Parabolic equations associated with symmetric nonlocal operators

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Introduction

Motivation

In the middle of the 20th century De Giorgi [DG57] and Nash [Nas57] independently proved that weak solutions u of a linear partial differential equation of the form

$$\operatorname{div}\left(A(x)\nabla u(x)\right) = 0 \tag{I.1}$$

satisfy an a priori Hölder estimate. More precisely, under the assumption that the operator in (I.1) is uniformly elliptic, i.e. if the symmetric matrix A is bounded, measurable and satisfies for some $\lambda, \Lambda > 0$

$$\lambda\,|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda\,|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d\,, \tag{I.2}$$

they showed that weak solutions u of (I.1) are Hölder continuous – i.e. $u \in C^{0,\beta}$ – where the Hölder exponent $\beta \in (0,1)$ and the corresponding seminorm depend only on the dimension and the constants $\lambda, \Lambda > 0$ in (I.2). A few years later Moser [Mos61] established a stronger result – namely an elliptic Harnack inequality – for weak solutions to (I.1). This Harnack inequality implies Hölder regularity for weak solutions and thereby Moser gave a third proof of the a priori Hölder estimate.

This result – which is often referred to as the De Giorgi-Nash-Moser result – applies to minimizers of nonlinear variational integrals of the form

$$\int F(\nabla w(x))\,\mathrm{d}x\,, \tag{I.3}$$

where F is a convex C^2-function. To be specific, the partial derivatives $\partial_i w$ of minimizers w of (I.3) are weak solutions to (I.1), where $a_{ij}(x) = \partial_i\partial_j F(\nabla w(x))$, and hence the De Giorgi-Nash-Moser result shows that $w \in C^{1,\beta}$ under appropriate assumptions on F that ensure (I.2). The assertion $w \in C^{1,\beta}$ for minimizers w was the most important contribution to the solution of Hilbert's 19th problem, who raised in his famous collection of problems (see [Hil00]) the question whether regular variational integrals such as (I.3) only allow for minimizers that are smooth. Thanks to the De Georgi-Nash-Moser result – and some results which had been established earlier – it was finally possible to give a positive answer to Hilbert's 19th problem.

Already Nash [Nas57] and later Moser [Mos64, Mos67, Mos71] proved Hölder regularity of weak solutions $u = u(t,x)$ of the parabolic equation associated to (I.1), i.e.

$$\partial_t u(t,x) - \operatorname{div}\left(A(t,x)\nabla u(t,x)\right) = 0\,, \tag{I.4}$$

under the assumption that the operator is uniformly elliptic. Again, Moser deduced the Hölder regularity from a parabolic Harnack inequality[1] (Theorem 7.6).

The present work extends Moser's results including a weak Harnack inequality (Theorem 4.4) and Hölder regularity (Theorem 4.5) to parabolic equations of the type

$$\partial_t u(t,x) - \text{p.v.} \int_{\mathbb{R}^d} [u(t,y) - u(t,x)]\, k_t(x,y)\, \mathrm{d}y = 0\,, \tag{I.5}$$

where $k_t(x,y)$ is a symmetric kernel that has a certain singularity at the diagonal $x = y$. A Harnack inequality in the form of Theorem 4.4 is called a *weak* Harnack inequality. If one could replace $\|u\|_{L^1(Q_\ominus)}$ by $\sup_{Q_\ominus} u$ therein, then one would call this type of inequality (cf. Theorem 7.6) a *strong* Harnack inequality or simply Harnack inequality.
In the special case[2] $k_t(x,y) = 2\mathcal{A}_{d,-\alpha} |x-y|^{-d-\alpha}$, $\alpha \in (0,2)$, equation (I.5) becomes

$$\partial_t u(t,x) + (-\Delta)^{\alpha/2} u(t,x) = 0\,, \tag{I.6}$$

where $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian – the pseudo-differential operator with symbol $|\xi|^\alpha$, see Section 2.7. This operator can be seen as a prototype of a nonlocal operator. The standing assumptions (see Section 4.1) that are imposed on the kernel k_t to prove the main theorems imply that the properties of the bilinear form associated to the nonlocal operator in (I.5) are in a certain sense comparable to those of the bilinear form associated to the fractional Laplacian.

In a very similar way as the classical De Giorgi-Nash-Moser result applies to minimizers of variational integrals (I.3), the a priori Hölder estimate for weak solutions to (I.5) applies to nonlocal, nonlinear variational integrals: Caffarelli, Chan and Vasseur [CCV11] showed that their Hölder regularity estimate for solutions to (I.5) implies that the minimizers w of

$$\iint \phi(w(x) - w(y)) K(x-y)\, \mathrm{d}x\, \mathrm{d}y$$

belong to $C^{1,\beta}$, where ϕ is a convex and even functional of class $C^2(\mathbb{R})$ with $\lambda \le \phi'' \le \Lambda$ and K a symmetric function satisfying $K(x) \asymp |x|^{-d-\alpha}$.
From this point of view, Hölder regularity estimates for (I.5) such as Theorem 4.5 can be considered as the central tool in proving regularity of minimizers of nonlocal, nonlinear variational integrals.

Another reason why regularity results for parabolic equations are interesting is the application to the potential theory of Markov processes: This relation is explained by the following observations: At least in the case $k_t(x,y) = k(x,y)$ and $k(x,y) \asymp |x-y|^{-d-\alpha}$ for small values of $|x-y|^{-d-\alpha}$, it is possible to show that there corresponds a regular, symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ to the nonlocal operator

$$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^d} [u(y) - u(x)]\, k(x,y)\, \mathrm{d}y\,.$$

[1] Around twenty years later, Fabes and Stroock [FS86] reproved Moser's parabolic Harnack inequality by means of Nash's ideas.
[2] The factor $2\mathcal{A}_{d,-\alpha}$ is just a norming constant that is specified in Section 2.4.

By the general theory of symmetric Dirichlet forms ([FŌT94]) $\mathcal{L}$ is then the infinitesimal generator of a Hunt process $\mathcal{X}$. In this situation, the density function $p_\alpha(t,x,y)$ (also called *heat kernel*) of the associated semigroup is the fundamental solution of (I.5). A priori Hölder continuity then implies that $p_\alpha(t,x,y)$ is continuous and thus the associated Hunt process $\mathcal{X}$ may be redefined to start in every point $x \in \mathbb{R}^d$, i.e. $\mathcal{X}$ is a strong Markov process on $\mathbb{R}^d$. This Markov process has discontinuous paths and does not possess second moments.

Even in the most simple case $\mathcal{L} = -(-\Delta)^{\alpha/2}$ (i.e. if (I.5) reduces to (I.6)) no explicit expression for $p_\alpha(t,x,y)$ is known except for the case $\alpha = 1$. Therefore heat kernel bounds are a matter of particular interest. A special case of a result by Chen and Kumagai [CK08] yields for $\mathcal{L} = -(-\Delta)^{\alpha/2}$ the two-sided heat kernel estimate

$$p_\alpha(t,x,y) \asymp t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{\alpha+d} \qquad t>0,\ x,y \in \mathbb{R}^d\,. \tag{I.7}$$

This result should be compared with the corresponding estimate for the heat kernel corresponding to (I.1): There are constants $c_1,\dots,c_4 > 0$ depending only on d, and λ, Λ in (I.2) such that for all $t>0$ and $x,y \in \mathbb{R}^d$

$$\frac{c_1}{(4\pi t)^{d/2}} \exp\left(c_2 \frac{-|x-y|^2}{4t}\right) \le p_2(t,x,y) \le \frac{c_3}{(4\pi t)^{d/2}} \exp\left(c_4 \frac{-|x-y|^2}{4t}\right)\,. \tag{I.8}$$

This estimate was obtained by Aronson [Aro67] and a central tool in his proof was Moser's parabolic Harnack inequality. Reversely, Fabes and Stroock [FS86] showed that the estimate (I.8) implies Moser's parabolic Harnack inequality for solutions to (I.4). Hence, Harnack inequalities for parabolic equations are closely related to heat kernel estimates of the associated process. This relation is still true and even more interesting if the state space is no longer the Euclidean space, but a manifold or a graph, see the introduction in Barlow, Grigor'yan and Kumagai [BGK12].

Related results

The proofs within this thesis use only analytical methods, in other words (I.5) is considered from the point of view of PDE theory. Therefore, this short survey starts with results that share this purely analytic point of view.

One may consider Komatsu's articles [Kom88, Kom95] as a starting point in the regularity theory of weak solutions to (I.5). The author proves Hölder regularity following Nash's method under the condition of pointwise comparability of $k_t(x,y)$ with $|x-y|^{-d-\alpha}$ and assuming continuity in t.
Kassmann [Kas09] established a Moser scheme for nonlocal elliptic equations leading to Hölder regularity for weak solutions. Due to the nonlocality of the operator the assumption of non-negativity of the solution in some domain in $\mathbb{R}^d$ is not strong enough to prove a classical elliptic Harnack inequality. A counterexample violating the global non-negativity

was given in [Kas07a]. An alternative formulation of Harnack's inequality which is equivalent to the classical one in case of a second order operator was proposed in [Kas11].
Caffarelli, Chan and Vasseur proved Hölder regularity for solutions to (I.5) following De Giorgi's method. This method yields an a priori Hölder estimate where the C^{β}-norm of a solution u is controlled by $\|u\|_{L^2}$ instead of $\|u\|_{L^\infty}$ as in Theorem 4.5. They provide a very interesting proof of how the regularity of weak solutions to linear, nonlocal equations apply to regularity of minimizers of nonlinear, nonlocal variational integrals. The constants in their results blow up if $\alpha \to 2-$.
There are some related results which are robust for $\alpha \to 2-$, but these apply to nonlocal operators in non-divergence form[3]: Chang Lara and Dávila [CLD12] proved Hölder regularity for viscosity solutions to fully nonlinear parabolic equations. Robust results for fully nonlinear elliptic equations were also found by Caffarelli and Silvestre [CS11] as well as by Guillen and Schwab [GS12].

As already explained previously in this introduction there is a huge interplay between analytic and probabilistic methods for parabolic equations such as (I.5). From a probabilistic point of view, the Harnack inequality for solutions to (I.6) is readily established by using the explicit expression for the exit time of a rotationally invariant α-stable Lévy process. The generalization to other kernels opened a large field of research starting with the article by Bass and Levin [BL02]. In the case $k_t(x,y) = k(x,y)$ and $k(x,y) \asymp |x-y|^{-d-\alpha}$ they use both probabilistic and analytical methods for their proof of a Harnack inequality and pointwise bounds on the heat kernel.
This approach was further generalized by Chen and Kumagai [CK03]. On a general d-set (F,ν) they showed that – under the assumption $k_t(x,y) = k(x,y)$ and $k(x,y) \asymp |x-y|^{-d-\alpha}$ – there exists a Feller process associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ given by

$$\mathcal{E}(u,v) = \iint\limits_{F\,F} [u(x)-u(y)]\,[v(x)-v(y)]\,k(x,y)\nu(\,\mathrm{d}x)\nu(\,\mathrm{d}y),$$
$$\mathcal{F} = \left\{u \in L^2(F,\nu) \colon \mathcal{E}(u,u) < \infty\right\} .$$

They showed that the heat kernel exists and satisfies (I.7) for $0 < t \leq 1$. In particular, their set-up allows to work with the weak formulation of (I.5).
In [SV04], Song and Vondraček list three abstract conditions on a general class of Markov processes that are sufficient to establish a Harnack inequality.
A situation where the pointwise comparability to one fixed order of singularity is not satisfied is studied in Barlow, Bass, Chen and Kassmann [BBCK09]. It is assumed that $k_t(x,y) = k(x,y)$ and that there are constants $c_1, c_2 > 0$ and $0 < \alpha \leq \beta < 2$ such that for all $|x-y| \leq 1$

$$c_1\,|x-y|^{-d-\alpha} \leq k(x,y) \leq c_2\,|x-y|^{-d-\beta} .$$

They show the existence of the heat kernel together with upper and lower bounds for it. Generally, it is important that the equation satisfies a certain scaling behavior in order to deduce regularity results. Such a scaling property is not satisfied for the equations in the situation of [BBCK09]; the authors provide an example of a discontinuous function u satisfying $\mathcal{L}u = 0$.

[3] A detailed discussion on the distinction between divergence form and non-divergence form in the case of nonlocal operators may be found in [KS13] or [Caf12]

This approach of combining probabilistic and analytical aspects has been extended further, also to much more general state spaces. A complete list of all related results in this area would go far beyond the scope of this introduction.

Local vs. Nonlocal

Due to the nonlocality of the operator in (I.5), there are several differences compared to the situation of a second-order parabolic equation. Two of them may be of particular interest: Firstly, very similar to (I.2), the conditions on k_t in this work ensure the non-degeneracy of the operator in the parabolic equation. However, a strong Harnack inequality cannot be expected to hold for solutions of (I.5). A counterexample was given by Bogdan and Sztonyk [BS05].
Secondly, the implication from Harnack inequality to Hölder regularity is more involved. For this reason, a weak Harnack inequality is established for nonnegative weak supersolutions to

$$\partial_t u(t,x) - \text{p.v.} \int_{\mathbb{R}^d} [u(t,y) - u(t,x)]\, k_t(x,y)\, \mathrm{d}y = f(t,x) \qquad \text{on } (0,T) \times \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain and $f \in L^\infty((0,T)\times\Omega)$. This weak Harnack inequality then implies an estimate on the oscillation of a solution u, which in turn yields Hölder regularity in the nonlocal setting. The method how to deduce Hölder regularity from the estimate on the oscillation of u was found by Silvestre [Sil06].

Main features of the approach

The presented work establishes the main results by modifying Moser's classical approach to the case of nonlocal parabolic equations. The advantages of this method are explained by the following features:

I) *Local regularity results:* In contrast to De Giorgi's method, Moser's technique uses local methods to derive regularity results. This means that (I.5) is only assumed to hold on a bounded region $I \times \Omega \subset \mathbb{R}^{d+1}$ in order to derive the weak Harnack inequality and the Hölder regularity result.
The weak Harnack inequality is of its own interest since these a priori inequalities play an important role[4] in partial differential equations and have applications that go beyond questions of regularity. Only recently, Jarohs and Weth [JW13] applied Theorem 4.4 in their results on asymptotic symmetry of solutions to nonlinear fractional reaction-diffusion equations. Also Erdős and Yau [EY12] as well as Jin and Xiong [JX11] apply a Harnack inequality for fractional order equations in their results.

II) *Mild assumptions on the kernel:* The techniques applied in this thesis allow for relatively mild assumptions on the underlying kernel k_t. In particular, no pointwise bounds and no regularity in the variables is required for k_t. This property is essential for the application to nonlocal, nonlinear variational integrals. Thus, (I.5) can be

[4] see the survey on Harnack inequalities [Kas07b].

seen as a nonlocal equation with bounded, measurable coefficients. Moreover, k_t may vanish on a large part around the diagonal, see Example 4.8 for an illustration. Up to the present, all related results that prove Hölder regularity for solutions to (I.5) have used stronger conditions on the underlying kernels.
Another important aspect in this context is explained in Kassmann and Schwab [KS13], where the authors use this approach to provide the main results in the case where the kernel of the nonlocal operator is not absolutely continuous with respect to the Lebesgue measure.

III) Robustness for $\alpha \to 2-$: All constants that appear in the main results are independent of $\alpha \in (\alpha_0, 2)$, i.e. the a priori estimates do not depend on the order of differentiability of the underlying operator – provided the order is bounded from below by a universal constant $\alpha_0 > 0$. In particular, the estimates hold uniformly for a sequence of solutions (u_n) to orders α_n approaching 2 from below, see Example 4.7. However, it will not be shown in this thesis that Moser's classical results can be obtained as a limit case.

Outline

Chapter 1 contains a short review on integration of vector-valued functions. The focus therein lies on the application to parabolic equations. Also the results in Chapter 2 serve as a basis for the theory of parabolic equations of fractional order. Except of Section 2.4 all results and proofs in Chapter 1 and Chapter 2 are collected from the literature. Detailed references are given therein.
In Chapter 3, the parabolic initial boundary value problem for both local and nonlocal operators is studied from a functional analytic point of view. The notion of weak solution is elaborated there and the well-posedness of the problem is established by means of Hilbert space methods.
The main theorems of this thesis and the framework for these results are presented in Chapter 4. Chapter 5 collects all technical tools that are needed to apply Moser's technique. This technique is applied in Chapter 6, where the proofs of the main results are given.
Finally, in Chapter 7, Moser's classical results for local operators are reviewed and reproved. The structure in this chapter is in one-to-one correspondence with the structure in Chapter 6 in order to facilitate a comparison between Moser's technique for second order and fractional order parabolic equations.
A short list of notation is given on pp. 117-118.

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Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

Den Inhalt der Kapitel 4-6 hat der Autor dieser Dissertation in einer Arbeit [FK13] gemeinsam mit seinem Betreuer, Moritz Kaßmann, veröffentlicht. Diese Arbeit wurde von der Zeitschrift *Communications in Partial Differential Equations* zur Veröffentlichung angenommen. Die Ergebnisse in Abschnitt 5.3, Abschnitt 5.4 und Abschnitt 6.6 gehen auf Moritz Kaßmanns frühere Arbeiten über elliptische Gleichungen zurück. Außerdem stammt die Idee, die punktweisen Schranken an den Kern $k_t(x, y)$ durch Integralbedingungen zu ersetzen – siehe Abschnitt 4.1 – von ihm.

Part I

Basics

1 Integration theory & Lebesgue spaces

In this chapter we review the construction of the Bochner-Lebesgue integral as well as some central results in this theory. The concept of Bochner measurability and integrability can be considered as an extension of Lebesgue's integration theory to functions that take values in some Banach space. In this presentation we focus on those parts of this concept that will be needed in the functional analytic theory of parabolic equations. Additionally, some important results in this chapter such as the theory of L^p-spaces are provided in an integrated way both for real-valued and vector-valued functions.

Throughout the whole chapter the triplet $(M, \mathcal{A}, \mu)$ denotes a complete[1] measure space. In this thesis the results are applied to the situation where $(M, \mathcal{A}, \mu)$ is the completion of the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field and λ^d the d-dimensional Lebesgue measure.

$(V, \|\cdot\|)$ denotes a real Banach space.

1.1 Measurable functions

Let us recall that a function $f\colon M \to \mathbb{R}$ is said to be Lebesgue measurable if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}(\mathbb{R})$.

The following definition generalizes the concept of measurability to the case where f maps into a general Banach space V. It is a summary of the definitions and remarks in [AE09, Section X.1] and [Zei90, Appendix].

Definition 1.1. Let $f\colon M \to V$.

(i) f is called *step function* (or *simple function*) if there is $k \in \mathbb{N}$, $(v_j, A_j) \in V \times \mathcal{A}$ with $\mu(A_j) < \infty$ for $j = 1, \dots, k$, such that

$$f = \sum_{j=1}^{k} v_j \mathbb{1}_{A_j}$$

with $v_j \neq 0$ for all $j = 1, \dots, k$ and $v_j \neq v_i$, $A_j \cap A_i = \emptyset$ for $j \neq i$. It is easy to see that this representation is unique.

[1]A measure space is called *complete* if every subset of a set of measure zero is again measurable. Note that this property is depending on both the σ-field and the measure. See Remark 1.2 for a discussion on this assumption.

(ii) f is called *Bochner measurable* (or *strongly measurable*) if there is a sequence $(f_j)_{j\in\mathbb{N}}$ of simple functions such that for μ-almost every $m \in M$

$$\lim_{j\to\infty} \|f_j(m) - f(m)\| = 0.$$

(iii) f is called *weakly measurable* if the real-valued functions $\xi \mapsto \langle g, f(\xi)\rangle$ are Lebesgue measurable for every $g \in V^*$.

(iv) f is called *almost surely separably valued* if there is $N \subset M$ with $\mu(N) = 0$ such that $f(N^c) \subset V$ is separable.

Remark 1.2. A well-known result (e.g. [AE09, Theorem X.1.14]) states that if a sequence (f_n) of strongly measurable functions converges to a function f almost everywhere, then f is again strongly measurable. The following example, which is taken from [AE09, Remark X.1.15], shows that this property fails if we do not assume the measure space to be complete:

The measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^1)$ is non-complete ([AE09, IX.5.29]). In particular, there is a subset N of the Cantor set C such that $N \notin \mathcal{B}(\mathbb{R})$. Define $f_n = \mathbb{1}_C$ and $f = \mathbb{1}_N$. C – as a compact set – is measurable and thus f_n is strongly measurable. Since $\lambda^1(C) = 0$, f_n converges λ^1-a.e. to f. However, f is not measurable since $\{f > 0\} = N \notin \mathcal{B}(\mathbb{R})$.

Of course, if g is defined by $g(t) = \lim_{n\to\infty} f_n(t)$, then $g = \mathbb{1}_C$, which is a measurable function. The result mentioned at the beginning of this remark asserts that every function that can be identified as a pointwise limit of a sequence of strongly measurable functions is again measurable – provided the underlying measure space is complete. ♦

Let us mention another rather simple observation: If $f\colon M \to V$ is strongly measurable then the real valued function $m \mapsto \|f(m)\|$ is Lebesgue measurable, see e.g. [Růž04, Lemma 2.1.7] for a proof.

The following theorem provides a characterization of strong measurability. It is due to Pettis [Pet38], see also [Yos80, Section V.4] for a proof.

Theorem 1.3 (Pettis' theorem)**.** *A function $f\colon M \to V$ is strongly measurable if and only if it is weakly measurable and almost surely separately valued.*

Since every subset of a separable normed space is again separable, we see that $f(M)$ is separable if V is itself separable. As an immediate consequence of Pettis' theorem we obtain[2]:

Corollary 1.4. *Let V be a separable Banach space. Then $f\colon M \to V$ is strongly measurable if and only if f is weakly measurable.*

We will frequently use this result without citing it explicitly.

[2]In literature this corollary is sometimes referred to as Pettis' theorem.

1.2 The Bochner-Lebesgue integral

The presentation in this section collects results on Bochner integration from [AE09, Chapter X], [GGZ74, §IV.1] and [Zei90, Section 23.2], see also [Emm04, Abschnitt 7.1] and [Růž04, Kapitel 2] for similar summaries on this topic.

Let us give a very short overview on the construction of the Lebesgue integral of real-valued functions: For every measurable function $f: M \to [0, \infty)$ there is a sequence of step functions (φ_n) such that $\varphi_n \to f$ a.e. The integral of f is then defined as the limit of the integrals of φ_n, i.e.

$$\int_M f(m)\,\mathrm{d}\mu(m) = \lim_{n\to\infty} \int_M \varphi_n(m)\,\mathrm{d}\mu(m) = \lim_{n\to\infty} \sum_{j=1}^{k_n} x_{j,n}\mu(A_{j,n})\,,$$

where $\varphi_n = \sum_{j=1}^{k_n} x_{j,n}\mathbb{1}_{A_{j,n}}$. A general function $f: M \to \mathbb{R}$ is called integrable if $\int f_+$ and $\int f_-$ exist and we set

$$\int_M f\,\mathrm{d}\mu = \int_M f_+\,\mathrm{d}\mu - \int_M f_-\,\mathrm{d}\mu\,.$$

This is just a minimal overview. Details can be found in the monograph [Bau01].

The next definition generalizes this concept to functions with values in a Banach space $(V, \|\cdot\|)$.

Definition 1.5. Let $\varphi: M \to V$ be a simple function as in Definition 1.1(i). The *integral of* φ is defined by

$$\int_M \varphi\,\mathrm{d}\mu = \sum_{j=1}^{k} v_j\,\mu(A_j) \in V\,.$$

Let $f: M \to V$ be a Bochner measurable function with approximating sequence (φ_n). We say that f is *Bochner integrable* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n, k \geq N$

$$\int_M \|\varphi_n(m) - \varphi_k(m)\|\,\mathrm{d}\mu(m) < \varepsilon\,. \tag{1.1}$$

In this case we define the *integral of* f by

$$\int_M f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_M \varphi_n\,\mathrm{d}\mu \in V\,. \tag{1.2}$$

Furthermore, we define for $A \in \mathcal{A}$

$$\int_A f\,\mathrm{d}\mu = \int_M \mathbb{1}_A f\,\mathrm{d}\mu\,.$$

A few comments are necessary in order to explain that this definition is meaningful. Firstly, the real-valued functions $m \mapsto \|\varphi_n(m)\|$ are measurable since φ_n is strongly measurable. In particular, the integral in (1.1) is a Lebesgue integral of real-valued functions $(M, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Secondly, it is easy to see that

$$\left\| \int_M \varphi(m) \, \mathrm{d}\mu(m) \right\| \leq \int_M \|\varphi(m)\| \, \mathrm{d}\mu(m) \quad \text{for every simple function } \varphi.$$

By this inequality we may deduce from (1.1) that the sequence

$$\left(\int_M \varphi_n \, \mathrm{d}\mu \colon n \in \mathbb{N} \right) \quad \text{is a Cauchy sequence in the Banach space } V,$$

and thus the limit on right-hand side of (1.2) exists. Thirdly, an elementary proof shows that the definition of the integral of f is independent of the choice of the approximating sequence. For details we refer to [AE09, Section X.2].

As can be seen from this definition and Corollary 1.4, the concept of integration of vector valued functions extends the Lebesgue integration theory in a consistent way, i.e. the two concepts coincide if $V = \mathbb{R}$.

Properties of the integral

Let us give a short summary of results on the theory of Bochner integrable functions.

The Bochner integral is linear, i.e. for all $\alpha, \beta \in \mathbb{R}$ and integrable $f, g \colon M \to V$

$$\int \alpha f + \beta g \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu.$$

A strongly measurable function $f \colon M \to V$ is Bochner integrable if and only if the real-valued function $m \mapsto \|f(m)\|$ is Lebesgue integrable. This fact is often referred to as *Bochner's theorem*. Moreover, for every $A \in \mathcal{A}$ and every integrable f we have

$$\left\| \int_A f \, \mathrm{d}\mu \right\| \leq \int_A \|f\| \, \mathrm{d}\mu \,. \tag{1.3}$$

The dominated convergence theorem is one of the most important properties of the Bochner-Lebesgue integral. A proof can be found in [AE09, Chapter X].

Theorem 1.6 (Dominated convergence)**.** *Let (f_n) be a sequence of measurable functions $f_n \colon M \to V$ such that $\lim_{n\to\infty} f_n = f$ a.e. for a function $f \colon M \to V$. Furthermore assume that there is a function $g \colon M \to \mathbb{R}$ such that $\|f_n\| \leq g$ a.e. for all $n \in \mathbb{N}$ and $\int |g| \, \mathrm{d}\mu < \infty$. Then*

$$\lim_{n\to\infty} \int_M \|f - f_n\| \, \mathrm{d}\mu = 0$$

and in particular

$$\lim_{n\to\infty} \int_M f_n \, \mathrm{d}\mu = \int_M \lim_{n\to\infty} f_n \, \mathrm{d}\mu < \infty.$$

We give another two convergence results for the special case of real-valued functions. For proofs we refer to [AE09, Chapter X].

Theorem 1.7 (Monotone convergence). *Assume that (f_n) is a sequence of measurable functions $f_n\colon M \to [0,\infty)$ such that $f_n \le f_{n+1}$ a.e. on M for every $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \int_M f_n \, \mathrm{d}\mu = \int_M \lim_{n\to\infty} f_n \, \mathrm{d}\mu \,. \tag{1.4}$$

Theorem 1.8 (Fatou's lemma). *Assume that (f_n) is a sequence of measurable functions $f_n\colon M \to [0,\infty)$. Then*

$$\int_M \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int_M f_n \, \mathrm{d}\mu \,. \tag{1.5}$$

Note that we interpret (1.4) and (1.5) as (in)equalities in $[0,\infty]$, i.e. we allow both sides to be equal to $+\infty$.

1.3 Spaces of integrable functions

As in the integration theory of real-valued functions we call two functions $f, g\colon M \to V$ *equivalent* if $f = g$ μ-a.e. on M. In what follows we will consider spaces that contain equivalence classes $[f]$ of integrable functions. We will denote these equivalence classes again by f without mentioning it explicitly.

From now on we will assume that the underlying measure space is an open subset Ω of $\mathbb{R}^d$ endowed with its natural σ-field and the restriction of the Lebesgue measure λ^d to Ω. Recall that $\|\cdot\|$ stands for the norm of the Banach space V.

Definition 1.9. For $p \in [1,\infty]$ we define the linear space

$$L^p(\Omega;V) = \Big\{ f\colon \Omega \to V\colon f \text{ is strongly measurable and } \|f\|_{L^p(\Omega;V)} < \infty \Big\}\,, \tag{1.6a}$$

where

$$\begin{aligned} \|f\|_{L^p(\Omega;V)} &= \left(\int_\Omega \|f(x)\|^p \, \mathrm{d}x \right)^{1/p} \qquad \text{if } 1 \le p < \infty\,, \\ \|f\|_{L^\infty(\Omega;V)} &= \operatorname*{ess\text{-}sup}_{x\in\Omega} \|f(x)\| = \inf\Big\{ \alpha \ge 0\colon \lambda^d(\{\|f\| > \alpha\}) = 0 \Big\}\,. \end{aligned} \tag{1.6b}$$

A function $f \in L^\infty(\Omega;V)$ is called *essentially bounded*. In the case $V = \mathbb{R}$ we simply write $L^p(\Omega) = L^p(\Omega;\mathbb{R})$. If $\Omega = (a,b) \subset \mathbb{R}$ we write $L^p(a,b;V) = L^p((a,b);V)$.

The following proposition collects some properties of L^p-spaces. Proofs of these assertions can be found in [AE09, Section X.4], [Emm04, Satz 7.1.23], [GGZ74, §IV.1.3] and [Zei90, Chapter 23].

Proposition 1.10.

(i) *For $1 \le p \le \infty$ the linear space $L^p(\Omega; V)$ endowed with the norm given by* (1.6b) *is a Banach space.*

(ii) *The set of simple functions is dense in $L^p(\Omega; V)$, $1 \le p < \infty$.*

(iii) *If V is a separable Banach space then so is $L^p(\Omega; V)$ for $1 \le p < \infty$.*

(iv) *If H is a Hilbert space then so is $L^2(\Omega; H)$, where the scalar product is given by*

$$(f, g)_{L^2(\Omega;H)} = \int_\Omega (f(x), g(x))_H \,\mathrm{d}x\,. \tag{1.7}$$

(v) *Let $f \in L^p(\Omega; V)$ and $g \in L^q(\Omega; V^*)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. Then* Hölder's inequality *holds:*

$$\int_\Omega \langle g(x), f(x)\rangle_V \,\mathrm{d}x \le \|g\|_{L^q(\Omega;V^*)} \|f\|_{L^p(\Omega;V)}\,. \tag{1.8}$$

(vi) *If V is reflexive and separable then $L^p(\Omega; V)$ is reflexive for $1 < p < \infty$. In this case there is an isometric isomorphism $(L^p(\Omega; V))^* \cong L^q(\Omega; V^*)$ where q is the dual exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The dual pairing is then given by*

$$\langle F, f\rangle_{L^q(\Omega;V^*)} = \int_\Omega \langle F(x), f(x)\rangle_V \,\mathrm{d}x \qquad \text{for } f \in L^p(\Omega; V)\,. \tag{1.9}$$

(vii) *Let W be another Banach space such that $V \hookrightarrow W$ and assume that Ω is bounded. Then for $1 \le p \le r \le \infty$ one has $L^r(\Omega; V) \hookrightarrow L^p(\Omega; W)$.*

Assertion (vi) remains true if we only assume that V is reflexive or V^* separable. A proof of this can be found in [Edw65]. We will not make use of this.

Another property of the spaces $L^p(\Omega; V)$ will turn out to be useful in the next section when dealing with averages of abstract functions. The proof of this property in the case of vector-valued functions defined on $\mathbb{R}^d$ is not very common in the literature. Therefore we provide an elementary proof.

Theorem 1.11 (Continuity of translations in L^p)**.** *Let $f\colon \mathbb{R}^d \to V$ and $1 \le p \le \infty$. For $h \in \mathbb{R}^d$ define $T_h f(x) = f(x + h)$. If $f \in L^p(\mathbb{R}^d; V)$ then also $T_h f \in L^p(\mathbb{R}^d; V)$. Moreover, if $p < \infty$, we have*

$$\lim_{h\to 0} \|T_h f - f\|_{L^p(\mathbb{R}^d;V)} = 0\,. \tag{1.10}$$

Proof. If the sequence (φ_n) of simple functions approximates f a.e. on $\mathbb{R}^d$ in the norm of V, then the sequence $(T_h\varphi_n)$, whose members are still simple functions, approximates $T_h f$. This shows that $T_h f$ is strongly measurable. Furthermore, the translation invariance of the Lebesgue measure and the definition of ess-sup, respectively, show that $T_h f \in L^p(\Omega; V)$ for $p < \infty$ and $p = \infty$. In particular, $\|f\|_{L^p(\Omega;V)} = \|T_h f\|_{L^p(\Omega;V)}$ for all $h \in \mathbb{R}^d$.

Now assume that $f = v\,\mathbb{1}_U$ for some measurable set $U \subset \mathbb{R}^d$ and some $v \in V$. Then

$$\|T_h f - f\|^p_{L^p(\mathbb{R}^d;V)} = \|v\|^p \int_{\mathbb{R}^d} [\mathbb{1}_U(x+h) - \mathbb{1}_U(x)]\,\mathrm{d}x$$
$$= \lambda^d((U \cup U+h) \setminus (U \cap U+h)) \xrightarrow{h\to 0} 0.$$

Property (1.10) is then readily checked if f is assumed to be a simple function. Now we prove the assertion for general $f \in L^p(\Omega;V)$. Let $\varepsilon > 0$. By Proposition 1.10(ii) there is a simple function φ such that $\|\varphi - f\|_{L^p(\Omega;V)} < \varepsilon$. For this simple function we find $\delta > 0$ such that for all $h \in B_\delta(0)$ we have $\|T_h\varphi - \varphi\|_{L^p(\Omega;V)} < \varepsilon$. Then for all $h \in B_\delta(x)$ (setting $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega;V)}$)

$$\|T_h f - f\|_p = \|T_h(f-\varphi) + T_h\varphi - \varphi + \varphi - f\|_p \le \|f-\varphi\|_p + \|T_h - \varphi\|_p + \|\varphi - f\|_p$$
$$< 3\varepsilon.$$

This finishes the proof. □

The solution of a parabolic equation is a function $u(t,x)$, where we interpret the variable t as time variable and the variable x as space variable. Generally, we seek for a solution u that is defined for t in a finite time interval, for instance $(0,T)$, and for x belonging to some domain $\Omega \subset \mathbb{R}^d$. In the functional analytic treatment of parabolic problems it is a common strategy to treat the differentiation with respect to t in a different way than the derivatives in space. In other words, one associates to each function $u\colon (0,T)\times\Omega \to \mathbb{R}$ an abstract function

$$U\colon (0,T) \to V_\Omega, \quad U(t) = u(t,\cdot).$$

In this situation V_Ω is a space of functions operating on Ω. The following result answers the question if one can identify the function u belonging to some space of (Lebesgue) integrable functions operating on $(0,T)\times\Omega$ with the space of (Bochner) integrable abstract functions U operating on $(0,T)$ with values in V_Ω. This result as well as its proof can be found in [Emm04, Section 7.1], [Růž04, Section 2.1.1] and [Zei90, Example 23.4].

Proposition 1.12. *For $1 \le p < \infty$ the mapping $u \mapsto U$ is an isometric isomorphism between $L^p((0,T)\times\Omega)$ and $L^p(0,T;L^p(\Omega))$.*

This proposition justifies that we shall denote the abstract function U again by u. Its proof uses the density of step functions in L^p. This density argument is not true if $p = \infty$. Indeed (cf. [Emm04, Satz 7.1.26]):

Lemma 1.13. *One has $L^\infty(0,T;L^\infty(\Omega)) \subsetneq L^\infty((0,T)\times\Omega)$.*

The example [Emm04, Beispiel 7.1.27] considers a function $f\colon (0,1)\times(0,1) \to \mathbb{R}$ defined by $f(x,y) = 1$ if $x \ge y$ and $f(x,y) = 0$ if $x < y$. Clearly, this function belongs to $L^\infty((0,1)\times(0,1))$. However, the strong measurability fails since for given $x_0 \in (0,1)$ one cannot approximate in the norm of $L^\infty(0,1)$ the function $f(x_0,\cdot)$ with a sequence of step functions. Details can be found in the mentioned reference.

1.4 Steklov averages

Steklov averages are needed to mollify (vector-valued) functions u defined on a time interval. In this section we give some technical results on these averaged functions in a general framework. Some of these results are stated in [LSU68, II.§4] and [DGV11, Section 2.5.3]. The application of Steklov averages to parabolic equations is explained in full detail in Section 5.2.

Let $I = (T_1, T_2)$, $Q = I \times \Omega$. For $v \in L^1(Q)$ and $0 < h < T_2 - T_1$ define

$$v_h(t,\cdot) = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} v(\cdot,s)\,\mathrm{d}s & \text{for } T_1 < t < T_2 - h, \\ 0, & \text{for } t \geq T_2 - h. \end{cases}$$

Lemma 1.14. *Let V be a Banach space and let $v \in L^p(I;V)$ for some $p \in [1,\infty]$ and $I' = (t_1,t_2) \subset I$ with $t_2 < T_2$. Then*

(i) $v_h \in C(\overline{I'};V)$ *for every* $h \in (0, T_2 - t_2)$.

(ii) $\|v_h\|_{L^p(I';V)} \leq \|v\|_{L^p(I';V)}$ *for every* $h \in (0, T_2 - t_2)$.

(iii) If $p < \infty$ *then* $\|v_h - v\|_{L^p(I';V)} \to 0$ *for* $h \to 0+$.

(iv) If $v \in C(I; L^p(\Omega))$ *for some* $p < \infty$, *then* $\|v_h(t,\cdot) - v(t,\cdot)\|_{L^p(\Omega)} \xrightarrow{h\to 0+} 0$ *for every* $t \in I'$.

