Regularity Results for Nonlocal Fully Nonlinear Elliptic Equations

DISSERTATION

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Contents

1 Introduction

In this thesis we consider nonlocal fully nonlinear elliptic operators derived from a certain class of linear integro-differential operators with kernels having anisotropic lower bounds. We obtain regularity estimates for solutions to corresponding nonlocal fully nonlinear elliptic equations. Our results extend the main result of [CS09]. An interesting feature of our approach and the one in [CS09] is the fact that the constants in the main results remain strictly positive and bounded if the singularity of the kernels converges to the critical diffusion limit.

1.1 Motivation

Nonlocal equations appear in a natural way in the theory of jump processes and have a great number of applications in physics, ecology, engineering and economics. Considering any n-dimensional Lévy process $X = (X_t)_{t\geq 0}$, it is well-known from general theory on semigroups that the infinitesimal generator of X exists for all functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ (cf. [Sat99, Theorem 31.5]). By the Lévy-Khintchine formula (cf. [Sat99, Theorem 8.1]), the generator of X is given by

$$\begin{aligned} Lu(x) =& \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u(x) + \sum_{i=1}^{n} b_i\partial_i u(x) \\ &+ \int_{\mathbb{R}^n} \big(u(x+y) - u(x) - (\nabla u(x)\cdot y)\mathbb{1}_{\{|y|\leq 1\}}\big)\mu(dy) \end{aligned} \tag{1.1}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, where the positive definite symmetric matrix $A = (a_{ij})_{1\leq i,j\leq n}$, the vector $b = (b_i)_{1\leq i\leq n} \in \mathbb{R}^n$ and the measure μ are the elements of the characteristic triple of X. So the first term on the right hand side of (1.1) belongs to the diffusion, the second to the drift, and the third to the jump part of the process X. Note from the general theory of Lévy processes that the Lévy measure μ satisfies $\mu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} (1 \wedge |y|^2)\,\mu(dy) < \infty.$$

In the following, we only consider operators of the form (1.1) without diffusion and drift part. Moreover, we restrict ourselves to measures μ of the form $\mu(dy) = K(y)\,dy$, where $K : \mathbb{R}^n \to [0,\infty)$ is a nonnegative measurable function which is symmetric, i.e.,

$K(y) = K(-y)$ for every $y \in \mathbb{R}^n$. Due to the symmetry of K and the general assumptions from above, the operators in (1.1) can be written in the form

$$\begin{aligned} Lu(x) &= \lim_{\varepsilon \searrow 0} \Bigg(\int\limits_{\{|y| \geq \varepsilon\}} (u(x+y) - u(x)) K(y)\, dy \Bigg) \\ &= \frac{1}{2} \lim_{\varepsilon \searrow 0} \Bigg(\int\limits_{\{|y| \geq \varepsilon\}} (u(x+y) - u(x)) K(y)\, dy + \int\limits_{\{|y| \geq \varepsilon\}} (u(x-y) - u(x)) K(-y)\, dy \Bigg) \\ &= \frac{1}{2} \int\limits_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y)\, dy. \end{aligned} \tag{1.2}$$

The operator L described in (1.2) is a *linear integro-differential operator*. Note that $Lu(x)$ is well-defined for $x \in \mathbb{R}^n$ if $u : \mathbb{R}^n \to \mathbb{R}$ is globally bounded and twice continuously differentiable in a neighborhood of x.

The main result of this thesis is a Hölder regularity result for solutions u to an equation of the form $\mathcal{I}u = 0$ in some bounded domain $\Omega \subset \mathbb{R}^n$, where $\mathcal{I}$ is a special type of fully *nonlinear* integro-differential operator. An example would be the operator

$$\mathcal{I}u(x) = \sup_{a \in J} L_a u(x), \tag{1.3}$$

where J is an arbitrary index set and L_a is of the form (1.2) for each $a \in J$. This operator appears in stochastic control theory (see [Son86]). In game theory, more complicated operators of the form

$$\mathcal{I}u(x) = \inf_{b \in J_2} \sup_{a \in J_1} L_{ab} u(x) \quad \text{or} \quad \mathcal{I}u(x) = \sup_{b \in J_2} \inf_{a \in J_1} L_{ab} u(x) \tag{1.4}$$

appear, where L_{ab} is of the form (1.2) for each choice of $a \in J_1, b \in J_2$. Note that all of these operators have the following property in common:

$$\inf_{ab} L_{ab}(u - v)(x) \leq \mathcal{I}u(x) - \mathcal{I}v(x) \leq \sup_{ab} L_{ab}(u - v)(x) \tag{1.5}$$

for bounded C^2 functions $u, v : \mathbb{R}^n \to \mathbb{R}$. This is an easy consequence of the linearity of the operators L_{ab}. (1.5) provides a nice connection to the local theory of fully nonlinear second order uniformly elliptic equations. Indeed, if we consider the uniformly elliptic extremal Pucci operators $\mathcal{M}^+$, $\mathcal{M}^-$ (cf. Section 2.2) and if we have

$$\mathcal{M}^-(D^2(u - v)(x)) \leq \mathcal{I}u(x) - \mathcal{I}v(x) \leq \mathcal{M}^+(D^2(u - v)(x)) \tag{1.6}$$

for every $x \in \Omega$ and all functions $u, v \in C^2(\Omega)$, then $\mathcal{I}$ must be an uniformly elliptic second order differential operator. We will discuss this property in Section 3.1. Hence, (1.6) can be used as a replacement for the concept of ellipticity. Now (1.5) can be seen as a nonlocal version of (1.6) for operators like the ones in (1.3) and (1.4) by replacing the Pucci extremal operators in (1.6) with suitable nonlocal extremal operators. We therefore obtain a concept of ellipticity for fully nonlinear integro-differential equations (cf. Definition 3.5). This connection allows us to adapt some ideas and results of the local ellipticity regularity theory to our nonlocal theory.

1.2 Known Regularity Results for Fully Nonlinear Integro-Differential Equations

There exist two main approaches for proving Hölder regularity for integro-differential equations:

- Either by Harnack inequalities (see Silvestre [Sil06], Caffarelli and Silvestre [CS09])
- Or by Ishii-Lions's method [IL90] (see Barles, Chasseigne and Imbert [BCI11]).

Both methods deal with different classes of equations and cannot treat all the examples given in [Sil06, CS09] and [BCI11] simultaneously. As mentioned above, the first method is based on Harnack inequalities which lead to Hölder regularity. It is possible to obtain further regularity such as $C^{1,\beta}$, but this requires some integrability condition of the measure $\mu(dy) = K(y)\,dy$ in (1.2) at infinity (see Section 3.8). The second method deals with a large class of second order fully nonlinear elliptic integro-differential equations, using viscosity methods which apply under weaker ellipticity conditions as in [CS09]. This allows measures that are only bounded at infinity, but do not seem to yield to further regularity. We do not consider this second approach further. Instead, we summarise the main ideas and results of the first method, introduced by Caffarelli and Silvestre in [CS09].

They consider the class of all linear integro-differential operators of the form (1.2) with corresponding positive symmetric kernels "comparable" to the respective kernel of the fractional Laplacian

$$-(-\Delta)^{\alpha/2}u(x) = c_{n,\alpha}\ \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+\alpha}}\,dy,$$

where $\alpha \in (0,2)$ and the constant $c_{n,\alpha}$ is comparable to $\alpha(2-\alpha)$. To be precise: Fixing constants $0 < \lambda \le \Lambda$ and letting $\alpha \in (0,2)$, they consider all positive measurable symmetric kernels satisfying

$$(2-\alpha)\frac{\lambda}{|y|^{n+\alpha}} \le K(y) \le (2-\alpha)\frac{\Lambda}{|y|^{n+\alpha}},\ y \in \mathbb{R}^n \setminus \{0\}. \tag{1.7}$$

The factor $(2-\alpha)$ plays a very important role in their theory. In fact, it allows their results to stay uniform when $\alpha \nearrow 2$ and therefore they can extend the existing local theory for fully nonlinear second order uniformly elliptic equations to the case of discontinuous processes. This extension has not been possible in earlier results about Harnack inequalities and Hölder estimates for integro-differential equations (as in [Sil06]). Note that the factor α does not appear in the lower and upper bound of the kernels K in (1.7) because the main results are stated for $\alpha \in (\alpha_0, 2)$, where $\alpha_0 \in (0,2)$ is any fixed number.
Using this class of kernels they prove among other things

- a nonlocal version of the Aleksandrov-Bakelman-Pucci estimate,

- the Harnack inequality for translation-invariant fully nonlinear integro-differential equations with kernels of the form (1.7) (which can be very discontinuous),
- a Hölder regularity result for the same class of equations as the Harnack inequality,
- an interior $C^{1,\beta}$ regularity result for translation-invariant fully nonlinear integro-differential equations with more restrictive kernels than the ones that are used to obtain the Hölder regularity result.

For details, we refer to [CS09].

Hölder regularity results are established in [KL12, KL13, CLD12], too. These contributions extend the results of [CS09] to cases, where K in (1.2) is not necessarily symmetric. [CS11] extends the results in [CS09] to the case of integro-differential equations that are not necessarily translation-invariant. A different approach is taken in [GS12], leading also to Aleksandrov-Bakelman-Pucci estimates and Hölder regularity. This approach is closer to the one in the (local) classical case $\alpha = 2$ and imposes some mild restrictions on the admissible kernels K in comparison to (1.7).

The main results of this thesis are also derived in [BCF12] because the assumed structure of our kernels in (1.8) from below satisfies the more general Assumptions 2.1. and 2.2. in [BCF12]. Let us comment on the differences between this work and [BCF12]:

- We derive the most important technical results, Lemma 3.22 and Lemma 3.24, taking explicit advantage of the special structure of the kernels in (1.8). The corresponding results in [BCF12] can be applied under more general assumptions on the basic structure of the kernels.
- The method of constructing the bump function in Section 3.5 is significantly different to the corresponding method in [BCF12, Section 3.3.].
- As explained in Remark 3.49, a strong formulation of the Harnack inequality does not hold in general under (1.8). This phenomenon seems to be neglected in [BCF12, Theorem 3.14].

1.3 Setting and Outline

We want to introduce a larger class of nonnegative measurable symmetric kernels in comparison to (1.7) by allowing the kernels to vanish in certain areas. To be precise: Fix $0 < \lambda \leq \Lambda$ and let $\alpha \in (0, 2)$. Let $I \subset \mathbb{S}^{n-1}$ be of the form $I = (B_\varrho(\xi_0) \cup B_\varrho(-\xi_0)) \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, $\xi_0 \in \mathbb{S}^{n-1}$ and $\varrho > 0$. Let $k : \mathbb{S}^{n-1} \to [0, 1]$ be a nonnegative measurable symmetric function with $k(\xi) = 1$ if $\xi \in I$. We consider the class $\mathcal{K}_0$ of all nonnegative measurable symmetric kernels $K : \mathbb{R}^n \to [0, \infty)$ satisfying

$$(2 - \alpha)k(\tfrac{y}{|y|})\frac{\lambda}{|y|^{n+\alpha}} \leq K(y) \leq (2 - \alpha)\frac{\Lambda}{|y|^{n+\alpha}}, \; y \in \mathbb{R}^n \setminus \{0\} \tag{1.8}$$

and denote by $\mathcal{L}_0 = \mathcal{L}_0(n, \lambda, \Lambda, k, \alpha)$ the collection of all corresponding linear integro-differential operators L of the form (1.2) with kernels $K \in \mathcal{K}_0$. Note that we do not impose a lower bound on K on the set $\{y \in \mathbb{R}^n : \frac{y}{|y|} \notin I\}$ due to the fact that we allow $k(\frac{y}{|y|}) = 0$ whenever $\frac{y}{|y|} \notin I$. The assumptions made above allow us to adapt most of the results in [CS09] to our situation. However, due to the anisotropy of the lower bound in (1.8) introduced by k, we obtain some technical difficulties in estimates relating to this lower bound.

As in [CS09], we bound α from below to derive our main regularity results, i.e., $\alpha \in (\alpha_0, 2)$ for any fixed number $\alpha_0 \in (0, 2)$. We prove interior Hölder regularity for solutions to equations of the form $\mathcal{I}u = 0$, where $\mathcal{I}$ (like the ones in (1.4)) is a translation-invariant nonlocal elliptic operator in the sense of Definition 3.5 with respect to the class $\mathcal{L}_0 = \mathcal{L}_0(n, \lambda, \Lambda, k, \alpha)$, i.e., $\mathcal{I}$ satisfies

$$\inf_{L \in \mathcal{L}_0} L[u - v](x) \leq \mathcal{I}u(x) - \mathcal{I}v(x) \leq \sup_{L \in \mathcal{L}_0} L[u - v](x)$$

in every point $x \in \mathbb{R}^n$ whenever the functions $u, v : \mathbb{R}^n \to \mathbb{R}$ are bounded and twice continuously differentiable in a neighborhood of x. Our main result is the following a priori Hölder regularity estimate in Section 3.7.

Theorem 1.1. *Let $\alpha_0 \in (0, 2)$ and consider any $\alpha \in (\alpha_0, 2)$. Assume that the bounded function $u : \mathbb{R}^n \to \mathbb{R}$ is continuous in $\overline{B_1(0)}$ and satisfies*

$$\mathcal{I}u = 0 \quad \text{in } B_1(0) \text{ in the viscosity sense,}$$

where $\mathcal{I}$ is a translation-invariant nonlocal elliptic operator with respect to $\mathcal{L}_0$. There exist $\beta \in (0, 1)$ and $C \geq 1$ depending on $n, \lambda, \Lambda, \alpha_0$ and $|I|$ such that $u \in C^\beta(\overline{B_{1/2}})$ and

$$\|u\|_{C^\beta(\overline{B_{1/2}})} \leq C\left(\sup_{\mathbb{R}^n} |u| + |\mathcal{I}0|\right),$$

where $\mathcal{I}0$ is the value we obtain when applying $\mathcal{I}$ to the constant function that is equal to zero.

The proof of Theorem 1.1 is based on a result that links a pointwise estimate with an estimate in measure to prove a decay of oscillation of the solution. The result itself is proved by combining a nonlocal version of the Aleksandrov-Bakelman-Pucci estimate adapted to kernels of the form (1.8) (see Section 3.4) and a special bump function (see Section 3.5). Our results generalise the corresponding ones in [CS09] (where $k \equiv 1$ in $\mathbb{S}^{n-1}$). As in [CS09], our main results remain uniform when $\alpha \nearrow 2$.

The outline of the thesis is as follows. In Chapter 2, we give a summary of the regularity theory for fully nonlinear second order uniformly elliptic equations. Since our theory (which will be presented in Chapter 3) can be seen as an extension of this local theory, we will present tools and techniques that will also be useful in the nonlocal theory. After giving basic definitions and examples regarding fully nonlinear elliptic equations, we discuss the concept of viscosity solutions in Section 2.1. This concept is crucial for the whole regularity theory. In Section 2.2, we present the Pucci extremal operators and corresponding

classes S of solutions to uniformly elliptic equations. Afterwards, we obtain regularity results for these solutions. The main technical tool will be an Aleksandrov-Bakelman-Pucci (ABP) estimate adapted to viscosity solutions. We derive such an estimate in Section 2.3 and use it to prove a Harnack inequality for viscosity solutions in Section 2.4.1. As a consequence of the Harnack inequality, we obtain interior Hölder regularity and also global Hölder regularity for solutions in $S(0)$ (see Section 2.4.2). The whole summary is based on [CC95].
In Chapter 3, we present the main part of this thesis. We first introduce the appropriate definitions of viscosity sub- and supersolutions for fully nonlinear integro-differential equations. After that, we define a concept of ellipticity for nonlocal operators with respect to a class of linear integro-differential operators by comparing its increments with a suitable maximal and minimal operator. In Section 3.2, we introduce the family of all kernels satisfying (1.8) and provide basic properties regarding the class $\mathcal{L}_0$ of linear integro-differential operators. In Section 3.3, we prove existence of solutions to the nonlocal Dirichlet problem with respect to operators as in (1.4). Afterwards, we present the regularity theory for solutions to nonlocal fully nonlinear elliptic equations derived from the class $\mathcal{L}_0$. We prove a nonlocal version of the Aleksandrov-Bakelman-Pucci estimate for the class $\mathcal{L}_0$ in Section 3.4. In Section 3.5, we construct a special function which will be used together with the nonlocal ABP estimate, to obtain some pointwise estimates in Section 3.6 which shall be useful in proving Hölder estimates in Section 3.7. Using these estimates, we prove an interior $C^{1,\beta}$ regularity result for a more restrictive class of linear integro-differential operators $\mathcal{L}_1 \subset \mathcal{L}_0$ with kernels satisfying (1.8) plus a certain integrability condition. This will be done in Section 3.8. Finally, we summarise the main results in Chapter 4.

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1.4 Notation

We introduce the notation that will often be used in the sequel. Let Ω always be a bounded domain in $\mathbb{R}^n$, i.e., a bounded open and connected set. We use $|\cdot|$ for the absolute value, the Euclidean norm, the n-dimensional Lebesgue measure and the $(n-1)$-dimensional surface measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ at the same time. The maximum norm of a vector $x \in \mathbb{R}^n$ is defined by $|x|_\infty = \max_{1 \le i \le n} |x_i|$.

Cubes and balls will be important geometrical objects in the following chapters. For $x_0 \in \mathbb{R}^n$ and $r > 0$, define

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \quad \text{and} \quad Q_r(x_0) = \left\{x \in \mathbb{R}^n \,:\, |x - x_0|_\infty < \tfrac{r}{2}\right\}.$$

Here, r denotes the radius of the (open) ball $B_r(x_0)$ and the edge length of the (open) cube $Q_r(x_0)$. If $x_0 = 0$ we just write B_r and Q_r instead of $B_r(0)$ and $Q_r(0)$. For $t > 1$, the enlarged cube $tQ_r(x_0)$ is defined by

$$tQ_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0|_\infty < \tfrac{tr}{2}\}.$$

Note that $2Q_1 \neq \{2x \in \mathbb{R}^n : x \in Q_1\}$. For a nonempty set $A \subset \mathbb{R}^n$, we define the diameter of A by $\operatorname{diam} A = \sup\{|x - y| : x, y \in A\}$. Moreover, the distance of a point $x \in \mathbb{R}^n$ to A is defined by $\operatorname{dist}(x, A) = \inf\{|x - y| : y \in A\}$.

Remark 1.2.

(i) The diameter of the cubes from above is

$$\operatorname{diam} Q_r(x_0) = r\sqrt{n}.$$

(ii) The following relation holds: $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{3\sqrt{n}/2} \subset B_{2\sqrt{n}} \subset Q_{4\sqrt{n}}$.

The volume of the unit ball in $\mathbb{R}^n$ is denoted by ω_n. Given a function f, we denote by f^+ and f^- the positive and negative parts of f. The support of f will be denoted by $\operatorname{supp}(f)$. We often use the symbols $\wedge$ and $\vee$ in the following sense: $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Given $u \in L^p(\Omega)$, $1 \le p \le \infty$, we denote by $\|u\|_{L^p(\Omega)}$ the L^p norm of u, i.e.,

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}, & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|, & p = \infty. \end{cases}$$

For $k \in \mathbb{N}_0$, $C^{k,\beta}(\overline{\Omega})$ denotes Hölder spaces if $0 < \beta < 1$ and Lipschitz spaces if $\beta = 1$. For $u \in C^{k,\beta}(\overline{\Omega})$, the norm $\|u\|_{C^{k,\beta}(\overline{\Omega})}$ is defined by

$$\|u\|_{C^{k,\beta}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + \sum_{|s|=k} [D^s u]_{C^\beta(\overline{\Omega})},$$

where

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|s|\le k} \left(\sup_{x\in\overline{\Omega}} |D^s u(x)| \right) \quad \text{and} \quad [u]_{C^\beta(\overline{\Omega})} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\beta}.$$

Here, $D^s u = \partial_1^{s_1} \dots \partial_n^{s_n} u$ denotes the s-th partial derivative of u for $s = (s_1, \dots, s_n) \in \mathbb{N}_0^n$. For $u \in C(\overline{\Omega}) := C^0(\overline{\Omega})$, we write $\|u\|_\infty$ instead of $\|u\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)|$.

Finally, the second differences of a function $u : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ are defined as the function $y \mapsto \Delta u(x; y) = u(x+y) + u(x-y) - 2u(x)$ for $y \in \mathbb{R}^n$.

2 Regularity Estimates for Local Fully Nonlinear Elliptic Equations

The aim of this chapter is to present basic results and techniques from the regularity theory of local fully nonlinear elliptic equations. Some of the results will be helpful in the nonlocal setting introduced in Chapter 3. For all of the details of this chapter, we refer to [CC95] which will be the main source for this summary.
We consider equations of the form

$$F(D^2u(x), x) = f(x) \tag{2.1}$$

where $x \in \Omega$, $u : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$. D^2u denotes the Hessian of u and $F(M, x)$ is a real valued function defined on $\mathcal{S} \times \Omega$, where $\mathcal{S}$ is the space of all real $n \times n$ symmetric matrices. $\|M\|$ will always denote the spectral norm of $M \in \mathcal{S}$, i.e., $\|M\| = \sup_{|x|\leq 1} |Mx|$.

Definition 2.1 ([CC95, Definition 2.1]). $F : \mathcal{S} \times \Omega \to \mathbb{R}$ is *uniformly elliptic* if there are two positive constants $0 < \lambda \leq \Lambda$ such that for every $M \in \mathcal{S}$ and $x \in \Omega$

$$\lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\| \text{ for all } N \geq 0, \tag{2.2}$$

where we write $N \geq 0$ whenever N is a nonnegative definite symmetric matrix.

The constants λ, Λ are called ellipticity constants. Note that $\|N\|$ is equal to the largest eigenvalue of N whenever $N \geq 0$. Note further that the condition of uniform ellipticity implies that $F(M, x)$ is monotone increasing and Lipschitz continuous in $M \in \mathcal{S}$. To prove the latter assertion, use the fact that every matrix $N \in \mathcal{S}$ can uniquely be composed as $N = N^+ - N^-$, where $N^+, N^- \geq 0$ and $N^+N^- = 0$. Therefore,

$$\|N^+ + N^-\|_F = \sqrt{\operatorname{tr}((N^+ + N^-)^2)} = \sqrt{\operatorname{tr}((N^+ - N^-)^2)} = \|N\|_F ,$$

where $\operatorname{tr}(\cdot)$ denotes the trace and $\|\cdot\|_F$ the Frobenius norm of an $n \times n$ matrix. Using this result and (2.2), we obtain for every $M, N \in \mathcal{S}$ and $x \in \Omega$

$$\begin{aligned} |F(M + N, x) - F(M, x)| &\leq \Lambda(\|N^+\| + \|N^-\|) \leq \Lambda \operatorname{tr}(N^+ + N^-) \\ &\leq \Lambda\sqrt{n}\, \|N^+ + N^-\|_F = \Lambda\sqrt{n}\, \|N\|_F \\ &\leq n\Lambda \|N\| , \end{aligned}$$

which proves the assertion.
Throughout the chapter, we will assume that F in (2.1) is uniformly elliptic. Under these assumptions, equations of the form (2.1) are called fully nonlinear second order uniformly elliptic equations. In addition, F and the right hand side f are always assumed to be continuous in each $x \in \Omega$ (unless otherwise stated).

Example 2.2. Let $A : \Omega \to \mathcal{S}$, $A(x) = (a_{ij}(x))_{ij}$, where $(a_{ij}(x))_{ij}$ is a symmetric $n \times n$ matrix with eigenvalues in $[\lambda, \Lambda]$ for some constants $0 < \lambda \leq \Lambda$ independent of x. In this case, the linear partial differential equation

$$F(D^2u(x), x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) = f(x)$$

is uniformly elliptic with ellipticity constants λ and $n\Lambda$. To prove this assertion, let $M \in \mathcal{S}$, $x \in \Omega$ and $N \geq 0$ be arbitrary. Then

$$F(M+N, x) - F(M, x) = F(N, x) = \operatorname{tr}(A(x)N). \tag{2.3}$$

We use the following result:

Lemma 2.3 ([WKH86, Lemma 1])**.** *Let A and N be symmetric $n \times n$ matrices and assume $N \geq 0$. Then*

$$\lambda_{\min}(A)\operatorname{tr}(N) \leq \operatorname{tr}(AN) \leq \lambda_{\max}(A)\operatorname{tr}(N),$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum resp. minimum eigenvalue of A.

By applying Lemma 2.3 to (2.3) and using the fact that

$$\lambda_{\max}(A(x))\operatorname{tr}(N) \leq n\Lambda \|N\| \quad \text{and} \quad \lambda_{\min}(A(x))\operatorname{tr}(N) \geq \lambda \|N\|,$$

we prove our assertion.

Let us provide the proof of Lemma 2.3 for completeness.

Proof of Lemma 2.3. Since A is symmetric, we can find an orthogonal matrix U and a diagonal matrix $D_A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ containing the eigenvalues λ_j of A, such that $U^{\mathrm{T}}AU = D_A$. Hence,

$$\operatorname{tr}(AN) = \operatorname{tr}(U^{\mathrm{T}}ANU) = \operatorname{tr}(U^{\mathrm{T}}AUU^{\mathrm{T}}NU) = \operatorname{tr}(D_AU^{\mathrm{T}}NU).$$

Define $C = U^{\mathrm{T}}NU$. Since $N \geq 0$, we also have $C \geq 0$ because

$$x^{\mathrm{T}}Cx = (Ux)^{\mathrm{T}}NUx \geq 0 \quad \text{for every } x \neq 0.$$

Moreover, C is symmetric and its diagonal elements satisfy $c_{jj} \geq 0$ for every $j = 1, \ldots, n$ because $0 \leq e_j^{\mathrm{T}}Ce_j = c_{jj}$, where e_j denotes the j-th unit vector in $\mathbb{R}^n$. Hence,

$$\operatorname{tr}(AN) = \operatorname{tr}(D_AC) = \sum_{j=1}^{n}\lambda_jc_{jj} \leq \lambda_{\max}(A)\operatorname{tr}(C) = \lambda_{\max}(A)\operatorname{tr}(N) \quad \text{and}$$
$$\operatorname{tr}(AN) \geq \lambda_{\min}(A)\operatorname{tr}(N). \qquad \square$$

Important examples for fully nonlinear second order elliptic equations are Pucci's Equations, Bellman Equations, Isaacs Equations and the Monge-Ampère Equation (see [CC95, Section 2.3]).

2.1 Viscosity Solutions

The theory of viscosity solutions for nonlinear partial differential equations was introduced by Crandall and Lions (see [CL85]) and turned out to be very useful in proving existence of solutions. In fact, it can be considered as a different notion of weak solutions. Before giving the precise definition, we would like to give an intuition where the idea comes from.
Assume that we have a classical solution of $\Delta u = 0$ in Ω (where Δ is the Laplacian), i.e., $u \in C^2(\Omega)$ and $\Delta u(x) = 0$ for every $x \in \Omega$. Consider a function $\varphi \in C^2(\Omega)$ touching u from above at some point x_0 in Ω, i.e., $\varphi(x) \geq u(x)$ for every $x \in \Omega$ and $\varphi(x_0) = u(x_0)$. Then $u - \varphi$ has a local maximum at the point x_0. Therefore, $u - \varphi$ looks locally concave around x_0, which implies

$$0 \geq \Delta(u - \varphi)(x_0) = \Delta u(x_0) - \Delta\varphi(x_0) = -\Delta\varphi(x_0),$$

i.e., $\Delta\varphi(x_0) \geq 0$. Considering a function $\varphi \in C^2(\Omega)$ touching u from below, we obtain the reverse inequality at the minimum points, i.e., $\Delta\varphi(x_0) \leq 0$ for every point x_0 where $u - \varphi$ has a local minimum.
When the function u is not C^2, we will use the above properties for C^2 functions touching from above and below to say whether u solves equation (2.1) in a weak sense.

Definition 2.4 ([CC95, Definition 2.3])**.** Let $u : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ be continuous.

(i) u is a *viscosity subsolution* of (2.1) at a point $x_0 \in \Omega$, if for every test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, then

$$F(D^2\varphi(x_0), x_0) \geq f(x_0).$$

(ii) u is a *viscosity supersolution* of (2.1) at a point $x_0 \in \Omega$, if for every test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, then

$$F(D^2\varphi(x_0), x_0) \leq f(x_0).$$

(iii) u is a *viscosity solution* of (2.1) in Ω if u is a viscosity subsolution and a viscosity supersolution at every $x_0 \in \Omega$.

(iv) We write $F(D^2u, x) \geq [\leq, =]\, f(x)$ in Ω in the viscosity sense whenever u is a viscosity subsolution [supersolution, solution] of (2.1) in Ω.

Remark 2.5. In order to check the condition for viscosity subsolution (resp. viscosity supersolution), it is enough to require the function u in Definition 2.4 to be upper semicontinuous (resp. lower semicontinuous). Note that upper (lower) semicontinuous functions attain their maximum (minimum) on compact sets.

We say that P is a *paraboloid of opening* M if

$$P(x) = l_0 + l(x) \pm \tfrac{M}{2} |x|^2 \tag{2.4}$$

where $M > 0$, $l_0 \in \mathbb{R}$ and $l : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Note that P is convex when we have "+" in (2.4) and concave when we have "−" in (2.4).
The following proposition will be useful when working with the concept of viscosity solutions. It is also motivated by the same introduction which lead to Definition 2.4. For the easy proof we refer to [CC95].

Proposition 2.6 ([CC95, Proposition 2.4]). *Let $u : \Omega \to \mathbb{R}$ be a continuous function. The following statements are equivalent:*

(i) u is a viscosity subsolution of (2.1) in Ω.

(ii) For every $x_0 \in \Omega$ and every testfunction $\varphi \in C^2(N)$ satisfying

$$\varphi \geq u \text{ in } N \quad \text{and} \quad \varphi(x_0) = u(x_0)$$

("φ touches u from above at x_0 in N"),
where $N \subset \Omega$ is any open neighborhood of x_0, we have

$$F(D^2\varphi(x_0), x_0) \geq f(x_0).$$

(iii) For every $x_0 \in \Omega$ and every paraboloid P satisfying

$$P \geq u \text{ in } N \quad \text{and} \quad P(x_0) = u(x_0),$$

where $N \subset \Omega$ is any open neighborhood of x_0, we have

$$F(D^2P(x_0), x_0) \geq f(x_0).$$

A corresponding result to Proposition 2.6 also holds for viscosity supersolutions.

Example 2.7. Let $n = 1$, $\Omega = (-1, 1)$ and consider again the equation

$$\Delta u = u'' = 0 \quad \text{in } \Omega. \tag{2.5}$$

Using Proposition 2.6 (and its corresponding version for supersolutions), we show that $u(x) = |x|$ is a viscosity subsolution of (2.5) in Ω but no viscosity supersolution.
Let $x_0 \in \Omega$, $N \subset \Omega$ be any open neighborhood of x_0 and $\varphi \in C^2(N)$ such that

$$u(x_0) = \varphi(x_0) \quad \text{and} \quad \varphi(x) \geq u(x) \text{ for each } x \in N.$$

There are two cases to consider:

- $x_0 \neq 0$. In this case, u is differentiable at x_0 and therefore

$$u''(x_0) - \varphi''(x_0) = (u(x_0) - \varphi(x_0))'' \leq 0$$

just because $u - \varphi$ has a local maximum at x_0. Thus

$$\Delta\varphi(x_0) = \varphi''(x_0) \geq u''(x_0) = 0.$$

- $x_0 = 0$. We have
$$|x| = u(x) - u(0) \leq \varphi(x) - \varphi(0),$$
near 0, which implies
$$-1 \geq \frac{\varphi(x) - \varphi(0)}{x} \text{ for } x < 0 \quad \text{and} \quad 1 \leq \frac{\varphi(x) - \varphi(0)}{x} \text{ for } x > 0.$$
Therefore, we obtain for the left and right derivative of φ by passing to the limit from above and below: $D^-\varphi(0) \leq -1$ and $D^+\varphi(0) \geq 1$ which is not possible for a C^2 function. So there can not be a C^2 function touching $u(x) = |x|$ from above, which means that *(ii)* in Proposition 2.6 is trivially verified.

Hence, $u(x) = |x|$ is a viscosity subsolution of (2.5).
To show that $u(x) = |x|$ is not a supersolution, consider the C^2 function $\varphi(x) = x^2$. We have
$$\varphi(0) = u(0) \quad \text{and} \quad \varphi(x) \leq u(x) \text{ for each } x \in N,$$
where $N \subset \Omega$ is any open neighborhood of 0. Then
$$\Delta\varphi(0) = \varphi''(0) = 2 > 0.$$
Hence, *(ii)* in the corresponding version of Proposition 2.6 for viscosity supersolutions is not satisfied.

The next result says that the notion of viscosity solutions is consistent with classical solutions.

Lemma 2.8 ([CC95, Corollary 2.6]). *Let $u \in C^2(\Omega)$. u is a viscosity subsolution of (2.1) in Ω iff $F(D^2u(x), x) \geq f(x)$ for each $x \in \Omega$.*

Proof. Let $u \in C^2(\Omega)$.
"⇒": Follows easily by choosing u itself as testfunction.
"⇐": Let $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0. This implies $D^2(u - \varphi)(x_0) \leq 0$ (nonpositive definit). Hence, using the fact that $F(M, x)$ is monotone increasing in $M \in \mathcal{S}$,
$$f(x_0) \leq F(D^2u(x_0), x_0) \leq F(D^2\varphi(x_0), x_0)$$
which proves that u is a viscosity subsolution of (2.1) at x_0. □

The following result is very important for Section 2.3.

Proposition 2.9 ([CC95, Proposition 2.8]). *Let Ω_1 and Ω_2 be bounded domains such that $\overline{\Omega}_1 \subset \Omega_2 \subset \mathbb{R}^n$. Assume that $u_2 \in C(\Omega_2)$ satisfies $F(D^2u_2, x) \leq f_2(x)$ in Ω_2 in the viscosity sense, $u_1 \in C(\overline{\Omega}_1)$ satisfies $F(D^2u_1, x) \leq f_1(x)$ in Ω_1 in the viscosity sense and $u_1 \geq u_2$ in $\partial\Omega_1 \cap \Omega_2$. Define*
$$w(x) = \begin{cases} u_2(x), & x \in \Omega_2 \setminus \overline{\Omega}_1 \\ \inf(u_1(x), u_2(x)), & x \in \overline{\Omega}_1 \end{cases} \text{ and } h(x) = \begin{cases} f_2(x), & x \in \Omega_2 \setminus \Omega_1 \\ \sup(f_1(x), f_2(x)), & x \in \Omega_1. \end{cases}$$
Then $w \in C(\Omega_2)$ satisfies $F(D^2w, x) \leq h(x)$ in Ω_2 in the viscosity sense.

This result shows that, under certain conditions, viscosity supersolutions can be extended to viscosity supersolutions in larger sets.

Proof. Note that $w \in C(\Omega_2)$ since u_1, u_2 are continuous functions in their domains and $u_1 \geq u_2$ in $\partial\Omega_1 \cap \Omega_2$. We use Proposition 2.6. So let $x_0 \in \Omega_2$ and φ be a C^2 function in some open neighborhood $N \subset \Omega_2$ of x_0 that touches w from below at x_0. We consider two cases:

- $w(x_0) = u_2(x_0)$: φ also touches u_2 from below at x_0 in N since $w \leq u_2$ in Ω_2. Since u_2 is a viscosity supersolution in Ω_2, using Proposition 2.6, we obtain

$$F(D^2\varphi(x_0), x_0) \leq f_2(x_0) \leq h(x_0).$$

- $w(x_0) < u_2(x_0)$: By definition, $w(x_0) = u_1(x_0)$ which implies $x_0 \in \Omega_1$ since $u_1 \geq u_2$ in $\partial\Omega_1 \cap \Omega_2$. Furthermore, φ touches u_1 from below at x_0 in $N \cap \Omega_1$ because $w \leq u_1$ in Ω_1. Since u_1 is a viscosity supersolution in Ω_1, using again Proposition 2.6, we obtain

$$F(D^2\varphi(x_0), x_0) \leq f_1(x_0) \leq h(x_0). \qquad \square$$

For more details regarding basic properties of viscosity solutions, we refer to [CC95] and [CL85].

2.2 The Class S of Solutions to Special Types of Uniformly Elliptic Equations

In this section, we introduce special classes of uniformly elliptic equations which are important for obtaining regularity results. At a first step, we define *Pucci's extremal operators*. We will see that these operators are uniformly elliptic.

Let $0 < \lambda \leq \Lambda$. For $M \in \mathcal{S}$, we define

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$
$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i \in \mathbb{R}$, $i = 1, \dots, n$, are the eigenvalues of the real $n \times n$ symmetric matrix M. We claim that $\mathcal{M}^-$ and $\mathcal{M}^+$ are uniformly elliptic with ellipticity constants $\lambda, n\Lambda$. To prove this, we note as a first step that for every $N \geq 0$ (nonnegative definite symmetric matrix)

$$\begin{aligned} \lambda \|N\| &\leq \mathcal{M}^-(N) = \lambda \operatorname{tr}(N) \leq n\lambda \|N\|, \\ \Lambda \|N\| &\leq \mathcal{M}^+(N) = \Lambda \operatorname{tr}(N) \leq n\Lambda \|N\|. \end{aligned} \tag{2.6}$$

Let A be a symmetric matrix whose eigenvalues are contained in the interval $[\lambda, \Lambda]$, i.e., $\lambda |\xi|^2 \leq \xi^{\mathrm{T}} A \xi \leq \Lambda |\xi|^2$ for every $\xi \in \mathbb{R}^n$. In this case, we will write $\lambda I \leq A \leq \Lambda I$, where I is the $n \times n$ identity matrix. Recall that every $M \in \mathcal{S}$ can uniquely be composed as $M = M^+ - M^-$, where $M^+, M^- \geq 0$ and $M^+ M^- = 0$. Hence, for every A such that $\lambda I \leq A \leq \Lambda I$ and every $M \in \mathcal{S}$,

$$\operatorname{tr}(AM) = \operatorname{tr}(AM^+) - \operatorname{tr}(AM^-) \leq \Lambda \operatorname{tr}(M^+) - \lambda \operatorname{tr}(M^-) = \mathcal{M}^+(M).$$

Analogously, $\operatorname{tr}(AM) \geq \mathcal{M}^-(M)$. We can obtain equalities instead of inequalities as follows: For $M \in \mathcal{S}$ choose an orthogonal matrix O such that $M = OD_1O^{\mathrm{T}}$, where $D_1 = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ contains the eigenvalues σ_i of M. Define the matrix $A = OD_2O^{\mathrm{T}}$ with the same orthogonal matrix O as before and let $D_2 = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with

$$\lambda_i = \begin{cases} \Lambda & \text{if } \sigma_i \geq 0 \\ \lambda & \text{if } \sigma_i < 0. \end{cases}$$

We see that $\lambda I \leq A \leq \Lambda I$ and

$$\operatorname{tr}(AM) = \operatorname{tr}(D_1 D_2) = \mathcal{M}^+(M).$$

If we choose

$$\lambda_i = \begin{cases} \lambda & \text{if } \sigma_i \geq 0 \\ \Lambda & \text{if } \sigma_i < 0, \end{cases}$$

we obtain in the same way that $\operatorname{tr}(AM) = \mathcal{M}^-(M)$. As a consequence,

$$\mathcal{M}^+(M) = \sup_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AM), \quad \mathcal{M}^-(M) = \inf_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AM). \tag{2.7}$$

Using (2.7), we obtain

$$\begin{aligned} \mathcal{M}^+(M_1) + \mathcal{M}^-(M_2) \leq \mathcal{M}^+(M_1 + M_2) \leq \mathcal{M}^+(M_1) + \mathcal{M}^+(M_2), \\ \mathcal{M}^-(M_1) + \mathcal{M}^-(M_2) \leq \mathcal{M}^-(M_1 + M_2) \leq \mathcal{M}^-(M_1) + \mathcal{M}^+(M_2) \end{aligned} \tag{2.8}$$

for every $M_1, M_2 \in \mathcal{S}$. Finally, combining (2.6) and (2.8), we conclude that

$$\lambda \|N\| \leq \mathcal{M}^-(N) \leq \mathcal{M}^\pm(M + N) - \mathcal{M}^\pm(M) \leq \mathcal{M}^+(N) \leq n\Lambda \|N\|$$

for every $M \in \mathcal{S}$ and $N \geq 0$, i.e., the Pucci extremal operators are uniformly elliptic with ellipticity constants λ and $n\Lambda$. We can now define the class S.

Definition 2.10. [CC95, Definition 2.11] Let $f : \Omega \to \mathbb{R}$ be continuous and $0 < \lambda \leq \Lambda$. We denote by

(i) $\underline{S}(\lambda, \Lambda, f)$ the space of continuous functions $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{M}^+(D^2 u, \lambda, \Lambda) \geq f$$

in Ω in the viscosity sense;

(ii) $\overline{S}(\lambda, \Lambda, f)$ the space of continuous functions $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f$$

in Ω in the viscosity sense;

(iii) $S(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f)$ and $S^*(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|)$.

The following property of the class $\overline{S}(\lambda, \Lambda, f)$ is important for the next section.

