Dissertation zur Erlangung des Doktorgrades
der Mathematik (Dr.Math.)
der Universität Bielefeld

The homotopy categories of injective modules of derived discrete algebras

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April 2013

Gedruckt auf alterungsbeständigem Papier nach DIN–ISO 9706

Abstract

We study the homotopy category $K(\operatorname{Inj} A)$ of all injective A-modules $\operatorname{Inj} A$ and derived category $D(\operatorname{Mod} A)$ of the category $\operatorname{Mod} A$ of all A-modules, where A is finite dimensional algebra over an algebraically closed field. We are interested in the algebra with discrete derived category (derived discrete algebra. For a derived discrete algebra A, we get more concrete properties of $K(\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$. The main results we obtain are as following.

Firstly, we consider the generic objects in compactly generated triangulated categories, specially in $D(\operatorname{Mod} A)$. We construct some generic objects in $D(\operatorname{Mod} A)$ for A derived discrete and not derived hereditary. Consequently, we give a characterization of algebras with generically trivial derived categories. Moreover, we establish some relations between the locally finite triangulated category of compact objects of $D(\operatorname{Mod} A)$, which is equivalent to the category $K^b(\operatorname{proj} A)$ of perfect complexes and the generically trivial derived category $D(\operatorname{Mod} A)$. Generic objects in $K(\operatorname{Inj} A)$ were also considered.

Secondly, we study $K(\operatorname{Inj} A)$ for some derived discrete algebra A and give a classification of indecomposable objects in $K(\operatorname{Inj} A)$ for A radical square zero self-injective algebra. The classification is based on the fully faithful triangle functor from $K(\operatorname{Inj} A)$ to the stable module category $\underline{\operatorname{Mod}}\,\hat{A}$ of repetitive algebra $\hat{A}$ of A. In general, there is no explicit description of this functor. However, we use the covering technique to describe the image of indecomposable objects in $K(\operatorname{Inj} A)$. This leads to a full classification of indecomposable objects in $K(\operatorname{Inj} A)$. Moreover, these indecomposable objects are endofinite. Thus we give a description of the Ziegler spectrum of $K(\operatorname{Inj} A)$ according to the classification.

Acknowledgment

It is my pleasure to express my great appreciation to all those who have helped me in carrying out this thesis. First of all, I would like to thank my supervisor, Professor Henning Krause, for his consistent support and encouragement; for his patient explanation to me a lot of knowledge which I did not understand and his constructive suggestions.

I would like to thank Claus Ringel and Dieter Vossieck for their stimulating explanations and viewpoints. I am very grateful to Bin Zhu for leading me to study representation theory and for his constant encouragement. I would also like to thank Yong Jiang for his suggestions on writing. I wish to thank Nils Mahrt for his discussions and help. I would also like to thank Dong Yang and Greg Stevensen for their helpful discussions. I also wish to thank Xiaowu Chen, Xueping Huang, Jue Le, Zhiwei Li, Fei Xu and Goudong Zhou for their help on my academics and life.

I would like to thank to Department of Mathematics of University of Bielefeld and Birep for giving me an active atmosphere in studying representation theory. I acknowledge the financial support from CSC (China Scholarship Council) during the term of this degree.

Finally, I wish to thank my family, especially my parents, for their support throughout my study. Last but not least, I owe the great debt to my wife, Ruijuan Yin. She gives me a comfortable home and our lovely daughter who keeps me going.

Bielefeld, April 2013 Zhe Han

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Chapter 0

Introduction

0.1 General overview

During the development of representation theory of finite dimensional algebras, the triangulated categories associated with these algebras are extensively studied. For example, the bounded derived category of finitely generated A-modules $D^b(\operatorname{mod} A)$ is an important triangulated category which indicates the homological properties of A-modules.

In this thesis, we investigate some compactly generated triangulated categories associated with finite dimensional algebras A. In particular, we are interested in the unbounded derived category $D(\operatorname{Mod} A)$ of all A-modules and the homotopy category $K(\operatorname{Inj} A)$ of all injective A-modules. In context of triangulated categories, $K(\operatorname{Inj} A)$ could be viewed as the 'compactly generated completion' of $D^b(\operatorname{mod} A)$ for any finite dimensional algebra A. We are interested in finding out some connections between the behaviors of $D^b(\operatorname{mod} A)$ and $K(\operatorname{Inj} A)$. More precisely, we concern the discreteness of $D^b(\operatorname{mod} A)$, and try to relate it to the behavior of $K(\operatorname{Inj} A)$.

We are interested in the following aspects of categories $K(\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$.

(1) The behaviors of generic objects in these compactly generated triangulated categories.

Analogous to generic modules in module categories, there are generic objects in a compactly generated triangulated category. This category is generically trivial if it does not contain any generic object. Our result about the generically trivial triangulated category is the following: We characterize the algebras with generically trivial derived categories and show the relation between locally triangulated categories and generically trivial triangulated categories.

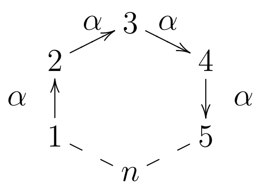
(2) The indecomposable objects in $K(\operatorname{Inj} A)$ for derived discrete algebras A.

Roughly speaking, an algebra A is derived discrete if the derived category $D^b(\operatorname{mod} A)$ does not contain continuous families of non-isomorphic indecomposable objects in $D^b(\operatorname{mod} A)$. For finite dimensional algebras A, the subcategory of all compact objects in $K(\operatorname{Inj} A)$ is equivalent to $D^b(\operatorname{mod} A)$. We expect that the category $K(\operatorname{Inj} A)$ is easy to control in case A is derived discrete. We give a classification of indecomposable objects in $K(\operatorname{Inj} A)$ for radical square zero algebras A which are derived discrete. In this case, the indecomposable pure injective objects in $K(\operatorname{Inj} A)$ are completely determined.

Setup

Throughout this thesis, the field k is algebraic closed. Let $A = kQ/I$ be a path algebra of a finite connected quiver Q with relations ρ and $I = (\rho)$ the ideal generated by ρ. The category of all left A-modules is denoted by $\operatorname{Mod} A$, which contains a subcategory $\operatorname{mod} A$ of finitely generated modules. The unbounded derived category of $\operatorname{Mod} A$ is $D(\operatorname{Mod} A)$. We use $D^b(\operatorname{mod} A)$ to denote the bounded derived category of $\operatorname{mod} A$. The category $K(\operatorname{Inj} A)$ is the homotopy category of all injective A-modules. For an algebra A, the repetitive algebra $\hat{A}$ is a self-injective infinite dimensional algebra with complete idempotents. The stable module categories $\underline{\operatorname{Mod}}\,\hat{A}$ and $\underline{\operatorname{mod}}\,\hat{A}$ both are triangulated categories.

We consider all radical square zero self-injective algebras of the form $kC_n/I_n, n \geq 1$, where the quiver C_n is given by



with relations α^2 for all α. From [8], every radical square zero self-injective algebra is obtained in this way.

Let $\mathcal{A}$ be a abelian category. we have the category of complexes $C(\mathcal{A})$ over $\mathcal{A}$. For a complex

$$X = \cdots \longrightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots,$$

the truncation functors $\sigma_{\geq n}, \sigma_{\leq n} : C(\mathcal{A}) \to C(\mathcal{A})$ are defined as: $(\sigma_{\geq n} X)^i$ is X^i if $i \geq n$ and 0 if $i < n$; $(\sigma_{\leq n} X)^i$ is X^i if $i \leq n$ and 0 if $i > n$.

Let $\mathcal{A}$ be a small additive category and Ab be the category of all abelian groups. We denote the category of contravariant additive functors from $\mathcal{A}$ to Ab by $\operatorname{Mod} \mathcal{A}$, where Ab is the category of all abelian groups. A functor $F \in \operatorname{Mod}\mathcal{A}$ is called *finitely presented* if there exists an exact sequence $\mathcal{A}(-,Y) \to \mathcal{A}(-,X) \to F \to 0$ in $\operatorname{Mod}\mathcal{A}$. The finitely presented $\mathcal{A}$-modules form an additive category with cokernels, denoted by $\operatorname{mod}\mathcal{A}$.

For a compactly generated triangulated category $\mathcal{T}$, the subcategory of all compact objects is denoted by $\mathcal{T}^c$. In order to consider the purity of $\mathcal{T}$, we need to study the restricted Yoneda functor

$$h_{\mathcal{T}} : \mathcal{T} \to \operatorname{Mod}\mathcal{T}^c, \quad X \mapsto H_X = \operatorname{Hom}_{\mathcal{T}}(-X)|_{\mathcal{T}^c}.$$

An object $X \in \mathcal{T}$ is call *pure injective* if H_X is injective in $\operatorname{Mod}\mathcal{T}^c$. The category $\mathcal{T}$ is called *pure semisimple* if every object in $\mathcal{T}$ is pure injective. The pure injective objects have some nice properties [48].

Analogous generic modules in module category, we could define generic object in $\mathcal{T}$. An object X in T is called *endofinite* if $\operatorname{Hom}_{\mathcal{T}}(C,X)$ is finite length as $\operatorname{End}_{\mathcal{T}} X$-module. An endofinite object X is called *generic* if it is indecomposable and not in $\mathcal{T}^c$. The category $\mathcal{T}$ is called *generically trivial* if $\mathcal{T}$ does not contain any generic objects.

0.2 The homotopy categories of injective modules over algebras

In general, for an additive category $\mathcal{A}$, the homotopy category $K(\mathcal{A})$ is a triangulated category. For a Noetherian ring A, the homotopy category $K(\operatorname{Inj} A)$ of injective A-modules is a compactly generated triangulated category [51]. There is a closed relation between $K(\operatorname{Inj} A)$ and the unbounded derived category $D(\operatorname{Mod} A)$ of all A-modules. We denote by $K_{ac}(\operatorname{Inj} A)$ the category of acyclic complexes in $K(\operatorname{Inj} A)$. There is a recollement between these triangulated categories [51]

$$K_{ac}(\operatorname{Inj} A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} K(\operatorname{Inj} A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(\operatorname{Mod} A).$$

There is a fully faithful functor from $D(\operatorname{Mod} A)$ to $K(\operatorname{inj} A)$ which takes the injective resolution of objects in $D(\operatorname{Mod} A)$.

The compact objects are very important objects in the compactly generated triangulated categories. They are introduced by Neeman in [56, 58]. The compactly generated triangulated category can be regarded as a infinite completion of the subcategory of compact objects. Compact objects in triangulated category could be viewed as the role of finitely presented objects in locally finitely presented abelian category [49]. In abelian category, it turns out that it is very important to consider the properties of finitely presented objects in order to study the whole category. Similarly, for compactly generated triangulated category $\mathcal{T}$, we can investigate the properties of $\mathcal{T}$ via the properties of $\mathcal{T}^c$. Conversely, we can determine the some properties of $\mathcal{T}^c$ via the properties of $\mathcal{T}$.

In general, the subcategory of compact objects in $D(\operatorname{Mod} A)$ are the perfect complexes $K^b(\operatorname{proj} A)$, not the bounded derived category $D^b(\operatorname{mod} A)$ of finitely generated modules. For the subcategory of compact objects in $K(\operatorname{Inj} A)$, we have an equivalence of triangulated categories

$$D^b(\operatorname{mod} A) \cong K^c(\operatorname{Inj} A).$$

Based on the study of $D^b(\operatorname{mod} A)$, we try to understand the category $K(\operatorname{Inj} A)$. We wish to investigate the properties of compactly generated categories $K(\operatorname{Inj} A)$ or $D(\operatorname{Mod} A)$ for algebras A with discrete derived categories $D^b(\operatorname{mod} A)$.

From the recollement, the triangulated category $K_{ac}(\operatorname{Inj} A)$ is compactly generated and the subcategory of compact objects $K^c_{ac}(\operatorname{Inj} A)$ is triangle equivalent to the singularity category $D^b_{sg}(A)$ of A [51]. If A has finite global dimension, then there is an equivalent of triangulated categories $K(\operatorname{Inj} A) \cong D(\operatorname{Mod} A)$. This suggests that $K(\operatorname{Inj} A)$ contains more information than $D(\operatorname{Mod} A)$ in general.

The method that we study the category $K(\operatorname{Inj} A)$ is based on the following constructions of functors. In [38], Happel gave a construction of full faithful triangle functor $F : D^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\, \hat{A}$. When the global dimension of A is finite, the functor F is a triangle equivalence. We can study the derived category $D^b(\operatorname{mod} A)$ using the stable module category $\underline{\operatorname{mod}}\, \hat{A}$. Krause and Le extended the functor to a full faithful triangle functor $F : K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\, \hat{A}$ [52]. This suggests that we can study $K(\operatorname{Inj} A)$ by the module category $\underline{\operatorname{Mod}}\, \hat{A}$. In general, there is no explicit description of this functor. However, we will give an explicit description of the image of indecomposable objects in $K(\operatorname{Inj} A)$ for some special algebras A.

0.3 Derived discrete algebras

There is an interesting phenomenon in representation theory of finite dimensional algebras. There are either finitely many indecomposable modules up to isomorphism or infinite continuous families of indecomposable modules up to isomorphism. If we consider $D^b(\operatorname{mod} A)$ of an algebra A, there is some intermediate behavior between the above cases. It is possible that there are infinitely many indecomposable objects up to shift and isomorphism in $D^b(\operatorname{mod} A)$, but they do not form continuous families.

In [71], Vossieck studied these algebras and gave a complete classification of them.

From [71], we know that two classes of algebras which have discrete derived category: the algebras of derived equivalent to hereditary algebras of Dynkin type and the algebras of derived equivalent to a gentle algebra with one cycle not satisfying the clock condition. For an algebra A in the first class, we have that $K(\operatorname{inj} A)$ is triangle equivalent to $D(\operatorname{Mod} A)$ and the structure of latter category is known. The difference between these two triangulated categories occurs when algebras have infinite global dimension which lie in the second class.

For a derived discrete algebra A, the indecomposable objects in $D^b(\operatorname{mod} A)$ are classified in [11]. Based on this result, Bobiński gave a description of the almost split triangles for $K^b(\operatorname{proj} A)$ [17].

0.4 Purity of triangulated categories

The notion of purity of a compactly generated triangulated category $\mathcal{T}$ are defined via the Yoneda functor $h_{\mathcal{T}} : \mathcal{T} \to \operatorname{Mod} \mathcal{T}^c$, where $\operatorname{Mod} \mathcal{T}^c$ is the category of all contravariant additive functor from $\mathcal{T}^c$ to Ab [48]. The category $\operatorname{Mod} \mathcal{T}^c$ is a locally coherent Grothendieck category. It is worthing to point out that the functor is neither faithful nor full on morphisms in general [12, Theorem 11.8].

Pure injective objects of a triangulated category exactly correspond to injective objects of $\operatorname{Mod} \mathcal{T}^c$ under the functor $h_{\mathcal{T}}$. The Ziegler spectrum $\operatorname{Zg} \mathcal{T}$ of a compactly generated triangulated category $\mathcal{T}$ is defined as the Ziegler spectrum of $\operatorname{Mod} \mathcal{T}^c$.In particular, the endofinite objects are pure injective. Moreover, these endofinite objects have a nice decomposition theorem [46].

In order to study the purity of $\mathcal{T}$, we need to study the restricted Yoneda functor $h_{\mathcal{T}}$. According to the result of Krause [48], $\mathcal{T}$ is pure semisimple if and only if the functor $h_{\mathcal{T}}$ is fully faithful.

In order to determine the Ziegler spectrum of $K(\operatorname{Inj} A)$ for some derived discrete algebra A, we give a classification of all indecomposable objects in $K(\operatorname{Inj} A)$. Thus we know all the indecomposable pure injective objects and all endofinite objects. After classifying the indecomposable objects of $K(\operatorname{Inj} A)$, we determine the Ziegler spectrum of $K(\operatorname{Inj} A)$ and the subcategory of endofinite objects of $K(\operatorname{Inj} A)$.

0.5 Main results

We have seen that there are closed relationship between $K(\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$ for an algebra A. We shall investigate these categories in the further results. For an

Artin algebra A, it is representation finite if and only if there is no generic module in $\operatorname{Mod} A$. Similarly, there exists generic objects in a compactly generated triangulated categories $\mathcal{T}$. The derived category $D(\operatorname{Mod} A)$ is called generically trivial if it does not contain generic objects. Generically triviality is a finiteness condition on the derived category $D(\operatorname{Mod} A)$. We will describe the generic objects in $D(\operatorname{Mod} A)$ for all finite dimensional algebra A. From the description of generic objects, we have the result about the algebras which have generically trivial derived categories, as stated in Theorem 2.3.15.

Theorem 0.5.1. *Let A be a finite dimensional k-algebra, then $D(\operatorname{Mod} A)$ is generically trivial if and only if A is derived equivalent to a hereditary algebra of Dynkin type .*

A compactly generated triangulated category $\mathcal{T}$ is called pure semisimple if every object is pure injective object. The derived category $D(\operatorname{Mod} A)$ is not pure semisimple in general. From [12, Theorem 12.20], the generic objects in $D(\operatorname{Mod} A)$ also characterize the pure semisimplicity of $D(\operatorname{Mod} A)$. Our result, Corollary 2.3.16, shows the characterization.

Theorem 0.5.2. *Let A be a finite dimensional k-algebra, then $D(\operatorname{Mod} A)$ is pure semisimple if and only if $D(\operatorname{Mod} A)$ is generically trivial.*

Concerning the pure semisimplicity of the category $K(\operatorname{Inj} A)$, we analogue the result in [12, Corollary 11.20] and have the following result.

Theorem 0.5.3. *For an artin algebra A, $K(\operatorname{Inj} A)$ is pure semisimple if and only if A is derived equivalence of a hereditary algebra of Dynkin type .*

The locally finite triangulated category was introduced in [72]. A k-linear triangulated category $\mathcal{C}$ is *locally finite* if $\operatorname{supp}\operatorname{Hom}(X,-)$ contains only finitely many indecomposable objects for every indecomposable $X \in \mathcal{C}$, where $\operatorname{supp}\operatorname{Hom}(X,-)$ is the subcategory generated by indecomposable objects Y in $\mathcal{C}$ with $\operatorname{Hom}(X,Y) \neq 0$. There is an equivalent characterization of locally finite triangulated category $\mathcal{C}$, which says that the functor $\operatorname{Hom}_{\mathcal{C}}(-,X)$ is finite length in $\operatorname{mod}\mathcal{C}$ for every object $X \in \mathcal{C}$. We established the relationship between the locally finite triangulated category $K^b(\operatorname{proj} A)$ and generically trivial derived category $D(\operatorname{Mod} A)$. The results was stated in Proposition 2.3.20.

Theorem 0.5.4. *Let A be a finite dimensional k-algebra. Then $D(\operatorname{Mod} A)$ is generically trivial if and only if $K^b(\operatorname{proj} A)$ is locally finite.*

The category $K(\operatorname{Inj} A)$ is a compactly generated triangulated category with the subcategory of compact objects $K^c(\operatorname{Inj} A) \cong D^b(\operatorname{mod} A)$. It has a complicated structure in general. But the indecomposable objects of $D^b(\operatorname{mod} A)$ are classified in [11]. We expect to classify the indecomposable objects in $K(\operatorname{Inj} A)$. The radical square zero self-injective algebras are derived discrete algebras. We investigate the homotopy category $K(\operatorname{Inj} A)$ for these algebras.

We know that each radical square zero self-injective algebra A are of the form kC_n/I_n. In order to classify the indecomposable objects in $K(\operatorname{Inj} A)$, we firstly classify the indecomposable objects in $K(\operatorname{Inj} \Lambda)$ where $\Lambda = k[x]/(x^2)$. This is just the special case of $n = 1$ for algebras kC_n/I_n. Then we apply covering theory to classify the general case of $K(\operatorname{Inj} A)$ since there is a covering map $kC_n/I_n \to \Lambda$ for every $n > 1$. For an algebra $A \cong kC_n/I_n$ Theorem 3.3.9 gives an explicit description of the indecomposable objects in $K(\operatorname{Inj} A)$.

Theorem 0.5.5. *If A is of form kC_n/I_n, then indecomposable objects of $K(\operatorname{Inj} A)$ are exactly the truncations $\sigma_{\leq m}\sigma_{\geq l}I^\bullet[i], i \in \mathbb{Z} \cup \{\pm\infty\}$, where $I^\bullet$ is the periodic complex*

$$\cdots \xrightarrow{d} I_n \xrightarrow{d} I_1 \xrightarrow{d} I_2 \xrightarrow{d} \cdots \xrightarrow{d} I_n \xrightarrow{d} I_1 \xrightarrow{d} \cdots,$$

the differential d is given by the canonical morphism between indecomposable non-isomorphic projective-injective A-modules $I_k, k = 1, \ldots, n$.

Because the category $K(\operatorname{Inj} A)$ is not pure semisimple, there are always some objects which are not pure injective. However, we can description the Ziegler spectrum of $K(\operatorname{Inj} A)$ for the algebra A of the form kC_n/I_n. For an indecomposable pure injective object in $K(\operatorname{Inj} A)$, we denote $[X]$ the corresponding point in $\operatorname{Zg}(K(\operatorname{Inj} A))$. For an triangulated category $\mathcal{T}$, an open basis of $\operatorname{Zg}(\mathcal{T})$ was given by a family of sets $\mathcal{O}(C) = \{[M] \in \operatorname{Zg} \mathcal{T} | \operatorname{Hom}_{\mathcal{T}}(C, M) \neq 0\}$ for $C \in \operatorname{mod} \mathcal{T}^c$. If we know all the points in $\operatorname{Zg}(K(\operatorname{Inj} A))$, we could calculate these open sets in the space $\operatorname{Zg}(K(\operatorname{Inj} A))$. We give the following results in Theorem 3.4.1 and Corollary 3.4.2.

Theorem 0.5.6. *Let $A = kC_n/I_n$ for some $n \in \mathbb{N}^*$. Every indecomposable object $I_{l,m}[r]$ of $K(\operatorname{Inj} A_n)$ is an endofinite object. Then the Ziegler spectrum $\operatorname{Zg}(K(\operatorname{Inj} A))$ consists of the point $[I^A_{m,n}[r]]$ for each indecomposable object $I^A_{m,n}[r] \in K(\operatorname{Inj} A)$ up to isomorphism.*

0.6 Structure of the thesis

The thesis is organized as follows. In Chapter 1 we review representation theory of algebras and triangulated categories arising in module category of algebra. We consider the bounded derived category of finitely generated modules and the homotopy category of injective modules over finite dimensional k-algebra. There are also some basic properties of compactly generated triangulated categories. Chapter 2 is about the purity of triangulated categories. We investigate the endofinite object in $D(\mathrm{Mod}\, A)$ for some algebra A. Using generic objects in $D(\mathrm{Mod}\, A)$, we give a criterion for an algebra A admitting generically trivial derived category $D(\mathrm{Mod}\, A)$ in Theorem 2.3.15. This theorem deduces a relation between the locally finite triangulated category $K^b(\mathrm{proj}\, A)$ and generically trivial derived category $D(\mathrm{Mod}\, A)$. We also analysis generic objects in $K(\mathrm{Inj}\, A)$. In chapter 3, we first classify all indecomposable objects in $K(\mathrm{Inj}\, \Lambda)$ with $\Lambda = k[x]/(x^2)$. Based on this result, Theorem 3.3.9 describes all indecomposable objects in $K(\mathrm{Inj}\, A)$ for a radical square zero self-injective algebra A. Furthermore, we show all these indecomposable objects are endofinite objects. The last part is an appendix. We survey some foundations about group graded algebras and triangulated categories.

Chapter 1

Background and foundations

In this chapter we review some background materials. We introduce the representation theory of quivers and finite dimensional algebras. Then we consider some triangulated categories raised from representation theory. We recall some results about compactly generated triangulated categories in section 4. In the last section we will describe the repetitive algebra of a finite dimensional algebra.

1.1 Representations of quivers

A finite dimensional algebra over field k has very nice structure. There is also a systematic representation theory about finite dimensional algebras . We will describe the representation theory of finite dimensional algebras. It is remarkable that the quiver representation plays a very important role in this theory.

A *k-algebra* A is a k-vector space equipped with a multiplication which is compatible with the structure of vector space. If A as k-vector space is finite dimensional, then it is a finite dimensional algebra.

We will denote the category of all left A-modules by $\operatorname{Mod} A$, and the full subcategory of $\operatorname{Mod} A$ consisting of finite dimensional modules by $\operatorname{mod} A$.

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and s, t are two maps $Q_1 \to Q_0$ indicating the source and target vertices of an arrow. A quiver Q is *finite* if Q_0 and Q_1 are finite sets. A quiver Q is *connected* if the underlying graph is a connected graph.

A *path* α in the quiver Q is a sequence of arrows $\alpha_n \dots \alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1}), 1 \leq i < n$. We define paths of form $\{e_i = (i|i)\}_{i \in Q_0}$ to be trivial paths. Let kQ be the k-vector space with the paths of Q as a basis, We define a multiplication on kQ: for any two paths α, β,

$$\beta\alpha = \begin{cases} \beta\alpha & \text{if } t(\alpha) = s(\beta), \\ 0 & \text{otherwise} \end{cases}$$

Then kQ has a natural structure of k-algebra, called the *path algebra* of the quiver Q.

A *relation* σ on quiver Q is a k-linear combination of paths $\sigma = \sum x_i p_i$ with $\{p_i\}$ having the same source and same target. Let R_Q be the ideal of kQ generated by all the arrows in Q. A set of relations $\{\rho_i\}$ is called admissible if it generates the ideal (ρ_i) satisfying that $R_Q^m \subset (\rho_i) \subset R_Q^2$ for some integer $m \geq 2$. The pair (Q, ρ_i) is called a quiver with relations $\rho = (\rho_i)$, or a *bounded quiver*. We associate an algebra $k(Q, \rho) = kQ/(\rho)$ to the pair (Q, ρ). The algebra $k(Q, \rho)$ is an associative k-algebra.

Theorem 1.1.1. *Let A ba an finite dimensional basic k-algebra, then there is a quiver Q with relations ρ such that $A \cong k(Q, \rho)$.*

Proof. See [8, Chapter III, Corollary 1.10] or [3, Chapter II, Theorem 3.7] . □

Thus every basic finite dimensional algebra over k is the quotient algebra of the path algebra of some quiver. It has some advantages to consider the path algebra of quiver. Actually, we can associate the modules of a finite dimensional algebra with the representations of its associated quiver.

A *representation* $V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1})$ of a given quiver Q over k assigns to each vertices $i \in Q_0$ a vector space V_i, and each arrow $\alpha : i \to j$ a k-linear map $V_\alpha : V_i \to V_j$.

A *morphism* between two representations V, W is given by a family of linear maps $f = (f_i)_{i \in Q_0}$, where $f_i : V_i \to W_i$ such that for any arrow $\alpha : i \to j$, $W_\alpha f_i = f_j V_\alpha$.

We call W a *subrepresentation* of V, if W_i is a subspace of V_i for each vertex $i \in Q_0$ and $W_\alpha(x) = V_\alpha(x)$ for each arrow α and $w \in W_{s(\alpha)}$.

Given a morphism $f : W \to V$, its *kernel* $\operatorname{Ker} f$ is by definition the subrepresentation of W with $(\operatorname{Ker} f)_i = \operatorname{Ker} f_i$ for each vertex i. The *image* $\operatorname{Im} f$ is the subrepresentation of V with $(\operatorname{Im} f)_i = \operatorname{Im} f_i$. The *cokernel* $\operatorname{Coker} f$ is the quotient of V with $(\operatorname{Coker} f)_i = \operatorname{Coker} f_i$.

From the definition of kernel, cokernel and image, we get that f is a *monomorphism* if and only if $\operatorname{Ker} f = 0$, while f is an *epimorphism* if and only if $\operatorname{Coker} f = 0$. The morphism f is an *isomorphism* if each f_i is an isomorphism.

The *dimension vector* of a finite dimensional representation V is the vector $\underline{\dim} V$ in $\mathbb{Z}^{Q_0}$ with

$$(\underline{\dim} V)_i = \dim V_i \quad (i \in Q_0).$$

It is called finite dimensional if every V_i for $i \in Q_0$ is finite k-dimension.

A representation V of Q is called *indecomposable* if it is not isomorphic to the direct sum of two non-zero subrepresentations. Any non-zero representation can be decomposed into a sum of indecomposables.

The *support* of a representation V is a full subquiver $suppV$ of Q with $(suppV)_0 = \{i \in Q_0 | V_i \neq 0\}$. A representation V of quiver Q is called *thin*, if $\dim V_i \leq 1$ for every vertex $i \in Q_0$. We define the category $Rep_k(Q, \rho)$ of all representations over k of (Q, ρ). We denote by $rep_k(Q, \rho)$ the subcategory of $Rep_k(Q, \rho)$ consisting of the finite dimensional representations. Both of them are abelian categories. Moreover, there is an categories equivalence between the module category of $k(Q, \rho)$ and the category of representations of quiver (Q, ρ).

Theorem 1.1.2. *Let $A = k(Q, \rho)$, where Q is a finite connected quiver. There is a k-linear equivalence of categories*

$$\operatorname{Mod} A \xrightarrow{\sim} Rep_k(Q, \rho)$$

which restricts to an equivalence $\operatorname{mod} A \xrightarrow{\sim} rep_k(Q, \rho)$.

