Obstruction groups for extending deformations of subdiagrams to deformations of diagrams in the categories of ringed topoi and schemes

vorgelegt von

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Bielefeld, den 13. August 2013

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Abstract

In this thesis we exhibit an obstruction group for the following deformation theoretic problem. Given a diagram and a subdiagram of ringed topoi over a fixed ringed topos $\mathcal{S}$ and given a deformation ξ of the subdiagram, what is an obstruction group for finding a deformation of the diagram reducing to the given deformation ξ of the subdiagram? We will calculate this obstruction group for a large variety of interesting cases as explicitly as possible. The results hold as well in the case of diagrams and subdiagrams of schemes over a fixed scheme S.

Contents

Introduction

Deformation theory of diagrams of schemes is the infinitesimal study of a family of diagrams in the neighborhood of a given point. For instance, if we are given a morphism of schemes $f\colon X \to Y$ considered as a diagram over a fixed parameter scheme T, we may think of f as a family of diagrams $f_t\colon X_t \to Y_t$ for $t \in T$. Fixing a fibre $f_0\colon X_0 \to Y_0$ over some point $0 \in T$, deformation theory of diagrams helps us to study infinitesimal properties of the family near the fibre f_0 such as smoothness and properness.

Moreover, there is a connection between deformation theory of diagrams of schemes and the moduli problem of classifying isomorphism classes of diagrams of schemes of a fixed type with certain additional conditions. If k is a field, we may ask the question whether there is a coarse or even a fine moduli space parametrising isomorphism classes of diagrams $f\colon X \to Y$ over Spec k with prescribed conditions such as, for example, the dimension, the Euler characteristic and the regularity of X and Y, amongst others. Deformation theory gives us some insight into properties of the moduli space and is still very powerful concerning questions on the classification problem even if there is no moduli space.

It turns out that deformation theoretic questions concerning diagrams of schemes over a fixed scheme S can be answered by constructing a ringed topos to the diagram. This is the reason why we generalize our framework by considering diagrams and subdiagrams of ringed topoi over a fixed ringed topos $\mathcal{S}$. Formally, a diagram is a functor $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ from the dual of a finite index category $\mathfrak{C}$ to the category of ringed topoi over $\mathcal{S}$ and a subdiagram is the restriction of I to a subcategory $\mathfrak{D}$ of $\mathfrak{C}$. As already mentioned, it is possible to associate a ringed topos $\mathcal{X}$ to the diagram and a ringed topos $\mathcal{Y}$ to the subdiagram together with morphisms of ringed topoi $m_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ and $m_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{S}$ as shown in [Buc81, Chapitre II.2.1.]. Let $L_{\mathcal{X}/\mathcal{S}}$ and $L_{\mathcal{Y}/\mathcal{S}}$ be the cotangent complexes of $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{S}$, respectively, as defined in [Ill71, Equation II.1.2.7.1]. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are flat over $\mathcal{S}$ and let

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_{\mathcal{S}} \to 0$$

be an extension of $\mathcal{O}_{\mathcal{S}}$ giving rise to a closed embedding $\mathcal{S} \to \mathcal{S}'$. Then there is an obstruction in the group $\mathbb{Ext}^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J})$ whose vanishing is necessary and sufficient for the existence of a deformation of $\mathcal{X}$ over $\mathcal{S}'$. If the obstruction vanishes, the set of isomorphism classes of deformations of $\mathcal{X}$ over $\mathcal{S}'$ is a torsor under $\mathbb{Ext}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J})$ by [Ill71, Théorème III.2.1.7.] and the analog statements hold for $\mathcal{Y}$.

Fix a diagram and a subdiagram with associated ringed topoi $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let us pose the following two questions.

i) Given a deformation ξ of $\mathcal{Y}$ over $\mathcal{S}'$, what is an obstruction group for the existence of a deformation of $\mathcal{X}$ over $\mathcal{S}'$ inducing the given deformation ξ of $\mathcal{Y}$?

ii) If the obstruction in this group vanishes, how many different isomorphism classes of deformations of $\mathcal{X}$ inducing ξ are there?

These questions are answered for arbitrary diagrams and subdiagrams in the following result.

Theorem 2.13. *There is an exact sequence of abelian groups*

$$0 \to \mathbb{E}xt^0_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}xt^0_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}xt^0_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\to \mathbb{E}xt^1_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \xrightarrow{\sigma} \mathbb{E}xt^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \xrightarrow{\tau} \mathbb{E}xt^1_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\xrightarrow{\omega} \mathbb{E}xt^2_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}xt^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}xt^2_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\to \ldots$$

where $\mathrm{Cone}(m)$ *is the cone of a certain morphism* m *of complexes of* $\mathcal{O}_{\mathcal{X}}$-*modules. The morphism* τ *is the forgetful morphism sending a deformation of* $\mathcal{X}$ *to the induced deformation of* $\mathcal{Y}$. *Given a deformation* ξ *of the subdiagram* $\mathcal{Y}$ *over* $\mathcal{S}'$, *there is an obstruction*

$$\omega(\xi) \in \mathbb{E}xt^2_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ *over* $\mathcal{S}'$ *reducing to* ξ. *If the obstruction* $\omega(\xi)$ *is zero, then the set of isomorphism classes of deformations of* $\mathcal{X}$ *over* $\mathcal{S}'$ *reducing to* ξ *is a torsor under the image of*

$$\sigma \colon \mathbb{E}xt^1_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}xt^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}).$$

The above exact sequence holds as well for the case of diagrams and subdiagrams of schemes over a fixed scheme S.

Section 1 gives some preliminaries concerning the category of ringed topoi, the cotangent complex and deformations of ringed topoi and schemes, amongst others.

In Section 2 we will construct a left adjoint of the forgetful functor $u \colon \mathcal{X} \to \mathcal{Y}$ which we will use in the proof of Theorem 2.13. Under weak conditions on the subdiagram $\mathcal{Y}$, the obstruction group $\mathbb{E}xt^2_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J})$ is isomorphic to

$$\mathbb{E}xt^2_{\mathcal{X}}(L_h, m_{\mathcal{X}}^* \mathcal{J})$$

as shown in Theorem 2.20 where L_h is the cotangent complex of a certain ring morphism $h \colon u^{-1} \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ as defined in Proposition 2.16.

Since L_h is the cotangent complex of a ring morphism, it consists of free $\mathcal{O}_{\mathcal{X}}$-modules in contrast to $\mathrm{Cone}(m)$ and the obstruction group may be calculated more explicitly for many interesting cases as done in Section 3. We will introduce several types of subdiagrams such as well-positioned and full subdiagrams in the first two subsections. For example, if we consider the diagram

in the category of schemes over a fixed scheme S together with the three subdiagrams

subdiagram 1	subdiagram 2	subdiagram 3

2

then each subdiagram is well-positioned and full.

Given a subdiagram $\mathcal{Y}$, we may associate its complementary subdiagram $\overline{\mathcal{Y}}$ as defined in Definition 3.17. We will see in Corollary 3.26 that for well-positioned and full subdiagrams $\mathcal{Y}$, the obstruction group is given by

$$\mathbb{E}\mathrm{xt}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$$

where $\overline{u}\colon \mathcal{X} \to \overline{\mathcal{Y}}$ is the forgetful functor and $\mathbf{R}G$ is the total right derived functor of the functor G in Lemma 3.23. Hence the obstruction group is only concentrated on the complementary subdiagram $\overline{\mathcal{Y}}$ of $\mathcal{Y}$.

Continuing the above example, if t_X is the structure morphism of X over S, then the respective complementary subdiagrams and obstruction groups are given as follows.

complementary subdiagram 1	complementary subdiagram 2	complementary subdiagram 3
	Y	
X		Z
obstruction group 1	obstruction group 2	obstruction group 3
$\mathbb{E}\mathrm{xt}^2_X(L_{X/Y}, t^*_X\mathcal{J}))$	$\mathbb{E}\mathrm{xt}^2_Y(L_{Y/Z}, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$	$\mathbb{E}\mathrm{xt}^2_Z(L_{Z/S}, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$

Subsection 3.4 deals with subdiagrams obtained from the diagram by omitting a target, a source or a bridge in which case the obstruction group in concentrated on the omitted ringed topos. In the above example, Z is a target, X is a source and Y is a bridge of the commutative triangle.

The other extreme of the discrete subdiagram is treated as well. Here, the subdiagram is obtained from the diagram by keeping all ringed topoi, but by omitting all morphisms except for the identities.

Section 4 gives an overview of all the subdiagrams of a single morphism and of a commutative triangle of schemes together with their respective obstruction groups. We will get back some obstruction groups already known for particular cases. Moreover, we will derive the cotangent braid of [Buc81, Diagramme II.2.4.3.2] by a different approach. Considering the second subdiagram of the above example again, we will give a sufficient criterion for the obstruction group to vanish if $h\colon X \to Z$ is the Albanese map of a nonsingular projective variety X and Y is the image of the Albanese map.

Section 5 deals with deformations of diagrams and subdiagrams of $\mathcal{O}_{\mathcal{S}}$-modules for a fixed ringed topos $\mathcal{S}$. We will see that there is a relation between the notions of "graded extensions of diagrams of ringed topoi" and "deformations of diagrams of $\mathcal{O}_{\mathcal{S}}$-modules" which is a generalization of the ideas in [Ill71, Chapitre IV.3.1.]. This relation allows us to answer the two analogue questions for diagrams of $\mathcal{O}_{\mathcal{S}}$-modules in Theorem 5.10. Basically, the answer is given by a graded analogue of the above long exact sequence.

1 Preliminaries

All sheaves of rings on any topos and in particular the structure sheaves of all ringed topoi occuring in the following are assumed to be associative, commutative and unitary.

1.1 The categories of topoi and ringed topoi

In this section we recall the notion of topoi and ringed topoi according to [SGA41, Exposé IV.], where all of the following definitions can be found. Let us fix once and for all a universe $\mathcal{U}$.

Definition 1.1. A category $\mathcal{X}$ is called a topos if there is a site $\mathcal{C}$ such that $\mathcal{X}$ is equivalent to the category of sheaves of sets on $\mathcal{C}$. A ringed topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a pair consisting of a topos $\mathcal{X}$ and a sheaf of rings $\mathcal{O}_\mathcal{X}$ of $\mathcal{X}$, called the structure sheaf of $\mathcal{X}$.

Example 1.2. Let X be a topological space and let $\mathcal{C}$ be the small site associated to X, i.e., the underlying category of the site $\mathcal{C}$ has all open subsets of X as objects and all inclusions of open sets as morphisms. Given an open subset U of X, a family of inclusions $\{V_i \subseteq U\}_{i \in I}$ is a covering family of U if and only if $U = \bigcup_{i \in I} V_i$. Then the topos $\mathcal{X}$, the category of sheaves of sets on $\mathcal{C}$, is the category of sheaves of sets on X in the usual sense. We call $\mathcal{X}$ the associated topos of the topological space X.

Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{X}$ be the topos associated to the topological space X. We take $\mathcal{O}_\mathcal{X}$ to be the contravariant functor $\mathcal{O}_X$ from $\mathcal{C}$ to the category of sets. Then $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a ringed topos.

If the structure sheaf $\mathcal{O}_\mathcal{X}$ of $\mathcal{X}$ is clear from the context, we will omit it and just write $\mathcal{X}$ for the ringed topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Similarly, if A is a sheaf of abelian groups (of rings, of $\mathcal{O}_\mathcal{X}$-modules, etc.) of $\mathcal{X}$, we will usually omit the word sheaf and just write that A is an abelian group (a ring, an $\mathcal{O}_\mathcal{X}$-module, etc.) of $\mathcal{X}$.

Definition 1.3. A morphism of topoi $v\colon \mathcal{X} \to \mathcal{Y}$ of a topos $\mathcal{X}$ to a topos $\mathcal{Y}$ is a triple $v = (v_*, v^{-1}, \varphi)$ where

$$v_*\colon \mathcal{X} \to \mathcal{Y} \quad \text{and} \quad v^{-1}\colon \mathcal{Y} \to \mathcal{X}$$

are adjoint functors such that v^{-1} is left exact, i.e., v^{-1} commutes with finite limits, and

$$\varphi_{A,B}\colon \mathrm{Hom}_\mathcal{X}(v^{-1}B, A) \xrightarrow{\cong} \mathrm{Hom}_\mathcal{Y}(B, v_*A)$$

is an isomorphism of adjunction, bifunctorial in $B \in \mathrm{ob}(\mathcal{Y})$ and $A \in \mathrm{ob}(\mathcal{X})$. This means that for every morphism $\alpha\colon A \to A'$ in $\mathcal{X}$ and for every morphism $\beta\colon B \to B'$ in $\mathcal{Y}$, the square

$$
\begin{array}{ccc}
\mathrm{Hom}_\mathcal{X}(v^{-1}B', A) & \xrightarrow{\varphi_{A,B'}} & \mathrm{Hom}_\mathcal{Y}(B', v_*A) \\
{\scriptstyle \mathrm{Hom}_\mathcal{X}(v^{-1}\beta, \alpha)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_\mathcal{Y}(\beta, v_*\alpha)} \\
\mathrm{Hom}_\mathcal{X}(v^{-1}B, A') & \xrightarrow{\varphi_{A',B}} & \mathrm{Hom}_\mathcal{Y}(B, v_*A')
\end{array}
$$

is commutative.

A morphism of ringed topoi $v\colon (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is given by a quadruple $v = (v_*, v^{-1}, \varphi, \theta)$ where (v_*, v^{-1}, φ) is a morphism of the underlying topoi and $\theta\colon v^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is a morphism of sheaves of rings.

By abuse of notation we will denote a morphism of ringed topoi by $v \colon \mathcal{X} \to \mathcal{Y}$. Notice that giving a morphism of rings $\theta \colon v^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is equivalent to giving a morphism of rings $\theta^{\mathrm{ad}} \colon \mathcal{O}_{\mathcal{Y}} \to v_*\mathcal{O}_{\mathcal{X}}$ by the adjunction of v_* and v^{-1}.

Example 1.4. If $f \colon X \to Y$ is a continuous map of topological spaces, then as explained in [SGA41, Exposé IV.4.1.1.], it is possible to construct a morphism of topoi between the associated topoi $\mathcal{X}$ and $\mathcal{Y}$ of X and Y, respectively, which we denote $f \colon \mathcal{X} \to \mathcal{Y}$ by abuse of notation.

Similarly, if $f \colon X \to Y$ is a morphism of schemes, it is possible to construct a morphism of ringed topoi between the associated ringed topoi $\mathcal{X}$ and $\mathcal{Y}$ of X and Y, respectively. Again by abuse of notation we denote it by $f \colon \mathcal{X} \to \mathcal{Y}$.

Remark 1.5. If $v^{-1} \colon \mathcal{Y} \to \mathcal{X}$ is a functor between topoi and commutes with arbitrary colimits and with finite limits, then v^{-1} has a right adjoint $v_* \colon \mathcal{X} \to \mathcal{Y}$ which is unique up to unique isomorphism of functors. Then after choosing a right adjoint v_*, the functor v^{-1} may be regarded as the inverse image functor of a morphism of topoi $v \colon \mathcal{X} \to \mathcal{Y}$.

Similarly, if $v_* \colon \mathcal{X} \to \mathcal{Y}$ is a functor between topoi and commutes with arbitrary limits, then v_* has a left adjoint $v^{-1} \colon \mathcal{Y} \to \mathcal{X}$ which is unique up to unique isomorphism. If in addition v^{-1} commutes with finite limits, then v_* may be regarded as the direct image functor of a morphism of topoi $v \colon \mathcal{X} \to \mathcal{Y}$ by [SGA41, Exposé IV.3.1.3.].

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topos. Then the category $\mathcal{O}_{\mathcal{X}}$-mod of $\mathcal{O}_{\mathcal{X}}$-modules is an abelian category having enough injectives by [Sta13, Theorem 18.9.4.]. Let

$$(v_*, v^{-1}, \varphi, \theta) \colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

be a morphism of ringed topoi, let further M be an $\mathcal{O}_{\mathcal{X}}$-module and N be an $\mathcal{O}_{\mathcal{Y}}$-module. Then v_*M is a $v_*\mathcal{O}_{\mathcal{X}}$-module and we define the $\mathcal{O}_{\mathcal{Y}}$-module structure of v_*M by the map $\theta^{\mathrm{ad}} \colon \mathcal{O}_{\mathcal{Y}} \to v_*\mathcal{O}_{\mathcal{X}}$. On the other hand, $v^{-1}N$ is a $v^{-1}\mathcal{O}_{\mathcal{Y}}$-module which is not an $\mathcal{O}_{\mathcal{X}}$-module in general. Using $\theta \colon v^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ we define $v^*N = \mathcal{O}_{\mathcal{X}} \otimes_{v^{-1}\mathcal{O}_{\mathcal{Y}}} v^{-1}N$ which is an $\mathcal{O}_{\mathcal{X}}$-module. With these definitions we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(v^*N, M) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-mod}}(N, v_*M),$$

bifunctorial in $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ and $N \in \mathrm{ob}(\mathcal{O}_{\mathcal{Y}}\text{-mod})$ by [SGA41, Proposition IV.13.4].

1.2 Extensions of algebras

We will present the notion of extensions of algebras. Let $\mathcal{T}$ be a topos, let A and C be sheaves of rings of $\mathcal{T}$ and let $\varphi \colon A \to C$ be a morphism of sheaves of rings which will be fixed for the whole section. Usually, we just speak of a morphism of rings instead of a morphism of sheaves of rings.

Definition 1.6. An A-extension of C by I is an A-algebra E together with a surjective map of A-algebras $p \colon E \to C$ whose kernel I is an ideal of square zero. We denote such an extension by $\underline{E} \colon (0 \to I \to E \xrightarrow{p} C \to 0)$. Given another A-extension $\underline{F} \colon (0 \to I \to F \xrightarrow{q} C \to 0)$ of C by I, an isomorphism from $\underline{E}$ to $\underline{F}$ is given by an isomorphism of A-algebras $g \colon E \to F$ such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E & \xrightarrow{p} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & F & \xrightarrow{q} & C & \longrightarrow & 0
\end{array}
$$

is commutative. If $A = C$ and the A-algebra structure of C is given by the identity, we just speak of an extension of C by I.

Since $I^2 = 0$ we may give I a structure of C-module by setting $ci = ei$ for $c \in C$, $i \in I$ and $e \in E$ such that $p(e) = c$.

Let I be a C-module. Let $C\tilde{\oplus}I$ be the A-algebra whose underlying abelian group is $C \oplus I$, whose ring multiplication is given by

$$
(C\tilde{\oplus}I) \times (C\tilde{\oplus}I) \to C\tilde{\oplus}I \ , \quad \big((c_1, i_1), (c_2, i_2)\big) \mapsto (c_1 c_2, c_1 i_2 + c_2 i_1)
$$

and whose A-algebra structure is defined by $(\varphi|0) \colon A \to C\tilde{\oplus}I$. The sequence

$$
0 \to I \xrightarrow{(0|\mathrm{id})} C\tilde{\oplus}I \xrightarrow{\mathrm{pr}_1} C \to 0
$$

is an A-extension of C by I where pr_1 is the projection to the first direct summand.

The set of isomorphism classes of A-extensions of C by a C-module I will be denoted $\mathrm{Exal}_A(C, I)$ which by [Ill71, Chapitre III.1.1.5.1] is an abelian group whose zero element is the class of the above trivial extension defined by $C\tilde{\oplus}I$.

Now let $\underline{E} \colon (0 \to I \to E \xrightarrow{p} C \to 0)$ be an A-extension of C and $f \colon B \to C$ a morphism of A-algebras. Then there is an induced A-extension

$$
\underline{E} * f \colon \ (0 \to I \to E \times_C B \to B \to 0)
$$

of B by I and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E \times_C B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I & \longrightarrow & E & \xrightarrow{p} & C & \longrightarrow & 0.
\end{array}
$$

Proposition 1.7. [Ill71, Chapitre III.1.1.5.2] *Let $f \colon B \to C$ be a morphism of A-algebras. Then*

$$
\mathrm{Exal}_A(C, I) \to \mathrm{Exal}_A(B, I), \ \underline{E} \mapsto \underline{E} * f
$$

is a well-defined group homomorphism.

1.3 The cotangent complex and its main properties

Let $\mathfrak{A}$ be an abelian category. By a theorem of Dold-Kan (see for example [Wei94, Dold-Kan Theorem 8.4.1]), the category of simplicial objects in $\mathfrak{A}$ is equivalent to the category of chain complexes $C.$ in $\mathfrak{A}$ with $C_n = 0$ for $n < 0$. This category is equivalent to the category of cochain complexes $C^{\cdot}$ in $\mathfrak{A}$ with $C^n = 0$ for $n > 0$.

Let $\mathcal{T}$ be a topos and let $\psi\colon A \to B$ be a morphism of sheaves of rings of $\mathcal{T}$. Then we may associate the cotangent complex $L_{\psi} = L_{B/A}$ of B over A which is a bounded above cochain complex

$$L_{B/A}\colon \quad \cdots \to (L_{B/A})^{-n-1} \to (L_{B/A})^{-n} \to (L_{B/A})^{-n+1} \to \ \ldots \ \to (L_{B/A})^{-1} \to (L_{B/A})^0$$

consisting of free B-modules $(L_{B/A})^{-n}$. Its definition can be found in [Ill71, Chapitre II.1.2.3.1] where the cotangent complex is introduced as a simplicial object in the category of B-modules. Using the above equivalence of categories, we will always think of it as a cochain complex.

Let $v\colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of ringed topoi with ring morphism $\theta\colon v^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$. Then we define the cotangent complex of v to be $L_{\mathcal{X}/\mathcal{Y}} = L_\theta = L_{\mathcal{O}_{\mathcal{X}}/v^{-1}\mathcal{O}_{\mathcal{Y}}}$.

If $\mathbf{Ch}(\mathfrak{A})$ is the category of cochain complexes of objects of $\mathfrak{A}$, if $L, M \in \mathrm{ob}(\mathbf{Ch}(\mathfrak{A}))$ and $i \in \mathbb{Z}$ and if $\mathbf{D}(\mathfrak{A})$ is the derived category of $\mathfrak{A}$, we define

$$\mathbb{E}\mathrm{xt}^i_{\mathfrak{A}}(L, M) = \mathbb{H}\mathrm{om}_{\mathfrak{A}}(L, M[i]) = \mathrm{Hom}_{\mathbf{D}(\mathfrak{A})}(L, M[i])$$

to be the morphisms in $\mathbf{D}(\mathfrak{A})$ from L to the complex $M[i]$ where $M[i]^n = M^{i+n}$ for all $n \in \mathbb{Z}$. If $\mathfrak{A}$ is the category B-mod of modules over a ring B, we will write $\mathbb{E}\mathrm{xt}^i_B(-, -)$ instead of $\mathbb{E}\mathrm{xt}^i_{B\text{-mod}}(-, -)$.

The cotangent complex has several relations to the theory of extensions and to deformation theory.

Theorem 1.8. *The cotangent complex $L_{B/A}$ possesses the following properties.*

  i) There is a natural morphism of complexes $L_{B/A} \to \Omega^1_{B/A}$ which is a quasi-isomorphism if B is the symmetric algebra of a flat A-module.

  ii) Given a commutative diagram

$$
\begin{array}{ccc}
B & \longrightarrow & B' \\
\uparrow & & \uparrow \\
A & \longrightarrow & A'
\end{array}
\tag{1.1}
$$

  of sheaves of rings of $\mathcal{T}$, there is a natural morphism of complexes of B-modules

$$L_{B/A} \to L_{B'/A'}$$

  and by adjunction a natural morphism of complexes of B'-modules

$$L_{B/A} \otimes_B B' \to L_{B'/A'}.$$

  If $f\colon \tilde{\mathcal{T}} \to \mathcal{T}$ is a morphism of topoi and $A \to B$ a morphism of sheaves of rings of $\mathcal{T}$, there is a natural isomorphism of complexes of $f^{-1}B$-modules

$$f^{-1}L_{B/A} \xrightarrow{\cong} L_{f^{-1}B/f^{-1}A},$$

functorial in $A \to B$ in the sense that if

$$
\begin{array}{ccc}
f^{-1}B & \longrightarrow & f^{-1}B' \\
\uparrow & & \uparrow \\
f^{-1}A & \longrightarrow & f^{-1}A'
\end{array}
$$

is the pullback of Square (1.1) under f, then

$$
\begin{array}{ccc}
f^{-1}L_{B/A} & \xrightarrow{\cong} & L_{f^{-1}B/f^{-1}A} \\
\downarrow & & \downarrow \\
f^{-1}L_{B'/A'} & \xrightarrow{\cong} & L_{f^{-1}B'/f^{-1}A'}
\end{array}
$$

is commutative.

iii) For each B-module M, there are canonical isomorphisms

$$
\operatorname{Der}_A(B, M) \to \mathbb{Ext}^0_B(L_{B/A}, M)
$$

and

$$
\operatorname{Exal}_A(B, M) \to \mathbb{Ext}^1_B(L_{B/A}, M)
$$

where $\operatorname{Der}_A(B, M)$ is the group of A-derivations of B to M.

iv) If $A \to B \to C$ are morphisms of sheaves of rings of $\mathcal{T}$, there is a distinguished triangle

$$
\begin{array}{c}
L_{C/B} \\
\swarrow \qquad \nwarrow \\
L_{B/A} \otimes_B C \longrightarrow L_{C/A}
\end{array}
$$

in the derived category $\mathbf{D}(C)$ of the category of C-modules.

v) If $A \to B \to C$ are morphisms of sheaves of rings of $\mathcal{T}$ and $F \in \operatorname{ob}(\mathbf{D}(C))$, then for each $n \in \mathbb{Z}$ there is a natural isomorphism of groups

$$
\mathbb{Ext}^n_C(L_{B/A} \otimes_B C, F) \to \mathbb{Ext}^n_B(L_{B/A}, F),
$$

functorial in F.

vi) Let

$$
\begin{array}{ccc}
B_1 & \longrightarrow & C \\
\uparrow & & \uparrow \\
A & \longrightarrow & B_2
\end{array}
$$

with $C = B_1 \otimes_A B_2$ be a cocartesian diagram of sheaves of rings of $\mathcal{T}$. Let furthermore $L_{B_j/A} \otimes_{B_j} C \to L_{C/A}$ be the natural morphisms of complexes of C-modules for $j = 1, 2$. Assume $\operatorname{Tor}^A_i(B_1, B_2) = 0$ for all $i > 0$. Then the sum morphism

$$
(L_{B_1/A} \otimes_{B_1} C) \oplus (L_{B_2/A} \otimes_{B_2} C) \to L_{C/A}
$$

is a quasi-isomorphism.

Proof. All proofs can be found in [Ill71].

   i) [Ill71, Proposition II.1.2.4.4.].

  ii) [Ill71, Equation II.1.2.3.2] and [Ill71, Equation II.1.2.3.5].

 iii) [Ill71, Corollaire II.1.2.4.3.] and [Ill71, Théorème III.1.2.3.].

 iv) [Ill71, Proposition II.2.1.2.].

  v) [Ill71, Proposition I.3.3.4.4.].

 vi) [Ill71, Corollaire II.2.2.3.].

$\square$

The following theorem will help us several times to calculate the groups $\mathbb{Ext}^i_B(L_{B/A}, M)$ if M is a complex of B-modules. Let $\mathfrak{A}$ be an abelian category with enough injectives and let $\mathbf{K}(\mathfrak{A})$ be the category whose objects are cochain complexes of objects of $\mathfrak{A}$ and whose morphisms are morphisms of complexes up to homotopy. Let $\mathbf{K}^+(\mathfrak{A})$ the full subcategory of $\mathbf{K}(\mathfrak{A})$ whose objects are bounded below complexes. If $L \in \mathrm{ob}(\mathbf{K}(\mathfrak{A}))$ and $M \in \mathrm{ob}(\mathbf{K}^+(\mathfrak{A}))$, then by [Wei94, Theorem 10.7.4] there are canonical isomorphisms

$$\mathbb{Ext}^n_{\mathfrak{A}}(L, M) \cong H^n\big(\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathfrak{A}}(L, M)\big)$$

for all $n \geq 0$. Thus in order to calculate the groups $\mathbb{Ext}^n_B(L_{B/A}, M)$, we may use the total right derived functor of $\mathrm{Hom}^{\cdot}_B(L_{B/A}, -)$ and the following theorem which is a weak version of [Wei94, Composition Theorem 10.8.2].

Theorem 1.9. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be abelian categories having enough injectives and suppose given morphisms of triangulated categories*

$$\mathbf{K}^+(\mathfrak{A}) \xrightarrow{\quad G \quad} \mathbf{K}^+(\mathfrak{B}).$$

with FG and F mapping into $\mathbf{K}(\mathfrak{C})$.

Then there is a natural transformation

$$\zeta \colon \mathbf{R}(FG) \to \mathbf{R}(F) \circ \mathbf{R}(G)$$

of functors from $\mathbf{D}^+(\mathfrak{A})$ to $\mathbf{D}(\mathfrak{C})$. Suppose furthermore that there is a triangulated subcategory $\mathbf{K}$ of $\mathbf{K}^+(\mathfrak{B})$ with the following properties:

  i) Each object of $\mathbf{K}^+(\mathfrak{B})$ admits a quasi-isomorphism to an object of $\mathbf{K}$.

 ii) Every exact complex L of $\mathbf{K}$ is F-acyclic, i.e., $H^i(F(L)) = 0$ for all i.

 iii) The full subcategory of $\mathbf{K}^+(\mathfrak{A})$ consisting of complexes of injectives is sent to $\mathbf{K}$ by G.

Then $\zeta \colon \mathbf{R}(FG) \to \mathbf{R}(F) \circ \mathbf{R}(G)$ is an isomorphism.

1.4 Deformations of ringed topoi and schemes

Suppose given morphisms

$$
\begin{array}{c}
\mathcal{X} \\
{\scriptstyle f} \downarrow \\
\mathcal{S} \xrightarrow{\ j\ } \mathcal{S}'
\end{array}
$$

of ringed topoi where f is flat, i.e., $\mathcal{O}_\mathcal{X}$ is a flat $f^{-1}\mathcal{O}_\mathcal{S}$-module via the ring morphism $f^{-1}\mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{X}$ of f, and j is a closed embedding induced by an extension

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_\mathcal{S} \to 0$$

of $\mathcal{O}_\mathcal{S}$ by an $\mathcal{O}_\mathcal{S}$-module $\mathcal{J}$.

Definition 1.10. Given the above situation, a deformation of $\mathcal{X}$ over $\mathcal{S}'$ is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ i\ } & \mathcal{X}' \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle f'} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
$$

of ringed topoi such that f' is flat, i is a closed embedding and the morphism $\mathcal{X} \to \mathcal{X}' \times_{\mathcal{S}'} \mathcal{S}$ induced by the above square is an isomorphism. Two deformations

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ i_1\ } & \mathcal{X}'_1 \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle f'_1} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ i_2\ } & \mathcal{X}'_2 \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle f'_2} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
$$

of $\mathcal{X}$ over $\mathcal{S}'$ are isomorphic if there is an isomorphism $g\colon \mathcal{X}'_1 \to \mathcal{X}'_2$ of ringed topoi such that

$$
\begin{array}{ccc}
 & \mathcal{X} & \\
{\scriptstyle i_1} \swarrow & & \searrow {\scriptstyle i_2} \\
\mathcal{X}'_1 & \xrightarrow{\ g\ } & \mathcal{X}'_2 \\
{\scriptstyle f'_1} \searrow & & \swarrow {\scriptstyle f'_2} \\
 & \mathcal{S}' &
\end{array}
$$

is commutative.

In general there is an obstruction for the existence of a deformation of $\mathcal{X}$ over $\mathcal{S}'$. Using the cotangent complex $L_{\mathcal{X}/\mathcal{S}}$, it is possible to give an obstruction group and to determine all possible isomorphism classes of deformations of $\mathcal{X}$ over $\mathcal{S}'$.

Theorem 1.11. [Ill71, Théorème III.2.1.7.] *Assume given the above situation.*

   i) There is an obstruction for the existence of a deformation of $\mathcal{X}$ over $\mathcal{S}'$ lying in

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, f^*\mathcal{J}).$$

   ii) If this obstruction vanishes, then the set of isomorphism classes of deformations of $\mathcal{X}$ over $\mathcal{S}'$ is a torsor under

$$\mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, f^*\mathcal{J}).$$

   iii) The automorphism group of any fixed deformation of $\mathcal{X}$ over $\mathcal{S}'$ is canonically isomorphic to

$$\mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, f^*\mathcal{J}).$$

Remark 1.12. Suppose that $X \xrightarrow{f} S$ is a morphism of schemes and that $\mathcal{J}$ is a quasi-coherent $\mathcal{O}_S$-module. If $\mathcal{X} \xrightarrow{f} \mathcal{S}$ is the associated morphism of ringed topoi as in Example 1.4, then by [Ill71, Chapitre III.2.1.9.] the ringed topos $\mathcal{S}'$ "is" a scheme, i.e., $\mathcal{S}'$ is the ringed topos associated to a scheme S' and $\mathcal{S} \xrightarrow{j} \mathcal{S}'$ is induced from a morphism of schemes $S \xrightarrow{j} S'$. Each deformation $\mathcal{X}'$ of $\mathcal{X}$ over $\mathcal{S}'$ comes from a scheme X' and the morphisms of ringed topoi

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ i\ } & \mathcal{X}' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
$$

are the morphisms associated to morphisms

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
S & \xrightarrow{\ j\ } & S'
\end{array}
$$

of schemes. In particular, if $\mathcal{J}$ is a quasi-coherent $\mathcal{O}_S$-module, then each deformation in the category of schemes is again a scheme and Theorem 1.11 still holds for schemes.

Let $\mathcal{T}$ be a topos and let $A \to B \xrightarrow{f} C$ be morphisms of sheaves of rings of $\mathcal{T}$. By Theorem 1.8.iv) these morphisms yield a distinguished triangle

$$
\begin{array}{ccc}
 & L_{C/B} & \\
\nearrow & & \nwarrow \\
L_{B/A} \otimes_B C & \longrightarrow & L_{C/A}
\end{array}
$$

in the derived category $\mathbf{D}(C)$. Let M be a C-module. Together with the canonical isomorphism $\mathbb{E}\mathrm{xt}^n_C(L_{B/A}\otimes_B C, M) \xrightarrow{\cong} \mathbb{E}\mathrm{xt}^n_B(L_{B/A}, M)$ in Theorem 1.8.v), the triangle induces an exact sequence

$$
\begin{aligned}
0 \to \ & \mathbb{E}\mathrm{xt}^0_C(L_{C/B}, M) \to \mathbb{E}\mathrm{xt}^0_C(L_{C/A}, M) \to \mathbb{E}\mathrm{xt}^0_B(L_{B/A}, M) \\
\to \ & \mathbb{E}\mathrm{xt}^1_C(L_{C/B}, M) \to \mathbb{E}\mathrm{xt}^1_C(L_{C/A}, M) \to \mathbb{E}\mathrm{xt}^1_B(L_{B/A}, M) \\
\to \ & \mathbb{E}\mathrm{xt}^2_C(L_{C/B}, M) \to \dots
\end{aligned}
$$

of abelian groups.

Proposition 1.13. [Ill71, Equation III.1.2.5.3] *Using the isomorphisms in Theorem 1.8.iii), the above exact sequence is given by*

$$0 \to \mathrm{Der}_B(C, M) \xrightarrow{\alpha} \mathrm{Der}_A(C, M) \xrightarrow{\beta} \mathrm{Der}_A(B, M)$$

$$\xrightarrow{\gamma} \mathrm{Exal}_B(C, M) \xrightarrow{\delta} \mathrm{Exal}_A(C, M) \xrightarrow{\varepsilon} \mathrm{Exal}_A(B, M)$$

$$\to \mathbb{Ext}^2_C(L_{C/B}, M) \to \dots$$

whose first five morphisms are defined as follows.

- α *considers a B-derivation from C to M as an A-derivation.*
- β *sends an A-derivation $C \to M$ to the A-derivation $B \to C \to M$.*
- γ *sends an A-derivation $u\colon B \to M$ to the isomorphism class of the B-extension*

$$0 \to M \xrightarrow{(0|\mathrm{id})} C \oplus M \xrightarrow{\mathrm{pr}_1} C \to 0$$

  of C by M whose B-algebra structure is given by $(f| - u)\colon B \to C \oplus M$. Here pr_1 is the projection to the first direct summand and the ring structure of $C \oplus M$ is the ring structure of the trivial C-algebra $C \tilde{\oplus} M$ as defined in Subsection 1.2.
- δ *considers a B-extension of C by M as an A-extension.*
- ε *is the group homomorphism*

$$\mathrm{Exal}_A(C, M) \to \mathrm{Exal}_A(B, M), \ \underline{E} \mapsto \underline{E} * f$$

  in Proposition 1.7.

1.5 Diagrams of ringed topoi and schemes

Now we come to the notion of diagrams of ringed topoi and their associated topoi.

Definition 1.14. Let $\mathfrak{C}$ be a finite category, i.e., the morphisms of $\mathfrak{C}$ and thereby the objects of $\mathfrak{C}$ are finite sets. Let further $\mathcal{S}$ be a ringed topos. A diagram of ringed topoi over $\mathcal{S}$ of type $\mathfrak{C}$ is a contravariant functor I from $\mathfrak{C}$ to the category $\mathfrak{RTop}/\mathcal{S}$ of ringed topoi over $\mathcal{S}$, denoted $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$.

Thus for any $c \in \mathrm{ob}(\mathfrak{C})$ there is a ringed topos $\mathcal{X}_c = I(c)$ over $\mathcal{S}$ and for any $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ there is an $\mathcal{S}$-morphism $f_\alpha = I(\alpha)\colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ of ringed topoi such that for any composition $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ in $\mathfrak{C}$ we have $f_{\beta\alpha} = f_\alpha \circ f_\beta\colon \mathcal{X}_{c_3} \to \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$.

Let $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ be a diagram. Following [Buc81, Chapitre II.2.1.5], we may associate a category $\mathcal{X}_I$ to I.

An object $\mathcal{F}$ of $\mathcal{X}_I$ consists of the following data.

i) For any $c \in \mathrm{ob}(\mathfrak{C})$ there is a sheaf $\mathcal{F}_c \in \mathrm{ob}(\mathcal{X}_c)$.

ii) For any $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ yielding the morphism $f_\alpha\colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ of ringed topoi, there is a morphism of sheaves $\mathcal{F}_\alpha\colon f_\alpha^{-1}\mathcal{F}_{c_1} \to \mathcal{F}_{c_2}$ such that for any composition $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ in $\mathfrak{C}$, the diagram

$$
\begin{array}{ccc}
f_{\beta\alpha}^{-1}\mathcal{F}_{c_1} = f_\beta^{-1}f_\alpha^{-1}\mathcal{F}_{c_1} & \xrightarrow{\;f_\beta^{-1}\mathcal{F}_\alpha\;} & f_\beta^{-1}\mathcal{F}_{c_2} \\
& \searrow{\scriptstyle \mathcal{F}_{\beta\alpha}} \qquad \swarrow{\scriptstyle \mathcal{F}_\beta} & \\
& \mathcal{F}_{c_3} &
\end{array}
\tag{1.2}
$$

is commutative.

A morphism $l \in \mathrm{Hom}_{\mathcal{X}_I}(\mathcal{F}, \mathcal{G})$ from an object $\mathcal{F}$ to an object $\mathcal{G}$ of $\mathcal{X}_I$ is given, for any $c \in \mathrm{ob}(\mathfrak{C})$, by a morphism of sheaves $l_c\colon \mathcal{F}_c \to \mathcal{G}_c$ of $\mathcal{X}_c$ such that for any $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, the diagram

$$
\begin{array}{ccc}
f_\alpha^{-1}\mathcal{F}_{c_1} & \xrightarrow{\;\mathcal{F}_\alpha\;} & \mathcal{F}_{c_2} \\
{\scriptstyle f_\alpha^{-1}l_{c_1}}\downarrow & & \downarrow{\scriptstyle l_{c_2}} \\
f_\alpha^{-1}\mathcal{G}_{c_1} & \xrightarrow{\;\mathcal{G}_\alpha\;} & \mathcal{G}_{c_2}
\end{array}
$$

is commutative.

Notice that giving a morphism of sheaves $\mathcal{F}_\alpha\colon f_\alpha^{-1}\mathcal{F}_{c_1} \to \mathcal{F}_{c_2}$ of $\mathcal{X}_{c_2}$ is equivalent to giving a morphism $\mathcal{F}_\alpha^{\mathrm{ad}}\colon \mathcal{F}_{c_1} \to f_{\alpha*}\mathcal{F}_{c_2}$ of $\mathcal{X}_{c_1}$ by the adjunction of f_α^{-1} and $f_{\alpha*}$, since f_α is a morphism of ringed topoi.

Now by [Ill72, Chapitre VI.5.2.,VI.5.3.] we have the following result.

Proposition 1.15. *For any diagram $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ the category $\mathcal{X}_I$ is a topos, called the associated topos to I.*

We make $\mathcal{X}_I$ into a ringed topos by taking the structure sheaf $\mathcal{O}_{\mathcal{X}_I}$ to be the collection of all structure sheaves $\mathcal{O}_{\mathcal{X}_c}$ for $c \in \mathrm{ob}(\mathfrak{C})$ together with the ring morphisms $\theta_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{c_1}} \to \mathcal{O}_{\mathcal{X}_{c_2}}$ belonging to the morphism of ringed topoi $f_\alpha\colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$. By abuse of notation we also call $(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}_I})$ a diagram.

14

Notation 1.16. For any diagram $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$, the ringed topoi $\mathcal{X}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ are called the levels of $\mathcal{X}_I$. If $l \in \mathrm{Hom}_{\mathcal{X}_I}(\mathcal{F}, \mathcal{G})$, then we say that l is given levelwise by the morphisms $l_c\colon \mathcal{F}_c \to \mathcal{G}_c$ for all $c \in \mathrm{ob}(\mathfrak{C})$.

By [Buc81, Chapitre II.2.1.6] the topos $\mathcal{X}_I$ is an $\mathcal{S}$-topos.

Lemma 1.17. *For any diagram $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$, there is a canonical morphism of ringed topoi $m\colon \mathcal{X}_I \to \mathcal{S}$.*

Let us describe the functors $m_*\colon \mathcal{X}_I \to \mathcal{S}$ and $m^{-1}\colon \mathcal{S} \to \mathcal{X}_I$ as well as the ring morphism $\theta\colon m^{-1}\mathcal{O}_\mathcal{S} \to \mathcal{O}_{\mathcal{X}_I}$:

Let $t_c\colon \mathcal{X}_c \to \mathcal{S}$ be the structure morphisms of the diagram for $c \in \mathrm{ob}(\mathcal{X}_I)$. If $\mathcal{F} \in \mathrm{ob}(\mathcal{X}_I)$, we define $m_*\mathcal{F} = \varprojlim t_{c*}\mathcal{F}_c$ to be the limit of the system consisting of the sheaves $t_{c*}\mathcal{F}_c \in \mathrm{ob}(\mathcal{S})$ and the sheaf morphisms $t_{c_1*}\mathcal{F}^{\mathrm{ad}}_\alpha\colon t_{c_1*}\mathcal{F}_{c_1} \to t_{c_1*}f_{\alpha*}\mathcal{F}_{c_2} = t_{c_2*}\mathcal{F}_{c_2}$ for $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$.

For $\mathcal{G} \in \mathrm{ob}(\mathcal{S})$ let $m^{-1}\mathcal{G} \in \mathrm{ob}(\mathcal{X}_I)$ be the collection of all $t_c^{-1}\mathcal{G} \in \mathrm{ob}(\mathcal{X}_c)$ with identity as sheaf morphisms.

The ring morphism $\theta\colon m^{-1}\mathcal{O}_\mathcal{S} \to \mathcal{O}_{\mathcal{X}_I}$ is given, for $c \in \mathrm{ob}(\mathfrak{C})$, by the ring morphisms $\theta_c\colon t_c^{-1}\mathcal{O}_\mathcal{S} \to \mathcal{O}_{\mathcal{X}_c}$ belonging to t_c. In particular, if each $t_c\colon \mathcal{X}_c \to \mathcal{S}$ is flat, it follows that $m\colon \mathcal{X}_I \to \mathcal{S}$ is flat as well.

Lemma 1.18. *Let $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ be a diagram and let $j\colon \mathcal{S} \to \mathcal{S}'$ be a closed embedding of ringed topoi induced by an extension*

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_\mathcal{S} \to 0$$

of $\mathcal{O}_\mathcal{S}$. Let $t_c\colon \mathcal{X}_c \to \mathcal{S}$ be flat for every $c \in \mathrm{ob}(\mathfrak{C})$. Then giving a deformation

$$
\begin{array}{ccc}
\mathcal{X}_I & \xrightarrow{\ i\ } & \mathcal{X}'_I \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
\tag{1.3}
$$

of $\mathcal{X}_I$ over $\mathcal{S}'$ is equivalent to giving, for each $c \in \mathrm{ob}(\mathfrak{C})$, a deformation

$$
\begin{array}{ccc}
\mathcal{X}_c & \xrightarrow{\ i_c\ } & \mathcal{X}'_c \\
{\scriptstyle t_c}\downarrow & & \downarrow{\scriptstyle t'_c} \\
\mathcal{S} & \xrightarrow{\ j\ } & \mathcal{S}'
\end{array}
\tag{1.4}
$$

of $\mathcal{X}_c$ over $\mathcal{S}'$ together with, for each $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$, a morphism of ringed topoi $f'_\alpha\colon \mathcal{X}'_{c_2} \to \mathcal{X}'_{c_1}$ such that

$$
\tag{1.5}
$$

is commutative.

Proof. Giving Diagram (1.3) where $\mathcal{X}'_I$ is a deformation of $\mathcal{X}_I$ is equivalent to giving a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & m^*\mathcal{J} & \longrightarrow & \mathcal{O}_{\mathcal{X}'_I} & \longrightarrow & \mathcal{O}_{\mathcal{X}_I} & \longrightarrow & 0 \\
& & \uparrow & & \theta'\uparrow & & \theta\uparrow & & \\
0 & \longrightarrow & m^{-1}\mathcal{J} & \longrightarrow & m^{-1}\mathcal{O}_{\mathcal{S}'} & \longrightarrow & m^{-1}\mathcal{O}_{\mathcal{S}} & \longrightarrow & 0
\end{array}
$$

whose rows are $m^{-1}\mathcal{O}_{\mathcal{S}}$-extensions, whose right and middle vertical morphisms are flat and whose square of rings is cocartesian (see [Ill71, Diagramme III.2.1.6.2]). By definition of the ringed topos $\mathcal{X}_I$ and the morphism $m\colon \mathcal{X}_I \to \mathcal{S}$, this is equivalent to giving, for each $c \in \mathrm{ob}(\mathfrak{C})$, a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & t_c^*\mathcal{J} & \longrightarrow & \mathcal{O}_{\mathcal{X}'_c} & \longrightarrow & \mathcal{O}_{\mathcal{X}_c} & \longrightarrow & 0 \\
& & \uparrow & & \theta'_c\uparrow & & \theta_c\uparrow & & \\
0 & \longrightarrow & t_c^{-1}\mathcal{J} & \longrightarrow & t_c^{-1}\mathcal{O}_{\mathcal{S}'} & \longrightarrow & t_c^{-1}\mathcal{O}_{\mathcal{S}} & \longrightarrow & 0
\end{array}
$$

whose rows are $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-extensions, whose right and middle vertical morphisms are flat and whose square of rings is cocartesian, together with, for each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, a morphism $\theta'_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}'_{c_1}} \to \mathcal{O}_{\mathcal{X}'_{c_2}}$ such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f_\alpha^{-1}t_{c_1}^*\mathcal{J} & \longrightarrow & f_\alpha^{-1}\mathcal{O}_{\mathcal{X}'_{c_1}} & \longrightarrow & f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{c_1}} & \longrightarrow & 0 \\
& & \downarrow\uparrow & & \theta'_\alpha\downarrow & & \theta_\alpha\downarrow & & \\
0 & \longrightarrow & t_{c_2}^*\mathcal{J} & \longrightarrow & \mathcal{O}_{\mathcal{X}'_{c_2}} & \longrightarrow & \mathcal{O}_{\mathcal{X}_{c_2}} & \longrightarrow & 0 \\
& & \uparrow & & \theta'_{c_2}\uparrow \; \theta'_{c_1} & & \theta_{c_2}\uparrow \; \theta_{c_1} & & \\
0 & \longrightarrow & t_c^{-1}\mathcal{J} & \longrightarrow & t_c^{-1}\mathcal{O}_{\mathcal{S}'} & \longrightarrow & t_c^{-1}\mathcal{O}_{\mathcal{S}} & \longrightarrow & 0
\end{array}
$$

is commutative. But giving these data is equivalent to giving Diagrams (1.4) and (1.5) where $\mathcal{X}'_c$ is a deformation of $\mathcal{X}_c$. $\qquad\square$

It follows that deformation theory of diagrams of ringed topoi is accessible to calculations using Theorem 1.11 where we have to take the cotangent complex $L_{\mathcal{X}_I/\mathcal{S}} = L_{\mathcal{O}_{\mathcal{X}_I}/m^{-1}\mathcal{O}_{\mathcal{S}}}$ of m_I. From Remark 1.12 we get the analogue result for schemes.

Corollary 1.19. *In the situation of the above lemma, assume that the diagram consists of schemes and morphisms of schemes over a fixed scheme S. Let furthermore $\mathcal{J}$ be a quasi-coherent $\mathcal{O}_S$-module. Then each deformation of the diagram over S' is a diagram consisting of schemes and morphisms of schemes over S'.*

2 Deformations of diagrams and subdiagrams of ringed topoi

We fix the following notation.

General assumption 2.1. Let $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ be a diagram with associated ringed topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in the sense of Proposition 1.15 where the index I is omitted for simplicity. Let further $\mathfrak{D}$ be a subcategory of $\mathfrak{C}$ and let $J\colon \mathfrak{D}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ be the diagram obtained by restricting I to $\mathfrak{D}^{\mathrm{op}}$. Let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be the ringed topos associated to J. By abuse of notation we call $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ the diagram and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ the subdiagram (with respect to I and J, respectively). For $c \in \mathrm{ob}(\mathfrak{C})$ let $t_c\colon \mathcal{X}_c \to \mathcal{S}$ be the structure morphisms with ring morphisms $\theta_c\colon t_c^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}_c}$. For $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ let $\theta_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{c_1}} \to \mathcal{O}_{\mathcal{X}_{c_2}}$ be the ring morphisms belonging to the morphism of ringed topoi $f_\alpha\colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$. Let $m_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ and $m_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{S}$ be the morphisms of ringed topoi by Lemma 1.17 with ring morphisms $\theta_{\mathcal{X}}\colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}}$ and $\theta_{\mathcal{Y}}\colon m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{Y}}$, respectively. Assume that all structure morphisms $t_c\colon \mathcal{X}_c \to \mathcal{S}$ are flat for $c \in \mathrm{ob}(\mathfrak{C})$. Let further

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_{\mathcal{S}} \to 0$$

be an extension of $\mathcal{O}_{\mathcal{S}}$ by an $\mathcal{O}_{\mathcal{S}}$-module $\mathcal{J}$.

We will answer the following two questions on deformation theory of diagrams of ringed topoi:

i) Given a deformation ξ of the subdiagram $\mathcal{Y}$, what is an obstruction group for the existence of a deformation of the diagram $\mathcal{X}$ inducing the given deformation ξ of $\mathcal{Y}$?

ii) If the obstruction in this group vanishes, how many different isomorphism classes of deformations of $\mathcal{X}$ inducing ξ are there?

For this purpose we construct a left adjoint to the forgetful functor in Subsection 2.1. This construction will be applied, amongst others, to the forgetful functor $\mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$ from the category of $\mathcal{O}_{\mathcal{X}}$-modules to the category of $\mathcal{O}_{\mathcal{Y}}$-modules in order to get a long exact sequence of abelian groups in Theorem 2.13 where we will read off the answers to the above two questions in Subsection 2.2. Since we do not know the cone of m in the obstruction group

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^*\mathcal{J})$$

in general, we will derive, under some mild conditions on the subdiagram, another long exact sequence in Theorem 2.20 in Subsection 2.3. The complex L_h occurring in the obstruction group

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m_{\mathcal{X}}^*\mathcal{J})$$

there has the advantage of being the cotangent complex of a ring morphism h as defined in Proposition 2.16. In particular, L_h consists of free $\mathcal{O}_{\mathcal{X}}$-modules in contrast to $\mathrm{Cone}(m)$ which will help us to determine the obstruction group more explicitly in Section 3.

Finally Subsection 2.4 deals with the relationship between a diagram $\mathcal{X}$, a subdiagram $\mathcal{Y}$ of $\mathcal{X}$ and a subdiagram $\mathcal{Z}$ of $\mathcal{Y}$. We will see in Proposition 2.22 that there is a braid containing the long exact sequences for the pairs $(\mathcal{X}, \mathcal{Y})$, $(\mathcal{X}, \mathcal{Z})$ and $(\mathcal{Y}, \mathcal{Z})$.

2.1 The construction of a left adjoint to the forgetful functor

Assume given the situation of General assumption 2.1. Let $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ yielding an $\mathcal{S}$-morphism of ringed topoi $f_\alpha \colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$. If A_{c_1} is a sheaf of $t_{c_1}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras, then applying f_α^{-1} to the given ring morphism

$$t_{c_1}^{-1}\mathcal{O}_{\mathcal{S}} \to A_{c_1}$$

and using $f_\alpha^{-1} t_{c_1}^{-1} = t_{c_2}^{-1}$ we get a ring morphism $t_{c_2}^{-1}\mathcal{O}_{\mathcal{S}} \to f_\alpha^{-1} A_{c_1}$, thus $f_\alpha^{-1} A_{c_1}$ is a sheaf of $t_{c_2}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras.

Similarly, if $\mathcal{R}$ is a sheaf of rings of $\mathcal{X}$ and if M_{c_1} is a module or a complex of modules over $\mathcal{R}_{c_1}$, then $f_\alpha^{-1} M_{c_1}$ is a module or a complex of modules over $f_\alpha^{-1}\mathcal{R}_{c_1}$, respectively. Using the given ring morphism $\mathcal{R}_\alpha \colon f_\alpha^{-1}\mathcal{R}_{c_1} \to \mathcal{R}_{c_2}$, we see that

$$f_\alpha^{-1} M_{c_1} \otimes_{f_\alpha^{-1}\mathcal{R}_{c_1}} \mathcal{R}_{c_2}$$

is a module or a complex of modules over $\mathcal{R}_{c_2}$. In the special case $\mathcal{R} = \mathcal{O}_{\mathcal{X}}$ we write $f_\alpha^* M_{c_1}$ for this module or complex of modules, respectively.

Let $\mathfrak{A}_{\mathcal{X}}$ be one of the following categories:

i) The category $\mathcal{X}$,

ii) the category of sheaves of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{X}$,

iii) the category of $\mathcal{R}$-modules for a given sheaf of rings $\mathcal{R}$ of $\mathcal{X}$ or

iv) the category of complexes of $\mathcal{R}$-modules for a given sheaf of rings $\mathcal{R}$ of $\mathcal{X}$.

In any of these four cases an object A of $\mathfrak{A}_{\mathcal{X}}$ is a collection of objects $A_c \in \mathrm{ob}(\mathfrak{A}_c)$ for $c \in \mathrm{ob}(\mathfrak{C})$ where $\mathfrak{A}_c$ is

i) the category $\mathcal{X}_c$,

ii) the category of sheaves of $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{X}_c$,

iii) the category of $\mathcal{R}_c$-modules for the given sheaf of rings $\mathcal{R}_c$ of $\mathcal{X}_c$ or

iv) the category of complexes of $\mathcal{R}_c$-modules for the given sheaf of rings $\mathcal{R}_c$ of $\mathcal{X}_c$, respectively,

together with, for each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, a morphism $A_\alpha \colon f_\alpha^{-1} A_{c_1} \to A_{c_2}$ in $\mathfrak{A}_{c_2}$ such that for any composition $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ in $\mathfrak{C}$, the triangle

$$f_{\beta\alpha}^{-1} A_{c_1} = f_\beta^{-1} f_\alpha^{-1} A_{c_1} \xrightarrow{f_\beta^{-1} A_\alpha} f_\beta^{-1} A_{c_2}$$
$$A_{\beta\alpha} \searrow \qquad \swarrow A_\beta$$
$$A_{c_3}$$

is commutative in $\mathfrak{A}_{c_3}$.

A morphism $l \in \mathrm{Hom}_{\mathfrak{A}_{\mathcal{X}}}(A, B)$ from an object A to an object B of $\mathfrak{A}_{\mathcal{X}}$ is given, for any $c \in \mathrm{ob}(\mathfrak{C})$, by a morphism $l_c \colon A_c \to B_c$ in $\mathfrak{A}_c$ such that for any $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, the diagram

$$
\begin{array}{ccc}
f_\alpha^{-1} A_{c_1} & \xrightarrow{A_\alpha} & A_{c_2} \\
{\scriptstyle f_\alpha^{-1} l_{c_1}} \downarrow & & \downarrow {\scriptstyle l_{c_2}} \\
f_\alpha^{-1} B_{c_1} & \xrightarrow{B_\alpha} & B_{c_2}
\end{array}
$$

is commutative in $\mathfrak{A}_{c_2}$.

If $\mathfrak{A}_{\mathcal{X}}$ is the category of $\mathcal{R}$-modules or the category of complexes of $\mathcal{R}$-modules, we must replace $f_\alpha^{-1}A_{c_1}$ by $f_\alpha^{-1}A_{c_1} \otimes_{f_\alpha^{-1}\mathcal{R}_{c_1}} \mathcal{R}_{c_2}$ and similarly for $f_\alpha^{-1}B_{c_1}$, $f_{\beta\alpha}^{-1}A_{c_1}$ and $f_\beta^{-1}A_{c_2}$. In order to treat the four cases simultaneously, we will keep on writing f^{-1} by abuse of notation.

The category $\mathfrak{A}_{\mathcal{Y}}$ is defined analogously, we restrict to those categories $\mathfrak{A}_c$ such that $c \in \mathrm{ob}(\mathfrak{D})$ and to all morphisms $\alpha \in \mathrm{Hom}_{\mathfrak{D}}(c_1, c_2)$. Let

$$u\colon \mathfrak{A}_{\mathcal{X}} \to \mathfrak{A}_{\mathcal{Y}}$$

be the forgetful functor which maps an object $A \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$ to the collection of all $A_c \in \mathrm{ob}(\mathfrak{A}_c)$ such that $c \in \mathrm{ob}(\mathfrak{D})$ together with all morphisms $A_\alpha\colon f_\alpha^{-1}A_{c_1} \to A_{c_2}$ in $\mathfrak{A}_{c_2}$ whenever $\alpha \in \mathrm{Hom}_{\mathfrak{D}}(c_1, c_2)$. The aim of this subsection is to construct a left adjoint

$$u^{-1}\colon \mathfrak{A}_{\mathcal{Y}} \to \mathfrak{A}_{\mathcal{X}}$$

of u. The construction depends on the index categories $\mathfrak{C}^{\mathrm{op}}$ and $\mathfrak{D}^{\mathrm{op}}$. For simplicity we omit the brackets and write uA for the image of $A \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$ under u.

Notice that if $\mathfrak{A}_{\mathcal{X}} = \mathcal{X}$ and $\mathfrak{A}_{\mathcal{Y}} = \mathcal{Y}$, then since u commutes with arbitrary limits, it has a left adjoint by [SGA41, Exposé IV.3.1.3.] which is unique up to unique isomorphism. Nevertheless, in any of the above four cases for the category $\mathfrak{A}_{\mathcal{X}}$, we will give an explicit construction of u^{-1} which we will use for all following considerations, in particular for the explicit calculations of the obstruction group.

Proposition 2.2. *The functor $u^{-1}\colon \mathfrak{A}_{\mathcal{Y}} \to \mathfrak{A}_{\mathcal{X}}$ in Proposition 2.11 is left adjoint to the forgetful functor $u\colon \mathfrak{A}_{\mathcal{X}} \to \mathfrak{A}_{\mathcal{Y}}$, i.e., for every $A \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$ and every $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$, there is a natural isomorphism of adjunction*

$$\mathrm{Hom}_{\mathfrak{A}_{\mathcal{X}}}(u^{-1}B, A) \cong \mathrm{Hom}_{\mathfrak{A}_{\mathcal{Y}}}(B, uA),$$

bifunctorial in B and A.

The rest of this subsection is devoted to construct the functor u^{-1} in Proposition 2.11 and to give the proof of the above proposition on page 26.

Definition 2.3. Let $c \in \mathrm{ob}(\mathfrak{C})$. We define the category $\underline{c}$ as follows.
   i) The objects of $\underline{c}$ are morphisms $\gamma\colon d \to c$ in $\mathfrak{C}$ such that $d \in \mathrm{ob}(\mathfrak{D})$.
   ii) The morphisms from an object $\gamma_1\colon d_1 \to c$ to an object $\gamma_2\colon d_2 \to c$ of $\underline{c}$ are those morphisms $\varphi\colon d_1 \to d_2$ in $\mathfrak{D}$ such that

is commutative in $\mathfrak{C}$.

Notice that $\underline{c}$ is a comma category (see for example [Mac71, Chapter II.6.]). If $i\colon \mathfrak{D} \to \mathfrak{C}$ is the inclusion functor, then $\underline{c}$ is the category $(i \downarrow c)$ of objects i-over c, but we will keep on writing $\underline{c}$ for this category.

Now fix $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$ and let $c \in \mathrm{ob}(\mathfrak{C})$. Then for any $(\gamma \colon d \to c) \in \mathrm{ob}(\underline{c})$ there is a morphism of ringed topoi $f_\gamma \colon \mathcal{X}_c \to \mathcal{X}_d$ and we get an object $f_\gamma^{-1} B_d$ of $\mathfrak{A}_c$. For every $\varphi \in \mathrm{Hom}_{\underline{c}}\big((\gamma_1 \colon d_1 \to c), (\gamma_2 \colon d_2 \to c)\big)$ we have a commutative diagram

$$
\begin{array}{ccc}
 & \mathcal{X}_c & \\
{\scriptstyle f_{\gamma_1}} \swarrow & & \searrow {\scriptstyle f_{\gamma_2}} \\
\mathcal{X}_{d_1} & \xleftarrow{\ f_\varphi\ } & \mathcal{X}_{d_2}
\end{array}
$$

of ringed topoi over $\mathcal{S}$. Applying $f_{\gamma_2}^{-1}$ to the given morphism $B_\varphi \colon f_\varphi^{-1} B_{d_1} \to B_{d_2}$ in $\mathfrak{A}_{d_2}$, we get a morphism

$$
f_{\gamma_2}^{-1} B_\varphi \colon f_{\gamma_1}^{-1} B_{d_1} \to f_{\gamma_2}^{-1} B_{d_2}
$$

in $\mathfrak{A}_c$. If $(\gamma_1 \colon d_1 \to c) \xrightarrow{\varphi} (\gamma_2 \colon d_2 \to c) \xrightarrow{\psi} (\gamma_3 \colon d_3 \to c)$ is a composition in $\underline{c}$, then

$$
\begin{array}{ccc}
f_{\gamma_1}^{-1} B_{d_1} & \xrightarrow{\ f_{\gamma_2}^{-1} B_\varphi\ } & f_{\gamma_2}^{-1} B_{d_2} \\
{\scriptstyle f_{\gamma_3}^{-1} B_{\psi\varphi}} \searrow & & \swarrow {\scriptstyle f_{\gamma_3}^{-1} B_\psi} \\
 & f_{\gamma_3}^{-1} B_{d_3} &
\end{array}
$$

is commutative in $\mathfrak{A}_c$ because

$$
\begin{array}{ccccc}
 & & \mathcal{X}_c & & \\
{\scriptstyle f_{\gamma_3}} \swarrow & {\scriptstyle f_{\gamma_2}}\downarrow & & \searrow {\scriptstyle f_{\gamma_1}} & \\
\mathcal{X}_{d_1} & \xleftarrow{\ f_\varphi\ } & \mathcal{X}_{d_2} & \xleftarrow{\ f_\psi\ } & \mathcal{X}_{d_3} \\
 & & \xleftarrow{\ f_{\psi\varphi}\ } & &
\end{array}
$$

is commutative by definition.