Lemma 2.11. *Let $u \in \overline{S}(\lambda, \Lambda, f)$, $\Phi \in C^2(\Omega)$ and $\mathcal{M}^-(D^2\Phi(x), \lambda, \Lambda) \geq g(x)$ for every $x \in \Omega$, where $g : \Omega \to \mathbb{R}$ is a continuous function. Then*

$$u - \Phi \in \overline{S}(\lambda, \Lambda, f - g).$$

Proof. We use Proposition 2.6. Let $x_0 \in \Omega$, $N \subset \Omega$ be an open neighborhood of x_0 and $\varphi \in C^2(N)$ such that

$$\varphi(x) \leq (u - \Phi)(x) \text{ for every } x \in N \quad \text{and} \quad \varphi(x_0) = u(x_0) - \Phi(x_0).$$

We claim that

$$\mathcal{M}^-(D^2\varphi(x_0), \lambda, \Lambda) \leq f(x_0) - g(x_0).$$

Since $u \in \overline{S}(\lambda, \Lambda, f)$ and $\varphi + \Phi$ touches u from below at x_0 in N,

$$\mathcal{M}^-(D^2\varphi(x_0) + D^2\Phi(x_0), \lambda, \Lambda) \leq f(x_0).$$

Using (2.8), this implies $\mathcal{M}^-(D^2\varphi(x_0), \lambda, \Lambda) + \mathcal{M}^-(D^2\Phi(x_0), \lambda, \Lambda) \leq f(x_0)$. Finally, using the assumption for Φ, we finish the proof. □

For a similar lemma, regarding viscosity subsolutions, we refer to [CC95, Lemma 2.12]. The usefulness of the classes S lies in the following fact: One can show that any result for functions in the classes S is also valid for solutions to fully nonlinear uniformly elliptic equations. This is due to the fact that a viscosity solution of (2.1) (where F is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$) belongs to the class $S(\frac{\lambda}{n}, \Lambda, f(x) - F(0, x))$ (cf. [CC95, Proposition 2.13]). For this fact and more properties of the operators and the classes described above, we refer to [CC95].

2.3 Aleksandrov-Bakelman-Pucci Estimate

Consider any smooth solution u to the linear partial differential equation in Example 2.2. The classical Aleksandrov-Bakelman-Pucci (ABP) estimate states that the supremum of u in Ω is bounded in terms of the supremum of u in $\partial\Omega$ and the $L^n(\Omega)$-norm of f. We refer to [Jos07] for a full review of this classical result. The aim of this section is to present the ABP estimate adapted to viscosity solutions. This estimate is the main tool in the regularity theory presented in Section 2.4. The difficulty in the transformation process of the classical ABP theorem to the viscosity case lies in the fact that viscosity supersolutions (resp. subsolutions) u may be very singular. However, we will show that a very important tool, the convex envelope of $-u^-$, is regular enough to obtain the desired ABP estimate. Before stating the result we need to introduce some technical tools.

Definition 2.12. Let $U \subseteq \mathbb{R}^n$ be an open set. A function $\varphi : U \to \mathbb{R}$ is said to be $C^{1,1}$ *at the point* $x \in U$, and we write $\varphi \in C^{1,1}(x)$, if there exist some vector $v \in \mathbb{R}^n$ and numbers $A > 0$, $r > 0$ such that

$$|\varphi(z) - \varphi(x) - (z - x) \cdot v| \leq A |z - x|^2 \text{ for all } z \in B_r(x). \tag{2.9}$$

For a set $W \subseteq U$, we write $\varphi \in C^{1,1}[W]$, if $\varphi \in C^{1,1}(x)$ for every point $x \in W$ and the constant A in (2.9) is independent of x.

Remark 2.13.

(i) Note that Definition 2.12 is equivalent to the definition given in [CC95, Chapter 1], as mentioned in [KL12].

(ii) If φ is $C^{1,1}$ at a point $x \in \mathbb{R}^n$ then φ is differentiable at x. Moreover, v is uniquely determined by $v = \nabla\varphi(x)$.

(iii) Let $U \subset \mathbb{R}^n$ be any open set and $\varphi \in C^2(\overline{U})$. Then – by Taylor expansion – we have $\varphi \in C^{1,1}(x)$ for every $x \in U$, where we choose $r = \operatorname{dist}(x, \partial U)$, $v = \nabla\varphi(x)$ and $A = \max_{\xi \in \overline{B_r(x)}} \left\| D^2\varphi(\xi) \right\| < \infty$ to satisfy (2.9).
Moreover, if U is bounded, we can find a uniform constant $A > 0$ such that (2.9) holds for every $x \in U$ (substitute $\overline{B_r(x)}$ by $\overline{U}$ in the choice of A above). Hence, $\varphi \in C^{1,1}[U]$.

We want to justify the previous definition and terminology by giving the following result which can be proved by combining the arguments from the proofs of [CC95, Proposition 1.1] and [CC95, Proposition 1.2].

Proposition 2.14. *Let $u : \Omega \to \mathbb{R}$ be a continuous function and $B \neq \emptyset$ be a convex domain such that $\overline{B} \subset \Omega$. Assume there exist constants $A > 0$ and $\varepsilon > 0$ such that for every $x \in \overline{B}$ there is some vector $v \in \mathbb{R}^n$ such that the inequality*

$$|u(z) - u(x) - (z - x) \cdot v| \leq A |z - x|^2 \tag{2.10}$$

holds for every $z \in B_\varepsilon(x) \cap \Omega$.
Then u belongs to the Lipschitz space $C^{1,1}(\overline{B})$ and

$$|\nabla u(x) - \nabla u(y)| \leq 4nA |x - y| \quad \text{for every } x, y \in \overline{B}. \tag{2.11}$$

Proof. Since $u \in C^{1,1}(x)$ for every $x \in \overline{B}$, we know that u is differentiable in $\overline{B}$ (see Remark 2.13). We have to show that the partial derivatives $\partial_i u$ are Lipschitz continuous and (2.11) to complete the proof. For this purpose, we claim that

$$D^\gamma u \in L^\infty(B) \quad \text{and} \quad \|D^\gamma u\|_{L^\infty(B)} \leq 4A$$

for every multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$ with $|\gamma| = \gamma_1 + \ldots + \gamma_n = 2$, where $D^\gamma u$ denotes the γ^{th} weak derivative of u. We prove this claim as follows:
Since $(L^1(B))^* = L^\infty(B)$, it is sufficient to prove that

$$\left| \int_B u(x) \partial_{ij}\varphi(x)\, dx \right| \leq 4A \|\varphi\|_{L^1(B)} \tag{2.12}$$

for every testfunction $\varphi \in C_c^\infty(B)$ and all indices $i, j \in \{1, \dots, n\}$ (cf. [Bre11, Proposition 8.3]). To prove (2.12), it is sufficient to prove

$$\left| \int_B u(x) \partial_{ii} \varphi(x)\, dx \right| \leq 2A \, \|\varphi\|_{L^1(B)} \tag{2.13}$$

for every testfunction $\varphi \in C_c^\infty(B)$ and every index $i \in \{1, \dots, n\}$, since

$$\partial_{ij}\varphi = \tfrac{1}{2}(2\partial_{vv}\varphi - \partial_{ii}\varphi - \partial_{jj}\varphi),$$

where $v = \frac{e_i + e_j}{\sqrt{2}}$ and $\{e_i\}$ is the canonical basis in $\mathbb{R}^n$. We prove (2.13): Denote the second differential quotients of u at x by

$$\Psi_h u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{|h|^2},$$

where $h \in \mathbb{R}^n$ such that $x + h$ and $x - h$ belong to Ω. Using (2.10), it is easy to see that

$$|\Psi_h u(x)| \leq 2A \tag{2.14}$$

for each $x \in \overline{B}$ and $h \in \mathbb{R}^n$ such that $|h| < \varepsilon \wedge \operatorname{dist}(\overline{B}, \partial\Omega) =: d$. Let $\varphi \in C_c^\infty(B)$ be any testfunction with $\operatorname{supp}(\varphi) = K$, where $K \subset B$ is a compact set. Then

$$\begin{aligned}
\int_B u(x)\partial_{ii}\varphi(x)\, dx &= \int_K u(x)\partial_{ii}\varphi(x)\, dx \\
&= \lim_{\delta \searrow 0} \int_K u(x) \Psi_{\delta e_i}\varphi(x)\, dx \\
&= \lim_{\delta \searrow 0} \int_K (\Psi_{\delta e_i} u(x))\varphi(x)\, dx.
\end{aligned}$$

If $\delta < d \wedge \operatorname{dist}(K, \partial B)$, (2.14) implies that $|\Psi_{\delta e_i} u(x)| \leq 2A$ for each $x \in K \subset B$; this proves (2.13). Hence, $\partial_i u \in W^{1,\infty}(B)$ for every $i = 1, \dots, n$, where $W^{1,\infty}(B)$ denotes the Sobolev space of functions which belong, together with their weak derivatives of order 1, to $L^\infty(B)$. Since $B \subset \Omega$ is bounded and convex, we have $W^{1,\infty}(B) \subset C^{0,1}(\overline{B})$ (cf. [AF03, Lemma 4.28]; note that B has locally Lipschitz boundary) which implies the Lipschitz continuity of $\partial_i u$ in $\overline{B}$ and

$$\begin{aligned}
\partial_i u(x) - \partial_i u(y) &= \int_0^1 \frac{d}{dt} \partial_i u(tx + (1-t)y)\, dt \\
&= \sum_{j=1}^n \int_0^1 \partial_{ij} u(tx + (1-t)y)\, dt \; (x_j - y_j)
\end{aligned}$$

for every $i \in \{1, \dots, n\}$ and $x, y \in \overline{B}$. Finally, we use that $\|D^\gamma u\|_{L^\infty(B)} \leq 4A$ for each $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = 2$ and conclude (2.11). □

Next, we introduce the concept of convex envelope of a continuous function. A function $L : \mathbb{R}^n \to \mathbb{R}$ is said to be *affine* if

$$L(x) = l_0 + l(x),$$

where $l_0 \in \mathbb{R}$ and $l : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Let $U \subset \mathbb{R}^n$, $x_0 \in U$ and consider a function $w : U \to \mathbb{R}$. Assume that the affine function L touches w from below at x_0 in U, i.e., $L(x) \leq w(x)$ for every $x \in U$ and $L(x_0) = w(x_0)$. In this situation, L is called a *supporting hyperplane* for w at x_0 in U.

Remark 2.15. Let $U \subset \mathbb{R}^n$ be an open convex set and $w : U \to \mathbb{R}$ be a convex function. Applying the theorem of Hahn-Banach (cf. [Wer00, Theorem III.2.4]) to the open convex set $\{(x, y) \in U \times \mathbb{R} : y > w(x)\}$ and the convex set $\{(x_0, w(x_0))\}$, we obtain the existence of a supporting hyperplane for w at x_0 (in U) for every $x_0 \in U$. Note that this hyperplane may not be unique.

Definition 2.16 ([CC95, Definition 3.1])**.** Let $v : U \to \mathbb{R}$ be continuous in an open convex set $U \subset \mathbb{R}^n$. The *convex envelope* of v in U is defined by

$$\begin{aligned}\Gamma(v)(x) = \Gamma_v(x) &= \sup\{w(x) : w \leq v \text{ in } U,\ w \text{ convex in } U\}\\ &= \sup\{L(x) : L \leq v \text{ in } U,\ L \text{ is affine}\}\end{aligned}$$

for $x \in U$. Note that the second equality is an immediate consequence of Remark 2.15.

Note that $\Gamma(v)$ is a convex function in U, since the supremum of a family of convex functions, is again convex. The set

$$\{v = \Gamma(v)\} = \{x \in U : v(x) = \Gamma(v)(x)\}$$

is called the (*lower*) *contact set* of v. The points in the contact set are called *contact points*. We will see that the contact set contains very important information about the function v.

We now state the ABP theorem adapted to viscosity supersolutions. Throughout the section, we fix some numbers $0 < \lambda \leq \Lambda$.

Theorem 2.17 ([CC95, Theorem 3.2])**.** *Let $r > 0$ and $f : B_r \to \mathbb{R}$ be a continuous and bounded function. Assume that $u \in C(\overline{B_r})$ is nonnegative on ∂B_r and belongs to the space $\overline{S}(\lambda, \Lambda, f)$ in B_r. We extend u by zero in $\overline{B_{2r}} \setminus \overline{B_r}$. Then*

$$\sup_{B_r} u^- \leq C_1 r \left(\int_{B_r \cap \{u = \Gamma_u\}} (f^+(x))^n \, dx \right)^{1/n}, \tag{2.15}$$

where Γ_u is the convex envelope of $-u^-$ in B_{2r} and $C_1 = C_1(\lambda, \Lambda, n) \geq 1$.

We always assume that $u^- \not\equiv 0$ which implies $\Gamma_u \not\equiv 0$. Otherwise (2.15) is trivially satisfied.

Remark 2.18. $-u^-$ is continuous in $\overline{B_{2r}}$ since $u \geq 0$ on ∂B_r and $u \equiv 0$ in $\overline{B_{2r}} \setminus \overline{B_r}$. In order to have $\Gamma_u \in C(\overline{B_{2r}})$, we extend $\Gamma_u \equiv 0$ on ∂B_{2r}; using Remark 2.15, there exists a supporting hyperplane for Γ_u at every point in $\overline{B_{2r}}$.

In order to prove Theorem 2.17, we will show that Γ_u belongs to the Lipschitz space $C^{1,1}(\overline{B_r})$. Then the classical proof of the ABP estimate for smooth functions can be applied. The following lemma deals with the regularity of the convex envelope Γ_u of $-u^-$ at the contact points. Since the proof is very technical and not applicable to the nonlocal setting in the next chapter, we skip it and refer to [CC95].

Lemma 2.19 ([CC95, Lemma 3.3])**.** *Let $u \in \overline{S}(\lambda, \Lambda, f)$ in $B_r(x_0)$, where $r > 0$ and $x_0 \in \mathbb{R}^n$. Let $f : B_r(x_0) \to \mathbb{R}$ be bounded (not necessarily continuous) and let φ be a convex function in $B_r(x_0)$ such that $0 \leq \varphi \leq u$ in $B_r(x_0)$ and $0 = \varphi(x_0) = u(x_0)$. Then*

$$\varphi(x) \leq C_2 (\sup_{B_r(x_0)} f^+) \, |x - x_0|^2 \quad \text{for every } x \in B_{\delta r}(x_0), \tag{2.16}$$

where $\delta = \delta(\lambda, \Lambda, n) \in (0,1)$ and $C_2 = C_2(\lambda, \Lambda, n) \geq 1$.

We illustrate the connection between Lemma 2.19 and the regularity of Γ_u in the contact points: Consider any $x_0 \in \overline{B_r} \cap \{u = \Gamma_u\}$, where u, Γ_u and f are as in Theorem 2.17. Let L be a supporting hyperplane for Γ_u at x_0 in B_{2r}. Applying Proposition 2.9 (with $F = \mathcal{M}^-$, $\Omega_1 = B_r$, $\Omega_2 = B_{2r}$, $u_1 = u$, $u_2 = 0$, $f_1 = f$, $f_2 = 0$), we obtain

$$-u^- = \inf(u, 0) \in \overline{S}(\lambda, \Lambda, h) \text{ in } B_{2r}, \text{ where } h(x) = \begin{cases} 0, & x \in B_{2r} \setminus B_r \\ f^+(x), & x \in B_r \;. \end{cases}$$

Note that $\Gamma_u - L$ is convex in B_{2r} (since L is affine) and $-u^- - L \in \overline{S}(\lambda, \Lambda, h)$ in B_{2r}, where we have used Lemma 2.11 with $\Phi = L$ and $g = 0$. In addition, $B_{r'}(x_0) \subset B_{2r}$ for every $r' \in (0, r]$ which implies – by the definition of Γ_u – that

$$0 \leq \Gamma_u - L \leq -u^- - L \text{ in } B_{r'}(x_0) \quad \text{and} \quad 0 = \Gamma_u(x_0) - L(x_0) = -u^-(x_0) - L(x_0).$$

Applying Lemma 2.19 (with $\varphi = \Gamma_u - L$, $u = -u^- - L$), we obtain

$$L(x) \leq \Gamma_u(x) \leq L(x) + C_2 (\sup_{B_{r'}(x_0)} h^+) \, |x - x_0|^2 \text{ for every } x \in B_{\delta r'}(x_0), \tag{2.17}$$

where δ and C_2 are the positive constants in Lemma 2.19.
Using (2.17), we obtain

$$|\Gamma_u(x) - \Gamma_u(x_0) - (x - x_0) \cdot \underbrace{\nabla \Gamma(x_0)}_{= \nabla L}| \leq C_2 (\sup_{B_r} f^+) \, |x - x_0|^2 =: A_1 \, |x - x_0|^2 \tag{2.18}$$

for every $x \in B_{\delta r}(x_0) \subset B_{2r}$. Hence, $\Gamma_u \in C^{1,1}(x_0)$, where $v = \nabla \Gamma_u(x_0) = \nabla L$ in Definition 2.12. Note that A_1 and $r_1 = \delta r$ are independent of x_0. We have proved the following corollary:

Corollary 2.20. *Under the conditions of Theorem 2.17, Γ_u is $C^{1,1}$ at every point $x_0 \in \overline{B_r} \cap \{u = \Gamma_u\}$ with constants $A > 0$ and $r > 0$ in (2.9) independent of x_0.*

The next lemma deals with the regularity of Γ_u outside the contact points. The previous result will be crucial for the proof. This stresses the importance of the contact set $\{u = \Gamma_u\}$.

Lemma 2.21. *Under the conditions of Theorem 2.17, Γ_u is $C^{1,1}$ at each point $x_0 \in \overline{B_r} \setminus \{u = \Gamma_u\}$ with constants $A > 0$ and $r > 0$ in (2.9) independent of x_0.*

Proof. We follow the proof of [CC95, Lemma 3.5]. Let $x_0 \in \overline{B_r} \setminus \{u = \Gamma_u\}$ and let L be a supporting hyperplane for Γ_u at x_0 in $\overline{B_{2r}}$. The proof is done in two steps:

Step 1. We divide this step into two claims:

a) Let $\operatorname{conv}(U)$ denote the convex hull of a set $U \subset \mathbb{R}^n$. We claim that $x_0 \in \mathcal{H} = \operatorname{conv}(\{x_1, \ldots, x_{n+1}\})$ for some points $x_1, \ldots, x_{n+1}$ which do not need to be distinct and belong to the contact set $B_r \cap \{u = \Gamma_u\}$, except for possible one, x_{n+1}, which is in ∂B_{2r}. Moreover, $L = \Gamma_u$ in $\mathcal{H}$.

b) Using a), we can write $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$, where $\lambda_i \geq 0$ for each $i = 1, \ldots, n+1$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Then $\lambda_i \geq \frac{1}{3n}$ for at least one index i for which $x_i \in B_r \cap \{u = \Gamma_u\}$.

Step 2. Using Step 1, we can find an estimate similar to (2.17) but in a smaller ball:

$$L(x) \leq \Gamma_u(x) \leq L(x) + 3nC_2(\sup_{B_r} f^+)\,|x - x_0|^2 \quad \text{for each } x \in B_{\delta r/(3n)}(x_0), \tag{2.19}$$

where δ and C_2 are as in (2.17). By the same arguments leading to (2.18), choosing again $v = \nabla \Gamma_u(x_0) = \nabla L$ in Definition 2.12, we obtain

$$|\Gamma_u(x) - \Gamma_u(x_0) - (x - x_0) \cdot v| \leq 3nC_2(\sup_{B_r} f^+)\,|x - x_0|^2 =: A_2\,|x - x_0|^2$$

for every $x \in B_{\delta r/(3n)}(x_0)$, where A_2 and $r_2 = \delta r/(3n)$ are again independent of x_0.

To prove a) in Step 1, note that

$$L(x_0) = \Gamma_u(x_0) = \sup\{\widetilde{L}(x_0) : \widetilde{L} \leq -u^- \text{ in } \overline{B_{2r}},\ \widetilde{L} \text{ is affine}\},$$

so L is the $\widetilde{L}$ that realises the supremum at x_0. This implies the existence of at least one point $x \in \overline{B_{2r}}$ such that $L(x) = -u^-(x)$. To prove this, assume that the statement is not true; so we can define $\widetilde{L} = L + \varepsilon$, where $\varepsilon = \min_{\overline{B_{2r}}}(-u^- - L) > 0$. Then we have $\widetilde{L} \geq L$ and $\widetilde{L} \leq -u^-$ in $\overline{B_{2r}}$. Contradiction to the maximality of L at x_0. Hence,

$$E = \{x \in \overline{B_{2r}} : L(x) = -u^-(x)\} \neq \emptyset.$$

This implies that

$$\widetilde{E} = \overline{\operatorname{conv}(E)} \neq \emptyset.$$

One can easily prove that $x_0 \in \widetilde{E}$ (by assuming $x_0 \notin \widetilde{E}$ and using the geometric version of the theorem of Hahn-Banach (cf. [Wer00, Theorem III.2.5]) which leads to a contradiction to the maximality of L at x_0). Using Carthéodory's theorem for convex hulls (cf. [Rud73, Theorem 3.25] and the following lemma), we have that x_0 is a convex combination of $n+1$ points $x_1, \dots, x_{n+1}$ in E, i.e., $x_0 \in \mathcal{H} = \operatorname{conv}(\{x_1, \dots, x_{n+1}\})$. It is easy to see that $L = \Gamma_u$ in $\mathcal{H}$, since for every $x \in \mathcal{H}$, $x = \sum_{i=1}^{n+1} \lambda_i x_i$ (where $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$), we have

$$L(x) = L\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \sum_{i=1}^{n+1} \lambda_i L(x_i) = \sum_{i=1}^{n+1} \lambda_i \Gamma_u(x_i) \geq \Gamma_u\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \Gamma_u(x).$$

Since $L(x) \leq \Gamma_u(x)$ for each $x \in \overline{B_{2r}}$, we obtain $L = \Gamma_u$ in $\mathcal{H}$. The remaining claim in a) is an immediate consequence of the fact that $\Gamma_u \not\equiv 0$ according to our general assumption. To prove b), consider two cases:

- All x_i belong to $B_r \cap \{u = \Gamma_u\}$: Then $\lambda_i \geq \frac{1}{n+1} \geq \frac{1}{3n}$ for at least one index i.
- $x_{n+1} \in \partial B_{2r}$: Assume that $\lambda_i < \frac{1}{3n}$ for each $i = 1, \dots, n$. Then $\lambda_{n+1} > \frac{2}{3}$ which implies

$$|x_0| > \tfrac{2}{3} |x_{n+1}| - \sum_{i=1}^{n} \tfrac{1}{3n} |x_i| > \tfrac{4}{3} r - \tfrac{1}{3} r = r.$$

  Contradiction.

We now prove (2.19) by using Step 1 and (2.17). Take any $h \in B_{\delta r/(3n)}$. Using Step 1, we can relabel x_i such that $x_1 \in B_r \cap \{u = \Gamma_u\}$ and $\lambda_1 \geq \frac{1}{3n}$. We write

$$x_0 + h = \lambda_1 (x_1 + \tfrac{h}{\lambda_1}) + \lambda_2 x_2 + \ldots + \lambda_{n+1} x_{n+1}.$$

Using the convexity of Γ_u, we obtain

$$L(x_0 + h) \leq \Gamma_u(x_0 + h) \leq \lambda_1 \Gamma_u(x_1 + \tfrac{h}{\lambda_1}) + \lambda_2 \Gamma_u(x_2) + \ldots + \lambda_{n+1} \Gamma_u(x_{n+1}).$$

Since $\frac{|h|}{\lambda_1} < \delta r$, we can apply (2.17) to estimate $\Gamma_u(x_1 + \frac{h}{\lambda_1})$ in the inequality above and obtain

$$\begin{aligned} L(x_0 + h) &\leq \Gamma_u(x_0 + h) \\ &\leq \lambda_1 \left(L(x_1 + \tfrac{h}{\lambda_1}) + C_2 (\sup_{B_r} f^+) \left|\tfrac{h}{\lambda_1}\right|^2\right) + \lambda_2 \underbrace{\Gamma_u(x_2)}_{=L(x_2)} + \ldots + \lambda_{n+1} \underbrace{\Gamma_u(x_{n+1})}_{=L(x_{n+1})} \\ &= L(x_0 + h) + \frac{C_2 (\sup_{B_r} f^+)}{\lambda_1} |h|^2 \leq L(x_0 + h) + 3n C_2 (\sup_{B_r} f^+) |h|^2 . \end{aligned}$$

This proves (2.19). □

Under the conditions of Theorem 2.17, we finally conclude – using Corollary 2.20, Lemma 2.21 and Proposition 2.14 (where $\Omega = B_{2r}$, $B = B_r$, $\varepsilon = \delta r/(3n)$ and $A = A_2$) – that $\Gamma_u \in C^{1,1}(\overline{B_r})$. We can now prove Theorem 2.17 like in the classical ABP setting for smooth functions.

Proof of Theorem 2.17. We follow the proof of [CC95, Lemma 3.4]. Recall that $u^- \not\equiv 0$. Since $u^- = 0$ on ∂B_r, we have

$$M = \sup_{B_r} u^- = u^-(x_0) > 0$$

for some $x_0 \in B_r$ because u^- is continuous.

We consider the function $k_{x_0} : \overline{B_{3r}(x_0)} \to \mathbb{R}$, whose graph is the cone in $\mathbb{R}^n \times \mathbb{R}$ with vertex $(x_0, -M) = (x_0, -u^-(x_0))$ and base $\partial B_{3r}(x_0) \times \{0\}$, i.e.,

$$k_{x_0}(x) = -M\left(1 - \frac{|x - x_0|}{3r}\right).$$

For each $\xi \in \mathbb{R}^n$ with

$$|\xi| < \frac{M}{3r},$$

the hyperplane $H = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} = L(x) = -M + \xi \cdot (x - x_0)\}$ is a supporting hyperplane for k_{x_0} at x_0 in $B_{3r}(x_0)$, i.e.,

$$L(x_0) = k_{x_0}(x_0) \quad \text{and} \quad L(x) \le k_{x_0}(x) \text{ for each } x \in B_{3r}(x_0).$$

Since $u^- \equiv 0$ outside B_r and $\overline{B_{2r}} \subset B_{3r}(x_0)$, it follows that H has a parallel hyperplane H' which is a supporting hyperplane for $-u^-$ in B_{2r} at some point $x^* \in B_r$.

We sketch the construction of H': Choose $b_1 > 0$ sufficiently large such that the affine function $L'_{b_1} = L - b_1$ satisfies $-u^-(x) > L'_{b_1}(x)$ for each $x \in B_{2r}$. As we decrease b_1, let $b \in [0, b_1)$ be the first value when the graphs of $-u^-$ and L'_b touch at a point x^*. Our desired hyperplane H' is the graph of L'_b. Indeed, we have $L'_b \le -u^-$ in B_{2r} and $x^* \in B_r$ because if we assume $x^* \in B_{2r} \setminus B_r$, we obtain the contradiction $L'_b(x^*) = -u^-(x^*) = 0 > k_{x_0}(x^*) \ge L(x^*)$.

By the definition of convex envelope, we have $\Gamma_u(x^*) = L'_b(x^*)$ and $\Gamma_u(x) \ge L'_b(x)$ for every $x \in B_{2r}$. Since Γ_u is differentiable in B_r, it follows that H' is the tangent hyperplane to the graph of Γ_u at x^* which implies $\xi = \nabla\Gamma_u(x^*)$. Hence,

$$B_{M/(3r)} \subset \nabla\Gamma_u(B_r).$$

This implies the existence of a constant $c = c(n) > 0$ such that

$$c\tfrac{M^n}{r^n} \le |\nabla\Gamma_u(B_r)|. \tag{2.20}$$

Applying the area formula for Lipschitz maps (cf. [CC95, Theorem 1.3]) to $\nabla\Gamma_u$ (recall that $\Gamma_u \in C^{1,1}(\overline{B_r})$), we obtain that $\nabla\Gamma_u$ is differentiable almost everywhere in B_r and

$$|\nabla\Gamma_u(B_r)| \le \int_U \left|\det D^2\Gamma_u(x)\right| \, dx,$$

where $U \subset B_r$ such that $|B_r \setminus U| = 0$. Using (2.20) and the fact that $D^2\Gamma_u(x)$ is nonnegative definite for each $x \in U$ because of the convexity of Γ_u in U, we obtain the following estimate:

$$c\tfrac{M^n}{r^n} \le \int_U \det D^2\Gamma_u(x)\, dx. \tag{2.21}$$

To conclude (2.15) from (2.21), we have to show that

$$\det D^2\Gamma_u(x) = 0 \quad \text{a.e. } x \in B_r \setminus \{u = \Gamma_u\} \tag{2.22}$$

and

$$\det D^2\Gamma_u(x) \le C_1 f^+(x)^n \quad \text{a.e. } x \in B_r \cap \{u = \Gamma_u\} \tag{2.23}$$

for some constant $C_1 = C_1(\lambda, \Lambda, n) \ge 1$.
(2.22) is an immediate consequence of a) in Step 1 in the proof of Lemma 2.21, since a) implies the existence of an open interval of a line through x on which Γ_u is affine (note that $\Gamma_u = L$ in the simplex mentioned in a)). Using in addition that Γ_u is second order differentiable almost everywhere in B_r, we obtain (2.22).
Finally, (2.23) is a consequence of (2.17) by letting $r' \searrow 0$ and using the fact that f is continuous. This finishes the proof of Theorem 2.17. □

We can extend Theorem 2.17 to arbitrary bounded domains:

Theorem 2.22 ([CC95, Theorem 3.6]). *Let $f : \Omega \to \mathbb{R}$ be a continuous and bounded function. Assume that $u \in \overline{S}(\lambda, \Lambda, f)$ in Ω, $u \in C(\overline{\Omega})$ and $u \ge 0$ on $\partial\Omega$. Extend u by zero outside Ω. Then*

$$\sup_\Omega u^- \le C_3 d \left\|f^+\right\|_{L^n(\Omega\cap\{u=\Gamma_u\})}, \tag{2.24}$$

where $C_3 = C_3(\lambda, \Lambda, n) \ge 1$, $d = \operatorname{diam}(\Omega)$ is the diameter of Ω, Γ_u is the convex envelope of $-u^-$ in B_{2d} and B_d is a ball of radius d such that $\Omega \subset B_d$.

Proof. As always we assume $u^- \not\equiv 0$ which implies $\Gamma_u \not\equiv 0$. We use Theorem 2.17 and Proposition 2.9: Since $\Omega \subset B_d \subset B_{2d}$, we apply Proposition 2.9 (with $F = \mathcal{M}^-$, $\Omega_1 = \Omega$, $\Omega_2 = B_{2d}$, $u_1 = u$, $u_2 = 0$, $f_1 = f$, $f_2 = 0$) and obtain $-u^- \in \overline{S}(\lambda, \Lambda, h)$ in B_d, where

$$h(x) = \begin{cases} 0, & x \in B_{2d} \setminus \Omega \\ f^+(x), & x \in \Omega. \end{cases}$$

We want to apply Theorem 2.17 to $-u^-$. Since $\Gamma_u \not\equiv 0$, the contact points belong to Ω because the existence of a point $x_0 \in (B_{2d} \setminus \Omega) \cap \{u = \Gamma_u\}$ would immediately imply $\Gamma_u \equiv 0$ by the definition of Γ_u. Note that $-u^- = 0$ on ∂B_d and $-u^- \in C(\overline{B_d})$ since $u \geq 0$ on $\partial\Omega$ and $u \equiv 0$ outside Ω. The only condition left to check in order to apply Theorem 2.17 to $-u^-$ is the continuity of h in B_d which we do not have in general. However, the last part of the proof of Theorem 2.17, leading to (2.23), shows that we can replace the condition that f is continuous in Theorem 2.17 by the condition

$$\sup_{B_{r'}(x_0)} f^+ \searrow f^+(x_0) \quad \text{as } r' \searrow 0 \tag{2.25}$$

for a.e. $x_0 \in B_r(0) \cap \{u = \Gamma_u\}$. Since h is continuous in Ω and every contact point belongs to Ω, h satisfies (2.25) and therefore we can apply Theorem 2.17 which leads to (2.24). □

As an immediate consequence of Theorem 2.22, we obtain the *maximum principle* for viscosity solutions which will be useful in the next section.

Corollary 2.23 ([CC95, Corollary 3.7])**.** *Let* $u \in C(\overline{\Omega})$*.*

(i) If $u \in \overline{S}(\lambda, \Lambda, 0)$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$*, then* $u \geq 0$ *in* Ω*.*

(ii) If $u \in \underline{S}(\lambda, \Lambda, 0)$ *in* Ω *and* $u \leq 0$ *on* $\partial\Omega$*, then* $u \leq 0$ *in* Ω*.*

Proof. (i) follows directly from Theorem 2.22. (ii) follows from (i) and the fact that $u \in \underline{S}(\lambda, \Lambda, f) \Rightarrow -u \in \overline{S}(\lambda, \Lambda, -f)$. □

2.4 Harnack Inequality and Hölder Regularity for Viscosity Solutions and the class S

The Krylov and Safonov Harnack inequality (see [GT01, Section 9.8]) for any nonnegative solution u to the following uniformly elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) + b(x) \cdot \nabla u(x) = f(x), \quad x \in B_1,$$

where $\lambda I \leq (a_{ij}(x))_{ij} \leq \Lambda I$ for all $x \in B_1$, $b \in L^n(B_1)$ and $f \in L^n(B_1)$, states that the supremum of u in $B_{1/2}$ is controlled by the infimum of u in $B_{1/2}$ plus the L^n-norm of f:

$$\sup_{B_{1/2}} u \leq c(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)}).$$

The constant $c > 0$ depends only on n, λ, Λ and $\|b\|_{L^n(B_1)}$. We summarise the adaption of the Krylov and Safonov Harnack inequality to viscosity supersolutions and the class S. We will use this Harnack inequality to prove an interior Hölder regularity result which can be extended up to the boundary by using a barrier argument when the boundary data is Hölder continuous. Fix $0 < \lambda \leq \Lambda$ throughout the section.

2.4.1 Harnack Inequality

The following theorem is the Harnack inequality for viscosity solutions. Recall the definition of cubes given in Section 1.4.

Theorem 2.24 ([CC95, Theorem 4.3])**.** *Let $f : Q_1 \to \mathbb{R}$ be a continuous and bounded function, where $Q_1 \subset \mathbb{R}^n$ is the open cube centered at 0 with edge length 1.*
Assume that $u : Q_1 \to \mathbb{R}$ is nonnegative in Q_1 and belongs to $S^*(\lambda, \Lambda, f)$ in Q_1. Then

$$\sup_{Q_{1/2}} u \leq C_4(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)}), \tag{2.26}$$

where the constant $C_4 \geq 1$ depends only on n, λ and Λ.

Remark 2.25. Theorem 2.24 can be extended to any cube $Q_R(x_0)$, where $R > 0$ and $x_0 \in \mathbb{R}^n$. To be precise: Let $f : Q_R(x_0) \to \mathbb{R}$ be continuous and bounded. Let $v \in S^*(\lambda, \Lambda, f)$ in $Q_R(x_0)$ be a nonnegative function in $Q_R(x_0)$. Define

$$u(x) = v(Rx + x_0), \quad x \in Q_1.$$

Then u is nonnegative in Q_1 and belongs to $S^*(\lambda, \Lambda, R^2 f(Rx + x_0))$ in Q_1. We apply Theorem 2.24 to u and obtain (after rescaling)

$$\sup_{Q_{R/2}(x_0)} v \leq C_5(\inf_{Q_{R/2}(x_0)} v + \|f\|_{L^n(Q_R(x_0))}),$$

where $C_5 = C_5(n, \lambda, \Lambda, R) \geq 1$.

Theorem 2.24 is an immediate consequence of the following lemma.

Lemma 2.26 ([CC95, Lemma 4.4])**.** *Let $f : Q_{4\sqrt{n}} \to \mathbb{R}$ be a continuous and bounded function. Let $u \in S^*(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ be nonnegative in $Q_{4\sqrt{n}}$ and continuous in $\overline{Q_{4\sqrt{n}}}$. Assume that $\inf_{Q_{1/4}} u \leq 1$. Then there exist constants $\varepsilon_0 \in (0, 1)$ and $C_6 \geq 1$ (depending only on λ, Λ and n) such that*

$$\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0 \Rightarrow \sup_{Q_{1/4}} u \leq C_6. \tag{2.27}$$

We obtain Theorem 2.24 from Lemma 2.26 in the following way: Let $f : Q_{4\sqrt{n}} \to \mathbb{R}$ be continuous and bounded. Let $u \in S^*(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ be nonnegative in $Q_{4\sqrt{n}}$ and continuous in $\overline{Q_{4\sqrt{n}}}$. For $\delta > 0$, let

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \delta + (\|f\|_{L^n(Q_{4\sqrt{n}})} / \varepsilon_0)},$$

where ε_0 is the (sufficiently small) number in Lemma 2.26 which only depends on λ, Λ and n. Since $\mathcal{M}^\pm(aM, \lambda, \Lambda) = a\mathcal{M}^\pm(M, \lambda, \Lambda)$ for each $a \geq 0$, $M \in \mathcal{S}$ and $u \in S^*(\lambda, \Lambda, f)$, we have

$$u_\delta \in S^*\big(\lambda, \Lambda, \widetilde{f}\,\big)$$

for each $\delta > 0$, where $\widetilde{f} = \frac{f}{\inf_{Q_{1/4}} u+\delta+(\|f\|_{L^n(Q_{4\sqrt{n}})}/\varepsilon_0)}$. It is easy to see that u_δ and $\widetilde{f}$ satisfy all the conditions in Lemma 2.26. We apply Lemma 2.26 and obtain (after letting $\delta \searrow 0$)

$$\sup_{Q_{1/4}} u \leq C_7(\inf_{Q_{1/4}} u + \|f\|_{L^n(Q_{4\sqrt{n}})}), \tag{2.28}$$

where $C_7 = \frac{C_6}{\varepsilon_0} \geq 1$.
We use (2.28) and a covering argument to conclude Theorem 2.24: Let $u : Q_1 \to \mathbb{R}$ and $f : Q_1 \to \mathbb{R}$ be as in Theorem 2.24. Fix $r = \frac{1}{8\sqrt{n}}$. Choose $m = m(n) \in \mathbb{N}$ and $x_1, \ldots, x_m \in Q_{1/2}$ such that

$$\bigcup_{i=1}^{m} Q_{r/4}(x_i) \supset Q_{1/2}\,.$$

Consider the function

$$\widetilde{u}(x) = u(rx + x_i),\ \ x \in \overline{Q_{4\sqrt{n}}}.$$

Note that $rx + x_i \in \overline{Q_{1/2}(x_i)} \subset Q_1$ whenever $x \in \overline{Q_{4\sqrt{n}}}$ (by triangle inequality). We have $\widetilde{u} \in S^*(\lambda, \Lambda, r^2 f(rx + x_i))$ in $Q_{4\sqrt{n}}$, $\widetilde{u} \geq 0$ in $Q_{4\sqrt{n}}$ and $\widetilde{u} \in C(\overline{Q_{4\sqrt{n}}})$. Using (2.28) and rescaling leads to

$$\sup_{Q_{r/4}(x_i)} u \leq C_7(\inf_{Q_{r/4}(x_i)} u + \|f\|_{L^n(Q_1)}) \tag{2.29}$$

for every $i = 1, \ldots, m$, where C_7 is as in (2.28). We finally obtain Theorem 2.24 as an immediate consequence of (2.29), using the following chain argument:
For any points $x, y \in Q_{1/2}$ we can find cubes $Q^1, Q^2, \ldots, Q^s \in \{Q_{r/4}(x_i)\}_{i=1}^m$, $1 \leq s \leq m$, such that

$$x \in Q^1,\ Q^1 \cap Q^2 \neq \emptyset,\ Q^2 \cap Q^3 \neq \emptyset,\ \ldots,\ Q^{s-1} \cap Q^s \neq \emptyset,\ y \in Q^s.$$

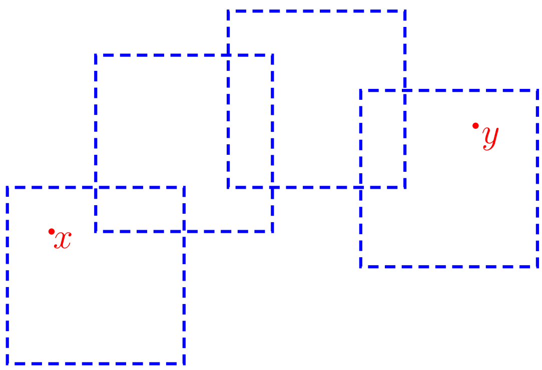

Figure 2.1: The cubes $Q^1, \ldots, Q^4$.

We apply (2.29) repeatedly and obtain

$$u(x) \le sC_7^s(\inf_{Q^s} u + \|f\|_{L^n(Q_1)}) \le mC_7^m(u(y) + \|f\|_{L^n(Q_1)}).$$

Hence,

$$\sup_{Q_{1/2}} u \le C_4(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)}), \quad \text{where } C_4 = mC_7^m,$$

which proves (2.26).

To prove Lemma 2.26, we need several lemmas. Some of them will also be useful in a nonlocal setting. From now on $f : Q_{4\sqrt{n}} \to \mathbb{R}$ will always be bounded and continuous.