Proof. See [3, Chapter III, Theorem 1.6] □

From this theorem, to study the modules of a finite dimensional algebra equals to study the representations of the corresponding quiver with relations. Sometimes, we will not distinguish these two settings in the context.

There is a categorical language for the path algebras of bounded quivers. We can identify each associate algebra with a bounded category. First, a *locally bounded category* R is a k-category satisfying

1. distinguish objects are non-isomorphic;
2. $R(x, x)$ is local for each $x \in R$;
3. $\sum_{y \in R} \dim_k R(x, y) + \dim_k R(y, x) < \infty$ for each $x \in R$.

There is an important characterization of locally bounded categories in [18], like basic k-algebras can be expressed as path algebras of bounded quivers. Each locally bounded category R is isomorphic to a bounded quiver category kQ/I, where Q is a locally finite quiver, i.e for every vertex, there are only finitely many arrows going in and out.

A locally bounded category R is called *bounded*, if it has finitely many objects. We can associate a finite dimensional k-algebra $\oplus R = \oplus_{x,y \in R} R(x, y)$ to each bounded category R. Note that if R is locally bounded, the algebra $\oplus R$ is possibly infinite dimensional. In this setting, the R-module M is defined as a contravariant k-linear functor from R to the category of vector spaces. If R is bounded, the module category $\operatorname{mod} R$ is equivalent to the module category of the finite dimensional k-algebra $\oplus R$. Thus we can view a basic finite dimensional algebra as the path algebra of a bounded quiver or a bounded category.

1.2 Gentle algebras

Gentle algebras have many interesting properties. They appeared in the context of classification algebras of Dynkin type $\mathbb{A}_n$ [2] and type $\tilde{\mathbb{A}}_n$ [4] up to derived equivalence. They are Gorenstein algebras in the sense of [32]. Moreover, Gentle algebras are closed under derived equivalence [67]. Let Q be a (not necessarily finite) quiver, ρ be a set of relations for Q. The bounded quiver (Q, ρ) is called *special biserial* if it satisfies

(1) For any vertex in Q, there are at most two arrows starting at and ending in this vertex.

(2) Given an arrow β, there is at most one arrow α with $t(\alpha) = s(\beta)$ and $\beta\alpha \notin \rho$, and there is at most one arrow γ with $t(\beta) = s(\gamma)$ and $\gamma\beta \notin \rho$.

(3) Each infinite path in Q contains a subpath which is in ρ.

A bounded quiver (Q, ρ) is *gentle* if it is special biserial and the following additional conditions hold,

(4) All elements in ρ are paths of length 2;

(5) Given an arrow β, there is at most one arrow α' with $t(\alpha') = s(\beta)$ and $\beta\alpha \in \rho$, and there is at most one arrow γ' with $t(\beta) = s(\gamma')$ and $\gamma'\beta \in \rho$.

A k-algebra $A \cong kQ/(\rho)$ is called *special bieserial* or *gentle* if (Q, ρ) is special biserial or gentle, respectively. A k-algebra $A \cong kQ/(\rho)$ is called *string algebra* if it is special biserial and the relation is generated by zero relations. The gentle algebras can be characterized by functions $\sigma, \tau : Q_1 \to \{-1, +1\}$ as follows.

Lemma 1.2.1. *Let $A = kQ/I$, where I is generated by paths of length 2, then A is a gentle algebra if and only if there exist two functions $\sigma, \tau : Q_1 \to \{-1, +1\}$ satisfying*

1. *if $s(\alpha) = s(\beta)$ and $\sigma(\alpha) = \sigma(\beta)$, then $\alpha = \beta$.*

2. *if* $t(\alpha) = t(\beta)$ *and* $\tau(\alpha) = \tau(\beta)$*, then* $\alpha = \beta$.

3. *if* $t(\alpha) = s(\beta)$*, then* $\tau(\alpha) = \sigma(\beta)$ *iff* $\beta\alpha \in I$.

Proof. See [32, Lemma 2.2] or [21, Section 3]. □

A gentle algebra A is a *Gorenstein algebra* in the sense that it has finite injective dimension both as a left and a right A-module. It has been proved by calculating the injective resolutions of string modules over a gentle algebra [32, Theorem 3.4].

For a special biserial algebra A, the non-projective modules of A can be viewed as modules of some string algebra $\bar{A}$ [28, Chapter 2]. It is very useful because Butler and Ringel have given a complete classification of finite dimensional modules over any string algebra [21]. The indecomposable modules are parameterised by the strings and bands of the corresponding quiver.

Let Q be an arbitrary quiver, $\beta \in Q_1$, denote β^{-1} the formal inverse of β, $s(\beta^{-1}) = t(\beta)$ and $t(\beta^{-1}) = s(\beta)$. A *word* w is a sequence $w_1 w_2 \dots w_n$ where $w_i \in Q_1 \cup Q_1^{-1}$ and $s(w_i) = t(w_{i+1})$ for $1 \leq i \leq n$. The inverse of a word $w = w_1 w_2 \dots w_n$ is $w^{-1} = w_n^{-1} \dots w_2^{-1} w_1^{-1}$. If $s(w) = t(w)$, a rotation of $w = w_1 w_2 \dots w_n$ is a word of form $w_{i+1} \dots w_n w_1 \dots w_i$. On the set of words, we consider two relations as following: The relation $\sim$ identifies the w with w^{-1} and $\sim_r$ is the relation identifies w with its rotations and inverse.

Assume $A = kQ/I$ to be a string algebra. Let $\mathcal{S}$ be the set of representatives of words w under the relation $\sim$ which either $w = w_1 w_2 \dots w_n$ and $w_i \neq w_{i+1}^{-1}$ for $1 \leq i \leq n$ and no subpath of w belongs to I, or w is a trivial path 1_x for some vertex $x \in Q_0$. The elements of $\mathcal{S}$ are called strings. For a word $w = w_1 w_2 \dots w_n$ of length $n > 0$, such that w^m can be defined for any $m \in \mathbb{N}$ and w is not a power of path with length smaller then n. Let $\mathcal{B}$ be the set of representatives of such words under the relation $\sim_r$. The elements of $\mathcal{B}$ are called bands. Let $\mathcal{I}$ be the disjoint union of equivalence class of strings (i.e the set $\mathcal{S}$) and equivalence of bands (i.e the set $\mathcal{B}$). For any word $w \in \mathcal{I}$, we define the corresponding string module or band module $M(w)$ as in [21, Section 3].

Theorem 1.2.2. *[21] Let* $A = kQ/I$ *be a string algebra, then the modules* $\{M(w) : w \in \mathcal{I}\}$ *form a complete set of pairwise non-isomorphic indecomposable finite dimensional* A*-modules.*

Remark 1.2.3. *If the string algebra* $A = kQ/I$ *is of infinite representation type, then there exist infinite strings. There is a canonical indecomposable representation associated to every infinite string [45].*

A remarkable connection between gentle algebras and special biserial algebras was introduced in [67]. Gentle algebras can be obtained from a module without self-extension of special biserial algebra. Given a gentle algebra, not only itself is special biserial but also its repetitive algebra.

Theorem 1.2.4. *Let A be a special biserial algebra, and M be an A-module without self-extension, then the stable endomorphism algebra $\underline{\mathrm{End}}_A(M)$ is a gentle algebra.*

Proof. See [67, Section 3]. □

Remark 1.2.5. *Given a gentle algebra $A = kQ/I$, the repetitive algebra of A is a special biserial algebra $\hat{A} = k\hat{Q}/\hat{I}$. We can get a string algebra $\bar{\hat{A}} = k\hat{Q}/\bar{\hat{I}}$ obtained from $\hat{A}$ by modulo the socle of all projective-injective modules [65, Section 7]. The indecomposable $\bar{\hat{A}}$-modules are just the non-projective-injective indecomposable $\hat{A}$-modules.*

1.3 Triangulated categories in representation theory

In representation theory of finite dimensional k-algebra, the triangulated categories appear as the homotopy categories and derived categories of module categories or subcategories of module categories, also the stable category of module category over self injective algebra. In derived category, we can transfer the homological information to the complexes. Under this setting, many homological results become more apparent. In this section we mainly consider the homological properties of finite dimensional algebra.

1.3.1 Derived equivalence and stable categories

In general, for an additive category $\mathcal{A}$, there is corresponding category of complexes $C(\mathcal{A})$ and homotopy category $K(\mathcal{A})$. The homotopy category $K(\mathcal{A})$ is not an abelian category, even if $\mathcal{A}$ is abelian. But $K(\mathcal{A})$ is a triangulated category, an additive category with a special automorphism satisfying some axioms. If $\mathcal{A}$ is an abelian category, we can form the derived category $D(\mathcal{A})$ by localising $K(\mathcal{A})$ with respect to quasi-isomorphisms. It is also a triangulated category.

For an algebra A, we denote $D^b(\mathrm{mod}\, A)$ the bounded derived category of $\mathrm{mod}\, A$. It is an triangulated category.

Given two algebras A and B, if $D^b(\mathrm{mod}\, A)$ and $D^b(\mathrm{mod}\, B)$ are equivalent as triangulated categories, then A and B are said to be *derived equivalent.*

Happelintroduced the tilting theory under the derived category setting, which is a special case of derived equivalent [38] . In [64], Rickard proved a more general theorem.

Theorem 1.3.1. *Let A and B be two finite dimensional algebras. The following are equivalent.*

(1) The algebras A and B are derived equivalent.

(2) $K^b(\operatorname{proj} A)$ and $K^b(\operatorname{proj} B)$ are equivalent as triangulated categories.

(3) the algebra $B \cong \operatorname{End}_{K^b(\operatorname{proj} A)}(T)$, where $T \in K^b(\operatorname{proj} A)$ satisfying

(a) For integer $n \neq 0$, we have $\operatorname{Hom}(T, \Sigma^n T) = 0$.

(b) The smallest full triangulated category of $K^b(\operatorname{proj} A)$ containing T and closed under forming direct summands equals $K^b(\operatorname{proj} A)$.

Proof. See [64, Theorem 6.4, Corollary 8.3]. □

Let $\mathcal{A}$ be a Frobenius category, more details referred to Appendix B.1. The stable category $\underline{\mathcal{A}}$ associated with $\mathcal{A}$ is the category with the same objects of $\mathcal{A}$. The morphism $\operatorname{Hom}_{\underline{\mathcal{A}}}(X, Y)$ between objects X, Y of $\underline{\mathcal{A}}$ is defined by the quotient of $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ with respect to the ideal $\mathcal{I}$ of maps which factor through an injective I. The composition of $\underline{\mathcal{A}}$ is induced by that of $\mathcal{A}$. Moreover $\mathcal{A}$ is a triangulated category, see Appendix B.1.

The shift functor $\Sigma : \mathcal{A} \to \mathcal{A}$ is defined for each object $X \in \mathcal{A}$ as follows,

$$0 \longrightarrow X \xrightarrow{i_X} I_X \xrightarrow{\pi_X} \Sigma X \longrightarrow 0$$

where I_X is an injective object, and for any morphism $f : X \to Y$ in $\mathcal{A}$, the standard triangle $X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X$ in $\underline{\mathcal{A}}$ is defined by the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & I_X & \xrightarrow{\pi_X} & \Sigma X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & Y & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma X & \longrightarrow & 0 \end{array}.$$

Moreover, given a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the induced triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle in $\underline{\mathcal{A}}$ and every distinguished triangle in $\underline{\mathcal{A}}$ is obtained in this way.

Example 1. *(1) If $\mathcal{A}$ is an additive category, the complex category $C(\mathcal{A})$ with the set of all degree-wise split exact sequences is a Frobenius category. The stable category of $C(\mathcal{A})$ is nothing but the homotopy category $K(\mathcal{A})$.*

(2) If R is a self-injective ring, then the module category $\operatorname{Mod} R$ with all exact sequences is a Frobenius category and the stable category $\underline{\operatorname{Mod}}\, R$ is a triangulated category.

An *algebraic triangulated category* $\mathcal{T}$ is a triangulated category $\mathcal{T}$ which is triangle equivalent to a stable category of some Frobenius category. All triangulated categories that we consider in this thesis are algebraic triangulated categories. Since they arise from the categories of complexes, which are Frobenius categories under some exact structure.

Remark 1.3.2. *There are triangulated categories which are not algebraic. For example, the stable homotopy category of spectra is not algebraic.*

1.3.2 The classification of derived discrete algebras

The algebra with discrete derived category was introduced by Vossieck [71]. We call these algebras as *derived discrete algebra*, followed by [9]. In [33], Geiss and Krause introduced the tameness for derived categories based on the tameness definitions of module categories . The basic technique is to transfer the objects in the derived category to the modules over the repetitive algebra. The derived tame algebras contain the derived discrete algebra introduced in [71]. An important class of derived discrete algebra is the gentle algebra. The indecomposable complexes in $D^b(\operatorname{mod} A)$ for a gentle algebras are completely classified [11]. It is similar the classification of indecomposable modules over string algebras. Bobiński gave a explicitly description of Auslander-Reiten triangles for $K^b(\operatorname{proj} A)$ [17]. We shall study these derived behaviors of algebras.

For an algebra A, let $D^b(\operatorname{mod} A)$ be the bounded derived category of $\operatorname{mod} A$. For an complex $X \in D^b(\operatorname{mod} A)$, we define the *homology dimension* of X to be the vector $h\dim(X) = (\dim H^i(X))_{i\in\mathbb{Z}}$.

Definition 1.3.3. *[33] A locally bounded k-category A is called derived discrete if for every vector n there exist a localization $R = k[t]_f$ with respect to some $f \in k[t]$ and a finite number of bounded complexes of R-A-bimodules $X_1, \dots, X_n$ such that X_j^i is finitely generated free over R and all but finitely many indecomposable objects of cohomology dimension n in $D^b(\operatorname{mod} A)$ are of form $S \otimes_R X_i$ for $i = 1, \dots, n$ and some simple R-module S.*

Algebras with this property that there are only finitely many indecomposable objects in the derived category with the some homological dimension were studied in [71].

Definition 1.3.4. *A derived category $D^b(\mathrm{mod}\, A)$ is discrete if for every vector $d = (d_i)_{i\in\mathbb{Z}}$ with $d_i \geq 0$, there are only finitely many isomorphism classes of indecomposables $X \in D^b(\mathrm{mod}\, A)$ such that $h\mathrm{dim}X = d$. And we call the algebra A is derived discrete.*

The discreteness or tameness of a derived category $D^b(\mathrm{mod}\, A)$ is a derived invariant.

Proposition 1.3.5. *[33, 71] If two finite dimensional algebras A and B are derived equivalent, then A is derived tame (resp. discrete) if and only if B is derived tame (resp. discrete).*

The derived tameness is similar the tameness of module categories. Bekkert and Drozd proved that every finite dimensional algebra over algebraic closed field is either derived tame or derived wild [70].

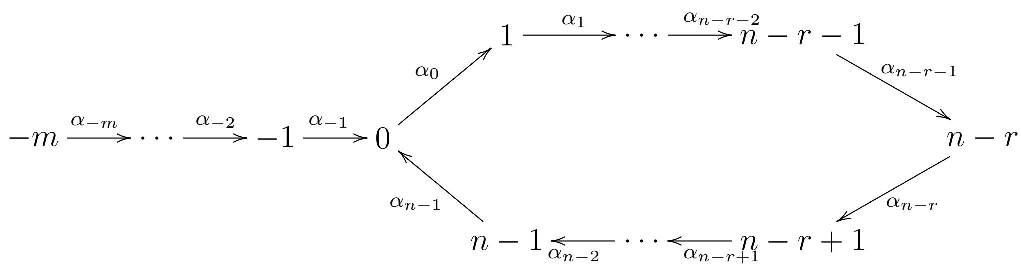
Now, it is reasonable to classification the derived discrete algebras even the derived tame algebras. In [71], Vossieck gave a complete classification of the algebras which are derived discrete. Let (Q, I) be a finite gentle quiver and C be a non-oriented cycle in Q. We call C satisfies *clock condition* if the numbers of clockwise and of counterclockwise paths of length two in C that belong to I are same.

Theorem 1.3.6. *If A is a connected finite dimensional k-algebra. Then A is derived discrete if and only if A is either derived hereditary of Dynkin type or is Morita equivalent to the path algebra of a gentle bounded quiver with exactly one cycle not satisfying the clock condition.*

Proof. See [71, Theorem 2.1]. □

Based on this theorem, the classification of derived equivalence classes of derived discrete algebras was completed [16].

Let Ω be the set of all triples (r, n, m) of integers such that $n \geq r \geq 1$ and $m \geq 0$. For each $(r, n, m) \in \Omega$ consider the quiver $Q(r, n, m)$ of the form



The ideal $I(r,n,m)$ in the path algebra $kQ(r,n,m)$ is generated by the paths $\alpha_0\alpha_{n-1}, \alpha_{n-1}\alpha_{n-2}, \dots, \alpha_{n-r+1}\alpha_{n-r}$, and let $L(r,n,m) = kQ(r,n,m)/I(r,n,m)$. It is an easy observation that $gl.\dim L(r,n,m) = \infty$ for $n = r$.

Theorem 1.3.7. *[16, Theorem A] Let A be a connected algebra and assume that A is not derived hereditary of Dynkin type. The following are equivalent:*

(1) $D^b(\mathrm{mod}\, A)$ *is discrete.*

(2) $D^b(\mathrm{mod}\, A) \cong D^b(\mathrm{mod}\, L(r,n,m))$*, for some* $(r,n,m) \in \Omega$.

(3) A is tilting-cotilting equivalent to $L(r,n,m)$*, for* $(r,n,m) \in \Omega$.

From Proposition 1.3.5, when we study properties of $D^b(\mathrm{mod}\, A)$ for some derived discrete algebra A, it is enough to study the algebras of the forms as following list.

1. The derived hereditary of Dynkin type, $\mathbb{A}_m (m \geq 1), \mathbb{D}_n (n \geq 4), \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.

2. The form $L(r,n,m)$ for $(r,n,m) \in \Omega$.

By the theorem 1.2.4 and the embedding functor $D^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}\, \hat{A}$, we have that the gentle algebras are closed under derived equivalence. And the derived equivalent class of a gentle algebra contains finitely many algebras. The tilting-coltilting equivalence is the same as derived equivalence if the algebra is derived of type Dynkin, Euclidean or tubular canonical [4, 5].

Remark 1.3.8. *By the embedding* $D^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}\, \hat{A}$ *of triangulated categories, if the category* $\underline{\mathrm{mod}}\, \hat{A}$ *is tame, then the derived category* $D^b(\mathrm{mod}\, A)$ *should be tame. There are some papers [5, 6, 26, 27], about when the repetitive algebra* $\hat{A}$ *is tame.*

1.4 Compactly generated triangulated categories

1.4.1 Basic definitions

There are triangulated categories with arbitrary coproducts. For example, compactly generated triangulated categories are triangulated categories with arbitrary coproducts. We consider a class of special objects, compact objects in compactly generated triangulated categories, like small objects in abelian category. Neeman introduced the compact objects and compactly generated triangulated category [58, 57]. He proved that the compactly generated triangulated category satisfying the Brown representability [58, Theorem 3.1]. This property has many applications in the study of the structure of triangulated categories. There are also compactly generated triangulated categories associated representation of finite dimensional algebras. We shall expose the topic in this section.

Definition 1.4.1. *Let $\mathcal{T}$ be a triangulated category with infinite coproducts. An object T in $\mathcal{T}$ is compact if* $\mathrm{Hom}_{\mathcal{T}}(T,-)$ *preserves all coproducts. $\mathcal{T}$ is called compactly generated if there is a set $\mathbb{S}$ of compact objects such that* $\mathrm{Hom}_{\mathcal{T}}(T,\Sigma^i X) = 0$ *for all $T \in \mathbb{S}$ and $i \in \mathbb{Z}$ implies $X = 0$.*

Now we assume that the triangulated category $\mathcal{T}$ has infinite coproducts. It is easy to show that compact objects are closed under shifts and triangles. Thus the full subcategory of compact objects also has a triangulated structure.

Let $\mathcal{T}$ be a compactly generated triangulated category. We denote the full subcategory $\mathcal{T}^c$ of $\mathcal{T}$ consisting of compact objects. $\mathcal{T}^c$ is a triangulated category.

If compact objects in a triangulated category are known, we can get the triangulated category form the infinite completion of all compact objects.

Definition 1.4.2. *Let $\mathcal{R}$ be a full triangulated subcategory of $\mathcal{T}$. If $\mathcal{R}$ is closed under coproducts and the inclusion $\mathcal{R} \subset \mathcal{T}$ preserves coproducts, then $\mathcal{R}$ is called a localizing subcategory of $\mathcal{T}$.*

Lemma 1.4.3. *[58, Lemma 3.2] Let $\mathcal{T}$ be a compactly generated triangulated category, with $\mathbb{S}$ a generating set. If $\mathcal{R}$ is the localizing subcategory containing $\mathbb{S}$, then there is a triangulated equivalence $\mathcal{R} \cong \mathcal{T}$.*

The following result is a consequence of [14, Lemma 4.5] or follows directly from Lemma 1.4.3.

Lemma 1.4.4. *Let $F, G : \mathcal{T} \to \mathcal{S}$ be two triangle functors preserving coproducts between two k-linear compactly generated triangulated categories. If there is a natural isomorphism $F(X) \cong G(X)$ for all compact object $X \in \mathcal{T}$, then $F \cong G$.*

From the above results, we know that the triangulated category $\mathcal{T}$ and $\mathcal{T}^c$ are determined by each other. In some cases, we can completely determine the compact objects of a triangulated category.

Theorem 1.4.5. *Let $\mathcal{T}$ be a compactly generated triangulated category. If a homological functor $H : \mathcal{T}^{op} \to \mathrm{Ab}$ preserves coproducts, then $H \cong \mathrm{Hom}_{\mathcal{T}}(-, X)$ for some $X \in \mathcal{T}$.*

Proof. [58, Theorem 3.1]. □

There is a characterization of representable functors by Brown representability. Using this theorem, we can construct adjoint functors of exact functor $F : \mathcal{T} \to \mathcal{S}$ with $\mathcal{T}$ compactly generated.

Corollary 1.4.6. *Let $F : \mathcal{T} \to \mathcal{S}$ with $\mathcal{T}$ be a an exact functor between triangulated categories with $\mathcal{T}$ compactly generated.*

(1) The functor F has a right adjoint if and only if F preserves all coproducts.

(2) The functor F has a left adjoint if and only if F preserves all products.

Proof. For (1), we consider the functor $\mathrm{Hom}_{\mathcal{S}}(F(-), X) : \mathcal{T}^{op} \to \mathrm{Ab}$ for any $X \in \mathcal{S}$. The functor preserves coproducts. By Theorem 1.4.4, the functor is representable. We have a $G(X) \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{S}}(F(Y), X) = \mathrm{Hom}_{\mathcal{T}}(Y, G(X))$ for any $Y \in \mathcal{T}$. For (2), see [59, Theorem 8.6.1]. □

1.4.2 Recollement

It is well-known that the derived category $D(\mathrm{Mod}\, A)$ is compactly generated, with the full subcategory of compact objects $K^b(\mathrm{proj}\, A)$, the full subcategory of perfect complexes.

In [51], Krause proved that the homotopy category $K(\mathcal{A})$ of a locally noetherian Grothendieck category $\mathcal{A}$ is compactly generated and determined all the compact objects. In particular, the result applies for $\mathcal{A} = \mathrm{Mod}\, A$ for a finite dimensional algebra A or more general, a Noetherian ring A.

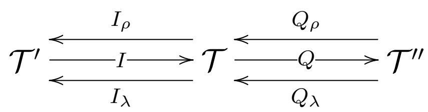
Theorem 1.4.7. *If A is a finite dimensional k-algebra, then the category $K(\mathrm{Inj}\, A)$ is compactly generated, and there is a natural triangle equivalence*

$$K^c(\mathrm{Inj}\, A) \cong D^b(\mathrm{mod}\, A).$$

Proof. See [51, Section 2]. □

Proposition 1.4.8. *[15, Section 8] The category $K(\mathrm{Inj}\, A)$ is derived invariant, i.e if $D^b(\mathrm{mod}\, A) \cong D^b(\mathrm{mod}\, B)$, then we have $K(\mathrm{Inj}\, A) \cong K(\mathrm{Inj}\, B)$.*

We are interested in the triangulated categories $D(\mathrm{Mod}\, A)$ and $K(\mathrm{Inj}\, A)$. There are close relations between them. Recall that a *recollement* [10] is a sequence of triangulated categories and functors between them.



satisfying the following conditions.

(1) I_λ is a left adjoint and I_ρ a right adjoint of I;

(2) Q_λ is a left adjoint and Q_ρ a right adjoint of Q;

(3) $I_\lambda I \cong Id_{\mathcal{T}'} \cong I_\rho I$ and $QQ_\rho \cong Id_{\mathcal{T}''} \cong QQ_\lambda$;

(4) $QX = 0$ if and only if $X \cong IX'$ for some X' in $\mathcal{T}'$.

Consider the canonical functors

$$I : K_{ac}(\operatorname{Inj} A) \to K(\operatorname{Inj} A) \quad \text{and} \quad Q : K(\operatorname{Inj} A) \xrightarrow{inc} K(\operatorname{Mod} A) \xrightarrow{can} D(\operatorname{Mod} A),$$

we have that I and Q have right and left adjoints.

Firstly, we show that Q has right adjoint Q_ρ. This is equivalent to the functor I has right adjoint I_ρ [51, Lemma 3.2]. Let $K_{\text{inj}}(A)$ be the smallest triangulated category of $K(\operatorname{Mod} A)$ closed under taking products and contains $\operatorname{Inj} A$. The inclusion functor $K_{\text{inj}}(A) \to K(\operatorname{Inj} A)$ preserves products, and has a left adjoint $i : K(\operatorname{Mod} A) \to K_{\text{Inj}}(A)$ by Lemma 1.4.6. The functor i induces an equivalence

$$D(\operatorname{Mod} A) \xrightarrow{\sim} K_{\text{inj}}(A).$$

By the natural isomorphism $\operatorname{Hom}_{D(\operatorname{Mod} A)}(X, Y) \cong \operatorname{Hom}_{K(\operatorname{Mod} A)}(X, iY)$, we can take the right adjoint Q_ρ of Q as the composition

$$D(\operatorname{Mod} A) \xrightarrow{i} K_{\text{inj}}(A) \longrightarrow K(\operatorname{Inj} A)\,.$$

Let $\mathcal{K}$ be the localizing subcategory of $K(\operatorname{Inj} A)$, generated by all compact objects $X \in K(\operatorname{Inj} A)$ such that QX is compact in $D(\operatorname{Mod} A)$. Then we have that $Q_{\mathcal{K}} : \mathcal{K} \to D(\operatorname{Mod} A)$ is an equivalence. Fix a left adjoint $L : D(\operatorname{Mod} A) \to \mathcal{K}$, the composition $D(\operatorname{Mod} A) \xrightarrow{L} \mathcal{K} \xrightarrow{inc} K(\operatorname{Inj} A)$ is a left adjoint of Q. The fully faithful functor $Q_\lambda : D(\operatorname{Mod} A) \to K(\operatorname{Inj} A)$ identifies $D(\operatorname{Mod} A)$ with the localizing subcategory of $K(\operatorname{Inj} A)$ which is generated by all compact objects in the image of Q_λ. The functor Q_λ identifies $D(A)$ with the localizing subcategory of $K(\operatorname{Inj} A)$ which is generated by the injective resolution I_A of A.

Summarize the above, there is a recollement [51, Section 4]

$$K_{ac}(\operatorname{Inj} A) \underset{\xleftarrow{I_\lambda}}{\overset{\xleftarrow{I_\rho}}{\xrightarrow{I}}} K(\operatorname{Inj} A) \underset{\xleftarrow{Q_\lambda}}{\overset{\xleftarrow{Q_\rho}}{\xrightarrow{Q}}} D(\operatorname{Mod} A)$$

There is an important reason to study the category $K(\operatorname{Inj} A)$. We have the equivalence $K^c(\operatorname{Inj} A) \cong D^b(\operatorname{mod} A)$. There are also close relations between the subcategories of compact ojects of $K_{ac}(\operatorname{Inj} A), K((\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$.

Proposition 1.4.9. *The functor $I_\lambda \cdot Q_\rho : D(\operatorname{Mod} A) \to K_{ac}(\operatorname{Inj} A)$ induces an equivalence*

$$D^b(\operatorname{mod} A)/D^c(\operatorname{Mod} A) \xrightarrow{\sim} K^c_{ac}(\operatorname{Inj} A).$$

Proof. See [51, Corollary 5.4] □

Remark 1.4.10. *For a finite dimensional algebra A, the triangulated categories $D^b(\operatorname{mod} A)$ and $D^c(\operatorname{Mod} A) \cong K^b(\operatorname{proj} A)$ both are Hom-finite. But $K^c_{ac}(\operatorname{Inj} A)$ is not Hom-finite in general.*

1.5 Repetitive algebras

The repetitive algebra is considered as a Galois covering of the trivial extension of a finite dimensional algebra [40]. In [38], Happel proved that the bounded derived category of a finite dimensional algebra can be embedded into the stable module category of its repetitive algebra. We can study the derived category through the representations of repetitive algebras. The generalization the embedding to an infinite setting was contained the paper [52]. We introduce the definition and some basic properties of repetitive algebras.