Proof. *ad (i):* For every $t \in \overline{I'}$ and $h \in (0, T_2 - t_2)$ we have

$$\begin{aligned}\lim_{\Delta t\to 0} \|v_h(t+\Delta t) - v_h(t)\| &\leq \lim_{\Delta t\to 0} \int_{I'} \|v(s)\| \left|\mathbb{1}_{[t+\Delta t, t+\Delta t+h]}(s) - \mathbb{1}_{[t,t+h]}(s)\right| \mathrm{d}s \\ &= \int_{I'} \|v(s)\| \lim_{\Delta t\to 0} \left|\mathbb{1}_{[t+\Delta t, t+\Delta t+h]}(s) - \mathbb{1}_{[t,t+h]}(s)\right| \mathrm{d}s = 0\ .\end{aligned}$$

We can interchange limit and integration due to dominated convergence theorem (Theorem 1.6).

ad (ii): By Jensen's inequality, inequality (1.3), and Fubini's theorem we obtain

$$\begin{aligned}\|v_h\|^p_{L^p(I';V)} = \int_{I'} \|v_h(t)\|^p \,\mathrm{d}t &\leq \frac{1}{h}\int_{I'}\int_t^{t+h} \|v(s)\|^p \,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{I'} \|v(s)\|^p \frac{1}{h}\int_{I'} \mathbb{1}_{(t,t+h)}(s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{I'} \|v(s)\|^p \frac{1}{h}\int_{I'} \mathbb{1}_{(s-h,s)}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \|v\|^p_{L^p(I';V)}\ .\end{aligned}$$

This proves (ii) in the case $p < \infty$. In the case $p = \infty$ we use again (1.3) to deduce that for almost every $t \in I'$

$$\|v(t)\| \le \frac{1}{h} \int_t^{t+h} \|v(s)\| \, \mathrm{d}s \le \|v\|_{L^\infty(I';V)} \, .$$

This proves (ii) for $p = \infty$.

ad (iii): Observe that for a.e. $t \in I'$

$$\begin{aligned} \|v_h(t) - v(t)\|^p &= \left\| \frac{1}{h} \int_t^{t+h} v(s) - v(t) \, \mathrm{d}s \right\|^p = \left\| \int_0^1 v(t+sh) - v(t) \, \mathrm{d}s \right\|^p \\ &\le \int_0^1 \|v(t+sh) - v(t)\|^p \, \mathrm{d}s \, . \end{aligned} \tag{1.11}$$

Hence,

$$\begin{aligned} \|v_h - v\|^p_{L^p(I';V)} &= \int_{I'} \|v_h(t) - v(t)\|^p \, \mathrm{d}t \le \int_{I'} \int_0^1 \|v(t+sh) - v(t)\|^p \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_0^1 \int_{I'} \|v(t+sh) - v(t)\|^p \, \mathrm{d}t \, \mathrm{d}s \le \sup_{0<s<h} \|v(\cdot + s) - v(\cdot)\|^p_{L^p(I';V)} \, . \end{aligned}$$

The right-hand side tends to zero for $h \to 0$ due to uniform continuity of translations in $L^p(I';V)$, $1 \le p < \infty$ (Theorem 1.11). This proves assertion (iii).

ad (iv): Let $t \in I'$ and $\varepsilon > 0$. By assumption we find $\delta > 0$ such that for all $h \in (0, \delta)$ we have $\|v(t+h) - v(t)\|_{L^p(\Omega)} < \varepsilon$. By (1.11)

$$\|v_h(t) - v(t)\|^p_{L^p(\Omega)} \le \int_0^1 \|v(t+sh) - v(t)\|^p_{L^p(\Omega)} \, \mathrm{d}t < \varepsilon^p.$$

This shows assertion (iv). □

2 Distributions & Sobolev spaces

Sobolev spaces arise in a natural way as spaces of weak solutions to partial differential equations. Generally speaking, these spaces consist of functions that belong together with all generalized partial derivatives up to a certain order to some L^p-space. In this chapter we define generalized derivatives in the context of distributions. We shall define both integer order and fractional order Sobolev spaces as spaces of functions with a domain being an arbitrary open set $\Omega \subset \mathbb{R}^d$. In Section 2.6 we shall present – in the case $\Omega = \mathbb{R}^d$ – some results that connect the Fourier transform to Sobolev spaces. Finally, we introduce the fractional Laplacian as a prototype of a nonlocal operator that is closely related to fractional Sobolev spaces.

Throughout the whole chapter we denote by Ω a – bounded or unbounded – open set in $\mathbb{R}^d$.

2.1 The spaces $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ and generalized derivatives

In this section fundamental results and definitions concerning distributions are provided. In this presentation we concentrate on the main aspects of distributions that are necessary to study partial differential equations and Sobolev spaces.

The way of presentation in this section is largely influenced by [AF03, HT08]. Detailed references for the results and definitions will be given below.

Unless otherwise stated we consider functions with values in the complex plane $\mathbb{C}$ and by a linear space we mean a complex vector space. We recall that a measurable function $f\colon \Omega \to \mathbb{C}$ with $f = u + iv$, $u, v\colon \Omega \to \mathbb{R}$, belongs to $L^p(\Omega;\mathbb{C})$ if $|f| = (u^2 + v^2)^{1/2}$ belongs to $L^p(\Omega;\mathbb{R})$. For $f \in L^1(\Omega;\mathbb{C})$ the integral of f is defined by

$$\int f(x)\,\mathrm{d}x = \int u(x)\,\mathrm{d}x + i\int v(x)\,\mathrm{d}x\,.$$

In the following definition we follow the lines of [HT08, Section 2.1]:

Definition 2.1. Let $f \in C_{loc}(\Omega)$.

(i) *The support of* f is the set

$$\mathrm{supp}[f] = \overline{\{x \in \Omega\colon f(x) \neq 0\}}, \tag{2.1}$$

where the closure is taken with respect to any norm in $\mathbb{R}^d$. Note that supp$[f]$ may contain points $x \in \Omega^c$. The reason why this definition is given only for $f \in C_{loc}(\Omega)$ is explained by Example 2.5.

(ii) f is said *to have compact support (in Ω)* if $\operatorname{supp}[f]$ is bounded and $\operatorname{supp}[f] \subset \Omega$.

(iii) The linear space $C_c^\infty(\Omega)$ is defined by

$$C_c^\infty(\Omega) = \{\phi \in C^\infty(\Omega) \colon \phi \text{ has compact support in } \Omega\}.$$

In literature, distributions are also referred to as *generalized functions* in the sense that they are considered as mappings $T \colon C_c^\infty(\Omega) \to \mathbb{R}$. But for the notion of continuity (i.e. a notion of T being a functional) it is necessary to endow this vector space with a certain topology. This topology, which we denote by $\mathcal{T}$ from now on, has to be chosen in a reasonable way in the sense that at least the following requirements are satisfied:

a) $\mathcal{T}$ should be a Hausdorff topology on $C_c^\infty(\Omega)$ and addition and scalar multiplication should be continuous operations. A vector space that is endowed with a topology and shares these two properties is called a *topological vector space*, cf. [Rud91, Section 1.6].

b) The topological vector space $(C_c(\Omega), \mathcal{T})$ and especially its dual space should allow for a functional analytic treatment parallel to the one of normed spaces. This is to a certain extent possible for *locally convex* topological vector spaces. A topological vector space is called *locally convex* if there is a local base[1] whose members are convex, cf. [Rud91, Section 1.8].

c) The topology $\mathcal{T}$ should be compatible with a notion of convergence of elements $\phi_n \in C_c^\infty(\Omega)$. In particular, a mapping $T \colon (C_c^\infty(\Omega), \mathcal{T}) \to \mathbb{R}$ should be continuous if and only if T is sequentially continuous – with respect to the before mentioned notion of convergence in $(C_c^\infty(\Omega), \mathcal{T})$.

We collect the following assertions from [Rud91, Theorem 6.4(b)], [Rud91, Theorem 6.5(f)] and [Rud91, Theorem 6.6], where also the proofs can be found.

Proposition 2.2. *There is a topology $\mathcal{T}$ on $C_c^\infty(\Omega)$ such that*

(i) $(C_c^\infty(\Omega), \mathcal{T})$ is a locally convex topological vector space,

(ii) $\phi_n \to 0$ in $(C_c^\infty(\Omega), \mathcal{T})$ if and only if there is a compact $K \subset \Omega$ such that $\operatorname{supp}[\phi_n] \subset K$ for every $n \in \mathbb{N}$ and $\partial^\alpha \phi_n \to 0$ uniformly as $n \to \infty$ for every multi-index $\alpha \in \mathbb{N}_0^d$,

(iii) a linear mapping $T \colon (C_c^\infty(\Omega), \mathcal{T}) \to Y$, where Y is a normed space, is continuous if and only if $\|T(\phi_n)\|_Y \to 0$, $n \to \infty$, for every sequence (ϕ_n) with $\phi_n \to 0$ in $(C_c^\infty(\Omega), \mathcal{T})$.

From now on we fix this topology[2] $\mathcal{T}$. We make the following definitions:

---

[1] Let (X, τ) be a topological space. A family $U(x, \tau)$ of neighborhoods of a point $x \in X$ is called a *local base* if every neighborhood of x contains a member of $U(x, \tau)$, [Rud91, p. 7].

[2] It can be shown (see [Rud91, Remark 6.9] or [Leo09, Exercise 9.9]) that the topology $\mathcal{T}$ of $\mathcal{D}(\Omega)$ is not metrizable.

Definition 2.3.

(i) The topological vector space $(C_c^\infty(\Omega), \mathcal{T})$ is denoted by $\mathcal{D}(\Omega)$. $\phi \in \mathcal{D}(\Omega)$ is called a *test function.*

(ii) *A distribution* or *generalized function* is a continuous linear functional $T\colon \mathcal{D}(\Omega) \to \mathbb{R}$. The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.

The terminology "generalized functions" is justified by the following observation: Every $f \in L^1_{loc}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ via

$$T_f(\phi) = \int_\Omega f(x)\phi(x)\,\mathrm{d}x \ . \tag{2.2}$$

Indeed, T_f is obviously linear and continuous due to the characterization of continuity in Proposition 2.2 (ii),(iii).

It is also clear that not every distribution can be represented as in (2.2). For example, for $x \in \Omega$, $\phi \mapsto \phi(x)$ defines a distribution in $\mathcal{D}'(\Omega)$. It is obvious that there is no function f allowing for a representation (2.2). The following definition singles out the distributions having such a representation:

(iii) A distribution $T \in \mathcal{D}'(\Omega)$ is called *regular* if there is $f \in L^1_{loc}(\Omega)$ such that $T = T_f$, where T_f is given by (2.2).

(iv) The *restriction* $T|_{\Omega_0} \in \mathcal{D}'(\Omega_0)$ of a distribution $T \in \mathcal{D}'(\Omega)$ to an open subset $\Omega_0 \subset \Omega$ is defined by

$$T|_{\Omega_0}(\phi) = T(\phi) \qquad \text{for } \phi \in \mathcal{D}(\Omega_0).$$

(v) The *support of a distribution* $T \in \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{supp} T = \big\{x \in \Omega\colon T|_{\Omega \cap B_\varepsilon(x)} \neq 0 \text{ for any } \varepsilon > 0\big\} \,. \tag{2.3}$$

We cite the following proposition from [HT08, Proposition 2.7], which is often called *fundamental lemma of calculus of variations.* For a proof we refer to the mentioned reference.

Proposition 2.4. *Let $f \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega f(x)\phi(x)\,\mathrm{d}x = 0 \qquad \textit{for all } \phi \in \mathcal{D}(\Omega),$$

then $f = 0$ almost everywhere on Ω.

One consequence of this result is that for $f \in C_{loc}(\Omega)$ the support of f as in (2.1) and the support of f interpreted as a distribution $T_f \in \mathcal{D}'(\Omega)$ in (2.3) coincide. The following example shows that this is not true if f is not continuous:

Example 2.5 ([HT08, Remark 2.23])**.** This example shows that the generalization of (2.1) to functions $f \in L^1_{loc}(\Omega)$ would have the consequence that the support of f – interpreted as a function – would differ from the support of f interpreted as a regular distribution:

Let f be the Dirichlet function, i.e. the function $f\colon \mathbb{R} \to [0,1]$ defined by $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$. Clearly, $f = 0$ in $L^1(\mathbb{R})$ and hence $\operatorname{supp} T_f = \emptyset$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have

$$\overline{\{x \in \mathbb{R}\colon f(x) \neq 0\}} = \mathbb{R}.$$

♦

Generalized derivatives

Definition 2.6 (Derivative of a distribution)**.** Let $\alpha \in \mathbb{N}_0^d$ and $T \in \mathcal{D}'(\Omega)$. The *derivative* $\partial^\alpha T$ is defined by

$$(\partial^\alpha T)(\phi) = (-1)^{|\alpha|}\ T(\partial^\alpha \phi) \qquad \text{for } \phi \in \mathcal{D}(\Omega).$$

Since $\phi_n \to 0$ in $\mathcal{D}(\Omega)$ implies $\partial^\alpha \phi \to 0$ uniformly (cf. Proposition 2.2(ii)), we obtain that $|\partial^\alpha T(\phi_n)| \to 0$, i.e. $\partial^\alpha T \in \mathcal{D}'(\Omega)$ for every $\alpha \in \mathbb{N}_0^d$.

As a consequence of this definition, every function $f \in L^1_{loc}(\Omega)$ possesses a distributional derivative of every order, simply by interpreting f as a distribution $T_f \in \mathcal{D}'(\Omega)$. The following definition singles out the cases where the distributional derivative of f is a regular distribution:

Definition 2.7 (Generalized derivatives)**.** Let $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. If $\partial^\alpha T_f$ is a regular distribution, i.e. if there is a function $g \in L^1_{loc}(\Omega)$ such that

$$\partial^\alpha T_f = T_g, \tag{2.4}$$

then g is called *the generalized derivative (corresponding to $\alpha \in \mathbb{N}_0^d$).* If such a function g exists, we write $\partial^\alpha f = g$.

Remark 2.8.

1. Note that (2.4) means

$$\int_\Omega f(x)\ \partial^\alpha \phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g(x)\ \phi(x)\,\mathrm{d}x \qquad \text{for all } \phi \in \mathcal{D}(\Omega). \tag{2.5}$$

   Therefore, Proposition 2.4 implies that if there are g_1, g_2 with $\partial^\alpha T_f = T_{g_1} = T_{g_2}$, then $g_1 = g_2$ almost everywhere on Ω.

2. (2.5) is an integration by parts formula, which holds for sufficiently smooth functions f and g. In particular, we can deduce that the generalized derivative and the classical derivative coincide – provided both exist. In other words, the concept of generalized derivatives extends the concept of classical derivatives in a consistent way.

3. The two foregoing remarks justify the notation $g = \partial^\alpha f$, which we use both for the classical and the generalized derivative of f.

4. From (2.5) we can also deduce that piecewise continuous derivatives of a function $f\colon [a,b] \to \mathbb{R}$ are generalized derivatives, see [Zei90, p. 232], where also a proof can be found.

5. Clearly, the notion of generalized derivatives of $f \in L^p(\Omega)$ is well-defined since $L^p(\Omega) \subset L^1_{loc}(\Omega)$ for every (bounded or unbounded) open set$\Omega \subset \mathbb{R}^d$. ♦

The following short example ([GS64, Section 2.2, Ex. 3]) illustrates the relation between generalized derivatives and distributional derivatives.

Example 2.9. Let $\Omega = (0,\infty)$ and $\lambda \in (0,1)$. The function $f(t) = t^{-\lambda}$ clearly belongs to $L^1_{loc}(\Omega)$ and thus defines a distribution $T_f \in \mathcal{D}'(\Omega)$ via (2.2). Its classical derivative $-\lambda t^{-\lambda-1}$ does not belong to $L^1_{loc}(\Omega)$. However, we can compute the distributional derivative T'_f of f: Let $\phi \in \mathcal{D}(\Omega)$. By (2.4) and partial integration

$$T'_f(\phi) = -\int_0^\infty t^{-\lambda}\phi'(t)\,\mathrm{d}t = \lim_{r\to 0}\Big[-t^{-\lambda}\left(\phi(t)-\phi(0)\right)\Big]_{t=r}^{\infty} + \lambda\int_0^\infty t^{-\lambda-1}\left(\phi(t)-\phi(0)\right)\,\mathrm{d}t.$$

The limit on the right-hand side is equal to zero and the integral on the right-hand side exists since $\phi(t)-\phi(0) = o(t)$ for $t \to 0+$. Hence, T'_f is given by

$$T'_f(\phi) = \int_\Omega \lambda t^{-\lambda-1}\left(\phi(t)-\phi(0)\right)\,\mathrm{d}t\ . \qquad \blacklozenge$$

2.2 The spaces $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$ and the Fourier transform

Unless otherwise stated we consider functions with values in the complex plane $\mathbb{C}$ and by a linear space we mean a complex vector space.

In this section we follow [HT08, Section 2.5].

In Section 2.1 we have introduced – in the special case $\Omega = \mathbb{R}^d$ – the space $\mathcal{D}(\mathbb{R}^d)$ and its topological dual $\mathcal{D}'(\mathbb{R}^d)$. As it was elaborated in the previous section these spaces are perfectly suited for the theory of generalized derivatives. However, the space $\mathcal{D}(\mathbb{R}^d)$ is too small to develop the theory of Fourier transform. It is easy to see that the Fourier transform is well-defined for functions $u \in \mathcal{D}(\mathbb{R}^d)$, but the transformed function in general does not belong to $\mathcal{D}(\mathbb{R}^d)$, in other words the space $\mathcal{D}'(\mathbb{R}^d)$ is too large. As will turn out, the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ are optimal to overcome this problem.

Definition 2.10. Set

$$\mathcal{S}(\mathbb{R}^d) = \left\{\phi \in C^\infty(\mathbb{R}^d)\colon\ \|\phi\|_{k,l} < \infty \text{ for all } k,l \in \mathbb{N}_0\right\},$$

where

$$\|\phi\|_{k,l} = \sup_{x\in\mathbb{R}^d}\left(1+|x|^2\right)^{k/2}\sum_{|\alpha|\le l}|\partial^\alpha\phi(x)|\,. \tag{2.6}$$

The space $\mathcal{S}(\mathbb{R}^d)$ is called the *Schwartz space* and a function $\phi \in \mathcal{S}(\mathbb{R}^d)$ is called a *Schwartz function* or *rapidly decreasing* function.

Remark 2.11.

a) From this definition it is clear that

$$\partial^\alpha\phi \in \mathcal{S}(\mathbb{R}^d) \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d) \text{ and } \alpha \in \mathbb{N}_0^d. \tag{2.7}$$

b) A local base in $\mathcal{S}(\mathbb{R}^d)$ can be defined by means of the seminorm in (2.6) and therefore $\mathcal{S}(\mathbb{R}^d)$ is a locally convex topological vector space[3] with the topology induced by the family of seminorms in (2.6), cf. [Rud91, Theorem 1.37]. It this sense the task of finding a suitable topology on the space of rapidly decreasing functions is easier than in the situation of compactly supported functions. In particular, it can be shown that the topology on $\mathcal{S}(\mathbb{R}^d)$ is metrizable, cf. [HT08, Exercise 2.35, Note 2.9.3].

In particular, a sequence (ϕ_n) in $\mathcal{S}(\mathbb{R}^d)$ is convergent to ϕ in $\mathcal{S}(\mathbb{R}^d)$ if and only if

$$\|\phi_n - \phi\|_{k,l} \xrightarrow{n\to\infty} 0 \qquad \text{for all } k, l \in \mathbb{N}_0. \tag{2.8}$$

c) It is clear that $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ and $\phi_n \to \phi$ in $\mathcal{D}(\mathbb{R}^d)$ implies $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^d)$.

Of course there are rapidly decreasing functions which are not compactly supported, for example $\phi(x) = e^{-|x|^2}$. ♦

In the same way as we introduced $\mathcal{D}'(\mathbb{R}^d)$ we define now $\mathcal{S}'(\mathbb{R}^d)$:

Definition 2.12. Set

$$\mathcal{S}'(\mathbb{R}^d) = \Big\{ T\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}\colon T \text{ is linear and continuous} \Big\}.$$

The elements of $\mathcal{S}'(\mathbb{R}^d)$ are called *tempered distributions*.

Remember that by (2.8) T is continuous if and only if for all sequences (ϕ_n) in $\mathcal{S}(\mathbb{R}^d)$

$$\phi_n \to \phi \text{ in } \mathcal{S}(\mathbb{R}^d) \quad \Rightarrow \quad |T(\phi_n) - T(\phi)| \to 0.$$

Fourier transform

Definition 2.13. For $u \in \mathcal{S}(\mathbb{R}^d)$ the *Fourier transform* $\widehat{u}$ of u is defined by

$$\widehat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, u(x)\, \mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

The *inverse Fourier transform* $\check{u}$ of $u \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$(\check{u})(x) = \widehat{u}(-x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} u(\xi)\, \mathrm{d}\xi, \qquad x \in \mathbb{R}^d.$$

It is well known (see [Gra08, Proposition 2.2.11, Theorem 2.2.14]) that for all $u \in \mathcal{S}(\mathbb{R}^d)$

$$\widehat{u}, \check{u} \in \mathcal{S}(\mathbb{R}^d), \quad \check{\widehat{u}} = u = \widehat{\check{u}}, \quad \text{and } \|u\|_{L^2(\mathbb{R}^d)} = \|\widehat{u}\|_{L^2(\mathbb{R}^d)} = \|\check{u}\|_{L^2(\mathbb{R}^d)}. \tag{2.9}$$

The last identity is known as *Plancherel's identity*.

[3] cf. p. 14

In the previous section we observed that every function $f \in L^1_{loc}(\Omega)$ defines $T_f \in \mathcal{D}'(\Omega)$ via (2.2). A similar statement holds in the context of tempered distibutions (see [HT08, Corollary 2.50]): If $u \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$ then $T_u \in \mathcal{S}'(\mathbb{R}^d)$, where T_u is defined by

$$T_u(\phi) = \int_{\mathbb{R}^d} u(x)\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{S}(\mathbb{R}^d)\,. \tag{2.10}$$

The Fourier transforms $\widehat{\ }$ and $\check{\ }$ on $\mathcal{S}(\mathbb{R}^d)$ can be extended to operators $\mathscr{F}$ and $\mathscr{F}^{-1}$ on the space of tempered distributions:

Definition 2.14. Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Define $\mathscr{F}T$ and $\mathscr{F}^{-1}T$ by

$$(\mathscr{F}T)(u) = T(\widehat{u}) \quad \text{and} \quad (\mathscr{F}^{-1}T)(u) = T(\check{u}), \qquad u \in \mathcal{S}(\mathbb{R}^d).$$

Note that this definition makes sense due to the first property in (2.9). Moreover,

$$\mathscr{F}T, \mathscr{F}^{-1}T \in \mathcal{S}'(\mathbb{R}^d), \qquad \text{for } T \in \mathcal{S}'(\mathbb{R}^d).$$

It is easy to see that $\mathscr{F}(T_u) = T_{\widehat{u}}$ and $\mathscr{F}^{-1}(T_u) = T_{\check{u}}$ for $u \in \mathcal{S}(\mathbb{R}^d)$, thus $\mathscr{F}$ and $\mathscr{F}^{-1}$ are indeed extensions from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ consistent with the (inverse) Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ as in Definition 2.13. The operators are bijective mappings from $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$ and satisfy

$$\mathscr{F}\mathscr{F}^{-1}T = T = \mathscr{F}^{-1}\mathscr{F}T, \qquad T \in \mathcal{S}'(\mathbb{R}^d),$$

see [HT08, Section 2.7].

For the following result, which we cite from [HT08, Section 2.8], we interpret the spaces $L^p(\mathbb{R}^d)$ as subspaces of $\mathcal{S}'(\mathbb{R}^d)$ in the sense of (2.10) and do not distinguish between the function $u \in L^p(\mathbb{R}^d)$ and its distribution $T_u \in \mathcal{S}'(\mathbb{R}^d)$.

Theorem 2.15 (Fourier transform in $L^p(\mathbb{R}^d)$, Plancherel's theorem)**.** *If $u \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$, then the tempered distribution $\mathscr{F}u$ is regular. Furthermore, the restrictions of $\mathscr{F}$ and $\mathscr{F}^{-1}$, respectively, to $L^2(\mathbb{R}^d)$ generate unitary operators in $L^2(\mathbb{R}^d)$, in particular*

$$\|\mathscr{F}f\|_{L^2(\mathbb{R}^d)} = \left\|\mathscr{F}^{-1}f\right\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}\,, \qquad f \in L^2(\mathbb{R}^d). \tag{2.11}$$

Moreover, if $u \in L^1(\mathbb{R}^d)$ then

$$\mathscr{F}u(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

2.3 Sobolev spaces of integer order

From now on we consider functions with values in $\mathbb{R}$ and by Banach spaces (or linear spaces) we mean linear spaces over $\mathbb{R}$.

In the following definition we follow the lines of [Eva10, Section 5.2.2].

Definition 2.16 (Sobolev space)**.** Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$. The *Sobolev space* $W^{k,p}(\Omega)$ consists of all functions $f \in L^p(\Omega)$ such that for each multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ the generalized derivative $\partial^\alpha f$ exists and belongs to $L^p(\Omega)$. In compact form

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega) \colon \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k \right\}.$$

Furthermore, for $f \in W^{1,p}(\Omega)$ set

$$\nabla f = (\partial_1 f, \ldots, \partial_d f)\,.$$

Note that $W^{0,p}(\Omega) = L^p(\Omega)$.

The following result states elementary properties of Sobolev spaces. For a proof we refer to [AF03, Theorems 3.3, 3.6], [Eva10, Section 5.2.3, Theorem 2], [Wlo87, Theorem 3.1], [Zei90, Proposition 21.10].

Proposition 2.17. *Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$. Endowed with the norm*

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$
$$\|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty(\Omega)}\,,$$

the linear space $W^{k,p}(\Omega)$ is a Banach space. $W^{k,p}(\Omega)$ is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$.

Note that for all $k \in \mathbb{N}_0$ the norm on $W^{k,2}(\Omega)$ is induced by a scalar product. Therefore, the spaces $W^{k,2}(\Omega)$ are separable Hilbert spaces and we define[4]

$$H^k(\Omega) = W^{k,2}(\Omega)$$

and the scalar product on these spaces by

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}\,.$$

[4]In literature, one finds also the notation $H^{k,p}(\Omega)$. This space is defined as the completion of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. It was shown in [MS64] that $H^{k,p}(\Omega) = W^{k,p}(\Omega)$ for every open set $\Omega \subset \mathbb{R}^d$ and $1 \leq p < \infty$, see also [AF03, Theorem 3.17]. If $p = \infty$ we have $H^{m,p}(\Omega) \subsetneq W^{m,p}(\Omega)$, see [AF03, Corollary 3.4, Example 3.18].

The space H_0^k and its dual

Definition 2.18. Let $k \in \mathbb{N}_0$ and $1 \le p \le \infty$.

(i) The space $W_0^{k,p}(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. Furthermore, set $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

(ii) The Banach space $H^{-1}(\Omega)$ is defined as the dual space of $H_0^1(\Omega)$ endowed with the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup\left\{\langle f, u\rangle : u \in H_0^1(\Omega), \|u\|_{H^1(\Omega)} \le 1\right\}.$$

The following result, which we cite from [Eva10, Section 5.9.1, Theorem 1] and [Bre11, Proposition 9.20], gives a characterization of the dual space $H^{-1}(\Omega)$.

Proposition 2.19. *Let $F \in H^{-1}(\Omega)$. Then there are functions $f_0, f_1, \ldots, f_d \in L^2(\Omega)$ such that for all $v \in H_0^1(\Omega)$*

$$\langle F, v\rangle = (f_0, v)_{L^2(\Omega)} + \sum_{i=1}^{d} (f_i, \partial_i v)_{L^2(\Omega)} \quad \text{and} \quad \|F\|_{H^{-1}(\Omega)} = \max_{1\le i\le d} \|f_i\|_{L^2(\Omega)}.$$

Remark 2.20.

a) Since we do not need the dual spaces of general spaces $W^{k,p}$ we omit their definition here and restrict ourselves to the case $(k,p) = (1,2)$. For detailed characterizations of the normed duals of $W^{k,p}$ similar to Proposition 2.19 we refer to [AF03, pp. 62-65].

b) A function f belongs to $W_0^{k,p}(\Omega)$ if and only if there is a sequence (f_n) in $C_c^\infty(\Omega)$ such that $\|f_n - f\|_{W^{k,p}(\Omega)} \to 0$ as $n \to \infty$.

c) Note that $W_0^{0,p}(\Omega) = L^p(\Omega)$ if $1 \le p < \infty$, since $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

d) Due to the fact that $C_c^\infty(\Omega)$ is a linear subset of $W^{m,p}(\Omega)$ it is clear that $W_0^{k,p}(\Omega)$ is a closed linear subspace. Hence, also the spaces $W_0^{k,p}(\Omega)$ are Banach spaces with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ and the spaces $H_0^k(\Omega)$ are Hilbert spaces with scalar product $(\cdot,\cdot)_{H^k(\Omega)}$. The assertions on separability and reflexivity in Proposition 2.17 hold in the same way for $W_0^{k,p}(\Omega)$.

e) Obviously, for all $u \in C_c^\infty(\Omega)$ we have $\|u\|_{W^{k,p}(\Omega)} = \|u\|_{W^{k,p}(\mathbb{R}^d)}$. Therefore, we can define the spaces $W_0^{k,p}(\Omega)$ in an equivalent way as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{k,p}(\mathbb{R}^d)}$.

f) Note that we *do not* identify the dual space $H^{-1}(\Omega)$ with H_0^1. The reason for this is that we shall consider the Gelfand triplet

$$H_0^1(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-1}(\Omega) \tag{2.12}$$

in the functional analytic treatment of linear evolution equations in Chapter 3. A simultaneous identification of L^2 with its dual as well as of H_0^1 with H^{-1} would make (2.12) senseless. See also the discussion in [Bre11, Section 5.2]. ♦

2.4 The constant $\mathcal{A}_{d,-2s}$

In this section we compute the norming constant that ensures that the Fourier symbol of the fractional Laplacian (cf. Section 2.7) is $|\xi|^{2s}$. More precisely, we prove the following identity:

Lemma 2.21. *Let $0 < s < 1$, $d \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$. The following identity holds:*

$$\int_{\mathbb{R}^d} \frac{\left|e^{i\xi\cdot h} - 1\right|^2}{|h|^{d+2s}} \,\mathrm{d}h = |\xi|^{2s}\, 2^{1-2s}\pi^{d/2} \frac{|\Gamma(-s)|}{\Gamma\left(\frac{d+2s}{2}\right)} \,. \tag{2.13}$$

Up to the author's knowledge there is no standard reference where one may find a detailed computation of the exact value of this constant, although its precise value is well-established. The proof of this lemma extends the sketch of proof in [FLS08, Lemma 3.1] and is based only on classical results.

For further reference we set

$$\mathcal{A}_{d,-2s} = \frac{2^{2s-1}}{\pi^{d/2}} \frac{\Gamma\left(\frac{d+2s}{2}\right)}{|\Gamma(-s)|} \,, \qquad s \in (0,1),\ d \in \mathbb{N}. \tag{2.14}$$

Sometimes, this constant is defined in a different way, namely as the reciprocal of

$$\int_{\mathbb{R}^d} \frac{1 - \cos(h_1)}{|h|^{d+2s}} \,\mathrm{d}h \,.$$

The relation to (2.13) and (2.14), respectively, is the following:

$$\int_{\mathbb{R}^d} \frac{\left|e^{i\xi\cdot h} - 1\right|^2}{|h|^{d+2s}} \,\mathrm{d}h = 2\,|\xi|^{2s} \int_{\mathbb{R}^d} \frac{1 - \cos(h_1)}{|h|^{d+2s}} \,\mathrm{d}h \,, \tag{2.15}$$

i.e.

$$\int_{\mathbb{R}^d} \frac{1 - \cos(h_1)}{|h|^{d+2s}} \,\mathrm{d}h = \frac{1}{2}\mathcal{A}_{d,-2s}^{-1} = 2^{-2s}\pi^{d/2} \frac{|\Gamma(-s)|}{\Gamma\left(\frac{d+2s}{2}\right)} \,. \tag{2.16}$$

The proof of this identity is given right after the proof of Lemma 2.21.

Let us start with the definition of Bessel functions. We define it in terms of the Taylor series around zero, cf. [AS70, 9.1.10] or [Wat66, III.1.(8)].

Definition 2.22. The *Bessel function $\mathscr{J}_\nu$ of the first kind of order $\nu \in \mathbb{R}$* is defined by

$$\begin{aligned} \mathscr{J}_\nu(t) &= \sum_{j=0}^{\infty} (-1)^j \frac{t^{\nu+2j}}{2^{\nu+2j}\, j!\, \Gamma(\nu+j+1)} \\ &= \pi^{-1/2} \left(\frac{t}{2}\right)^{\nu} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j+\nu+1)} \frac{t^{2j}}{(2j)!} \,, \qquad t \in \mathbb{R} \,. \end{aligned} \tag{2.17}$$

If $\nu > -\frac{1}{2}$, the representation as a Poisson integral

$$\begin{aligned} \mathscr{J}_\nu(t) &= \frac{\left(\frac{t}{2}\right)^\nu}{\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\nu-\frac{1}{2}}\,\mathrm{d}s \\ &= \frac{\left(\frac{t}{2}\right)^\nu}{\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} \cos(ts)(1-s^2)^{\nu-\frac{1}{2}}\,\mathrm{d}s, \qquad t\in\mathbb{R}, \end{aligned} \tag{2.18}$$

holds, see [AS70, 9.1.20] or [Gra08, Appendix B.1]. Note that it is possible to consider Bessel functions of complex order $\nu \in \mathbb{C}$. However, for our purposes we may assume $\nu \in \mathbb{R}$.

The following identity was found in the 19th century ([Son80, p. 39], [Sch87, p. 161]), see also [Wat66, XIII.24.(1)]: If $\nu > -\frac{3}{2}$ and $-\frac{1}{2} < \operatorname{Re} z < \nu + 1$ then

$$\int_0^\infty t^{-z}\,\mathscr{J}_\nu(t)\,\mathrm{d}t = 2^{-z}\frac{\Gamma\left(\frac{\nu+1-z}{2}\right)}{\Gamma\left(\frac{\nu+1+z}{2}\right)}\ . \tag{2.19}$$

Consider this as an identity of functions in the complex variable z. It is possible to determine the analytic continuation of the integral on the left-hand side to the strip $\nu+1 < \operatorname{Re} z < \nu+3$. This method is standard, cf. [GS64, Section I.3].

Lemma 2.23. *Let $\nu \in (0,\infty)$ and define a_ν as the coefficient of t^ν in (2.17), i.e.*

$$a_\nu = \frac{1}{2^\nu\,\Gamma(\nu+1)}\ .$$

The following identity holds for $\nu+1 < \operatorname{Re} z < \nu+3$:

$$\int_0^\infty t^{-z}\left(\mathscr{J}_\nu(t) - a_\nu t^\nu\right)\,\mathrm{d}t = 2^{-z}\frac{\Gamma\left(\frac{\nu+1-z}{2}\right)}{\Gamma\left(\frac{\nu+1+z}{2}\right)}\ . \tag{2.20}$$

Proof. Decompose the integral in (2.19) in the following way:

$$\int_0^\infty t^{-z}\,\mathscr{J}_\nu(t)\,\mathrm{d}t = \int_0^1 t^{-z}\left(\mathscr{J}_\nu(t) - a_\nu t^\nu\right)\,\mathrm{d}t + \int_0^1 a_\nu t^{\nu-z}\,\mathrm{d}t + \int_1^\infty t^{-z}\,\mathscr{J}_\nu(t)\,\mathrm{d}t\ . \tag{2.21}$$

The first term on the right-hand side exists for $\operatorname{Re} z < \nu+3$ since (2.17) implies $|\mathscr{J}_\nu(t) - a_\nu t^\nu| = \mathcal{O}(t^{\nu+2})$ for $t \to 0+$. The second term exists for all $\operatorname{Re} z > \nu+1$, whereas the third term is well-defined for all $\operatorname{Re} z > -\frac{1}{2}$, cf. (2.19). Hence, the right-hand side of (2.21) is an analytic continuation of the integral on the left-hand side to the the strip $\nu+1 < \operatorname{Re} z < \nu+3$. By the identity theorem of complex analysis (see [FB09, Corollary III.3.2]) and the property

$$\int_0^1 a_\nu t^{\nu-z}\,\mathrm{d}t = \frac{a_\nu}{\nu+1-z} = -\int_1^\infty t^{-z}\left(a_\nu t^\nu\right)\,\mathrm{d}t$$

we finally deduce from (2.19)

$$\int_0^\infty t^{-z}\left(\mathscr{J}_\nu(t) - a_\nu t^\nu\right)\,\mathrm{d}t = 2^{-z}\frac{\Gamma\left(\frac{\nu+1-z}{2}\right)}{\Gamma\left(\frac{\nu+1+z}{2}\right)}\ ,$$

which is valid for $\nu+1 < \operatorname{Re} z < \nu+3$. □

One could use the decomposition as in (2.21) with higher order terms in order to determine the analytic continuation to any strip of the form $\nu + 2k + 1 < \operatorname{Re} z < \nu + 2k + 3$, $k \in \mathbb{N}$. For our purposes the continuation to $\nu + 1 < \operatorname{Re} z < \nu + 3$ is sufficient.

Proof of Lemma 2.21. From a change of variable $z = |\xi|\, h$ and $\left|e^{it} - 1\right|^2 = 2(1 - \cos t)$, $t \in \mathbb{R}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\left|e^{i\xi\cdot h} - 1\right|^2}{|h|^{d+2s}}\, \mathrm{d}h &= |\xi|^{2s} \int_{\mathbb{R}^d} \frac{\left|\exp\left(i\frac{\xi}{|\xi|}\cdot z\right) - 1\right|^2}{|z|^{d+2s}}\, \mathrm{d}z \\ &= 2\,|\xi|^{2s} \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{\xi}{|\xi|}\cdot h\right)}{|h|^{d+2s}}\, \mathrm{d}h\ . \end{aligned} \tag{2.22}$$

Writing $\zeta = \frac{\xi}{|\xi|}$ and changing to spherical coordinates yields

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{\xi}{|\xi|}\cdot h\right)}{|h|^{d+2s}}\, \mathrm{d}h &= \int_0^\infty r^{d-1} \int_{\mathbb{S}^{d-1}} \frac{1 - \cos(r\zeta\cdot\theta)}{r^{d+2s}}\, \mathrm{d}\theta\, \mathrm{d}r \\ &= \int_0^\infty r^{-1-2s} \int_{\mathbb{S}^{d-1}} 1 - \cos(r\zeta\cdot\theta)\, \mathrm{d}\theta\, \mathrm{d}r\ . \end{aligned} \tag{2.23}$$

Due to a well-known formula on spherical integration (see [Gra08, Appendix D.3])

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \cos(r\zeta\cdot\theta)\, \mathrm{d}\theta &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-1}^1 \cos(rs)(1 - s^2)^{\frac{d-3}{2}}\, \mathrm{d}s \\ &= (2\pi)^{\frac{d}{2}} r^{-\frac{d-2}{2}} \mathscr{J}_{\frac{d-2}{2}}(r)\,, \end{aligned}$$

where the last identify follows from (2.18). We apply this identity and $\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ in (2.23) to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{\xi}{|\xi|}\cdot h\right)}{|h|^{d+2s}}\, \mathrm{d}h &= \int_0^\infty r^{-\frac{d}{2}-2s} \left[\frac{2\pi^{d/2}}{\Gamma(d/2)} r^{\frac{d-2}{2}} - (2\pi)^{\frac{d}{2}} \mathscr{J}_{\frac{d-2}{2}}(r)\right] \mathrm{d}r \\ &= (2\pi)^{\frac{d}{2}} \int_0^\infty r^{-\frac{d}{2}-2s} \left[\frac{2^{1-d/2}}{\Gamma(d/2)} r^{\frac{d-2}{2}} - \mathscr{J}_{\frac{d-2}{2}}(r)\right] \mathrm{d}r\ . \end{aligned}$$

Finally, we use Lemma 2.23 with $\nu = \frac{d-2}{2}$ and $z = \frac{d}{2} + 2s \in (\nu + 1, \nu + 3)$ to evaluate the integral, which proves the assertion (2.13):

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\left|e^{i\xi\cdot h} - 1\right|^2}{|h|^{d+2s}}\, \mathrm{d}h &= |\xi|^{2s} \left(-2^{1-2s}\pi^{d/2} \frac{\Gamma(-s)}{\Gamma(\frac{d}{2} + s}\right) \\ &= |\xi|^{2s}\, 2^{1-2s}\pi^{d/2} \frac{|\Gamma(-s)|}{\Gamma(\frac{d}{2} + s)}\ . \end{aligned}$$

□

Let us now prove (2.15) (see [DNPV12, p. 531-532]): By (2.22) it remains to show that

$$\int_{\mathbb{R}^d} \frac{1-\cos\left(\frac{\xi}{|\xi|}\cdot h\right)}{|h|^{d+2s}}\,\mathrm{d}h = \int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|h|^{d+2s}}\,\mathrm{d}h\ .$$

Given $\xi \in \mathbb{R}^d$ define an orthogonal mapping $O\colon \mathbb{R}^d \to \mathbb{R}^d$ with $O(\xi/|\xi|) = e_1$. Using $O^t = O^{-1}$ in the scalar product, $|O(h)| = |h|$ and $\mathrm{d}O(h) = \mathrm{d}h$ we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1-\cos\left(\frac{\xi}{|\xi|}\cdot h\right)}{|h|^{d+2s}}\,\mathrm{d}h &= \int_{\mathbb{R}^d} \frac{1-\cos\left(O^{-1}(e_1)\cdot h\right)}{|h|^{d+2s}}\,\mathrm{d}h = \int_{\mathbb{R}^d} \frac{1-\cos(e_1\cdot O(h))}{|h|^{d+2s}}\,\mathrm{d}h\\ &= \int_{\mathbb{R}^d} \frac{1-\cos(e_1\cdot h')}{|h'|^{d+2s}}\,\mathrm{d}h' = \int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|h|^{d+2s}}\,\mathrm{d}h\ .\end{aligned}$$

The following result provides the asymptotic behavior of the constant $\mathcal{A}_{d,-2s}$ for $s \to 0+$ and $s \to 1-$, respectively. It is a restatement of [DNPV12, Corollary 4.2] but the proof here is different.