Lemma 2.27 ([CC95, Lemma 4.5])**.** *There exist constants $\varepsilon_0 > 0$, $\mu \in (0,1)$ and $M > 1$ depending only on λ, Λ and n, such that if $u \in \overline{S}(\lambda, \Lambda, |f|)$ in $Q_{4\sqrt{n}}$, $u \in C(\overline{Q_{4\sqrt{n}}})$ and f satisfy*

(i) $u \ge 0$ *in* $Q_{4\sqrt{n}}$,

(ii) $\inf_{Q_3} u \le 1$ *and*

(iii) $\|f\|_{L^n(Q_{4\sqrt{n}})} \le \varepsilon_0$,

then

$$|\{u \le M\} \cap Q_1| > \mu. \tag{2.30}$$

The proof of Lemma 2.27 is based on the construction of the following *barrier function*: There exists $\varphi \in C^\infty(\mathbb{R}^n)$ and constants $C_8 \ge 1$ and $M > 1$ (note that M is the constant which will be needed in Lemma 2.27) depending only on λ, Λ and n such that

$$\varphi \ge 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\sqrt{n}}, \tag{2.31}$$

$$\varphi \le -2 \text{ in } Q_3 \text{ and} \tag{2.32}$$

$$\mathcal{M}^+(D^2\varphi, \lambda, \Lambda) \le C_8\xi \text{ in } \mathbb{R}^n, \tag{2.33}$$

where $0 \le \xi \le 1$ is a continuous function in $\mathbb{R}^n$ with $\operatorname{supp} \xi \subset \overline{Q_1}$. Moreover,

$$\varphi \ge -M \quad \text{in } \mathbb{R}^n. \tag{2.34}$$

We do not prove the existence of φ at this point because we will construct functions with similar properties in Chapter 3. Instead, we refer to [CC95, Lemma 4.1].

Proof of Lemma 2.27. The main idea of the proof is to add the barrier function φ from above to our nonnegative supersolution u and apply the ABP estimate from the previous section. Define $w : \overline{Q_{4\sqrt{n}}} \to \mathbb{R}$, $w = u + \varphi$. Note that the supremum of the negative part of w in $B_{2\sqrt{n}}$ is bounded from below (because of φ) which will be the key to prove (2.30). Recall that $Q_{4\sqrt{n}} \supset B_{2\sqrt{n}} \supset Q_3$. Using (2.33) from above, the fact that

$\mathcal{M}^+(N) = -\mathcal{M}^-(-N)$ for every $N \in \mathcal{S}$, and $\varphi \in C^\infty(\mathbb{R}^n)$, we can apply Lemma 2.11 (with $\Omega = B_{2\sqrt{n}}$) and obtain

$$w \in \overline{S}(\lambda, \Lambda, |f| + C_8\xi) \quad \text{in } B_{2\sqrt{n}}.$$

In addition, $w \geq 0$ on $\partial B_{2\sqrt{n}}$ because of (i) and (2.31). Moreover, using (ii) and (2.32), $\inf_{Q_3} w \leq -1$ ($\Rightarrow \sup_{B_{2\sqrt{n}}} w^- \geq 1$); finally, $w \in C(\overline{B_{2\sqrt{n}}})$. So we can apply Theorem 2.17 and obtain a constant $C_1 \geq 1$ such that

$$\begin{aligned} 1 &\leq C_1 2\sqrt{n} \Bigg(\int_{\{w=\Gamma_w\}\cap B_{2\sqrt{n}}} (|f(x)| + C_8\xi(x))^n \, dx \Bigg)^{1/n} \\ &\leq c_1 \|f\|_{L^n(Q_{4\sqrt{n}})} + c_1 \left|\{w = \Gamma_w\} \cap Q_1\right|^{1/n}, \end{aligned}$$

where Γ_w is the convex envelope of $-w^-$ in $B_{4\sqrt{n}}$ and $c_1 \geq 1$ depends only on n, λ and Λ. Note that the second estimate is due to the fact that $0 \leq \xi \leq 1$ and $\operatorname{supp} \xi \subset \overline{Q_1}$. Choosing $\varepsilon_0 = \frac{1}{2c_1}$, (iii) and the estimates from above imply

$$\frac{1}{2} \leq c_1 \left|\{w = \Gamma_w\} \cap Q_1\right|^{1/n}. \tag{2.35}$$

For each $x \in \{w = \Gamma_w\}$, we have $w(x) \leq 0$ (by definition of Γ_w) and therefore

$$u(x) \leq -\varphi(x) \leq M$$

with $M > 1$ as in (2.34). Using this fact and (2.35), we conclude

$$\frac{1}{2} \leq c_1 \left|\{u \leq M\} \cap Q_1\right|^{1/n},$$

which proves (2.30) for any positive μ satisfying $\mu < \frac{1}{(2c_1)^n}$. □

The next lemma (which will also be useful in the next chapter) uses Lemma 2.27 to obtain estimates similar to (2.30) but involving M^k, $k \in \mathbb{N}_0$, where $M > 1$ is the number in Lemma 2.27. As a consequence, we will obtain some power decay for the distribution function of u in Q_1 (with u as in Lemma 2.27), i.e., the function $\lambda_u : (0, \infty) \to [0, 1]$, $\lambda_u(t) = |\{u > t\} \cap Q_1|$ will be bounded by $dt^{-\varepsilon}$ for each $t > 0$, where $d > 1$ and $\varepsilon > 0$ only depend on λ, Λ and n.

Lemma 2.28 ([CC95, Lemma 4.6])**.** *Let u be as in Lemma 2.27. Then the following estimate holds for every $k \in \mathbb{N}_0$:*

$$\Big|\{u > M^k\} \cap Q_1\Big| \leq (1 - \mu)^k, \tag{2.36}$$

where M and μ are as in Lemma 2.27. As a consequence,

$$|\{u \geq t\} \cap Q_1| \leq dt^{-\varepsilon} \quad \textit{for every } t > 0, \tag{2.37}$$

where $d > 1$ and $\varepsilon > 0$ only depend on λ, Λ and n.

To prove this lemma, we need a corollary of the Calderón-Zygmund cube decomposition (cf. [GT01, Section 9.3]):

Consider the unit cube Q_1 and split it into 2^n cubes of half diameter. As a next step, split each one of these 2^n cubes in the same way as before and iterate this procedure. The cubes obtained in this way are called *dyadic cubes*. If Q is a dyadic cube different from Q_1, we say that $\widetilde{Q}$ is the predecessor of Q if Q is one of the 2^n cubes obtained from dividing $\widetilde{Q}$.

Lemma 2.29 ([CC95, Lemma 4.2]). *Let $A \subset B \subset Q_1$ be measurable sets and $\delta \in (0,1)$ such that*

(i) $|A| \leq \delta$ and

(ii) If Q is a dyadic cube such that $|A \cap Q| > \delta |Q|$ then $\widetilde{Q} \subset B$.

Then $|A| \leq \delta |B|$.

Proof of Lemma 2.28. For $k = 0$, (2.36) is trivial since $|Q_1| = 1$. For $k = 1$, (2.36) is the estimate from Lemma 2.27:

$$\begin{aligned} &\big|Q_1 \cap \big(\{u \leq M\} \cup \{u > M\}\big)\big| = |Q_1| = 1 \\ \Rightarrow\ &|Q_1 \cap \{u \leq M\}| + |Q_1 \cap \{u > M\}| = 1 \\ \Rightarrow\ &|Q_1 \cap \{u > M\}| = 1 - |Q_1 \cap \{u \leq M\}| \leq 1 - \mu. \end{aligned}$$

Assume that (2.36) holds for $k-1$, $k \geq 2$, and let

$$A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1.$$

We prove (2.36) for k by showing that

$$|A| \leq (1-\mu)\,|B|\,. \tag{2.38}$$

We want to apply Lemma 2.29: So let us check whether conditions (i) and (ii) in Lemma 2.29 are satisfied. Clearly $A \subset B \subset Q_1$ and $|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$. It remains to prove condition (ii): We need to show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube for some $x_0 \in Q_1$ and $i \in \mathbb{N}$ satisfying

$$|A \cap Q| > (1-\mu)\,|Q|\,, \tag{2.39}$$

then $\widetilde{Q} \subset B$. Assume $\widetilde{Q} \not\subset B$ and choose

$$\widetilde{x} \in \widetilde{Q} \quad \text{such that} \quad u(\widetilde{x}) \leq M^{k-1}. \tag{2.40}$$

For $y \in \mathbb{R}^n$ consider the transformation

$$\tau_i(y) = x_0 + \frac{1}{2^i}y.$$

Note that $y \in Q_1 \Leftrightarrow \tau_i(y) \in Q = Q_{1/2^i}(x_0)$. We define $\widetilde{u} : \overline{Q_{4\sqrt{n}}} \to \mathbb{R}$,

$$\widetilde{u}(y) = \frac{u(\tau_i(y))}{M^{k-1}}.$$

We claim that $\widetilde{u}$ satisfies all the conditions in Lemma 2.27:

- Let $y \in Q_{4\sqrt{n}}$. Note that $\tau_i(y) = x_0 + \frac{1}{2^i} y \in Q_{4\sqrt{n}/2^i}(x_0) \subset Q_{4\sqrt{n}}$. Therefore, using the definition of $\widetilde{u}$ and the chain rule,

$$\mathcal{M}^-(D^2\widetilde{u}(y), \lambda, \Lambda) \leq \frac{1}{M^{k-1}2^{2i}} f(x_0 + \tfrac{1}{2^i} y) =: \widetilde{f}(y),$$

where we have assumed for simplicity that $\widetilde{u} \in C^2(N)$ for some open neighborhood N of y. Hence, $\widetilde{u} \in \overline{S}(\lambda, \Lambda, \widetilde{f}\,)$ in $Q_{4\sqrt{n}}$.

- Since $u \geq 0$ in $Q_{4\sqrt{n}}$ and $u \in C(\overline{Q_{4\sqrt{n}}})$, we have the same properties for $\widetilde{u}$.
- Let $\widetilde{x} \in \widetilde{Q}$ be the point in (2.40). Since $\widetilde{Q}$ is the predecessor of $Q = Q_{1/2^i}(x_0)$, we have $\widetilde{Q} \subset Q_{3/2^i}(x_0)$ which implies $z = 2^i(\widetilde{x} - x_0) \in Q_3$. Moreover, $\widetilde{u}(z) = \frac{u(\widetilde{x})}{M^{k-1}}$. Hence, using (2.40),

$$\inf_{Q_3} \widetilde{u} \leq \frac{u(\widetilde{x})}{M^{k-1}} \leq 1.$$

- Finally,

$$\|\widetilde{f}\,\|_{L^n(Q_{4\sqrt{n}})} = \frac{2^i}{2^{2i}M^{k-1}} \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0.$$

Using Lemma 2.27, we obtain

$$\mu < |\{y \in Q_1 : \widetilde{u}(y) \leq M\}| = 2^{in} \left|\{x \in Q : u(x) \leq M^k\}\right|.$$

Hence, $|Q \setminus A| > \mu\,|Q|$. At the same time, we obtain from (2.39)

$$|Q \setminus A| = |Q| - |A \cap Q| < |Q| - (1-\mu)\,|Q| = \mu\,|Q|\,.$$

Contradiction.

(2.37) follows immediately from (2.36) by choosing $d = (1-\mu)^{-1}$ and $\varepsilon = \frac{\log(1/(1-\mu))}{\log(M)}$. Note that the choice of ε implies $1 - \mu = M^{-\varepsilon}$. We prove (2.37):
For $0 < t \leq 1$, (2.37) is trivial since $|\{u \geq t\} \cap Q_1| \leq 1 \leq dt^{-\varepsilon}$.
For $t > 1$, choose $k \in \mathbb{N}_0$ such that $M^k < t \leq M^{k+1}$. Therefore,

$$\begin{aligned} |\{u \geq t\} \cap Q_1| &\leq \left|\{u > M^k\} \cap Q_1\right| \leq (1-\mu)^k = d(1-\mu)^{k+1} \\ &= d(M^{-\varepsilon})^{k+1} = d(M^{k+1})^{-\varepsilon} \leq dt^{-\varepsilon}. \end{aligned}$$

□

Using similar arguments as in the proof of Lemma 2.28, we obtain the following result. Once again, we refer to [CC95] for a proof of this result because we will not need it in the next chapter.

Lemma 2.30 ([CC95, Lemma 4.7]). *Let $u \in \underline{S}(\lambda, \Lambda, -|f|)$ in $Q_{4\sqrt{n}}$. Assume that f satisfies (iii) in Lemma 2.27 and u satisfies* (2.37).
There exist constants $M_0 > 1$ and $\sigma > 0$ depending only on λ, Λ and n such that, for ε as in (2.37) *and $\nu = \frac{M_0}{M_0 - \frac{1}{2}} > 1$, the following hold:*

For each $j \in \mathbb{N}$ and each $x_0 \in \overline{Q_{1/2}}$ satisfying

$$u(x_0) \geq \nu^{j-1} M_0, \tag{2.41}$$

the relations

$$Q^j := Q_{l_j}(x_0) \subset Q_1 \quad \textit{and} \quad \sup_{Q^j} u \geq \nu^j M_0$$

hold, where $l_j = \sigma M_0^{-\varepsilon/n} \nu^{-\varepsilon j/n}$.

We can finally prove Lemma 2.26.

Proof of Lemma 2.26. Let u and f be as in Lemma 2.26. Take ε_0 as in the proof of Lemma 2.27 and assume that

$$\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0.$$

Then u and f satisfy the conditions of Lemma 2.27 and Lemma 2.28 (note that $S^*(\lambda, \Lambda, f) \subset \overline{S}(\lambda, \Lambda, |f|)$ by definition) and hence of Lemma 2.30 (note that $S^*(\lambda, \Lambda, f) \subset \underline{S}(\lambda, \Lambda, -|f|)$ by definition). Let M_0, ν and l_j, $j \in \mathbb{N}$, be as in Lemma 2.30. Since $\nu > 1$, we can find $j_0 \in \mathbb{N}$ (depending only on n, λ and Λ) such that

$$\sum_{j \geq j_0} l_j \leq \frac{1}{4}. \tag{2.42}$$

We claim that

$$\sup_{Q_{1/4}} u \leq \nu^{j_0 - 1} M_0,$$

which will finish the proof. We prove the claim by contradiction: Let us assume that the claim is not true. This implies the existence of a point $x_{j_0} \in \overline{Q_{1/4}}$ such that

$$u(x_{j_0}) \geq \nu^{j_0 - 1} M_0.$$

We apply Lemma 2.30 to obtain a point x_{j_0+1} such that

$$|x_{j_0+1} - x_{j_0}|_\infty \leq \tfrac{l_{j_0}}{2} \quad \text{and} \quad u(x_{j_0+1}) \geq \nu^{j_0} M_0.$$

By induction, we construct a sequence $(x_j)_{j \geq j_0}$ such that for each $j \geq j_0$

$$|x_{j+1} - x_j|_\infty \leq \tfrac{l_j}{2} \quad \text{and} \quad u(x_{j+1}) \geq \nu^j M_0. \tag{2.43}$$

We already constructed the beginning where $j = j_0$. Assume we have constructed the sequence up to $j \geq j_0$. In order to apply Lemma 2.30 to x_{j+1}, we only need to check whether $x_{j+1} \in \overline{Q_{1/2}}$, since we already have $u(x_{j+1}) \geq \nu^j M_0$ by induction hypothesis. Using (2.42) and (2.43), we obtain

$$\begin{aligned} |x_{j+1}|_\infty &\leq |x_{j_0}|_\infty + \sum_{k=j_0}^{j} |x_{k+1} - x_k|_\infty \\ &\overset{(2.43)}{\leq} \tfrac{1}{8} + \sum_{k \geq j_0} \tfrac{l_k}{2} \overset{(2.42)}{\leq} \tfrac{1}{4}. \end{aligned}$$

So we can apply Lemma 2.30 and obtain a point x_{j+2} which satisfies

$$|x_{j+2} - x_{j+1}|_\infty \leq \tfrac{l_{j+1}}{2} \quad \text{and} \quad u(x_{j+2}) \geq \nu^{j+1} M_0.$$

We easily see that $(x_j)_{j \geq j_0}$ is a Cauchy sequence in $\overline{Q_{1/2}}$ because $\nu > 1$ and $x_j \in \overline{Q_{1/2}}$ for each $j \geq j_0$, which implies the existence of a point $x_0 \in \overline{Q_{1/2}}$ such that

$$|x_j - x_0|_\infty \xrightarrow[]{\substack{j\to\infty,\\ j\geq j_0}} 0.$$

Using the fact that u is continuous in $\overline{Q_{1/2}}$ and $\nu > 1$, we finally obtain the contradiction

$$u(x_0) \xleftarrow[]{\substack{j\to\infty,\\ j\geq j_0}} u(x_j) \geq \nu^{j-1} M_0 \xrightarrow[]{\substack{j\to\infty,\\ j\geq j_0}} \infty,$$

i.e., $u(x_0) \geq c$ for every $c > 0$. □

2.4.2 Hölder Regularity

We repeat the definition of Hölder spaces from Section 1.4.

Definition 2.31.

(i) A function $f : \Omega \to \mathbb{R}$ is called *Hölder continuous with exponent* $\alpha \in (0, 1]$ if

$$[f]_{C^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

(ii) We define the *space of Hölder continuous functions with exponent* $\alpha \in (0, 1]$ by

$$C^\alpha(\overline{\Omega}) = \{f \in C(\overline{\Omega}) \,:\, [f]_{C^\alpha(\overline{\Omega})} < \infty\}.$$

Then, $C^\alpha(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|f\|_{C^\alpha(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + [f]_{C^\alpha(\overline{\Omega})}.$$

We apply the previous Harnack inequality to obtain an *interior Hölder continuity* of solutions. The *oscillation* of a function $f : \mathbb{R}^n \to \mathbb{R}$ in a set $A \subset \mathbb{R}^n$ is defined by

$$\operatorname{osc}_A f = \sup\{|f(x) - f(y)| : x, y \in A\}.$$

Proposition 2.32 ([CC95, Proposition 4.10])**.** *Let $f : Q_1 \to \mathbb{R}$ be a continuous and bounded function and $u \in S^*(\lambda, \Lambda, f)$ in Q_1.*

a) There exists a constant $\mu = \mu(\lambda, \Lambda, n) \in (0, 1)$ such that

$$\operatorname{osc}_{Q_{1/2}} u \leq \mu \operatorname{osc}_{Q_1} u + 2 \|f\|_{L^n(Q_1)}.$$

b) There exist constants $\alpha = \alpha(\lambda, \Lambda, n) \in (0,1)$ *and* $C_9 = C_9(\lambda, \Lambda, n) \geq 1$ *such that* $u \in C^\alpha(\overline{Q_{1/2}})$ *and*

$$\|u\|_{C^\alpha(\overline{Q_{1/2}})} \leq C_9(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}).$$

Proof. For $r > 0$, define $m_r = \inf_{Q_r} u$, $M_r = \sup_{Q_r} u$ and $o_r = \operatorname{osc}_{Q_r} u$. Note that the functions $u_1 = u - m_1$ and $u_2 = M_1 - u$ are nonnegative functions in Q_1 and belong to $S^*(\lambda, \Lambda, f)$ in Q_1. Therefore, we can apply the Harnack inequality (Theorem 2.24) to u_1 and u_2 and obtain

$$\begin{aligned} M_{1/2} - m_1 &\leq C_4(m_{1/2} - m_1 + \|f\|_{L^n(Q_1)}) \quad \text{and} \\ M_1 - m_{1/2} &\leq C_4(M_1 - M_{1/2} + \|f\|_{L^n(Q_1)}), \end{aligned}$$

where $C_4 = C_4(\lambda, \Lambda, n) \geq 1$. Adding both inequalities leads to

$$o_{1/2} + o_1 \leq C_4(o_1 - o_{1/2} + 2\|f\|_{L^n(Q_1)}).$$

This implies

$$o_{1/2} \leq \frac{C_4 - 1}{C_4 + 1} o_1 + \frac{2C_4}{C_4 + 1} \|f\|_{L^n(Q_1)},$$

which proves a).
We prove b) by imitating the proof of [GT01, Lemma 8.23]. Note that for every $R \in (0, \frac{1}{2}]$ and every $x_0 \in \overline{Q_{1/2}}$ we have $Q_R(x_0) \subset Q_1$ and therefore, using Remark 2.25 and a),

$$\operatorname{osc}_{Q_{R/2}(x_0)} u \leq \mu \operatorname{osc}_{Q_R(x_0)} u + 2R\|f\|_{L^n(Q_R(x_0))}, \tag{2.44}$$

where μ is the same number as in a). Let $x_0 \in \overline{Q_{1/2}}$. For $R \in (0, \frac{1}{2}]$, we write o_R to denote $\operatorname{osc}_{Q_R(x_0)} u$. Consider any $R_1 \in (0, \frac{1}{2}]$. Iteration of (2.44) gives for every $m \in \mathbb{N}$

$$\begin{aligned} o_{(1/2)^m R_1} &\leq \mu^m o_{R_1} + 2R_1 \|f\|_{L^n(Q_1)} \sum_{k=0}^{m-1} \mu^k \\ &\leq \mu^m o_{1/2} + \tfrac{2R_1\|f\|_{L^n(Q_1)}}{1-\mu}. \end{aligned}$$

For every $R \in (0, R_1]$, we can choose $m \in \mathbb{N}$ such that

$$(\tfrac{1}{2})^m R_1 < R \leq (\tfrac{1}{2})^{m-1} R_1.$$

Hence,

$$\begin{aligned} o_R \leq o_{(1/2)^{m-1}R_1} &\leq \mu^{m-1} o_{1/2} + \tfrac{2R_1\|f\|_{L^n(Q_1)}}{1-\mu} \\ &\leq \tfrac{1}{\mu}\left(\tfrac{R}{R_1}\right)^{\log\mu/\log(1/2)} o_{1/2} + \tfrac{2R_1\|f\|_{L^n(Q_1)}}{1-\mu}. \end{aligned}$$

Let $\gamma \in (0,1)$ be arbitrary for the moment. For $R \in (0, \frac{1}{2}]$, let $R_1 = (\frac{1}{2})^{1-\gamma} R^\gamma$. We have $R \le R_1 \le \frac{1}{2}$ and therefore, using the previous estimates, we obtain

$$o_R \le \tfrac{1}{\mu}(2R)^{(1-\gamma)(\log \mu / \log(1/2))} o_{1/2} + \tfrac{(2R)^\gamma \|f\|_{L^n(Q_1)}}{1-\mu}$$

for every $R \in (0, \frac{1}{2}]$. Finally, choose γ large enough such that $\alpha = (1-\gamma)\frac{\log \mu}{\log 1/2} \in (0,1)$ and $\gamma > \alpha$. Note that the choice of γ depends only on λ, Λ and n. Therefore, we obtain the following estimate: For every $x_0 \in \overline{Q_{1/2}}$ and every $R \in (0, \frac{1}{2}]$, the inequality

$$\operatorname{osc}_{Q_R(x_0)} u \le c_1 R^\alpha (\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}) \tag{2.45}$$

holds, where $c_1 = c_1(\lambda, \Lambda, n) = 4 \max\{\frac{1}{\mu}, \frac{1}{1-\mu}\} > 4$.

Let $x_0, y \in \overline{Q_{1/2}}$, $x_0 \ne y$. There exists $k \in \mathbb{N}$ such that $2^{-k-1} < |x_0 - y|_\infty \le 2^{-k}$. Using (2.45) and the fact that $\operatorname{osc}_{Q_1} u \le 2\|u\|_{L^\infty(Q_1)} \le c_1(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)})$, we conclude

$$\begin{aligned} |u(x_0) - u(y)| &\le \operatorname{osc}_{Q_{2^{-k+1}}(x_0) \cap Q_1} u \\ &\le c_1 2^{(-k+1)\alpha} (\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}) \\ &\le 4c_1 2^{(-k-1)\alpha} (\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}) \\ &\le 4c_1 (\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}) \, |x_0 - y|_\infty^\alpha . \end{aligned}$$

This proves b) with $C_9 = 4c_1 + 1$. □

Remark 2.33. Using a similar covering argument as in Section 2.4.1, one can state Proposition 2.32 for balls B_1, $B_{1/2}$ instead of cubes.

The following result is a Hölder continuity estimate at boundary points for solutions in $S(\lambda, \Lambda, 0)$. It will be used (in combination with Proposition 2.32) to obtain *global Hölder continuity* of solutions in $S(\lambda, \Lambda, 0)$.

Proposition 2.34 ([CC95, Proposition 4.12])**.** *Let $u \in S(\lambda, \Lambda, 0)$ in B_1. Assume that $u \in C(\overline{B_1})$ and $u_{|\partial B_1} = \varphi$, where $\varphi \in C^\beta(\partial B_1)$ for some $\beta \in (0,1)$.*
For every $x_0 \in \partial B_1$, u is Hölder continuous at x_0 with exponent $\beta/2$, and

$$\sup_{x \in \overline{B_1}} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\beta/2}} \le 2^{\beta/2} \sup_{x \in \partial B_1} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|^\beta}. \tag{2.46}$$

Proof. For convenience, we replace B_1 and ∂B_1 by $B_1(y)$ and $\partial B_1(y)$, where $y = e_n = (0, \dots, 0, 1)$ and prove (2.46) just for $x_0 = 0$. In addition, we assume that

$$\varphi(x_0) = \varphi(0) = 0.$$

Define $K = \sup_{x \in \partial B_1(y)} \frac{|\varphi(x)|}{|x|^\beta}$. For every $x \in \partial B_1(y)$,

$$x_1^2 + x_2^2 + \ldots + (x_n - 1)^2 = 1, \quad \text{which implies} \quad |x|^2 = 2x_n.$$

Therefore,

$$x \in \partial B_1(y) \ \Rightarrow u(x) = \varphi(x) \leq K\,|x|^{\beta} = 2^{\beta/2} K x_n^{\beta/2}. \tag{2.47}$$

Define $h : B_1(y) \to \mathbb{R}$, $h(x) = x_n^{\beta/2}$. Note that $D^2 h(x) = \operatorname{diag}\left(0, \ldots, 0, \frac{\beta}{2}(\frac{\beta}{2} - 1)x_n^{\beta/2-2}\right)$ with eigenvalues 0 and $\frac{\beta}{2}(\frac{\beta}{2} - 1)x_n^{\beta/2-2} < 0$. Therefore,

$$\mathcal{M}^+(D^2h(x), \lambda, \Lambda) = \lambda \tfrac{\beta}{2}(\tfrac{\beta}{2} - 1)x_n^{\beta/2-2} < 0$$

in $B_1(y)$. Using the corresponding version of Lemma 2.11 for the class $\underline{S}$ (cf. [CC95, Lemma 2.12]), we obtain $u - 2^{\beta/2}Kh \in \underline{S}(\lambda, \Lambda, 0)$ in $B_1(y)$. Since this function is continuous in $\overline{B_1(y)}$ and nonpositive on $\partial B_1(y)$ by (2.47), we can apply Corollary 2.23 and obtain

$$u(x) \leq 2^{\beta/2} K h(x) = 2^{\beta/2} K x_n^{\beta/2} \leq 2^{\beta/2} K\,|x|^{\beta/2}$$

for every $x \in B_1(y)$. Replacing u by $-u$ in the previous result, we obtain

$$|u(x)| \leq 2^{\beta/2} K\,|x|^{\beta/2}$$

for every $x \in B_1(y)$, which implies (2.46). □

We can now state a *global Hölder continuity* result of solutions in $S(\lambda, \Lambda, 0)$.

Proposition 2.35 ([CC95, Proposition 4.13])**.** *Let $u \in S(\lambda, \Lambda, 0)$ in B_1. Assume that $u \in C(\overline{B_1})$ and $u_{|\partial B_1} = \varphi$, where $\varphi \in C^{\beta}(\partial B_1)$ for some $\beta \in (0, 1)$. Then $u \in C^{\gamma}(\overline{B_1})$ and*

$$\|u\|_{C^{\gamma}(\overline{B_1})} \leq C_{10}\, \|\varphi\|_{C^{\beta}(\partial B_1)}\,, \tag{2.48}$$

where $C_{10} = C_{10}(\lambda, \Lambda, n) \geq 1$ and $\gamma = \min\{\alpha, \beta/2\}$ with $\alpha = \alpha(\lambda, \Lambda, n) \in (0, 1)$ as in Proposition 2.32.

Proof. Using Corollary 2.23 and the fact that $u = \varphi$ on ∂B_1, we obtain

$$\inf_{\partial B_1} \varphi \leq u \leq \sup_{\partial B_1} \varphi \text{ in } B_1.$$

Hence, $\|u\|_{C(\overline{B_1})} \leq \|\varphi\|_{C^{\beta}(\partial B_1)}$. We show that $[u]_{C^{\gamma}(\overline{B_1})}$ is also controlled by $\|\varphi\|_{C^{\beta}(\partial B_1)}$. Let $x, y \in B_1$, $x \neq y$. Let $d_x = \operatorname{dist}(x, \partial B_1)$ and $d_y = \operatorname{dist}(y, \partial B_1)$. Without loss of generality we assume $d_y \leq d_x$ and choose $x_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $y_0 \in \partial B_1$ such that $|y - y_0| = d_y$. We consider two cases:

Assume first that $|x - y| \leq \frac{d_x}{2}$. Then $y \in \overline{B_{d_x/2}(x)} \subset B_{d_x}(x) \subset B_1$. We apply Proposition 2.32 (for balls and properly scaled; see Remark 2.33 and Section 2.4.1 for such a scaling argument) to $u - u(x_0)$ in $B_{d_x}(x)$ and use the fact that $\gamma \leq \alpha$:

$$d_x^{\gamma} \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \leq d_x^{\alpha} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C_9\, \|u - u(x_0)\|_{L^{\infty}(B_{d_x}(x))}\,, \tag{2.49}$$

where $C_9 = C_9(\lambda, \Lambda, n) \geq 1$. Using (2.46), we can estimate the right hand side of the previous inequality: Take any $z \in B_{d_x}(x)$. Then

$$\begin{aligned} |u(z) - u(x_0)| &= \frac{|u(z) - u(x_0)|}{|z - x_0|^{\beta/2}} |z - x_0|^{\beta/2} \leq 2^{\beta/2} \|\varphi\|_{C^\beta(\partial B_1)} |z - x_0|^{\beta/2} \\ &\leq 2^{\beta/2}(2d_x)^{\beta/2} \|\varphi\|_{C^\beta(\partial B_1)} \leq 2d_x^{\beta/2} \|\varphi\|_{C^\beta(\partial B_1)} . \end{aligned}$$

Hence,

$$\|u - u(x_0)\|_{L^\infty(B_{d_x}(x))} \leq 2d_x^{\beta/2} \|\varphi\|_{C^\beta(\partial B_1)} . \tag{2.50}$$

Since $\gamma \leq \frac{\beta}{2}$ and $d_x \leq 1$, we obtain from (2.49) and (2.50)

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq 2C_9 d_x^{\beta/2-\gamma} \|\varphi\|_{C^\beta(\partial B_1)} \leq 2C_9 \|\varphi\|_{C^\beta(\partial B_1)} .$$

Assume finally that $d_y \leq d_x \leq 2|x - y|$. If $|x - y| \geq 1$ then

$$|u(x) - u(y)| \leq 2 \|\varphi\|_{C^\beta(\partial B_1)} |x - y|^\gamma .$$

So it remains to consider the case where $\frac{d_x}{2} \leq |x - y| \leq 1$. Using again (2.46), we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\leq 2(d_x^{\beta/2} + |x_0 - y_0|^{\beta/2} + d_y^{\beta/2}) \|\varphi\|_{C^\beta(\partial B_1)} . \end{aligned}$$

Moreover, using the assumption of this case, we have

$$|x_0 - y_0| \leq d_x + |x - y| + d_y \leq 5|x - y|$$

and therefore,

$$|u(x) - u(y)| \leq 18|x - y|^{\beta/2} \|\varphi\|_{C^\beta(\partial B_1)} \leq 18|x - y|^\gamma \|\varphi\|_{C^\beta(\partial B_1)} ,$$

where the last inequality holds since $|x - y| \leq 1$ and $\gamma \leq \frac{\beta}{2}$.
In any case, $[u]_{C^\gamma(\overline{B_1})}$ and $\|u\|_{C(\overline{B_1})}$ are controlled by $\|\varphi\|_{C^\beta(\partial B_1)}$ which implies

$$\|u\|_{C^\gamma(\overline{B_1})} \leq C_{10} \|\varphi\|_{C^\beta(\partial B_1)} ,$$

where $C_{10} = \max\{2C_9, 18\} + 1$. □

3 Regularity Estimates for Nonlocal Fully Nonlinear Elliptic Equations

3.1 Motivation and Basic Definitions

For functions $u : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ consider the linear integro-differential operator

$$Lu(x) = \int_{\mathbb{R}^n} \big(u(x+y) - u(x) - (\nabla u(x) \cdot y)\mathbb{1}_{\{|y|\leq 1\}}\big) K(y)\, dy, \tag{3.1}$$

where $K : \mathbb{R}^n \to [0, \infty)$ is a nonnegative measurable symmetric kernel satisfying

$$\int_{\mathbb{R}^n} (1 \wedge |y|^2) K(y)\, dy < \infty. \tag{3.2}$$

Recall from Chapter 1 that the operators in (3.1) are infinitesimal generators of purely Lévy jump processes for all functions u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In this situation, the kernel K determines the frequency and size of jumps of the Lévy process in each direction.

Due to the symmetry of K, we can rewrite (3.1) in the following way:

$$\begin{aligned} Lu(x) &= \text{p.v.} \int_{\mathbb{R}^n} (u(x+y) - u(x)) K(y)\, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y)\, dy. \end{aligned} \tag{3.3}$$

In order to simplify the notation, we define

$$\varDelta u(x; y) = u(x+y) + u(x-y) - 2u(x).$$

As a consequence, the expression for L can be written as

$$Lu(x) = \int_{\mathbb{R}^n} \varDelta u(x; y) K(y)\, dy \tag{3.4}$$

for some kernel K (which would be the half of the one in (3.1)). Note that $Lu(x)$ is well-defined in a point $x \in \mathbb{R}^n$ if $u : \mathbb{R}^n \to \mathbb{R}$ is bounded and $u \in C^{1,1}(x)$ (cf. Remark 3.1). These conditions are sufficient, but not necessary (see the discussion in [CS11], where the authors allow functions having linear growth at infinity).

Remark 3.1. Recall Definition 2.12 in Section 2.3. Let K be as in (3.2). If $u : \mathbb{R}^n \to \mathbb{R}$ is bounded and $u \in C^{1,1}(x)$ for some point $x \in \mathbb{R}^n$, $Lu(x)$ is well-defined due to the symmetry of K, i.e., the integral in (3.4) exists and is finite. To prove this fact, let $0 < r \leq 1$, $v \in \mathbb{R}^n$ and $A > 0$ such that for each $|y| \leq r$

$$|u(x+y) - u(x) - v \cdot y| \leq A\,|y|^2 .$$

Set $\mu(dy) = K(y)\,dy$. Then

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |\Delta u(x;y)|\,\mu(dy) &= \int\limits_{\{|y|\leq r\}} |\Delta u(x;y)|\,K(y)\,dy + \int\limits_{\{|y|>r\}} |\Delta u(x;y)|\,K(y)\,dy \\ &\leq 2A \int\limits_{\{|y|\leq r\}} |y|^2\,K(y)\,dy + 4\,\|u\|_\infty \int\limits_{\{|y|>r\}} K(y)\,dy < \infty,\end{aligned}$$

where we used the fact that u is bounded, $\mu(\{|y| > \varepsilon\}) < \infty$ for each $\varepsilon > 0$ and (3.2).

The aim of this chapter is to obtain regularity results for solutions to special types of fully nonlinear integro-differential equations. Recall that Pucci's extremal operators from Chapter 2 played an important role in the regularity theory of second order elliptic equations. We introduce some important fully nonlinear integro-differential operators which can be seen as a nonlocal analogue. Let J be some index set and define the family

$$\mathcal{K} = (K_\alpha)_{\alpha \in J}, \tag{3.5}$$

where for each $\alpha \in J$, $K_\alpha : \mathbb{R}^n \to [0,\infty)$ is a nonnegative measurable symmetric kernel satisfying (3.2). We assume that

$$\int\limits_{\mathbb{R}^n} (1 \wedge |y|^2) K(y)\,dy < \infty \quad \text{with } K(y) = \sup_{\alpha \in J} K_\alpha(y). \tag{3.6}$$

By $\mathcal{L}$ we denote the collection of all corresponding linear integro-differential operators L_α of the form (3.4) with kernels $K_\alpha \in \mathcal{K}$, $\alpha \in J$. The (fully nonlinear) maximal and minimal operator with respect to $\mathcal{L}$ are defined as

$$M^+_{\mathcal{L}} u(x) = \sup_{L \in \mathcal{L}} Lu(x) = \sup_{\alpha \in J} L_\alpha u(x), \tag{3.7}$$

$$M^-_{\mathcal{L}} u(x) = \inf_{L \in \mathcal{L}} Lu(x) = \inf_{\alpha \in J} L_\alpha u(x). \tag{3.8}$$

As in Chapter 2, we want to introduce the concept of ellipticity for a general family $\mathcal{L}$ of linear integro-differential operators. As mentioned in Chapter 1, this concept is motivated by the local case in the following sense: Consider the extremal Pucci operators $\mathcal{M}^+$ and $\mathcal{M}^-$ from Section 2.2 with constants $0 < \lambda \leq \Lambda$. Since these operators are uniformly elliptic (see Section 2.2), it is easy to see that if we have

$$\mathcal{M}^-(D^2(u-v)(x), \lambda, \Lambda) \leq F(D^2u(x), x) - F(D^2v(x), x) \leq \mathcal{M}^+(D^2(u-v)(x), \lambda, \Lambda)$$

for every $x \in \Omega$ and for all functions $u, v \in C^2(\Omega)$, then $F : \mathcal{S} \times \Omega \to \mathbb{R}$ is uniformly elliptic with ellipticity constants λ and $n\Lambda$ (using (2.6)). Instead of the Pucci extremal operators we use the maximal and minimal operators with respect to $\mathcal{L}$ to define a concept of ellipticity in the nonlocal setting. We use the definitions in [CS09, CS11].

Definition 3.2. We say that $\mathcal{I}$ is a *nonlocal operator* if

- $\mathcal{I}$ assigns a well-defined value $\mathcal{I}u(x) \in \mathbb{R}$ to a function $u : \mathbb{R}^n \to \mathbb{R}$ at every point $x \in \mathbb{R}^n$ as long as u is bounded and $u \in C^{1,1}(x)$,
- $x \mapsto \mathcal{I}u(x)$ is continuous for $x \in \Omega$ whenever u is bounded and $u \in C^{1,1}[\Omega]$.

Remark 3.3. Recall that $u \in C^{1,1}[\Omega]$, if $u : \mathbb{R}^n \to \mathbb{R}$ satisfies (2.9) in Definition 2.12 for every $x \in \Omega$ with a constant $A > 0$ independent of x.

Remark 3.4. It is possible to replace the condition "u is bounded" with the condition "$u \in L^1(\mathbb{R}^n, \omega)$", i.e., $\int_{\mathbb{R}^n} |u(y)|\,\omega(y)\,dy < \infty$, where ω is a suitable chosen absolutely continuous weight. An important example is the weight $\omega(y) = \frac{1}{1+|y|^{n+\alpha}}$ for $\alpha \in (0, 2)$. We refer to [CS11] for more details.

Definition 3.5. Let $\mathcal{K}$ be a family of kernels of the form (3.5) satisfying (3.6) and let $\mathcal{L}$ be the corresponding class of linear integro-differential operators. We call a nonlocal operator $\mathcal{I}$ *(uniformly) elliptic with respect to* $\mathcal{L}$, if the inequalities

$$M^-_{\mathcal{L}}(u - v)(x) \le \mathcal{I}u(x) - \mathcal{I}v(x) \le M^+_{\mathcal{L}}(u - v)(x) \tag{3.9}$$

hold in every point $x \in \mathbb{R}^n$ whenever the functions u, v are bounded and $u, v \in C^{1,1}(x)$.

Example 3.6. Let $\mathcal{L}$ be a class of linear integro-differential operators and assume that it contains only operators $L_{\alpha\beta}$ of the form (3.4) with associated nonnegative measurable symmetric kernels $K_{\alpha\beta}$ satisfying

$$\int_{\mathbb{R}^n} (1 \wedge |y|^2) K(y)\,dy < \infty, \quad \text{where } K(y) = \sup_{\alpha\beta} K_{\alpha\beta}(y).$$

(i) Fix any $L_{\alpha\beta} \in \mathcal{L}$. Then the linear operator $\mathcal{I}u(x) = L_{\alpha\beta}u(x)$ is elliptic with respect to $\mathcal{L}$ as a consequence of the linearity of $\mathcal{I}$ and [CS09, Lemma 4.2].

(ii) The nonlocal fully nonlinear operator

$$\mathcal{I}u(x) = \inf_{b \in J_1} \sup_{a \in J_2} L_{ab}u(x),$$

where $L_{ab} \in \mathcal{L}$ for every choice of $b \in J_1$ and $a \in J_2$, is elliptic with respect to $\mathcal{L}$ (cf. [CS09, Lemma 3.2 and Lemma 4.2]). The operator $\mathcal{I}$ can be found in nonlocal Isaacs equation and plays an important role in stochastic control problems (see [Son86]).

Next we give a definition of viscosity sub- and supersolutions for integro-differential equations which is only slightly different from Proposition 2.6 (resp. Definition 2.4) in Chapter 2 because of the nonlocal structure of the operators involved in this chapter.