1.5.1 Basic properties of repetitive algebras

Let A be a finite dimension basic k-algebra. $D = \operatorname{Hom}_k(-, k)$ is the standard duality on $\operatorname{mod} A$. $Q = DA$ is a A-A-module via $a', a'' \in A, \varphi \in Q, (a'\varphi a'')(a) = \varphi(a'aa'')$. The *trivial extension algebra* $T(A) = A \times DA$ of A is the symmetric algebra whose k-vector space is $A \oplus DA$, and the multiplication is given by $(a, f)\cdot(b, g) = (ab, ag + fb)$. for $a, b \in A, f, g \in DA$.

Definition 1.5.1. *The repetitive algebra $\hat{A}$ of A is defined as following, the underlying vector space is given by*

$$\hat{A} = (\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} Q)$$

denote the element of $\hat{A}$ by $(a_i, \varphi_i)_i$, almost all a_i, φ_i being zero. The multiplication is defined by

$$(a_i, \varphi_i)_i \cdot (b_i, \phi_i)_i = (a_i b_i, a_{i+1}\phi_i + \varphi b_i)_i$$

In fact $\hat{A}$ can be viewed as a double infinite matrix algebra, without identity

$$\hat{A} = \begin{pmatrix} \ddots & & & & \\ \ddots & A_{i-1} & & & 0 \\ & Q_{i-1} & A_i & & \\ & & Q_i & A_{i+1} & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

where each $A_i = A$ is (i,i)-th element $Q_i = Q$ is the $(i+1,i)$-th element, and all the remaining elements are zero. The multiplication is induced from the canonical maps

$$A \otimes_A Q \to Q, Q \otimes_A \to Q, \text{ and } Q \otimes_A Q \to 0.$$

The multiplication is just the usual matrix multiplication.

If $1_A = e_1 + \ldots + e_n$ is the decomposition of the identity 1_A of A into a sum of orthogonal primitive idempotent $e_1, \ldots, e_n$, then $1_{A_m} = e_{m,1} + \ldots + e_{m,n}, e_{m,i} = e_i$ for all $m \in \mathbb{Z}, a \leq i \leq n$.

We define the automorphism $\mu_{\hat{A}} : \hat{A} \to \hat{A}, e_{m,i} \mapsto e_{m+1,i}$ for $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, is called the Nakayama automorphism of $\hat{A}$. Then we have the Galois covering $F : \hat{A} \to \hat{A}/G$, where $G = (\mu_{\hat{A}})$ is generated by $\mu_{\hat{A}}$, and $\hat{A}/G \cong T(A)$. The covering induces a $\mathbb{Z}$-grading on $T(A)$, with $degA = 0, degDA = 1$, such that $\operatorname{mod} \hat{A} \cong \operatorname{mod}^{\mathbb{Z}} T(A)$. We can view the associated push down functor $F_\lambda : \operatorname{mod} \hat{A} \to \operatorname{mod} T(A)$ as a forgetful functor $\operatorname{mod}^{\mathbb{Z}} T(A) \to \operatorname{mod} T(A)$. In general, the push down functor is not dense, see Appendix A.1.

A repetitive algebra $\hat{A}$ is *locally support finite* if for each indecomposable projective $\hat{A}$-module P, the set of indecomposable projective $\hat{A}$-modules

$$\{P' \in \operatorname{mod} \hat{A} | \operatorname{Hom}(P, M) \neq 0 \neq \operatorname{Hom}(P', M) \text{ for some } M \in \operatorname{mod} \hat{A}\}$$

is finite. Under the assumption of locally support finite, the push down functor is dense [26].

If we have a path algebra $kQ/I$of a quiver Q with relations I, then we can construct the quiver $\hat{Q}$ and the relations $\hat{I}$ of the repetitive algebra of kQ/I [66, 65]. Let ρ be the set of relations generating I. A path $\alpha_1 \ldots \alpha_n$ of length ≥ 1 in (Q, ρ) is called *maximal* if $\beta\alpha_1 \ldots \alpha_n$ and $\alpha_1 \ldots \alpha_n\gamma$ are not in I for each β with $t(\beta) = s(\alpha_1)$ and each γ with $t(\alpha_n) = s(\gamma)$. Denote $\mathscr{M}(Q, \rho)$ the set of all maximal paths in (Q, ρ). If (Q, ρ) is locally bounded, we construct the quiver $\hat{Q} = (\hat{Q}_0, \hat{Q}_1)$ as follows,

Take countable many copies $Q[i]$ of Q where $i \in \mathbb{Z}$, thus for every $a \in Q$ there is a vertex $a[i]$ in $Q[i]$ for all $i \in \mathbb{Z}$, and for every arrow $\alpha : a \to b$ in Q, there is an arrow $\alpha[i] : a[i] \to b[i]$ in $Q[i]$ for all $i \in \mathbb{Z}$. In addition, for every maximal path

$p : a \to b \in \mathscr{M}(Q,\rho)$ there is an arrow $p'[i] : b[i] \to a[i-1]$ for all $i \in \mathbb{Z}$, these are called *connecting arrows*. The resulting quiver is denotes by $(\hat{Q}_0, \hat{Q}_1)$. Let $p = p_1p_2$ be a maximal path in (Q,ρ), the path of form $p_2[i]p'[i]p_1[i-1]$ is called a *full path* in $\hat{Q}$.

Now, we define a set of relations for $\hat{Q}$:

R1 Let p, p_1, p_2 be paths in Q, if $p \in \rho$(resp. $p_1 - p_2 \in \rho$) then $p[i] \in hat\rho$(resp.$p_1[i] - p_2[i] \in \hat{\rho}$) for all $i \in \mathbb{Z}$.

R2 Let p be a path containing a connecting arrow. If p is not a subpath of a full path, then $p \in \hat{\rho}$.

R3 Let $p = p_1p_2p_3$ and $q = q_1q_2q_3$ be a maximal path in (Q,ρ) with $p_2 = q_2$, then $p_3[i]p'[i]p_1[i-1] - q_3q'[i]q_1[i-1] \in \hat{\rho}$ for all $i \in \mathbb{Z}$.

Under the above construction of repetitive quiver, there is a characterization of gentle algebra [66, 65].

Theorem 1.5.2. *Let (Q,ρ) be a locally bounded quiver, then $k\hat{Q}/(\hat{\rho})$ is the repetitive algebra of $kQ/(\rho)$. If $A = kQ/(\rho)$ is a finite dimensional k-algebra, then A is gentle if and only if $\hat{A} = k\hat{Q}/(\hat{\rho})$ is special biserial.*

1.5.2 The embedding functors

For a finite dimensional algebra A, there are close relations between $K(\operatorname{Inj} A)$ and $\underline{\operatorname{Mod}}\,\hat{A}$, as well as $D^b(\operatorname{mod} A)$ and $\underline{\operatorname{mod}}\,\hat{A}$. Happel introduced the embedding functor $D^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\,A$ [37]. The functor was extended to $K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\,A$ of unbounded complexes in [52]. The embedding functor is very useful because the module category $\operatorname{mod}\hat{A}$ is well-understood, even $\operatorname{Mod}\hat{A}$ is well-understood. By the embedding functors, the behavior of $D^b(\operatorname{mod} A)$ can be controlled by $\underline{\operatorname{mod}}\,\hat{A}$ and the behavior of $K(\operatorname{Inj} A)$ can be controlled by $\underline{\operatorname{Mod}}\,\hat{A}$. If $\hat{A}$ is tame, it is possible to describe the category $D^b(\operatorname{mod} A)$ and $K(\operatorname{Inj} A)$ more explicitly.

An $\hat{A}$-module is given by $M = (M_i, f_i)_{i\in\mathbb{Z}}$, where M_i are A-modules and $f_i : Q \otimes_A M_i \to M_{i+1}$ such that $f_{i+1} \circ (1 \otimes f_i) = 0$. Given $\hat{A}$-modules $M = (M_i, f_i)$ and $N = (N_i, g_i)$, the morphism $h : M \to N$ is a sequence $h = (h_i)_{i\in\mathbb{Z}}$ such that

$$\begin{array}{ccc} Q \otimes_A M_i & \xrightarrow{f_i} & M_{i+1} \\ \downarrow{\scriptstyle 1\otimes h_i} & & \downarrow{\scriptstyle h_{i+1}} \\ Q \otimes N_i & \xrightarrow{g_i} & N_{i+1} \end{array}$$

commutes. Sometimes, we write $(M_i, f_i')_{i\in\mathbb{Z}}$ as

$$\cdots M_{-1} \sim^{f_{-1}} M_0 \sim^{f_0} M_1 \sim \cdots .$$

Let $\operatorname{Mod}\hat{A}$ be the category of all left $\hat{A}$-modules, and $\operatorname{mod}\hat{A}$ be the subcategory of finite dimensional modules. They both are Frobenius categories since $\hat{A}$ is a self-injective algebra. Thus the associated stable categories $\underline{\operatorname{Mod}}\,\hat{A}$ and $\underline{\operatorname{mod}}\,\hat{A}$ are triangulated categories. Moreover, $\underline{\operatorname{mod}}\,\hat{A}$ is the full subcategory of compact objects in $\underline{\operatorname{Mod}}\,\hat{A}$.

Lemma 1.5.3. *[38, Lemma II.4.2] There exist an exact functor $I : \operatorname{mod}\hat{A} \to \operatorname{mod}\hat{A}$ and a monomorphism $\mu : id \to I$ such that I(X) is injective for each $X \in \operatorname{mod}\hat{A}$.*

We can use this lemma to construct suspension functor for the triangulated category $\underline{\operatorname{mod}}\,\hat{A}$ as a stable category of Frobenius category $\operatorname{mod}\hat{A}$.

Theorem 1.5.4. *[38, Theorem II.4.9] There is an exact functor $F : K^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\,\hat{A}$, and if $f : X \to Y$ is a quasi-isomorphism. Then $F(f)$ is an isomorphism in $\underline{\operatorname{mod}}\,\hat{A}$ in $K^b(A)$. Thus F induces an exact functor $F : D^b(A) \to \underline{\operatorname{mod}}\,\hat{A}$. If $gl.\dim A < \infty$, then the functor F is an equivalence of triangulated categories.*

We call the functor $F : D^b(A) \to \underline{\operatorname{mod}}\,\hat{A}$ as Happel's functor. Now, we extended the result to the category of infinite dimensional modules over $\hat{A}$ as in [52]. The bimodule${}_A A_{\hat{A}}$induces an adjoint pair of functors between

$$\operatorname{Mod} A \underset{\operatorname{Hom}_{\hat{A}}({}_A A,-)}{\overset{-\otimes_A A_{\hat{A}}}{\rightleftarrows}} \operatorname{Mod}\hat{A}$$

also induced adjoint functors between $K(\operatorname{Mod} A)$ and $K(\operatorname{Mod}\hat{A})$. Note that the functor $-\otimes_A A_{\hat{A}}$ sends a module $X \in \operatorname{Mod} A$ to $(X_n, f_n) \in \operatorname{Mod}\hat{A}$ with $X_0 = X$ and $X_n = 0$ for $N \neq 0$ and the functor $\operatorname{Hom}_{\hat{A}}(A, -)$ takes injective $\hat{A}$-modules to injective A-modules. Thus it induces a functor from $K(\operatorname{Inj}\hat{A})$ to $K(\operatorname{Inj} A)$. This functor preserves products and has therefore a left adjoint $\phi : K(\operatorname{Inj} A) \to K(\operatorname{Inj}\hat{A})$, by Corollary 1.4.6. The functor $\operatorname{Hom}_{\hat{A}}(A, -)$ preserves the coproducts since A is finitely generated over $\hat{A}$.

The inclusion $K(\operatorname{Inj} A) \to K(\operatorname{Mod} A)$ preserves the products and therefore has a left adjoint j_A. We have $\phi \circ j_A = j_{\hat{A}} \circ (-\otimes_A A)$. It follows that ϕ takes the injective resolution of a A-module to the injective resolution of the A-module $M \otimes_A A$. This shows that ϕ^c corresponding to $-\otimes_A A$.

The inclusion $K_{ac}(\operatorname{Inj}\hat{A}) \to K(\operatorname{Inj}\hat{A})$ has a left adjoint ψ, which can be explicitly described as $pM \to iM \to \psi M \to pM[a]$ which is a triangle in $K(\operatorname{Inj}\hat{A})$. The following theorem [52, Theorem 7.2] shows that there exists a fully faithful functor between $K(\operatorname{Inj} A)$ and $\underline{\operatorname{Mod}}\,\hat{A}$ extending Happel's functor.

Theorem 1.5.5. *There is a fully faithful triangle functor F which is the composition of*

$$K(\operatorname{Inj} A) \xrightarrow{\psi\circ\phi} K_{ac}(\operatorname{Inj} \hat{A}) \xrightarrow{\sim} \underline{\operatorname{Mod}}\,\hat{A}$$

extending Happel's functor

$$D^b(\operatorname{mod} A) \xrightarrow{-\otimes_A A_{\hat{A}}} D^b(\operatorname{mod} \hat{A}) \longrightarrow \underline{\operatorname{mod}}\,\hat{A}\,.$$

The functor F admits a right adjoint G which is the composition

$$\underline{\operatorname{Mod}}\,\hat{A} \xrightarrow{\sim} K_{ac}(\operatorname{Inj} \hat{A}) \xrightarrow{\operatorname{Hom}_{\hat{A}}(A,-)} K(\operatorname{Inj} A).$$

The image of the embedding functor $F : D^b(\operatorname{Mod} A) \to \underline{\operatorname{Mod}}\, A$ was considered in [33]. Given $i \in \mathbb{Z}$, the i-th syzygy of $\hat{A}$-module X is denoted by $\Omega^i X$. The assignment $X \mapsto \Omega^i X$ induces an equivalence between $\underline{\operatorname{Mod}}\,\hat{A} \to \underline{\operatorname{Mod}}\,\hat{A}$. The cosyzygy functor Ω^{-1} is a translation functor of the triangulated category $\underline{\operatorname{Mod}}\,\hat{A}$.

Lemma 1.5.6. *[33, Lemma 2.1] Let X be a $\hat{A}$ module and $X_i = 0$ for $i < r$, and $i > s$. Then*

(1)$(\Omega X)_i = 0$ for $i < r$ and $i > s + 1$.

(2)$(\Omega^{-1} X)_i = 0$ for $i < r - 1$ and $i > s$.

(3)If $pdX_r = n$(resp.$idX_s = n$), then $(\Omega^{n+1} X)_r = 0$(resp.$(\Omega^{-n-1} X)_s = 0$).

From this lemma, we know that $X = (X_i, f_i) \in \operatorname{Im} F$ if and only if there exists $n \geq 0$ such that$(\Omega_{\hat{A}}^{-n} X)_j = 0$ for $j < 0$, i.e $pd_A X_j < \infty$. Actually, we have a concrete consequence about the $\operatorname{Im} F$.

Proposition 1.5.7. *[22, Theorem 4.1] Let A be a Gorenstein algebra, then*

$$\operatorname{Im} F = \{(X_i, f_i) \in \operatorname{mod} \hat{A} | pd_A X_i < \infty \text{ for } i \neq 0\}.$$

If $X \in D^b(\operatorname{Mod} A)$, then the image $F(X)$ is support finite $\hat{A}$-module. If A has finite global dimension, then support-finite modules in $\operatorname{Mod} \hat{A}$ has preimages in $D^b(\operatorname{Mod} A)$. The following propositions [33, Lemma 3.4,3.5] show these relations between $D^b(\operatorname{Mod} A)$ and $\operatorname{Mod} \hat{A}$.

Lemma 1.5.8. *Let A be a finite dimensional algebra and F be the Happel's embedding functor.*

(1) *Let* $X \in D^b(\mathrm{Mod}\, A)$*, with* $H^i(X) = 0$ *for* $|i| > n$*, then* $F(X)_i = 0$ *for* $|i| > 2(n+1)$.

(2) *Assume that* A *has finite global dimension* d*. Let* $X \in D^b(\mathrm{Mod}\, A)$ *with* $F(X)_i = 0$ *for* $|i| > n$*. Then* $H^i(X) = 0$ *for* $|i| > (n+1)(d+1)$.

Chapter 2

Generic objects in derived categories

In this chapter we recall some background of functor categories and purity of triangulated categories. We investigate the purity of $K(\text{Inj}\, A)$ for an algebra A. Furthermore, we study the generic objects in derived categories and construct some generic objects in derived categories. Consequently we give a criterion for an algebra which has generically trivial derived categories. We also consider the generic objects in $K(\text{Inj}\, A)$.

2.1 Functor categories

Let $\mathcal{A}$ be a small additive category, $\mathcal{B}$ be an arbitrary additive category and $F_1, F_2 : \mathcal{A} \to \mathcal{B}$ be additive functors from $\mathcal{A}$ to $\mathcal{B}$. A morphism from F_1 to F_2 is determined by a certain element of product $\prod_{X \in \mathcal{A}} \text{Hom}_{\mathcal{B}}(F_1(X), F_2(X))$ (a subclass of $\prod_{X \in \mathcal{A}} \text{Hom}_{\mathcal{B}}(F_1(X), F_2(X))$), forming a set denoted by $\text{Hom}(F_1, F_2)$. Now, we denote the category of contravariant additive functors from $\mathcal{A}$ to $\mathcal{B}$ by $\text{Fun}(\mathcal{A}, \mathcal{B})$. With the composition of functors and the group structure on $\text{Hom}(F, G)$, $\text{Fun}(\mathcal{A}, \mathcal{B})$ becomes an additive category (sometimes we denote $\text{Fun}(\mathcal{A}, \text{Ab})$ by $\text{Mod}\, \mathcal{A}$). Let all indecomposable objects in $\mathcal{A}$ be $ind\mathcal{A} = \mathcal{A}'$, for $F, G \in \text{Fun}(\mathcal{A}, \mathcal{B})$ a morphism $u : F \to G$ is uniquely determined by its restriction $u|_{\mathcal{A}'} : F|_{\mathcal{A}'} \to G|_{\mathcal{A}'}$ to $\mathcal{A}'$. Moreover, the restriction map

$$\text{Hom}(F, G) \to \text{Hom}(F|_{\mathcal{A}'}, G|_{\mathcal{A}'}), \quad u \mapsto u|_{\mathcal{A}'}$$

is an isomorphism of groups and

$$\text{Fun}(\mathcal{A}, \mathcal{B}) \to \text{Fun}(\mathcal{A}', \mathcal{B}), \quad F \mapsto F|_{\mathcal{A}'}$$

is an equivalence of categories.

Now, we mainly concern the case $\mathcal{B} = \mathrm{Ab}$ the category of abelian groups. The functor which is isomorphic to the Hom-functor $H_X = \mathrm{Hom}_{\mathcal{A}}(-, X) : \mathcal{A} \to \mathrm{Ab}$ is called a *representable functor*.

There are some basic properties of the additive category $\mathrm{Mod}\,\mathcal{A}$. We refer to the reference [41, Appendix B] or [62, Chapter 3].

Lemma 2.1.1 (Yoneda's Lemma). *For any additive functor $F : \mathcal{A} \to \mathrm{Ab}$, the mapping*

$$\mathrm{Hom}(H_X, F) \to F(X), \quad u \mapsto u_A(1_A)$$

is an isomorphism of abelian groups with inverse

$$e \mapsto (u_Y)_{Y \in \mathcal{A}}$$

where $u_Y(f) = F(f)(e)$.

Proof. See [34, Theorem II.3.3] or [29, Theorem 5.32]. □

This lemma shows that every additive category $\mathcal{A}$ could be embedded in a functor category $\mathrm{Mod}\,\mathcal{A}$. The category $\mathcal{A}$ is equivalent to the full subcategory of $\mathrm{Mod}\,\mathcal{A}$ formed by representation functors. A family of objects $\{U_i\}_{i \in I}$ is called a *family of generators* for an additive category $\mathcal{A}$ if for each nonzero morphism $\alpha : A \to B$ in $\mathcal{A}$ there is a morphism $u : U_i \to A$ such that $\alpha u = 0$.

Theorem 2.1.2. *Let $\mathcal{A}$ be a small additive category. Then $\mathrm{Mod}\,\mathcal{A}$ is Grothendieck category. For every functor $F \in \mathrm{Mod}\,\mathcal{A}$, there exists an injective envelop.*

Proof. See [29, Propostition 5.21, Theorem 6.25]. □

Remark 2.1.3. *The set of all representation functors $\{H_X\}_{X \in \mathcal{A}}$ form a family of projective generators in $\mathrm{Mod}\,\mathcal{A}$ [68, Corollary 7.4]. Since for every non-zero morphism $u : F_1 \to F_2$ in $\mathrm{Mod}\,\mathcal{A}$, there exists $X \in \mathcal{A}$ such that $u_X : F_1(X) \to F_2(X)$ is not zero. By Yoneda's Lemma, there is an element $s \in F_1(X) \cong \mathrm{Hom}(H_X, F_1)$ such that $us \neq 0$. If an abelian category has a generator then the family of subobjects of any objects is a set [68, Chapter IV, Proposition 6.6].*

A functor $F \in \mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$ is *finitely presented* if there exists an exact sequence $\mathcal{H}_Y \to H_X \to F \to 0$ in $\mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$. A functor $F \in \mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$ is *coherent* if every finitely generated subobject is finitely presented. The Grothendieck category $\mathcal{C}$ is *locally coherent* if every object of $\mathcal{C}$ is a direct limit of coherent object. We denote the full subcategory of all coherent functors in $\mathcal{A}$ as *coh*$\mathcal{A}$. By [39, Proposition 2.1], the category $\operatorname{Mod}\mathcal{A}$ is locally coherent for small additive category $\mathcal{A}$.

Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between small additive categories, we have the restriction functor $F_* : \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}, \quad M \mapsto M \circ F$, with right adjoint functor $F^* : \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{D}$ sends $\operatorname{Hom}(-, C)$ to $\operatorname{Hom}(-, F(C))$ for every $C \in \mathcal{C}$. By Yoneda's lemma,

$$\operatorname{Hom}(F^*(\operatorname{Hom}(-, C), M) = M(F(C)) = F_*(M)(C) \cong \operatorname{Hom}(\operatorname{Hom}(-, C), F_*(M)).$$

Remark 2.1.4. *The functor category* $\operatorname{Mod}\mathcal{A}$ *is a generalization of module category over some ring [55].*

Each ring R *can be viewed as an additive category* $\mathcal{R}$ *with only one object* Δ *and* $\operatorname{Hom}_{\mathcal{R}}(\Delta, \Delta) = R$*(composition given by multiplication of* R*). Then* $\operatorname{Mod}R \cong \operatorname{Fun}(\mathcal{R}, \operatorname{Ab})$*. The additive functor* $M : \mathcal{R} \to \operatorname{Ab}$ *in* $\operatorname{Mod}\mathcal{R}$ *could be viewed as a* R*-module as the following way. Let* $M(\Delta) = G \in \operatorname{Ab}$*, we have the* R*-action on* G *as* $\rho : M(R) \to \operatorname{End} G$*. Conversely, every* R*-module* $N \in \operatorname{Mod}R$ *is a functor* $M_N \in \operatorname{Mod}\mathcal{R}$ *defined by* $M_N(\Delta) = N$ *and* $M_N(\operatorname{Hom}(\Delta, \Delta)) \to \operatorname{End} N$ *induced by the* R*-action on* N*. For a small additive category* $\mathcal{A}$*, if* X *is an object in* $\mathcal{A}$ *then the set* $\operatorname{Hom}_{\mathcal{A}}(X, X)$ *is a ring. The category* $\mathcal{A}$ *can be viewed as a ring* R *with several objects. Thus the functor category* $\operatorname{Fun}(\mathcal{A}, \operatorname{Ab})$ *can be viewed as the module category* $\operatorname{Mod}R$*, where* $R = \oplus_{X \in \mathcal{A}} \operatorname{Hom}_{\mathcal{A}}(X, X)$ *and the functors in* $\operatorname{Fun}(\mathcal{A}, \operatorname{Ab})$ *are called* $\mathcal{A}$*-modules. The finitely presented* $\mathcal{A}$*-modules form an additive category with cokernels, denoted by* $\operatorname{mod}\mathcal{A}$*.* $\operatorname{mod}\mathcal{A}$ *is abelian if and only if every map* $M \to N$ *in* $\mathcal{A}$ *has a weak kernel* $L \to M$*, i.e the sequence* $\operatorname{Hom}_{\mathcal{A}}(-, L) \to \operatorname{Hom}_{\mathcal{A}}(-, M) \to \operatorname{Hom}_{\mathcal{A}}(-, N)$ *is exact in* $\operatorname{mod}\mathcal{A}$*.*

2.2 Purity of triangulated categories

In [39], Herzog considered the Ziegler spectrum of locally coherent Grothendieck category and applied to the case $\operatorname{Mod}(\operatorname{mod}R)^{op}$, the additive covariant functors from $\operatorname{mod}R$ to Ab. Let $\mathcal{C}$ be a locally coherent Grothendieck category, the *Ziegler spectrum* $\operatorname{Zg}(\mathcal{C})$ is the set of indecomposable injective objects in $\mathcal{C}$ with the open sets indexed by Serre subcategories of *coh*$\mathcal{C}$, all coherent objects in $\mathcal{C}$. The Ziegler spectrums of locally coherent Grothendieck categories play an important role in studying the Ziegler spectrum of triangulated category.

Let us recall some basic concepts of Ziegler spectrums of locally coherent Grothendieck categories. Let $\mathcal{C}$ be a Grothendieck category, i.e an abelian category with a generator, that colimits exist in $\mathcal{C}$ and direct limits are exact. A Grothendieck category $\mathcal{C}$ is locally coherent if every object of $\mathcal{C}$ is a direct limit of coherent objects. We have some examples of locally coherent Grothendieck categories. One of the important examples is the functor category $\mathrm{Mod}(\mathrm{mod}\, R)^{op})$. The category $\mathrm{Mod}(\mathrm{mod}\, R)^{op})$ can be viewed as a generalized left R-modules via the fully faithful right exact functor $\mathrm{Mod}\, R \to \mathrm{Mod}(\mathrm{mod}\, R)^{op}), M \to - \otimes_R M$. It is well known that every right exact functor $G \in \mathrm{Mod}(\mathrm{mod}\, R)^{op})$ is of the form $- \otimes_R G(R)$. Thus $\mathrm{Mod}\, R$ can be recovered from $\mathrm{Mod}(\mathrm{mod}\, R)^{op})$ as the subcategory consisting of object $M \in \mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(C, M) = 0$ for each $C \in coh\mathcal{C}$, i.e coh-injective objects.

For every locally coherent Grothendieck category $\mathcal{C}$, we associate a set $\mathrm{Zg}(\mathcal{C})$, the set of indecomposable injective objects of $\mathcal{C}$. For an arbitrary subcategory $\mathcal{X} \subset coh\mathcal{C}$, we associate the subset of $\mathrm{Zg}(\mathcal{C})$,

$$\mathcal{O}(\mathcal{X}) = \{E \in \mathrm{Zg}\,\mathcal{C} | \exists C \in \mathcal{X}, \mathrm{Hom}_{\mathcal{C}}(C, E) \neq 0\}.$$

Given an subcategory $\mathcal{X} \in coh\mathcal{C}$, the Serre category closure $\hat{\mathcal{X}}$ of $\mathcal{X}$ is denoted by

$$\hat{\mathcal{X}} = \cap\{\mathcal{S} \subset coh\mathcal{C} : \mathcal{S} \supset \mathcal{X} \text{ is Serre}\}.$$

For more details, we refer to [39, Section 3].

Theorem 2.2.1. *The collection of subsets of* $\mathrm{Zg}(\mathcal{C})$

$$\{\mathcal{O}(\mathcal{S}) : \mathcal{S} \subset coh\mathcal{C} \textit{ is a Serre subcategory}\},$$

satisfies the axioms for the open sets of a topology on $\mathrm{Zg}(\mathcal{C})$.

The collection of open subsets $\{\mathcal{O}(C) : C \in coh\mathcal{C}\}$ *satisfies the axioms for basis of open subsets of the Ziegler spectrum.*

Proof. See [73, Theorem 4.9] or [39, Theorem 3.4]. □

Example 2. *Let R be a ring, and $\mathcal{C} = \mathrm{Mod}(\mathrm{mod}\, R)$. The Ziegler spectrums $\mathrm{Zg}\, R = \mathrm{Zg}(\mathcal{C})$ is the definition of Ziegler spectrum of ring R. To consider $\mathrm{Zg}\,\mathcal{C}$ is equivalent to consider the indecomposable pure injective object in* $\mathrm{Mod}\, R$.