Proposition 2.24 (Asymptotics of the constant $\mathcal{A}_{d,-2s}$)**.** *For any $d \in \mathbb{N}$ we have*

$$\lim_{s\to 0+} \frac{\mathcal{A}_{d,-2s}}{s(1-s)} = \frac{1}{|\mathbb{S}^{d-1}|} \quad \textit{and} \quad \lim_{s\to 1-} \frac{\mathcal{A}_{d,-2s}}{s(1-s)} = \frac{2d}{|\mathbb{S}^{d-1}|}\ . \tag{2.24}$$

Recall that $\left|\mathbb{S}^{d-1}\right|$ denotes the $(d-1)$-dimensional measure of the unit sphere, i.e.

$$\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$

Let us emphasize that the precise value of the constants on the right-hand sides of (2.24) is secondary; the main aspect is that $\mathcal{A}_{d,-2s}$ is asymptotically equivalent to $s(1-s)$ for $s \to 0+$ and $s \to 1-$.

Proof. By definition of $\mathcal{A}_{d,-2s}$ in (2.14) and the properties of the Gamma function we have

$$\frac{\mathcal{A}_{d,-2s}}{s(1-s)} = \frac{2^{2s-1}\Gamma\left(\frac{d+2s}{2}\right)}{\pi^{d/2}(1-s)\,|(-s)\Gamma(-s)|} = \frac{2^{2s-1}\Gamma\left(\frac{d+2s}{2}\right)}{\pi^{d/2}(1-s)\Gamma(1-s)} = \frac{2^{2s-1}\Gamma\left(\frac{d+2s}{2}\right)}{\pi^{d/2}\Gamma(2-s)}\ .$$

Hence,

$$\lim_{s\to 0+} \frac{\mathcal{A}_{d,-2s}}{s(1-s)} = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}} = \frac{1}{|\mathbb{S}^{d-1}|} \quad \text{and} \quad \lim_{s\to 1-} \frac{\mathcal{A}_{d,-2s}}{s(1-s)} = \frac{d\,\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} = \frac{2d}{|\mathbb{S}^{d-1}|}\ . \qquad \square$$

2.5 Sobolev spaces of fractional order

There are several ways to define (fractional order) Sobolev spaces, for example via Fourier transform (cf. Section 2.6) or via interpolation theory[5]. We will comment below on the relation to the definition via Fourier transform. To distinguish the spaces from possible other definitions, the spaces in Definition 2.25 are sometimes called *Sobolev-Slobodeckij* spaces and the seminorm in (2.28) as *Slobodeckij*, *Aronszajn* or *Gagliardo* seminorm.

Definition 2.25 (Fractional order Sobolev space)**.** Let Ω be an arbitrary open set in $\mathbb{R}^d$ and $1 \leq p < \infty$. The linear space $W^{s,p}(\Omega)$, $0 < s < 1$, is defined by

$$W^{s,p}(\Omega) = \left\{ f \in L^2(\Omega) \colon \frac{|f(x) - f(y)|}{|x-y|^{d/p+s}} \in L^p(\Omega \times \Omega) \right\} . \tag{2.25}$$

Let $r = k + s$ with $k \in \mathbb{N}_0$ and $0 \leq s < 1$. The linear space $W^{r,p}(\Omega)$, $0 \leq r < \infty$, is defined by

$$W^{r,p}(\Omega) = \left\{ f \in W^{k,p}(\Omega) \colon \partial^\alpha f \in W^{s,p}(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| = k \right\} . \tag{2.26}$$

Obviously, $W^{r,p}(\Omega)$ in Definition 2.25 coincides with $W^{k,p}(\Omega)$ in Definition 2.16 if $r = k \in \mathbb{N}_0$.

Proposition 2.26. *Let $1 \leq p < \infty$, Ω an open set in $\mathbb{R}^d$, $r = k + s$ with $k \in \mathbb{N}_0$ and $0 \leq s < 1$. Endowed with the norm*

$$\|f\|_{W^{r,p}(\Omega)} = \left(\|f\|^p_{W^{k,p}} + \sum_{|\alpha|=k} [\partial^\alpha f]^p_{W^{s,p}(\Omega)} \right)^{1/p} , \tag{2.27}$$

where for $u \in W^{s,p}(\Omega)$

$$[u]^p_{W^{s,p}} = \mathcal{A}_{d,-2s} \left\| \frac{u(x) - u(y)}{|x-y|^{d/p+s}} \right\|^p_{L^p(\Omega \times \Omega)} = \mathcal{A}_{d,-2s} \iint\limits_{\Omega\,\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x \,\mathrm{d}y , \tag{2.28}$$

the linear space $W^{r,p}(\Omega)$ is a separable Banach space.

Remark 2.27.

a) For $p = 2$ the norm in (2.27) is induced by a scalar product and therefore the spaces $W^{k,2}(\Omega)$ are separable Hilbert spaces and we define

$$H^r(\Omega) = W^{r,2}(\Omega)$$

and the scalar product on these spaces by

$$(f, g)_{H^r(\Omega)} = (f, g)_{H^k(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha f, \partial^\alpha g]_{H^s(\Omega)} , \tag{2.29}$$

[5] When defining Sobolev spaces as interpolation spaces, the Sobolev spaces of fractional order are considered as special cases of *Besov spaces*, cf. [AF03, Chapter 7].

where for $u, v \in H^s(\Omega)$

$$[u,v]_{H^s(\Omega)} = \mathcal{A}_{d,-2s} \iint\limits_{\Omega\,\Omega} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \;. \tag{2.30}$$

b) The role of the constant $\mathcal{A}_{d,-2s}$ in (2.28) and (2.30) is not evident at this point. Defining the seminorm in (2.30) and the fractional Laplacian in (2.43) with exactly this constant has the following desirable consequences in the case $\Omega = \mathbb{R}^d$:

- The seminorm induced by (2.30) converges to the norm on $L^2(\mathbb{R}^d)$ and to the seminorm on $H^1(\mathbb{R}^d)$ for $s \to 0+$ and $s \to 1-$, respectively[6]. We refer to Proposition 2.36 for a precise statement and the proof.
- The symbol of the fractional Laplacian as defined in (2.43) is exactly $|\xi|^{2s}$, i.e. for $u \in \mathcal{S}(\mathbb{R}^d)$
$$\mathscr{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\, \mathscr{F}u \;,$$
see Proposition 2.39.
- Moreover, the operator $(-\Delta)^s$ converges for $s \to 1-$ and $s \to 0+$ to the classical Laplace operator $-\Delta$ and to the identity operator, respectively. More precisely – cf. Proposition 2.41 – for $u \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$
$$\lim_{s\to 1-} (-\Delta)^s u(x) = (-\Delta u)(x), \qquad \text{and} \qquad \lim_{s\to 0+} (-\Delta)^s u(x) = u(x). \qquad \blacklozenge$$

The following proof is based on [DD12, Proposition 4.24], [Dob06, Satz 6.33] and [Wlo87, Theorem 3.1].

Proof. Without loss of generality we may assume $\mathcal{A}_{d,-2s} = 1$ in the whole proof. It is clear that (2.27) defines a norm on $W^{r,p}(\Omega)$. It remains to show that $W^{r,p}(\Omega)$ is complete. We prove the completeness of $W^{s,p}(\Omega)$, $0 < s < 1$, which immediately implies that $W^{r,p}(\Omega)$, $0 \le r < \infty$, are Banach spaces.

Let (f_n) be a Cauchy sequence with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Set

$$v_n(x,y) = \frac{f_n(x) - f_n(y)}{|x-y|^{d/p+s}}.$$

Then, by definition of $\|\cdot\|_{W^{s,p}(\Omega)}$ and the completeness of L^p-spaces, (f_n) converges to some f in the norm of $L^p(\Omega)$. We may chose a subsequence f_{n_k} that converges a.e. to f. Then v_{n_k} converges a.e. in $\Omega \times \Omega$ to the function

$$v(x,y) = \frac{f(x) - f(y)}{|x-y|^{d/p+s}} \;.$$

[6] A similar statement is also true when $p \neq 2$, see [BBM01] and [MS02].

By Fatou's lemma (Theorem 1.8),

$$\iint\limits_{\Omega\,\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}x\,\mathrm{d}y \le \liminf_{k\to\infty} \iint\limits_{\Omega\,\Omega} |v_{n_k}(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \le \sup_{k\in\mathbb{N}} \|v_{n_k}\|^p_{L^p(\Omega\times\Omega)}\,.$$

Since v_n is a Cauchy sequence (and hence bounded) in $L^p(\Omega\times\Omega)$, this shows that $[f]_{W^{s,p}(\Omega)} < \infty$, i.e. $f \in W^{s,p}(\Omega)$. Another application of Fatou's lemma shows that

$$\begin{aligned}[][f_{n_k} - f]^p_{W^{s,p}(\Omega)} &= \iint\limits_{\Omega\,\Omega} |v_{n_k}(x,y) - v(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\ &\le \liminf_{l\to\infty} \iint\limits_{\Omega\,\Omega} |v_{n_k}(x,y) - v_{n_l}(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \xrightarrow{k\to\infty} 0.\end{aligned}$$

This shows that $\|f_{n_k} - f\|_{W^{s,p}(\Omega)} \to 0$ for $k\to\infty$ for the subsequence (n_k) chosen above and thus $\|f_n - f\|_{W^{s,p}(\Omega)} \to 0$ as $n\to\infty$, since (f_n) was assumed to be a Cauchy sequence. The completeness of $W^{r,p}(\Omega)$ is proved.

The mapping $\mathcal{I}$

$$\begin{aligned}\mathcal{I}\colon W^{r,p}(\Omega) &\to \bigtimes_{|\alpha|\le k} L^2(\Omega) \times \bigtimes_{|\alpha|=k} L^2(\Omega\times\Omega), \\ \mathcal{I}(f) &= \left(\partial^\alpha f\ [|\alpha|\le k], \frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{d/p+s}}\ [|\alpha| = k]\right),\end{aligned} \tag{2.31}$$

is isometric due to the definition of the norm in $W^{r,p}(\Omega)$ in (2.27). Having shown the completeness of $W^{r,p}(\Omega)$ we obtain that $\mathcal{I}(W^{r,p}(\Omega))$ is a closed subspace of the Cartesian product on the right-hand side of (2.31). This product is separable and so is $W^{r,p}(\Omega)$, cf. [Wlo87, Lemma 3.1]. □

The next result shows that in the case $\Omega = \mathbb{R}^d$ there is an equivalent norm to (2.27) in terms of the difference operator Δ_h, $h\in\mathbb{R}^d$, defined by

$$\Delta_h f(x) = f(x+h) - f(x)\,. \tag{2.32}$$

Lemma 2.28. *Let $1 \le p < \infty$ and $r = k+s$ with $k\in\mathbb{N}_0$ and $0 < s < 1$. There is a constant $c = c(d,s,p) > 0$ such that for all $f\in W^{r,p}(\mathbb{R}^d)$*

$$c\,\|f\|^p_{W^{r,p}(\mathbb{R}^d)} \le \|f\|^p_{W^{k,p}(\mathbb{R}^d)} + \mathcal{A}_{d,-2s} \sum_{|\alpha|=k} \int_{|h|\le 1} \frac{\|\Delta_h(\partial^\alpha f)\|^p_{L^p(\mathbb{R}^d)}}{|h|^{ps}} \frac{\mathrm{d}h}{|h|^d} \le \|f\|^p_{W^{r,p}(\mathbb{R}^d)}\,. \tag{2.33}$$

Note that the analogous assertion to (2.33) for the seminorm $[\,\cdot\,]_{W^{r,p}(\mathbb{R}^d)}$ is not true in general. This can be seen in the proof below. Let us also mention that the constant c tends to zero if $s\to 0+$.

The ideas of this proof are taken from [HT08, p. 65]. In contrast to the computations therein we consider general $p \in [1, \infty)$ and we stress the fact that the constant will depend on $s \in (0, 1)$ (and of course on the dimension $d \in \mathbb{N}$).

Proof. Fix $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$ and write $\partial^\alpha f = g$. We use the coordinate transform $y = x + h$, $h \in \mathbb{R}^d$, and the decomposition $\mathbb{R}^d = \{|h| \le 1\} \cup \{|h| > 1\}$ to obtain

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|g(x + h) - g(x)|^p}{|h|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}h$$

$$= \int\limits_{|h|\le 1} \frac{\|\Delta_h g\|^p_{L^p(\mathbb{R}^d)}}{|h|^{ps}} \frac{\mathrm{d}h}{|h|^d} + \int\limits_{|h|>1} \int\limits_{\mathbb{R}^d} \frac{|\Delta_h g|^p}{|h|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}h\,. \tag{2.34}$$

Now observe that $\|\Delta_h g\|^p_{L^p(\mathbb{R}^d)} \le 2^{p-1} \|g\|^p_{L^p(\mathbb{R}^d)}$ for all $h \in \mathbb{R}^d$ and

$$\int\limits_{|h|>1} |h|^{-d-ps} \,\mathrm{d}h = \frac{d\,|B_1(0)|}{ps}.$$

This implies that the lower bound in (2.33) holds, where $c = c(d, s, p)$ can be chosen as

$$c = \left(2 + \frac{2^p\, d\, |B_1(0)|}{ps}\right)^{-1}.$$

Estimating the second term in (2.34) from below by zero we establish the upper estimate in (2.33). □

The space $H_0^s(\Omega)$ and its dual

Definition 2.29. Let $1 \le p < \infty$ and $0 \le r < \infty$ with $r = k + s$, $k \in \mathbb{N}_0$, $0 \le s < 1$.

(i) The space $W_0^{r,p}(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{r,p}(\mathbb{R}^d)}$. Furthermore, set $H_0^r(\Omega) = W_0^{r,2}(\Omega)$.

(ii) The Banach space $H^{-s}(\Omega)$, $0 < s < 1$, is defined as the dual space of $H_0^s(\Omega)$ endowed with the norm

$$\|f\|_{H^{-s}(\Omega)} = \sup\left\{\langle f, u\rangle : u \in H_0^s(\Omega), \|u\|_{H^s(\mathbb{R}^d)} \le 1\right\}.$$

Remark 2.30.

a) Let us emphasize that the closure in the foregoing definition is taken with respect to the norm $\|\cdot\|_{H^r(\mathbb{R}^d)}$. To give sense to $\|u\|_{H^r(\mathbb{R}^d)}$, the functions $u \in C_c^\infty(\Omega)$ are extended in the natural way, namely by zero, outside of Ω.

b) Note that the two spaces $W_0^{r,p}(\Omega)$ in Definition 2.29 and $W_0^{k,p}(\Omega)$ in Definition 2.18 coincide if $r = k \in \mathbb{N}_0$, see Remark 2.20e.

c) The reason for defining $H_0^r(\Omega)$ in this way is given by the easy observation that for a given function $u \in H_0^r(\Omega)$ we have $\widetilde{u} \in H^r(\mathbb{R}^d)$, where

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \Omega^c. \end{cases} \tag{2.35}$$

This would not hold in general if we defined $H_0^r(\Omega)$ as the completion with respect to $\|\cdot\|_{H^r(\Omega)}$. However, in most cases the latter definition is equivalent to the one given Definition 2.29. We state this result, which is taken from [McL00, Theorem 3.33], in a proposition below.

d) Again, we do not identify $H_0^s(\Omega)$ with its dual, cf. Remark 2.20f). ♦

Proposition 2.31. *Let $0 \le r < \infty$ and assume that Ω is a Lipschitz domain. Then*

$$H_0^r(\Omega) = \left\{ u \in L^2(\Omega) \colon \widetilde{u} \in H^r(\mathbb{R}^d) \right\},$$

where $\widetilde{u}$ is given by (2.35). *If additionally $r \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}$, then*

$$H_0^r(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^r(\Omega)}} .$$

2.6 Characterization of (fractional) Sobolev spaces by Fourier transform

In the previous section Sobolev spaces of fractional order, in particular $H^r(\Omega)$ for arbitrary $r \in (0,\infty)$, were introduced. This subsection is designed to explain why the extension to $H^r(\Omega)$ with non-integer $r \in (0,\infty)$ can be seen as a generalized definition of $H^k(\Omega)$, $k \in \mathbb{N}_0$.

To start with, let us recall Definition 2.25, which is valid for $\Omega = \mathbb{R}^d$, i.e. for $r = k + s$ with $k \in \mathbb{N}_0$ and $0 \le s < 1$

$$H^r(\mathbb{R}^d) = \left\{ f \in H^k(\mathbb{R}^d) \colon \frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{d/2+s}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \text{ for all } |\alpha| = k \right\}, \tag{2.36}$$

$$\|f\|_{H^r(\mathbb{R}^d)}^2 = \|f\|_{H^k(\mathbb{R}^d)}^2 + \mathcal{A}_{d,-2s} \sum_{|\alpha|=k} \iint_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y. \tag{2.37}$$

The following result is taken from [HT08, Section 3.2], see also [LM72, Theorem 1.2], [Eva10, Section 5.8.5] and [McL00, Theorem 3.16]. It states that $\mathscr{F}$ is an isometric map from $H^r(\mathbb{R}^d)$ into a certain weighted L^2-space. For $r = 0$ this is a restatement of Plancherel's theorem, Theorem 2.15.

Theorem 2.32 (Characterization of $H^r(\mathbb{R}^d)$ via Fourier-transform)**.** *Let $r = k + s$ with $k \in \mathbb{N}_0$, $0 \le s < 1$. A function $f \in L^2(\mathbb{R}^d)$ belongs to $H^r(\mathbb{R}^d)$ if and only if the function $\xi \mapsto (1 + |\xi|^2)^{r/2}(\mathscr{F} f)(\xi)$ belongs to $L^2(\mathbb{R}^d)$, i.e.*

$$H^r(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \colon \left(1 + |\xi|^2\right)^{r/2} \mathscr{F} f \in L^2(\mathbb{R}^d) \right\}. \tag{2.38}$$

Moreover, for $u \in H^s(\mathbb{R}^d)$ we have

$$[u, u]_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} \, |\mathscr{F} u(\xi)|^2 \, \mathrm{d}\xi \tag{2.39}$$

and the norm $\|\cdot\|_{\mathscr{F},r}$ defined by

$$\|f\|_{\mathscr{F},r} = \left\| \left(1 + |\xi|^2\right)^{r/2} \mathscr{F} f \right\|_{L^2(\mathbb{R}^d)}$$

is an equivalent norm to $\|\cdot\|_{H^r(\mathbb{R}^d)}$ in (2.37).

The following remark explains one reason for the occurrence of the factor $\mathcal{A}_{d,-2s}$ in the definition of the seminorms on $H^r(\mathbb{R}^d)$:

Remark 2.33. The equivalence of the norms, which is stated in the above theorem, is robust for $s \in (0, 1)$, i.e. the constants c, C in the relation

$$c \|\cdot\|_{H^r(\mathbb{R}^d)} \le \|\cdot\|_{\mathscr{F},r} \le C \|\cdot\|_{H^r(\mathbb{R}^d)}$$

depend on d and k, but not on $s \in (0, 1)$. ♦

Concerning the case $\Omega \neq \mathbb{R}^d$ we have the following result, see [McL00, Theorem 3.18].

Proposition 2.34. *Let $0 \le r < \infty$ and Ω a domain. If Ω is an extension domain*[7] *then*

$$H^r(\Omega) = \left\{ T \in \mathcal{D}'(\Omega) \colon T = f|_\Omega \text{ for some } f \in H^r(\mathbb{R}^d) \right\}$$

and

$$\|T\|_{\mathscr{F},r,\Omega} = \inf \left\{ \|f\|_{H^r(\mathbb{R}^d)} : f|_\Omega = T \right\}$$

is an equivalent norm to $\|\cdot\|_{H^s(\Omega)}$.

We shall present the proof of Theorem 2.32 following the lines of [LM72, Theorem 1.2], [HT08, Theorems 3.11, 3.24] and [McL00, Theorem 3.16]. All these references use an upper and lower bound for $\sum_{|\alpha| \le k} \xi^\alpha$, which they do not prove. We give a proof of this elementary estimate as a lemma below.

Lemma 2.35. *Let $d \ge 1$ and $k \in \mathbb{N}_0$. There are constants $c, C > 0$ such that for all $y \in \mathbb{R}^d$*

$$c(1 + |y|^2)^k \le \sum_{|\alpha| \le k} y^{2\alpha} \le C(1 + |y|^2)^k, \tag{2.40}$$

where – as usual – $|y|^2 = y_1^2 + \ldots + y_d^2$ and the sum is taken over all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$.

[7] Ω is an *extension domain* if there is a continuous linear operator $E_\Omega \colon H^r(\Omega) \to H^r(\mathbb{R}^d)$ such that $Eu|_\Omega = u$ for all $u \in H^s(\Omega)$.

Proof. The assertion is easily verified if $k = 0$ or $y = 0$. From now on we assume $k \in \mathbb{N}$ and $y \neq 0$. Next observe that the function

$$\zeta \mapsto \sum_{|\alpha|\leq k} \zeta^{2\alpha}$$

is a continuous function on the compact set $\mathbb{S}^{d-1}$. Therefore, this function attains its maximum, which we denote by $M > 0$. Of course, M depends on d and k.

For any given $y \neq 0$ write $y = |y|\,\zeta$ for $\zeta = \frac{y}{|y|} \in \mathbb{S}^{d-1}$. Then

$$\sum_{|\alpha|\leq k} (|y|\,\zeta)^{2\alpha} = \sum_{|\alpha|\leq k} |y|^{2|\alpha|}\,\zeta^{2\alpha} \leq M(1 + |y|^{2k}) \leq M(1 + |y|^2)^k.$$

This proves the upper bound in (2.40). To prove the lower bound we note that

$$\{(0,\dots,0),(k,0,\dots,0),(0,k,0,\dots),\dots,(0,\dots,0,k)\} \subset \left\{\alpha \in \mathbb{N}_0^d\colon\ |\alpha| \leq k\right\}.$$

Hence,

$$\begin{aligned}\sum_{|\alpha|\leq k} y^{2\alpha} &\geq 1 + y_1^{2k} + \ldots + y_d^{2k} \geq 1 + d^{1-k}\,|y|^{2k} \geq d^{1-k}(1 + |y|^{2k}) \\ &\geq (2d)^{1-k}\left(1 + |y|^2\right)^k ,\end{aligned}$$

where we have applied Jensen's inequality to the convex function $t \mapsto t^k$ and the trivial estimate $(a+b)^k \leq 2^{k-1}(a^k + b^k)$. This finishes the proof of Lemma 2.35. □

Proof of Theorem 2.32. The assertion is proved in two steps:

I. Assume $r = k \in \mathbb{N}$. Let $f \in H^k(\mathbb{R}^d)$. By Theorem 2.15 and the property $\widehat{\partial^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi)$, which is easily checked for $u \in \mathcal{S}(\mathbb{R}^d)$ and extends to $\mathscr{F}$ on $L^2(\mathbb{R}^d)$, we obtain

$$\|\partial^\alpha f\|_{L^2(\mathbb{R}^d)} = \|\xi^\alpha \mathscr{F} f\|_{L^2(\mathbb{R}^d)} \qquad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k.$$

By (2.40) we then have

$$\|f\|_{H^k(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \Big(\sum_{|\alpha|\leq k} \xi^{2\alpha}\Big)\,|(\mathscr{F} f)(\xi)|^2\,\mathrm{d}\xi \asymp \int_{\mathbb{R}^d} (1 + |\xi|^2)^k\,|(\mathscr{F} f)(\xi)|\,\mathrm{d}\xi = \|f\|_{\mathscr{F},k}^2 .$$

This proves the result if $r = k \in \mathbb{N}_0$.

II. Now assume $r = s \in (0,1)$. The transformation $y = x + h$ and the definition of Δ_h in (2.32) yield

$$\begin{aligned}\iint_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}h \\ &= \int_{\mathbb{R}^d} \frac{\|\Delta_h f\|_{L^2(\mathbb{R}^d)}^2}{|h|^{d+2s}}\,\mathrm{d}h .\end{aligned}$$

Changing the order of integration, using the fact that

$$\mathscr{F}(\Delta_h u)(\xi) = \left(e^{i\xi\cdot h} - 1\right)(\mathscr{F}f)(\xi)$$

and Plancherel's theorem we obtain

$$\int\limits_{\mathbb{R}^d} \frac{\|\Delta_h f\|^2_{L^2(\mathbb{R}^d)}}{|h|^{d+2s}}\,\mathrm{d}h = \int_{\mathbb{R}^d} |\mathscr{F}f(\xi)|^2 \int_{\mathbb{R}^d} \frac{\left|e^{i\xi\cdot h} - 1\right|^2}{|h|^{d+2s}}\,\mathrm{d}h\,\mathrm{d}\xi\ .$$

From this and Lemma 2.21 we deduce (2.39). Furthermore,

$$\begin{aligned}\|f\|^2_{W^s(\mathbb{R}^d)} &= \|\mathscr{F}f\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\xi|^{2s}\,|\mathscr{F}f(\xi)|^2\,\mathrm{d}\xi\\ &\asymp \int_{\mathbb{R}^d} (1+|\xi|^2)^s\,|\mathscr{F}f(\xi)|^2\,\mathrm{d}\xi,\end{aligned}$$

where the constants that are contained in the notation "$\asymp$" do not depend on $s \in (0,1)$.

This proves the result if $r = s \in (0,1)$. For general $r = k+s \in (0,\infty)$ apply step II to the functions $\partial^\alpha f$, where $|\alpha| = k$. □

Combining (2.39) and the asymptotics of the constant $\mathcal{A}_{d,-2s}$ in Proposition 2.24 we are now able to prove the convergence of the seminorms on $H^s(\mathbb{R}^d)$ to the norm on $L^2(\mathbb{R}^d)$ and the seminorm on $H^1(\mathbb{R}^d)$, respectively.

Proposition 2.36.

(i) For $u \in \bigcup\limits_{s\in(0,1)} H^s(\mathbb{R}^d)$ *we have*

$$\lim_{s\to 0+} s \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y = |\mathbb{S}^{d-1}|\,\|u\|_{L^2(\mathbb{R}^d)}\,. \tag{2.41}$$

(ii) For $u \in H^1(\mathbb{R}^d)$ *we have*

$$\lim_{s\to 1-} (1-s) \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y = \frac{|\mathbb{S}^{d-1}|}{2d}\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)}\,. \tag{2.42}$$

A generalization of these statements to the case $p \neq 2$ holds true, see [BBM01] and [MS02].

Proof. Both assertions follow immediately from (2.39), Proposition 2.24 and Plancherel's theorem:

$$\begin{aligned}\lim_{s\to 0+} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \lim_{s\to 0+} s\,\mathcal{A}^{-1}_{d,-2s} \int_{\mathbb{R}^d} |\xi|^{2s}\,|\mathscr{F}u(\xi)|^2\,\mathrm{d}\xi\\ &= |\mathbb{S}^{d-1}|\,\|u\|^2_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

$$\lim_{s\to 1-}(1-s)\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y = \lim_{s\to 1-}(1-s)\mathcal{A}_{d,-2s}^{-1}\int_{\mathbb{R}^d}|\xi|^{2s}\,|\mathscr{F}u(\xi)|^2\,\mathrm{d}\xi$$
$$= \frac{|\mathbb{S}^{d-1}|}{2d}\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)}\,. \qquad \square$$

The following proposition justifies why the dual space of $H_0^r(\mathbb{R}^d) = H^r(\mathbb{R}^d)$ was denoted by $H^{-r}(\mathbb{R}^d)$ in Definition 2.29. It is taken from [DD12, Proposition 4.10], where also a proof can be found.

Proposition 2.37. *Let $0 < r < \infty$. Then*

$$H^{-r}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d)\colon (1+|\xi|^2)^{-r/2}\mathscr{F}f \in L^2(\mathbb{R}^d)\right\}\,.$$

2.7 The fractional Laplacian

The fractional Laplacian can be considered as the prototype of a nonlocal operator. In this section we provide the definition of $(-\Delta)^{\alpha/2}$ and explain the connection to fractional order Sobolev spaces.

Definition 2.38. Let $0 < s < 1$ and $u \in \mathcal{S}(\mathbb{R}^d)$. The *fractional Laplacian* $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = 2\,\mathcal{A}_{d,-2s}\ \text{p. v.}\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y \qquad \text{for } x\in\mathbb{R}^d. \tag{2.43}$$

For $0 < s < \frac{1}{2}$ it can be shown ([DNPV12, p. 528-529]) that

$$\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|}{|x-y|^{d+2s}}\,\mathrm{d}y < \infty,$$

i.e. (2.43) has to be understood in the principal value sense only if $\frac{1}{2} \le s < 1$. Moreover, the fractional Laplacian can be defined equivalently via second order differences ([DNPV12, Lemma 3.2]):

$$(-\Delta)^s u(x) = -\mathcal{A}_{d,-2s}\int_{\mathbb{R}^d}\frac{u(x+h)+u(x-h)-2u(x)}{|h|^{d+2s}}\,\mathrm{d}h \qquad \text{for } x\in\mathbb{R}^d. \tag{2.44}$$

Proposition 2.39. *Let $0 < s < 1$. For any $u \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$*

$$\mathscr{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s}\,\mathscr{F}u(\xi)\,. \tag{2.45}$$

Proof. Applying (2.44) and Fubini's theorem we obtain

$$\mathscr{F}\left((-\Delta)^s u\right)(\xi) = -\mathcal{A}_{d,-2s}\int_{\mathbb{R}^d}\frac{\mathscr{F}\left[u(x+h)+u(x-h)-2u(x)\right](\xi)}{|h|^{d+2s}}\,\mathrm{d}h$$

$$
\begin{aligned}
&= -\mathcal{A}_{d,-2s}\,\mathscr{F}u(\xi)\int_{\mathbb{R}^d}\frac{e^{i\xi\cdot h}+e^{-i\xi\cdot h}-2}{|h|^{d+2s}}\,\mathrm{d}h\\
&= \mathcal{A}_{d,-2s}\,\mathscr{F}u(\xi)\int_{\mathbb{R}^d}\frac{\left|e^{i\xi\cdot h}-1\right|^2}{|h|^{d+2s}}\,\mathrm{d}h\\
&= |\xi|^{2s}\,\mathscr{F}u(\xi)\,.
\end{aligned}
$$

The last identity follows from Lemma 2.21 and the application of Fubini's theorem in the first identity is justified by the fact that

$$\frac{|u(x+h)+u(x-h)-2u(x)|}{|h|^{d+2s}} \in L^1(\mathbb{R}^d\times\mathbb{R}^d),$$

cf. [DNPV12, p.531]. □

Proposition 2.40. *Let $0 < s < 1$ and $u \in H^s(\mathbb{R}^d)$. The following identity holds:*

$$[u,u]_{H^s(\mathbb{R}^d)} = \Big((-\Delta)^{s/2}u, (-\Delta)^{s/2}u\Big)_{L^2(\mathbb{R}^d)}\ . \tag{2.46}$$

Proof. Plancherel's theorem, Proposition 2.39 and (2.39) imply

$$\Big((-\Delta)^{s/2}u, (-\Delta)^{s/2}u\Big)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|\xi|^{2s}\,|\mathscr{F}u(\xi)|^2\,\mathrm{d}\xi = [u,u]_{H^s(\mathbb{R}^d)}\ . \qquad \square$$

Concerning the asymptotics of $(-\Delta)^s$ we cite the following proposition from [DNPV12, Proposition 4.4].

Proposition 2.41. *Let $d \in \mathbb{N}$ and $u \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\lim_{s\to 0+}(-\Delta)^s u = u \qquad \text{and} \qquad \lim_{s\to 1-}(-\Delta)^s u = -\Delta u\ . \tag{2.47}$$

Part II

Existence & Uniqueness

3 Existence and uniqueness of solutions to local and nonlocal parabolic differential equations

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $0 < T < \infty$. Set $Q_T = (0,T) \times \Omega$.

For $s \in (0,1]$ we consider operators $\mathscr{L}^s$ of the form

$$(\mathscr{L}^s u)(t,x) = \begin{cases} \operatorname{div}(A\nabla u)(t,x) & \text{if } s = 1, \\ 2 \text{ p. v.} \displaystyle\int_{\mathbb{R}^d} [u(t,y) - u(t,x)]\, k_t(x,y)\, \mathrm{d}y & \text{if } s \in (0,1), \end{cases} \tag{3.1}$$

where

- $A = (a_{ij})_{1\le i,j\le d}$ denotes a symmetric matrix of functions $a_{ij} \colon (0,T) \times \Omega \to \mathbb{R}$,
- k denotes a symmetric kernel $k \colon (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, $(t,x,y) \mapsto k_t(x,y)$, which typically has a certain singularity at the diagonal $x = y$.

The detailed assumptions on A and k will be given in Lemma 3.16 and Lemma 3.17, respectively.

Note that in both cases $(\mathscr{L}^s u)$ may not exist even for functions $u \in C_c^\infty(\Omega)$ or $u \in \mathcal{S}(\Omega)$. Since we do not treat classical solutions but weak solutions to problem (3.2), we work with bilinear forms instead of the operator itself. Only then it will be necessary to specify the domain of the corresponding bilinear forms.

For $s \in (0,1]$ we study initial boundary value problems of the type

$$\begin{cases} \partial_t u - \mathscr{L}^s u = f & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } [0,T] \times \Omega^c, \\ u = u_0 & \text{on } \{0\} \times \Omega, \end{cases} \tag{3.2}$$

under the conditions

$$f \in L^2(Q_T), \qquad u_0 \in L^2(\Omega). \tag{3.3}$$

The aim of this chapter is to prove that there is a unique weak solution $u \in \mathcal{W}(0,T)$ to problem (3.2). The well-posedness in the case of a local operator – i.e. $s = 1$ in (3.2) – has been common knowledge since the 1950's, see for example [Lad54, Višˇ54, LV58]. A detailed discussion on the development of existence and uniqueness results concerning weak solutions of linear parabolic equations with bounded and measurable coefficients can be found in the introduction of [Lad85].

3.1 Generalized derivatives of abstract functions

The following definition extends the concept of generalized derivatives (cf. Section 2.1) of real-valued functions to vector-valued functions. Similarly to Section 2.1, this could be done in the context of vector-valued distributions. But for the sake of shortness we omit the notion of vector-valued distributions.

Definition 3.1. Let V be a Banach space and $u \in L^1(0,T;V)$. If there is a function $v \in L^1(0,T;V)$ such that

$$\int_0^T u(t)\phi'(t)\,\mathrm{d}t = -\int_0^T v(t)\phi(t)\,\mathrm{d}t \qquad \text{for all } \phi \in C_c^\infty(0,T)\,, \tag{3.4}$$

then v is called the *generalized derivative of* u and we write $u' = v$.

If (3.4) is satisfied for two functions v_1 and v_2, then $v_1(t) = v_2(t)$ for almost every $t \in (0,T)$. This follows from the following variational lemma for vector-valued functions, see [Zei90, Propositions 23.10,23.18] for a proof.

Lemma 3.2. *Let V be a Banach space and $u \in L^1(0,T;V)$ such that*

$$\int_0^t u(t)\phi(t)\,\mathrm{d}t = 0 \qquad \textit{for all } \phi \in C_c^\infty(0,T)\,.$$

Then $u(t) = 0$ for almost every $t \in (0,T)$.

The following lemma is taken from [Emm04, Satz 8.1.5]:

Lemma 3.3. *Let V be a Banach space and $u, v \in L^1(0,T;V)$. The following are equivalent:*

(i) g is the generalized derivative of u: $u' = v$.

(ii) There is $u_0 \in V$ such that

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s \qquad \textit{for a.e. } t \in (0,T).$$

(iii) For all $f \in V^*$ the real-valued function $t \mapsto \langle f, v(t)\rangle_V$ is the generalized derivative (in the sense of Definition 2.7) of the function $t \mapsto \langle f, u(t)\rangle_V$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f, u(t)\rangle_V = \langle f, v(t)\rangle_V \qquad \textit{for all } f \in V^*.$$

The equivalent formulation in part (iii) is of special interest if we take into account the following characterization (see [EG92, Section 4.9.1]) of absolutely continuous functions on a finite interval such as $(0,T)$. We will use this characterization to prove the equivalence of several weak formulations of the parabolic problem, cf. Proposition 3.13.

Recall that an absolutely continuous function $f\colon (0,T) \to \mathbb{R}$ is almost everywhere on $(0,T)$ differentiable.

Theorem 3.4. *Let $1 \le p < \infty$ and $f\colon (0,T) \to \mathbb{R}$.*

(i) If $f \in W^{1,p}(0,T)$ then there is a unique $\overline{f} \in C[0,T]$ belonging to the equivalence class of f. $\overline{f}$ is absolutely continuous on $[0,T]$ and $\overline{f}' \in L^p(0,T)$.

(ii) Conversely, if the equivalence class of $f \in L^p(0,T)$ contains a function g which is absolutely continuous on $[0,T]$ with $g' \in L^p(0,T)$, then we have $f \in W^{1,p}(0,T)$.

3.2 Evolution triplets and the space $\mathcal{W}(0,T)$

As already mentioned in Section 1.3 one particularity in the functional analytic study of parabolic equations is that one treats the derivative in time in a different way than the derivatives in space. In particular, one considers the solution $u\colon (0,T) \times \Omega \to \mathbb{R}$ as an abstract function $u(t)$ with values in V, which is a Banach space of functions acting on Ω. On the one hand, V should provide enough regularity such that all expressions without time derivative in the weak formulation are well-defined. On the other hand, it would be much too restrictive to require that also the time derivative u' takes values in V. This explains why one considers a second space H with $V \subset H$.

The following result is a preparation to the definition of an evolution triplet.

Proposition 3.5. *Let V and W be two Banach spaces. Assume that $V \overset{d}{\hookrightarrow} W$. Then*

(i) $W^* \hookrightarrow V^*$ and $\langle w^*, v\rangle_V = \langle w^*, v\rangle_W$ for all $w^* \in W^*$, $v \in V$.

(ii) W^* is dense in V^* if V is assumed to be reflexive.

We give the proof following the argumentation in [Emm04, pp. 205–206] and [Zei90, Problem 18.6]:

Proof. *ad (i):* For $w^* \in W^*$ we define $J(w^*)$ to be the restriction of the functional w^* to the subset V of W. Of course, $J(w^*)$ is again a linear functional on V. Moreover, since $V \hookrightarrow W$ there is $c > 0$ such that $\|v\|_W \le c\,\|v\|_V$ for all $v \in V$. Thus

$$\|J(w^*)\|_{V^*} = \sup_{v \in V\setminus\{0\}} \frac{\langle J(w^*), v\rangle_V}{\|v\|_V} \le c \sup_{v \in W\setminus\{0\}} \frac{\langle w^*, v\rangle_W}{\|v\|_W} = c\,\|w^*\|_{W^*}\,.$$

This means that J defines a continuous mapping $W^* \to V^*$. It remains to show that $J\colon W^* \to V^*$ is injective, i.e.

$$\langle J(w^*), v\rangle_V = 0 \quad \text{for all } v \in V \quad \Rightarrow \quad \langle w^*, w\rangle_W = 0 \quad \text{for all } w \in W.$$

But since V is dense in W this is an immediate consequence (see e.g. [Bre11, Corollary 1.8]) of the Hahn-Banach theorem.