Definition 3.7 ([CS09, Definition 2.2])**.** Let $\mathcal{I}$ be a nonlocal elliptic operator with respect to some class $\mathcal{L}$ of integro-differential operators and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. A bounded function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be a *viscosity subsolution (supersolution)* of the equation $\mathcal{I}u = f$ in Ω, and we write "$\mathcal{I}u \geq f$ ($\mathcal{I}u \leq f$) in Ω in the viscosity sense", if u is upper (lower) semicontinuous in $\overline{\Omega}$ and

$$\mathcal{I}v(x) \geq f(x) \quad (\mathcal{I}v(x) \leq f(x)) \tag{3.10}$$

for every $x \in \Omega$ and every function $v : \mathbb{R}^n \to \mathbb{R}$ of the form

$$v(y) = \begin{cases} \varphi(y) \ , & y \in N \\ u(y) \ , & y \in \mathbb{R}^n \setminus N, \end{cases}$$

where $N \subset \Omega$ is any open neighborhood of x and $\varphi \in C^2(N)$ is an arbitrary function satisfying

$$\varphi(x) = u(x) \quad \text{and} \quad \varphi(y) \geq u(y) \quad (\varphi(y) \leq u(y)) \text{ for every } y \in N.$$

A *viscosity solution* is a function u that is both a viscosity sub- and supersolution.

Remark 3.8.

(i) Note that the test function v from Definition 3.7 is $C^{1,1}$ at x (using Remark 2.13).

(ii) For the concept of viscosity sub- and supersolutions in Ω, it would be enough to require u in Definition 3.7 to be upper (resp. lower) semicontinuous in Ω. However, in view of Section 3.4, it is necessary to assume the resp. semicontinuity of u in $\overline{\Omega}$.

We will obtain regularity results (see Section 3.7 and Section 3.8) for equations of the form $\mathcal{I}u = 0$, where $\mathcal{I}$ is a translation invariant nonlocal elliptic operator with respect to a special class of linear integro-differential operators.

We conclude this section with two technical results for the nonlocal elliptic operators introduced in Definition 3.5. These results will be needed in the following sections. The first result deals with classical evaluations of nonlocal elliptic operators. The second result is important for obtaining a comparison principle which can be used in order to prove existence of solutions to the nonlocal Dirichlet problem (see Section 3.3). We refer to [CS09] for the proofs.

Lemma 3.9 ([CS09, Lemma 4.3])**.** *Let $\mathcal{I}$ be a nonlocal elliptic operator with respect to some class $\mathcal{L}$ of linear integro-differential operators. Let $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $\mathcal{I}u \geq f$ in Ω in the viscosity sense for some function $f : \mathbb{R}^n \to \mathbb{R}$. Assume that the bounded function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,1}$ at the point $x \in \Omega$ and touches u from above at x. Then $\mathcal{I}\varphi(x)$ is defined in the classical sense and $\mathcal{I}\varphi(x) \geq f(x)$.*

Lemma 3.10 ([CS09, Theorem 5.9])**.** *Let $\mathcal{I}$ be a nonlocal elliptic operator with respect to some class $\mathcal{L}$ of linear integro-differential operators. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a bounded function satisfying $\mathcal{I}v \geq f$ in Ω in the viscosity sense, where $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a bounded function satisfying $\mathcal{I}w \leq g$ in Ω in the viscosity sense, where $g : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then $M^+_{\mathcal{L}}(v - w) \geq f - g$ in Ω in the viscosity sense.*

3.2 A special Class of Operators

For fixed $\varrho > 0$ and $\xi_0 \in \mathbb{S}^{n-1}$ (throughout the rest of the chapter) define the set $I = I_{\varrho,\xi_0} = (B_\varrho(\xi_0) \cup B_\varrho(-\xi_0)) \cap \mathbb{S}^{n-1}$. Let $k : \mathbb{S}^{n-1} \to [0,1]$ be a measurable symmetric function with

$$k(\xi) = 1 \quad \text{if } \xi \in I. \tag{3.11}$$

Note that we allow k to be zero outside I. An example for a function satisfying (3.11) would be $k(\xi) = \mathbb{1}_I(\xi),\ \xi \in \mathbb{S}^{n-1}$.

Let $\alpha \in (0,2)$ and fix two reals λ, Λ such that $0 < \lambda \leq \Lambda$ throughout the rest of the chapter. A class that will be used for regularity results in the later sections is given by the class $\mathcal{L}_0 = \mathcal{L}_0(n, \lambda, \Lambda, k, \alpha)$ of **all** operators L of the form (3.4) with corresponding symmetric kernels $K \in \mathcal{K}_0$ satisfying

$$(2-\alpha)k(\tfrac{y}{|y|})\frac{\lambda}{|y|^{n+\alpha}} \leq K(y) \leq (2-\alpha)\frac{\Lambda}{|y|^{n+\alpha}},\ y \in \mathbb{R}^n \setminus \{0\}. \tag{3.12}$$

In the case where $k \equiv 1$ on $\mathbb{S}^{n-1}$, (3.12) can be seen as a natural ellipticity condition of order α for linear integro-differential operators of the form (3.4), since these kernels are comparable to the one of the fractional Laplacian $-(-\Delta)^{\alpha/2}$, which is the most basic elliptic linear integro-differential operator. Note that the family of kernels from above satisfies (3.6). An interesting observation lies in the fact that (3.11) implies the existence of a number $\vartheta \in (0, \frac{\pi}{2}]$ such that

$$K(y) \geq (2-\alpha)\frac{\lambda}{|y|^{n+\alpha}} \quad \text{for all } y \in \mathbb{R}^n \setminus \{0\} \text{ satisfying } \frac{|y \cdot \xi_0|}{|y|} > \cos\vartheta. \tag{3.13}$$

In [CS09], the authors consider the case where $k \equiv 1$ on $\mathbb{S}^{n-1}$. In the case of (3.12), we do not impose a lower bound on K on the set $\{y \in \mathbb{R}^n : \frac{y}{|y|} \notin I\}$ due to the fact that k may be zero outside the set I.

For bounded $u \in C^{1,1}(x)$, the maximal and minimal operators $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ have the following simple form:

$$M^+_\alpha u(x) = M^+_{\mathcal{L}_0} u(x) = (2-\alpha) \int\limits_{\mathbb{R}^n} \big(\Lambda \varDelta u(x;y)^+ - \lambda k(\tfrac{y}{|y|}) \varDelta u(x;y)^-\big)\, \mu(dy), \tag{3.14}$$

$$M^-_\alpha u(x) = M^-_{\mathcal{L}_0} u(x) = (2-\alpha) \int\limits_{\mathbb{R}^n} \big(\lambda k(\tfrac{y}{|y|}) \varDelta u(x;y)^+ - \Lambda \varDelta u(x;y)^-\big)\, \mu(dy), \tag{3.15}$$

where $\mu(dy) = |y|^{-n-\alpha}\, dy$. $\varDelta u(x;\cdot)^\pm$ denotes the positive resp. negative part of $\varDelta u(x;\cdot)$. Note that the condition "u is bounded" can be replaced by the condition

$$\int\limits_{\mathbb{R}^n} |u(y)| \frac{1}{1+|y|^{n+\alpha}}\, dy < \infty. \tag{3.16}$$

In this situation, we allow u to have some growth at infinity. However, we will formulate our main results for bounded u.

The factor $(2-\alpha)$ in (3.12) is important when $\alpha \nearrow 2$ since second order uniformly elliptic operators can be recovered as limits of integro-differential operators. To demonstrate this fact, take any bounded function $u \in C^2(\mathbb{R}^n)$ and consider for $x \in \mathbb{R}^n$ the integro-differential operator

$$\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \frac{(2-\alpha)a(\frac{y}{|y|})}{|y|^{n+\alpha}}\, dy, \tag{3.17}$$

where $a : \mathbb{S}^{n-1} \to [0, \Lambda]$ is a measurable symmetric function satisfying $a(\xi) \geq \lambda k(\xi)$ for $\xi \in \mathbb{S}^{n-1}$. Note that $K(y) = \frac{(2-\alpha)a(y/|y|)}{|y|^{n+\alpha}}$ satisfies (3.12). Using Taylor expansion, we can write

$$\begin{aligned} u(x+y) &= u(x) + y \cdot \nabla u(x) + \tfrac{1}{2} y^{\mathrm{T}} D^2 u(x) y + \varphi_1(y) \quad \text{and} \\ u(x-y) &= u(x) - y \cdot \nabla u(x) + \tfrac{1}{2} y^{\mathrm{T}} D^2 u(x) y + \varphi_2(y), \end{aligned}$$

where $\varphi_i(y) = o(|y|^2)$ for $y \to 0$, $i = 1, 2$. Note that there exists a small $\beta > 0$ (independent of i) such that $\varphi_i(y) = \mathcal{O}(|y|^{2+\beta})$ for $y \to 0$. Hence, we can choose $R > 0$ and $c_1 > 0$ (independent of i) such that $|\varphi_i(y)| \leq c_1 |y|^{2+\beta}$ for $|y| < R$. Thus

$$\begin{aligned} |\mathcal{J}_\alpha| := \left| \int_{B_R} (\varphi_1(y) + \varphi_2(y))\, K(y)\, dy \right| &\leq 2(2-\alpha)\Lambda c_1 \int_{B_R} |y|^{2+\beta-n-\alpha}\, dy \\ = 2\Lambda c_1 \frac{2-\alpha}{2+\beta-\alpha} n\omega_n R^{2+\beta-\alpha} &\xrightarrow{\alpha \nearrow 2} 0. \end{aligned}$$

Let us split the integral in (3.17) into the domains B_R and $\mathbb{R}^n \setminus B_R$. For the second part, we have

$$\int_{\mathbb{R}^n \setminus B_R} (u(x+y) + u(x-y) - 2u(x)) \frac{(2-\alpha)a(\frac{y}{|y|})}{|y|^{n+\alpha}}\, dy \leq c_2(2-\alpha) \xrightarrow{\alpha \nearrow 2} 0,$$

where $c_2 = 4\|u\|_\infty \Lambda n\omega_n \frac{R^{-\alpha}}{\alpha}$. Note that the integral from above still vanishes as $\alpha \nearrow 2$ if u is not bounded but satisfies (3.16) for some $\alpha_0 \in (0, 2)$.
For the first part, we use the expansion of u from above:

$$\begin{aligned} &\int_{B_R} (u(x+y) + u(x-y) - 2u(x)) K(y)\, dy \\ &\quad = \int_{B_R} (y^{\mathrm{T}} D^2 u(x) y + \varphi_1(y) + \varphi_2(y)) \frac{(2-\alpha)a(\frac{y}{|y|})}{|y|^{n+\alpha}}\, dy \\ &\quad = \int_{B_R} y^{\mathrm{T}} D^2 u(x) y \frac{(2-\alpha)a(\frac{y}{|y|})}{|y|^{n+\alpha}}\, dy + \mathcal{J}_\alpha \end{aligned}$$

$$= \int_0^R \left(\frac{r^2}{r^{n+\alpha}} r^{n-1} (2-\alpha) \int_{\mathbb{S}^{n-1}} a(s) s^{\mathrm{T}} D^2 u(x) s \, \sigma(ds) \right) dr + \mathcal{J}_\alpha$$

$$= (2-\alpha) \int_0^R r^{1-\alpha} dr \int_{\mathbb{S}^{n-1}} a(s) s^{\mathrm{T}} D^2 u(x) s \, \sigma(ds) + \mathcal{J}_\alpha$$

$$= R^{2-\alpha} \int_{\mathbb{S}^{n-1}} a(s) s^{\mathrm{T}} D^2 u(x) s \, \sigma(ds) + \mathcal{J}_\alpha,$$

where σ denotes the surface measure on the unit sphere. Hence,

$$\lim_{\alpha \nearrow 2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \frac{(2-\alpha) a(\frac{y}{|y|})}{|y|^{n+\alpha}} \, dy = \int_{\mathbb{S}^{n-1}} a(s) s^{\mathrm{T}} D^2 u(x) s \, \sigma(ds),$$

which is a linear operator in $D^2 u$. For a symmetric $n \times n$ matrix M, we now define

$$F(M) = \int_{\mathbb{S}^{n-1}} a(s) s^{\mathrm{T}} M s \, \sigma(ds).$$

Set $c_3 = c_3(n, \varrho) = \inf_{\xi \in \mathbb{S}^{n-1}} \int_{I_{\varrho,\xi}} s_1^2 \, \sigma(ds) > 0$. We claim that F is uniformly elliptic in the sense of Definition 2.1 with ellipticity constants depending only on n, λ, Λ and ϱ. Indeed, for every symmetric $n \times n$ matrix M and every nonnegative definite symmetric $n \times n$ matrix N,

$$F(M+N) - F(M) = F(N) \le \Lambda \int_{\mathbb{S}^{n-1}} |Ns| \, \sigma(ds) \le \Lambda \left| \mathbb{S}^{n-1} \right| \|N\| =: \overline{\Lambda} \, \|N\| \, .$$

Recall that $\|N\| = \sup_{|s|=1} |Ns|$ is equal to the largest eigenvalue of N. Moreover, writing $N = SDS^{\mathrm{T}}$ with an orthogonal matrix S and a diagonal matrix $D = \mathrm{diag}(\tau_1, \ldots, \tau_n)$ containing the (nonnegative) eigenvalues of N with $\tau_1 = \|N\|$,

$$F(N) \ge \int_{\mathbb{S}^{n-1}} \lambda k(s) (S^{\mathrm{T}} s)^{\mathrm{T}} D (S^{\mathrm{T}} s) \, \sigma(ds) \ge \lambda \int_{I_{\varrho,\tilde{\xi}}} \tilde{s}^{\mathrm{T}} D \tilde{s} \, \sigma(d\tilde{s})$$

$$\ge \lambda \|N\| \int_{I_{\varrho,\tilde{\xi}}} \tilde{s}_1^2 \, \sigma(d\tilde{s}) \ge \lambda c_3 \|N\| =: \overline{\lambda} \, \|N\| \, ,$$

where $\widetilde{\xi} = S^{\mathrm{T}} \xi_0$. Hence, F is uniformly elliptic with ellipticity constants $\overline{\lambda}$ and $\overline{\Lambda}$.

Similar as above, one can prove that the operators M_2^+ and M_2^- defined by

$$M_2^+ u(x) = \lim_{\alpha \nearrow 2} M_\alpha^+ u(x), \tag{3.18}$$

$$M_2^- u(x) = \lim_{\alpha \nearrow 2} M_\alpha^- u(x) \tag{3.19}$$

– where we assume for simplicity that $u \in C^2(\mathbb{R}^n)$ and bounded – are second order uniformly elliptic operators of the form $M_2^+ u(x) = G(D^2 u(x))$ and $M_2^- u(x) = H(D^2 u(x))$ with

$$G(M) = \int_{\mathbb{S}^{n-1}} \Lambda(s^{\mathrm{T}} M s)^+ - \lambda k(s)(s^{\mathrm{T}} M s)^- \, \sigma(ds) \quad \text{and} \tag{3.20}$$

$$H(M) = \int_{\mathbb{S}^{n-1}} \lambda k(s)(s^{\mathrm{T}} M s)^+ - \Lambda(s^{\mathrm{T}} M s)^- \, \sigma(ds). \tag{3.21}$$

Again, the ellipticity constants of G, $\widetilde{\lambda}$ and $\widetilde{\Lambda}$, depend on n, λ, Λ and ϱ: Let M be a symmetric $n \times n$ matrix and N be a nonnegative definite symmetric $n \times n$ matrix. Define the sets

$$\begin{aligned} O &= \{s \in \mathbb{S}^{n-1} : -s^{\mathrm{T}} N s \le s^{\mathrm{T}} M s < 0\}, \\ P &= \{s \in \mathbb{S}^{n-1} : s^{\mathrm{T}} M s \ge -s^{\mathrm{T}} N s\}, \\ Q &= \{s \in \mathbb{S}^{n-1} : s^{\mathrm{T}} M s < -s^{\mathrm{T}} N s\}. \end{aligned}$$

Then

$$G(M+N) - G(M) = \int_O (\Lambda - \lambda k(s)) s^{\mathrm{T}} M s \, \sigma(ds) + \int_{\mathbb{S}^{n-1}} (\Lambda \mathbb{1}_P + \lambda k(s) \mathbb{1}_Q) s^{\mathrm{T}} N s \, \sigma(ds).$$

Since the first integral from the line above is nonpositive, we obtain

$$G(M+N) - G(M) \le (\Lambda + \lambda) \int_{\mathbb{S}^{n-1}} s^{\mathrm{T}} N s \, \sigma(ds) \le (\Lambda + \lambda) \left|\mathbb{S}^{n-1}\right| \|N\| =: \widetilde{\Lambda} \|N\| .$$

Moreover,

$$\begin{aligned} G(M+N) - G(M) &\ge \int_O (\lambda k(s) - \Lambda) s^{\mathrm{T}} N s \, \sigma(ds) + \int_{\mathbb{S}^{n-1}} (\Lambda \mathbb{1}_P + \lambda k(s) \mathbb{1}_Q) s^{\mathrm{T}} N s \, \sigma(ds) \\ &\ge \int_{\mathbb{S}^{n-1}} \lambda k(s) s^{\mathrm{T}} N s \, \sigma(ds) \ge \overline{\lambda} \|N\| =: \widetilde{\lambda} \|N\| . \end{aligned}$$

Hence, G is uniformly elliptic with ellipticity constants $\widetilde{\lambda}, \widetilde{\Lambda}$. In addition, the following relation holds as a consequence of [CC95, Lemma 2.2] and the fact that $G(0) = 0$:

$$M_2^+ u(x) \le \mathcal{M}^+\big(D^2 u(x), \tfrac{\widetilde{\lambda}}{n}, \widetilde{\Lambda}\big) \quad \text{for every } x \in \mathbb{R}^n, \tag{3.22}$$

where $\mathcal{M}^+$ denotes the Pucci operator in Section 2.2.
The corresponding relations hold for M_2^- (resp. H).

In terms of regularity, we need the factor $(2 - \alpha)$ for the estimates not to blow up as $\alpha \nearrow 2$. To be precise: Our estimates will be uniform with respect to $\alpha \in (\alpha_0, 2)$, where

$\alpha_0 \in (0,2)$ is some lower bound for α. As a consequence, one may prove regularity results for solutions of second order elliptic equations by proving regularity results for solutions of integro-differential equations.

The family of operators $\mathcal{L}_0$ with nonnegative measurable symmetric kernels satisfying (3.12) has some important properties which are stated in the next lemmas.

Lemma 3.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $M^+_{\mathcal{L}_0}$ be the operator given in* (3.14). *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function satisfying $M^+_{\mathcal{L}_0} u \geq f$ in Ω in the viscosity sense. Let $x \in \Omega$. Assume there exists a function $\varphi \in C^2(\mathbb{R}^n)$ such that*

$$\varphi(x) = u(x) \quad \textit{and} \quad \varphi(y) \geq u(y) \quad \textit{for all } y \in \mathbb{R}^n.$$

Then $M^+_{\mathcal{L}_0} u(x)$ is defined in the classical sense and $M^+_{\mathcal{L}_0} u(x) \geq f(x)$.
In other words: The nonlinear operator $M^+_{\mathcal{L}_0}$ can be evaluated classically in u at those points $x \in \Omega$ where u can be touched from above by a C^2 function.

Proof. The proof can be obtained by following the proof of [CS09, Lemma 3.3]. Let $d_x = \operatorname{dist}(x, \partial\Omega) > 0$. We write M^+ to denote $M^+_{\mathcal{L}_0}$. For $r \in (0, d_x)$ define

$$v_r(z) = \begin{cases} \varphi(z), & z \in B_r(x) \\ u(z), & z \in \mathbb{R}^n \setminus B_r(x). \end{cases}$$

Then

$$M^+ v_r(x) \geq f(x)$$

because u is a subsolution. Hence,

$$\infty > (2-\alpha) \int\limits_{\mathbb{R}^n} \frac{\Lambda \Delta v_r(x;y)^+}{|y|^{n+\alpha}} - k\big(\tfrac{y}{|y|}\big) \frac{\lambda \Delta v_r(x;y)^-}{|y|^{n+\alpha}} \, dy \geq f(x).$$

As a first step, we show that $\frac{\Delta u(x;y)^+}{|y|^{n+\alpha}}$ is integrable w.r.t. y:
Since φ touches u from above at x, we have

$$\Delta v_r(x;y) \geq \Delta u(x;y) \text{ for each } y \in \mathbb{R}^n.$$

Since $v_r \in C^{1,1}(x)$ and bounded, $\frac{|\Delta v_r(x;y)|}{|y|^{n+\alpha}}$ is integrable w.r.t. y, which implies that $\frac{\Delta u(x;y)^+}{|y|^{n+\alpha}}$ is also integrable w.r.t. y because

$$|\Delta v_r(x;y)| \geq \Delta v_r(x;y)^+ \geq \Delta u(x;y)^+.$$

In the next step, we show that $k(\frac{y}{|y|})\frac{\Delta u(x;y)^-}{|y|^{n+\alpha}}$ is integrable w.r.t. y:
We have

$$(2-\alpha) \int\limits_{\mathbb{R}^n} k\big(\tfrac{y}{|y|}\big) \frac{\lambda \Delta v_r(x;y)^-}{|y|^{n+\alpha}} \, dy \leq (2-\alpha) \int\limits_{\mathbb{R}^n} \frac{\Lambda \Delta v_r(x;y)^+}{|y|^{n+\alpha}} \, dy - f(x).$$

Fix some number $r_0 \in (0, d_x)$. Since φ touches u from above at x, the function $r \mapsto \Delta v_r(x;y)$ is decreasing as $r \searrow 0$. Therefore, for every $r < r_0$

$$(2-\alpha)\int_{\mathbb{R}^n} k\left(\tfrac{y}{|y|}\right)\frac{\lambda \Delta v_r(x;y)^-}{|y|^{n+\alpha}}\,dy \le (2-\alpha)\int_{\mathbb{R}^n} \frac{\Lambda \Delta v_{r_0}(x;y)^+}{|y|^{n+\alpha}}\,dy - f(x). \tag{3.23}$$

But $r \mapsto \Delta v_r(x;y)^-$ is monotone increasing as $r \searrow 0$ and $\Delta v_r(x;y)^- \nearrow \Delta u(x;y)^-$ as $r \searrow 0$. By the monotone convergence theorem:

$$\lim_{r\searrow 0}(2-\alpha)\int_{\mathbb{R}^n} k\left(\tfrac{y}{|y|}\right)\frac{\lambda \Delta v_r(x;y)^-}{|y|^{n+\alpha}}\,dy = (2-\alpha)\int_{\mathbb{R}^n} k\left(\tfrac{y}{|y|}\right)\frac{\lambda \Delta u(x;y)^-}{|y|^{n+\alpha}}\,dy. \tag{3.24}$$

Using (3.23), we obtain

$$(2-\alpha)\int_{\mathbb{R}^n} k\left(\tfrac{y}{|y|}\right)\frac{\lambda \Delta u(x;y)^-}{|y|^{n+\alpha}}\,dy \le (2-\alpha)\int_{\mathbb{R}^n} \frac{\Lambda \Delta v_{r_0}(x;y)^+}{|y|^{n+\alpha}}\,dy - f(x) < +\infty.$$

Therefore, $k(\frac{y}{|y|})\frac{\Delta u(x;y)^-}{|y|^{n+\alpha}}$ is integrable w.r.t. y and so we conclude (together with the first part of the proof) that $M^+u(x)$ is computable in the classical sense.

Moreover, the convergence

$$\int_{\mathbb{R}^n} \frac{\Lambda \Delta v_r(x;y)^+}{|y|^{n+\alpha}}\,dy \quad\to\quad \int_{\mathbb{R}^n} \frac{\Lambda \Delta u(x;y)^+}{|y|^{n+\alpha}}\,dy \tag{3.25}$$

as $r \searrow 0$ follows by dominated convergence. Since the inequality $M^+v_r(x) \ge f(x)$ holds for every $r > 0$, we finally conclude – using (3.24) and (3.25) – that $M^+u(x) \ge f(x)$ by passing to the limit $r \searrow 0$. □

Before showing another important property of the operators $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ we need a technical real analysis lemma. We refer to [CS09] for a proof.

Lemma 3.12 ([CS09, Lemma 4.1])**.** *Let J be an arbitrary index set, $f \in L^\infty(\mathbb{R}^n)$ and $\{g_\beta\}_{\beta\in J}$ be a family of functions. Assume there exists a function $g \in L^1(\mathbb{R}^n)$ such that for every $\beta \in J$ the inequality $|g_\beta(x)| \le g(x)$ holds for every $x \in \mathbb{R}^n$. Then the family $\{f * g_\beta\}_{\beta\in J}$ is equicontinuous in every compact set.*

Using this Lemma we obtain the following result.

Lemma 3.13. *Let $M^+_{\mathcal{L}}$ and $M^-_{\mathcal{L}}$ be as in* (3.7) *and* (3.8) *and assume that* (3.6) *holds. Let $v \in C^{1,1}[\Omega]$ be bounded. Then $M^+_{\mathcal{L}}v$ and $M^-_{\mathcal{L}}v$ are continuous in Ω.*

Proof. We provide the proof of [CS09, Lemma 4.2] for completeness. Let $\mathcal{K}$ be defined as in (3.5) and assume that (3.6) holds. Let $\mathcal{L}$ be the corresponding class of linear integro-differential operators, where each $L_\beta \in \mathcal{L}$, $\beta \in J$, is of the form (3.4). As in

(3.6), we set $K = \sup_{\beta \in J} K_\beta$. Let $\varepsilon > 0$, $x_0 \in \Omega$ and $d_{x_0} = \operatorname{dist}(x_0, \partial\Omega)$. Consider any $x \in B_{d_{x_0}/2}(x_0) \subset \Omega$. Since $v \in C^{1,1}[\Omega]$, there is a constant $A > 0$ (independent of x as above) such that for every $|y| < \frac{d_{x_0}}{2}$

$$|\Delta v(x;y)| \leq A\,|y|^2 . \tag{3.26}$$

Choose $r_0 \in (0, d_{x_0}/2)$ small enough such that

$$\int\limits_{B_{r_0}} A\,|y|^2\, K(y)\, dy \leq \frac{\varepsilon}{3} \tag{3.27}$$

and note that this choice of r_0 is independent of $x \in B_{d_{x_0}/2}(x_0)$. For every $\beta \in J$ and every $x \in B_{d_{x_0}/2}(x_0)$, we have

$$\begin{aligned} L_\beta v(x) &= \int\limits_{\mathbb{R}^n} \Delta v(x;y) K_\beta(y)\, dy \\ &= \int\limits_{B_{r_0}} \Delta v(x;y) K_\beta(y)\, dy + \int\limits_{\mathbb{R}^n \setminus B_{r_0}} \Delta v(x;y) K_\beta(y)\, dy \\ &= w_\beta^1(x) + w_\beta^2(x). \end{aligned}$$

Using (3.26) and (3.27), we obtain

$$\left|\omega_\beta^1(x)\right| \leq \int\limits_{B_{r_0}} A\,|y|^2\, K(y)\, dy \leq \frac{\varepsilon}{3}$$

for every $\beta \in J$ and every $x \in B_{d_{x_0}/2}(x_0)$.
We rewrite w_β^2 to apply Lemma 3.12:

$$\begin{aligned} w_\beta^2(x) &= \int\limits_{\mathbb{R}^n \setminus B_{r_0}} (v(x+y) + v(x-y) - 2v(x)) K_\beta(y)\, dy \\ &= (v * g_\beta)(x) + (v * \hat{g}_\beta)(x) - 2v(x)(1 * g_\beta)(x), \end{aligned}$$

where $g_\beta(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{r_0}}(y) K_\beta(y)$ and $\hat{g}_\beta(y) = g_\beta(-y)$. Note that for every $\beta \in J$ and $y \in \mathbb{R}^n$, $|g_\beta(y)| \leq g(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{r_0}}(y) K(y)$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a L^1 function. Therefore, using Lemma 3.12, $\{w_\beta^2\}_{\beta \in J}$ is equicontinuous in Ω. Hence, we may choose a number $\delta = \delta(\varepsilon, x_0) \in (0, d_{x_0}/2)$ small enough, such that for every $\beta \in J$ and every $x \in B_\delta(x_0)$ the inequality $\left|w_\beta^2(x) - w_\beta^2(x_0)\right| \leq \frac{\varepsilon}{3}$ holds. Combining the previous results, we conclude that for every $x \in B_\delta(x_0)$

$$|L_\beta v(x) - L_\beta v(x_0)| \leq \left|w_\beta^1(x)\right| + \left|w_\beta^1(x_0)\right| + \left|w_\beta^2(x) - w_\beta^2(x_0)\right| \leq \varepsilon$$

uniformly in β. Thus $\left|M_{\mathcal{L}}^{\pm} v(x) - M_{\mathcal{L}}^{\pm} v(x_0)\right| \leq \varepsilon$ every time $|x - x_0| \leq \delta$. □

Remark 3.14. Since (3.6) holds for the family of kernels satisfying (3.12), $M^+_{\mathcal{L}_0}u$ and $M^-_{\mathcal{L}_0}u$ are continuous functions in Ω as long as $u \in C^{1,1}[\Omega]$ is bounded.

The following result will be important for Section 3.5.

Corollary 3.15. *Let G be a compact set such that $G \subset \Omega$ and let $\alpha_0 \in (0,2)$. Let $v \in C^{1,1}[\Omega]$ be bounded. There exists $c_1 \geq 1$ such that for every $\alpha \in (\alpha_0, 2)$ the following inequality holds:*

$$\sup_{x\in G} \left|M^-_\alpha v(x)\right| \leq c_1(A + \|v\|_\infty),$$

where $A > 0$ is the constant in Definition 2.12.

Proof. Let $\alpha \in (\alpha_0, 2)$. Using Lemma 3.13, $M^-_\alpha v$ is continuous in Ω. This implies the existence of a point $x_0 \in G$ such that $|M^-_\alpha v(x)| \leq |M^-_\alpha v(x_0)|$ for every $x \in G$. Since $v \in C^{1,1}[\Omega]$, we can find $A > 0$ such that

$$|v(x_0 + y) - v(x_0) - \nabla v(x_0) \cdot y| \leq A\,|y|^2 \text{ for every } y \in B_r,$$

where $r = 1 \wedge \operatorname{dist}(G, \partial\Omega)$. Recall that A is independent of x_0. Hence,

$$\begin{aligned}\left|M^-_\alpha v(x_0)\right| &\leq (2-\alpha)\Lambda \int\limits_{B_r} \frac{|\Delta v(x_0;y)|}{|y|^{n+\alpha}}\,dy + (2-\alpha)\Lambda \int\limits_{B_r^c} \frac{|\Delta v(x_0;y)|}{|y|^{n+\alpha}}\,dy \\ &\leq 2A(2-\alpha)\Lambda n\omega_n \int\limits_0^r s^{1-\alpha}\,ds + 4\,\|v\|_\infty\,(2-\alpha)\Lambda n\omega_n \int\limits_r^\infty s^{-1-\alpha}\,ds \\ &\leq 2A\Lambda n\omega_n + \frac{8}{\alpha_0}\,\|v\|_\infty\,\Lambda n\omega_n r^{-2} \leq \frac{8}{\alpha_0}\Lambda n\omega_n r^{-2}(A + \|v\|_\infty).\end{aligned}$$

□

3.3 Existence of Solutions to the Nonlocal Dirichlet Problem

Recall the setting at the beginning of Section 3.2. The aim of this section is to give a brief overview how to prove existence of viscosity solutions $u \in C(\overline{\Omega}) \cap L^\infty(\mathbb{R}^n)$ to the following nonlocal Dirichlet problem

$$\begin{cases}\mathcal{I}u(x) = 0, & x \in \Omega\\ \quad u(x) = g(x), & x \in \mathbb{R}^n \setminus \Omega.\end{cases} \tag{3.28}$$

Here, $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ is a globally bounded function which is continuous in every $x \in \partial\Omega$ and $\mathcal{I}$ is a nonlocal fully nonlinear elliptic operator of the form (cf. Example 3.6)

$$\mathcal{I}u(x) = \mathcal{I}_\alpha u(x) = \inf_{b\in J_1}\sup_{a\in J_2} L_{ab}u(x) \tag{3.29}$$

with linear integro-differential operators $L_{ab} \in \mathcal{L}_0 = \mathcal{L}_0(n, \lambda, \Lambda, k, \alpha)$ for each choice of $a \in J_2$ and $b \in J_1$, where J_1, J_2 denote arbitrary index sets. Note that $\mathcal{I}0 = 0$, where

$\mathcal{I}0$ is the value we obtain when applying $\mathcal{I}$ to the constant function that is equal to zero. Hence, $M_\alpha^- v(x) \le \mathcal{I}v(x) \le M_\alpha^+ v(x)$ for every $x \in \mathbb{R}^n$ whenever v is bounded and $v \in C^{1,1}(x)$.

Fix any $\alpha_0 \in (0,2)$ throughout the section. We prove the following existence result.

Theorem 3.16. *Let $\alpha \in (\alpha_0, 2)$ and assume that the bounded domain Ω satisfies the exterior ball condition. Then there exists a unique viscosity solution $u \in C(\overline{\Omega}) \cap L^\infty(\mathbb{R}^n)$ of* (3.28).

Existence of solutions under similar assertions has been established in [CLD12, BCF12]. We adapt their argumentation to our setting introduced in (3.12).

Remark 3.17.

(i) Ω satisfies the *exterior ball condition*, if there is $r_0 > 0$ such that for every $\omega \in \partial\Omega$ there exists $x \in \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{B_{r_0}(x)} \cap \overline{\Omega} = \{\omega\}$.
For example, Ω with C^1-boundary satisfies the exterior ball condition.

(ii) The lower bound for α in Theorem 3.16 is important to construct suitable barriers on $\partial\Omega$ as explained below. These barriers will be independent of $\alpha \in (\alpha_0, 2)$.

The proof of Theorem 3.16 is based on Perron's method which originated in local theory to prove existence of solutions to the Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

by combining the explicit formula in the special case $\Omega = B_R$ and the maximum principle. We refer to [GT01] for a general overview of this technique.

As main technical tool, we prove a comparison principle which will be crucial for the construction of the solutions to (3.28). The corresponding result in [CLD12] is [CLD12, Theorem 4.10].

Theorem 3.18. *Let $\alpha \in (\alpha_0, 2)$ and let $\mathcal{I}$ be as in* (3.29). *Assume that the bounded functions $v : \mathbb{R}^n \to \mathbb{R}$ and $w : \mathbb{R}^n \to \mathbb{R}$ satisfy*

(i) $\mathcal{I}v \ge f$ and $\mathcal{I}w \le f$ in Ω in the viscosity sense for some $f \in C(\Omega)$ and

(ii) $v \le w$ in $\mathbb{R}^n \setminus \Omega$.

Then $v \le w$ in Ω.

The proof of the comparison principle in [CLD12, Theorem 4.10] is based on the construction of a suitable bump function (cf. [CLD12, Lemma 4.11]). [CLD12, Lemma 4.11] has to be adapted to our setting.

Lemma 3.19. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(x) = \min\big(1, \frac{|x|^2}{4}\big)$. There exists $\delta > 0$ such that for every $\alpha \in (\alpha_0, 2)$*

$$M_\alpha^- \varphi \ge \delta \quad \text{in } B_1.$$

Proof. Note that φ is regular enough to evaluate $M_\alpha^- \varphi$ classically in B_1. Let $x \in B_1$. We claim that $\Delta\varphi(x;y) \geq 0$ for every $y \in \mathbb{R}^n$. Indeed, if we assume $x \pm y \in B_2$ or $x \pm y \notin B_2$ then the claim is easily verified. For example, if $x \pm y \in B_2$ then

$$\Delta\varphi(x;y) = \frac{|x+y|^2}{4} + \frac{|x-y|^2}{4} - \frac{|x|^2}{2} = \frac{|y|^2}{2} \geq 0. \tag{3.30}$$

If only $x + y \in B_2$, we use that $\varphi(x) \leq \frac{1}{4}$ and obtain

$$\Delta\varphi(x;y) = \varphi(x+y) + 1 - 2\varphi(x) \geq \tfrac{1}{2}.$$

The same holds true if only $x - y \in B_2$. So, $\Delta\varphi(x;y) \geq 0$ for every $y \in \mathbb{R}^n$. Using this fact and (3.30), we obtain for every $\alpha \in (\alpha_0, 2)$:

$$\begin{aligned}
M_\alpha^- \varphi(x) &\geq (2-\alpha) \int_{B_1} \lambda k\big(\tfrac{y}{|y|}\big) \Delta\varphi(x;y)\, \mu(dy) \\
&= \tfrac{2-\alpha}{2} \lambda \int_{B_1} k\big(\tfrac{y}{|y|}\big)\, |y|^{-n+2-\alpha}\; dy \\
&\geq \tfrac{2-\alpha}{2} \lambda \int_0^1 r^{n-1} \int_I r^{-n+2-\alpha}\, \sigma(dy)\, dr \\
&= \tfrac{2-\alpha}{2} \lambda \sigma(I) \int_0^1 r^{1-\alpha}\, dr = \tfrac{1}{2} \lambda\, |I| =: \delta,
\end{aligned}$$

where σ denotes the surface measure on $\mathbb{S}^{n-1}$ and $I \subset \mathbb{S}^{n-1}$ is as in Section 3.2. This completes the proof. □

Proof of Theorem 3.18. We provide the proof of [CLD12, Theorem 4.10] for completeness. Define $g = v - w$. Using Lemma 3.10,

$$M_\alpha^+ g \geq 0 \quad \text{in } \Omega \tag{3.31}$$

in the viscosity sense. We show that $\sup\limits_{\Omega} g \leq \sup\limits_{\mathbb{R}^n \setminus \Omega} g =: N$ which will prove the theorem since $g \leq 0$ in $\mathbb{R}^n \setminus \Omega$.

Choose $R \geq 1$ large enough such that $\Omega \subset B_R$ and define $\psi(x) = \varphi(x/R)$, where φ is the bump function in Lemma 3.19. For every $x \in \Omega$,

$$\begin{aligned}
M_\alpha^- \psi(x) &= \sup_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta\varphi\big(\tfrac{x}{R}; \tfrac{y}{R}\big) K(y)\, dy = R^n \sup_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta\varphi\big(\tfrac{x}{R}; z\big) K(Rz)\, dz \\
&= R^n R^{-n-\alpha} M_\alpha^- \varphi(x/R) \geq R^{-\alpha} \delta,
\end{aligned}$$

where $\delta > 0$ is as in Lemma 3.19. Fix any $\varepsilon > 0$ and consider

$$\psi_\varepsilon(x) = N + \varepsilon(1 - \psi(x)),$$

which satisfies $M^+_\alpha \psi_\varepsilon = -\varepsilon M^-_\alpha \psi \leq -\varepsilon R^{-\alpha}\delta < 0$ in Ω.
We claim that $\psi_\varepsilon \geq g$ in Ω. Assume this is not true. Therefore, $\inf_\Omega(\psi_\varepsilon - g) < 0$ which implies the existence of some $d > 0$ such that $\psi_\varepsilon + d$ touches g from above at some $x \in \Omega$. Because of (3.31) and Lemma 3.9, $0 \leq M^+_\alpha(\psi_\varepsilon + d)(x) = M^+_\alpha \psi_\varepsilon(x)$. Contradiction. Therefore, $\psi_\varepsilon \geq g$ in Ω which leads to $\sup_\Omega g \leq N$ by letting $\varepsilon \searrow 0$. □

The next technical result deals with the construction of suitable barriers to attain the boundary data in (3.28).

Lemma 3.20. *There exist constants $\gamma > 0$ and $c_0 \geq 1$ such that the continuous and nonnegative function $\Phi : \mathbb{R}^n \to \mathbb{R}$,*

$$\Phi(x) = \min(1, c_0((|x| - 1)^+)^\gamma)$$

satisfies

$$M^+_\alpha \Phi \leq 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B_1} \text{ for every } \alpha \in (\alpha_0, 2). \tag{3.32}$$

Remark 3.21. Note that Φ in Lemma 3.20 satisfies $\Phi = 0$ in $\overline{B_1}$ and $\Phi = 1$ in $\mathbb{R}^n \setminus B_2$.

Proof. The proof can be obtained by direct computation similar to the approach in Section 3.5. However, we will avoid lengthy computations at this point by adapting [CS11, Lemma 1 and Corollary 31] to our setting. For $\xi \in \mathbb{S}^{n-1}$ let $I_\xi = I_{\xi,\varrho}$ be as in Section 3.2, where $\varrho > 0$ is fixed. For $\alpha \in (0,2)$ let $\mathcal{K}' = \mathcal{K}'(n, \lambda, \Lambda, \alpha, \varrho)$[1] denote the class of all measurable symmetric kernels K satisfying the following condition: There exists $\xi \in \mathbb{S}^{n-1}$ (which may depend on K) such that

$$(2-\alpha)\mathbb{1}_{I_\xi}\big(\tfrac{y}{|y|}\big)\lambda\, |y|^{-n-\alpha} \leq K(y) \leq (2-\alpha)\Lambda\, |y|^{-n-\alpha} \quad \text{for every } y \in \mathbb{R}^n \setminus \{0\}.$$

Note that $\mathcal{K}_0(\alpha) \subset \mathcal{K}'(\alpha)$, where $\mathcal{K}_0(\alpha)$ is as in Section 3.2. Let $\mathcal{L}'(\alpha)$ denote the corresponding class of all linear-integro differential operators of the form (3.4) with kernels $K \in \mathcal{K}'(\alpha)$. We just stress the dependence on α here. So if we prove the assertion for $M^+_{\mathcal{L}'(\alpha)}\Phi$, then this implies the assertion for $M^+_{\mathcal{L}_0(\alpha)}\Phi$.
For $\gamma \in (0, \frac{\alpha_0}{2})$ define $\varphi_\gamma : \mathbb{R}^n \to \mathbb{R}$,

$$\varphi_\gamma(x) = ((|x| - 1)^+)^\gamma.$$

Note that for every $r \in (0,1)$, we can find an open set $U \supset \partial B_{1+r}$ such that $\varphi_\gamma|_U \in C^2(U)$, which implies – together with the fact that (3.16) holds for φ_γ with $\alpha \in (\alpha_0, 2)$ arbitrary – that $M^+_{\mathcal{L}'(\alpha)}\varphi_\gamma(x)$ is well-defined for every $\alpha \in (\alpha_0, 2)$ and every $x \in \partial B_{1+r}$.