Definition 2.4. For $c \in \mathrm{ob}(\mathfrak{C})$ and $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$ we define

$$
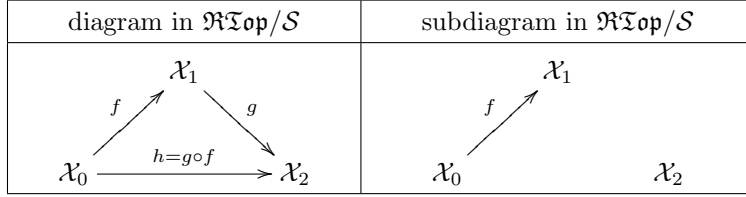(u^{-1}B)_c = \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d
$$

to be the colimit in $\mathfrak{A}_c$ of the system consisting of all $f_\gamma^{-1} B_d$ for $(\gamma \colon d \to c) \in \mathrm{ob}(\underline{c})$ together with the morphisms $f_{\gamma_2}^{-1} B_\varphi \colon f_{\gamma_1}^{-1} B_{d_1} \to f_{\gamma_2}^{-1} B_{d_2}$ for $\varphi \in \mathrm{Hom}_{\underline{c}}\big((\gamma_1 \colon d_1 \to c), (\gamma_2 \colon d_2 \to c)\big)$.

Remark 2.5. Notice that in each of the four cases considered, the colimit always exists: In the category $\mathcal{X}_c$ each finite colimit exists by [SGA41, Théorème II.4.1.].

If A is an associative, commutative and unitary ring in the usual sense, then the category of associative, commutative and unitary A-algebras has all finite coproducts and all coequalizers by [Eis95, Proposition A6.7.], and since each finite colimit is the coequalizer of two morphisms between finite coproducts by [Eis95, Proposition A6.1.], it follows that the category of A-algebras has all finite colimits. Now if we consider the category of sheaves of $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras on the ringed topos $\mathcal{X}_c$, we may take the colimit of the respective system as a presheaf of $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras and its sheafification is the colimit in the category of sheaves of $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{X}_c$.

Finally in the category of $\mathcal{R}_c$-modules and in the category of complexes of $\mathcal{R}_c$-modules, each finite colimit exists by [Sta13, Lemma 17.14.2.]. Notice that if $\underline{c}$ is the empty category, then $(u^{-1}B)_c$ is the initial object of $\mathfrak{A}_c$, i.e., the empty set in the category $\mathcal{X}_c$, the algebra $t_c^{-1}\mathcal{O}_\mathcal{S}$ in the category of $t_c^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}_c$, the zero module in the category of $\mathcal{R}_c$-modules and the zero complex in the category of complexes of $\mathcal{R}_c$-modules.

Example 2.6. Let $\mathfrak{A}_\mathcal{X}$ be the category of $m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}$ and thus $\mathfrak{A}_\mathcal{Y}$ the category of $m_\mathcal{Y}^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{Y}$. Consider the example

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1$ with $f\colon \mathcal{X}_0 \to \mathcal{X}_1$, $g\colon \mathcal{X}_1 \to \mathcal{X}_2$, $h=g\circ f\colon \mathcal{X}_0 \to \mathcal{X}_2$	$\mathcal{X}_1$ with $f\colon \mathcal{X}_0 \to \mathcal{X}_1$, and $\mathcal{X}_0$, $\mathcal{X}_2$

with no other morphisms involved except for the identities and denote the objects of $\mathcal{Y}$ as $\left(\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2, f^{-1}\mathcal{F}_1 \to \mathcal{F}_0\right)$ where $\mathcal{F}_i$ is a sheaf of $\mathcal{X}_i$. Let

$$\mathcal{O}_\mathcal{Y} = \left(\mathcal{O}_{\mathcal{X}_0}, \mathcal{O}_{\mathcal{X}_1}, \mathcal{O}_{\mathcal{X}_2}, f^{-1}\mathcal{O}_{\mathcal{X}_1} \xrightarrow{\theta_f} \mathcal{O}_{\mathcal{X}_0}\right)$$

be the structure sheaf of $\mathcal{Y}$ which is an object of $\mathfrak{A}_\mathcal{Y}$ by the ring morphism $\theta_\mathcal{Y}\colon m_\mathcal{Y}^{-1}\mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{Y}$ given by

$$\left(t_0^{-1}\mathcal{O}_\mathcal{S}, t_1^{-1}\mathcal{O}_\mathcal{S}, t_2^{-1}\mathcal{O}_\mathcal{S}, f^{-1}t_1^{-1}\mathcal{O}_\mathcal{S} \xrightarrow{\mathrm{id}} t_0^{-1}\mathcal{O}_\mathcal{S}\right) \xrightarrow{(\theta_0,\theta_1,\theta_2)} \left(\mathcal{O}_{\mathcal{X}_0}, \mathcal{O}_{\mathcal{X}_1}, \mathcal{O}_{\mathcal{X}_2}, f^{-1}\mathcal{O}_{\mathcal{X}_1} \xrightarrow{\theta_f} \mathcal{O}_{\mathcal{X}_0}\right).$$

We will calculate the object $(u^{-1}\mathcal{O}_\mathcal{Y})_c$ of $\mathfrak{A}_c$, the category of $t_c^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}_c$, for each $c = 0,1,2$.

By definition $(u^{-1}\mathcal{O}_\mathcal{Y})_0$ is the colimit in the category of $t_0^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}_0$ of the system

$$
\begin{array}{cc}
f^{-1}\mathcal{O}_{\mathcal{X}_1} & h^{-1}\mathcal{O}_{\mathcal{X}_2}, \\
\Big\downarrow{\scriptstyle\theta_f} & \\
\mathcal{O}_{\mathcal{X}_0} &
\end{array}
$$

that is, $(u^{-1}\mathcal{O}_\mathcal{Y})_0 = h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_0}$. Similarly, $(u^{-1}\mathcal{O}_\mathcal{Y})_1$ is the colimit in the category of $t_1^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}_1$ of the system

$$g^{-1}\mathcal{O}_{\mathcal{X}_2} \qquad \mathcal{O}_{\mathcal{X}_1},$$

that is, $(u^{-1}\mathcal{O}_\mathcal{Y})_1 = g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_1}$. Finally, $(u^{-1}\mathcal{O}_\mathcal{Y})_2$ is the colimit in the category of $t_2^{-1}\mathcal{O}_\mathcal{S}$-algebras of $\mathcal{X}_2$ of the system consisting only of $\mathcal{O}_{\mathcal{X}_2}$, whence $(u^{-1}\mathcal{O}_\mathcal{Y})_2 = \mathcal{O}_{\mathcal{X}_2}$.

Let us proceed with the construction of u^{-1}. On each level $\mathcal{X}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$, we have defined an object $(u^{-1}B)_c$ of $\mathfrak{A}_c$. In order to get a well defined object $u^{-1}B$ of $\mathfrak{A}_\mathcal{X}$, we have to define, for each $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1,c_2)$, a morphism $(u^{-1}B)_\alpha\colon f_\alpha^{-1}(u^{-1}B)_{c_1} \to (u^{-1}B)_{c_2}$ in $\mathfrak{A}_{c_2}$ subject to the compatibility condition (1.2) on page 14.

So let $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$. For any $\big(\gamma \colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ let $p_1^\gamma \colon f_\gamma^{-1} B_d \to \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d$ be the natural morphisms. Then by the universal property of $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d$ there is a unique morphism

$$p(\alpha)\colon \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d \ \to \ f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d$$

in $\mathfrak{A}_{c_2}$ such that

$$
\begin{array}{ccc}
f_\alpha^{-1} f_\gamma^{-1} B_d & \xrightarrow{\ \ \mathrm{id}\ \ } & f_\alpha^{-1} f_\gamma^{-1} B_d \\
\downarrow & & \downarrow{\scriptstyle f_\alpha^{-1} p_1^\gamma} \\
\varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d & \xrightarrow{\ p(\alpha)\ } & f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d
\end{array}
$$

is commutative for every $\big(\gamma \colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ where the left vertical maps are the natural morphisms. Since f_α is a morphism of ringed topoi, we have that f_α^{-1} is left adjoint to $f_{\alpha *}$, hence f_α^{-1} commutes with arbitrary colimits by [Sta13, Lemma 4.23.3.]. It follows that $p(\alpha)$ is an isomorphism. Now notice that each $f_\alpha^{-1} f_\gamma^{-1} B_d$ is part of the system in $\mathfrak{A}_{c_2}$ defined by $\underline{c_2}$, so there are natural morphisms $f_\alpha^{-1} f_\gamma^{-1} B_d \to \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d$. By the universal property of $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d$ there is a unique morphism

$$q(\alpha)\colon \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d \ \to \ \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d$$

in $\mathfrak{A}_{c_2}$ such that

$$
\begin{array}{ccccc}
f_\alpha^{-1} f_\gamma^{-1} B_d & \xrightarrow{\ \ \mathrm{id}\ \ } & f_\alpha^{-1} f_\gamma^{-1} B_d & \xrightarrow{\ \ \mathrm{id}\ \ } & f_\alpha^{-1} f_\gamma^{-1} B_d \\
\downarrow{\scriptstyle f_\alpha^{-1} p_1^\gamma} & & \downarrow & & \downarrow \\
f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d & \xrightarrow{\ p(\alpha)^{-1}\ } & \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d & \xrightarrow{\ q(\alpha)\ } & \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d
\end{array}
\tag{2.1}
$$

commutes for every $\big(\gamma \colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ where the middle and right vertical maps are the natural morphisms.

Definition 2.7. For $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ and $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$ we define

$$(u^{-1} B)_\alpha \colon f_\alpha^{-1} (u^{-1} B)_{c_1} = f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d \to \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d = (u^{-1} B)_{c_2}$$

to be the composition of the two lower horizontal morphisms $q(\alpha) \circ p(\alpha)^{-1}$ in $\mathfrak{A}_{c_2}$.

Lemma 2.8. *Let $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$. The collection $u^{-1} B$ of all objects $(u^{-1} B)_c \in \mathrm{ob}(\mathfrak{A}_c)$ for $c \in \mathrm{ob}(\mathfrak{C})$ together with the morphisms $(u^{-1} B)_\alpha \colon f_\alpha^{-1} (u^{-1} B)_{c_1} \to (u^{-1} B)_{c_2}$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ is a well-defined object of $\mathfrak{A}_{\mathcal{X}}$.*

Proof. Let $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ be a composition in $\mathfrak{C}$. We have to show that

$$
\begin{array}{ccc}
f_{\beta\alpha}^{-1} (u^{-1} B)_{c_1} = f_\beta^{-1} f_\alpha^{-1} (u^{-1} B)_{c_1} & \xrightarrow{\ f_\beta^{-1} (u^{-1} B)_\alpha\ } & f_\beta^{-1} (u^{-1} B)_{c_2} \\
 & \searrow{\scriptstyle (u^{-1} B)_{\beta\alpha}} \qquad \swarrow{\scriptstyle (u^{-1} B)_\beta} & \\
 & (u^{-1} B)_{c_3} &
\end{array}
$$

is commutative in $\mathfrak{A}_{c_3}$. By definition of all the occurring morphisms, the above diagram is commutative if and only if

$$f_\beta^{-1} f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d \xrightarrow{f_\beta^{-1} p(\alpha)^{-1}} f_\beta^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d \xrightarrow{f_\beta^{-1} q(\alpha)} f_\beta^{-1} \varinjlim_{\delta\in\mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d$$

with the left vertical morphism $p(\beta\alpha)$, the right side giving $\varinjlim_{\delta\in\mathrm{ob}(\underline{c_2})} f_\beta^{-1} f_\delta^{-1} B_d$ via $p(\beta)^{-1}$ and $q(\beta)$,

$$\varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d \xrightarrow{q(\beta\alpha)} \varinjlim_{\varepsilon\in\mathrm{ob}(\underline{c_3})} f_\varepsilon^{-1} B_d$$

is commutative. For any $\big(\gamma\colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ let

$$f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d \to \varinjlim_{\varepsilon\in\mathrm{ob}(\underline{c_3})} f_\varepsilon^{-1} B_d \qquad \text{and} \qquad f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d \to \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d$$

be the natural morphisms. Now by construction both of the above morphisms in $\mathfrak{A}_{c_3}$ from $\varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d$ to $\varinjlim_{\varepsilon\in\mathrm{ob}(\underline{c_3})} f_\varepsilon^{-1} B_d$ form a commutative triangle

$$
\begin{array}{ccc}
 & f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d & \\
 \swarrow & & \searrow \\
\varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d & \longrightarrow & \varinjlim_{\varepsilon\in\mathrm{ob}(\underline{c_3})} f_\varepsilon^{-1} B_d
\end{array}
$$

for every $\big(\gamma\colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$, thus they are equal by the universal property of the colimit

$$\varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\beta^{-1} f_\alpha^{-1} f_\gamma^{-1} B_d.$$

$\square$

Example 2.9. Let us continue Example 2.6 with the diagrams

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1$ $f\nearrow \quad \searrow g$ $\mathcal{X}_0 \xrightarrow{h=g\circ f} \mathcal{X}_2$	$\mathcal{X}_1$ $f\nearrow$ $\mathcal{X}_0 \qquad \mathcal{X}_2$

with no other morphisms involved except for the identities. We have already seen that

$$(u^{-1}\mathcal{O}_\mathcal{Y})_0 = h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_0}\ , \quad (u^{-1}\mathcal{O}_\mathcal{Y})_1 = g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_1} \quad \text{and} \quad (u^{-1}\mathcal{O}_\mathcal{Y})_2 = \mathcal{O}_{\mathcal{X}_2}.$$

Remember that $f^{-1}t_1^{-1}\mathcal{O}_\mathcal{S} = t_0^{-1}\mathcal{O}_\mathcal{S}$, $g^{-1}t_2^{-1}\mathcal{O}_\mathcal{S} = t_1^{-1}\mathcal{O}_\mathcal{S}$ and $h^{-1}t_2^{-1}\mathcal{O}_\mathcal{S} = t_0^{-1}\mathcal{O}_\mathcal{S}$ by definition of $m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S}$ in Lemma 1.17. The morphism $f^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_1 \to (u^{-1}\mathcal{O}_\mathcal{Y})_0$ is given by

$$f^{-1}\big(g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_1}\big) \cong h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_\mathcal{S}} f^{-1}\mathcal{O}_{\mathcal{X}_1} \xrightarrow{\mathrm{id}\otimes\theta_f} h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_0},$$

the morphism $g^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_2 \to (u^{-1}\mathcal{O}_\mathcal{Y})_1$ is given by

$$g^{-1}\mathcal{O}_{\mathcal{X}_2} \xrightarrow{-\otimes 1} g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_1}$$

and the morphism $h^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_2 \to (u^{-1}\mathcal{O}_\mathcal{Y})_0$ is given by

$$h^{-1}\mathcal{O}_{\mathcal{X}_2} \xrightarrow{-\otimes 1} h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_0}.$$

We see that $h^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_2 \to (u^{-1}\mathcal{O}_\mathcal{Y})_0$ is equal to the composition

$$h^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_2 = f^{-1}g^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_2 \to f^{-1}(u^{-1}\mathcal{O}_\mathcal{Y})_1 \to (u^{-1}\mathcal{O}_\mathcal{Y})_0.$$

We come back to the construction of the functor u^{-1}. So far we have defined an association $B \mapsto u^{-1}B$ from $\mathrm{ob}(\mathfrak{A}_\mathcal{Y})$ to $\mathrm{ob}(\mathfrak{A}_\mathcal{X})$. Now fix a morphism $l\colon B \to \overline{B}$ in $\mathfrak{A}_\mathcal{Y}$. Let $c \in \mathrm{ob}(\mathfrak{C})$ and $(\gamma\colon d \to c) \in \mathrm{ob}(\underline{c})$. Then the given morphism

$$l_d\colon B_d \to \overline{B}_d$$

in $\mathfrak{A}_d$ defines a morphism $f_\gamma^{-1}l_d\colon f_\gamma^{-1}B_d \to f_\gamma^{-1}\overline{B}_d$ in $\mathfrak{A}_c$. If $\varphi \in \mathrm{Hom}_{\underline{c}}\big((\gamma_1\colon d_1 \to c), (\gamma_2\colon d_2 \to c)\big)$, then there is a commutative diagram

$$
\begin{array}{ccc}
f_\varphi^{-1}B_{d_1} & \xrightarrow{B_\varphi} & B_{d_2} \\
{\scriptstyle f_\varphi^{-1}l_{d_1}}\downarrow & & \downarrow{\scriptstyle l_{d_2}} \\
f_\varphi^{-1}\overline{B}_{d_1} & \xrightarrow{\overline{B}_\varphi} & \overline{B}_{d_2}
\end{array}
$$

in $\mathfrak{A}_{d_2}$ by definition of $l\colon B \to \overline{B}$. Applying $f_{\gamma_2}^{-1}$ we get a commutative diagram

$$
\begin{array}{ccc}
f_{\gamma_1}^{-1}B_{d_1} & \xrightarrow{f_{\gamma_2}^{-1}B_\varphi} & f_{\gamma_2}^{-1}B_{d_2} \\
{\scriptstyle f_{\gamma_1}^{-1}l_{d_1}}\downarrow & & \downarrow{\scriptstyle f_{\gamma_2}^{-1}l_{d_2}} \\
f_{\gamma_1}^{-1}\overline{B}_{d_1} & \xrightarrow{f_{\gamma_2}^{-1}\overline{B}_\varphi} & f_{\gamma_2}^{-1}\overline{B}_{d_2} \\
& \searrow \quad \swarrow & \\
& (u^{-1}\overline{B})_c &
\end{array}
$$

in $\mathfrak{A}_c$ where the diagonal morphisms are the natural ones. By the universal property of $(u^{-1}B)_c$ there is a unique morphism

$$(u^{-1}l)_c\colon (u^{-1}B)_c \to (u^{-1}\overline{B})_c$$

in $\mathfrak{A}_c$ such that

$$
\begin{array}{ccc}
f_\gamma^{-1}B_d & \xrightarrow{f_\gamma^{-1}l_d} & f_\gamma^{-1}\overline{B}_d \\
\downarrow & & \downarrow \\
\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1}B_d & \xrightarrow{(u^{-1}l)_c} & \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1}\overline{B}_d
\end{array}
\tag{2.2}
$$

commutes for every $(\gamma\colon d \to c) \in \mathrm{ob}(\underline{c})$ where the vertical morphisms are the natural ones.

Lemma 2.10. *Let $l\colon B \to \overline{B}$ be a morphism in $\mathfrak{A}_\mathcal{Y}$. Then the collection of all morphisms $(u^{-1}l)_c$ in $\mathfrak{A}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ defines a morphism $u^{-1}l\colon u^{-1}B \to u^{-1}\overline{B}$ in $\mathfrak{A}_\mathcal{X}$.*

Proof. If $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$ we have to show that

$$
\begin{array}{ccc}
f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d & \xrightarrow{(u^{-1}B)_\alpha} & \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} B_d \\
{\scriptstyle f_\alpha^{-1}(u^{-1}l)_{c_1}} \downarrow & & \downarrow {\scriptstyle (u^{-1}l)_{c_2}} \\
f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} \overline{B}_d & \xrightarrow{(u^{-1}\overline{B})_\alpha} & \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} \overline{B}_d
\end{array}
$$

is commutative in $\mathfrak{A}_{c_2}$. By definition both morphisms from $f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d$ to $\varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} \overline{B}_d$ fit into a commutative triangle

$$
\begin{array}{ccc}
 & f_\alpha^{-1} f_\gamma^{-1} B_d & \\
{\scriptstyle f_\alpha^{-1} p_1^\gamma} \swarrow & & \searrow \\
f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d & \longrightarrow & \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} \overline{B}_d
\end{array}
$$

for every $\big(\gamma\colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ where $p_1^\gamma\colon f_\gamma^{-1} B_d \to \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d$ is the natural morphism and $f_\alpha^{-1} f_\gamma^{-1} B_d \to \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} \overline{B}_d$ is the composition of

$$
f_\alpha^{-1} f_\gamma^{-1} l_d\colon f_\alpha^{-1} f_\gamma^{-1} B_d \to f_\alpha^{-1} f_\gamma^{-1} \overline{B}_d
$$

and the natural morphism $f_\alpha^{-1} f_\gamma^{-1} \overline{B}_d \to \varinjlim_{\delta \in \mathrm{ob}(\underline{c_2})} f_\delta^{-1} \overline{B}_d$. Thus by the universal property of $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\alpha^{-1} f_\gamma^{-1} B_d \cong f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{c_1})} f_\gamma^{-1} B_d$ they are equal. $\qquad\square$

Proposition 2.11. *There is a functor $u^{-1}\colon \mathfrak{A}_\mathcal{Y} \to \mathfrak{A}_\mathcal{X}$ mapping an object B of $\mathfrak{A}_\mathcal{Y}$ to the object $u^{-1}B$ of $\mathfrak{A}_\mathcal{X}$ as defined in Lemma 2.8 and mapping a morphism $l\colon B \to \overline{B}$ in $\mathfrak{A}_\mathcal{Y}$ to the morphism $u^{-1}l\colon u^{-1}B \to u^{-1}\overline{B}$ in $\mathfrak{A}_\mathcal{X}$ as defined in Lemma 2.10.*

Proof. If $l\colon B \to B$ is the identity in $\mathfrak{A}_\mathcal{Y}$, then $u^{-1}l\colon u^{-1}B \to u^{-1}B$ is the identity in $\mathfrak{A}_\mathcal{X}$. Moreover, if

$$
B \xrightarrow{l} \overline{B} \xrightarrow{k} \hat{B}
$$

is a composition in $\mathfrak{A}_\mathcal{Y}$, we get, for every fixed $c \in \mathrm{ob}(\mathfrak{C})$ and for every $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$, a diagram

$$
\begin{array}{ccccc}
f_\gamma^{-1} B_d & \xrightarrow{f_\gamma^{-1} l_d} & f_\gamma^{-1} \overline{B}_d & \xrightarrow{f_\gamma^{-1} k_d} & f_\gamma^{-1} \hat{B}_d \\
\downarrow & & \downarrow & & \downarrow \\
\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d & \xrightarrow{(u^{-1}l)_c} & \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} \overline{B}_d & \xrightarrow{(u^{-1}k)_c} & \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} \hat{B}_d
\end{array}
$$

with the curved arrow $(u^{-1}(kl))_c$ from $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d$ to $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} \hat{B}_d$

in $\mathfrak{A}_c$ where the vertical morphisms are the natural ones. By the universal property of $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d$, both morphisms from $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d$ to $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} \hat{B}_d$ are equal. Consequently, the composition is sent to

$$
u^{-1}(kl)\colon u^{-1}B \xrightarrow{u^{-1}l} u^{-1}\overline{B} \xrightarrow{u^{-1}k} u^{-1}\hat{B}
$$

and so $u^{-1}\colon \mathfrak{A}_\mathcal{Y} \to \mathfrak{A}_\mathcal{X}$ is indeed a functor. $\qquad\square$

Lemma 2.12. *For every object B of $\mathfrak{A}_{\mathcal{Y}}$, there is a natural morphism*

$$n^B \colon B \to uu^{-1}B$$

in $\mathfrak{A}_{\mathcal{Y}}$, functorial in B. Similarly, for every object A of $\mathfrak{A}_{\mathcal{X}}$, there is a natural morphism

$$m^A \colon u^{-1}uA \to A$$

in $\mathfrak{A}_{\mathcal{X}}$, functorial in A. The compositions

$$uA \xrightarrow{n^{uA}} uu^{-1}uA \xrightarrow{um^A} uA \qquad and \qquad u^{-1}B \xrightarrow{u^{-1}n^B} u^{-1}uu^{-1}B \xrightarrow{m^{u^{-1}B}} u^{-1}B \qquad (2.3)$$

are the identities for every object A of $\mathfrak{A}_{\mathcal{X}}$ and every object B of $\mathfrak{A}_{\mathcal{Y}}$, respectively. In particular, the functor u^{-1} is left adjoint to u as claimed in Proposition 2.2.

Proof. We will divide the proof in four steps. In the first two steps we define the morphisms n^B and m^A and in the last two steps we verify that the compositions

$$uA \xrightarrow{n^{uA}} uu^{-1}uA \xrightarrow{um^A} uA \qquad and \qquad u^{-1}B \xrightarrow{u^{-1}n^B} u^{-1}uu^{-1}B \xrightarrow{m^{u^{-1}B}} u^{-1}B$$

are the identities.

Step 1: Notice that for every $d \in \mathrm{ob}(\mathfrak{D})$, the identity $d \to d$ is an object of $\underline{d}$. Hence for every $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$ and $d \in \mathrm{ob}(\mathfrak{D})$ there is a natural morphism $n_d^B \colon B_d \to \varinjlim_{\gamma \in \mathrm{ob}(\underline{d})} f_\gamma^{-1} B_{d'}$ in $\mathfrak{A}_d$ where we write $\gamma \colon d' \to d$ in order to avoid confusion with the fixed d. If $\alpha \in \mathrm{Hom}_{\mathfrak{D}}(d_1, d_2)$, then we have a diagram

$$
\begin{array}{ccc}
f_\alpha^{-1} B_{d_1} & \xrightarrow{\quad B_\alpha \quad} & B_{d_2} \\
{\scriptstyle f_\alpha^{-1} n_{d_1}^B} \downarrow & \searrow & \downarrow {\scriptstyle n_{d_2}^B} \\
f_\alpha^{-1} \varinjlim_{\gamma \in \mathrm{ob}(\underline{d_1})} f_\gamma^{-1} B_{d'} & \xrightarrow{\ (u^{-1}B)_\alpha\ } & \varinjlim_{\delta \in \mathrm{ob}(\underline{d_2})} f_\delta^{-1} B_{d'}
\end{array}
$$

in $\mathfrak{A}_{d_2}$ whose diagonal morphism is the natural one. The right upper triangle is commutative by definition of $\varinjlim_{\delta \in \mathrm{ob}(\underline{d_2})} f_\delta^{-1} B_{d'}$ and the left lower triangle is commutative by definition of $(u^{-1}B)_\alpha$. Hence the square is commutative as well and the collection of all n_d^B for $d \in \mathrm{ob}(\mathfrak{D})$ defines a natural morphism

$$n^B \colon B \to uu^{-1}B$$

in $\mathfrak{A}_{\mathcal{Y}}$. The commutativity of Diagram (2.2) on page 24 in the special case $\gamma = \mathrm{id}$ in the upper horizontal morphism shows that n^B is functorial in B.

Step 2: If $A \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$ and $c \in \mathrm{ob}(\mathfrak{C})$, then for every $\varphi \in \mathrm{Hom}_{\underline{c}}\big((\gamma_1 \colon d_1 \to c), (\gamma_2 \colon d_2 \to c)\big)$ there is a commutative square

$$
\begin{array}{ccc}
f_{\gamma_1}^{-1} A_{d_1} = f_{\gamma_2}^{-1} f_\varphi^{-1} A_{d_1} & \xrightarrow{\ f_{\gamma_2}^{-1} A_\varphi\ } & f_{\gamma_2}^{-1} A_{d_2} \\
{\scriptstyle A_{\gamma_1}} \searrow & & \swarrow {\scriptstyle A_{\gamma_2}} \\
& A_c &
\end{array}
$$

in $\mathfrak{A}_c$. Consequently, by the universal property of $\varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} A_d$ there is a unique morphism $m_c^A\colon \varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} A_d \to A_c$ in $\mathfrak{A}_c$ such that

$$
\begin{array}{ccc}
f_\gamma^{-1} A_d & & \\
\downarrow & \searrow^{A_\gamma} & \\
\varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} A_d & \xrightarrow[m_c^A]{} & A_c
\end{array}
\tag{2.4}
$$

is commutative for every $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$. If $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, we have a diagram

$$
\begin{array}{ccc}
f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d & \xrightarrow{(u^{-1}uA)_\alpha} & \varinjlim_{\delta\in\mathrm{ob}(\underline{c_2})} f_\delta^{-1} A_d \\
{\scriptstyle f_\alpha^{-1} m_{c_1}^A}\downarrow & & \downarrow{\scriptstyle m_{c_2}^A} \\
f_\alpha^{-1} A_{c_1} & \xrightarrow{A_\alpha} & A_{c_2}
\end{array}
$$

in $\mathfrak{A}_{c_2}$ both of its morphisms from $f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d$ to A_{c_2} fit into a commutative square

$$
\begin{array}{ccc}
f_\alpha^{-1} f_\gamma^{-1} A_d & \xrightarrow{\mathrm{id}} & f_{\alpha\gamma}^{-1} A_d \\
{\scriptstyle f_\alpha^{-1} p_1^\gamma}\downarrow & & \downarrow{\scriptstyle A_{\alpha\gamma}} \\
f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d & \longrightarrow & A_{c_2}
\end{array}
$$

for every $\big(\gamma\colon d \to c_1\big) \in \mathrm{ob}(\underline{c_1})$ where $p_1^\gamma\colon f_\gamma^{-1} A_d \to \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d$ is the natural morphism. Hence both morphisms from $f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d$ to A_{c_2} are equal by the universal property of $f_\alpha^{-1} \varinjlim_{\gamma\in\mathrm{ob}(\underline{c_1})} f_\gamma^{-1} A_d$. Thus the collection of all m_c^A for $c \in \mathrm{ob}(\mathfrak{C})$ defines a natural morphism

$$
m^A\colon u^{-1}uA \to A
$$

in $\mathfrak{A}_{\mathcal{X}}$. If $l\colon A \to \overline{A}$ is a morphism in $\mathfrak{A}_{\mathcal{X}}$, then for every $c \in \mathrm{ob}(\mathfrak{C})$ the diagram

$$
\begin{array}{ccc}
\varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} A_d & \xrightarrow{(u^{-1}ul)_c} & \varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} \overline{A}_d \\
{\scriptstyle m_c^A}\downarrow & & \downarrow{\scriptstyle m_c^{\overline{A}}} \\
A_c & \xrightarrow{l_c} & \overline{A}_c
\end{array}
$$

commutes by the universal property of $\varinjlim_{\gamma\in\mathrm{ob}(\underline{c})} f_\gamma^{-1} A_d$. It follows that m^A is functorial in A.

Step 3: Given $A \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$, we may apply u to the morphism $m^A\colon u^{-1}uA \to A$ in $\mathfrak{A}_{\mathcal{X}}$ to get a morphism $um^A\colon uu^{-1}uA \to uA$ in $\mathfrak{A}_{\mathcal{Y}}$. The object $uA \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$ yields a morphism

$$
n^{uA}\colon uA \to uu^{-1}uA
$$

in $\mathfrak{A}_{\mathcal{Y}}$ and the composition

$$
uA \xrightarrow{n^{uA}} uu^{-1}uA \xrightarrow{um^A} uA
$$

is the identity since by definition it is levelwise given, for $d \in \mathrm{ob}(\mathfrak{D})$, by the composition

$$
A_d \xrightarrow{n_d^{uA}} \varinjlim_{\gamma\in\mathrm{ob}(\underline{d})} f_\gamma^{-1} A_{d'} \xrightarrow{m_d^A} A_d
$$

27

which is the identity as follows from Triangle (2.4) for $\gamma = \mathrm{id}\colon d \to d$.

Step 4: Similarly, given $B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{Y}})$, we may apply u^{-1} to the morphism $n^B\colon B \to uu^{-1}B$ in $\mathfrak{A}_{\mathcal{Y}}$ to get a morphism $u^{-1}B \xrightarrow{u^{-1}n^B} u^{-1}uu^{-1}B$ in $\mathfrak{A}_{\mathcal{X}}$. The object $u^{-1}B \in \mathrm{ob}(\mathfrak{A}_{\mathcal{X}})$ yields a morphism

$$m^{u^{-1}B}\colon u^{-1}uu^{-1}B \to u^{-1}B$$

in $\mathfrak{A}_{\mathcal{X}}$ and it remains to show that the composition

$$u^{-1}B \xrightarrow{u^{-1}n^B} u^{-1}uu^{-1}B \xrightarrow{m^{u^{-1}B}} u^{-1}B$$

is the identity. For $c \in \mathrm{ob}(\mathfrak{C})$ the composition is given by

$$(u^{-1}B)_c \xrightarrow{(u^{-1}n^B)_c} \varinjlim_{(\delta\colon d'\to c)\in\mathrm{ob}(\underline{c})} f_\delta^{-1}(u^{-1}B)_{d'} \xrightarrow{m_c^{u^{-1}B}} (u^{-1}B)_c. \qquad (2.5)$$

Now fix $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$. There is a diagram

$$
\begin{array}{ccccc}
f_\gamma^{-1}B_d & \xrightarrow{\ f_\gamma^{-1}n_d^B\ } & f_\gamma^{-1}(u^{-1}B)_d & \xrightarrow{\quad (u^{-1}B)_\gamma \quad} & (u^{-1}B)_c \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
(u^{-1}B)_c & \xrightarrow{(u^{-1}n^B)_c} & \varinjlim_{(\delta\colon d'\to c)\in\mathrm{ob}(\underline{c})} f_\delta^{-1}(u^{-1}B)_{d'} & \xrightarrow{\quad m_c^{u^{-1}B} \quad} & (u^{-1}B)_c
\end{array}
$$

all of whose three nameless morphisms are the respective natural ones. The left hand square is commutative by definition of n^B, the right hand square is commutative as follows from Triangle (2.4) for $A = u^{-1}B$ and the top part is commutative by definition of $(u^{-1}B)_\gamma$. Hence the identity on $(u^{-1}B)_c$ and the morphism in Equation (2.5) fit into a commutative diagram

$$
\begin{array}{ccc}
 & f_\gamma^{-1}B_d & \\
 \swarrow & & \searrow \\
(u^{-1}B)_c & \xrightarrow{\hspace{3cm}} & (u^{-1}B)_c
\end{array}
$$

for every $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$, so they are equal by the universal property of the colimit $(u^{-1}B)_c$.

Finally, it follows that u^{-1} is left adjoint to u because by [Mac71, Theorem IV.1.2.], it suffices to give the above adjunction morphisms such that the compositions in Equation (2.3) are the identities. $\qquad \square$

2.2 The long exact sequence associated to the diagram and its subdiagram

We keep the notations of General assumption 2.1. Let $v\colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$ be the forgetful functor from the category of $\mathcal{O}_{\mathcal{X}}$-modules to the category of $\mathcal{O}_{\mathcal{Y}}$-modules. By Proposition 2.2 there is a left adjoint functor

$$v^*\colon \mathcal{O}_{\mathcal{Y}}\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$$

of v and an isomorphism of adjunction

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(v^*L, M) \cong \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-mod}}(L, vM) \tag{2.6}$$

for $M \in \operatorname{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ and $L \in \operatorname{ob}(\mathcal{O}_{\mathcal{Y}}\text{-mod})$. If $L \in \operatorname{ob}(\mathcal{O}_{\mathcal{Y}}\text{-mod})$, then v^*L is given levelwise, for $c \in \operatorname{ob}(\mathfrak{C})$, by

$$(v^*L)_c = \varinjlim_{\gamma \in \operatorname{ob}(\underline{c})} \left(f_\gamma^{-1} L_d \otimes_{f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c} \right)$$

where the colimit is taken in the category $\mathcal{O}_{\mathcal{X}_c}\text{-mod}$. By abuse of notation we denote by v and v^* the functors between the categories $\mathbf{Ch}(\mathcal{X})$ and $\mathbf{Ch}(\mathcal{Y})$ of complexes of $\mathcal{O}_{\mathcal{X}}$-modules and $\mathcal{O}_{\mathcal{Y}}$-modules, respectively, as well. Remember that $vL_{\mathcal{X}/\mathcal{S}} = L_{\mathcal{Y}/\mathcal{S}}$ because by Corollary A.3 the complex $L_{\mathcal{X}/\mathcal{S}}$ is given levelwise by the collection of all complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules $L_{\mathcal{X}_c/\mathcal{S}}$ for $c \in \operatorname{ob}(\mathfrak{C})$ together with the natural morphisms $f_\alpha^* L_{\mathcal{X}_{c_1}/\mathcal{S}} \to L_{\mathcal{X}_{c_2}/\mathcal{S}}$ for $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(c_1, c_2)$.

Now we may state the long exact sequence for the diagram and its subdiagram.

Theorem 2.13. *Assume given the situation of General assumption 2.1. Let*

$$m\colon v^* v L_{\mathcal{X}/\mathcal{S}} = v^* L_{\mathcal{Y}/\mathcal{S}} \to L_{\mathcal{X}/\mathcal{S}}$$

be the adjunction morphism in Lemma 2.12 of the cotangent complex $L_{\mathcal{X}/\mathcal{S}}$ and let $\operatorname{Cone}(m)$ be the cone of m. Then the distinguished triangle

in the derived category $\mathbf{D}(\mathcal{X})$ yields a long exact sequence

$$0 \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(\operatorname{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}^0_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(\operatorname{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \xrightarrow{\sigma} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \xrightarrow{\tau} \mathbb{E}\mathrm{xt}^1_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\xrightarrow{\omega} \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\operatorname{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J})$$

$$\to \ldots$$

of abelian groups. The morphism τ is the forgetful morphism sending a deformation of $\mathcal{X}$ to the induced deformation of $\mathcal{Y}$. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\operatorname{Cone}(m), m_{\mathcal{X}}^* \mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ. If the obstruction $\omega(\xi)$ is zero, then the set of isomorphism classes of deformations of $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ is a torsor under the image of

$$\sigma \colon \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}).$$

In the next subsection we will see two important cases when the cone of m is isomorphic in $\mathbf{D}(\mathcal{X})$ to the cotangent complex of a certain ring morphism. This will help a lot to control the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m), m_{\mathcal{X}}^* \mathcal{J})$ better. The rest of this section deals with the proof of the above theorem on page 32.

The adjunction in Equation (2.6) gives rise to an isomorphism

$$\mathrm{Hom}^{\cdot}_{\mathcal{X}}(v^* L, M) \cong \mathrm{Hom}^{\cdot}_{\mathcal{Y}}(L, vM)$$

of complexes of abelian groups, functorial in $M \in \mathrm{ob}(\mathbf{Ch}(\mathcal{X}))$ and $L \in \mathrm{ob}(\mathbf{Ch}(\mathcal{Y}))$, where $\mathrm{Hom}^{\cdot}(-,-)$ is the complex as defined in [Har66, Chapter I.§6.]. Hence if $\mathfrak{Ab}$ is the category of abelian groups and if $\mathbf{K}^+(\mathcal{X})$ and $\mathbf{K}^+(\mathcal{Y})$ are the categories of bounded below complexes of $\mathcal{O}_{\mathcal{X}}$-modules and $\mathcal{O}_{\mathcal{Y}}$-modules, respectively, up to homotopy, then the diagram

is commutative up to natural isomorphism of functors for fixed $L \in \mathrm{ob}(\mathbf{Ch}(\mathcal{Y}))$.

Proposition 2.14. *Let $L \in \mathrm{ob}(\mathbf{Ch}^-(\mathcal{Y}))$ be a bounded above complex consisting of free $\mathcal{O}_{\mathcal{Y}}$-modules. Then there is a natural isomorphism of functors*

$$\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(v^* L, -) \cong \mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{Y}}(L, v(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathfrak{Ab})$. In particular, for each $M \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X}))$ and $i \in \mathbb{Z}$, there are natural isomorphisms of abelian groups

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{X}}(v^* L, M) \cong \mathbb{E}\mathrm{xt}^i_{\mathcal{Y}}(L, vM)$$

which are functorial in M.

Proof. By Theorem 1.9 there is a natural transformation of functors

$$\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(v^* L, -) \to \mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{Y}}(L, \mathbf{R}v(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathfrak{Ab})$. But since the functor v is exact, the natural transformation of functors $v(-) \to \mathbf{R}v(-)$ is an isomorphism. Hence there is a natural transformation of functors

$$\zeta \colon \mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(v^* L, -) \to \mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{Y}}(L, v(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathfrak{Ab})$. Now we take $\mathbf{K} = \mathbf{K}^+(\mathcal{Y})$ in the notations of Theorem 1.9. In order to show that ζ is an isomorphism, it remains to show that each exact complex E of $\mathbf{K}^+(\mathcal{Y})$ is $\mathrm{Hom}_{\mathcal{Y}}^{\textbf{.}}(L, -)$-acyclic, i.e., $H^i\big(\mathrm{Hom}_{\mathcal{Y}}^{\textbf{.}}(L, E)\big) = 0$ for all i. But

$$H^i\big(\mathrm{Hom}_{\mathcal{Y}}^{\textbf{.}}(L, E)\big) = \mathrm{Hom}_{\mathbf{K}(\mathcal{Y})}(L, E[i])$$

by definition of $\mathrm{Hom}_{\mathcal{Y}}^{\textbf{.}}(L, E)$ (see for example [Wei94, Chapter 10.7]) and $\mathrm{Hom}_{\mathbf{K}(\mathcal{Y})}(L, E[i])$ is zero because each morphism of complexes from a bounded above complex of free modules to an exact complex is homotopic to the zero morphism.

Now if $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$, we get a natural isomorphism

$$\mathbb{Ext}_{\mathcal{X}}^i(v^*L, M) \cong \mathbb{Ext}_{\mathcal{Y}}^i(L, vM)$$

of abelian groups by taking the i-th cohomology of $\mathbf{R}\mathrm{Hom}_{\mathcal{X}}^{\textbf{.}}(v^*L, M) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_{\mathcal{Y}}^{\textbf{.}}(L, vM)$ by [Wei94, Theorem 10.7.4]. $\qquad\square$

Notice that the above proposition holds in particular if $L = L_{\mathcal{Y}/\mathcal{S}}$ is the cotangent complex of the subdiagram $\mathcal{Y}$ over $\mathcal{S}$.

If $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$, then applying the functor $\mathbb{Ext}_{\mathcal{X}}^i(-, M)$ to the morphism of complexes of $\mathcal{O}_{\mathcal{X}}$-modules

$$m \colon v^* v L_{\mathcal{X}/\mathcal{S}} = v^* L_{\mathcal{Y}/\mathcal{S}} \to L_{\mathcal{X}/\mathcal{S}}$$

in Theorem 2.13, we get a morphism of abelian groups

$$\tau_i \colon \mathbb{Ext}_{\mathcal{X}}^i(L_{\mathcal{X}/\mathcal{S}}, M) \to \mathbb{Ext}_{\mathcal{X}}^i(v^* L_{\mathcal{Y}/\mathcal{S}}, M) \xrightarrow{\cong} \mathbb{Ext}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, vM)$$

where the last isomorphism is the one of the above proposition.

Lemma 2.15. *For each $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ and $i \in \mathbb{Z}$ the morphism*

$$\tau_i \colon \mathbb{Ext}_{\mathcal{X}}^i(L_{\mathcal{X}/\mathcal{S}}, M) \to \mathbb{Ext}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, vM)$$

is the forgetful morphism, i.e., if $l \in \mathbb{Ext}_{\mathcal{X}}^i(L_{\mathcal{X}/\mathcal{S}}, M)$ is represented by

$$
\begin{array}{ccc}
L_{\mathcal{X}/\mathcal{S}} & & M[i] \\
& \searrow{\scriptstyle s} \quad \swarrow{\scriptstyle s'} & \\
& N &
\end{array}
$$

where N is a complex of $\mathcal{O}_{\mathcal{X}}$-modules, s is a morphism of complexes of $\mathcal{O}_{\mathcal{X}}$-modules and s' is a quasi-isomorphism, then $\tau_i(l) \in \mathbb{Ext}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, vM)$ is represented by

$$
\begin{array}{ccc}
L_{\mathcal{Y}/\mathcal{S}} = v L_{\mathcal{X}/\mathcal{S}} & & vM[i]. \\
& \searrow{\scriptstyle vs} \quad \swarrow{\scriptstyle vs'} & \\
& vN &
\end{array}
$$

Proof. Let $l \in \mathbb{Ext}^i_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, M)$ be represented by

$$L_{\mathcal{X}/\mathcal{S}} \qquad\qquad M[i].$$
$$\searrow{\scriptstyle s} \qquad \swarrow{\scriptstyle s'}$$
$$N$$

Then $\mathbb{Ext}^i_{\mathcal{X}}(m, M)(l) \in \mathbb{Ext}^i_{\mathcal{X}}(v^* L_{\mathcal{Y}/\mathcal{S}}, M)$ is represented by

$$v^* L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\ m\ } L_{\mathcal{X}/\mathcal{S}} \qquad\qquad M[i]$$
$$\searrow{\scriptstyle s} \qquad \swarrow{\scriptstyle s'}$$
$$N$$

and $\tau_i(l) \in \mathbb{Ext}^i_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, vM)$ is represented by

$$L_{\mathcal{Y}/\mathcal{S}} = vL_{\mathcal{X}/\mathcal{S}} \xrightarrow{\ n^{vL_{\mathcal{X}/\mathcal{S}}}\ } vv^* L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\ vm\ } vL_{\mathcal{X}/\mathcal{S}} = L_{\mathcal{Y}/\mathcal{S}} \qquad\qquad vM[i]$$
$$\searrow{\scriptstyle vs} \qquad \swarrow{\scriptstyle vs'}$$
$$vN$$

by definition of the adjunction between v and v^* in Lemma 2.12. But

$$L_{\mathcal{Y}/\mathcal{S}} = vL_{\mathcal{X}/\mathcal{S}} \xrightarrow{\ n^{vL_{\mathcal{X}/\mathcal{S}}}\ } vv^* L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\ vm\ } vL_{\mathcal{X}/\mathcal{S}} = L_{\mathcal{Y}/\mathcal{S}}$$

is the identity by Lemma 2.12. $\qquad\square$

Proof of Theorem 2.13. We apply the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(-, m^*_{\mathcal{X}}\mathcal{J})$ to the distinguished triangle

$$\mathrm{Cone}(m)$$
$$\swarrow \qquad\qquad \nwarrow$$
$$v^* L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\qquad m \qquad} L_{\mathcal{X}/\mathcal{S}}$$

in $\mathbf{D}(\mathcal{X})$ to get a long exact sequence

$$0 \to \mathbb{Ext}^0_{\mathcal{X}}(\mathrm{Cone}(m), m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^0_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^0_{\mathcal{X}}(v^* L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J})$$
$$\to \mathbb{Ext}^1_{\mathcal{X}}(\mathrm{Cone}(m), m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^1_{\mathcal{X}}(v^* L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J})$$
$$\to \mathbb{Ext}^2_{\mathcal{X}}(\mathrm{Cone}(m), m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{Ext}^2_{\mathcal{X}}(v^* L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J})$$
$$\to \dots$$

of abelian groups. We have $vm^*_{\mathcal{X}}\mathcal{J} = m^*_{\mathcal{Y}}\mathcal{J}$ since v is the forgetful functor. By Proposition 2.14 there are natural isomorphisms

$$\mathbb{Ext}^i_{\mathcal{X}}(v^* L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathbb{Ext}^i_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, vm^*_{\mathcal{X}}\mathcal{J}) = \mathbb{Ext}^i_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{Y}}\mathcal{J})$$

for each $i \in \mathbb{Z}$. Let

$$\tau' \colon \mathrm{Exal}_{m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{X}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathrm{Exal}_{m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{Y}}, m^*_{\mathcal{Y}}\mathcal{J})$$

be the group homomorphism which maps the isomorphism class of an $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-extension

$$0 \to m_{\mathcal{X}}^*\mathcal{J} \to \mathcal{A} \to \mathcal{O}_{\mathcal{X}} \to 0$$

by restricting to the isomorphism class of the $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-extension

$$0 \to m_{\mathcal{Y}}^*\mathcal{J} \to v\mathcal{A} \to \mathcal{O}_{\mathcal{Y}} \to 0.$$

Then using the canonical isomorphisms of abelian groups

$$\mathrm{Exal}_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J}) \xrightarrow{\cong} \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \quad \text{and} \quad \mathrm{Exal}_{m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{Y}}, m_{\mathcal{Y}}^*\mathcal{J}) \xrightarrow{\cong} \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^1(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J})$$

in Theorem 1.8, the diagram

$$
\begin{array}{ccc}
\mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) & \xrightarrow{\ \tau\ } & \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^1(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J}) \\
\cong \big\uparrow & & \big\uparrow \cong \\
\mathrm{Exal}_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J}) & \xrightarrow{\ \tau'\ } & \mathrm{Exal}_{m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{Y}}, m_{\mathcal{Y}}^*\mathcal{J})
\end{array}
$$

is commutative because the left vertical isomorphism restricts to the right vertical isomorphism as follows from its definition in [Ill71, Théorème III.1.2.3.]. Together with Lemma 2.15 this shows that τ maps the isomorphism class of a deformation of $\mathcal{X}$ to the isomorphism class of the deformation of $\mathcal{Y}$ induced by restricting. All deformation theoretic assertions follow from the exactness of the long exact sequence. $\qquad\square$

2.3 The simplification of the obstruction group

We keep the notations of General assumption 2.1. In the last subsection we have seen that $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m), m^*_{\mathcal{X}}\mathcal{J})$ is an obstruction group for extending a given deformation of the subdiagram to a deformation of the diagram. We will see in Theorem 2.20 that $\mathrm{Cone}(m)$ is isomorphic in $\mathbf{D}(\mathcal{X})$ to the cotangent complex L_h of a certain ring morphism defined in Proposition 2.16 if some additional weak assumption on the left adjoint of the forgetful functor is satisfied. Thus the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J})$ may be calculated more efficiently in a large variety of interesting cases as done in Section 3.

Let $u\colon m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}\text{-alg} \to m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}\text{-alg}$ be the forgetful functor from the category of $m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{X}$ to the category of $m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{Y}$. By Proposition 2.2 there is a left adjoint functor

$$u^{-1}\colon m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}\text{-alg} \to m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}\text{-alg}$$

of u which is given levelwise, for $c \in \mathrm{ob}(\mathfrak{C})$, by

$$(u^{-1}B)_c = \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f^{-1}_{\gamma}B_d$$

where $B \in \mathrm{ob}(m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}\text{-alg})$ and the colimit is taken in $t^{-1}_c\mathcal{O}_{\mathcal{S}}\text{-alg}$, the category of $t^{-1}_c\mathcal{O}_{\mathcal{S}}$-algebras.

Proposition 2.16. *With the above notations there is a natural factorization*

of $\theta_{\mathcal{X}}$ all of whose morphisms are ring morphisms in $\mathcal{X}$.

Proof. Taking $A = \mathcal{O}_{\mathcal{X}}$ in the adjunction morphism m^A in Lemma 2.12, we get a morphism

$$h\colon u^{-1}u\mathcal{O}_{\mathcal{X}} = u^{-1}\mathcal{O}_{\mathcal{Y}} \xrightarrow{m^{\mathcal{O}_{\mathcal{X}}}} \mathcal{O}_{\mathcal{X}}$$

and we take $l\colon m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}} \to u^{-1}\mathcal{O}_{\mathcal{Y}}$ to be the structure morphism of $u^{-1}\mathcal{O}_{\mathcal{Y}}$ as an $m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}$-algebra. $\square$

By definition the morphism $h\colon u^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is given levelwise, for $c \in \mathrm{ob}(\mathfrak{C})$, by the unique $t^{-1}_c\mathcal{O}_{\mathcal{S}}$-algebra morphism

$$h_c\colon \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f^{-1}_{\gamma}\mathcal{O}_{\mathcal{X}_d} \to \mathcal{O}_{\mathcal{X}_c}$$

such that

 (2.7)

is commutative in the category of $t^{-1}_c\mathcal{O}_{\mathcal{S}}$-algebras for every $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$.

Remark 2.17. The triangle in Proposition 2.16 can be described alternatively as follows. We have $u^{-1}m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$ by definition since $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$ is the collection of all $t_c^{-1}\mathcal{O}_{\mathcal{S}}$ for $c \in \mathrm{ob}(\mathfrak{D})$ together with identities as sheaf morphisms for each $\alpha \in \mathrm{Hom}_{\mathfrak{D}}(c_1, c_2)$. We may apply u^{-1} to the ring morphism $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} \xrightarrow{\theta_{\mathcal{Y}}} \mathcal{O}_{\mathcal{Y}} = u\mathcal{O}_{\mathcal{X}}$ to get a ring morphism

$$l = u^{-1}\theta_{\mathcal{Y}} \colon\ m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} = u^{-1}m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} \to u^{-1}u\mathcal{O}_{\mathcal{X}}$$

which is just the structure morphism of $u^{-1}u\mathcal{O}_{\mathcal{X}}$ as an $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebra. Hence the triangle in Proposition 2.16 is given by

$$
\begin{array}{ccc}
u^{-1}m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} & \xrightarrow{\ \theta_{\mathcal{X}}\ } & \mathcal{O}_{\mathcal{X}}. \\
{}_{l=u^{-1}\theta_{\mathcal{Y}}} \searrow & & \nearrow {}_{h=m^{\mathcal{O}_{\mathcal{X}}}} \\
& u^{-1}u\mathcal{O}_{\mathcal{X}} &
\end{array}
$$

By Lemma 2.12 we have a commutative diagram

$$
\begin{array}{ccc}
m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} & \xrightarrow{\ \theta_{\mathcal{Y}}\ } & \mathcal{O}_{\mathcal{Y}} \\
{}_{ul}\searrow\ \ {}_{\theta_{\mathcal{Y}}}\searrow & {}_{uh}\nearrow & \nearrow {}_{\mathrm{id}} \\
& uu^{-1}\mathcal{O}_{\mathcal{Y}} & \\
& \uparrow {}_{n^{\mathcal{O}_{\mathcal{Y}}}} & \\
& \mathcal{O}_{\mathcal{Y}} &
\end{array}
$$

(2.8)

of $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras.

Example 2.18. Let us continue Example 2.9 with the diagrams

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\begin{array}{ccc} & \mathcal{X}_1 & \\ {}^{f}\nearrow & & \searrow {}^{g} \\ \mathcal{X}_0 & \xrightarrow{h=g\circ f} & \mathcal{X}_2 \end{array}$	$\begin{array}{ccc} & \mathcal{X}_1 & \\ {}^{f}\nearrow & & \\ \mathcal{X}_0 & & \mathcal{X}_2 \end{array}$

with no other morphisms involved except for the identities. We have already seen that

$$(u^{-1}\mathcal{O}_{\mathcal{Y}})_0 = h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_{\mathcal{X}_0}, \quad (u^{-1}\mathcal{O}_{\mathcal{Y}})_1 = g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_{\mathcal{X}_1} \quad \text{and} \quad (u^{-1}\mathcal{O}_{\mathcal{Y}})_2 = \mathcal{O}_{\mathcal{X}_2}.$$

In this example $h \colon u^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is given levelwise as follows. $h_0 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_0 \to \mathcal{O}_{\mathcal{X}_0}$ is the unique morphism such that

$$
\begin{array}{ccc}
& & \mathcal{O}_{\mathcal{X}_0} \\
& {}^{\theta_h}\nearrow\ \ {}^{h_0}\nearrow & \\
h^{-1}\mathcal{O}_{\mathcal{X}_2} \longrightarrow h^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_0^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_{\mathcal{X}_0} & & \\
\uparrow {}_{h^{-1}\theta_2} \qquad\qquad \uparrow & \searrow {}_{\mathrm{id}} & \\
t_0^{-1}\mathcal{O}_{\mathcal{S}} \xrightarrow{\ \theta_0\ } \mathcal{O}_{\mathcal{X}_0} & &
\end{array}
$$

is commutative in $\mathcal{X}_0$. Similarly, $h_1\colon (u^{-1}\mathcal{O}_\mathcal{Y})_1 \to \mathcal{O}_{\mathcal{X}_1}$ is the unique morphism such that

$$
\begin{array}{ccc}
 & & \mathcal{O}_{\mathcal{X}_1} \\
 & \theta_g \nearrow & \uparrow h_1 \\
g^{-1}\mathcal{O}_{\mathcal{X}_2} \longrightarrow & g^{-1}\mathcal{O}_{\mathcal{X}_2} \otimes_{t_1^{-1}\mathcal{O}_\mathcal{S}} \mathcal{O}_{\mathcal{X}_1} & \\
\uparrow g^{-1}\theta_2 & \uparrow & \mathrm{id} \\
t_1^{-1}\mathcal{O}_\mathcal{S} \xrightarrow{\ \theta_1\ } & \mathcal{O}_{\mathcal{X}_1} &
\end{array}
$$

is commutative in $\mathcal{X}_1$. Finally, $h_2\colon (u^{-1}\mathcal{O}_\mathcal{Y})_2 \to \mathcal{O}_{\mathcal{X}_2}$ is the identity $\mathcal{O}_{\mathcal{X}_2} \to \mathcal{O}_{\mathcal{X}_2}$.

We come back to the general case. The composition of rings $\theta_\mathcal{X}\colon m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S} \xrightarrow{l} u^{-1}\mathcal{O}_\mathcal{Y} \xrightarrow{h} \mathcal{O}_\mathcal{X}$ yields a distinguished triangle

$$
\begin{array}{ccc}
 & L_h & \\
 \swarrow & & \nwarrow \\
L_l \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} & \longrightarrow & L_{\mathcal{X}/\mathcal{S}}
\end{array}
\qquad (2.9)
$$

in $\mathbf{D}(\mathcal{X})$ by Theorem 1.8.iv).

After stating the following lemma, we will formulate the main result of this subsection.

Lemma 2.19. *There is a natural morphism of complexes of $\mathcal{O}_\mathcal{X}$-modules*

$$
a\colon v^* L_{\mathcal{Y}/\mathcal{S}} \to L_l \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}
$$

such that

$$
\begin{array}{ccc}
v^* L_{\mathcal{Y}/\mathcal{S}} & \xrightarrow{\ m\ } & L_{\mathcal{X}/\mathcal{S}} \\
a \downarrow & & \| \\
L_l \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} & \longrightarrow & L_{\mathcal{X}/\mathcal{S}}
\end{array}
$$

is commutative in the category of complexes of $\mathcal{O}_\mathcal{X}$-modules.

Besides the forgetful functors $u\colon m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S}\text{-alg} \to m_\mathcal{Y}^{-1}\mathcal{O}_\mathcal{S}\text{-alg}$ and $v\colon \mathcal{O}_\mathcal{X}\text{-mod} \to \mathcal{O}_\mathcal{Y}\text{-mod}$, let $w\colon \mathcal{X} \to \mathcal{Y}$ be the forgetful functor from the ringed topos $\mathcal{X}$ to the ringed topos $\mathcal{Y}$.

Theorem 2.20. *Assume given the situation of General assumption 2.1. Let further*

$$
a\colon v^* L_{\mathcal{Y}/\mathcal{S}} \to L_l \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}
$$

be the natural morphism of complexes of $\mathcal{O}_\mathcal{X}$-modules in Lemma 2.19. Assume that one of the following two conditions holds:

  1 *The left inverse $w^{-1}\colon \mathcal{Y} \to \mathcal{X}$ of the forgetful functor $w\colon \mathcal{X} \to \mathcal{Y}$ commutes with finite limits (satisfied for example if the category $\underline{c}$ in Definition 2.3 is filtered for each $c \in \mathrm{ob}(\mathfrak{C})$).*

  2 *For each $c \in \mathrm{ob}(\mathfrak{C})$ we have $(u^{-1}\mathcal{O}_\mathcal{Y})_c = \bigotimes_{t_c^{-1}\mathcal{O}_\mathcal{S}} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ where the tensor product over $t_c^{-1}\mathcal{O}_\mathcal{S}$ is taken over a (possibly empty) set of objects $\big(\gamma\colon d \to c\big) \in \mathrm{ob}(\underline{c})$.*

Then a is a quasi-isomorphism.

If a is a quasi-isomorphism, then for each $i \in \mathbb{Z}$ and $M \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X}))$ it induces a natural isomorphism of abelian groups

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{X}}(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, M) \cong \mathbb{E}\mathrm{xt}^i_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, vM),$$

functorial in M. Furthermore, if a is a quasi-isomorphism, then the application of the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(-, m^*_{\mathcal{X}}\mathcal{J})$ to the distinguished triangle

yields a long exact sequence

$$0 \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{E}\mathrm{xt}^0_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{Y}}\mathcal{J})$$
$$\to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \xrightarrow{\sigma} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \xrightarrow{\tau} \mathbb{E}\mathrm{xt}^1_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{Y}}\mathcal{J})$$
$$\xrightarrow{\omega} \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{Y}}\mathcal{J})$$
$$\to \ldots$$

of abelian groups. The morphism τ is the forgetful morphism sending a deformation of $\mathcal{X}$ to the induced deformation of $\mathcal{Y}$. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ. If the obstruction $\omega(\xi)$ is zero, then the set of isomorphism classes of deformations of $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ is a torsor under the image of

$$\sigma \colon \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}).$$

Remark 2.21. Using the natural isomorphisms

$$\mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathrm{Exal}_{u^{-1}\mathcal{O}_{\mathcal{Y}}}(\mathcal{O}_{\mathcal{X}}, m^*_{\mathcal{X}}\mathcal{J}) \quad \text{and} \quad \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{S}}, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathrm{Exal}_{m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{X}}, m^*_{\mathcal{X}}\mathcal{J})$$

in Theorem 1.8.iii), we see from Proposition 1.13 that the image of σ may be described as the subgroup of $\mathrm{Exal}_{m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{X}}, m^*_{\mathcal{X}}\mathcal{J})$ consisting of all those $m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}$-extensions of $\mathcal{O}_{\mathcal{X}}$ by $m^*_{\mathcal{X}}\mathcal{J}$ which are already $u^{-1}\mathcal{O}_{\mathcal{Y}}$-extensions of $\mathcal{O}_{\mathcal{X}}$ by $m^*_{\mathcal{X}}\mathcal{J}$ under the ring morphism $l \colon m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}} \to u^{-1}\mathcal{O}_{\mathcal{Y}}$.

We use the rest of this subsection to give the proof of the above lemma and theorem.

Proof of Lemma 2.19. By Corollary A.3 we know that the complexes $L_{\mathcal{X}/\mathcal{S}}$ and $L_{\mathcal{Y}/\mathcal{S}}$ are given levelwise by the collection of all $L_{\mathcal{X}_c/S}$ and $L_{\mathcal{X}_d/S}$ for $c \in \mathrm{ob}(\mathfrak{C})$ and $d \in \mathrm{ob}(\mathfrak{D})$, respectively. Now fix $c \in \mathrm{ob}(\mathfrak{C})$. We have

$$(v^*L_{\mathcal{Y}/\mathcal{S}})_c = \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(f_\gamma^{-1} L_{\mathcal{X}_d/S} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c} \right),$$

the colimit being taken in the category of complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules. Again by Corollary A.3 we know that the complex of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules $(L_l)_c$ is given by

$$(L_l)_c = L_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c/t_c^{-1}\mathcal{O}_{\mathcal{S}}} = L_{l_c}.$$

Let $\varphi \in \operatorname{Hom}_{\underline{c}}\big((\gamma_1\colon d_1 \to c),(\gamma_2\colon d_2 \to c)\big)$. By definition of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$ we know that

is commutative where the two right hand side diagonal morphisms are the natural ones. Moreover, both of the above compositions from $t_c^{-1}\mathcal{O}_{\mathcal{S}}$ to $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$ are l_c. The above diagram gives rise to a commutative diagram

of morphisms of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules. Whence by the universal property of the colimit there is a unique morphism of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules

$$b_c\colon \varinjlim_{\gamma\in\operatorname{ob}(\underline{c})} \big(L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_c\big) \to L_{l_c} \qquad (2.10)$$

such that

is commutative for every $\big(\gamma\colon d \to c\big) \in \operatorname{ob}(\underline{c})$ where the colimit is taken in the category of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules and the vertical morphisms are the natural ones.