Theorem 2.2.2. *Let $\mathcal{C}$ be a locally coherent Grothendieck category. There is an inclusion preserving bijection correspondence between the Serre subcategories $\mathcal{S}$ of $coh\mathcal{C}$ and the open subsets of $\mathcal{O}$ of $\mathrm{Zg}\,\mathcal{C}$. The correspondence is given by*

$$\mathcal{S} \mapsto \mathcal{O}(\mathcal{S}) := \cup_{C \in \mathcal{S}} \mathcal{O}(C), \mathcal{O} \mapsto \mathcal{S}_{\mathcal{O}} := \{C \in coh\mathcal{C} | \mathcal{O}(C) \subset \mathcal{O}\}$$

Proof. See [73, Theorem 4.9] or [39, Corollary 3.5]. □

The above setting about locally coherent Grothendieck category can be applied in the context of triangulated category. Since for a compactly generated triangulated category $\mathcal{T}$, the category $\operatorname{Mod}\mathcal{T}^c$ is a locally coherent Grothendieck category.

From now on, we fix $\mathcal{T}$ a compactly generated triangulated category and $\mathcal{T}^c$ the subcategory of compact objects.

Definition 2.2.3. *A triangle*

$$L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow \Sigma L$$

in $\mathcal{T}$ is pure-exact if for every object C in $\mathcal{T}^c$ the induced sequence

$$0 \longrightarrow \mathcal{T}(C, L) \longrightarrow \mathcal{T}(C, M) \longrightarrow \mathcal{T}(C, N) \longrightarrow 0$$

is exact. we call the map f a pure monomorphism. The object L of $\mathcal{T}$ is pure-injective if every pure-exact triangle

$$L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow \Sigma L$$

splits.

A map $f : X \to Y$ is called a *phantom map* if the induced map $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$ is zero for all $C \in \mathcal{T}^c$.

Remark 2.2.4. *The phantom map in compactly generated triangulated category was considered by Neeman in [56]. He pointed out the phantomless triangulated category is rare and gave necessary and sufficient conditions on $\mathcal{T}$ such that $\mathcal{T}$ is phantomless.*

Lemma 2.2.5. *For a triangle $\Delta : X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in $\mathcal{T}$, the followings are equivalent,*

1. *α is a phantom map;*
2. *β is a pure monomorphism;*
3. *γ is pure epimorphism;*
4. *the shifted triangle $Y \longrightarrow Z \longrightarrow X[1] \longrightarrow \Sigma Y$ is pure exact.*

Proof. For any compact object $C \in \mathcal{T}^c$, we apply the exact functor $\operatorname{Hom}(C, -)$ to the triangle Δ. We have the following long exact sequence

$$\operatorname{Hom}(C, X) \xrightarrow{\alpha^*} \operatorname{Hom}(C, Y) \xrightarrow{\beta^*} \operatorname{Hom}(C, Z) \xrightarrow{\gamma^*} \operatorname{Hom}(C, \Sigma X) \xrightarrow{\Sigma\alpha^*} \cdots .$$

α is a phantom map if and only if $\mathrm{Hom}(C,\alpha)=0$ if and only if $\alpha^*=\mathrm{Hom}(C,\Sigma\alpha)=0$ if and only if γ is pure epimorphism if and only if β is pure monomorphism. □

Now we consider the Yoneda embedding

$$h_{\mathcal{T}}:\mathcal{T}\to \mathrm{Mod}\,\mathcal{T}^c,\quad X\mapsto H_X=\mathrm{Hom}(-,X)|_{\mathcal{T}^c}.$$

The category $\mathrm{Mod}\,\mathcal{T}^c$ of $\mathrm{Mod}\,\mathcal{T}^c$ is a locally coherent Grothendieck category. The subcategory $\mathrm{mod}\,\mathcal{T}^c$ consisting of all finitely presented objects is abelian category since $\mathcal{T}^c$ is a triangulated category [59, Proposition 5.1.10]. Moreover, the category $\mathrm{Flat}\,\mathcal{T}^c$ of flat functors in $\mathrm{Mod}\,\mathcal{T}^c$ is exactly the category of all cohomological functors from $\mathcal{T}^c$ to Ab [56, Lemma 2.1].

A functor $F:\mathcal{T}\to \mathrm{Ab}$ is *coherent* if there exists an exact sequence $\mathcal{T}(X,-)\to \mathcal{T}(Y,-)\to F\to 0$, where $X,Y\in\mathcal{T}^c$. We denote $coh\mathcal{T}$ the collection of all coherent functors from $\mathcal{T}$ to Ab, which is an abelian category. There is an equivalence of categories [49, Lemma 7.2]

$$(\mathrm{mod}\,\mathcal{T}^c)^{op}\xrightarrow{\sim} coh\mathcal{T}.$$

Indeed, for each $F\in\mathrm{mod}\,\mathcal{T}^c$, we have a presentation $H_X\to H_Y\to F\to 0$. Then we complete the corresponding map $X\to Y$ to a triangle $X\to Y\to Z\to X[1]$. Naturally, there is a coherent functor G with presentation $\mathrm{Hom}(A[1],-)\to \mathrm{Hom}(C,-)\to G\to 0$ by Yoneda's Lemma.

The following characterizations of pure injective objects in triangulated category can be found in [48]. For reader's convenience, we give a proof here.

Proposition 2.2.6. *Let $\mathcal{T}$ be a compactly generated triangulated category, the following are equivalent,*

(1) An object X in $\mathcal{T}$ is pure-injective.

(2) $H_X=\mathrm{Hom}(-,X)|_{\mathcal{T}^c}$ is injective in $\mathrm{Mod}\,\mathcal{T}^c$.

(3) If $\phi:Y\to X$ is a phantom map, then $\phi=0$.

(4) For every set I, the summation map $X^{(I)}\to X$ factors through the canonical map $X^{(I)}\to X^I$ from the coproduct to the product.

Proof. See [48, Theorem 1.8]. □

If the triangulated category $\mathcal{T}$ is k-linear, then every compact object is pure injective. Since the duality $D=\mathrm{Hom}_k(-,k):\mathrm{Mod}\,k\to\mathrm{Mod}\,k$ induces a natural map $H\to$

D^2H for each $H \in \operatorname{Mod}\mathcal{T}^c$, and this map is isomorphism for $H = H_X$ for some $X \in \mathcal{T}^c$. By [7, Proposition I.3.8], H_X is an injective object in $\operatorname{Mod}\mathcal{T}^c$. By the above Proposition, X is pure injective in $\mathcal{T}$.

Definition 2.2.7. *A compactly generated triangulated category $\mathcal{T}$ is pure-semisimple if every object of $\mathcal{T}$ is pure injective.*

The following theorem generalize the Bass's characterization of perfect ring, [41].

Theorem 2.2.8. *Let $\mathcal{T}$ be compactly generated triangulated category, the following are equivalent*

(1) $\mathcal{T}$ is pure semisimple;

(2) Every object in $\mathcal{T}$ is a coproduct of indecomposable objects with local endomorphism rings;

(3) The restricted Yoneda functor $h_{\mathcal{T}} : \mathcal{T} \to \operatorname{Mod}\mathcal{T}^c, X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{T}^c}$ is fully faithful;

Proof. See [48, Theorem 2.10]. □

Now, we consider when the category $K(\operatorname{Inj} A)$ for some algebra A is pure semisimple. Consider the restricted Yoneda functor $h_{\mathcal{T}} : \mathcal{T} \to \operatorname{Mod}\mathcal{T}^c, X \mapsto H_X = \operatorname{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c}$, it is not faithful in general. And $h_{\mathcal{T}}$ is faithful implies that $h_{\mathcal{T}}$ is full implies that $h_{\mathcal{T}}$ is dense [12]. If $h_{\mathcal{T}}$ is faithful, we call $\mathcal{T}$ *phantomless*.

Theorem 2.2.9. *For an Artin algebra A, the following are equivalent*

(1) $D(\operatorname{Mod} A)$ is pure semisimple.

(2) A is derived equivalence of algebra of Dynkin type.

Proof. See [12, Theorem 12.20]. □

Theorem 2.2.10. *For an artin algebra A, $K(\operatorname{Inj} A)$ is pure semisimple if and only if A is derived equivalence of algebra of Dynkin type .*

Proof. The inclusion $i : K^b(\operatorname{proj} A) \hookrightarrow D^b(\operatorname{mod} A)$ induces a functor

$$i* : \operatorname{Mod}(D^b(\operatorname{mod} A)) \to \operatorname{Mod}(K^b(\operatorname{proj} A)),$$

which has fully faithful right adjoint functor $i^!$. $K(\operatorname{Inj} A)$ is pure semisimple iff $\operatorname{Mod}(D^b(\operatorname{mod} A))$ is locally Noetherian [12, Theorem 9.3]. By [62, Chapter 5, Corollary 8.4], this implies $\operatorname{Mod}(K^b(\operatorname{proj} A))$ is locally Noetherian. In this case, $D(A)$ is pure semisimple, thus A is derived equivalence of algebra of Dynkin type. □

Remark 2.2.11. *In general, the pure injective object contains a superdecomposable (without indecomposable direct summand) pure injective direct summand, which is difficult to control. So it is hopeless to find out all pure injective objects in this case. But for a pure semisimple triangulated category $\mathcal{T}$, the structure of pure injective objects is explicit by the above theorem.*

The structure of injective object in a Grothendieck category $\mathcal{A}$ is explicit [62]. For each injective object $I \in \mathcal{A}$, there is a unique (up to isomorphism) decomposition of I,

$$I = E(\oplus_{\lambda\in\Lambda} I_\lambda) \oplus I_s$$

where each I_λ is indecomposable injective, ($E(M)$ is the injective envelope of a object M) and I_s is a superdecomposable injective. We have a similar structure property for the pure injective object in a triangulated category.

Proposition 2.2.12. *Let I be a pure injective object in $\mathcal{T}$, then*

$$I \cong E(\oplus_{\lambda\in\Lambda} I_\lambda) \oplus I_s$$

where each I_λ is indecomposable pure injective, and I_s is a superdecomposable pure injective object.

Proof. Consider the embedding $\mathcal{T} \to \operatorname{Mod}\mathcal{T}^c$. If I is a pure injective object in $\mathcal{T}$, then the image H_I is a injective object in $\operatorname{Mod}\mathcal{T}^c$, Proposition 2.2.6. By the fact that $\operatorname{Mod}\mathcal{T}^c$ is Grothendieck category and the structure of the injective object in $\operatorname{Mod}\mathcal{T}^c$. We have the structure of I as

$$I = E(\oplus_{\lambda\in\Lambda} I_\lambda) \oplus I_s.$$

□

The Ziegler spectrum $\operatorname{Zg}\mathcal{T}$ of a compactly generated triangulated category $\mathcal{T}$ have its points as the indecomposable injective objects in $\operatorname{Mod}\mathcal{T}^c$, coinciding with the indecomposable pure injective objects in $\mathcal{T}$ by Proposition 2.2.6. We will give the topological basis of $\operatorname{Zg}\mathcal{T}$, which are determined by the finitely presented object in $\operatorname{Mod}\mathcal{T}^c$.

We can define the open subsets of Ziegler spectrum of $\mathcal{T}$ inducing from the Serre subcategories of $coh\mathcal{T}$ or the Serre subcategories of $\operatorname{mod}\mathcal{T}^c$. There is a dual between these two categories and the open subsets defined by the two types Serre subcategories coincide. For a Serre subcategory $\mathcal{S}$ of $coh\mathcal{T}$, we define the subset of form

$$\mathcal{O}(\mathcal{S}) = \{X \in \operatorname{Zg}\mathcal{T} | C(X) \neq 0, \forall C \in \mathcal{S}\}$$

as the open subset in $\operatorname{Zg}\mathcal{T}$. The sets of this form satisfy the axioms of open subsets in space $\operatorname{Zg}\mathcal{T}$. Similar in the module category case, we can find out a topological basis of $\operatorname{Zg}\mathcal{T}$.

Lemma 2.2.13. *The collection of subsets of* $\operatorname{Zg}\mathcal{T}$

$$\mathcal{O}(C) = \{M \in \operatorname{Zg}\mathcal{T} | C(M) \neq 0, C \in coh\mathcal{T}\}$$

with $C \in coh\mathcal{T}$ *satisfies the axioms of open subsets of* $\operatorname{Zg}\mathcal{T}$.

Proof. See [49, Main theorem]. □

Corollary 2.2.14. *The collection of open subsets of* $\operatorname{Zg}\mathcal{T}$

$$\mathcal{O}(C) = \{M \in \operatorname{Zg}\mathcal{T} | \operatorname{Hom}(C, H_M) \neq 0, C \in \operatorname{mod}\mathcal{T}^c\}$$

with $C \in \operatorname{mod}\mathcal{T}^c$ *is a basis of open subsets of* $\operatorname{Zg}\mathcal{T}$.

Proof. It is obvious that all these sets are open. Each open subset in $\operatorname{Zg}\mathcal{T}$ is of the form

$$\mathcal{O}(\mathcal{S}) = \{N \in \operatorname{Zg}\mathcal{T} | C(N) \neq 0 \text{ for some } C \in \mathcal{S}\}$$

with $\mathcal{S}$ is a Serre subcategory in $coh\mathcal{T}$. Then $\mathcal{O}(\mathcal{S}) = \cup_{C \in \mathcal{S}} \mathcal{O}(C)$. □

A map $\phi : X \to Y$ in $\mathcal{T}$ is called a *pure injective envelope* of X if Y is pure injective and a composition $\psi \cdot \phi$ with a map $\psi : Y \to Z$ is a pure monomorphism if and only if ψ is a pure monomorphism. It is straightforward that ϕ is a pure injective envelope if and only if $H_\phi : H_X \to H_Y$ is an injective envelope in $\operatorname{Mod}\mathcal{T}^c$ [47]. The following property is from [31], analogue the result of module category [39].

Proposition 2.2.15. *If* $\mathcal{T}^c$ *is a Krull-Schmidt category, then the set*

$$\mathcal{S} = \{\hat{M} \textit{ pure injective envelope of } M | M \in \mathcal{T}^c \textit{ is indecomposable.}\}$$

is a dense subset of $\operatorname{Zg}\mathcal{T}$.

Proof. We need to show every open subsets intersecting with $\mathcal{S}$. It suffices to consider the open sets of topological basis $\{\mathcal{O}(C) : C \in \operatorname{mod} T^c\}$. Let $C \in \operatorname{mod}\mathcal{T}^c$, there is a monomorphism $\alpha : C \to H_M$ in $\operatorname{mod}\mathcal{T}^C$ with $M \in \mathcal{T}^c$, since $\operatorname{mod}\mathcal{T}^c$ has enough injective objects. By the Krull-Schmidt property of $\mathcal{T}^c$, $H_M \cong \coprod_{i=1}^n H_{M_i}$ with every H_{M_i} a uniform object in $\operatorname{Mod}\mathcal{T}^c$, then $\operatorname{Hom}(C, H_{M_i}) \neq 0$ for some H_{M_i}. Therefore $\hat{M}_i \in \mathcal{O}(C)$ □

For a finite dimensional k-algebra A, it is well-known that the bounded derived category $D^b(\operatorname{mod} A)$ and the category of perfect compelexes are Krull-Schmidt and Hom-finite. We have the following result about the Ziegler spectrum of $K(\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$.

Corollary 2.2.16. *If A be a finite dimensional k-algebra,*

(1) The injective resolutions of indecomposable objects in $D^b(\operatorname{mod} A)$ consist a dense subset of the Ziegler spectrum $\operatorname{Zg}(K(\operatorname{Inj} A))$.

(2) The indecomposable perfect complexes consist a dense subset of the Ziegler spectrum $\operatorname{Zg}(D(\operatorname{Mod} A))$.

The relation between Ziegler spectrum $\operatorname{Zg} R$ of a ring R and $\operatorname{Zg}(D(\operatorname{Mod} R))$ was investigated in [31]. Consider the inclusion of $\operatorname{Mod} R$ to $D(\operatorname{Mod} R)$, the inclusion induces a map between $\operatorname{Zg} R$ and $\operatorname{Zg}(D(\operatorname{Mod} R))$. We denote the image of $\operatorname{Zg} R$ by $(\operatorname{Zg} R)[n]$. The following theorem gives a description of the relationship between $\operatorname{Zg} R$ and $\operatorname{Zg}(D(\operatorname{Mod} R))$.

Theorem 2.2.17. *Let R be any ring,*

(1) The Ziegler spectrum $(\operatorname{Zg} R)[n]$ of R is a closed subset of $\operatorname{Zg}(D(\operatorname{Mod} R))$.

(2) $\cup_n(\operatorname{Zg} R)[n]$ is a closed subset of $\operatorname{Zg}(D(\operatorname{Mod} R))$. It open complement $\mathcal{O}$ consists of the indecomposable pure-injective complexes having at least two non-zero cohomology groups.

Proof. See [31, Theorem 7.3]. □

In general, the disjoint union $\cup_n(\operatorname{Zg} R)[n]$ is a proper closed subset of $\operatorname{Zg}(D(\operatorname{Mod} R))$. If R is right hereditary or von Neumann ring then the above sets are homeomorphic [31].

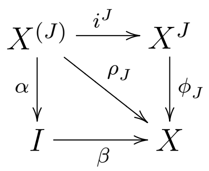
If $gl.\dim R < \infty$, then there is a triangle equivalence $D(\operatorname{Mod} R) \cong K(\operatorname{Inj} R)$. Thus there is a homeomorphic between the Ziegler spectrums.

For a Gorenstein ring R, we denote $G\operatorname{Inj} R$ the full subcategory consisting of Gorenstein injective modules of $\operatorname{Mod} R$ and $G\operatorname{Inj} \underline{R}$ the stable category of $G\operatorname{Inj} R$, i.e $G\operatorname{Inj} R \cap \underline{\operatorname{mod}} R$.

Proposition 2.2.18. *If R is a Gorenstein ring, An R-module $X \in G\operatorname{Inj} R$ is pure injective if and only if X is a pure injective object in $G\operatorname{Inj} \underline{R}$.*

Proof. The "only if" part is easy from the characterization of pure injective module in $\operatorname{Mod} R$ [41] (or the definition of pure injective module). X is pure injective iff for every set J, the summation map $X^{(J)} \to X$ factors through the canonical map $X^{(J)} \to X^J$ from the coproduct to the product.

For the "if" part. X is a pure injective object in $G \operatorname{Inj} \underline{R}$ if and only if the summation map $\rho_J : X^{(J)} \to X$ factors through $i^J : X^{(J)} \to X^J$ in $G \operatorname{Inj} \underline{R}$. There is a diagram



where I is an injective R-module and such that $\rho_J = \phi_J \circ i^J + \beta \circ \alpha$. By the injectivity of I, the map $\alpha : X^{(J)} \to I$ factors through the monomorphism i^J, that is $\alpha = \alpha' \circ i^J$ for $\alpha' : X^J \to I$. Thus

$$\rho_J = \phi_J \circ i_J + \beta \circ \alpha = (\phi_J + \beta \circ \alpha') \circ i^J.$$

We show that ρ_J factors through i^J in $\operatorname{Mod} R$. The proof is finished. □

Theorem 2.2.19. *Assume that R is a Gorenstein ring. Then the triangle equivalence* $K_{ac}(\operatorname{Inj} R) \cong G \operatorname{Inj} \underline{R}$ *induces the Zielger spectrum* $\operatorname{Zg}(G \operatorname{Inj} \underline{R})$ *as a closed subset of* $\operatorname{Zg}(K(\operatorname{Inj} R))$.

Proof. For some $M \in \operatorname{Mod} R$, we denote iM the injective resolution of an R-module M. We have equivalences of functors

$$\operatorname{Hom}_{K(\operatorname{Inj} R)}(iM, -) \cong \operatorname{Hom}_{K(R)}(M, -) \cong H^0(\operatorname{Hom}_R(M, -)).$$

It is obvious that $H_{iR} = \operatorname{Hom}(-, iR) \in \operatorname{mod}(K(\operatorname{Inj} R)^c)$. The subset

$$\mathcal{I}(H_{iR}) = \{X \in \operatorname{Zg} K(\operatorname{Inj} A) | \operatorname{Hom}(H_{iR}, H_X) = \operatorname{Hom}(iR, X) = 0\},$$

is closed in $\operatorname{Zg}(K(\operatorname{Inj} R))$ Moreover, $\mathcal{I}(H_{iR})$ contains all indecomposable acyclic complexes in $K(\operatorname{Inj} R)$. So is the closed subset $\mathcal{I}(H_{iR[i]})$ for all $i \in \mathbb{Z}$, where $iR[i]$ is the i-th shift of complex iR.

By the definition of acyclic complexes, we have that the closed subset $\cap_{i\in\mathbb{Z}}\mathcal{I}(H_{iR[i]})$ only contains the indecomposable acyclic complexes. This implies that the Zielger spectrum of $K_{ac}(\operatorname{Inj} R)$ is a closed subset of $\operatorname{Zg}(K(\operatorname{Inj} R))$.

By the triangle equivalence $K_{ac}(\operatorname{Inj} R) \cong G \operatorname{Inj} \underline{R}$ [51, Proposition 7.2], we have that $\operatorname{Zg}(G \operatorname{Inj} \underline{R})$ is a closed subset of $\operatorname{Zg}(K(\operatorname{Inj} R))$. □

Remark 2.2.20. *If R is (quasi-)Frobenius ring, then $X \in \operatorname{Mod} R$ is pure injective if and only if $X \in \underline{\operatorname{Mod}}\, R$ is pure injective [48]. By the triangle equivalence $\underline{\operatorname{Mod}}\, R \cong K_{ac}(\operatorname{Inj} R)$, and the fact that every pure injective object in $K_{ac}(\operatorname{Inj} R)$ is a pure injective object in $K(\operatorname{Inj} R)$, we have that $\operatorname{Zg}(\underline{\operatorname{Mod}}\, R)$ is a closed subset of $\operatorname{Zg}(K(\operatorname{Inj} R))$. Since the morphism from R to every acyclic complexes is zero in $K(\operatorname{Inj} R)$, the closed subset $\cap_{i\in\mathbb{Z}}\mathcal{I}(H_{R[i]})$ is exactly the set of all acyclic complexes.*

2.3 Generically trivial derived categories

The endolength of a module was introduced by Crawley-Boevey [25]. Modules of finite endolength (endofinite modules) can be used to characterise the representation type of an algebra. In [46], Krause generalized the notion of endofinite modules to endofinite objects in compactly generated triangulated categories. Endofinite objects have a unique decomposition into indecomposable objects with local endomorphism rings. Thus endofinite objects are completely determined by these indecomposable objects. Moreover, the full subcategory of endofinite objects is determined by the full subcategory of compact objects. We will show that endofinite objects play an important role in the structure of triangulated categories.

We mainly concern endofinite objects in the special triangulated categories $K(\operatorname{Inj} A)$ and $D(\operatorname{Mod} A)$, for some finite dimensional k-algebra A .

Definition 2.3.1. *Let $\mathcal{T}$ be a compactly generated triangulated category.An object $E \in \mathcal{T}$ is endofinite if the $\operatorname{End}_{\mathcal{T}} E$-module $\operatorname{Hom}(X, E)$ has finite length for any $X \in \mathcal{T}^c$.*

Example 3. *Let A be a finite dimensional k-algebra. The homotopy category of injective A-modules $K(\operatorname{Inj} A)$ is compactly generated with $K^c(\operatorname{Inj} A) \xrightarrow{\sim} D^b(\operatorname{mod} A)$. We have that any object in $D^b(\operatorname{mod} A)$ is endofinite since the category is Hom-finite.*

Endofinite objects of triangulated categories have some nice decomposition properties.

Theorem 2.3.2. *Let $\mathcal{T}$ be a compactly generated triangulated category. An endofinite object $X \in \mathcal{T}$ has a decomposition $X = \coprod_i X_i$ into indecomposable objects with $\operatorname{End} X_i$ is local, and the decomposition is unique up to isomorphism.*

Proof. See [46, Proposition 1.2]. □

There is a characterization of endofinite modules, which can be found in [25], originally due to Garavaglia. An indecomposable module M has finite endolength if and only if every product of copies of M is isomorphic to a direct sum of copies of M. There is a similar result about endofinite objects in triangulated category [53, Corollary 3.8].

Proposition 2.3.3. *An indecomposable object X is endofinite if and only if every product of copies of X is a coproduct of copies of X.*

For a finite dimensional self-injective algebra A, the stable module category $\underline{\mathrm{Mod}}\, A$ is a compactly generated triangulated category with $(\underline{\mathrm{Mod}}\, A)^c \cong \underline{\mathrm{mod}}\, A$. We know that A is generically trivial if and only if A is of finite representation type. From the characterization of endofinite objects in $\underline{\mathrm{Mod}}\, A$, we will describe the similar result for $\underline{\mathrm{Mod}}\, A$.

Lemma 2.3.4. *[13, Proposition 2.1] Let A be a finite dimensional self injective algebra. The following conditions are equivalent.*

(1) M is an endofinite module in $\mathrm{Mod}\, A$.

(2) M is an endofinite object in $\underline{\mathrm{Mod}}\, A$,

(3) $\underline{\mathrm{Hom}}_A(S, X)$ *is finite length over* $\underline{\mathrm{End}}_A X$ *for every simple A-module S.*

Proof. The implication (2) $\Rightarrow$ (3) is easily from the definition of endofinite object. It suffices to show that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) It follows that $\underline{\mathrm{Hom}}_A(X, M)$ has finite length over $\underline{\mathrm{End}}_A M$ since has finite length over $\mathrm{End}_A\, M$ for every $X \in \mathrm{mod}\, A$.

(3) $\Rightarrow$ (1) Since the direct limit of injective modules is injective, we can apply Zorn's lemma. For every module $M \in \mathrm{Mod}\, A$, there is a maximal injective submodule I of M. Let $M = I \oplus M'$. For any simple module S

$$\underline{\mathrm{Hom}}_A(S, M) = \underline{\mathrm{Hom}}_A(S, M') = \mathrm{Hom}_A(S, M')$$

By hypothesis, this module is finite length over $\mathrm{End}_A\, M$ since it is finite over $\underline{\mathrm{End}}_A M \cong \underline{\mathrm{End}}_A M'$.

Thus $\mathrm{Hom}_A(A, M')$ is finite length over $\mathrm{End}_A\, M'$ since A is finite length over A. That is M' is endofinite module. Since every injective module is endofinite, we have that $M = I \oplus M'$ is endofinite module. ☐

We consider the derived category $D(\mathrm{Mod}\, A)$ of unbounded complexes of A-modules. It is compactly generated with $D^c(\mathrm{Mod}\, A)$ being exactly $K^b(\mathrm{proj}\, A)$. For every $X \in D(\mathrm{Mod}\, A)$ and $i \in \mathbb{Z}$, the i-th cohomology group $H^i(X) = \mathrm{Hom}(A, X[i])$ has a natural $\mathrm{End}\, X$-module structure. We have the following characterization for the endofinite objects in $D(\mathrm{Mod}\, A)$.

Lemma 2.3.5. *Assume that A is a finite dimensional algebra.*

(1) A complex $X \in D(\operatorname{Mod} A)$ is an endofinite object if and only if $H^i(X)$ has finite composition length as an $\operatorname{End} X$-module for every $i \in \mathbb{Z}$.

(2) Let X be an endofinite complex in $D(\operatorname{Mod} A)$. Then $H^i(X)$ is an endofinite A-module for all $i \in \mathbb{Z}$.

Proof. See [50, Lemma 4.1,4.2]. □

The relation between endofinite objects in a triangulated category $\mathcal{T}$ and its localizing subcategory is established by the following result.

Lemma 2.3.6. *Let $\mathcal{S}$ be a localizing subcategory of $\mathcal{T}$ which is generated by compact objects from $\mathcal{T}$, and $q : \mathcal{T} \to \mathcal{S}$ be a right adjoint of the inclusion $i : \mathcal{S} \to \mathcal{T}$.*

(1) $\mathcal{S}$ is a compactly generated triangulated category.

(2) If X is an endofinite object in $\mathcal{T}$, then $q(X)$ is endofinite in $\mathcal{S}$.

Proof. See [46, Lemma 1.3] □

In general, an endofinite object X in $\mathcal{S}$ is not necessary an endofinite object in $\mathcal{T}$.

Now, we consider the fully faithful triangle functor $F : K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\,\hat{A}$ in Theorem 1.5.5 and G the right adjoint of F.

Corollary 2.3.7. *If $X \in \underline{\operatorname{Mod}}\,\hat{A}$ is an endofinite object, then $G(X) \in K(\operatorname{Inj} A)$ is an endofinite object.*

Lemma 2.3.8. *If $X \in K(\operatorname{Inj} A)^c$, then the image $F(X) \in \underline{\operatorname{Mod}}\,\hat{A}$ can be represented by a finitely generated $\hat{A}$-module M.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} K(\operatorname{Inj} A) & \xrightarrow{F} & \underline{\operatorname{Mod}}\,\hat{A}\,, \\ {\scriptstyle i}\uparrow & & \uparrow \\ D^b(\operatorname{mod} A) & \xrightarrow{H} & \underline{\operatorname{mod}}\,\hat{A} \end{array}$$

where i is the injective resolution of complex. The result is a direct consequence by Proposition 1.5.8. □

Remark 2.3.9. *We consider the case that A is derived discrete.*

If A is a hereditary algebra of Dynkin type, then we know that all indecomposable endofinite objects of $D(\operatorname{Mod} A)$ are shifts of the indecomposable A-modules.