[1]We thank R. Schwab for proposing this class of kernels.

Claim 1: Choosing $\gamma \in (0, \frac{\alpha_0}{2})$ and $r \in (0,1)$ sufficiently small, we obtain

$$M^+_{\mathcal{L}'(\alpha)}\varphi_\gamma(x) \leq 0 \text{ for every } x \in \partial B_{1+r} \text{ and } \alpha \in (\alpha_0, 2).$$

Since the operator $M^+_{\mathcal{L}'(\alpha)}$ is rotational invariant, it is sufficient to prove the claim for $x_0 = (1+r)e_1$, where e_1 denotes the first unit vector. We consider two steps:

Step 1: In this step, we choose the number $r \in (0,1)$ appropriately. Define a function $l_N : \mathbb{R}^n \to \mathbb{R}$,

$$l_N(x) = \begin{cases} -N, & |x| \leq 1 \\ \log(|x|-1) \vee -N, & |x| > 1, \end{cases}$$

where $N > 0$ will be chosen below. Note that $M^+_{\mathcal{L}'(\alpha)} l_N(x_0)$ is well-defined for every $r \in (0,1)$ if $N \geq -\log \frac{r}{2}$, because $l_N|_{U'} \in C^2(U')$ for some open set $U' \supset \partial B_{1+r}$, and l_N satisfies (3.16) for arbitrary $\alpha > 0$.

Choose any $r \in (0,1)$ and $N = N(r) \geq -\log \frac{r}{2}$. For $\xi \in \mathbb{S}^{n-1}$ define

$$M^+_{2,\xi} l_N(x_0) = \lim_{\alpha \nearrow 2} M^+_{\alpha,\xi} l_N(x_0),$$

where $M^+_{\alpha,\xi} l_N(x_0) = (2-\alpha)\int_{\mathbb{R}^n}(\Lambda \Delta l_N(x_0;y)^+ - \lambda \mathbb{1}_{I_\xi}(\frac{y}{|y|})\Delta l_N(x_0;y)^-)\mu(dy)$ (cf. (3.18)). It is easy to see that $M^+_{\mathcal{L}'(\alpha)} l_N(x_0) = \sup_{\xi \in \mathbb{S}^{n-1}} M^+_{\alpha,\xi} l_N(x_0)$. Using (3.22) with $k = \mathbb{1}_{I_\xi}$ in (3.20), we obtain

$$M^+_{2,\xi} l_N(x_0) \to -\infty \quad \text{as } |x_0| \searrow 1 \text{ (and } N \to \infty)$$

uniformly in $\xi \in \mathbb{S}^{n-1}$, because

$$M^+_{2,\xi} l_N(x_0) \leq \mathcal{M}^+(D^2 l_N(x_0), \tfrac{\widetilde{\lambda}}{n}, \widetilde{\Lambda}) \text{ and } \mathcal{M}^+(D^2 \log(|x_0|-1), \tfrac{\widetilde{\lambda}}{n}, \widetilde{\Lambda}) \to -\infty \text{ as } |x_0| \searrow 1.$$

Recall that $\mathcal{M}^+$ denotes the maximal Pucci operator and $\widetilde{\lambda} = \lambda c_3$, $\widetilde{\Lambda} = (\Lambda + \lambda)\left|\mathbb{S}^{n-1}\right|$ with $c_3 = \inf_{\xi \in \mathbb{S}^{n-1}} \int_{I_\xi} s_1^2 \, \sigma(ds) > 0$, where σ denotes the surface measure on $\mathbb{S}^{n-1}$.

So we may choose $r \in (0,1)$ sufficiently small (depending on λ, Λ, n and ϱ) and take any $N \geq -\log \frac{r}{2}$ such that there exists $\alpha_1 \in (\alpha_0, 2)$ with $M^+_{\mathcal{L}'(\alpha)} l_N(x_0) < -1$ for every $\alpha \in (\alpha_1, 2]$. By choosing $N \geq -\log \frac{r}{2}$ sufficiently large, we can also make sure that $M^+_{\mathcal{L}'(\alpha)} l_N(x_0) < -1$ for every $\alpha \in [\alpha_0, \alpha_1]$:

For $\xi \in \mathbb{S}^{n-1}$ set $\mathcal{S}_\xi = \{y \in \mathbb{R}^n : \frac{y}{|y|} \in I_\xi \wedge x_0 + y \in B_1\}$. After another possible decrease of r (depending on ϱ), we can find $\nu > 0$ independent of ξ such that $|\mathcal{S}_\xi| \geq \nu$. Let $\alpha \in [\alpha_0, \alpha_1]$ and $K \in \mathcal{K}'(\alpha)$. We write

$$\begin{aligned} L\, l_N(x_0) &= \int_{\mathbb{R}^n} \Delta l_N(x_0;y)^+ \, K(y)\, dy - \int_{\mathbb{R}^n} \Delta l_N(x_0;y)^- \, K(y)\, dy \\ &=: \mathfrak{I}_1 + \mathfrak{I}_2 \end{aligned}$$

and estimate $-\mathfrak{I}_2$: According to the definition of $\mathcal{K}'$, there exists $\xi \in \mathbb{S}^{n-1}$ such that

$$-\mathfrak{I}_2 \geq 2(2-\alpha)\lambda \int_{\mathcal{S}_\xi} (-N - \log r)^- \, |y|^{-n-\alpha} \, dy = 2(2-\alpha)\lambda(N + \log r) \int_{\mathcal{S}_\xi} |y|^{-n-\alpha} \, dy$$
$$\geq 2(2-\alpha_1)\lambda(N + \log r)\frac{\nu}{(2+r)^{n+2}},$$

where we note that $y \in \mathcal{S}_\xi$ implies $r \leq |y| \leq 2+r$. Hence, $\mathfrak{I}_2 \to -\infty$ as $N \to \infty$ uniformly in ξ. $\mathfrak{I}_1$ on the other side can be estimated from above by some number $c = c(n, \Lambda, \alpha_0) \geq 1$ because $l_N\,|_{U'} \in C^2(U')$ and l_N satisfies (3.16) for arbitrary $\alpha > 0$.

Hence, we can choose $N \geq -\log \frac{r}{2}$ large enough (depending on $n, \lambda, \Lambda, \varrho, \alpha_0$ and α_1) such that $M^+_{\mathcal{L}'(\alpha)} l_N(x_0) < -1$ for every $\alpha \in [\alpha_0, \alpha_1]$. The choice of r and N is now complete.

Step 2: It remains to choose $\gamma \in (0, \frac{\alpha_0}{2})$. Let $\gamma \in (0, \frac{\alpha_0}{2})$ be arbitrary for the moment and let N, r be as in the end of Step 1. Define $v_\gamma : \mathbb{R}^n \to \mathbb{R}$,

$$v_\gamma(x) = \frac{\varphi_\gamma(x) - 1}{\gamma} \vee -N.$$

Then

$$\lim_{\gamma \searrow 0} v_\gamma(x) = \log(|x| - 1) \vee -N = l_N(x)$$

for every $|x| > 1$ and this convergence holds locally uniformly. Note that $\frac{a^\gamma - 1}{\gamma} \xrightarrow{\gamma \searrow 0} \log a$ for every $a > 0$.

Assume that Claim 1 does not hold. Then there exists a sequence $(\gamma_j)_{j\in\mathbb{N}}$, $\gamma_j \in (0, \frac{\alpha_0}{2})$, converging to zero and a sequence $(\alpha_j)_{j\in\mathbb{N}}$, $\alpha_j \in (\alpha_0, 2)$, such that $M^+_{\mathcal{L}'(\alpha_j)} \varphi_{\gamma_j}(x_0) \geq 0$, and thus $M^+_{\mathcal{L}'(\alpha_j)} v_{\gamma_j} \geq 0$ in $\Omega = B_{2(1+r)} \setminus B_{1+r}$. Take a subsequence (α_j) (which we do not relabel) such that $\alpha_j \to \overline{\alpha} \in [\alpha_0, 2]$ as $j \to \infty$. Using the fact that $v_{\gamma_j} \to l_N$ locally uniformly as $j \to \infty$, there exists $j_0 \in \mathbb{N}$ with the property that for every $j \geq j_0$ there exists a small number $\delta_j \in \mathbb{R}$ (converging to zero as $j \to \infty$) such that $l_N + \delta_j$ touches v_{γ_j} from above at a point $x_j \in B_{r/2}(x_0) \cap \Omega$. Set $r_j = \frac{r}{2} - |x_j - x_0|$ and note that $r_j \to \frac{r}{2}$ as $j \to \infty$. We can now proceed as in the proof of [BCF12, Theorem 4.6]: Assume first that $\overline{\alpha} < 2$. Hence, there exists $j_1 \geq j_0$ and a number $\widetilde{\alpha} \in (\alpha_0, 2)$ such that $\alpha_j \leq \widetilde{\alpha}$ for every $j \geq j_1$. Since $M^+_{\mathcal{L}'(\alpha_j)} v_{\gamma_j}(x_j) = \sup_{\xi \in \mathbb{R}^n} M^+_{\alpha_j, \xi} v_{\gamma_j}(x_j) \geq 0$, we see that for every $\varepsilon > 0$ there exists a sequence (ξ_j) with $\xi_j \in \mathbb{S}^{n-1}$ such that for every $j \geq j_1$

$$\begin{aligned} -\varepsilon &\leq (2-\alpha_j) \int_{\mathbb{R}^n} \frac{\Lambda \Delta v_{\gamma_j}(x_j; y)^+ - \lambda \mathbb{1}_{I_{\xi_j}}(\frac{y}{|y|}) \Delta v_{\gamma_j}(x_j; y)^-}{|y|^{n+\alpha_j}} \, dy \\ &\leq (2-\alpha_j) \int_{B_{r_j}} \frac{\Lambda \Delta \eta_j(x_j; y)^+ - \lambda \mathbb{1}_{I_{\xi_j}}(\frac{y}{|y|}) \Delta \eta_j(x_j; y)^-}{|y|^{n+\alpha_j}} \, dy \\ &\quad + (2-\alpha_j) \int_{\mathbb{R}^n \setminus B_{r_j}} \frac{\Lambda \Delta v_{\gamma_j}(x_j; y)^+ - \lambda \mathbb{1}_{I_{\xi_j}}(\frac{y}{|y|}) \Delta v_{\gamma_j}(x_j; y)^-}{|y|^{n+\alpha_j}} \, dy, \end{aligned} \tag{3.33}$$

where $\eta_j = l_N + \delta_j$. Note that $\Delta\eta_j(x_j; y) = \Delta l_N(x_j; y)$ for every y. Since $\mathbb{S}^{n-1}$ is compact, we may take a subsequence (ξ_j) (which we again do not relabel) converging to some $\xi' \in \mathbb{S}^{n-1}$. The first integrand from above is bounded by the integrable function $A\,|y|^{2-\widetilde{\alpha}-n}$ for some $A > 0$ since $l_N|_{B_{r/2}(x_0)} \in C^2(B_{r/2}(x_0))$ and $B_{r_j}(x_j) \subset B_{r/2}(x_0)$. Moreover, we can bound the second integrand by an integrable function g since

$$\frac{|v_{\gamma_j}(z)|}{1+|z|^{n+\alpha_j}} \le \left[N + \sup_{0\le\gamma\le\frac{\alpha_0}{2}} \tfrac{(|z|-1)^{\gamma}-1}{\gamma}\mathbb{1}_{\{|z|\ge 2\}}\right]\left(\frac{1}{1+|z|^{n+\alpha_0}} + \frac{1}{1+|z|^{n+2}}\right)$$
$$\le \left[N + \tfrac{2}{\alpha_0}(|z|-1)^{\alpha_0/2}\mathbb{1}_{\{|z|\ge 2\}}\right]\left(\frac{1}{1+|z|^{n+\alpha_0}} + \frac{1}{1+|z|^{n+2}}\right) =: g(z)$$

for every $z \in \mathbb{R}^n$ and every $j \in \mathbb{N}$. Since $\xi_j \to \xi'$, $r_j \to \frac{r}{2}$ and $x_j \to x_0$ as $j \to \infty$, the dominated convergence theorem implies

$$-\varepsilon \le M^+_{\overline{\alpha},\xi'} l_N(x_0) \le M^+_{\mathcal{L}'(\overline{\alpha})} l_N(x_0).$$

As ε is arbitrary we conclude $M^+_{\mathcal{L}'(\overline{\alpha})} l_N(x_0) \ge 0$.

If $\overline{\alpha} = 2$ we start again with (3.33) and argue in a similar way as in the discussion following (3.17) to obtain

$$-\varepsilon \le \lim_{j\to\infty} M^+_{\alpha_j,\xi_j} v_{\gamma_j}(x_j) = M^+_{2,\xi'} l_N(x_0) \le \sup_{\xi\in\mathbb{S}^{n-1}} M^+_{2,\xi} l_N(x_0),$$

which gives $\sup_{\xi\in\mathbb{S}^{n-1}} M^+_{2,\xi} l_N(x_0) \ge 0$ since ε is arbitrary.
But both cases ($\overline{\alpha} < 2$ and $\overline{\alpha} = 2$) contradict the result in Step 1. Claim 1 is now proved.

Claim 2: Let r, γ be as in Claim 1. If $1 < |x| < 1 + r$, we obtain the same result as in Claim 1.

We prove Claim 2 by the same scaling argument as in the proof of [CLD12, Lemma 4.15]. Let x be a point such that $(1+r)x = (1+sr)x_0$, where $s \in (0,1)$ and $x_0 \in \partial B_{1+r}$. Hence, $|x| = 1 + sr$. Define $s_0 = \frac{1+sr}{1+r}$ and set

$$v(y) = s_0^{-\gamma}\varphi_\gamma(s_0 y) = ((|y| - s_0{}^{-1})^+)^\gamma.$$

Note that $v(y) \le \varphi_\gamma(y)$ for every $y \in \mathbb{R}^n$. We translate v such that it remains below φ_γ but touches it in a whole ray passing through x and x_0. We still denote this translation v. Let $\alpha \in (\alpha_0, 2)$. Then

$$\begin{aligned}
M^+_{\mathcal{L}'(\alpha)}\varphi_\gamma(x) &= \sup_{K\in\mathcal{K}'(\alpha)} \int_{\mathbb{R}^n} (\varphi_\gamma(x+y) + \varphi_\gamma(x-y) - 2\varphi_\gamma(x))K(y)\,dy \\
&= s_0^\gamma \sup_{K\in\mathcal{K}'(\alpha)} \int_{\mathbb{R}^n} \big(v(\tfrac{x+y}{s_0}) + v(\tfrac{x-y}{s_0}) - 2v(\tfrac{x}{s_0})\big)K(y)\,dy \\
&\le s_0^{n+\gamma} \sup_{K\in\mathcal{K}'(\alpha)} \int_{\mathbb{R}^n} \big(\varphi_\gamma(\tfrac{x}{s_0}+z) + \varphi_\gamma(\tfrac{x}{s_0}-z) - 2\varphi_\gamma(\tfrac{x}{s_0})\big)K(s_0 z)\,dz \\
&= s_0^{n+\gamma} s_0^{-n-\alpha} M^+_{\mathcal{L}'(\alpha)}\varphi_\gamma(\tfrac{x}{s_0}) = s_0^{\gamma-\alpha} M^+_{\mathcal{L}'(\alpha)}\varphi_\gamma(x_0) \le 0,
\end{aligned}$$

where the last inequality holds because of Claim 1. Claim 2 is now proved.

We finish the proof by choosing $c_0 = r^{-\gamma}$ in the definition of Φ, where $r \in (0,1)$ and $\gamma \in (0, \frac{\alpha_0}{2})$ are as in Claim 1. Because of Claims 1 and 2, (3.32) holds for $x \in \overline{B_{1+r}} \setminus \overline{B_1}$. Since Φ attains its global maximum at every point $x \in \mathbb{R}^n \setminus B_{1+r}$, (3.32) also holds for $x \in \mathbb{R}^n \setminus \overline{B_{1+r}}$. □

The proof of Theorem 3.16 now follows exactly as in [CLD12, Theorem 4.12]. We again provide the proof for completeness.

Proof of Theorem 3.16. We denote by $\mathcal{C}^+(\Omega)$ the set of all functions $v : \mathbb{R}^n \to \mathbb{R}$ which are upper semicontinuous in Ω. Let $\alpha \in (\alpha_0, 2)$ and consider the operator $\mathcal{I} = \mathcal{I}_\alpha$ in (3.29). Let S be the set of all viscosity subsolutions of the equation $\mathcal{I}v = 0$ in Ω with boundary data smaller than g:

$$S = \{v \in \mathcal{C}^+(\Omega) \cap L^\infty(\mathbb{R}^n) \, : \, \mathcal{I}v \geq 0 \text{ in } \Omega \text{ in the viscosity sense and } v \leq g \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

S is non-empty because $v = -\|g\|_\infty \in S$. Now set $u(x) = \sup_{v \in S} v(x)$ and define the upper semicontinuous envelope of u in Ω by

$$\overline{u}(x) = \begin{cases} \lim\limits_{r \searrow 0} \sup\{u(\xi) \, : \, \xi \in \overline{B_r(x)} \cap \Omega\}, & x \in \Omega \\ u(x), & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

$\overline{u}$ is the smallest function $w \in \mathcal{C}^+(\Omega)$ such that $w \geq u$ in $\mathbb{R}^n$. Analogously, the lower semicontinuous envelope of u in Ω is defined by

$$\underline{u}(x) = \begin{cases} \lim\limits_{r \searrow 0} \inf\{u(\xi) \, : \, \xi \in \overline{B_r(x)} \cap \Omega\}, & x \in \Omega \\ u(x), & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Note that [CLD12, Theorem 4.13 and Theorem 4.14] are applicable to our situation. Using [CLD12, Theorem 4.13], we obtain $\mathcal{I}\,\overline{u} \geq 0$ in Ω in the viscosity sense which implies $\overline{u} \in S$ and therefore $\overline{u} = u$ by the definition of u. [CLD12, Theorem 4.14] implies that $\mathcal{I}\,\underline{u} \leq 0$ in Ω in the viscosity sense since $u \in S$ is the biggest subsolution. Using the comparison principle (Theorem 3.18), the aforementioned result implies that $\underline{u} \geq u$ in Ω. Therefore, $\underline{u} = u = \overline{u}$ in $\mathbb{R}^n$ which implies $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$ and $\mathcal{I}u = 0$ in Ω in the viscosity sense.

We now prove that the boundary values are attained in a continuous way. We claim that for every $\varepsilon > 0$ and every $x \in \mathbb{R}^n \setminus \Omega$, we can find bounded barriers $v : \mathbb{R}^n \to \mathbb{R}$ and $w : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\mathcal{I}w \leq 0$ and $\mathcal{I}v \geq 0$ in Ω in the viscosity sense,

(ii) $w \geq g$ and $v \leq g$ in $\mathbb{R}^n \setminus \Omega$,

(iii) $w(x) \leq g(x) + \varepsilon$ and $v(x) \geq g(x) - \varepsilon$.

Assume for now that we already proved this claim. Note that $v \in S$ and – in combination with the comparison principle – we obtain from (i) and (ii) that $v \leq u \leq w$ in $\mathbb{R}^n$. This proves that $u = g$ in $\mathbb{R}^n \setminus \Omega$ by letting $\varepsilon \searrow 0$ in (iii). In particular, $u \in C(\overline{\Omega}) \cap L^\infty(\mathbb{R}^n)$ and u satisfies (3.28).

We prove (i)-(iii) for w. If $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we define

$$w(y) = \begin{cases} \|g\|_\infty, & y \neq x \\ g(x), & y = x. \end{cases}$$

w is lower semicontinuous in $\mathbb{R}^n$ and $\mathcal{I}w \leq 0$ in Ω in the viscosity sense. (ii) and (iii) are trivially satisfied.

Now assume that $x \in \partial\Omega$ and let $\varepsilon > 0$. Since Ω satisfies the exterior ball condition, there exist $r_0 \in (0,1)$ (independent of $x \in \partial\Omega$) and $\xi \in \mathbb{S}^{n-1}$ such that $\overline{B_{r_0}(x + r_0\xi)} \cap \overline{\Omega} = \{x\}$. Note that $\operatorname{dist}(\Omega, x + r_0\xi) \geq r_0$. Define

$$w(y) = 2\|g\|_\infty \Phi\left(\tfrac{y-(x+r\xi)}{r}\right) + g(x) + \varepsilon,$$

where $0 < r < r_0$ and Φ is the function in Lemma 3.20. We will choose $r \in (0, r_0)$ sufficiently small such that w satisfies (i)-(iii).
(i) is satisfied (for each choice of $r \in (0, r_0)$) because every $\widetilde{x} \in \Omega$ satisfies the inequality $|\widetilde{x} - (x + r\xi)| > r$, which implies

$$\mathcal{I}w(\widetilde{x}) \leq M_\alpha^+ w(\widetilde{x}) = 2\|g\|_\infty r^{-\alpha} M_\alpha^+ \Phi\left(\tfrac{\widetilde{x}-(x+r\xi)}{r}\right) \leq 0$$

by Lemma 3.20 and the ellipticity of $\mathcal{I}$ with respect to $\mathcal{L}_0$. (iii) is trivially satisfied since $\Phi = 0$ on $\mathbb{S}^{n-1}$ by Remark 3.21. It remains to prove (ii). Since g is continuous on $\partial\Omega$, we can choose $\delta > 0$ such that

$$|g(z) - g(x)| \leq \varepsilon \quad \text{whenever } z \in B_\delta(x) \cap (\mathbb{R}^n \setminus \Omega). \tag{3.34}$$

Recall that g is only defined on $\mathbb{R}^n \setminus \Omega$. Choose $r \in (0, r_0)$ small enough such that $B_{2r}(x + r\xi) \subset B_\delta(x)$. Let $y \in \mathbb{R}^n \setminus \Omega$. We consider two cases:

- Assume that $y \in B_\delta(x) \cap (\mathbb{R}^n \setminus \Omega)$. Then $w(y) \geq g(x) + \varepsilon \geq g(y)$. Note that the first inequality holds because Φ is a nonnegative function and the second inequality holds because of (3.34).
- Assume that $y \in (\mathbb{R}^n \setminus B_\delta(x)) \cap (\mathbb{R}^n \setminus \Omega)$. Then $y \in (\mathbb{R}^n \setminus B_{2r}(x + r\xi)) \cap (\mathbb{R}^n \setminus \Omega)$ by the choice of r from above, which implies that $\frac{y-(x+r\xi)}{r} \notin B_2$. Therefore, by Remark 3.21, $w(y) \geq \|g\|_\infty \geq g(y)$.

The choice of v in the respective situations is now obvious.

It remains to prove uniqueness of the solutions to (3.28). Assume that there are two solutions u_1 and u_2 of (3.28). Since $\mathcal{I}u_1 \geq 0$ and $\mathcal{I}u_2 \leq 0$ in Ω in the viscosity sense and $u_1 = u_2 = g$ in $\mathbb{R}^n \setminus \Omega$, we can apply Theorem 3.18 and obtain $u_1 \leq u_2$ in $\mathbb{R}^n$. Interchanging the roles of u_1 and u_2 gives the reverse inequality. Hence, $u_1 = u_2$ in $\mathbb{R}^n$. □

3.4 A nonlocal Aleksandrov-Bakelman-Pucci Estimate

As in the local setting, we need a specific tool to prove Hölder regularity. We already observed in Chapter 2 that the ABP estimate (see Section 2.3) leads to a Harnack inequality which gives an interior Hölder regularity result. Taking a look back at the proof of the ABP estimate, we observe that the second derivatives of the concave envelope of the negative part of the solution u are controlled by the equation itself. We can not expect our integro-differential equations to provide such a control since the order of the equations is less than 2. However, we will obtain an estimate in this section that can be seen as a nonlocal version of the ABP estimate.

We repeat the setting introduced in Section 3.2: Fix some $\varrho > 0$, $\xi_0 \in \mathbb{S}^{n-1}$ and define the set $I = (B_\varrho(\xi_0) \cup B_\varrho(-\xi_0)) \cap \mathbb{S}^{n-1}$. Let $k : \mathbb{S}^{n-1} \to [0,1]$ be a measurable symmetric function satisfying (3.11). Fix numbers $0 < \lambda \leq \Lambda$ and let $\alpha \in (0,2)$. Consider the class $\mathcal{L}_0 = \mathcal{L}_0(n, \lambda, \Lambda, k, \alpha)$ defined by condition (3.12). Set $\mu(dy) = |y|^{-n-\alpha}\, dy$.
Throughout this section we work with the following assumptions:
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable, bounded and positive function. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable and bounded function that is upper semicontinuous in $\overline{B_1}$ with $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and $u^+ \not\equiv 0$. Assume further that u satisfies $M_\alpha^+ u \geq -f$ in B_1 in the viscosity sense. Let $\Gamma : \mathbb{R}^n \to \mathbb{R}$ be the concave envelope of u^+ in B_3 defined by

$$\Gamma(x) = \begin{cases} \inf\{p(x) : p \text{ is affine and } p \geq u^+ \text{ in } B_3\}, & x \in B_3 \\ 0, & x \in \mathbb{R}^n \setminus B_3. \end{cases}$$

Define the contact set of u and Γ in B_1 by $\Sigma = \{u = \Gamma\} \cap B_1$. The next lemma is the key tool for obtaining the nonlocal ABP estimate.

Lemma 3.22. *Let $\rho_0 \in (0,1)$ and $r_j = \rho_0 2^{-\frac{1}{2-\alpha}-j}$, $j \in \mathbb{N}_0$. For $x \in \mathbb{R}^n$ define the rings $R_j(x) = B_{r_j}(x) \setminus B_{r_{j+1}}(x)$ and the subsets $R_j^I(x) = \left\{y \in R_j(x) : \frac{y-x}{|y-x|} \in I\right\}$.*
There is a constant $C_0 = C_0(n, |I|, \rho_0, \lambda) \geq 1$ such that for every $x \in \Sigma$ and every $A > 0$ there is an index $j \in \mathbb{N}_0$ with

$$\left|R_j^I(x) \cap \{z \in \mathbb{R}^n : u(z) < u(x) + (z-x)\cdot\nabla\Gamma(x) - Ar_j^2\}\right| \leq C_0 \frac{f(x)}{A} \left|R_j^I(x)\right|, \quad (3.35)$$

where $\nabla\Gamma(x) \in \mathbb{R}^n$ is an arbitrary vector satisfying $\Gamma(z) \leq \Gamma(x) + (z-x)\cdot\nabla\Gamma(x)$ for every $z \in B_3$.

We remark at this point that the concave envelope Γ will not be as regular as the corresponding envelope in the local setup introduced in Chapter 2.

Remark 3.23. Note that $\nabla\Gamma(x) = \nabla u(x)$ for $x \in \Sigma$ if u is differentiable at the contact point x.

Proof of Lemma 3.22. We adapt the proof of [CS09, Lemma 8.1] to our setting. According to Remark 2.15, there exists a supporting hyperplane for $-\Gamma$ at every point $x \in B_3$

since B_3 is an open convex set and $-\Gamma$ is a convex function in B_3. Hence, for every $x \in B_3$ there exists an affine function p such that

$$p(x) = \Gamma(x) \quad \text{and} \quad p(y) \geq \Gamma(y) \quad \text{for every } y \in B_3.$$

Let $x \in \Sigma$. Since u can be touched by a supporting hyperplane from above at x (since $u(x) = \Gamma(x) = p(x)$ for an affine function p with $p \geq u^+$ in B_3), Lemma 3.11 implies that $M_\alpha^+ u(x)$ is defined classically and

$$M_\alpha^+ u(x) = (2-\alpha) \int\limits_{\mathbb{R}^n} (\Lambda \Delta u(x;y)^+ - \lambda k(\tfrac{y}{|y|}) \Delta u(x;y)^-)\, \mu(dy) \geq -f(x).$$

Note that if both $x+y \in B_3$ and $x-y \in B_3$, then

$$\Delta u(x;y) \leq p(x+y) + p(x-y) - 2p(x) = 2p(x) - 2p(x) = 0.$$

Moreover, if either $x+y \notin B_3$ or $x-y \notin B_3$, then $x+y \notin B_1$ and $x-y \notin B_1$. Thus $u(x+y) \leq 0$ and $u(x-y) \leq 0$ according to our general assumptions. Hence, $\Delta u(x;y) \leq 0$ for every $y \in \mathbb{R}^n$ which implies

$$\begin{aligned} -f(x) \leq M_\alpha^+ u(x) &= -(2-\alpha) \int\limits_{\mathbb{R}^n} \lambda k(\tfrac{y}{|y|}) \Delta u(x;y)^- \, \mu(dy) \\ &\leq -(2-\alpha) \int\limits_{B_{r_0}} \lambda k(\tfrac{y}{|y|}) \Delta u(x;y)^- \, \mu(dy). \end{aligned}$$

Recall that $r_0 = \rho_0 2^{-1/(2-\alpha)}$. Since $\bigcup\limits_{j=0}^{\infty} R_j^I(0) \subset B_{r_0}$ and $R_j^I(0) \cap R_l^I(0) = \emptyset$ for $j \neq l$, we obtain from the inequality above

$$f(x) \geq (2-\alpha)\lambda \sum_{j=0}^{\infty} \int\limits_{R_j^I(0)} k(\tfrac{y}{|y|}) \Delta u(x;y)^- \, \mu(dy) = (2-\alpha)\lambda \sum_{j=0}^{\infty} \int\limits_{R_j^I(0)} \Delta u(x;y)^- \, \mu(dy), \tag{3.36}$$

where the last equality holds since $k(\frac{y}{|y|}) = 1$ for each $y \in R_j^I(0)$, $j \in \mathbb{N}_0$. We want to estimate the integrals appearing in (3.36). First note that for each $y \in R_j^I(0)$

$$0 \leq \Delta u(x;y)^- = -\Delta u(x;y) = -[u(x+y) - u(x) - y \cdot \nabla\Gamma(x)] - [u(x-y) - u(x) + y \cdot \nabla\Gamma(x)]$$

and each term in the brackets above is nonpositive because of the concavity of Γ. For example,

$$u(x+y) - u(x) - y \cdot \nabla\Gamma(x) \leq \Gamma(x+y) - \Gamma(x) - y \cdot \nabla\Gamma(x) \leq 0 \quad \text{for all } y \in R_j^I(0).$$

We use the argument from above to estimate each integral in (3.36):

$$\begin{aligned}\int_{R_j^I(0)} \Delta u(x;y)^- \,\mu(dy) &= -\int_{R_j^I(0)} \Delta u(x;y)\,\mu(dy)\\ &\geq -\int_{R_j^I(0)} (u(x+y)-u(x)-y\cdot\nabla\Gamma(x))\,\mu(dy).\end{aligned}$$

Let us assume that the assertion of Lemma 3.22 fails, i.e., for every $C_0 \geq 1$ there are $x \in \Sigma$ and $A > 0$ with the property that for every $j \in \mathbb{N}_0$

$$\left|R_j^I(0) \cap D_j\right| > C_0 \tfrac{f(x)}{A} \left|R_j^I(0)\right|, \tag{3.37}$$

where $D_j = \{y \in \mathbb{R}^n : u(x+y) < u(x) + y\cdot\nabla\Gamma(x) - Ar_j^2\}$. Take any $C_0 \geq 1$. Choose $x \in \Sigma$ and $A > 0$ such that (3.37) holds for every $j \in \mathbb{N}_0$. Then

$$\begin{aligned}&-\int_{R_j^I(0)} (u(x+y)-u(x)-y\cdot\nabla\Gamma(x))\,\mu(dy) \geq -\int_{R_j^I(0)\cap D_j} \tfrac{u(x+y)-u(x)-y\cdot\nabla\Gamma(x)}{|y|^{n+\alpha}}\,dy\\ &\geq Ar_j^2 \tfrac{1}{r_j^{n+\alpha}} \left|R_j^I(0)\cap D_j\right| \overset{(3.37)}{>} Ar_j^{2-n-\alpha} C_0 \tfrac{f(x)}{A} \left|R_j^I(0)\right|\\ &= r_j^{2-n-\alpha} C_0 f(x) c_0 \left|R_j(0)\right| = r_j^{2-n-\alpha} C_0 f(x) c_0 \left|B_1\right| r_j^n (1-2^{-n})\\ &= c_1 r_j^{2-\alpha} C_0 f(x),\end{aligned}$$

where $c_0 = c_0(n, |I|) \in (0,1]$ such that $\left|R_j^I(0)\right| = c_0 \left|R_j(0)\right|$ and $c_1 = c_0 \left|B_1\right|(1-2^{-n})$. Hence,

$$\begin{aligned}f(x) &\geq (2-\alpha)\lambda \sum_{j=0}^{\infty} \int_{R_j^I(0)} \Delta u(x;y)^-\,\mu(dy) \geq c_1(2-\alpha)\lambda\, C_0\, f(x) \sum_{j=0}^{\infty} r_j^{2-\alpha}\\ &= c_1\,(2-\alpha)\lambda\, C_0\, f(x) \sum_{j=0}^{\infty} \rho_0^{2-\alpha}\, 2^{-1-j\,(2-\alpha)}\\ &= \frac{c_1}{2}\,\rho_0^{2-\alpha}\,(2-\alpha)\lambda\, C_0\, f(x) \sum_{j=0}^{\infty} (2^{-(2-\alpha)})^j\\ &\geq \frac{c_1}{2}\lambda\rho_0^2 \frac{2-\alpha}{1-2^{-(2-\alpha)}}\, C_0\, f(x) \geq c_2\, C_0\, f(x),\end{aligned}$$

where the last inequality holds with a constant $c_2 = c_2(n, |I|, \rho_0, \lambda) > 0$ because $\alpha \mapsto \frac{2-\alpha}{1-2^{-(2-\alpha)}}$ remains bounded below for $\alpha \in (0,2)$. By choosing C_0 large enough, we obtain a contradiction. □

The goal of this section is to find a specific covering of the contact set $\{u = \Gamma\} \cap B_1$ by a finite number of cubes. This covering will be used in Section 3.6 to prove a result similar to Lemma 2.27. We need the following lemma:

Lemma 3.24. *Define* $R = B_1 \setminus B_{1/2}$ *and* $R^I = \left\{ y \in R : \frac{y}{|y|} \in I \right\}$. *There exists* $l = l(n, |I|) \in (0, \frac{1}{2})$ *such that for every concave function* $G : B_1 \to \mathbb{R}$ *and* $h > 0$ *satisfying*

$$\left| \{ z \in R^I : G(z) < G(0) + z \cdot \nabla G(0) - h \} \right| \leq l \left| R^I \right| ,$$

the inequality

$$G(y) \geq G(0) + y \cdot \nabla G(0) - h$$

holds for every $y \in B_l$.

Remark 3.25. Note that the assertion of this result is weaker than the corresponding one of [CS09, Lemma 8.4]. This is due to the geometric restriction imposed by the set I.

Proof. Choose $l_0 \in (0, \frac{1}{2})$ sufficiently small such that for every $y \in B_{l_0}$ one can find two points $y_1, y_2 \in R^I$ such that

- $y = (y_1 + y_2)/2$,
- $B_{l_0}(y_1)$ and $B_{l_0}(y_2)$ are contained in R^I.

(*Construction*: Start with center zero and choose suitable $y_0 \in \partial B_{\frac{3}{4}} \cap R^I$ such that for sufficiently small $l_0 \in (0, \frac{1}{2})$, $B_{2l_0}(y_0)$ and $B_{2l_0}(-y_0)$ are completely contained in R^I. Hence, for every $y \in B_{l_0}$ one can find y_1 and y_2 as above (see Figure 3.1).)

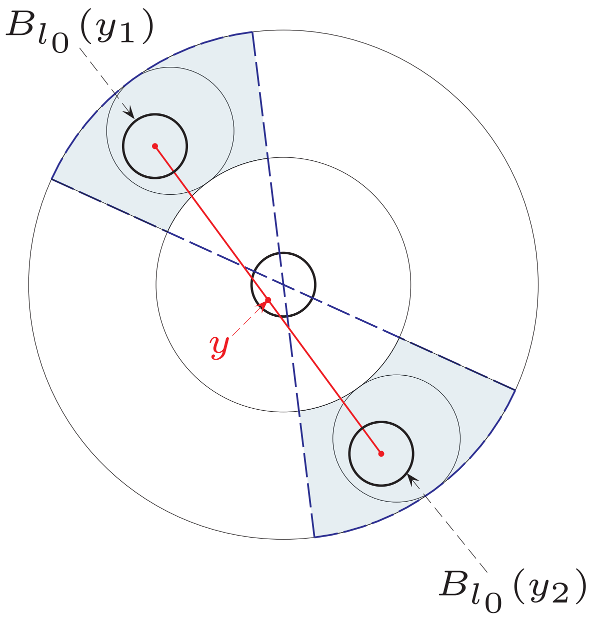

Figure 3.1: The balls $B_{l_0}(y_1)$ and $B_{l_0}(y_2)$.

We claim that $l \in (0, l_0)$ can be chosen small enough, such that for every $y \in B_l$, y_1 and y_2 as above, every concave function $G : B_1 \to \mathbb{R}$ and every $h > 0$ satisfying

$$\left| \{ z \in R^I : G(z) < G(0) + z \cdot \nabla G(0) - h \} \right| \leq l \left| R^I \right| , \tag{3.38}$$

there will be two points $z_1 \in B_{l_0}(y_1)$ and $z_2 \in B_{l_0}(y_2)$ with the following properties:

(i) $y = (z_1 + z_2)/2$,

(ii) $G(z_1) \geq G(0) + z_1 \cdot \nabla G(0) - h$, and

(iii) $G(z_2) \geq G(0) + z_2 \cdot \nabla G(0) - h$.

We prove the claim as follows: Choose $l \in (0, l_0)$ sufficiently small such that $l \left|R^I\right| < \frac{|B_{l_0}|}{2}$. Let $y \in B_l$. Choose $y_1, y_2 \in R^I$ with $y = (y_1 + y_2)/2$ and $B_{l_0}(y_1) \subset R^I, B_{l_0}(y_2) \subset R^I$. Let $G : B_1 \to \mathbb{R}$ be a concave function and $h > 0$. We define

$$D_1 = \{z_1 \in B_{l_0}(y_1) : G(z_1) \geq G(0) + z_1 \cdot \nabla G(0) - h\} \subset R^I,$$
$$D_2 = \{z_2 \in B_{l_0}(y_2) : G(z_2) \geq G(0) + z_2 \cdot \nabla G(0) - h\} \subset R^I.$$

Using (3.38) and the choice of l from above, we obtain $|D_1| > \frac{|B_{l_0}|}{2}$ and $|D_2| > \frac{|B_{l_0}|}{2}$. It is clear that for every point $z_1 \in B_{l_0}(y_1)$ there exists a point $z_2 \in B_{l_0}(y_2)$ such that $y = \frac{z_1+z_2}{2}$. We want to find points $z_1 \in D_1$ and $z_2 \in D_2$ such that $y = \frac{z_1+z_2}{2}$. Let us assume that this is not possible. Hence, for every $z_1 \in D_1$ we can only find a point $z_2 \in B_{l_0}(y_2) \setminus D_2$ such that $y = \frac{z_1+z_2}{2}$. This implies that

$$|B_{l_0}(y_2) \setminus D_2| \geq |D_1| > \frac{|B_{l_0}|}{2}.$$

This is a contradiction to the fact that $|D_2| > \frac{|B_{l_0}|}{2}$. This proves our claim. For $z_1 \in B_{l_0}(y_1)$ and $z_2 \in B_{l_0}(y_2)$ satisfying (i)-(iii), we finally have

$$\begin{aligned} G(y) = G\left(\frac{z_1 + z_2}{2}\right) &\geq \frac{1}{2}G(z_1) + \frac{1}{2}G(z_2) \\ &\geq G(0) + \frac{1}{2}(z_1 + z_2) \cdot \nabla G(0) - h = G(0) + y \cdot \nabla G(0) - h. \end{aligned}$$ □

A simple scaling argument leads to the following generalisation of Lemma 3.24:

Corollary 3.26. *For $x \in \mathbb{R}^n$ and $r > 0$ define $R_r(x) = B_r(x) \setminus B_{r/2}(x)$ and the subset $R_r^I(x) = \left\{y \in R_r(x) : \frac{y-x}{|y-x|} \in I\right\}$. For every concave function $G : B_r(x) \to \mathbb{R}$ and $h > 0$ satisfying*

$$\left|\{z \in R_r^I(x) : G(z) < G(x) + (z - x) \cdot \nabla G(x) - h\}\right| \leq l \left|R_r^I(x)\right|, \tag{3.39}$$

the inequality

$$G(y) \geq G(x) + (y - x) \cdot \nabla G(x) - h$$

holds for every $y \in B_{lr}(x)$, where $l \in (0, \frac{1}{2})$ is as in Lemma 3.24.