Having shown that J is a continuous, injective mapping from W^* to V^* we may now identify each element $J(w^*) \in V^*$ in the image of J with $w^* \in W^*$. In this sense we have $W^* \hookrightarrow V^*$ and

$$\langle w^*, v\rangle_V = \langle w^*, v\rangle_W \qquad \text{for all } w^* \in W^*, v \in V\,. \tag{3.5}$$

ad (ii): We prove this by contradiction: Assume that the closure of W^* w.r.t. the norm of V^* is a proper closed subset of V^*. Then by the Hahn-Banach theorem there is $F \in (V^*)^*$ such that $F(w^*) = 0$ for all $w^* \in W^*$ with $F \neq 0$ as element of $(V^*)^*$. But since V is reflexive we may find $v_F \in V$ such that

$$F(v^*) = \langle v^*, v_F \rangle_V \qquad \text{for all } v^* \in V^*.$$

By the assumptions on F and (3.5) this implies

$$F(w^*) = \langle w^*, v_F \rangle = 0 \qquad \text{for all } w^* \in W^*.$$

Since $v_F \in V \subset W$ this implies $v_F = 0$ and hence $F = 0$ on W, which contradicts $F \neq 0$. □

Definition 3.6 (Evolution triplet)**.** A triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ is called an *evolution triplet* (or a *Gelfand triplet*) if $\mathcal{V}$ is a reflexive, separable Banach space and $\mathcal{H}$ a separable Hilbert space such that $\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H}$.

Remark 3.7. a) By the Riesz-Fréchet representation theorem (see e.g. [Bre11, Theorem 5.5]) we may identify $\mathcal{H}$ with its dual $\mathcal{H}^*$. Proposition 3.5 then shows that

$$\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \cong \mathcal{H}^* \overset{d}{\hookrightarrow} \mathcal{V}^* \quad \text{and} \quad \langle h, v \rangle_{\mathcal{V}} = (h, v)_{\mathcal{H}} \text{ for all } h \in \mathcal{H}, v \in \mathcal{V}.$$

In particular, for all $v, w \in \mathcal{V}$

$$\langle v, w \rangle_{\mathcal{V}} = (v, w)_{\mathcal{H}} = (w, v)_{\mathcal{H}} = \langle w, v \rangle_{\mathcal{V}} \,. \tag{3.6}$$

b) Note that we do not identify $\mathcal{V}$ with its dual $\mathcal{V}^*$ even in the cases when $\mathcal{V}$ is itself a Hilbert space. A very detailed discussion as well as an example concerning the question when to identify or not to identify Hilbert spaces is given in [Bre11, Ch. 5, Remark 3]. ♦

From now on the calligraphic letters $\mathcal{V}$, $\mathcal{V}^*$ and $\mathcal{H}$ are used to denote that these spaces form an evolution triplet in the sense of Definition 3.6.

Definition 3.8. Let $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$ be an evolution triplet. The linear space $\mathcal{W}(0, T; \mathcal{V}, \mathcal{H})$ is defined by

$$\mathcal{W}(0, T; \mathcal{V}, \mathcal{H}) = \left\{ u \in L^2(0, T; \mathcal{V}) \colon u' \text{ exists and } u' \in L^2(0, T; \mathcal{V}^*) \right\} \,. \tag{3.7}$$

The spaces $\mathcal{V}$ and $\mathcal{H}$ are often fixed and then we simply write $\mathcal{W}(0, T)$ to denote this space.

Note that by Proposition 1.10(vi) we have $L^2(0, T; \mathcal{V}^*) \cong (L^2(0, T; \mathcal{V}))^*$.

Furthermore, for a general Banach space V, we define the linear space $C([0, T]; V)$ by

$$C([0, T]; V) = \{ u \colon [0, T] \to V \colon u \text{ is continuous on } [0, T] \} \,. \tag{3.8}$$

The following results are taken from [Zei90, Propositions 23.2, 23.23]. For proofs we refer the reader to this reference or to [Emm04, Satz 8.1.9, Korollar 8.1.10].

Proposition 3.9. *Let V be a Banach space and $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$ an evolution triplet.*

(i) $C([0,T];V)$ *is a Banach space with norm*

$$\|u\|_{C([0,T];V)} = \max_{0\le t\le T} \|u(t)\|_V$$

(ii) $\mathcal{W}(0,T)$ *is a Banach space with norm*

$$\|u\|_{\mathcal{W}(0,T)} = \|u\|_{L^2(0,T;\mathcal{V})} + \left\|u'\right\|_{L^2(0,T;\mathcal{V}^*)} .$$

(iii) $\mathcal{W}(0,T;\mathcal{V},\mathcal{H}) \hookrightarrow C([0,T];\mathcal{H})$. *More precisely, each equivalence class in* $\mathcal{W}(0,T;\mathcal{V},\mathcal{H})$ *contains a uniquely determined function that belongs to* $C([0,T];\mathcal{H})$. *Moreover, there is* $c > 0$ *such that*

$$\max_{0\le t\le T} \|u(t)\|_{\mathcal{H}} \le c\,\|u\|_{\mathcal{W}(0,T)} \qquad \text{for all } u \in \mathcal{W}(0,T).$$

(iv) The following integration by parts formula *holds: For* $u, v \in \mathcal{W}(0,T)$ *and* s, t *with* $0 \le s \le t \le T$ *we have*

$$(u(t), v(t))_{\mathcal{H}} - (u(s), v(s))_{\mathcal{H}} = \int_s^t \left\langle u'(\tau), v(\tau)\right\rangle_{\mathcal{V}} + \left\langle v'(\tau), u(\tau)\right\rangle_{\mathcal{V}} \,\mathrm{d}\tau\,. \tag{3.9}$$

(v) For $u \in \mathcal{W}(0,T)$ *we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u(t), u(t))_{\mathcal{H}} = \left\langle u'(t), u(t)\right\rangle_{\mathcal{V}}$$

in the sense of generalized derivatives and a.e. on $(0,T)$*, respectively.*

3.3 Hilbert space methods for parabolic equations

Recall that $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$ stands for an evolution triplet.

For the rest of this chapter we denote by $\mathcal{B}(t;u,v)$ a mapping $\mathcal{B}\colon (0,T)\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ that satisfies the following properties:

For all $t \in (0,T)$ the mapping $\mathcal{B}(t;\cdot,\cdot)$ is a bilinear form on $\mathcal{V}$, (3.10)

For all $u, v \in \mathcal{V}$ the mapping $t \mapsto \mathcal{B}(t;u,v)$ is measurable on $(0,T)$, (3.11)

there is $M > 0$ such that for all $t \in (0,T)$ and $u, v \in \mathcal{V}$ we have

$$|\mathcal{B}(t;u,v)| \le M\,\|u\|_{\mathcal{V}}\,\|v\|_{\mathcal{V}}\,, \tag{3.12}$$

there are $m > 0$ and $L_0 \ge 0$ such that for all $t \in (0,T)$ and $u \in \mathcal{V}$ we have

$$\mathcal{B}(t;u,u) \ge m\,\|u\|_{\mathcal{V}}^2 - L_0\,\|u\|_{\mathcal{H}}^2\,. \tag{3.13}$$

Remark 3.10. (3.13) is also referred to as Gårding's inequality. If $L_0 = 0$ then (3.13) states that $\mathcal{B}$ is a uniformly (in t) strongly positive bilinear form.

If the bilinear form $\mathcal{B}$ satisfies (3.13) with $L_0 > 0$ one can apply the transformation $\widetilde{u}(t) = e^{-L_0 t}u(t)$ and analogously for $\widetilde{f}$ and $\widetilde{\mathcal{B}}$, such that the transformed problem involves a bilinear form $\widetilde{\mathcal{B}}$ that is uniformly strongly positive. Therefore, it suffices to prove results like Theorem 3.14 assuming that $\mathcal{B}$ is strongly positive, i.e. $\mathcal{B}$ satisfies (3.13) with $L_0 = 0$. For more details on this transformation see [Emm04, p. 218] and [Zei90, Remark 23.25]. ♦

Consider the following abstract initial value problem (cf. [Zei90, Section 23.7]):

Problem 3.11. *For given $u_0 \in \mathcal{H}$ and $f \in L^2(0,T;\mathcal{V}^*)$ find $u \in \mathcal{W}(0,T)$ such that*

$$\text{for all } v \in \mathcal{V}: \quad \frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)_{\mathcal{H}} + \mathcal{B}(t;u(t),v) = \langle f(t), v\rangle_{\mathcal{V}} \quad \text{in } \mathcal{D}'(0,T)\,, \tag{3.14a}$$

$$u(0) = u_0\,. \tag{3.14b}$$

Remark 3.12.

1. By an identity in $\mathcal{D}'(0,T)$ we mean that the appearing derivatives have to be understood in the sense of generalized derivatives of real-valued functions as in Definition 2.7, i.e. (3.14a) is satisfied if for all $v \in \mathcal{V}$ and for all $\phi \in C_c^\infty(0,T)$

$$-\int_0^T (u(t),v)_{\mathcal{H}}\phi'(t)\,\mathrm{d}t + \int_0^T \mathcal{B}(t;u(t),v)\phi(t)\,\mathrm{d}t = \int_0^T \langle f(t),v\rangle_{\mathcal{V}}\,\phi(t)\,\mathrm{d}t. \tag{3.15}$$

2. The initial condition (3.14b) is well-defined since u is a.e. equal to a function in $C([0,T];\mathcal{H})$, see Proposition 3.9(iii). ♦

There are several equivalent formulations of (3.14a), cf. [Emm04, pp. 215–216]. We state some of them in the following proposition.

Proposition 3.13. *Let $u \in \mathcal{W}(0,T)$ and $f \in L^2(0,T;\mathcal{V}^*)$. Then the following conditions are both equivalent to* (3.14a)*:*

(i) For all $v \in \mathcal{V}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t),v\rangle_{\mathcal{V}} + \mathcal{B}(t;u(t),v) = \langle f(t),v\rangle_{\mathcal{V}}\,.$$

(ii) For all $v \in \mathcal{V}$

$$\left\langle u'(t),v\right\rangle_{\mathcal{V}} + \mathcal{B}(t;u(t),v) = \langle f(t),v\rangle_{\mathcal{V}}\,.$$

The identities can be interpreted as identities in $\mathcal{D}'(0,T)$ and as identities that hold almost everywhere[1] *on $(0,T)$. The assumptions on f and u ensure that all appearing terms belong to $L^2(0,T)$.*

[1] In this case the quantifiers have to be understood in the following way: The exceptional set, i.e. the subset of $(0,T)$ where the identities may not hold, has to be independent of $v \in \mathcal{V}$.

Proof. The proof collects arguments from [Emm04, pp. 215–216]. Since $u(t) \in \mathcal{V}$ for almost every $t \in (0,T)$ we obtain by (3.6) that $\langle u(t), v\rangle_{\mathcal{V}} = (u(t), v)_{\mathcal{H}}$ for every $v \in \mathcal{V}$ almost everywhere on $(0,T)$. This proves the equivalence of (i) and (3.14a). Moreover, $u' \in L^2(0,T;\mathcal{V}^*)$ is assumed to be the generalized derivative of $u \in L^2(0,T;\mathcal{V}) \subset L^2(0,T;\mathcal{V}^*)$ and we may interpret $v \in \mathcal{V}$ as element in $(\mathcal{V}^*)^* \cong \mathcal{V}$. Thus, by Lemma 3.3(iii) and (3.6),

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle u(t), v\rangle_{\mathcal{V}} = \left\langle u'(t), v\right\rangle_{\mathcal{V}} \qquad \text{in } \mathcal{D}'(0,T).$$

This shows the equivalence of (3.14a) and (ii).

That all identities may also be considered as identities that hold a.e. on $(0,T)$ follows immediately from the characterization of $W^{1,2}(0,T)$-functions on the real line in Theorem 3.4. To apply this result it remains to show that the functions $t \mapsto \langle u'(t), v\rangle_{\mathcal{V}}$ and $t \mapsto (u(t), v)_{\mathcal{H}}$ belong to $L^2(0,T)$ for every $v \in \mathcal{V}$:

$$\int_0^T \left|\left\langle u'(t), v\right\rangle_{\mathcal{V}}\right|^2 \,\mathrm{d}t \le \int_0^T \left\|u'(t)\right\|_{\mathcal{V}^*}^2 \|v\|_{\mathcal{V}}^2 \,\mathrm{d}t = \left\|u'\right\|_{L^2(0,T;\mathcal{V}^*)}^2 \|v\|_{\mathcal{V}}^2 \,,$$

$$\int_0^T |(u(t), v)_{\mathcal{H}}|^2 \,\mathrm{d}t \le \int_0^T \|u(t)\|_{\mathcal{H}}^2 \|v\|_{\mathcal{H}}^2 \,\mathrm{d}t = \|u\|_{L^2(0,T;\mathcal{H})}^2 \|v\|_{\mathcal{H}}^2 \,.$$

Finally, we show that also $t \mapsto \mathcal{B}(t; u(t), v)$ belongs to $L^2(0,T)$. The measurability follows from (3.11) and the square integrability from (3.12):

$$\int_0^T |\mathcal{B}(t; u(t), v)|^2 \,\mathrm{d}t \le M^2 \int_0^T \|u(t)\|_{\mathcal{V}}^2 \|v\|_{\mathcal{V}}^2 = M^2 \|u\|_{L^2(0,T;\mathcal{V})}^2 \|v\|_{\mathcal{V}}^2 \,.$$

This finishes the proof of Proposition 3.13. □

The following theorem, which can be found for instance in [Chi00, Theorem 11.7] or [Zei90, Theorem 23.A and Corollary 23.26], asserts that there is a unique solution to Problem 3.11.

Theorem 3.14. *Assume that* $\mathcal{B}$ *satisfies* (3.10)-(3.13). *Then Problem 3.11 has a unique solution* $u \in \mathcal{W}(0,T)$. *This solution depends continuously on the initial data. More precisely, there is a constant* $C > 0$ *such that for all* $u_0 \in \mathcal{H}$ *and* $f \in L_2(0,T,\mathcal{V}^*)$

$$\|u\|_{\mathcal{W}(0,T)} \le C \left(\|u_0\|_{\mathcal{H}} + \|f\|_{L^2(0,T;\mathcal{V}^*)}\right). \tag{3.16}$$

For a proof of this theorem we refer to [Chi00, Section 11.3], [GGZ74, §VI.1] or [Zei90, Section 23.9].

Remark 3.15.

1. A common technique to prove existence of solutions is a Galerkin approximation. The general idea is to solve (3.14) on finite dimensional subspaces and then pass to the limit. The solution on the finite dimensional subspaces is obtained via existence & regularity theorems for systems of ordinary differential equations.

   For similar results on existence of solutions to abstract initial value problems like Problem 3.11 via Galerkin approximation we refer also to [Eva10, Section 7.1], [GGZ74, Chapter VI], [LSU68, Chapter III] and [Wlo87, Chapter IV].

2. Let us emphasize that the bilinear form $\mathcal{B}$ in Problem 3.11 may depend on the time variable t. In (3.12),(3.13) we require the constants to be independent of $t \in (0,T)$. In the sense of estimates deduced from (3.12) & (3.13), the change from time-independent forms to forms as in (3.14) causes no particular difficulties.

   However, the systems of ordinary differential equations obtained by the Galerkin method are not the same in the cases of time-dependent and time-independent forms $\mathcal{B}$, cf. [Zei90, p. 439]. ♦

3.4 The bilinear forms associated to $\mathscr{L}^s$

As usual in the functional analytic theory of weak solutions, the bilinear form that will be part of the weak formulation is obtained as L^2-product of $(-\mathscr{L}^s u)$ with a smooth function $v \in C_c^\infty(\Omega)$. We assume the coefficients, the kernel and the solution to be regular enough such that all terms are well-defined in the derivation of the bilinear form. When inspecting the structural properties of these forms in Lemma 3.16 and Lemma 3.17, we provide formal assumptions on the domain and on the coefficients and kernels, respectively.

3.4.1 The local case $s = 1$:

Let $v \in C_c^\infty(\Omega)$. An integration by parts yields

$$\big(-\mathscr{L}^1 u(t,\cdot), v(\cdot)\big)_{L^2(\mathbb{R}^d)} = -\int_{\mathbb{R}^d} \operatorname{div}(A\nabla u)(t,x)v(x)\,\mathrm{d}x = \int_\Omega A(t,x)\nabla u(t,x)\cdot\nabla v(x)\,\mathrm{d}x.$$

Under certain conditions on A the right-hand side of this identity defines a continuous bilinear form on $H_0^1(\Omega)$ that satisfies Gårding's inequality:

Lemma 3.16. *Assume that $A = (a_{ij})_{1\le i,j\le d}$ is a quadratic matrix of functions $a_{ij} \in L^\infty((0,T);L^\infty(\Omega))$ such that $a_{ij} = a_{ji}$ for all $1 \le i,j \le d$. Furthermore assume that there is $\gamma > 0$ such that for all $\xi \in \mathbb{R}^d$ and $(t,x) \in Q_T$*

$$\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j = A(t,x)\xi\cdot\xi \ge \gamma\,|\xi|^2\,. \tag{3.17}$$

Then $\mathcal{E}^1\colon (0,T)\times H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}^1(t;u,v) = \int_\Omega A(t,x)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x$$

satisfies the properties (3.10)-(3.13)*, where $\mathcal{V} = H_0^1(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and $m = L_0 = \gamma$ in* (3.13)*.*

The proof is straightforward, cf. [Eva10, p. 318] or [Zei90, Proposition 23.30]. We present this proof here for the sake of completeness:

Proof. Property (3.10) is obvious. Since all functions a_{ij} are measurable, we easily obtain the measurability of $t \mapsto \mathcal{E}^1(t; u, v)$ for all $u, v \in H_0^1(\Omega)$. To prove the boundedness, observe that for all $t \in (0, T)$

$$\left|\mathcal{E}^1(t; u, v)\right| \leq \max_{1 \leq i,j \leq d} \|a_{ij}\|_{L^\infty(Q_T)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} .$$

Hence, (3.12) is satisfied with $M = \max_{1 \leq i,j \leq d} \|a_{ij}\|_{L^\infty(Q_T)}$, which is independent of t, u, v.

Finally, to prove that $\mathcal{E}^1$ satisfies (3.13), we apply (3.17): For all $u \in H_0^1(\Omega)$ we have

$$\mathcal{E}^1(t; u, u) + \gamma \|u\|^2_{L^2(\Omega)} \geq \int_\Omega \gamma \left|\nabla u(x)\right|^2 \,\mathrm{d}x + \gamma \|u\|^2_{L^2(\Omega)} = \gamma \|u\|^2_{H^1(\Omega)} . \qquad \square$$

3.4.2 The nonlocal case $s \in (0, 1)$

Let $v \in C_c^\infty(\Omega)$ and $D_n = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \colon |x - y| \geq 1/n\right\}$ for $n \in \mathbb{N}$. By definition of the principal value we obtain

$$\begin{aligned}(-\mathscr{L}^s u(t,\cdot), v(\cdot))_{L^2(\mathbb{R}^d)} &= 2 \int_{\mathbb{R}^d} v(x) \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} [u(t, x) - u(t, y)]\, k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= 2 \lim_{n \to \infty} \iint\limits_{D_n} [u(t, x) - u(t, y)]\, v(x) k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x.\end{aligned}$$

Since the kernel is symmetric we have

$$\iint\limits_{D_n} [u(t, x) - u(t, y)]\, v(x) k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x = \iint\limits_{D_n} [u(t, y) - u(t, x)]\, v(y) k_t(x, y) \,\mathrm{d}x \,\mathrm{d}y.$$

Fubini's theorem then implies

$$\begin{aligned}2\iint\limits_{D_n} [u(t, x) - u(t, y)]\, v(x) k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x &= \iint\limits_{D_n} [u(t, x) - u(t, y)]\, v(x) k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &\quad + \iint\limits_{D_n} [u(t, y) - u(t, x)]\, v(y) k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \iint\limits_{D_n} [u(t, x) - u(t, y)][v(x) - v(y)]\, k_t(x, y) \,\mathrm{d}y \,\mathrm{d}x.\end{aligned}$$

If we assume that the form $\mathcal{E}^s(u, u)$ is in some sense comparable to the seminorm in $H^s(\mathbb{R}^d)$ then the limit as $n \to \infty$ of the right-hand side expression exists, and this limit defines a continuous bilinear form on $H_0^s(\Omega)$ that satisfies Gårding's inequality. More precisely we have the following result:

Lemma 3.17. *Define* $\mathcal{E}^s \colon (0,T) \times H^s_0(\Omega) \times H^s_0(\Omega) \to \mathbb{R}$ *by*

$$\mathcal{E}^s(t;u,v) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [u(x)-u(y)]\,[v(x)-v(y)]\,k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Assume that $(t,x,y) \mapsto k_t(x,y)$ *is a nonnegative, measurable function on* $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ *satisfying* $k_t(x,y) = k_t(y,x)$ *for a.e.* t,x,y. *If we assume that there is a constant* $c_1 = c_1(s,d) > 0$ *such that for every* $u \in H^s_0(\Omega)$

$$\mathcal{E}^s(u,u) \leq c_1 \left\|u\right\|^2_{H^s(\mathbb{R}^d)}, \tag{3.18}$$

then $\mathcal{E}^s(t;u,v)$ *satisfies the properties* (3.10)-(3.12), *where* $\mathcal{V} = H^s_0(\Omega)$, $\mathcal{H} = L^2(\Omega)$. *Moreover,*

$$\mathcal{E}^s(t;u,v) = \lim_{n\to\infty} \iint\limits_{D_n} [u(x)-u(y)]\,[v(x)-v(y)]\,k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x. \tag{3.19}$$

If we additionally assume that there are constants $c_2, c_3 > 0$ *that may depend on* s *and* d, *such that for every* $u \in H^s_0(\Omega)$

$$[u,u]_{H^s(\mathbb{R}^d)} \leq c_2\,\mathcal{E}^s(u,u) + c_3 \left\|u\right\|^2_{L^2(\Omega)}, \tag{3.20}$$

then $\mathcal{E}^s$ *satisfies property* (3.13).

Remark 3.18.

a) Clearly, all assumptions in the previous lemma are satisfied if we assume that there are constants $\lambda, \Lambda > 0$ such that

$$\lambda\left|x-y\right|^{-d-2s} \leq k_t(x,y) \leq \Lambda\left|x-y\right|^{-d-2s} \qquad \text{for almost every } (t,x,y) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d.$$

b) The assumptions (3.18) and (3.20) state that the kernel k_t has to be chosen in such a way that the corresponding bilinear form is comparable to the seminorm on $H^s(\mathbb{R}^d)$. It is currently an interesting field of research (cf.[DK11]) in the area of nonlocal forms to find quite mild conditions on the kernel k_t that are sufficient for this comparability in the sense of (3.18) and (3.20), respectively. In particular, in [DK11, Appendix], the authors show that no essential conditions on the behavior of $k_t(x,y)$ for large values of $|x-y|$ have to be imposed in order to guarantee (global) comparability.

c) Let us give two sufficient conditions on the kernel such that (3.18) and (3.20) hold. Condition (3.22) is very restrictive in comparison to those in [DK11, Appendix]. However, they are easy to compare with the assumptions (K_1) and (K_2), which we shall use to prove regularity results in Part III of this thesis.

   Assume that there is $s \in (0,1)$, $\Lambda \geq 1$ and $U \Supset \Omega$ bounded such that the following properties hold: for all $t \in (0,T)$, $x_0 \in \mathbb{R}^d$, $\rho \in (0,1)$ and $v \in H^s(U)$

$$\rho^{-2}\int_{|x_0-y|\leq\rho} |x_0-y|^2\,k_t(x_0,y)\,\mathrm{d}y + \int_{|x_0-y|>\rho} k_t(x_0,y)\,\mathrm{d}y \leq \Lambda\rho^{-2s}, \tag{3.21}$$

$$\begin{aligned}\Lambda^{-1} \iint\limits_{U\,U} [v(x)-v(y)]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y &\leq (1-s) \iint\limits_{U\,U} \frac{[v(x)-v(y)]^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \Lambda \iint\limits_{U\,U} [v(x)-v(y)]^2\, k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.22}$$

For simplicity we assume that $\operatorname{dist}(\Omega, U) \geq 1$ and remark that we may replace 1 by any fixed number greater than zero.

To prove (3.18) let $u \in H_0^s(\Omega)$. By (3.21), (3.22) and the fact that $U^c \cap B_1(x) = \emptyset$ for every $x \in \Omega$ we obtain

$$\begin{aligned}\mathcal{E}^s(t;u,u) &= \iint\limits_{U\,U} [u(x)-u(y)]^2\, k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\int\limits_{U} u^2(x) \int\limits_{U^c} k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \Lambda(1-s) \iint\limits_{U\,U} \frac{[u(x)-u(y)]^2}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x + 2\int\limits_{\Omega} u^2(x) \int\limits_{B_1(x)^c} k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \Lambda(1-s)\, \|u\|^2_{H^s(\mathbb{R}^d)} + 2\Lambda\, \|u\|^2_{L^2(\Omega)} \\ &\leq 2\Lambda\, \|u\|^2_{H^s(\mathbb{R}^d)}\,.\end{aligned}$$

To see that also (3.20) holds let $u \in H_0^s(\Omega)$. We apply the second inequality of (3.22):

$$\begin{aligned}[u]^2_{H^s(\mathbb{R}^d)} &= \iint\limits_{U\,U} \frac{[u(x)-u(y)]^2}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x + 2\int\limits_{\Omega} u^2(x) \int\limits_{B_1(x)^c} |x-y|^{-d-2s}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{\Lambda}{1-s}\,\mathcal{E}^s(u,u) + 2C\, \|u\|^2_{L^2(\Omega)}\,,\end{aligned}$$

where $C = C(d,s)$ is some constant such that $\int_{B_1(x)^c} |x-y|^{-d-2s}\,\mathrm{d}y \leq C$ for every $x \in \Omega$. ♦

Proof of Lemma 3.17. It is easily seen that for every $t \in (0,T)$ the mapping $u, v \mapsto \mathcal{E}^s(t;u,v)$ is bilinear. Property (3.11) follows from the measurability of $t \mapsto k_t(\cdot,\cdot)$.

By Hölder's inequality we have $|\mathcal{E}^s(t;u,v)| \leq \sqrt{\mathcal{E}^s(t;u,u)}\sqrt{\mathcal{E}^s(t;v,v)}$ for every $t \in (0,T)$ and $u, v \in H_0^s(\Omega)$. This shows that (3.12) is satisfied with $M = c_1(d,s)$. Due to the monotone convergence theorem, the existence of the limit and the representation in (3.19) is now an easy consequence of the boundedness.

We can write (3.20) in an equivalent way:

$$\|u\|^2_{H_0^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\mathbb{R}^d)} \leq c_1\mathcal{E}^s(u,u) + (c_2+1)\, \|u\|^2_{L^2(\Omega)}\,,$$

which shows that (3.13) is satisfied with $m = c_2(d,s)$ and $L_0 = c_3(d,s) + 1$. □

3.5 Weak formulation of the initial boundary value problem

We will now introduce two weak formulations of problem (3.2). The difference between the two formulations is the role of the partial derivative with respect to time. On the one hand, we can require the test function to possess (a weak) derivative with respect to time, which in turn implies that there is no such restriction to the solution, cf. Problem 3.19. On the other hand we may require that the solution possesses at least a generalized derivative in some suitable space, which allows us to use test functions with less regularity in the time variable, cf. Problem 3.20.

In view of the the abstract approach in Section 3.3 we consider for each $s \in (0,1]$ the Gelfand triplet

$$\mathcal{V} = H_0^s(\Omega), \quad \mathcal{H} = L^2(\Omega), \quad \mathcal{V}^* = H^{-s}(\Omega).$$

Hence, in addition to the dependence of the underlying function spaces on the domain where the initial boundary value problem is stated, these spaces also depend on the order $s \in (0,1]$ of the underlying operator $\mathscr{L}^s$.

The presentation in this section uses several ideas from [Trö10, Chapter 3].

Problem 3.19. *Find $u \in L^2(0,T;\mathcal{V})$ satisfying for all*

$$\phi \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \text{ with } \phi(T) = 0$$

the variational equality

$$-(u_0, \phi(0))_{\mathcal{H}} - \int_0^T \big(u(t), \phi'(t)\big)_{\mathcal{H}} \,\mathrm{d}t + \int_0^T \mathcal{E}^s(t; u(t), \phi(t)) \,\mathrm{d}t = \int_0^T (f(t), \phi(t))_{\mathcal{H}} \,\mathrm{d}t \,. \tag{3.23}$$

Note that all expressions in (3.23) are finite due to the requirements on u and ϕ. In particular, the terms $\phi(0)$ and $\phi(T)$ are well-defined due to the embedding $H^1((0,T);\mathcal{H}) \hookrightarrow C([0,T];\mathcal{H})$, see Proposition 3.9(iii).

Problem 3.20. *Find $u \in \mathcal{W}(0,T)$ satisfying for all*

$$\phi \in L^2(0,T;\mathcal{V})$$

the variational equality

$$\int_0^T \big\langle u'(t), \phi(t)\big\rangle_{\mathcal{V}^*} \,\mathrm{d}t + \int_0^T \mathcal{E}^s(t; u(t), \phi(t)) \,\mathrm{d}t = \int_0^T (f(t), \phi(t))_{\mathcal{H}} \,\mathrm{d}t \tag{3.24a}$$

and

$$u(0) = u_0. \tag{3.24b}$$

Again, all expressions in (3.24a) are finite and the initial condition (3.24b) makes sense due to the embedding $\mathcal{W}(0,T) \hookrightarrow C([0,T];\mathcal{H})$, see Proposition 3.9(iii).

Remark 3.21. Note that (3.24a) can be seen as an integrated version of the abstract formulation in (3.14a): Assume that $u \in \mathcal{W}(0,T)$ satisfies (3.14a). Define for a given test function $\phi \in L^2(0,T;\mathcal{V})$

$$I_0(\phi) = \{t \in (0,T) \colon \phi(t) \notin \mathcal{V}\}\,.$$

By the properties of the function space $L^2(0,T;\mathcal{V})$ the set $I_0(v)$ is contained in a measurable set of measure zero. Denote by $I_1 \subset (0,T)$ the set of $t \in (0,T)$ such that (3.14a) does not hold. Similarly, I_1 is contained in a set of measure zero. On $(0,T) \setminus (I_0 \cup I_1)$ we may apply (3.14a), and integrating over $(0,T)$ yields (3.24a). ♦

The following two propositions provide the link between the two formulations. Proposition 3.22 says that $u \in \mathcal{W}(0,T)$ is no essential requirement for u as in Problem 3.19. Proposition 3.23 asserts that the formulation in Problem 3.19 is weaker.

Proposition 3.22. *If u satisfies Problem 3.19 then $u \in \mathcal{W}(0,T)$.*

The proof of this assertion in the case $s = 1$ and $a_{ij} = \delta_{ij}$, i.e. $\mathscr{L}^s = \Delta$, can be found in [Trö10, Section 3.4.4]. Following the method there we provide a proof in the general setting.

Proof. Assume u satisfies Problem 3.19. In (3.23) we apply a test function ϕ of the form $\phi(t,x) = v(x)\chi(t)$, where $v \in \mathcal{V}$ and $\chi \in C_c^\infty((0,T))$. Using $\chi(0) = 0$ we obtain

$$-\int_0^T \big(u(t)\chi'(t), v\big)_{\mathcal{H}}\,\mathrm{d}t = -\int_0^T \chi(t)\mathcal{E}^s(u(t),v)\,\mathrm{d}t + \int_0^T \chi(t)\,(f(t),v)_{\mathcal{H}}\,\mathrm{d}t. \tag{3.25}$$

For every fixed $t \in (0,T)$ – possibly after redefining $u(t) = 0$ on a subset of $(0,T)$ of measure zero – the integrands on the right-hand side define linear functionals $F_1(t), F_2(t)\colon \mathcal{V} \to \mathbb{R}$, namely

$$F_1(t)\colon v \mapsto \mathcal{E}^s(u(t),v) \quad \text{and} \quad F_2(t)\colon v \mapsto (f(t),v)_{\mathcal{H}}\,.$$

Obviously, $F_1(t)$ and $F_2(t)$ are bounded and hence $F_1(t), F_2(t) \in \mathcal{V}^*$ for every $t \in (0,T)$. Furthermore, for every $t \in (0,T)$ there is a constant $C = C(A,\Lambda,s)$ such that

$$\|F_1(t)\|_{\mathcal{V}^*} + \|F_2(t)\|_{\mathcal{V}^*} \le C\left(\|u(t)\|_{\mathcal{V}} + \|f(t)\|_{\mathcal{H}}\right).$$

Since the right-hand side of this inequality belongs to $L^2(0,T)$, this shows that $F_1, F_2 \in L^2(0,T;\mathcal{V}^*)$. Setting $F = F_1 + F_2$ we can rewrite (3.25) and obtain that for all $v \in \mathcal{V}$ and $\chi \in C_c^\infty((0,T))$

$$-\int_0^T \big(u(t)\chi'(t), v\big)_{\mathcal{H}}\,\mathrm{d}t = -\int_0^T \langle u(t)\chi'(t), v\rangle_{\mathcal{V}}\,\mathrm{d}t = \int_0^T \langle F(t)\chi(t), v\rangle_{\mathcal{V}}\,\mathrm{d}t$$

This means that for all $\chi \in C_c^\infty((0,T))$

$$-\int_0^T u(t)\chi'(t)\,\mathrm{d}t = \int_0^T F(t)\chi(t)\,\mathrm{d}t \quad \in \mathcal{V}^*\,,$$

i.e. $u' = F \in L^2(0,T;\mathcal{V}^*)$. This finishes the proof of Proposition 3.22. □

Proposition 3.23. *If u satisfies Problem 3.20 then u also satisfies Problem 3.19.*

Proof. Let u satisfy Problem 3.20 and let

$$\phi \in H^1((0,T);\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \text{ with } \phi(T) = 0,$$

i.e. ϕ is an arbitrary test function as in Problem 3.19. In particular, $\phi \in \mathcal{W}(0,T)$. Therefore we can apply the integration by parts formula (3.9) to the first term in (3.24), which implies

$$(u(T)\phi(T))_{\mathcal{H}} - (u(0)\phi(0))_{\mathcal{H}} - \int_0^T \langle u(t), \phi'(t)\rangle_{\mathcal{V}} \,\mathrm{d}t + \int_0^T \mathcal{E}^s(u,\phi)\,\mathrm{d}t = \int_0^T (f(t), \phi(t))_{\mathcal{H}}\,\mathrm{d}t.$$

Finally, we use $\phi(T) = 0$, the initial condition (3.24b) and the property $\langle u(t), \phi'(t)\rangle_{\mathcal{V}} = (u(t), \phi'(t))_{\mathcal{H}}$ to conclude that u satisfies Problem 3.19. □

3.6 Well-posedness result

As an easy consequence of the results in Section 3.4, which show that the bilinear forms $\mathcal{E}^s(t; u, v)$ satisfy all conditions in Theorem 3.14, we are now able to deduce the well-posedness of Problem 3.20.

Theorem 3.24. *If $s = 1$ let A satisfy the assumptions of Lemma 3.16. If $s \in (0,1)$ assume that k_t satisfy the assumptions of Lemma 3.17. Then there is a unique weak solution $u \in \mathcal{W}(0,T)$ to Problem 3.20. This solution depends continuously on the given data: For all $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$ there is a constant $C > 0$ such that*

$$\|u\|_{\mathcal{W}(0,T)} \le C\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}\right). \tag{3.26}$$

The constant C depends on $s \in (0,1]$, on the dimension and on the constants that appear in the assumptions on $\mathcal{E}^s$ in Lemma 3.16 and Lemma 3.17, respectively.

Note that $\mathcal{W}(0,T) = \mathcal{W}(0,T; H_0^s(\Omega), L^2(\Omega))$.

By Proposition 3.23 the previous theorem proves also the existence of a solution to Problem 3.19. However, the question of uniqueness (and the derivation of an estimate comparable to (3.26)) is technically more involved. This is explained by the fact that one may not apply the solution u as test function in Problem 3.19. There are two possibilities to overcome this problem: One may derive an energy estimate similar to (3.26) for the solutions in the Galerkin scheme and then pass to the limit. Another possibility is to use differentiable approximations of the solution u, such as Steklov averages. We refer the reader to [LSU68, Chapter III] for details.

Part III

Local regularity of solutions to the parabolic equation

4 Set-up & Main results

By Ω we denote a bounded domain in $\mathbb{R}^d$ and by I an open, bounded interval in $\mathbb{R}$.

In this last part (Chapters 4-7) of the thesis we study local properties of weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to the equation

$$\partial_t u(t,x) - \mathscr{L}^{\alpha/2} u(t,x) = f(t,x), \qquad (t,x) \in I \times \Omega, \tag{4.1}$$

where $\mathscr{L}^{\alpha/2}$ is an operator of the form[1] (cf. Chapter 3)

$$(\mathscr{L}^{\alpha/2} u)(t,x) = \begin{cases} \operatorname{div}(A \nabla u)(t,x) & \text{if } \alpha = 2, \\ 2 \text{ p.v.} \displaystyle\int_{\mathbb{R}^d} [u(t,y) - u(t,x)]\, k_t(x,y)\, \mathrm{d}y & \text{if } \alpha \in (0,2)\,. \end{cases} \tag{4.2}$$

In the two cases $\alpha = 2$ and $\alpha \in (0,2)$ we use the following notation:

- $A = (a_{ij})_{1 \le i,j \le d}$ denotes a symmetric matrix of functions $a_{ij} \colon (0,T) \times \Omega \to \mathbb{R}$. We assume that $a_{ij} \in L^\infty(I; L^\infty(\Omega))$ for every $i,j = 1, \dots, d$ and that there is $\lambda > 0$ such that $A(t,x)\xi \cdot \xi \ge \lambda |\xi|^2$. Thus we may assume that there are $0 < \lambda \le \Lambda < \infty$ such that for all $(t,x) \in I \times \Omega$ and for all $\xi \in \mathbb{R}^d$

  $$\lambda |\xi|^2 \le A(t,x)\xi \cdot \xi \le \Lambda |\xi|^2\,. \tag{4.3}$$

  We will refer to the equation

  $$\partial_t u(t,x) - \mathscr{L}^1 u(t,x) = f(t,x), \qquad (t,x) \in I \times \Omega, \tag{PE$_2$}$$

  as *second order parabolic equation.* Local regularity results for this equation were shown by Moser. We restate and reprove these results in Chapter 7.

- k denotes a symmetric kernel $k \colon (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, $(t,x,y) \mapsto k_t(x,y)$, which typically has a certain singularity at the diagonal $x = y$. The detailed assumptions on $k_t(x,y)$ will be given below. We will refer to the equation

  $$\partial_t u(t,x) - \mathscr{L}^{\alpha/2} u(t,x) = f(t,x), \qquad (t,x) \in I \times \Omega, \tag{PE$_\alpha$}$$

  as *α-order* or *fractional order parabolic equation.* The proofs of the regularity results for this equation are given in Chapter 6. These results have already been published by the author in a joint publication [FK13] with M. Kassmann.

[1] We recall from Chapter 3 that in both cases the operator may not exist even for smooth functions u.

Note that in the case

$$k_t(x,y) = \frac{\mathcal{A}_{d,-\alpha}}{|x-y|^{d+\alpha}}$$

with $\mathcal{A}_{d,-\alpha}$ defined in (2.14), the integro-differential operator $\mathscr{L}^s$ defined by (4.2) is equal to the pseudo-differential operator $(-\Delta)^{\alpha/2}$ with symbol $|\xi|^\alpha$, cf. Section 2.7. Thus the operator in equation (PE$_\alpha$) can be seen as an integro-differential operator of order α with bounded measurable coefficients. From this point of view it is natural to denote the order of the equation rather by $\alpha \in (0,2)$ than by $s \in (0,1)$. This is one of several reasons why we stick to this notation for the rest of the thesis. The results from the previous sections are applied to $s = \alpha/2$. The notation sup and inf are also used to denote ess-sup and ess-inf, respectively.

4.1 Assumptions on $k_t(x,y)$

Let us specify the class of admissible kernels. We assume that the kernels k are of the form

$$k_t(x,y) = a(t,x,y)k_0(x,y)$$

for measurable functions $k_0 \colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ and $a \colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \left[\frac{1}{2}, 1\right]$, which are symmetric with respect to x and y.