Considering the ring morphism $h_c\colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \to \mathcal{O}_{\mathcal{X}_c}$, the functor $-\otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}$ from the category of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules to the category of complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules is left adjoint to the functor which considers a complex of $\mathcal{O}_{\mathcal{X}_c}$-modules as a complex of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules via h_c. It follows that $-\otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}$ commutes with arbitrary colimits by [Sta13, Lemma 4.23.3.] and tensoring b_c with $\mathcal{O}_{\mathcal{X}_c}$ yields a morphism

$$\varinjlim_{\gamma\in\operatorname{ob}(\underline{c})} \big(L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}\big) \to L_{l_c} \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}.$$

By Diagram (2.7) on page 34 the composition $f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \to (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \xrightarrow{h_c} \mathcal{O}_{\mathcal{X}_c}$ is θ_γ for each $(\gamma\colon d \to c) \in \mathrm{ob}(\underline{c})$, hence the above morphism simplifies to

$$\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c}\right) \to L_{l_c} \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}.$$

Finally, by Theorem 1.8.ii) for each $(\gamma\colon d \to c) \in \mathrm{ob}(\underline{c})$ there is a natural isomorphism of complexes of $f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$-modules $f_\gamma^{-1}L_{\mathcal{X}_d/\mathcal{S}} \to L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}}$. Tensoring with $\mathcal{O}_{\mathcal{X}_c}$, we get isomorphisms of complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules $f_\gamma^{-1}L_{\mathcal{X}_d/\mathcal{S}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c} \to L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c}$ which are compatible with the morphisms defined by the index category $\underline{c}$. Consequently, for each $c \in \mathrm{ob}(\mathfrak{C})$ there is a natural morphism

$$a_c\colon \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(f_\gamma^{-1}L_{\mathcal{X}_d/\mathcal{S}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c}\right) \to L_{l_c} \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}$$

and the naturalness and functoriality of all steps used in their construction show that their collection defines a morphism of complexes of $\mathcal{O}_{\mathcal{X}}$-modules

$$a\colon v^*L_{\mathcal{Y}/\mathcal{S}} \to L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

From the universal property of the colimit $\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})}\left(f_\gamma^{-1}L_{\mathcal{X}_d/\mathcal{S}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c}\right)$ it follows that the square

$$
\begin{array}{ccc}
v^*L_{\mathcal{Y}/\mathcal{S}} & \xrightarrow{\ \ m\ \ } & L_{\mathcal{X}/\mathcal{S}} \\
{\scriptstyle a}\downarrow & & \Big\| \\
L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} & \longrightarrow & L_{\mathcal{X}/\mathcal{S}}
\end{array}
$$

is commutative. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2.20. If a is a quasi-isomorphism and thus an isomorphism in $\mathbf{D}(\mathcal{X})$, then it induces natural isomorphisms

$$\mathbb{Ext}_{\mathcal{X}}^i(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, M) \cong \mathbb{Ext}_{\mathcal{X}}^i(v^*L_{\mathcal{Y}/\mathcal{S}}, M) \cong \mathbb{Ext}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, vM)$$

of abelian groups, functorial in $M \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X}))$, where the second isomorphism is the one in Proposition 2.14. Applying the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(-, m_{\mathcal{X}}^*\mathcal{J})$ to the distinguished triangle (2.9) on page 36, we get a long exact sequence

$$0 \to \mathbb{Ext}_{\mathcal{X}}^0(L_h, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^0(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^0(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J})$$
$$\to \mathbb{Ext}_{\mathcal{X}}^1(L_h, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^1(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J})$$
$$\to \mathbb{Ext}_{\mathcal{X}}^2(L_h, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^2(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^2(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J})$$
$$\to \dots$$

of abelian groups. For each $i \in \mathbb{Z}$ the composition

$$\mathbb{Ext}_{\mathcal{X}}^i(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{Ext}_{\mathcal{X}}^i(L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, m_{\mathcal{X}}^*\mathcal{J}) \xrightarrow{\cong} \mathbb{Ext}_{\mathcal{X}}^i(v^*L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \xrightarrow{\cong} \mathbb{Ext}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J})$$

is the forgetful morphism as follows from Lemma 2.15 and the commutativity of the square

$$
\begin{array}{ccc}
v^* L_{\mathcal{Y}/\mathcal{S}} & \xrightarrow{\ m\ } & L_{\mathcal{X}/\mathcal{S}} \\
{\scriptstyle a}\big\downarrow & & \big\| \\
L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} & \longrightarrow & L_{\mathcal{X}/\mathcal{S}}.
\end{array}
$$

It remains to show that a is a quasi-isomorphism if condition **1** or **2** is satisfied.

Assume that condition **1** is satisfied. Then $w \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of topoi and since w^{-1} commutes with arbitrary colimits and with finite limits, it preserves, amongst others, algebra structures (see for example [SGA41, Exposé IV.3.1.2.]). This means that if B is a sheaf of $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{Y}$, then $w^{-1}B$ is a sheaf of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras of $\mathcal{X}$. It follows that we may identify the functor $u^{-1} \colon m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-alg $\to m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-alg with the restriction of w^{-1} to the subcategory $m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-alg of $\mathcal{Y}$ because both functors are left adjoint to the forgetful functor $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-alg $\to m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}$-alg. Taking $h \colon w^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ as ring morphism, we see that $w \colon \mathcal{X} \to \mathcal{Y}$ is even a morphism of ringed topoi.

Since w^{-1} preserves module structures we have that $w^{-1}M$ is a complex of $w^{-1}\mathcal{O}_{\mathcal{Y}}$-modules whenever M is a complex of $\mathcal{O}_{\mathcal{Y}}$-modules, hence $w^*M = w^{-1}M \otimes_{w^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$ is a complex of $\mathcal{O}_{\mathcal{X}}$-modules. Hence we may identify the functors w^* and v^* from $\mathbf{Ch}(\mathcal{Y})$ to $\mathbf{Ch}(\mathcal{X})$ since they are both left adjoint to the forgetful functor $\mathbf{Ch}(\mathcal{X}) \to \mathbf{Ch}(\mathcal{Y})$. Now since w is a morphism of ringed topoi we know from Theorem 1.8.ii) that the natural morphism of complexes of $w^{-1}\mathcal{O}_{\mathcal{Y}}$-modules

$$
w^{-1}L_{\mathcal{Y}/\mathcal{S}} = w^{-1}L_{\mathcal{O}_{\mathcal{Y}}/m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}} \to L_{w^{-1}\mathcal{O}_{\mathcal{Y}}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} = L_l
$$

is an isomorphism and tensoring with $\mathcal{O}_{\mathcal{X}}$ shows that

$$
a \colon w^*L_{\mathcal{Y}/\mathcal{S}} = w^{-1}L_{\mathcal{Y}/\mathcal{S}} \otimes_{w^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \to L_l \otimes_{w^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}
$$

is an isomorphism as well. Notice that there is a commutative diagram of ringed topoi

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ w\ } & \mathcal{Y} \\
{\scriptstyle m_{\mathcal{X}}}\searrow & & \swarrow{\scriptstyle m_{\mathcal{Y}}} \\
& \mathcal{S} &
\end{array}
$$

and the distinguished triangle of the ring composition in Proposition 2.16 is by definition the distinguished triangle

$$
\begin{array}{ccc}
& L_{\mathcal{X}/\mathcal{Y}} & \\
\nearrow & & \nwarrow \\
w^*L_{\mathcal{Y}/\mathcal{S}} & \longrightarrow & L_{\mathcal{X}/\mathcal{S}}
\end{array}
$$

of the above composition of morphisms of ringed topoi.

If the category $\underline{c}$ in Definition 2.3 is filtered for each $c \in \mathrm{ob}(\mathfrak{C})$, then w^{-1} commutes with finite limits because w^{-1} is given levelwise by taking the colimit over the categories $\underline{c}$ and because filtered colimits commute with finite limits by [Sta13, Lemma 4.18.2.].

Now assume that condition **2** is satisfied and let $c \in \mathrm{ob}(\mathfrak{C})$. Then $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c = \bigotimes_{\Gamma_c} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ is a coproduct in the category $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-alg where $\Gamma_c \subseteq \mathrm{ob}(\underline{c})$ is a possibly empty subset of $\mathrm{ob}(\underline{c})$ depending on c. Since the left adjoint $\mathfrak{A}_{\mathcal{Y}} \to \mathfrak{A}_{\mathcal{X}}$ of any of the forgetful functors $\mathfrak{A}_{\mathcal{X}} \to \mathfrak{A}_{\mathcal{Y}}$ at the beginning of Subsection 2.1 is defined levelwise by taking the colimit over the index category $\underline{c}$ for $c \in \mathrm{ob}(\mathfrak{C})$, it follows that it is given levelwise by taking the coproduct over Γ_c in the respective category. In particular, for every $c \in \mathrm{ob}(\mathfrak{C})$ the colimit

$$\varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \right),$$

taken in the category of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules, is a coproduct, i.e., a finite direct sum, and the morphism

$$b_c \colon \bigoplus_{\Gamma_c} \left(L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \right) \to L_{l_c}$$

in Equation (2.10) on page 38 is given by the sum morphism of the morphisms

$$L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \to L_{l_c}$$

for $(\gamma \colon d \to c) \in \Gamma_c$. But since all structure morphisms $t_c \colon \mathcal{X}_c \to \mathcal{S}$ for $c \in \mathrm{ob}(\mathfrak{C})$ are flat by General assumption 2.1 it follows that b_c is a quasi-isomorphism for each $c \in \mathrm{ob}(\mathfrak{C})$ by Proposition A.1. Since b_c is a quasi-isomorphism between complexes consisting of free $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$-modules, we may tensor with $\mathcal{O}_{\mathcal{X}_c}$ and still get a quasi-isomorphism by [Ill71, Lemme I.3.3.2.1.]. It follows that

$$a_c \colon \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(f_\gamma^{-1} L_{\mathcal{X}_d/\mathcal{S}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c} \right) \to L_{l_c} \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_c} \mathcal{O}_{\mathcal{X}_c}$$

is a quasi-isomorphism for every $c \in \mathrm{ob}(\mathfrak{C})$, hence a is a quasi-isomorphism. $\qquad \square$

2.4 Relations between a diagram, a subdiagram and a subsubdiagram

Under General assumption 2.1 let furthermore $\mathfrak{E}$ be a subcategory of $\mathfrak{D}$ and let $\mathcal{Z}$ be the ringed topos associated to the restriction of $J\colon \mathfrak{D}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ to $\mathfrak{E}^{\mathrm{op}}$. Let $m_{\mathcal{Z}}\colon \mathcal{Z} \to \mathcal{S}$ be the morphism of ringed topoi in Lemma 1.17. Thus we have the diagram $\mathcal{X}$ with index category $\mathfrak{C}^{\mathrm{op}}$, its subdiagram $\mathcal{Y}$ with index category $\mathfrak{D}^{\mathrm{op}}$ and its subsubdiagram $\mathcal{Z}$ with index category $\mathfrak{E}^{\mathrm{op}}$. We will derive a commutative braid for the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

Proposition 2.22. *Assume given the above situation. For distinction of the three different morphisms m in Theorem 2.13, let $m_{\mathcal{XY}}$ be the morphism for the pair $(\mathcal{X}, \mathcal{Y})$, let $m_{\mathcal{XZ}}$ be the morphism for the pair $(\mathcal{X}, \mathcal{Z})$ and $m_{\mathcal{YZ}}$ be the morphism for the pair $(\mathcal{Y}, \mathcal{Z})$.*

Then there is a commutative braid
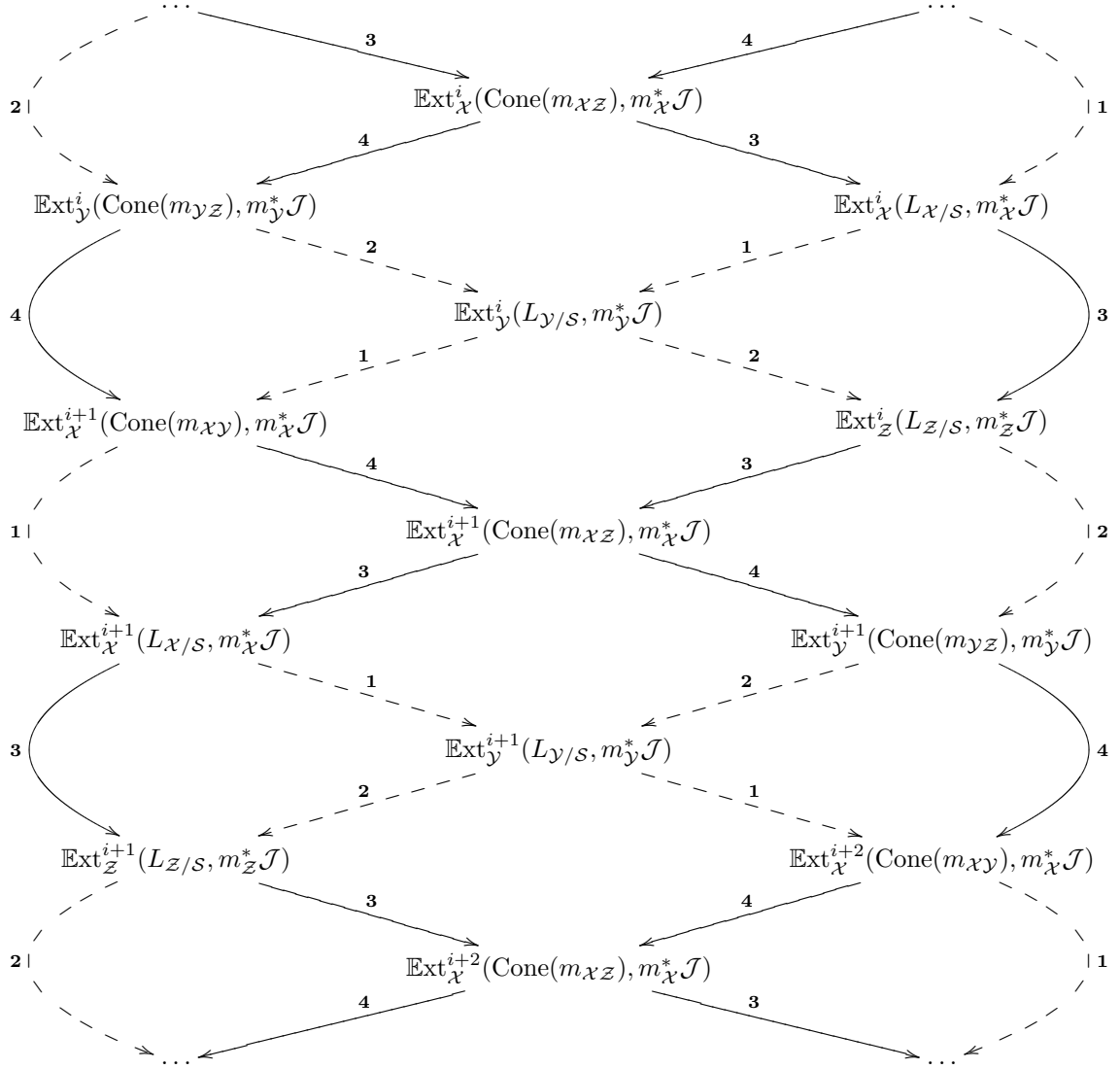
*containing four long exact sequences of abelian groups. Sequences **1**, **2** and **3** are the long exact sequences of the pairs $(\mathcal{X}, \mathcal{Y})$, $(\mathcal{Y}, \mathcal{Z})$ and $(\mathcal{X}, \mathcal{Z})$ in Theorem 2.13, respectively.*

Proof. For distinction let $v_{\mathcal{X}\mathcal{Y}}$ be the forgetful functor from $\mathbf{Ch}(\mathcal{X})$ to $\mathbf{Ch}(\mathcal{Y})$, let $v_{\mathcal{X}\mathcal{Z}}$ be the forgetful functor from $\mathbf{Ch}(\mathcal{X})$ to $\mathbf{Ch}(\mathcal{Z})$ and let $v_{\mathcal{Y}\mathcal{Z}}$ be the forgetful functor from $\mathbf{Ch}(\mathcal{Y})$ to $\mathbf{Ch}(\mathcal{Z})$. Then there are three distinguished triangles

$$v_{\mathcal{X}\mathcal{Y}}^{*}L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{m_{\mathcal{X}\mathcal{Y}}} L_{\mathcal{X}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} \mathrm{Cone}(m_{\mathcal{X}\mathcal{Y}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{X}\mathcal{Y}}^{*}L_{\mathcal{Y}/\mathcal{S}}[1],$$

$$v_{\mathcal{X}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{m_{\mathcal{X}\mathcal{Z}}} L_{\mathcal{X}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} \mathrm{Cone}(m_{\mathcal{X}\mathcal{Z}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{X}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}}[1],$$

$$v_{\mathcal{Y}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{m_{\mathcal{Y}\mathcal{Z}}} L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} \mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{Y}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}}[1],$$

the first and the second in $\mathbf{K}(\mathcal{X})$ and the third in $\mathbf{K}(\mathcal{Y})$, all of whose displayed morphisms are in $\mathbf{Ch}(\mathcal{X})$ and $\mathbf{Ch}(\mathcal{Y})$, respectively. Since $v_{\mathcal{X}\mathcal{Y}}^{*}\colon \mathbf{Ch}(\mathcal{Y}) \to \mathbf{Ch}(\mathcal{X})$ is induced from the left adjoint $\mathcal{O}_{\mathcal{Y}}\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$ of the forgetful functor $\mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$, we may apply $v_{\mathcal{X}\mathcal{Y}}^{*}$ to the last of the above triangles and still get a distinguished triangle

$$v_{\mathcal{X}\mathcal{Y}}^{*}v_{\mathcal{Y}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{v_{\mathcal{X}\mathcal{Y}}^{*}m_{\mathcal{Y}\mathcal{Z}}} v_{\mathcal{X}\mathcal{Y}}^{*}L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} v_{\mathcal{X}\mathcal{Y}}^{*}\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{X}\mathcal{Y}}^{*}v_{\mathcal{Y}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}}[1]$$

in $\mathbf{K}(\mathcal{X})$. Since both functors $v_{\mathcal{X}\mathcal{Y}}^{*}v_{\mathcal{Y}\mathcal{Z}}^{*}$ and $v_{\mathcal{X}\mathcal{Z}}^{*}$ from $\mathbf{Ch}(\mathcal{Z})$ to $\mathbf{Ch}(\mathcal{X})$ are left adjoint to the forgetful functor $\mathbf{Ch}(\mathcal{X}) \to \mathbf{Ch}(\mathcal{Z})$ there is a natural isomorphism of functors between them and we will identify $v_{\mathcal{X}\mathcal{Y}}^{*}v_{\mathcal{Y}\mathcal{Z}}^{*}$ with $v_{\mathcal{X}\mathcal{Z}}^{*}$. Furthermore the composition

$$v_{\mathcal{X}\mathcal{Z}}^{*}L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{v_{\mathcal{X}\mathcal{Y}}^{*}m_{\mathcal{Y}\mathcal{Z}}} v_{\mathcal{X}\mathcal{Y}}^{*}L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{m_{\mathcal{X}\mathcal{Y}}} L_{\mathcal{X}/\mathcal{S}}$$

is $m_{\mathcal{X}\mathcal{Z}}$ by definition.

Visualizing the three distinguished triangles in $\mathbf{K}(\mathcal{X})$ in an octahedron

we see that there are morphisms

$$f\colon v_{\mathcal{X}\mathcal{Y}}^{*}\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \to \mathrm{Cone}(m_{\mathcal{X}\mathcal{Z}}) \quad \text{and} \quad g\colon \mathrm{Cone}(m_{\mathcal{X}\mathcal{Z}}) \to \mathrm{Cone}(m_{\mathcal{X}\mathcal{Y}})$$

in $\mathbf{K}(\mathcal{X})$ such that

$$v_{\mathcal{X}\mathcal{Y}}^*\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \xrightarrow{f} \mathrm{Cone}(m_{\mathcal{X}\mathcal{Z}}) \xrightarrow{g} \mathrm{Cone}(m_{\mathcal{X}\mathcal{Y}}) \xrightarrow{(\mathrm{id}|0)[1]\circ\mathrm{pr}_2} v_{\mathcal{X}\mathcal{Y}}^*\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}})[1]$$

is a distinguished triangle in $\mathbf{K}(\mathcal{X})$ by the octohedral axiom (see for example [Har66, Chapter I.1. Axiom (TR4)]). Applying $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(-, m_{\mathcal{X}}^*\mathcal{J})$ to the triangle

$$v_{\mathcal{X}\mathcal{Y}}^* v_{\mathcal{Y}\mathcal{Z}}^* L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{v_{\mathcal{X}\mathcal{Y}}^* m_{\mathcal{Y}\mathcal{Z}}} v_{\mathcal{X}\mathcal{Y}}^* L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} v_{\mathcal{X}\mathcal{Y}}^*\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{X}\mathcal{Y}}^* v_{\mathcal{Y}\mathcal{Z}}^* L_{\mathcal{Z}/\mathcal{S}}[1]$$

in $\mathbf{D}(\mathcal{X})$ and applying $\mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(-, m_{\mathcal{Y}}^*\mathcal{J})$ to the triangle

$$v_{\mathcal{Y}\mathcal{Z}}^* L_{\mathcal{Z}/\mathcal{S}} \xrightarrow{m_{\mathcal{Y}\mathcal{Z}}} L_{\mathcal{Y}/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} \mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}) \xrightarrow{\mathrm{pr}_2} v_{\mathcal{Y}\mathcal{Z}}^* L_{\mathcal{Z}/\mathcal{S}}[1]$$

in $\mathbf{D}(\mathcal{Y})$ yield a commutative diagram

of abelian groups where the first and forth horizontal isomorphisms are the ones in Proposition 2.14 for the functor $v_{\mathcal{X}\mathcal{Y}}$ and for the complex $L_{\mathcal{Y}/\mathcal{S}}$ and where the second and fifth horizontal isomorphisms are the ones in Proposition 2.14 for the functor $v_{\mathcal{X}\mathcal{Z}}$ and for the complex $L_{\mathcal{Z}/\mathcal{S}}$. From the exactness of the two columns it follows that

$$\mathbb{Ext}_{\mathcal{X}}^{i+1}(v_{\mathcal{X}\mathcal{Y}}^*\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}), m_{\mathcal{X}}^*\mathcal{J}) \longrightarrow \mathbb{Ext}_{\mathcal{Y}}^{i+1}(\mathrm{Cone}(m_{\mathcal{Y}\mathcal{Z}}), m_{\mathcal{Y}}^*\mathcal{J})$$

is an isomorphism as well for every $i \in \mathbb{Z}$. Consequently, the braid in the proposition is obtained by applying the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(-, m_{\mathcal{X}}^*\mathcal{J})$ to the above octohedron where all morphisms are considered in $\mathbf{D}(\mathcal{X})$. $\qquad\square$

Notice that the isomorphisms in Proposition 2.14 do not hold a priori for the complex $\mathrm{Cone}(m_{\mathcal{YZ}})$ because it does not consist of free $\mathcal{O}_{\mathcal{Y}}$-modules in general. Therefore we have to argue as done above.

From the commutativity and the exactness of the four sequences in the braid of Proposition 2.22 we immediately get a proof of the following plausible idea. Given a deformation $\xi_{\mathcal{Z}}$ of $\mathcal{Z}$ over $\mathcal{S}'$ such that the obstruction in $\mathbb{E}\mathrm{xt}^2_{\mathcal{Y}}(\mathrm{Cone}(m_{\mathcal{YZ}}), m^*_{\mathcal{Y}}\mathcal{J})$ vanishes, we may choose a deformation $\xi_{\mathcal{Y}}$ of $\mathcal{Y}$ over $\mathcal{S}'$ extending $\xi_{\mathcal{Z}}$. Assuming further that the obstruction in $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m_{\mathcal{XY}}), m^*_{\mathcal{X}}\mathcal{J})$ for finding a deformation $\xi_{\mathcal{X}}$ of $\mathcal{X}$ over $\mathcal{S}'$ extending $\xi_{\mathcal{Y}}$ vanishes, we may choose such a deformation $\xi_{\mathcal{X}}$. Then the obstruction in $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m_{\mathcal{XZ}}), m^*_{\mathcal{X}}\mathcal{J})$ for finding a deformation $\xi_{\mathcal{X}}$ of $\mathcal{X}$ over $\mathcal{S}'$ extending $\xi_{\mathcal{Z}}$ vanishes as well.

Remark 2.23. It follows that if we are only interested in extending a given deformation of the subdiagram to a deformation of the diagram, then we may always assume that $\mathfrak{D}$ is obtained from $\mathfrak{C}$ either by omitting exactly one level $0 \in \mathrm{ob}(\mathfrak{C})$ (and all morphisms from and to 0) or by omitting exactly one morphism in $\mathfrak{C}$ (such that $\mathfrak{D}$ is a category).

The following section deals with the calculation of the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J})$ found in Theorem 2.20 for many particular cases.

3 Calculations of the obstruction group for particular cases

We keep the notations of General assumption 2.1. Assuming that the morphism a in Lemma 2.19 is a quasi-isomorphism, the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J})$ for the problem of extending a given deformation of $\mathcal{Y}$ to a deformation of $\mathcal{X}$ will be calculated more explicitly in this section for some particular cases. Therefore we have to introduce several types of subdiagrams.

The first three subsections are kept rather abstract and explain the techniques necessary for the simplification. We will apply these results to many concrete cases in the last two subsections.

Subsection 3.1 considers well-positioned subdiagrams, characterised by the property that each morphism between two levels not belonging to the subdiagram must not factor through some level of the subdiagram. Many subdiagrams possess this property, for example all subdiagrams such that $\mathrm{ob}(\mathfrak{D}) = \mathrm{ob}(\mathfrak{C})$.

Subsection 3.2 explains the notion of a full subdiagram in which case $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$. Corollary 3.16 shows that if $\mathcal{Y}$ is a full and well-positioned subdiagram, then the obstruction group is given by
$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^*_h, m^*_{\mathcal{X}}\mathcal{J})$$
where $(L^*_h)_c = (L_h)_c$ for all $c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, but $(L^*_h)_d = 0$ is the zero complex for all $d \in \mathrm{ob}(\mathfrak{D})$.

After introducing the complementary subdiagram $\overline{\mathcal{Y}}$ of a given subdiagram $\mathcal{Y}$ in Subsection 3.3, we will see in Corollary 3.26 that if $\mathcal{Y}$ is full and well-positioned, then the obstruction group simplifies to
$$\mathbb{E}\mathrm{xt}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$$
where $\overline{u}\colon \mathcal{X} \to \overline{\mathcal{Y}}$ is the forgetful functor and $\mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J})$ is a certain complex of $\mathcal{O}_{\overline{\mathcal{Y}}}$-modules. Thus we see that the obstruction group is actually concentrated on the ringed topos $\overline{\mathcal{Y}}$ instead of the ringed topos $\mathcal{X}$. Appendix B describes a procedure for finding certain injective resolutions of modules on a diagram which are adequate to calculate $\mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J})$ in many cases.

As already mentioned above, the next subsections consider concrete examples. Subsection 3.4 treats subdiagrams obtained by omitting a single level of the diagram and all morphisms from and to this level. The obstruction group is then concentrated on the omitted ringed topos. The other extreme is considered as well where the subdiagram is obtained from the diagram by keeping all levels, but by omitting all morphisms except for the identities. We call this subdiagram the discrete subdiagram of $\mathcal{X}$.

3.1 Well-positioned subdiagrams

This subsection treats some technical results on well-positioned subdiagrams which we will need in Subsections 3.2 and 3.3. We will see that if $\mathcal{X}$ is a diagram and if L is the cotangent complex of a ring morphism of $\mathcal{X}$ such that L_d is an exact complex for each $d \in \mathrm{ob}(\mathfrak{D})$, then we may replace L in the derived category $\mathbf{D}(\mathcal{X})$ by a complex L^* such that L_d^* is the zero complex for each $d \in \mathrm{ob}(\mathfrak{D})$. This will help us to calculate the obstruction group $\mathbb{Ext}_{\mathcal{X}}^2(L_h, m_{\mathcal{X}}^* \mathcal{J})$ of Theorem 2.20 more explicitly in Subsection 3.3.

Definition 3.1. Let $\mathcal{X}$ be a diagram. A subdiagram $\mathcal{Y}$ of $\mathcal{X}$ is called well-positioned (with respect to $\mathcal{X}$) if each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ between objects $c_1, c_2 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$ does not factor through some object $d \in \mathrm{ob}(\mathfrak{D})$, i.e., for each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ such that $c_1, c_2 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, there is no triangle

$$
\begin{array}{ccc}
c_1 & \xrightarrow{\ \ \alpha\ \ } & c_2 \\
 & \searrow \quad \nearrow & \\
 & d &
\end{array}
$$

in $\mathfrak{C}$ such that $d \in \mathrm{ob}(\mathfrak{D})$.

Example 3.2. Consider the example

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 1 in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$, $h = g \circ f$	$\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \qquad \mathcal{X}_2$
subdiagram 2 in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 3 in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_0 \xrightarrow{\ h\ } \mathcal{X}_2$	$\mathcal{X}_1$

with no other morphisms involved except for the identities. Then subdiagram 1 which we have already considered in Example 2.18 is well-positioned since $\mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$ is the empty set. Subdiagram 2 is also well-positioned but subdiagram 3 is not well-positioned because $h \colon \mathcal{X}_0 \to \mathcal{X}_2$ factors through $\mathcal{X}_1$.

Now let $\mathcal{Y}$ be a well-positioned subdiagram of $\mathcal{X}$ and let $A \to B$ a morphism of sheaves of rings of $\mathcal{X}$. By Corollary A.3 the cotangent complex L of $A \to B$ is given by the collection of all cotangent complexes $L_c = L_{B_c/A_c}$ of the ring morphisms $A_c \to B_c$ of $\mathcal{X}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ together with, for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, the natural morphisms

$$
n_\alpha \colon f_\alpha^{-1} L_{c_1} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \to L_{c_2}
$$

of complexes of B_{c_2}-modules. We will construct two complexes of B-modules L' and L^* and two morphisms

$$
p \colon L \to L' \quad \text{and} \quad q \colon L^* \to L'
$$

of complexes of B-modules such that L_d^* is the zero complex for each $d \in \mathrm{ob}(\mathfrak{D})$. The construction of L' and $p \colon L \to L'$ is very similar to the construction of L^* and $q \colon L^* \to L'$, respectively.

Definition 3.3. Let L be the cotangent complex of a ring morphism $A \to B$ in $\mathcal{X}$. For $c \in \mathrm{ob}(\mathfrak{C})$ let

$$
L_c' = \begin{cases}
L_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\
L_c & \text{if } c \in \mathrm{ob}(\mathfrak{D}) \text{ and } \mathrm{Hom}_{\mathfrak{C}}(c', c) \text{ is empty for every } c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\
0 & \text{else}
\end{cases}
$$

and for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ let

$$
\left(f_\alpha^{-1} L_{c_1}' \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \xrightarrow{n_\alpha'} L_{c_2}' \right) = \begin{cases}
0 \to L_{c_2}' & \text{if } L_{c_1}' = 0 \\
f_\alpha^{-1} L_{c_1}' \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \to 0 & \text{if } L_{c_2}' = 0 \\
f_\alpha^{-1} L_{c_1} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \xrightarrow{n_\alpha} L_{c_2} & \text{else.}
\end{cases}
$$

Remark 3.4. Let $\mathcal{Y}$ be a well-positioned subdiagram of $\mathcal{X}$. Let $c_1 \in \mathrm{ob}(\mathfrak{D})$ such that $\mathrm{Hom}_{\mathfrak{C}}(c', c_1)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$. By definition we have $L_{c_1}' = 0$. Now let $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$. If $c_2 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$ then we have a factorization $c' \to c_1 \xrightarrow{\alpha} c_2$ between two objects not in $\mathrm{ob}(\mathfrak{D})$ through the object $c_1 \in \mathrm{ob}(\mathfrak{D})$. But since $\mathcal{Y}$ is well-positioned, we must have $c_2 \in \mathrm{ob}(\mathfrak{D})$ and the composition $c' \to c_1 \xrightarrow{\alpha} c_2$ shows that $L_{c_2}' = 0$.

Lemma 3.5. *Let $\mathcal{Y}$ be a well-positioned subdiagram of $\mathcal{X}$. The collection of all L_c' for $c \in \mathrm{ob}(\mathfrak{C})$ together with all morphisms*

$$
n_\alpha' \colon f_\alpha^{-1} L_{c_1}' \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \to L_{c_2}'
$$

of complexes of B_{c_2}-modules for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ defines a complex L' of B-modules. There is a natural morphism

$$
p \colon L \to L'
$$

of complexes of B-modules.

Proof. Let $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ be morphisms in $\mathfrak{C}$. We have to show that the morphisms

$$
f_{\beta\alpha}^{-1} L_{c_1}' \otimes_{f_{\beta\alpha}^{-1} B_{c_1}} B_{c_3} \xrightarrow{n_{\beta\alpha}'} L_{c_3}'
$$

and

$$
f_{\beta\alpha}^{-1} L_{c_1}' \otimes_{f_{\beta\alpha}^{-1} B_{c_1}} f_\beta^{-1} B_{c_2} \otimes_{f_\beta^{-1} B_{c_2}} B_{c_3} \xrightarrow{\left(f_\beta^{-1} n_\alpha' \otimes \mathrm{id} \right)} f_\beta^{-1} L_{c_2}' \otimes_{f_\beta^{-1} B_{c_2}} B_{c_3} \xrightarrow{n_\beta'} L_{c_3}'
$$

are equal.

If $c_1 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_1)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L_{c_1}' = 0$ and the above morphisms agree since they have the zero complex as source.

If $L_{c_1}' = L_{c_1}$, we have to examine L_{c_2}'.

If $c_2 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_2)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L_{c_2}' = 0$. By Remark 3.4 it follows $L_{c_3}' = 0$. Thus the above morphisms are equal since they have the zero complex as target.

If $L_{c_2}' = L_{c_2}$, we have to examine L_{c_3}'.

If $L'_{c_3} = 0$, the above morphisms are equal since they have the zero complex as target.

If $L'_{c_3} = L_{c_3}$, then by definition the above morphisms are the ones belonging to L. Since L is a well-defined complex of B-modules, the above morphisms are equal as well.

Now for $c \in \mathrm{ob}(\mathfrak{C})$ we define

$$
\left(p_c \colon L_c \to L'_c \right) = \begin{cases} L_c \xrightarrow{\mathrm{id}} L_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ L_c \xrightarrow{\mathrm{id}} L_c & \text{if } c \in \mathrm{ob}(\mathfrak{D}) \text{ and } \mathrm{Hom}_{\mathfrak{C}}(c', c) \text{ is empty for every } c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ L_c \to 0 & \text{else.} \end{cases}
$$

For each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ the square

$$
\begin{array}{ccc}
f_\alpha^{-1} L_{c_1} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} & \xrightarrow{\;\;n_\alpha\;\;} & L_{c_2} \\
\downarrow{\scriptstyle f_\alpha^{-1} p_{c_1} \otimes \mathrm{id}} & & \downarrow{\scriptstyle p_{c_2}} \\
f_\alpha^{-1} L'_{c_1} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} & \xrightarrow{\;\;n'_\alpha\;\;} & L'_{c_2}
\end{array}
$$

is commutative as follows again by a distinction of cases.

If $c_2 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_2)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L'_{c_2} = 0$ and the above square is commutative since it has the zero complex as target.

If $L'_{c_2} = L_{c_2}$ and thus $p_{c_2} = \mathrm{id}$, we have to examine L'_{c_1}.

If $c_1 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_1)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L'_{c_1} = 0$. By Remark 3.4 it follows $L'_{c_2} = 0$. Thus the above square is commutative since it has the zero complex as target.

If $L'_{c_1} = L_{c_1}$, we have $p_{c_1} = \mathrm{id}$ and $n'_\alpha = n_\alpha$. It follows that the above square is commutative as well. $\qquad\square$

Definition 3.6. Let L be the cotangent complex of a ring morphism $A \to B$ in $\mathcal{X}$. For $c \in \mathrm{ob}(\mathfrak{C})$ let

$$
L_c^* = \begin{cases} L_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ 0 & \text{else} \end{cases}
$$

and for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ let

$$
\left(f_\alpha^{-1} L_{c_1}^* \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \xrightarrow{n_\alpha^*} L_{c_2}^* \right) = \begin{cases} 0 \to L_{c_2}^* & \text{if } L_{c_1}^* = 0 \\ f_\alpha^{-1} L_{c_1}^* \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \to 0 & \text{if } L_{c_2}^* = 0 \\ f_\alpha^{-1} L_{c_1} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \xrightarrow{n_\alpha} L_{c_2} & \text{else.} \end{cases}
$$

Lemma 3.7. *Let $\mathcal{Y}$ be a well-positioned subdiagram of $\mathcal{X}$. The collection of all L_c^* for $c \in \mathrm{ob}(\mathfrak{C})$ together with all morphisms*

$$
n_\alpha^* \colon f_\alpha^{-1} L_{c_1}^* \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \to L_{c_2}^*
$$

of complexes of B_{c_2}-modules for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ defines a complex L^* of B-modules. There is a natural morphism

$$
q \colon L^* \to L'
$$

of complexes of B-modules.

Proof. Let $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ be morphisms in $\mathfrak{C}$. We have to show that the morphisms

$$f_{\beta\alpha}^{-1} L_{c_1}^* \otimes_{f_{\beta\alpha}^{-1} B_{c_1}} B_{c_3} \xrightarrow{n_{\beta\alpha}^*} L_{c_3}^*$$

and

$$f_{\beta\alpha}^{-1} L_{c_1}^* \otimes_{f_{\beta\alpha}^{-1} B_{c_1}} f_\beta^{-1} B_{c_2} \otimes_{f_\beta^{-1} B_{c_2}} B_{c_3} \xrightarrow{\left(f_\beta^{-1} n_\alpha^* \otimes \mathrm{id}\right)} f_\beta^{-1} L_{c_2}^* \otimes_{f_\beta^{-1} B_{c_2}} B_{c_3} \xrightarrow{n_\beta^*} L_{c_3}^*$$

are equal.

If $c_1 \in \mathrm{ob}(\mathfrak{D})$, we have $L_{c_1}^* = 0$ and the above morphisms agree since they have the zero complex as source.

If $c_1 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then $L_{c_1}^* = L_{c_1}$ and we have to examine $L_{c_2}^*$.

If $c_2 \in \mathrm{ob}(\mathfrak{D})$, we have $L_{c_2}^* = 0$. Assume $c_3 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$. Then $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ is a factorization between the objects $c_1, c_3 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$ through the object $c_2 \in \mathrm{ob}(\mathfrak{D})$. But since $\mathcal{Y}$ is well-positioned we must have $c_3 \in \mathrm{ob}(\mathfrak{D})$, hence $L_{c_3}^* = 0$. Thus the above morphisms are equal since they have the zero complex as target.

If $c_2 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then $L_{c_2}^* = L_{c_2}$ and we have to examine $L_{c_3}^*$.

If $c_3 \in \mathrm{ob}(\mathfrak{D})$, we have $L_{c_3}^* = 0$ and the above morphisms are equal since they have the zero complex as target.

If $c_3 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then $L_{c_3}^* = L_{c_3}$. By definition the above morphisms are the ones belonging to L. Since L is a well-defined complex of B-modules, the above morphisms are equal as well.

Now for $c \in \mathrm{ob}(\mathfrak{C})$ we define

$$\left(q_c \colon L_c^* \to L_c'\right) = \begin{cases} L_c \xrightarrow{\mathrm{id}} L_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ 0 \to L_c' & \text{else.} \end{cases}$$

For each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ the square

$$\begin{array}{ccc}
f_\alpha^{-1} L_{c_1}^* \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} & \xrightarrow{\;\;n_\alpha^*\;\;} & L_{c_2}^* \\
\Big\downarrow{\scriptstyle f_\alpha^{-1} q_{c_1} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle q_{c_2}} \\
f_\alpha^{-1} L_{c_1}' \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} & \xrightarrow{\;\;n_\alpha'\;\;} & L_{c_2}'
\end{array}$$

is commutative as follows again by a distinction of cases.

If $c_2 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_2)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L_{c_2}' = 0$ and the above square is commutative since it has the zero complex as target.

If $c_2 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then $L_{c_2}^* = L_{c_2}' = L_{c_2}$ and $q_{c_2} = \mathrm{id}$. We have to examine c_1.

If $c_1 \in \mathrm{ob}(\mathfrak{D})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c', c_1)$ is nonempty for some $c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, we have $L_{c_1}' = 0$. By Remark 3.4 it follows $L_{c_2}' = 0$. Thus the above square is commutative since it has the zero complex as target.

If $c_1 \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then $L_{c_1}^* = L_{c_1}' = L_{c_1}$ and $q_{c_1} = \mathrm{id}$ and $n_\alpha^* = n_\alpha' = n_\alpha$. It follows that the above square is commutative as well. $\qquad\square$

Remark 3.8. Let $\mathcal{Y}$ be a well-positioned subdiagram of $\mathcal{X}$. Then the above constructions are in particular valid for the cotangent complex L_h of the ring morphism $h\colon u^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ in Proposition 2.16. Thus there are complexes of $\mathcal{O}_{\mathcal{X}}$-modules L'_h and L^*_h and morphisms

$$p\colon L_h \to L'_h \quad \text{and} \quad q\colon L^*_h \to L'_h$$

of complexes of $\mathcal{O}_{\mathcal{X}}$-modules by Lemma 3.5 and Lemma 3.7, respectively, such that $(L^*_h)_d$ is the zero complex for each $d \in \mathrm{ob}(\mathfrak{D})$. We will need these morphisms in Proposition 3.15 where they are quasi-isomorphisms.

Example 3.9. Consider the example

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\begin{array}{c} \mathcal{X}_1 \\ {\scriptstyle f}\nearrow \quad \searrow{\scriptstyle g} \\ \mathcal{X}_0 \xrightarrow{\;h=g\circ f\;} \mathcal{X}_2 \end{array}$	$\mathcal{X}_0 \xrightarrow{\;\;h\;\;} \mathcal{X}_2$

with no other morphisms involved except for the identities and let L be the cotangent complex of $\theta_{\mathcal{X}}\colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}}$, denoted

$$L = \left(L_{\mathcal{X}_0},\ L_{\mathcal{X}_1},\ L_{\mathcal{X}_2},\ f^*L_{\mathcal{X}_1} \to L_{\mathcal{X}_0},\ g^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_1},\ h^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_0} \right)$$

where $L_{\mathcal{X}_j}$ is the cotangent complex of $\theta_j\colon t_j^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}_j}$. Then $\mathcal{Y}$ is a well-positioned subdiagram of $\mathcal{X}$ as we have seen in Example 3.2 and the complexes of $\mathcal{O}_{\mathcal{X}}$-modules L' and L^* are given by

$$L' = \left(0\ ,\ L_{\mathcal{X}_1},\ L_{\mathcal{X}_2},\ f^*L_{\mathcal{X}_1} \to 0\ ,\ g^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_1},\ h^*L_{\mathcal{X}_2} \to 0 \right)$$

and

$$L^* = \left(0\ ,\ L_{\mathcal{X}_1},\ 0\ ,\ f^*L_{\mathcal{X}_1} \to 0\ ,\ g^*\, 0\ \to L_{\mathcal{X}_1},\ h^*\, 0\ \to 0 \right),$$

respectively. The morphisms of $\mathcal{O}_{\mathcal{X}}$-modules $p = (p_0, p_1, p_2)\colon L \to L'$ and $q = (q_0, q_1, q_2)\colon L^* \to L'$ are given by

$$L = \left(L_{\mathcal{X}_0},\ L_{\mathcal{X}_1},\ L_{\mathcal{X}_2},\ f^*L_{\mathcal{X}_1} \to L_{\mathcal{X}_0},\ g^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_1},\ h^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_0} \right)$$
$$\Big\downarrow {\scriptstyle p=(0,\mathrm{id},\mathrm{id})}$$
$$L' = \left(0\ ,\ L_{\mathcal{X}_1},\ L_{\mathcal{X}_2},\ f^*L_{\mathcal{X}_1} \to 0\ ,\ g^*L_{\mathcal{X}_2} \to L_{\mathcal{X}_1},\ h^*L_{\mathcal{X}_2} \to 0 \right)$$
$$\Big\uparrow {\scriptstyle q=(0,\mathrm{id},0)}$$
$$L^* = \left(0\ ,\ L_{\mathcal{X}_1},\ 0\ ,\ f^*L_{\mathcal{X}_1} \to 0\ ,\ g^*\, 0\ \to L_{\mathcal{X}_1},\ h^*\, 0\ \to 0 \right).$$

Notice in particular that $(0, \mathrm{id}, 0)\colon L \to L^*$ is no well-defined morphism of complexes of $\mathcal{O}_{\mathcal{X}}$-modules because

$$\begin{array}{ccc} g^*L_{\mathcal{X}_2} & \longrightarrow & L_{\mathcal{X}_1} \\ {\scriptstyle g^*0}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ g^*0 & \longrightarrow & L_{\mathcal{X}_1} \end{array}$$

is not commutative. Similarly, $(0, \mathrm{id}, 0) \colon L^* \to L$ is no well-defined morphism of complexes of $\mathcal{O}_{\mathcal{X}}$-modules either because

$$
\begin{array}{ccc}
f^* L_{\mathcal{X}_1} & \longrightarrow & 0 \\
{\scriptstyle f^*\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle 0} \\
f^* L_{\mathcal{X}_1} & \longrightarrow & L_{\mathcal{X}_0}
\end{array}
$$

is not commutative. This example shows that in general there is neither a morphism $L \to L^*$ nor a morphism $L^* \to L$ of complexes of $\mathcal{O}_{\mathcal{X}}$-modules, even if $\mathcal{Y}$ is well-positioned.

Example 3.10. Consider the example

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\begin{array}{c} \mathcal{X}_1 \\ {\scriptstyle f}\nearrow \quad \searrow {\scriptstyle g} \\ \mathcal{X}_0 \xrightarrow{\ h=g\circ f\ } \mathcal{X}_2 \end{array}$	$\mathcal{X}_1$

with no other morphisms involved except for the identities and let again

$$
L = \big(L_{\mathcal{X}_0},\ L_{\mathcal{X}_1},\ L_{\mathcal{X}_2},\ f^* L_{\mathcal{X}_1} \to L_{\mathcal{X}_0},\ g^* L_{\mathcal{X}_2} \to L_{\mathcal{X}_1},\ h^* L_{\mathcal{X}_2} \to L_{\mathcal{X}_0}\big)
$$

be the cotangent complex of $\theta_{\mathcal{X}} \colon m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}}$. This time $\mathcal{Y}$ is not a well-positioned subdiagram of $\mathcal{X}$ as pointed out in Example 3.2. By definition we have

$$
L^* = L' = \big(L_{\mathcal{X}_0},\ 0,\ L_{\mathcal{X}_2},\ f^* 0 \xrightarrow{0} L_{\mathcal{X}_0},\ g^* L_{\mathcal{X}_2} \xrightarrow{0} 0,\ h^* L_{\mathcal{X}_2} \to L_{\mathcal{X}_0}\big)
$$

which is not a well-defined complex of $\mathcal{O}_{\mathcal{X}}$-modules since $h^* L_{\mathcal{X}_2} \to L_{\mathcal{X}_0}$ is not the composition $f^* g^* L_{\mathcal{X}_2} \xrightarrow{f^*0} f^* 0 \xrightarrow{0} L_{\mathcal{X}_0}$.

3.2 Full subdiagrams

We will see that if $\mathcal{Y}$ is a full and well-positioned subdiagram of $\mathcal{X}$, then the cotangent complex L_h of the ring morphism h in Proposition 2.16 may be replaced in $\mathbf{D}(\mathcal{X})$ by the complex L_h^* as defined in Lemma 3.7. The replacement of L_h by L_h^* will facilitate the calculation of the obstruction group $\mathbb{E}\mathrm{xt}_\mathcal{X}^2(L_h, m_\mathcal{X}^*\mathcal{J})$ as pointed out in Proposition 3.25 if the morphism a in Lemma 2.19 is a quasi-isomorphism.

Definition 3.11. Let $\mathfrak{D}$ be a full subcategory of $\mathfrak{C}$, i.e., $\mathrm{Hom}_\mathfrak{D}(d_1, d_2) = \mathrm{Hom}_\mathfrak{C}(d_1, d_2)$ for any objects d_1, d_2 of $\mathfrak{D}$. Then we call $\mathcal{Y}$ a full subdiagram of $\mathcal{X}$.

If $\mathfrak{D}$ is obtained from $\mathfrak{C}$ by omitting exactly one object $0 \in \mathrm{ob}(\mathfrak{C})$ and all morphisms from and to 0, then $\mathcal{Y}$ is a full subdiagram of $\mathcal{X}$. Notice that this is one of the two types of subdiagrams in Remark 2.23 we have to consider if we want to extend a given deformation of an arbitrary subdiagram "step by step" to a deformation of the given diagram, whence it is important to understand the resulting obstruction group.

Example 3.12. Consider the diagram and subdiagrams

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 1 in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1$ $f \nearrow \quad \searrow g$ $\mathcal{X}_0 \xrightarrow{h = g \circ f} \mathcal{X}_2$	$\mathcal{X}_1$ $f \nearrow$ $\mathcal{X}_0 \qquad \mathcal{X}_2$
subdiagram 2 in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 3 in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_0 \xrightarrow{h} \mathcal{X}_2$	$\mathcal{X}_1$

of Example 3.2 with no other morphisms involved except for the identities. Then subdiagram 1 which we have already considered in Example 2.18 is not a full subdiagram because the morphism g does not occur between the ringed topoi $\mathcal{X}_1$ and $\mathcal{X}_2$. Subdiagram 2 and Subdiagram 3 are full subdiagrams.

Lemma 3.13. *Let $\mathcal{Y}$ be a full subdiagram of $\mathcal{X}$. Then*

$$(u^{-1}\mathcal{O}_\mathcal{Y})_d = \mathcal{O}_{\mathcal{X}_d}$$

for every $d \in \mathrm{ob}(\mathfrak{D})$. Furthermore, if $d_1, d_2 \in \mathrm{ob}(\mathfrak{D})$ and $\alpha \in \mathrm{Hom}_\mathfrak{C}(d_1, d_2) = \mathrm{Hom}_\mathfrak{D}(d_1, d_2)$, then the morphism of $t_{d_2}^{-1}\mathcal{O}_\mathcal{S}$-algebras

$$(u^{-1}\mathcal{O}_\mathcal{Y})_\alpha \colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} \to \mathcal{O}_{\mathcal{X}_{d_2}}$$

in Definition 2.7 is the ring morphism $\theta_\alpha \colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} \to \mathcal{O}_{\mathcal{X}_{d_2}}$ of $f_\alpha \colon \mathcal{X}_{d_2} \to \mathcal{X}_{d_1}$.

Proof. Since $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$, each $\big(\gamma \colon d' \to d\big) \in \mathrm{ob}(\underline{d})$ is an element of $\mathrm{Hom}_\mathfrak{D}(d', d)$. In particular, we have $\gamma \in \mathrm{Hom}_{\underline{d}}\big((\gamma \colon d' \to d), (\mathrm{id} \colon d \to d)\big)$ and so the ring morphism

$$\theta_\gamma \colon f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} \to \mathcal{O}_{\mathcal{X}_d}$$

belongs to the system of all $f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}}$ defined by $\underline{d}$. It follows that

$$(u^{-1}\mathcal{O}_\mathcal{Y})_d = \varinjlim_{\gamma \in \mathrm{ob}(\underline{d})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} = \mathcal{O}_{\mathcal{X}_d}$$

because $\mathcal{O}_{\mathcal{X}_d}$ is the final object in the system of all $f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}}$ defined by $\underline{d}$.

Now if $d_1, d_2 \in \mathrm{ob}(\mathfrak{D})$ and $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(d_1, d_2) = \mathrm{Hom}_{\mathfrak{D}}(d_1, d_2)$, then Diagram (2.1) on page 22 is given by

$$
\begin{array}{ccccc}
f_\alpha^{-1}f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} & \xrightarrow{\mathrm{id}} & f_\alpha^{-1}f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} & \xrightarrow{\mathrm{id}} & f_\alpha^{-1}f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} \\
\downarrow{\scriptstyle f_\alpha^{-1}\theta_\gamma} & & \downarrow{\scriptstyle f_\alpha^{-1}\theta_\gamma} & & \downarrow{\scriptstyle \theta_{\alpha\gamma}} \\
f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} & \xrightarrow{\mathrm{id}} & f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} & \xrightarrow{\theta_\alpha} & \mathcal{O}_{\mathcal{X}_{d_2}}
\end{array}
$$

and is commutative for every $\left(\gamma\colon d' \to d_1\right) \in \mathrm{ob}(\underline{d_1})$. Since $(u^{-1}\mathcal{O}_\mathcal{Y})_\alpha$ is the composition of the two lower horizontal morphisms by Definition 2.7 we see that the morphism of $t_{d_2}^{-1}\mathcal{O}_\mathcal{S}$-algebras

$$(u^{-1}\mathcal{O}_\mathcal{Y})_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} = f_\alpha^{-1}\varinjlim_{\gamma\in\mathrm{ob}(\underline{d_1})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{d'}} \to \varinjlim_{\delta\in\mathrm{ob}(\underline{d_2})} f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{d'}} = \mathcal{O}_{\mathcal{X}_{d_2}}$$

is the ring morphism $\theta_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_{d_1}} \to \mathcal{O}_{\mathcal{X}_{d_2}}$ of $f_\alpha\colon \mathcal{X}_{d_2} \to \mathcal{X}_{d_1}$ $\qquad\square$

Thus if $\mathcal{Y}$ is a full subdiagram of $\mathcal{X}$, the morphism of $m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S}$-algebras

$$h\colon u^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$$

in Proposition 2.16 is given by the identity $h_d\colon \mathcal{O}_{\mathcal{X}_d} \xrightarrow{\mathrm{id}} \mathcal{O}_{\mathcal{X}_d}$ for $d \in \mathrm{ob}(\mathfrak{D})$.

Lemma 3.14. *Let $\mathcal{Y}$ be a full subdiagram of $\mathcal{X}$ and let L_h be the cotangent complex of the ring morphism $h\colon u^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ in Proposition 2.16. Then the cotangent complex $(L_h)_d$ is exact for each $d \in \mathrm{ob}(\mathfrak{D})$.*

Proof. By Corollary A.3 the cotangent complex L_h is given by the collection of all cotangent complexes of $h_c\colon (u^{-1}\mathcal{O}_\mathcal{Y})_c \to \mathcal{O}_{\mathcal{X}_c}$ for $c \in \mathrm{ob}(\mathfrak{C})$. But h_d is the identity for $d \in \mathrm{ob}(\mathfrak{D})$ and the cotangent complex of the identity is exact. $\qquad\square$

Proposition 3.15. *Let $\mathcal{Y}$ be a full and well-positioned subdiagram of $\mathcal{X}$. Then the natural morphisms of complexes of $\mathcal{O}_\mathcal{X}$-modules*

$$p\colon L_h \to L_h' \quad and \quad q\colon L_h^* \to L_h'$$

of Remark 3.8 are quasi-isomorphisms. In particular, there is a natural isomorphism

$$L_h \xrightarrow{p} L_h' \xrightarrow{q^{-1}} L_h^*$$

in the derived category $\mathbf{D}(\mathcal{X})$ which yields, for each cochain complex $M \in \mathrm{ob}(\mathbf{Ch}(\mathcal{X}))$ and each $i \in \mathbb{Z}$, a natural functorial isomorphism

$$\mathbb{Ext}_\mathcal{X}^i(L_h, M) \cong \mathbb{Ext}_\mathcal{X}^i(L_h^*, M)$$

of abelian groups.

Proof. Let $c \in \mathrm{ob}(\mathfrak{C})$. Remember that by definition we have

$$\big(p_c \colon (L_h)_c \to (L'_h)_c\big) = \begin{cases} (L_h)_c \xrightarrow{\mathrm{id}} (L_h)_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ (L_h)_c \xrightarrow{\mathrm{id}} (L_h)_c & \text{if } c \in \mathrm{ob}(\mathfrak{D}) \text{ and } \mathrm{Hom}_{\mathfrak{C}}(c', c) \text{ is empty} \\ & \quad\quad \text{for every } c' \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ (L_h)_c \to 0 & \text{else} \end{cases}$$

and

$$\big(q_c \colon (L_h^*)_c \to (L'_h)_c\big) = \begin{cases} (L_h)_c \xrightarrow{\mathrm{id}} (L_h)_c & \text{if } c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D}) \\ 0 \to (L'_h)_c & \text{else.} \end{cases}$$

If $c \in \mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{D})$, then p_c and q_c are the identity and in particular quasi-isomorphisms of complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules.

If $c \in \mathrm{ob}(\mathfrak{D})$, then since $\mathcal{Y}$ is a full subdiagram it follows that $(L_h)_c$ is an exact complex of $\mathcal{O}_{\mathcal{X}_c}$-modules by Lemma 3.14.

If $(L'_h)_c = (L_h)_c$, then $p_c \colon (L_h)_c \to (L_h)_c$ is the identity and $q_c \colon 0 \to (L_h)_c$ is a quasi-isomorphism.

If $(L'_h)_c = 0$, then $p_c \colon (L_h)_c \to 0$ is a quasi-isomorphim and $q_c \colon 0 \to 0$ is the identity. $\qquad\square$

Corollary 3.16. *In the situation of General assumption 2.1, let $\mathcal{Y}$ be a full and well-positioned subdiagram of $\mathcal{X}$. Assume that the morphism a in Lemma 2.19 is a quasi-isomorphism. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction*

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h^*, m_{\mathcal{X}}^* \mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. By Theorem 2.20 we know that $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m_{\mathcal{X}}^* \mathcal{J})$ is an obstruction group and by Proposition 3.15 there is a natural isomorphism of abelian groups $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m_{\mathcal{X}}^* \mathcal{J}) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h^*, m_{\mathcal{X}}^* \mathcal{J})$. $\qquad\square$

Since $(L_h^*)_d$ is the zero complex in $\mathbf{D}(\mathcal{X}_d)$ for each $d \in \mathrm{ob}(\mathfrak{D})$, we may calculate $\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(L_h^*, m_{\mathcal{X}}^* \mathcal{J})$ better than $\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(L_h, m_{\mathcal{X}}^* \mathcal{J})$. Thus using the natural isomorphism of groups

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h^*, m_{\mathcal{X}}^* \mathcal{J}) \cong H^2\big(\mathbf{R}\mathrm{Hom}^{\cdot}_{\mathcal{X}}(L_h^*, m_{\mathcal{X}}^* \mathcal{J})\big)$$

by [Wei94, Theorem 10.7.4], it is possible to calculate the obstruction group more explicitly in the case of full and well-positioned subdiagrams. This is the subject of the next subsection.

3.3 Complementary subdiagrams

In order to calculate the obstruction group $\mathbb{E}xt^2_{\mathcal{X}}(L^*_h, m^*_{\mathcal{X}}\mathcal{J})$ in Corollary 3.16, we will introduce the notion of complementary subdiagrams.

Definition 3.17. Let $\mathfrak{D}$ be a subcategory of $\mathfrak{C}$ such that $\mathrm{ob}(\mathfrak{D}) \neq \mathrm{ob}(\mathfrak{C})$ and let $\overline{\mathfrak{D}}$ be the full subcategory of $\mathfrak{C}$ with objects $\mathrm{ob}(\mathfrak{C}) \backslash \mathrm{ob}(\mathfrak{D})$. Let $\overline{\mathcal{Y}}$ be the ringed topos associated to the restriction of $I \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ to $\overline{\mathfrak{D}}^{\mathrm{op}}$. Then we call $\overline{\mathcal{Y}}$ the complementary subdiagram of $\mathcal{Y}$.

Notice that if $\mathcal{Y}$ is a full subdiagram of $\mathcal{X}$ and if $\overline{\mathcal{Y}}$ is the complementary subdiagram of $\mathcal{Y}$, then $\mathcal{Y}$ is the complementary subdiagram of $\overline{\mathcal{Y}}$ by definition.

Example 3.18. Let $f \colon \mathcal{X}_0 \to \mathcal{X}_1$ be a morphism of ringed topoi over $\mathcal{S}$ and let $\mathcal{X}$ be the associated ringed topos. Then the subdiagrams $\mathcal{X}_0$ and $\mathcal{X}_1$ of $\mathcal{X}$ are complementary to each other.

We denote the objects $\mathcal{F}$ of $\mathcal{X}$ in the form $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, f^{-1}\mathcal{F}_1 \to \mathcal{F}_0)$ where $\mathcal{F}_j$ is a sheaf of $\mathcal{X}_j$. Letting $\mathcal{Y}$ be the full and well-positioned subdiagram $\mathcal{X}_0$, we have

$$u^{-1}\mathcal{O}_{\mathcal{Y}} = (\mathcal{O}_{\mathcal{X}_0}, t_1^{-1}\mathcal{O}_{\mathcal{S}}, f^{-1}t_1^{-1}\mathcal{O}_{\mathcal{S}} = t_0^{-1}\mathcal{O}_{\mathcal{S}} \xrightarrow{\theta_f} \mathcal{O}_{\mathcal{X}_0})$$

by definition, hence condition **2** in Theorem 2.20 is fulfilled. L^*_h is given by

$$L^*_h = (0, L_{\mathcal{X}_1/\mathcal{S}}, f^*L_{\mathcal{X}_1/\mathcal{S}} \to 0).$$

To calculate $\mathbb{E}xt^2_{\mathcal{X}}(L^*_h, m^*_{\mathcal{X}}\mathcal{J})$ we may first consider $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{X})}(L^*_h, m^*_{\mathcal{X}}\mathcal{J})$. By definition this group consists of all morphisms of complexes of $\mathcal{O}_{\mathcal{X}_0}$-modules $a \colon 0 \to t_0^*\mathcal{J}$ and all morphisms of complexes of $\mathcal{O}_{\mathcal{X}_1}$-modules $b \colon L_{\mathcal{X}_1/\mathcal{S}} \to t_1^*\mathcal{J}$ such that

$$\begin{array}{ccc} f^*L_{\mathcal{X}_1/\mathcal{S}} & \longrightarrow & 0 \\ \scriptstyle{f^*b} \downarrow & & \downarrow \scriptstyle{a} \\ f^*t_1^*\mathcal{J} & \xrightarrow{\mathrm{id}} & t_0^*\mathcal{J} \end{array}$$

is commutative. Hence if $\mathcal{K}$ is the kernel of the morphism of complexes of $\mathcal{O}_{\mathcal{X}_1}$-modules $t_1^*\mathcal{J} \to f_*t_0^*\mathcal{J}$ corresponding to $f^*t_1^*\mathcal{J} \xrightarrow{\mathrm{id}} t_0^*\mathcal{J}$ by adjunction, it follows that

$$\mathrm{Hom}_{\mathbf{Ch}(\mathcal{X})}(L^*_h, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathrm{Hom}_{\mathbf{Ch}(\mathcal{X}_1)}(L_{\mathcal{X}_1/\mathcal{S}}, \mathcal{K}).$$

We would like to have a similar isomorphism

$$\mathbb{E}xt^2_{\mathcal{X}}(L^*_h, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathbb{E}xt^2_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{S}}, \tilde{\mathcal{K}})$$

for the obstruction group and we will see in Corollary 3.26 at the end of this section that there is indeed such an isomorphism where $\tilde{\mathcal{K}}$ is a certain complex of $\mathcal{O}_{\mathcal{X}_1}$-modules. Notice that $\mathcal{X}_0$ is the subdiagram considered here and the obstruction group is concentrated on the complementary subdiagram $\mathcal{X}_1$. Notice further that $uL^*_h = 0$ where $u \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{X}_0}\text{-mod}$ is the forgetful functor.

We come back to the general case. Let $\mathcal{Y}$ be a subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$, let

$$u \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod} \quad \text{and} \quad \overline{u} \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}$$

be the forgetful functors from the category of $\mathcal{O}_{\mathcal{X}}$-modules to the category of $\mathcal{O}_{\mathcal{Y}}$-modules and to the category of $\mathcal{O}_{\overline{\mathcal{Y}}}$-modules, respectively. Let further $L \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ such that $uL = 0$.

We will construct a functor $G: \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}$ in Lemma 3.23, sending a module $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ to a certain module $\overline{M} \in \mathrm{ob}(\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod})$, and an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M) \cong \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M})$$

of abelian groups.

Fix $c \in \mathrm{ob}(\overline{\mathfrak{D}})$. Let $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ such that $d \in \mathrm{ob}(\mathfrak{D})$. The morphism of $\mathcal{O}_{\mathcal{X}_d}$-modules

$$M_\gamma: f_\gamma^* M_c \to M_d$$

corresponds by adjunction to a morphism of $\mathcal{O}_{\mathcal{X}_c}$-modules

$$M_\gamma^{\mathrm{ad}}: M_c \to f_{\gamma*} M_d.$$

Definition 3.19. Let $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ and $c \in \mathrm{ob}(\overline{\mathfrak{D}})$. We define

$$\overline{M}_c = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,d), \\ d \in \mathrm{ob}(\mathfrak{D})}} \ker\left(M_\gamma^{\mathrm{ad}}: M_c \to f_{\gamma*} M_d\right) \subseteq M_c$$

to be the $\mathcal{O}_{\mathcal{X}_c}$-module obtained by intersecting in M_c all kernels of M_γ^{ad} for $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ such that $d \in \mathrm{ob}(\mathfrak{D})$. This is a submodule of M_c.

Notice that if $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and if $\mathrm{Hom}_{\mathfrak{C}}(c, d)$ is empty for every $d \in \mathrm{ob}(\mathfrak{D})$, then $\overline{M}_c = M_c$.