Proof. Let R and R^I be as in Lemma 3.24. Consider any concave function $G : B_r(x) \to \mathbb{R}$ and $h > 0$ satisfying (3.39). Define the functions $\Phi : R \to R_r(x)$, $\Phi(y) = ry + x$ and $\widetilde{G} : B_1 \to \mathbb{R}$, $\widetilde{G}(y) = G(ry + x)$. Since

$$r^n \left|R^I\right| = \left|\Phi(R^I)\right| = \left|R_r^I(x)\right|,$$

(3.39) is equivalent to

$$\left|\{z \in R^I : \widetilde{G}(z) < \widetilde{G}(0) + z \cdot \nabla \widetilde{G}(0) - h\}\right| \leq l \left|R^I\right|.$$

We finish the proof by using Lemma 3.24 and rescaling. □

Lemma 3.22 and Corollary 3.26 lead to the following result:

Corollary 3.27. *Let $\rho_0 \in (0,1)$ be arbitrary and $l \in (0, \frac{1}{2})$ be as in Lemma 3.24. There exists a constant $C_1 = C_1(n, |I|, \rho_0, \lambda) \geq 1$ and for every $x \in \Sigma$ there is $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ such that*

$$\frac{\left|\{y \in R^I_r(x) : u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - C_1 f(x)(lr)^2\}\right|}{|R^I_r(x)|} \leq l \tag{3.40}$$

and

$$\left|\nabla\Gamma(B_{lr/2}(x))\right| \leq (8C_1)^n f(x)^n \left|B_{lr/2}(x)\right|, \tag{3.41}$$

where $R^I_r(x)$ is defined as in Corollary 3.26.

Proof. Because of Lemma 3.22, there is a constant $C_0 = C_0(n, |I|, \rho_0, \lambda) \geq 1$ and for every $x \in \Sigma$ and every $A > 0$ there exists some $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ such that

$$\left|\{y \in R^I_r(x) : \, u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - Ar^2\}\right| \leq C_0 \,\frac{f(x)}{A} \left|R^I_r(x)\right|.$$

By choosing $A = \frac{C_0 f(x)}{l}$, we obtain (3.40) with $C_1 = \frac{C_0}{l^3}$.

Now let us prove (3.41). First note that for every $x \in \Sigma$ and every $h > 0$, the set

$$\{y \in \mathbb{R}^n : \, \Gamma(y) < \Gamma(x) + (y-x) \cdot \nabla\Gamma(x) - h\}$$

is a subset of

$$\{y \in \mathbb{R}^n : \, u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - h\}.$$

Using this relation and (3.40), we conclude that there is a constant $C_1 \geq 1$ and for every $x \in \Sigma$ there is some $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ such that

$$\frac{\left|\{y \in R^I_r(x) : \, \Gamma(y) < \Gamma(x) + (y-x) \cdot \nabla\Gamma(x) - C_1 f(x)(lr)^2\}\right|}{|R^I_r(x)|} \leq l. \tag{3.42}$$

Let $x \in \Sigma$ be arbitrary and choose $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ such that (3.42) holds. Because of the concavity of Γ and (3.42), we may apply Corollary 3.26 for $G = \Gamma$ and $h = C_1 f(x)(lr)^2$. We obtain

$$\Gamma(y) \geq \Gamma(x) + (y-x) \cdot \nabla\Gamma(x) - C_1 f(x)(lr)^2$$

for every $y \in B_{lr}(x)$. At the same time,

$$\Gamma(y) \leq \Gamma(x) + (y-x) \cdot \nabla\Gamma(x)$$

for every $y \in B_{lr}(x)$ because of the concavity of Γ. Hence,

$$|\Gamma(y) - \Gamma(x) - (y-x) \cdot \nabla\Gamma(x)| \leq C_1 f(x)(lr)^2 \text{ for every } y \in B_{lr}(x).$$

Recall that f is a positive function. Lemma 3.28(ii) from below completes the proof. □

Lemma 3.28. *(i) Let $G : B_R(x) \to \mathbb{R}$ be a concave function. Then*

$$\sup_{y \in B_{R/2}(x)} |\nabla G(y)| \leq \frac{4}{R} \sup_{y \in B_R(x)} |G(y)| \ . \tag{3.43}$$

(ii) Let $G : B_R(x) \to \mathbb{R}$ be a concave function satisfying

$$|G(y) - G(x) - (y-x) \cdot \nabla G(x)| \leq KR^2 \tag{3.44}$$

for every $y \in B_R(x)$ with some $K > 0$. Then

$$\left|\nabla G(B_{R/2}(x))\right| \leq (8K)^n \left|B_{R/2}(x)\right| \ . \tag{3.45}$$

Proof. (i) Without loss of generality we assume $x = 0$. Set $M = \sup_{y \in B_R} |G(y)|$.
Let $y \in B_{R/2}$. Given $h \neq 0$, choose numbers $s < 0 < t$ such that $|y + sh| = |y + th| = R$. Using the concavity of G, we have

$$-M \leq G(y + sh) \leq G(y) + sh \cdot \nabla G(y) \leq M + sh \cdot \nabla G(y).$$

In addition,

$$|sh| \geq |y + sh| - |y| \geq R - \frac{R}{2} = \frac{R}{2}.$$

These estimates still hold when we replace s by t. Hence,

$$\nabla G(y) \cdot h \leq -\frac{2M}{s} = -\frac{2M\,|h|}{s\,|h|} = \frac{2M\,|h|}{|sh|} \leq \frac{2M\,|h|}{R/2} \text{ and } \nabla G(y) \cdot h \geq -\frac{2M}{t} \geq -\frac{2M\,|h|}{R/2}.$$

This implies $|\nabla G(y) \cdot h| \leq \frac{2M|h|}{R/2} = \frac{4M}{R}\,|h|$ which is equivalent to $\frac{|\nabla G(y)\cdot h|}{|h|} \leq \frac{4M}{R}$. Since this estimate holds for every $h \neq 0$, we conclude

$$|\nabla G(y)| \leq \frac{4M}{R}.$$

(ii) For $y \in B_R(x)$ define $\widehat{G}(y) = G(y) - G(x) - (y-x) \cdot \nabla G(x)$. Note that $\widehat{G}$ is a concave function in $B_R(x)$. Let $z \in B_{R/2}(x)$. Using (3.43) and (3.44), we obtain

$$|\nabla G(z) - \nabla G(x)| = \left|\nabla \widehat{G}(z)\right| \leq \frac{4}{R} \sup_{y \in B_R(x)} |G(y) - G(x) - (y-x) \cdot \nabla G(x)| \leq 8K\tfrac{R}{2}.$$

Therefore,

$$\nabla G(B_{R/2}(x)) \subset B_{8K(R/2)}(\nabla G(x))$$

which implies

$$\left|\nabla G(B_{R/2}(x))\right| \leq \left|B_{8K(R/2)}(x)\right| = (8K)^n \left|B_{R/2}(x)\right|,$$

which proves (3.45). □

As a consequence of this corollary, we derive a theorem which can be considered as a nonlocal version of the ABP estimate from Section 2.3, cf. Theorem 2.17. Recall the assumptions on u and f made at the beginning of this section.

Theorem 3.29. *Let $l \in (0, \frac{1}{2})$ be as in Lemma 3.24 and assume $0 < \rho_0 \leq \frac{l}{16n}$. There are constants $C_2 = C_2(|I|, \lambda, \rho_0, n) \geq 1$, $\nu = \nu(|I|, n) \in (0,1)$ and a disjoint family of open cubes $(Q^j)_{j=1,\ldots,m}$, $m \in \mathbb{N}$, with diameters $0 < d_j \leq \rho_0 2^{-1/(2-\alpha)}$, covering the contact set Σ in a way that the following properties hold for every $j = 1, \ldots, m$:*

(i) $\Sigma \cap \overline{Q^j} \neq \emptyset$, *(ii)* $\left|\nabla\Gamma\left(\overline{Q^j}\right)\right| \leq C_2(\sup_{\overline{Q^j}} f)^n \left|Q^j\right|$,

(iii) $\left|\{y \in \eta Q^j : u(y) \geq \Gamma(y) - C_2(\sup_{\overline{Q^j}} f) d_j^2\}\right| \geq \nu \left|\eta Q^j\right|$, *where* $\eta = (1+\frac{8}{l})\sqrt{n}$.

Remark 3.30.

(i) Recall the definition of enlarged cubes tQ, $t > 1$, provided in Section 1.4.

(ii) Note that Theorem 3.29 is formulated for subsolutions in B_1. Using a scaling argument, a similar assertion holds when considering subsolutions in $B_{2\sqrt{n}}$ as in Section 3.6. In this case ρ_0 is replaced by $2\sqrt{n}\rho_0$. To be precise: Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable and bounded function that is upper semicontinuous in $\overline{B_{2\sqrt{n}}}$ with $v \leq 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$ and $v^+ \not\equiv 0$. Assume further that v satisfies $M^+_\alpha v \geq -f$ in $B_{2\sqrt{n}}$ in the viscosity sense, where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable, bounded and positive function. Let Γ_v be the concave envelope of v^+ in $B_{6\sqrt{n}}$. Define $u(x) = v(2\sqrt{n}x)$ and $\Gamma_u(x) = \Gamma_v(2\sqrt{n}x)$. Hence,

$$u \leq 0 \text{ in } \mathbb{R}^n \setminus B_1 \quad \text{and} \quad M^+_\alpha u \geq -\widetilde{f} \text{ in } B_1 \text{ in the viscosity sense,}$$

where $\widetilde{f}(x) = (2\sqrt{n})^\alpha f(2\sqrt{n}x)$. We apply Theorem 3.29 to u: Let (Q^j), (d_j), $C_2 \geq 1$ and $\nu > 0$ be as in Theorem 3.29. Define the cubes $\mathcal{Q}^j = \{2\sqrt{n}x : x \in Q^j\}$ with corresponding diameters $D_j = 2\sqrt{n}d_j$. Using (ii) and (iii) in Theorem 3.29, we obtain (after rescaling)

$$\begin{aligned}\left|\nabla\Gamma_v\left(\overline{\mathcal{Q}^j}\right)\right| &= \left|\nabla\Gamma_u\left(\overline{Q^j}\right)\right| \leq C_2(\sup_{\overline{Q^j}} \widetilde{f}\,)^n \left|Q^j\right| \leq 4nC_2(\sup_{\overline{\mathcal{Q}^j}} f)^n \left|\mathcal{Q}^j\right| \\ &=: \widetilde{C_2}(\sup_{\overline{\mathcal{Q}^j}} f)^n \left|\mathcal{Q}^j\right|\end{aligned}$$

and

$$\left|\{y \in \eta\mathcal{Q}^j : v(y) \geq \Gamma_v(y) - \widetilde{C_2}(\sup_{\overline{\mathcal{Q}^j}} f) D_j^2\,\}\right| \geq \nu \left|\eta\mathcal{Q}^j\right|.$$

Proof of Theorem 3.29. The proof follows the one of [CS09, Theorem 8.7]. In our context, the main constants additionally depend on $|I|$. Let $C_1 \geq 1$ be as in Corollary 3.27. Set $c_1 = (8C_1)^n$ and $c_2 = 16C_1$. We prove the assertion of the theorem with $C_2 = c_1\eta^n$ and $\nu = (1-l)\frac{|R^I|}{|B_1|}(8\sqrt{n})^{-n}$, where R^I is as in Lemma 3.24. Let $\mathcal{Q}_1$ be a finite disjoint family of open cubes Q with diameter $d_1 = \rho_0 2^{-1/(2-\alpha)}$ and the property $B_1 \subset \bigcup_{\mathcal{Q}_1} \overline{Q}$. Let $\mathcal{Q}_1' \subset \mathcal{Q}_1$ be the subfamily of all cubes Q with $\overline{Q} \cap \Sigma \neq \emptyset$. We decompose every cube in $\mathcal{Q}_1'$ which does not satisfy both conditions (ii) and (iii) from above into 2^n subcubes of half diameter. Now, let $\mathcal{Q}_2$ be the family of these newly created subcubes plus those cubes from $\mathcal{Q}_1'$ that do satisfy both conditions (ii) and (iii) from above (and hence were not decomposed). We repeat this procedure and obtain a sequence of families

$$\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \dots$$

We claim that there is an index $k \in \mathbb{N}$ with $\mathcal{Q}_k = \mathcal{Q}_{k+i}$ for all $i \in \mathbb{N}$. In this case, we set $m = \#\mathcal{Q}_k$. Let us assume that no such index $k \in \mathbb{N}$ exists. Then there exists a sequence of cubes Q^j with diameter $d_j = 2^{-j+1}d_1$ and for every $j \in \mathbb{N}$ the following properties hold:

a) $Q^j \supset Q^{j+1}$,

b) $\overline{Q^j} \cap \Sigma \neq \emptyset$,

c) Q^j violates (ii) or (iii).

Let $x_0 \in \bigcap_{j\in\mathbb{N}} \overline{Q^j}$. We prove that $x_0 \in \Sigma$: First of all, it is clear that $x_0 \in \overline{B_1}$. To prove the fact that $x_0 \in \{u = \Gamma\}$, we consider a sequence $(x_j)_{j\in\mathbb{N}}$ with

$$x_1 \in \overline{Q^1} \cap \Sigma, \quad x_2 \in \overline{Q^2} \cap \Sigma, \quad \dots .$$

(x_j) is a Cauchy sequence because of properties a) and b). Hence, $x_j \xrightarrow{j\to\infty} x_0$. Using the upper semicontinuity of u in $\overline{B_1}$ and the fact that $x_j \in \Sigma$ for every $j \in \mathbb{N}$, we obtain

$$u(x_0) \geq \limsup_{j\to\infty} u(x_j) = \limsup_{j\to\infty} \Gamma(x_j) = \Gamma(x_0).$$

At the same time, $u(x_0) \leq \Gamma(x_0)$ because Γ is the concave envelope of u^+. Thus, $x_0 \in \{u = \Gamma\}$. Finally, if $x_0 \in \mathbb{S}^{n-1}$ then $\Gamma(x_0) = u(x_0) = 0$ which implies $\Gamma \equiv 0$ and $u^+ \equiv 0$ (contradiction). We conclude $x_0 \in \Sigma$.

We now derive a contradiction by showing that one of the cubes of the sequence from above satisfies (ii) and (iii).

Using Corollary 3.27, there is a number r with $0 < r < \rho_0 2^{-1/(2-\alpha)}$ such that

$$\frac{\left|\{y \in R_r^I(x_0) : u(y) < u(x_0) + (y - x_0)\cdot\nabla\Gamma(x_0) - C_1 f(x_0)(lr)^2\}\right|}{|R_r^I(x_0)|} \leq l \tag{3.46}$$

and

$$\left|\nabla\Gamma(B_{lr/2}(x_0))\right| \leq c_1 f(x_0)^n \left|B_{lr/2}(x_0)\right|. \tag{3.47}$$

Fix an index $j_0 \in \mathbb{N}$ such that

$$\frac{lr}{4} \le d_{j_0} < \frac{lr}{2}.$$

Therefore,

$$B_{lr/2}(x_0) \supset \overline{Q^{j_0}}, \quad B_r(x_0) \subset \eta Q^{j_0}. \tag{3.48}$$

We prove (3.48). Note that we may assume that x_0 is a vertex of $\overline{Q^{j_0}}$.

- Since $d_{j_0} < \frac{lr}{2}$, we easily see that $\overline{Q^{j_0}} \subset B_{lr/2}(x_0)$.
- To prove the second inclusion, let $\widetilde{x}$ be the center of the cube Q^{j_0} and $c > 0$. If

$$c\frac{d_{j_0}}{2\sqrt{n}} \ge r + \frac{d_{j_0}}{2\sqrt{n}} \Longleftrightarrow c \ge 1 + \frac{2\sqrt{n}r}{d_{j_0}},$$

we ensure, that the larger cube cQ^{j_0} with center $\widetilde{x}$ contains $B_r(x_0)$. Note that

$$1 + \frac{2\sqrt{n}r}{d_{j_0}} \le 1 + \frac{2\sqrt{n}r}{lr/4} = 1 + \frac{8\sqrt{n}}{l} \quad \text{since } d_{j_0} \ge \frac{lr}{4}\ .$$

So we may choose $c = \eta = (1 + \frac{8}{l})\sqrt{n}$ which proves the second inclusion. Note that $\frac{d_{j_0}}{2\sqrt{n}}$ is half of the edge length of Q^{j_0}. See Figure 3.2 for a visualisation of the previous arguments.

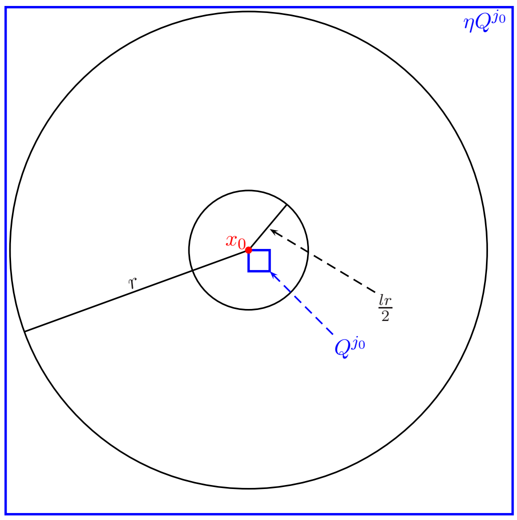

Figure 3.2: Visualisation of (3.48) with $d_{j_0} = \frac{lr}{4}$

Note that $\eta Q^{j_0} \subset B_3$ since

$$\eta d_{j_0} \le (1 + \tfrac{8}{l})\sqrt{n}\rho_0 \le (1 + \tfrac{8}{l})\sqrt{n}\tfrac{l}{16n} \le \tfrac{9}{16} < 1$$

and $x_0 \in B_1$. Recall that $\Gamma(y) \le u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0)$ for every $y \in B_3$ because of the concavity of Γ and the fact that $\Gamma(x_0) = u(x_0)$. Using (3.46), (3.48) and the relation between d_{j_0} and r, we obtain

$$\begin{aligned}
&\left|\{y \in \eta Q^{j_0} : u(y) \ge \Gamma(y) - C_2(\sup\nolimits_{\overline{Q^{j_0}}} f) d_{j_0}^2\}\right| \\
&\quad \ge \left|\{y \in \eta Q^{j_0} : u(y) \ge u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0) - c_2 f(x_0)\tfrac{(lr)^2}{16}\}\right| \\
&\quad \ge \left|\{y \in B_r(x_0) \setminus B_{r/2}(x_0) : u(y) \ge u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0) - C_1 f(x_0)(lr)^2\}\right| \\
&\quad \ge \left|\{y \in R_r^I(x_0) : u(y) \ge u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0) - C_1 f(x_0)(lr)^2\}\right| \\
&\quad \ge \left|R_r^I(x_0)\right| - l\left|R_r^I(x_0)\right| = (1 - l)\left|R_r^I(x_0)\right| \ge \nu\left|\eta Q^{j_0}\right|.
\end{aligned}$$

Moreover, using (3.47) and (3.48), we obtain

$$\begin{aligned}
\left|\nabla\Gamma\big(\overline{Q^{j_0}}\,\big)\right| &\le \left|\nabla\Gamma(B_{lr/2}(x_0))\right| \le c_1 f(x_0)^n \left|B_{lr/2}(x_0)\right| \\
&\le c_1 (\sup_{\overline{Q^{j_0}}} f)^n \left|\eta Q^{j_0}\right| = C_2 (\sup_{\overline{Q^{j_0}}} f)^n \left|Q^{j_0}\right|.
\end{aligned}$$

Therefore, Q^{j_0} satisfies (i)-(iii) with C_2, ν from above. Contradiction. □

Remark 3.31. Assume the positive function $f : \mathbb{R}^n \to \mathbb{R}$ in Theorem 3.29 to be continuous in addition. In this case, letting $\alpha \nearrow 2$, we obtain the classical ABP estimate (Theorem 2.17) as a limit of Riemann sums (see also [CS09, Remark 8.8]): Note first that the upper bound $\rho_0 2^{-1/(2-\alpha)}$ for the diameters of the cubes in Theorem 3.29 decreases when α approaches 2 from below. Applying Theorem 3.29, we obtain for every $\alpha \in (0, 2)$

$$\begin{aligned}
|\nabla\Gamma(\Sigma)| = |\nabla\Gamma(\{u = \Gamma\} \cap B_1)| &\le \Big|\bigcup_{j=1}^{m} \nabla\Gamma(\overline{Q^j})\Big| \\
&\le \sum_{j=1}^{m} \left|\nabla\Gamma(\overline{Q^j})\right| \le C_2 \sum_{j=1}^{m} (\sup_{\overline{Q^j}} f)^n \left|Q^j\right|,
\end{aligned}$$

where $m \in \mathbb{N}$, the family of cubes $(Q^j)_{j=1,\dots,m}$ and the constant C_2 are as in Theorem 3.29. Recall that $C_2 \ge 1$ does not depend on α. As $\alpha \nearrow 2$, the cube covering of the contact set $\{u = \Gamma\} \cap B_1$ is getting closer and closer to $\{u = \Gamma\} \cap B_1$ and the estimate from above leads to

$$|\nabla\Gamma(\Sigma)| \le C_2 \int_{\{u=\Gamma\}\cap B_1} (f(x))^n \, dx. \tag{3.49}$$

Modifying the proof of Theorem 2.17 leading to (2.20), we find a constant $c(n) \ge 1$ such that (together with (3.49) and Remark 3.32 from below)

$$\begin{aligned}
\sup_{B_1} u^+ &\le c(n) \left|\nabla\Gamma(B_1)\right|^{1/n} = c(n) \left|\nabla\Gamma(\{u = \Gamma\} \cap B_1)\right|^{1/n} \\
&\le c_1 \Bigg(\int_{\{u=\Gamma\}\cap B_1} (f(x))^n \, dx \Bigg)^{1/n} \quad \text{with } c_1 = c(n) C_2^{1/n}.
\end{aligned}$$

Remark 3.32. Note that

$$\nabla\Gamma(B_1 \cap \{u = \Gamma\}) = \nabla\Gamma(B_1). \tag{3.50}$$

This can be proven as follows: Let $x_0 \in B_1$ such that $\nabla\Gamma(x_0) = a$ for some $a \in \mathbb{R}^n$. We want to find a point $x \in B_1 \cap \{u = \Gamma\}$ such that $\nabla\Gamma(x) = a$. Since $\nabla\Gamma(x_0) = a$, there exists – by definition of the concave envelope – an affine function p of the form $p(y) = a \cdot y + b$, where $b \in \mathbb{R}$, which is a supporting hyperplane of Γ at x_0 in B_3, i.e.,

$$p(x_0) = \Gamma(x_0) \quad \text{and} \quad \Gamma(y) \le p(y) \text{ for every } y \in B_3.$$

We consider two cases:

- Assume there exists a point $x \in B_3$ such that $u^+(x) = p(x)$. Since $u \le 0$ in $\mathbb{R}^n \setminus B_1$, $0 \le u^+ \le \Gamma \le p$ in B_3 and $\Gamma \not\equiv 0$, we conclude that $x \in B_1$, $u^+(x) = u(x) > 0$ and $\Gamma(x) = u(x) = p(x)$. This implies $x \in B_1 \cap \{u = \Gamma\}$ and $\nabla\Gamma(x) = \nabla p(x) = a$.
- Assume that $u^+ < p$ in B_3. Hence, $\inf_{B_3}(p - u^+) > 0$ which implies the existence of some $\varepsilon > 0$ such that $\widetilde{p} = p - \varepsilon \ge u^+$ in B_3. Since $\widetilde{p} < p$ in B_3, we derive a contradiction to the fact that p is the affine function that realizes the infimum in the definition of the concave envelope Γ at x_0. So this case can not occur.

This proves (3.50).

3.5 Bump Functions

In this section we construct a special function with similar properties as the one in [CC95, Lemma 4.1]. We will use this function in Section 3.6 in combination with the ABP estimate from the previous section. The construction is based on an idea used in [CS09]. However, the necessary modifications are technically involved due to the fact that the lower bound of the kernels under consideration is not rotational invariant. We begin with some basic observations.

For $p > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$ let $f(x) = f_p(x) = |x|^{-p}$. Recall that

$$\Delta f(x; y) = f(x + y) + f(x - y) - 2f(x).$$

For $x \in \mathbb{R}^n \setminus \{0\}$ define the set

$$\mathfrak{O}(x) = \{\xi \in \mathbb{S}^{n-1} : \xi \cdot x = 0\}.$$

Fix some $x \neq 0$ and $\xi \in \mathfrak{O}(x)$. For $0 < r < |x|$ and $\beta \in [0, \frac{\pi}{2}]$ define

$$g_r(\beta) = \Delta f(x; y_\beta),$$

where y_β is any element of ∂B_r with $\frac{|y_\beta \cdot \xi|}{|y_\beta|} = \cos\beta$. Note that for any r as above, the function g_r is well-defined because $\Delta f(x; y) = \Delta f(x; z)$ for every $y, z \in \mathbb{R}^n$ with $|y| = |z|$ and $|y \cdot \xi| = |z \cdot \xi|$.

Lemma 3.33. *For $0 < r < |x|$ the function $g_r : [0, \frac{\pi}{2}] \to \mathbb{R}$ is strictly increasing and satisfies $g_r(0) < 0$.*

Proof. Let $0 < r < |x|$ and $\beta \in [0, \frac{\pi}{2}]$. Let $y_\beta \in \partial B_r$ such that $\frac{|y_\beta \cdot \xi|}{|y_\beta|} = \cos\beta$. Thus

$$g_r(\beta) = (|x|^2 + r^2 + 2\,|x|\,r\cos(\tfrac{\pi}{2} - \beta))^{-\frac{p}{2}} + (|x|^2 + r^2 - 2\,|x|\,r\cos(\tfrac{\pi}{2} - \beta))^{-\frac{p}{2}} - 2\,|x|^{-p}\,.$$

Then

$$g_r(0) = 2((|x|^2 + r^2)^{-\frac{p}{2}} - |x|^{-p}) < 0.$$

Moreover, g_r is differentiable at every $\beta \in (0, \frac{\pi}{2})$ and

$$g_r'(\beta) = p\,|x|\,r\sin(\tfrac{\pi}{2} - \beta)\Big[(\,|x|^2 + r^2 - 2r\,|x|\cos(\tfrac{\pi}{2} - \beta))^{-\frac{p+2}{2}} - (|x|^2 + r^2 + 2r\,|x|\cos(\tfrac{\pi}{2} - \beta))^{-\frac{p+2}{2}}\Big].$$

Since the term from above is strictly positive for each $\beta \in (0, \frac{\pi}{2})$, we see that g_r is strictly increasing. This implies in particular $g_r(0) \le g_r(\beta)$ for every $\beta \in [0, \frac{\pi}{2}]$. □

For $\xi \in \mathfrak{O}(x)$, $\beta \in (0, \frac{\pi}{2}]$ and $r > 0$ define

$$S_{\beta,r} = S_{\beta,r}(\xi) = \big\{y \in \partial B_r : \tfrac{|y\cdot\xi|}{|y|} > \cos\beta\big\}.$$

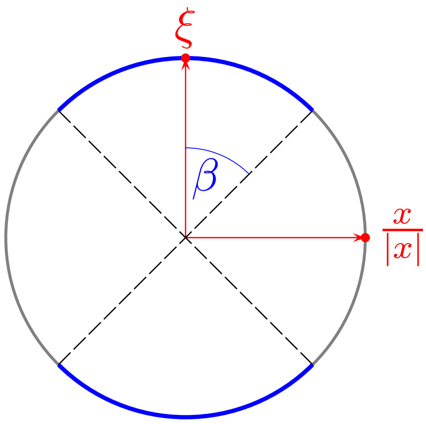

Figure 3.3: The set $S_{\beta,1}$.

The following result is an immediate consequence of Lemma 3.33.

Corollary 3.34. *Let $0 < r < |x|$. For $y, z \in \partial B_r$ with $|y \cdot \xi| \ge |z \cdot \xi|$ we have*

$$\Delta f(x; y) \le \Delta f(x; z).$$

In particular:

i) The function $y \mapsto \Delta f(x;y)^-$ *for* $y \in \partial B_r$ *attains its maximum for* $y = r\xi$.

ii) Let $\beta \in (0, \frac{\pi}{2}]$. *The function* $y \mapsto \Delta f(x;y)^+$ *for* $y \in \partial B_r \setminus S_{\beta,r}$ *attains its minimum for* $y = y_\beta$ *with* $y_\beta \in \partial B_r$ *arbitrary and* $\frac{|y_\beta \cdot \xi|}{|y_\beta|} = \cos\beta$.

Lemma 3.35. *Let* $\beta \in (0, \frac{\pi}{2}]$ *and* $K \geq 0$. *There exist* $p_0 = p_0(\beta, |x|, K) > 0$ *and* $r_0 = r_0(\beta, |x|) > 0$ *such that for every* $r \in (0, r_0]$, $p \geq p_0$ *and* $y \in \partial B_r$ *with* $\frac{|y\cdot\xi|}{|y|} = \cos\beta$

$$\Delta f_p(x;y)^+ - K\Delta f_p(x;r\xi)^- \geq C_3 p\,|x|^{-p}\,r^2,$$

where $C_3 = C_3(\beta, |x|, K) > 0$.

Proof. Let $0 < r < |x|$. Note that $\Delta f(x;r\xi)^- = 2(|x|^{-p} - (|x|^2 + r^2)^{-p/2}) > 0$. Let $y \in \partial B_r$ with $\frac{|y\cdot\xi|}{|y|} = \cos\beta$. Set $\vartheta = \frac{\pi}{2} - \beta$. Define

$$\begin{aligned} h(r) &= \Delta f(x;y) - K\Delta f(x;r\xi)^- \\ &= (|x|^2 + r^2 - 2r\,|x|\cos\vartheta)^{-p/2} + (|x|^2 + r^2 + 2r\,|x|\cos\vartheta)^{-p/2} - 2\,|x|^{-p} \\ &\quad - 2K(|x|^{-p} - (|x|^2 + r^2)^{-p/2}). \end{aligned}$$

We have $\Delta f(x;y)^+ - K\Delta f(x;r\xi)^- \geq h(r)$. We calculate the derivatives of h:

$$\begin{aligned} h'(r) =& -p(|x|^2 + r^2 - 2r\,|x|\cos\vartheta)^{-\frac{p+2}{2}}(r - |x|\cos\vartheta) \\ &- p(|x|^2 + r^2 + 2r\,|x|\cos\vartheta)^{-\frac{p+2}{2}}(r + |x|\cos\vartheta) \\ &- 2Kp(|x|^2 + r^2)^{-\frac{p+2}{2}}r, \\ h''(r) =& \,p(|x|^2 + r^2 - 2r\,|x|\cos\vartheta)^{-\frac{p+4}{2}}\left[(p+2)(r - |x|\cos\vartheta)^2 - (|x|^2 + r^2 - 2r\,|x|\cos\vartheta)\right] \\ &+ p(|x|^2 + r^2 + 2r\,|x|\cos\vartheta)^{-\frac{p+4}{2}}\left[(p+2)(r + |x|\cos\vartheta)^2 - (|x|^2 + r^2 + 2r\,|x|\cos\vartheta)\right] \\ &+ 2Kp(p+2)(|x|^2 + r^2)^{-\frac{p+4}{2}}r^2 - 2Kp(|x|^2 + r^2)^{-\frac{p+2}{2}}. \end{aligned}$$

Now set $r_0 = \frac{1}{2}\,|x|\cos\vartheta$ and choose $p_0 \geq 1$ large enough such that

$$(p_0 + 2)(|x|\cos\vartheta - r_0)^2 > (2K+1)(|x| + r_0)^2. \tag{3.51}$$

Let $r \in (0, r_0]$ and $p \geq p_0$. Using Taylor expansion and (3.51), there is $\zeta \in (0, r)$ such that

$$\begin{aligned} h(r) &= \underbrace{h(0)}_{=0} + \underbrace{h'(0)}_{=0}r + h''(\zeta)\tfrac{r^2}{2} \\ &\geq \tfrac{r^2}{2}\Bigg(p(|x|^2 + \zeta^2 - 2\zeta\,|x|\cos\vartheta)^{-\frac{p+4}{2}}\left[(p_0+2)(\zeta - |x|\cos\vartheta)^2 - (2K+1)(|x| + \zeta)^2\right] \\ &\qquad + p(|x|^2 + \zeta^2 + 2\zeta\,|x|\cos\vartheta)^{-\frac{p+4}{2}}\left[(p_0+2)(\zeta + |x|\cos\vartheta)^2 - (|x| + \zeta)^2\right]\Bigg) \\ &\geq \tfrac{1}{2}\,|x|^{-4}\left[(p_0+2)(|x|\cos\vartheta - r_0)^2 - (2K+1)(|x| + r_0)^2\right]p\,|x|^{-p}\,r^2 =: C_3 p\,|x|^{-p}\,r^2. \qquad \square \end{aligned}$$

As an immediate consequence of the previous results, we can identify a certain area where $\Delta f(x;y)^- = 0$.

Corollary 3.36. *Let* $\beta \in (0, \frac{\pi}{2}]$. *There exist* $p_1 = p_1(\beta, |x|) > 0$ *and* $r_0 = r_0(\beta, |x|) > 0$ *such that for every* $r \in (0, r_0]$, $p \geq p_1$ *and* $y \in \partial B_r \setminus S_{\beta,r}$

$$\Delta f_p(x;y)^- = 0.$$

Proof. Because of Lemma 3.35 (with $K = 0$), we may choose $r_0 > 0$ and $p_1 > 0$ such that for every $r \in (0, r_0]$, $p \geq p_1$ and $y_\beta \in \mathbb{S}^{n-1}$ with $|y_\beta \cdot \xi| = \cos\beta$

$$\Delta f(x; ry_\beta)^+ \geq C_3 p \, |x|^{-p} \, r^2 > 0.$$

Note that $C_3 > 0$ is as in Lemma 3.35 with $K = 0$. Take any $r \in (0, r_0]$ and $p \geq p_1$. Using Corollary 3.34 ii) with β as above, we have $\Delta f(x;y)^+ \geq \Delta f(x; ry_\beta)^+ > 0$ for every $y \in \partial B_r \setminus S_{\beta,r}$. □

Let us define a bounded approximation of the function f. For $\gamma > 0$, $p > 0$ and $x \neq 0$ set

$$f_\gamma(x) = f_{\gamma,p}(x) = \gamma^{-p} \wedge |x|^{-p} .$$

Our aim is to prove Proposition 3.40. Recall the setting at the beginning of Section 3.2. We begin with two auxiliary results.

Lemma 3.37. *Let* $0 < R \leq 1$, $x \in \partial B_R$, $\alpha \in (0, 2)$, $K > 0$ *and* $\gamma \in (0, \frac{R}{2}]$. *There exist* $r_0 \in (0, 1)$ *and* $p_0 > 0$ *(depending only on* $n, R, |I|$ *and* K*) such that for every* $p \geq p_0$

$$\int\limits_{B_{r_0}} k\big(\tfrac{y}{|y|}\big) \Delta f_{\gamma,p}(x;y)^+ - K \Delta f_{\gamma,p}(x;y)^- \, \mu(dy) \geq 0, \tag{3.52}$$

where $\mu(dy) = |y|^{-n-\alpha} \, dy$, $I \subset \mathbb{S}^{n-1}$ *and* $k : \mathbb{S}^{n-1} \to [0,1]$ *are as in Section 3.2.*

Remark 3.38. If $I = \mathbb{S}^{n-1}$ then the assertion can be easily obtained by choosing p large and γ sufficiently small. This approach is used in [CS09] but cannot be applied here due to the anisotropy given by k or I respectively.

Proof. Recall that I is of the form $I = (B_\varrho(\xi_0) \cup B_\varrho(-\xi_0)) \cap \mathbb{S}^{n-1}$. Choose $\vartheta_I \in (0, \frac{\pi}{2}]$ such that

$$I \supset S_{\vartheta_I,1}(\xi_0).$$

Let us first assume that $\xi_0 \in \mathfrak{O}(x)$. Choose $\beta \in (0, \vartheta_I)$ small enough such that

$$|S_{\vartheta_I,1}(\xi_0) \setminus S_{\beta,1}(\xi_0)| \geq |S_{\beta,1}(\xi_0)| . \tag{3.53}$$

Set $r_0 = \frac{R\cos(\frac{\pi}{2}-\beta)}{2}$. Note that for every $y \in B_{r_0}$ (and $p > 0$)

$$\Delta f_\gamma(x;y) = |x+y|^{-p} + |x-y|^{-p} - 2R^{-p}.$$

Using Lemma 3.35, there exists $p_0 > 0$ such that for every $r \in (0, r_0]$, $p \geq p_0$ and $y_\beta \in \mathbb{S}^{n-1}$ with $|y_\beta \cdot \xi_0| = \cos\beta$

$$\Delta f_\gamma(x; ry_\beta)^+ - K\Delta f_\gamma(x; r\xi_0)^- \geq C_3 p\,|x|^{-p}\,r^2, \tag{3.54}$$

where the constant $C_3 = C_3(|I|, R, K) > 0$ is as in Lemma 3.35. Note that these choices of r_0 and p_0 imply

$$\Delta f_\gamma(x; y)^- = 0$$

for each $r \in (0, r_0]$, $p \geq p_0$ and $y \in \partial B_r \setminus S_{\beta,r}(\xi_0)$ by Corollary 3.36, since $p_0 \geq p_1$ and p_1 is as in Corollary 3.36.
Hence,

$$\begin{aligned}
&\int\limits_{B_{r_0}} \frac{k(\frac{y}{|y|})\Delta f_\gamma(x; y)^+ - K\Delta f_\gamma(x; y)^-}{|y|^{n+\alpha}}\,dy \qquad (3.55)\\
&\geq \int\limits_0^{r_0} r^{n-1}\Bigg(\int\limits_{I\setminus S_{\beta,1}(\xi_0)} k(y)\frac{\Delta f_\gamma(x; ry)^+}{r^{n+\alpha}}\,\sigma(dy) - \int\limits_{S_{\beta,1}(\xi_0)} \frac{K\Delta f_\gamma(x; ry)^-}{r^{n+\alpha}}\,\sigma(dy)\Bigg)\,dr\\
&\geq \int\limits_0^{r_0} \frac{\Delta f_\gamma(x; ry_\beta)^+\,|I \setminus S_{\beta,1}(\xi_0)| - K\Delta f_\gamma(x; r\xi_0)^-\,|S_{\beta,1}(\xi_0)|}{r^{1+\alpha}}\,dr\\
&\overset{(3.53)}{\geq} |I \setminus S_{\beta,1}(\xi_0)| \int\limits_0^{r_0} \frac{\Delta f_\gamma(x; ry_\beta)^+ - K\Delta f_\gamma(x; r\xi_0)^-}{r^{1+\alpha}}\,dr\\
&\overset{(3.54)}{\geq} \tfrac{1}{2-\alpha}\,|I \setminus S_{\beta,1}(\xi_0)|\,C_3 pR^{-p} r_0^{2-\alpha} \geq 0,
\end{aligned}$$

where σ denotes the surface measure on the unit sphere.
If $\xi_0 \notin \mathfrak{O}(x)$, consider any $\xi \in \mathfrak{O}(x)$ and take β, r_0 and p_0 as above. Clearly,

$$|I \setminus S_{\beta,1}(\xi)| \geq |I \setminus S_{\beta,1}(\xi_0)| \geq |S_{\beta,1}(\xi_0)| = |S_{\beta,1}(\xi)|\,.$$

So we can estimate the integral in (3.55) in the same way as above, using $S_{\beta,1}(\xi)$ instead of $S_{\beta,1}(\xi_0)$. This finishes the proof. □

Lemma 3.39. *Let $0 < R \leq 1$, $x \in \partial B_R$, $K > 0$, $\gamma \in (0, \frac{R}{2}]$ and $\alpha_0 \in (0, 2)$. Let $r_0 \in (0, 1)$ and $p_0 > 0$ be as in Lemma 3.37. There exists $p \geq p_0$ (depending on $n, |I|, \alpha_0, K$ and R) such that for every $\alpha \in (\alpha_0, 2)$*

$$\int\limits_{B_{r_0}} k\big(\tfrac{y}{|y|}\big)\Delta f_{\gamma,p}(x; y)^+\,\mu(dy) \geq K \int\limits_{\mathbb{R}^n\setminus B_{r_0}} \Delta f_{\gamma,p}(x; y)^-\,\mu(dy). \tag{3.56}$$

Proof. We start by estimating the right hand side of (3.56). For every $p \geq p_0$, we have

$$K \int\limits_{\mathbb{R}^n\setminus B_{r_0}} \frac{\Delta f_\gamma(x; y)^-}{|y|^{n+\alpha}}\,dy \leq K2R^{-p}n\omega_n \int\limits_{r_0}^{\infty} \tfrac{1}{r^{1+\alpha}}\,dr \leq 2KR^{-p}n\omega_n \tfrac{r_0^{-2}}{\alpha_0} =: c(p).$$

Let $\xi \in \mathfrak{O}(x)$ and let $\beta \in (0, \vartheta_I)$ be as in the previous proof. Using Lemma 3.35, we obtain for every $p \geq p_0$

$$\int_{B_{r_0}} k\left(\tfrac{y}{|y|}\right) \frac{\Delta f_\gamma(x;y)^+}{|y|^{n+\alpha}}\, dy \geq C_3 R^{-p} p \int_0^{r_0} \int_{I \setminus S_{\beta,1}(\xi)} \frac{1}{r^{-1+\alpha}}\, \sigma(dy)\, dr$$
$$\geq \tfrac{1}{2} C_3 R^{-p} p \left| I \setminus S_{\beta,1}(\xi_0) \right| r_0^2 =: h(p),$$

where the constant C_3 is as in Lemma 3.35. Choosing $p \geq p_0$ large enough such that

$$p \tfrac{C_3}{2} \left| I \setminus S_{\beta,1}(\xi_0) \right| r_0^2 \geq 2 K n \omega_n \tfrac{r_0^{-2}}{\alpha_0},$$

we obtain $h(p) \geq c(p)$ which finishes the proof. □

We summarise the previous results:

Proposition 3.40. *Let $0 < R \leq 1$, $\alpha_0 \in (0,2)$ and $\gamma \in (0, \frac{R}{2}]$. There exists $p > 0$ (depending on $n, |I|, \alpha_0, R, \lambda$ and Λ) such that for every $\alpha \in (\alpha_0, 2)$ and $|x| \geq R$*

$$M_\alpha^- f_{\gamma,p}(x) \geq 0. \tag{3.57}$$

Proof. Let $\alpha \in (\alpha_0, 2)$. We firstly consider the case $x \in \partial B_R$. Set $K = \frac{2\Lambda}{\lambda}$. Choose $r_0 > 0$ and $p_0 > 0$ as in Lemma 3.37 and Lemma 3.39. For $p \geq p_0$ we write

$$\begin{aligned} M_\alpha^- f_\gamma(x) &= (2-\alpha) \int_{\mathbb{R}^n} \lambda k\left(\tfrac{y}{|y|}\right) \Delta f_\gamma(x;y)^+ - \Lambda \Delta f_\gamma(x;y)^- \,\mu(dy) \\ &\geq \tfrac{(2-\alpha)\lambda}{2} \int_{B_{r_0}} k\left(\tfrac{y}{|y|}\right) \Delta f_\gamma(x;y)^+ - \tfrac{2\Lambda}{\lambda} \Delta f_\gamma(x;y)^- \,\mu(dy) \\ &\quad + \tfrac{(2-\alpha)\lambda}{2} \left(\int_{B_{r_0}} k\left(\tfrac{y}{|y|}\right) \Delta f_\gamma(x;y)^+ \,\mu(dy) - \int_{\mathbb{R}^n \setminus B_{r_0}} \tfrac{2\Lambda}{\lambda} \Delta f_\gamma(x;y)^- \,\mu(dy) \right) \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

Lemma 3.37 implies $\mathcal{I}_1 \geq 0$ and Lemma 3.39 implies $\mathcal{I}_2 \geq 0$ provided $p \geq p_0$ is chosen sufficiently large in dependence of $n, |I|, \alpha_0, R, \lambda$ and Λ.

