieving subject who was observed to have severe difficulties impeding successful learning. Rather than building coherent knowledge structures guided by the instruction, Terri was likely to invent her own, often flawed "theories" and procedures. Interventions in classroom instruction sometimes made her arrive at an "insight" which could turn out to have been an ephemeral one the very next day. The interview following her solution of the gray levels task reveals some deep insights into her thinking styles, which can serve to raise some hypotheses about the structures and mechanisms of this child's mathematical knowledge. At the time of the interview, the interviewer had known Terri from daily classroom contacts and other interviews for more than one year and was quite familiar with her idiosyncratic styles of thinking.

COMMENTED INTERVIEW TRANSCRIPT

In the following transcript is displayed some of the inconsistencies and misconceptions in Terri's knowledge of fraction equivalence and rational number ordering. In her behavior there is found indication for several conflicting "frames" or "theories" serving as bases for her decisions on comparisons of the fractions she was presented with. From earlier observations, Terri was known to persistingly call two fractions - when presented to her as written or spoken symbols - equivalent if they had the same denominator.

In the present problem situation, Terri had attached the fractions 6/15 and 12/15 to different gray levels (90% and right of 100%), apparently following some kind of "lexical" ordering bearing on the whole number symbols in the 12 fractions. This fact raises doubts over whether Terri had understood at all the interpreation of fraction symbols by means of gray levels. However, at least in the beginning it seemed to have been clear to her as will be seen from the following dialog. After she has placed all fraction cards at the gray level scale, Terri is asked at first why she has put 0/20 at the beginning of the scale (white, i.e. 0%). Terri explains:

0. TERRI: Because there'd be no black ink, no black ink so it would be clear water.

After a short dialog about 4/8 and 4/6 which Terri calls about equally dark, but 4/8 still a little bit darker than 4/6, Terri is asked about the two fractions 6/15 and 12/15.

1. INTERVIEWER: Now, Terri, what about 6/15 and 12/15?
2. TERRI: They're equal, like [laughs].
3. INTERVIEWER: OK, but you put them in different positions, though, why did you do that?

By this question, Terri's attention is called back to the task situation where she, not necesearily through an interpretation in terms of gray levels but presumably through her strategy of "lexical ordering", has rated 6/15 and 12/15 as being different. This is in contradiction to her momentary opinion that these two fractions are equal. She responds:

4. TERRI: Because! That's the way I thought I should do it! [moves and messes up chart].

That is, confrontation of her momentary opinion with her previous one results in a cognitive conflict which Terri apparently tries to escape from by destroying the solution she constructed. After a short dialog (Terri should have been asked more questions to other cards) the interviewer continues (without Terri's solution being further displayed):

5. INTERVIEWER: I would still like to know: you say six-fifteenths and twelve-fifteenths are equal?

The interviewer returns to this question to find out which was Terri's reason for calling the fractions equal before; besides, he is interested

now in which of her opinions will persist after the conflict.

6. TERRI: Right.

7. INTERVIEWER: But you put them on different parts ...

The conflict is evoked another time; Terri's response confirms that she had in mind, without making any connection to the gray levels, an ordering strategy guided by the whole number relationsships in the fraction symbols:

8. TERRI: 'Cause six comes before twelve so I thought that's the way you do it ...

Her thinking was that this was the way to make sense of the task. Now the interviewer wants to find out whether gray levels played any part at all in her doing (Terri's explanation on placing the 0/20 card had actually made reference to gray levels).

9. INTERVIEWER: OK, did you think in terms of darkness when you did that?

10. TERRI: Yeah, sorta like ...

Terri's answer does not sound convincing. Even when gray levels have played a part in the beginning, one is tempted to assume that later she has focused on a whole number ordering strategy. The next question is to find out whether Terri, in the situational context of gray levels, realizes that 12/15 represents a darker mixture than 6/15 does.

11. INTERVIEWER: Which would be darker? Six-fifteenths or twelve-fifteenths?

12. TERRI: Twelve-fifteenths.

She does rate 12/15 as darker than 6/15, but can she reach a conclusion on the ordering of the fractions 6/15 and 12/15 from this?

13. INTERVIEWER: OK, and which fraction would be bigger?

14. TERRI: Twelve-fifteenths.

Terri apparently infers that 12/15 should be the greater fraction of the two. This inference is based on an interpretation of the fraction symbols which grounds its meaning on gray levels, but it already states a "greater" (and no longer "darker") relationship between the two fractions. The inferred statement, however, continues to be in conflict with Terri's earlier opinion about the relationship between 6/15 and 12/15 which apparently resulted from her flawed "theory" of when two fractions should be equivalent (i.e., when presented to her in a purely symbolical context, Terri calls same denominator fractions equivalent).

The interviewer's next question is to find out whether Terri's opinion inferred meaningfully (12/15 > 6/15) outweighs her earlier opinion which was based on her "theory" of symbolical fraction equivalence. A critical section in the interview begins here. Through careful wording (namely, as it had been used in a stereotypical fashion in repeated interviews presenting fraction comparisons in a symbolical setting[1]), the interviewer on purpose attempts to trigger Terri's "theory" of symbolical fraction equivalence.

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1) The original wording in these interviews was like "One-fifth and one-sixth, are they equal or is one less? - Which one is less? - Tell me how you know!"

15. INTERVIEWER: And if I ask you, six-fifteenths, twelve-fifteenths, are they equal or is one less?

If Terri's "theory" were activated by this, she should reply 'They are equal'.