If A is a gentle algebra with one cycle not satisfying the clock condition and $gl.\dim A < \infty$, then the indecomposable endofinite objects of $K(\mathrm{Inj}\, A) \cong D(\mathrm{Mod}\, A)$ are the indecomposable endofinite complexes in $D(\mathrm{mod}\, A)$.

2.3.1 Generic objects in derived categories

Generic modules in module category characterize the representation type of an artin algebra. We show that the generic objects in a derived category characterize the behaviour of this derived category.

Definition 2.3.10. *Let $\mathcal{T}$ be a compactly generated triangulated category, an object $E \in \mathcal{T}$ is called generic if it is an indecomposable endofinite object and not compact object. $\mathcal{T}$ is called generically trivial if it does not have any generic objects.*

We give some examples to show the generic objects in derived categories.

Example 4. *Let A be a finite dimensional k-algebra and $D(\mathrm{Mod}\, A)$ be a compactly generated triangulated category. If there exists a generic module $M \in \mathrm{Mod}\, A$, then M viewed as a complex concentrated in one degree, is a generic object in $D(\mathrm{Mod}\, A)$. If $gl.\dim A = \infty$, an indecomposable object X in $D^b(\mathrm{mod}\, A)$ not quasi-isomorphic to a perfect complex is a generic object in $D(\mathrm{Mod}\, A)$.*

From the characterization of endofinite objects in $\underline{\mathrm{Mod}}\, A$ for A a self-injective algebra, we show that when $\underline{\mathrm{Mod}}\, A$ is generically trivial.

Proposition 2.3.11. *Let A be a finite dimensional self-injective algebra. Then $\underline{\mathrm{Mod}}\, A$ is generically trivial if and only if A is of finite representation type.*

Proof. If A is a representation finite self-injective algebra then every module is a direct sum of indecomposable finitely generated A-modules. Thus every A-modules is endofinite and every indecomposable endofinite module is finitely generated. It implies that $\underline{\mathrm{Mod}}\, A$ is generically trivial. Conversely, we assume that $\underline{\mathrm{Mod}}\, A$ is generically trivial. If A is not representation finite, then there exist generic A-module G. By Lemma 2.3.4, we have that G is an generic object in $\underline{\mathrm{Mod}}\, A$. □

We know that there are some similar characterizations between derived discrete algebras and algebras of finite representation type. There generic objects in $D(\mathrm{Mod}\, A)$ for an algebra A with finite global dimension characterize the derived discreteness of A.

Theorem 2.3.12. *Let A be a finite dimensional k-algebra with finite global dimension. Then A is derived discrete if and only if $D(\mathrm{Mod}\, A)$ does not contain a generic object Y such that $F(Y)$ is support-finite .*

Proof. Since A has finite global dimension, there is a triangle equivalence $F : D(\operatorname{Mod} A) \to \underline{\operatorname{Mod}}\,\hat{A}$. The functor F restricted to the subcategory of compact objects is again a triangle equivalence between full subcategories of compact objects.

Suppose that A is not derived discrete, then there is a family of infinitely many indecomposable compact objects $\{X_i\}_{i\in I} \in D(\operatorname{Mod} A)$ with the same homological dimension $d = (d_i)_{i\in\mathbb{Z}}, d_i \in \mathbb{N}^A$. By [33, lemma 3.6], the objects $F(X_i)$ in the family $\{F(X_i)\}_{i\in I}$ are endofinite objects in $\underline{\operatorname{Mod}}\,\hat{A}$. Moreover, the family $\{F(X_i)\}_{i\in I}$ can be expressed as support-finite $\hat{A}$-modules with the same bounded by Proposition 1.5.8 .

Let $\hat{Q}_A$ be the quiver of $\hat{A}$, and $1 = e_1 + \ldots + e_n$ be the decomposition of identity of primitive idempotent of A. Then $\hat{A}$ has primitive idempotent $\mathcal{E}_j(e_i)$ for $j \in \mathbb{Z}$. There are bijection between the vertices of $\hat{Q}_A$ and the set $\{\mathcal{E}_j(e_i)\}$. Let $f_j = \sum_{i=1}^{n} \mathcal{E}_j(e_i)$, and $\hat{A}_{m,n} = (\sum_{j=m}^{n} f_j)\hat{A}(\sum_{j=m}^{n} f_j)$ be a finite dimensional algebra. Thus $F(X_i)$ could be viewed as $\hat{A}_{m,n}$-module $Y = Y^{red}$ for some $Y \in \operatorname{Mod} \hat{A}_{m,n}$, for some $m, n \in \mathbb{Z}$.

There exists a generic module M in $\operatorname{Mod} \hat{A}_{m,n}$, since there are infinitely many indecomposable endofinite $\hat{A}_{m,n}$-modules with the same endolength [24, Theorem 7.3] . We can view M as an $\hat{A}$-module with finite endolength and support-finite. By Corollary 2.3.7, the complex $G(M) \in D(\operatorname{Mod} A)$ is an endofinite object, moreover is generic object, since the functor G is a triangle equivalence. The object $FG(M) \in \underline{\operatorname{Mod}}\,\hat{A}$ is exactly M, and is support-finite.

Conversely, assume that there exists a generic object $Y \in D(\operatorname{Mod} A)$ such that $F(Y)$ is support-finite, we show that A is not derived discrete. For any compact object $C \in D(\operatorname{Mod} A)$, we have $\operatorname{Hom}(C, Y) \cong \underline{\operatorname{Hom}}_{\hat{A}}(FC, FY)$ and $\operatorname{End}(Y) \cong \underline{\operatorname{End}}(FY)$. Thus $F(Y) \in \underline{\operatorname{Mod}}\,\hat{A}$ is a generic object. It corresponds a generic module over $\hat{A}_{m,n}$ for some finite dimensional algebra $\hat{A}_{m,n}$. Therefore, there are infinitely many finitely generated $\hat{A}_{m,n}$-modules has the same endolength with $F(Y)$. This contradicts to the fact that $\hat{A}$ is representation discrete [71]. $\square$

Assume that $A = kQ/I$ is a gentle algebra, where Q has one cycle not satisfying the clock condition and $I = (\rho)$. Choose a generating set ρ of I, by [65, Proposition 4], the bounded quiver $(\hat{Q}, \mathbb{Z}\rho)$ is an expanded gentle quiver. This means that the vertices a are of the following two cases: either a is a crossing vertex: there are exactly two arrows ending in a and two arrows starting in a, or else it is a transition vertex: there just one arrow α ending in a and just one arrow β, starting in a, and $\alpha\beta \notin \rho$. Moreover, if (Q, ρ) is expanded, and p is any path of length at least one in (Q, ρ), then there exactly one arrow α and exactly one arrow β such that $\alpha p \beta$ is a path in (Q, ρ).

Let $\hat{A} = k\hat{Q}/\hat{I}$ be the repetitive algebra of A. There is a string algebra $\bar{\hat{A}} = k(\hat{Q}, \bar{\hat{\rho}})$, where $\bar{\hat{\rho}}$ is the union of $\hat{\rho}$ and the set of all full paths. The algebra $\bar{\hat{A}}$ is the quotient algebra of $\hat{A}$. By the construction of $(\hat{Q}, \bar{\hat{\rho}})$, there is a unique maximal path starting at and ending in each transition vertex. There are precisely two maximal paths starting at and ending in each crossing vertex.

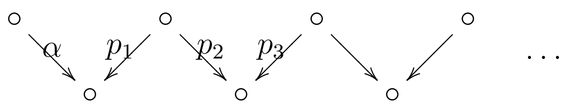
Let $A = kQ/I, I = (\rho)$ be a gentle one cycle algebra not satisfying the clock condition, and $\hat{A} = k\hat{Q}/\hat{I}$ be the repetitive algebra of A. Denote by μ the shift map on $(\hat{Q}, \hat{\rho})$: $a[i] \mapsto a[i+1]$, and $\alpha[i] \mapsto \alpha[i+1]$, where $a \in Q_0$ and $\alpha \in Q_1$. Then μ is an automorphism of $(\hat{Q}, \hat{\rho})$. There is a canonical embedding $\iota : (Q, \rho) \to (\hat{Q}, \hat{\rho})$ given by $a \mapsto a[0]$ and $\alpha \mapsto \alpha[0]$. Denote by $\mathcal{R}$ the finite subset of arrows in $(\hat{Q}, \hat{\rho})$ consisting of all arrows $\alpha[0]$ with $\alpha \in Q_1$ and the connecting arrows starting at $a[0]$. Then every arrow in $\hat{Q}$ is a μ-shift of an element in $\mathcal{R}$.

The following result is known for experts. We give an explicit proof as following for reader's convenience.

Lemma 2.3.13. *If $A = kQ/I$ is a gentle one cycle algebra not satisfying the clock condition, then the quiver $(\hat{Q}, \hat{\rho})$ of the repetitive algebra $\hat{A}$ has a string of infinite length.*

Proof. Let $\hat{A} = k\hat{Q}/\hat{I}$ be the repetitive algebra of A. We know that every vertex in $\hat{Q}_0$ is either a crossing or a transition vertex. Given any arrow $\alpha \in \hat{Q}_1$, we shall find an infinite string starting in $s(\alpha)$.

If $t(\alpha)$ is a crossing vertex, there exists a unique minimal direct path p in $\hat{A}$ with $t(p) = t(\alpha)$ and $s(p)$ crossing. We denote it by p_1. Consider the vertex $s(p_1)$, there is a unique minimal direct path $p_2 \notin (\hat{\rho})$ with $s(p_2) = s(p_1)$. Moreover, $t(p_2)$ is a crossing vertex. Repeating this procedure, we get a sequence $\ldots p_n p_{n-1} \ldots p_2 p_1 \alpha$ with $t(p_{2i}) = t(p_{2i+1}), s(p_{2i}) = s(p_{2i-1})$ for $i \in \mathbb{N}$ and $t(\alpha) = t(p_1)$.



If $t(\alpha)$ is a transition vertex, there exists a unique minimal direct path q such that $s(q) = t(\alpha), p\alpha \notin (\hat{\rho})$ and $t(q)$ is a crossing vertex. Then we get a sequence $\ldots p_n p_{n-1} \ldots p_2 p_1 q\alpha$ with $t(p_{2i}) = t(p_{2i+1}), s(p_{2i}) = s(p_{2i-1})$ for $i \in \mathbb{N}$ and $t(q) = t(p_1)$.

There exist at least two arrows α_1 and α_2 in the above sequences with $\alpha_1 = \gamma[j]$ and $\alpha_2 = \gamma[k]$ where $\gamma \in \mathcal{R}$ since $\mathcal{R}$ is a finite set. If (Q, ρ) does not satisfy the clock condition, then $i \neq k$ for any pair of arrows (α_1, α_2). Otherwise there exists a non-oriented cycle $C = \gamma \ldots \gamma \notin (\hat{\rho})$ [4, Theorem B]. From the finite sequence

$\gamma[i]q'\gamma[k]$ and the automorphism μ of $\hat{Q}$, we get an sequence of infinite length

$$\dots\gamma[2i-k]q'[i-k]\gamma[i]q'\gamma[k]q'[k-i]\gamma[2k-i]\dots.$$

This sequence corresponds to a string of infinite length in $(\hat{Q},\hat{\rho})$. □

The existence of infinite strings of $\hat{A}$ implies the existence of generic objects in $\underline{\mathrm{Mod}}\,\hat{A}$.

Lemma 2.3.14. *If $A = kQ/I$ is a gentle one cycle algebra not satisfying the clock condition, then there is a generic object in $\underline{\mathrm{Mod}}\,\hat{A}$.*

Proof. By Lemma 2.3.13, there exists a string of infinite length in $(\hat{Q},\hat{\rho})$, and the string corresponds to an indecomposable representation of $(\hat{Q},\hat{\rho})$ [45]. Moreover, the corresponding string $\hat{A}$-module M_s is locally finite, i.e we have that

$$\dim \mathrm{Hom}_{\hat{A}}(P, M_s) < \infty,$$

for every indecomposable projective $\hat{A}$-module P. The Hom-space $\mathrm{Hom}(S, M_s)$ is finite dimensional for every simple $\hat{A}$-module S. This implies that for every finite dimensional $\hat{A}$-module N, $\mathrm{Hom}(N, M_s)$ is finite dimensional. Thus M_s is a generic object in $\underline{\mathrm{Mod}}\,\hat{A}$. □

We can prove our main result now.

Theorem 2.3.15. *Let A be a finite dimensional k-algebra.Then $D(\mathrm{Mod}\,A)$ is generically trivial if and only if A is derived equivalent to a hereditary algebra of Dynkin type .*

Proof. Assume there is no generic object in $D(\mathrm{Mod}\,A)$, we show that A is derived equivalent to a hereditary algebra of Dynkin type

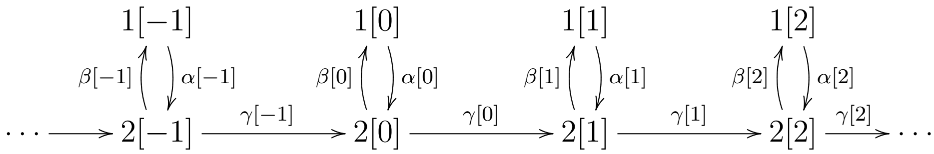
If A has infinite global dimension, then there exits objects in $D^b(\mathrm{mod}\,A)$ but not in $D^c(\mathrm{Mod}\,A) \cong K^b(\mathrm{proj}\,A)$. Since $D^b(\mathrm{mod}\,A)$ is a Hom-finite k-linear triangulated category, there exists an indecomposable object $M \in D^b(\mathrm{mod}\,A)$ not in $K^b(\mathrm{proj}\,A)$ which is a generic object in $D(\mathrm{Mod}\,A)$.

Assume $gl.\dim A < \infty$. If A is not derived discrete, then there is an triangle equivalence $F : D(\mathrm{Mod}\,A) \to \underline{\mathrm{Mod}}\,\hat{A}$. By Theorem 2.3.12, there exists a generic object in $\underline{\mathrm{Mod}}\,\hat{A}$. Generic objects are preserved under triangle equivalences. Thus there exists a generic object in $D(\mathrm{Mod}\,A)$. If A is derived discrete and not derived equivalent to an algebra of Dynkin type, then A is a gentle algebra with exactly one cycle in the quiver Q of A not satisfying the clock condition. By Lemma 2.3.14, there exists a generic object in $\underline{\mathrm{Mod}}\,\hat{A}$, therefore in $D(\mathrm{Mod}\,A)$. Thus A is derived equivalent to a hereditary algebra of Dynkin type. □

Corollary 2.3.16. *Let A be a finite dimensional algebra. Then $D(\mathrm{Mod}\,A)$ is pure semisimple if and only if $D(\mathrm{Mod}\,A)$ is generically trivial.*

Proof. It follows easily from Theorem 2.2.9 and Theorem 2.3.15. □

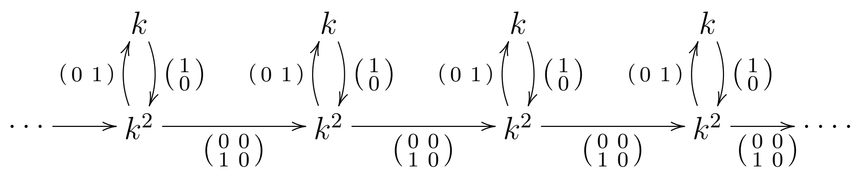
Example 5. *1. Let $A = kQ/(\rho)$ be the path algebra of the gentle bounded quiver (Q, ρ),which Q is the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with relation $\beta\alpha = 0$. It is derived discrete algebra and $gl.\dim A = 2$. The repetitive algebra $\hat{A}$ is the path algebra of locally bounded quiver $(\hat{Q}, \hat{\rho})$, where $\hat{Q}$ is*



with relations $\beta[i]\alpha[i] = 0, \gamma[i]\gamma[i-1] = 0$ and $\gamma[i]\alpha[i]\beta[i] = \alpha[i+1]\beta[i+1]\gamma[i]$ for all $i \in \mathbb{Z}$. There is an infinite string of the form

$$\cdots \beta^{-1}[i+1]\alpha^{-1}[i+1]\gamma[i]\beta^{-1}[i]\alpha^{-1}[i]\gamma[i-1]\cdots .$$

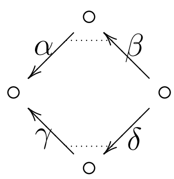
The corresponding representation M of $\hat{Q}$ is



where $M(\alpha[i]) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), M(\beta[i]) = \left(\begin{smallmatrix}0&1\end{smallmatrix}\right)$ and $M(\gamma[i]) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$.

Moreover, M is an endofinite object in $\underline{\mathrm{Mod}}\,\hat{A}$ and is indecomposable. Thus it is a generic object in $\underline{\mathrm{Mod}}\,\hat{A}$. By the triangle equivalence $D(\mathrm{Mod}\,A) \cong \underline{\mathrm{Mod}}\,\hat{A}$, there exists a generic object in $D(\mathrm{Mod}\,A)$.

2. Let $A = kQ/(\rho)$ be the path algebra of the gentle bounded quiver



with $\alpha\beta = 0, \delta\gamma = 0$. A is tilting equivalent to $A' = k\tilde{A}_3$. Thus we have a triangule equivalence $\Phi : D(\mathrm{Mod}\, A') \xrightarrow{\sim} D(\mathrm{Mod}\, A)$. For any generic A'-module $M \in \mathrm{Mod}\, A'$, it is a generic object in $D(\mathrm{Mod}\, A')$ by Lemma 2.3.5. Therefore $\Phi(M)$ is a generic object in $D(\mathrm{Mod}\, A)$.

2.3.2 Locally finite triangulated categories

The locally finite triangulated category was introduced in [72]. These categories are class of triangulated categories satisfying some finiteness conditions.

Definition 2.3.17. *A k-linear triangulated category $\mathcal{C}$ is locally finite if* $\mathrm{supp}\,\mathrm{Hom}(X, -)$ *contains only finitely many indecomposable objects for every indecomposable $X \in \mathcal{C}$, where* $\mathrm{supp}\,\mathrm{Hom}(X, -)$ *denotes the subcategory generated by indecomposable objects Y in $\mathcal{C}$ with* $\mathrm{Hom}(X, Y) \neq 0$.

If $\mathcal{C}$ is a locally finite k-linear triangulated categories, then $\mathcal{C}$ has Auslander-Reiten triangles. The following result characterizes the locally finite triangulated categories.

Theorem 2.3.18. *[72, Theorem 2.3.5] Let $\mathcal{C}$ be a locally finite triangulated categories. Then its Auslander-Reiten quiver $\Gamma_{\mathcal{C}}$ is isomorphic to $\mathbb{Z}\Delta/G$, where Δ is a diagram of Dynkin type and G an automorphism group of $\mathbb{Z}\Delta$.*

There is another characterization of locally finite triangulated categories by representable functors.

Proposition 2.3.19. *[43, Section 2] A k-linear triangulated category $\mathcal{C}$ is locally finite if and only if the functor* $\mathrm{Hom}_{\mathcal{C}}(-, X)$ *is finite length in* $\mathrm{mod}\,\mathcal{C}$ *for every object $X \in \mathcal{C}$.*

From the characterizations of locally finite triangulated categories, we have the following result.

Proposition 2.3.20. *Let A be a finite dimensional k-algebra. Then* $D(\mathrm{Mod}\, A)$ *is generically trivial if and only if* $K^b(\mathrm{proj}\, A)$ *is locally finite.*

Proof. By Theorem 2.3.15, we only need to show the 'if' part. If $K^b(\mathrm{proj}\, A)$ is locally finite, then the Auslander-Reiten quiver Γ of $K^b(\mathrm{proj}\, A)$ is isomorphic to $\mathbb{Z}\Delta/G$, where Δ is a diagram of Dynkin type and G is an automorphism group of $\mathbb{Z}\Delta$ by Theorem 2.3.18. Since $K^b(\mathrm{proj}\, A)$ has infinitely many indecomposable objects, thus G is trivial. It means that there exist algebra $B = k\Delta$ such that $K^b(\mathrm{proj}\, A) \cong K^b(\mathrm{proj}\, B)$. We have that A and B are derived equivalent. Thus $D(\mathrm{Mod}\, A) \cong D(\mathrm{Mod}\, B)$ is generically trivial. $\square$

2.4 Generic objects in $K(\mathrm{Inj}\, A)$

Firstly, we recall the following recollement

$$K_{ac}(\mathrm{Inj}\, A) \underset{I_\lambda}{\overset{I_\rho}{\leftleftarrows}} \!\!\xrightarrow{I}\!\! K(\mathrm{Inj}\, A) \underset{Q_\lambda}{\overset{Q_\rho}{\leftleftarrows}} \!\!\xrightarrow{Q}\!\! D(\mathrm{Mod}\, A)\ .$$

If the global dimension of A is finite then $D(\mathrm{Mod}\, A) \cong K(\mathrm{Inj}\, A)$. In this case, $K(\mathrm{Inj}\, A)$ is generically trivial if and only if $D(\mathrm{Mod}\, A)$ is generically trivial.

If the global dimension of A is infinite, then we can view $D(\mathrm{Mod}\, A)$ as a localizing subcategory of $K(\mathrm{Inj}\, A)$ by the adjoint functors. In this case, if X is an endofinite object in $K(\mathrm{Inj}\, A)$ then the object $Q(X)$ in $D(\mathrm{Mod}\, A)$ is endofinite by Lemma 2.3.6. Conversely, for an endofinite object $Y \in D(\mathrm{Mod}\, A)$, we do not know whether $Q_\rho(Y)$ or $Q_\lambda(\lambda)$ is endofinite.

Lemma 2.4.1. *Assume that $gl.\dim A = \infty$. If the singularity category $D^b_{sg}(A)$ of A is Hom-finite, then there exists generic object in $K(\mathrm{Inj}\, A)$.*

Proof. By the assumption, the category $K^c_{ac}(\mathrm{Inj}\, A)$ is Hom-finite. Given an object $X \in K_{ac}(\mathrm{Inj}\, A)$, we have that

$$\mathrm{Hom}_{K(\mathrm{Inj}\, A)}(C, X) \cong \mathrm{Hom}_{K_{ac}(\mathrm{Inj}\, A)}(I_\lambda C, X), \quad \text{for} \quad C \in K^c(\mathrm{Inj}\, A). \qquad (\dagger)$$

The left adjoint functor I_λ preserves compact objects. It follows that $I_\lambda(C) \in K^c_{ac}(\mathrm{Inj}\, A)$. We choose an indecomposable object $Z \in K^c_{ac}(\mathrm{Inj}\, A)$ which is not compact in $K(\mathrm{Inj}\, A)$. We claim that the object Z is a generic object in $K(\mathrm{Inj}\, A)$. In order to show this, we only need to check that Z is an endofinite object in $K(\mathrm{Inj}\, A)$. It follows from that the isomorphism (†) and $K^c_{ac}(\mathrm{Inj}\, A)$ is Hom-finite.

□

By [61] and [20], the singularity category $D^b_{sg}(\mathrm{mod}\, A)$ of an algebra A is the quotient category

$$D^b_{sg}(\mathrm{mod}\, A) = D^b(\mathrm{mod}\, A)/K^b(\mathrm{proj}\, A).$$

By Proposition 1.4.9, we have an equivalence up to direct factors $\Gamma : D^b_{sg}(\mathrm{mod}\, A) \to K^c_{ac}(\mathrm{Inj}\, A)$, i.e Γ is fully faithful and every object in $K^c_{ac}(\mathrm{Inj}\, A)$ is a direct factor of some objects in the image of Γ. The category $D^b_{sg}(A)$ vanishes if and only if the global dimension of A is finite. The categories $D^b(\mathrm{mod}\, A)$ and $K^b(\mathrm{proj}\, A)$ both are Hom-finite. However, $D^b_{sg}(A)$ is not Hom-finite in general. We know that $D^b_{sg}(A)$ is Hom-finite for Gorenstein algebra A. It follows that $K(\mathrm{Inj}\, A)$ contains generic objects if A is derived discrete and not derived hereditary of Dynkin type.

Lemma 2.4.2. *If A is of infinite representation type, and M is a generic A-module, then the minimal injective resolution iM is a generic object in $K(\operatorname{Inj} A)$.*

Proof. We need to show that $\operatorname{Hom}(C, iM)$ is finite length over $\operatorname{End}_{K(\operatorname{Inj} A)}(iM)$ for all compact objects C in $K(\operatorname{Inj} A)$, where iM is a injective resolution of an A-module M. It suffices to consider the isomorphism

$$\operatorname{Hom}_{K\operatorname{Inj} A}(iS, iM) \cong \operatorname{Hom}_{D(A)}(iS, M) \cong \operatorname{Hom}_{D(A)}(S, M) \cong \operatorname{Hom}_A(S, M),$$

for any simple A-module S.

For a projective resolution of S

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0 \ldots,$$

we obtain the following exact sequence by applying the functor $\operatorname{Hom}_A(-, M)$ to the resolution of S,

$$0 \to \operatorname{Hom}_A(S, M) \to \operatorname{Hom}_A(P_0, M) \to \operatorname{Hom}_A(P_1, M) \to \ldots.$$

Since $\operatorname{Hom}_{D(\operatorname{Mod} A)}(P_0, M)$ is finite length over $\operatorname{End}_{D(\operatorname{Mod} A)} \cong \operatorname{End}_{K(\operatorname{Inj} A)}(iM)$, thus $\operatorname{Hom}_A(S, M) \cong \operatorname{Hom}_{K(\operatorname{Inj} A)}(iS, iM)$ is finite length as $\operatorname{End}_{K(\operatorname{Inj} A)}(iM)$-module.

□

We summarize the above results in the following theorem.

Theorem 2.4.3. *(1) If $gl.\dim A < \infty$, then $K(\operatorname{Inj} A)$ is generically trivial if and only if A is derived hereditary of Dynkin type.*

(2) Assume that $gl.\dim A = \infty$ and A is representation infinite. then $K(\operatorname{Inj} A)$ has generic objects.

(3) Assume that A is Gorenstein and not derived hereditary of Dynkin type, then $K(\operatorname{Inj} A)$ has generic objects.

Chapter 3

The category $K(\operatorname{Inj} A)$ of a derived discrete algebra A

In this chapter, we review some results about the radical square zero algebras and show some derived discrete algebras which are radical square zero. Based on this, we classify all the indecomposable objects in $K(\operatorname{Inj} A)$ for these algebras. The main techniques that we use are the fully faithful triangle functor $K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\, \hat{A}$ and covering technique.

For some symmetric algebra A, we could find out another natural fully faithful triangle functor $K(\operatorname{Mod} A) \to \underline{\operatorname{Mod}}\, \hat{A}$. If $A \cong k[x]/(x^2)$, the embedding induces a functor $K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\, \hat{A}$ which can be applied to determine the indecomposable objects in $K(\operatorname{Inj} A)$. By covering theory, we shall classify all the indecomposable objects in $K(\operatorname{Inj} A)$ for a radical square zero self-injective algebra A which is derived discrete algebra. Moreover, all indecomposable objects are endofinite. Thus we can determine the Ziegler spectrum of $K(\operatorname{Inj} A)$.

3.1 Radical square zero algebras

In [8], there is a connection between an artin algebra A with $\operatorname{Rad}^2(A) = 0$ and a hereditary algebra with radical square zero, where $\operatorname{Rad}(A)$ is the Jacobison radical of A, denoted by J.

Now, assume that A' is any artin algebra, the algebra $A = A'/J'^2$ is a radical square zero algebra. We can associate a hereditary algebra $A^s = \begin{pmatrix} A/J & 0 \\ J & A/J \end{pmatrix}$ with radical square zero to A [8, Chapter X.2]. The radical of A^s is $\operatorname{rad}(A^s) = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}$. Let Q be a quiver with vertex set $Q_0 = 1, 2, \ldots, n$, the *separated quiver* Q^s of Q has $2n$ vertices $1, \ldots, n, 1', \ldots, n'$ and an arrow $l \to m'$ for every arrow $l \to m$ in Q. The ordinary quiver Q^s of A^s coincides with the separated quiver $(Q_A)^s$ of Q_A.

We can study the algebra A via the hereditary algebra A^s. For any $M \in \operatorname{Mod} A$, we define an A^s-module $[\,{}^{M/JM}_{JM}\,]$. There is a functor defined by the following

$$S : \operatorname{Mod} A \to \operatorname{Mod} A^s, \quad M \mapsto [\,{}^{M/JM}_{JM}\,].$$

The functor S has some nice properties , which could be generalized to all modules, see [8, Chapter X] or [41, Proposition 8.63].

Proposition 3.1.1. *Let A,A^s and S as above, then we have the following.*

(1) M and N in $\operatorname{Mod} A$ are isomorphic if and only if $S(M)$ and $S(N)$ are isomorphic in $\operatorname{Mod} A^s$.

(2) M is indecomposable in $\operatorname{Mod} A$ if and only if $S(M)$ is indecomposable in $\operatorname{Mod} A^s$.

(3) M is projective in $\operatorname{Mod} A$ if and only if $S(M)$ is projective in $\operatorname{Mod} A^s$.