Definition 4.1. Fix $\alpha_0 \in (0,2)$ and $\Lambda \geq \max(1, \alpha_0^{-1})$.

(i) We say that a kernel k *belongs to* $\mathcal{K}(\alpha_0, \Lambda)$, if there is $\alpha \in (\alpha_0, 2)$ such that k_0 satisfies the following properties: for every $x_0 \in \mathbb{R}^d$, $\rho \in (0,2)$ and $v \in H^{\alpha/2}(B_\rho(x_0))$

$$\rho^{-2} \int_{|x_0-y| \leq \rho} |x_0 - y|^2 \, k_0(x_0,y) \, \mathrm{d}y + \int_{|x_0-y|>\rho} k_0(x_0,y) \, \mathrm{d}y \leq \Lambda \rho^{-\alpha}, \tag{K$_1$}$$

$$\begin{aligned} \Lambda^{-1} \iint\limits_{B\,B} [v(x)-v(y)]^2 k_0(x,y) \, \mathrm{d}x \, \mathrm{d}y &\leq (2-\alpha) \iint\limits_{B\,B} \frac{[v(x)-v(y)]^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\ &\leq \Lambda \iint\limits_{B\,B} [v(x)-v(y)]^2 \, k_0(x,y) \, \mathrm{d}x \, \mathrm{d}y, \quad \text{where } B = B_\rho(x_0). \end{aligned} \tag{K$_2$}$$

(ii) We say that a kernel k *belongs to* $\mathcal{K}'(\alpha_0, \Lambda)$ if $k \in \mathcal{K}(\alpha_0, \Lambda)$ and if

$$\sup_{x \in B_2(0)} \int_{\mathbb{R}^d \setminus B_3(0)} |y|^{1/\Lambda} \, k_0(x,y) \, \mathrm{d}y \leq \Lambda \, . \tag{K$_3$}$$

Note that (K$_3$) is satisfied if $\int_{\mathbb{R}^d \setminus B_3(0)} |y|^\delta \, k_0(x,y) \, \mathrm{d}y$ is uniformly bounded in $B_2(0)$ for some $\delta > 0$. We use this notation in order to avoid a third constant to appear in the class $\mathcal{K}'$.

As we will show, the conditions (K$_1$) and (K$_2$) are sufficient to prove a weak Harnack inequality for nonnegative supersolutions of (PE$_\alpha$). The additional assumption (K$_3$) is needed for the proof of Hölder regularity.

4.2 Local weak solutions

In Chapter 3 we studied existence and uniqueness of solutions to the boundary value *problem* (3.2). In what follows we will prove local regularity results for functions that are only solutions to the *equation*, disregarding the boundary and initial values prescribed in (3.2). As a first consequence we obtain that a solution is no longer required to vanish on Ω^c. Thus a solution – written as abstract function – may take values in $H^{\alpha/2}(\Omega)$ (instead of $H_0^{\alpha/2}(\Omega)$ as in (3.2)). Furthermore, we only require the equation to hold locally, i.e. on every compact subset of $I \times \Omega$.

For both the second order and fractional order case we take the formulation in Problem 3.19 as a starting point. In particular, the assumption $u \in C_{loc}(I; L^2_{loc}(\Omega'))$ is motivated by Proposition 3.22 and the embedding

$$\mathcal{W}(I'; H_0^s(\Omega'), L^2(\Omega')) \hookrightarrow C(\overline{I'}; L^2(\Omega'))\,,$$

see Proposition 3.9(iii).

The definitions in this section follow [DGV11, Section 3.1].

4.2.1 Second order parabolic equation

Definition 4.2. Assume $Q = I \times \Omega \subset \mathbb{R}^{d+1}$ and $f \in L^\infty(Q)$. We say that u is a *supersolution of* (PE_2) *in* $Q = I \times \Omega$, if

(i) $u \in C_{loc}(I; L^2_{loc}(\Omega)) \cap L^2_{loc}(I; H^1_{loc}(\Omega))$,

(ii) for every subdomain $\Omega' \Subset \Omega$, for every subinterval $I' = [t_1, t_2] \subset I$ and for every nonnegative test function $\phi \in H^1(I'; L^2(\Omega')) \cap L^2(I'; H^1_0(\Omega'))$,

$$\int_{\Omega'} \phi(t_2, x) u(t_2, x)\,\mathrm{d}x - \int_{\Omega'} \phi(t_1, x) u(t_1, x)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_{\Omega'} u(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_{\Omega'} A(t,x)\nabla u(t,x)\cdot\nabla\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \geq \int_{t_1}^{t_2}\int_{\Omega'} f(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \quad (4.4)$$

From now on "$\partial_t u - \mathscr{L}^1 u \geq f$ in $I \times \Omega$" denotes that u is a supersolution in $I \times \Omega$ in the sense of this definition. Subsolutions and solutions[2] are defined analogously.

Note that the values of $\phi(t_1)$ and $\phi(t_2)$ are well-defined due to the embedding

$$H^1(I'; L^2(\Omega')) \hookrightarrow C(\overline{I}'; L^2(\Omega'))\,,$$

see Proposition 3.9(iii).

[2] In the definition of a solution there is no restriction on the sign of the test function ϕ.

4.2.2 Fractional order parabolic equation

Let us recall from Lemma 3.17 the bilinear form that is associated to the operator $\mathscr{L}^{\alpha/2}$:

$$\mathcal{E}^{\alpha/2}(t;u,v) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [u(x)-u(y)]\,[v(x)-v(y)]\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where we assume that the kernel k_t satisfies (K_1) and (K_2) on page 56. For simplicity we shall omit the upper index in the bilinear form and write $\mathcal{E}_t(u,v) = \mathcal{E}^{\alpha/2}(t;u,v)$ if there is no danger of confusion. We will reserve the letter $\mathcal{E}$ for the bilinear form in the the fractional order case.

Definition 4.3. Assume $Q = I\times\Omega \subset \mathbb{R}^{d+1}$ and $f \in L^\infty(Q)$. We say that $u \in L^\infty(I;L^\infty(\mathbb{R}^d))$ is a *supersolution of* (PE_α) *in* $Q = I\times\Omega$, if

(i) $u \in C_{loc}(I;L^2_{loc}(\Omega)) \cap L^2_{loc}(I;H^{\alpha/2}_{loc}(\Omega))$,

(ii) for every subdomain $\Omega' \Subset \Omega$, for every subinterval $I' = [t_1,t_2] \subset I$ and for every nonnegative test function $\phi \in H^1(I';L^2(\Omega')) \cap L^2(I';H^{\alpha/2}_0(\Omega'))$,

$$\int\limits_{\Omega'} \phi(t_2,x)u(t_2,x)\,\mathrm{d}x - \int\limits_{\Omega'} \phi(t_1,x)u(t_1,x)\,\mathrm{d}x \\ - \int\limits_{t_1}^{t_2}\int\limits_{\Omega'} u(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{t_1}^{t_2} \mathcal{E}_t(u,\phi)\,\mathrm{d}t \geq \int\limits_{t_1}^{t_2}\int\limits_{\Omega'} f(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.5}$$

From now on "$\partial_t u - \mathscr{L}^{\alpha/2}u \geq f$ in $I\times\Omega$" denotes that u is a supersolution in $I\times\Omega$ in the sense of this definition. Subsolutions and solutions[3] are defined analogously.

The values of $\phi(t_1)$ and $\phi(t_2)$ are well-defined due to the embedding

$$H^1(I';L^2(\Omega')) \hookrightarrow C(\overline{I}';L^2(\Omega')),$$

see Proposition 3.9(iii).

In comparison with Definition 4.2 we added the assumption $u \in L^\infty(I;L^\infty(\mathbb{R}^d))$, i.e. we allow only for bounded solutions. This is explained by the nonlocality of $\mathcal{E}_t$ in (4.5): Let ϕ be a test function as in Definition 4.3. We may extend ϕ by zero outside of Ω', by Remark 2.30c) this extension belongs to $H^{\alpha/2}(\mathbb{R}^d)$. Let $\Omega_0 \subset \mathbb{R}^d$ such that $\Omega' \Subset \Omega_0 \Subset \Omega$. Then, by (K_1) and (K_2), for almost every $t \in I'$

$$\begin{aligned} |\mathcal{E}_t(u,\phi)| &\leq \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(t,x)-u(t,y)|\,|\phi(t,x)-\phi(t,y)|\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{\Omega_0\,\Omega_0} |u(t,x)-u(t,y)|\,|\phi(t,x)-\phi(t,y)|\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

[3] In the definition of a solution there is no restriction on the sign of the test function ϕ.

$$
\begin{aligned}
&+ 2 \iint\limits_{\Omega_0^c\,\Omega_0} \phi(t,x)\,|u(t,x) - u(t,y)|\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \Lambda\,\|u(t)\|_{H^{\alpha/2}(\Omega_0)}\,\|\phi(t)\|_{H^{\alpha/2}(\mathbb{R}^d)} + 4\,\|u(t)\|_{L^\infty(\mathbb{R}^d)} \int_{\Omega'} \phi(t,x) \int_{\Omega_0^c} k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq C\,\|\phi(t)\|_{H^{\alpha/2}(\mathbb{R}^d)} \left(\|u(t)\|_{H^{\alpha/2}(\Omega_0)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)} \right),
\end{aligned}
$$

where the constant C depends on Λ, d and the distance of Ω' to Ω_0. The assumptions on u and ϕ ensure that the terms in the last line belong to $L^1(I')$.

4.3 Main results: Weak Harnack inequality and Hölder regularity for fractional order parabolic equations

In Section 6.5 we prove the following result:

Theorem 4.4 (Weak Harnack inequality)**.** *Let $k \in \mathcal{K}(\alpha_0, \Lambda)$ for some $\alpha_0 \in (0,2)$ and $\Lambda \geq 1$. Then there is a constant $C = C(d, \alpha_0, \Lambda)$ such that for every supersolution u of* (PE$_\alpha$) *on $Q = (-1,1) \times B_2(0)$ which is nonnegative in $(-1,1) \times \mathbb{R}^d$ the following inequality holds:*

$$
\|u\|_{L^1(U_\ominus)} \leq C \left(\inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right) \tag{HI}
$$

where $U_\oplus = \big(1 - (\frac{1}{2})^\alpha, 1\big) \times B_{1/2}(0)$, $U_\ominus = \big(-1, -1 + (\frac{1}{2})^\alpha\big) \times B_{1/2}(0)$.

Note that the domains $U_\oplus$, $U_\ominus$ can be replaced by $\big(\frac{3}{4}, 1\big) \times B_{1/2}(0)$, $\big(-1, -\frac{3}{4}\big) \times B_{1/2}(0)$, respectively.

The proof of the following interior Hölder regularity estimate will be given in Section 6.6.

Theorem 4.5 (Hölder regularity)**.** *Let $k \in \mathcal{K}'(\alpha_0, \Lambda)$ for some $\alpha_0 \in (0,2)$ and $\Lambda \geq 1$. Then there is a constant $\beta = \beta(d, \alpha_0, \Lambda)$ such that for every solution u of* (PE$_\alpha$) *in $Q = I \times \Omega$ with $f = 0$ and every $Q' \Subset Q$ the following estimate holds:*

$$
\sup_{(t,x),(s,y) \in Q'} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/\alpha}\right)^\beta} \leq \frac{\|u\|_{L^\infty(I \times \mathbb{R}^d)}}{\eta^\beta}\,, \tag{HC}
$$

with some constant $\eta = \eta(Q, Q') > 0$.

Similar to the previous result, $|t-s|^{1/\alpha}$ can be replaced by $|t-s|^{1/2}$.

Remark 4.6.

a) In this work we concentrate on the most simple characteristic setting in order to explain the main arguments. In particular, one can obtain Theorem 4.4 for supersolutions u in general domains in $\mathbb{R}^{d+1}$ by rescaling u to a function that is a solution in a standard cylinder $(-1,1)\times B_2(0)$ (cf. Lemma 5.1) and by applying some covering arguments (cf. [Mos71, Lemma 4]).

   Let us also mention that the $f \in L^\infty(Q)$ for the inhomogeneity f is not optimal in (HI). However, this assumption suffices to derive the Hölder regularity estimate from the weak Harnack inequality.

b) Note that a strong Harnack inequality, i.e. $\|u\|_{L^1(U_\ominus)}$ replaced by $\sup_{U_\ominus} u$ in (HI), cannot be obtained under our assumptions. A counterexample was found by Bogdan and Sztonyk [BS05, p. 148], see also the discussion in [KRS13, Section 7.A]. Thus, the strong formulation of Harnack's inequality fails although conditions (K_1) and (K_2) ensure non-degeneracy of the operator L in (4.2). In this sense the nonlocal case differs from the case of local diffusion operators. ♦

Example 4.7 illustrates the robustness for $\alpha \to 2-$. Example 4.8 shows that $k_t(x,y)$ may be zero on a large set around the diagonal $x = y$.

Example 4.7. Consider a sequence of kernels $(k^n)_{n\in\mathbb{N}}$ such that $k^n \in \mathcal{K}(\alpha_0,\Lambda)$ for every $n \in \mathbb{N}$ and some $\alpha_0 \in (0,2)$, $\Lambda \geq 1$ independent of $n \in \mathbb{N}$. For instance $k_t(x,y)$ defined[4] by

$$k_t^n(x,y) = (2-\alpha_n)\,|x-y|^{2-\alpha_n} \qquad \text{with } \alpha_n = 2 - \tfrac{1}{n+1} \tag{4.6}$$

belongs to $\mathcal{K}(1,\Lambda)$ for some $\Lambda = \Lambda(d) \geq 1$. Let (u_n) be a sequence of solutions to the corresponding equation (PE_α). Then (HI) holds true for the sequence (u_n) uniformly in $n \in \mathbb{N}$. Furthermore, if (k^n) additionally satisfies (K_3) uniformly in $n \in \mathbb{N}$ – such as the kernels in (4.6) – then also (HC) holds uniformly in $n \in \mathbb{N}$. Note that the theorems are interesting and new even if $\alpha_n = \alpha$ for some $\alpha \in (0,2)$ and all $n \in \mathbb{N}$. ♦

Example 4.8. Fix $\alpha_0 \in (0,2)$. Assume $k_t(x,y) = \frac{2-\alpha}{|x-y|^{d+\alpha}}$ for some $\alpha \in (\alpha_0, 2)$. Let $\zeta \in \mathbb{S}^{d-1}$ and $r \in (0,1)$. Set

$$S = \mathbb{S}^{d-1} \cap (B_r(\zeta) \cup B_r(-\zeta)) \qquad \text{and } k_t'(x,y) = k_t(x,y)\mathbb{1}_S\left(\tfrac{x-y}{|x-y|}\right).$$

Then we have $k' \in \mathcal{K}(\alpha_0,\Lambda)$ for some $\Lambda \geq 1$. ♦

[4] Note that the factor $(2-\alpha_n)$ in (4.6) is essential to find Λ and α_0 independent of $n \in \mathbb{N}$.

5 Auxiliary Results

In this chapter we provide all technical tools that are needed to apply Moser's iteration technique. The results concerning the fractional order case have already been published in a joint publication by M. Kassmann and the author [FK13].

Four basic tools for Moser's proof of the regularity results can be singled out, cf. the description in [SC95]:

1. In order to deduce Hölder regularity from the (weak) Harnack inequality, a certain *scaling property* of the equation is used. This scaling result is provided in Section 5.1.

2. The basic steps of Moser's iteration are proved by means of *Sobolev's inequality*, which is shown in Section 5.4.1.

3. A *Poincaré inequality* is needed in order to prove estimates for $\log u$. In contrast to Moser's proof for elliptic equations, a *weighted* Poincaré inequality is needed in the situation of a parabolic equation. Heuristically, this is explained by the observation that one has no information on the sign of $\partial_t v$, where v is the auxiliary function in the proof of Proposition 6.9 and Proposition 7.5. The weighted Poincaré inequality is established in Section 5.4.2.

4. The John-Nirenberg embedding, which ensures exponential integrability of BMO functions. Moser applied this embedding in [Mos61, Mos64, Mos67]. In Moser's own words, the proof of the parabolic generalization of the John-Nirenberg embedding "is quite intricate and it was desirable to avoid it entirely"[1]. Therefore, Moser showed in [Mos71] that indeed this argument can be avoided by adapting a lemma which was found by Bombieri and Giusti [BG72]. We reprove the *Bombieri-Giusti lemma* in Section 5.6.

5.1 Standard cylindrical domains and scaling property

Let us briefly explain the scaling behavior of equation (PE_α). Here and later we will use the following notation. Define

$$B_r(x_0) = \left\{ x \in \mathbb{R}^d \colon\ |x - x_0| < r \right\}, \qquad r > 0$$

and (cf. Figure 5.1)

$$\begin{aligned} I_r(t_0) &= (t_0 - r^\alpha, t_0 + r^\alpha), & Q_r(x_0, t_0) &= I_r(t_0) \times B_r(x_0), \\ I_\oplus(r) &= (0, r^\alpha), & Q_\oplus(r) &= I_\oplus(r) \times B_r(0), \\ I_\ominus(r) &= (-r^\alpha, 0), & Q_\ominus(r) &= I_\ominus(r) \times B_r(0). \end{aligned}$$

[1] [Mos71, p. 727]

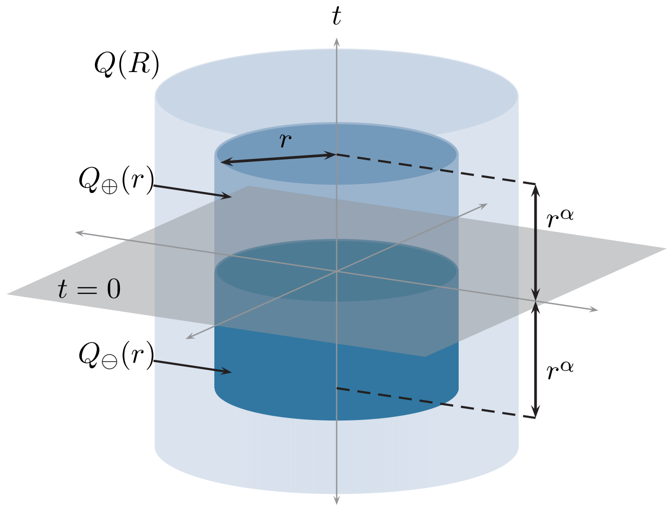

Figure 5.1: Standard cylindrical domains

Lemma 5.1 (Scaling property)**.** *Fix $\alpha_0 \in (0,2)$, $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$ and $r > 0$. Assume that there is $\alpha \in (\alpha_0, 2)$ and $\Lambda \geq \max(1, \alpha_0^{-1})$ such that the kernel $k_t(x,y) = a(t,x,y)k_0(x,y)$ satisfies the following properties: for every $x_0 \in \mathbb{R}^d$, $\rho \in (0, 2r)$ and $v \in H^{\alpha/2}(B_\rho(x_0))$*

$$\rho^{-2} \int_{|x_0-y|\leq\rho} |x_0 - y|^2 \, k_0(x_0, y) \,\mathrm{d}y + \int_{|x_0-y|>\rho} k_0(x_0, y) \,\mathrm{d}y \leq \Lambda \rho^{-\alpha} \,, \tag{5.1a}$$

$$\begin{aligned} \Lambda^{-1} \iint\limits_{B\,B} [v(x) - v(y)]^2 \, k_0(x, y) \,\mathrm{d}x \,\mathrm{d}y &\leq (2 - \alpha) \iint\limits_{B\,B} \frac{[v(x) - v(y)]^2}{|x - y|^{d+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \\ &\leq \Lambda \iint\limits_{B\,B} [v(x) - v(y)]^2 \, k_0(x, y) \,\mathrm{d}x \,\mathrm{d}y, \quad \textit{where } B = B_\rho(x_0) \,, \end{aligned} \tag{5.1b}$$

$$\sup_{x\in B_{2r}(\xi)} \int_{\mathbb{R}^d \setminus B_{3r}(\xi)} |y|^{1/\Lambda} \, k_0(x, y) \,\mathrm{d}y \leq \Lambda \, r^{1/\Lambda-\alpha} \,. \tag{5.1c}$$

Let u be a supersolution of equation (PE$_\alpha$) *in $Q \Supset Q_r(\xi, \tau)$ with a kernel $k_t(x,y)$ that satisfies* (5.1a)-(5.1c)*. Then $\widetilde{u}(t,x) = u(r^\alpha t + \tau, rx + \xi)$ satisfies*

$$\begin{aligned} &\int_{B_1} \phi(t,x)\widetilde{u}(t,x) \,\mathrm{d}x \Big|_{t=-1}^{1} - \int_{Q_1(0)} \widetilde{u}(t,x)\partial_t\phi(t,x) \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_{-1}^{1} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \, [\phi(t,x) - \phi(t,y)] \, \widetilde{k}_t(x,y) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \geq \int\limits_{Q_1(0)} r^\alpha \widetilde{f}(t,x)\phi(t,x) \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{5.2}$$

for every nonnegative test function

$$\phi \in H^1((-1,1); L^2(B_1(0))) \cap L^2((-1,1); H_0^{\alpha/2}(B_1(0))) \,, \tag{5.3}$$

where $\widetilde{f}(t,x) = f(tr^\alpha + \tau, rx + \xi)$ *and*

$$\widetilde{k}_t(x,y) = a(r^\alpha t + \tau, rx + \xi, ry + \xi)\, r^{d+\alpha}\, k_0(rx + \xi, ry + \xi).$$

In particular, $\widetilde{k}$ *belongs to* $\mathcal{K}'(\alpha_0, \Lambda)$.

It is readily checked that the second order parabolic equation satisfies an analog scaling property.

Proof. Let u be a supersolution of (PE_α) in $Q \Supset Q_r(\xi,\tau)$ with a kernel $k_t(x,y)$ that satisfies the conditions of Lemma 5.1. Let ϕ be a test function as in (5.3). For $r > 0$ and ξ, τ as in Lemma 5.1 define a diffeomorphism

$$\Psi\colon Q(1) \to Q_r(\xi,\tau), \qquad \Psi(t,x) = (r^\alpha t + \tau, rx + \xi)\,.$$

The two components of Ψ are denoted by $\Psi_\xi\colon B_1(0) \to B_r(\xi)$ and $\Psi_\tau\colon I_1(0) \to I_r(\tau)$. By the change-of-variables formula we find

$$\int\limits_{B_1} \phi(t,x)\widetilde{u}(t,x)\,\mathrm{d}x\Big|_{t=-1}^{1} = r^{-d} \int\limits_{B_r(\xi)} (\phi \circ \Psi^{-1})(t,x)u(t,x)\,\mathrm{d}x\Big|_{t=\Psi_\tau(-1)}^{\Psi_\tau(1)},$$

$$\int\limits_{Q_1(0)} \widetilde{u}(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t = r^{-d-\alpha} \int\limits_{Q_r(\xi,\tau)} u(t,x) r^\alpha \partial_t(\phi\circ\Psi^{-1})(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,$$

$$\int\limits_{-1}^{1} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [\widetilde{u}(t,x) - \widetilde{u}(t,y)]\,[\phi(t,x) - \phi(t,y)]\,\widetilde{k}_t(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$= r^{-2d-\alpha} \int\limits_{\tau - r^\alpha}^{\tau + r^\alpha} r^{d+\alpha}\mathcal{E}_t(u, \phi\circ\Psi^{-1})\,\mathrm{d}t\,,$$

$$\int\limits_{Q_1(0)} r^\alpha \widetilde{f}(t,x)\phi(t,x)\;\mathrm{d}x\,\mathrm{d}t = r^{-d+\alpha} \int_{Q_r(\xi,\tau)} r^\alpha f(t,x)(\phi\circ\Psi^{-1})(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Now observe that the factor r^{-d} cancels out and that $\phi\circ\Psi^{-1}$ is an admissible test function in (4.5). This proves (5.2).

It remains to verify (K_1)-(K_3) for $\widetilde{k}_t(x,y)$. To shorten notation we assume $\xi = 0$, $\tau = 0$. Let $x_0 \in \mathbb{R}^d$, $\rho \in (0,2)$. Then we may deduce from (5.1a)

$$\begin{aligned}
&\rho^{-2} \int_{B_\rho(x_0)} |x_0 - y|^2\, r^{d+\alpha} k_0(rx, ry)\,\mathrm{d}y + \int_{B_\rho(x_0)^c} r^{d+\alpha} k_0(rx, ry)\,\mathrm{d}y \\
&= (r\rho)^{-2} r^\alpha \int_{B_{r\rho}(rx_0)} |rx_0 - y|^2\, k_0(rx, y)\,\mathrm{d}y + r^\alpha \int_{B_{r\rho}(rx_0)^c} k_0(rx, y)\,\mathrm{d}y \\
&\le \Lambda r^\alpha (r\rho)^{-\alpha} = \Lambda \rho^\alpha\,.
\end{aligned}$$

(5.1b) implies for $B = B_\rho(x_0)$, $\widetilde{B} = B_{r\rho}(rx_0)$ and $v \in H^{\alpha/2}(B)$

$$\begin{aligned}\iint\limits_{B\,B} |v(x)-v(y)|^2\, r^{d+\alpha} k_0(rx,ry)\,\mathrm{d}x\,\mathrm{d}y &= \iint\limits_{\widetilde{B}\,\widetilde{B}} |v(x/r)-v(y/r)|^2\, r^{-d+\alpha} k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \Lambda(2-\alpha) r^{-d+\alpha} \iint\limits_{\widetilde{B}\,\widetilde{B}} \frac{|v(x/r)-v(y/r)|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \Lambda(2-\alpha) \iint\limits_{B\,B} \frac{|v(x)-v(y)|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

The proof of the lower bound in (K_2) for $\widetilde{k}_0$ is similar.

Finally, we deduce (K_3) from (5.1c):

$$\begin{aligned}&\sup_{x\in B_2(0)} \int_{B_3(0)^c} |y|^{1/\Lambda}\, r^{d+\alpha} k_0(rx,ry)\,\mathrm{d}y \\ &\qquad = r^{\alpha-\Lambda^{-1}} \sup_{x\in B_{2r}(0)} \int_{B_{3r}(0)^c} |y|^{1/\Lambda}\, r^{\alpha} k_0(x,y)\,\mathrm{d}y \le \Lambda\end{aligned}$$

This finishes the proof of Lemma 5.1. □

5.2 Alternative formulation in terms of Steklov averages

In the main proofs we do not use (4.5) directly. The starting point in these proofs is the inequality

$$\int_{\Omega'} \partial_t u(t,x)\phi(t,x)\,\mathrm{d}x + \mathcal{E}_t(u(t,\cdot),\phi(t,\cdot)) \ge \int_{\Omega'} f(t,x)\phi(t,x)\,\mathrm{d}x \qquad \text{for a.e. } t\in I, \quad (5.4)$$

where we apply test functions of the form $\phi(t,x) = \psi(x)u^{-q}(t,x)$, $q>0$, where u is a positive supersolution in $I\times\Omega$ and ψ a suitable cut-off function. In particular, we assume that u is a.e. differentiable in time.

The aim of this section is to justify the use of (5.4) instead of (4.5) in our main technical results, Proposition 6.1, Proposition 6.3 and Proposition 6.9. Thus, we can work with supersolutions u as if they were a.e. differentiable with respect to t. This approach is standard when proving regularity results for solutions to second order parabolic problems, cf. [AS67, Sec. 9]. Nevertheless, we provide details and show that the nonlocality (in space) of the underlying operator does not form a serious obstacle.

In the above mentioned proofs we multiply (5.4) with some piecewise differentiable function $\chi\colon \mathbb{R}\to[0,\infty)$ and integrate over some time interval $(t_1,t_2)\subset I$. This implies, together with the chain rule and partial integration,

$$\Big[\chi(t)\int_{\Omega'} \psi(x)w(t,x)\,\mathrm{d}x\Big]_{t=t_1}^{t_2} + \int_{t_1}^{t_2} \chi(t)\mathcal{E}_t(u,\phi)\,\mathrm{d}t$$

$$\geq \int_{t_1}^{t_2} \chi(t) \int_{\Omega'} f(t,x)\phi(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \chi'(t) \int_{\Omega'} \psi(x)w(t,x)\,\mathrm{d}x\,\mathrm{d}t, \quad (5.5)$$

where

$$w(t,x) = \begin{cases} \frac{1}{1-q} u^{1-q}(t,x) & \text{if } q \neq 1, \\ \log u(t,x) & \text{if } q = 1. \end{cases} \quad (5.6)$$

Inequality (5.5) is the main source for our estimates. Let us now show how to derive (5.5) from (4.5). To this end, we recall the concept of Steklov averages in Section 1.4: Let $I = (T_1, T_2)$, $Q = I \times \Omega$. For $v \in L^1(Q)$ and $0 < h < T_2 - T_1$ define

$$v_h(t,\cdot) = \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} v(\cdot, s)\,\mathrm{d}s & \text{for } T_1 < t < T_2 - h, \\ 0 & \text{for } t \geq T_2 - h. \end{cases}$$

Fix $t \in I$, $\Omega' \Subset \Omega$ and $h > 0$ such that $t + h \in I$. In (4.5) we choose $\phi(s,x) = \eta(x)$ with $\eta \in H_0^{\alpha/2}(\Omega')$, $t_1 = t$ and $t_2 = t + h$. Dividing by h we obtain

$$\int_{\Omega'} \partial_t u_h(t,x)\eta(x)\,\mathrm{d}x + \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(s,\cdot), \eta(\cdot))\,\mathrm{d}s \geq \int_{\Omega'} f_h(t,x)\eta(x)\,\mathrm{d}x\ , \quad (5.7)$$

valid for all $t \in I$ and $\eta \in H_0^{\alpha/2}(\Omega')$.

Next we choose in (5.7) (for fixed $t \in I$) test functions of the form $\eta = \chi(t)\psi u_h^{-q}(t,\cdot)$, $q > 0$, where ψ, χ are suitable cut-off functions. Then we integrate (5.7) over some time interval (t_1, t_2). Hence, with w as in (5.6),

$$\int_{t_1}^{t_2} \chi(t) \int_{\Omega'} \psi(x)\partial_t w_h(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \chi(t)\frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(s,\cdot), \psi(\cdot)u_h^{-q}(t,\cdot))\,\mathrm{d}s\,\mathrm{d}t$$
$$\geq \int_{t_1}^{t_2} \chi(t) \int_{\Omega'} f_h(t,x)\psi(x)u_h^{-q}(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

After partial integration in the first term we pass to the limit $h \to 0$. Lemma 1.14 and Lemma 5.2 from below then imply

$$\left[\chi(t)\int_{\Omega'} \psi(x)w(t,x)\,\mathrm{d}x\right]_{t=t_1}^{t_2} + \int_{t_1}^{t_2} \chi(t)\mathcal{E}_t(u(t,\cdot), \psi(\cdot)u^{-q}(t,\cdot))\,\mathrm{d}t$$
$$\geq \int_{t_1}^{t_2} \chi(t)\int_{\Omega'} f(t,x)u(t,x)\psi(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \chi'(t)\psi(x)w(t,x)\,\mathrm{d}x\,\mathrm{d}t\,, \quad (5.8)$$

which we wanted to show.

Let us mention that the situation of a second order parabolic equation is easier due to the structure of the form $A\nabla u \cdot \nabla\phi$ in (4.4), see [DGV11, Section 3.1.1] and [AS67, Section 9]: In the same way as we deduced (5.7) above we may deduce from (4.4)

$$\int_{\Omega'} \partial_t u_h(t,x)\eta(x)\,\mathrm{d}x + \int_{\Omega'} (A\nabla u)_h(t,x) \cdot \nabla\eta(x)\,\mathrm{d}x \geq \int_{\Omega'} f_h(t,x)\eta(x)\,\mathrm{d}x\,.$$

From this inequality we proceed as in the case of the fractional order equation and observe that the convergence of the term with space derivatives follows from (4.3) and Lemma 1.14. Thus we may take

$$\int_{\Omega'} \partial_t u(t,x)\phi(t,x)\,\mathrm{d}x + \int_{\Omega'} (A\nabla u\cdot\nabla\phi)(t,x)\,\mathrm{d}x \geq \int_{\Omega'} f(t,x)\phi(t,x)\,\mathrm{d}x \quad \text{for a.e. } t\in I \tag{5.9}$$

as a starting point in the proofs of Proposition 7.1 and Proposition 7.5.

It remains to prove the auxiliary result.

Lemma 5.2. *Let v be a positive supersolution to $\partial_t v - Lv = f$ in $Q = \Omega\times I$. Let ϕ be an admissible test function as in Definition 4.3 that is bounded and satisfies* $\operatorname{supp}[\phi(t,\cdot)] \subset B_R \Subset \Omega$ *for some $R>0$ and every $t\in I$. Then for every $I'\Subset I$*

$$\int_{I'} \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(v(s,\cdot),\phi_h(t,\cdot))\,\mathrm{d}s\,\mathrm{d}t \xrightarrow{h\to 0+} \int_{I'} \mathcal{E}_t(v,\phi)\,\mathrm{d}t. \tag{5.10}$$

Proof. Set $V(t,x,y) = a(t,x,y)\,(v(t,x)-v(t,y))$ and $\Phi(t,x,y) = \phi(t,x)-\phi(t,y)$. Since

$$\int_{I'} \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(v(s,\cdot),\phi_h(t,\cdot))\,\mathrm{d}s\,\mathrm{d}t = \int_{I'} \iint_{\mathbb{R}^d\,\mathbb{R}^d} V_h(t,x,y)\Phi_h(t,x,y)k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ ,$$

$$\int_{I'} \mathcal{E}_t(v,\phi)\,\mathrm{d}t = \int_{I'} \iint_{\mathbb{R}^d\,\mathbb{R}^d} V(t,x,y)\Phi(t,x,y)k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ ,$$

the convergence in (5.10) follows if we show both

$$\int_{I'} \iint_{\mathbb{R}^d\,\mathbb{R}^d} |V_h(t,x,y)-V(t,x,y)|\,|\Phi(t,x,y)|\,k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \xrightarrow{h\to 0+} 0, \tag{5.11a}$$

$$\int_{I'} \iint_{\mathbb{R}^d\,\mathbb{R}^d} |V_h(t,x,y)|\,|\Phi_h(t,x,y)-\Phi(t,x,y)|\,k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \xrightarrow{h\to 0+} 0. \tag{5.11b}$$

First we prove (5.11a). Define $B = B_{R+\varepsilon}$ for some fixed $\varepsilon>0$. A usual decomposition of the integral over $\mathbb{R}^d\times\mathbb{R}^d$ yields

$$\begin{aligned}
&\int_{I'} \iint_{\mathbb{R}^d\,\mathbb{R}^d} |V_h(t,x,y)-V(t,x,y)|\,|\phi(t,x)-\phi(t,y)|\,k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\quad= \int_{I'} \iint_{B\,B} |V_h(t,x,y)-V(t,x,y)|\,|\phi(t,x)-\phi(t,y)|\,k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad + 2\int_{I'}\int_B |\phi(t,x)| \int_{B^c} |V_h(t,x,y)-V(t,x,y)|\,k_0(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=: I_1(h)+I_2(h).
\end{aligned}$$

Hölder's inequality applied to $I_1(h)$ shows that $I_1(h) \to 0$:

$$\begin{aligned}\|(V_h - V)\Phi_h\, k_0\|_{L^1(I'\times B\times B)} &\leq \left\|(V_h - V)k_0^{\frac{1}{2}}\right\|_{L^2(I';L^2(B\times B))} \left\|\Phi k_0^{\frac{1}{2}}\right\|_{L^2(I';L^2(B\times B))} \\ &\leq \left\|(V_h - V)k_0^{\frac{1}{2}}\right\|_{L^2(I';L^2(B\times B))} \|\phi\|_{L^2(I';H^{\alpha/2}(B))}\,,\end{aligned}$$

where we have used (K_2) in the second inequality. Lemma 1.14(iii) implies that the first factor tends to zero since – again due to property (K_2) –

$$v \in L^2(I'; H^{\alpha/2}(B)) \Rightarrow V\, k_0^{\frac{1}{2}} \in L^2(I'; L^2(B \times B)).$$

In a similar way we obtain the convergence of $I_2(h)$:

$$\begin{aligned}&\int_{I'}\int_B |\phi(t,x)| \int_{B^c} |V_h(t,x,y) - V(t,x,y)|\, k_0(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leq \|\phi\|_{L^\infty(I'\times B)} \int_{I'} \|V_h(t,\cdot,\cdot) - V(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \iint\limits_{B_R\, B^c} k_0(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leq \Lambda\, \varepsilon^{-\alpha}\, |B_R|\, \|\phi\|_{L^\infty(I'\times B)} \|V_h - V\|_{L^1(I;L^\infty(\mathbb{R}^d\times\mathbb{R}^d))}\,,\end{aligned}$$

where we have applied (K_1) in the second inequality. The convergence of the last factor follows again from Lemma 1.14(iii).

Next, we prove (5.11b): Again, we use the decomposition

$$\begin{aligned}&\int_{I'} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |V_h(t,x,y)|\, |\Phi_h(t,x,y) - \Phi(t,x,y)|\, k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad= \int_{I'} \iint\limits_{B\,B} |V_h(t,x,y)|\, |\Phi_h(t,x,y) - \Phi(t,x,y)|\, k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad\quad + 2\int_{I'}\int_B |\phi_h(t,x) - \phi(t,x)| \int_{B^c} |V_h(t,x,y)|\, k_0(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad=: J_1(h) + J_2(h).\end{aligned}$$

The convergence $J_1(h) \xrightarrow{h\to 0+} 0$ follows by the same argument as we used for the convergence of $I_1(h)$. It remains to show that $J_2(h) \xrightarrow{h\to 0+} 0$:

$$\begin{aligned}\tfrac{1}{2} J_2(h) &\leq \int_{I'} \|\phi_h(t,\cdot) - \phi(t,\cdot)\|_{L^\infty(B_R)} \iint\limits_{B_R\, B^c} |V_h(t,x,y)|\, k_0(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\leq 2\varepsilon^{-\alpha}\, |B_R|\, \|v_h\|_{L^\infty(I';L^\infty(\mathbb{R}^d))} \|\phi_h - \phi\|_{L^1(I';L^\infty(\mathbb{R}^d))}\,.\end{aligned}$$

Finally, we apply Lemma 1.14(ii),(iii) to conclude that $J_2(h)$ converges to zero. This finishes the proof. □

5.3 Some algebraic inequalities

The following two results are tools for Lemma 5.5, which is essential for the proof of the basic steps in Section 6.1.

Proposition 5.3. *Let $f, g \in C^1([a,b])$. Then*

$$\frac{f(b)-f(a)}{b-a} + \left(\frac{g(b)-g(a)}{b-a}\right)^2 \le \max_{t\in[a,b]} \left[f'(t) + (g'(t))^2\right]. \tag{5.12}$$

Proof. Assume that (5.12) was not true and integrate the reversed inequality over $[a,b]$ resulting in

$$f(b)-f(a) + \frac{(g(b)-g(a))^2}{b-a} > f(b)-f(a) + \int_a^b (g'(t))^2\,\mathrm{d}t,$$

which is equivalent to

$$\left(\frac{g(b)-g(a)}{(b-a)}\right)^2 > \frac{1}{b-a}\int_a^b (g'(t))^2\,\mathrm{d}t.$$

This is a contradiction (Jensen's inequality) and hence Proposition 5.3 is proved. □

Lemma 5.4. *Let $q > 0, q \neq 1$ and $a, b > 0$. Then*

$$(b-a)(a^{-q} - b^{-q}) \ge \frac{4q}{(1-q)^2}\left(b^{\frac{1-q}{2}} - a^{\frac{1-q}{2}}\right)^2. \tag{5.13}$$

Proof. Setting $c(q) = \frac{(1-q)^2}{4q}$, (5.13) is equivalent to

$$c(q)\frac{b^{-q}-a^{-q}}{b-a} + \frac{\left(b^{\frac{1-q}{2}} - a^{\frac{1-q}{2}}\right)^2}{(b-a)^2} \le 0.$$

Proposition 5.3 with $f(t) = c(q)t^{-q}$ and $g(t) = t^{\frac{1-q}{2}}$ yields

$$c(q)\frac{b^{-q}-a^{-q}}{b-a} + \frac{\left(b^{\frac{1-q}{2}} - a^{\frac{1-q}{2}}\right)^2}{(b-a)^2} \le \max_{t\in[a,b]} t^{-1-q}\left(-\frac{(1-q)^2}{4} + \frac{(1-q)^2}{4}\right) = 0,$$

which proves inequality (5.13). □

Part (i) of the following lemma is taken from [Kas09, Lemma 2.5]. Part (ii) was derived in collaboration with M. Kassmann and R. Zacher.