Next, consider the case $|x| > R$. Define $\widetilde{f}_\gamma : \mathbb{R}^n \to \mathbb{R}$

$$\widetilde{f}_\gamma(y) = \left(|x|/R\right)^p f_\gamma\left((|x|/R) y\right),$$

with p as above. Note that $\widetilde{f}_\gamma(y) \geq f_\gamma(y)$ for every $y \in \mathbb{R}^n$. Thus

$$\begin{aligned} \Delta f_\gamma\left(R\tfrac{x}{|x|}; y\right) &= f_\gamma\left(R\tfrac{x}{|x|} + y\right) + f_\gamma\left(R\tfrac{x}{|x|} - y\right) - 2R^{-p} \\ &\leq \widetilde{f}_\gamma\left(R\tfrac{x}{|x|} + y\right) + \widetilde{f}_\gamma\left(R\tfrac{x}{|x|} - y\right) - 2\underbrace{\widetilde{f}_\gamma\left(R\tfrac{x}{|x|}\right)}_{=R^{-p}} = \Delta \widetilde{f}_\gamma\left(R\tfrac{x}{|x|}; y\right) \end{aligned}$$

for every $y \in \mathbb{R}^n$. This leads to

$$M_\alpha^- f_\gamma(x) = cM_\alpha^- \widetilde{f}_\gamma(R\tfrac{x}{|x|}) \geq cM_\alpha^- f_\gamma(R\tfrac{x}{|x|}) \geq 0,$$

where $c = (|x|/R)^{-(p+\alpha)}$. This completes the proof. □

We are now able to construct the function mentioned at the beginning of this section. It is almost identical to the one in [CS09, Corollary 9.3].

Corollary 3.41. *Assume* $0 < R \leq 1$ *and* $\alpha_0 \in (0,2)$. *There exists a continuous function* $\Phi_R : \mathbb{R}^n \to \mathbb{R}$ *with the following properties:*

(i) $\Phi_R(x) = 0$ *for every* $x \in \mathbb{R}^n \setminus B_{2\sqrt{n}}$.

(ii) $\Phi_R(x) > 2$ *for every* $x \in Q_3$.

(iii) There exists a bounded, nonnegative function $\psi_R : \mathbb{R}^n \to \mathbb{R}$, *supported in* $\overline{B_R}$, *such that* $M_\alpha^- \Phi_R(x) \geq -\psi_R(x)$ *for every* $x \in \mathbb{R}^n$ *and every* $\alpha \in (\alpha_0, 2)$.

Remark 3.42. Recall that $Q_r(x)$ denotes an open cube of the form

$$Q_r(x) = \{y \in \mathbb{R}^n \,:\, |y - x|_\infty < \tfrac{r}{2}\}$$

and $Q_r = Q_r(0)$. If we set $Q = Q_r(x)$ then $sQ = Q_{sr}(x)$ for $s > 0$.

Proof. Let $p > 0$ be as in Proposition 3.40 and $\alpha \in (\alpha_0, 2)$. Set $\gamma = \frac{R}{2}$, $a = \frac{p}{2}\gamma^{-p-2}$ and $b = \gamma^{-p}(1 + \frac{p}{2}) - (2\sqrt{n})^{-p} > 0$. Consider the function

$$\Phi_R(x) = c \begin{cases} 0, & x \in \mathbb{R}^n \setminus B_{2\sqrt{n}} \\ |x|^{-p} - (2\sqrt{n})^{-p}, & x \in B_{2\sqrt{n}} \setminus B_\gamma \\ q(x), & x \in B_\gamma, \end{cases}$$

where $c > 0$ will be chosen below and $q(x) = -a|x|^2 + b$. Note that, due to the choice of a and b, the paraboloid given by q extends the function $x \mapsto |x|^{-p} - (2\sqrt{n})^{-p}$ across ∂B_γ such that $\Phi_R \in C^{1,1}[B_{2\sqrt{n}}]$. Besides,

$$q(x) \geq \gamma^{-p} - (2\sqrt{n})^{-p} = f_\gamma(x) - (2\sqrt{n})^{-p}$$

for every $|x| \leq \gamma$. In order to satisfy *(ii)*, we choose the constant $c > 0$ such that $\Phi_R(x) > 2$ for $x \in Q_3$ (recall that $Q_3 \subset B_{3\sqrt{n}/2} \subset B_{2\sqrt{n}}$). It remains to prove *(iii)*. Observe that

$$\Phi_R(x) \geq c(f_\gamma(x) - (2\sqrt{n})^{-p})$$

for every $x \in \mathbb{R}^n$. Hence,

$$\begin{aligned} \Delta\Phi_R(x;y) &= \Phi_R(x+y) + \Phi_R(x-y) - 2\Phi_R(x) \\ &\geq c(f_\gamma(x+y) + f_\gamma(x-y) - 2(2\sqrt{n})^{-p}) - 2c(f_\gamma(x) - (2\sqrt{n})^{-p}) \\ &= c(f_\gamma(x+y) + f_\gamma(x-y) - 2f_\gamma(x)) = c\Delta f_\gamma(x;y) \end{aligned}$$

for every $x \in B_{2\sqrt{n}} \setminus B_\gamma$ and every $y \in \mathbb{R}^n$. Using this fact and Proposition 3.40, we obtain

$$M_\alpha^- \Phi_R(x) \geq c M_\alpha^- f_\gamma(x) \geq 0 \text{ for } R \leq |x| < 2\sqrt{n}.$$

Moreover,

$$M_\alpha^- \Phi_R(x) \geq 0 \text{ for every } x \in \mathbb{R}^n \setminus B_{2\sqrt{n}}$$

since $\Delta\Phi_R(x;y) = \Phi_R(x+y) + \Phi_R(x-y) \geq 0$ for every $y \in \mathbb{R}^n$. This implies

$$M_\alpha^- \Phi_R(x) \geq 0 \text{ for every } |x| \geq R.$$

On the other hand, using Corollary 3.15, there exist constants $c_1 \geq 1$ and $A > 0$ such that

$$\sup_{|x| \leq R} \left| M_\alpha^- \Phi_R(x) \right| \leq c_1(A + \|\Phi_R\|_\infty) =: c_2.$$

Note that c_1 and A do not depend on α. Hence, the nonnegative function

$$\psi_R(x) = \begin{cases} c_2, & |x| < R \\ 0, & |x| \geq R \end{cases}$$

satisfies (iii). □

3.6 Point Estimates

The key tool that shall be useful in proving a decay of oscillation and then Hölder regularity is a lemma that connects a pointwise estimate with an estimate in measure. The nonlocal ABP estimate in combination with our bump function from the previous section are important tools in the proof of this lemma which can be considered as a nonlocal version of Lemma 2.27. Recall the setting introduced in Section 3.2.

Lemma 3.43. *Let $\alpha_0 \in (0,2)$. There exist constants $\varepsilon_0 > 0$, $\kappa \in (0,1)$ and $A > 1$ (depending only on λ, Λ, n, $|I|$ and α_0) such that for every $\alpha \in (\alpha_0, 2)$ and every bounded function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying*

(i) $u \geq 0$ *in* $\mathbb{R}^n$,

(ii) $\inf_{Q_3} u \leq 1$,

(iii) $M_\alpha^- u \leq \varepsilon_0$ *in* $Q_{4\sqrt{n}}$ *in the viscosity sense,*

the following estimate holds:

$$|\{u \leq A\} \cap Q_1| > \kappa. \tag{3.58}$$

In order to prove Lemma 3.43, we introduce the following results in real analysis.

Lemma 3.44 ([LN03, Lemma 3.14])**.** *For $m \in \mathbb{N}$, let $\mathcal{A} = \{A_i : 1 \leq i \leq m\}$ be a covering of the bounded set $\emptyset \neq U \subset \mathbb{R}^n$ by open sets $\emptyset \neq A_i \subset \mathbb{R}^n$. We define the maximum overlapping number of this covering by $\omega(\mathcal{A}) = \max_{1 \leq i \leq m} \#\{A \in \mathcal{A} : A_i \cap A \neq \emptyset\}$. Then this system of open sets can be decomposed into at most $\omega(\mathcal{A})$ groups such that in each group, the sets A_i are mutually disjoint.*

Proof. Choose any $A_{i_1} \in \mathcal{A}$ which is intersected by $\omega(\mathcal{A})$ sets from $\mathcal{A}$. Remove A_{i_1} from $\mathcal{A}$ and put it into group 1. The subfamily of the remaining sets forms an open covering of the set $U \setminus A_{i_1}$. If we find a set A_{i_2} in this subfamily which is again intersected by $\omega(\mathcal{A})$ sets, we conclude that $A_{i_1} \cap A_{i_2} = \emptyset$. Remove A_{i_2} from the subfamily and put it into group 1. Repeating this procedure until there are no longer sets A_i being intersected by $\omega(\mathcal{A})$ sets, the resulting group 1 consists of mutually disjoint sets. We continue in the same way as above in the case where the sets A_i are intersected by $\omega(\mathcal{A}) - 1, \omega(\mathcal{A}) - 2, \dots, 1$ sets. This completes the proof. □

Lemma 3.45. *Let $t > 3$, $m \in \mathbb{N}$ and $U = \bigcup_{i=1}^m \overline{Q^i}$, where $Q^i = Q_{r_i}(x_i)$ with $r_i > 0$, $x_i \in \mathbb{R}^n$ are pairwise disjoint cubes. Consider the family $\mathcal{Q} = \{tQ^i : 1 \leq i \leq m\}$ which provides an open covering of the set U.*
There exists a subfamily $\mathcal{Q}' \subset \mathcal{Q}$ which still covers U and $\omega(\mathcal{Q}') \leq N$ for some number $N \in \mathbb{N}$ depending only on dimension, where $\omega(\mathcal{Q}')$ is defined as in Lemma 3.44.

Proof. $\mathcal{Q}'$ is the result of the following greedy algorithm:

I) Set $\mathcal{Q}' = \emptyset$.

II) If there exists $tQ \in \mathcal{Q} \setminus \mathcal{Q}'$ such that

$$\overline{Q} \cap \bigcap_{tQ' \in \mathcal{Q}'} (tQ')^c \neq \emptyset, \tag{3.59}$$

we add the largest cube $tQ \in \mathcal{Q} \setminus \mathcal{Q}'$ satisfying (3.59) to $\mathcal{Q}'$ (this cube may not be unique) and start again at II). Otherwise we stop.

This algorithm leads (after renumbering) to a subcovering $\mathcal{Q}' = \{tQ^i : 1 \leq i \leq m'\} \subset \mathcal{Q}$, $m' \leq m$, which still covers U. We may assume that there exists a point $x \in \mathbb{R}^n$ which is covered by $\omega(\mathcal{Q}')$ different cubes of the subcovering. Consider x to be the origin of a coordinate system (chosen so that its coordinate axes are parallel to the corresponding edges of the cubes). Note that this coordinate system consists of 2^n orthants. Let us fix one orthant $\mathfrak{A}$ [2]. We claim that if there exists a cube $tQ \ni x$ from our subcovering such that $Q \subset \mathfrak{A}$, then there will be no other cube tQ' from our subcovering with the property that both $x \in tQ'$ and $Q' \subset \mathfrak{A}$.
We prove this by contradiction: We assume that there are two cubes tQ and tQ' of the same size such that $x \in tQ \cap tQ'$ and $Q \dot{\cup} Q' \subset \mathfrak{A}$ (note that such cubes tQ, tQ' may exist since $t > 3$). Hence, $\overline{Q} \subset tQ'$ and $\overline{Q'} \subset tQ$ (see Figure 3.4).

[2] Formally, $\mathfrak{A}$ can be defined as follows: $\mathfrak{A} = \{y \in \mathbb{R}^n : e_i x_i < e_i y_i \text{ for every } i = 1, \dots, n\}$, where $e_i \in \{-1, 1\}$.

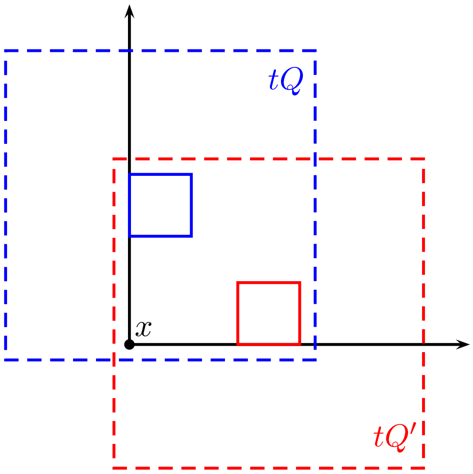

Figure 3.4: The cubes tQ and tQ'.

So one of these cubes would not have been chosen by the algorithm from above which proves our claim because proving the claim for cubes of the same size implies the general case.

We use the same argumentation as above to prove that if there exists $tQ \in \mathcal{Q}'$ with the property that $x \in tQ$ and Q has its center in $\mathfrak{A}$ but is contained in exactly 2^j different orthants, $1 \leq j \leq n$, then there will be no other cube tQ' from our subcovering containing x such that Q' has its center in $\mathfrak{A}$ and is contained in exactly the same orthants as Q. Hence, we can find at most $\binom{n}{j}$ cubes[3] from our subcovering containing x such that their "smaller versions" have their centers in $\mathfrak{A}$ and are contained in exactly 2^j orthants. This implies

$$\omega(\mathcal{Q}') \leq 2^n \sum_{j=0}^{n} \binom{n}{j} = 4^n =: N. \qquad \square$$

Proof of Lemma 3.43. The proof uses the same strategy as the one of [CS09, Lemma 10.1]. Let $l \in (0, \frac{1}{2})$ be as in Corollary 3.26. Set $R = \frac{l}{8}$ and $v = \Phi_R - u$, where Φ_R is the special function constructed in Corollary 3.41. Let us summarise properties of v:

- v is upper semicontinuous in $\overline{Q_{4\sqrt{n}}} \supset \overline{B_{2\sqrt{n}}}$.
- $v \leq 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$.
- For every $\alpha \in (\alpha_0, 2)$, $M^+_\alpha v \geq M^-_\alpha \Phi_R - M^-_\alpha u \geq -\psi_R - \varepsilon_0$ in $Q_{4\sqrt{n}} \supset B_{2\sqrt{n}}$ in the viscosity sense, where $\psi_R : \mathbb{R}^n \to \mathbb{R}$ is as in Corollary 3.41.

[3] Note that this number of cubes can be obtained by counting the number of possibilities to choose j among n axes and move a given cube which is completely contained in $\mathfrak{A}$ along these axes of the coordinate system such that the shifted cube still has its center in $\mathfrak{A}$ but is contained in 2^j orthants.

Let Γ be the concave envelope of v^+ in $B_{6\sqrt{n}}$ and let $\alpha \in (\alpha_0, 2)$. We can apply the rescaled version of Theorem 3.29 with $\rho_0 = 2\sqrt{n}\frac{l}{16n} = \frac{l}{8\sqrt{n}}$, stated in Remark 3.30(ii), to v: Let $(Q^j)_{j=1,\dots,m}$ be the family of cubes covering $\Sigma = \{v = \Gamma\} \cap B_{2\sqrt{n}}$ in the rescaled version of Theorem 3.29. Imitating the proof of Theorem 2.17 (leading to (2.20)) for v^+, we can find a constant $c_1 = c_1(n) \geq 1$ such that

$$\sup_{B_{2\sqrt{n}}} v^+ \leq c_1 \left|\nabla\Gamma(B_{2\sqrt{n}})\right|^{1/n}.$$

Using this result, Theorem 3.29 (rescaled) and Remark 3.32, we obtain

$$\begin{aligned}
\sup_{B_{2\sqrt{n}}} v \leq c_1 \left|\nabla\Gamma(B_{2\sqrt{n}})\right|^{1/n} &\leq c_1 \left(\sum_{j=1}^{m} \left|\nabla\Gamma(\overline{Q^j})\right|\right)^{1/n} \\
&\leq c_2 \left(\sum_{j=1}^{m} \big(\sup_{\overline{Q^j}}(\psi_R + \varepsilon_0)\big)^n \left|Q^j\right|\right)^{1/n} \\
&\leq c_3\varepsilon_0 + c_3 \left(\sum_{j=1}^{m} \big(\sup_{\overline{Q^j}} \psi_R\big)^n \left|Q^j\right|\right)^{1/n},
\end{aligned}$$

where $c_3 = c_3(\lambda, |I|, n) \geq c_2 \geq c_1$.
The properties $\inf_{Q_3} u \leq 1$ and $\Phi_R > 2$ in Q_3 imply $\sup_{B_{2\sqrt{n}}} v \geq 1$. Set $\varepsilon_0 = \frac{1}{2c_3}$. Since ψ_R is supported in $\overline{B_R}$, we obtain

$$\frac{1}{2} \leq c_3 \big(\sup_{\overline{B_R}} \psi_R\big) \Bigg(\sum_{\substack{j=1,\dots,m \\ Q^j \cap B_R \neq \emptyset}} \left|Q^j\right| \Bigg)^{1/n}.$$

Hence, there is $c_4 \in (0,1)$ (which only depends on $|I|, \lambda, \Lambda, \alpha_0$ and n) such that

$$\sum_{\substack{j=1,\dots,m \\ Q^j \cap B_R \neq \emptyset}} \left|Q^j\right| \geq c_4. \tag{3.60}$$

Set $c_5 = C_2(\|\psi_R\|_\infty + \varepsilon_0)$ with $C_2 \geq 1$ from Theorem 3.29 (rescaled) which we now apply for the second time: There is $\nu \in (0,1)$ such that for every $j = 1, \dots, m$

$$\begin{aligned}
&\left|\{y \in \eta Q^j : v(y) \geq \Gamma(y) - c_5 d_j^2\}\right| \\
&\geq \left|\{y \in \eta Q^j : v(y) \geq \Gamma(y) - C_2 \big(\sup_{\overline{Q^j}}(\psi_R + \varepsilon_0)\big) d_j^2\}\right| \geq \nu \left|\eta Q^j\right|
\end{aligned} \tag{3.61}$$

and $d_j^2 \leq \rho_0^2$. Recall that d_j denotes the diameter of the cube Q^j.
Let us consider the family

$$\mathcal{Q} = \{\eta Q^j : Q^j \cap B_R \neq \emptyset,\ 1 \leq j \leq m\},$$

which provides an open covering of the union

$$U = \bigcup_{\substack{j=1,\dots,m \\ Q^j \cap B_R \neq \emptyset}} \overline{Q^j}.$$

By Lemma 3.45, we may take a subfamily $\mathcal{Q}' \subset \mathcal{Q}$ which still covers the set U and whose maximum overlapping number $\omega(\mathcal{Q}')$ can be estimated by some number $N \geq \omega(\mathcal{Q}')$ which only depends on the dimension n. Note that the diameters d_j of all cubes Q^j are bounded by $\rho_0 2^{-1/(2-\alpha)}$, which is always smaller than $\frac{l}{8\sqrt{2n}}$. Therefore, $Q^j \cap B_R \neq \emptyset$ implies $\eta Q^j \subset B_{1/2}$ due to the choice of $\eta = (1 + \frac{8}{l})\sqrt{n}$.
Fix any $\kappa \in (0, \frac{\nu c_4}{N})$. Then it follows from Lemma 3.44, (3.60) and (3.61) that

$$\left|\{y \in B_{1/2} : v(y) \geq \Gamma(y) - c_5\rho_0^2\}\right| > \kappa. \tag{3.62}$$

To prove (3.62), we apply Lemma 3.44: Let G_i, $i = 1, \dots, \omega(\mathcal{Q}')$, be the groups consisting of all mutually disjoint cubes $\eta Q^j \in \mathcal{Q}'$ which have i intersections with elements from $\mathcal{Q}'$. Eventually, some of the groups are empty. Then

$$\begin{aligned}
&\left|\{y \in B_{1/2} : v(y) \geq \Gamma(y) - c_5\rho_0^2\}\right| \\
&\geq \left|\bigcup_{i=1}^{\omega(\mathcal{Q}')} \bigcup_{\eta Q^j \in G_i} \{y \in \eta Q^j : v(y) \geq \Gamma(y) - c_5 d_j^2\}\right| \\
&\geq \frac{1}{\omega(\mathcal{Q}')} \sum_{i=1}^{\omega(\mathcal{Q}')} \left|\bigcup_{\eta Q^j \in G_i} \{y \in \eta Q^j : v(y) \geq \Gamma(y) - c_5 d_j^2\}\right| \\
&= \frac{1}{\omega(\mathcal{Q}')} \sum_{i=1}^{\omega(\mathcal{Q}')} \sum_{\eta Q^j \in G_i} \left|\{y \in \eta Q^j : v(y) \geq \Gamma(y) - c_5 d_j^2\}\right| \\
&\overset{(3.61)}{\geq} \frac{\nu}{\omega(\mathcal{Q}')} \left|\bigcup_{\eta Q \in \mathcal{Q}'} \eta Q\right| \geq \frac{\nu}{\omega(\mathcal{Q}')} |U| = \frac{\nu}{\omega(\mathcal{Q}')} \sum_{\substack{j=1,\dots,m \\ Q^j \cap B_R \neq \emptyset}} \left|Q^j\right| \\
&\overset{(3.60)}{\geq} \frac{\nu c_4}{N} > \kappa.
\end{aligned}$$

Let $A_0 = \sup_{B_{1/2}} \Phi_R$. Since

$$\{y \in B_{1/2} : v(y) \geq \Gamma(y) - c_5\rho_0^2\} \subset \{y \in B_{1/2} : u(y) \leq A_0 + c_5\rho_0^2\},$$

we obtain from (3.62)

$$\left|\{y \in B_{1/2} : u(y) \leq A_0 + c_5\rho_0^2\}\right| > \kappa.$$

Let $A = A_0 + c_5\rho_0^2$. Since $B_{1/2} \subset Q_1$, we finally conclude

$$|\{y \in Q_1 : u(y) \leq A\}| > \kappa. \qquad \square$$

Lemma 3.43 is the key to the proof of the Hölder regularity result in the next section. Using Lemma 3.43, we obtain the same lemma as in the local case (cf. Lemma 2.28):

Lemma 3.46. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be as in Lemma 3.43. Then the following estimate holds for every $m \in \mathbb{N}_0$:*

$$|\{u > A^m\} \cap Q_1| \leq (1-\kappa)^m, \tag{3.63}$$

where A and κ are as in Lemma 3.43. As a consequence, for every $t > 0$

$$|\{u \geq t\} \cap Q_1| \leq dt^{-\varepsilon}, \tag{3.64}$$

where $d > 1$ and $\varepsilon > 0$ depend only on $\lambda, \Lambda, n, |I|$ and α_0.

Proof. We only have to show that the function $\widetilde{u}$, defined as in the proof of Lemma 2.28, is under the hypothesis of Lemma 3.43. In this case, (3.63) and (3.64) can be proven in the same way as in Lemma 2.28. Recall that

$$\widetilde{u}(y) = \frac{u(\tau_i(y))}{A^{m-1}}, \quad y \in \mathbb{R}^n$$

with

$$\tau_i(y) = x_0 + \frac{1}{2^i}y,$$

where $i \in \mathbb{N}$ and $x_0 \in Q_1$ are as in Lemma 2.28.

- Since $u \geq 0$ in $\mathbb{R}^n$, u bounded, we have the same properties for $\widetilde{u}$.
- $\inf_{Q_3} \widetilde{u} \leq 1$ as shown in the proof of Lemma 2.28.
- Let $\varepsilon_0 > 0$ be the number such that (3.58) in Lemma 3.43 holds. So we have $M_\alpha^- u \leq \varepsilon_0$ in $Q_{4\sqrt{n}}$ in the viscosity sense. Let $y \in Q_{4\sqrt{n}}$ and note that $\tau_i(y) \in Q_{4\sqrt{n}/2^i}(x_0) \subset Q_{4\sqrt{n}}$. Without loss of generality we assume $M_\alpha^- \widetilde{u}(y) \geq 0$ classically. Since $A > 1$, we obtain

$$M_\alpha^- \widetilde{u}(y) = \frac{M_\alpha^- u(\tau_i(y))}{2^{i\alpha} A^{m-1}} \leq M_\alpha^- u(\tau_i(y)) \leq \varepsilon_0.$$

We conclude that $\widetilde{u}$ is under the hypothesis of Lemma 3.43 and finish the proof. □

By a standard covering argument we obtain the following theorem:

Theorem 3.47. *Let $\alpha_0 \in (0, 2)$ and consider any $\alpha \in (\alpha_0, 2)$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded nonnegative function such that $u(0) \leq 1$ and $M_\alpha^- u \leq \varepsilon_0$ in B_2 in the viscosity sense, where $\varepsilon_0 > 0$ is the number in Lemma 3.43. Then*

$$|\{u \geq t\} \cap B_1| \leq C_4 t^{-\varepsilon} \quad \textit{for every } t > 0,$$

where the constants $C_4 \geq 1$ and $\varepsilon > 0$ depend on $\lambda, \Lambda, n, |I|$ and α_0.

Proof. Fix $r = \frac{1}{n}$ and note that $Q_{4\sqrt{n}r} \subset B_2$. Define $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}$,

$$\widetilde{u}(x) = u(rx).$$

Clearly, $\widetilde{u} \geq 0$ in $\mathbb{R}^n$, $\widetilde{u}$ bounded and $\inf\limits_{Q_3} \widetilde{u} \leq u(0) \leq 1$.
For $x \in Q_{4\sqrt{n}}$, we calculate

$$\begin{aligned} M_\alpha^- \widetilde{u}(x) &= \inf_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta\widetilde{u}(x;y)\, K(y)\, dy \\ &= \inf_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} (u(rx+ry) + u(rx-ry) - 2u(rx))\, K(y)\, dy \\ &= r^{n+\alpha}(2-\alpha) \int_{\mathbb{R}^n} (\lambda k\big(\tfrac{y}{|y|}\big) \Delta u(rx;y)^+ - \Lambda \Delta u(rx;y)^-) r^{-n}\, \mu(dy) \\ &= r^\alpha\, M_\alpha^- u(rx), \end{aligned}$$

where we have assumed (for simplicity) $\widetilde{u} \in C^{1,1}(x)$. Recall that $\mu(dy) = |y|^{-n-\alpha}\, dy$. Hence, $M_\alpha^- \widetilde{u} \leq \varepsilon_0$ in $Q_{4\sqrt{n}}$ in the viscosity sense. Using Lemma 3.46, we obtain

$$|\{\widetilde{u} \geq t\} \cap Q_1| \leq dt^{-\varepsilon} \text{ for every } t > 0,$$

where $d > 1$ and $\varepsilon > 0$ are the constants in Lemma 3.46. Rescaling leads to

$$|\{u \geq t\} \cap Q_r| \leq dt^{-\varepsilon} \text{ for every } t > 0 \tag{3.65}$$

(since $r \leq 1$). We now want to control the distribution in a larger domain, say

$$Q_r \cup Q_{r/4}(\tfrac{r}{2}e_1) \subset B_1,$$

where $e_1 \in \mathbb{R}^n$ denotes the first unit vector. It is important that the cubes Q_r and $Q_{r/4}(\frac{r}{2}e_1)$ have positive intersecting mass. Choose $A = t_0 \geq 1$ large enough such that

$$\frac{\big|Q_r \cap Q_{r/4}(\frac{r}{2}e_1)\big|}{2} > dt_0^{-\varepsilon}.$$

Using (3.65), we have

$$\big|\{u \geq t_0\} \cap (Q_r \cap Q_{r/4}(\tfrac{r}{2}e_1))\big| < \frac{\big|Q_r \cap Q_{r/4}(\frac{r}{2}e_1)\big|}{2}.$$

Therefore, we can find a point $x_1 \in Q_{r/4}(\frac{r}{2}e_1)$ such that $u(x_1) < A$. Now define

$$v(x) = \frac{u(\frac{r}{2}x + x_1)}{A},\ x \in \mathbb{R}^n.$$

We still have $v \geq 0$ in $\mathbb{R}^n$, $\inf_{Q_3} v \leq v(0) < 1$ and $M_\alpha^- v \leq \varepsilon_0$ in $Q_{4\sqrt{n}}$ in the viscosity sense. Note that $\frac{r}{2}x + x_1 \in Q_{4\sqrt{n}r/2}(x_1) \subset B_2$ for each $x \in Q_{4\sqrt{n}}$. We apply Lemma 3.46 to v and obtain (after rescaling)

$$\left|\{u \geq t\} \cap Q_{r/2}(x_1)\right| \leq A^\varepsilon d t^{-\varepsilon} \text{ for every } t > 0.$$

Since $Q_{r/2}(x_1) \supset Q_{r/4}(\frac{r}{2}e_1)$, this implies

$$\left|\{u \geq t\} \cap Q_{r/4}(\tfrac{r}{2}e_1)\right| \leq c_1 t^{-\varepsilon} \text{ for every } t > 0,$$

where $c_1 = A^\varepsilon d$. Hence,

$$\left|\{u \geq t\} \cap (Q_r \cup Q_{r/4}(\tfrac{r}{2}e_1))\right| \leq 2c_1 t^{-\varepsilon} \text{ for every } t > 0.$$

Note that $c_1 \geq d$ again only depends on $\lambda, \Lambda, n, |I|$ and α_0. So we also control the distribution of u in the larger domain $Q_r \cup Q_{r/4}(\frac{r}{2}e_1)$.
We continue the argumentation from above and obtain after a finite number of steps (which only depends on n) the ball B_1 where we originally wanted to control the distribution. □

Scaling the above theorem (which is possible because of the symmetry of our kernels), we obtain the following version:

Theorem 3.48. *Let $\alpha_0 \in (0, 2)$ and consider any $\alpha \in (\alpha_0, 2)$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded nonnegative function satisfying $M_\alpha^- u \leq C'$ in $B_{2r}(x)$ in the viscosity sense for some $r > 0$, $C' > 0$ and $x \in \mathbb{R}^n$. Then*

$$|\{u \geq t\} \cap B_r(x)| \leq C_5 r^n (u(x) + C' r^\alpha)^\varepsilon t^{-\varepsilon} \quad \textit{for every } t > 0,$$

where the constants $C_5 \geq 1$ and $\varepsilon > 0$ depend on $\lambda, \Lambda, n, |I|$ and α_0.

Proof. The proof uses a similar strategy as the proof before. Let $\varepsilon_0 > 0$ be as in Theorem 3.47 and assume without loss of generality that $\varepsilon_0 < 1$ (note from the proof of Lemma 3.43 that ε_0 may be as small as we wish). Define

$$v(z) = \frac{u(rz + x)}{u(x) + \frac{C' r^\alpha}{\varepsilon_0}}.$$

We show that v satisfies all the conditions in Theorem 3.47:

- $v \geq 0$ in $\mathbb{R}^n$, v bounded.
- $v(0) = \dfrac{u(x)}{u(x) + \frac{C' r^\alpha}{\varepsilon_0}} \leq 1.$

- Let $z \in B_2$. For simplicity, we assume $v \in C^{1,1}(z)$. Then we obtain, using the fact that $rz + x \in B_{2r}(x)$,

$$\begin{aligned}
M_\alpha^- v(z) &= \inf_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta v(z;y)\, K(y)\, dy \\
&= \frac{r^{n+\alpha}}{u(x) + \frac{C' r^\alpha}{\varepsilon_0}} (2-\alpha) \int_{\mathbb{R}^n} (\lambda k\big(\tfrac{y}{|y|}\big) \Delta u(rz+x;y)^+ - \Lambda \Delta u(rz+x;y)^-) r^{-n}\, \mu(dy) \\
&= \frac{r^\alpha}{u(x) + \frac{C' r^\alpha}{\varepsilon_0}} M_\alpha^- u(rz+x) \\
&= \varepsilon_0 \underbrace{\frac{r^\alpha}{\varepsilon_0 u(x) + C' r^\alpha} M_\alpha^- u(rz+x)}_{\le \frac{C'\, r^\alpha}{\varepsilon_0 u(x) + C' r^\alpha} \le 1} \le \varepsilon_0 .
\end{aligned}$$

  Thus, $M_\alpha^- v \le \varepsilon_0$ in B_2 in the viscosity sense.

Using Theorem 3.47, we obtain constants $C_4 \ge 1$ and $\varepsilon > 0$ such that

$$|\{v \ge t\} \cap B_1| \le C_4 t^{-\varepsilon} \text{ for each } t > 0.$$

Rescaling leads to

$$\Big|\{z \in B_r(x) :\, u(z) \ge t(u(x) + \tfrac{C' r^\alpha}{\varepsilon_0})\}\Big| \le C_4 r^n t^{-\varepsilon} \text{ for every } t > 0. \tag{3.66}$$

We finish the proof by using (3.66) and the fact that for every $t > 0$ there is $\widetilde{t} > 0$ such that $t = \widetilde{t}\,(u(x) + \frac{C' r^\alpha}{\varepsilon_0})$:

$$\begin{aligned}
|\{u \ge t\} \cap B_r(x)| &= \Big|\{u \ge \widetilde{t}\,(u(x) + \tfrac{C' r^\alpha}{\varepsilon_0})\} \cap B_r(x)\Big| \\
&\le C_4 r^n\, \widetilde{t}^{\,-\varepsilon} = C_4 r^n \left(u(x) + \frac{C' r^\alpha}{\varepsilon_0}\right)^\varepsilon t^{-\varepsilon} \\
&\le C_5 r^n (u(x) + C' r^\alpha)^\varepsilon t^{-\varepsilon},
\end{aligned}$$

where we have used that $\varepsilon_0 < 1$ and $C_5 = \frac{C_4}{\varepsilon_0^\varepsilon}$. □

Remark 3.49. Using the corresponding versions of Theorem 3.47 and Theorem 3.48 in [CS09], the authors prove a strong Harnack inequality (cf. [CS09, Theorem 11.1]). Under assumption (3.12), the strong Harnack inequality does not hold in general. Let us provide the example from [BS05, p. 148]: For $m \in \mathbb{N}$, define sets I_m of the form

$$I_m = B_{4^{-m}}(\xi_m) \cap \mathbb{S}^{n-1},$$

where $\xi_m \in \mathbb{S}^{n-1}$ are chosen such that the balls $B_{2^{-m}}(\xi_m)$ are pairwise disjoint.

Set $\mathcal{J} = \bigcup_{m \in \mathbb{N}} \big[I_m \cup (-I_m)\big]$, where $-I_m = B_{4^{-m}}(-\xi_m) \cap \mathbb{S}^{n-1}$. Finally, define the symmetric kernel $K : \mathbb{R}^n \to [0,\infty)$ by $K(0) = 0$ and $K(y) = \mathbb{1}_{\mathcal{J}}\big(\frac{y}{|y|}\big)\, |y|^{-n-\alpha}$ for $|y| \ne 0$

and some $\alpha \in (0,2)$. Note that K satisfies (3.12) with $\lambda = \Lambda = \frac{1}{2-\alpha}$ and $k = \mathbb{1}_{\mathcal{J}}$. Then it is shown in [BS05] that nonnegative solutions u to $Lu = 0$ (with L as in (3.1)) do not satisfy a Harnack inequality. This is due to the fact that the kernel K does not satisfy the Relative Kato condition in [BS05]. Hence, a strong formulation of the Harnack inequality does not hold in general under (3.12).

We specify the exact part of the proof of [CS09, Theorem 11.1] which breaks down under assumption (3.12): Note that the final expression in (11.2) on page 629 in [CS09] is bounded by providing a suitable lower bound of the form $g_\tau(x) = \tau(1-|4x|^2)$, $0 < \tau \leq 1$, for the solution u. As a consequence of the (maximal) choice of τ, there exists a point $x_1 \in B_{1/4}$ such that u touches g_τ from above at x_1. Then the property $M_\alpha^- u(x_1) \leq 1$ is sufficient in the setting of [CS09], i.e., under (1.7) in Chapter 1, to complete the proof of the strong Harnack inequality. This property is not sufficient in our setting because of the presence of k in (3.12).

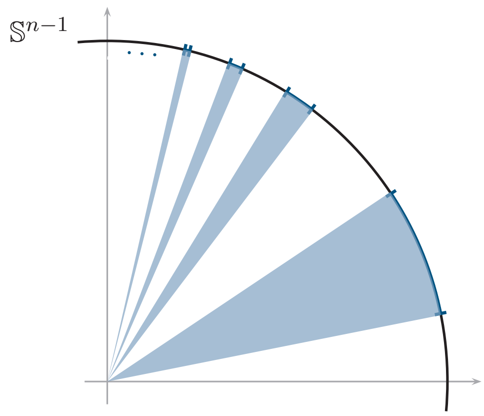

Figure 3.5: Schematic sketch of the sets $I_m \subset \mathbb{S}^{n-1}$ in Remark 3.49.

3.7 Hölder Regularity

Using the previous sections (which lead to Theorem 3.48) we are now able to prove a Hölder regularity result. The first result of this section (Lemma 3.50) deals with the decay of oscillation of functions which are sub- and supersolutions to some extremal equations. The importance of such a result lies in the fact that if the oscillation of a function decays geometrically in geometrically decaying balls, it implies a Hölder modulus of continuity at the center of such balls. By applying it at every point of a ball strictly contained in the domain, we obtain Hölder regularity (see Theorem 3.51). The point estimate (Theorem 3.48) will be crucial for the proof of Lemma 3.50. Recall the assumptions made in Section 3.2.

Lemma 3.50. *Let $\alpha_0 \in (0,2)$ and consider any $\alpha \in (\alpha_0, 2)$. Assume that $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $|u| \le \frac{1}{2}$ in $\mathbb{R}^n$ and*

$$M^+_\alpha u \ge -\varepsilon_0 \text{ in } B_1 \quad \text{and} \quad M^-_\alpha u \le \varepsilon_0 \text{ in } B_1$$

in the viscosity sense for some sufficiently small constant $\varepsilon_0 > 0$.
There are constants $\beta \in (0,1)$ and $C_6 \ge 1$ depending only on $n, \lambda, \Lambda, |I|$ and α_0 such that $u \in C^\beta$ at the origin, i.e., $|u(x) - u(0)| \le C_6 |x|^\beta$ for every $x \in \mathbb{R}^n$.

Proof. The proof uses the same strategy as the one of [CS09, Lemma 12.2]. By induction we construct sequences $(a_l)_{l\in\mathbb{Z}}$ (nondecreasing), $(A_l)_{l\in\mathbb{Z}}$ (nonincreasing) such that

(i) $a_l \le u \le A_l$ in $B_{8^{-l}}$ and
(ii) $A_l - a_l = 8^{-\beta l}$ with a number $\beta \in (0,1)$ chosen below.