Remark 3.20. If $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and $d_1, d_2 \in \mathrm{ob}(\mathfrak{D})$ and if there is a commutative triangle

$$
\begin{array}{ccc}
d_1 & \xrightarrow{\ \chi\ } & d_2 \\
& & \\
{\scriptstyle \gamma_1} \nwarrow & & \nearrow {\scriptstyle \gamma_2} \\
& c &
\end{array}
$$

in $\mathfrak{C}$, then the triangle

$$
\begin{array}{ccc}
f_{\gamma_1*} M_{d_1} & \xrightarrow{\ f_{\gamma_1*} M_\chi^{\mathrm{ad}}\ } & f_{\gamma_1*} f_{\chi*} M_{d_2} = f_{\gamma_2*} M_{d_2} \\
& & \\
{\scriptstyle M_{\gamma_1}^{\mathrm{ad}}} \nwarrow & & \nearrow {\scriptstyle M_{\gamma_2}^{\mathrm{ad}}} \\
& M_c &
\end{array}
$$

is commutative and the kernel of $M_{\gamma_1}^{\mathrm{ad}}$ is contained in the kernel of $M_{\gamma_2}^{\mathrm{ad}}$. Thus in the definition of $\overline{M}_c$ we may restrict to those $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$ which do not factor through some object $d' \in \mathrm{ob}(\mathfrak{D})$ except for the trivial factorization $c \xrightarrow{\gamma} d \xrightarrow{\mathrm{id}} d$.

Now let $c_1, c_2 \in \mathrm{ob}(\overline{\mathfrak{D}})$ and $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{D}}}(c_1, c_2)$. In order to get a well-defined module $\overline{M} \in \mathrm{ob}(\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod})$, we have to define a morphism

$$\overline{M}_\alpha: f_\alpha^* \overline{M}_{c_1} \to \overline{M}_{c_2}$$

of $\mathcal{O}_{\mathcal{X}_{c_2}}$-modules. By adjunction we may equivalently define a morphism

$$\overline{M}_\alpha^{\mathrm{ad}}: \overline{M}_{c_1} \to f_{\alpha*} \overline{M}_{c_2}$$

57

of $\mathcal{O}_{\mathcal{X}_{c_1}}$-modules. Since $f_\alpha \colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ is a morphism of ringed topoi, $f_{\alpha *} \colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ commutes with arbitrary limits. In particular the functor $f_{\alpha *} \colon \mathcal{O}_{\mathcal{X}_{c_2}}\text{-mod} \to \mathcal{O}_{\mathcal{X}_{c_1}}\text{-mod}$, again denoted $f_{\alpha *}$ by abuse of notation, is left exact. Consequently, if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{O}_{\mathcal{X}_{c_2}}\text{-mod}$, then $0 \to f_{\alpha *} M' \to f_{\alpha *} M \to f_{\alpha *} M''$ is exact in $\mathcal{O}_{\mathcal{X}_{c_1}}\text{-mod}$ by [Sta13, Lemma 11.5.1.]. It follows that we have naturally

$$
\begin{aligned}
f_{\alpha *} \overline{M}_{c_2} = f_{\alpha *} \bigcap_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_2, d), \\ d \in \mathrm{ob}(\mathfrak{D})}} & \ker\left(M_\delta^{\mathrm{ad}} \colon M_{c_2} \to f_{\delta *} M_d\right) \\
= \bigcap_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_2, d), \\ d \in \mathrm{ob}(\mathfrak{D})}} & f_{\alpha *} \ker\left(M_\delta^{\mathrm{ad}} \colon M_{c_2} \to f_{\delta *} M_d\right) \qquad (3.1) \\
= \bigcap_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_2, d), \\ d \in \mathrm{ob}(\mathfrak{D})}} & \ker\left(f_{\alpha *} M_\delta^{\mathrm{ad}} \colon f_{\alpha *} M_{c_2} \to f_{\alpha *} f_{\delta *} M_d\right) \subseteq f_{\alpha *} M_{c_2}.
\end{aligned}
$$

So to define a morphism $\overline{M}_\alpha^{\mathrm{ad}}$ from

$$
\overline{M}_{c_1} = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_1, d), \\ d \in \mathrm{ob}(\mathfrak{D})}} \ker\left(M_\gamma^{\mathrm{ad}} \colon M_{c_1} \to f_{\gamma *} M_d\right)
$$

to $f_{\alpha *} \overline{M}_{c_2}$, we take an element x of $\overline{M}_{c_1}$. Let y be the image of x under $M_\alpha^{\mathrm{ad}} \colon M_{c_1} \to f_{\alpha *} M_{c_2}$. Let $\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_2, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$ and set $\gamma = \delta \alpha$. The commutativity of

$$
\begin{array}{ccc}
M_{c_1} & \xrightarrow{\quad M_\gamma^{\mathrm{ad}} \quad} & f_{\gamma *} M_d = f_{\alpha *} f_{\delta *} M_d \\
\searrow{\scriptstyle M_\alpha^{\mathrm{ad}}} & \nearrow{\scriptstyle f_{\alpha *} M_\delta^{\mathrm{ad}}} & \\
& f_{\alpha *} M_{c_2} &
\end{array}
$$

shows that y is sent to zero under $f_{\alpha *} M_\delta^{\mathrm{ad}} \colon f_{\alpha *} M_{c_2} \to f_{\alpha *} f_{\delta *} M_d$ because x is sent to zero under $M_\gamma^{\mathrm{ad}} \colon M_{c_1} \to f_{\gamma *} M_d$. Since this is true for all $\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_2, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$ we see that $y \in f_{\alpha *} M_{c_2}$ is actually an element of the submodule $f_{\alpha *} \overline{M}_{c_2}$ of $f_{\alpha *} M_{c_2}$ by Equation (3.1).

Definition 3.21. Let $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$, let $c_1, c_2 \in \mathrm{ob}(\overline{\mathfrak{D}})$ and let $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{D}}}(c_1, c_2)$. We define

$$
\overline{M}_\alpha^{\mathrm{ad}} \colon \overline{M}_{c_1} \to f_{\alpha *} \overline{M}_{c_2}
$$

to be the restriction of $M_\alpha^{\mathrm{ad}} \colon M_{c_1} \to f_{\alpha *} M_{c_2}$ to the submodule $\overline{M}_{c_1}$ of M_{c_1}.

Lemma 3.22. *Let $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$. The collection of all $\mathcal{O}_{\mathcal{X}_c}$-modules*

$$
\overline{M}_c = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d), \\ d \in \mathrm{ob}(\mathfrak{D})}} \ker\left(M_\gamma^{\mathrm{ad}} \colon M_c \to f_{\gamma *} M_d\right) \subseteq M_c
$$

for $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ together with all morphisms of $\mathcal{O}_{\mathcal{X}_{c_1}}$-modules

$$
\overline{M}_\alpha^{\mathrm{ad}} \colon \overline{M}_{c_1} \to f_{\alpha *} \overline{M}_{c_2}
$$

for $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{D}}}(c_1, c_2)$ defines a module $\overline{M} \in \mathrm{ob}(\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod})$.

Proof. Let $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ be a composition in $\overline{\mathfrak{D}}$. The commutativity of

$$
\begin{array}{ccc}
\overline{M}_{c_1} & \xrightarrow{\ \overline{M}_{\beta\alpha}^{\mathrm{ad}}\ } & f_{\beta\alpha*}\overline{M}_{c_3} = f_{\alpha*}f_{\beta*}\overline{M}_{c_3} \\
& \searrow{\scriptstyle \overline{M}_{\alpha}^{\mathrm{ad}}} \quad \nearrow{\scriptstyle f_{\alpha*}\overline{M}_{\beta}^{\mathrm{ad}}} & \\
& f_{\alpha*}\overline{M}_{c_2} &
\end{array}
$$

follows from the commutativity of

$$
\begin{array}{ccc}
M_{c_1} & \xrightarrow{\ M_{\beta\alpha}^{\mathrm{ad}}\ } & f_{\beta\alpha*}M_{c_3} = f_{\alpha*}f_{\beta*}M_{c_3} \\
& \searrow{\scriptstyle M_{\alpha}^{\mathrm{ad}}} \quad \nearrow{\scriptstyle f_{\alpha*}M_{\beta}^{\mathrm{ad}}} & \\
& f_{\alpha*}M_{c_2} &
\end{array}
$$

because

$$
\overline{M}_{\alpha}^{\mathrm{ad}} \colon \overline{M}_{c_1} \to f_{\alpha*}\overline{M}_{c_2}, \quad \overline{M}_{\beta}^{\mathrm{ad}} \colon \overline{M}_{c_2} \to f_{\beta*}\overline{M}_{c_3} \quad \text{and} \quad \overline{M}_{\beta\alpha}^{\mathrm{ad}} \colon \overline{M}_{c_1} \to f_{\beta\alpha*}\overline{M}_{c_3}
$$

are by definition the restrictions of

$$
M_{\alpha}^{\mathrm{ad}} \colon M_{c_1} \to f_{\alpha*}M_{c_2}, \quad M_{\beta}^{\mathrm{ad}} \colon M_{c_2} \to f_{\beta*}M_{c_3} \quad \text{and} \quad M_{\beta\alpha}^{\mathrm{ad}} \colon M_{c_1} \to f_{\beta\alpha*}M_{c_3}
$$

to the submodules $\overline{M}_{c_1}$, $\overline{M}_{c_2}$ and $\overline{M}_{c_3}$ of M_{c_1}, M_{c_2} and M_{c_3}, respectively. $\qquad\square$

Now let $\nu \colon M_1 \to M_2$ be a morphism in $\mathcal{O}_{\mathcal{X}}$-mod. We will describe a morphism $\overline{\nu} \colon \overline{M_1} \to \overline{M_2}$ in $\mathcal{O}_{\overline{\mathcal{Y}}}$-mod. Let $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and let $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$. Then the commutativity of

$$
\begin{array}{ccc}
(M_1)_c & \xrightarrow{\ (M_1)_{\gamma}^{\mathrm{ad}}\ } & f_{\gamma*}(M_1)_d \\
{\scriptstyle \nu_c}\downarrow & & \downarrow{\scriptstyle f_{\gamma*}\nu_d} \\
(M_2)_c & \xrightarrow{\ (M_2)_{\gamma}^{\mathrm{ad}}\ } & f_{\gamma*}(M_2)_d
\end{array}
$$

shows that the restriction of $\nu_c \colon (M_1)_c \to (M_2)_c$ to the submodule $(\overline{M_1})_c$ defines a morphism of $\mathcal{O}_{\mathcal{X}_c}$-modules $\overline{\nu}_c \colon (\overline{M_1})_c \to (\overline{M_2})_c$.

If $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{D}}}(c_1, c_2)$, then the commutativity of

$$
\begin{array}{ccc}
(M_1)_{c_1} & \xrightarrow{\ (M_1)_{\alpha}^{\mathrm{ad}}\ } & f_{\alpha*}(M_1)_{c_2} \\
{\scriptstyle \nu_{c_1}}\downarrow & & \downarrow{\scriptstyle f_{\alpha*}\nu_{c_2}} \\
(M_2)_{c_1} & \xrightarrow{\ (M_2)_{\alpha}^{\mathrm{ad}}\ } & f_{\alpha*}(M_2)_{c_2}
\end{array}
$$

implies by restriction the commutativity of

$$
\begin{array}{ccc}
(\overline{M_1})_{c_1} & \xrightarrow{\ (\overline{M_1})_{\alpha}^{\mathrm{ad}}\ } & f_{\alpha*}(\overline{M_1})_{c_2} \\
{\scriptstyle \nu_{c_1}}\downarrow & & \downarrow{\scriptstyle f_{\alpha*}\nu_{c_2}} \\
(\overline{M_2})_{c_1} & \xrightarrow{\ (\overline{M_2})_{\alpha}^{\mathrm{ad}}\ } & f_{\alpha*}(\overline{M_2})_{c_2}
\end{array}
$$

which shows that the collection of all $\overline{\nu}_c \colon (\overline{M_1})_c \to (\overline{M_2})_c$ for $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ defines a morphism $\overline{\nu} \colon \overline{M_1} \to \overline{M_2}$ in $\mathcal{O}_{\overline{\mathcal{Y}}}$-mod.

Lemma 3.23. *Let $\mathcal{Y}$ be a subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$. The association*

$$M \mapsto \overline{M} \ , \quad \left(\nu\colon M_1 \to M_2\right) \mapsto \left(\overline{\nu}\colon \overline{M_1} \to \overline{M_2}\right)$$

defines an additive, left exact functor

$$G\colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}.$$

In particular, there is an induced additive functor $G\colon \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\overline{\mathcal{Y}})$, denoted again G by abuse of notation.

Proof. It follows by construction that the identity $M \to M$ is sent to the identity $\overline{M} \to \overline{M}$ and that a composition $M_1 \xrightarrow{\nu} M_2 \xrightarrow{\mu} M_3$ in $\mathcal{O}_{\mathcal{X}}$-mod is sent to the composition $\overline{M_1} \xrightarrow{\overline{\nu}} \overline{M_2} \xrightarrow{\overline{\mu}} \overline{M_3}$ in $\mathcal{O}_{\overline{\mathcal{Y}}}$-mod. Thus the association defines a functor

$$G\colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}.$$

Since all operations in the definition of $\overline{M}$ are compatible with addition, it follows that the maps

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(M_1, M_2) \to \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{M_1}, \overline{M_2})$$

are group homomorphisms for every $M_1, M_2 \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$, i.e., G is additive. It remains to show that G is left exact. Let

$$0 \to M_1 \xrightarrow{\nu} M_2 \xrightarrow{\mu} M_3 \to 0$$

be an exact sequence in $\mathcal{O}_{\mathcal{X}}$-mod. We show that the sequence

$$0 \to \overline{M_1} \xrightarrow{\overline{\nu}} \overline{M_2} \xrightarrow{\overline{\mu}} \overline{M_3}$$

is exact in $\mathcal{O}_{\overline{\mathcal{Y}}}$-mod. Since $\overline{M_1} \xrightarrow{\overline{\nu}} \overline{M_2}$ is the restriction of the injective morphism $M_1 \xrightarrow{\nu} M_2$, it follows that $\overline{\nu}$ is injective as well. Furthermore, since G is a functor and since $\mu \circ \nu$ is the zero morphism, we have that $\overline{\mu} \circ \overline{\nu}$ is zero as well, thus it is left to show that the kernel of $\overline{\mu}$ is contained in the image of $\overline{\nu}$.

So let $y \in \overline{M_2}$ such that $\overline{\mu}(y) = 0$. From the commutativity of

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \overline{M_1} & \xrightarrow{\overline{\nu}} & \overline{M_2} & \xrightarrow{\overline{\mu}} & \overline{M_3} \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\
0 & \longrightarrow & M_1 & \xrightarrow{\nu} & M_2 & \xrightarrow{\mu} & M_3
\end{array}
$$

whose vertical morphisms are the inclusions and from the exactness of the lower row, we get an element $x \in M_1$ such that $\nu(x) = i_2(y)$. Let $c \in \mathrm{ob}(\overline{\mathfrak{D}})$. Let $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$. Since y_c is sent to zero under $(M_2)_\gamma^{\mathrm{ad}}\colon (M_2)_c \to f_{\gamma*}(M_2)_d$ it follows from the commutativity of

$$
\begin{array}{ccccccc}
0 & \longrightarrow & (M_1)_c & \xrightarrow{\nu_c} & (M_2)_c & \xrightarrow{\mu_c} & (M_3)_c \\
 & & \downarrow{\scriptstyle (M_1)_\gamma^{\mathrm{ad}}} & & \downarrow{\scriptstyle (M_2)_\gamma^{\mathrm{ad}}} & & \downarrow{\scriptstyle (M_3)_\gamma^{\mathrm{ad}}} \\
0 & \longrightarrow & f_{\gamma*}(M_1)_d & \xrightarrow{f_{\gamma*}\nu_d} & f_{\gamma*}(M_2)_d & \xrightarrow{f_{\gamma*}\mu_d} & f_{\gamma*}(M_3)_d
\end{array}
$$

and the injectivity of $f_{\gamma*}\nu_d$ that x_c is sent to zero under $(M_1)_\gamma^{\mathrm{ad}}\colon (M_1)_c \to f_{\gamma*}(M_1)_d$ as well. This shows that $x \in \overline{M_1}$ and $\overline{\nu}(x) = y$.

Now $G\colon \mathcal{O}_{\mathcal{X}}$-mod $\to \mathcal{O}_{\overline{\mathcal{Y}}}$-mod induces a functor $\mathbf{Ch}(\mathcal{X}) \to \mathbf{Ch}(\overline{\mathcal{Y}})$ and since it preserves homotopy equivalences, there is an induced functor $G\colon \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\overline{\mathcal{Y}})$. $\qquad\square$

Now let $L \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}$-mod$)$ such that $uL = 0$ and let $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}$-mod$)$. We will construct a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M) \cong \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M})$$

of abelian groups, functorial in M.

Let $\psi \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M)$, given by a collection of elements $\psi_c \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}_c}\text{-mod}}(L_c, M_c)$ for $c \in \mathrm{ob}(\mathfrak{C})$. If $c \in \mathrm{ob}(\mathfrak{D})$, we have $L_c = 0$ since $uL = 0$, thus $\psi_c\colon 0 \to M_c$ is the zero morphism. If $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and if $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$, there is a commutative square

$$
\begin{array}{ccc}
L_c & \xrightarrow{\ L_\gamma^{\mathrm{ad}}\ } & f_{\gamma*}L_d = 0 \\
\downarrow{\scriptstyle \psi_c} & & \downarrow{\scriptstyle f_{\gamma*}\psi_d} \\
M_c & \xrightarrow{\ M_\gamma^{\mathrm{ad}}\ } & f_{\gamma*}M_d
\end{array}
$$

of $\mathcal{O}_{\mathcal{X}_c}$-modules. It follows that $\psi_c\colon L_c \to M_c$ factors through the kernel of M_γ^{ad} for each $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$ with $d \in \mathrm{ob}(\mathfrak{D})$, i.e., $\psi_c\colon L_c \to M_c$ factors through the submodule $\overline{M}_c$ of M_c. If $i_c\colon \overline{M}_c \to M_c$ is the inclusion, then there is a morphism of $\mathcal{O}_{\mathcal{X}_c}$-modules $\overline{\psi}_c\colon L_c \to \overline{M}_c$ such that

$$
\begin{array}{ccc}
L_c & \xrightarrow{\ \psi_c\ } & M_c \\
{\scriptstyle \overline{\psi}_c}\searrow & & \nearrow{\scriptstyle i_c} \\
& \overline{M}_c &
\end{array}
$$

is commutative.

If $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{D}}}(c_1, c_2)$, then the commutativity of

$$
\begin{array}{ccc}
L_{c_1} & \xrightarrow{\ L_\alpha^{\mathrm{ad}}\ } & f_{\alpha*}L_{c_2} \\
\downarrow{\scriptstyle \psi_{c_1}} & & \downarrow{\scriptstyle f_{\alpha*}\psi_{c_2}} \\
M_{c_1} & \xrightarrow{\ M_\alpha^{\mathrm{ad}}\ } & f_{\alpha*}M_{c_2}
\end{array}
$$

implies the commutativity of

$$
\begin{array}{ccc}
L_{c_1} & \xrightarrow{\ L_\alpha^{\mathrm{ad}}\ } & f_{\alpha*}L_{c_2} \\
\downarrow{\scriptstyle \overline{\psi}_{c_1}} & & \downarrow{\scriptstyle f_{\alpha*}\overline{\psi}_{c_2}} \\
\overline{M}_{c_1} & \xrightarrow{\ \overline{M}_\alpha^{\mathrm{ad}}\ } & f_{\alpha*}\overline{M}_{c_2}
\end{array}
$$

showing that the collection of all $\overline{\psi}_c\colon L_c \to \overline{M}_c$ for $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ defines an element

$$\overline{\psi} \in \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M}).$$

Lemma 3.24. *Let $\mathcal{Y}$ be a subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$, let $L \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$ such that $uL = 0$. Let further $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$. Then the map*

$$\mu_M \colon \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M) \to \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M}), \ \psi \mapsto \overline{\psi}$$

is an isomorphism of abelian groups. Furthermore, if $\nu \colon M_1 \to M_2$ is a morphism in $\mathcal{O}_{\mathcal{X}}\text{-mod}$, then

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M_1) & \xrightarrow{\ \mu_{M_1}\ } & \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M_1}) \\
{\scriptstyle \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L,\nu)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L,\overline{\nu})} \\
\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}(L, M_2) & \xrightarrow{\ \mu_{M_2}\ } & \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M_2})
\end{array}
\qquad (3.2)
$$

is commutative.

Proof. By definition μ_M sends ψ to the collection of those $\psi_c \colon L_c \to M_c$ such that $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and omits all irrelevant $\psi_c \colon 0 \to M_c$ for $c \in \mathrm{ob}(\mathfrak{D})$. Thus μ_M is an injective homomorphism of groups. On the other hand, given $\chi \in \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod}}(\overline{u}L, \overline{M})$, we take the composition $\psi_c \colon L_c \xrightarrow{\chi_c} \overline{M}_c \xrightarrow{i_c} M_c$ for $c \in \mathrm{ob}(\overline{\mathfrak{D}})$ and the zero morphisms $\psi_c \colon 0 \to M_c$ for $c \in \mathrm{ob}(\mathfrak{D})$ which together define a preimage of χ under μ_M. Thus μ_M is surjective.

Since $\overline{\nu} \colon \overline{M_1} \to \overline{M_2}$ is obtained by restricting $\nu \colon M_1 \to M_2$ to the submodule $\overline{M_1}$ we see that Square (3.2) is commutative. $\qquad\square$

Since $\overline{u}$ is the forgetful functor, μ_M is functorial in L for every $M \in \mathrm{ob}(\mathcal{O}_{\mathcal{X}}\text{-mod})$. It follows that there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ch}(\mathcal{X})}(L, M) \cong \mathrm{Hom}_{\mathbf{Ch}(\overline{\mathcal{Y}})}(\overline{u}L, \overline{M})$$

of abelian groups, functorial in $M \in \mathrm{ob}(\mathbf{Ch}(\mathcal{X}))$ and $L \in \mathrm{ob}(\mathbf{Ch}(\mathcal{X}))$ whenever $uL = 0$ in $\mathbf{Ch}(\mathcal{Y})$. For fixed $L \in \mathrm{ob}(\mathbf{Ch}(\mathcal{X}))$ such that $uL = 0$ in $\mathbf{Ch}(\mathcal{Y})$, there are natural isomorphisms

$$\mathrm{Hom}^{\cdot}_{\mathcal{X}}(L, M) \cong \mathrm{Hom}^{\cdot}_{\overline{\mathcal{Y}}}(\overline{u}L, \overline{M})$$

of complexes of abelian groups, functorial in M, i.e., if $\mathfrak{Ab}$ is the category of abelian groups, then the diagram

$$
\begin{array}{ccc}
\mathbf{K}^+(\mathcal{X}) & \xrightarrow{\quad G \quad} & \mathbf{K}^+(\overline{\mathcal{Y}}) \\
 & & \\
{\scriptstyle \mathrm{Hom}^{\cdot}_{\mathcal{X}}(L,-)} \searrow & & \swarrow {\scriptstyle \mathrm{Hom}^{\cdot}_{\overline{\mathcal{Y}}}(\overline{u}L,-)} \\
 & \mathbf{K}(\mathfrak{Ab}) &
\end{array}
$$

is commutative up to natural isomorphism of functors where $G \colon \mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(\overline{\mathcal{Y}})$ is the functor in Lemma 3.23.

Proposition 3.25. *Let $\mathcal{Y}$ be a subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$ and let*

$$u \colon \mathbf{Ch}(\mathcal{X}) \to \mathbf{Ch}(\mathcal{Y}) \quad \text{and} \quad \overline{u} \colon \mathbf{Ch}(\mathcal{X}) \to \mathbf{Ch}(\overline{\mathcal{Y}})$$

be the forgetful functors. Let $L \in \mathrm{ob}(\mathbf{Ch}^-(\mathcal{X}))$ be a bounded above cochain complex all of whose components are free $\mathcal{O}_{\mathcal{X}}$-modules such that $uL = 0$. Then there is a natural isomorphism of functors

$$\mathbf{R}\mathrm{Hom}_{\mathcal{X}}^{\cdot}(L, -) \cong \mathbf{R}\mathrm{Hom}_{\overline{\mathcal{Y}}}^{\cdot}(\overline{u}L, \mathbf{R}G(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathfrak{Ab})$. In particular, for each $M \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X}))$ and $i \in \mathbb{Z}$, there are natural isomorphisms of abelian groups

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^i(L, M) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Y}}}^i(\overline{u}L, \mathbf{R}G(M))$$

which are functorial in M.

Proof. We proceed similarly as in the proof of Proposition 2.14. By Theorem 1.9 there is a natural transformation of functors

$$\zeta \colon \mathbf{R}\mathrm{Hom}_{\mathcal{X}}^{\cdot}(L, -) \to \mathbf{R}\mathrm{Hom}_{\overline{\mathcal{Y}}}^{\cdot}(\overline{u}L, \mathbf{R}G(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathfrak{Ab})$. We take $\mathbf{K} = \mathbf{K}^+(\overline{\mathcal{Y}})$ in the notations of Theorem 1.9. In order to show that ζ is an isomorphism, it remains to show that each exact complex E of $\mathbf{K}^+(\overline{\mathcal{Y}})$ is $\mathrm{Hom}_{\overline{\mathcal{Y}}}^{\cdot}(\overline{u}L, -)$-acyclic, i.e., $H^i\big(\mathrm{Hom}_{\overline{\mathcal{Y}}}^{\cdot}(\overline{u}L, E)\big) = 0$ for all i. But

$$H^i\big(\mathrm{Hom}_{\overline{\mathcal{Y}}}^{\cdot}(\overline{u}L, E)\big) = \mathrm{Hom}_{\mathbf{K}(\overline{\mathcal{Y}})}(\overline{u}L, E[i])$$

which is zero because each morphism of complexes from a bounded above complex of free modules to an exact complex is homotopic to the zero morphism. $\square$

Corollary 3.26. *Under General assumption 2.1, let $\mathcal{Y}$ be a full and well-positioned subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$. Assume that the morphism a in Lemma 2.19 is a quasi-isomorphism. Then given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction*

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Y}}}^2(\overline{u}L_h, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. By Corollary 3.16 there is an obstruction in $\mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L_h^*, m_{\mathcal{X}}^*\mathcal{J})$. The complex L_h^* of Remark 3.8 satisfies the conditions of Proposition 3.25, furthermore, we have $\overline{u}L_h^* = \overline{u}L_h$ by definition of L_h^*. It follows from Proposition 3.25 that there is a natural isomorphism

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L_h^*, m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Y}}}^2(\overline{u}L_h, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$$

of abelian groups. $\square$

Hence for full and well-positioned subdiagrams $\mathcal{Y}$ such that the morphism a in Lemma 2.19 is a quasi-isomorphism, the obstruction group actually lives on the smaller diagram $\overline{\mathcal{Y}}$, at the expense of the complex of $\mathcal{O}_{\overline{\mathcal{Y}}}$-modules $\mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J})$ which is in general more difficult to handle then the $\mathcal{O}_{\mathcal{X}}$-module $m_{\mathcal{X}}^*\mathcal{J}$. Appendix B deals with the calculation of $\mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J})$ by using certain injective resolutions of $m_{\mathcal{X}}^*\mathcal{J}$. Corollary 3.36 will give an example where $\mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J})$ may be calculated explicitly.

Notice that we may not apply Proposition 3.25 to the complex $\mathrm{Cone}(m)$ in the obstruction group in Theorem 2.13 because it does not consist of free $\mathcal{O}_{\mathcal{X}}$-modules in general.

3.4 Particular cases

Using injective resolutions as in Corollary B.4, we are able to calculate the complexes $\mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J})$ of Corollary 3.26 for the following particular cases.

The first three subsections deal with subdiagrams obtained from the diagram by omitting exactly one level. For example, if we consider the diagram

$$
\begin{array}{ccc}
 & Y & \\
{}^{f}\nearrow & & \searrow^{g} \\
X & \xrightarrow[h=g\circ f]{} & Z,
\end{array}
$$

we will deal with the subdiagrams obtained by omitting the target Z, the source X and the bridge Y and their respective obstruction groups. The notions of target, source and bridge will be defined in the subsections. Motivated by deformation theoretic considerations, we will distinguish between cyclic and non-cyclic subdiagrams in Subsections 3.4.1 and 3.4.2.

Subsection 3.4.4 is devoted to another extreme case. If the subdiagram is obtained by keeping all levels, but by omitting all morphisms except for the identities, we get the discrete subdiagram.

We will consider several diagrams and subdiagrams with more than 3 levels, amongst them for example

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 1 in $\mathfrak{RTop}/\mathcal{S}$
subdiagram 2 in $\mathfrak{RTop}/\mathcal{S}$	subdiagram 3 in $\mathfrak{RTop}/\mathcal{S}$

with no other morphisms except for the identities where $j = h \circ f = i \circ g$.

3.4.1 Omitting a target

Definition 3.27. Let $\mathcal{X}$ be a diagram. A target of $\mathcal{X}$ is an element $0 \in \mathrm{ob}(\mathfrak{C})$ such that $\mathrm{Hom}_{\mathfrak{C}}(0,0)$ consists only of the identity and such that there are only morphisms from 0, i.e., $\mathrm{Hom}_{\mathfrak{C}}(c,0)$ is empty for $c \neq 0$. Equivalently, the corresponding level $\mathcal{X}_0$ has only morphisms of ringed topoi over $\mathcal{S}$ from the other levels and the only morphism from $\mathcal{X}_0$ is the identity. By abuse of notation we call $\mathcal{X}_0$ a target of $\mathcal{X}$.

A target $\mathcal{X}_0$ of $\mathcal{X}$ may be visualized in $\mathfrak{RTop}/\mathcal{S}$ as
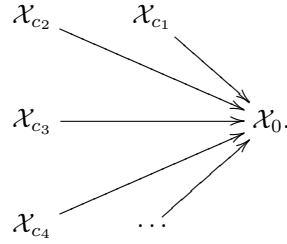
Proposition 3.28. *Assume given the situation of General assumption 2.1. Let $\mathcal{X}_0$ be a target of $\mathcal{X}$ and let $\mathfrak{D}$ be the full subcategory of $\mathfrak{C}$ with objects $\mathrm{ob}(\mathfrak{C}) \setminus \{0\}$. Then $\mathcal{Y}$ is a full and well-positioned subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$ equal to $\mathcal{X}_0$ and condition **2** in Theorem 2.20 is fulfilled.*

Let $L_{\mathcal{X}_0/\mathcal{S}}$ be the cotangent complex of $\mathcal{X}_0$ over $\mathcal{S}$ and let

$$G \colon \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\mathcal{X}_0)$$

be the functor in Lemma 3.23. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. Since $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$ it follows that $\mathcal{Y}$ is a full subdiagram of $\mathcal{X}$. Furthermore, $\mathcal{Y}$ is well-positioned in $\mathcal{X}$ since we have omitted a target of $\mathcal{X}$ and $\overline{\mathcal{Y}}$ is the ringed topos $\mathcal{X}_0$ because $\mathrm{Hom}_{\mathfrak{C}}(0,0) = \{\mathrm{id}\}$.

As 0 is a target of $\mathcal{X}$, the category $\underline{0}$ defined in Definition 2.3 is empty, whence

$$(u^{-1}\mathcal{O}_{\mathcal{Y}})_0 = t_0^{-1}\mathcal{O}_{\mathcal{S}}$$

by Remark 2.5. On the other hand we have $(u^{-1}\mathcal{O}_{\mathcal{Y}})_d = \mathcal{O}_{\mathcal{X}_d}$ for every $d \neq 0$ by Lemma 3.13. It follows that condition **2** in Theorem 2.20 is satisfied.

By Corollary 3.26 there is an obstruction lying in $\mathbb{Ext}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$. The ring morphism $h_0 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_0 \to \mathcal{O}_{\mathcal{X}_0}$ is just $\theta_0 \colon t_0^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}_0}$, the ring morphism of $t_0 \colon \mathcal{X}_0 \to \mathcal{S}$. Therefore we have $\overline{u}L_h = L_{h_0} = L_{\mathcal{X}_0/\mathcal{S}}$ an the obstruction lies in

$$\mathbb{Ext}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J})) = \mathbb{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$$

$\square$

In general, we are able to control the cotangent complex $L_{\mathcal{X}_0/\mathcal{S}}$ in practice since it is the cotangent complex of the structure morphism $t_0\colon \mathcal{X}_0 \to \mathcal{S}$. But it is more difficult to control $\mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J})$. We will use injective resolutions as in Corollary B.4.

Let $\mathcal{X}_0$ be a target of a diagram $\mathcal{X}$ and let M be an $\mathcal{O}_{\mathcal{X}}$-module. Then $G(M)$ is the submodule

$$\overline{M}_0 = \bigcap_{\substack{\gamma\in\mathrm{Hom}_{\mathfrak{C}}(0,d)\\d\neq 0}} \ker\left(M_\gamma^{\mathrm{ad}}\colon M_0 \to f_{\gamma*}M_d\right) \subseteq M_0$$

of M_0 by Definition 3.19.

Notation 3.29. Let Γ be the set of those morphisms $\gamma\colon 0 \to d$ in $\mathfrak{C}$ with $d \neq 0$ which do not factor in $\mathfrak{C}$ through some $d' \in \mathrm{ob}(\mathfrak{C})\setminus\{0\}$ except for the trivial factorization $0 \xrightarrow{\gamma} d \xrightarrow{\mathrm{id}} d$.

By Remark 3.20 we may restrict to all morphisms in Γ when calculating $\overline{M}_0$, i.e., we have

$$\overline{M}_0 = \bigcap_{\gamma\in\Gamma} \ker\left(M_\gamma^{\mathrm{ad}}\colon M_0 \to f_{\gamma*}M_d\right) \subseteq M_0.$$

For each $c \in \mathrm{ob}(\mathfrak{C})$ choose an injection $j_c^0\colon M_c \to J_c^0$ to an injective $\mathcal{O}_{\mathcal{X}_c}$-module J_c^0. Let I^0 be the injective $\mathcal{O}_{\mathcal{X}}$-module consisting of the modules

$$I_c^0 = \bigoplus_{\substack{\delta\in\mathrm{Hom}_{\mathfrak{C}}(c,c'),\\c'\in\mathrm{ob}(\mathfrak{C})}} f_{\delta*}J_{c'}^0$$

as in Lemma B.2 and $i^0\colon M \to I^0$ the injection as in Proposition B.3. Let Q be the cokernel of $i^0\colon M \to I^0$ and choose injections $j_c^1\colon Q_c \to J_c^1$ to injective $\mathcal{O}_{\mathcal{X}_c}$-modules J_c^1 as in the proof of Proposition B.3. Continuing this way, we get an injective resolution

$$0 \to M \xrightarrow{i^0} I^0 \xrightarrow{i^1} I^1 \xrightarrow{i^2} I^2 \xrightarrow{i^3} I^3 \to \dots$$

of M as in Corollary B.4. Now fix $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(0,d)$ such that $d \neq 0$ and $\gamma \in \Gamma$. Then the chosen injective resolution of M yields morphisms

$$\mathrm{pr}_\gamma^n\colon I_0^n = \bigoplus_{\substack{\delta\in\mathrm{Hom}_{\mathfrak{C}}(0,c'),\\c'\in\mathrm{ob}(\mathfrak{C})}} f_{\delta*}J_{c'}^n \to \bigoplus_{\substack{\varepsilon\in\mathrm{Hom}_{\mathfrak{C}}(d,c'),\\c'\in\mathrm{ob}(\mathfrak{C})}} f_{\varepsilon\gamma*}J_{c'}^n \cong f_{\gamma*}\left(\bigoplus_{\substack{\varepsilon\in\mathrm{Hom}_{\mathfrak{C}}(d,c'),\\c'\in\mathrm{ob}(\mathfrak{C})}} f_{\varepsilon*}J_{c'}^n\right) = f_{\gamma*}I_d^n$$

of $\mathcal{O}_{\mathcal{X}_0}$-modules given by projecting whenever $0 \xrightarrow{\delta} c'$ factors through γ as $0 \xrightarrow{\gamma} d \xrightarrow{\varepsilon} c'$ and a commutative diagram

of $\mathcal{O}_{\mathcal{X}_0}$-modules. Now for a finite family of $\mathcal{O}_{\mathcal{X}_0}$-modules $(B_\lambda)_{\lambda\in\Lambda}$ and $\mathcal{O}_{\mathcal{X}_0}$-linear morphisms $c_\lambda\colon A \to B_\lambda$ let

$$D(c_\lambda)\colon A \to \bigoplus_{\lambda\in\Lambda} B_\lambda$$

be the composition

$$A \xrightarrow{\text{diag}} \bigoplus_{\lambda \in \Lambda} A \xrightarrow{\oplus_{\lambda \in \Lambda} c_\lambda} \bigoplus_{\lambda \in \Lambda} B_\lambda , \quad a \mapsto (a, \ldots, a) \mapsto (c_\lambda(a))_{\lambda \in \Lambda}.$$

Taking $\Lambda = \Gamma$, we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \xrightarrow{i_0^0} & I_0^0 & \xrightarrow{i_0^1} & I_0^1 & \xrightarrow{i_0^2} & \cdots \\
& & \downarrow{\scriptstyle D(M_\gamma)} & & \downarrow{\scriptstyle D(\text{pr}_\gamma^0)} & & \downarrow{\scriptstyle D(\text{pr}_\gamma^1)} & & \\
0 & \longrightarrow & \bigoplus_\gamma f_{\gamma*} M_d & \xrightarrow{\oplus_\gamma f_{\gamma*} i_d^0} & \bigoplus_\gamma f_{\gamma*} I_d^0 & \xrightarrow{\oplus_\gamma f_{\gamma*} i_d^1} & \bigoplus_\gamma f_{\gamma*} I_d^1 & \xrightarrow{\oplus_\gamma f_{\gamma*} i_d^2} & \cdots
\end{array}
$$

whose first row is exact and whose second row is not exact in general. Let us exhibit the kernel and the cokernel of

$$D(\text{pr}_\gamma^n) \colon \ I_0^n \ = \bigoplus_{\substack{\delta \in \text{Hom}_{\mathfrak{C}}(0,c'), \\ c' \in \text{ob}(\mathfrak{C})}} f_{\delta*} J_{c'}^n \ \to \bigoplus_{\substack{\gamma \in \Gamma, \\ \gamma \colon 0 \to d}} \bigoplus_{\substack{\varepsilon \in \text{Hom}_{\mathfrak{C}}(d,c'), \\ c' \in \text{ob}(\mathfrak{C})}} f_{\varepsilon\gamma*} J_{c'}^n \ \cong \bigoplus_{\substack{\gamma \in \Gamma, \\ \gamma \colon 0 \to d}} f_{\gamma*} I_d^n.$$

Since each $f_{\delta*} J_{c'}^n$ for $\delta \in \text{Hom}_{\mathfrak{C}}(0,c')$ with $c' \neq 0$ occurs somewhere among the $f_{\varepsilon\gamma*} J_{c'}^n$, we see that the kernel of $D(\text{pr}_\gamma^n)$ is just the direct summand belonging to the identity $0 \to 0$, i.e.,

$$\ker(D(\text{pr}_\gamma^n)) = J_0^n. \tag{3.3}$$

For $\delta \in \text{Hom}_{\mathfrak{C}}(0,c')$ with $c' \neq 0$ let $\mu(\delta)$ be the number of factorizations

$$
\begin{array}{ccc}
0 & \xrightarrow{\ \delta\ } & c' \\
& {\scriptstyle \gamma} \searrow \quad \nearrow {\scriptstyle \varepsilon} & \\
& d &
\end{array}
\tag{3.4}
$$

in $\mathfrak{C}$ such that $d \neq 0$ and $\gamma \colon 0 \to d$ does not factor in $\mathfrak{C}$ through some $d' \in \text{ob}(\mathfrak{C}) \setminus \{0\}$ except for the trivial factorization $0 \xrightarrow{\gamma} d \xrightarrow{\text{id}} d$, i.e., such that $\gamma \in \Gamma$. These are the relevant γ we have restricted our attention to. Notice that $\mu(\delta) \geq 1$ and that the given $\delta \colon 0 \to c'$ might have several factorizations through the same $\gamma \colon 0 \to d$, i.e., there might be different $\varepsilon_1, \varepsilon_2 \colon d \to c'$ such that $\varepsilon_1 \gamma = \delta = \varepsilon_2 \gamma$. By construction $\mu(\delta)$ is equal to the number of times the direct summand $f_{\varepsilon\gamma*} J_{c'}^n$ occurs in $\bigoplus_\gamma f_{\gamma*} I_d^n$ whenever $\delta = \varepsilon\gamma$. Notice that for each $\delta \in \text{Hom}_{\mathfrak{C}}(0,c')$ with $c' \neq 0$, the direct summand $f_{\delta*} J_{c'}^n$ always occurs with multiplicity one in I_0^n.

Lemma 3.30. *The cokernel of $D(\text{pr}_\gamma^n)$ is isomorphic to*

$$\bigoplus_{\substack{\delta \in \text{Hom}_{\mathfrak{C}}(0,c') \\ c' \neq 0}} \bigoplus_{j=1}^{\mu(\delta)-1} f_{\delta*} J_{c'}^n.$$

In particular, $D(\text{pr}_\gamma^n)$ is surjective if and only if $\mu(\delta) = 1$ for all $\delta \in \text{Hom}_{\mathfrak{C}}(0,c')$ with $c' \neq 0$.

Proof. For every $\delta \in \text{Hom}_{\mathfrak{C}}(0,c')$ with $c' \neq 0$, the restriction of $D(\text{pr}_\gamma^n)$ to $f_{\delta*} J_{c'}^n$ is given by

$$f_{\delta*} J_{c'}^n \xrightarrow{\text{diag}} \bigoplus_{j=1}^{\mu(\delta)} f_{\delta*} J_{c'}^n , \quad x \mapsto (x, \ldots, x)$$

and has cokernel $\displaystyle\bigoplus_{j=1}^{\mu(\delta)-1} f_{\delta*} J_{c'}^n$. $\qquad \square$

We have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \overline{M}_0 & \xrightarrow{\;j_0^0|_{\overline{M}_0}\;} & J_0^0 & \xrightarrow{\;i_0^1|_{J_0^0}\;} & J_0^1 & \xrightarrow{\;i_0^2|_{J_0^1}\;} & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & M_0 & \xrightarrow{\;i_0^0\;} & I_0^0 & \xrightarrow{\;i_0^1\;} & I_0^1 & \xrightarrow{\;i_0^2\;} & \cdots \\
 & \downarrow{\scriptstyle D(M_\gamma)} & & \downarrow{\scriptstyle D(\mathrm{pr}_\gamma^0)} & & \downarrow{\scriptstyle D(\mathrm{pr}_\gamma^1)} & & \\
0 \longrightarrow & \bigoplus_\gamma f_{\gamma*}M_d & \xrightarrow{\;\oplus_\gamma f_{\gamma*}i_d^0\;} & \bigoplus_\gamma f_{\gamma*}I_d^0 & \xrightarrow{\;\oplus_\gamma f_{\gamma*}i_d^1\;} & \bigoplus_\gamma f_{\gamma*}I_d^1 & \xrightarrow{\;\oplus_\gamma f_{\gamma*}i_d^2\;} & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathrm{cok}(D(M_\gamma)) & \longrightarrow & \mathrm{cok}(D(\mathrm{pr}_\gamma^0)) & \longrightarrow & \mathrm{cok}(D(\mathrm{pr}_\gamma^1)) & \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}
\tag{3.5}
$$

of $\mathcal{O}_{\mathcal{X}_0}$-modules all of whose columns are exact. The direct sums are taken over the index set Γ. Furthermore, the second row

$$0 \to M_0 \xrightarrow{\;i_0^0\;} I_0^0 \xrightarrow{\;i_0^1\;} I_0^1 \to \dots$$

is exact because it is the part on the level 0 of the chosen injective resolution of M. Remember that the second row is not an injective resolution of M_0 since the I_0^n are no injective $\mathcal{O}_{\mathcal{X}_0}$-modules in general. Moreover, the first, third and forth row are not exact in general.

Since $0 \to M \xrightarrow{\;i^0\;} I^0 \xrightarrow{\;i^1\;} I^1 \to \dots$ is an injective resolution of M, we have

$$\mathbf{R}G(M) = \left(J_0^0 \xrightarrow{\;i_0^1|_{J_0^0}\;} J_0^1 \xrightarrow{\;i_0^2|_{J_0^1}\;} J_0^2 \xrightarrow{\;i_0^3|_{J_0^2}\;} J_0^3 \longrightarrow \dots \right).$$

In order to calculate this complex, we distinguish between two cases. The case $\mu(\delta) = 1$ for every $\delta \in \mathrm{Hom}_{\mathfrak{C}}(0, c')$ with $c' \neq 0$ is particularly interesting.

Definition 3.31. Let $\mathcal{X}_0$ be a target of $\mathcal{X}$. Then $\mathfrak{C}$ is called non-cyclic (with respect to 0) if every $\delta \in \mathrm{Hom}_{\mathfrak{C}}(0, c')$ with $c' \neq 0$ has a unique factorization in $\mathfrak{C}$ of the form $0 \xrightarrow{\;\gamma\;} d \xrightarrow{\;\varepsilon\;} c'$ where $d \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$ and $\gamma \in \Gamma$. The case $\varepsilon = \mathrm{id}$ is possible. Otherwise $\mathfrak{C}$ is called cyclic (with respect to 0). The diagram $\mathcal{X}$ is called non-cyclic (with respect to $\mathcal{X}_0$) if $\mathfrak{C}$ is non-cyclic with respect to 0. Otherwise $\mathcal{X}$ is called cyclic (with respect to $\mathcal{X}_0$).

So $\mathcal{X}$ is non-cyclic if and only if $\mu(\delta) = 1$ for every $\delta \in \mathrm{Hom}_{\mathfrak{C}}(0, c')$ with $c' \neq 0$.

Example 3.32. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $\mathfrak{C}_4$ be the categories
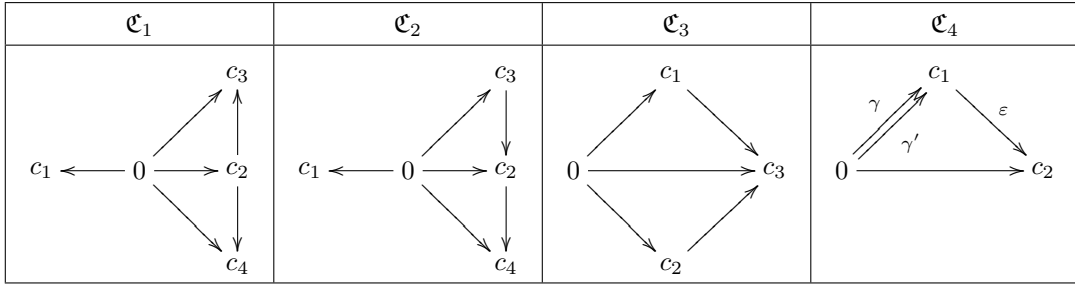
$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$
$c_1 \leftarrow 0 \rightarrow c_2$, with $0 \to c_3$, $c_2 \to c_3$, $0 \to c_4$, $c_2 \to c_4$	$c_1 \leftarrow 0 \rightarrow c_2$, with $0 \to c_3$, $c_3 \to c_2$, $0 \to c_4$, $c_2 \to c_4$	$0 \to c_1$, $0 \to c_3$, $c_1 \to c_3$, $0 \to c_2$, $c_2 \to c_3$	$0 \rightrightarrows c_1$ via γ, γ', $0 \to c_2$, $c_1 \to c_2$ via ε

with no other morphisms besides the identities such that γ and γ' in $\mathfrak{C}_4$ are distinct and such that in each category, the composition of any two consecutive morphisms is the displayed morphism. Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are non-cyclic and $\mathfrak{C}_3$ and $\mathfrak{C}_4$ are cyclic.

A non-cyclic category $\mathfrak{C}$ might be visualized as
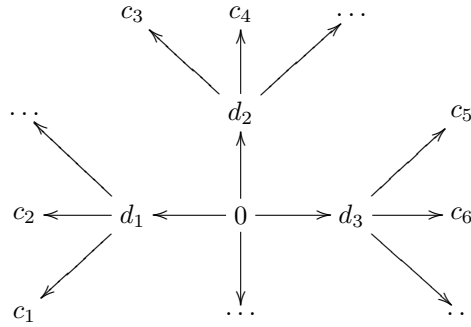
with no other morphisms involved except for the identities and the composition of any consecutive morphisms. Notice that the notion "non-cyclic with respect to 0" does not necessarily mean that the directed graph is a tree with root 0 because, for instance, the directed graph

$$0 \longrightarrow d \longleftarrow c$$

is not a tree, but the corresponding category $\mathfrak{C}$ is non-cyclic with respect to 0.

Our next aim is to show that in the case of non-cyclic diagrams, we may restrict to the relevant morphisms $\gamma \in \Gamma$ not only for calculating $G(M)$ as we have already seen in Remark 3.20, but also for calculating $\mathbf{R}G(M)$ which is not clear a priori. The idea behind the possibility of this restriction is the following. Suppose given the two pairs
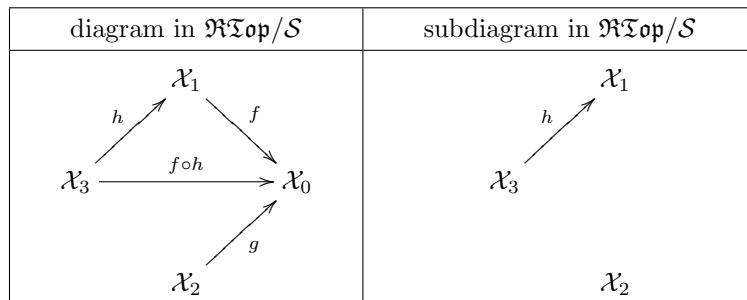
diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_3 \xrightarrow{h} \mathcal{X}_1 \xrightarrow{f} \mathcal{X}_0$, $\mathcal{X}_3 \xrightarrow{f \circ h} \mathcal{X}_0$, $\mathcal{X}_2 \xrightarrow{g} \mathcal{X}_0$	$\mathcal{X}_3 \xrightarrow{h} \mathcal{X}_1$, $\mathcal{X}_2$

and

69

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1 \xrightarrow{f} \mathcal{X}_0 \xleftarrow{g} \mathcal{X}_2$	$\mathcal{X}_1 \qquad \mathcal{X}_2$

with no other morphisms involved except for the identities where the subdiagrams are obtained by omitting the target $\mathcal{X}_0$ in both cases. Each diagram is non-cyclic with respect to $\mathcal{X}_0$. The obstruction for extending a given deformation of the subdiagram to a deformation of the diagram should be the same because in both cases we only have to find a deformation $\mathcal{X}_0'$ of $\mathcal{X}_0$ and deformations of f and g from the given deformations of $\mathcal{X}_1$ and $\mathcal{X}_2$ to $\mathcal{X}_0'$, respectively. On the other hand, if we are given the two pairs

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_3 \xrightarrow{g_1} \mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_0$, $\mathcal{X}_3 \to \mathcal{X}_0$, $\mathcal{X}_3 \xrightarrow{g_2} \mathcal{X}_2 \xrightarrow{f_2} \mathcal{X}_0$	$\mathcal{X}_3 \xrightarrow{g_1} \mathcal{X}_1$, $\mathcal{X}_3 \xrightarrow{g_2} \mathcal{X}_2$

and

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_0 \xleftarrow{f_2} \mathcal{X}_2$	$\mathcal{X}_1 \qquad \mathcal{X}_2$

where the first diagram is cyclic, then the obstructions should be different because in the first case we have to make sure that the commutativity still holds whereas in the second case this problem does not occur.

Definition 3.33. Let $\mathcal{X}$ be non-cyclic with respect to the target $\mathcal{X}_0$. We define $\mathfrak{C}_\Gamma$ to be the subcategory of $\mathfrak{C}$ with objects

$$\mathrm{ob}(\mathfrak{C}_\Gamma) = \{0\} \cup \{d \in \mathrm{ob}(\mathfrak{C}) \mid \text{there exists } \big(\gamma \colon 0 \to d\big) \in \Gamma\}$$

and morphisms

$$\mathrm{Hom}_{\mathfrak{C}_\Gamma}(0, d) = \{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(0, d) \mid \gamma \in \Gamma\}$$

for $d \neq 0$, besides the identities.

This category may be visualized as

$$\begin{array}{ccccccc} d_1 & & d_2 & & d_3 & \cdots & d_m \end{array}$$

$$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \quad \gamma_m$$

$$0$$
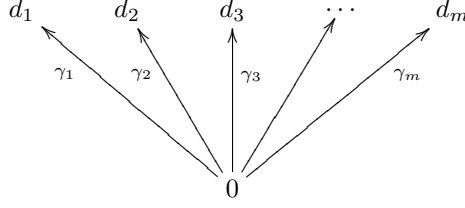
where the objects d_i do not have to be pairwise distinct, but if $d_i = d_j$ for $i \neq j$ then $\gamma_i \neq \gamma_j$. Let $\mathcal{X}_\Gamma$ be the ringed topos associated to the restriction of the diagram $I \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ to the subcategory $\mathfrak{C}^{\mathrm{op}}_\Gamma$ of $\mathfrak{C}^{\mathrm{op}}$. Let

$$u_\Gamma \colon \mathcal{X} \to \mathcal{X}_\Gamma \tag{3.6}$$

be the forgetful functor from $\mathcal{X}$ to the subdiagram $\mathcal{X}_\Gamma$ and let $G_\Gamma \colon \mathcal{O}_{\mathcal{X}_\Gamma}\text{-mod} \to \mathcal{O}_{\mathcal{X}_0}\text{-mod}$ be the functor in Lemma 3.23. Then by construction we have commutative diagrams

$$\mathcal{O}_\mathcal{X}\text{-mod} \xrightarrow{u_\Gamma} \mathcal{O}_{\mathcal{X}_\Gamma}\text{-mod} \qquad \qquad \mathbf{K}(\mathcal{X})^+ \xrightarrow{u_\Gamma} \mathbf{K}(\mathcal{X}_\Gamma)^+$$

$$G \searrow \qquad \swarrow G_\Gamma \qquad \text{and} \qquad \qquad G \searrow \qquad \swarrow G_\Gamma$$

$$\mathcal{O}_{\mathcal{X}_0}\text{-mod} \qquad \qquad \qquad \mathbf{K}(\mathcal{X}_0).$$

Proposition 3.34. *Let $\mathcal{X}$ be non-cyclic with respect to the target $\mathcal{X}_0$. Then there is a natural isomorphism of functors*

$$\mathbf{R}G(-) \cong \mathbf{R}G_\Gamma(u_\Gamma(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathcal{X}_0)$. In particular, for every $\mathcal{O}_\mathcal{X}$-module M there is a natural isomorphism $\mathbf{R}G(M) \cong \mathbf{R}G_\Gamma(u_\Gamma(M))$ in $\mathbf{D}(\mathcal{X}_0)$.

Proof. By Proposition 1.9 there is a natural transformation of functors

$$\mathbf{R}G(-) \to \mathbf{R}G_\Gamma(\mathbf{R}u_\Gamma(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathcal{X}_0)$. We have seen in Proposition B.3 that each $\mathcal{O}_\mathcal{X}$-module admits an injection to an injective $\mathcal{O}_\mathcal{X}$-module I of the form in Lemma B.2. Furthermore the natural morphism of functors $u_\Gamma(-) \to \mathbf{R}u_\Gamma(-)$ is an isomorphism because u_Γ is exact. Thus by [Wei94, Corollary 10.8.3] it is enough to show that each injective $\mathcal{O}_\mathcal{X}$-module I of the form in Lemma B.2 is sent under u_Γ to a G_Γ-acyclic object.

So let I be an injective $\mathcal{O}_\mathcal{X}$-module built from injective $\mathcal{O}_{\mathcal{X}_c}$-modules J_c for $c \in \mathrm{ob}(\mathfrak{C})$ as in Lemma B.2. Then $M = u_\Gamma(I)$ is given levelwise by

$$M_0 = \bigoplus_{\substack{\delta \in \mathrm{Hom}_\mathfrak{C}(0,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\delta *} J_{c'}$$

and

$$M_d = \bigoplus_{\substack{\varepsilon \in \mathrm{Hom}_\mathfrak{C}(d,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\varepsilon *} J_{c'}$$

71

for $d \in \mathrm{ob}(\mathfrak{C}_\Gamma)$, $d \neq 0$. For every $\gamma \in \mathrm{Hom}_{\mathfrak{C}_\Gamma}(0,d)$, $d \neq 0$, we have a morphism

$$M_\gamma \colon\ M_0\ =\ \bigoplus_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(0,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\delta *} J_{c'}\ \to\ \bigoplus_{\substack{\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(d,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\varepsilon\gamma *} J_{c'}\ \cong\ f_{\gamma *}\Big(\bigoplus_{\substack{\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(d,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\varepsilon *} J_{c'} \Big)\ =\ f_{\gamma *} M_d$$

given by projecting whenever δ factors as $0 \xrightarrow{\gamma} d \xrightarrow{\varepsilon} c'$ and these projections define a morphism

$$D(M_\gamma)\colon\ M_0\ =\ \bigoplus_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(0,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\delta *} J_{c'}\ \to\ \bigoplus_{\substack{\gamma \in \Gamma, \\ \gamma\colon 0 \to d}}\ \bigoplus_{\substack{\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(d,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\varepsilon\gamma *} J_{c'}\ \cong\ \bigoplus_{\gamma \in \Gamma} f_{\gamma *} M_d$$

as in Diagram (3.5) on page 68. Since $\mathcal{X}$ is non-cyclic with respect to the target $\mathcal{X}_0$, it follows that $D(M_\gamma)$ is surjective by Lemma 3.30. Moreover we have $\ker(D(M_\gamma)) = J_0$ as we have seen in Equation (3.3) on page 67. Now for every $d \in \mathrm{ob}(\mathfrak{C}_\Gamma)$ with $d \neq 0$ choose an injective resolution

$$0 \to M_d \xrightarrow{\hat{j}_d^0} \hat{J}_d^0 \xrightarrow{\hat{j}_d^1} \hat{J}_d^1 \xrightarrow{\hat{j}_d^2} \hat{J}_d^2 \to \dots$$

of M_d and choose an injection $\hat{j}_0^0 \colon M_0 \to \hat{J}_0^0$ of M_0 to an injective $\mathcal{O}_{\mathcal{X}_0}$-module $\hat{J}_0^0$. Let $\hat{I}^0$ be the injective $\mathcal{O}_{\mathcal{X}_\Gamma}$-module built from the $\hat{J}_d^0$ and $\hat{J}_0^0$, let $\hat{i}^0 \colon M \to \hat{I}^0$ be the injection of Proposition B.3 and let $\hat{Q}^0$ be the cokernel of $\hat{i}^0$. Since there are no morphisms from $d \neq 0$ in $\mathfrak{C}_\Gamma$ except for the identities, we have by definition

$$\hat{I}_d^0 = \hat{J}_d^0$$

for every $d \in \mathrm{ob}(\mathfrak{C}_\Gamma) \setminus \{0\}$ and $\hat{Q}_d^0$, the cokernel of $\hat{i}_d^0 \colon M_d \to \hat{I}_d^0$, is isomorphic to the cokernel of $\hat{j}_d^0 \colon M_d \to \hat{J}_d^0$. After choosing an injection $\hat{Q}_0^0 \to \hat{J}_0^1$ to an injective $\mathcal{O}_{\mathcal{X}_0}$-module $\hat{J}_0^1$, let $\hat{I}^1$ be the injective $\mathcal{O}_{\mathcal{X}_\Gamma}$-module built from the $\hat{J}_d^1$ and $\hat{J}_0^1$. Continuing this way and choosing $\hat{I}_d^n = \hat{J}_d^n$ for all $n \in \mathbb{N}$ and all $d \in \mathrm{ob}(\mathfrak{C}_\Gamma) \setminus \{0\}$, we get an injective resolution

$$0 \to M \xrightarrow{\hat{i}^0} \hat{I}^0 \xrightarrow{\hat{i}^1} \hat{I}^1 \xrightarrow{\hat{i}^2} \hat{I}^2 \to \dots$$

of M and for every $\gamma \in \mathrm{Hom}_{\mathfrak{C}_\Gamma}(0,d)$, $d \neq 0$, we have a commutative diagram

of $\mathcal{O}_{\mathcal{X}_0}$-modules which together induce a commutative diagram

whose first column is exact because $D(M_\gamma)$ is surjective, all of whose other columns are exact by Lemma 3.30 since $\mathcal{X}_\Gamma$ is non-cyclic with respect to the target $\mathcal{X}_0$, and whose middle row is exact because it is the part on the level $\mathcal{X}_0$ of the injective resolution of M.

Remember that each injective module is flasque by [SGA42, V.4.6.] and the direct image functor of a morphism of ringed topoi sends flasque modules to flasque modules by [SGA42, Proposition V.4.9.]. In particular, for each $d \in \mathrm{ob}(\mathfrak{C}_\Gamma)$ the $\mathcal{O}_{\mathcal{X}_d}$-module

$$M_d = \bigoplus_{\substack{\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(d, c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\varepsilon *} J_{c'}$$

is flasque. Thus the chosen injective resolutions

$$0 \to M_d \xrightarrow{\hat{j}_d^0} \hat{J}_d^0 \xrightarrow{\hat{j}_d^1} \hat{J}_d^1 \xrightarrow{\hat{j}_d^2} \hat{J}_d^2 \to \ldots$$

yield exact sequences

$$0 \to f_{\gamma *} M_d \xrightarrow{f_{\gamma *} \hat{j}_d^0} f_{\gamma *} \hat{J}_d^0 \xrightarrow{f_{\gamma *} \hat{j}_d^1} f_{\gamma *} \hat{J}_d^1 \xrightarrow{f_{\gamma *} \hat{j}_d^2} f_{\gamma *} \hat{J}_d^2 \to \ldots$$

for each $\gamma \in \mathrm{Hom}_{\mathfrak{C}_\Gamma}(0, d)$, $d \neq 0$. But since $\hat{I}_d^n = \hat{J}_d^n$ for all $n \in \mathbb{N}$ and all $d \in \mathrm{ob}(\mathfrak{C}_\Gamma) \setminus \{0\}$ by our choice, we see that the lower row of the above commutative diagram is exact as well. The exactness of the middle and the lower row imply the exactness of the first row, therefore the i-th cohomology

$$H^i\big(G_\Gamma(u_\Gamma(I))\big) = H^i\big(G_\Gamma(M)\big) = H^i\big(G_\Gamma(I^\bullet)\big) = H^i\big((\hat{J}_0^0 \to \hat{J}_0^1 \to \hat{J}_0^2 \to \ldots)\big)$$

is zero for $i > 0$, showing that $M = u_\Gamma(I)$ is acyclic for G_Γ. $\qquad\square$

Consequently, if we are only interested in calculating $\mathbf{R}G(M)$ for an $\mathcal{O}_{\mathcal{X}}$-module M, we may always assume that the category $\mathfrak{C}$ defining the given non-cyclic diagram $\mathcal{X}$ is equal to its subcategory $\mathfrak{C}_\Gamma$. In general this strongly simplifies the calculation of

$$\mathbf{R}G(M) = \left(J_0^0 \xrightarrow{i_0^1|_{J_0^0}} J_0^1 \xrightarrow{i_0^2|_{J_0^1}} J_0^2 \xrightarrow{i_0^3|_{J_0^2}} J_0^3 \longrightarrow \ldots \right)$$

from Diagram (3.5) on page 68 because if $\mathfrak{C} = \mathfrak{C}_\Gamma$, we may start with injective resolutions

$$0 \to M_c \xrightarrow{j_c^0} J_c^0 \xrightarrow{j_c^1} J_c^1 \xrightarrow{j_c^2} J_c^2 \to \ldots$$

of M_c for every $c \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$.