Lemma 5.5.

(i) Let $q > 1$, $a, b > 0$ and $\tau_1, \tau_2 \geq 0$. Set $\vartheta(q) = \max\left\{4, \frac{6q-5}{2}\right\}$. Then

$$(b-a)\left(\tau_1^{q+1}a^{-q} - \tau_2^{q+1}b^{-q}\right) \geq \frac{1}{q-1}(\tau_1\tau_2)\left[\left(\frac{b}{\tau_2}\right)^{\frac{1-q}{2}} - \left(\frac{a}{\tau_1}\right)^{\frac{1-q}{2}}\right]^2$$
$$- \vartheta(q)\,(\tau_2-\tau_1)^2\left[\left(\frac{b}{\tau_2}\right)^{1-q} + \left(\frac{a}{\tau_1}\right)^{1-q}\right]. \quad (5.14)$$

Since $1-q<0$ the division by $\tau_1 = 0$ or $\tau_2 = 0$ is allowed.

(ii) Let $q \in (0,1)$, $a, b > 0$ and $\tau_1, \tau_2 \geq 0$. Set $\zeta(q) = \frac{4q}{1-q}$, $\zeta_1(q) = \frac{1}{6}\zeta(q)$ and $\zeta_2(q) = \zeta(q) + \frac{9}{q}$. Then

$$(b-a)\left(\tau_1^2 a^{-q} - \tau_2^2 b^{-q}\right) \geq \zeta_1(q)\left(\tau_2 b^{\frac{1-q}{2}} - \tau_1 a^{\frac{1-q}{2}}\right)^2$$
$$- \zeta_2(q)(\tau_2-\tau_1)^2\left(b^{1-q} + a^{1-q}\right) \quad (5.15)$$

Proof. Here we only prove (5.15); for the proof of (5.14) we refer to [Kas09, pp. 5-6]. (5.15) is easily checked if $\tau_2 = 0$. If $\tau_1 = 0$ and $\tau_2 > 0$ the inequality reads

$$-b^{1-q} + ab^{-q} \geq (\zeta_1(q) - \zeta_2(q))\,b^{1-q} - \zeta_2(q)a^{1-q}.$$

This is true since $\zeta_1(q) - \zeta_2(q) < -1$.

Now we consider the case $\tau_1\tau_2 > 0$. We can assume $b \geq a$ due to symmetry. Setting $t = \frac{b}{a} \geq 1$, $s = \frac{\tau_2}{\tau_1} > 0$ and $\lambda = s^2 t^{-q}$, assertion (5.15) is equivalent to

$$\zeta_1(q)\left(\sqrt{\lambda t} - 1\right)^2 \leq (t-1)(1-\lambda) + \zeta_2(q)(s-1)^2\left(t^{1-q}+1\right) \quad (5.16)$$

We estimate

$$\left(\sqrt{\lambda t}-1\right)^2 \leq 2\left(\sqrt{\lambda t} - t^{\frac{1-q}{2}}\right)^2 + 2\left(t^{\frac{1-q}{2}}-1\right)^2 = 2(s-1)^2 t^{1-q} + 2\left(t^{\frac{1-q}{2}}-1\right)^2$$
$$\leq 2(s-1)^2 t^{1-q} + \frac{2}{\zeta(q)}(t-1)(1-t^{-q}),$$

where we have used Lemma 5.4 in the last inequality noting that $\frac{4q}{(1-q)^2} \geq \frac{4q}{1-q} = \zeta(q)$ for $q \in (0,1)$. We decompose the last factor of the above inequality as follows:

$$1 - t^{-q} = (1-\lambda) + (\lambda - t^{-q}) = (1-\lambda) + (s-1)^2 t^{-q} + 2(s-1)t^{-q}.$$

This implies

$$\left(\sqrt{\lambda t}-1\right)^2 \leq \left(2 + \frac{2}{\zeta(q)}\right)(s-1)^2 t^{1-q} + \frac{2}{\zeta(q)}(t-1)(1-\lambda) + \frac{4}{\zeta(q)}(t-1)(s-1)t^{-q}. \quad (5.17)$$

It remains to estimate the last term in (5.17). To this end we consider different ranges of $t \in [1,\infty)$ and $s \in (0,\infty)$:

a) $t > 1$, $s \in (1,2)$ and $t - 1 > \frac{4}{q} t(s-1)$: By the mean value theorem, there is $\xi \in (1,t)$ such that $t^q - 1 = q\xi^{q-1}(t-1)$. Then we can estimate

$$\frac{(s+2)(s-1)}{t-1} \le \frac{q(s+2)}{4t} \le \frac{q}{t} \le \frac{q}{t^{1-q}} \le q\xi^{q-1} = \frac{t^q-1}{t-1}.$$

Therefore

$$(s+2)(s-1) \le t^q - 1, \quad \text{or equivalently} \quad s - 1 \le t^q - s^2 = t^q(1-\lambda).$$

This implies $(t-1)(s-1)t^{-q} \le (t-1)(1-\lambda)$. We deduce from (5.17)

$$\tfrac{1}{6}\zeta(q)\left(\sqrt{\lambda t} - 1\right)^2 \le \left(\tfrac{1}{3}\zeta(q) + \tfrac{1}{3}\right)(s-1)^2 t^{1-q} + (t-1)(1-\lambda). \tag{5.18a}$$

b) $t > 1$, $s \in (1,2)$ and $t - 1 \le \frac{4}{q} t(s-1)$: In this case $(t-1)(s-1)t^{-q} \le \frac{4}{q} t^{1-q}(s-1)^2$ and – again by (5.17) –

$$\tfrac{1}{2}\zeta(q)\left(\sqrt{\lambda t} - 1\right)^2 \le \left(\zeta(q) + 1 + \tfrac{8}{q}\right)(s-1)^2 t^{1-q} + (t-1)(1-\lambda). \tag{5.18b}$$

c) $t = 1$ or $s \le 1$: Then obviously $(t-1)(s-1)t^{-q} \le 0$ and

$$\tfrac{1}{2}\zeta(q)\left(\sqrt{\lambda t} - 1\right)^2 \le \left(\zeta(q) + 1\right)(s-1)^2 t^{1-q} + (t-1)(1-\lambda). \tag{5.18c}$$

d) $t > 1$, $s \ge 2$: Using $s - 1 \le (s-1)^2$ we obtain $(t-1)(s-1)t^{-q} \le (s-1)^2 t^{1-q}$ and

$$\tfrac{1}{2}\zeta(q)\left(\sqrt{\lambda t} - 1\right)^2 \le \left(\zeta(q) + 3\right)(s-1)^2 t^{1-q} + (t-1)(1-\lambda). \tag{5.18d}$$

Combining inequalities (5.18a)-(5.18d) we obtain (5.16) since $3 < 1 + \frac{8}{q} < \frac{9}{q}$. This finishes the proof of Lemma 5.5. □

5.4 Sobolev and weighted Poincaré inequalities

5.4.1 Sobolev inequality

Proposition 5.6 (Sobolev inequality – H^1-version)**.**

(i) Let $d \ge 3$, $\theta = \frac{d}{d-2}$. Then there is a constant $S > 0$ such that for every $r > 0$ and $u \in H^1(B_r)$

$$\left(\int_{B_r} |u(x)|^{2\theta}\, \mathrm{d}x\right)^{1/\theta} \le S \int_{B_r} |\nabla u(x)|^2\, \mathrm{d}x + S r^{-2} \int_{B_r} u^2(x)\, \mathrm{d}x\,. \tag{5.19a}$$

(ii) Let $d = 1, 2$ and $\theta = 3$. Then there is a constant $S' > 0$ such that for every $r \in (0, 2)$ and $u \in H^1(B_r)$

$$\left(\int_{B_r} |u(x)|^{2\theta}\,\mathrm{d}x\right)^{1/\theta} \le S' \int_{B_r} |\nabla u(x)|^2\,\mathrm{d}x + S' r^{-2} \int_{B_r} u^2(x)\,\mathrm{d}x\,. \tag{5.19b}$$

Proof. *ad (i):* For $r = 1$ the inequality is a consequence of the Gagliardo-Nirenberg-Sobolev inequality, cf. [Eva10, §5.6.1]. The assertion for arbitrary $r > 0$ then follows by a scaling argument: Let $u \in H^1(B_r)$ and define the diffeomorphism $\Phi\colon B_1 \to B_r$, $\Phi(x) = rx$. Then $u \circ \Phi \in H^1(B_1)$ and

$$\begin{aligned}\left(\int_{B_r} |u(x)|^{2\theta}\,\mathrm{d}x\right)^{1/\theta} &= r^{d/\theta}\left(\int_{B_1} |(u\circ\Phi)(x)|^{2\theta}\,\mathrm{d}x\right)^{1/\theta}\\ &\le S\int_{B_1} r^{d/\theta}\,|\nabla(u\circ\Phi)(x)|^2 + r^{d/\theta}\,|u\circ\Phi(x)|^2\,\mathrm{d}x\\ &= S\int_{B_1} r^{d/\theta+2}\,|\nabla u(rx)|^2 + r^{d/\theta}\,|u(rx)|^2\,\mathrm{d}x\\ &= S\int_{B_r} r^{d/\theta+2-d}\,|\nabla u(x)|^2 + r^{d/\theta-d}\,|u(x)|^2\,\mathrm{d}x\,. \end{aligned}\tag{5.20}$$

This proves (5.19a) for general $r > 0$ since $d/\theta + 2 - d = 0$ and $d/\theta - d = -2$.

ad (ii): In this case we may consider u as a function of three variables – temporarily denoted by $\overline{u}$ – which is constant in the second (and third) variable. Then we may apply (5.19a) to $\overline{u}$, where $r = 1$ and the ball B_1 is replaced by the cube $R_1 = \{x \in \mathbb{R}^d\colon \max_{1\le k\le 3} |x_k| < 1\}$. In this way we establish (5.19b) for $r = 1$ with $\theta = \frac{3}{3-2} = 3$.

For $d = 1, 2$ and $\theta = 3$ we have $d/3 + 2 - d > 0$ and $d/3 - d > -2$ in (5.20). Therefore we can use the upper bound $r < 2$ to establish (5.19b). □

We provide a Sobolev inequality and a weighted Poincaré inequality for fractional Sobolev spaces with constants that are uniform for $\alpha \to 2-$:

Proposition 5.7 (Sobolev inequality – $H^{\alpha/2}$-version)**.**

(i) Let $d \ge 3$ and $\alpha_0 > 0$. Then there is a constant $S' > 0$ such that for any $\alpha \in (\alpha_0, 2)$, $\theta = \frac{d}{d-\alpha}$, $r > 0$ and $u \in H^{\alpha/2}(B_r)$ the following inequality holds:

$$\left(\int_{B_r} |u(x)|^{2\theta}\,\mathrm{d}x\right)^{1/\theta} \le (2-\alpha)S \iint_{B_r\,B_r} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y + S r^{-\alpha}\int_{B_r} u^2(x)\,\mathrm{d}x\,. \tag{5.21a}$$

(ii) Let $d = 1, 2$ and $\alpha_0 > 0$. Then there is a constant $S' > 0$ such that for any $\alpha \in (\alpha_0, 2)$, $\theta = \frac{3}{3-\alpha}$, $r \in (0, 2)$ and $u \in H^{\alpha/2}(B_r)$ the following inequality holds:

$$\left(\int_{B_r} |u(x)|^{2\theta} \,\mathrm{d}x \right)^{1/\theta} \leq (2-\alpha)S' \iint_{B_r\, B_r} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + S' r^{-\alpha} \int_{B_r} u^2(x)\,\mathrm{d}x\,. \tag{5.21b}$$

Proof. In [BBM02, Theorem 1] we find for $B = B_1$ and $u \in H^{\alpha/2}(B)$

$$\left\| u - \textstyle\fint_B u \right\|^2_{L^{2\theta}(B)} \leq c(d) \frac{2-\alpha}{d-\alpha} \iint_{B\,B} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y\,.$$

Since

$$\left\| \textstyle\fint_B u \right\|^2_{L^{2\theta}(B)} \leq |B|^{2/\theta+1} \|u\|^2_{L^2(B)} \leq c'(d)\, \|u\|^2_{L^2(B)}$$

and

$$\|u\|^2_{L^{2\theta}(B)} \leq \frac{1}{2} \left\| u - \textstyle\fint u \right\|^2_{L^{2\theta}(B)} + \frac{1}{2} \left\| \textstyle\fint u \right\|^2_{L^{2\theta}(B)}\,,$$

this proves (5.21a) in the case $r = 1$. The result for general $r > 0$ follows after a change of variables: Let $u \in H^{\alpha/2}(B_r)$ and define a diffeomorphism $\Phi\colon B_1 \to B_r$, $\Phi(x) = rx$. Then $u \circ \Phi \in H^{\alpha/2}(B_1)$ and

$$\begin{aligned}
\left(\int_{B_r} |u(x)|^{2\theta} \,\mathrm{d}x \right)^{1/\theta} &= r^{d/\theta} \left(\int_{B_1} |u \circ \Phi(x)|^{2\theta} \,\mathrm{d}x \right)^{1/\theta} \\
&\leq (2-\alpha) S r^{d/\theta} \iint_{B_1\,B_1} \frac{|u(rx)-u(ry)|^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + S r^{d/\theta} \int_{B_1} u^2(rx)\,\mathrm{d}x \\
&\leq (2-\alpha) S r^{d/\theta-2d} \iint_{B_r\,B_r} \frac{|u(x)-u(y)|^2}{|x/r-y/r|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + S r^{d/\theta-d} \int_{B_r} u^2(x)\,\mathrm{d}x \\
&= S(2-\alpha) \iint_{B_r\,B_r} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + S r^{-\alpha} \int_{B_r} u^2(x)\,\mathrm{d}x\,.
\end{aligned}$$

This proves (5.21a) for general $r > 0$. (5.21b) can be proved in the same way as is explained in the proof of (5.19b).

This finishes the proof of Proposition 5.7. □

5.4.2 Weighted Poincaré inequality

In order to derive estimates on $\log u$ in Section 6.4 we will need weighted Poincaré inequalities: A standard weighted Poincaré inequality in the case of a local operator and a fractional version in the nonlocal case.

We do not state these results in full generality but only in the special cases that apply in Section 6.4. To this end we fix $B = B_{3/2}(0)$ and define $\Psi\colon B \to [0,1]$ by $\Psi(x) = (\frac{3}{2} - |x|) \wedge 1$. Clearly, Ψ is Lipschitz continuous in B and its profile $\psi(r) = \Psi(|x|)$ is non-increasing.

The following result is a special case of [DK13, Corollary 3]. We refer also to [DGV11, Prop. 2.2.1], [Mos64, Lemma 3] or [SC02, Theorem 5.3.4] for similar results.

Lemma 5.8 (Weighted Poincaré inequality – H^1-version)**.** *There is a constant $C = C(d)$ such that for every $v \in L^1(B, \Psi(x)\,\mathrm{d}x)$*

$$\int_B \left(v(x) - v_B^\Psi\right)^2 \Psi(x)\,\mathrm{d}x \le C \int_B |\nabla v(x)|^2\, \Psi(x)\,\mathrm{d}x,$$

where $v_B^\Psi = \left(\int_B \Psi(x)\,\mathrm{d}x\right)^{-1} \int_B v(x)\Psi(x)\,\mathrm{d}x.$

Lemma 5.9 (Weighted Poincaré inequality – $H^{\alpha/2}$-version)**.** *Let $k \in \mathcal{K}(\alpha_0, \Lambda)$ for some $\alpha_0 \in (0,2)$ and $\Lambda \ge 1$. Then there is a positive constant $c_2(d, \alpha_0, \Lambda)$ such that for every $v \in L^1(B_{3/2}, \Psi(x)\,\mathrm{d}x)$*

$$\int_{B_{3/2}} [v(x) - v_\Psi]^2\, \Psi(x)\,\mathrm{d}x \le c_2 \iint_{B_{3/2}\,B_{3/2}} [v(x) - v(y)]^2\, k_t(x,y)\, (\Psi(x) \wedge \Psi(y))\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $v_\Psi = \left(\int_{B_{3/2}} \Psi(x)\,\mathrm{d}x\right)^{-1} \int_{B_{3/2}} v(x)\Psi(x)\,\mathrm{d}x.$

Proof. For $x \in B_{3/2} \setminus \overline{B_1}$ write $\Psi(x) = 2\int_1^{3/2} \mathbb{1}_{B_s}(x)\,\mathrm{d}s$. Then for some $\alpha \in (\alpha_0, 2)$

$$\begin{aligned}
&\int_{B_{3/2}} \int_{B_{3/2}} [v(x) - v(y)]^2\, k_t(x,y)\, (\Psi(x) \wedge \Psi(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{B_{3/2}} \int_{B_{3/2}} [v(x) - v(y)]^2\, k_t(x,y) 2 \int_1^{3/2} \mathbb{1}_{B_s}(x)\mathbb{1}_{B_s}(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = 2 \int_1^{3/2} \int_{B_{3/2}} \int_{B_{3/2}} [v(x) - v(y)]^2\, k_t(x,y)\mathbb{1}_{B_s}(x)\mathbb{1}_{B_s}(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \ge 2\Lambda^{-1}(2-\alpha) \int_1^{3/2} \int_{B_{3/2}} \int_{B_{3/2}} \frac{[v(x) - v(y)]^2}{|x-y|^{d+\alpha}} \mathbb{1}_{B_s}(x)\mathbb{1}_{B_s}(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \Lambda^{-1}(2-\alpha) \int_{B_{3/2}} \int_{B_{3/2}} \frac{[v(x) - v(y)]^2}{|x-y|^{d+\alpha}} (\Psi(x) \wedge \Psi(y))\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}$$

where we have applied (K_2) to obtain the inequality. The assertion of Lemma 5.9 follows now immediately from [DK13, Corollary 6]. □

5.5 Abstract Moser iteration

This section contains a restatement of Moser's iteration result. This iteration technique (Lemma 5.11 in combination with Proposition 7.1) was applied for the first time in [Mos61] for linear elliptic equations and in [Mos64] for parabolic equations. Since then, there have been many generalizations of this approach, such as the extension to quasilinear equations in [Tru67] and [Tru68] and to parabolic equations on manifolds (e.g. [SC95]) and on graphs (e.g. [Del99]).

Moser's iteration technique has extended into the literature on partial differential equations, e.g. in [GT01, Lie96, SC02]. In contrast to the mentioned references, the presentation here states these results in an abstract way that may apply in different contexts. To the author's knowledge this idea of abstraction goes back to [CZ04], see also [Zac10].

We start with a definition:

Definition 5.10. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We say that a family $\mathcal{U} = (U_r, 0 < r \leq 1)$ of subsets of Ω is *increasing* if

$$\mu(U_1) < \infty \quad \text{and} \quad U_r \subset U_{r'} \subset \Omega \text{ for all } 0 < r \leq r' \leq 1.$$

For a given increasing family $\mathcal{U}$ and a measurable function $f\colon U_1 \to \mathbb{R}$, $p \in (0,\infty)$ and $r \in (0,1)$ we use the notation

$$\mathcal{N}(f; r, p) = \left(\int_{U_r} |f|^p \, \mathrm{d}\mu \right)^{1/p} .$$

5.5.1 Abstract Moser iteration scheme – type I

Lemma 5.11 (Moser's iteration I)**.** *Let $\kappa > 1$, $p_0 \in [1,\infty)$ and $\mathcal{U} = (U_r, 0 < r \leq 1)$ be an increasing family of subsets of some measure space $(\Omega, \mathcal{A}, \mu)$. For $m \in \mathbb{N}_0$, $p \in (0, p_0]$ set $p_m = p\kappa^m$ and let $(r_m)_{m\in\mathbb{N}_0}$ be a sequence such that*

$$1 \geq \sigma = r_0 > r_1 > \ldots > r_j > r_{j+1} > \ldots > \rho > 0.$$

Let $f\colon U_1 \to \mathbb{R}$ be a measurable function with the property

$$\mathcal{N}(f; r_{j+1}, p_{j+1}) \leq A_j(p)^{1/p_j} \mathcal{N}(f; r_j, p_j) \qquad \text{for all } j \in \mathbb{N}_0 \tag{5.22}$$

for some family $(A_j(p))_{j\in\mathbb{N}_0}$ that may also depend on σ, ρ and κ. If there is $M(\sigma, \rho, p_0, \kappa) \geq 1$ such that for all $p \in (0, p_0]$

$$\prod_{j=0}^{\infty} A_j(p)^{1/\kappa^j} \leq M < \infty\,, \tag{5.23}$$

then

$$\sup_{U_\rho} |f| \leq M^{1/p} \left(\int_{U_\sigma} |f|^p \, \mathrm{d}\mu \right)^{1/p} \qquad \text{for all } p \in (0, p_0]. \tag{5.24}$$

Proof. Since f is fixed, we omit f in the argument of $\mathcal{N}$. Applying (5.22) repeatedly we obtain for $m \in \mathbb{N}_0$ the chain of inequalities

$$\mathcal{N}^p(\rho, p_{m+1}) \leq \mathcal{N}^p(r_{m+1}, p_{m+1}) \leq A_m(p)^{1/\kappa^m} \mathcal{N}^p(r_m, p_m) \leq \mathcal{N}^p(r_0, p) \prod_{j=0}^{m} A_j(p)^{1/\kappa^j} .$$

By the property $\lim_{m\to\infty} \mathcal{N}(\rho, p_m) = \sup_{U_\rho} |f|$ and (5.23) we see that

$$\sup_{U_\rho} |f| = \lim_{m\to\infty} \mathcal{N}(\rho, p_m) \leq \left(\prod_{j=0}^{\infty} A(j, p_j) \right)^{1/p} \mathcal{N}(r_0, p) \leq M^{1/p} \left(\int_{U_\sigma} |f|^p \, \mathrm{d}\mu \right)^{1/p}$$

for every $p \in (0, p_0]$, which finishes the proof of Lemma 5.11. □

5.5.2 Abstract Moser iteration scheme – type II

In some situations one cannot expect an inequality of the type (5.22) to hold for arbitrarily large values of p but only for a finite range of values. However, one can still prove an estimate in the spirit of (5.24) with $\sup f$ replaced by a p-norm of f.

Lemma 5.12 (Moser's iteration II)**.** *Let $\kappa > 1$, $p_0 \in (0, \kappa)$, and $\mathcal{U} = (U_r, 0 < r \leq 1)$ be an increasing family of subsets of some measure space $(\Omega, \mathcal{A}, \mu)$. For $m \in \mathbb{N}_0$ set $p_m = p_0 \kappa^{-m}$ and let $(r_m)_{m\in\mathbb{N}_0}$ be a sequence such that*

$$1 \geq \sigma = r_0 > r_1 > \ldots > r_j > r_{j+1} > \ldots > \rho > 0 .$$

Let $f \colon U_1 \to \mathbb{R}$ be a measurable function with the property that for all $n \in \mathbb{N}$

$$\mathcal{N}(f; r_j, p_{n-j}) \leq A_j^{1/p_{n-j+1}} \mathcal{N}(f; r_{j-1}, p_{n-j+1}) \quad \text{for all } j = 1, \ldots, n \tag{5.25}$$

for some family $(A_j)_{1\leq j\leq n}$. If there is $M = M(\sigma, \rho, \kappa) \geq 1$ such that

$$\prod_{j=1}^{n} A_j^{1/p_{n-j+1}} \leq M^{1/p_n - 1/p_0} \qquad \text{for all } n \in \mathbb{N}, \tag{5.26}$$

then for all $p \in \left(0, \frac{p_0}{\kappa}\right]$

$$\left(\int_{U_\rho} |f|^{p_0} \, \mathrm{d}\mu \right)^{1/p_0} \leq \left[M \left(1 \vee \mu(U_1) \right) \right]^{(1+\kappa)(1/p - 1/p_0)} \left(\int_{U_\sigma} |f|^p \, \mathrm{d}\mu \right)^{1/p} . \tag{5.27}$$

The following proof uses ideas from [Zac10, pp. 6-7].

Proof. Since f is fixed we omit f in the argument of $\mathcal{N}$. Let $n \in \mathbb{N}$. From (5.25) and (5.26) we deduce

$$\mathcal{N}(\rho, p_0) \leq \mathcal{N}(r_n, p_0) \leq A_n^{1/p_1} \mathcal{N}(r_{n-1}, p_1) \leq A_n^{1/p_1} A_{n-1}^{1/p_2} \mathcal{N}(r_{n-2}, p_2)$$

$$\le \mathcal{N}(r_0, p_n) \prod_{j=1}^{n} A_j^{1/p_{n-j+1}} \le M^{1/p_n - 1/p_0}\, \mathcal{N}(r_0, p_n)\,. \tag{5.28}$$

Let $p \in \left(0, \frac{p_0}{\kappa}\right]$ and fix $n \ge 2$ with the property $p_n < p \le p_{n-1}$. Thus

$$\frac{1}{p_n} - \frac{1}{p_0} = \frac{\kappa^n - 1}{p_0} \le \frac{\kappa^n + \kappa^{n-1} - \kappa - 1}{p_0} = \frac{(1+\kappa)(\kappa^{n-1}-1)}{p_0} = (1+\kappa)\left(\frac{1}{p_{n-1}} - \frac{1}{p_0}\right)$$
$$\le (1+\kappa)\left(\frac{1}{p} - \frac{1}{p_0}\right). \tag{5.29}$$

Additionally we have by Hölder's inequality

$$\mathcal{N}(r_0, p_n) = \mathcal{N}(\sigma, p_n) \le \mu(U_\sigma)^{\frac{1}{p_n} - \frac{1}{p}}\, \mathcal{N}(\sigma, p) \le \mu(U_1)^{\frac{1}{p_n} - \frac{1}{p}}\, \mathcal{N}(\sigma, p)$$
$$\le \left(1 \vee \mu(U_1)^{\frac{1}{p_n} - \frac{1}{p_0}}\right) \mathcal{N}(\sigma, p)\,.$$

Using (5.29) and the latter inequality in (5.28) proves the assertion. □

5.6 A lemma by Bombieri and Giusti

The following abstract lemma extends the idea of [BG72] to the parabolic case. It was first proved in [Mos71, pp. 731-733]. The version below can be found in [SC02, Section 2.2.3].

Lemma 5.13. *Let $(U(r))_{\theta \le r \le 1}$ be a increasing family of domains in some measure space $(\Omega, \mathcal{A}, \mu)$. Let m, c_0 be positive constants, $\theta \in [1/2, 1]$, $\eta \in (0,1)$ and $0 < p_0 \le \infty$. Furthermore assume that w is a positive, measurable function defined on $U(1)$ which satisfies*

$$\left(\int_{U(r)} w^{p_0}\, \mathrm{d}\mu\right)^{1/p_0} \le \left(\frac{c_0}{(R-r)^m\, |U(1)|}\right)^{1/p - 1/p_0} \left(\int_{U(R)} w^p\, \mathrm{d}\mu\right)^{1/p} < \infty. \tag{5.30}$$

for all $r, R \in [\theta, 1], r < R$ and for all $p \in (0, 1 \wedge \eta p_0)$.

Additionally suppose that

$$\forall s > 0\colon \quad \mu\left(U(1) \cap \{\log w > s\}\right) \le \frac{c_0}{s} \mu(U(1)). \tag{5.31}$$

Then there is a constant $C = C(\theta, \eta, m, c_0, p_0)$ such that

$$\left(\int_{U(\theta)} w^{p_0}\, \mathrm{d}\mu\right)^{1/p_0} \le C\, [\mu(U(1))]^{1/p_0}\ . \tag{5.32}$$

Proof. This proof follows the lines of [CZ04, Lemma 2.6].

Without loss of generality we can normalize $|U(1)| = 1$.

Define

$$\beta(r) = \log \left(\int_{U(r)} w^{p_0} \,\mathrm{d}\mu \right)^{1/p_0} = \log \left(\mathcal{N}(w; r, p_0) \right)$$

Note that $\beta(r) \leq \beta(R)$ for $r \leq R$.

Let $r, R \in [\theta, 1]$ with $r < R$. For $0 < p < p_0$ we make use of (5.31) and Hölder's inequality with exponents $q_1 = \frac{p_0}{p_0 - p}$ and $q_2 = \frac{p_0}{p}$ in the first term of the following equality and obtain

$$\begin{aligned}
\int_{U(R)} w^p \,\mathrm{d}\mu &= \int_{U(R)} \mathbb{1}_{\{\log w > \beta(R)/2\}} w^p \,\mathrm{d}\mu + \int_{U(R)} \mathbb{1}_{\{\log w \leq \beta(R)/2\}} w^p \,\mathrm{d}\mu \\
&\leq \Big(\mu \left[U(R) \cap \{\log w > \beta(R)/2\} \right] \Big)^{1-p/p_0} \left(\int_{U(R)} w^{p_0} \,\mathrm{d}\mu \right)^{p/p_0} + e^{p\beta(R)/2} \\
&\leq \left(\frac{2c_0}{\beta(R)} \right)^{p(1/p - 1/p_0)} e^{p\beta(R)} + e^{p\beta(R)/2} . \qquad (5.33)
\end{aligned}$$

If

$$\beta(R) > 2c_0 \quad \text{and} \quad (1/p - 1/p_0)^{-1} = \tfrac{2}{\beta(R)} \log \tfrac{\beta(R)}{2c_0}, \qquad (5.34)$$

the two terms on the right-hand side of (5.33) coincide and requiring

$$\beta(R) > c_2 \qquad \text{for some suitable constant } c_2 = c_2(c_0, p_0, \eta) \qquad (5.35)$$

ensures $\beta(R) > 2c_0$ and $(1/p - 1/p_0)^{-1} \leq \min(1, \eta p_0)$. For p chosen as in (5.34) and if (5.35) is satisfied we thus have

$$\mathcal{N}(w; R, p) \leq 2^{1/p} e^{\beta(R)/2}.$$

Now we apply (5.30) for p chosen as in (5.34):

$$\begin{aligned}
\beta(r) &\leq \log \left(\left(\tfrac{2c_0}{(R-r)^m} \right)^{1/p - 1/p_0} \mathcal{N}(w; R, p) \right) \leq \left(\tfrac{1}{p} - \tfrac{1}{p_0} \right) \log \left(\tfrac{2c_0}{(R-r)^m} \right) + \tfrac{\beta(R)}{2} + \tfrac{1}{p} \log 2 \\
&= \frac{\beta(R)}{2} \left(\frac{\log \left(\frac{2c_0}{(R-r)^m} \right)}{\log \left(\frac{\beta(R)}{2c_0} \right)} + 1 \right) + \frac{1}{p} \log 2.
\end{aligned}$$

If, in addition to the first requirement (5.35) on $\beta(R)$,

$$\beta(R) \geq \frac{8c_0^3}{(R-r)^{2m}} \qquad (5.36)$$

holds, then we have $\frac{\log \frac{2c_0}{(R-r)^m}}{\log \frac{\beta(R)}{2c_0}} \leq \frac{1}{2}$ and consequently $\beta(r) \leq \frac{3}{4} \beta(R) + \frac{1}{p} \log 2$.

On the other hand, if one of the requirements (5.35) or (5.36) is not satisfied, then

$$\beta(R) \leq \frac{\gamma_1}{(R-r)^{2m}} + c_2 \leq \frac{\gamma_1 + c_2}{(R-r)^{2m}} \qquad \text{with a constant } \gamma_1 = \gamma_1(c_0, m, \theta).$$

Since $\beta(r) \leq \beta(R)$ and $r, R \in [\theta, 1]$, $r < R$, were arbitrary we have in all cases

$$\forall r, R \in [\theta, 1]\colon \beta(r) \leq \frac{3}{4}\beta(R) + \frac{\gamma_2}{(R-r)^{2m}} \qquad \text{with a constant } \gamma_2 = \gamma_2(c_0, m, \theta).$$

By iteration we obtain

$$\beta(r_0) \leq \left(\frac{3}{4}\right)^k \beta(r_k) + \gamma_2 \sum_{j=0}^{k-1} \left(\frac{3}{4}\right)^j (r_{j+1} - r_j)^{-2m}$$

for any sequence of radii $\theta \leq r_0 < r_1 < \ldots < r_k \leq 1$.

Finally, we have $\beta(r_k) \leq \beta(1) < \infty$ and letting $k \to \infty$ we obtain for the sequence of radii defined by $r_j = 1 - \dfrac{1-\theta}{1+j}$

$$\beta(\theta) \leq \beta(r_0) \leq \gamma_2 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j (r_{j+1} - r_j)^{-2m} \leq 4\gamma_2 \left(\frac{1-\theta}{2}\right)^{-2m},$$

which finishes the proof. □

6 Proof of the main results for fractional order parabolic equations

In this chapter we give the proof for the main results, Theorems 4.4 and 4.5. These proofs are published in joint article of the author with M. Kassmann [FK13].

6.1 Basic step of Moser's iteration

Negative exponents

The following result generalizes Proposition 7.1(i) to the case of a parabolic equation with a nonlocal operator.

Proposition 6.1. *Let $\frac{1}{2} \leq r < R \leq 1$ and $p > 0$. Then every nonnegative supersolution u of* (PE$_\alpha$) *in $Q = I \times \Omega$, $Q \Supset Q_\ominus(R)$, with $u \geq \varepsilon > 0$ in Q satisfies the following inequality*

$$\left(\int_{Q_\ominus(r)} \widetilde{u}^{-\kappa p}(t,x) \,\mathrm{d}x \,\mathrm{d}t \right)^{1/\kappa} \leq A \int_{Q_\ominus(R)} \widetilde{u}^{-p}(t,x) \,\mathrm{d}x \,\mathrm{d}t \ , \tag{6.1}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$, $\kappa = 1 + \frac{\alpha}{d}$ ($\kappa = 1 + \frac{\alpha}{3}$ if $d = 1, 2$) and A can be chosen as

$$A = C(p+1)^2 \left((R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1} \right) \qquad \textit{with } C = C(d, \alpha_0, \Lambda). \tag{6.2}$$

Remark 6.2. Note that

$$\frac{1}{(R-r)^\alpha} + \frac{1}{(R^\alpha - r^\alpha)} \leq \begin{cases} \frac{2}{(R-r)^\alpha} & \text{for } \alpha \in [1,2], \\ \frac{2}{(R^\alpha - r^\alpha)} & \text{for } \alpha \in (0,1]. \end{cases} \tag{6.3}$$

Proof. Let u be a supersolution in Q with $u \geq \varepsilon > 0$ in Q. We set $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$. If $f = 0$ a.e. in Q we set $\widetilde{u} = u + \delta$ for some $\delta > 0$. The additional assumption $u \geq \delta > 0$ on $I \times \mathbb{R}^d \setminus \Omega$ is temporarily needed to ensure that the nonlocal term on the right-hand side of (6.4) is finite. Since (6.1) is a statement only on B_R, where the assumption ensures $u \geq \varepsilon > 0$, we may pass to the limit $\delta \to 0+$ in the end.

For $q > 1$ define

$$v(t,x) = \widetilde{u}^{\frac{1-q}{2}}(t,x), \qquad \phi(t,x) = \widetilde{u}^{-q}(t,x)\psi^{q+1}(x),$$

where $\psi \colon \mathbb{R}^d \to [0,1]$ is defined by $\psi(x) = \left(\frac{R-|x|}{R-r} \wedge 1 \right) \vee 0$. Obviously, $\psi^{q+1} \in H_0^{\alpha/2}(B_R)$.

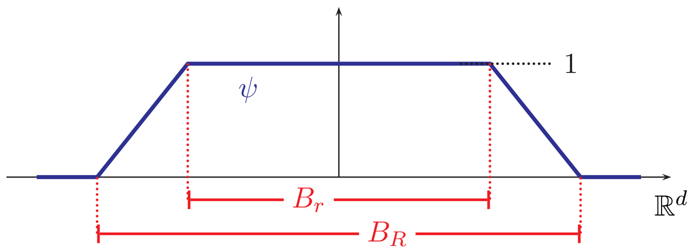

Figure 6.1: Profile of the cut-off function ψ in the space variable

We apply the test function ϕ in (5.4):

$$\int_{B_R} -\psi^{q+1}(x)\widetilde{u}^{-q}(t,x)\partial_t\widetilde{u}(t,x)\,\mathrm{d}x + \\ + \iint_{\mathbb{R}^d\,\mathbb{R}^d} [\widetilde{u}(t,x)-\widetilde{u}(t,y)]\left[\psi^{q+1}(y)\widetilde{u}^{-q}(t,y)-\psi^{q+1}(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ \leq \int_{B_R} -\psi^{q+1}(x)\widetilde{u}^{-q}(t,x)f(t,x)\,\mathrm{d}x.$$

Applying Lemma 5.5(i) (remember $\vartheta(q)=\max\left\{4,\frac{6q-5}{2}\right\}$ therein) and rewriting $\partial_t v^2=(1-q)\widetilde{u}^{-q}\partial_t\widetilde{u}$ yields[1]

$$\frac{1}{q-1}\int_{B_R}\psi^{q+1}\partial_t(v^2)\,\mathrm{d}x \\ + \frac{1}{q-1}\iint_{\mathbb{R}^d\,\mathbb{R}^d}\psi(x)\psi(y)\left[\left(\frac{\widetilde{u}(t,x)}{\psi(x)}\right)^{\frac{1-q}{2}}-\left(\frac{\widetilde{u}(t,y)}{\psi(y)}\right)^{\frac{1-q}{2}}\right]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ \leq \vartheta(q)\iint_{\mathbb{R}^d\,\mathbb{R}^d}[\psi(x)-\psi(y)]^2\left[\left(\frac{\widetilde{u}(t,x)}{\psi(x)}\right)^{1-q}+\left(\frac{\widetilde{u}(t,y)}{\psi(y)}\right)^{1-q}\right]k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ + \int_{B_R}\psi^{q+1}(x)\left|\widetilde{u}^{-q}(t,x)\right||f(t,x)|\,\mathrm{d}x, \quad (6.4)$$

The properties $\psi\equiv 1$ on B_r and $\sup_{x,y\in\mathbb{R}^d}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^2}\leq\frac{1}{(R-r)^2}$ result in the two estimates

$$\iint_{\mathbb{R}^d\,\mathbb{R}^d}[\psi(x)\psi(y)]\left[\left(\frac{\widetilde{u}(t,x)}{\psi(x)}\right)^{\frac{1-q}{2}}-\left(\frac{\widetilde{u}(t,y)}{\psi(y)}\right)^{\frac{1-q}{2}}\right]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ \geq \iint_{B_r\,B_r}[v(t,x)-v(t,y)]^2\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad (6.5)$$

[1] Note that the division by ψ is only a slight abuse of notation; we have $1-q<0$.