Moreover, the functor S induces a stable equivalence,

$$S : \underline{\operatorname{Mod}}\, A \to \underline{\operatorname{Mod}}\, A^s.$$

Let Q be any quiver, and V be a representation of Q, we define the *radical* $\operatorname{Rad} V$ of V to be the subrepresentation of V with $(\operatorname{Rad} V)_i = \sum_{\alpha : j \to i} \operatorname{Im} V_\alpha$, and $\operatorname{Rad}^{n+1} = \operatorname{Rad}(\operatorname{Rad}^n V)$. From this definition, $V/\operatorname{Rad} V$ is semisimple. The *Jacobison radical* $\operatorname{rad} V$ of V is the intersection of all maximal subrepresentations of X,and $\operatorname{rad}^{n+1} = \operatorname{rad}(\operatorname{rad}^n V)$.

In general, $\operatorname{rad} V$ is a subrepresentation of $\operatorname{Rad} V$, but if $\operatorname{Rad}^n(V) = 0$ for some $n \geq 1$, then $\operatorname{rad} V = \operatorname{Rad} V$. If a quiver with relations (Q, ρ) does not infinite length path, then $\operatorname{Rad}^m V = 0$ from some $m \in \mathbb{N}$. The representation V is called *radical square zero* if $\operatorname{Rad}^2 V = 0$. When we consider radical square zero representations of a quiver Q, it is equivalent to consider the module category $\operatorname{Mod} kQ/R_Q^2$, where R_Q is the ideal of kQ generated by all arrows.

There is another version for this correspondence via quiver representations [44]. There are two functors

$$T' : \mathit{Rep}(Q, k) \to \mathit{Rep}(Q^s, k)$$

and

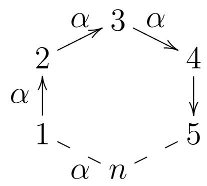
$$T : \mathit{Rep}(Q^s, k) \to \mathit{Rep}(Q, k)$$

which are defined as follows: Given a representation X of Q, let $(T'X)_i = (X/\operatorname{Rad} X)_i$ and $(T'X)_{i'} = (\operatorname{Rad} X/\operatorname{Rad}^2 X)_i$ for each vertex $i \in Q_0$. For each arrow $a : i \to j$ of Q, let $(T'X)_{\bar{\alpha}} : (T'X)_i \to (T'X)_{j'}$ be the map which is induced by X_α. Given a representation Y of Q^s, let $(TY)_i = Y_i \oplus Y_{i'}$ for each vertex $i \in Q_0$. For each arrow $\alpha \in Q_1$, let $(TY)_\alpha = \left(\begin{smallmatrix} 0 & 0 \\ Y_\alpha & 0 \end{smallmatrix}\right)$.

We call a representation X *separated* if $(\operatorname{Rad} X)_i = X_i$ for every sink i.

Proposition 3.1.2. *[44, Proposition 11.2.2] The functors T' and T induce mutually inverse bijections between the isomorphism classes of radical square zero representations of Q and the isomorphism classes of separated representations of Q^s.*

We consider the quiver C_n as following,



with relations $I_n = \alpha^2$. Set $A_n = kC_n/I_n$. The algebra of the form A_n is self-injective and $\operatorname{Rad}^2 A_n = 0$. There is a characterization of these algebras in [8, Chapter IV, Proposition 2.16].

Lemma 3.1.3. *Let A be a basic self-injective algebra which is not semisimple. Then* $\operatorname{Rad}^2 A = 0$ *if and only if $A \cong A_n$ for some n.*

There is a quiver morphism $\pi : (C_n, I_n) \to (C_1, I_1)$ by $\pi(i) = 1, i \in [1, n]$, and $\pi(\alpha) = \alpha$. The algebra A_n is the derived discrete algebra $L(n, n, 0)$ which occurs in Theorem 1.3.7.

Corollary 3.1.4. *For every quiver C_n, I_n as above, the algebra $A_n = kC_n/I_n$ is derived discrete algebra.*

3.2 Indecomposable objects of $K(\operatorname{Inj} k[x]/(x^2))$

Given a finite dimensional k-algebra A, the *trivial extension algebra* of A is the algebra $T(A) = A \ltimes D(A) = \{(a, \psi) | a \in A, \phi \in D(A)\}$, where the multiplication is given as following:

$$(a, f)(b, g) = (ab, fb + ag).$$

$T(A)$ is a $\mathbb{Z}$-graded algebra, namely as vector space we have $T(A) = (A, 0) \oplus (0, D(A))$, and the elements of $A \oplus 0$ (resp.$0 \oplus DA$) is of degree 0 (resp. degree 1).

The trivial extension $T(A)$ of a finite dimensional algebra A is a symmetric algebra. Since there is a symmetric bilinear pairing

$$(-,-) : T(A) \times T(A) \to T(A), \quad ((a,f),(b,g)) \mapsto f(b) + g(a)$$

satisfying associativity.

Remark 3.2.1. *M is a $\mathbb{Z}$-graded $T(A)$-module if and only if $M \cong \oplus_{i \in \mathbb{Z}} M_i$, satisfies $M_i \in \operatorname{Mod} A$, and there exists a homomorphism $f_i : D(A) \otimes_A M_i \to M_{i+1}$ for any $i \in \mathbb{Z}$. Denote $\operatorname{Mod}_{\mathbb{Z}} T(A)$ the category of $\mathbb{Z}$-graded $T(A)$-modules.*

Given any $M = \oplus_i M_i \in \operatorname{Mod}_{\mathbb{Z}} T(A)$, there is a module $M = (M_i, f_i) \in \operatorname{Mod} \hat{A}$. Given any $M = (M_i, f_i) \in \operatorname{Mod} \hat{A}$, it corresponds to the module $M = \oplus_i M_i \in \operatorname{Mod}_{\mathbb{Z}} T(A)$. This correspondence gives an equivalence of these two categories.

Proposition 3.2.2. *Let A be a finite dimensional k-algebra, them there is an equivalence of categories $\operatorname{Mod} \hat{A} \cong \operatorname{Mod}_{\mathbb{Z}} T(A)$.*

We already know the relations between $D^b(\operatorname{mod} A)$ and $\underline{\operatorname{mod}}\, \hat{A}$, which extends the embedding $\operatorname{mod} A \to \underline{\operatorname{mod}}\, \hat{A}$. Now if the algebra A is a symmetric algebra, then we can build some special relations between the complex category of A-modules $C(\operatorname{Mod} A)$ and $\operatorname{Mod} \hat{A}$.

If A is a symmetric algebra, we have that $D(A) \cong A$ as A-A-bimodules. Given a complex in $\operatorname{Mod} A$

$$\cdots \xrightarrow{d_{i-2}} X^{i-1} \xrightarrow{d_{i-1}} X^i \xrightarrow{d_i} X^{i+1} \xrightarrow{d_{i+1}} \cdots,$$

we can naturally view the complex as a $\mathbb{Z}$-graded $T(A)$-module $\oplus_{i \in \mathbb{Z}} X_i$ with the morphism $d_i : A \otimes_A X_i \to X_{i+1}$. Morphisms of complexes correspond to homomorphisms of $\mathbb{Z}$-graded $T(A)$-modules. Therefore, we have an embedding functor $S : C(\operatorname{Mod} A) \to \operatorname{Mod}_{\mathbb{Z}} T(A)$, $(X^i, d^i) \mapsto (X^i, d^i)_i$ and a forgetful functor $U : \operatorname{Mod}_{\mathbb{Z}} T(A) \to C(\operatorname{Mod} A)$. In this case, the functors S and U are equivalences between $C(\operatorname{Mod} A)$ and $\operatorname{Mod}_{\mathbb{Z}} T(A)$. By the equivalence $\operatorname{Mod} \hat{A} \cong \operatorname{Mod}_{\mathbb{Z}} T(A)$, we transform the complexes in $\operatorname{Mod} A$ to the $\hat{A}$-modules.

Proposition 3.2.3. *If A is a symmetric algebra, there is an equivalence of categories $C(\operatorname{Mod} A) \cong \operatorname{Mod} \hat{A}$.*

The following well-known characterization is helpful for us to study the homotopy category $K(\operatorname{Inj} A)$.

Lemma 3.2.4. *Let $\mathcal{A}$ be an additive category. Given complexes $X, Y \in C(\mathcal{A})$ and $f : X \to Y \in C(\mathcal{A})$, let C_f be the mapping cone of f. Then the following are equivalence*

(1) *The map f is null-homotopic.*

(2) *The canonical sequence*

$$0 \longrightarrow Y \longrightarrow C_f \longrightarrow X[1] \longrightarrow 0$$

splits.

(3) *The map f factors through C_{id_Z} for some complex Z.*

Proof. $1 \Rightarrow 2$, we need find out some morphism $p : C_f \to Y$ such that $p(0,1)^t = id_Y$. Since f is null-homotopic, we have $f = sd_X + d_Y s$, where $s = (s^i)$ and $s^i : X^i \to Y^{i-1}$. Let $p = (s, 1) : C_f \to Y$, this is a complexes morphism and $(s, 1)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = 1_Y$.

$2 \Rightarrow 3$, Assume the short exact sequence

$$0 \longrightarrow Y \longrightarrow C_f \longrightarrow X[1] \longrightarrow 0$$

splits. We have the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & C_{id_X} & \longrightarrow & X[1] & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & Y & \underset{\xleftarrow{p}}{\longrightarrow} & C_f & \longrightarrow & X[1] & \longrightarrow & 0 \end{array}$$

We get $f = (g \cdot p) \cdot i_X$.

$3 \Rightarrow 1$, Assume that $f = gh$, where $h = \left(\begin{smallmatrix}h_1\\h_2\end{smallmatrix}\right) : X \to C_{id_Z}$ and $g = (g_1, g_2) : C_{id_Z} \to Y$, then apply the commutativity of complexes maps to get the expression of h_1, g_2. From the relation $f = g_1 h_1 + g_2 h_2$, we have the null-homotopic relation $f = sd_X + d_Y s$. □

For any algebra A, the category $\mathrm{Mod}\,\hat{A}$ is a Frobenius category and the complex category $C(\mathrm{Inj}\,A)$ with the set of all degree-wise split exact sequences in $C(\mathrm{Inj}\,A)$ is also a Frobenius category. All indecomposable projective-injective objects in $C(\mathrm{Inj}\,A)$ are complexes of the form

$$0 \longrightarrow I \xrightarrow{id} I \longrightarrow 0$$

where I is an indecomposable injective A-module. The associated stable categories are $\underline{\mathrm{Mod}}\,\hat{A}$ and $K(\mathrm{Inj}\,A)$ respectively, by Lemma 3.2.4. If A is a symmetric algebra, then $C(\mathrm{Inj}\,A)$ is a full exact subcategory of $\mathrm{Mod}\,\hat{A}$, i.e. $C(\mathrm{Inj}\,A)$ is a full subcategory of $\mathrm{Mod}\,\hat{A}$ and closed under extensions.

Lemma 3.2.5. *Let A be a symmetric algebra, the equivalence $C(\operatorname{Mod} A) \to \alpha_{n-r+1} \operatorname{Mod} \hat{A}$ restrict to the embedding $\Psi : C(\operatorname{Inj} A) \to \operatorname{Mod} \hat{A}$ is an exact functor between. Moreover, the embedding induces a bijection between the indecomposable projective-injective objects in $C(\operatorname{Inj} A)$ and $\operatorname{Mod} \hat{A}$.*

Proof. To show the embedding is an exact functor, it suffices to show $C(\operatorname{Inj} A)$ is a full exact subcategory of $\operatorname{Mod} \hat{A}$. It is obvious that $C(\operatorname{Inj} A)$ is closed under extension and the conflations of $\operatorname{Mod} \hat{A}$ with terms in $C(\operatorname{Inj} A)$ split. Thus the embedding is a fully faithful exact functor.

Now, we consider projective-injective modules in $\operatorname{Mod} \hat{A} \cong \operatorname{Mod}_{\mathbb{Z}} T(A)$. Let $e_1, e_2, \ldots, e_n$ be the primitive idempotents of A, and $1 = \sum_{i=0}^{n} e_i$ be the unit of A. Let $\mathcal{E}_i(e_j)$ be the 'matrix' with (i,i) position is e_j, and the other positions are 0. Then $\{\mathcal{E}_i(e_j)\}_{i\in\mathbb{Z},1\leq j\leq n}$ are all primitive idempotents of $\hat{A}$.

All indecomposable projective-injective $\hat{A}$-modules are of form

$$\hat{P}_i = \hat{A}\mathcal{E}_i(e_j) \cong Ae_j \oplus D(A)e_j$$

with $f = id : D(A) \otimes Ae_j \to D(A)e_j$. Since A is a symmetric algebra, every indecomposable projective-injective module in $\operatorname{Mod} \hat{A}$ is of the form

$$\cdots 0 \sim Ae_j \sim^{1_{Ae_j}} Ae_j \sim 0 \cdots .$$

The indecomposable projective-injective (associated with the exact structure) objects in $C(\operatorname{Inj} A)$ are of form

$$0 \longrightarrow Ae_j \xrightarrow{id} Ae_j \longrightarrow 0 ,$$

for some j. Thus there is a natural bijection induced by the embedding $\Psi : C(\operatorname{Inj} A) \to \operatorname{Mod} \hat{A}$ between the indecomposable projective-injective modules in $\operatorname{Mod} \hat{A}$ and $C(\operatorname{Inj} A)$. $\square$

By the lemma, we have that the exact embedding functor $\Psi : C(\operatorname{Inj} A) \to \operatorname{Mod} \hat{A}$ induces an additive functor $\Phi : K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\, \hat{A}$, moreover, Φ is a triangle functor.

Proposition 3.2.6. *If A is a symmetric algebra, then there is a fully faithful triangle functor $\Phi : K(\operatorname{Inj} A) \to \underline{\operatorname{Mod}}\, \hat{A}$ induced by the exact embedding $\Psi : C(\operatorname{Inj} A) \to \operatorname{Mod} \hat{A}$.*

To proof the proposition, we need the following lemma which guarantees an exact functor between exact categories inducing a triangle functor between the associated stable categories.

Lemma 3.2.7. *[38, Chapter 1,Lemma 2.8] Let $F : \mathcal{B} \to \mathcal{A}$ be an exact functor between Frobenius categories $\mathcal{B}$ and $\mathcal{A}$ such that F transforms injective objects of $\mathcal{B}$ to injectives of $\mathcal{A}$. If there exists an invertible natural transformation $\alpha : FT \to TF$, then $\underline{F}$ is a triangle functor of triangulated categories $\underline{\mathcal{B}}$ and $\underline{\mathcal{A}}$.*

Proof of Proposition 3.2.6. Firstly, the embedding preserves injective objects by Lemma 3.2.5. We only need to show that there exists an invertible natural transformation $\alpha : \Psi\Sigma \to \Sigma\Psi$. For any object $X \in C(\operatorname{Inj} A)$, there is a exact sequence $0 \to X \to I(X) \to \Sigma X \to 0$. By the fact $\Psi(X) \cong X$ and $\Psi(I(X)) \cong I(\Psi(X))$, it is natural that $\Psi\Sigma X \to \Sigma\Psi(X)$. ☐

Remark 3.2.8. *(1) For a symmetric algebra,the functor $K(\operatorname{Mod} A)$ to $\underline{\operatorname{Mod}}\,\hat{A}$ induced by the equivalence $C(\operatorname{Mod} A) \cong \operatorname{Mod} \hat{A}$ is not fully faithful, since for any non-projective simple A-module S, the complex*

$$0 \longrightarrow S \xrightarrow{id} S \longrightarrow 0$$

in $C(\operatorname{Mod} A)$ is null-homotopic, but not a projective-injective module in $\operatorname{Mod} \hat{A}$.

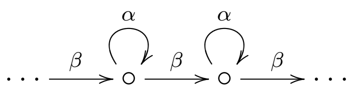
(2) The functor Φ restricted to the full subcategory $K^c(\operatorname{Inj} A)$ of compact objects of $K(\operatorname{Inj} A)$ is also a fully faithful triangle functor. But it is different from the fully faithful functor $D^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\,\hat{A}$ constructed by Happel [38].

Let Q be any quiver, and V be a representation of Q, we define the *radical* $\operatorname{Rad} V$ of V to be the subrepresentation of V with $(\operatorname{Rad} V)_i = \sum_{\alpha:j\to i} \operatorname{Im} V_\alpha$, and $\operatorname{Rad}^{n+1} = \operatorname{Rad}(\operatorname{Rad}^n V)$. From this definition, $V/\operatorname{Rad} V$ is semisimple. The *Jacobison radical* $\operatorname{rad} V$ of V is the intersection of all maximal subrepresentations of X,and $\operatorname{rad}^{n+1} = \operatorname{rad}(\operatorname{rad}^n V)$.

In general, $\operatorname{rad} V$ is a subrepresentation of $\operatorname{Rad} V$, but if $\operatorname{Rad}^n(V) = 0$ for some $n \geq 1$, then $\operatorname{rad} V = \operatorname{Rad} V$. If a bounded quiver (Q, I) does not infinite length path, then $\operatorname{Rad}^m V = 0$ from some $m \in \mathbb{N}$. The representation V is called *radical square zero* if $\operatorname{Rad}^2 V = 0$. When we consider radical square zero representations of a quiver Q, it is equivalent to consider the module category $\operatorname{Mod} kQ/R_Q^2$, where R_Q is the ideal generated by all arrows.

Since the algebra $\Lambda = k[x]/(x^2)$ is a symmetric algebra, the embedding $\Psi : C(\operatorname{Inj} \Lambda) \to \operatorname{Mod} \hat{\Lambda}$ induces fully faithful triangle functor $\Phi : K(\operatorname{Inj} \Lambda) \to \underline{\operatorname{Mod}}\,\hat{\Lambda}$. We will show that the relation between indecomposable objects of $K(\operatorname{Inj} \Lambda)$ and the radical square zero representations of quiver $\hat{Q}$ of $\hat{\Lambda}$, and determine all the indecomposable objects in $K(\operatorname{Inj} \Lambda)$.

The algebra Λ is the path algebra of the quiver $Q = C_1$ with the relation $\alpha^2 = 0$. The quiver of the repetitive algebra $\hat{\Lambda}$ of Λ is $\hat{Q}$



with relations $\alpha^2 = 0 = \beta^2, \alpha\beta = \beta\alpha$

Proposition 3.2.9. *Let $\Lambda = k[x]/(x^2)$, as above, $\hat{Q}$ be the quiver of the repetitive algebra $\hat{\Lambda}$. The image of indecomposable nonzero objects in $K(\operatorname{Inj}\Lambda)$ under Φ can be expressed as radical square zero representations of $\hat{Q}$.*

Proof. The objects X in $K(\operatorname{Inj}\Lambda)$ are of form,

$$\cdots \longrightarrow \Lambda^{m_{-1}} \xrightarrow{d^{-1}} \Lambda^{m_0} \xrightarrow{d^0} \Lambda^{m_1} \xrightarrow{d^1} \Lambda^{m_2} \xrightarrow{d^2} \cdots$$

where $d^i \in \operatorname{Hom}_\Lambda(\Lambda^{m_i}, \Lambda^{m_{i+1}})$ and satisfy $d^{i+1} \cdot d^i = 0$. The differential d^i can be expressed as a $m_{i+1} \times m_i$ matrix (d^i_{jk}) with entries in $\operatorname{Hom}_\Lambda(\Lambda, \Lambda)$ if m_{i+1}, m_i are both finite. We have that $\dim_k \operatorname{Hom}_\Lambda(\Lambda, \Lambda) = 2$, so we can choose a basis $\{1, x\}$ of $\operatorname{Hom}_\Lambda(\Lambda, \Lambda)$.

In particular, if the complex $X \in K(\operatorname{Inj}\Lambda)$ is indecomposable, we can choose that every entry d^i_{jk} of the matrix (d^i_{jk}) associated to d^i is in $\operatorname{Rad}(\Lambda, \Lambda)$. Assume that there is a component of morphism $d^i_{jk} : \Lambda \to \Lambda$ with $d^i_{jk} \notin \operatorname{Rad}(\Lambda, \Lambda)$. Without loss of generality, let $d^0_{jk} = 1_\Lambda$. Consider the following morphisms of complexes in $K(\operatorname{Inj} A)$

$$\begin{array}{cccccccccc}
 & \cdots \longrightarrow & \Lambda^{m_{-1}} & \xrightarrow{d^{-1}} & \Lambda^{m_0} & \xrightarrow{d^0} & \Lambda^{m_1} & \xrightarrow{d^1} & \Lambda^{m_2} & \xrightarrow{d^2} \cdots, \\
 & & g\!: \downarrow 0 & & \downarrow g_0 & & \downarrow g_1 & & \downarrow 0 & \\
Y: & \cdots \longrightarrow & 0 & \longrightarrow & \Lambda & \xrightarrow{1} & \Lambda & \xrightarrow{d^1} & 0 & \longrightarrow \cdots \\
 & & f\!: \downarrow 0 & & \downarrow f_0 & & \downarrow f_1 & & \downarrow 0 & \\
 & \cdots \longrightarrow & \Lambda^{m_{-1}} & \xrightarrow{d^{-1}} & \Lambda^{m_0} & \xrightarrow{d^0} & \Lambda^{m_1} & \xrightarrow{d^1} & \Lambda^{m_2} & \xrightarrow{d^2} \cdots
\end{array}$$

where g_0 is the k-th row of d^0, g_1 is the canonical projection on the j-th component, f_0 is the embedding to the k-th component and f_1 is the j-th column of d^0. We can check that the morphism $fg : X \to X$ is idempotent, and $gf = id_Y$. Thus fg splits in $K(\operatorname{Inj}\Lambda)$ since $K(\operatorname{Inj}\Lambda)$ is idempotent complete. That means the complex X has a direct summand of form Y which is null-homotopic.

We know that Λ as a Λ-module corresponds the quiver representation $k^2 \circlearrowright \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, thus we assign the homomorphism x to the morphism of representations

$$\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right): \quad \overset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\circlearrowright}\; k^2 \;\to\; \overset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\circlearrowright}\; k^2 \; .$$

Under the embedding functor $\Phi : K(\mathrm{Inj}\,\Lambda) \to \underline{\mathrm{Mod}}\,\hat{\Lambda}$ as in Proposition 3.2.6, the complex $0 \longrightarrow \Lambda \xrightarrow{1} \Lambda \longrightarrow 0$ corresponds to the following representation of $\hat{\Lambda}$

$$\overset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\circlearrowright}\; k^2 \xrightarrow{id} \overset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\circlearrowright}\; k^2 \;;$$

which is a projective-injective $\hat{\Lambda}$-module. Let $\bar{\hat{\Lambda}}$ be the factor algebra of $\hat{\Lambda}$ modulo its socle [65]. The algebra $\bar{\hat{\Lambda}}$ has quiver $\hat{Q}$ and with relations $\alpha^2 = 0 = \beta^2$ and $\alpha\beta = 0 = \beta\alpha$. Every indecomposable $\hat{\Lambda}$-module without projective summand could be expressed as an indecomposable $\bar{\hat{\Lambda}}$-module. Thus for any indecomposable complex

$$X' : \cdots \longrightarrow \Lambda^{m_0} \xrightarrow{d^0} \Lambda^{m_1} \xrightarrow{d^1} \Lambda^{m_2} \xrightarrow{d^2} \cdots$$

$\Phi(X') \in \underline{\mathrm{Mod}}\,\hat{\Lambda}$ could be expressed as an indecomposable $\bar{\hat{\Lambda}}$-module. It naturally corresponds to a radical square zero representation of $\hat{Q}$. □

From the quiver $\hat{Q}$ of $\hat{\Lambda}$, we know that the separated quiver $\hat{Q}^s$ is of type A_∞^∞ with the following orientation

$$\cdots \longrightarrow 1' \longleftarrow 1 \longrightarrow 2' \longleftarrow 2 \longrightarrow \cdots .$$

We denote this quiver by A_∞^∞.

The representations of A_∞^∞ are known for experts. For the convenience of readers, we summarize the result in the following proposition.

Proposition 3.2.10. *Any indecomposable representation of A_∞^∞ over an algebraically closed field k is thin. More precisely, all indecomposable representations are of the form $k_{ab}, a, b \in \mathbb{Z} \cup \{+\infty, -\infty\}$, where*

$$k_{ab}(i) = \begin{cases} k & \text{if } a \le i \le b, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$k_{ab}(\alpha) = \begin{cases} id_k & \text{if } a \leq s(\alpha), t(\alpha) \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, we have that any k_{ab} is indecomposable. If $a = b$, obviously k_{aa} is a simple representation and indecomposable, denoted by k_a.

If $a \neq b$, we assume that $a < b$. If $k_{ab} = V \oplus V'$, where V and V' are nonzero. Then $(suppV)_0 \cap (suppV')_0 = \emptyset$, where $suppV$ is a full subquiver of Q with $(suppV)_0 = \{i \in Q_0 | V_i \neq 0\}$. There exists a vertex $i \in (suppV)_0$, and $i+1$ or $i-1 \in (suppV')_0$. We have that $V_i = k, V_{i+1} = 0, V'(i) = 0, V'_{i+1} = k$, and the map $V_i \oplus V'_i \to V_{i+1} \oplus V'_{i+1}$ is zero. But the map $V_i \to V_{i+1}$ is identity. It is a contradiction.

Second, we need to show any indecomposable representation is of form k_{ab}. Suppose we have a indecomposable representation $V = (V_i, f_i)$ of A_∞^∞. If $|(\operatorname{supp} V)_0| = 1$, then V is the direct sum of simple modules. Since V is indecomposable, we have that $V \cong k_a$ for some $a \in \mathbb{Z}$.

If $|(\operatorname{supp} V)_0| > 1$, then $\operatorname{supp} V$ is connected. Suppose V has the following form

$$\cdots \longrightarrow V_{-2} \xleftarrow{f_{-2}} V_{-1} \xrightarrow{f_{-1}} V_0 \xleftarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xleftarrow{f_2} V_3 \longrightarrow \cdots$$

Now if all non-zero f_i are bijections, then V has the form k_{ab} for some $a, b \in \mathbb{Z} \cup \{+\infty, -\infty\}$. Since we have the following isomorphism of representations,

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & V_{-2} & \xleftarrow{f_{-2}} & V_{-1} & \xrightarrow{f_{-1}} & V_0 & \xleftarrow{f_0} & V_1 & \xrightarrow{f_1} & V_2 & \xleftarrow{f_2} & V_3 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_{-2}\circ f_{-1}} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_0\circ f_1^{-1}} & & \downarrow & \\
\cdots \longrightarrow & V_0 & \xleftarrow{1} & V_0 & \xrightarrow{1} & V_0 & \xleftarrow{1} & V_0 & \xrightarrow{1} & V_0 & \xleftarrow{1} & V_0 & \longrightarrow \cdots
\end{array}$$

Thus the lower representation can be decomposed as copies of the corresponding k_{ab}, and V must be isomorphic to k_{ab}.

Actually, all non-zero f_i are bijection. If there exists some f_i which is not an isomorphism. Without loss generality, we assume that f_0 is not injective. In this case, V_1 has a decomposition $V_1 = \operatorname{Im} f_0 \oplus \operatorname{Ker} f_0$ and $\operatorname{Ker} f_0 \neq 0$. We can choose a basis $\{e_i\}_{i \in I}$ of $\operatorname{Ker} f_0$ and $\{e'_j\}_{j \in J}$ of $\operatorname{Im} f_0$ such that $\{e_i\}_{i \in I} \bigcup \{e'_j\}_{j \in J}$ is a basis of V_1 and $f_1(\operatorname{Im} f_0) \bigcap f_1(\operatorname{Ker} f_0) = 0$ (f_1 is not zero, otherwise there is a non-zero direct summand $\operatorname{Ker} f_0$ of V).

Consider the map $f_1 : \operatorname{Im} f_0 \oplus \operatorname{Ker} f_0 \to V_2$. If $\operatorname{Ker} f_1 \neq 0$, we get $\operatorname{Ker} f_0 \bigcap \operatorname{Ker} f_1 = 0$, otherwise the intersection will be a non-zero direct summand of V with support containing only one vertex. We denote $\langle f_1(\operatorname{Ker} f_0) \rangle = V'_2$, the subspace spanned by the vectors in $f_1(\operatorname{Ker} f_0)$ and the complement of V'_2 in V_2 is V''_2. The map f_1 can be expressed as

$$\begin{pmatrix} f_1' & f_{12} \\ 0 & f_1'' \end{pmatrix} : \operatorname{Ker} f_0 \oplus \operatorname{Im} f_0 \to V_2' \oplus V_2''$$

. If $f_{12} = 0$, then we already have a decomposition. If $f_{12} \neq 0$, then we can choose a suitable basis of V_2 such that $f_{12} = 0$. In precisely, If $f_{12}(e_j') = \sum_{i \in I'} f_1(e_i) \in V_2'$, then replace e_j' by $e_j' - \sum_{i \in I'}(e_i)$, and we have $f_{12}(e_j' - \sum_{i \in I'}(e_i)) = 0$. Repeat this procedure, we get a new basis of $\operatorname{Im} f_0$ such that f_1 can be expressed as

$$\begin{pmatrix} f_1' & 0 \\ 0 & f_1'' \end{pmatrix} : \operatorname{Ker} f_0 \oplus \operatorname{Im} f_0 \to V_2' \oplus V_2''$$

We can choose the corresponding basis of V_2 such that $V_2 \cong V_2' \oplus V_2''$, and $f_1(\operatorname{Im} f_0) \cong V_2'$, $\quad f_1(\operatorname{Ker} f_0) \cong V_2''$. Thus we have the following representation isomorphism

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & V_0 & \xleftarrow{f_0} & V_1 & \xrightarrow{f_1} & V_2 & \xleftarrow{f_2} & V_3 & \longrightarrow \cdots \\
& \downarrow {\scriptstyle \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}} & & \downarrow {\scriptstyle \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}} & & \downarrow {\scriptstyle 1} & & \downarrow {\scriptstyle 1} & \\
\cdots \longrightarrow & Imf_0 \oplus \operatorname{Coker} f_0 & \xleftarrow{\begin{pmatrix} f_0 & 0 \\ 0 & 0 \end{pmatrix}} & \operatorname{Im} f_0 \oplus \operatorname{Ker} f_0 & \xrightarrow{\begin{pmatrix} f_1' & 0 \\ 0 & f_1'' \end{pmatrix}} & V_2' \oplus V_2'' & \xleftarrow{f_2} & V_3 & \longrightarrow \cdots
\end{array}$$

.