16. TERRI: It's less.

I.e. one is less, that is, Terri does not call them equal which is (surprisingly at the moment) not the answer anticipated from her "theory". Should the conclusion she inferred on the basis of gray levels have ultimately affected Terri's belief? The interviewer's next question ('which one is less') is posed even though Terri had previously named 12/15 as the greater fraction. This question corresponds - in wording and in the sequence of events - to the stereotypical situation of the interviews on symbolical fraction comparisons and thus again addresses (as is the interviewer's hypothesis at this point) Terri's "theory" on equivalence of fractions presented to her symbolically.

17. INTERVIEWER: Which one is less?

18. TERRI: Six... um... fifteenths.

And now the interviewer wants to know which of Terri's theories her opinion, after all, is based upon.

19. INTERVIEWER: And why did you say it's less?

20. TERRI: 'Cause it ...oh! [puts head in hand and sighs] No, they're equal. Because they have the same denominator.

DISCUSSION

We observe several instances of the crucial part which cognitive structures and mechanisms play in Terri's behavior.

1. Relevant knowledge can remain latent in a task situation, i.e. the subject "knows" but does not access, at any time, particular facts which apply to the situation. That is, the "momentary opinion" of the subject does not resemble the global knowledge she has acquired which is relevant to the task (were it so, the subject would have to become aware herself of the inconsistencies existing in her knowledge). Rather, we can conclude that her "momentary opinion" is based on a local subset of her knowledge being determined by her actual focus, i.e. the knowledge she has "in sight", and by the part of knowledge which she can access from this point.

Another point which has been observed is the possible lack of mutual accessibility of relevant knowledge units, e.g., in the context of one the subject may be able to access another one but the opposite may not be true. This is yet another instance in support of the fact that knowledge tends to develop in discrete units and that attention has to be given to the development of a proper access framework. Still more evidence for this point is found in other interview dialogs not presented here. Future work will elaborate on this.

2. We mention the crucial role of language cues (and of other cues possibly generated from a situation). As was shown in the dialog, certain language can serve to shift the subject's focus to access knowledge contained in other memory units while loosing sight of the factual situation focussed on previously. A striking instance of this is documented in lines 15 - 20 of the above transcript. The momentary opinion of Terri (lines 16 and 18) was obtained by the chain of inferences she had undergone before (lines 11 - 14) and was supported by the meaning constructed from the situational context of gray levels. Presumably, the resulting conclusion (6/15 < 12/15) is still present in Terri's short-term memory while the chain of inferences which made her arrive at this conclusion is no longer present in her short-term memory. But then the interviewer, again, calls for reasons while cueing her knowledge on symbolical fraction equivalence (line 19). Indeed, Terri's focus turns out to have shifted back to this realm: In order to give a reason, Terri has to make a new inference, based on her current focus. And - no way out of there - she comes up (line 20) with an according opinion (changed again!), together with an appropriate reason.

3. Cognitive restrictions can limit the use of relevant knowledge a subject possesses and can possibly intercept the change of incorrect beliefs. One is tempted to resign on the usefulness of a Socratic style of dialog and on whether incorrect beliefs of a subject can be changed through such a dialog. Admittedly, the example discussed is an extreme one and probably requires further analysis in terms of attitudinal patterns in the interaction of interviewer and subject. It shows, however, that a single intervention does not necessarily lead to an "insight" which becomes persistent instead of being a momentary one. Apparently, the access structures calling on Terri's flawed knowledge on symbolic fraction equivalence are much stronger than the connection made on the basis of a several-step inference which the subject is not likely to achieve all by herself. A point can be made that a more global consistency check and revision of acquired knowledge structures requires cognitive capabilities this child has not developed so far. A momentary and single "insight" is not sufficient for a long-term change in the cognitive structures manifested in Terri; it will need more than that.

CONCLUSION

At the beginning, we made a distinction between the memory units which knowledge is locally organized in, and the global organizational structure given by the connections established between such units. It is one thing that a subject can have acquired incorrect knowledge (e.g., Terri's "theory" on

fraction equivalence); it is yet another thing that relevant knowledge, whether or not it is correct, does not become activated in a situation when it should be. Moreover, the fact that incorrect as well as correct knowledge on the same topic can coexist in memory calls for particular attention on how instruction can help to improve access to the right piece of knowledge at the right time. Future work will be devoted to the critical part which mental representation plays in the learning of mathematics.

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2. The mental structures postulated of Terri in this paper have been modeled in a knowledge representation system implemented in a PROLOG machine. Parts of the dialog have been reproduced by the system. The author is grateful to Helmar Gust of Osnabrück University who has made available his PROLOG system MLOG and has helped implementing the model.

REFERENCES

Bauersfeld, H. Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In Bauersfeld, H., Bussmann, H., Krummheuer, G., Lorenz, J.H. & Voigt, J. (Eds.) Lernen und Lehren von Mathematik - Analysen zum Unterrichtshandeln II, Köln: Aulis Verlag Deubner & Co., 1983.

Davis, R.B. The postulation of certain specific, explicit, commonly-shared frames. The Journal of Mathematical Behavior 3, 1 (1980), 167-199.

Lawler, R.W. The progressive construction of mind. Cognitive Science 5, 1 (1981), 1-30.

Wachsmuth, I., Behr, M.J., & Post, T.R. Children's perception of fractions and ratios in grade 5. In R. Hershkowitz (Ed.) Proceedings of the Seventh International Conference for the Psychology of Mathematics Education, Rehovot (Israel): The Weizmann Institute of Science, 1983.