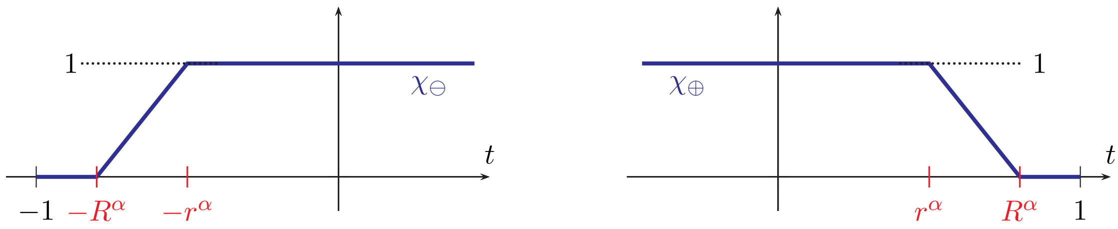

Figure 6.2: The two types of cut-off functions in the time variable

and

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [\psi(x)-\psi(y)]^2 \left[\left(\frac{\widetilde{u}(t,x)}{\psi(x)}\right)^{1-q} + \left(\frac{\widetilde{u}(t,y)}{\psi(y)}\right)^{1-q}\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le 2 \iint\limits_{B_R\,B_R} [\psi(x)-\psi(y)]^2\, \widetilde{u}^{1-q}(t,x)\, k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad + 4 \iint\limits_{B_R\,B_R^c} [\psi(x)-\psi(y)]^2\, \widetilde{u}^{1-q}(t,x) k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad\le c_1(d,\Lambda)(R-r)^{-\alpha} \int_{B_R} v^2(t,x)\,\mathrm{d}x,
\end{aligned} \tag{6.6}$$

where we have used (K_1) in the following way:

$$\begin{aligned}
&\int_{B_R} \widetilde{u}^{1-q}(t,x) \int_{B_R} [\psi(x)-\psi(y)]^2\, k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad\le \int_{B_R} \widetilde{u}^{1-q}(t,x) \Bigg(\int\limits_{|x-y|>R-r} k_0(x,y)\,\mathrm{d}y + (R-r)^{-2} \int\limits_{|x-y|\le R-r} |x-y|^2\, k_0(x,y)\,\mathrm{d}y \Bigg)\,\mathrm{d}x \\
&\quad\le \Lambda(R-r)^{-\alpha} \int_{B_R} v^2(t,x)\,\mathrm{d}x,
\end{aligned} \tag{6.7}$$

and similar for the second term.

Combining (6.5), (6.6) and the fact that $\|f/\widetilde{u}\|_{L^\infty(Q)} \le 1$ we obtain from (6.4)

$$\begin{aligned}
&\int_{B_R} \psi^{q+1}(x)\partial_t(v^2)(t,x)\,\mathrm{d}x + \iint\limits_{B_r\,B_r} [v(t,x)-v(t,y)]^2\, k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\le\ (q-1)\left(1+\vartheta(q)c_1(R-r)^{-\alpha}\right) \int_{B_R} v^2(x)\,\mathrm{d}x.
\end{aligned} \tag{6.8}$$

Now define a piecewise differentiable function $\chi_\ominus\colon \mathbb{R} \to [0,1]$ by $\chi_\ominus(t) = \left(\frac{t+R^\alpha}{R^\alpha - r^\alpha} \wedge 1\right) \vee 0$. Multiplying (6.8) with $\chi_\ominus^2$ we get

$$\partial_t \int_{B_R} \psi^{q+1}(x)\left[\chi_\ominus(t)v(t,x)\right]^2 \,\mathrm{d}x + \chi_\ominus^2(t) \iint\limits_{B_r\, B_r} \left[v(t,x)-v(t,y)\right]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\le c_2(q-1)\vartheta(q)(R-r)^{-\alpha}\chi_\ominus^2(t)\int_{B_R} v^2(t,x)\,\mathrm{d}x + 2\chi_\ominus(t)\left|\chi_\ominus'(t)\right|\int_{B_R} v^2(t,x) \quad (6.9)$$

and integrating this inequality from $-R^\alpha$ to some $t \in I_\ominus(r)$ yields

$$\int_{B_R}\psi^{q+1}(x)(\chi_\ominus(t)v(t,x))^2\,\mathrm{d}x + \int_{-R^\alpha}^t \chi_\ominus^2(s)\iint\limits_{B_r\, B_r}\left[v(s,x)-v(s,y)\right]^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$\le c_2(q-1)\vartheta(q)(R-r)^{-\alpha}\int_{-R^\alpha}^t \chi_\ominus^2(s)\int_{B_R} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s +$$
$$+\int_{-R^\alpha}^t 2\chi_\ominus(s)\left|\chi_\ominus'(s)\right|\int_{B_R} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s,$$

which implies, noting that $\left|\chi_\ominus'\right| \le \frac{1}{R^\alpha - r^\alpha}$,

$$\sup_{t\in I_\ominus(r)}\int_{B_r} v^2(t,x)\,\mathrm{d}x + \int_{Q_\ominus(r)}\int_{B_r}\left[v(s,x)-v(s,y)\right]^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$\le c_3(q-1)\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)\int_{Q_\ominus(R)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s. \quad (6.10)$$

In order to estimate the second term on the left-hand side from below we apply Hölder's inequality with exponents[2] $\theta = \frac{d}{d-\alpha}$, $\theta' = \frac{d}{\alpha}$ to the integrand $v^{2\kappa}$ and then we make use of Sobolev's inequality (5.21):

$$\int_{Q_\ominus(r)} v^{2\kappa}(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\ominus(r)} v^2(t,x)v^{2\alpha/d}(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_{I_\ominus(r)}\left(\int_{B_r} v^{2\theta}(t,x)\,\mathrm{d}x\right)^{1/\theta}\left(\int_{B_r} v^2(t,x)\,\mathrm{d}x\right)^{1/\theta'}\mathrm{d}t$$
$$\le S \sup_{t\in I_\ominus(r)}\left(\int_{B_r} v^2(t,x)\,\mathrm{d}x\right)^{1/\theta'} \times$$
$$\times\left[(2-\alpha)\iint\limits_{Q_\ominus(r)\, B_r}\frac{|v(s,x)-v(s,y)|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + r^{-\alpha}\int\limits_{Q_\ominus(r)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right],$$

where $S = S(d,\alpha_0)$. Using (6.10) twice, $r \ge \frac{1}{2}$ and (K_2) yields

$$\int_{Q_\ominus(r)} v^{2\kappa}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le c_4(d,\Lambda,\alpha_0)\left[(q-1)\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)\right]^{1/\theta'} \times$$

[2] These exponents are valid in the case $d \ge 3$. In view of (5.21b), the modifications for the case $d = 1, 2$ are obvious.

$$\times \left[(q-1)\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)+1\right]\left[\int_{Q_\ominus(R)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]^{1+1/\theta'} .$$

Finally, we can estimate the coefficient by

$$\begin{aligned} c_4(d,\Lambda,\alpha_0)&\left[(q-1)\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)\right]^{\frac{1}{\theta'}} \\ &\qquad + \left[(q-1)\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)\right]^{\kappa} \\ &\le c_5(d,\Lambda,\alpha_0)\left[q\vartheta(q)\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right)\right]^{\kappa} \\ &\le c_6(d,\Lambda,\alpha_0) q^{2\kappa}\left[(R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right]^{\kappa} \ , \end{aligned}$$

which finishes the proof of (6.1) by taking $p=q-1$ and resubstituting $v=u^{\frac{1-q}{2}}$. □

Small positive exponents

The following result generalizes Proposition 7.1(ii) to the case of a parabolic equation with a nonlocal operator.

The technique for the proof of the basic step for small positive exponents is very similar to the one used in the case of negative exponents. We state it separately and indicate the modifications that have to be made.

Proposition 6.3. *Let $\frac{1}{2}\le r<R\le 1$ and $p\in(0,\kappa^{-1}]$ with $\kappa=1+\frac{\alpha}{d}$ ($\kappa=1+\frac{\alpha}{3}$ if $d=1,2$). Then every nonnegative supersolution u of* (PE$_\alpha$) *in $Q=I\times\Omega$, $Q\Supset Q_\oplus(R)$, satisfies the following inequality*

$$\left(\int_{Q_\oplus(r)} \widetilde{u}^{\kappa p}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/\kappa} \le A'\int_{Q_\oplus(R)} \widetilde{u}^{p}(t,x)\,\mathrm{d}x\,\mathrm{d}t \ , \tag{6.11}$$

where $\widetilde{u}=u+\|f\|_{L^\infty(Q)}$ and A' can be chosen as

$$A' = C'\left((R-r)^{-\alpha}+(R^\alpha-r^\alpha)^{-1}\right) \qquad \textit{with } C'=C'(d,\alpha_0,\Lambda). \tag{6.12}$$

Proof. Let u be a supersolution in Q with $u\ge 0$ on $I\times\mathbb{R}^d$. We set $\widetilde{u}=u+\|f\|_{L^\infty(Q)}$. If $f=0$ a.e. in Q we set $\widetilde{u}=u+\varepsilon$ and pass to the limit $\varepsilon\to 0+$ in the end.

Set $q=1-p\in[1-\kappa^{-1},1)$ and define

$$v(t,x)=\widetilde{u}^{\frac{1-q}{2}}(t,x), \qquad \phi(t,x)=\widetilde{u}^{-q}(t,x)\psi^2(x)$$

with ψ as in the proof of Proposition 6.1, namely $\psi(x)=\left(\frac{R-|x|}{R-r}\wedge 1\right)\vee 0$.

From (5.4) we obtain for a.e. $t\in I$

$$\int_{B_R} -\psi^2(x)\widetilde{u}^{-q}(t,x)\partial_t\widetilde{u}(t,x)\,\mathrm{d}x$$

$$+ \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[\psi^2(y)\widetilde{u}^{-q}(t,y) - \psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq \int_{B_R} -\psi^2(x)\widetilde{u}^{-q}(t,x) f(t,x)\,\mathrm{d}x. \quad (6.13)$$

First we observe that for every small $h > 0$

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[\psi^2(y)\widetilde{u}^{-q}(t,y) - \psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \iint\limits_{B_{R+h}\,B_{R+h}} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[\psi^2(y)\widetilde{u}^{-q}(t,y) - \psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ 2 \iint\limits_{B_R\,B^c_{R+h}} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[-\psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x\,. \quad (6.14)$$

Using (K_1), the positivity of $\widetilde{u}$ and the fact that $\frac{(\psi(x)-\psi(y))^2}{|x-y|^2} \leq (R-r)^{-2}$ we can estimate as follows:

$$\iint\limits_{B_R\,B^c_{R+h}} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[-\psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$\geq - \int_{B_R} \widetilde{u}^{1-q}(t,x) \left[(R-r)^{-2} \int\limits_{|y-x|\leq R-r} |x-y|^2\, k_t(x,y)\,\mathrm{d}y + \int\limits_{|y-x|>R-r} k_t(x,y)\,\mathrm{d}y\right] \mathrm{d}x$$
$$\geq -\Lambda(R-r)^{-\alpha} \int_{B_R} v^2(t,x)\,\mathrm{d}x.$$

If $h \to 0$, this shows also that the decomposition in (6.14) is valid with $h = 0$. Rewriting $\partial_t v^2 = (1-q)\widetilde{u}^{-q}\partial_t\widetilde{u}$ and using $\|f/\widetilde{u}\|_{L^\infty(Q)} \leq 1$ we deduce from (6.13) and (6.14)

$$\frac{-1}{1-q} \int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x +$$
$$+ \iint\limits_{B_R\,B_R} [\widetilde{u}(t,x) - \widetilde{u}(t,y)] \left[\psi^2(y)\widetilde{u}^{-q}(t,y) - \psi^2(x)\widetilde{u}^{-q}(t,x)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq c_1\Lambda(R-r)^{-\alpha} \int_{B_R} v^2(t,x)\,\mathrm{d}x.$$

Remember

$$\zeta(q) = \frac{4q}{1-q}, \quad \zeta_1(q) = \frac{1}{6}\zeta(q), \quad \zeta_2(q) = \zeta(q) + \frac{9}{q}$$

from Lemma 5.5(ii). Applying this result we arrive at

$$\frac{-1}{1-q} \int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + \zeta_1(q) \iint\limits_{B_R\,B_R} [\psi(x)v(t,x) - \psi(y)v(t,y)]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\leq c_1 \Lambda (R-r)^{-\alpha} \int_{B_R} v^2(t,x)\,\mathrm{d}x + \zeta_2(q) \iint\limits_{B_R\,B_R} [\psi(x)-\psi(y)]^2 \left[v^2(t,x)+v^2(t,y)\right] k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

By the properties of ψ and (K_1) this implies (cf. (6.7))

$$-\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + (1-q)\zeta_1(q) \iint\limits_{B_r\,B_r} [v(t,x)-v(t,y)]^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq c_2(d,\Lambda)(1-q)(R-r)^{-\alpha}\big(1+\zeta_2(q)\big)\int_{B_R} v^2(t,x)\,\mathrm{d}x.$$

We multiply this inequality with χ^2, where $\chi_\oplus\colon \mathbb{R}\to[0,1]$ is defined by

$$\chi_\oplus(t) = \left(\frac{R^\alpha - t}{R^\alpha - r^\alpha} \wedge 1\right) \vee 0.$$

We integrate the resulting inequality from some $t \in I_\oplus(r)$ to R^α and apply the same technique that we used to deduce (6.10) from (6.8) on page 81 in the proof of Proposition 6.1. As a result we get

$$\sup_{t\in I_\oplus(r)} \int_{B_r} v^2(t,x)\,\mathrm{d}x + (1-q)\zeta_1(q) \int\limits_{Q_\oplus(r)} \int\limits_{B_r} [v(s,x)-v(s,y)]^2\, k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leq c_3(d,\Lambda)\left[(1-q)(1+\zeta_2(q))(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right] \int\limits_{Q_\oplus(R)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

We estimate the coefficients by

$$(1-q)\zeta_1(q) = \frac{2q}{3} \geq \frac{2}{3}\frac{\alpha_0}{d+2} = c_4(d,\alpha_0)$$
$$(1-q)(1+\zeta_2(q)) \leq 1 + (1-q)\zeta_2(q) \leq 1 + 4q + \frac{9(1-q)}{q} \leq 5 + 9\frac{d+2}{\alpha_0} = c_5(d,\alpha_0),$$

which implies

$$\sup_{t\in I_\oplus(r)} \int_{B_r} v^2(t,x)\,\mathrm{d}x + c_4 \int\limits_{Q_\oplus(r)} \int\limits_{B_r} [v(s,x)-v(s,y)]^2\, k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leq c_6(d,\Lambda,\alpha_0)\left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right] \int\limits_{Q_\oplus(R)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Applying Sobolev's inequality as in the proof of Proposition 6.1 we obtain

$$\int\limits_{Q_\oplus(r)} v^{2\kappa}(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq c_7(d,\Lambda,\alpha_0)\left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right]^{1/\theta'} \times$$

$$\times \left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1} + 1\right] \left[\int_{Q_\oplus(R)} v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]^{1+1/\theta'} .$$

We can estimate the coefficient by

$$\left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right]^{1/\theta'} + \left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right]^{\kappa} \\ \leq c_8(d,\Lambda,\alpha_0)\left[(R-r)^{-\alpha} + (R^\alpha - r^\alpha)^{-1}\right]^{\kappa} .$$

This finishes the proof of Proposition 6.3 by resubstituting $q = 1-p$ and $v = \widetilde{u}^{\frac{1-p}{2}}$. □

Note that κ^{-1}, the upper bound on p, can be replaced by any number less than 1.

6.2 An estimate for the infimum of supersolutions

Having established the basic step (Proposition 6.1) of Moser's iteration it is now possible to apply Lemma 5.11 to prove a lower estimate for a nonnegative supersolution u:

Theorem 6.4. *Let $\frac{1}{2} \leq r < R \leq 1$ and $0 < p \leq 1$. There is a constant $C = C(d,\alpha_0,\Lambda) > 0$ such that for every nonnegative supersolution u of (PE_α) in $Q = I \times \Omega$, $Q \Supset Q_\ominus(R)$, with $u \geq \varepsilon > 0$ in Q the following estimate holds:*

$$\sup_{Q_\ominus(r)} \widetilde{u}^{-1} \leq \left(\frac{C}{G_1(r,R)}\right)^{1/p} \left(\int_{Q_\ominus(R)} \widetilde{u}^{-p}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}, \tag{6.15}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$ and $G_1(r,R) = \begin{cases} (R-r)^{d+\alpha} & \text{if } \alpha \geq 1, \\ (R^\alpha - r^\alpha)^{(d+\alpha)/\alpha} & \text{if } \alpha < 1. \end{cases}$

In particular, $G_1(r,R) \geq (R-r)^{d+2} \wedge (\alpha_0(R-r))^{(d+2)/\alpha_0}$.

Remark 6.5. The proof that we shall present below establishes (6.15) in the case $d \geq 3$. Of course, Theorem 6.4 is true for $d = 1,2$, but the precise structure of $G_1(r,R)$ is determined in a different way: First, one proves (6.15) for $r = \frac{1}{2}$ and $R = 1$. A simple scaling argument then shows that there is a constant $C = C(d,\alpha_0,\Lambda) > 0$ such that

$$\sup_{Q_\ominus(\rho/2)} \widetilde{u}^{-1} \leq \left(\frac{C}{\rho^{d+\alpha}}\right)^{1/p} \left(\int_{Q_\ominus(\rho)} \widetilde{u}^{-p}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \quad \text{for all } \rho \in (0,1]. \tag{6.15'}$$

Next, for given r,R as in the theorem, set $\rho = R-r$ in the case $\alpha \geq 1$. Consider all possible translations

$$\mathcal{T}_\ominus(\rho;t_0,x_0) = \mathcal{T}_\ominus(t_0,x_0) = (t_0 - \rho^\alpha, t_0) \times B_\rho(x_0)$$

of the cylindrical domain $Q_\ominus(\rho)$, such that $\mathcal{T}_\ominus(\rho;t_0,x_0) \subset Q_\ominus(R)$. Clearly, (6.15') remains true with $Q_\ominus$ replaced by some $\mathcal{T}_\ominus(t_0,x_0)$. We denote by $\mathcal{C}$ the set of all possible centers (t_0,x_0) such that $\mathcal{T}_\ominus(t_0,x_0) \subset Q_\ominus(R)$.

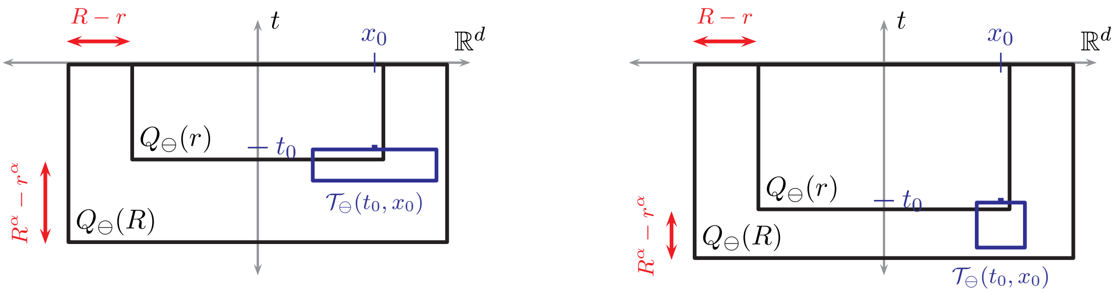

Figure 6.3: The different scaling behavior in the cases $\alpha \geq 1$ (on the left-hand side) and $\alpha < 1$ (on the right-hand side)

In the case $\alpha \geq 1$ we have $r + \rho = r$ and $r^\alpha + \rho^\alpha \leq R^\alpha$ (cf. Figure 6.3), which implies that every point $(t_0, x_0) \in Q_\ominus(r)$ is a center of a translated domain such that $\mathcal{T}_\ominus(t_0, x_0) \subset Q_\ominus(R)$, i.e. $Q_\ominus(r) \subset \mathcal{C}$. Clearly,

$$\mathcal{U} := \bigcup_{(t_0,x_0)\in Q_\ominus(r)} \mathcal{T}_\ominus(\rho/2; t_0, x_0) \supset Q_\ominus(r) \quad \text{and} \quad \bigcup_{(t_0,x_0)\in Q_\ominus(r)} \mathcal{T}_\ominus(\rho; t_0, x_0) \subset Q_\ominus(R).$$

Hence, by (6.15'),

$$\sup_{Q_\ominus(r)} \widetilde{u}^{-1} \leq \sup_{\mathcal{U}} \widetilde{u}^{-1} \leq \left(\frac{C}{(R-r)^{d+\alpha}}\right)^{1/p} \left(\int_{Q_\ominus(R)} \widetilde{u}^{-p}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p},$$

which shows (6.15) in the case $\alpha \geq 1$.

In the case $\alpha_0 < \alpha < 1$ we may choose $\rho = (R^\alpha - r^\alpha)^{1/\alpha}$. Then $r^\alpha + \rho^\alpha = R^\alpha$ and $r + \rho \leq R$ (cf. Figure 6.3), from where the same reasoning as above proves (6.15) in the case $\alpha_0 < \alpha < 1$.

The method that is explained in this remark can be considered as an alternative way to prove Theorem 6.4 because it applies to the case $d \geq 3$, too. ♦

Proof of Theorem 6.4. We apply Lemma 5.11. To this end choose $\kappa = 1 + \frac{\alpha}{d}$, $p_0 = 1$ and for $p \in (0, 1]$ and $m \in \mathbb{N}_0$ set $p_m = p\kappa^m$. Consider the increasing sequence $(Q_\ominus(r), 0 < r \leq 1)$ of subsets of $\mathbb{R}^{d+1}$. There is a slight difference in the choice of the sequence (r_m) in two cases[3]:

I. $1 \vee \alpha_0 \leq \alpha < 2$: Choose the sequence of radii $(r_m)_{m\in\mathbb{N}_0}$ defined by

$$r_m = r + \frac{R-r}{2^m}.$$

[3] Note that the second case is irrelevant if $\alpha_0 \geq 1$.

Note that $r_0 = R$, $r_m > r$ and $\lim_{m\to\infty} r_m = r$. Proposition 6.1 in combination with (6.3) states that for all $j \in \mathbb{N}_0$

$$\left(\int\limits_{Q_\ominus(r_{j+1})} \widetilde{u}^{-p_{j+1}}(t,x)\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p_{j+1}}} \le A_j(p)^{1/p_j} \left(\int\limits_{Q_\ominus(r_j)} \widetilde{u}^{-p_j}(t,x)\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p_j}} \tag{6.16}$$

with $A_j(p) = c_1(p_j+1)^2(r_j - r_{j+1})^{-\alpha}$ for some constant $c_1 = c_1(d,\alpha_0,\Lambda) \ge 1$. This means that condition (5.22) is satisfied and due to $p \le 1$ we may estimate

$$A_j(p) = c_1(p\kappa^j+1)^2(r_j - r_{j+1})^{-\alpha} \le c_1(2\kappa^j)^2 \left(\frac{2^{j+1}}{R-r} \right)^\alpha \le \frac{c_2^j}{(R-r)^\alpha}$$

for some constant $c_2 = c_2(d,\alpha_0,\Lambda) \ge 1$. Since

$$\sum_{j=0}^\infty \frac{j}{\kappa^j} \le \sum_{j=0}^\infty \frac{j}{(1+\alpha_0/d)^j} \le c_3(\alpha_0,d)$$

we can verify condition (5.23):

$$\prod_{j=0}^\infty A_j(p)^{1/\kappa^j} \le c_4(R-r)^{-\alpha\sum\limits_{j=0}^\infty \kappa^{-j}} = c_4(R-r)^{-\alpha\frac{\kappa}{\kappa-1}} = \frac{c_4}{(R-r)^{d+\alpha}}$$

for some constant $c_4 = c_4(d,\alpha_0,\Lambda) \ge 1$. This proves (6.15) in the case $\alpha \in [1,2)$.

II. $\alpha_0 < \alpha < 1$: In this case choose (r_m) defined by

$$r_m = \left(r^\alpha + \frac{R^\alpha - r^\alpha}{2^m} \right)^{1/\alpha}.$$

Again, $r_0 = R$, $r_m > r$ and $\lim_{m\to\infty} r_m = r$. Proposition 6.1 and (6.3) state that (6.16) holds with $A_j(p) = c_1(p_j+1)^2(r_j^\alpha - r_{j+1}^\alpha)^{-1}$. In the same way as above we may estimate

$$A_j(p) = c_1(p\kappa^j+1)^2 \frac{2^{j+1}}{R^\alpha - r^\alpha} \le \frac{c_2^j}{R^\alpha - r^\alpha}$$

and

$$\prod_{j=0}^\infty A_j(p)^{1/\kappa^j} \le \frac{c_4}{(R^\alpha - r^\alpha)^{(d+\alpha)/\alpha}}.$$

This proves (6.15) in the case $\alpha_0 < \alpha < 1$.

The lower bound on G_1 follows from the elementary inequalities

$$\begin{aligned} &(R-r)^{d+\alpha} \ge (R-r)^{d+2}, \\ &(R^\alpha - r^\alpha) \ge \alpha\, 1^{\alpha-1}(R-r) \ge \alpha_0(R-r) \quad \text{if } \alpha_0 < \alpha < 1. \end{aligned}$$

The proof of Theorem 6.4 is complete. □

6.3 An estimate for the L^1-norm of a supersolution

An application of Lemma 5.12 to the basic step (Proposition 6.3) yields the following result:

Theorem 6.6. *Let $\frac{1}{2} \le r < R \le 1$ and $p \in (0, \kappa^{-1})$ with $\kappa = 1 + \frac{\alpha}{d}$ ($\kappa = 1 + \frac{\alpha}{3}$ if $d = 1, 2$). Then there are constants $C, \omega_1, \omega_2 > 0$ depending only on d, α_0, Λ, such that for every nonnegative supersolution u of* (PE$_\alpha$) *in $Q = I \times \Omega$, $Q \Supset Q_\oplus(R)$, the following estimate holds:*

$$\int_{Q_\oplus(r)} \widetilde{u}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le \left(\frac{C}{(R-r)^{\omega_1} \wedge (\alpha_0(R-r))^{\omega_2}}\right)^{1/p-1} \left(\int_{Q_\oplus(R)} \widetilde{u}^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}, \tag{6.17}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(Q(R))}$. If $\alpha_0 \ge 1$ then $(R-r)^{\omega_1} \le (\alpha_0(R-r))^{\omega_2}$.

Proof. In view of Lemma 5.12 choose[4] $\kappa = 1 + \frac{\alpha}{d}$, $p_0 = 1$ and consider the sequence $(Q_\oplus(r), 0 < r \le 1)$ of increasing subsets of $\mathbb{R}^{d+1}$. For $m \in \mathbb{N}_0$ define $p_m = \kappa^{-m}$. Similar to the proof of Theorem 6.4 there are two cases, the second being irrelevant if $\alpha_0 \ge 1$:

I. $1 \vee \alpha_0 \le \alpha < 2$: Choose the sequence of radii $(r_m)_{m\in\mathbb{N}_0}$ defined by

$$r_m = r + \frac{R-r}{2^m}.$$

Note that $r_0 = R$, $r_m > r$ and $\lim_{m\to\infty} r_m = r$. Let $n \in \mathbb{N}$. Proposition 6.3 in combination with (6.3) states that for all $j = 1, \ldots, n$

$$\left(\int_{Q_\oplus(r_j)} \widetilde{u}^{p_{n-j}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_{n-j}}} \le A_j^{\frac{1}{p_{n-j+1}}} \left(\int_{Q_\oplus(r_{j-1})} \widetilde{u}^{p_{n-j+1}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_{n-j+1}}},$$

with $A_j = c_1(r_{j-1} - r_j)^{-\alpha}$ for some constant $c_1 = c_1(d, \alpha_0, \Lambda) \ge 1$. To verify condition (5.26) observe

$$\prod_{j=1}^n A_j^{1/p_{n-j+1}} = \prod_{j=1}^n A_{n-j+1}^{1/p_j} = \prod_{j=1}^n \left(\frac{c_1 2^{\alpha(n-j+1)}}{(R-r)^\alpha}\right)^{\kappa^j}.$$

Due to

$$\sum_{j=1}^n \kappa^j = \frac{\kappa}{\kappa-1}\left(\frac{1}{p_n} - 1\right) = \frac{d+\alpha}{\alpha}\left(\frac{1}{p_n} - 1\right), \qquad \text{and}$$

$$\sum_{j=1}^n (n-j+1)\kappa^j \le \frac{\kappa^3}{(\kappa-1)^3}\left(\frac{1}{p_n} - 1\right),$$

[4] This choice is appropriate in the case $d \ge 3$. The modifications for the case $d = 1, 2$ are obvious and we omit the details for the sake of brevity.

we may estimate

$$\prod_{j=1}^{n} A_j^{1/p_{n-j+1}} \leq \left(\frac{2^{\frac{\alpha\kappa^3}{(\kappa-1)^3}} c_1^{\frac{\kappa}{\kappa-1}}}{(R-r)^{d+\alpha}} \right)^{1/p_n - 1} \leq \left(\frac{c_2}{(R-r)^{d+\alpha}} \right)^{1/p_n - 1}$$

for some constant $c_2 = c_2(d, \alpha_0, \Lambda) \geq 1$. By Lemma 5.12 we obtain

$$\int_{Q_\oplus(r)} \widetilde{u}(t,x) \,\mathrm{d}x \,\mathrm{d}t \leq \left(\frac{c_2 \, |Q_\oplus(1)|}{(R-r)^{d+\alpha}} \right)^{(1+\kappa)/p} \left(\int_{Q_\oplus(R)} \widetilde{u}^p(t,x) \,\mathrm{d}x \,\mathrm{d}t \right)^{1/p}$$

for all $p \in (0, \kappa^{-1}]$. Since $(R-r)^{(1+\kappa)(d+\alpha)} \geq (R-r)^{2d+6+\frac{4}{d}}$, $|Q_\oplus(1)| = c_3(d)$ and $(1+\kappa) \leq 1 + \frac{2}{d}$, this proves (6.17) in the case $\alpha \in [1,2)$ with $\omega_1 = 2d + 6 + \frac{4}{d}$.

II. $\alpha_0 < \alpha < 1$: Choose the sequence of radii $(r_m)_{m \in \mathbb{N}_0}$ defined by

$$r_m = \left(r^\alpha + \frac{R^\alpha - r^\alpha}{2^m} \right)^{1/\alpha} .$$

Note that $r_0 = R$, $r_m > r$ and $\lim_{m\to\infty} r_m = r$. Proposition 6.3 and (6.3) imply that condition (5.25) is satisfied with

$$A_j = c_1 (r_{j-1}^\alpha - r_j^\alpha)^{-1} = \frac{c_1 2^j}{R^\alpha - r^\alpha}, \quad \text{with } c_1 \text{ as above.}$$

In a similar way to the computations in case I we obtain

$$\prod_{j=1}^{n} A_j^{1/p_{n-j+1}} \leq \left(\frac{2^{\frac{\kappa^3}{(\kappa-1)^3}} c_1^{\frac{\kappa}{\kappa-1}}}{(R^\alpha - r^\alpha)^{\frac{d+\alpha}{\alpha}}} \right)^{1/p_n - 1} \leq \left(\frac{c_3}{(R^\alpha - r^\alpha)^{\frac{d+\alpha}{\alpha}}} \right)^{1/p_n - 1} .$$

Observe that

$$(R^\alpha - r^\alpha)^{\frac{d+\alpha}{\alpha}} \geq [\alpha_0 (R-r)]^{(1+\kappa)\frac{d+\alpha}{\alpha}} \geq [\alpha_0 (R-r)]^{3 + \frac{1}{d} + \frac{2d}{\alpha_0}} ,$$

which proves (6.17) in the case $\alpha_0 < \alpha < 1$ with $\omega_2 = 3 + \frac{1}{d} + \frac{2d}{\alpha_0}$.

The additional assertion concerning the case $\alpha_0 \geq 1$ is obvious. The proof of Theorem 6.6 is complete. □

6.4 An inequality for $\log u$

The following lemma provides a lower bound for the nonlocal term in (4.5) when applying u^{-1} times some cut-off function as test function. It can be seen as the nonlocal analog to the inequality

$$\left(\nabla(\psi^2 \widetilde{u}^{-1}) \cdot A \nabla \widetilde{u} \right) \geq \frac{1}{2} \psi^2 \left(\nabla(\log \widetilde{u}) \cdot A \nabla(\log \widetilde{u}) \right) - 2 \left(\nabla \psi \cdot A \nabla \psi \right) ,$$

which is used in the proof of Proposition 7.5 to establish (7.23) when applying the test function $\psi^2 \widetilde{u}^{-1}$.

See [BBCK09, Proposition 4.9] for a similar result.

Lemma 6.7. *Let $I \subset \mathbb{R}$ and $\psi \colon \mathbb{R}^d \to [0,\infty)$ be a continuous function satisfying* $\operatorname{supp}[\psi] = \overline{B_R}$ *for some $R > 0$ and $\sup_{t\in I} \mathcal{E}_t(\psi,\psi) < \infty$. Then the following computation rule holds for $w \colon I \times \mathbb{R}^d \to [0,\infty)$:*

$$\mathcal{E}_t(w, -\psi^2 w^{-1}) \geq \iint\limits_{B_R\, B_R} \psi(x)\psi(y) \left(\log \frac{w(t,y)}{\psi(y)} - \log \frac{w(t,x)}{\psi(x)}\right)^2 k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y - 3\ \mathcal{E}_t(\psi,\psi)$$

Remark 6.8. We apply the rule above only in cases where all terms are finite. ♦

Proof. Fix $t \in I$. First of all we note that

$$\begin{aligned}
\mathcal{E}_t(w, -\psi^2 w^{-1}) &= \iint\limits_{\mathbb{R}^d\, \mathbb{R}^d} [w(t,y) - w(t,x)] \left[\psi^2(x) w^{-1}(t,x) - \psi^2(y) w^{-1}(t,y)\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \iint\limits_{B_R\, B_R} \psi(x)\psi(y) \left[\frac{\psi(x)w(t,y)}{\psi(y)w(t,x)} + \frac{\psi(y)w(t,x)}{\psi(x)w(t,y)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)}\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + 2 \iint\limits_{B_R\, B_R^c} [w(t,y) - w(t,x)] \left[\psi^2(x) w^{-1}(t,x) - \psi^2(y) w^{-1}(t,y)\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \iint\limits_{B_R^c\, B_R^c} [w(t,y) - w(t,x)] \left[\psi^2(x) w^{-1}(t,x) - \psi^2(y) w^{-1}(t,y)\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{6.18}$$

Because of $\operatorname{supp}[\psi] = \overline{B_R}$ the third term on the right-hand side vanishes.

To estimate the first term on the right-hand side we apply the inequality

$$\frac{(a-b)^2}{ab} = (a-b)\left(b^{-1} - a^{-1}\right) \geq (\log a - \log b)^2 \qquad \text{for } a,b > 0$$

to $a = A_t(x,y) = \frac{w(t,y)}{w(t,x)}$ and $b = B(x,y) = \frac{\psi(y)}{\psi(x)}$, $x,y \in B_R$:

$$\begin{aligned}
&\frac{\psi(x)w(t,y)}{\psi(y)w(t,x)} + \frac{\psi(y)w(t,x)}{\psi(x)w(t,y)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)} \\
&\qquad = \frac{A_t(x,y)}{B(x,y)} + \frac{B(x,y)}{A_t(x,y)} - 2 - \left(\sqrt{B(x,y)} - \frac{1}{\sqrt{B(x,y)}}\right)^2 \\
&\qquad \geq \left(\log \frac{w(t,y)}{\psi(y)} - \log \frac{w(t,x)}{\psi(x)}\right)^2 - \left(\frac{\psi(x)}{\psi(y)} + \frac{\psi(y)}{\psi(x)} - 2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\iint\limits_{B_R\, B_R} \psi(x)\psi(y) \left[\frac{\psi(x)w(t,y)}{\psi(y)w(t,x)} + \frac{\psi(y)w(t,x)}{\psi(x)w(t,y)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)}\right] k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad \geq \iint\limits_{B_R\, B_R} \psi(x)\psi(y) \left(\log \frac{w(t,y)}{\psi(y)} - \log \frac{w(t,x)}{\psi(x)}\right)^2 \mathrm{d}y\,\mathrm{d}x - \mathcal{E}_t(\psi,\psi).
\end{aligned} \tag{6.19}$$

Finally, we estimate the second term using the non-negativity of $w(t,\cdot)$ in $\mathbb{R}^d$:

$$\begin{aligned}
&\iint\limits_{B_R\,B_R^c} [w(t,y)-w(t,x)]\left[\psi^2(x)w^{-1}(t,x)-\psi^2(y)w^{-1}(t,y)\right]k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad=\iint\limits_{B_R\,B_R^c} [w(t,y)-w(t,x)]\left[\psi^2(x)w^{-1}(t,x)\right]k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad=\int_{B_R}\frac{\psi^2(x)}{w(t,x)}\int_{B_R^c} w(t,y)k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x-\int_{B_R}\psi^2(x)\int_{B_R^c}k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad\geq-\int_{B_R}\int_{B_R^c}[\psi(x)-\psi(y)]^2\,k_t(x,y)\,\mathrm{d}y\,\mathrm{d}x\geq-\mathcal{E}_t(\psi,\psi).
\end{aligned}\tag{6.20}$$

Applying the estimates (6.19) and (6.20) in (6.18) finishes the proof of Lemma 6.7. □

Proposition 6.9. *Assume $k\in\mathcal{K}(\alpha_0,\Lambda)$ for some $\alpha_0\in(0,2)$ and $\Lambda\geq 1$. Then there is $C=C(d,\alpha_0,\Lambda)>0$ such that for every supersolution u of (PE_α) in $Q=(-1,1)\times B_2(0)$ which satisfies $u\geq\varepsilon>0$ in $(-1,1)\times\mathbb{R}^d$, there is a constant $a=a(\widetilde{u})\in\mathbb{R}$ such that the following inequalities hold simultaneously:*

$$\forall s>0\colon(\,\mathrm{d}t\otimes\,\mathrm{d}x)\,(Q_\oplus(1)\cap\{\log\widetilde{u}<-s-a\})\leq\frac{C\,|B_1|}{s},\tag{6.21a}$$

$$\forall s>0\colon(\,\mathrm{d}t\otimes\,\mathrm{d}x)\,(Q_\ominus(1)\cap\{\log\widetilde{u}>s-a\})\leq\frac{C\,|B_1|}{s},\tag{6.21b}$$

where $\widetilde{u}=u+\|f\|_{L^\infty(Q)}$.

Proof. In the course of the proof we introduce constants c_1,c_2,c_3,c_4 that may depend on d, α_0, and Λ. We use the test function $\phi(t,x)=\psi^2(x)\widetilde{u}^{-1}(t,x)$ in (5.4), where

$$\psi^2(x)=\left[\left(\tfrac{3}{2}-|x|\right)\wedge 1\right]\vee 0,\qquad x\in\mathbb{R}^d,$$

and we write $v(t,x)=-\log\frac{\widetilde{u}(t,x)}{\psi(x)}$. Thus we have for a.e. $t\in(-1,1)$

$$\int_{B_{3/2}}\psi^2(x)\partial_t v(t,x)\,\mathrm{d}x+\mathcal{E}_t(\widetilde{u},-\psi^2\widetilde{u}^{-1})\leq-\int_{B_{3/2}}\psi^2(x)\widetilde{u}^{-1}(t,x)f(t,x)\,\mathrm{d}x.$$

Note that $\mathcal{E}_t(\widetilde{u},-\psi^2\widetilde{u}^{-1})$ is finite since $u(t,\cdot)\in H^{\alpha/2}_{loc}(B_2)$ for a.e. $t\in(-1,1)$ and $\operatorname{supp}\psi=\overline{B_{3/2}}$. Applying Lemma 6.7 and $\|f/\widetilde{u}\|_{L^\infty(Q)}\leq 1$ we obtain

$$\begin{aligned}
&\int_{B_{3/2}}\psi^2(x)\partial_t v(t,x)\,\mathrm{d}x\,+\\
&\qquad+\iint\limits_{B_{3/2}\,B_{3/2}}\psi(x)\psi(y)\,[v(t,y)-v(t,x)]^2\,k_t(x,y)\,\mathrm{d}x\,\mathrm{d}y\leq\left|B_{3/2}\right|+3\mathcal{E}_t(\psi,\psi).
\end{aligned}$$

Now we apply the weighted Poincaré-inequality, Lemma 5.9, to the second term and use the fact $\sup\limits_{t\in(-1,1)} \mathcal{E}_t(\psi,\psi) \leq C$ for some constant $C = C(d, \alpha_0, \Lambda)$. We obtain

$$\int_{B_{3/2}} \psi^2(x)\partial_t v(t,x)\,\mathrm{d}x + c_1 \int_{B_{3/2}} [v(t,x) - V(t)]^2\,\psi^2(x)\,\mathrm{d}x \leq c_2\,|B_1|\,, \tag{6.22}$$

where

$$V(t) = \frac{\int_{B_{3/2}} v(t,x)\psi^2(x)\,\mathrm{d}x}{\int_{B_{3/2}} \psi^2(x)\,\mathrm{d}x}.$$

The proof now proceeds as in the case of local operators. Our presentation uses ideas from [Mos64, pp. 120-123] and [SC02, Lemma 5.4.1].