Proposition 3.35. *Let $\mathcal{X}$ be non-cyclic with respect to the target $\mathcal{X}_0$. Let M be an $\mathcal{O}_{\mathcal{X}}$-module. If the third row of Diagram (3.5) is exact, then there is a natural isomorphism*

$$\mathbf{R}G(M) \cong \left(\overline{M}_0 \xrightarrow{0} \mathrm{cok}(D(M_\gamma)) \to 0 \to 0 \to \ldots \right)$$

in $\mathbf{D}(\mathcal{X}_0)$, where $\overline{M}_0$ is placed in degree 0. In particular, there is a natural isomorphism

$$\mathbb{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathbf{R}G(M)) \cong \mathbb{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \overline{M}_0) \oplus \mathbb{Ext}^1_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathrm{cok}(D(M_\gamma)))$$

Proof. Since $\mathcal{X}$ is non-cyclic, it follows that $\mathrm{cok}(D(\mathrm{pr}_\gamma^n)) = 0$ for all $n \geq 0$ by Lemma 3.30. Each column and the second and third row of Diagram (3.5) are exact. From the short exact sequence of complexes

$$0 \to J_0^\bullet \to I_0^\bullet \xrightarrow{D(\mathrm{pr}_\gamma^\bullet)} \bigoplus_\gamma f_{\gamma*} I_d^\bullet \to 0$$

defined by

we get a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(J_0^\bullet) \longrightarrow H^0(I_0^\bullet) \longrightarrow H^0(\bigoplus_\gamma f_{\gamma*} I_d^\bullet) \longrightarrow H^1(J_0^\bullet) \longrightarrow H^1(I_0^\bullet) \longrightarrow \cdots$$

and by exactness of the second and third row of Diagram (3.5) we have

$$H^n(I_0^\bullet) = H^n(\bigoplus_\gamma f_{\gamma*} I_d^\bullet) = 0$$

for all $n \geq 1$. From the exactness of the above cohomology sequence it follows $H^n(J_0^\bullet) = 0$ for all $n \geq 2$ and the cohomology sequence is given by

$$0 \longrightarrow \overline{M}_0 \longrightarrow M_0 \xrightarrow{D(M_\gamma)} \bigoplus_\gamma f_{\gamma*} M_d \longrightarrow \mathrm{cok}(D(M_\gamma)) \longrightarrow 0 \longrightarrow \cdots .$$

In particular, the complex $J_0^\bullet$ is exact except for the places J_0^0 and J_0^1 and the morphism of complexes

is a quasi-isomorphism, i.e., an isomorphism in $\mathbf{D}(\mathcal{X}_0)$. Hence we have

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathbf{R}G(M)) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \overline{M}_0) \oplus \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathrm{cok}(D(M_\gamma))[-1])$$
$$\cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \overline{M}_0) \oplus \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathrm{cok}(D(M_\gamma))).$$

$\square$

Corollary 3.36. *Let $\mathcal{X}$ be non-cyclic with respect to the target $\mathcal{X}_0$ and let M be an $\mathcal{O}_\mathcal{X}$-module. If $R^i f_{\gamma*} M_d = 0$ for all $i > 0$ and all $(\gamma: 0 \to d) \in \Gamma$, then there is a natural isomorphism*

$$\mathbf{R}G(M) \cong \left(\overline{M}_0 \xrightarrow{0} \mathrm{cok}(D(M_\gamma)) \to 0 \to 0 \to \dots \right)$$

in $\mathbf{D}(\mathcal{X}_0)$, where $\overline{M}_0$ is placed in degree 0. In particular, there is a natural isomorphism

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathbf{R}G(M)) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \overline{M}_0) \oplus \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, \mathrm{cok}(D(M_\gamma))).$$

Proof. By Proposition 3.34 we may assume $\mathfrak{C} = \mathfrak{C}_\Gamma$. Moreover in this case the construction of Diagram (3.5) may be started with injective resolutions

$$0 \to M_c \xrightarrow{j_c^0} J_c^0 \xrightarrow{j_c^1} J_c^1 \xrightarrow{j_c^2} J_c^2 \to \dots$$

of M_c for every $c \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$. Since $R^i f_{\gamma*} M_c = 0$ for all $i > 0$ and all $c \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$, the sequences

$$0 \to f_{\gamma*} M_c \xrightarrow{f_{\gamma*} j_c^0} f_{\gamma*} J_c^0 \xrightarrow{f_{\gamma*} j_c^1} f_{\gamma*} J_c^1 \xrightarrow{f_{\gamma*} j_c^2} f_{\gamma*} J_c^2 \to \dots$$

are exact. Hence the third row of Diagram (3.5) is exact and the claimed isomorphism follows from Proposition 3.35. $\qquad\qquad\square$

Example 3.37. Let us consider one example of a cyclic diagram. Assume given the pair
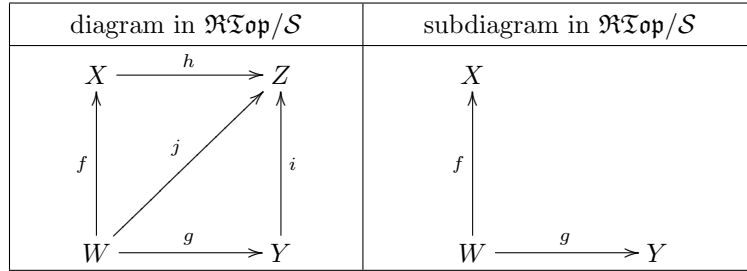
with no other morphisms besides the identities where $h \circ f = j = i \circ g$. The ringed topos Z is a target of $\mathcal{X}$ and $\mathcal{X}$ is cyclic with respect to Z since j has two different factorizations. Let $M = m_\mathcal{X}^* \mathcal{J}$. With the notations of the injective resolution on page 66, we choose injectives modules J_W^n, J_X^n, J_Y^n and J_Z^n over $\mathcal{O}_W$, $\mathcal{O}_X$, $\mathcal{O}_Y$ and $\mathcal{O}_Z$, respectively, and we have

$$I_W^n = J_W^n \ , \quad I_X^n = J_X^n \oplus f_* J_W^n \ , \quad I_Y^n = J_Y^n \oplus g_* J_W^n \quad \text{and} \quad I_Z^n = J_Z^n \oplus h_* J_X^n \oplus j_* J_W^n \oplus i_* J_Y^n$$

for all n. Diagram (3.5) on page 68 is given by

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 \longrightarrow \overline{M}_Z & \longrightarrow & J_Z^0 \longrightarrow \cdots \\
\downarrow & & \downarrow \\
0 \longrightarrow M_Z & \longrightarrow & J_Z^0 \oplus h_* J_X^0 \oplus j_* J_W^0 \oplus i_* J_Y^0 \longrightarrow \cdots \\
\downarrow {\scriptstyle D(M_\gamma)} & & \downarrow {\scriptstyle D(\mathrm{pr}_\gamma^0)} \\
0 \longrightarrow h_* M_X \oplus i_* M_Y & \longrightarrow & (h_* J_X^0 \oplus j_* J_W^0) \oplus (i_* J_Y^0 \oplus j_* J_W^0) \longrightarrow \cdots \\
\downarrow & & \downarrow \\
\mathrm{cok}(D(M_\gamma)) & \longrightarrow & j_* J_W^0 \longrightarrow \cdots \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

and the cokernel of $D(\mathrm{pr}_\gamma^n)$ is given by $j_* J_W^n$ for all n by Lemma 3.30. The second row of the above diagram is exact but the third row is not exact, even if $R^i h_* M_X = 0$ and $R^i i_* M_Y = 0$ for all $i > 0$ since we may not start with injective resolutions

$$0 \to M_X \to J_X^0 \to J_X^1 \to J_X^2 \to \ldots \quad \text{and} \quad 0 \to M_Y \to J_Y^0 \to J_Y^1 \to J_Y^2 \to \ldots$$

of M_X and M_Y, respectively. Altogether, it is very difficult to control the obstruction group even for one of the simplest cases of a cyclic diagram.

Nevertheless, under our General assumption 2.1, if we are given a deformation

$$
\begin{array}{ccccc}
X' & & & & X \\
{\scriptstyle f'}\uparrow & & \text{of} & & {\scriptstyle f}\uparrow \\
W' \xrightarrow{\ g'\ } Y' & & & & W \xrightarrow{\ g\ } Y
\end{array}
$$

over $\mathcal{S}'$, we may first try to find a deformation $j' \colon W' \to Z'$ of $j \colon W \to Z$ and then try to find a deformation $h' \colon X' \to Z'$ of $h \colon X \to Z$ and a deformation $i' \colon Y' \to Z'$ of $i \colon Y \to Z$ such that $h' \circ f' = j' = i' \circ g'$. We will calculate the corresponding obstruction groups in Proposition 4.5 and Proposition 4.8.

3.4.2 Omitting a source

Definition 3.38. Let $\mathcal{X}$ be a diagram. A source of $\mathcal{X}$ is an element $0 \in \mathrm{ob}(\mathfrak{C})$ such that $\mathrm{Hom}_{\mathfrak{C}}(0,0)$ consists only of the identity and such that there are only morphisms to 0, i.e., $\mathrm{Hom}_{\mathfrak{C}}(0,c)$ is empty for $c \neq 0$. Equivalently, the corresponding level $\mathcal{X}_0$ has only morphisms of ringed topoi over $\mathcal{S}$ to the other levels and the only morphism to $\mathcal{X}_0$ is the identity. By abuse of notation we call $\mathcal{X}_0$ a source of $\mathcal{X}$.

A source $\mathcal{X}_0$ of $\mathcal{X}$ may be visualized in $\mathfrak{RTop}/\mathcal{S}$ as
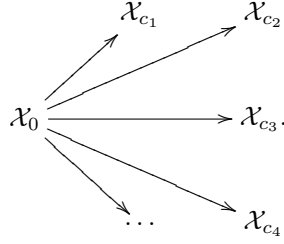
Proposition 3.39. *Assume given the situation of General assumption 2.1. Let $\mathcal{X}_0$ be a source of $\mathcal{X}$ and let $\mathfrak{D}$ be the full subcategory of $\mathfrak{C}$ with objects $\mathrm{ob}(\mathfrak{C}) \backslash \{0\}$. Then $\mathcal{Y}$ is a full and well-positioned subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$ equal to $\mathcal{X}_0$.*

Assume furthermore that the morphism a in Lemma 2.19 is a quasi-isomorphism and let L be the cotangent complex of the ring morphism

$$h_0 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_0 = \varinjlim_{\gamma \in \mathrm{ob}(\underline{0})} f_{\gamma}^{-1}\mathcal{O}_{\mathcal{X}_d} \to \mathcal{O}_{\mathcal{X}_0}$$

in Proposition 2.16. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{Ext}^2_{\mathcal{X}_0}(L, t_0^*\mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. Similarly as in the proof of Proposition 3.28, we have that $\mathcal{Y}$ is a full subdiagram since $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$, furthermore $\mathcal{Y}$ is well-positioned since we have omitted a source of $\mathcal{X}$ and the complementary subdiagram $\overline{\mathcal{Y}}$ is $\mathcal{X}_0$ because $\mathrm{Hom}_{\mathfrak{C}}(0,0) = \{\mathrm{id}\}$.

Now we come to the obstruction group. By Corollary 3.26 there is an obstruction lying in $\mathbb{Ext}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$. Since $\overline{\mathcal{Y}} = \mathcal{X}_0$ we have that $\overline{u}L_h = L$, the cotangent complex of $h_0 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_0 \to \mathcal{O}_{\mathcal{X}_0}$. The functor

$$G \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Y}}}\text{-mod} = \mathcal{O}_{\mathcal{X}_0}\text{-mod}$$

in Lemma 3.23 maps an $\mathcal{O}_{\mathcal{X}}$-module M to the $\mathcal{O}_{\mathcal{X}_0}$-module

$$\overline{M}_0 = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(0,d), \\ d \neq 0}} \ker\left(M_{\gamma}^{\mathrm{ad}} \colon M_0 \to f_{\gamma*}M_d\right) \subseteq M_0.$$

But $\mathrm{Hom}_{\mathfrak{C}}(0,d)$ is empty for $d \neq 0$ since $\mathcal{X}_0$ is a source of $\mathcal{X}$. It follows that $\overline{M}_0 = M_0$ and G is just the forgetful functor $\overline{u} \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{X}_0}\text{-mod}$. In particular, G is exact and the natural

morphism of functors $G(-) \to \mathbf{R}G(-)$ from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}^+(\mathcal{X}_0)$ is an isomorphism. Consequently the obstruction lies in

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L, \mathbf{R}G(m_{\mathcal{X}}^* \mathcal{J})) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L, G(m_{\mathcal{X}}^* \mathcal{J})) = \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L, t_0^* \mathcal{J}).$$

$\square$

Notice that in contrast to the analogue statement in Proposition 3.28 for omitting a target, we have to assume that the morphism a in Lemma 2.19 is a quasi-isomorphism. We will see in Proposition 3.43 that condition **2** in Theorem 2.20 is satisfied for a certain class of subdiagrams obtained by omitting a source, hence for this class the morphism a is a quasi-isomorphism.

In practice, we are generally able to control $t_0^* \mathcal{J}$ because $t_0 \colon \mathcal{X}_0 \to \mathcal{S}$ is the given structure morphism of the source. But the cotangent complex L of h_0 might be hard to control. In order to calculate L more precisely, we continue similarly as in the case of omitting a target in Subsection 3.4.1.

Notation 3.40. Let $\mathcal{X}$ be a diagram having a source $\mathcal{X}_0$. Let Γ be the set of those morphisms $\gamma \colon d \to 0$ in $\mathfrak{C}$ with $d \neq 0$ which do not factor in $\mathfrak{C}$ through some $d' \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$ except for the trivial factorization $d \xrightarrow{\mathrm{id}} d \xrightarrow{\gamma} 0$.

Definition 3.41. Let $\mathcal{X}_0$ be a source of $\mathcal{X}$. Then $\mathfrak{C}$ is called non-cyclic (with respect to 0) if every $\delta \in \mathrm{Hom}_{\mathfrak{C}}(c', 0)$ with $c' \neq 0$ has a unique factorization in $\mathfrak{C}$ of the form $c' \xrightarrow{\varepsilon} d \xrightarrow{\gamma} 0$ where $d \in \mathrm{ob}(\mathfrak{C}) \setminus \{0\}$ and $\gamma \in \Gamma$. The case $\varepsilon = \mathrm{id}$ is possible. Otherwise $\mathfrak{C}$ is called cyclic (with respect to 0). The diagram $\mathcal{X}$ is called non-cyclic (with respect to $\mathcal{X}_0$) if $\mathfrak{C}$ is non-cyclic with respect to 0. Otherwise $\mathcal{X}$ is called cyclic (with respect to $\mathcal{X}_0$).

Example 3.42. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $\mathfrak{C}_4$ be the categories
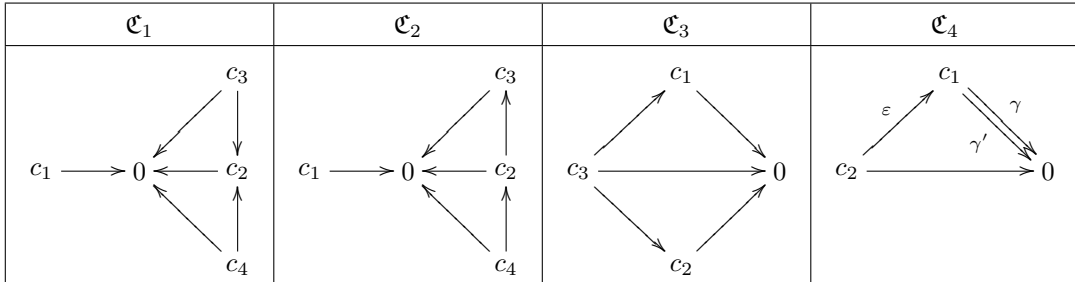
with no other morphisms besides the identities such that γ and γ' in $\mathfrak{C}_4$ are distinct and such that in each category the composition of any two consecutive morphisms is the displayed morphism. Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are non-cyclic and $\mathfrak{C}_3$ and $\mathfrak{C}_4$ are cyclic.

A non-cyclic category $\mathfrak{C}$ might be visualized as

$$
\begin{array}{ccccc}
& c_3 & c_4 & \cdots & \\
& \searrow & \downarrow & \swarrow & \\
\cdots & & d_2 & & c_5 \\
\searrow & & \downarrow & & \swarrow \\
c_2 \longrightarrow d_1 \longrightarrow & 0 & \longleftarrow d_3 \longleftarrow & c_6 \\
\nearrow & & \uparrow & & \nwarrow \\
c_1 & & \cdots & & \cdots
\end{array}
$$
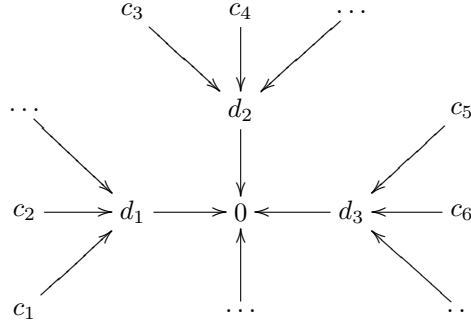
with no other morphisms involved except for the identities and the composition of any consecutive morphisms.

If $(\gamma \colon d \to 0) \in \Gamma$, we may consider γ as on object of the category $\underline{0}$. By definition of Γ we have

$$
\operatorname{Hom}_{\underline{0}}\big((\gamma_1 \colon d_1 \to 0), (\gamma_2 \colon d_2 \to 0)\big) = \begin{cases} \operatorname{id} & \text{if } \gamma_1 = \gamma_2 \\ \emptyset & \text{else} \end{cases}
$$

for each $(\gamma_1 \colon d_1 \to 0), (\gamma_2 \colon d_2 \to 0) \in \Gamma$. Hence we may consider Γ as a discrete subcategory of $\underline{0}$ and we write again Γ for this subcategory by abuse of notation.

We will see that in the case of a non-cyclic diagram $\mathcal{X}$, we may restrict to all morphisms $\gamma \in \Gamma$ for the calculation of

$$
(u^{-1}\mathcal{O}_{\mathcal{Y}})_0 = \varinjlim_{\gamma \in \operatorname{ob}(\underline{0})} f_{\gamma}^{-1}\mathcal{O}_{\mathcal{X}_d}.
$$

The idea behind this possibility comes again from deformation theoretic considerations. Suppose given the two pairs

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

and

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

with no other morphisms involved except for the identities where the subdiagrams are obtained by omitting the source $\mathcal{X}_0$ in both cases. Each diagram is non-cyclic with respect to $\mathcal{X}_0$. The obstruction for extending a given deformation of the subdiagram to a deformation of the diagram should be the same because in both cases we only have to find a deformation $\mathcal{X}_0'$ of $\mathcal{X}_0$ and deformations of f and g from $\mathcal{X}_0'$ to the given deformations of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. On the other hand, if we are given the two pairs
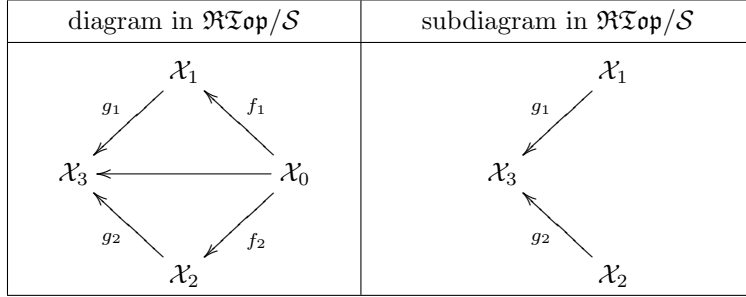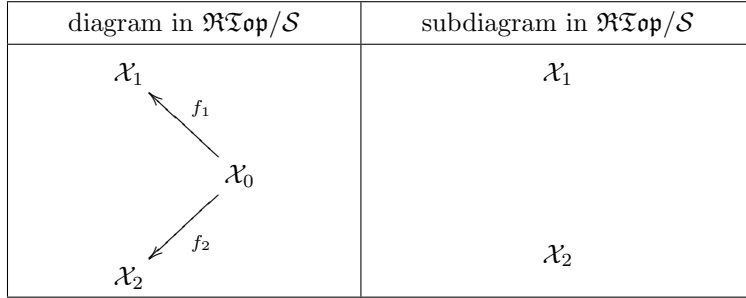
diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

and

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

where the first diagram is cyclic, then the obstructions should be different because in the first case we have to ensure that the commutativity still holds whereas in the second case this problem does not occur.

Proposition 3.43. *Let $\mathcal{X}_0$ be a source of $\mathcal{X}$ and assume that $\mathcal{X}$ is non-cyclic with respect to $\mathcal{X}_0$. Then condition **2** in Theorem 2.20 is satisfied and there is a natural isomorphism*

$$\varinjlim_{\gamma \in \mathrm{ob}(\underline{0})} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_d} \cong \varinjlim_{\gamma \in \mathrm{ob}(\Gamma)} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_d} = \bigotimes_{\gamma \in \Gamma} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_d}$$

of $t_0^{-1} \mathcal{O}_\mathcal{S}$-algebras. In particular, the cotangent complex L of $h_0 \colon (u^{-1}\mathcal{O}_\mathcal{Y})_0 \to \mathcal{O}_{\mathcal{X}_0}$ is isomorphic to the cotangent complex of the natural ring morphism

$$\bigotimes_{\gamma \in \Gamma} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_d} \to \mathcal{O}_{\mathcal{X}_0}$$

defined by all $t_0^{-1}\mathcal{O}_\mathcal{S}$-algebra morphisms $\theta_\gamma \colon f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \to \mathcal{O}_{\mathcal{X}_0}$ for $\bigl(\gamma \colon d \to 0\bigr) \in \mathrm{ob}(\Gamma)$, the tensor product being taken over $t_0^{-1}\mathcal{O}_\mathcal{S}$.

Proof. We know that $(u^{-1}\mathcal{O}_\mathcal{Y})_d = \mathcal{O}_{\mathcal{X}_d}$ for $d \neq 0$ by Lemma 3.13 since $\mathcal{Y}$ is a full and well-positioned subdiagram of $\mathcal{X}$ by Proposition 3.39. Thus once the isomorphism $(u^{-1}\mathcal{O}_\mathcal{Y})_0 \cong \bigotimes_{\gamma \in \Gamma} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ is shown, it follows that condition **2** in Theorem 2.20 is satisfied.

By the universal property of the colimit $\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ there is a unique morphism of $t_0^{-1}\mathcal{O}_{\mathcal{S}}$-algebras $\zeta\colon \varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \to \varinjlim_{\gamma\in\mathrm{ob}(\underline{0})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ such that

$$
\begin{array}{ccc}
f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} & \xrightarrow{\ \mathrm{id}\ } & f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \\
\downarrow & & \downarrow \\
\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} & \xrightarrow{\ \zeta\ } & \varinjlim_{\gamma\in\mathrm{ob}(\underline{0})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}
\end{array}
$$

is commutative for every $\big(\gamma\colon d \to 0\big) \in \mathrm{ob}(\Gamma)$ where the vertical morphisms are the natural ones. If $\big(\delta\colon c' \to 0\big) \in \mathrm{ob}(\underline{0})$, then since $\mathcal{X}$ is non-cyclic with respect to $\mathcal{X}_0$ there is a unique factorization

$$
\begin{array}{ccc}
c' & \xrightarrow{\ \delta\ } & 0 \\
{\scriptstyle \varepsilon}\searrow & & \nearrow{\scriptstyle \gamma} \\
 & d &
\end{array}
$$

in $\mathfrak{C}$ for some $d \neq 0$ such that $\big(\gamma\colon d \to 0\big) \in \mathrm{ob}(\Gamma)$. Since γ, ε and d depend uniquely on δ, we write $\gamma = \gamma(\delta)$, $\varepsilon = \varepsilon(\delta)$ and $d = d(\delta)$. The factorization yields a morphism

$$
f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}} = f_\gamma^{-1}f_\varepsilon^{-1}\mathcal{O}_{\mathcal{X}_{c'}} \xrightarrow{\ f_\gamma^{-1}\theta_\varepsilon\ } f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}
$$

of $t_0^{-1}\mathcal{O}_{\mathcal{S}}$-algebras. If $\varphi \in \mathrm{Hom}_{\underline{0}}\big((\delta_1\colon c_1' \to 0),(\delta_2\colon c_2' \to 0)\big)$ and if $c_2' \xrightarrow{\varepsilon} d \xrightarrow{\gamma} 0$ is the unique factorization of $c_2' \xrightarrow{\delta_2} 0$, then it follows from the commutativity of

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad\delta_1\quad} & & \\
c_1' & \xrightarrow{\varphi} c_2' & \xrightarrow{\ \delta_2\ } & & 0 \\
 & {\scriptstyle \varepsilon\varphi}\searrow & {\scriptstyle \varepsilon}\searrow & \nearrow{\scriptstyle \gamma} & \\
 & & d & &
\end{array}
$$

and the uniqueness of the factorization of $c_1' \xrightarrow{\delta_1} 0$ that either $\varepsilon = \mathrm{id}$ or $\varphi = \mathrm{id}$. In any case we have $\gamma(\delta_1) = \gamma(\delta_2) = \gamma$, $d(\delta_1) = d(\delta_2) = d$ and $\varepsilon(\delta_2) = \varepsilon$, $\varepsilon(\delta_1) = \varepsilon\varphi$. Hence the triangle

$$
\begin{array}{ccc}
f_{\delta_1}^{-1}\mathcal{O}_{\mathcal{X}_{c_1'}} & \xrightarrow{\ f_{\delta_2}^{-1}\theta_\varphi\ } & f_{\delta_2}^{-1}\mathcal{O}_{\mathcal{X}_{c_2'}} \\
{\scriptstyle f_\gamma^{-1}\theta_{\varepsilon\varphi}}\searrow & & \swarrow{\scriptstyle f_\gamma^{-1}\theta_\varepsilon} \\
 & f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} &
\end{array}
$$

is commutative which shows that the morphisms

$$
f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}} \xrightarrow{\ f_{\gamma(\delta)}^{-1}\theta_{\varepsilon(\delta)}\ } f_{\gamma(\delta)}^{-1}\mathcal{O}_{\mathcal{X}_{d(\delta)}} \to \varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}
$$

are compatible with the morphisms defined by $\underline{0}$ for every $\big(\delta\colon c' \to 0\big) \in \mathrm{ob}(\underline{0})$. Consequently, by the universal property of the colimit $\varinjlim_{\delta\in\mathrm{ob}(\underline{0})} f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}}$ there is a unique morphism of

$t_0^{-1}\mathcal{O}_\mathcal{S}$-algebras $\xi\colon \varinjlim_{\delta\in\mathrm{ob}(\underline{0})} f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}} \to \varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ such that

$$
\begin{array}{ccc}
f_{\gamma(\delta)}^{-1}\mathcal{O}_{\mathcal{X}_{d(\delta)}} & \xleftarrow{\ f_{\gamma(\delta)}^{-1}\theta_{\varepsilon(\delta)}\ } & f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}} \\
\downarrow & & \downarrow \\
\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} & \xleftarrow{\ \xi\ } & \varinjlim_{\delta\in\mathrm{ob}(\underline{0})} f_\delta^{-1}\mathcal{O}_{\mathcal{X}_{c'}}
\end{array}
$$

is commutative for every $\big(\delta\colon c'\to 0\big) \in \mathrm{ob}(\underline{0})$ where the vertical morphisms are the natural ones. By definition both morphisms id and $\xi\zeta$ from $\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ to $\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ fit into a commutative diagram

$$
\begin{array}{ccc}
f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} & \xrightarrow{\ \ \mathrm{id}\ \ } & f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \\
\downarrow & & \downarrow \\
\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} & \longrightarrow & \varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}
\end{array}
$$

for every $\big(\gamma\colon d\to 0\big) \in \mathrm{ob}(\Gamma)$, hence $\mathrm{id} = \xi\zeta$ by the universal property of $\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$. Similarly, it follows $\mathrm{id} = \zeta\xi$. Hence there is a natural isomorphism

$$
\varinjlim_{\gamma\in\mathrm{ob}(\underline{0})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} \cong \varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}
$$

and since Γ is a discrete subcategory of $\underline{0}$, we have $\varinjlim_{\gamma\in\mathrm{ob}(\Gamma)} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} = \bigotimes_{\gamma\in\Gamma} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$. $\qquad\square$

Our next aim is to show that the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L, t_0^*\mathcal{J})$ in Proposition 3.39 is always part of an exact sequence different from the exact sequence in Theorem 2.20.

In any case, whether $\mathcal{X}$ is non-cyclic or cyclic with respect to $\mathcal{X}_0$, the factorization

$$
\begin{array}{ccc}
t_0^{-1}\mathcal{O}_\mathcal{S} & \xrightarrow{\ \ \theta_0\ \ } & \mathcal{O}_{\mathcal{X}_0} \\
{}_s\searrow & & \nearrow_{h_0} \\
& (u^{-1}\mathcal{O}_\mathcal{Y})_0 &
\end{array}
$$

of θ_0, where s is the structure morphism of $(u^{-1}\mathcal{O}_\mathcal{Y})_0$ as a $t_0^{-1}\mathcal{O}_\mathcal{S}$-algebra, yields a distinguished triangle

$$
\begin{array}{ccc}
& L & \\
\swarrow & & \nwarrow \\
L_s \otimes_{(u^{-1}\mathcal{O}_\mathcal{Y})_0} \mathcal{O}_{\mathcal{X}_0} & \longrightarrow & L_{\mathcal{X}_0/\mathcal{S}}
\end{array}
\tag{3.7}
$$

in the derived category $\mathbf{D}(\mathcal{X}_0)$ by Theorem 1.8.iv). This triangle gives rise to an exact sequence

$$
\begin{aligned}
0 \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}_0}(L, t_0^*\mathcal{J}) &\xrightarrow{\alpha} \mathbb{E}\mathrm{xt}^0_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \xrightarrow{\beta} \mathbb{E}\mathrm{xt}^0_{\mathcal{X}_0}(L_s \otimes_{(u^{-1}\mathcal{O}_\mathcal{Y})_0} \mathcal{O}_{\mathcal{X}_0}, t_0^*\mathcal{J}) \\
&\xrightarrow{\gamma} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_0}(L, t_0^*\mathcal{J}) \xrightarrow{\delta} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \xrightarrow{\varepsilon} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_0}(L_s \otimes_{(u^{-1}\mathcal{O}_\mathcal{Y})_0} \mathcal{O}_{\mathcal{X}_0}, t_0^*\mathcal{J}) \\
&\to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L, t_0^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_0}(L_s \otimes_{(u^{-1}\mathcal{O}_\mathcal{Y})_0} \mathcal{O}_{\mathcal{X}_0}, t_0^*\mathcal{J}) \\
&\to \dots
\end{aligned}
\tag{3.8}
$$

of abelian groups and the morphisms $\alpha, \beta, \gamma, \delta$ and ε may be described as in Proposition 1.13.

Now if $\mathcal{X}$ is non-cyclic with respect to $\mathcal{X}_0$, we may assume that $\Gamma = \underline{0}$ by Proposition 3.43, hence we may assume that the diagram and the subdiagram are given by

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
	$\mathcal{X}_1 \qquad \mathcal{X}_2 \qquad \mathcal{X}_3 \qquad \cdots \qquad \mathcal{X}_m$
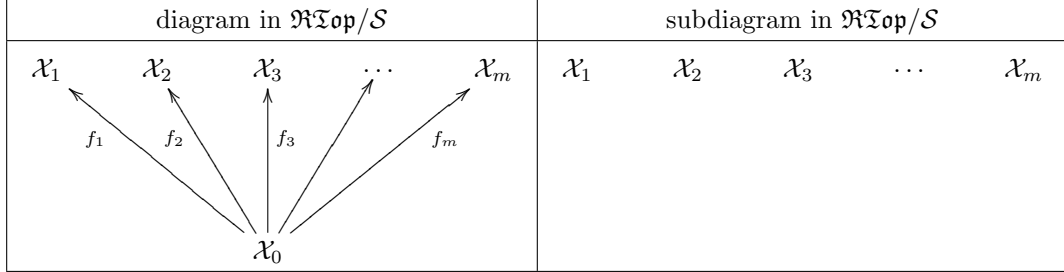
with no other morphisms besides the identities where the levels $\mathcal{X}_i$ do not have to be pairwise distinct, but if $\mathcal{X}_i = \mathcal{X}_j$ for $i \neq j$, then $f_i \neq f_j$. Since all structure morphisms $t_c \colon \mathcal{X}_c \to \mathcal{S}$ are flat by our General assumption 2.1, it follows from Proposition A.1 that the natural morphism

$$\bigoplus_{j=1}^{m} L_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}/t_0^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}} (u^{-1}\mathcal{O}_{\mathcal{Y}})_0 \;\to\; L_s$$

of complexes of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_0$-modules is a quasi-isomorphism. Since both complexes of the above quasi-isomorphism consist of free $(u^{-1}\mathcal{O}_{\mathcal{Y}})_0$-modules, we may tensor with $\mathcal{O}_{\mathcal{X}_0}$ and still get a quasi-isomorphism

$$\bigoplus_{j=1}^{m} f_j^* L_{\mathcal{X}_j/\mathcal{S}} \cong \bigoplus_{j=1}^{m} L_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}/t_0^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}} \mathcal{O}_{\mathcal{X}_0} \;\to\; L_s \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_0} \mathcal{O}_{\mathcal{X}_0} \tag{3.9}$$

by [Ill71, Lemme I.3.3.2.1.] where the first isomorphism is induced from the natural isomorphism $f_j^* L_{\mathcal{X}_j/\mathcal{S}} \cong L_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}/t_0^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_j^{-1}\mathcal{O}_{\mathcal{X}_j}} \mathcal{O}_{\mathcal{X}_0}$ by Theorem 1.8.ii). Consequently, we have that

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{i}(L_s \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_0} \mathcal{O}_{\mathcal{X}_0}, t_0^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{i}(\bigoplus_{j=1}^{m} f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J}) \cong \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{i}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J})$$

for all $i \in \mathbb{Z}$ and there is an exact sequence

$$0 \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{0}(L, t_0^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{0}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{0}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{1}(L, t_0^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{1}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{1}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J}) \tag{3.10}$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{2}(L, t_0^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{2}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{2}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J})$$

$$\to \dots .$$

Now if

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{2}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^*\mathcal{J}) = 0 \quad \text{and} \quad \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{1}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J}) = 0, \tag{3.11}$$

then $\mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{2}(L, t_0^*\mathcal{J}) = 0$ by the exactness of the sequence.

We will see in Example 3.60 that $\bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^{1}(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^*\mathcal{J})$ is an obstruction group for the pair

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1 \quad \mathcal{X}_2 \quad \mathcal{X}_3 \quad \cdots \quad \mathcal{X}_m$ $f_1 \quad f_2 \quad f_3 \qquad f_m$ $\mathcal{X}_0$	$\mathcal{X}_1 \quad \mathcal{X}_2 \quad \mathcal{X}_3 \quad \cdots \quad \mathcal{X}_m$ $\mathcal{X}_0$

and we know from Theorem 1.11 that $\mathbb{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{S}}, t_0^* \mathcal{J})$ is an obstruction group for finding a deformation of $\mathcal{X}_0$ over $\mathcal{S}'$. Whence if Equation (3.11) holds and if we are given a deformation of the subdiagram $\mathcal{Y}$ obtained by omitting the source $\mathcal{X}_0$, then we may choose a deformation $\mathcal{X}_0'$ of $\mathcal{X}_0$ and then a deformation $f_j' \colon \mathcal{X}_0' \to \mathcal{X}_j'$ of $f_j \colon \mathcal{X}_0 \to \mathcal{X}_j$ for every j where $\mathcal{X}_j'$ is the given deformation of $\mathcal{X}_j$. We have already encountered Sequence (3.10) in the disguise of the long exact sequence **4** of the braid in Proposition 2.22. The diagram there is the local diagram $\mathcal{X}$, the subdiagram there is the local diagram $\mathcal{Y} \coprod \mathcal{X}_0$ and the subsubdiagram there is the local diagram $\mathcal{Y}$.

Nevertheless, if $\mathcal{X}$ is non-cyclic or cyclic with respect to $\mathcal{X}_0$ and if the morphism a in Lemma 2.19 is a quasi-isomorphism, we might get some information about the obstruction group $\mathbb{Ext}^2_{\mathcal{X}_0}(L, t_0^* \mathcal{J})$ from Sequence (3.8) on page 82.

Example 3.44. Let us consider one example of a cyclic diagram, similarly as in Example 3.37. Assume given the pair

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$X \xrightarrow{\ h\ } Z$ $f \uparrow \quad {}^{j} \nearrow \quad \uparrow i$ $W \xrightarrow{\ g\ } Y$	$X \xrightarrow{\ h\ } Z$ $\uparrow i$ Y

with no other morphisms except for the identities such that $h \circ f = j = i \circ g$. Then W is a source of $\mathcal{X}$ and $\mathcal{X}$ is cyclic with respect to W since j has two factorizations. Let $t_W \colon W \to \mathcal{S}$ and $t_Z \colon Z \to \mathcal{S}$ be the structure morphisms of W and Z over $\mathcal{S}$, respectively. By definition $u^{-1}\mathcal{O}_{\mathcal{Y}}$ is given on the level W by the colimit of the system

$$
\begin{array}{ccc}
 & & f^{-1}\mathcal{O}_X \\
 & {}^{f^{-1}\theta_h}\nearrow & \\
j^{-1}\mathcal{O}_Z & & \\
 & {}_{g^{-1}\theta_i}\searrow & \\
 & & g^{-1}\mathcal{O}_Y
\end{array}
$$

in the category of $t_W^{-1}\mathcal{O}_{\mathcal{S}}$-algebras where $\theta_h \colon h^{-1}\mathcal{O}_Z \to \mathcal{O}_X$ and $\theta_i \colon i^{-1}\mathcal{O}_Z \to \mathcal{O}_Y$ are the ring morphisms of h and i, respectively. This colimit is $f^{-1}\mathcal{O}_X \otimes_{j^{-1}\mathcal{O}_Z} g^{-1}\mathcal{O}_Y$ considered as a $t_W^{-1}\mathcal{O}_{\mathcal{S}}$-algebra. It follows that condition **2** in Theorem 2.20 is not fulfilled. Condition **1** is not fulfilled either because this type of colimit does not commute with finite limits.

But there is a possibility to find a remedy in this example. If we are given a deformation

$$
\begin{array}{ccc}
X' & \xrightarrow{\ h'\ } & Z' \\
 & & \uparrow{\scriptstyle i'} \\
 & & Y'
\end{array}
\qquad \text{of the subdiagram} \qquad
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Z \\
 & & \uparrow{\scriptstyle i} \\
 & & Y
\end{array}
$$

over $\mathcal{S}'$, we may fix the given extension

$$
0 \to t_Z^* \mathcal{J} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0 \tag{3.12}
$$

of $\mathcal{O}_Z$. Assume that $h\colon X \to Z$, $i\colon Y \to Z$ and $j\colon W \to Z$ are flat. Then finding an extension

$$
\begin{array}{ccc}
X' & \xrightarrow{\ h'\ } & Z' \\
{\scriptstyle f'}\uparrow \ \nearrow{\scriptstyle j'} & & \uparrow{\scriptstyle i'} \\
W' & \xrightarrow[\ g'\ ]{} & Y'
\end{array}
\qquad \text{of the given deformation} \qquad
\begin{array}{ccc}
X' & \xrightarrow{\ h'\ } & Z' \\
 & & \uparrow{\scriptstyle i'} \\
 & & Y'
\end{array}
$$

of $\mathcal{Y}$ to $\mathcal{X}$ over $\mathcal{S}'$ is equivalent to finding an extension of

$$
\begin{array}{ccc}
X' & & \\
 & & \\
 & & \\
Y' & &
\end{array}
\qquad \text{to a deformation} \qquad
\begin{array}{ccc}
 & X' & \\
{\scriptstyle f'}\uparrow & & \\
W' & \xrightarrow{\ g'\ } & Y'
\end{array}
$$

over Z' with respect to Extension (3.12) where we have to consider $h\colon X \to Z$, $i\colon Y \to Z$ and $j\colon W \to Z$ as structure morphisms. Hence if these structure morphisms are flat and if we consider Extension (3.12), then we may assume the situation

diagram in $\mathfrak{RTop}/Z$	subdiagram in $\mathfrak{RTop}/Z$
$\begin{array}{ccc} X & & \\ \uparrow & & \\ {\scriptstyle f} & & \\ W & \xrightarrow{\ g\ } & Y \end{array}$	$\begin{array}{c} X \\ \\ \\ Y \end{array}$

where the subdiagram is now non-cyclic with respect to the source W and we know from the non-cyclic case in Proposition 3.43 that the corresponding obstruction group is given by

$$
\mathbb{Ext}_W^2(L, j^* t_Z^* \mathcal{J}) = \mathbb{Ext}_W^2(L, t_W^* \mathcal{J}).
$$

Here L is the cotangent complex of the ring morphism

$$
f^{-1}\mathcal{O}_X \otimes_{j^{-1}\mathcal{O}_Z} g^{-1}\mathcal{O}_Y \to \mathcal{O}_W \ , \quad a \otimes b \mapsto \theta_f(a) \cdot \theta_g(b)
$$

where $\theta_f\colon f^{-1}\mathcal{O}_X \to \mathcal{O}_W$ and $\theta_g\colon g^{-1}\mathcal{O}_Y \to \mathcal{O}_W$ are the ring morphisms of f and g, respectively.

Notice that the additional assumptions made are satisfied if $Z = \mathcal{S}$ and if $t_Z\colon Z \to \mathcal{S}$ is the identity.

3.4.3 Omitting a bridge

Definition 3.45. Let $\mathcal{X}$ be a diagram. A bridge of $\mathcal{X}$ is a quintuple $2 \xrightarrow{\alpha} 1 \xrightarrow{\beta} 0$ in $\mathfrak{C}$ with $\operatorname{Hom}_{\mathfrak{C}}(1,1) = \{\mathrm{id}\}$ such that each morphism in $\mathfrak{C}$ to 1 apart from the identity factors through α and each morphism in $\mathfrak{C}$ from 1 apart from the identity factors through β. The identity $1 \to 1$ does not factor. Equivalently, denoting $f = f_\beta$ and $g = f_\alpha$ and regarding the corresponding quintuple $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ over $\mathcal{S}$, each morphism occuring in the diagram other than the identity from $\mathcal{X}_1$ factors through g and each morphism occuring in the diagram other than the identity to $\mathcal{X}_1$ factors through f, while the identity $\mathcal{X}_1 \to \mathcal{X}_1$ is the only $\mathcal{S}$-morphism from $\mathcal{X}_1$ to itself and does not factor. By abuse we call $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ a bridge of $\mathcal{X}$.

A bridge $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ of $\mathcal{X}$ may be visualized in $\mathfrak{RTop}/\mathcal{S}$ as
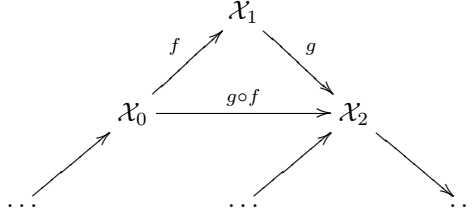
Proposition 3.46. *Assume given the situation of General assumption 2.1. Let $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ be a bridge of $\mathcal{X}$ and let $\mathfrak{D}$ be the full subcategory of $\mathfrak{C}$ with objects $\operatorname{ob}(\mathfrak{C}) \setminus \{1\}$. Then $\mathcal{Y}$ is a full and well-positioned subdiagram of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Y}}$ equal to $\mathcal{X}_1$ and condition **2** in Theorem 2.20 is satisfied.*

Let $G \colon \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\mathcal{X}_1)$ be the functor of Lemma 3.23 and let $L_{\mathcal{X}_1/\mathcal{X}_2}$ be the cotangent complex of $\mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{Ext}^2_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{X}_2}, \mathbf{R}G(m_{\mathcal{X}}^* \mathcal{J}))$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. $\mathcal{Y}$ is a full subdiagram since $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$. Furthermore, $\mathcal{Y}$ is well-positioned since $\operatorname{ob}(\mathfrak{C}) \setminus \operatorname{ob}(\mathfrak{D}) = \{1\}$ and the only morphism $1 \to 1$ in $\mathfrak{C}$ is the identity which by assumption does not factor in $\mathfrak{C}$. The complementary diagram $\overline{\mathcal{Y}}$ is $\mathcal{X}_1$ because $\operatorname{Hom}_{\mathfrak{C}}(1,1) = \{\mathrm{id}\}$.

Since $\mathcal{Y}$ is a full subdiagram of $\mathcal{X}$ we have that $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c = \mathcal{O}_{\mathcal{X}_c}$ for all $c \in \operatorname{ob}(\mathfrak{C}) \setminus \{1\}$ by Lemma 3.13. Hence in order to verify condition **2** in Theorem 2.20 we have to calculate

$$(u^{-1}\mathcal{O}_{\mathcal{Y}})_1 = \varinjlim_{\gamma \in \operatorname{ob}(\underline{1})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}.$$

Since $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ is a bridge of $\mathcal{X}$, each morphism $(\gamma \colon d \to 1) \in \operatorname{ob}(\underline{1})$ admits a factorization
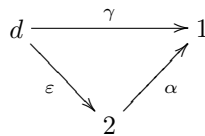
through α. Because $\mathfrak{D}$ is the full subcategory of $\mathfrak{C}$ with objects $\mathrm{ob}(\mathfrak{C}) \setminus \{1\}$ it follows that $\varepsilon \in \mathrm{Hom}_{\underline{1}}\big((\gamma\colon d \to 1), (\alpha\colon 2 \to 1)\big)$. Moreover, for each $f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}$ with $(\gamma\colon d \to 1) \in \mathrm{ob}(\underline{1})$, there is a morphism of $t_1^{-1}\mathcal{O}_\mathcal{S}$-algebras

$$f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} = f_\alpha^{-1}f_\varepsilon^{-1}\mathcal{O}_{\mathcal{X}_d} \xrightarrow{\ f_\alpha^{-1}\theta_\varepsilon\ } f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_2}$$

belonging to the system defined by the index category $\underline{1}$. Consequently,

$$(u^{-1}\mathcal{O}_\mathcal{Y})_1 = \varinjlim_{\gamma \in \mathrm{ob}(\underline{1})} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d} = f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_2}.$$

Now we come to the obstruction group. By Corollary 3.26 there is an obstruction lying in $\mathbb{E}\mathrm{xt}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m_\mathcal{X}^*\mathcal{J}))$. Since $\overline{\mathcal{Y}} = \mathcal{X}_1$ it follows that $\overline{u}L_h$ is the cotangent complex of $h_1\colon (u^{-1}\mathcal{O}_\mathcal{Y})_1 \to \mathcal{O}_{\mathcal{X}_1}$ and h_1 is just the ring morphism $\theta_\alpha\colon f_\alpha^{-1}\mathcal{O}_{\mathcal{X}_2} \to \mathcal{O}_{\mathcal{X}_1}$, whence $\overline{u}L_h = L_{\mathcal{X}_1/\mathcal{X}_2}$. $\qquad\qquad\square$

Let us consider the two pairs

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

and

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

with $h = g \circ f$ and with no other morphisms except for the identities and except for the composition of any two consecutive morphisms. Then in both cases the obstruction for extending a given deformation of the subdiagram to a deformation of the diagram should be the same since in both cases, if $h'\colon \mathcal{X}_0' \to \mathcal{X}_2'$ is the given deformation of $h\colon \mathcal{X}_0 \to \mathcal{X}_2$ over $\mathcal{S}'$, we have to find a deformation $\mathcal{X}_1'$ of $\mathcal{X}_1$ and deformations $f'\colon \mathcal{X}_0' \to \mathcal{X}_1'$ and $g'\colon \mathcal{X}_1' \to \mathcal{X}_2'$ of $f\colon \mathcal{X}_0 \to \mathcal{X}_1$ and $g\colon \mathcal{X}_1 \to \mathcal{X}_2$, respectively, such that

is commutative. These considerations are similar to the ones we have stated in Subsections 3.4.1 and 3.4.2, but this time we do not have to distinguish between two cases which we called non-cyclic and cyclic. The obstruction group only depends on the bridge $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$, made precise by the following lemma and corollary.

Lemma 3.47. *Assume given the assumptions and notations of Proposition 3.46. Let $\mathcal{T}(f)$ be the ringed topos associated to the diagram $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1$ over $\mathcal{S}$ and let $u\colon \mathcal{O}_{\mathcal{X}}$-mod $\to \mathcal{O}_{\mathcal{T}(f)}$-mod be the forgetful functor. If $G'\colon \mathcal{O}_{\mathcal{T}(f)}$-mod $\to \mathcal{O}_{\mathcal{X}_1}$-mod is the functor in Lemma 3.23, then there are commutative diagrams*

$$
\begin{array}{ccc}
\mathcal{O}_{\mathcal{X}}\text{-mod} \xrightarrow{\quad u \quad} \mathcal{O}_{\mathcal{T}(f)}\text{-mod} & \quad and \quad & \mathbf{K}(\mathcal{X})^+ \xrightarrow{\quad u \quad} \mathbf{K}(\mathcal{T}(f))^+ \\
\searrow^{G} \quad \swarrow_{G'} & & \searrow^{G} \quad \swarrow_{G'} \\
\mathcal{O}_{\mathcal{X}_1}\text{-mod} & & \mathbf{K}(\mathcal{X}_1)
\end{array}
$$

and there is a natural isomorphism of functors

$$\mathbf{R}G(-) \cong \mathbf{R}G'(u(-))$$

from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathcal{X}_1)$. In particular, if $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ is a bounded below cochain complex, then the complex $\mathbf{R}G(M)$ only depends on $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1$ and its part uM, up to natural isomorphism in $\mathbf{D}(\mathcal{X}_1)$.

Proof. The functor $G\colon \mathcal{O}_{\mathcal{X}}$-mod $\to \mathcal{O}_{\mathcal{X}_1}$-mod is given by

$$M \ \mapsto \ \overline{M}_1 = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(1,d), \\ d \neq 1}} \ker\left(M_\gamma^{\mathrm{ad}}\colon M_1 \to f_{\gamma *}M_d\right) \subseteq M_1$$

by definition in Lemma 3.23. But since $2 \xrightarrow{\alpha} 1 \xrightarrow{\beta} 0$ is a bridge of $\mathcal{X}$, each morphism $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(1,d)$ with $d \neq 1$ factors through β, hence by Remark 3.20 we have that

$$\overline{M}_1 = \ker\left(M_\beta^{\mathrm{ad}}\colon M_1 \to f_{\beta *}M_0\right) \subseteq M_1.$$

On the other hand, since $f = f_\beta$ the functor $G'\colon \mathcal{O}_{\mathcal{T}(f)}$-mod $\to \mathcal{O}_{\mathcal{X}_1}$-mod sends the $\mathcal{O}_{\mathcal{T}(f)}$-module uM to the same submodule $\ker\left(M_\beta^{\mathrm{ad}}\colon M_1 \to f_{\beta *}M_0\right)$ of M_1. It follows that

$$
\begin{array}{c}
\mathcal{O}_{\mathcal{X}}\text{-mod} \xrightarrow{\quad u \quad} \mathcal{O}_{\mathcal{T}(f)}\text{-mod} \\
\searrow^{G} \qquad \swarrow_{G'} \\
\mathcal{O}_{\mathcal{X}_1}\text{-mod}
\end{array}
$$

is indeed commutative and

$$
\begin{array}{c}
\mathbf{K}(\mathcal{X})^+ \xrightarrow{\quad u \quad} \mathbf{K}(\mathcal{T}(f))^+ \\
\searrow^{G} \qquad \swarrow_{G'} \\
\mathbf{K}(\mathcal{X}_1)
\end{array}
$$

yields an isomorphism of functors $\mathbf{R}G(-) \cong \mathbf{R}G'(u(-))$ from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}(\mathcal{X}_1)$ exactly as in Proposition 3.34. $\qquad\square$

Corollary 3.48. *Under the assumptions and notations of Proposition 3.46, the obstruction group* $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{X}_2}, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$ *only depends on the bridge* $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$.

Proof. The first entry of the obstruction group $L_{\mathcal{X}_1/\mathcal{X}_2}$ only depends on $\mathcal{X}_1 \xrightarrow{g} \mathcal{X}_2$ and the second entry $\mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J})$ only depends on $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1$ by the above lemma. $\qquad\square$

Corollary 3.49. *Under the assumptions and notations of Proposition 3.46, set* $M = m^*_{\mathcal{X}}\mathcal{J}$ *and let* K *and* Q *be kernel and cokernel of*

$$M^{\mathrm{ad}}_{\beta} \colon M_1 \to f_{\beta*}M_0,$$

respectively. If $R^i f_{\beta*} M_0 = 0$ *for all* $i > 0$*, then there is a natural isomorphism of abelian groups*

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{X}_2}, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J})) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{X}_2}, K) \oplus \mathbb{E}\mathrm{xt}^1_{\mathcal{X}_1}(L_{\mathcal{X}_1/\mathcal{X}_2}, Q).$$

Proof. By Lemma 3.47 and its notations the complex $\mathbf{R}G(M)$ only depends on $\mathcal{X}_0 \xrightarrow{f} \mathcal{X}_1$ and uM and we may calculate it in $\mathbf{D}(\mathcal{X}_1)$ as $\mathbf{R}G'(uM)$. But by Corollary 3.36 there is a natural isomorphism

$$\mathbf{R}G'(uM) \cong \left(K \xrightarrow{0} Q \to 0 \to 0 \to \dots \right)$$

in $\mathbf{D}(\mathcal{X}_1)$ which gives rise to the claimed isomorphism.

$\qquad\square$

3.4.4 The discrete subdiagram

In this subsection we consider another extreme case. The subdiagram is obtained from the diagram by keeping all levels $\mathcal{X}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ and by omitting all morphisms between them except for the identities. We will show that condition **2** in Theorem 2.20 is satisfied for this type of subdiagram and we will simplify the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J})$ in Proposition 3.55 and Corollary 3.59. Four examples exemplifying the new notions will be given at the end of the subsection. Moreover, we will derive a result of [Ran89] in one of the examples.

Definition 3.50. Let $\mathcal{X}$ be a diagram. If $\mathfrak{D}$ is the discrete subcategory of $\mathfrak{C}$, i.e., $\mathrm{ob}(\mathfrak{D}) = \mathrm{ob}(\mathfrak{C})$ and the only morphisms in $\mathfrak{D}$ are the identities, then $\mathcal{Y}$ is called the discrete subdiagram of $\mathcal{X}$.

Lemma 3.51. *Let $\mathcal{Y}$ be the discrete subdiagram of a diagram $\mathcal{X}$. Then condition **2** in Theorem 2.20 is satisfied.*

Proof. Let $c \in \mathrm{ob}(\mathfrak{C})$. Since $\mathcal{Y}$ is discrete we have that the category $\underline{c}$ is discrete, hence

$$(u^{-1}\mathcal{O}_{\mathcal{Y}})_c = \bigotimes_{\gamma \in \mathrm{ob}(\underline{c})} f^{-1}_{\gamma}\mathcal{O}_{\mathcal{X}_{c'}} = \bigotimes_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c', c), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f^{-1}_{\gamma}\mathcal{O}_{\mathcal{X}_{c'}}$$

by definition of $u^{-1}\mathcal{O}_{\mathcal{Y}}$ in Proposition 2.16 where the tensor product is taken over $t^{-1}_c\mathcal{O}_{\mathcal{S}}$. $\square$

Let $\mathcal{Y}$ be the discrete subdiagram of a diagram $\mathcal{X}$. Our aim is to show that there is a complex L^d of free $\mathcal{O}_{\mathcal{X}}$-modules and a natural isomorphism $L^d[1] \cong L_h$ in $\mathbf{D}(\mathcal{X})$ such that the obstruction group

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L_h, m^*_{\mathcal{X}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^d[1], m^*_{\mathcal{X}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L^d, m^*_{\mathcal{X}}\mathcal{J})$$

in Theorem 2.20 may be calculated better by using the complex L^d. The superscript d stands for discrete.

By Diagram (2.8) on page 35 there are morphisms of $m^{-1}_{\mathcal{Y}}\mathcal{O}_{\mathcal{S}}$-algebras

$$\mathcal{O}_{\mathcal{Y}} \xrightarrow{n^{\mathcal{O}_{\mathcal{Y}}}} uu^{-1}\mathcal{O}_{\mathcal{Y}} \xrightarrow{uh} \mathcal{O}_{\mathcal{Y}}$$

whose composition is the identity. Because $\mathcal{Y}$ is discrete this just means that we have, for each $c \in \mathrm{ob}(\mathfrak{C})$, morphisms of $t^{-1}_c\mathcal{O}_{\mathcal{S}}$-algebras $\mathcal{O}_{\mathcal{X}_c} \to (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \xrightarrow{h_c} \mathcal{O}_{\mathcal{X}_c}$ where the first morphism is the natural one. For each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ the diagram

$$
\begin{array}{ccc}
f^{-1}_{\alpha}\mathcal{O}_{\mathcal{X}_{c_1}} & \xrightarrow{\quad\theta_{\alpha}\quad} & \mathcal{O}_{\mathcal{X}_{c_2}} \\
\downarrow & & \downarrow \\
f^{-1}_{\alpha}\bigotimes_{\gamma \in \mathrm{ob}(\underline{c_1})} f^{-1}_{\gamma}\mathcal{O}_{\mathcal{X}_{c'}} & \xrightarrow{(u^{-1}\mathcal{O}_{\mathcal{Y}})_{\alpha}} & \bigotimes_{\delta \in \mathrm{ob}(\underline{c_2})} f^{-1}_{\delta}\mathcal{O}_{\mathcal{X}_{c'}}
\end{array}
$$

whose vertical morphisms are the natural ones is commutative by definition of $(u^{-1}\mathcal{O}_{\mathcal{Y}})_{\alpha}$. Hence the collection of all $\mathcal{O}_{\mathcal{X}_c} \to (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \xrightarrow{h_c} \mathcal{O}_{\mathcal{X}_c}$ for $c \in \mathrm{ob}(\mathfrak{C})$ defines a morphism of $m^{-1}_{\mathcal{X}}\mathcal{O}_{\mathcal{S}}$-algebras

$$\mathcal{O}_{\mathcal{X}} \to u^{-1}\mathcal{O}_{\mathcal{Y}} \xrightarrow{h} \mathcal{O}_{\mathcal{X}}$$

whose composition is the identity. Now for $c \in \mathrm{ob}(\mathfrak{C})$ we define

$$\mathcal{T}_c = \bigotimes_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_{c'}}$$

to be the $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebra obtained from $(u^{-1}\mathcal{O}_{\mathcal{Y}})_c$ by omitting the factor corresponding to the identity. There is a natural morphism of $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras $\mathcal{T}_c \rightarrow (u^{-1}\mathcal{O}_{\mathcal{Y}})_c$ and a cocartesian diagram

$$
\begin{array}{ccc}
\mathcal{T}_c & \longrightarrow & (u^{-1}\mathcal{O}_{\mathcal{Y}})_c \\
\uparrow & & \uparrow \\
t_c^{-1}\mathcal{O}_{\mathcal{S}} & \longrightarrow & \mathcal{O}_{\mathcal{X}_c}.
\end{array}
$$

Since the tensor product is a special colimit, we may imitate the construction given in Definition 2.7 to see that for each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ there is a morphism of $t_{c_2}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras $f_\alpha^{-1}\mathcal{T}_{c_1} \rightarrow \mathcal{T}_{c_2}$ fulfilling the compatibility condition (1.2) on page 14 for a composition $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ in $\mathfrak{C}$.

Definition 3.52. We define $\mathcal{T}$ to be the $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebra which is given levelwise by the $t_c^{-1}\mathcal{O}_{\mathcal{S}}$-algebras

$$\mathcal{T}_c = \bigotimes_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_\gamma^{-1} \mathcal{O}_{\mathcal{X}_{c'}}$$

together with the morphisms $f_\alpha^{-1}\mathcal{T}_{c_1} \rightarrow \mathcal{T}_{c_2}$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ as described above.

The construction in Definition 2.7 shows that the collection of all natural morphisms $\mathcal{T}_c \rightarrow (u^{-1}\mathcal{O}_{\mathcal{Y}})_c$ defines a natural morphism

$$\mathcal{T} \rightarrow u^{-1}\mathcal{O}_{\mathcal{Y}}$$

of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras. Consequently there is a commutative diagram

$$
\begin{array}{ccc}
 & & \mathcal{O}_{\mathcal{X}} \\
 & \nearrow \; h \; \nearrow & \uparrow \\
\mathcal{T} & \longrightarrow u^{-1}\mathcal{O}_{\mathcal{Y}} \quad \mathrm{id} & \\
\uparrow & \uparrow & \\
m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} & \xrightarrow{\;\theta_{\mathcal{X}}\;} \mathcal{O}_{\mathcal{X}} &
\end{array}
$$

of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras whose square is cocartesian.

Lemma 3.53. *In the situation of General assumption 2.1, let $\mathcal{Y}$ be the discrete subdiagram of $\mathcal{X}$. Then there is a natural isomorphism*

$$\left(L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}} \right)[1] \cong L_h$$

in $\mathbf{D}(\mathcal{X})$.

Proof. The factorization $\mathcal{O}_{\mathcal{X}} \to u^{-1}\mathcal{O}_{\mathcal{Y}} \xrightarrow{h} \mathcal{O}_{\mathcal{X}}$ of the identity yields a distinguished triangle

$$
\begin{array}{c}
L_h \\
\swarrow \qquad \nwarrow \\
L_{u^{-1}\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}} \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \xrightarrow{\hspace{3cm}} L_{\mathrm{id}}
\end{array}
$$

in $\mathbf{D}(\mathcal{X})$. Since L_{id} is an exact complex, it follows that L_{id} is isomorphic to the zero complex in $\mathbf{D}(\mathcal{X})$, hence the natural morphism $L_h \to \left(L_{u^{-1}\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}} \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}\right)[1]$ in $\mathbf{D}(\mathcal{X})$ is an isomorphism. Because $m_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$ is flat we have that the natural morphism of complexes of $u^{-1}\mathcal{O}_{\mathcal{Y}}$-modules

$$
L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} u^{-1}\mathcal{O}_{\mathcal{Y}} \to L_{u^{-1}\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}}
$$

is a quasi-isomorphism by [Ill71, Corollaire II.2.2.3.]. This is a quasi-isomorphism between complexes of free $u^{-1}\mathcal{O}_{\mathcal{Y}}$-modules, so tensoring with $\mathcal{O}_{\mathcal{X}}$ yields a quasi-isomorphism

$$
L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}} \to L_{u^{-1}\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}} \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}
$$

of complexes of $\mathcal{O}_{\mathcal{X}}$-modules by [Ill71, Lemme I.3.3.2.1.]. The composition

$$
\left(L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}}\right)[1] \to \left(L_{u^{-1}\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}} \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}\right)[1] \to L_h
$$

is an isomorphism in $\mathbf{D}(\mathcal{X})$. $\qquad\square$

Our next aim is to simplify the complex $L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}}$ by defining the complex L^d announced at the beginning of this subsection. Fix $c \in \mathrm{ob}(\mathfrak{C})$. Then for each $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c', c)$, $\gamma \neq \mathrm{id}$, the morphism

$$
t_c^{-1}\mathcal{O}_{\mathcal{S}} = f_\gamma^{-1} t_{c'}^{-1}\mathcal{O}_{\mathcal{S}} \xrightarrow{f_\gamma^{-1}\theta_{c'}} f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}
$$

is flat and it follows from Proposition A.1 that the natural morphism

$$
\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}} \mathcal{T}_c \; \to \; L_{\mathcal{T}_c/t_c^{-1}\mathcal{O}_{\mathcal{S}}}
$$

is a quasi-isomorphism. Since this is a quasi-isomorphism between complexes of free $\mathcal{T}_c$-modules, we get a quasi-isomorphism

$$
\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}} \mathcal{O}_{\mathcal{X}_c} \; \to \; L_{\mathcal{T}_c/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}_c} \mathcal{O}_{\mathcal{X}_c}
$$

by tensoring with $\mathcal{O}_{\mathcal{X}_c}$ by [Ill71, Lemme I.3.3.2.1.] where we have used the natural isomorphism

$$
\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}} \mathcal{O}_{\mathcal{X}_c} \cong \Big(\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}} \mathcal{T}_c \Big) \otimes_{\mathcal{T}_c} \mathcal{O}_{\mathcal{X}_c}
$$

of complexes of $\mathcal{O}_{\mathcal{X}_c}$-modules. Since f_γ is a morphism of ringed topoi, the natural morphism of complexes $f_\gamma^{-1} L_{\mathcal{X}_{c'}/\mathcal{S}} \to L_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_{c'}}/t_c^{-1}\mathcal{O}_{\mathcal{S}}}$ is an isomorphism by Theorem 1.8.ii). Hence for each $c \in \mathrm{ob}(\mathfrak{C})$ there is a quasi-isomorphism

$$
\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_\gamma^* L_{\mathcal{X}_{c'}/\mathcal{S}} \; \to \; L_{\mathcal{T}_c/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}_c} \mathcal{O}_{\mathcal{X}_c}.
$$

Abbreviating $L_c^d = \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_\gamma^* L_{\mathcal{X}_{c'}/\mathcal{S}}$ for $c \in \mathrm{ob}(\mathfrak{C})$, we may again imitate the construction in Definition 2.7 since the direct sum is a special colimit.

Definition 3.54. We define the complex of $\mathcal{O}_{\mathcal{X}}$-modules L^d to be the collection of all complexes

$$L_c^d = \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_\gamma^* L_{\mathcal{X}_{c'}/\mathcal{S}}$$

of $\mathcal{O}_{\mathcal{X}_c}$-modules for $c \in \mathrm{ob}(\mathfrak{C})$ together with, for each $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, the morphisms

$$f_\alpha^* L_{c_1}^d \to L_{c_2}^d$$

of complexes of $\mathcal{O}_{\mathcal{X}_{c_2}}$-modules in Definition 2.7.