Now we have a nonzero direct summand of V as follows,

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ker} f_0 \longrightarrow V_2'' \longrightarrow \cdots$$

It is contradict with that V is indecomposable.

Use the same argument we can show that if f_0 is not surjective, then there also is an nonzero direct summand of V.

Given a representation of Q, we can decompose it as direct sum of indecomposable representations by the procedure and each indecomposable representation has endomorphism ring k. By Krull-Schmidt-Azumaya Theorem [1], this decomposition is unique. □

Corollary 3.2.11. *Let $\Lambda = k[x]/(x^2)$, and Q be the quiver of Λ, $\hat{Q}$ be the quiver of $\hat{\Lambda}$. Then every indecomposable object in $K(\operatorname{Inj} \Lambda)$ corresponds to an indecomposable representation of $\hat{Q}^s$.*

Proof. By Proposition 3.2.9, the functor Φ sends every indecomposable object in $K(\operatorname{Inj} A)$ to a radical square zero representation of $\hat{Q}$. From Proposition 3.1.2, there exists a bijection between radical square zero representations of $\hat{Q}$ and separated representations of $\hat{Q}^s = A_\infty^\infty$. □

Let $C^\bullet$ be a complex and $n \in \mathbb{Z}$. We form a new complex

$$\sigma_{\leq n} C^\bullet \cdots \to C^{n-1} \to C^n \to 0 \to \cdots .$$

The canonical inclusion $\sigma_{\leq n} C^\bullet \to C^\bullet$ induces an exact sequence

$$\sigma_{\leq n} C^\bullet \to C^\bullet \to C^\bullet / \sigma_{\leq n} C^\bullet .$$

We denote by $\sigma_{\geq n} = C^\bullet / \sigma_{\leq n-1} C^\bullet$. In convenient, $\sigma_{\leq +\infty} C^\bullet$ and $\sigma_{\geq -\infty} C^\bullet$ represent that we do nothing truncations to the complex $C^\bullet$.

We denote by $I^\bullet_\Lambda$ the following acyclic object in $K(\operatorname{Inj}\Lambda)$

$$\cdots \longrightarrow \Lambda \xrightarrow{x} \cdots \xrightarrow{x} \Lambda \xrightarrow{x} \cdots \xrightarrow{x} \Lambda \xrightarrow{x} \cdots .$$

For any two integers $m, n \in \mathbb{Z}, n \geq m$, we denote the truncation $\sigma_{\leq m} \sigma_{\geq l} I^\bullet_\Lambda$ by $I^\Lambda_{m,n}$. Now we give the main result in this part which describes the indecomposable objects of $K(\operatorname{Inj} A)$.

Proposition 3.2.12. *Let $\Lambda = k[x]/(x^2)$, then every indecomposable object in $K(\operatorname{Inj}\Lambda)$ is of the form $I^\Lambda_{m,n}$.*

Proof. For an indecomposable object $X \in K(\operatorname{Inj}\Lambda)$, we consider the embedding functor $\Phi : K(\operatorname{Inj}\Lambda) \to \underline{\operatorname{Mod}}\,\hat{\Lambda}$ as in Proposition 3.2.6. We have that $\Phi(X)$ is an indecomposable radical square zero $\hat{A}$-module by Proposition 3.2.9. By Corollary 3.2.11, every indecomposable object in $K(\operatorname{Inj}\Lambda)$ corresponds to an indecomposable representation of $\hat{Q}^s = A^\infty_\infty$. From Proposition 3.1.2 and 3.2.10, the dimension of $\operatorname{Hom}_{\hat{\Lambda}}(\hat{\Lambda} e[i], \Phi(X))$ over k is at most 2 , where $\{e[i]\}_{i\in\mathbb{Z}}$ are all primitive idempotents of $\hat{\Lambda}$. Thus we only have choices Λ and 0 for each X^i. X is of the form $I^\Lambda_{m,n}$, since it is indecomposable. □

3.3 The category $K(\operatorname{Inj} A)$ of a radical square zero self-injective algebra A

Covering theory has many applications in representation theory of finite dimensional k-algebras. Now, we use the covering technique to classify all indecomposable objects of $K(\operatorname{Inj} A)$ for radical square zero self-injective algebras A.

Let $\pi : (Q', I') \to (Q, I)$ be the covering of bounded quiver (Q, I). π can be extended to a surjective homomorphism of algebra $\pi : kQ'/I' \to kQ/I$. We set that $A' = kQ'/I'$ and $A = kQ/I$. This induces the push down functor $F_\lambda : \operatorname{Mod} A' \to \operatorname{Mod} A$ [30] .

In general, let G be a group, and $A = \oplus_{g\in G} A_g$ be a G-graded algebra. We define the covering algebra $\tilde{A}$ associated to the G-grading as follows [23, 60, 36]: $\tilde{A}$ is the $G \times G$ matrices $(a_{g,h})$, where $a_{g,h} \in A_{gh^{-1}}$ and all but a finite number of $a_{g,h}$ are 0. Then $\tilde{A}$ is a ring via matrix multiplication and addition. Set $\mathcal{E} = \{e_g\}_{g\in G}$, where e_g is the matrix with 1 in the (g, g)-entry and 0 in all other entries. For more details, we refer to Appendix A.1.

Proposition 3.3.1. *[23, 60] Let G be a group and $A = \oplus_{g\in G} A_g$ is a G-graded algebra. The covering algebra of A associated to the G-grading is $\tilde{A}$. Then $\tilde{A}$ is a locally bounded k-algebra. Moreover, the category of finitely generated graded A-modules* $\operatorname{mod}_G A$ *is equivalent to* $\operatorname{mod} \tilde{A}$.

The forgetful functor $F_\lambda : \operatorname{mod}_G A \to \operatorname{mod} A$ is the functor sending X to X, viewed as an A-module. This functor is exactly the pushdown functor $F_\lambda : \operatorname{mod} \tilde{A} \to \operatorname{mod} A$. The functor F_λ is exact [35, Proposition 2.7] or Appendix A.2. By the exactness of functor F_λ, we have the induced functor $F_\lambda : D^b(\operatorname{mod} \tilde{A}) \to D^b(\operatorname{mod} A)$ between the corresponding derived categories. Let G be a group and $\tilde{A}$ be the covering algebra associated to a G-graded algebra A. Then the forgetful functor $F_\lambda : \operatorname{mod} \tilde{A} \to \operatorname{mod} A$ induces a triangle functor $F_\lambda : D^b(\operatorname{mod} \tilde{A}) \to D^b(\operatorname{mod} A)$ such that the following diagram commutes,

$$\begin{array}{ccc} \operatorname{mod} \tilde{A} & \xrightarrow{can} & D^b(\operatorname{mod} \tilde{A}) \\ \downarrow{\scriptstyle F_\lambda} & & \downarrow{\scriptstyle F_\lambda} \\ \operatorname{mod} A & \xrightarrow{can} & D^b(\operatorname{mod} A) \end{array}$$

where the functor $can : \operatorname{mod} A \to D^b(\operatorname{mod} A)$ is the canonical embedding functor.

Now, assume that the group G is finite, A is a G-graded algebra and the induced action of G on Q_A is free , B is the covering algebra associated to the group G. Then the covering functor $B \to A$ induces a covering functor between the corresponding repetitive algebras $\hat{B} \to \hat{A}$. The forgetful functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{A}$ induces a functor $F_\lambda : \underline{\operatorname{mod}}\, \hat{B} \to \underline{\operatorname{mod}}\, \hat{A}$. Consider the full embedding $\operatorname{mod} \Lambda \to \underline{\operatorname{mod}}\, \hat{\Lambda}$ for any algebra Λ [38], we have following commutative diagram

$$\begin{array}{ccc} \operatorname{mod} B & \longrightarrow & \underline{\operatorname{mod}}\, \hat{B} \\ \downarrow{\scriptstyle F_\lambda} & & \downarrow{\scriptstyle F_\lambda} \\ \operatorname{mod} A & \longrightarrow & \underline{\operatorname{mod}}\, \hat{A} \end{array}$$

Given an algebra A, there is the Happel functor $F^A : D^b(\operatorname{mod} A) \to \underline{\operatorname{mod}}\, \hat{A}$ embedding the derived category to the stable module category of the repetitive algebra.

Lemma 3.3.2. *For the covering of algebras $B \to A$ with group G, the functors F_λ, F^A, F^B as above, we have that*

$$F_\lambda \cdot F^B \cong F^A \cdot F_\lambda. \qquad (a)$$

Proof. For any module $M \in \mathrm{mod}\, B$, we have seen that $F_\lambda \cdot F^B(M) \cong F^A \cdot F_\lambda(M)$. For a bounded complex $X \in D^b(\mathrm{mod}\, B)$, we denote $l(X) = max\{|p-q|, X^p \neq 0 \neq X^q\}$. and proceed by the induction on $l(X)$ for every bounded complex $X \in D^b(\mathrm{mod}\, B)$. For $i > 1$, we assume that for any $X \in D^b(\mathrm{mod}\, B)$ with $l(X) \leq i-1$, there is $F_\lambda \cdot F^B(X) \cong F^A \cdot F_\lambda(X)$. Now we show that $F_\lambda \cdot F^B(X) \cong F^A \cdot F_\lambda(X)$ holds for $l(X) = i$. There is a natural number $n \in \mathbb{N}$ such that $X^n \neq 0$ and $X^m = 0$ for $m > n$. There is a distinguished triangle in $D^b(\mathrm{mod}\, B)$

$$X' \to X^n \to X \to \Sigma X',$$

with $l(X') < i, X^n \in \mathrm{mod}\, B$. By the assumption, (a) holds for all $X \in D^b(\mathrm{mod}\, B)$. □

We summarize the above construction. For an algebra A, Let $F^A : K(\mathrm{Inj}\, A) \hookrightarrow \underline{\mathrm{Mod}}\, \hat{A}$ be the functor constructed in Proposition 1.5.5 and the functor extends the Happel's functor $F^A : D^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}\, \hat{A}$. If $\pi : B \to A$ is the covering of algebra with group G , then there is a covering functor $F_\lambda : \mathrm{Mod}\, B \to \mathrm{Mod}\, A$. In this case, the functor F_λ induces a functor $K(\mathrm{Mod}\, B) \to K(\mathrm{Mod}\, A)$ which preserves injective modules. Thus F_λ induces a functor denoted by $F_\lambda : K(\mathrm{Inj}\, B) \to K(\mathrm{Inj}\, A)$.

Proposition 3.3.3. *Let G be a finite group, A be a G-graded algebra and B be the covering algebra of A associated to G. Let the functors F^A, F^B and F_λ be as above. Then we have that $F^A \cdot F_\lambda \cong F_\lambda \cdot F^B$, i.e the following commutative diagram*

$$\begin{array}{ccc} K(\mathrm{Inj}\, B) & \xrightarrow{F^B} & \underline{\mathrm{Mod}}\, \hat{B} \\ \downarrow{\scriptstyle F_\lambda} & & \downarrow{\scriptstyle F_\lambda} \\ K(\mathrm{Inj}\, A) & \xrightarrow{F^A} & \underline{\mathrm{Mod}}\, \hat{A}. \end{array}$$

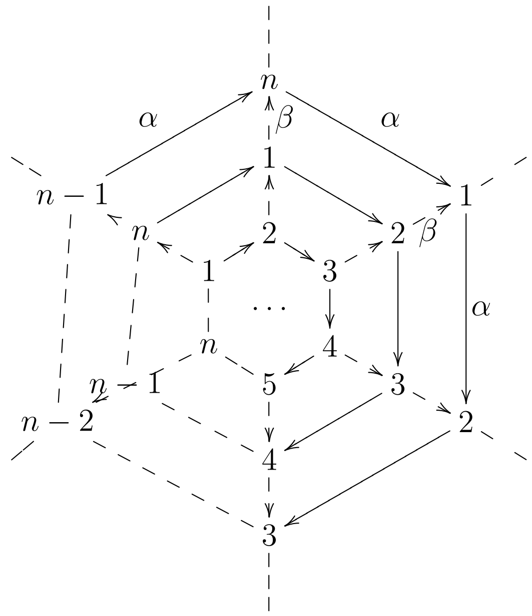
Proof. Let $L_1 = F^A \cdot F_\lambda$ and $L_2 = F_\lambda \cdot F^B$ be two exact functors from $K(\mathrm{Inj}\, B)$ to $\underline{\mathrm{Mod}}\, \hat{A}$.

The given diagram restricted to the subcategory of compact objects is commutative, i.e $L_1(X) \cong L_2(X)$ for all compact object $X \in K^c(\mathrm{Inj}\, B)$, by Lemma 3.3.2. We have that

$$\mathrm{Im}\, L_2|_{K^c(\mathrm{Inj}\, B)} \cong \mathrm{Im}\, L_1|_{K^c(\mathrm{Inj}\, B)} \subset \underline{\mathrm{Mod}}\, \hat{A}.$$

By Lemma 1.4.4, we only need to show L_1 and L_2 preserves coproducts. By the properties of functors F_λ and F, we know that L_1 and L_2 preserves coproducts. □

The repetitive quiver $\hat{C}_n$ of C_n is given by the following infinite quiver



We label all the original arrows in quiver C_n as α and all connecting arrows (i.e dash arrows) as β. The relation $\hat{I}_n$ is generated by all the possible α^2, β^2 and $\alpha\beta = \beta\alpha$. The path algebra of quiver $(\hat{C}_n, \hat{I}_n)$ is the repetitive algebra $\hat{A}_n$ of A_n.

Indecomposable objects in $\underline{\operatorname{Mod}}\, \hat{A}_n$ can be expressed as the indecomposable modules of $\bar{\hat{A}}_n$, which is the factor algebra of $\hat{A}_n$ modulo its socle [65, 28]. More precisely, the algebra $\bar{\hat{A}}_n$ is given by the quiver $\hat{C}_n$ and relations $\alpha^2 = 0 = \beta^2$ and $\alpha\beta = \beta\alpha = 0$, which is radical square zero.

There is a quiver morphism $\pi : (C_n, I_n) \to (C_1, I_1)$ by $\pi(i) = 0, i \in [1, n]$, and $\pi(\alpha) = \alpha$. In order to understand $\hat{A}_n$-modules, we need to study the representations of the quiver $\hat{C}_n$.

Proposition 3.3.4. *Consider the covering of bounded quivers $\pi : (C_n, I_n) \to (C_1, I_1)$, let $(\hat{C}_n, \hat{I}_n)$ and $(\hat{C}_1, \hat{I}_1)$ be the repetitive quivers of (C_n, I_n) and (C_1, I_1) respectively. Then $(\hat{C}_n, \hat{I}_n) \to (\hat{C}_1, \hat{I}_1)$ is a covering of bounded quivers with group $G = \mathbb{Z}_n$.*

Remark 3.3.5. *In general, if there is a covering of bounded quivers $\pi : (Q, \rho) \to (Q', \rho')$ with finite abelian group G, let $(\hat{Q}, \hat{\rho})$ and $(\hat{Q}', \hat{\rho}')$ be the repetitive quivers. Then π induces a covering of bounded quivers $(\hat{Q}, \hat{\rho}) \to (\hat{Q}', \hat{\rho}')$ with group G.*

Proposition 3.3.6. *Let $A_n = kC_n/I_n$ be the algebra as above. Every indecomposable object in $K(\operatorname{Inj} A_n)$ corresponds to an indecomposable module of $\bar{\hat{A}}_n$.*

Proof. The fully faithful embedding $F : K(\mathrm{Inj}\, A_n) \to \underline{\mathrm{Mod}}\, \hat{A}_n$ identifies $K(\mathrm{Inj}\, A_n)$ with a localizing subcategory of $\underline{\mathrm{Mod}}\, \hat{A}_n$. Thus every indecomposable object can be viewed as an indecomposable object in $\underline{\mathrm{Mod}}\, \hat{A}_n$ under the functor F. It is suffice to consider the indecomposable objects in $\underline{\mathrm{Mod}}\, \hat{A}$. We know that two $\hat{A}_n$-modules M, N, are isomorphic in $\underline{\mathrm{Mod}}\, \hat{A}_n$ if and only if there exist projective-injective $\hat{A}_n$-modules P, Q such that $M \oplus P \cong N \oplus Q$. Furthermore, the indecomposable $\bar{\hat{A}}$-modules are just the non- projective indecomposable $\hat{A}$-modules. It follows that indecomposable objects in $\underline{\mathrm{Mod}}\, \hat{A}_n$ corresponds to indecomposable modules of $\bar{\hat{A}}_n$. ☐

Lemma 3.3.7. *Let* $F_\lambda : \mathrm{Mod}\, \hat{A}_n \to \mathrm{Mod}\, \hat{A}_1$ *be the forgetful functor induced by the covering of bounded quivers* $\pi : (C_n, I_n) \to (C_1, I_1)$. *If* X *is an indecomposable module in* $\mathrm{Mod}\, \bar{\hat{A}}_n$, *then the module* $Y = F_\lambda X$ *is indecomposable in* $\mathrm{Mod}\, \bar{\hat{A}}_1$.

Proof. Since $\bar{\hat{A}}_n$ is a radical square zero algebra, therefore there is an bijection between indecomposable modules in $\mathrm{Mod}\, \bar{\hat{A}}_n$ and indecomposable modules in $\mathrm{Mod}\, \bar{\hat{A}}_n^s$, where $\bar{\hat{A}}_n^s$ is the separated algebra of $\bar{\hat{A}}_n$ by Proposition 3.1.1. The separated quiver $\hat{C}_n^s$ of $(\hat{C}_n, \bar{\hat{I}}_n)$ is just the union of n-copies of quiver A_∞^∞. Every indecomposable representation of $\hat{C}_n^s$ is an indecomposable representation of A_∞^∞, which corresponds to an indecomposable module in $\mathrm{Mod}\, \bar{\hat{A}}_1$ by Corollary 3.2.11.

☐

Proposition 3.3.8. *The pushdown functor* $F_\lambda : K(\mathrm{Inj}\, A_n) \to K(\mathrm{Inj}\, A_1)$ *preserves indecomposable objects.*

Proof. By Proposition 3.3.3, we have $F_\lambda F^{A_n} \cong F^{A_1} F_\lambda : K(\mathrm{Inj}\, A_n) \to \underline{\mathrm{Mod}}\, \hat{A}_1$. Assume that $X \in K(\mathrm{Inj}\, A_n)$ is indecomposable and $Y = F_\lambda(X) = Y_1 \oplus Y_2$ is a decomposition of $Y \in K(\mathrm{Inj}\, A_n)$ with $Y_i \neq 0$ for $i = 1, 2$. We have that $F_\lambda F^{A_n}(X) \in \underline{\mathrm{Mod}}\, \hat{A}_1$ is indecomposable by Proposition 3.3.6 and Lemma 3.3.7. On the other hand, $F_\lambda F^{A_n}(X) \cong F^{A_1} F_\lambda(X) = F^{A_1}(Y_1 \oplus Y_2)$ is decomposable in $\underline{\mathrm{Mod}}\, \hat{A}_1$. This is a contradiction. ☐

Let $I_k, k \in [1, n]$ be the indecomposable injective modules of A_n, and $I^\bullet$ be the periodic complex with the left I_1 in the degree 0,

$$\cdots \longrightarrow I_n \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_1 \longrightarrow \cdots$$

where the differentials are the canonical morphism between them.

Define a family of complexes $I_{l,m} = \sigma_{\leq m} \sigma_{\geq l} I^\bullet$, where $l \leq m \in \mathbb{Z} \cup \{\pm\infty\}$, where σ is the truncation functor.

Theorem 3.3.9. *The indecomposable objects in the homotopy category* $K(\operatorname{Inj} A_n)$ *are of form* $I_{l,m}[r]$ *where* $r \in \mathbb{Z}$. *Moreover,* $I_{l,m}[r] = I_{l',m'}[r']$ *if and only if* $l' = l + nk, m' = m + nk$ *and* $r' = r + nk$ *where* $k \in \mathbb{Z}$.

Proof. If $X \in K(\operatorname{Inj} A_n)$ with $X^i = \oplus_k I_k$ for some $i \in \mathbb{Z}$, then $F_\lambda(X) \in K(\operatorname{Inj} A_1)$ with $F_\lambda(X)^i$ being $\oplus_k A_1$. Therefore $F_\lambda(X)$ is decomposable in $K(\operatorname{Inj} A_1)$ by Proposition 3.2.12. From Proposition 3.3.8, this implies that X is decomposable in $K(\operatorname{Inj} A_n)$. Thus $X \in K(\operatorname{Inj} A_n)$ is indecomposable, we have that $F_\lambda(X^i)$ is either A_1 or 0 for any $i \in \mathbb{Z}$. It follows that X^i is either I_k or 0 for some $\leq k \leq n$ and any $i \in \mathbb{Z}$.

Consider homomorphisms between indecomposable injective A_n-modules I_j for $1 \leq j \leq n$, we have that

$$\operatorname{Hom}_{A_1}(I_j, I_k) = \begin{cases} (id_{I_j}) & \text{if } j = k; \\ (d_j) & \text{if } j+1 = k; \text{ or } j = n, k = 1; \\ 0 & \text{otherwise,} \end{cases}$$

where $d_j : I_j \to I_{j-1}$ is the composition of $I_j \to I_j/\operatorname{soc} I_j \cong \operatorname{rad} I_{j-1} \hookrightarrow I_{j-1}$. From this, the periodic complex $I^\bullet$ defined by the following with I_1 in the degree 0,

$$\cdots \longrightarrow I_n \xrightarrow{d_n} I_{n-1} \xrightarrow{d_{n-1}} I_{n-2} \xrightarrow{d_{n-2}} \cdots \longrightarrow I_1 \xrightarrow{d_1} I_n \longrightarrow \cdots$$

and its shifts $I^\bullet[r]$ for $r \in \mathbb{Z}$ are exactly the unbounded indecomposable complexes in $K(\operatorname{Inj} A_n)$. Moreover, $I^\bullet[r] = I^\bullet[r+n]$ for $r \in \mathbb{Z}$.

Assume that $X \in K(\operatorname{Inj} A)$ is indecomposable and bounded below but not above, i.e there exists $l \in Z$ such that $X^p \neq 0$ for all $p \geq l$ and $X^q = 0$ for all $q < l$. With out loss generality, let $X^l = I_1$, we have that $X = \sigma_{\geq l} I^\bullet$. In general, X is of the form $\sigma_{\geq l}(I^\bullet[r])$, where $0 \leq r \leq n-1$, denoted by $I_{l',+\infty}$. Similarly, if $X \in K(\operatorname{Inj} A_n)$ is indecomposable and bounded above but not below, then $X = \sigma_{\leq m}(I^\bullet[r])$ where $0 \leq r \leq n-1$,denoted by $I_{-\infty,m'}$.

Assume that $X \in K(\operatorname{Inj} A_n)$ is indecomposable and bounded. It follows that $X^i \neq 0$ if and only if $l \leq i \leq m$ for some $l, m \in \mathbb{Z}$. Without loss of generality,let $X^l = I - 1$, we have that $X = \sigma_{\leq m}\sigma_{\geq l}(I^\bullet[-l])$. In general, X is of the form $I_{l,m}[r] = \sigma_{\leq m}\sigma_{\geq l}(I^\bullet[r])$, where $0 \leq r \leq n-1$. ☐

3.4 The Ziegler spectrum of $K(\operatorname{Inj} A)$

Let $\mathcal{T}$ be a compactly generated triangulated category, for every coherent functor $C \in coh\mathcal{T}$, the subset $\mathcal{O}(C) = \{M \in \operatorname{Zg} \mathcal{T} | C(M) \neq 0\}$ with $C \in coh\mathcal{T}$ is open in

$\operatorname{Zg}\mathcal{T}$. The corresponding closed subset is $I(C) = \{M \in \operatorname{Zg}\mathcal{T} | C(M) = 0\}$. By the equivalence of $(\operatorname{mod}\mathcal{T}^c)^{op} = coh\mathcal{T}$, the set

$$I(C) = \{M \in \operatorname{Zg}\mathcal{T}| \operatorname{Hom}(C, H_M) = 0\}, \quad C \in \operatorname{mod}\mathcal{T}^c$$

is also a closed set in $\operatorname{Zg}\mathcal{T}$. We apply this to compute the open subsets of some Ziegler spectrum $\operatorname{Zg}\mathcal{T}$. The subsets $\mathcal{O}(C)$ with $C \in coh\mathcal{T}$ form a basis of open subets of $\operatorname{Zg}\mathcal{T}$ by Corollary 2.2.14.

Theorem 3.4.1. *Keep the notions in section 3. Then every indecomposable object $I_{l,m}[r]$of $K(\operatorname{Inj} A_n)$ is an endofinite object.*

Proof. Based on the calculation of Hom-space $\operatorname{Hom}_{K(\operatorname{Inj} A)}(C, I_{l,m}[r])$ for all $C \in K^c(\operatorname{Inj} A)$. For indecomposable objects $I, I' \in K(\operatorname{Inj} A)$, $\dim_k \operatorname{Hom}_{K(\operatorname{Inj} A)}(I', I) \leq 1$ and $\operatorname{Hom}_{K(\operatorname{Inj} A)}(I, I) \cong k$. By the fact that $K^c(\operatorname{Inj} A)$ is Krull-Schmidt k-linear triangulated category, we know that Hom-space $\operatorname{Hom}_{K(\operatorname{Inj} A)}(C, I_{l,m}[r])$ is finite k-dimensional for all $C \in K^c(\operatorname{Inj} A)$. □

Every endofinite object in $K(\operatorname{Inj} A)$ is pure injective, thus the Ziegler spectrum of $K(\operatorname{Inj} A)$ is explicit.

Corollary 3.4.2. *Let $A = kC_n/I_n$ for some $n \in \mathbb{N}^*$. Then the Ziegler spectrum $\operatorname{Zg}(K(\operatorname{Inj} A))$ consists of the point $[I^A_{m,n}[r]]$ for each indecomposable object $I^A_{m,n}[r] \in K(\operatorname{Inj} A)$ up to isomorphism.*

Example 6. *If $\mathcal{T} = K(\operatorname{Inj} A)$, where $A = k[x]/(x^2)$. It is known that $K^c(\operatorname{Inj} A) \cong D^b(\operatorname{mod} A)$. Denote $A_{m,n}, A_{-\infty,n}, A_{m,+\infty}$ be the complexes with A in degree m(resp.$-\infty, m$) to n(resp.$n, +\infty$) and the differential is $d : A \to A, a \mapsto xa$. We know that the non zero morphism between complexes in $K^b(\operatorname{inj} A)$ is the linear combinations of the forms:*

(1)

$$\begin{array}{ccccccccccccc}
0 & \to & \cdots & \to & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \to & 0 \\
 & & & & & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x} & & \\
 & & & & & & 0 & \to & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \to & 0
\end{array}$$

(2)

$$\begin{array}{ccccccccccccc}
0 & \to & \cdots & \to & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \to & 0 \\
 & & & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & & & & & \\
 & & 0 & \to & A & \xrightarrow{x} & \cdots & \xrightarrow{x} & A & \to & 0 & & & &
\end{array}$$

(3)

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & \longrightarrow & A & \xrightarrow{x} \cdots \xrightarrow{x} & A & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & A & \xrightarrow{x} \cdots \xrightarrow{x} & & & A & \xrightarrow{x} \cdots \xrightarrow{x} & A & \longrightarrow & 0
\end{array}$$

(4)

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & A & \xrightarrow{x} \cdots \xrightarrow{x} & A & \longrightarrow & 0 & & \\
 & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x} & & & & \\
0 & \longrightarrow \cdots \longrightarrow & & & A & \xrightarrow{x} \cdots \xrightarrow{x} & A & \xrightarrow{x} \cdots \xrightarrow{x} & A & \xrightarrow{x} & 0
\end{array}$$

By the description of morphisms, every map $A_{m,n} \to A_{-\infty,+\infty}$ is 0 in $\mathcal{T}$. Thus $A_{-\infty,+\infty} \notin \mathcal{O}(A_{m,n})$. The functor $H_{A_{m,n}} = \operatorname{Hom}(-, A_{m,n})|_{\mathcal{T}^c} \in \operatorname{mod} \mathcal{T}^c$, the open subset $\mathcal{O}(H_{A_{m,n}})$ consists of the functors represented by complexes $A_{0,m}, A_{-n,0}, A_{-\infty,0}$ and $A_{0,\infty}$ where $m, n \in \mathbb{N}$. The union of open subsets $\mathcal{O} = \cup_{X \in ind(K^b(\operatorname{inj} A))} \mathcal{O}(H_X)$ is open. We get that the complement of $\mathcal{O}$ has only one point $A_{+\infty,-\infty}$. The point $A_{+\infty,-\infty}$ is not an isolated point in $\operatorname{Zg}(\mathcal{T})$ [31, Proposition 4.5]. Thus it is an accumulation point of the indecomposable objects in $K^b(\operatorname{inj} A)$. It is also the accumulation point of the set $\cup_{i \in \mathbb{Z}} \mathcal{O}(A[i])$, thus is an accumulation point of set $\{A[i] : i \in \mathbb{Z}\}$.