Using (i) and (ii), we obtain the theorem with $C_6 = 8^\beta$ because for every $x \in \mathbb{R}^n$ we can find a number $l \in \mathbb{Z}$ such that $8^{-l-1} \le |x| \le 8^{-l}$ which implies

$$|u(x) - u(0)| \le A_l - a_l = 8^{-\beta l} = 8^\beta (8^{-l-1})^\beta \le C_6 |x|^\beta .$$

For $l \le 0$ choose $a_l = -\frac{1}{2}$ and $A_l = a_l + 8^{-\beta l}$, where $\beta \in (0,1)$ is arbitrary for the moment. Since $|u| \le \frac{1}{2}$ in $\mathbb{R}^n$, (i) and (ii) are satisfied because $A_l \ge \frac{1}{2}$.
Assume we have constructed a_l and A_l up to $l \ge 0$. We show that we can continue the sequences by finding a_{l+1} and A_{l+1}.
In the ball $B_{8^{-l-1}}$, either $u \ge \frac{A_l + a_l}{2}$ in at least half of the points (in measure), or $u \le \frac{A_l + a_l}{2}$ in at least half of the points. For now we assume

$$\left| \left\{ u \ge \frac{A_l + a_l}{2} \right\} \cap B_{8^{-l-1}} \right| \ge \frac{|B_{8^{-l-1}}|}{2}. \tag{3.67}$$

Consider the bounded function

$$v(x) = \frac{u(8^{-l}x) - a_l}{(A_l - a_l)/2}.$$

Let us discuss some properties of v:

I) $v \ge 0$ in B_1 by inductive hypothesis.
II) For $x \in B_1^c$ we still have the following lower bound:

$$v(x) \ge -2(|8x|^\beta - 1) \quad \text{(where } \beta \in (0,1) \text{ is still unspecified)}.$$

Note that this inequality is trivially satisfied when $l = 0$. To prove II) for $l \ge 1$, we use the inductive hypothesis and observe that for every $j \in \mathbb{N}$, the following estimate holds for every $x \in B_{8^j}$:

$$\begin{aligned} v(x) &\ge \frac{a_{l-j} - a_l}{(A_l - a_l)/2} \ge \frac{a_{l-j} - A_{l-j} + A_l - a_l}{(A_l - a_l)/2} \\ &= -2 \cdot 8^{-\beta(l-j)} \cdot 8^{\beta l} + 2 = 2(1 - 8^{\beta j}). \end{aligned}$$

For each $x \in B_1^c$ there is a number $j \in \mathbb{N}$ such that $8^{j-1} \le |x| \le 8^j$ ($\Rightarrow 8^j \le |8x| \le 8^{j+1}$) and so we conclude from the estimate above:

$$v(x) \ge -2(8^{\beta j} - 1) \ge -2(|8x|^\beta - 1).$$

III) Using (3.67), we obtain the following estimate:

$$\left|\{v \ge 1\} \cap B_{1/8}\right| = 8^{ln} \left| \left\{ y \in B_{8^{-l-1}} \,:\, u(y) \ge \frac{A_l + a_l}{2} \right\} \right| \ge \frac{|B_{1/8}|}{2}.$$

IV) Since $M_\alpha^- u \le \varepsilon_0$ in B_1 in the viscosity sense (where $\varepsilon_0 > 0$ will be chosen below), it is easy to see that

$$M_\alpha^- v \le 2\varepsilon_0 \quad \text{in } B_{8^l} \text{ in the viscosity sense,} \quad \text{if } \beta \text{ is chosen less than } \alpha_0.$$

To prove IV), let $x \in B_{8^l}$. For simplicity, we assume $v \in C^{1,1}(x)$ and obtain

$$\begin{aligned} M_\alpha^- v(x) &= \inf_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta v(x;y) K(y)\, dy \\ &= \frac{1}{(A_l - a_l)/2} \inf_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} \Delta u(8^{-l}x; 8^{-l}y) K(y)\, dy \\ &= \frac{1}{(A_l - a_l)/2} (2-\alpha) \int_{\mathbb{R}^n} \frac{\lambda k\left(\frac{y}{|y|}\right) \Delta u(8^{-l}x;y)^+ - \Lambda \Delta u(8^{-l}x;y)^-}{|8^l y|^{n+\alpha}} 8^{ln}\, dy \\ &= \frac{8^{-l\alpha}}{(A_l - a_l)/2} M_\alpha^- u(\underbrace{8^{-l}x}_{\in B_1}) \\ &\le \frac{8^{-l\alpha}\varepsilon_0}{(A_l - a_l)/2} = 2\varepsilon_0 8^{-l(\alpha-\beta)} \le 2\varepsilon_0. \end{aligned}$$

So we see that $M_\alpha^- v \le 2\varepsilon_0$ in B_{8^l} in the viscosity sense, provided β is chosen less than α_0 (***first global requirement on*** β).

Now define $w = \max(v, 0) = v^+$. If we choose β small enough (***second global requirement on*** β), we obtain the following estimate:

$$M_\alpha^- w \le M_\alpha^- v + 2\varepsilon_0 \quad \text{in } B_{3/4} \text{ in the viscosity sense.} \tag{3.68}$$

We prove (3.68): First note that $w = v^+ = v + v^-$ and thus $M_\alpha^- w \le M_\alpha^- v + M_\alpha^+ v^-$. Now consider any $x \in B_{3/4}$. We estimate $M_\alpha^+ v^-(x)$ and assume for simplicity $v^- \in C^{1,1}(x)$.

Note that $v^-(x) = 0$ because of property I). Hence,

$$\begin{aligned}
M_\alpha^+ v^-(x) &= \sup_{K \in \mathcal{K}_0} \int_{\mathbb{R}^n} (v^-(x+y) + v^-(x-y) - 2v^-(x))\, K(y)\, dy \\
&= 2(2-\alpha)\Lambda \Bigg(\int_{\{y\in\mathbb{R}^n : x+y \in B_1\}} \underbrace{\tfrac{v^-(x+y)}{|y|^{n+\alpha}}}_{=0} dy + \int_{\{y\in\mathbb{R}^n : x+y \notin B_1\}} \tfrac{v^-(x+y)}{|y|^{n+\alpha}} dy \Bigg) \\
&\le 2(2-\alpha)\Lambda \int_{\{y\in\mathbb{R}^n : x+y \notin B_1\}} \tfrac{2(|8(x+y)|^\beta - 1)}{|y|^{n+\alpha}} dy \quad \text{(property II))} \\
&\le 4(2-\alpha)\Lambda \int_{\{y\in\mathbb{R}^n : x+y \notin B_1\}} \tfrac{|8(x+y)|^\beta - 1}{(|x+y|-3/4)^{n+\alpha}} dy \\
&= 4(2-\alpha)\Lambda n\omega_n \int_1^\infty \tfrac{8^\beta r^\beta - 1}{(r-3/4)^{n+\alpha}} r^{n-1}\, dr \\
&\le (2-\alpha)\tfrac{8^\beta 4^{n+1+\alpha}\Lambda n\omega_n}{\alpha}\left[\tfrac{1}{1-\frac{\beta}{\alpha}} - 8^{-\beta}\right] \le 2\tfrac{8^\beta 4^{n+3}\Lambda n\omega_n}{\alpha_0}\left[\tfrac{1}{1-\frac{\beta}{\alpha_0}} - 8^{-\beta}\right] \xrightarrow{\beta\searrow 0} 0.
\end{aligned}$$

So we can choose β small enough (and independent of $\alpha \in (\alpha_0, 2)$) such that $M_\alpha^+ v^- \le 2\varepsilon_0$ in $B_{3/4}$ in the viscosity sense. Hence, (3.68) holds.

Using property IV), (3.68) implies

$$M_\alpha^- w \le 4\varepsilon_0 \quad \text{in } B_{3/4} \text{ in the viscosity sense.} \tag{3.69}$$

Note that we still have property III) when replacing v by w.

Let $x \in B_{1/8}$. Then $B_{1/2}(x) \subset B_{3/4}$, so we have $M_\alpha^- w \le 4\varepsilon_0$ in $B_{1/2}(x)$ in the viscosity sense. Using Theorem 3.48 (in $B_{1/2}(x)$), there are constants $C_5 \ge 1$ and $\varepsilon > 0$ (which only depend on $|I|, \lambda, \Lambda, n$ and α_0) such that

$$C_5(w(x) + 4\varepsilon_0)^\varepsilon \ge \left|\{w \ge 1\} \cap B_{1/4}(x)\right| \ge \left|\{w \ge 1\} \cap B_{1/8}\right| \ge \frac{|B_{1/8}|}{2}.$$

Now choose ε_0 small enough such that

$$\widetilde{\theta} = \left(\frac{|B_{1/8}|}{2C_5}\right)^{1/\varepsilon} - 4\varepsilon_0 > 0 \quad (\textit{global definition of } \varepsilon_0).$$

Thus $v = w \ge \widetilde{\theta} > 0$ in $B_{1/8}$ where we have used property I). Now set

$$A_{l+1} = A_l, \quad a_{l+1} = a_l + \theta \frac{A_l - a_l}{2}, \; 0 < \theta \le \widetilde{\theta}$$

and choose β and θ small enough such that β satisfies all previous global requirements and

$$\left(1 - \frac{\theta}{2}\right) = 8^{-\beta}.$$

We shall prove that these choices give us the desired result:

(i) $u \le A_{l+1}$ in $B_{8^{-l-1}}$ by inductive hypothesis.
For every $x \in B_{1/8}$

$$a_{l+1} \le a_l + v(x)\frac{A_l - a_l}{2} = u(8^{-l}x).$$

Since this estimate holds for every $x \in B_{1/8}$, we obtain $a_{l+1} \le u$ in $B_{8^{-l-1}}$.

(ii) $A_{l+1} - a_{l+1} = A_l - a_l - \theta\frac{A_l - a_l}{2} = (A_l - a_l)\left(1 - \frac{\theta}{2}\right) = 8^{-\beta l}8^{-\beta} = 8^{-\beta(l+1)}$.

If $\left|\left\{u \le \dfrac{A_l + a_l}{2}\right\} \cap B_{8^{-l-1}}\right| \ge \dfrac{|B_{8^{-l-1}}|}{2}$, we define

$$v(x) = \frac{A_l - u(8^{-l}x)}{(A_l - a_l)/2}$$

and continue in the same way as above, using that $M_\alpha^+ u \ge -\varepsilon_0$ in B_1 in the viscosity sense. □

By a simple scaling argument, we finally obtain the following Hölder regularity result:

Theorem 3.51. *Let $\alpha_0 \in (0,2)$ and consider any $\alpha \in (\alpha_0, 2)$. Assume that the bounded function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $M_\alpha^+ u \ge -C'$ in B_1 and $M_\alpha^- u \le C'$ in B_1 in the viscosity sense for some constant $C' > 0$.*
There are constants $\beta \in (0,1)$ and $C_7 \ge 1$ depending only on $n, \lambda, \Lambda, |I|$ and α_0 such that $u \in C^\beta(\overline{B_{1/2}})$ and

$$\|u\|_{C^\beta(\overline{B_{1/2}})} \le C_7\big(\|u\|_\infty + C'\big). \tag{3.70}$$

Remark 3.52. The regularity result in Theorem 3.51 still holds when we replace $\overline{B_{1/2}}$ by any set compactly contained in B_1, modifying the constant C_7 accordingly.

The following corollary is an immediate consequence of the previous theorem. It states that we have Hölder regularity for viscosity solutions of an equation $\mathcal{I}u = 0$ in B_1, where $\mathcal{I}$ is a nonlocal elliptic operator with respect to $\mathcal{L}_0$. We will assume in addition that $\mathcal{I}$ is translation invariant, meaning that if u solves $\mathcal{I}u = 0$ in B_1, then $v = u(\cdot - x)$, $x \in \mathbb{R}^n$, solves $\mathcal{I}v = 0$ in $B_1(x)$.

Theorem 3.53. *Let $\alpha_0 \in (0,2)$ and consider any $\alpha \in (\alpha_0, 2)$. Assume that the bounded function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\mathcal{I}u = 0 \quad \textit{in } B_1 \textit{ in the viscosity sense,}$$

where $\mathcal{I}$ is a nonlocal elliptic operator with respect to $\mathcal{L}_0 = \mathcal{L}_0(n, \alpha, \lambda, \Lambda, k)$. Let $\beta \in (0,1)$ and $C_7 \ge 1$ be as in Theorem 3.51. Then $u \in C^\beta(\overline{B_{1/2}})$ and

$$\|u\|_{C^\beta(\overline{B_{1/2}})} \le C_7\big(\|u\|_\infty + |\mathcal{I}0|\big),$$

where $\mathcal{I}0$ is the value we obtain when applying $\mathcal{I}$ to the constant function that is equal to zero.

Proof. The result follows immediately from Theorem 3.51 with $C' = |\mathcal{I}0|$, using Definition 3.5. □

Proof of Theorem 3.51. Let $x_0 \in \overline{B_{1/2}}$. Define

$$v(x) = \frac{u(\frac{1}{2}x + x_0)}{2\|u\|_\infty + \frac{C'(\frac{1}{2})^\alpha}{\varepsilon_0}},$$

where $\varepsilon_0 > 0$ is the constant in Lemma 3.50. We show that v satisfies all the conditions in Lemma 3.50:

- Clearly, $|v| \le \frac{1}{2}$ in $\mathbb{R}^n$.
- Let $x \in B_1$. For simplicity, we assume that $v \in C^{1,1}(x)$ and obtain

$$\begin{aligned} M_\alpha^- v(x) &= \frac{(\frac{1}{2})^\alpha}{2\|u\|_\infty + \frac{C'(\frac{1}{2})^\alpha}{\varepsilon_0}} M_\alpha^- u(\underbrace{\tfrac{1}{2}x + x_0}_{\in B_1}) \\ &\le \varepsilon_0 \frac{C'(\frac{1}{2})^\alpha}{2\|u\|_\infty \varepsilon_0 + C'(\frac{1}{2})^\alpha} \le \varepsilon_0. \end{aligned}$$

  Thus, $M_\alpha^- v \le \varepsilon_0$ in B_1 in the viscosity sense. Analogously, $M_\alpha^+ v \ge -\varepsilon_0$ in B_1 in the viscosity sense.

Using Lemma 3.50, we obtain constants $\beta \in (0,1)$ and $C_6 \ge 1$ depending only on $n, \lambda, \Lambda, |I|$ and α_0 such that for each $x \in \mathbb{R}^n$

$$|v(x) - v(0)| \le C_6 |x|^\beta.$$

This implies

$$\left|u(\tfrac{1}{2}x + x_0) - u(x_0)\right| \le c_1 \left|\tfrac{1}{2}x + x_0 - x_0\right|^\beta \text{ for each } x \in \mathbb{R}^n, \tag{3.71}$$

where $c_1 = 2^\beta C_6(2\|u\|_\infty + \frac{C'}{\varepsilon_0})$. Substituting $y = \frac{1}{2}x + x_0$ proves that u is C^β at every $x_0 \in \overline{B_{1/2}}$. Thus $u \in C^\beta(\overline{B_{1/2}})$. It remains to prove (3.70): Using (3.71), we can estimate $\|u\|_{C^\beta(\overline{B_{1/2}})}$ as follows:

$$\begin{aligned} \|u\|_{C^\beta(\overline{B_{1/2}})} &\le \|u\|_\infty + c_1 = \|u\|_\infty + 2^\beta C_6(2\|u\|_\infty + \tfrac{C'}{\varepsilon_0}) \\ &\le (4C_6 + 1)\|u\|_\infty + \tfrac{2C_6}{\varepsilon_0} C' \le \max\left\{4C_6 + 1, \tfrac{2C_6}{\varepsilon_0}\right\}(\|u\|_\infty + C') \\ &=: C_7(\|u\|_\infty + C'). \end{aligned}$$

□

3.8 $C^{1,\beta}$ Regularity

In the case of translation invariant equations, $C^{1,\beta}$ regularity can be obtained by proving C^β regularity for the incremental quotients of a given solution. We briefly sketch the idea: Assume that u is a solution of some equation in B_r and assume that this implies C^β regularity of u in $\overline{B_{r-\gamma}}$. If we can prove C^β regularity for the incremental quotient

$$w_{\beta,h} = \frac{u(\cdot + h) - u(\cdot)}{|h|^\beta}, \quad h \in \mathbb{R}^n \setminus \{0\},$$

the regularity of u will be improved from C^β to $C^{2\beta}$ in some smaller ball $\overline{B_{r-2\gamma}}$ (using Lemma 3.55 from below). Iterating this procedure, one can prove Lipschitz regularity $C^{0,1}$ and then $C^{1,\beta}$ regularity of u after a finite number of steps. We refer to [CC95, Section 5.3] for a general overview over this technique.

If we want to adapt the idea from above to our situation, a difficulty arises from the fact that, in each step, the incremental quotients are not uniformly bounded in $\mathbb{R}^n$ which would be necessary in order to apply Theorem 3.51 to obtain C^β regularity. The Hölder regularity of our solution only guaranties such an uniform boundedness of the incremental quotients in a ball $B_{r-\gamma}$, given that the equation is satisfied in B_r. We solve this problem by assuming some extra regularity for the family of operators $\mathcal{L}_0$ introduced in Section 3.2 (cf. [CS09, Section 13]).

For $\rho \in (0,1)$ and $C_8 > 0$ define the class $\mathcal{L}_1 = \mathcal{L}_1(\lambda, \Lambda, \alpha, n, k, \rho, C_8) \subset \mathcal{L}_0(\lambda, \Lambda, \alpha, n, k)$ of all linear integro-differential operators of the form (3.4) with corresponding measurable symmetric nonnegative kernels K satisfying (3.12) and

$$\sup_{h \in B_{\rho/2}} \int\limits_{\mathbb{R}^n \setminus B_\rho} \frac{|K(y) - K(y-h)|}{|h|}\, dy \le C_8. \tag{3.72}$$

A sufficient condition for (3.72) to hold (with $\rho \in (0,1)$ arbitrary and $C_8 = \Lambda 2^{3+n} n\omega_n \rho^{-3}$) would be that $|\nabla K(y)| \le \Lambda\, |y|^{-1-n-\alpha}$ for every $y \in \mathbb{R}^n \setminus \{0\}$. Indeed, for every $y \in \mathbb{R}^n \setminus B_\rho$ and $h \in B_{\rho/2}$, the mean value theorem leads to

$$\begin{aligned}\frac{|K(y) - K(y-h)|}{|h|} &= \frac{\left|\int_0^1 \nabla K(y - th)\, dt\right| |h|}{|h|} \le \Lambda \int\limits_0^1 |y - th|^{-1-n-\alpha}\, dt \\ &\le \Lambda(|y| - \tfrac{\rho}{2})^{-1-n-\alpha} \le \Lambda 2^{1+n+\alpha}\, |y|^{-1-n-\alpha}.\end{aligned}$$

Hence, for each $h \in B_{\rho/2}$

$$\begin{aligned}\int\limits_{\mathbb{R}^n \setminus B_\rho} \frac{|K(y) - K(y-h)|}{|h|}\, dy &\le \Lambda 2^{1+n+\alpha} \int\limits_{\mathbb{R}^n \setminus B_\rho} |y|^{-1-n-\alpha}\, dy \\ &= \Lambda 2^{1+n+\alpha} n\omega_n \int\limits_\rho^\infty r^{-2-\alpha}\, dr \\ &\le \Lambda 2^{3+n} n\omega_n \rho^{-3} = C_8.\end{aligned}$$

Throughout this section we work with the following assumptions: Let $\mathcal{I} = \mathcal{I}_\alpha$ be an arbitrary translation invariant nonlocal elliptic operator with respect to $\mathcal{L}_1$. Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a bounded function, which is continuous in $\overline{B_1}$ and satisfies

$$\mathcal{I}u = 0 \text{ in } B_1 \text{ in the viscosity sense.}$$

The following result is the desired interior $C^{1,\beta}$ regularity result.

Theorem 3.54. *Let $\alpha_0 \in (0,2)$ and assume $\alpha \in (\alpha_0, 2)$. There exist $\rho \in (0,1)$ and $\beta \in (0,1)$ (depending on $n, \lambda, \Lambda, |I|$ and α_0) such that if u is a viscosity solution to $\mathcal{I}u = 0$ in B_1, then $u \in C^{1,\beta}(\overline{B_{1/2}})$ and*

$$\|u\|_{C^{1,\beta}(\overline{B_{1/2}})} \leq C_9\big(\|u\|_\infty + |\mathcal{I}0|\big)$$

for some constant $C_9 \geq 1$ depending on $\lambda, \Lambda, n, |I|, \alpha_0$ and C_8.

In order to prove this theorem, we need the following auxiliary result. It is a standard telescopic sum argument which is used to improve the regularity of our solution from C^β to $C^{2\beta}$ and so forth all the way up to $C^{0,1}$.

Lemma 3.55 ([CC95, Lemma 5.6])**.** *Let $\beta_1 \in (0,1)$, $\beta_2 \in (0,1]$ and $K > 0$. Let $\varphi \in L^\infty([-1,1])$ satisfy $\|\varphi\|_{L^\infty([-1,1])} \leq K$. For $h \in \mathbb{R}$ with $0 < |h| \leq 1$, define*

$$v_{\beta_2,h}(x) = \frac{\varphi(x+h) - \varphi(x)}{|h|^{\beta_2}},$$

where $x \in I_h = [-1, 1-h]$ if $h > 0$ and $x \in I_h = [-1-h, 1]$ if $h < 0$. Assume that $v_{\beta_2,h} \in C^{\beta_1}(I_h)$ and $\|v_{\beta_2,h}\|_{C^{\beta_1}(I_h)} \leq K$ for every $0 < |h| \leq 1$. We then have:

(i) If $\beta_1 + \beta_2 < 1$ then $\varphi \in C^{\beta_1+\beta_2}([-1,1])$ and $\|\varphi\|_{C^{\beta_1+\beta_2}([-1,1])} \leq cK$;

(ii) If $\beta_1 + \beta_2 > 1$ then $\varphi \in C^{0,1}([-1,1])$ and $\|\varphi\|_{C^{0,1}([-1,1])} \leq cK$,

where the constants $c \geq 1$ in (i) and (ii) depend only on $\beta_1 + \beta_2$.

Proof. It is enough to bound $|\varphi(x+h) - \varphi(x)|$ for $x \in [-1, 0]$, $h > 0$ and $x + h \leq 1$.
Chose $l \in \mathbb{N}_0$ large enough such that $x + 2^l h \leq 1 < x + 2^{l+1}h$ and define $\tau_0 = 2^l h$. We have that $-1 \leq x < x + \tau_0 \leq 1 < x + 2\tau_0$ and therefore

$$\frac{1}{2} < \tau_0 \leq 2. \tag{3.73}$$

For $\tau \in (0, \tau_0]$, we define $w(\tau) = \varphi(x+\tau) - \varphi(x)$.
Using the assumption that $\|v_{\beta_2,h}\|_{C^{\beta_1}([-1,1-\tau/2])} \leq K$ (since $0 < \tau/2 \leq \tau_0/2 \leq 1$), we have

$$\begin{aligned}|w(\tau) - 2w(\tau/2)| &= |\varphi(x+\tau) - 2\varphi(x+\tau/2) + \varphi(x)| \\ &= \left(\tfrac{\tau}{2}\right)^{\beta_2} \left|v_{\beta_2,\tau/2}(x+\tau/2) - v_{\beta_2,\tau/2}(x)\right| \leq K\left(\tfrac{\tau}{2}\right)^{\beta_1+\beta_2}.\end{aligned}$$

Using the previous inequality repeatedly, we obtain

$$\begin{aligned}
|w(\tau_0) - 2w(\tau_0/2)| &\le c_1 K \tau_0^{\beta_1+\beta_2},\\
\left|2w(\tau_0/2) - 2^2 w(\tau_0/2^2)\right| &\le c_1 K 2^{1-(\beta_1+\beta_2)} \tau_0^{\beta_1+\beta_2},\\
&\vdots\\
\left|2^{l-1} w(\tau_0/2^{l-1}) - 2^l w(\tau_0/2^l)\right| &\le c_1 K 2^{(l-1)(1-(\beta_1+\beta_2))} \tau_0^{\beta_1+\beta_2},
\end{aligned}$$

where $c_1 = 2^{-(\beta_1+\beta_2)}$. Using a telescopic sum argument, we obtain

$$\begin{aligned}
&\left|w(\tau_0) - 2^l w(h)\right| = \left|w(\tau_0) - 2^l w(\tau_0/2^l)\right|\\
&\le |w(\tau_0) - 2w(\tau_0/2)| + \left|2w(\tau_0/2) - 2^2 w(\tau_0/2^2)\right| + \ldots + \left|2^{l-1} w(\tau_0/2^{l-1}) - 2^l w(\tau_0/2^l)\right|\\
&\le c_1 K \tau_0^{\beta_1+\beta_2} \sum_{j=0}^{l-1} 2^{j(1-(\beta_1+\beta_2))}.
\end{aligned}$$

Since $2^{-l} = \tau_0^{-1} h \le 2h$ (because of (3.73)) and $\|\varphi\|_{L^\infty([-1,1])} \le K$, we have

$$\begin{aligned}
|w(h)| &\le 2^{-l}\,|w(\tau_0)| + c_1 K 2^{-l} \tau_0^{\beta_1+\beta_2} \sum_{j=0}^{l-1} 2^{j(1-(\beta_1+\beta_2))}\\
&\le 4Kh + c_1 K h \tau_0^{\beta_1+\beta_2-1} \sum_{j=0}^{l-1} 2^{j(1-(\beta_1+\beta_2))}.
\end{aligned}$$

(i) Let $\beta_1+\beta_2 < 1$. Using the previous estimate and the definition of τ_0, we obtain

$$\begin{aligned}
|w(h)| &\le 4Kh + c_1 K h \tau_0^{\beta_1+\beta_2-1} \tfrac{2^{l(1-(\beta_1+\beta_2))}}{2^{1-(\beta_1+\beta_2)}-1} = 4Kh + \tfrac{c_1}{2^{1-(\beta_1+\beta_2)}-1} K h^{\beta_1+\beta_2}\\
&\le cKh^{\beta_1+\beta_2},
\end{aligned}$$

where $c = 4 \cdot 2^{1-(\beta_1+\beta_2)} + \frac{c_1}{2^{1-(\beta_1+\beta_2)}-1}$.

(ii) Let $\beta_1+\beta_2 > 1$. Then

$$|w(h)| \le 4Kh + \tfrac{c_1}{1-2^{1-(\beta_1+\beta_2)}} K h \underbrace{\tau_0^{\beta_1+\beta_2-1}}_{\le 2^{\beta_1+\beta_2-1}} \le cKh,$$

where $c = 4 + \frac{2^{\beta_1+\beta_2-1} c_1}{1-2^{1-(\beta_1+\beta_2)}}$. □

Proof of Theorem 3.54. The proof follows exactly as in [CS09, Theorem 13.1]. Consider any $\rho \in (0,1)$ for the moment and assume that $\mathcal{L}_1 \ne \emptyset$. Since $\mathcal{L}_1 \subset \mathcal{L}_0$ and $\mathcal{I}u = 0$ in B_1 in the viscosity sense,

$$M^+_{\mathcal{L}_0} u \ge M^+_{\mathcal{L}_1} u \ge \mathcal{I}u - \mathcal{I}0 \ge -|\mathcal{I}0| \quad \text{and} \quad M^-_{\mathcal{L}_0} u \le M^-_{\mathcal{L}_1} u \le |\mathcal{I}0|$$

in B_1 in the viscosity sense. By Theorem 3.51 and Remark 3.52, there exists $\beta \in (0,1)$ depending on $\lambda, \Lambda, n, |I|$ and α_0 such that for every $\gamma \in (0,1)$, $u \in C^{\beta}(\overline{B_{1-\gamma}})$ and $\|u\|_{C^{\beta}(\overline{B_{1-\gamma}})} \leq C_7(\|u\|_{\infty} + |\mathcal{I}0|)$, where the constant $C_7 \geq 1$ depends on $\lambda, \Lambda, n, |I|, \alpha_0$ and γ. Note that we can assume the existence of a number $l \in \mathbb{N}$ such that $l\beta < 1 < (l+1)\beta$ by making β smaller if necessary. We want to improve the obtained regularity of u by applying Theorem 3.51 repeatedly to the respective incremental quotients of u until we obtain Lipschitz regularity in a finite number of steps (using a rescaled version of Lemma 3.55).

Set $\gamma = \frac{1}{4(l+1)}$ and fix a unit vector $e \in \mathbb{R}^n$. Define the incremental quotient

$$w_h(x) = w_{h,\beta}(x) = \frac{\tau_h u(x) - u(x)}{|h|^{\beta}} = \frac{u(x+he) - u(x)}{|h|^{\beta}}, \quad x \in \mathbb{R}^n,\ h \in \mathbb{R} \setminus \{0\}. \tag{3.74}$$

Let $|h| \in (0, \gamma)$. We have $\mathcal{I}\tau_h u = 0$ in $B_{1-\gamma}$ in the viscosity sense since $\mathcal{I}$ is translation invariant. Therefore, using Lemma 3.10 (with $\mathcal{L} = \mathcal{L}_1$, $f = g = 0$) and the fact that $M^+_{\mathcal{L}_1} w_h = -M^-_{\mathcal{L}_1}(-w_h)$, we obtain

$$M^+_{\mathcal{L}_1} w_h \geq 0 \quad \text{and} \quad M^-_{\mathcal{L}_1} w_h \leq 0 \tag{3.75}$$

in $B_{1-\gamma}$ in the viscosity sense. Since $u \in C^{\beta}(\overline{B_{1-\gamma}})$, we see that w_h is uniformly bounded in $\overline{B_{1-\gamma}}$. Outside $\overline{B_{1-\gamma}}$, w_h is not uniformly bounded and so we can not apply Theorem 3.51 directly. However, (3.72) allows for a different approach:
Set $r = 1 - \gamma$. Let η be a smooth cutoff function supported in $\overline{B_r}$ such that $\eta \equiv 1$ in $B_{r-\gamma/4}$. Using η, we can write $w_h = w_h^{(1)} + w_h^{(2)}$, where

$$w_h^{(1)}(x) = \eta(x) w_h(x) \quad \text{and} \quad w_h^{(2)}(x) = (1 - \eta(x)) w_h(x).$$

Note that $w_h^{(2)} \equiv 0$ in $B_{r-\gamma/2}$ due to the choice of η, which implies $w_h(x) = w_h^{(1)}(x)$ for all $x \in B_{r-\gamma/2}$. We show that $w_h^{(1)} \in C^{\beta}(\overline{B_{r-\gamma}})$ for every $|h| \in (0, \frac{\gamma}{16})$.

Let $|h| \in (0, \frac{\gamma}{16})$. Using (3.75), we have

$$M^+_{\mathcal{L}_0} w_h^{(1)} \geq M^+_{\mathcal{L}_1} w_h^{(1)} = M^+_{\mathcal{L}_1}(w_h - w_h^{(2)}) \geq 0 - M^+_{\mathcal{L}_1} w_h^{(2)}, \tag{3.76}$$
$$M^-_{\mathcal{L}_0} w_h^{(1)} \leq M^-_{\mathcal{L}_1} w_h^{(1)} = M^-_{\mathcal{L}_1}(w_h - w_h^{(2)}) \leq 0 - M^-_{\mathcal{L}_1} w_h^{(2)} \tag{3.77}$$

in B_r in the viscosity sense. We show that we can bound the expressions $\left| M^{\pm}_{\mathcal{L}_1} w_h^{(2)} \right|$ in $B_{r-\gamma/2}$ by some universal constant. Consider any $L \in \mathcal{L}_1$ and let $x \in B_{r-\gamma/2}$. Since

$w_h^{(2)}(x) = 0$, we obtain

$$\begin{aligned}
\left|Lw_h^{(2)}(x)\right| &= \left|\int_{\mathbb{R}^n} w_h^{(2)}(x+y)K(y)\,dy\right| \\
&\leq \left|\int_{\mathbb{R}^n} \frac{(1-\eta(x+y+he))u(x+y+he)-(1-\eta(x+y))u(x+y)}{|h|^\beta}K(y)\,dy\right| \\
&\quad + \left|\int_{\mathbb{R}^n} \frac{(\eta(x+y+he)-\eta(x+y))u(x+y+he)}{|h|^\beta}K(y)\,dy\right| \\
&=: \mathfrak{I}_1 + \mathfrak{I}_2.
\end{aligned}$$

Choose and fix $\rho = \frac{\gamma}{8}$.
We can estimate $\mathfrak{I}_1$ by using (3.72) and the fact that $(1-\eta(x+y))u(x+y) = 0$ for every $|y| < \frac{\gamma}{8} = \rho$:

$$\begin{aligned}
\mathfrak{I}_1 &\leq \int_{\mathbb{R}^n\setminus B_\rho} |1-\eta(x+y)|\,|u(x+y)|\,\frac{|K(y-he)-K(y)|}{|h|^\beta}\,dy \\
&\leq |h|^{1-\beta}\,\|u\|_\infty \int_{\mathbb{R}^n\setminus B_\rho} \frac{|K(y-he)-K(y)|}{|h|}\,dy \leq C_8\,\|u\|_\infty\,.
\end{aligned}$$

Using the mean value theorem and the fact that $\eta(x+y+he) = \eta(x+y) = 1$ for every $|y| < \frac{\gamma}{8}$ (recall that $|h| \in (0, \frac{\gamma}{16})$), we obtain

$$\begin{aligned}
\mathfrak{I}_2 &\leq \int_{\mathbb{R}^n\setminus B_\rho} \frac{\left|\int_0^1 \nabla\eta(x+y+\tau he)\cdot he\,d\tau\right|\,|u(x+y+he)|}{|h|^\beta}K(y)\,dy \\
&\leq 2\,|h|^{1-\beta}\,\|u\|_\infty\,\|\eta\|_\infty \int_{\mathbb{R}^n\setminus B_\rho} \Lambda\,|y|^{-n-\alpha}\,dy \\
&\leq c_0\,\|u\|_\infty\,,
\end{aligned}$$

where $c_0 > 1$ depends on $\lambda, \Lambda, n, \alpha_0$ and $|I|$ but not on h.
It then follows from (3.76) and (3.77) that

$$M^+_{\mathcal{L}_0} w_h^{(1)} \geq -C'\,\|u\|_\infty \quad \text{and} \quad M^-_{\mathcal{L}_0} w_h^{(1)} \leq C'\,\|u\|_\infty$$

in $B_{r-\gamma/2}$ in the viscosity sense and for each $0 < |h| < \frac{\gamma}{16}$, where $C' = C_8 + c_0$. Moreover, the family $\{w_h^{(1)}\}_{|h|\in(0,\gamma/16)}$ is uniformly bounded in $\mathbb{R}^n$ as seen above. So we can apply Theorem 3.51, which leads to

$$\begin{aligned}
\|w_h\|_{C^\beta(\overline{B_{r-\gamma}})} = \left\|w_h^{(1)}\right\|_{C^\beta(\overline{B_{r-\gamma}})} &\leq C_7\left(\sup_{\overline{B_r}}\left|w_h^{(1)}\right| + C'\,\|u\|_\infty\right) \\
&\leq C_7\left(\|u\|_{C^\beta(\overline{B_r})} + C'\,\|u\|_\infty\right) \leq c_1(\|u\|_\infty + |\mathcal{I}0|)
\end{aligned} \tag{3.78}$$

for every $0 < |h| < \frac{\gamma}{16}$, where $c_1 = C_7(C_7 + C')$.
Because of (3.78), we can apply for every $e \in \mathbb{R}^n$ as above Lemma 3.55 (rescaled and with $\beta_1 = \beta$) on segments parallel to e and obtain

$$\|u\|_{C^{2\beta}(\overline{B_{1-2\gamma}})} \leq c_2(\|u\|_\infty + |\mathcal{I}0|),$$

where $c_2 \geq 1$ only depends on $\lambda, \Lambda, n, |I|, C_8$ and α_0. We can repeat this process (with suitable incremental quotients w_h of the form (3.74) with β replaced by $2\beta, 3\beta, \ldots, l\beta$) since $l\beta < 1 < (l+1)\beta$ to obtain $u \in C^{3\beta}(\overline{B_{1-3\gamma}}), \ldots, u \in C^{l\beta}(\overline{B_{1-l\gamma}})$ and finally, by (ii) in Lemma 3.55,

$$\|u\|_{C^{0,1}(\overline{B_{3/4}})} \leq c_3(\|u\|_\infty + |\mathcal{I}0|)$$

with a new universal constant $c_3 \geq 1$ depending again on $\lambda, \Lambda, n, |I|, C_8$ and α_0. By using the same reasoning from above (leading to (3.78)) one more time for the difference quotient $w_h(x) = \frac{u(x+he)-u(x)}{h}$ for every $0 < |h| < \frac{\gamma}{16}$ and every unit vector $e \in \mathbb{R}^n$, we obtain

$$\|w_h\|_{C^\beta(\overline{B_{1/2}})} \leq c_4 \left(\|u\|_{C^{0,1}(\overline{B_{3/4}})} + C'' \|u\|_\infty \right) \leq c_5(\|u\|_\infty + |\mathcal{I}0|)$$

and conclude that $u \in C^{1,\beta}(\overline{B_{1/2}})$ and $\|u\|_{C^{1,\beta}(\overline{B_{1/2}})} \leq C_9(\|u\|_\infty + |\mathcal{I}0|)$. □

4 Conclusion

We study regularity properties of solutions to equations of the form $\mathcal{I}u = 0$, where $\mathcal{I}$ is a nonlocal fully nonlinear elliptic operator with respect to the class $\mathcal{L}_0$ of all linear integro-differential operators of the form

$$Lu(x) = \int_{\mathbb{R}^n} \big(u(x+y) - u(x) - (\nabla u(x) \cdot y)\mathbb{1}_{\{|y|\le 1\}}\big) K(y)\, dy$$

with corresponding measurable symmetric kernels $K : \mathbb{R}^n \to [0,\infty)$ satisfying

$$(2-\alpha)k\big(\tfrac{y}{|y|}\big)\frac{\lambda}{|y|^{n+\alpha}} \le K(y) \le (2-\alpha)\frac{\Lambda}{|y|^{n+\alpha}}, \quad y \in \mathbb{R}^n \setminus \{0\}, \tag{4.1}$$

where $0 < \lambda \le \Lambda$, $\alpha \in (0,2)$ and $k : \mathbb{S}^{n-1} \to [0,1]$ is a measurable symmetric function with $k(\xi) = 1$ if $\xi \in I$ for some fixed set I of the form $I = (B_\varrho(\xi_0) \cup B_\varrho(-\xi_0)) \cap \mathbb{S}^{n-1}$, $\varrho > 0$, $\xi_0 \in \mathbb{S}^{n-1}$.

The main result, Theorem 3.53, states that these solutions are Hölder continuous. This regularity result is robust with respect to $\alpha \nearrow 2$, provided we bound α from below. By assuming some extra regularity for the class of operators $\mathcal{L}_0$, resulting in the class $\mathcal{L}_1 \subset \mathcal{L}_0$, we obtain $C^{1,\beta}$ regularity in Theorem 3.54. We therefore extend the corresponding regularity results in [CS09].

Let us explain the significance of our results with regards to the literature. Results for linear equations involving nonlocal operators have been studied by several authors using the corresponding Markov jump processes (see [BL02, SV04, KM13]). They consider linear operators $\mathcal{A}$ of the form

$$\mathcal{A}u(x) = \int_{\mathbb{R}^n} (u(x+h) - u(x) - (\nabla u(x) \cdot h)\mathbb{1}_{\{|h|\le 1\}}) n(x,h)\, dh,$$

for bounded functions $u \in C^2(\mathbb{R}^n)$. Assume that $n\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is a measurable function with $n(x,h) = n(x,-h)$ and

$$c_1|h|^{-n-\alpha} \le n(x,h) \le c_2|h|^{-n-\alpha} \tag{4.2}$$

for two fixed positive reals $c_1 < c_2$, for every $h \in \mathbb{R}^n \setminus \{0\}$ and for every $x \in \mathbb{R}^n$. In [BL02] it is shown that harmonic functions with respect to $\mathcal{A}$ satisfy a Harnack inequality and Hölder regularity estimates. These Hölder estimates follow from Theorem 3.51 because $n(x,\cdot)$ satisfies (4.1) with $\lambda = \frac{c_1}{2-\alpha}$ and $\Lambda = \frac{c_2}{2-\alpha}$ for every $x \in \mathbb{R}^n$, which implies that

$M_\alpha^- u(x) \le \mathcal{A}u(x) \le M_\alpha^+ u(x)$ for every $x \in \mathbb{R}^n$. Recall the definition of the extremal operators M_α^+, M_α^- in (3.14) and (3.15). Since $\mathcal{A}u = 0$ in B_1, we have $M_\alpha^- u \le C'$ and $M_\alpha^+ u \ge -C'$ in B_1 for any $C' > 0$ and Theorem 3.51 leads to Hölder regularity of u in every subdomain $\Omega \Subset B_1$. In this sense, Theorem 3.51 extends the results of [BL02, SV04, KM13] on linear operators to the fully nonlinear case. Note that, concerning anisotropy, assumption (4.1) is less restrictive than the corresponding assumption in [KM13]. Moreover, different from [BL02, SV04, KM13], Theorem 3.51 provides estimates which are uniform with respect to $\alpha \nearrow 2$.

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