The collection of all the above quasi-isomorphisms $L_c^d \to L_{\mathcal{T}_c/t_c^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}_c} \mathcal{O}_{\mathcal{X}_c}$ for $c \in \mathrm{ob}(\mathfrak{C})$ defines a quasi-isomorphism

$$L^d \to L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}}$$

of complexes of $\mathcal{O}_{\mathcal{X}}$-modules.

Proposition 3.55. *Assume given the situation of General assumption 2.1 and let $\mathcal{Y}$ be the discrete subdiagram of $\mathcal{X}$. Then there is a natural isomorphism $L_h \cong L^d[1]$ in $\mathbf{D}(\mathcal{X})$. In particular, there is an exact sequence*

$$0 \to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^0(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L^d, m_{\mathcal{X}}^*\mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^1(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J}) \xrightarrow{\omega} \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L^d, m_{\mathcal{X}}^*\mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^2(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^*\mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L^d, m_{\mathcal{X}}^*\mathcal{J})$$

$$\to \dots$$

of abelian groups. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L^d, m_{\mathcal{X}}^*\mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ.

Proof. By Lemma 3.53 there is a natural isomorphism $\left(L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}}\right)[1] \cong L_h$ in $\mathbf{D}(\mathcal{X})$ and we have seen above that there is a natural quasi-isomorphism $L^d[1] \to \left(L_{\mathcal{T}/m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \otimes_{\mathcal{T}} \mathcal{O}_{\mathcal{X}}\right)[1]$, whence $L_h \cong L^d[1]$ in $\mathbf{D}(\mathcal{X})$.

The long exact sequence is obtained from the long exact sequence in Theorem 2.13. Since condition **2** in Theorem 2.20 is fulfilled by Lemma 3.51 we may use the isomorphisms

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^i(\mathrm{Cone}(m), m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}}^i(L_h, m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}}^i(L^d[1], m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}}^{i-1}(L^d, m_{\mathcal{X}}^*\mathcal{J})$$

of abelian groups for each $i \in \mathbb{Z}$. Furthermore, we have

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L_h, m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L^d[1], m_{\mathcal{X}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}}^{-1}(L^d, m_{\mathcal{X}}^*\mathcal{J}) = 0.$$

$\square$

Since $\mathcal{Y}$ is the discrete subdiagram of $\mathcal{X}$ we have natural isomorphisms

$$\mathbb{Ext}^i_{\mathcal{Y}}(L_{\mathcal{Y}/\mathcal{S}}, m^*_{\mathcal{Y}}\mathcal{J}) \cong \bigoplus_{c \in \mathrm{ob}(\mathfrak{C})} \mathbb{Ext}^i_{\mathcal{X}_c}(L_{\mathcal{X}_c/\mathcal{S}}, t^*_c\mathcal{J}) \qquad (3.13)$$

of abelian groups for each $i \in \mathbb{Z}$ because the topos $\mathcal{Y}$ is the disjoint union of the topoi $\mathcal{X}_c$.

Before giving some examples we will simplify the obstruction group $\mathbb{Ext}^1_{\mathcal{X}}(L^d, m^*_{\mathcal{X}}\mathcal{J})$ in Proposition 3.55 even further. We will see that it is easier to calculate the obstruction group by using $L^d[1]$ rather than L_h. The replacement of L_h by $L^d[1]$ for the discrete subdiagram is similar to the replacement of L_h by L^*_h for full and well-positioned subdiagrams as in Proposition 3.15.

Let us recall the situation in the case of full and well-positioned subdiagrams $\mathcal{Y}$. If $\overline{\mathcal{Y}}$ is the complementary subdiagram of $\mathcal{Y}$ and if $u \colon \mathcal{X} \to \mathcal{Y}$ and $\overline{u} \colon \mathcal{X} \to \overline{\mathcal{Y}}$ are the forgetful functors, then we have $uL^*_h = 0$ by definition of L^*_h and we know that the obstruction lives in

$$\mathbb{Ext}^2_{\overline{\mathcal{Y}}}(\overline{u}L_h, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$$

by Corollary 3.26. Similarly, if $\mathcal{Y}$ is the discrete subdiagram of $\mathcal{X}$, we will define a certain subdiagram $\mathcal{Z}$ of $\mathcal{X}$ with complementary subdiagram $\overline{\mathcal{Z}}$ and forgetful functors $w \colon \mathcal{X} \to \mathcal{Z}$ and $\overline{w} \colon \mathcal{X} \to \overline{\mathcal{Z}}$ such that $wL^d = 0$ and such that the obstruction lives in

$$\mathbb{Ext}^1_{\overline{\mathcal{Z}}}(\overline{w}L^d, m^*_{\overline{\mathcal{Z}}}\mathcal{J})$$

in Corollary 3.59.

If $c \in \mathrm{ob}(\mathfrak{C})$, then

$$L^d_c = \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c',c), \\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f^*_\gamma L_{\mathcal{X}_{c'}/\mathcal{S}}$$

is the zero complex if and only if the only morphism to c is the identity.

Notation 3.56. Let $\mathcal{Y}$ be the discrete subdiagram of $\mathcal{X}$. Let $\mathfrak{E}$ be the discrete subcategory of $\mathfrak{C}$ with objects

$$\mathrm{ob}(\mathfrak{E}) = \{c \in \mathrm{ob}(\mathfrak{C}) \mid \text{the only morphism to } c \text{ in } \mathfrak{C} \text{ is the identity}\}$$

and let $\overline{\mathfrak{E}}$ be the full subcategory of $\mathfrak{C}$ with objects $\mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{E})$. Let $\mathcal{Z}$ be the ringed topos associated to the restriction of $I \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ to $\mathfrak{E}$ and let $\overline{\mathcal{Z}}$ be the ringed topos associated to the restriction of $I \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ to $\overline{\mathfrak{E}}$.

Notice that we may assume $\mathrm{ob}(\mathfrak{C}) \setminus \mathrm{ob}(\mathfrak{E}) \neq \emptyset$ because otherwise $\mathfrak{C}$ would be discrete itself and we would have $\mathcal{X} = \mathcal{Y}$. Notice further that $\overline{\mathcal{Z}}$ is the complementary subdiagram of $\mathcal{Z}$.

Let $w \colon \mathcal{X} \to \mathcal{Z}$ and $\overline{w} \colon \mathcal{X} \to \overline{\mathcal{Z}}$ be the forgetful functors. By abuse of notation we denote $w \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Z}}\text{-mod}$ and $\overline{w} \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Z}}}\text{-mod}$ the forgetful functors between the categories of modules as well. Let $G \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\overline{\mathcal{Z}}}\text{-mod}$ be the functor in Lemma 3.23.

Lemma 3.57. *The functors* $G\colon \mathcal{O}_\mathcal{X}$-mod $\to \mathcal{O}_{\overline{\mathcal{Z}}}$-mod *and* $\overline{w}\colon \mathcal{O}_\mathcal{X}$-mod $\to \mathcal{O}_{\overline{\mathcal{Z}}}$-mod *are naturally isomorphic. In particular, G is exact.*

Proof. The functor G maps an $\mathcal{O}_\mathcal{X}$-module M to the $\mathcal{O}_{\overline{\mathcal{Z}}}$-module $\overline{M}$ defined levelwise by

$$\overline{M}_c = \bigcap_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{E}}(c,d), \\ d \in \mathrm{ob}(\mathfrak{E})}} \ker\left(M_\gamma^{\mathrm{ad}}\colon M_c \to f_{\gamma*}M_d\right) \subseteq M_c$$

for $c \in \mathrm{ob}(\overline{\mathfrak{E}})$. Since $\mathrm{Hom}_{\mathfrak{E}}(c,d)$ is empty for $d \in \mathrm{ob}(\mathfrak{E})$ and $c \in \mathrm{ob}(\overline{\mathfrak{E}})$ by definition of $\mathfrak{E}$, it follows that $\overline{M}_c = M_c$ for every $c \in \mathrm{ob}(\overline{\mathfrak{E}})$. Since the morphism

$$\overline{M}_\alpha^{\mathrm{ad}}\colon \overline{M}_{c_1} \to f_{\alpha*}\overline{M}_{c_2}$$

for $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{E}}}(c_1, c_2)$ in Lemma 3.22 is defined to be the restriction of the given morphism $M_\alpha^{\mathrm{ad}}\colon M_{c_1} \to f_{\alpha*}M_{c_2}$, it follows that $\overline{M}_\alpha^{\mathrm{ad}} = M_\alpha^{\mathrm{ad}}$ for all $\alpha \in \mathrm{Hom}_{\overline{\mathfrak{E}}}(c_1, c_2)$. Hence we see that G and $\overline{w}$ are naturally isomorphic. The exactness of $\overline{w}$ implies the exactness of G. $\qquad\square$

Corollary 3.58. *Let $\mathcal{X}$ be a diagram and let $\mathcal{Y}$ be the discrete subdiagram of $\mathcal{X}$. Then for each $M \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X}))$ and for each $i \in \mathbb{Z}$ there are natural isomorphisms*

$$\mathbb{E}\mathrm{xt}_\mathcal{X}^i(L^d, M) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^i(\overline{w}L^d, \overline{w}M)$$

of abelian groups which are functorial in M.

Proof. By definition of L^d we have $wL^d = 0$. Moreover, L^d is a bounded above complex consisting of free $\mathcal{O}_\mathcal{X}$-modules. Hence L^d satisfies the conditions in Proposition 3.25 and applying this proposition to L^d, the subdiagram $\mathcal{Z}$ and the complementary subdiagram $\overline{\mathcal{Z}}$, we get natural isomorphisms

$$\mathbb{E}\mathrm{xt}_\mathcal{X}^i(L^d, M) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^i(\overline{w}L^d, \mathbf{R}G(M)),$$

functorial in M. Since G is exact by the above lemma, the natural morphism of functors $G(-) \to \mathbf{R}G(-)$ from $\mathbf{D}^+(\mathcal{X})$ to $\mathbf{D}^+(\overline{\mathcal{Z}})$ is an isomorphism. Moreover, G and $\overline{w}$ are naturally isomorphic, hence there are natural and functorial isomorphisms

$$\mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^i(\overline{w}L^d, \mathbf{R}G(M)) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^i(\overline{w}L^d, \overline{w}M).$$

$\qquad\square$

Corollary 3.59. *Assume given the situation of General assumption 2.1 and let $\mathcal{Y}$ be the discrete subdiagram of $\mathcal{X}$. Given a deformation ξ of the subdiagram $\mathcal{Y}$ over $\mathcal{S}'$, there is an obstruction*

$$\omega(\xi) \in \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^1(\overline{w}L^d, m_{\overline{\mathcal{Z}}}^*\mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a deformation of the diagram $\mathcal{X}$ over $\mathcal{S}'$ reducing to ξ. Here $\overline{\mathcal{Z}}$ is the ringed topos in Notation 3.56.

Proof. By Proposition 3.55 there is an obstruction in $\mathbb{E}\mathrm{xt}_\mathcal{X}^1(L^d, m_\mathcal{X}^*\mathcal{J})$ and by the above corollary there is a natural isomorphism

$$\mathbb{E}\mathrm{xt}_\mathcal{X}^1(L^d, m_\mathcal{X}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}_{\overline{\mathcal{Z}}}^1(\overline{w}L^d, \overline{w}m_\mathcal{X}^*\mathcal{J})$$

of abelian groups. Since $\overline{w}\colon \mathcal{X} \to \overline{\mathcal{Z}}$ is the forgetful functor we have $\overline{w}m_\mathcal{X}^*\mathcal{J} = m_{\overline{\mathcal{Z}}}^*\mathcal{J}$. $\qquad\square$

Let us consider four examples.

Example 3.60. Assume given the situation

diagram in $\mathfrak{RTop}/\mathcal{S}$					subdiagram in $\mathfrak{RTop}/\mathcal{S}$				
$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\cdots$	$\mathcal{X}_m$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\cdots$	$\mathcal{X}_m$
	$f_1 \quad f_2 \quad f_3 \qquad f_m$								
		$\mathcal{X}_0$					$\mathcal{X}_0$		

with no other morphisms involved except for the identities. The ringed topoi $\mathcal{X}_j$ do not have to be pairwise distinct, but if $\mathcal{X}_j = \mathcal{X}_k$ for some $j \neq k$, then we assume that $f_j \neq f_k$. By definition we have $L_c^d = 0$ for $c = 1, \ldots, m$, hence $\overline{\mathcal{Z}} = \mathcal{X}_0$ and

$$\overline{w}L^d = L_0^d = \bigoplus_{j=1}^{m} f_j^* L_{\mathcal{X}_j/\mathcal{S}} \quad \text{as well as} \quad \overline{w}M = M_0$$

for every $\mathcal{O}_{\mathcal{X}}$-module M. Thus we have naturally

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^i(L^d, M) \cong \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^i\Big(\bigoplus_{j=1}^{m} f_j^* L_{\mathcal{X}_j/\mathcal{S}}, M_0\Big) \cong \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^i(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, M_0)$$

for every complex $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ and every $i \in \mathbb{Z}$ by Corollary 3.58.

The long exact sequence in Proposition 3.55 is given by

$$0 \to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^0(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^0(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^1(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^1(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{Y}}^2(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^2(f_j^* L_{\mathcal{X}_j/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \ldots .$$

and we have

$$\mathbb{E}\mathrm{xt}_{\mathcal{Y}}^i(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \cong \bigoplus_{j=0}^{m} \mathbb{E}\mathrm{xt}_{\mathcal{X}_c}^i(L_{\mathcal{X}_c/\mathcal{S}}, t_c^* \mathcal{J})$$

for all $i \in \mathbb{Z}$ by Equation (3.13) on page 94.

In particular, for $m = 1$ the above sequence is given by

$$0 \to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^0(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^0(L_{\mathcal{X}_0/\mathcal{S}}, t_0^* \mathcal{J}) \oplus \mathbb{E}\mathrm{xt}_{\mathcal{X}_1}^0(L_{\mathcal{X}_1/\mathcal{S}}, t_1^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^0(f_1^* L_{\mathcal{X}_1/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^1(L_{\mathcal{X}_0/\mathcal{S}}, t_0^* \mathcal{J}) \oplus \mathbb{E}\mathrm{xt}_{\mathcal{X}_1}^1(L_{\mathcal{X}_1/\mathcal{S}}, t_1^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^1(f_1^* L_{\mathcal{X}_1/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^2(L_{\mathcal{X}_0/\mathcal{S}}, t_0^* \mathcal{J}) \oplus \mathbb{E}\mathrm{xt}_{\mathcal{X}_1}^2(L_{\mathcal{X}_1/\mathcal{S}}, t_1^* \mathcal{J}) \to \mathbb{E}\mathrm{xt}_{\mathcal{X}_0}^2(f_1^* L_{\mathcal{X}_1/\mathcal{S}}, t_0^* \mathcal{J})$$

$$\to \ldots$$

which has been described in [Ran89, Section 2.2.2] if $f_1 \colon \mathcal{X}_0 \to \mathcal{X}_1$ is a morphism of ringed spaces.

Example 3.61. Next we consider the dual problem. Assume given the situation

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
$\mathcal{X}_1 \quad \mathcal{X}_2 \quad \mathcal{X}_3 \quad \cdots \quad \mathcal{X}_m$ $f_1 \quad f_2 \quad f_3 \qquad f_m$ $\mathcal{X}_0$	$\mathcal{X}_1 \qquad \mathcal{X}_2 \qquad \mathcal{X}_3 \qquad \cdots \qquad \mathcal{X}_m$ $\mathcal{X}_0$
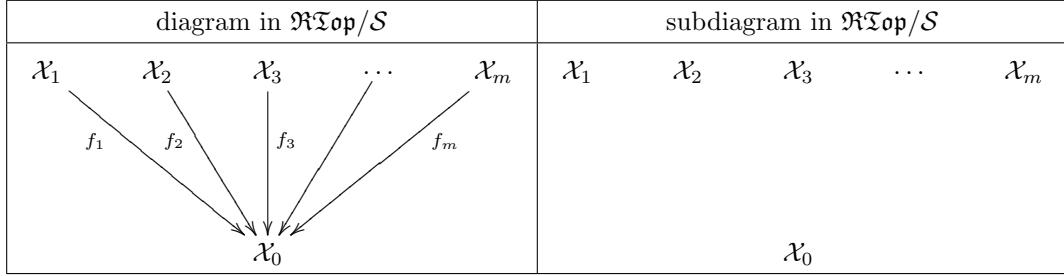
with no other morphisms involved except for the identities. Again the ringed topoi $\mathcal{X}_j$ do not have to be pairwise distinct, but if $\mathcal{X}_j = \mathcal{X}_k$ for some $j \neq k$, then we assume that $f_j \neq f_k$. By definition we have $L_0^d = 0$ and the ringed topos

$$\overline{\mathcal{Z}} = \coprod_{j=1}^{m} \mathcal{X}_j$$

is the disjoint union of the topoi $\mathcal{X}_j$ for $j = 1, \ldots, m$. Moreover,

$$\overline{w}L^d = \left(f_1^* L_{\mathcal{X}_0/\mathcal{S}}, \ldots, f_m^* L_{\mathcal{X}_0/\mathcal{S}}\right) \quad \text{and} \quad \overline{w}M = (M_1, \ldots, M_m)$$

for every $\mathcal{O}_{\mathcal{X}}$-module M. Thus we have naturally

$$\mathbb{Ext}_{\mathcal{X}}^i(L^d, M) \cong \mathbb{Ext}_{\overline{\mathcal{Z}}}^i\left((f_1^* L_{\mathcal{X}_0/\mathcal{S}}, \ldots, f_m^* L_{\mathcal{X}_0/\mathcal{S}}), (M_1, \ldots, M_m)\right) \cong \bigoplus_{j=1}^{m} \mathbb{Ext}_{\mathcal{X}_j}^i(f_j^* L_{\mathcal{X}_0/\mathcal{S}}, M_j)$$

for every complex $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ and every $i \in \mathbb{Z}$ by Corollary 3.58.

The long exact sequence in Proposition 3.55 is given by

$$0 \to \mathbb{Ext}_{\mathcal{X}}^0(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{Ext}_{\mathcal{Y}}^0(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{Ext}_{\mathcal{X}_j}^0(f_j^* L_{\mathcal{X}_0/\mathcal{S}}, t_j^* \mathcal{J})$$

$$\to \mathbb{Ext}_{\mathcal{X}}^1(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{Ext}_{\mathcal{Y}}^1(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{Ext}_{\mathcal{X}_j}^1(f_j^* L_{\mathcal{X}_0/\mathcal{S}}, t_j^* \mathcal{J})$$

$$\to \mathbb{Ext}_{\mathcal{X}}^2(L_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^* \mathcal{J}) \to \mathbb{Ext}_{\mathcal{Y}}^2(L_{\mathcal{Y}/\mathcal{S}}, m_{\mathcal{Y}}^* \mathcal{J}) \to \bigoplus_{j=1}^{m} \mathbb{Ext}_{\mathcal{X}_j}^2(f_j^* L_{\mathcal{X}_0/\mathcal{S}}, t_j^* \mathcal{J})$$

$$\to \ldots .$$

Example 3.62. Now let us consider the pair

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$
Y $f \quad\quad g$ $X \xrightarrow{\;h=g\circ f\;} Z$	Y $X \qquad\qquad Z$
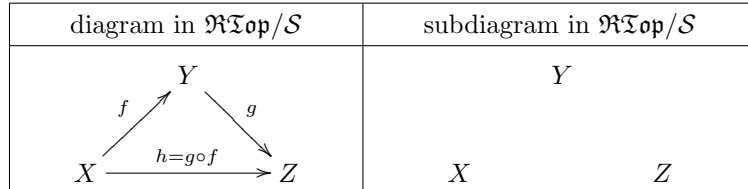
with no other morphisms except for the identities and let t_X, t_Y and t_Z be the structure morphisms of X, Y and Z over $\mathcal{S}$, respectively. By definition $\overline{\mathcal{Z}}$ is the ringed topos associated to the diagram $X \xrightarrow{f} Y$. We denote the objects of $\overline{\mathcal{Z}}$ in the form

$$(\mathcal{F}_X, \mathcal{F}_Y, f^{-1}\mathcal{F}_Y \to \mathcal{F}_X)$$

where $\mathcal{F}_X$ and $\mathcal{F}_Y$ are sheaves of X and Y, respectively, and $f^{-1}\mathcal{F}_Y \to \mathcal{F}_X$ is a sheaf morphism of X. If $\overline{w}\colon \mathcal{X} \to \overline{\mathcal{Z}}$ is the forgetful functor, then the complex $\overline{w}L^d$ is given by

$$(h^*L_{Z/\mathcal{S}} \oplus f^*L_{Y/\mathcal{S}}, g^*L_{Z/\mathcal{S}}, f^*g^*L_{Z/\mathcal{S}} = h^*L_{Z/\mathcal{S}} \xrightarrow{(\mathrm{id}|0)} h^*L_{Z/\mathcal{S}} \oplus f^*L_{Y/\mathcal{S}})$$

which naturally decomposes as $L_1 \oplus L_2$ in the category of complexes of $\mathcal{O}_{\overline{\mathcal{Z}}}$-modules where

$$L_1 = (f^*L_{Y/\mathcal{S}}, 0, f^*0 \to f^*L_{Y/\mathcal{S}})$$

and

$$L_2 = (h^*L_{Z/\mathcal{S}}, g^*L_{Z/\mathcal{S}}, f^*g^*L_{Z/\mathcal{S}} = h^*L_{Z/\mathcal{S}} \xrightarrow{\mathrm{id}} h^*L_{Z/\mathcal{S}}).$$

Hence the obstruction group in Corollary 3.59 is given by

$$\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(\overline{w}L^d, m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L_1, m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L_2, m^*_{\overline{\mathcal{Z}}}\mathcal{J}).$$

We may apply Proposition 3.25 to the diagram $\overline{\mathcal{Z}}$, the subdiagram Y and the complex L_1 to see that there is a natural isomorphism $\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L_1, m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathrm{Ext}^1_X(f^*L_{Y/\mathcal{S}}, t^*_X\mathcal{J})$. It remains to simplify the second direct summand.

If $M \in \mathrm{ob}(\mathbf{Ch}(\overline{\mathcal{Z}}))$, then by definition the group $\mathrm{Hom}_{\mathbf{Ch}(\overline{\mathcal{Z}})}(L_2, M)$ consists of all morphisms

$$a \in \mathrm{Hom}_{\mathbf{Ch}(X)}(h^*L_{Z/\mathcal{S}}, M_X) \quad \text{and} \quad b \in \mathrm{Hom}_{\mathbf{Ch}(Y)}(g^*L_{Z/\mathcal{S}}, M_Y)$$

such that

$$
\begin{array}{ccc}
f^*g^*L_{Z/\mathcal{S}} = h^*L_{Z/\mathcal{S}} & \xrightarrow{\ \mathrm{id}\ } & h^*L_{Z/\mathcal{S}} \\
{\scriptstyle f^*b}\downarrow & & \downarrow{\scriptstyle a} \\
f^*M_Y & \longrightarrow & M_X
\end{array}
$$

is commutative in $\mathbf{Ch}(X)$. It follows that b uniquely determines a, whence we have naturally

$$\mathrm{Hom}_{\mathbf{Ch}(\overline{\mathcal{Z}})}(L_2, M) \cong \mathrm{Hom}_{\mathbf{Ch}(Y)}(g^*L_{Z/\mathcal{S}}, M_Y),$$

functorial in M, and there is a diagram of functors

$$
\begin{array}{ccc}
\mathbf{K}^+(\overline{\mathcal{Z}}) & \xrightarrow{\quad\quad v \quad\quad} & \mathbf{K}^+(Y) \\
& & \\
{\scriptstyle \mathrm{Hom}^{\cdot}_{\overline{\mathcal{Z}}}(L_2, -)}\searrow & & \swarrow{\scriptstyle \mathrm{Hom}^{\cdot}_Y(g^*L_{Z/\mathcal{S}}, -)} \\
& \mathbf{K}(\mathfrak{Ab}) &
\end{array}
$$

commutative up to natural isomorphism of functors where $\mathfrak{Ab}$ is the category of abelian groups and v is the forgetful functor. Now exactly as shown in Proposition 2.14 we get natural isomorphisms $\mathbb{E}\mathrm{xt}^i_{\overline{\mathcal{Z}}}(L_2, M) \cong \mathrm{Ext}^i_Y(g^*L_{Z/\mathcal{S}}, M_Y)$ for each $i \in \mathbb{Z}$, functorial in $M \in \mathrm{ob}(\mathbf{D}^+(\overline{\mathcal{Z}}))$. In particular, we have

$$\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L_2, m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathrm{Ext}^1_Y(g^*L_{Z/\mathcal{S}}, t^*_Y\mathcal{J})$$

and the obstruction group is given by $\mathbb{E}\mathrm{xt}^1_X(f^*L_{Y/\mathcal{S}}, t_X^*\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_Y(g^*L_{Z/\mathcal{S}}, t_Y^*\mathcal{J})$. The first summand contains the obstruction for finding a deformation $f' \colon X' \to Y'$ of $f \colon X \to Y$ over $\mathcal{S}'$ between the given deformations X' and Y' of X and Y, respectively, and the second summand contains the obstruction for finding a deformation $g' \colon Y' \to Z'$ of $g \colon Y \to Z$ over $\mathcal{S}'$ between the given deformations Y' and Z' of Y and Z, respectively. The composition of f' and g' is a deformation h' of h.

Example 3.63. Now let us consider the pair

diagram in $\mathfrak{RTop}/\mathcal{S}$	subdiagram in $\mathfrak{RTop}/\mathcal{S}$

with no other morphisms except for the identities such that $h \circ f = j = i \circ g$. Let t_W, t_X, t_Y and t_Z be the structure morphisms of W, X, Y and Z over $\mathcal{S}$, respectively. By definition $\overline{\mathcal{Z}}$ is the ringed topos associated to the diagram

$$
\begin{array}{c}
X \\
\;{\scriptstyle f}\big\uparrow\; \\
W \xrightarrow{\;g\;} Y.
\end{array}
$$

Let $\overline{w} \colon \mathcal{X} \to \overline{\mathcal{Z}}$ the forgetful functor. Denoting the objects of $\overline{\mathcal{Z}}$ as

$$(\mathcal{F}_W, \mathcal{F}_X, \mathcal{F}_Y, f^{-1}\mathcal{F}_X \to \mathcal{F}_W, g^{-1}\mathcal{F}_Y \to \mathcal{F}_W)$$

where $\mathcal{F}_W, \mathcal{F}_X$ and $\mathcal{F}_Y$ are sheaves of W, X and Y, respectively, $\overline{w}L^d$ is by definition given by

$$(j^*L_{Z/\mathcal{S}} \oplus f^*L_{X/\mathcal{S}} \oplus g^*L_{Y/\mathcal{S}}, h^*L_{Z/\mathcal{S}}, i^*L_{Z/\mathcal{S}},$$

$$f^*h^*L_{Z/\mathcal{S}} = j^*L_{Z/\mathcal{S}} \xrightarrow{(\mathrm{id}|0|0)} j^*L_{Z/\mathcal{S}} \oplus f^*L_{X/\mathcal{S}} \oplus g^*L_{Y/\mathcal{S}},$$

$$g^*i^*L_{Z/\mathcal{S}} = j^*L_{Z/\mathcal{S}} \xrightarrow{(\mathrm{id}|0|0)} j^*L_{Z/\mathcal{S}} \oplus f^*L_{X/\mathcal{S}} \oplus g^*L_{Y/\mathcal{S}}).$$

There is a natural decomposition $\overline{w}L^d = L(f) \oplus L(g) \oplus L(j)$ of $\overline{w}L^d$ as complexes of $\mathcal{O}_{\overline{\mathcal{Z}}}$-modules where

$$L(f) = (f^*L_{X/\mathcal{S}}, 0, 0, f^*0 \to f^*L_{X/\mathcal{S}}, g^*0 \to f^*L_{X/\mathcal{S}}),$$

$$L(g) = (g^*L_{Y/\mathcal{S}}, 0, 0, f^*0 \to g^*L_{Y/\mathcal{S}}, g^*0 \to g^*L_{Y/\mathcal{S}}),$$

$$L(j) = (j^*L_{Z/\mathcal{S}}, h^*L_{Z/\mathcal{S}}, i^*L_{Z/\mathcal{S}}, f^*h^*L_{Z/\mathcal{S}} = j^*L_{Z/\mathcal{S}} \xrightarrow{\mathrm{id}} j^*L_{Z/\mathcal{S}}, g^*i^*L_{Z/\mathcal{S}} = j^*L_{Z/\mathcal{S}} \xrightarrow{\mathrm{id}} j^*L_{Z/\mathcal{S}}).$$

Hence the obstruction group $\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(\overline{w}L^d, m_{\overline{\mathcal{Z}}}^*\mathcal{J})$ in Corollary 3.59 is given by

$$\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(\overline{w}L^d, m_{\overline{\mathcal{Z}}}^*\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(f), m_{\overline{\mathcal{Z}}}^*\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(g), m_{\overline{\mathcal{Z}}}^*\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(j), m_{\overline{\mathcal{Z}}}^*\mathcal{J}).$$

Let us simplify the direct summands further. If $\mathcal{V}$ is the ringed topos associated to the diagram consisting of X and Y, we may apply Proposition 3.25 to the diagram $\overline{\mathcal{Z}}$, the subdiagram $\mathcal{V}$ and to the complex $L(f)$. Since $G\colon \mathcal{O}_{\overline{\mathcal{Z}}}$-mod $\to \mathcal{O}_W$-mod is the forgetful functor $\mathcal{O}_{\overline{\mathcal{Z}}}$-mod $\to \mathcal{O}_W$-mod and since the forgetful functor is exact, it follows from Proposition 3.25 that there is a natural isomorphism $\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(f), m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_W(f^*L_{X/\mathcal{S}}, t^*_W\mathcal{J})$. Similarly, there is a natural isomorphism $\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(g), m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_W(g^*L_{Y/\mathcal{S}}, t^*_W\mathcal{J})$, whence the obstruction group is given by

$$\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(\overline{w}L^d, m^*_{\overline{\mathcal{Z}}}\mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_W(f^*L_{X/\mathcal{S}}, t^*_W\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_W(g^*L_{Y/\mathcal{S}}, t^*_W\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(j), m^*_{\overline{\mathcal{Z}}}\mathcal{J}).$$

Let us exhibit these three direct summands. If we are given deformations W', X', Y' and Z' of W, X, Y and Z over $\mathcal{S}'$, respectively, then the first direct summand contains the obstruction for extending W', X' to a deformation $f'\colon W' \to X'$ of $f\colon W \to X$ and the second direct summand contains the obstruction for extending W', Y' to a deformation $g'\colon W' \to Y'$ of $g\colon W \to Y$ by Example 3.60. The third direct summand contains the remaining obstruction for extending

By definition the group $\mathrm{Hom}_{\mathbf{Ch}(\overline{\mathcal{Z}})}(L(j), m^*_{\overline{\mathcal{Z}}}\mathcal{J})$ consists of all morphisms

$$a \in \mathrm{Hom}_{\mathbf{Ch}(W)}(j^*L_{Z/\mathcal{S}}, t^*_W\mathcal{J}), \ b \in \mathrm{Hom}_{\mathbf{Ch}(X)}(h^*L_{Z/\mathcal{S}}, t^*_X\mathcal{J}), \ c \in \mathrm{Hom}_{\mathbf{Ch}(Y)}(i^*L_{Z/\mathcal{S}}, t^*_Y\mathcal{J})$$

such that

$$
\begin{array}{ccccc}
f^*h^*L_{Z/\mathcal{S}} = j^*L_{Z/\mathcal{S}} & \xrightarrow{\ \mathrm{id}\ } & j^*L_{Z/\mathcal{S}} & \xleftarrow{\ \mathrm{id}\ } & j^*L_{Z/\mathcal{S}} = g^*i^*L_{Z/\mathcal{S}} \\
\ \ \downarrow{\scriptstyle f^*b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle g^*c} \\
f^*t^*_X\mathcal{J} = t^*_W\mathcal{J} & \xrightarrow{\ \mathrm{id}\ } & t^*_W\mathcal{J} & \xleftarrow{\ \mathrm{id}\ } & t^*_W\mathcal{J} = g^*t^*_Y\mathcal{J}
\end{array}
$$

is commutative in $\mathbf{Ch}(W)$. Hence b uniquely determines a and c uniquely determines a and we must have $f^*b = g^*c$. Thus if

$$f^*\colon \mathrm{Hom}_{\mathbf{Ch}(X)}(h^*L_{Z/\mathcal{S}}, t^*_X\mathcal{J}) \to \mathrm{Hom}_{\mathbf{Ch}(W)}(j^*L_{Z/\mathcal{S}}, t^*_W\mathcal{J})$$

and

$$g^*\colon \mathrm{Hom}_{\mathbf{Ch}(Y)}(i^*L_{Z/\mathcal{S}}, t^*_Y\mathcal{J}) \to \mathrm{Hom}_{\mathbf{Ch}(W)}(j^*L_{Z/\mathcal{S}}, t^*_W\mathcal{J})$$

denote the pullback morphisms by abuse of notation, we have that $\mathrm{Hom}_{\mathbf{Ch}(\overline{\mathcal{Z}})}(L(j), m^*_{\overline{\mathcal{Z}}}\mathcal{J})$ is the kernel of the difference morphism

$$\mathrm{Hom}_{\mathbf{Ch}(X)}(h^*L_{Z/\mathcal{S}}, t^*_X\mathcal{J}) \oplus \mathrm{Hom}_{\mathbf{Ch}(Y)}(i^*L_{Z/\mathcal{S}}, t^*_Y\mathcal{J}) \to \mathrm{Hom}_{\mathbf{Ch}(W)}(j^*L_{Z/\mathcal{S}}, t^*_W\mathcal{J}),$$

defined by $(b, c) \mapsto f^*b - g^*c$. But it is not clear a priori how to simplify $\mathbb{E}\mathrm{xt}^1_{\overline{\mathcal{Z}}}(L(j), m^*_{\overline{\mathcal{Z}}}\mathcal{J})$ further.

4 Diagrams with at most three levels

This section considers diagrams consisting of at most three levels. The first subsection deals with a single morphism $f\colon X \to Y$ whereas the second subsection treats a commutative triangle

$$
\begin{array}{ccc}
 & Y & \\
 {\scriptstyle f}\nearrow & & \searrow {\scriptstyle g} \\
 X & \xrightarrow[\;h=g\circ f\;]{} & Z.
\end{array}
$$

We will derive the cotangent braid of a morphism $f\colon X \to Y$ of ringed topoi over a ringed topos S and show that it coincides with the cotangent braid in [Buc81, Diagramme II.2.4.2.1] if $S = \operatorname{Spec} k$ is the spectrum of a field k. For this purpose, we consider the three subdiagrams X, Y and (X, Y) of the ringed topos $\mathcal{Z}$ associated to the diagram $f\colon X \to Y$. We will get two morphisms of distinguished triangles between the three corresponding distinguished triangles in $\mathbf{D}(\mathcal{Z})$. The braid in [Buc81, Diagramme II.2.4.2.1] is, however, obtained from four distinguished triangles which are part of a certain octahedron, similarly as we obtained the braid in Proposition 2.22 in Subsection 2.4.

Afterwards, we consider all subdiagrams of a single morphism $f\colon X \to Y$ and the respective obstruction groups using results of Section 3. We will restrict to the case of schemes in order to replace certain cotangent complexes by modules, if some additional conditions on X, Y or the morphism f are satisfied. For example, if f is a smooth morphism of schemes, then the natural morphism of complexes of $\mathcal{O}_X$-modules $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism by [Ill71, Proposition III.3.1.2.].

It turns out that there are fifteen non-empty proper subdiagrams of the above commutative triangle. The obstruction groups for ten of them may be derived easily from the obstruction groups of a single morphism, but for the other five subdiagrams, this is not possible. Four of these five subdiagrams are even not full. Nevertheless, we will calculate the obstruction groups as explicitly as possible.

Finally, we will give an example of a triangle where the obstruction group for the subdiagram obtained by omitting Y vanishes. Let X be a nonsingular projective variety over the complex numbers $\mathbb{C}$, let $h\colon X \to Z = \operatorname{Alb}(X)$ be the Albanese map of X and let Y denote the image of h. Considering only extensions where $\mathcal{J} = \mathcal{O}_{\operatorname{Spec} \mathbb{C}}$, it turns out that the obstruction group for extending a deformation of h to a deformation of the triangle is zero if f is a flat morphism such that $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$ and if Y a product of nonsingular projective curves.

4.1 A single morphism

4.1.1 The cotangent braid of a morphism of ringed topoi

Let $f\colon X \to Y$ be a morphism of ringed topoi over a ringed topos S such that the structure morphisms $t_X\colon X \to S$ and $t_Y\colon Y \to S$ are flat. Let furthermore $j\colon S \to S'$ be a closed embedding induced by an extension

$$0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$$

of $\mathcal{O}_S$ by an $\mathcal{O}_S$-module $\mathcal{J}$. Using the results of Sections 2 and 3, we will derive the cotangent braid in [Buc81, Diagramme II.2.4.3.2] which can also be found in [GLS07, Corollary C.5.2.].

Let $\mathcal{Z}$ be the ringed topos associated to the diagram $X \xrightarrow{f} Y$. We denote the objects of $\mathcal{Z}$ in the form

$$\left(F_X \xleftarrow{v} F_Y\right)$$

where F_X is a sheaf of X, F_Y is a sheaf of Y and $v\colon f^{-1}F_Y \to F_X$ is a sheaf morphism. If v is clear from the context we will omit it and simply write $\left(F_X \leftarrow F_Y\right)$.

The three different subdiagrams X, Y and (X,Y) of $\mathcal{Z}$ yield three $m_{\mathcal{Z}}^{-1}\mathcal{O}_S$-algebras

$$u^{-1}\mathcal{O}_X = \left(\mathcal{O}_X \leftarrow t_Y^{-1}\mathcal{O}_S\right), u^{-1}\mathcal{O}_Y = \left(f^{-1}\mathcal{O}_Y \leftarrow \mathcal{O}_Y\right) \text{ and } u^{-1}\mathcal{O}_{(X,Y)} = \left(\mathcal{O}_X \otimes_{t_X^{-1}\mathcal{O}_S} f^{-1}\mathcal{O}_Y \leftarrow \mathcal{O}_Y\right),$$

respectively, by Proposition 2.16 where we have written u for the forgetful functor in all three cases by abuse of notation. Notice that

$$u^{-1}\mathcal{O}_{(X,Y)} = u^{-1}\mathcal{O}_X \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_S} u^{-1}\mathcal{O}_Y$$

by [Ill71, Equation III.4.3.1] since the tensor product commutes with the inverse image functor of a morphism of ringed topoi. Furthermore, letting $l(X)$, $l(Y)$, $l(X,Y)$ and $h(X)$, $h(Y)$, $h(X,Y)$ denote the morphisms l and h in Proposition 2.16 for the different subdiagrams, the diagram

$$
\begin{array}{ccccc}
m_{\mathcal{Z}}^{-1}\mathcal{O}_S & \xrightarrow{\ l(X)\ } & u^{-1}\mathcal{O}_X & \xrightarrow{\ h(X)\ } & \mathcal{O}_{\mathcal{Z}} \\
\| & & \downarrow{\scriptstyle -\otimes 1} & & \| \\
m_{\mathcal{Z}}^{-1}\mathcal{O}_S & \xrightarrow{\ l(X,Y)\ } & u^{-1}\mathcal{O}_{(X,Y)} & \xrightarrow{\ h(X,Y)\ } & \mathcal{O}_{\mathcal{Z}} \\
\| & & \uparrow{\scriptstyle 1\otimes -} & & \| \\
m_{\mathcal{Z}}^{-1}\mathcal{O}_S & \xrightarrow{\ l(Y)\ } & u^{-1}\mathcal{O}_Y & \xrightarrow{\ h(Y)\ } & \mathcal{O}_{\mathcal{Z}}
\end{array}
\tag{4.1}
$$

of $m_{\mathcal{Z}}^{-1}\mathcal{O}_S$-algebras is commutative. Since X and Y are flat over S we have that $l(X)\colon m_{\mathcal{Z}}^{-1}\mathcal{O}_S \to u^{-1}\mathcal{O}_X$ and $l(Y)\colon m_{\mathcal{Z}}^{-1}\mathcal{O}_S \to u^{-1}\mathcal{O}_Y$ are flat as well. Hence by Theorem 1.8.vi) the natural morphism of complexes

$$\left(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} u^{-1}\mathcal{O}_{(X,Y)}\right) \oplus \left(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} u^{-1}\mathcal{O}_{(X,Y)}\right) \to L_{l(X,Y)}$$

is a quasi-isomorphism. Since both of the complexes in its definition consist of free $u^{-1}\mathcal{O}_{(X,Y)}$-modules, we still get a quasi-isomorphism

$$\left(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}\right) \oplus \left(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}\right) \to L_{l(X,Y)} \otimes_{u^{-1}\mathcal{O}_{(X,Y)}} \mathcal{O}_{\mathcal{Z}}$$

by tensoring with $\mathcal{O}_{\mathcal{Z}}$ by [Ill71, Lemma I.3.3.2.1.]. By [Ill71, Chapitre II.2.1.] Diagram (4.1) induces two morphisms of distinguished triangles

$$L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}} \xrightarrow{(\mathrm{id}|0)} \left(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}\right) \oplus \left(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}\right) \xleftarrow{(0|\mathrm{id})} L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}$$

$$L_{l(X,Y)} \otimes_{u^{-1}\mathcal{O}_{(X,Y)}} \mathcal{O}_{\mathcal{Z}}$$

$$L_{\mathcal{Z}/S} = L_{\mathcal{Z}/S} = L_{\mathcal{Z}/S}$$

$$L_{h(X)} \longrightarrow L_{h(X,Y)} \longleftarrow L_{h(Y)}$$

$$(4.2)$$

in $\mathbf{D}(\mathcal{Z})$. Notice that condition **2** in Theorem 2.20 is satisfied for all three subdiagrams.

Proposition 4.1. *The above morphisms of distinguished triangles induce a commutative diagram of abelian groups*

$$(4.3)$$

whose columns are the long exact sequences associated to the subdiagrams X, Y and (X,Y) of $\mathcal{Z}$ by Theorem 2.20, respectively.

Proof. We apply the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{Z})}(-, m_{\mathcal{Z}}^* \mathcal{J})$ to Diagram (4.2) and use the natural isomorphism

$$\mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i(L_{l(X,Y)} \otimes_{u^{-1}\mathcal{O}_{(X,Y)}} \mathcal{O}_{\mathcal{Z}}, m_{\mathcal{Z}}^* \mathcal{J})$$

$$\downarrow \cong$$

$$\mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i\big((L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}) \oplus (L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}), m_{\mathcal{Z}}^* \mathcal{J}\big)$$

$$\downarrow \cong$$

$$\mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}, m_{\mathcal{Z}}^* \mathcal{J}) \oplus \mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}, m_{\mathcal{Z}}^* \mathcal{J})$$

induced from $\big(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}\big) \oplus \big(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}\big) \xrightarrow{\cong} L_{l(X,Y)} \otimes_{u^{-1}\mathcal{O}_{(X,Y)}} \mathcal{O}_{\mathcal{Z}}$ and the natural isomorphisms

$$\mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i(L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}, m_{\mathcal{Z}}^* \mathcal{J}) \cong \mathbb{E}\mathrm{xt}_X^i(L_{X/S}, t_X^* \mathcal{J})$$

and

$$\mathbb{E}\mathrm{xt}_{\mathcal{Z}}^i(L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}, m_{\mathcal{Z}}^* \mathcal{J}) \cong \mathbb{E}\mathrm{xt}_Y^i(L_{Y/S}, t_Y^* \mathcal{J})$$

in Theorem 2.20. $\qquad\square$

Now we are able to derive the cotangent braid.

Corollary 4.2. *Let $f \colon X \to Y$ be a morphism of ringed topoi over a ringed topos S such that the structure morphisms $t_X \colon X \to S$ and $t_Y \colon Y \to S$ are flat and let $\mathcal{Z}$ be the ringed topos associated to the diagram $X \xrightarrow{f} Y$. Let $j \colon S \to S'$ be a closed embedding induced by an extension*

$$0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$$

of $\mathcal{O}_S$ by an $\mathcal{O}_S$-module $\mathcal{J}$. Then there is a commutative braid, the cotangent braid

$$\cdots \qquad \cdots$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i}(L_{h(X,Y)}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i}(L_{h(X)}, m_{\mathcal{Z}}^*\mathcal{J}) \qquad \mathbb{Ext}_{\mathcal{Z}}^{i}(L_{h(Y)}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i}(L_{\mathcal{Z}/S}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{Y}^{i}(L_{Y/S}, t_Y^*\mathcal{J}) \qquad \mathbb{Ext}_{X}^{i}(L_{X/S}, t_X^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+1}(L_{h(X,Y)}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+1}(L_{h(Y)}, m_{\mathcal{Z}}^*\mathcal{J}) \qquad \mathbb{Ext}_{\mathcal{Z}}^{i+1}(L_{h(X)}, m_{\mathcal{Z}}^*\mathcal{J}) \qquad (4.4)$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+1}(L_{\mathcal{Z}/S}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{X}^{i+1}(L_{X/S}, t_X^*\mathcal{J}) \qquad \mathbb{Ext}_{Y}^{i+1}(L_{Y/S}, t_Y^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+2}(L_{h(X,Y)}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+2}(L_{h(X)}, m_{\mathcal{Z}}^*\mathcal{J}) \qquad \mathbb{Ext}_{\mathcal{Z}}^{i+2}(L_{h(Y)}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\mathbb{Ext}_{\mathcal{Z}}^{i+2}(L_{\mathcal{Z}/S}, m_{\mathcal{Z}}^*\mathcal{J})$$
$$\cdots \qquad \cdots$$

*containing four long exact sequences. The sequences **1** and **2** are the long exact sequences of Diagram (4.3) in Proposition 4.1, i.e., the long exact sequences associated to the subdiagrams X and Y of $\mathcal{Z}$ by Theorem 2.20, respectively.*

Proof. We define the morphisms of the third sequence **3** to be certain compositions in Diagram (4.3). Take $\mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J}) \xrightarrow{\mathbf{3}} \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J})$ to be the composition of the curved arrows in

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J}) \longleftarrow \mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \longrightarrow \mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{h(Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} \quad\quad \xrightarrow{\mathrm{id}} \quad\quad\quad \xrightarrow{\mathrm{id}} \quad\quad \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) =\!=\!= \mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) =\!=\!= \mathbb{E}\mathrm{xt}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \xleftarrow{\mathrm{pr}_1} \mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \xrightarrow{\mathrm{pr}_2} \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

let $\mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \xrightarrow{\mathbf{3}} \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J})$ be the composition of the curved arrows in

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \xleftarrow{\mathrm{pr}_1} \mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \xrightarrow{\mathrm{pr}_2} \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} \quad\quad\quad \xleftarrow[(0|\mathrm{id})]{} \quad\quad \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J}) \longleftarrow \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \longrightarrow \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

and set $\mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \xrightarrow{\mathbf{3}} \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J})$ to be the curved arrow in

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \xleftarrow{\mathrm{pr}_1} \mathbb{E}\mathrm{xt}^i_X(L_{X/S}, t^*_X\mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \xrightarrow{\mathrm{pr}_2} \mathbb{E}\mathrm{xt}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J}) \longleftarrow \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \longrightarrow \mathbb{E}\mathrm{xt}^{i+1}_{\mathcal{Z}}(L_{h(Y)}, m^*_{\mathcal{Z}}\mathcal{J}). \\
\Big\downarrow{\scriptstyle\mathbf{1}} & & & & \Big\downarrow{\scriptstyle\mathbf{2}} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

The exactness of sequence **3** follows from the exactness of the columns of Diagram (4.3) and all parts of the braid containing morphisms of Sequences **1**, **2** and **3** are commutative by definition. By interchanging the roles of X and Y we define the forth sequence **4** analogously. $\qquad\square$

106

Remark 4.3. Notice that backwards one may derive Diagram (4.3) from the braid (4.4). We take sequences **1** and **2** of the braid to be the left and right column of Diagram (4.3), respectively. For $\mathbb{Ext}^i_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}) \rightsquigarrow \mathbb{Ext}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J})$ we take either of the compositions in

$$\mathbb{Ext}^i_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J})$$

$$\overset{\mathbf{3}}{\swarrow} \qquad\qquad \overset{\mathbf{4}}{\searrow}$$

$$\mathbb{Ext}^i_{\mathcal{Z}}(L_{h(X)}, m^*_{\mathcal{Z}}\mathcal{J}) \qquad\qquad \mathbb{Ext}^i_{\mathcal{Z}}(L_{h(Y)}, m^*_{\mathcal{Z}}\mathcal{J})$$

$$\overset{\mathbf{1}}{\searrow} \qquad\qquad \overset{\mathbf{2}}{\swarrow}$$

$$\mathbb{Ext}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J})$$

and $\mathbb{Ext}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) \rightsquigarrow \mathbb{Ext}^i_X(L_{X/S}, t^*_X\mathcal{J}) \oplus \mathbb{Ext}^i_Y(L_{Y/S}, t^*_Y\mathcal{J})$ is defined to have component morphisms

$$\mathbb{Ext}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) \overset{\mathbf{1}}{\longrightarrow} \mathbb{Ext}^i_X(L_{X/S}, t^*_X\mathcal{J}) \quad \text{and} \quad \mathbb{Ext}^i_{\mathcal{Z}}(L_{\mathcal{Z}/S}, m^*_{\mathcal{Z}}\mathcal{J}) \overset{\mathbf{2}}{\longrightarrow} \mathbb{Ext}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}),$$

respectively. Finally, $\mathbb{Ext}^i_X(L_{X/S}, t^*_X\mathcal{J}) \oplus \mathbb{Ext}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \rightsquigarrow \mathbb{Ext}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J})$ is the sum morphism of

$$\mathbb{Ext}^i_Y(L_{Y/S}, t^*_Y\mathcal{J}) \qquad\qquad \mathbb{Ext}^i_X(L_{X/S}, t^*_X\mathcal{J})$$

$$\overset{\mathbf{3}}{\searrow} \qquad\qquad \overset{\mathbf{4}}{\swarrow}$$

$$\mathbb{Ext}^{i+1}_{\mathcal{Z}}(L_{h(X,Y)}, m^*_{\mathcal{Z}}\mathcal{J}).$$

Hence giving the braid (4.4) is equivalent to giving Diagram (4.3) and both are obtained from the two morphisms of distinguished triangles in Diagram (4.2) on page 103.

Proposition 4.4. *If $S = \operatorname{Spec} k$ is the spectrum of a field k, then the cotangent braid (4.4) is naturally isomorphic to the cotangent braid in [Buc81, Diagramme II.2.4.3.2].*

Proof. Some of the six complexes of $\mathcal{O}_{\mathcal{Z}}$-modules $L_{\mathcal{Z}/S}$, $L_{h(Y)}$, $L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}}$, $L_{h(X)}$, $L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}}$ and $L_{h(X,Y)}$ may be simplified in $\mathbf{D}(\mathcal{Z})$ by results of Section 3 as shown in the following table.

cotangent complex	naturally isomorphic to	by
$L_{\mathcal{Z}/S} = \bigl(L_{X/S} \leftarrow L_{Y/S}\bigr)$	$\bigl(L_{X/S} \leftarrow L_{Y/S}\bigr)$	
$L_{h(Y)} = \bigl(L_{X/Y} \leftarrow L_{Y/Y}\bigr)$	$\bigl(L_{X/Y} \leftarrow 0\bigr)$	Proposition 3.15
$L_{l(X)} \otimes_{u^{-1}\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}} = \bigl(L_{X/S} \leftarrow L_{Y/Y}\bigr)$	$\bigl(L_{X/S} \leftarrow 0\bigr)$	Proposition 3.15
$L_{h(X)} = \bigl(L_{X/X} \leftarrow L_{Y/S}\bigr)$	$\bigl(0 \leftarrow L_{Y/S}\bigr)$	Proposition 3.15
$L_{l(Y)} \otimes_{u^{-1}\mathcal{O}_Y} \mathcal{O}_{\mathcal{Z}} = \bigl(f^*L_{Y/S} \leftarrow L_{Y/S}\bigr)$	$\bigl(f^*L_{Y/S} \leftarrow L_{Y/S}\bigr)$	
$L_{h(X,Y)} = \bigl(L_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X} \leftarrow L_{Y/Y}\bigr)$	$\bigl(f^*L_{Y/S} \leftarrow 0\bigr)[1]$	Proposition 3.55.

If $S = \operatorname{Spec} k$, then the six complexes of the middle column are just the cotangent complexes $L^0, \ldots, L^5$ of [Buc81, Chapitre II.2.4.2] and the morphisms ψ^{ij} of [Buc81, Diagramme II.2.4.2.1] coincide, up to natural isomorphism, with the corresponding morphisms in Diagram (4.2) on page 103. $\qquad\square$

4.1.2 The subdiagrams of a single morphism of schemes

We will list all subdiagrams of a single morphism of schemes and detect the corresponding obstruction groups in Theorem 2.20 using the results of Section 3. Assume given the situation in General assumption 2.1. The obstruction groups will be calculated as explicitly as possible for the different cases.

We restrict to the case of schemes in order to replace some cotangent complexes by modules under certain conditions. Let $f\colon X \to Y$ be a morphism of schemes over the fixed scheme S with structure morphisms $t_X\colon X \to S$ and $t_Y\colon Y \to S$ which are assumed to be flat by General assumption 2.1. Assume furthermore that $\mathcal{J}$ is quasi-coherent such that each deformation of $f\colon X \to Y$ over $\mathcal{S}'$ is given by a morphism of schemes $f'\colon X' \to Y'$ by Remark 1.12.

Proposition 4.5. *The following chart is a list of all subdiagrams of*

$$X \xrightarrow{\ f\ } Y$$

and the corresponding obstruction groups as found in Theorem 2.20:

	subdiagram	condition or notation	obstruction group
1	X	$\mathbf{R}G$ as in Proposition 3.25 for $G\colon \mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(Y)$ as in Lemma 3.23	$\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$
		t_Y smooth, $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$, $\mathcal{K}$ and $\mathcal{Q}$ kernel and cokernel of $t_Y^*\mathcal{J} \to f_*(t_X^*\mathcal{J})$, respectively	$\mathrm{Ext}^2_Y(\Omega^1_{Y/S}, \mathcal{K})$ $\oplus$ $\mathrm{Ext}^1_Y(\Omega^1_{Y/S}, \mathcal{Q})$
2	Y	none	$\mathbb{E}\mathrm{xt}^2_X(L_{X/Y}, t_X^*\mathcal{J})$
		f smooth	$\mathrm{Ext}^2_X(\Omega^1_{X/Y}, t_X^*\mathcal{J})$
		f a closed embedding with regular ideal sheaf $\mathcal{I}$	$\mathrm{Ext}^1_X(\mathcal{I}/\mathcal{I}^2, t_X^*\mathcal{J})$
3	$X \qquad Y$	none	$\mathrm{Ext}^1_X(f^*L_{Y/S}, t_X^*\mathcal{J})$
		t_Y smooth	$\mathrm{Ext}^1_X(f^*\Omega^1_{Y/S}, t_X^*\mathcal{J})$

Proof. If $g\colon W \to Z$ is a smooth morphism of schemes, then the natural morphism of complexes $L_{W/Z} \to \Omega^1_{W/Z}$ is a quasi-isomorphism by [Ill71, Proposition III.3.1.2.]. We will use this result several times.

Subdiagram 1: By Proposition 3.28 the obstruction group is given by

$$\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, \mathbf{R}G(m_{\mathcal{X}}^*\mathcal{J}))$$

without any supplementary conditions. If $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$, then the obstruction group is isomorphic to $\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, \mathcal{K}) \oplus \mathbb{E}\mathrm{xt}^1_Y(L_{Y/S}, \mathcal{Q})$ by Corollary 3.36 where $\mathcal{K}$ and $\mathcal{Q}$ are kernel and cokernel of $t_Y^*\mathcal{J} \to f_*(t_X^*\mathcal{J})$, respectively. If t_Y is smooth, then $L_{Y/S} \to \Omega^1_{Y/S}$ is a quasi-isomorphism.

Subdiagram 2: Similarly, without any further conditions, the obstruction group is

$$\mathbb{E}\mathrm{xt}^2_X(L_{X/Y}, t_X^*\mathcal{J})$$

108

by Proposition 3.43 and Proposition 3.39. If f is smooth, then $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism and if f is a closed embedding with regular ideal sheaf $\mathcal{I}$, then the natural morphism of complexes $L_{X/Y} \to \mathcal{I}/\mathcal{I}^2[1]$ is a quasi-isomorphism by [Ill71, Proposition III.3.2.4.].

Subdiagram 3: The obstruction group is

$$\mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/S}, t^*_X \mathcal{J})$$

by Corollary 3.59 or by Example 3.60 for $m = 1$. If t_Y is smooth, then $L_{Y/S} \to \Omega^1_{Y/S}$ is a quasi-isomorphism and $\Omega^1_{Y/S}$ is locally free of finite type by [Ill71, Proposition III.3.1.2.], in particular $\Omega^1_{Y/S}$ is a flat $\mathcal{O}_Y$-module. Since the complex $L_{Y/S}$ consists of free $\mathcal{O}_Y$-modules, it follows that the induced morphism $f^* L_{Y/S} \to f^* \Omega^1_{Y/S}$ is still a quasi-isomorphism by [Ill71, Lemme I.3.3.2.1.]. $\quad\square$

The obstruction groups $\mathbb{E}\mathrm{xt}^2_X(L_{X/Y}, t^*_X \mathcal{J})$ and $\mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/S}, t^*_X \mathcal{J})$ for subdiagrams 2 and 3 have been calculated in [Ill71, Théorème III.2.1.7.] and [Ill71, Proposition III.2.2.4.], respectively, by using different argumentations. But the obstruction group $\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, \mathbf{R}G(m^*_X \mathcal{J}))$ for subdiagram 1 does not occur there.

Remark 4.6. Notice that the above chart contains the obstruction groups for extending the given deformation of the respective subdiagram to a deformation of the diagram in one step. It is also possible to try to extend the given deformation of the respective subdiagram to a deformation of the diagram in two steps, at least for subdiagrams 1 und 2. For example, if we consider the subdiagram X and if we fix a deformation X' of X over S', we may first search for a deformation of Y (with obstruction in $\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, t^*_Y \mathcal{J})$ by Theorem 1.11) and then, having chosen a deformation Y' of Y if possible, search for a deformation of f from X' to Y' (with obstruction in $\mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/S}, t^*_X \mathcal{J})$ by the above proposition). The problem of this "two-step-obstruction" is the choice of Y' because for some choices of Y', the two-step-obstruction in $\mathbb{E}\mathrm{xt}^2_Y(L_{Y/S}, t^*_Y \mathcal{J}) \oplus \mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/S}, t^*_X \mathcal{J})$ might vanish and for other choices of Y', it might not.

Example 4.7. Let $S = \mathrm{Spec}\, k$ be the spectrum of a field k, let $\mathcal{J} = \mathcal{O}_{\mathrm{Spec}\, k}$ and let $f\colon X \to Y$ be a closed embedding of schemes such that Y is nonsingular and projective. Assume that the ideal sheaf $\mathcal{I}$ of the embedding is regular. Then $t^*_X \mathcal{J} = \mathcal{O}_X$ and $t^*_Y \mathcal{J} = \mathcal{O}_Y$ and $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$ since f is a closed embedding. Moreover, the morphism $t^*_Y \mathcal{J} \to f_*(t^*_X \mathcal{J})$ is just the ring morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ and since $\Omega^1_{Y/k}$ is locally free, the obstruction group for the subdiagram X is given by

$$\mathrm{Ext}^2_Y(\Omega^1_{Y/k}, \mathcal{K}) = \mathrm{Ext}^2_Y(\Omega^1_{Y/k}, \mathcal{I}) \cong \mathrm{Ext}^2_Y(\mathcal{O}_Y, \mathcal{T}_{Y/k} \otimes \mathcal{I}) \cong H^2(Y, \mathcal{T}_{Y/k} \otimes \mathcal{I})$$

where $\mathcal{T}_{Y/k} = \mathcal{H}om_Y(\Omega^1_{Y/k}, \mathcal{O}_Y)$ is the tangent sheaf of Y over k. This result is stated in [Ser06, Proposition 3.4.23.]. If additionally X is smooth such that $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_X$-module and if $\mathcal{N} = \mathcal{H}om_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ is the normal sheaf of X in Y, then the obstruction group for the subdiagram Y is given by

$$\mathrm{Ext}^1_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \cong \mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{N}) \cong H^1(X, \mathcal{N})$$

which is as well stated in [Ser06, Proposition 3.4.23.].

4.2 A commutative triangle

4.2.1 The subdiagrams of a commutative triangle of schemes

We will list all subdiagrams of a commutative triangle

$$
\begin{array}{ccc}
Y & \xrightarrow{\;g\;} & Z \\
\Big\uparrow{\scriptstyle f} & \nearrow{\scriptstyle h} & \\
X & &
\end{array}
$$

of schemes over a fixed scheme S and detect the corresponding obstruction groups in Theorem 2.20 using some results of Section 3. Assume given the situation in General assumption 2.1 and let $t_X\colon X \to S$, $t_Y\colon Y \to S$ and $t_Z\colon Z \to S$ be the structure morphisms of X, Y and Z over S, respectively. Assume that $\mathcal{J}$ is quasi-coherent.

There are 15 non-empty proper subdiagrams of this commutative triangle, numbered in the following table.

Y X **1**	Y **2**	Z **3**	Y $\uparrow f$ X **4**	Y $\quad Z$ $\uparrow f$ X **5**
Y X **6**	$Y \quad Z$ **7**	$Y \quad Z$ X **8**	$Y \xrightarrow{g} Z$ **9**	$Y \xrightarrow{g} Z$ X **10**
Z X **11**	Z $\nearrow h$ X **12**	$Y \quad Z$ $\nearrow h$ X **13**	$Y \xrightarrow{g} Z$ $\nearrow h$ X **14**	$Y \quad Z$ $\uparrow f \;\nearrow h$ X **15**

The calculation of the obstruction groups for the first 10 subdiagrams may be traced back to the calculations for the subdiagrams of a single morphism, as exhibited in Proposition 4.5. For example, suppose given the subdiagram X of the triangle. If $\mathcal{T}(g)$ denotes the ringed topos associated to the diagram $Y \xrightarrow{g} Z$ with structure morphism $m_{\mathcal{T}(g)}\colon \mathcal{T}(g) \to S$, then the commutative triangle gives rise to a morphism of ringed topoi

$$
X \xrightarrow{\;(f,h)\;} \mathcal{T}(g)
$$

over S by [Ill71, Chapitre III.4.12.]. Extending a given deformation X' of X over S' to a deformation of the triangle is equivalent to extending X' to a deformation of the morphism $X \xrightarrow{(f,h)} \mathcal{T}(g)$.

Hence from Proposition 4.5 we see that the obstruction lives in $\mathbb{E}\mathrm{xt}^2_{\mathcal{T}(g)}(L_{\mathcal{T}(g)/S}, \mathbf{R}G(m^*_{\mathcal{X}}\mathcal{J}))$ with $G\colon \mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(\mathcal{T}(g))$ as in Lemma 3.23. Subdiagram 8 has been treated in Example 3.62.

Consequently, it suffices to calculate the obstruction group for subdiagrams 11 to 15. Let $\mathcal{T}(f)$ and $\mathcal{T}(g)$ be the ringed topoi associated to the diagrams

$$X \xrightarrow{\ f\ } Y \quad \text{and} \quad Y \xrightarrow{\ g\ } Z$$

over S, respectively. The commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\ g\ } & Z
\end{array}
$$

yields a morphism of ringed topoi $(f, g)\colon \mathcal{T}(f) \to \mathcal{T}(g)$ over S by [Ill71, Chapitre III.4.12.].