Moreover, the Ziegler spectrum $\operatorname{Zg} \mathcal{T}$ is not quasi-compact. Consider the object $H_{A_{n,+\infty}} \in \operatorname{mod} \mathcal{T}^c$, the open subset

$$\mathcal{O}(H_{A_{n,+\infty}}) = \{A_{m,+\infty}, A_{-\infty,m} (m \geq n), A_{m,r} (m \geq n \text{ or } r \geq n), A_{-\infty,+\infty}\}.$$

The open subsets $\{\mathcal{O}(H_{A_{n,+\infty}})\}_{n \in \mathbb{Z}}$ is an open covering of $\operatorname{Zg} \mathcal{T}$. But it has no finite subcovering of $\operatorname{Zg} \mathcal{T}$.

Appendix A

Covering theory and group graded algebras

A.1 Covering theory

The covering technique for finite dimensional algebras was introduced in [30, 18]. It provides an efficient way to describe indecomposable modules over a finite dimensional algebra. It reduces the study of representation-finite algebras to the study of representation-finite simply connected algebras [18], which are well-understood. At the beginning, the covering technique mainly concerns the representation finite algebra A and the covering of its Auslander-Reiten quiver Γ_A. In general, there are finitely many coverings of Γ_A and each covering is also an Auslander-Reiten quiver of some representation-finite algebra. A triangular algebra is simply connected if and only if it admits no proper Galois coverings [54]. For any representation finite algebra A, indecomposable A-modules can be lifted to indecomposable modules over a simply connected algebra B contained in the Galois covering of A, called the standard form of A [19]. Every finite dimensional algebra has an universal covering [54]. The covering technique for algebras in some sense is grading of algebra [35]. We will use argument in the context.

Definition A.1.1. *A k-linear functor $F : \mathcal{A} \to \mathcal{B}$ between two k-categories $\mathcal{A}, \mathcal{B}$ is called a covering functor if the induced maps*

$$\oplus_{Fy=a} \operatorname{Hom}_{\mathcal{A}}(x, y) \to \operatorname{Hom}_{\mathcal{B}}(Fx, a), \oplus_{Fy=a} \operatorname{Hom}_{\mathcal{A}}(y, x) \to \operatorname{Hom}_{\mathcal{B}}(a, Fx)$$

are bijection for all $x \in \mathcal{A}$ and $a \in \mathcal{B}$.

Definition A.1.2. *A bounded quiver morphism $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ is called a covering morphism if*

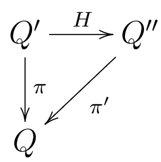
1. *$\pi^{-1}(x) \neq \emptyset, \forall x \in Q_0$.*
2. *For every $x \in Q_0$ and $x' \in \pi^{-1}(x)$, the map induces bijection $x'^{+} \to x^{+}$ and $x'^{-} \to x^{-}$.*
3. *For every $x, y \in Q_0$, every relation $\rho \in I(x, y)$ and every $x' \in \pi^{-1}(x)$, there exists $y' \in \pi^{-1}(y)$ and $\rho' \in \tilde{I}(x', y')$ such that $\pi(\rho') = \rho$.*

The covering morphism $\pi : (Q', I') \to (Q, I)$is called a universal cover of (Q, I) if for any other cover $\pi' : (\bar{Q}, \bar{I}) \to (Q, I)$, there exists a covering morphism $\eta : (Q', I') \to (\bar{Q}, \bar{I})$ satisfying $\pi = \pi'\eta$.

The covering map $\pi : (Q', I') \to (Q, I)$ is called a Galois covering defined by the action of a group G of automorphism of (Q', I') if

1. *$\pi g = \pi, \forall g \in G$.*
2. *$\pi^{-1}(x) = G\pi^{-1}(x)$ and $\pi^{-1}(\alpha) = G\pi^{-1}(\alpha)$ for $x \in Q_0, \alpha \in Q_1$.*
3. *G acts freely on Q', i.e $gx \neq x$ if $g \neq 1$.*

We denote $Cov(Q, \rho)$ the category whose objects are $\pi : (Q', \rho') \to (Q, \rho)$ such that π is morphism of quiver with relations. The morphisms in $Cov(Q, \rho)$ are morphisms $H : \pi \to \pi'$ where $\pi : (Q', \rho') \to (Q, \rho)$ and $\pi' : (Q'', \rho'') \to (Q, \rho)$ satisfying that $H : Q' \to Q''$ is covering such that $\pi = \pi' \cdot H$, i.e the following commutative diagram



For a locally finite quiver (not necessary finite) (Q, ρ) with relations , there exists an unique universal object in $Cov(Q, \rho)$. We show the construction of this universal object for some finite dimensional algebra $A = kQ/I$.

In [54],it was proved that every finite dimensional algebra $A = kQ/I$ has a universal cover, which is a Galois covering. The construction of the covering is given as follows[54]: Let $Q' \to Q$ be the universal cover of Q, constructed as in [18], $\{\rho_v\}$ is a set of zero relations in kQ which generates the ideal I. Lift this set to a set $\{\rho'_v\}$ of zero relations of kQ'. Let I' be the ideal generated by ρ'_v, then we have a covering $\pi : (Q', I') \to (Q, I)$ of bounded quivers. This is a Galois covering with fundamental group $\Pi_1(Q, x_0)$ for some fixed vertex x_0.

If the ideal I is generated by relations containing the form $\rho = \sum_{i=1}^{m} a_i u_i$ with $a_i \in k$ and $u_1, \dots u_m, m \geq 2$ are paths in Q with the same starting point and end point.

The relation $\rho = \sum a_i u_i$ is called *minimal* if $m \geq 2$, and for any proper subset $J \subset 1, \ldots, m$, $\sum_{i \in J} a_i u_i \notin I$. Denote $m(I)$ the set of minimal relations generating the ideal I, and $N(Q, m(I), x_0)$ be the normal subgroup of $\Pi_1(Q, x_0)$ generated by the homotopy classes $wuv^{-1}w^{-1}$, where w is a path from x_0 to $s(u) = s(v)$ and $u, v \in \{u_i\}_{1 \leq i \leq m}$ for a minimal relation $\rho = \sum a_i u_i \in m(I)$. Then

$$\Pi_1(Q, I) = \Pi_1(Q, x_0)/N(Q, m(I), x_0)$$

The quiver Q' is obtained as the orbit quiver $\bar{Q}/N(Q, m(I), x_0)$ for universal quiver $\bar{Q}$. The fundamental group $\Pi_1(Q, I)$ acts on Q', and Q can be viewed as the orbit quiver $\bar{Q}/\Pi_1(Q, I)$. Lift the relations from $m(I)$ of I to kQ', we get a Galois covering $\pi : (Q', I') \to (Q, I)$ of bounded quivers with Galois group $\Pi_1(Q, I)$.

Let A be a finite dimensional k-algebra, there is a quiver Q_A associated to A, and exists a surjective morphism of algebras $\psi : kQ_A \to A$, with kernel $I_\psi = \operatorname{Ker} \psi$. The pair (Q_A, I_ψ) is called a presentation of A. In general, the presentation of A is not unique. Therefore, the fundamental group $\Pi_1(Q_A, I_\psi)$ depends on the presentations ψ.

Now, let $\pi : (Q', I') \to (Q, I)$ be the covering of bounded quiver (Q, I), π can be extended to a surjective algebra homomorphism $\pi : kQ'/I' \to kQ/I$. Consider π as a functor $F : (Q', I') \to (Q, I)$ between locally bounded categories . The functor F induces a functor

$$F_\lambda : \operatorname{mod} kQ'/I' \to \operatorname{mod} kQ/I$$

by $F_\lambda(M)_a = \oplus_{\pi(x)=a} M_x$ for $M \in \operatorname{mod} kQ'/I'$ and $a \in Q_0$. The module structure is given as follows: For an arrow $\alpha : a \to b \in Q_0$, for any $x \in Q'_0$ with $\pi(x) = a$ there is a unique arrow α' of Q' such that $\pi(\alpha') = \alpha$, then $F_\lambda(M)_\alpha : F_\lambda(M)_a \to F_\lambda(M)_b$ given by $(m) \mapsto (M_{\alpha'}(m))$. It is called the *push down functor*.

If the covering π is a Galois covering with group G which is torsion free, Gabriel [30]proved that F_λ is exact, preserves indecomposables and Auslander-Reiten sequence and induces a bijection between the G-orbits of isoclasses of indecomposable kQ'/I'-module and of indecomposable kQ/I-modules.

There is a right adjoint of F_λ, the pull-up functor $F_\bullet : \operatorname{mod} kQ/I \to \operatorname{mod} kQ'/I'$ defined by $F_\bullet M(x) = M(Fx)$ for x either a vertex in Q_0 or an arrow in Q_1.

The translation of modules: Let $M \in rep_k(Q', I')$, the translation $g.M$ is the representation of (Q', I') with $(g.M)_x = M_{g^{-1}x}, (g.M)_\alpha = M_{g^{-1}\alpha}$ for $x \in Q'_0, \alpha \in Q'_1$.

Lemma A.1.3. *[30, Lemma 3.2,Lemma 3.5](1)For each* $M \in \operatorname{Mod} kQ'$ *and each* $g \in G$, *we have* $F_\lambda(M_g) \cong F_\lambda M$ *and* $\coprod_{h \in G} M_h \cong F_\bullet F_\lambda M$ *canonically.*

(2)Let $M \in \operatorname{Mod} kQ'$ *be indecomposable finite dimensional and* $M_g \neq M$ *if* $1 \neq g \in G$. *Then* $F_\lambda M$ *is indecomposable. Moreover, each* $L \in \operatorname{Mod} kQ'$ *such that*

$F_\lambda L \cong F_\lambda M$ is isomorphic to some $M_g, g \in G$.

A.2 Group graded algebras

Let R be a commutative Artin ring and let $\mathcal{I}$ be an index set. For any $i, j \in \mathcal{I}$, we have that K_i is an Artin R-algebra and M_{ij} is an K_i-K_i-bimodule, which is finite length over R with $M_{ii} = K_i$. For $i, j, l \in \mathcal{I}$, there is a K_i-K_l-bimodule homomorphism $\mu_{ijl} : M_{ij} \otimes_{K_j} M_{jl} \to M_{il}$ satisfies the associativity:

$$\mu_{ijk} \otimes 1_{M_{kl}} \mu_{ikl} = (1_{M_{ij}} \otimes \mu_{jkl})\mu_{ijl} \qquad \text{for} \quad i, j, k, l \in \mathcal{I}.$$

We suppose that for each $i \in \mathcal{I}$, the bimodule $M_{ij} = 0$ for all but finitely many $j \in \mathcal{I}$ and for each $j \in \mathcal{I}$, the bimodule $M_{ij} = 0$ for all but finitely many $i \in \mathcal{I}$. We can view Γ be the set of all $\mathcal{I} \times \mathcal{I}$ matrices, (m_{ij}), such that $m_{ij} \in M_{ij}$ for $i, j \in \mathcal{I}$, and all but a finite number of $m_{ij} = 0$. The matrix addition and multiplication defined via μ_{ijk} induce an R-algebra structure on Γ. For $i \in \mathcal{I}$, let e_i be in Γ having $1 \in K_i$ in (i, i)-entry and 0 all other entries and denote $\mathcal{E} = \{e_i\}_{i \in \mathcal{I}}$. The elements in $\mathcal{E}$ are all primary orthogonal idempotents in Γ. The algebra Γ has enough orthogonal idempotents since $\Gamma = \oplus_{i \in \mathcal{I}} \Gamma e_i = \oplus_{i \in \mathcal{I}} e_i \Gamma$. The pair $(\Gamma, \mathcal{E})$ is called a *locally finite R-algebra with respect to $\mathcal{I}$*. If $\mathcal{I}$ is a finite set, then Γ is an Artin R-algebra. Let $K = \prod_{i \in \mathcal{I}} K_i$ and $M = \coprod_{i,j \in \mathcal{I}} M_{ij}$. We have that M is a R-bimodule. The tensor algebra

$$T_R(M) = R \oplus M \oplus M \otimes_R M \oplus M \otimes_R M \otimes_R M \oplus \dots$$

is isomorphic to the algebra Γ.

A k-algebra $A = \coprod_{g \in G} A_g$ is a G-grading algebra if for all $g, h \in G$, we have $A_G A_h \subset A_{gh}$. We denote $\mathrm{Mod}_G\, A$ ($\mathrm{mod}_G\, A$) the category of all (finite dimensional) G-graded modules. Now assume that A is a G-graded k-algebra. We construct a covering algebra A' associated to the G-grading. Let A_G be the set of $G \times G$-matrices $(x_{g,h})_{g,h \in G}$, with entries in A which satisfies the following conditions:

(C1) $x_{g,h} \in A_{gh^{-1}}$.

(C2) $x_{g,h} = 0$ for all but a finite number of $g, h \in G$.

The set A_G has a ring structure induced by the matrix addition and multiplication. If G is a finite group, then A_G has an identity element, the matrix $(x_{g,h})$ where $x_{g,g} = 1_A$ and $x_{g,h} = 0$ if $g \neq h$. The algebra A_G is called a *covering algebra associated to the graded algebra A*.

Remark A.2.1. *The covering algebra A_G is just the smash product $A\sharp kG^*$ of the algebras A and kG^*, where kG^8 is the dual of the group algebra kG, [23]. Let* $\mathrm{Mod}(A\sharp kG^*)$ *be the category of all unital $A\sharp kG^*$-modules and their homomorphisms. By [23, Theorem 2.2], there is an equivalence of categories*

$$\mathrm{Mod}(A\sharp kG^*) \to \mathrm{Mod}_G A.$$

Therefore, we have the same equivalence relation between categories $\mathrm{Mod}\, A_G$ *and* $\mathrm{Mod}_G A$.

Let $A = kQ/I$ be the path algebra of quiver Q modulo the ideal I. Let G be a group and $f : Q_1 \to G$ be a map of sets. The map f is called *weight function.* If $p = \alpha_n \dots \alpha_1$ is a path in Q with $\alpha_i \in Q_1$ then the weight of p is $f(\alpha_n)\dots f(\alpha_2)f(\alpha_1) \in G$. We set the weight of a vertex to be 1_G. If $x = \sum_{i=1}^{n} c_i p_i$, with c_i non-zero in k and p_i a path, then x is called *weight homogenous* if $f(p_1) = f(p_i)$ for all i. The G-grading on kQ induces a G-grading on A if and only if I can be generated by weight homogenous elements.

Now suppose that the ideal I can be generated by weight homogenous elements. We define a new quiver Q_f by $(Q_f)_0 = Q_0 \times G$ and $(Q_f)_1 = Q_1 \times G$. We denote the vertices of Q_f by v_g if $v \in Q_0$ and $g \in G$, and the arrow α_g of Q_f for $\alpha \in Q_1$ and $g \in G$. The arrows in Q_f are defined as follows: if $\alpha : v \to w$ is an arrow in Q_1 and $g \in G$ then $\alpha_g : v_g \to w_g$. There exists an ideal I_f in kQ_f such that the algebra $A_G = kQ_f/I_f$ is the covering algebra of A associated to the group G.

Now we recover the push down functor and pull up functor from the equivalence $\mathrm{Mod}_G A \xrightarrow{\sim} \mathrm{Mod}\, A_G$. The push down functor F_λ is the composition of functors $\mathrm{Mod}\, A_G \xrightarrow{\sim} \mathrm{Mod}_G A \xrightarrow{F} \mathrm{Mod}\, A$, where F is the forgetful functor. The pull up functor $F_\bullet : \mathrm{Mod}\, A \to \mathrm{Mod}\, A_G$ is just the composition of functors $\mathrm{Mod}\, A \xrightarrow{\Psi} \mathrm{Mod}_G A \xrightarrow{\sim} \mathrm{Mod}\, A_G$. The functor $\Psi : \mathrm{Mod}\, A \to \mathrm{Mod}_G A$ is defined as follows. If $M \in ModA$ Then $\Psi(M) = X = X_g$, where for each $g \in G$, $X_g = M$; If $x = m_g \in X$ and $a \in A_h$, then $r \cdot x = rm_{h^{-1}g}$. Moreover, the functor Ψ is right adjoint functor of F. Applying the equivalence $\mathrm{Mod}_G A \xrightarrow{\sim} \mathrm{Mod}\, A_G$. We have that the functor $F_\bullet$ is right adjoint of F_λ.

Proposition A.2.2. *[35, Theorem 3.2] Let $\pi : (Q', \rho') \to (Q, \rho)$ be an object in $Cov(Q, \rho)$. Let G be group associate this covering. Then there is a weight function $f : Q_1 \to G$ such that $A = kQ/(\rho)$ can be given the G-grading induced by f. Under this grading, there is a natural equivalence of categories between* $\mathrm{mod}_G A$ *and*

$rep(Q', \rho')$ such that the following diagram commutes:

$$\begin{array}{ccc} rep(Q',\rho') & \xrightarrow{\sim} & \operatorname{mod}_G A \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ rep(Q,\rho) & \xrightarrow{\sim} & \operatorname{mod} A. \end{array}$$

Appendix B

Triangulated categories

B.1 Triangulated categories

Let $\mathcal{A}$ be an additive category, a *cochain complex* X over $\mathcal{A}$ is a family (X^n, d_X^n) of objects $X^n \in \mathcal{A}$ and morphisms $d_X^n : X^n \to X^{n+1}$ such that $d_{n+1} \circ d_n = 0, \forall n \in \mathbb{Z}$. A *morphism of cochain complexes* $f : X \to Y$ is a family of morphisms $(f^n : X^n \to Y^n)_n$ satisfying $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$, i.e the following diagram commutes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

We denote $C(\mathcal{A})$ the category of cochain complexes. If $\mathcal{A}$ is an additive (abelian) category then $C(\mathcal{A})$ is an additive (abelian) category.

Morphisms $f, g : X \to Y$ in the category $C(\mathcal{A})$ are called *homotopic* if there exist a family $(s^n)_{n\in\mathbb{Z}}$ of morphism $s_n : X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}s^n + s^{n+1}d_X^n$ for all $n \in \mathbb{Z}$. If $g = 0$, we call f is null-homotopic. For each pair $X, Y \in C(\mathcal{A})$, we denote $Htp(X, Y)$ a subset of $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$ consisting of null-homotopic morphisms. It is a additive subgroup of $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$. we define the *homotopy category* $K(\mathcal{A})$ of $\mathcal{A}$ which has the same objects with the category $C(\mathcal{A})$, for each pair of objects $X, Y \in K(\mathcal{A})$, the homomorphism

$$\mathrm{Hom}_{K(\mathcal{A})(X,Y)} := \mathrm{Hom}_{C(\mathcal{A})}(X, Y)/Htp(X, Y).$$

Definition B.1.1. *Given an additive category $\mathcal{T}$, together with an autofunctor $\Sigma : \mathcal{T} \to \mathcal{T}$, we define a sextuple (X, Y, Z, u, v, w) given by objects $X, Y, Z \in \mathcal{T}$ and morphisms u, v, w between them ,i.e a sequence $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, a morphism* *of sextuples from* (X, Y, Z, u, v, w) *to* (X', Y', Z', u', v', w') *is a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

If the morphisms f, g, h *are isomorphism in* $\mathcal{T}$*, then the morphism is called isomorphism.*

Definition B.1.2. *A triangulated category is an additive category* $\mathcal{T}$ *with autofunctor as above, and a collection of sextuples which are called distinguished triangles, satisfying axioms TR0-TR4,*

TR0 *Any sextuple isomorphic to a distinguished triangle is a distinguished triangle. The sextuple* $(X, X, 0, 1_X, 0, 0)$ *is a distinguished triangle.*

TR1 *For any morphism* $f : X \to Y$ *in* $\mathcal{T}$*, there exist a distinguished triangle* $X \xrightarrow{f} Y \to Z \to \Sigma X$.

TR2 *If* (X, Y, Z, u, v, w) *is a distinguished triangle, then* $(Y, Z, \Sigma X, v, w, -\Sigma u)$ *is a distinguished triangle.*

TR3 *For any commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$ *which make the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

commutative.

TR4 *For any two morphisms* $u : X \to Y, v : Y \to Z$*, We embed the morphism* u, v *and* vu *to the distinguished triangles* $(X, Y, Z', u, i, i'), (Y, Z, X', v, j, j')$ *and*

(X, Z, Y', vu, k, k') *respectively. There exists a commutative diagram*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & \Sigma X \\
\downarrow{=} & & \downarrow{v} & & \downarrow{f} & & \downarrow{=} \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & \Sigma X \\
 & & \downarrow{j} & & \downarrow{g} & & \\
 & & X' & \xrightarrow{=} & X' & & \\
 & & \downarrow{j'} & & \downarrow{\Sigma i j'} & & \\
 & & \Sigma Y & \xrightarrow{\Sigma i} & \Sigma Z' & &
\end{array}$$

where rows and columns are distinguished triangles.

In the category of complexes $C(\mathcal{A})$, a complex X is called bounded below if $X^n = 0$ for $n \ll 0$, bounced above of $X^n = 0$ for $n \gg 0$, and bounded if $X^{=}0$ for $n \gg 0$ and $n \ll 0$. We denote $C^-(\mathcal{A}), C^+(\mathcal{A})$ and $C^b(\mathcal{A})$ by the full subcategory of $C(\mathcal{A})$ consisting of bounded below, bounded above and bounded complexes respectively. Similar, we can define corresponding homotopy categories $K^-(\mathcal{A}), K^+(\mathcal{A})$ and $K^b(\mathcal{A})$.

Let $X \in C(\mathcal{A})$ be a complex, the n-th cohomology of X is defined by $H_n(X) := \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1}$. The complex X is called *acyclic* if $H^n(X) = 0$ for all $n \in \mathbb{Z}$. A morphism of complexes $f : X \to Y$ is called *quasi-isomorphism* if it induces isomorphisms in cohomology i.e $H^n(f) : H^n(X) \to H^n(Y)$ are isomorphism for all $n \in \mathbb{Z}$.

Let $K_{ac}(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ consisting of acyclic complexes.

Definition B.1.3. *Let $\mathcal{A}$ be an abelian categories, for $* = -, +, b$ or nothing, we define the corresponding derived category of $\mathcal{A}$ is the Verdier quotient*

$$D^*(\mathcal{A}) = K^*(\mathcal{A})/(K_{ac}(\mathcal{A}) \cap K^*(\mathcal{A})).$$

Theorem B.1.4. *Let $\mathcal{A}$ be an additive categories, for $* = -, +, b$ or nothing, then $K^*(\mathcal{A})$ are triangulated categories. Moreover, if $\mathcal{A}$ is an abelian category, then $D^*(\mathcal{A})$ are triangulated categories.*

Proof. See [69, Theorem 1.2.3] □

Example 7. *If A is a finite dimensional algebra, the derived category $D^b(\operatorname{mod} A)$ of finite dimensional modules over A is triangulated category. The unbounded derived category $D(\operatorname{Mod} A)$ of all A-modules is triangulated category. Let* $\operatorname{Inj} A$ *be the full subcategory of* $\operatorname{Mod} A$ *consisting of all injective modules, The homotopy* *category $K(\mathrm{Inj}\, A)$ is a triangulated category. We are interested in these triangulated categories for any finite dimensional algebra.*

B.2 Stable categories of Frobenius categories

In order to construct a triangulated structure over the module categories of self injective algebras, we consider the exact category.

A pair morphism $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z$ in an additive category $\mathcal{A}$ is called *exact* if α is a kernel of β and β is a cokernel of α. An *exact category* is an additive category $\mathcal{A}$ endowed with a class $\mathcal{E}$ of exact pair (α, β) closed under extension and satisfying some axioms, see [63] or [42]. We can view the exact category $\mathcal{A}$ as an extension closed full additive category of an abelian category $\mathcal{C}$. The class $\mathcal{E}$ is just the class of triples in $\mathcal{A}$ that are exact in $\mathcal{C}$. A morphism $\alpha : X \to Y$ in $\mathcal{A}$ is called *admissible monomorphism* if there exists an exact sequence $0 \to X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \to 0$ in $\mathcal{E}$. A morphism $\beta : Y \to Z$ in $\mathcal{A}$ is called *admissible epimorphism* if there exists an exact sequence $0 \to X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \to 0$ in $\mathcal{E}$. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is *extension-closed* if the midterm of each exact sequence of $\mathcal{A}$ with end terms in $\mathcal{B}$ belongs to $\mathcal{B}$.

A *full exact subcategory* of an exact category $\mathcal{A}$ is a full additive subcategory $\mathcal{B} \subset \mathcal{A}$ which is extension-closed. If $\mathcal{A}$ and $\mathcal{B}$ are exact categories, an additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *exact functor* if F preserves exact sequences.

An object I in $\mathcal{A}$ is $\mathcal{E}$*-injective* (resp. $\mathcal{E}$*-projective*) if the induced sequence

$$\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I) \text{ (resp. } \mathrm{Hom}_{\mathcal{A}}(I, Y) \to \mathrm{Hom}_{\mathcal{A}}(I, Z))$$

is injective (resp. surjective) for every exact sequence in $\mathcal{E}$. The category $\mathcal{A}$ has *enough injective* if every object X admits an admissible monomorphism $X \to I$ with I $\mathcal{E}$-injective. The category $\mathcal{A}$ has *enough projective* if every object Z admits an admissible monomorphism $Y \to Z$ with Y $\mathcal{E}$-projective. If an exact category $\mathcal{A}$ has enough projectives and injectives and the classes of projectives and injectives coincide, then $\mathcal{A}$ is called a *Frobenius category*.

Let $\mathcal{A}$ be a Frobenius category. For objects $X, Y \in \mathcal{A}$, denote by $I(X, Y)$ the set of all morphisms $f : X \to Y$ in $\mathcal{B}$ that can be factored through a $\mathcal{E}$-injective object. Define the corresponding *stable category* $\underline{\mathcal{A}}$ by setting $Ob\underline{\mathcal{A}} = Ob\mathcal{A}, \mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)/I(X, Y)$. The category $\underline{\mathcal{A}}$ is an additive category.

Now we define the translation functor $\Sigma : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ which is an automorphism of $\underline{\mathcal{A}}$. The category $\underline{\mathcal{A}}$ with the functor Σ is a triangulated category. Let

$$0 \longrightarrow X \longrightarrow I' \longrightarrow X' \longrightarrow 0 \qquad (*)$$

and

$$0 \longrightarrow X \longrightarrow I'' \longrightarrow X'' \longrightarrow 0$$

be in $\mathcal{E}$ such that I', I'' are $\mathcal{E}$-injective. We have that X' and X'' are isomorphic in $\underline{\mathcal{A}}$. Given any object $X \in \mathcal{A}$, $[X]$ denotes the isomorphism class of X in $\underline{\mathcal{A}}$. For a sequence $(*)$ with I $\mathcal{E}$-injective, there is a bijection $\gamma_X : [X] \to [X']$ for any $X \in \mathcal{A}$. We note that this bijection does not depend on the choice of the sequence $(*)$. For all $X \in \mathcal{A}$, we choose elements

$$0 \longrightarrow X \xrightarrow{i_X} I_X \xrightarrow{\pi_X} \Sigma X \longrightarrow 0$$

in $\mathcal{E}$, where I_X is an injective object. For any morphism $f : X \to Y$ in $\mathcal{A}$, there is a induced morphism $\Sigma(f) : \Sigma X \to \Sigma Y$. We can check that Σ is a functor from $\underline{\mathcal{A}}$ to $\underline{\mathcal{A}}$. The standard triangle $X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X$ in $\underline{\mathcal{A}}$ is defined by the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & I_X & \xrightarrow{\pi_X} & \Sigma X & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & Y & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma X & \longrightarrow & 0 \end{array}$$

Any triangle isomorphic to a standard one is called a distinguished triangle. Given a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the induced triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle in $\underline{\mathcal{A}}$ and every distinguished triangle in $\underline{\mathcal{A}}$ is obtained in this way.

Theorem B.2.1. *The category $\underline{\mathcal{A}}$ with Σ as translation functor and distinguished triangles defined as above is triangulated category.*

Proof. See [38, Theorem I.2.6] □

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