Proposition 4.8. *The following chart is a list of subdiagrams 11 to 15 of the above table of the diagram*

$$
\begin{array}{ccc}
Y & \xrightarrow{\;g\;} & Z \\
{\scriptstyle f}\uparrow & \nearrow_{h} & \\
X & &
\end{array}
$$

and the corresponding obstruction groups as found in Theorem 2.13 or Theorem 2.20.

	subdiagram	condition or notation	obstruction group
11	Z X	$\theta_h\colon h^{-1}\mathcal{O}_Z \to \mathcal{O}_X$ the ring morphism of h, L the cotangent complex of $h^{-1}\mathcal{O}_Z \otimes_{t_X^{-1}\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X$, $a \otimes b \mapsto \theta_h(a)b$	$\mathbb{E}\mathrm{xt}^2_{\mathcal{T}(f)}\big((L, L_{Y/Z}),(t_X^*\mathcal{J}, t_Y^*\mathcal{J})\big)$
12	Z $\nearrow_{h}$ X	$\mathbf{R}G$ as in Proposition 3.25 for $G\colon \mathbf{K}^+(\mathcal{T}(f)) \to \mathbf{K}^+(Y)$ as in Lemma 3.23	$\mathbb{E}\mathrm{xt}^2_Y(L_{Y/Z}, \mathbf{R}G(m_{\mathcal{T}(f)}^*\mathcal{J}))$
		g smooth, $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$, $\mathcal{K}$ and $\mathcal{Q}$ kernel and cokernel of $t_Y^*\mathcal{J} \to f_*(t_X^*\mathcal{J})$, respectively	$\mathrm{Ext}^2_Y(\Omega^1_{Y/Z}, \mathcal{K})$ $\oplus$ $\mathrm{Ext}^1_Y(\Omega^1_{Y/Z}, \mathcal{Q})$
		g a closed embedding with regular ideal sheaf $\mathcal{I}$, $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$, $\mathcal{K}$ and $\mathcal{Q}$ kernel and cokernel of $t_Y^*\mathcal{J} \to f_*(t_X^*\mathcal{J})$, respectively	$\mathrm{Ext}^1_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{K})$ $\oplus$ $\mathrm{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q})$
13	$Y \quad Z$ $\nearrow_{h}$ X	$L_{\mathcal{T}(g)/S} = (L_{Y/S}, L_{Z/S}, g^*L_{Z/S} \to L_{Y/S})$ the cotangent complex of $m_{\mathcal{T}(g)}\colon \mathcal{T}(g) \to S$	$\mathbb{E}\mathrm{xt}^1_{\mathcal{T}(f)}\big((f,g)^*L_{\mathcal{T}(g)/S}, m_{\mathcal{T}(f)}^*\mathcal{J}\big)$
14	$Y \xrightarrow{\;g\;} Z$ $\nearrow_{h}$ X	none	$\mathbb{E}\mathrm{xt}^1_X(f^*L_{Y/Z}, t_X^*\mathcal{J})$
		g smooth	$\mathrm{Ext}^1_X(f^*\Omega^1_{Y/Z}, t_X^*\mathcal{J})$
		g a closed embedding with regular ideal sheaf $\mathcal{I}$ and $\bullet\ \mathcal{I}/\mathcal{I}^2$ flat over $\mathcal{O}_Y$ or $\ \bullet\ f$ flat	$\mathrm{Hom}_X(f^*\mathcal{I}/\mathcal{I}^2, t_X^*\mathcal{J})$
15	$Y \quad Z$ ${\scriptstyle f}\uparrow \nearrow_{h}$ X	$\mathbf{R}G$ as in Proposition 3.25 for $G\colon \mathbf{K}^+(\mathcal{T}(f)) \to \mathbf{K}^+(Y)$ as in Lemma 3.23	$\mathbb{E}\mathrm{xt}^1_Y(g^*L_{Z/S}, \mathbf{R}G(m_{\mathcal{T}(f)}^*\mathcal{J}))$
		t_Z smooth, $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$, $\mathcal{K}$ and $\mathcal{Q}$ kernel and cokernel of $t_Y^*\mathcal{J} \to f_*(t_X^*\mathcal{J})$, respectively	$\mathrm{Ext}^1_Y(g^*\Omega^1_{Z/S}, \mathcal{K})$ $\oplus$ $\mathrm{Hom}_Y(g^*\Omega^1_{Z/S}, \mathcal{Q})$

Notice that only the twelfth subdiagram is well-positioned and full. Thus for the other subdiagrams Corollary 3.26 is not applicable. We will use similar argumentations pointed out in Section 3 for the simplification of the respective obstruction group. Condition **2** in Theorem 2.20 will always be fulfilled except for subdiagram 14.

Proof. If $l\colon W \to W'$ is a smooth morphism of schemes (respectively a closed embedding of schemes with regular ideal sheaf $\mathcal{I}$), then the natural morphism of complexes $L_{W/W'} \to \Omega^1_{W/W'}$ (respectively $L_{W/W'} \to \mathcal{I}/\mathcal{I}^2[1]$) is a quasi-isomorphism by [Ill71, Proposition III.3.1.2.] (respectively by [Ill71, Proposition III.3.2.4.]). We will use these results several times.

Let $h\colon u^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ be the ring morphism in Proposition 2.16. We will denote the objects of $\mathcal{X}$ in the form $(\mathcal{F}_X, \mathcal{F}_Y, \mathcal{F}_Z)$ if the sheaf morphisms $f^{-1}\mathcal{F}_Y \to \mathcal{F}_X$, $g^{-1}\mathcal{F}_Z \to \mathcal{F}_Y$ and $h^{-1}\mathcal{F}_Z \to \mathcal{F}_X$ are clear from the context.

Subdiagram 11: By definition we have $u^{-1}\mathcal{O}_{\mathcal{Y}} = \big(h^{-1}\mathcal{O}_Z \otimes_{t_X^{-1}\mathcal{O}_S} \mathcal{O}_X, g^{-1}\mathcal{O}_Z, \mathcal{O}_Z\big)$, hence condition **2** is satisfied and

$$L_h = \big(L, L_{Y/Z}, L_{Z/Z}\big)$$

by Corollary A.3. The morphism

$$\big(\mathrm{id}, \mathrm{id}, 0\big)\colon \big(L, L_{Y/Z}, 0\big) \to \big(L, L_{Y/Z}, L_{Z/Z}\big)$$

in $\mathbf{Ch}(\mathcal{X})$ is a quasi-isomorphism by the exactness of $L_{Z/Z}$. It follows that we may replace L_h by $\big(L, L_{Y/Z}, 0\big)$ in $\mathbf{D}(\mathcal{X})$ for the calculation of the obstruction group. Since the forgetful functor $\mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(\mathcal{T}(f))$ is exact and since there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ch}(\mathcal{X})}\big((L, L_{Y/Z}, 0), m_{\mathcal{X}}^*\mathcal{J}\big) \cong \mathrm{Hom}_{\mathbf{Ch}(\mathcal{T}(f))}\big((L, L_{Y/Z}), m_{\mathcal{T}(f)}^*\mathcal{J}\big)$$

of abelian groups, we get a natural isomorphism

$$\mathbb{E}\mathrm{xt}_{\mathcal{X}}^2\big((L, L_{Y/Z}, 0), m_{\mathcal{X}}^*\mathcal{J}\big) \cong \mathbb{E}\mathrm{xt}_{\mathcal{T}(f)}^2\big((L, L_{Y/Z}), m_{\mathcal{T}(f)}^*\mathcal{J}\big)$$

by the same argument as used in the proof of Proposition 3.25.

Subdiagram 12: Since $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a bridge of $\mathcal{X}$ we know by Proposition 3.46 and Lemma 3.47 that the obstruction lives in $\mathbb{E}\mathrm{xt}_Y^2(L_{Y/Z}, \mathbf{R}G(m_{\mathcal{T}(f)}^*\mathcal{J}))$ for $G\colon \mathbf{K}^+(\mathcal{T}(f)) \to \mathbf{K}^+(Y)$ as in Lemma 3.23. If $R^i f_*(t_X^*\mathcal{J}) = 0$ for all $i > 0$ then we have naturally

$$\mathbb{E}\mathrm{xt}_Y^2(L_{Y/Z}, \mathbf{R}G(m_{\mathcal{T}(f)}^*\mathcal{J})) \cong \mathbb{E}\mathrm{xt}_Y^2(L_{Y/Z}, \mathcal{K}) \oplus \mathbb{E}\mathrm{xt}_Y^1(L_{Y/Z}, \mathcal{Q})$$

by Corollary 3.49. If g is smooth (respectively if g is a closed embedding with regular ideal sheaf $\mathcal{I}$), then the natural morphism of complexes

$$L_{Y/Z} \to \Omega^1_{Y/Z} \quad (\text{respectively } L_{Y/Z} \to \mathcal{I}/\mathcal{I}^2[1])$$

is a quasi-isomorphism.

Subdiagram 13: By definition we have

$$u^{-1}\mathcal{O}_{\mathcal{Y}} = \left(f^{-1}\mathcal{O}_Y \otimes_{t_X^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_X, g^{-1}\mathcal{O}_Z \otimes_{t_Y^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_Y, \mathcal{O}_Z \right),$$

hence condition **2** is fulfilled and

$$L_h = \left(L_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, L_{Z/Z} \right).$$

The same argument as in the proof for subdiagram 11 shows that we may replace L_h by

$$\left(L_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0 \right)$$

in $\mathbf{D}(\mathcal{X})$. Now as in the proof of Proposition 3.55 this complex is naturally isomorphic in $\mathbf{D}(\mathcal{X})$ to

$$\left(f^*L_{Y/S}, g^*L_{Z/S}, 0 \right)[1]$$

where the morphism of complexes of $\mathcal{O}_X$-modules $f^*g^*L_{Z/S} \to f^*L_{Y/S}$ is the pullback under f of the natural morphism of complexes of $\mathcal{O}_Y$-modules $g^*L_{Z/S} \to L_{Y/S}$. Since the forgetful functor $\mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(\mathcal{T}(f))$ is exact and since there is a natural isomorphism

$$\operatorname{Hom}_{\mathbf{Ch}(\mathcal{X})}\left((f^*L_{Y/S}, g^*L_{Z/S}, 0), m_{\mathcal{X}}^*\mathcal{J} \right) \cong \operatorname{Hom}_{\mathbf{Ch}(\mathcal{T}(f))}\left((f^*L_{Y/S}, g^*L_{Z/S}), m_{\mathcal{T}(f)}^*\mathcal{J} \right)$$

of abelian groups, we get a natural isomorphism

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}\left((f^*L_{Y/S}, g^*L_{Z/S}, 0)[1], m_{\mathcal{X}}^*\mathcal{J} \right) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{T}(f)}\left((f^*L_{Y/S}, g^*L_{Z/S})[1], m_{\mathcal{T}(f)}^*\mathcal{J} \right)$$

by the same argument as used in the proof of Proposition 3.25. Moreover, we have naturally

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{T}(f)}\left((f^*L_{Y/S}, g^*L_{Z/S})[1], m_{\mathcal{T}(f)}^*\mathcal{J} \right) \cong \mathbb{E}\mathrm{xt}^1_{\mathcal{T}(f)}\left((f^*L_{Y/S}, g^*L_{Z/S}), m_{\mathcal{T}(f)}^*\mathcal{J} \right)$$

$$\cong \mathbb{E}\mathrm{xt}^1_{\mathcal{T}(f)}\left((f,g)^*L_{\mathcal{T}(g)/S}, m_{\mathcal{T}(f)}^*\mathcal{J} \right).$$

Subdiagram 14: By definition we have $u^{-1}\mathcal{O}_{\mathcal{Y}} = \left(f^{-1}\mathcal{O}_Y \otimes_{h^{-1}\mathcal{O}_Z} \mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_Z \right)$, hence condition **2** is not satisfied. If $w \colon \mathcal{X} \to \mathcal{Y}$ is the forgetful functor with left adjoint $w^{-1} \colon \mathcal{Y} \to \mathcal{X}$ as in Proposition 2.11 and if $B \in \mathrm{ob}(\mathcal{Y})$, then $(w^{-1}B)_X$, the part of $w^{-1}B$ on the level X, is given by the colimit in X of the system

$$
\begin{array}{ccc}
 & & B_X \\
 & \nearrow & \\
h^{-1}B_Z & & \\
 & \searrow & \\
 & & f^{-1}B_Y
\end{array}
$$

and this colimit does not commute with finite limits. Thus condition **1** is not satisfied either. Nevertheless, we are able to calculate the obstruction group by Theorem 2.13. Assume given a deformation

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Z' \\
 & \nearrow_{h'} & \\
X' & &
\end{array}
\qquad \text{of the subdiagram} \qquad
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
 & \nearrow_{h} & \\
X & &
\end{array}
$$

over S' and let
$$0 \to t_Z^* \mathcal{J} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$
be the extension of $\mathcal{O}_Z$ where $t_Z \colon Z \to S$ is the structure morphism. Now we consider $h \colon X \to Z$ and $g \colon Y \to Z$ as (not necessarily flat) structure morphisms. We define $\mathcal{X}$ to be the ringed topos associated to the diagram $f \colon X \to Y$ in the category $\mathfrak{RTop}/Z$ of ringed topoi over Z. Similarly, let $\mathcal{Y}$ be the ringed topos associated to the diagram (X, Y) in the category $\mathfrak{RTop}/Z$ of ringed topoi over Z and let $v \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$ be the forgetful functor with left adjoint $v^* \colon \mathcal{O}_{\mathcal{Y}}\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$ as in Proposition 2.11. Letting
$$c \colon f^* L_{Y/Z} \to L_{X/Z}$$
be the natural morphism of complexes of $\mathcal{O}_X$-modules induced from the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ and denoting the objects of $\mathcal{X}$ in the form
$$(\mathcal{F}_X, \mathcal{F}_Y, f^{-1}\mathcal{F}_Y \to \mathcal{F}_X),$$
we have
$$v^* L_{\mathcal{Y}/Z} = (f^* L_{Y/Z} \oplus L_{X/Z}, L_{Y/Z}, f^* L_{Y/Z} \xrightarrow{(\mathrm{id}|c)} f^* L_{Y/Z} \oplus L_{X/Z})$$
and
$$L_{\mathcal{X}/Z} = (L_{X/Z}, L_{Y/Z}, f^* L_{Y/Z} \xrightarrow{c} L_{X/Z})$$
by definition of v^* and by Corollary A.3. Now the morphism $m \colon v^* L_{\mathcal{Y}/Z} \to L_{\mathcal{X}/Z}$ of complexes of $\mathcal{O}_{\mathcal{X}}$-modules in Theorem 2.13 is given levelwise by
$$m_X \colon f^* L_{Y/Z} \oplus L_{X/Z} \to L_{X/Z}, \ (a, b) \mapsto c(a) + b \quad \text{and} \quad m_Y \colon L_{Y/Z} \xrightarrow{\mathrm{id}} L_{Y/Z}.$$

Since m_X is surjective with kernel isomorphic to $f^* L_{Y/Z}$ and since the cone of the identity is homotopic to the zero complex, we have
$$\mathrm{Cone}(m) = (f^* L_{Y/Z}[1], 0, f^* 0 \to f^* L_{Y/Z}[1])$$
in $\mathbf{D}(\mathcal{X})$. By Theorem 2.13 there is an obstruction for finding an extension

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Z' \\
{\scriptstyle f'} \big\uparrow & \nearrow {\scriptstyle h'} & \\
X' & &
\end{array}
\qquad \text{of} \qquad
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Z' \\
& \nearrow {\scriptstyle h'} & \\
X' & &
\end{array}
$$

with respect to the extension
$$0 \to t_Z^* \mathcal{J} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$
of $\mathcal{O}_Z$ in the group $\mathbb{E}\mathrm{xt}_{\mathcal{X}}^2(\mathrm{Cone}(m), m_{\mathcal{X}}^* t_Z^* \mathcal{J})$ where $m_{\mathcal{X}} \colon \mathcal{X} \to Z$ is the morphism of ringed topoi in Proposition 1.17 (we did not need the flatness of the structure morphisms $h \colon X \to Z$ and $g \colon Y \to Z$ in Theorem 2.13).

Now considering Y as a subdiagram of $\mathcal{X}$ with the complementary subdiagram X, the functor $G \colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_X\text{-mod}$ in Lemma 3.23 is the forgetful functor $\mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_X\text{-mod}$, in particular, it is exact. Since $\mathrm{Cone}(m)$ is a bounded above complex consisting of free $\mathcal{O}_{\mathcal{X}}$-modules it follows

from Proposition 3.25, applied to the diagram $\mathcal{X}$, the subdiagram Y and the complementary subdiagram X, that we have naturally

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(\mathrm{Cone}(m), m^*_{\mathcal{X}} t^*_Z \mathcal{J}) \cong \mathbb{E}\mathrm{xt}^2_X(f^* L_{Y/Z}[1], h^* t^*_Z \mathcal{J})$$

which may be simplified to $\mathbb{E}\mathrm{xt}^2_X(f^* L_{Y/Z}[1], t^*_X \mathcal{J}) \cong \mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/Z}, t^*_X \mathcal{J})$.

Notice that if $Z = S$ and t_Z is the identity, then this is just the obstruction group of Subdiagram 3 in Proposition 4.5 and the proof there shows the simplification of the obstruction group if g is smooth. If g is a closed embedding with regular ideal sheaf $\mathcal{I}$, then the natural morphism of complexes of $\mathcal{O}_Y$-modules $L_{Y/Z} \to \mathcal{I}/\mathcal{I}^2[1]$ is a quasi-isomorphism. If $\mathcal{I}/\mathcal{I}^2$ is a flat $\mathcal{O}_Y$-module or if f is flat, then the induced morphism $f^* L_{Y/Z} \to f^* \mathcal{I}/\mathcal{I}^2[1]$ is still a quasi-isomorphism by [Ill71, Lemme I.3.3.2.1.] because $L_{Y/Z}$ consists of free $\mathcal{O}_Y$-modules.

Subdiagram 15: By definition we have $u^{-1}\mathcal{O}_{\mathcal{Y}} = \big(\mathcal{O}_X, g^{-1}\mathcal{O}_Z \otimes_{t_Y^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{O}_Y, \mathcal{O}_Z\big)$, hence condition **2** is satisfied and

$$L_h = \big(L_{X/X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, L_{Z/Z}\big)$$

which may be replaced in $\mathbf{D}(\mathcal{X})$ by $\big(0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0\big)$ using the quasi-isomorphisms

$$(\mathrm{id}, \mathrm{id}, 0)\colon \big(L_{X/X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0\big) \to \big(L_{X/X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, L_{Z/Z}\big)$$

and

$$(0, \mathrm{id}, 0)\colon \big(L_{X/X}, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0\big) \to \big(0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0\big).$$

Since the forgetful functor $\mathbf{K}^+(\mathcal{X}) \to \mathbf{K}^+(\mathcal{T}(f))$ is exact and since there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ch}(\mathcal{X})}\big((0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0), m^*_{\mathcal{X}}\mathcal{J}\big) \cong \mathrm{Hom}_{\mathbf{Ch}(\mathcal{T}(f))}\big((0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}), m^*_{\mathcal{T}(f)}\mathcal{J}\big)$$

of abelian groups, we get a natural isomorphism

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}\big((0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, 0), m^*_{\mathcal{X}}\mathcal{J}\big) \cong \mathbb{E}\mathrm{xt}^2_{\mathcal{T}(f)}\big((0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}), m^*_{\mathcal{T}(f)}\mathcal{J}\big)$$

by the same argument as used in the proof of Proposition 3.25. Now if $G\colon \mathbf{K}^+(\mathcal{T}(f)) \to \mathbf{K}^+(Y)$ is the functor in Lemma 3.23 we get a natural isomorphism

$$\mathbb{E}\mathrm{xt}^2_{\mathcal{T}(f)}\big((0, L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}), m^*_{\mathcal{T}(f)}\mathcal{J}\big) \cong \mathbb{E}\mathrm{xt}^2_Y(L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y}, \mathbf{R}G(m^*_{\mathcal{T}(f)}\mathcal{J}))$$

by Proposition 3.25. By an analogue argumentation as given in the proof of Proposition 3.55 there is a natural isomorphism $L_{\mathcal{O}_Y/g^{-1}\mathcal{O}_Z \otimes \mathcal{O}_Y} \cong g^* L_{Z/S}[1]$ in $\mathbf{D}(Y)$, hence the obstruction lives in

$$\mathbb{E}\mathrm{xt}^1_Y(g^* L_{Z/S}, \mathbf{R}G(m^*_{\mathcal{T}(f)}\mathcal{J})).$$

The simplification of the obstruction group if t_Z is smooth and if $R^i f_*(t^*_X \mathcal{J}) = 0$ for all $i > 0$ is shown as in the case of subdiagrams 12 and 14. $\qquad\square$

Remark 4.9. The obstruction groups for Subdiagrams 11 and 13 are difficult to control in practice since they are concentrated on the ringed topos $\mathcal{T}(f)$. In this case we may consider "two-step" obstruction groups similarly as in Remark 4.6.

Assume given deformations X' and Z' of X and Z, respectively. Let us consider an example of a two-step obstruction. There is an obstruction for finding a deformation $h' \colon X' \to Z'$ of $h \colon X \to Z$ lying in $\mathbb{E}\mathrm{xt}^1_X(h^* L_{Z/S}, t_X^* \mathcal{J})$ by Proposition 4.5 Subdiagram 3. If it vanishes, there is an obstruction in $\mathbb{E}\mathrm{xt}^2_Y(L_{Y/Z}, \mathbf{R}G(m^*_{\mathcal{T}(f)} \mathcal{J}))$ for extending $h' \colon X' \to Z'$ to a deformation of the triangle by Proposition 4.8 Subdiagram 12.

Now assume given a deformation $Y', h' \colon X' \to Z'$ of Subdiagram 13. There is, for example, an obstruction in $\mathbb{E}\mathrm{xt}^1_Y(g^* L_{Z/S}, t_Y^* \mathcal{J})$ for finding a deformation $g' \colon Y' \to Z'$ of $g \colon Y \to Z$ by Proposition 4.5 Subdiagram 3. If it vanishes, there is an obstruction in $\mathbb{E}\mathrm{xt}^1_X(f^* L_{Y/Z}, t_X^* \mathcal{J})$ for extending $h' \colon X' \to Z'$ and $g' \colon Y' \to Z'$ to a deformation of the triangle by Proposition 4.8 Subdiagram 14.

4.2.2 Deformations of the image of the Albanese map

We refer to [Băd01, Chapter 5] for the definition of the Albanese variety and its properties.

We consider only extensions of $\mathcal{O}_S$ by the module $\mathcal{J} = \mathcal{O}_S$, i.e., those extensions of $\mathcal{O}_S$ of the form $0 \to \mathcal{O}_S \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$. Let $S = \operatorname{Spec} \mathbb{C}$ be the spectrum of the field $\mathbb{C}$ of complex numbers. By a variety over $\mathbb{C}$ we mean an integral separated scheme of finite type over $\mathbb{C}$.

Let X be a nonsingular projective variety over $\mathbb{C}$ and let $Z = \operatorname{Alb}(X)$ be the Albanese variety of X. Then Z is a nonsingular projective abelian variety of dimension $q = \dim H^1(X, \mathcal{O}_X)$. Fixing a closed point $x_0 \in X$, we may consider the Albanese map

$$h\colon X \to Z = \operatorname{Alb}(X)$$

of X and its image Y. The morphism h and the image Y of h are independent of the chosen closed point x_0, up to translation in the abelian variety Z, so we need not care about this choice.

Now if $\mathcal{I}$ is the kernel of the ring morphism $\theta_h\colon \mathcal{O}_Z \to h_*\mathcal{O}_X$, we make Y into a scheme by taking the structure sheaf $\mathcal{O}_Y = \mathcal{O}_A/\mathcal{I}$. This is just the reduced induced subscheme structure of Y in Z. We get two morphisms of schemes $f\colon X \to Y$ and $g\colon Y \to Z$ with ring morphisms θ_f and θ_g and two short exact sequences of sheaves

$$0 \to \mathcal{O}_Y \xrightarrow{\theta_f} f_*\mathcal{O}_X \to \mathcal{Q} \to 0 \quad \text{and} \quad 0 \to \mathcal{I} \to \mathcal{O}_Z \xrightarrow{\theta_g} g_*\mathcal{O}_Y \to 0.$$

By definition there is a commutative triangle

of schemes over $\mathbb{C}$.

Remember that if $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$ and if Y is smooth, then $\operatorname{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q})$ is an obstruction group for extending a given deformation of $h\colon X \to Z$ to a deformation of the triangle by Proposition 4.8 Subdiagram 12.

The aim of this subsection is to proof the following result.

Proposition 4.10. *With the notations at the beginning of this section, assume that the following conditions hold.*

i) f is flat.

ii) $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$.

iii) $Y = C_1 \times \ldots \times C_m$ is a finite product of nonsingular projective curves C_j of genus $g(C_j) \geq 2$ for all j.

Then

$$\operatorname{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q}) = 0,$$

hence each deformation $h'\colon X' \to Z'$ of $h\colon X \to Z$ extends to a deformation

118

We need a preliminary lemma before giving the proof.

Lemma 4.11. *With the notations at the beginning of this subsection, assume only that Y is a nonsingular variety and let $\mathrm{Alb}(Y)$ be the Albanese variety of Y. Then $g\colon Y \to \mathrm{Alb}(X)$ induces an isomorphism*

$$\mathrm{Alb}_g\colon \mathrm{Alb}(Y) \xrightarrow{\cong} \mathrm{Alb}(X)$$

of abelian varieties.

Proof. Since $\mathrm{Alb}(X)$ is projective and since Y is a closed subscheme of $\mathrm{Alb}(X)$, it follows that Y is projective as well. Thus Y is nonsingular and projective, hence we may consider its Albanese variety $\mathrm{Alb}(Y)$.

For distinction, let $\alpha_X\colon X \to \mathrm{Alb}(X)$ and $\alpha_Y\colon Y \to \mathrm{Alb}(Y)$ be the Albanese maps of X and Y with respect to the closed points $x_0 \in X$ and $f(x_0) \in Y$, respectively. We have $\alpha_X(x_0) = 0$ and $\alpha_Y(f(x_0)) = 0$ by definition of the Albanese map. Now since $g\colon Y \to \mathrm{Alb}(X)$ is a morphism from Y to the Abelian variety $\mathrm{Alb}(X)$ and since $g(f(x_0)) = 0$, by the universal property of $\mathrm{Alb}(Y)$ there is a unique morphism of abelian varieties $\mathrm{Alb}_g\colon \mathrm{Alb}(Y) \to \mathrm{Alb}(X)$ such that $\mathrm{Alb}_g(0) = 0$ and such that

is commutative. Similarly, since the composition $c\colon X \xrightarrow{f} Y \xrightarrow{\alpha_Y} \mathrm{Alb}(Y)$ is a morphism from X to the Abelian variety $\mathrm{Alb}(Y)$ and since $c(x_0) = 0$, by the universal property of $\mathrm{Alb}(X)$ there is a unique morphism of abelian varieties $\mathrm{Alb}_c\colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that $\mathrm{Alb}_c(0) = 0$ and such that

is commutative. Now both morphisms $\mathrm{Alb}_g \circ \mathrm{Alb}_c$ and the identity from $\mathrm{Alb}(X)$ to $\mathrm{Alb}(X)$ are morphisms of abelian varieties sending 0 to 0, furthermore,

is commutative, hence $\mathrm{Alb}_g \circ \mathrm{Alb}_c = \mathrm{id}$ by the universal property of $\mathrm{Alb}(X)$. If $y \in Y$, we may choose $x \in X$ such that $f(x) = y$ since f is surjective. Thus

$$\mathrm{Alb}_c(\mathrm{Alb}_g(\alpha_Y(y))) = \mathrm{Alb}_c(g(y)) = \mathrm{Alb}_c(g(f(x))) = \mathrm{Alb}_c(\alpha_X(x)) = \alpha_Y(f(x)) = \alpha_Y(y),$$

whence $\mathrm{Alb}_c \circ \mathrm{Alb}_g \circ \alpha_Y = \alpha_Y$. Now similarly, since both morphisms $\mathrm{Alb}_c \circ \mathrm{Alb}_g$ and the identity from $\mathrm{Alb}(Y)$ to $\mathrm{Alb}(Y)$ are morphisms of abelian varieties sending 0 to 0, it follows that $\mathrm{Alb}_c \circ \mathrm{Alb}_g = \mathrm{id}$ by the universal property of $\mathrm{Alb}(Y)$, whence $\mathrm{Alb}_g \colon \mathrm{Alb}(Y) \to \mathrm{Alb}(X)$ is an isomorphism of abelian varieties. $\qquad\square$

Proof of Proposition 4.10. Let $\mathrm{pr}_j \colon Y \to C_j$ be the projection to C_j and let $f_j \colon X \to C_j$ be the composition $X \xrightarrow{f} Y \xrightarrow{\mathrm{pr}_j} C_j$. We will divide the proof into three steps. We will see that $f_{j*}\mathcal{O}_X = \mathcal{O}_{C_j}$ and $\mathrm{pr}_{j*}\mathcal{Q} = 0$ for all j. Finally, we conclude that $\mathrm{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q}) = 0$.

Step 1: Let us show that $f_{j*}\mathcal{O}_X = \mathcal{O}_{C_j}$ for all j. Consider the Stein factorization

of f_j where $C_j' = \mathbf{Spec}\, f_{j*}\mathcal{O}_X$ and f_j' has connected fibres and c_j is a finite morphism. f_j' is surjective and C_j' is normal by [Sta13, Lemma 33.36.1.]. Since f_j' is surjective and X is irreducible, we have that C_j' is irreducible as well. Since C_j' is a normal curve, it is nonsingular, furthermore, C_j' is projective because C_j is projective by assumption and c_j is finite. We conclude that C_j' is a nonsingular projective variety with Albanese variety $\mathrm{Alb}(C_j')$.

Then abbreviating $Y' = C_1' \times \ldots \times C_m'$, there is a commutative diagram

of schemes. Since c_j is surjective for all j, we have that $c_1 \times \ldots \times c_m$ is surjective as well. Similarly, since f is surjective, we conclude that $(f_1' | \ldots | f_m')$ is surjective as well because Y' and Y are irreducible of the same dimension m.

Consequently, the pullback morphisms

$$H^0(Y, \Omega^1_{Y/\mathbb{C}}) \to H^0(Y', \Omega^1_{Y'/\mathbb{C}}) \quad \text{and} \quad H^0(Y', \Omega^1_{Y'/\mathbb{C}}) \to H^0(X, \Omega^1_{X/\mathbb{C}})$$

are injective, but their composition $H^0(Y, \Omega^1_{Y/\mathbb{C}}) \to H^0(Y', \Omega^1_{Y'/\mathbb{C}}) \to H^0(X, \Omega^1_{X/\mathbb{C}})$ is an isomorphism by Lemma 4.11, thus each pullback morphism is an isomorphism. In particular, we have $\dim \mathrm{Alb}(Y) = h^0(Y, \Omega^1_{Y/\mathbb{C}}) = h^0(Y', \Omega^1_{Y'/\mathbb{C}}) = \dim \mathrm{Alb}(Y')$.

It follows that c_j is an isomorphism for every j. Otherwise, without loss of generality, c_1 would have degree at least 2. Since $g(C_1) \geq 2$ by assumption, it follows from Hurwitz formula that

$g(C_1') > g(C_1)$ (see for example [Har77, Example IV.2.5.4.]) and we get the contradiction

$$\dim \mathrm{Alb}(Y) = \sum_{j=1}^{m} \dim \mathrm{Alb}(C_j) = \sum_{j=1}^{m} g(C_j) < \sum_{j=1}^{m} g(C_j') = \sum_{j=1}^{m} \dim \mathrm{Alb}(C_j') = \dim \mathrm{Alb}(Y').$$

Whence from the Stein factorization we conclude that $f_{j*}\mathcal{O}_X = \mathcal{O}_{C_j}$ for all j.

Step 2: We show that $\mathrm{pr}_{j*}\mathcal{Q} = 0$ for all j. Applying pr_{j*} to the short exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{\theta_f} f_*\mathcal{O}_X \to \mathcal{Q} \to 0$$

of sheaves on Y, we get a long exact sequence

$$0 \to \mathrm{pr}_{j*}\mathcal{O}_Y \xrightarrow{\mathrm{pr}_{j*}\theta_f} \mathrm{pr}_{j*}f_*\mathcal{O}_X \to \mathrm{pr}_{j*}\mathcal{Q} \to R^1\mathrm{pr}_{j*}\mathcal{O}_Y \to R^1\mathrm{pr}_{j*}(f_*\mathcal{O}_X) \to R^1\mathrm{pr}_{j*}\mathcal{Q} \to \dots$$

of sheaves on C_j. We have $\mathrm{pr}_{j*}\mathcal{O}_Y = \mathcal{O}_{C_j}$ and $\mathrm{pr}_{j*}f_*\mathcal{O}_X = f_{j*}\mathcal{O}_X = \mathcal{O}_{C_j}$ by step one, moreover, from the five-term-exact-sequence

$$0 \to R^1\mathrm{pr}_{j*}(f_*\mathcal{O}_X) \to R^1 f_{j*}\mathcal{O}_X \to \mathrm{pr}_{j*}R^1 f_*\mathcal{O}_X \to R^2\mathrm{pr}_{j*}(f_*\mathcal{O}_X) \to R^2 f_{j*}\mathcal{O}_X$$

of the Leray spectral sequence and from the hypothesis $R^1 f_*\mathcal{O}_X = 0$, it follows that $R^1\mathrm{pr}_{j*}(f_*\mathcal{O}_X) \to R^1 f_{j*}\mathcal{O}_X$ is an isomorphism. Hence the long exact sequence simplifies to

$$0 \to \mathcal{O}_{C_j} \xrightarrow{\mathrm{id}} \mathcal{O}_{C_j} \to \mathrm{pr}_{j*}\mathcal{Q} \to R^1\mathrm{pr}_{j*}\mathcal{O}_Y \to R^1 f_{j*}\mathcal{O}_X \to R^1\mathrm{pr}_{j*}\mathcal{Q} \to \dots \ .$$

It remains to show that $R^1\mathrm{pr}_{j*}\mathcal{O}_Y \to R^1 f_{j*}\mathcal{O}_X$ is injective for each j. We treat the case $j = 1$, the case $j \neq 1$ is analogous.

The fibre of pr_1 over $y \in C_1$ is $F = C_2 \times \dots \times C_m$, whence it is independent of y and the function $y \mapsto h^1(F, \mathcal{O}_F)$ is constant. Since pr_1 is flat, it follows that $R^1\mathrm{pr}_{1*}\mathcal{O}_Y$ is locally free and that the natural morphism

$$R^1\mathrm{pr}_{1*}\mathcal{O}_Y \otimes \mathbb{C}(y) \to H^1(F, \mathcal{O}_F)$$

is an isomorphism for each $y \in C_1$ (see for example [Har77, Corollary III.12.9]).

For $y \in C_1$ let $f_1^{-1}(y) \subseteq X$ be the fibre of f_1 over y. Define

$$U = \{y \in C_1 \mid h^1(f_1^{-1}(y), \mathcal{O}_{f_1^{-1}(y)}) \text{ is minimal}\}.$$

Then U is a non-empty open subset of C_1 because the function $C_1 \to \mathbb{N}_0, y \mapsto h^1(f_1^{-1}(y), \mathcal{O}_{f_1^{-1}(y)})$ is upper semicontinuous. Since f is flat by assumption, we deduce that $R^1 f_{1*}\mathcal{O}_X|_U$ is locally free and that the natural morphism

$$R^1 f_{1*}\mathcal{O}_X|_U \otimes \mathbb{C}(y) \to H^1(f_1^{-1}(y), \mathcal{O}_{f_1^{-1}(y)})$$

is an isomorphism for each $y \in U$.

Consequently, on U the morphism $R^1\mathrm{pr}_{1*}\mathcal{O}_Y \to R^1 f_{1*}\mathcal{O}_X$ is given fibrewise by the natural morphism $H^1(F, \mathcal{O}_F) \to H^1(f_1^{-1}(y), \mathcal{O}_{f_1^{-1}(y)})$ which by complex conjugation is given by the pullback morphism $H^0(F, \Omega^1_{F/\mathbb{C}}) \to H^0(f_1^{-1}(y), \Omega^1_{f_1^{-1}(y)/\mathbb{C}})$. Since the morphism $f_1^{-1}(y) \to F$ is

surjective for all $y \in C_1$ as follows from the commutativity of

$$
\begin{array}{ccc}
 & & C_1 \\
 & \nearrow^{f_1} & \uparrow^{\mathrm{pr}_1} \\
X \xrightarrow{\quad f \quad} & Y = C_1 \times \ldots \times C_m & \\
 & \searrow & \downarrow^{\mathrm{pr}_2 \times \ldots \times \mathrm{pr}_m} \\
 & F = C_2 \times \ldots \times C_m,
\end{array}
$$

we see that $H^0(F, \Omega^1_{F/\mathbb{C}}) \to H^0(f_1^{-1}(y), \Omega^1_{f_1^{-1}(y)/\mathbb{C}})$ is injective. Hence $R^1\mathrm{pr}_{1*}\mathcal{O}_Y\big|_U \to R^1 f_{1*}\mathcal{O}_X\big|_U$ is injective. It follows that the support of the kernel $\mathcal{K}$ of $R^1\mathrm{pr}_{1*}\mathcal{O}_Y \to R^1 f_{1*}\mathcal{O}_X$ is contained in the finite set $C_1 \setminus U$, hence $\mathcal{K}$ is a torsion sheaf. But since $R^1\mathrm{pr}_{1*}\mathcal{O}_Y$ is locally free, it follows that $\mathcal{K} = 0$ and so $R^1\mathrm{pr}_{1*}\mathcal{O}_Y \to R^1 f_{1*}\mathcal{O}_X$ is injective. We conclude that $\mathrm{pr}_{j*}\mathcal{Q} = 0$ for all j.

Step 3: Finally, we show that $\mathrm{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q}) = 0$. Since $Y = C_1 \times \ldots \times C_m$ we have that $\mathrm{Alb}(Y) = \mathrm{Alb}(C_1) \times \ldots \times \mathrm{Alb}(C_m)$ and we know from Lemma 4.11 that $\mathrm{Alb}(X) \cong \mathrm{Alb}(Y)$ naturally, hence the normal bundle N_Y of Y in $Z = \mathrm{Alb}(X)$ is given by $N_Y \cong \bigoplus_{j=1}^m \mathrm{pr}_j^* N_{C_j}$ where N_{C_j} is the normal bundle of C_j in $\mathrm{Alb}(C_j)$. It follows

$$
\mathrm{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q}) \cong \mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{Q} \otimes N_Y) \cong H^0(Y, \mathcal{Q} \otimes N_Y) \cong H^0\big(Y, \mathcal{Q} \otimes \big(\bigoplus_{j=1}^m \mathrm{pr}_j^* N_{C_j}\big)\big)
$$

$$
\cong \bigoplus_{j=1}^m H^0(Y, \mathcal{Q} \otimes \mathrm{pr}_j^* N_{C_j}) \cong \bigoplus_{j=1}^m H^0\big(C_j, \mathrm{pr}_{j*}\big(\mathcal{Q} \otimes \mathrm{pr}_j^* N_{C_j}\big)\big).
$$

But since N_{C_j} is locally free, $\mathrm{pr}_{j*}(\mathcal{Q} \otimes \mathrm{pr}_j^* N_{C_j}) \cong \mathrm{pr}_{j*}\mathcal{Q} \otimes N_{C_j}$ by projection formula which is zero because $\mathrm{pr}_{j*}\mathcal{Q} = 0$ by step 2. We conclude that $\mathrm{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{Q}) = 0$. $\qquad\square$

5 Deformations of diagrams of modules

In Section 2 we have dealt with deformations of diagrams and subdiagrams of ringed topoi. Similarly, if $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is a ringed topos, we will now describe deformations of diagrams and subdiagrams of $\mathcal{O}_{\mathcal{S}}$-modules. We will use many notions and some results of [Ill71, Chapitre IV], for example graded extensions, the graded cotangent complex, the derived category of graded modules over a graded ring.

In the first subsection we describe a way of turning the given diagram of modules into a diagram of ringed topoi. Taking the ringed topos $\mathcal{X}$ associated to the diagram of ringed topoi as in Proposition 1.15, we will show that giving a deformation of the diagram of modules is equivalent to giving a certain graded extension of the structure sheaf $\mathcal{O}_{\mathcal{X}}$.

In the second subsection we will answer the two analogue questions for deformations of diagrams and subdiagrams of modules posed at the beginning of Section 2. Basically the answers are given by a graded analogue of Theorem 2.13 where all complexes occurring there are replaced by their graded analogues.

Finally in the last subsection we give an overview of all subdiagrams of a single morphism of modules and the respective obstruction groups.

For the whole section we fix an extension

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_{\mathcal{S}} \to 0$$

of $\mathcal{O}_{\mathcal{S}}$ by an $\mathcal{O}_{\mathcal{S}}$-module $\mathcal{J}$.

Definition 5.1. Let $\mathcal{F}$ be a flat $\mathcal{O}_{\mathcal{S}}$-module. A deformation of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{S}'}$ is a short exact sequence of $\mathcal{O}_{\mathcal{S}'}$-modules

$$\underline{\mathcal{F}} \colon (0 \to \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F} \to \mathcal{F}' \xrightarrow{p} \mathcal{F} \to 0)$$

such that $\mathcal{F}'$ is a flat $\mathcal{O}_{\mathcal{S}'}$-module and such that the morphism of $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{F}' \otimes_{\mathcal{O}_{\mathcal{S}'}} \mathcal{O}_{\mathcal{S}} \to \mathcal{F}$ corresponding to p by adjunction is an isomorphism.

Given another deformation

$$\underline{\tilde{\mathcal{F}}} \colon (0 \to \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F} \to \tilde{\mathcal{F}}' \xrightarrow{\tilde{p}} \mathcal{F} \to 0)$$

of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{S}'}$, an isomorphism of deformations $\underline{\mathcal{F}} \to \underline{\tilde{\mathcal{F}}}$ is an isomorphism of $\mathcal{O}_{\mathcal{S}'}$-modules $\mathcal{F}' \to \tilde{\mathcal{F}}'$ such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F} & \longrightarrow & \mathcal{F}' & \xrightarrow{p} & \mathcal{F} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F} & \longrightarrow & \tilde{\mathcal{F}}' & \xrightarrow{\tilde{p}} & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

is commutative.

Proposition 5.2. [Ill71, Proposition IV.3.1.5.] *Let $\mathcal{F}$ be a flat $\mathcal{O}_{\mathcal{S}}$-module.*

   i) There is an obstruction for the existence of a deformation of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{S}'}$ lying in

$$\operatorname{Ext}^2_{\mathcal{S}}(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F}).$$

  ii) If this obstruction vanishes, then the set of isomorphism classes of deformations of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{S}'}$ is a torsor under

$$\operatorname{Ext}^1_{\mathcal{S}}(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F}).$$

 iii) The automorphism group of any fixed deformation of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{S}'}$ is canonically isomorphic to

$$\operatorname{Hom}_{\mathcal{S}}(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{F}).$$

Now we come to the notion of diagrams of $\mathcal{O}_{\mathcal{S}}$-modules.

Definition 5.3. Let $\mathfrak{C}$ be a finite category, i.e., the morphisms of $\mathfrak{C}$ and thereby the objects of $\mathfrak{C}$ are finite sets. Let further $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ be a ringed topos. A diagram of sheaves of $\mathcal{O}_{\mathcal{S}}$-modules of type $\mathfrak{C}$ is a covariant functor I from $\mathfrak{C}$ to the category of $\mathcal{O}_{\mathcal{S}}$-modules, denoted $I \colon \mathfrak{C} \to \mathcal{O}_{\mathcal{S}}$-mod.

Thus for any $c \in \mathrm{ob}(\mathfrak{C})$ there is an $\mathcal{O}_{\mathcal{S}}$-module $I(c)$ and for any $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(c_1, c_2)$ there is an $\mathcal{O}_{\mathcal{S}}$-linear morphism $I(\alpha) \colon I(c_1) \to I(c_2)$ such that for any composition $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ in $\mathfrak{C}$ we have $I(\beta\alpha) = I(\beta) \circ I(\alpha) \colon I(c_1) \to I(c_2) \to I(c_3)$. Notice that in contrast to the case of diagrams of ringed topoi in Definition 1.14, we define this functor I to be covariant from $\mathfrak{C}$ to $\mathcal{O}_{\mathcal{S}}$-mod. The reason will become clear in Definition 5.6.

Definition 5.4. Let $I \colon \mathfrak{C} \to \mathcal{O}_{\mathcal{S}}$-mod be a diagram of $\mathcal{O}_{\mathcal{S}}$-modules. Assume that each $\mathcal{O}_{\mathcal{S}}$-module $I(c)$ is flat for $c \in \mathrm{ob}(\mathfrak{C})$. A deformation of I over $\mathcal{O}_{\mathcal{S}'}$ is a functor $I' \colon \mathfrak{C} \to \mathcal{O}_{\mathcal{S}'}$-mod together with, for each $c \in \mathrm{ob}(\mathfrak{C})$, a deformation

$$0 \to \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} I(c) \to I'(c) \xrightarrow{p_c} I(c) \to 0$$

of $I(c)$ such that for every $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(c_1, c_2)$ the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} I(c_1) & \longrightarrow & I'(c_1) & \xrightarrow{p_{c_1}} & I(c_1) & \longrightarrow & 0 \\
& & {\scriptstyle \mathrm{id}\otimes I(\alpha)}\downarrow & & {\scriptstyle I'(\alpha)}\downarrow & & \downarrow{\scriptstyle I(\alpha)} & & \\
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} I(c_2) & \longrightarrow & I'(c_2) & \xrightarrow{p_{c_2}} & I(c_2) & \longrightarrow & 0
\end{array}
$$

is commutative. Two deformations I' and $\tilde{I}'$ of I over $\mathcal{O}_{\mathcal{S}'}$ are called isomorphic if there is a natural isomorphism of functors $\varphi \colon I' \to \tilde{I}'$ such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} I(c) & \longrightarrow & I'(c) & \xrightarrow{p_c} & I(c) & \longrightarrow & 0 \\
& & \| & & {\scriptstyle \varphi(c)}\downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} I(c) & \longrightarrow & \tilde{I}'(c) & \xrightarrow{\tilde{p}_c} & I(c) & \longrightarrow & 0
\end{array}
$$

is commutative for every $c \in \mathrm{ob}(\mathfrak{C})$.

124

5.1 The relation between diagrams of modules and diagrams of ringed topoi

We restrict to the following case for the rest of this section.

General assumption 5.5. Let $(\mathcal{S}, \mathcal{O}_\mathcal{S})$ be a ringed topos and $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_\mathcal{S} \to 0$ an extension of $\mathcal{O}_\mathcal{S}$. Let $I \colon \mathfrak{C} \to \mathcal{O}_\mathcal{S}$-mod be a diagram of $\mathcal{O}_\mathcal{S}$-modules such that $\mathcal{F}_c = I(c)$ is flat over $\mathcal{O}_\mathcal{S}$ for every $c \in \mathrm{ob}(\mathfrak{C})$ and let $\mathfrak{D}$ a subcategory of $\mathfrak{C}$. By restricting I to $\mathfrak{D}$ we get a subdiagram $J \colon \mathfrak{D} \to \mathcal{O}_\mathcal{S}$-mod of I.

We may ask the analogue two questions at the beginning of Section 2 for the case of diagrams of $\mathcal{O}_\mathcal{S}$-modules:

i) Given a deformation J' of the subdiagram J, what is an obstruction group for the existence of a deformation of the diagram I inducing the given deformation J' of J?

ii) If the obstruction in this group vanishes, how many different isomorphism classes of deformations of I inducing the given one of J are there?

To answer these questions we will construct for I, as already mentioned at the beginning of this section, a diagram of ringed topoi over $\mathcal{S}$ with associated ringed topos $\mathcal{X}$ such that giving a deformation of the diagram of modules over $\mathcal{O}_{\mathcal{S}'}$ is equivalent to giving a graded extension of $\mathcal{O}_\mathcal{X}$ over $m_\mathcal{X}^{-1}\mathcal{O}_{\mathcal{S}'}$.

For each $c \in \mathrm{ob}(\mathfrak{C})$ the $\mathcal{O}_\mathcal{S}$-module $\mathcal{F}_c = I(c)$ gives rise to the $\mathcal{O}_\mathcal{S}$-algebra $\mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c$ which is a graded $\mathcal{O}_\mathcal{S}$-algebra having $\mathcal{O}_\mathcal{S}$ in degree 0 and $\mathcal{F}_c$ in degree 1 with ring multiplication

$$(\mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c) \times (\mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c) \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c \ , \quad \big((s_1, f_1), (s_2, f_2)\big) \mapsto (s_1 s_2, s_1 f_2 + s_2 f_1)$$

and with structure morphism

$$(\mathrm{id}|0) \colon \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c.$$

Each $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$ yielding the $\mathcal{O}_\mathcal{S}$-linear morphism

$$\mathcal{F}_\alpha = I(\alpha) \colon \mathcal{F}_{c_1} \to \mathcal{F}_{c_2}$$

gives rise to a morphism of graded $\mathcal{O}_\mathcal{S}$-algebras $\theta_\alpha \colon \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_{c_1} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_{c_2}$ defined by $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathcal{F}_\alpha \end{pmatrix}$.

For each $c \in \mathrm{ob}(\mathfrak{C})$ there is a morphism of ringed topoi

$$t_c \colon (\mathcal{S}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c) \to (\mathcal{S}, \mathcal{O}_\mathcal{S})$$

whose underlying morphism of topoi is the identity and whose ring morphism is the above morphism $(\mathrm{id}|0) \colon t_c^{-1}\mathcal{O}_\mathcal{S} = \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c$.

Definition 5.6. We define $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ to be the ringed topos associated to the following diagram of ringed topoi over $\mathcal{S}$ of type $\mathfrak{C}$ (in the sense of Proposition 1.15): The levels are the ringed topoi $(\mathcal{S}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_c)$ for $c \in \mathrm{ob}(\mathfrak{C})$, the morphisms of topoi $f_\alpha \colon \mathcal{S} \to \mathcal{S}$ are the identities for each $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$ and the ring morphisms are the above morphisms of graded $\mathcal{O}_\mathcal{S}$-algebras

$$\theta_\alpha \colon f_\alpha^{-1}\big(\mathcal{O}_\mathcal{S} \oplus \mathcal{F}_{c_1}\big) = \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_{c_1} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}_{c_2}$$

for $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$. The ringed topos $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is defined analogously be restricting to the objects and morphisms of $\mathfrak{D}$.

Notice that there are natural morphisms of ringed topoi $m_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ and $m_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{S}$ by Lemma 1.17 and the ring morphisms $\theta_{\mathcal{X}}\colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}}$ and $\theta_{\mathcal{Y}}\colon m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{Y}}$ are morphisms of graded rings since levelwise they are given by the morphisms of graded rings $(\mathrm{id}|0)\colon \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ and $c \in \mathrm{ob}(\mathfrak{D})$, respectively. Moreover, the collection $\mathcal{F}$ of all $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{F}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$ together with the $\mathcal{O}_{\mathcal{S}}$-linear morphisms $\mathcal{F}_{\alpha}\colon \mathcal{F}_{c_1} \to \mathcal{F}_{c_2}$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ is a module over $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$ and $\theta_{\mathcal{X}}$ may be written as

$$\theta_{\mathcal{X}} = (\mathrm{id}|0)\colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}$$

where $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$ is placed in degree 0 and $\mathcal{F}$ in degree 1.

Lemma 5.7. *In the situation of General assumption 5.5, the morphisms $m_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ and $m_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{S}$ are flat. In particular, $\mathcal{X}$ and $\mathcal{Y}$ satisfy all conditions of General assumption 2.1.*

Proof. For each $c \in \mathrm{ob}(\mathfrak{C})$ there is a short exact sequence of $\mathcal{O}_{\mathcal{S}}$-modules

$$0 \to \mathcal{F}_c \xrightarrow{(0|\mathrm{id})} \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}_c \xrightarrow{\mathrm{pr}_1} \mathcal{O}_{\mathcal{S}} \to 0.$$

Since $\mathcal{F}_c$ is a flat $\mathcal{O}_{\mathcal{S}}$-module by assumption and since $\mathcal{O}_{\mathcal{S}}$ is a flat $\mathcal{O}_{\mathcal{S}}$-module, it follows that $(\mathrm{id}|0)\colon \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}_c$ is flat as well. $\qquad\square$

Applying $m_{\mathcal{X}}^{-1}$ to the fixed extension $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{S}'} \to \mathcal{O}_{\mathcal{S}} \to 0$ of $\mathcal{O}_{\mathcal{S}}$ we get an extension

$$0 \to m_{\mathcal{X}}^{-1}\mathcal{J} \to m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'} \to m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to 0$$

of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$. Furthermore, by definition of the ringed topos $\mathcal{X}$, giving (an isomorphism class of) a deformation of the diagram $I\colon \mathfrak{C} \to \mathcal{O}_{\mathcal{S}}$-mod of $\mathcal{O}_{\mathcal{S}}$-modules over $\mathcal{O}_{\mathcal{S}'}$ in the sense of Definition 5.4 is equivalent to giving (an isomorphism class of) a deformation of the $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-module $\mathcal{F}$ over $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$ in the sense of Definition 5.1.

Now if A is a graded sheaf of rings on a topos $\mathcal{T}$ and if B is a graded A-algebra, we define the graded cotangent complex $L_{B/A}^{\mathrm{gr}}$ as in [Ill71, Equation IV.2.2.1]. Taking the concept of graded extensions and graded deformations as presented in [Ill71, Section IV.2.4.], it is possible to define the derived category $\mathbf{D}(A)_{\mathrm{gr}}$ of the category of graded A-modules as in [Ill71, Section IV.1.2.] and abelian groups

$$\mathrm{Exal}_A(B, M)_{\mathrm{gr}} \quad \text{and} \quad \mathbb{E}\mathrm{xt}_B^1(L_{B/A}^{\mathrm{gr}}, M)_{\mathrm{gr}}$$

for M a graded B-module which are naturally isomorphic by [Ill71, Equation IV.2.4.2] just as their analogues $\mathrm{Exal}_A(B, M) \cong \mathbb{E}\mathrm{xt}_B^1(L_{B/A}, M)$ in the ungraded case in Theorem 1.8.iii). The group $\mathrm{Exal}_A(B, M)_{\mathrm{gr}}$ is by definition the group of isomorphism classes of graded A-extensions of B by the fixed graded B-module M.

The crucial point is that there is a relation between the two notions "deformations of the $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-module $\mathcal{F}$ over $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$" in the sense of Definition 5.1 and "$m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$-extensions of $\mathcal{O}_{\mathcal{X}}$" in the sense of Definition 1.6, stated after the following lemma.

Lemma 5.8. *Let $\mathcal{T}$ be a topos, let $A = \bigoplus_{n \geq 0} A_n$ be a graded ring of $\mathcal{T}$ and let $B = \bigoplus_{n \geq 0} B_n$ be a graded A-algebra. For $M, L \in \mathrm{ob}(\mathbf{D}(A)_{\mathrm{gr}})$ and $i \in \mathbb{Z}$ let*

$$\mathbb{E}\mathrm{xt}_A^i(M, L)_{\mathrm{gr}} = \mathrm{Hom}_{\mathbf{D}(A)_{\mathrm{gr}}}(M, L[i]).$$

i) For each $E \in \mathrm{ob}(\mathbf{D}(A)_{\mathrm{gr}})$ and $F \in \mathrm{ob}(\mathbf{D}(A_0))$ and for each $i, n \in \mathbb{Z}$ there is a natural isomorphism

$$\mathbb{E}\mathrm{xt}_{A_0}^i(E_n, F) \cong \mathbb{E}\mathrm{xt}_A^i(E, F(-n))_{\mathrm{gr}}$$

of abelian groups, functorial in E and F, where E_n is the homogeneous part of degree n of E.

ii) There is a natural functorial isomorphism

$$(L_{B/A}^{\mathrm{gr}})_0 \cong L_{B_0/A_0}$$

in $\mathbf{D}(B_0)$. If $A = A_0$ and $B_0 = A_0$, then there are natural isomorphisms

$$(L_{B/A}^{\mathrm{gr}})_0 \cong 0 \quad and \quad (L_{B/A}^{\mathrm{gr}})_1 \cong B_1$$

in $\mathbf{D}(A)$, functorial in B in the sense that if $f \colon B = \bigoplus_{n \geq 0} B_n \to C = \bigoplus_{n \geq 0} C_n$ is a morphism of graded A-algebras such that $B_0 = A_0 = C_0$, then the degree 1 morphism $(L_{B/A}^{\mathrm{gr}})_1 \to (L_{C/A}^{\mathrm{gr}})_1$ of the natural morphism of graded B-modules $L_{B/A}^{\mathrm{gr}} \to L_{C/A}^{\mathrm{gr}}$ is identified with the degree 1 morphism $f_1 \colon B_1 \to C_1$ of $f \colon B \to C$.

Proof. Assertion *i)* is shown in [Ill71, Equation IV.1.2.2.1] and assertion *ii)* in [Ill71, Equation IV.2.2.4] and [Ill71, Equation IV.2.2.5] together with the functoriality of [Ill71, Equation IV.2.1.5]. $\square$

Proposition 5.9. *Under the conditions of General assumption 5.5, let $\mathcal{X}$ be the ringed topos associated to I as in Definition 5.6. Then for each $i \in \mathbb{Z}$ there is a natural isomorphism of abelian groups*

$$\mathrm{Ext}_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}}^i(\mathcal{F}, m_{\mathcal{X}}^{-1} \mathcal{J} \otimes_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}} \mathcal{F}) \cong \mathrm{Ext}_{\mathcal{X}}^i(L_{\mathcal{X}/\mathcal{S}}^{\mathrm{gr}}, m_{\mathcal{X}}^{-1} \mathcal{J} \otimes_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}} \mathcal{F})_{\mathrm{gr}}$$

where $m_{\mathcal{X}}^{-1} \mathcal{J} \otimes_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}} \mathcal{F}$ on the right hand side is considered as a graded $\mathcal{O}_{\mathcal{X}}$-module concentrated in degree 1.

In particular, for $i = 1$, it follows that there is a natural one-to-one correspondence between isomorphism classes of deformations of $\mathcal{F}$ over $m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}'}$ and isomorphism classes of graded extensions of $\mathcal{O}_{\mathcal{X}}$ over $m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}'}$ by the graded $\mathcal{O}_{\mathcal{X}}$-module $m_{\mathcal{X}}^{-1} \mathcal{J} \otimes_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}} \mathcal{F}$, considered as module concentrated in degree 1.

Before giving the proof, let us describe the above natural one-to-one correspondence which is proved in [Ill71, Equation IV.3.1.2] together with [Ill71, Equation IV.2.4.2]: Let

$$0 \to m_{\mathcal{X}}^{-1} \mathcal{J} \to m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}'} \xrightarrow{q} m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}} \to 0$$

be the extension of $m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}$ considered. Given the isomorphism class of a deformation

$$0 \to m_{\mathcal{X}}^{-1} \mathcal{J} \otimes_{m_{\mathcal{X}}^{-1} \mathcal{O}_{\mathcal{S}}} \mathcal{F} \xrightarrow{i} \mathcal{F}' \xrightarrow{p} \mathcal{F} \to 0$$

of $\mathcal{F}$ over $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$, we get an induced isomorphism class of a graded $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$-extension of $\mathcal{O}_{\mathcal{X}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}$ by the graded $\mathcal{O}_{\mathcal{X}}$-module $0 \oplus m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$ which is represented by

$$0 \longrightarrow 0 \oplus m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}} m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}' \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & p \end{pmatrix}} m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F} \longrightarrow 0.$$

with arrows $(q|0)$ and $(q|0)$ from $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$.

Here we have written explicitly $0 \oplus m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$ to indicate that this module is concentrated in degree 1. On the other hand, given the isomorphism class of a graded $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$-extension of $\mathcal{O}_{\mathcal{X}}$ by the graded $\mathcal{O}_{\mathcal{X}}$-module $0 \oplus m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$ represented by

$$0 \longrightarrow 0 \oplus m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F} \longrightarrow m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}' \longrightarrow m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F} \longrightarrow 0,$$

with arrow $(q|0)$ from $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$.

we get an isomorphism class of a deformation of $\mathcal{F}$ over $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}'}$ by taking the class represented by the degree 1 sequence

$$0 \to m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0.$$

Proof of Proposition 5.9. By Lemma 5.8.i) we have naturally

$$\mathbb{Ext}^i_{\mathcal{X}}(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F})_{\mathrm{gr}} \cong \mathbb{Ext}^i_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}\big((L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}})_1, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}\big)$$

because $m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$ is concentrated in degree 1. By Lemma 5.8.ii) there is a natural isomorphism $(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}})_1 \cong \mathcal{F}$ in $\mathbf{D}(m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}})$, hence

$$\mathbb{Ext}^i_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}\big((L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}})_1, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}\big) \cong \mathbb{Ext}^i_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{F}, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F})$$

$$\cong \mathrm{Ext}^i_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{F}, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F})$$

$\square$

5.2 The long exact sequence of a diagram and a subdiagram of modules

Assume given the situation of General assumption 5.5 and let $\mathcal{X}$ and $\mathcal{Y}$ be the ringed topoi for I and J as in Definition 5.6. As done in Subsections 2.2 and 2.3 for the ungraded case, we will derive a long exact sequence for diagrams and subdiagrams of modules containing all groups relevant for deformation theoretic considerations.

Let $\mathbb{N}_0$ be the set of nonnegative integers. If A is an $\mathbb{N}_0$-graded ring of a topos $\mathcal{T}$, we denote by A-modgr the category of $\mathbb{N}_0$-graded A-modules whose morphisms are A-linear morphisms preserving the grading. We will omit $\mathbb{N}_0$ in the notations and just speak of a graded module. The category A-modgr has all finite colimits. To see this, notice that if we are given a finite system of graded modules over A, we may disregard the grading and take the colimit in the category of modules. This colimit has a natural grading and regarding it as a graded module, we see that the category A-modgr has all finite colimits.

Let $\mathbf{D}(A)_{\mathrm{gr}}$ be the derived category of the category of graded A-modules as defined in [Ill71, Section IV.1.2.]. For $L, M \in \mathrm{ob}(\mathbf{D}(A)_{\mathrm{gr}})$ and $i \in \mathbb{Z}$ we define

$$\mathbb{E}\mathrm{xt}^i_A(L, M)_{\mathrm{gr}} = \mathrm{Hom}_{\mathbf{D}(A)_{\mathrm{gr}}}(L, M[i]).$$

Now let

$$v \colon \mathcal{O}_{\mathcal{X}}\text{-modgr} \to \mathcal{O}_{\mathcal{Y}}\text{-modgr}$$

be the forgetful functor. As in the ungraded case, there is a left adjoint

$$v^* \colon \mathcal{O}_{\mathcal{Y}}\text{-modgr} \to \mathcal{O}_{\mathcal{X}}\text{-modgr}$$

of v. For $L \in \mathrm{ob}(\mathcal{O}_{\mathcal{Y}}\text{-modgr})$ it is defined levelwise by

$$(v^*L)_c = \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} \left(f_\gamma^{-1} L_d \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c} \right)$$

for $c \in \mathrm{ob}(\mathfrak{C})$ where the colimit is taken in the category $\mathcal{O}_{\mathcal{X}_c}$-modgr. This makes sence because if L_d is a graded $\mathcal{O}_{\mathcal{X}_d}$-module for $(\gamma\colon d \to c) \in \mathrm{ob}(\underline{c})$, then $f_\gamma^{-1}L_d \otimes_{f_\gamma^{-1}\mathcal{O}_{\mathcal{X}_d}} \mathcal{O}_{\mathcal{X}_c}$ is a graded $\mathcal{O}_{\mathcal{X}_c}$-module. By abuse of notation we denote by v and v^* the functors between the categories $\mathbf{Ch}(\mathcal{X})_{\mathrm{gr}}$ and $\mathbf{Ch}(\mathcal{Y})_{\mathrm{gr}}$ of complexes of graded $\mathcal{O}_{\mathcal{X}}$-modules and graded $\mathcal{O}_{\mathcal{Y}}$-modules, respectively, as well.

Let $L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$ be the graded cotangent complex of the morphism of graded rings

$$\theta_{\mathcal{X}} = (\mathrm{id}|0) \colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{X}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}$$

of $m_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$ as defined in [Ill71, Equation IV.2.2.1].

Theorem 5.10. *Assume given the situation of General assumption 5.5 and let $\mathcal{X}$ and $\mathcal{Y}$ be the ringed topoi for I and J as in Definition 5.6. Let*

$$m^{\mathrm{gr}} \colon v^*v L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}} = v^* L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}} \to L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$$

be the adjunction morphism in Lemma 2.12 of the graded cotangent complex $L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$ and let $\mathrm{Cone}(m^{\mathrm{gr}})$ be the cone of m^{gr}. Let $M = m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$, considered as graded $\mathcal{O}_{\mathcal{X}}$-module concentrated in

degree 1. Then the distinguished triangle

$$\text{Cone}(m^{\text{gr}})$$

$$v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\quad m^{\text{gr}} \quad} L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}$$

in the derived category $\mathbf{D}(\mathcal{X})_{\text{gr}}$ *yields a long exact sequence*

$$0 \to \mathbb{E}\text{xt}^0_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^0_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^0_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$$

$$\to \mathbb{E}\text{xt}^1_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \xrightarrow{\sigma} \mathbb{E}\text{xt}^1_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \xrightarrow{\tau} \mathbb{E}\text{xt}^1_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$$

$$\xrightarrow{\omega} \mathbb{E}\text{xt}^2_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^2_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^2_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$$

$$\to \dots$$

of abelian groups. The morphism τ *is the forgetful morphism sending a deformation of the diagram I of* $\mathcal{O}_{\mathcal{S}}$-*modules to the induced deformation of J. Given a deformation J′ of the subdiagram J over* $\mathcal{O}_{\mathcal{S}'}$*, there is an obstruction*

$$\omega(J') \in \mathbb{E}\text{xt}^2_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}}$$

whose vanishing is necessary and sufficient for the existence of a deformation I′ of the diagram I over $\mathcal{O}_{\mathcal{S}'}$ *reducing to J′. If the obstruction* $\omega(J')$ *is zero, then the set of isomorphism classes of deformations I′ of I over* $\mathcal{O}_{\mathcal{S}'}$ *reducing to J′ is a torsor under the image of*

$$\sigma \colon \mathbb{E}\text{xt}^1_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^1_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}}.$$

Proof. We apply the functor $\text{Hom}_{\mathbf{D}(\mathcal{X})_{\text{gr}}}(-, M)$ to the above distinguished triangle to get an exact sequence

$$0 \to \mathbb{E}\text{xt}^0_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^0_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^0_{\mathcal{X}}(v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, M)_{\text{gr}}$$

$$\to \mathbb{E}\text{xt}^1_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^1_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^1_{\mathcal{X}}(v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, M)_{\text{gr}}$$

$$\to \mathbb{E}\text{xt}^2_{\mathcal{X}}(\text{Cone}(m^{\text{gr}}), M)_{\text{gr}} \to \mathbb{E}\text{xt}^2_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^2_{\mathcal{X}}(v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, M)_{\text{gr}}$$

$$\to \dots$$

of abelian groups. As in the ungraded case in Theorem 2.13, one shows that for each $i \in \mathbb{Z}$ *there is a natural isomorphism*

$$\mathbb{E}\text{xt}^i_{\mathcal{X}}(v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, M)_{\text{gr}} \cong \mathbb{E}\text{xt}^i_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$$

and that the composition

$$\mathbb{E}\text{xt}^i_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}} \to \mathbb{E}\text{xt}^i_{\mathcal{X}}(v^*L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, M)_{\text{gr}} \cong \mathbb{E}\text{xt}^i_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$$

is the forgetful morphism. We know that $\mathbb{E}\text{xt}^1_{\mathcal{X}}(L^{\text{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\text{gr}}$ *and* $\mathbb{E}\text{xt}^1_{\mathcal{Y}}(L^{\text{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\text{gr}}$ *classify isomorphism classes of deformations of I and J over* $\mathcal{O}_{\mathcal{S}'}$*, respectively, by Proposition 5.9 and Theorem 5.2. All deformation theoretic assertions follow from the exactness of the sequence.* $\square$

Our next aim is to state a graded analogue of Theorem 2.20. If A is an $\mathbb{N}_0$-graded ring of a topos $\mathcal{T}$, we denote by A-alggr the category of $\mathbb{N}_0$-graded A-algebras with morphisms of A-algebras preserving the grading. Again we omit the prefix $\mathbb{N}_0$ in the notations. The category A-alggr has all finite colimits. Similarly as in the case of the category A-modgr, if we are given a finite system of graded A-algebras, we may omit the grading and take the colimit in the category of algebras. This colimit has a natural grading as A-algebra, so it is the colimit in the category A-alggr.

Now let $u\colon m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr} \to m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr}$ be the forgetful functor. This functor has a left adjoint

$$u^{-1}\colon m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr} \to m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr}$$

which is given levelwise, for $c \in \mathrm{ob}(\mathfrak{C})$, by

$$(u^{-1}B)_c = \varinjlim_{\gamma \in \mathrm{ob}(\underline{c})} f_\gamma^{-1} B_d$$

where $B \in \mathrm{ob}(m_{\mathcal{Y}}^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr})$ and the colimit is taken in the category $t_c^{-1}\mathcal{O}_{\mathcal{S}}\text{-alggr}$. Remember that $t_c\colon (\mathcal{S}, \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}_c) \to (\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ has the identity as underlying morphism of topoi, thus this colimit is taken in the category $\mathcal{O}_{\mathcal{S}}\text{-alggr}$. Imitating the construction in Proposition 2.16, we get the following result.

Corollary 5.11. *With the above notations there is a natural factorization*

of the graded ring morphism $\theta_{\mathcal{X}}$ all of whose morphisms are morphisms of graded $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras.

Before stating the graded analogue of Theorem 2.20, let us examine three different examples.

Example 5.12. Consider the examples

diagram in $\mathcal{O}_{\mathcal{S}}$-mod	subdiagram 1 in $\mathcal{O}_{\mathcal{S}}$-mod
$\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$	$\mathcal{F}$
subdiagram 2 in $\mathcal{O}_{\mathcal{S}}$-mod	subdiagram 3 in $\mathcal{O}_{\mathcal{S}}$-mod
$\mathcal{G}$	$\mathcal{F} \qquad \mathcal{G}$

with no other morphisms involved except for the identities. Here $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_{\mathcal{S}}$-modules and $\varphi\colon \mathcal{F} \to \mathcal{G}$ is $\mathcal{O}_{\mathcal{S}}$-linear. The corresponding diagram in $\mathfrak{RTop}/\mathcal{S}$ is given by

$$f\colon (X, \mathcal{O}_X) := (\mathcal{S}, \mathcal{O}_{\mathcal{S}} \oplus \mathcal{G}) \to (\mathcal{S}, \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}) =: (Y, \mathcal{O}_Y)$$

whose underlying morphism of topoi is the identity and whose ring morphism is

$$\mathrm{id} \oplus \varphi\colon \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F} \to \mathcal{O}_{\mathcal{S}} \oplus \mathcal{G}.$$

Let further $t_X\colon (X, \mathcal{O}_X) \to (\mathcal{S}, \mathcal{O}_\mathcal{S})$ and $t_Y\colon (Y, \mathcal{O}_Y) \to (\mathcal{S}, \mathcal{O}_\mathcal{S})$ be the structure morphisms whose underlying morphisms of topoi are the identity and whose ring morphisms are $(\mathrm{id}|0)\colon \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{G}$ and $(\mathrm{id}|0)\colon \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S} \oplus \mathcal{F}$, respectively. Now let $\mathcal{X}$ be the ringed topos associated to the diagram $f\colon X \to Y$ in $\mathfrak{RTop}/\mathcal{S}$ as defined in Definition 5.6. We denote the objects of $\mathcal{X}$ in the form

$$(\mathcal{H}_X, \mathcal{H}_Y, \mathcal{H}_Y \to \mathcal{H}_X)$$

where $\mathcal{H}_X$ and $\mathcal{H}_Y$ are sheaves of X and Y, respectively. The structure sheaf of $\mathcal{X}$ is given by

$$\mathcal{O}_\mathcal{X} = \big(\mathcal{O}_\mathcal{S} \oplus \mathcal{G}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F} \xrightarrow{\mathrm{id} \oplus \varphi} \mathcal{O}_\mathcal{S} \oplus \mathcal{G}\big).$$

Subdiagram 1: By definition we have

$$u^{-1}\mathcal{O}_\mathcal{Y} = \big(\mathcal{O}_\mathcal{S} \oplus \mathcal{F}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F} \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathcal{O}_\mathcal{S} \oplus \mathcal{F}\big)$$

and $h\colon u^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is given by $h = \big(\mathrm{id} \oplus \varphi, \mathrm{id} \oplus \mathrm{id}\big)$.

Subdiagram 2: Similarly, we have

$$u^{-1}\mathcal{O}_\mathcal{Y} = \big(\mathcal{O}_\mathcal{S} \oplus \mathcal{G}, \mathcal{O}_\mathcal{S}, \mathcal{O}_\mathcal{S} \xrightarrow{(\mathrm{id}|0)} \mathcal{O}_\mathcal{S} \oplus \mathcal{G}\big)$$

and $h\colon u^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is given by $h = \big(\mathrm{id} \oplus \mathrm{id}, (\mathrm{id}|0)\big)$.

Subdiagram 3: This time $(u^{-1}\mathcal{O}_\mathcal{Y})_X$ is the tensor product of $\mathcal{O}_\mathcal{S} \oplus \mathcal{F}$ and $\mathcal{O}_\mathcal{S} \oplus \mathcal{G}$ over $\mathcal{O}_\mathcal{S}$, considered as a graded $\mathcal{O}_\mathcal{S}$-algebra, i.e.,

$$(u^{-1}\mathcal{O}_\mathcal{Y})_X = \mathcal{O}_\mathcal{S} \oplus (\mathcal{F} \oplus \mathcal{G}) \oplus (\mathcal{F} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{G}).$$

whose degree 0 part is $\mathcal{O}_\mathcal{S}$, whose degree 1 part is $\mathcal{F} \oplus \mathcal{G}$ and whose degree 2 part is $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{G}$. Hence we have

$$u^{-1}\mathcal{O}_\mathcal{Y} = \big(\mathcal{O}_\mathcal{S} \oplus (\mathcal{F} \oplus \mathcal{G}) \oplus (\mathcal{F} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{G}), \mathcal{O}_\mathcal{S} \oplus \mathcal{F}, \mathcal{O}_\mathcal{S} \oplus \mathcal{F} \xrightarrow{\mathrm{id} \oplus (\mathrm{id}|0)} \mathcal{O}_\mathcal{S} \oplus (\mathcal{F} \oplus \mathcal{G}) \oplus (\mathcal{F} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{G})\big).$$

If $l\colon \mathcal{F} \oplus \mathcal{G} \to \mathcal{G}$ is the morphism $(f, g) \mapsto \varphi(f) + g$, then $h\colon u^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is given by $h = \big(\mathrm{id} \oplus l \oplus 0, \mathrm{id} \oplus \mathrm{id}\big)$.

We come back to the general case. Each composition of graded rings gives rise to a distinguished triangle between the graded cotangent complexes by [Ill71, Diagramme IV.2.3.4]. Hence as in the ungraded case, we have a distinguished triangle

in $\mathbf{D}(\mathcal{X})_{\mathrm{gr}}$ induced from the composition of graded rings $\theta_\mathcal{X}\colon m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S} \xrightarrow{l} u^{-1}\mathcal{O}_\mathcal{Y} \xrightarrow{h} \mathcal{O}_\mathcal{X}$ in Corollary 5.11.

Now as in Lemma 2.19 we define a natural morphism of complexes of graded $\mathcal{O}_\mathcal{X}$-modules

$$a^{\mathrm{gr}}\colon v^* L_{\mathcal{Y}/\mathcal{S}}^{\mathrm{gr}} \to L_l^{\mathrm{gr}} \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$$

such that

$$v^*L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}} \xrightarrow{\;m^{\mathrm{gr}}\;} L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$$

$$a^{\mathrm{gr}} \downarrow \qquad\qquad \parallel$$

$$L^{\mathrm{gr}}_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \longrightarrow L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$$

is commutative in the category $\mathbf{Ch}(\mathcal{X})_{\mathrm{gr}}$ of complexes of graded $\mathcal{O}_{\mathcal{X}}$-modules.

Theorem 5.13. *Assume given the situation of General assumption 5.5 and let $\mathcal{X}$ and $\mathcal{Y}$ be the ringed topoi for I and J as in Definition 5.6. Let $M = m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}$, considered as graded $\mathcal{O}_{\mathcal{X}}$-module concentrated in degree 1. If a^{gr} is a quasi-isomorphism, then for each $i \in \mathbb{Z}$ it induces a natural isomorphism of abelian groups*

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{X}}(L^{\mathrm{gr}}_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}, N)_{\mathrm{gr}} \cong \mathbb{E}\mathrm{xt}^i_{\mathcal{Y}}(L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}}, vN)_{\mathrm{gr}},$$

functorial in $N \in \mathrm{ob}(\mathbf{D}^+(\mathcal{X})_{\mathrm{gr}})$. Furthermore, if a^{gr} is a quasi-isomorphism, then the application of the functor $\mathrm{Hom}_{\mathbf{D}(\mathcal{X})_{\mathrm{gr}}}(-, M)$ to the distinguished triangle

$$L^{\mathrm{gr}}_h$$

$$L^{\mathrm{gr}}_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \longrightarrow L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}$$

yields a long exact sequence

$$0 \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}} \to \mathbb{E}\mathrm{xt}^0_{\mathcal{X}}(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\mathrm{gr}} \to \mathbb{E}\mathrm{xt}^0_{\mathcal{Y}}(L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\mathrm{gr}}$$

$$\to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}} \xrightarrow{\sigma} \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\mathrm{gr}} \xrightarrow{\tau} \mathbb{E}\mathrm{xt}^1_{\mathcal{Y}}(L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\mathrm{gr}}$$

$$\xrightarrow{\omega} \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}} \to \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\mathrm{gr}} \to \mathbb{E}\mathrm{xt}^2_{\mathcal{Y}}(L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}}, vM)_{\mathrm{gr}}$$

$$\to \dots$$

of abelian groups. The morphism τ is the forgetful morphism sending a deformation of the diagram I of $\mathcal{O}_{\mathcal{S}}$-modules to the induced deformation of J. Given a deformation J' of the subdiagram J over $\mathcal{O}_{\mathcal{S}'}$, there is an obstruction

$$\omega(J') \in \mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}}$$

whose vanishing is necessary and sufficient for the existence of a deformation I' of the diagram I over $\mathcal{O}_{\mathcal{S}'}$ reducing to J'. If the obstruction $\omega(J')$ is zero, then the set of isomorphism classes of deformations I' of I over $\mathcal{O}_{\mathcal{S}'}$ reducing to J' is a torsor under the image of

$$\sigma \colon \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}} \to \mathbb{E}\mathrm{xt}^1_{\mathcal{X}}(L^{\mathrm{gr}}_{\mathcal{X}/\mathcal{S}}, M)_{\mathrm{gr}}.$$

Proof. We just imitate the proof of Theorem 2.20. $\qquad\square$

Notice that $a^{\mathrm{gr}} \colon v^* L^{\mathrm{gr}}_{\mathcal{Y}/\mathcal{S}} \to L^{\mathrm{gr}}_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$ is a quasi-isomorphism of complexes of graded $\mathcal{O}_{\mathcal{X}}$-modules if and only if the underlying morphism $a \colon v^* L_{\mathcal{Y}/\mathcal{S}} \to L_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$ of complexes of $\mathcal{O}_{\mathcal{X}}$-modules disregarding the grading in Lemma 2.19 is a quasi-isomorphism by [Ill71, Section IV.1.2.1.]. Hence we may use the sufficient conditions **1** or **2** in Theorem 2.20 to verify if a^{gr} is a quasi-isomorphism.

Notice further that

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}} = \mathbb{E}\mathrm{xt}^i_{\mathcal{X}}(L^{\mathrm{gr}}_h, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F})_{\mathrm{gr}} \cong \mathbb{E}\mathrm{xt}^i_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}}\big((L^{\mathrm{gr}}_h)_1, m_{\mathcal{X}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}\big)$$

for all $i \in \mathbb{Z}$ by Lemma 5.8.i). Thus in contrast to the ungraded case in Theorem 2.20, we do not need to know the graded cotangent complex L^{gr}_h to understand the obstruction group $\mathbb{E}\mathrm{xt}^2_{\mathcal{X}}(L^{\mathrm{gr}}_h, M)_{\mathrm{gr}}$, but only its degree 1 component $(L^{\mathrm{gr}}_h)_1$. The following lemma is a generalization of [Ill71, Equation IV.3.2.10].

Lemma 5.14. *Let $h_1 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_1 \to \mathcal{F}$ be the degree 1 morphism of the morphism*

$$h \colon u^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}$$

of graded $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-algebras in Corollary 5.11. Consider h_1 as a morphism of $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-modules. If $\mathrm{Cone}(h_1)$ is the cone of h_1, then there is an isomorphism

$$(L^{\mathrm{gr}}_h)_1 \cong \mathrm{Cone}(h_1)$$

in $\mathbf{D}(m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}})$.

Notice that the $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$-linear morphism $h_1 \colon (u^{-1}\mathcal{O}_{\mathcal{Y}})_1 \to \mathcal{F}$ and thereby its cone depend only on the chosen diagram $I \colon \mathfrak{C} \to \mathcal{O}_{\mathcal{S}}$-mod and the chosen subdiagram $J \colon \mathfrak{D} \to \mathcal{O}_{\mathcal{S}}$-mod of $\mathcal{O}_{\mathcal{S}}$-modules.

Proof. The distinguished triangle

in $\mathbf{D}(\mathcal{X})_{\mathrm{gr}}$ induced by the composition of graded rings $m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \xrightarrow{l} u^{-1}\mathcal{O}_{\mathcal{Y}} \xrightarrow{h} \mathcal{O}_{\mathcal{X}}$ gives rise to a distinguished triangle

in $\mathbf{D}(m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}})$ by taking the degree 1 components. Remember that $(u^{-1}\mathcal{O}_{\mathcal{Y}})_0 = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}$ and $\mathcal{O}_{\mathcal{X}} = m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}$, hence we have

$$
\begin{aligned}
(L^{\mathrm{gr}}_l \otimes_{u^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}})_1 &= \big((L^{\mathrm{gr}}_l)_0 \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_0} (\mathcal{O}_{\mathcal{X}})_1\big) \oplus \big((L^{\mathrm{gr}}_l)_1 \otimes_{(u^{-1}\mathcal{O}_{\mathcal{Y}})_0} (\mathcal{O}_{\mathcal{X}})_0\big) \\
&= \big((L^{\mathrm{gr}}_l)_0 \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}\big) \oplus \big((L^{\mathrm{gr}}_l)_1 \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}\big) \\
&\cong \big((L^{\mathrm{gr}}_l)_0 \otimes_{m_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{F}\big) \oplus (L^{\mathrm{gr}}_l)_1.
\end{aligned}
$$

By Lemma 5.8 ii) there are natural functorial isomorphism

$$(L_l^{\mathrm{gr}})_0 \cong 0 \ , \quad (L_l^{\mathrm{gr}})_1 \cong (u^{-1}\mathcal{O}_\mathcal{Y})_1 \quad \text{and} \quad (L_{\mathcal{X}/\mathcal{S}}^{\mathrm{gr}})_1 \cong \mathcal{F}$$

in $\mathbf{D}(m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S})$. Thus $(L_l^{\mathrm{gr}} \otimes_{u^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X})_1 \cong (u^{-1}\mathcal{O}_\mathcal{Y})_1$ and the functoriality of all isomorphisms used shows that there is a distinguished triangle

$$
\begin{array}{ccc}
 & (L_h^{\mathrm{gr}})_1 & \\
 \nearrow & & \nwarrow \\
(u^{-1}\mathcal{O}_\mathcal{Y})_1 & \xrightarrow{\quad h_1 \quad} & \mathcal{F}
\end{array}
$$

in $\mathbf{D}(m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S})$. In particular, there is an isomorphism $(L_h^{\mathrm{gr}})_1 \cong \mathrm{Cone}(h_1)$ in $\mathbf{D}(m_\mathcal{X}^{-1}\mathcal{O}_\mathcal{S})$. $\qquad\square$

5.3 The subdiagrams of a single morphism of modules

In this example we keep the situation of General assumption 5.5. We consider two flat $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{F}$ and $\mathcal{G}$ and the diagram

$$\varphi\colon \mathcal{F} \to \mathcal{G}$$

of $\mathcal{O}_{\mathcal{S}}$-modules in Example 5.12. Similarly as in Subsection 4.1.2, we treat all three possible subdiagrams and calculate the obstruction groups for the problem of extending a given deformation of the subdiagram to a deformation of the diagram as explicitly as possible.

We keep the notations of Example 5.12, in particular the corresponding diagram

$$f\colon (X, \mathcal{O}_X) := (\mathcal{S}, \mathcal{O}_{\mathcal{S}} \oplus \mathcal{G}) \to (\mathcal{S}, \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F}) =: (Y, \mathcal{O}_Y)$$

in $\mathfrak{RTop}/\mathcal{S}$ whose underlying morphism of topoi is the identity and whose ring morphism is

$$\mathrm{id} \oplus \varphi\colon \mathcal{O}_{\mathcal{S}} \oplus \mathcal{F} \to \mathcal{O}_{\mathcal{S}} \oplus \mathcal{G}.$$

Let $\mathcal{Z}$ be the ringed topos associated to the diagram $f\colon X \to Y$ in $\mathfrak{RTop}/\mathcal{S}$.

Proposition 5.15. *The following chart is a list of all subdiagrams of $\varphi\colon \mathcal{F} \to \mathcal{G}$ and the corresponding obstruction groups as found in Theorem 5.13:*

	subdiagram	condition or notation	obstruction group
1	$\mathcal{F}$	$C(\varphi)$ the cone of φ	$\mathbb{Ext}^2_{\mathcal{S}}(C(\varphi), \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})$
		φ surjective with kernel $\mathcal{K}$	$\mathrm{Ext}^1_{\mathcal{S}}(\mathcal{K}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})$
		φ injective with cokernel $\mathcal{Q}$	$\mathrm{Ext}^2_{\mathcal{S}}(\mathcal{Q}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})$
2	$\mathcal{G}$	$\mathcal{H} = (\mathcal{G}, \mathcal{F}, \mathcal{F} \xrightarrow{\varphi} \mathcal{G})$ $\mathcal{F}' = (0, \mathcal{F}, \mathcal{F} \to 0)$	$\mathrm{Ext}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{F}', m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H})$
3	$\mathcal{F} \qquad \mathcal{G}$	none	$\mathrm{Ext}^1_{\mathcal{S}}(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})$

Proof. First notice that considering the subdiagrams $\mathcal{F}$, $\mathcal{G}$ and $(\mathcal{F}, \mathcal{G})$ of $\varphi\colon \mathcal{F} \to \mathcal{G}$ corresponds to considering the subdiagrams Y, X and (X, Y) of $f\colon X \to Y$, respectively. We have seen in the proof of Proposition 4.5 that condition **2** in Theorem 2.20 is satisfied in each of the three cases.

We will use some graded versions of results of Section 3.

Subdiagram 1: Keeping $\mathcal{F}$ and omitting $\mathcal{G}$ corresponds to omitting the source $(X, \mathcal{O}_X)$ of f. By a graded analogue of the calculations for the obstruction group for subdiagram 2 in Proposition 4.5, there is an obstruction in

$$\mathbb{Ext}^2_X(L^{\mathrm{gr}}_{X/Y}, t_X^{-1}\mathcal{J} \otimes_{t_X^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}} = \mathbb{Ext}^2_X(L^{\mathrm{gr}}_{X/Y}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}}$$

for extending a given graded extension of $(Y, \mathcal{O}_Y)$ to a graded extension of f. Here $\mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}$ is considered as graded $\mathcal{O}_X$-module concentrated in degree 1. By Lemma 5.8.i) we have naturally

$$\mathbb{Ext}^2_X(L^{\mathrm{gr}}_{X/Y}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}} \cong \mathbb{Ext}^2_{\mathcal{S}}((L^{\mathrm{gr}}_{X/Y})_1, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}).$$

We have seen in Example 5.12 that the morphism h_1 in Lemma 5.14 is given by

$$h_1 = (\varphi, \mathrm{id})\colon (\mathcal{F}, \mathcal{F}, \mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}) \to (\mathcal{G}, \mathcal{F}, \mathcal{F} \xrightarrow{\varphi} \mathcal{G})$$

and we have $(L^{\mathrm{gr}}_{X/Y})_1 = \big((L^{\mathrm{gr}}_h)_1\big)_X \cong \big(\mathrm{Cone}(h_1)\big)_X = C(\varphi)$ in $\mathbf{D}(\mathcal{S})$ by Lemma 5.14. If φ is surjective with kernel $\mathcal{K}$ (respectively if φ is injective with cokernel $\mathcal{Q}$) it follows that $C(\varphi) \cong \mathcal{K}[1]$ (respectively $C(\varphi) \cong \mathcal{Q}$) naturally in $\mathbf{D}(\mathcal{S})$.

Subdiagram 2: Abbreviating $\mathcal{H} = (\mathcal{G}, \mathcal{F}, \mathcal{F} \xrightarrow{\varphi} \mathcal{G})$, we know from Theorem 5.13 that the obstruction group is given by $\mathbb{E}\mathrm{xt}^2_{\mathcal{Z}}(L^{\mathrm{gr}}_h, m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H})_{\mathrm{gr}}$ where $m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}$ is concentrated in degree 1. By Lemma 5.8.i) this group is isomorphic to

$$\mathbb{E}\mathrm{xt}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}((L^{\mathrm{gr}}_h)_1, m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}).$$

By Example 5.12 the morphism h_1 is given by

$$h_1 = (\mathrm{id}, 0) \colon (\mathcal{G}, 0, 0 \to \mathcal{G}) \to (\mathcal{G}, \mathcal{F}, \mathcal{F} \xrightarrow{\varphi} \mathcal{G})$$

and we see that $\mathrm{Cone}(h_1) = (0, \mathcal{F}, \mathcal{F} \to 0)$ in $\mathbf{D}(m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}})$ since the cone of the identity $\mathcal{G} \to \mathcal{G}$ is homotopic to zero. Hence abbreviating $\mathcal{F}' = (0, \mathcal{F}, \mathcal{F} \to 0)$, it follows from Lemma 5.14 that

$$\begin{aligned}
\mathbb{E}\mathrm{xt}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}((L^{\mathrm{gr}}_h)_1, m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}) &\cong \mathbb{E}\mathrm{xt}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}((0, \mathcal{F}, \mathcal{F} \to 0), m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}) \\
&= \mathbb{E}\mathrm{xt}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{F}', m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}) \\
&\cong \mathbb{E}\mathrm{xt}^2_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}}(\mathcal{F}', m_{\mathcal{Z}}^{-1}\mathcal{J} \otimes_{m_{\mathcal{Z}}^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{H}).
\end{aligned}$$

Subdiagram 3: Keeping both $\mathcal{F}$ and $\mathcal{G}$ but omitting φ corresponds to keeping $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ but omitting f. By a graded analogue of the calculations for the obstruction group for subdiagram 3 in Proposition 4.5, there is an obstruction in

$$\mathbb{E}\mathrm{xt}^1_X(f^* L^{\mathrm{gr}}_{Y/S}, t_X^{-1}\mathcal{J} \otimes_{t_X^{-1}\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}} = \mathbb{E}\mathrm{xt}^1_X(f^* L^{\mathrm{gr}}_{Y/S}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}}$$

for extending a given graded extension of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ to a graded extension of f where $\mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}$ is again concentrated in degree 1. By Lemma 5.8.i) we have naturally

$$\mathbb{E}\mathrm{xt}^1_X(f^* L^{\mathrm{gr}}_{Y/S}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}} \cong \mathbb{E}\mathrm{xt}^1_S\big((f^* L^{\mathrm{gr}}_{Y/S})_1, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}\big).$$

By definition we have isomorphisms of $\mathcal{O}_{\mathcal{S}}$-modules

$$(f^* L^{\mathrm{gr}}_{Y/S})_1 = (L^{\mathrm{gr}}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_X)_1 \cong \big((L^{\mathrm{gr}}_{Y/S})_0 \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}\big) \oplus (L^{\mathrm{gr}}_{Y/S})_1.$$

From Lemma 5.8.ii) it follows that $(L^{\mathrm{gr}}_{Y/S})_0 \cong 0$ and $(L^{\mathrm{gr}}_{Y/S})_1 \cong \mathcal{F}$ in $\mathbf{D}(\mathcal{S})$, whence

$$\mathbb{E}\mathrm{xt}^1_X(f^* L^{\mathrm{gr}}_{Y/S}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G})_{\mathrm{gr}} \cong \mathbb{E}\mathrm{xt}^1_S(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}) \cong \mathrm{Ext}^1_S(\mathcal{F}, \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{G}).$$

<div align="right">□</div>

Remark 5.16. The obstruction groups for subdiagrams 1 and 3 can also be found in [Ill71, Proposition IV.3.2.3.] and [Ill71, Proposition IV.3.2.12.], respectively, where they are calculated without using the corresponding obstruction groups for subdiagrams 1 and 3 in Proposition 4.5. But the obstruction group for subdiagram 2 does not occur in [Ill71].

A   Further properties of the cotangent complex

All sheaves of rings of any topos are assumed to be associative, commutative and unitary.

Proposition A.1. *Let $\mathcal{T}$ be a topos, A a sheaf of rings of $\mathcal{T}$ and $B_1, \ldots, B_n$ sheaves of A-algebras of $\mathcal{T}$. Let $C_n = B_1 \otimes_A \ldots \otimes_A B_n$ and $L_{B_j/A} \otimes_{B_j} C_n \to L_{C_n/A}$ the natural morphisms of complexes of C_n-modules. Assume that each B_j is flat over A. Then the sum morphism*

$$\zeta_n \colon \bigoplus_{j=1}^{n} \big(L_{B_j/A} \otimes_{B_j} C_n \big) \to L_{C_n/A}$$

is a quasi-isomorphism.

Proof. We proceed by induction on n. For $n = 1$ we have $C_1 = B_1$ and ζ_1 is the identity. So let $n \geq 1$. Since $\operatorname{Tor}_i^A(C_{n-1}, B_n) = 0$ for all $i > 0$ by the flatness of $A \to B_n$, the cocartesian diagram

$$
\begin{array}{ccc}
C_{n-1} & \longrightarrow & C_n \\
\uparrow & & \uparrow \\
A & \longrightarrow & B_n
\end{array}
$$

yields a quasi-isomorphism

$$\psi \colon \big(L_{C_{n-1}/A} \otimes_{C_{n-1}} C_n \big) \oplus \big(L_{B_n/A} \otimes_{B_n} C_n \big) \to L_{C_n/A}$$

by Theorem 1.8.vi). By induction hypothesis the sum morphism

$$\zeta_{n-1} \colon \bigoplus_{j=1}^{n-1} \big(L_{B_j/A} \otimes_{B_j} C_{n-1} \big) \to L_{C_{n-1}/A}$$

is a quasi-isomorphism of complexes of C_{n-1}-modules. Since this is a quasi-isomorphism of complexes consisting of free modules, the morphism

$$\zeta_{n-1} \otimes_{C_{n-1}} C_n \colon \bigoplus_{j=1}^{n-1} \big(L_{B_j/A} \otimes_{B_j} C_n \big) \to L_{C_{n-1}/A} \otimes_{C_{n-1}} C_n$$

obtained by tensoring with C_n is a quasi-isomorphism as well by [Ill71, Lemme I.3.3.2.1.]. But the composition of $\big(\zeta_{n-1} \otimes_{C_{n-1}} C_n \big) \oplus \operatorname{id}$ and ψ is ζ_n, showing that ζ_n is a quasi-isomorphism. $\qquad\square$

Lemma A.2. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be the ringed topos associated to the diagram $f \colon \mathcal{X}_0 \to \mathcal{X}_1$ in $\mathfrak{RTop}/\mathcal{S}$ in the sense of Proposition 1.15. Denote the objects of $\mathcal{X}$ in the form*

$$C = (C_0, C_1, f^{-1}C_1 \xrightarrow{c} C_0)$$

where C_j is an object of $\mathcal{X}_j$ and $f^{-1}C_1 \xrightarrow{c} C_0$ is a sheaf morphism of $\mathcal{X}_0$. Let

$$A = (A_0, A_1, f^{-1}A_1 \xrightarrow{a} A_0) \quad and \quad B = (B_0, B_1, f^{-1}B_1 \xrightarrow{b} B_0)$$

be sheaves of rings of $\mathcal{X}$ and let $q \colon A \to B$ be a ring morphism of $\mathcal{X}$, given by the ring morphisms $q_j \colon A_j \to B_j$ of $\mathcal{X}_j$ together with the commutative diagram

$$
\begin{array}{ccc}
f^{-1}A_1 & \xrightarrow{\;a\;} & A_0 \\
{\scriptstyle f^{-1}q_1}\big\downarrow & & \big\downarrow{\scriptstyle q_0} \\
f^{-1}B_1 & \xrightarrow{\;b\;} & B_0.
\end{array}
\tag{A.1}
$$

Let $f^{-1}L_{B_1/A_1} \otimes_{f^{-1}B_1} B_0 \xrightarrow{n} L_{B_0/A_0}$ be the natural morphism of complexes of B_0-modules defined by the above square as in Theorem 1.8.ii). Then there is a natural isomorphism

$$L_{B/A} \to (L_{B_0/A_0}, L_{B_1/A_1}, f^{-1}L_{B_1/A_1} \otimes_{f^{-1}B_1} B_0 \xrightarrow{n} L_{B_0/A_0})$$

in the derived category $\mathbf{D}(B)$ of the category of B-modules.

Proof. Let $(\mathcal{X}_0, A_0)$, $(\mathcal{X}_0, B_0)$, $(\mathcal{X}_1, A_1)$, $(\mathcal{X}_1, B_1)$, $(\mathcal{X}, A)$ and $(\mathcal{X}, B)$ be the ringed topoi whose underlying topoi is the respective first entry and whose structure sheaf is the respective second entry. Then the commutative square (A.1) defines a commutative square of ringed topoi

$$
\begin{array}{ccc}
(\mathcal{X}_0, B_0) & \xrightarrow{(f,b)} & (\mathcal{X}_1, B_1) \\
\scriptstyle(\mathrm{id},q_0)\downarrow & & \downarrow\scriptstyle(\mathrm{id},q_1) \\
(\mathcal{X}_0, A_0) & \xrightarrow{(f,a)} & (\mathcal{X}_1, A_1)
\end{array}
$$

where the first entries of the maps are the underlying morphisms of topoi and the second entries of the maps are the ring morphisms of the structure sheaves. By [Ill71, Chapitre III.4.12.] there is a commutative diagram

$$
\begin{array}{ccccc}
(\mathcal{X}_0, B_0) & \xrightarrow{(i_{\mathcal{X}_0},\mathrm{id})} & (\mathcal{X}, B) & \xleftarrow{(i_{\mathcal{X}_1},\mathrm{id})} & (\mathcal{X}_1, B_1) \\
\scriptstyle(\mathrm{id},q_0)\downarrow & & \scriptstyle(\mathrm{id},q)\downarrow & & \downarrow\scriptstyle(\mathrm{id},q_1) \\
(\mathcal{X}_0, A_0) & \xrightarrow{(i_{\mathcal{X}_0},\mathrm{id})} & (\mathcal{X}, A) & \xleftarrow{(i_{\mathcal{X}_1},\mathrm{id})} & (\mathcal{X}_1, A_1)
\end{array}
$$

of ringed topoi where

$$i_{\mathcal{X}_j}^{-1}\colon \mathcal{X} \to \mathcal{X}_j, \quad (C_0, C_1, f^{-1}C_1 \xrightarrow{c} C_0) \mapsto C_j$$

are the forgetful functors. The assertion follows from [Ill71, Proposition III.4.12.2.]. $\qquad\square$

Corollary A.3. *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be the ringed topos associated to the diagram $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ in the sense of Proposition 1.15, let $q\colon A \to B$ be a ring morphism of $\mathcal{X}$ consisting of ring morphisms $q_c\colon A_c \to B_c$ of $\mathcal{X}_c$ for $c \in \mathrm{ob}(\mathfrak{C})$. For $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$ the commutative diagram*

$$
\begin{array}{ccc}
f_\alpha^{-1}A_{c_1} & \xrightarrow{A_\alpha} & A_{c_2} \\
\scriptstyle f_\alpha^{-1}q_{c_1}\downarrow & & \downarrow\scriptstyle q_{c_2} \\
f_\alpha^{-1}B_{c_1} & \xrightarrow{B_\alpha} & B_{c_2}
\end{array}
$$

yields natural morphisms of complexes of B_{c_2}-modules

$$n_\alpha\colon f_\alpha^{-1}L_{B_{c_1}/A_{c_1}} \otimes_{f_\alpha^{-1}B_{c_1}} B_{c_2} \to L_{B_{c_2}/A_{c_2}}.$$

Then the cotangent complex $L_q = L_{B/A}$ is given, up to natural isomorphism in $\mathbf{D}(B)$, by the collection of all cotangent complexes $L_{q_c} = L_{B_c/A_c}$ for $c \in \mathrm{ob}(\mathfrak{C})$ together with the natural morphisms of complexes of B_{c_2}-modules $n_\alpha\colon f_\alpha^{-1}L_{B_{c_1}/A_{c_1}} \otimes_{f_\alpha^{-1}B_{c_1}} B_{c_2} \to L_{B_{c_2}/A_{c_2}}$ for $\alpha \in \mathrm{Hom}_\mathfrak{C}(c_1, c_2)$.

Proof. For fixed $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(c_1, c_2)$ denote by $\mathcal{X}_\alpha$ the ringed topos associated to the diagram $f_\alpha \colon \mathcal{X}_{c_2} \to \mathcal{X}_{c_1}$ in $\mathfrak{RTop}/\mathcal{S}$. Let

$$A^\alpha = (A_{c_2}, A_{c_1}, f_\alpha^{-1} A_{c_1} \xrightarrow{A_\alpha} A_{c_2}) \quad \text{and} \quad B^\alpha = (B_{c_2}, B_{c_1}, f_\alpha^{-1} B_{c_1} \xrightarrow{B_\alpha} B_{c_2})$$

and let $(\mathcal{X}_\alpha, A^\alpha)$, $(\mathcal{X}_\alpha, B^\alpha)$, $(\mathcal{X}, A)$ and $(\mathcal{X}, B)$ be the ringed topoi whose underlying topoi is the respective first entry and whose structure sheaf is the respective second entry. Since the forgetful functor $\mathcal{X} \to \mathcal{X}_\alpha$ commutes with arbitrary limits and colimits, we may choose a right adjoint $i_{\mathcal{X}_\alpha *} \colon \mathcal{X}_\alpha \to \mathcal{X}$ of it by Remark 1.5 and we may regard the forgetful functor as the inverse image functor $i_{\mathcal{X}_\alpha}^{-1} \colon \mathcal{X} \to \mathcal{X}_\alpha$ of a morphism of topoi $i_{\mathcal{X}_\alpha} \colon \mathcal{X}_\alpha \to \mathcal{X}$. There is a commutative diagram of ringed topoi

$$
\begin{array}{ccc}
(\mathcal{X}_\alpha, B^\alpha) & \xrightarrow{(i_{\mathcal{X}_\alpha}, \mathrm{id})} & (\mathcal{X}, B) \\
{\scriptstyle (\mathrm{id}, (q_{c_2}, q_{c_1}))} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}, q)} \\
(\mathcal{X}_\alpha, A^\alpha) & \xrightarrow{(i_{\mathcal{X}_\alpha}, \mathrm{id})} & (\mathcal{X}, A).
\end{array}
$$

Now by [Ill71, Proposition III.4.12.2.] the natural morphism of complexes of B^α-modules

$$i_{\mathcal{X}_\alpha}^* L_{B/A} \to L_{B^\alpha/A^\alpha}$$

is a quasi-isomorphism, whence an isomorphism in the derived category $\mathbf{D}(B^\alpha)$ of the category of B^α-modules. Notice that this isomorphism does not depend on the chosen right adjoint $i_{\mathcal{X}_\alpha *} \colon \mathcal{X}_\alpha \to \mathcal{X}$ of the forgetful functor $i_{\mathcal{X}_\alpha}^{-1} \colon \mathcal{X} \to \mathcal{X}_\alpha$.

By Lemma A.2 there is a natural isomorphism

$$L_{B^\alpha/A^\alpha} \cong (L_{B_{c_2}/A_{c_2}}, L_{B_{c_1}/A_{c_1}}, f_\alpha^{-1} L_{B_{c_1}/A_{c_1}} \otimes_{f_\alpha^{-1} B_{c_1}} B_{c_2} \xrightarrow{n_\alpha} L_{B_{c_2}/A_{c_2}})$$

in $\mathbf{D}(B^\alpha)$. Hence for every $c \in \mathrm{ob}(\mathfrak{C})$ we may choose a morphism $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(c, c_2)$ to get an isomorphism $(L_{B/A})_c \cong L_{B_c/A_c}$ in $\mathbf{D}(B_c)$ and it suffices to show that the collection of these isomorphisms $(L_{B/A})_c \cong L_{B_c/A_c}$ in $\mathbf{D}(B_c)$ for $c \in \mathrm{ob}(\mathfrak{C})$ we have constructed so far is independent of the choice of α. But if $c \in \mathrm{ob}(\mathfrak{C})$ and if α and β are morphisms in $\mathfrak{C}$ having c either as source or target, then the diagram of functors

$$
\begin{array}{ccc}
 & \mathcal{X} & \\
 \swarrow & & \searrow \\
\mathcal{X}_\alpha & & \mathcal{X}_\beta \\
 \searrow & & \swarrow \\
 & \mathcal{X}_c &
\end{array}
$$

all of whose functors are the forgetful functors is commutative. It follows that $(L_{B/A})_c \cong L_{B_c/A_c}$ is independent of the choice of α. $\qquad\square$

B Injective resolutions of modules on a diagram

Let $\mathcal{X}$ be the ringed topos associated to a diagram $I\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{RTop}/\mathcal{S}$ in the sense of Proposition 1.15. Given an $\mathcal{O}_{\mathcal{X}}$-module M, we will describe an injective resolution of it in terms of injective $\mathcal{O}_{\mathcal{X}_c}$-modules for $c \in \mathrm{ob}(\mathfrak{C})$ which we will use to calculate the complex $\mathbf{R}G(m_{\mathcal{X}}^* \mathcal{J})$ in Corollary 3.26. Remember that if $(\mathcal{T}, \mathcal{O}_{\mathcal{T}})$ is an arbitrary ringed topos, then the category of $\mathcal{O}_{\mathcal{T}}$-modules has enough injectives by [Sta13, Theorem 18.9.4.].

For each $c \in \mathrm{ob}(\mathfrak{C})$ let J_c be an injective $\mathcal{O}_{\mathcal{X}_c}$-module. Define

$$I_c \;=\; \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma *} J_{c'} \;=\; \Big(\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'),\\ c' \in \mathrm{ob}(\mathfrak{C}), \gamma \neq \mathrm{id}}} f_{\gamma *} J_{c'} \Big) \oplus J_c$$

to be the $\mathcal{O}_{\mathcal{X}_c}$-module obtained by all not necessarily injective modules $f_{\gamma *} J_{c'}$ for $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c')$ and $c' \in \mathrm{ob}(\mathfrak{C})$. For $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ we have a natural isomorphism

$$f_{\alpha *} I_{c_2} \;=\; f_{\alpha *} \Big(\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_2,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma *} J_{c'} \Big) \;\cong\; \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_2,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma \alpha *} J_{c'}$$

of $\mathcal{O}_{\mathcal{X}_{c_1}}$-modules. If $\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c')$ factors as $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\gamma} c'$, we have $f_{\delta *} = f_{\gamma \alpha *}$ and we define

$$I_\alpha\colon \; I_{c_1} \;=\; \bigoplus_{\substack{\delta \in \mathrm{Hom}_{\mathfrak{C}}(c_1,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\delta *} J_{c'} \;\to\; \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_2,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma \alpha *} J_{c'} \;\cong\; f_{\alpha *} \Big(\bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_2,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma *} J_{c'} \Big) \;=\; f_{\alpha *} I_{c_2}$$

to be the composition of the projection and the natural isomorphism.

If $c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$ is a composition in $\mathfrak{C}$, then the triangle

$$
\begin{array}{ccc}
I_{c_1} & \xrightarrow{\quad I_{\beta\alpha} \quad} & f_{\beta\alpha *} I_{c_3} = f_{\alpha *} f_{\beta *} I_{c_3} \\
& \raise0.5ex\hbox{$\scriptstyle I_\alpha$} \searrow \qquad \nearrow \raise0.5ex\hbox{$\scriptstyle f_{\alpha *} I_\beta$} & \\
& f_{\alpha *} I_{c_2} &
\end{array}
$$

of morphisms of $\mathcal{O}_{\mathcal{X}_{c_1}}$-modules is commutative because the morphisms are defined by projecting.

Example B.1. Consider the diagram

$$
\begin{array}{ccc}
& \mathcal{X}_1 & \\
{\scriptstyle f} \nearrow & & \searrow {\scriptstyle g} \\
\mathcal{X}_0 & \xrightarrow{\ h = g \circ f\ } & \mathcal{X}_2
\end{array}
$$

in $\mathfrak{RTop}/\mathcal{S}$ with no other morphisms except for the identities and choose an injective $\mathcal{O}_{\mathcal{X}_j}$-module J_j for $j = 0,1,2$. Then we have

$$I_0 = J_0 \ , \quad I_1 = f_* J_0 \oplus J_1 \quad \text{and} \quad I_2 = h_* J_0 \oplus g_* J_1 \oplus J_2$$

and the projections

$$I_1 = f_* J_0 \oplus J_1 \to f_* J_0 = f_* I_0 \ , \quad I_2 = h_* J_0 \oplus g_* J_1 \oplus J_2 \to h_* J_0 = h_* I_0$$

and

$$I_2 = h_* J_0 \oplus g_* J_1 \oplus J_2 \to h_* J_0 \oplus g_* J_1 \cong g_*(f_* J_0 \oplus J_1) = g_* I_1.$$

Lemma B.2. *For each $c \in \mathrm{ob}(\mathfrak{C})$ let J_c be an injective $\mathcal{O}_{\mathcal{X}_c}$-module. Then the collection I of all $\mathcal{O}_{\mathcal{X}_c}$-modules*

$$I_c \;=\; \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma*}J_{c'}$$

for $c \in \mathrm{ob}(\mathfrak{C})$ together with the morphisms of $\mathcal{O}_{\mathcal{X}_{c_1}}$-modules $I_\alpha \colon I_{c_1} \to f_{\alpha*}I_{c_2}$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$ is an injective $\mathcal{O}_{\mathcal{X}}$-module.

Proof. Let $0 \to M \xrightarrow{s} N$ be an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules and let $p \colon M \to I$ be a morphism of $\mathcal{O}_{\mathcal{X}}$-modules. We have to show that there is a morphism of $\mathcal{O}_{\mathcal{X}}$-modules $q \colon N \to I$ such that

$$
\begin{array}{ccc}
 & N & \\
s \nearrow & & \searrow q \\
M & \xrightarrow{\quad p \quad} & I
\end{array}
$$

is commutative. For $c \in \mathrm{ob}(\mathfrak{C})$ and $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c')$ let

$$\mathrm{pr}_c^\gamma \colon \; I_c \;=\; \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'), \\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma*}J_{c'} \;\to\; f_{\gamma*}J_{c'}$$

be the projection to the summand corresponding to γ. By definition of I_γ the triangle

$$
\begin{array}{ccc}
I_c & \xrightarrow{\quad I_\gamma \quad} & f_{\gamma*}I_{c'} \\
\mathrm{pr}_c^\gamma \searrow & & \swarrow f_{\gamma*}\mathrm{pr}_{c'}^{\mathrm{id}} \\
 & f_{\gamma*}J_{c'} &
\end{array}
$$

is commutative for every $c \in \mathrm{ob}(\mathfrak{C})$ and $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c')$. Since J_c is an injective $\mathcal{O}_{\mathcal{X}_c}$-module, we may choose, for every $c \in \mathrm{ob}(\mathfrak{C})$, a morphism of $\mathcal{O}_{\mathcal{X}_c}$-modules $q_c^{\mathrm{id}} \colon N_c \to J_c$ such that

$$
\begin{array}{ccc}
N_c & & \\
s_c \uparrow & \searrow q_c^{\mathrm{id}} & \\
M_c \xrightarrow{p_c} I_c & \xrightarrow{\quad \mathrm{pr}_c^{\mathrm{id}} \quad} & J_c
\end{array}
$$

is commutative. Now fix $c \in \mathrm{ob}(\mathfrak{C})$. Then for every $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c')$ the diagram

$$
\begin{array}{ccccc}
M_c & \xrightarrow{\quad M_\gamma \quad} & & & f_{\gamma*}M_{c'} \\
\downarrow^{p_c} \; {}^{s_c}\searrow & & & {}^{f_{\gamma*}s_{c'}}\searrow & \downarrow^{f_{\gamma*}p_{c'}} \\
 & N_c & \xrightarrow{\quad N_\gamma \quad} & & f_{\gamma*}N_{c'} \\
I_c & \xrightarrow{\quad I_\gamma \quad} & & & f_{\gamma*}I_{c'} \\
\mathrm{pr}_c^{\mathrm{id}} \downarrow \; {}^{q_c^{\mathrm{id}}}\nearrow & & & {}^{f_{\gamma*}q_{c'}^{\mathrm{id}}}\nearrow & \downarrow^{f_{\gamma*}\mathrm{pr}_{c'}^{\mathrm{id}}} \\
J_c & & & & f_{\gamma*}J_{c'}
\end{array}
$$

of morphisms of $\mathcal{O}_{\mathcal{X}_c}$-modules is commutative by assumption. Thus the morphism

$$q_c^\gamma \colon N_c \xrightarrow{N_\gamma} f_{\gamma*}N_{c'} \xrightarrow{f_{\gamma*}q_{c'}^{\mathrm{id}}} f_{\gamma*}J_{c'}$$

makes the triangle

commutative. We define $q_c \colon N_c \to I_c$ to be the direct sum of all morphisms q_c^γ for $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c')$. Then q_c is a morphism of $\mathcal{O}_{\mathcal{X}_c}$-modules. Now fix $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$. It remains to show that

$$
\begin{array}{ccc}
N_{c_1} & \xrightarrow{\ N_\alpha\ } & f_{\alpha*}N_{c_2} \\
{\scriptstyle q_{c_1}}\big\downarrow & & \big\downarrow{\scriptstyle f_{\alpha*}q_{c_2}} \\
I_{c_1} & \xrightarrow{\ I_\alpha\ } & f_{\alpha*}I_{c_2}
\end{array}
\tag{B.1}
$$

is commutative because then the collection of all q_c for $c \in \mathrm{ob}(\mathfrak{C})$ defines a morphism $q \colon N \to I$ of $\mathcal{O}_{\mathcal{X}}$-modules which makes the triangle

commutative by definition.

Let us show that Diagram (B.1) is commutative. Let $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c')$. If γ does not factor through α, then the restriction of $I_\alpha \colon I_{c_1} \to f_{\alpha*}I_{c_2}$ to the direct summand $f_{\gamma*}J_{c'}$ is the zero morphism by definition of I_α. If γ factors through α as $\gamma \colon c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\varepsilon} c'$, then

is commutative because the upper square commutes since N is an $\mathcal{O}_{\mathcal{X}}$-module and the lower square commutes since $f_{\alpha*}f_{\varepsilon*} = f_{\gamma*}$. Since $q_{c_1} \colon N_{c_1} \to I_{c_1}$ and $q_{c_2} \colon N_{c_2} \to I_{c_2}$ are the direct sum of all morphisms $q_{c_1}^\gamma$ for $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c')$ and $q_{c_2}^\varepsilon$ for $\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(c_2, c')$, respectively, it follows that Diagram (B.1) is commutative. $\qquad\square$

Proposition B.3. *Let M be an $\mathcal{O}_{\mathcal{X}}$-module, let $j_c \colon M_c \to J_c$ be an injection of $\mathcal{O}_{\mathcal{X}_c}$-modules for $c \in \mathrm{ob}(\mathfrak{C})$ such that J_c is injective. If I is the injective $\mathcal{O}_{\mathcal{X}}$-module built from the J_c as in Lemma B.2, then there is an injection $i \colon M \to I$ of $\mathcal{O}_{\mathcal{X}}$-modules.*

Proof. Let $c \in \mathrm{ob}(\mathfrak{C})$ and $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$. We define

$$i_c\colon\ M_c \to I_c\ =\ \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} f_{\gamma*} J_{c'}$$

to be the composition of

$$M_c\ \to\ \bigoplus_{\substack{\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c,c'),\\ c' \in \mathrm{ob}(\mathfrak{C})}} M_c,\ \ m \mapsto (m, \dots, m)$$

and $\bigoplus M_c \xrightarrow{\oplus M_\gamma} \bigoplus f_{\gamma*} M_{c'} \xrightarrow{\oplus f_{\gamma*} j_{c'}} \bigoplus f_{\gamma*} J_{c'}$ where the direct sums are taken over all $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$ with $c' \in \mathrm{ob}(\mathfrak{C})$. The component morphism of $i_c\colon M_c \to I_c$ of the identity $\gamma = \mathrm{id}\colon c \to c$ is just the injection $j_c\colon M_c \to J_c$ which shows that $i_c\colon M_c \to I_c$ is injective.

If $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(c_1, c_2)$, then

$$
\begin{array}{ccc}
M_{c_1} & \xrightarrow{\ M_\alpha\ } & f_{\alpha*} M_{c_2} \\
\ \ \downarrow{\scriptstyle i_{c_1}} & & \ \ \downarrow{\scriptstyle f_{\alpha*} i_{c_2}} \\
I_{c_1} & \xrightarrow{\ I_\alpha\ } & f_{\alpha*} I_{c_2}
\end{array}
$$

is commutative by definition of I_α. Thus the collection of the injective morphisms $i_c\colon M_c \to I_c$ of $\mathcal{O}_{\mathcal{X}_c}$-modules for $c \in \mathrm{ob}(\mathfrak{C})$ defines an injective morphism $i\colon M \to I$ of $\mathcal{O}_{\mathcal{X}}$-modules. $\qquad\square$

Corollary B.4. *Let M be an $\mathcal{O}_{\mathcal{X}}$-module. Then there is an injective resolution*

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

of M where all injective $\mathcal{O}_{\mathcal{X}}$-modules I^n are modules of the form I in Lemma B.2.

Proof. For each $c \in \mathrm{ob}(\mathfrak{C})$ chooce an injection $M_c \to J_c^0$ of M_c to an injective $\mathcal{O}_{\mathcal{X}_c}$-module J_c^0. By Proposition B.3 there is an induced injection $M \to I^0$ of M to the injective $\mathcal{O}_{\mathcal{X}}$-modules I^0 built from the J_c^0. Let Q be the cokernel of $M \to I^0$. For each $c \in \mathrm{ob}(\mathfrak{C})$ chooce an injection $Q_c \to J_c^1$ of Q_c to an injective $\mathcal{O}_{\mathcal{X}_c}$-module J_c^1 and let I^1 be the injective $\mathcal{O}_{\mathcal{X}}$-module built from the J_c^1. By Proposition B.3 there is an injection $Q \to I^1$. Repeating this procedure we get an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

of M of the desired form. $\qquad\square$

We will see in Example B.6 that in general it is not possible to start with an injective resolution

$$0 \to M_c \to J_c^0 \to J_c^1 \to J_c^2 \to \dots$$

of M_c for every $c \in \mathrm{ob}(\mathfrak{C})$ and to build an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

of M such that I^n is the injective $\mathcal{O}_{\mathcal{X}}$-module built from the J_c^n as in Lemma B.2 because in general, the cokernel of $M_c \to J_c^0$ is not the cokernel of $M_c \to I_c^0$.

Corollary B.5. *Let $M \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ be a bounded below complex and let $a \in \mathbb{Z}$ such that $M^n = 0$ for all $n < a$. Then there is a bounded below complex $I \in \mathrm{ob}(\mathbf{Ch}^+(\mathcal{X}))$ such that $I^n = 0$ for all $n < a$ and a quasi-isomorphism $l \colon M \to I$ such that each $l^m \colon M^m \to I^m$ is injective and such that each I^m is an injective $\mathcal{O}_{\mathcal{X}}$-module of the form in Lemma B.2 for all $m \geq a$.*

Proof. This is a special case of [Sta13, Lemma 12.15.4.]. $\qquad\square$

Example B.6. Let $\mathcal{X}$ be the ringed topos associated to the diagram

$$f \colon \mathcal{X}_a \to \mathcal{X}_b$$

in $\mathfrak{RTop}/\mathcal{S}$. Let M be an $\mathcal{O}_{\mathcal{X}}$-module, denoted in the form $M = (M_a, \ M_b, \ M_b \xrightarrow{p} f_* M_a)$ where M_k is an $\mathcal{O}_{\mathcal{X}_k}$-module. Now choose injections

$$j_a^0 \colon M_a \to J_a^0 \qquad \text{and} \qquad j_b^0 \colon M_b \to J_b^0$$

where J_k^0 is an injective $\mathcal{O}_{\mathcal{X}_k}$-module. Then the injective $\mathcal{O}_{\mathcal{X}}$-module I^0 built from the J_k^0 is given by

$$I^0 = (J_a^0, \ f_* J_a^0 \oplus J_b^0, \ f_* J_a^0 \oplus J_b^0 \xrightarrow{\mathrm{pr}_1} f_* J_a^0)$$

and the injection $i^0 \colon M \to I^0$ in Proposition B.3 is given levelwise by

$$i_a^0 = j_a^0 \colon M_a \to J_a^0 \qquad \text{and} \qquad i_b^0 = (f_* j_a^0 \circ p \mid j_b^0) \colon M_b \to f_* J_a^0 \oplus J_b^0,$$

respectively, and the square

$$
\begin{array}{ccc}
0 \longrightarrow M_b \xrightarrow{(f_* j_a^0 \circ p \ \mid \ j_b^0)} & f_* J_a^0 \oplus J_b^0 \\
\downarrow{\scriptstyle p} & \downarrow{\scriptstyle \mathrm{pr}_1} \\
0 \longrightarrow f_* M_a \xrightarrow{\quad f_* j_a^0 \quad} & f_* J_a^0
\end{array}
$$

is commutative and has exact rows. Let $Q^0 = (Q_a^0, \ Q_b^0, \ Q_b^0 \xrightarrow{q^0} f_* Q_a^0)$ be the cokernel of $i^0 \colon M \to I^0$. We have exact sequences

$$0 \to M_a \xrightarrow{i_a^0} I_a^0 \xrightarrow{l} Q_a^0 \to 0 \qquad \text{and} \qquad 0 \to M_b \xrightarrow{i_b^0} I_b^0 \to Q_b^0 \to 0$$

of $\mathcal{O}_{\mathcal{X}_a}$-modules and $\mathcal{O}_{\mathcal{X}_b}$-modules, respectively, and a commutative square

$$
\begin{array}{ccccc}
0 \longrightarrow M_b \xrightarrow{(f_* j_a^0 \circ p \ \mid \ j_b^0)} & f_* J_a^0 \oplus J_b^0 \longrightarrow & Q_b^0 \longrightarrow 0 \\
\downarrow{\scriptstyle p} & \downarrow{\scriptstyle \mathrm{pr}_1} & \downarrow{\scriptstyle q^0} \\
0 \longrightarrow f_* M_a \xrightarrow{f_* j_a^0} & f_* J_a^0 \xrightarrow{f_* l} f_* Q_a^0 \longrightarrow & R^1 f_* M_a \longrightarrow \cdots
\end{array}
$$

of $\mathcal{O}_{\mathcal{X}_b}$-modules with exact rows. Now choose injections

$$\hat{j}_a^1 \colon Q_a^0 \to J_a^1 \qquad \text{and} \qquad \hat{j}_b^1 \colon Q_b^0 \to J_b^1$$

where J_k^1 is an injective $\mathcal{O}_{\mathcal{X}_k}$-module. The injective $\mathcal{O}_{\mathcal{X}}$-module I^1 built from the J_k^1 is given by

$$I^1 = (J_a^1, \ f_* J_a^1 \oplus J_b^1, \ f_* J_a^1 \oplus J_b^1 \xrightarrow{\mathrm{pr}_1} f_* J_a^1)$$

and the injection $\hat{i}^1 : Q^0 \to I^1$ in Proposition B.3 is given levelwise by

$$\hat{i}_a^1 = \hat{j}_a^1 : Q_a^0 \to J_a^1 \qquad \text{and} \qquad \hat{i}_b^1 = (f_*\hat{j}_a^1 \circ q^0 \mid \hat{j}_b^1) : Q_b^0 \to f_*J_a^1 \oplus J_b^1,$$

respectively, and the square

$$
\begin{array}{ccc}
0 \longrightarrow Q_b^0 & \xrightarrow{(f_*\hat{j}_a^1 \circ q^0 \ \mid\ \hat{j}_b^1)} & f_*J_a^1 \oplus J_b^1 \\
\downarrow{\scriptstyle q^0} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
0 \longrightarrow f_*Q_a^0 & \xrightarrow{f_*\hat{j}_a^1} & f_*J_a^1
\end{array}
$$

is commutative and has exact rows. So far, letting i^1 be the composition $I^0 \to Q \xrightarrow{\hat{i}^1} I^1$, we have found an exact sequence

$$0 \to M \xrightarrow{i^0} I^0 \xrightarrow{i^1} I^1$$

of $\mathcal{O}_{\mathcal{X}}$-modules which may be completed to an injective resolution of M. Since $i_a^0 = j_a^0$ the injective module J_a^1 may be chosen to be part of an injective resolution

$$0 \to M_a \xrightarrow{j_a^0} J_a^0 \xrightarrow{j_a^1} J_a^1 \to J_a^2 \to \dots$$

of M_a. If we choose J_a^1 like that, then we see from the commutativity of

$$
\begin{array}{ccccc}
f_*J_a^0 \oplus J_b^0 & \longrightarrow & Q_b^0 & \xrightarrow{\hat{i}_b^1} & f_*J_a^1 \oplus J_b^1 \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle q^0} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
f_*J_a^0 & \longrightarrow & f_*Q_a^0 & \xrightarrow{f_*\hat{i}_a^1} & f_*J_a^1
\end{array}
$$

that we have a commutative diagram with exacts rows

$$
\begin{array}{ccccccc}
0 \longrightarrow M_b & \longrightarrow & f_*J_a^0 \oplus J_b^0 & \longrightarrow & f_*J_a^1 \oplus J_b^1 \\
\downarrow{\scriptstyle p} & \begin{pmatrix} f_*j_a^0 \circ p \\ j_b^0 \end{pmatrix} & \downarrow{\scriptstyle \mathrm{pr}_1} & \begin{pmatrix} f_*j_a^1 & 0 \\ *_1 & *_2 \end{pmatrix} & \downarrow{\scriptstyle \mathrm{pr}_1} \\
0 \longrightarrow f_*M_a & \xrightarrow{f_*j_a^0} & f_*J_a^0 & \xrightarrow{f_*j_a^1} & f_*J_a^1
\end{array}
$$

where the unknown entries $*_1$ and $*_2$ are the restrictions of $f_*J_a^0 \oplus J_b^0 \to Q_b^0 \xrightarrow{\hat{j}_b^1} J_b^1$ to $f_*J_a^0$ and J_b^0, respectively.

Now let

$$0 \to M_a \xrightarrow{j_a^0} J_a^0 \xrightarrow{j_a^1} J_a^1 \to J_a^2 \to \dots \qquad \text{and} \qquad 0 \to M_b \xrightarrow{j_b^0} J_b^0 \xrightarrow{j_b^1} J_b^1 \to J_b^2 \to \dots$$

be injective resolutions of M_k. We have seen in Proposition B.3 that we may take the injective $\mathcal{O}_{\mathcal{X}}$-module I^0 built from the J_k^0 as in Lemma B.2 as the beginning of an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

of M. But in this example it is not possible to take I^1 to be the injective $\mathcal{O}_{\mathcal{X}}$-module built from the J^1_k as in Lemma B.2. If Q^0_b denotes again the cokernel of $i^0_b = (f_* j^0_a \circ p \mid j^0_b) \colon M_b \to f_* J^0_a \oplus J^0_b$ and if $\tilde{Q}^0_b$ is the cokernel of $j^0_b \colon M^0_b \to J^0_b$, the commutativity of

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & f_* J & = & f_* J & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_b & \xrightarrow{(f_* j^0_a \circ p \mid j^0_b)} & f_* J^0_a \oplus J^0_b & \longrightarrow & Q^0_b & \longrightarrow & 0 \\
& & \| & & \downarrow{\mathrm{pr}_2} & & \downarrow & & \\
0 & \longrightarrow & M_b & \xrightarrow{j^0_b} & J^0_b & \longrightarrow & \tilde{Q}^0_b & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

shows that $Q^0_b \to \tilde{Q}^0_b$ is not injective, hence we cannot continue with the given embedding $\tilde{Q}^0_b \to J^1_b$.

List of notations

148

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Danksagung

Ich danke Sönke Rollenske für die Vergabe dieses Promotionsthemas und die Betreuung der Arbeit. Die intensiven Gespräche über Mathematik allgemein haben mir sehr geholfen. Insbesondere bin ich dankbar für seinen Vorschlag, das deformationstheoretische Problem mit dem Kotangentialkomplex zu lösen.

Ferner danke ich Jan Arthur Christophersen für seine Bereitschaft, als Zweitbetreuer diese Arbeit zu begutachten und für die Diskussionen mit ihm in Oslo. Durch ihn gewann ich neue Einblicke in geometrischer Invariantentheorie.

Mein Dank gilt auch Gregory Stevenson für die Gespräche über den Kotangentialkomplex, besonders über die Frage, ob der Kotangentialkomplex mit der Bildung von Differenzkokernen verträglich ist.