Integrating the above inequality over $[t_1, t_2] \subset (-1,1)$ yields

$$\left[\int_{B_{3/2}} \psi^2(x)v(t,x)\,\mathrm{d}x\right]_{t=t_1}^{t_2} + c_1 \int_{t_1}^{t_2}\int_{B_{3/2}} [v(t,x) - V(t)]^2\,\psi^2\,\mathrm{d}x \leq c_2(t_2-t_1)\,|B_1|\ . \tag{6.23}$$

Dividing by $\int_{B_{3/2}} \psi^2$, using $\int_{B_{3/2}} \psi^2 \leq 2^d\,|B_1|$ and $\psi = 1$ in B_1, we obtain

$$V(t_2) - V(t_1) + \frac{c_3^{-1}}{|B_1|}\int_{t_1}^{t_2}\int_{B_1} [v(t,x) - V(t)]^2\,\mathrm{d}x\,\mathrm{d}t \leq c_2(t_2 - t_1) \quad \text{with } c_3 = \frac{2^d}{c_1}, \tag{6.24}$$

or equivalently

$$\frac{V(t_2) - V(t_1)}{t_2 - t_1} + \frac{c_3^{-1}}{|B_1|\,(t_2 - t_1)}\int_{t_1}^{t_2}\int_{B_1} [v(t,x) - V(t)]^2\,\mathrm{d}x\,\mathrm{d}t \leq c_2 \tag{6.25}$$

Assume that $V(t)$ is differentiable. Taking the limit $t_2 \to t_1$ the above inequality yields

$$V'(t) + \frac{c_3^{-1}}{|B_1|}\int_{B_1} [v(t,x) - V(t)]^2\,\mathrm{d}x \leq c_2, \qquad \text{for a.e. } t \in (-1,1). \tag{6.26}$$

Now set

$$w(t,x) = v(t,x) - c_2 t, \qquad W(t) = V(t) - c_2 t,$$

such that (6.26) reads

$$W'(t) + \frac{c_3^{-1}}{|B_1|}\int_{B_1} [w(t,x) - W(t)]^2\,\mathrm{d}x \leq 0 \quad \text{for a.e. } t \in (-1,1), \qquad W(0) = a, \tag{6.27}$$

where a is a constant depending on u. Note that by the latter inequality W is non-increasing in $(-1,1)$.

We work out here the details for the proof of (6.21a). It is straightforward to mimic the arguments for the proof of (6.21b). Define for $t \in (0,1)$ and $s > 0$ the set

$$L_s^{\oplus}(t) = \{x \in B_1(0)\colon w(t,x) > s + a\}\,. \tag{6.28}$$

Noting that $W(t) \le a$ for a.e. $t \in (0,1)$, we obtain for such t and $x \in L_s^{\oplus}(t)$

$$w(t,x) - W(t) \ge s + a - W(t) > 0.$$

Using this in (6.27) yields

$$W'(t) + \frac{c_3^{-1}}{|B_1|} \left|L_s^{\oplus}(t)\right| (s + a - W(t))^2 \le 0,$$

which is equivalent to

$$\frac{-c_3\, W'(t)}{(s + a - W(t))^2} \ge \frac{|L_s^{\oplus}(t)|}{|B_1|}\,.$$

Integrating this inequality over $t \in (0,1)$ we obtain

$$\frac{c_3}{s} \ge \left[\frac{c_3}{s + a - W(\tau)}\right]_{\tau=0}^{1} \ge \frac{1}{|B_1|} \int_0^1 \left|L_s^{\oplus}(t)\right| \, \mathrm{d}t = \frac{|Q_{\oplus}(1) \cap \{w > s + a\}|}{|B_1|}$$

and replacing w again by $w(t,x) = v(t,x) - c_2 t = -\log \widetilde{u} - c_2 t$ in $Q_{\oplus}(1)$ yields

$$|Q_{\oplus}(1) \cap \{\log \widetilde{u} + c_2 t < -s - a\}| \le \frac{c_3\, |B_1|}{s}. \tag{6.29}$$

Finally,

$$\begin{aligned} |Q_{\oplus}(1) \cap \{\log \widetilde{u} < -s - a\}| &\le \left|Q_{\oplus}(1) \cap \left\{\log \widetilde{u} + c_2 t < \frac{-s}{2} - a\right\}\right| + \left|Q_{\oplus}(1) \cap \left\{c_2 t > \frac{s}{2}\right\}\right| \\ &\le \frac{2c_3}{s}\, |B_1| + \left(1 - \frac{s}{2c_2}\right) |B_1| \le \frac{c_4}{s}\,. \end{aligned}$$

In case that V is only continuous in $(-1,1)$ we derive the result in a different manner, cf. [Lie96, Lemma 6.21]: For $\varepsilon_0 > 0$ there is $\delta > 0$ such that for $t_2 < t_1 + \delta$

$$|v(t,x) - V(t)|^2 \le 2\,|v(t,x) - V(t_2)|^2 + 2\,|V(t_2) - V(t)|^2 \le 2\,|v(t,x) - V(t_2)|^2 + 2\varepsilon_0^2.$$

Hence, by (6.24) we obtain for $t_2 < t_1 + \delta$

$$V(t_2) - V(t_1) + \frac{c_3^{-1}}{|B_1|} \int_{t_1}^{t_2} \int_{B_1} [v(t,x) - V(t_2)]^2 \, \mathrm{d}x \, \mathrm{d}t \le \left(2c_2 + 2c_3^{-1}\varepsilon_0^2\right)(t_2 - t_1).$$

Defining

$$w(t,x) = v(t,x) - \left(2c_2 + 2c_3^{-1}\varepsilon_0^2\right) t, \qquad W(t) = V(t) - \left(2c_2 + 2c_3^{-1}\varepsilon_0^2\right) t,$$

the latter inequality reads

$$W(t_2) - W(t_1) + \frac{c_3^{-1}}{|B_1|} \int_{t_1}^{t_2} \int_{B_1} \left[w(t,x) - W(t_2) + (2c_2 + 2c_3^{-1}\varepsilon_0^2)(t_2 - t)\right]^2 \, \mathrm{d}x \, \mathrm{d}t \le 0.$$

Using the fact that for $t, t_2 \in (0,1)$ and $x \in L_s^{\oplus}(t)$ we have $w(t,x) - W(t_2) > s + a - W(t_2) \geq 0$, we can omit the term $(2c_2 + 2c_3^{-1}\varepsilon_0^2)(t_2 - t)$ in the integral and deduce that for $t_2 < t_1 + \delta$

$$\frac{W(t_2) - W(t_1)}{(s + a - W(t_2))^2} + \frac{c_3^{-1}}{|B_1|} \int_{t_1}^{t_2} \left|L_s^{\oplus}(t)\right| \,\mathrm{d}t \leq 0.$$

Again, since W is non-increasing, this implies

$$\begin{aligned} \frac{c_3^{-1}}{|B_1|} \int_{t_1}^{t_2} \left|L_s^{\oplus}(t)\right| \,\mathrm{d}t &\leq \frac{W(t_1) - W(t_2)}{(s + a - W(t_1))(s + a - W(t_2))} \\ &= \frac{1}{s + a - W(t_1)} - \frac{1}{s + a - W(t_2)}\ . \end{aligned} \tag{6.30}$$

Choosing $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$, writing

$$\int_0^1 \left|L_s^{\oplus}(t)\right| \,\mathrm{d}t = \sum_{j=0}^{k-1} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left|L_s^{\oplus}(t)\right| \,\mathrm{d}t,$$

and applying (6.30) in each summand, we establish (6.29). Using the same arguments as above we establish (6.21a). This finishes the proof of Proposition 6.9. □

6.5 Proof of the weak Harnack inequality

The aim of this section is to prove Theorem 4.4. The proof uses the well-known idea of Bombieri and Giusti, Lemma 5.13. Let us recall Theorem 4.4:

Theorem 4.4 (Weak Harnack inequality). *Let $k \in \mathcal{K}(\alpha_0, \Lambda)$ for some $\alpha_0 \in (0,2)$ and $\Lambda \geq 1$. Then there is a constant $C = C(d, \alpha_0, \Lambda)$ such that for every supersolution u of* (PE$_\alpha$) *on $Q = (-1,1) \times B_2(0)$ which is nonnegative in $(-1,1) \times \mathbb{R}^d$ the following inequality holds:*

$$\|u\|_{L^1(U_\ominus)} \leq C \left(\inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right) \tag{HI}$$

where $U_\oplus = \left(1 - (\frac{1}{2})^\alpha, 1\right) \times B_{1/2}(0)$, $U_\ominus = \left(-1, -1 + (\frac{1}{2})^\alpha\right) \times B_{1/2}(0)$.

Proof of Theorem 4.4. Let u as in the assumption and define $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$. If $f = 0$ a.e. on Q we set $\widetilde{u} = u + \varepsilon$ and pass to the limit $\varepsilon \to 0+$ in the end.

Furthermore, set $w = e^{-a}\widetilde{u}^{-1}$ and $\widehat{w} = w^{-1} = e^a \widetilde{u}$, where $a = a(\widetilde{u})$ is chosen according to Proposition 6.9, i.e. there is $c_1 > 0$ such that for every $s > 0$

$$|Q_\oplus(1) \cap \{\log w > s\}| \leq \frac{c_1\,|B_1|}{s}, \quad \text{and} \quad |Q_\ominus(1) \cap \{\log \widehat{w} > s\}| \leq \frac{c_1\,|B_1|}{s}. \tag{6.31}$$

The strategy of the proof is to apply Lemma 5.13 twice: on the one hand to w and a family of domains $\mathcal{U} = (U(r))_{\theta \le r \le 1}$ – and on the other hand to $\widehat{w}$ and a family of domains $\widehat{\mathcal{U}} = (\widehat{U}(r))_{\widehat{\theta} \le r \le 1}$. We consider the case $\alpha \ge 1$ first and define the families $\mathcal{U}$ and $\widehat{\mathcal{U}}$ by (cf. Figure 6.4)

$$U(1) = Q_{\oplus}(1), \qquad \theta = \frac{1}{2}, \qquad U(r) = (1 - r^{\alpha}, 1) \times B_r,$$
$$\widehat{U}(1) = Q_{\ominus}(1), \qquad \widehat{\theta} = \frac{1}{2} \qquad \widehat{U}(r) = (-1, -1 + r^{\alpha}) \times B_r$$

By virtue of (6.31) we see that condition (5.31) is satisfied for both w and $\widehat{w}$.

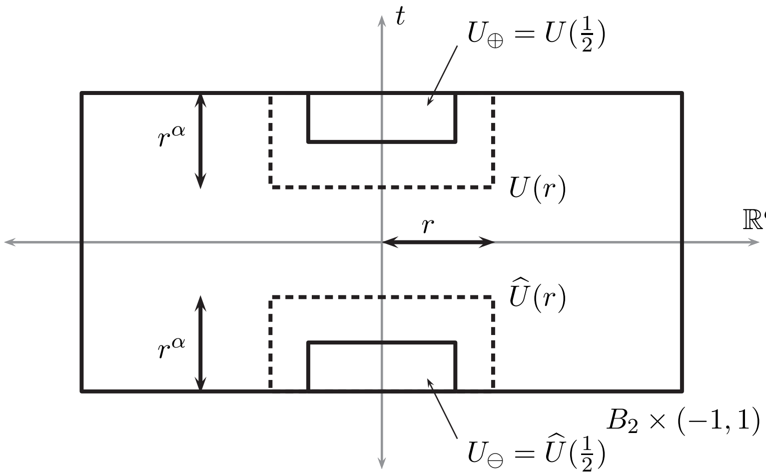

Figure 6.4: Sketch of the domains $\mathcal{U}$ and $\widehat{\mathcal{U}}$ in the case $\alpha \ge 1$

We apply Theorem 6.4 to $(w, \mathcal{U})$ with $p_0 = \infty$ and arbitrary η. We also apply Theorem 6.6 to $(\widehat{w}, \widehat{\mathcal{U}})$ with $\widehat{p}_0 = 1$ and $\widehat{\eta} = \frac{d}{d+2} \le \kappa^{-1}$. In both cases condition (5.30) of Lemma 5.13 is satisfied. Note that the domains $U(r)$ and $\widehat{U}(r)$ are obtained from $Q_{\ominus}(r)$ and $Q_{\oplus}(r)$, respectively, by shifting in time, i.e. transformations of the type $(t, x) \mapsto (t + \tau, x)$, which do not affect neither (6.15) nor (6.17).

All in all, application of Lemma 5.13 yields

$$\sup_{U(\theta)} w = e^{-a} \sup_{U(\theta)} \widetilde{u}^{-1} \le C \qquad \text{and} \qquad \|\widehat{w}\|_{L^1(\widehat{U}(\widehat{\theta}))} = e^{a} \|\widetilde{u}\|_{L^1(\widehat{U}(\widehat{\theta}))} \le \widehat{C}.$$

Multiplying these two inequalities eliminates a and yields

$$\|\widetilde{u}\|_{L^1(\widehat{U}(\widehat{\theta}))} \le c_2 \inf_{U(\theta)} \widetilde{u}$$

for a constant $c_2 = C\,\widehat{C}$ that depends only on d, α_0 and Λ. This proves (HI) in the case $\alpha \ge 1$ observing that $U_{\oplus} = U(\theta)$, $U_{\ominus} = \widehat{U}(\widehat{\theta})$ and

$$\|u\|_{L^1(U_{\ominus})} \le \|\widetilde{u}\|_{L^1(U_{\ominus})} \le c_2 \left(\inf_{U_{\oplus}} u + \|f\|_{L^{\infty}(Q)} \right).$$

If $\alpha < 1$, we define the domains $\mathcal{U}$ and $\widehat{\mathcal{U}}$ slightly differently, namely

$$U(1) = Q_\oplus(1), \qquad \theta = \left(\tfrac{1}{2}\right)^\alpha, \qquad U(r) = (1-r,1) \times B_{r^{1/\alpha}},$$
$$\widehat{U}(1) = Q_\ominus(1), \qquad \widehat{\theta} = \left(\tfrac{1}{2}\right)^\alpha, \qquad \widehat{U}(r) = (-1,-1+r) \times B_{r^{1/\alpha}}.$$

The same reasoning as above applies to these domains and hence (HI) is proved for all $\alpha \in (\alpha_0, 2)$. □

6.6 Proof of Hölder regularity

In this section we deduce Theorem 4.5 from Theorem 4.4. This step is not trivial and differs from the proof in the case of a local differential operator because the (super-)solutions in Theorem 4.4 are assumed to be nonnegative in the whole spatial domain. Note that the auxiliary functions of the type $M(t,x) = \sup_Q u - u(t,x)$ and $m(t,x) = u - \inf_Q u$ used in [Mos64, Section 2] are nonnegative in Q but not in all of $\mathbb{R}^d$. The key idea to overcome this problem is to derive Lemma 6.11 from the Harnack inequality. Lemma 6.11 then implies Theorem 4.5. This step is carried out in [Sil06] for elliptic equations.

The following corollary will be used to derive Hölder continuity.

Corollary 6.10. *Let $\sigma \in (0,1)$ and $D_\ominus = (-2,-2+\left(\frac{1}{2}\right)^\alpha) \times B_{1/2}$, $D_\oplus = (-\left(\frac{1}{2}\right)^\alpha, 0) \times B_{1/2}$. There exist $\varepsilon_0, \delta \in (0,1)$ such that for every function w satisfying*

$$\begin{cases} w \geq 0 & a.e. \text{ in } (-2,0) \times \mathbb{R}^d, \\ \partial_t w - \mathscr{L} w \geq -\varepsilon_0 & \text{in } (-2,0) \times B_2, \\ \left| D_\ominus \cap \{w \geq 1\} \right| \geq \sigma \left| D_\ominus \right|, & \end{cases}$$

the following estimate holds:

$$w \geq \delta \qquad a.e. \text{ in } D_\oplus. \tag{6.32}$$

The constants ε_0 and δ depend on $\sigma, \alpha_0, \Lambda, d$ but not on $\alpha \in (\alpha_0, 2)$.

Proof. Application of Theorem 4.4 to w yields

$$\sigma \leq \fint_{D_\ominus} w(t,x)\, \mathrm{d}x\, \mathrm{d}t \leq c \left(\inf_{D_\oplus} w + \varepsilon_0 \right)$$

for a constant $c = c(d, \alpha_0, \Lambda)$. Choosing $\varepsilon_0 < \frac{\sigma}{c}$ and $\delta = \frac{\sigma - c\varepsilon_0}{c}$ we obtain

$$\inf_{D_\oplus} w \geq \delta,$$

which is the desired inequality. □

Define for $(t, x) \in \mathbb{R}^{d+1}$ a distance function

$$\widehat{\rho}((t,x)) = \begin{cases} \max\left(\frac{1}{3}|x|, \frac{1}{2}(-t)^{1/\alpha}\right) & \text{if } t \in (-2,0], \\ \infty & \text{if } t \notin (-2,0]. \end{cases}$$

Note that $\widehat{\rho}((x,t)) \neq \widehat{\rho}(-(x,t))$. We define (cf. Figure 6.5)

$$\widehat{D}_r((x_0,t_0)) = \left\{(t,x) \in \mathbb{R}^{d+1} \mid \widehat{\rho}((t,x)-(t_0,x_0)) < r\right\}, \qquad I_1 = (-2,0)$$

and note

$$\widehat{D}_r((x_0,t_0)) = (t_0 - 2r^\alpha, t_0) \times B_{3r}(x_0) \quad \text{and} \quad \bigcup_{r>0} \widehat{D}_r((0,0)) = I_1 \times \mathbb{R}^d.$$

To simplify notation we write $\widehat{D}(r) = \widehat{D}_r((0,0))$. Additionally, we define

$$D(r) = (-2r^\alpha, 0) \times B_{2r}(0)$$

and recall the definitions of $D_\oplus$ and $D_\ominus$ in Corollary 6.10.

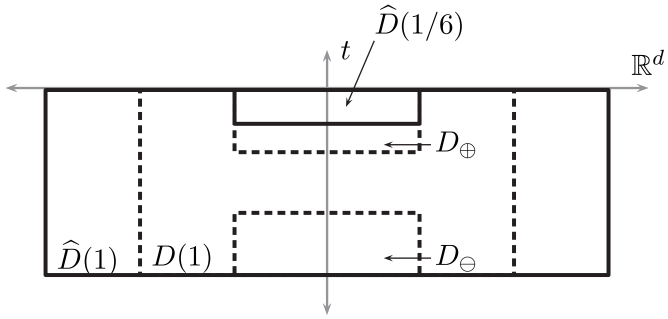

Figure 6.5: Sketch of the domains D and $\widehat{D}$

Lemma 6.11. *Assume that $\mathscr{L}$ is defined by* (4.2) *with a kernel k belonging to some $\mathcal{K}'(\alpha_0,\Lambda)$. Then there exist $\beta_0 \in (0,1)$ and $\delta \in (0,1)$ depending on d, α_0 and Λ such that for every function w with the properties*

$$w \geq 0 \qquad \text{a.e. in } \widehat{D}(1), \tag{6.33a}$$

$$\partial_t w - \mathscr{L} w \geq 0 \qquad \text{in } \widehat{D}(1), \tag{6.33b}$$

$$|D_\ominus \cap \{w \geq 1\}| \geq \frac{1}{2}|D_\ominus|, \tag{6.33c}$$

$$w \geq 2\left[1 - (6\,\widehat{\rho}(t,y))^{\beta_0}\right] \qquad \text{a.e. in } I_1 \times (\mathbb{R}^d \setminus B_3), \tag{6.33d}$$

the following inequality holds:

$$w \geq \delta \qquad \text{a.e. in } D_\oplus.$$

Proof. The conditions (6.33a) and (6.33b) imply $\partial_t w^+ - \mathscr{L} w^+ \geq -f$ in $D(1)$, where

$$f(t,x) = (\mathscr{L} w^-)(t,x) \qquad \text{for } (t,x) \in D(1).$$

Note that since $|x-y| \geq 1$ for $x \in B_2$ and $y \in \mathbb{R}^d \setminus B_3$

$$\|f\|_{L^\infty(D(1))} = \sup_{(t,x)\in D(1)} \int_{\mathbb{R}^d \setminus B_3(0)} w^-(t,y) k_t(x,y)\,\mathrm{d}y < \infty\ .$$

Next, from condition (6.33d) we deduce

$$w^-(t,y) \leq 2\,[6\,\widehat{\rho}(t,y)]^{\beta_0} - 2 \leq 2\left(4^{\beta_0}\,|y|^{\beta_0} - 1\right) \qquad \text{a.e. in } I_1 \times (\mathbb{R}^d \setminus B_3).$$

Our aim is to show $\|f\|_{L^\infty(D(1))} \leq \varepsilon_0$ with ε_0 as in Corollary 6.10 for $\sigma = \frac{1}{2}$. Note that for every $R > 3$

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B_3(0)} \left(4^{\beta_0}\,|y|^{\beta_0} - 1\right) k_t(x,y)\,\mathrm{d}y = &\int_{\mathbb{R}^d \setminus B_R(0)} \left(4^{\beta_0}\,|y|^{\beta_0} - 1\right) k_t(x,y)\,\mathrm{d}y \\ &+ \int_{B_R \setminus B_3(0)} \left(4^{\beta_0}\,|y|^{\beta_0} - 1\right) k_t(x,y)\,\mathrm{d}y.\end{aligned}$$

Because of (K_3) it is possible to choose R sufficiently large and $\beta_0 \in (0,1)$ sufficiently small in dependence of ε_0 and Λ such that $\|f\|_{L^\infty(D(1))} \leq \varepsilon_0$.

Condition (6.33c) ensures that Corollary 6.10 can be applied. □

Theorem 6.12 (Oscillation decay)**.** *Assume that $\mathscr{L}$ is defined by (4.2) with a kernel k belonging to some $\mathcal{K}'(\alpha_0, \Lambda)$. Then there exists $\beta \in (0,1)$ depending on d, α_0 and Λ such that every solution u to $\partial_t u - \mathscr{L} u = 0$ in $\widehat{D}(1)$ satisfies for all $\nu \in \mathbb{Z}$*

$$\operatorname*{osc}_{\widehat{D}(6^{-\nu})} u \leq 2\|u\|_{L^\infty(I_1 \times \mathbb{R}^d)} 6^{-\nu\beta}, \tag{6.34}$$

where $\operatorname{osc}_Q u = \sup_Q u - \inf_Q u$.

Proof. Set $K = M_0 - m_0$ where $M_0 = \sup_{I_1 \times \mathbb{R}^d} u$, $m_0 = \inf_{I_1 \times \mathbb{R}^d} u$. Let $\delta, \beta_0 \in (0,1)$ be the constants from Lemma 6.11. Define

$$\beta = \min\Big(\beta_0, \frac{\log(\frac{2}{2-\delta})}{\log 6}\Big) \quad \Longrightarrow \quad 1 - \frac{\delta}{2} < 6^{-\beta}. \tag{6.35}$$

We will construct inductively an increasing sequence $(m_\nu)_{\nu\in\mathbb{Z}}$ and a decreasing sequence $(M_\nu)_{\nu\in\mathbb{Z}}$ such that for every $\nu \in \mathbb{Z}$

$$\begin{aligned} m_\nu \leq u \leq M_\nu \qquad &\text{a.e. in } \widehat{D}(6^{-\nu}), \\ M_\nu - m_\nu = K 6^{-\nu\beta}. &\end{aligned} \tag{6.36}$$

Obviously, (6.36) implies (6.34). For $n \in \mathbb{N}$ set $M_{-n} = M_0$, $m_{-n} = m_0$. Assume we have constructed M_n and m_n for $n \leq k-1$ and define

$$v(t,x) = \left[u\left(\frac{t}{6^{\alpha(k-1)}}, \frac{x}{6^{k-1}}\right) - \frac{M_{k-1} + m_{k-1}}{2}\right] \frac{2 \cdot 6^{\beta(k-1)}}{K}\ .$$

Clearly, v satisfies

$$\partial_t v - \mathscr{L} v = 0 \text{ in } \widehat{D}(1) \quad \text{and} \quad |v| \leq 1 \text{ in } \widehat{D}(1) \text{ (by induction hypothesis).} \tag{6.37}$$

On $I_1 \times (\mathbb{R}^d \setminus B_3)$ we can estimate v in the following way: For $(t,y) \in I_1 \times (\mathbb{R}^d \setminus B_3)$ fix $j \in \mathbb{N}$ such that

$$6^{j-1} \leq \widehat{\rho}(t,y) < 6^j, \quad \text{or equivalently } (t,y) \in \widehat{D}(6^j) \setminus \widehat{D}(6^{j-1}).$$

Then

$$\begin{aligned}
\frac{K}{2 \cdot 6^{(k-1)\beta}} v(t,y) &= \left(u\left(\frac{t}{6^{\alpha(k-1)}}, \frac{y}{6^{k-1}} \right) - \frac{M_{k-1} + m_{k-1}}{2} \right) \\
&\leq \left(M_{k-j-1} - m_{k-j-1} + m_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2} \right) \\
&\leq \left(M_{k-j-1} - m_{k-j-1} - \frac{M_{k-1} - m_{k-1}}{2} \right) \\
&= \left(K 6^{-(k-j-1)\beta} - \frac{K}{2} 6^{-(k-1)\beta} \right), \\
\Rightarrow v(t,y) &\leq 2 \cdot 6^{j\beta} - 1 \qquad \text{for a.e. } (t,y) \in \widehat{D}(6^j) \setminus \widehat{D}(6^{j-1}) \\
\Rightarrow v(t,y) &\leq 2 \left[6\, \widehat{\rho}(t,y) \right]^{\beta} - 1 \qquad \text{for a.e. } (t,y) \in I_1 \times (\mathbb{R}^d \setminus B_3).
\end{aligned} \tag{6.38}$$

Analogously, we can estimate v from below by

$$v(t,y) \geq 1 - 2 \left[6\, \widehat{\rho}(t,y) \right]^{\beta} \qquad \text{for a.e. } (t,y) \in I_1 \times (\mathbb{R}^d \setminus B_3). \tag{6.39}$$

Now there are two cases. In the first case v is non-positive in at least half of the set $D_\ominus$, i.e.

$$|D_\ominus \cap \{v \leq 0\}| \geq \frac{1}{2} |D_\ominus| . \tag{6.40}$$

Set $w = 1 - v$. w satisfies conditions (6.33a)-(6.33d) of Lemma 6.11 and hence

$$w \geq \delta \quad \text{a.e. in } D_\oplus, \qquad \text{or equivalently} \quad v \leq 1 - \delta \quad \text{a.e. in } D_\oplus.$$

Noting that $\widehat{D}(1/6) \subset D_\oplus$ this estimate has the following consequence for u: For a.e. $(t,x) \in \widehat{D}(6^{-k})$ we have

$$\begin{aligned}
u(t,x) &= \frac{K}{2 \cdot 6^{(k-1)\beta}}\, v\left(6^{\alpha(k-1)} t, 6^{k-1} x \right) + \frac{M_{k-1} + m_{k-1}}{2} \\
&\leq \frac{K(1-\delta)}{2 \cdot 6^{(k-1)\beta}} + m_{k-1} + \frac{M_{k-1} - m_{k-1}}{2} \\
&\leq \frac{K(1-\delta)}{2 \cdot 6^{(k-1)\beta}} + m_{k-1} + \frac{K}{2 \cdot 6^{(k-1)\beta}} = m_{k-1} + \left(1 - \frac{\delta}{2} \right) K 6^{-(k-1)\beta} \\
&\leq m_{k-1} + K 6^{-k\beta},
\end{aligned}$$

where we apply (6.35) in the last inequality. By choosing $m_k = m_{k-1}$ and $M_k = m_{k-1} + K 6^{-k\beta}$ we obtain sequences (m_n) and (M_n) satisfying (6.36). In the second case v is positive in at least half of the set $D_\ominus$ and hence $w = 1 + v$ satisfies all conditions of Lemma 6.11. Therefore, we obtain

$$w \geq \delta \quad \text{a.e. in } D_\oplus, \qquad \text{or equivalently} \quad v \geq -1 + \delta \quad \text{a.e. in } D_\oplus.$$

Adopting the computations above we see that $M_k = M_{k-1}$ and $m_k = M_{k-1} - K6^{-k\beta}$ lead to the desired result.

This proves (6.36). □

Having established Theorem 6.12 we are now able to prove Theorem 4.5 providing a priori estimates of Hölder norms of solutions. Let us recall Theorem 4.5:

Theorem 4.5 (Hölder regularity). *Let $k \in \mathcal{K}'(\alpha_0, \Lambda)$ for some $\alpha_0 \in (0, 2)$ and $\Lambda \geq 1$. Then there is a constant $\beta = \beta(d, \alpha_0, \Lambda)$ such that for every solution u of* (PE$_\alpha$) *in $Q = I \times \Omega$ with $f = 0$ and every $Q' \Subset Q$ the following estimate holds:*

$$\sup_{(t,x),(s,y)\in Q'} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/\alpha}\right)^\beta} \leq \frac{\|u\|_{L^\infty(I\times\mathbb{R}^d)}}{\eta^\beta}, \tag{HC}$$

with some constant $\eta = \eta(Q, Q') > 0$.

Proof. Let u as in the assumption, $Q' \Subset Q$ and define

$$\eta(Q', Q) = \eta = \sup\left\{ r \in (0, \tfrac{1}{2}] \;\middle|\; \forall (t,x) \in Q' \colon \widehat{D}_r(t,x) \subset Q \right\}.$$

Fix $(t,x), (s,y) \in Q'$. Without loss of generality we may take $t \leq s$. At first, assume that

$$\widehat{\rho}((t,x) - (s,y)) < \eta \tag{6.41}$$

and choose $n \in \mathbb{N}_0$ such that

$$\frac{\eta}{6^{n+1}} \leq \widehat{\rho}((t,x) - (s,y)) < \frac{\eta}{6^n}.$$

Now set $\overline{u}(t,x) = u(\eta^\alpha t + s, \eta x + y)$. By assumption $\overline{u}$ is a solution of $\partial_t \overline{u} - \mathscr{L}\overline{u} = 0$ in $\widehat{D}(1)$. Accordingly, applying Theorem 6.12 to $\overline{u}$ we obtain

$$\begin{aligned}
|u(t,x) - u(s,y)| &= \left|\overline{u}(\eta^{-\alpha}(t-s), \eta^{-1}(x-y)) - \overline{u}(0,0)\right| \\
&\leq 2\,\|\overline{u}\|_{L^\infty(I_1\times\mathbb{R}^d)}\, 6^{-n\beta} \\
&\leq 2\,\|u\|_{L^\infty(I\times\mathbb{R}^d)} \left(6^{-n-1}\right)^\beta 6^\beta \\
&\leq 12\,\|u\|_{L^\infty(I\times\mathbb{R}^d)} \left(\frac{\widehat{\rho}((t,x)-(s,y))}{\eta}\right)^\beta \\
&\leq 12\,\|u\|_{L^\infty(I\times\mathbb{R}^d)} \left(\frac{|x-y| + (s-t)^{1/\alpha}}{\eta}\right)^\beta.
\end{aligned}$$

Hence, for all $(t,x),(y,s) \in Q'$ subject to (6.41)

$$\frac{|u(t,x)-u(s,y)|}{\left(|x-y|+|t-s|^{1/\alpha}\right)^{\beta}} \leq \frac{12\,\|u\|_{L^\infty(I\times\mathbb{R}^d)}}{\eta^\beta}\,.$$

If $\widehat{\rho}((t,x)-(s,y)) \geq \eta$ then the Hölder estimate follows directly:

$$\begin{aligned}|u(t,x)-u(s,y)| \leq 2\,\|u\|_{L^\infty(I\times\mathbb{R}^d)} &\leq \frac{2\,\|u\|_{L^\infty(I\times\mathbb{R}^d)}\left[\max\left(|x-y|\,,|t-s|^{1/\alpha}\right)\right]^{\beta}}{\eta^\beta}\\ &\leq \frac{2\,\|u\|_{L^\infty(I\times\mathbb{R}^d)}}{\eta^\beta}\left(|x-y|+|t-s|^{1/\alpha}\right)^{\beta}.\end{aligned}$$

Hence,

$$\sup_{(t,x),(s,y)\in Q'} \frac{|u(t,x)-u(s,y)|}{\left(|x-y|+|t-s|^{1/\alpha}\right)^{\beta}} \leq \frac{12\,\|u\|_{L^\infty(I\times\mathbb{R}^d)}}{\eta^\beta},$$

which had to be shown. □

7 Proof of the main results for second order parabolic equations

7.1 Basic step of Moser's iteration

The results in this section provide the basic steps for Moser's iteration for negative exponents. The case of a local operator, i.e. a parabolic equation in divergence form with bounded and measurable coefficients, is studied in this section. The result and the proof are contained in [Mos64] and [Mos71].

Proposition 7.1. *Let* $\frac{1}{2} \le r < R \le 1$ *and* $\kappa = 1 + \frac{2}{d}$ *(*$\kappa = \frac{5}{3}$ *if* $d = 1, 2$*).*

(i) Let $p < 0$*. Then every* supersolution u *of* (PE$_2$) *in* $Q = I \times \Omega$, $Q \Supset Q_{\ominus}(R)$, *with* $u \ge \varepsilon > 0$ *in* Q *satisfies*

$$\left(\int_{Q_{\ominus}(r)} \widetilde{u}^{\kappa p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \right)^{1/\kappa} \le A \int_{Q_{\ominus}(R)} \widetilde{u}^{p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \ . \tag{7.1}$$

(ii) Let $p \in (0,1)$*. Then every nonnegative* supersolution u *of* (PE$_2$) *in* $Q = I \times \Omega$, $Q \Supset Q_{\oplus}(R)$*, satisfies*

$$\left(\int_{Q_{\oplus}(r)} \widetilde{u}^{\kappa p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \right)^{1/\kappa} \le A \int_{Q_{\oplus}(R)} \widetilde{u}^{p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \ . \tag{7.2}$$

(iii) Let $p > 1$*. Then every nonnegative* subsolution u *of* (PE$_2$) *in* $Q = I \times \Omega$, $Q \Supset Q_{\ominus}(R)$*, satisfies*

$$\left(\int_{Q_{\ominus}(r)} \widetilde{u}^{\kappa p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \right)^{1/\kappa} \le A \int_{Q_{\ominus}(R)} \widetilde{u}^{p}(t,x) \,\mathrm{d}x\,\mathrm{d}t \ . \tag{7.3}$$

In (7.1)-(7.3) *we have used the notation* $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$*. The constant* A *can be chosen as*

$$A = \frac{C}{(R-r)^2} \left(\frac{\sigma}{\varepsilon} + |p| + 1 \right)^{\frac{\kappa+1}{\kappa}} \tag{7.4}$$

with $\varepsilon = \frac{1}{2}\left|1 - \frac{1}{p}\right|$, $\sigma = \Lambda + \lambda^{-1}$ *and a constant* $C = C(d)$*.*

Note that (7.1) is an estimate on $\widetilde{u}$ from below since the exponent is negative there.

Before starting the proof let us state and prove an immediate consequence of the preceding result:

Corollary 7.2. *Let* $p > 0$, $p \neq 1$. *Then every* solution u *of* (PE_2) *in* $Q = I \times \Omega$, $Q \Supset Q(R)$, *satisfies*

$$\left(\int_{Q(r)} \widetilde{u}^{\kappa p}(t,x) \, \mathrm{d}x \, \mathrm{d}t \right)^{1/\kappa} \leq A \int_{Q(R)} \widetilde{u}^{p}(t,x) \, \mathrm{d}x \, \mathrm{d}t \ . \tag{7.5}$$

where

$$A = \frac{C}{(R-r)^2} \left(\frac{\sigma}{\varepsilon} + |p| + 1 \right)^{\frac{\kappa+1}{\kappa}}$$

with the same notation as in Proposition 7.1.

Proof of Corollary 7.2. Solutions are invariant under the translation $(t,x) \mapsto (t \pm r^2, x)$. Taking into account the inclusions

$$(-r^2, R^2 - r^2) \times B_R \subset Q(R) \qquad \text{and} \qquad (-R^2 + r^2, r^2) \times B_R \subset Q(R)$$

we conclude from (7.2) and (7.3) that

$$\left(\int_{Q(r)} \widetilde{u}^{\kappa p}(t,x) \, \mathrm{d}x \, \mathrm{d}t \right)^{1/\kappa} \leq 2A \int_{Q(R)} \widetilde{u}^{p}(t,x) \, \mathrm{d}x \, \mathrm{d}t.$$

The proof of Corollary 7.2 is complete. □

Proof of Proposition 7.1. Let $p \in \mathbb{R}$, $p \neq 0$, $p \neq 1$, and u some function as in part (i)-(iii). Set $\widetilde{u} = u + \|f\|_{L^\infty(Q(R))}$ and

$$v(t,x) = \widetilde{u}^{\frac{p}{2}}(t,x), \qquad \phi(t,x) = \widetilde{u}^{p-1}(t,x)\psi^2(x),$$

where $\psi \colon \Omega \to [0,1]$ is defined by[1] $\psi(x) = \left(\frac{R-|x|}{R-r} \wedge 1 \right) \vee 0$. Obviously, $\psi^2 \in H_0^1(\Omega)$. Note that

$$\nabla v = \frac{p}{2} \, \widetilde{u}^{\frac{p}{2}-1} \nabla \widetilde{u}, \qquad \partial_t v^2 = p \, \widetilde{u}^{p-1} \partial_t \widetilde{u}$$

and

$$\nabla \phi = (p-1) \, \widetilde{u}^{p-2} \, \psi^2 \nabla \widetilde{u}(t,x) + 2\psi \, \widetilde{u}^{p-1} \, \nabla \psi \ .$$

We prove part (i) in full detail and indicate the modifications that are necessary for the proof of part (ii) and (iii) afterwards.

For the proof of part (i) we proceed in two steps: First, we establish a Caccioppoli-type estimate. In the second step we use these estimates in a space-time Sobolev-type embedding to obtain (7.1).

[1] cf. Figure 6.1 on page 80

I: Let u be a supersolution in Q with $u \geq \varepsilon > 0$ in Q. Since $p < 0$ this assumption is needed to guarantee the boundedness of u^p. We apply the test function ϕ in (5.9): For a.e. $t \in I$

$$\int_{B_R} \widetilde{u}^{p-1}(t,x)\psi^2(x)\partial_t \widetilde{u}(t,x)\,\mathrm{d}x + \int_{B_R} (p-1)\psi^2(x)\widetilde{u}^{p-2}(t,x)\left(\nabla\widetilde{u}\cdot A\nabla\widetilde{u}\right)(t,x)\,\mathrm{d}x$$
$$\geq -2\int_{B_R} \psi(x)\widetilde{u}^{p-1}(t,x)\left(\nabla\psi\cdot A\nabla\widetilde{u}\right)(t,x)\,\mathrm{d}x + \int_{B_R} f(t,x)\psi^2(x)\widetilde{u}^{p-1}(t,x)\,\mathrm{d}x\,.$$

In terms of v this inequality reads

$$\frac{1}{p}\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + \frac{4(p-1)}{p^2}\int_{B_R} \psi^2(x)\left(\nabla v\cdot A\nabla v\right)(t,x)\,\mathrm{d}x$$
$$\geq -\frac{4}{p}\int_{B_R} \psi(x)v(t,x)\left(\nabla\psi\cdot A\nabla v\right)(t,x)\,\mathrm{d}x + \int_{B_R} f(t,x)\psi^2(x)\widetilde{u}^{-p-1}\,\mathrm{d}x\,, \quad (7.6)$$

or equivalently

$$\frac{1}{4}\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + \left(1-\frac{1}{p}\right)\int_{B_R} \psi^2(x)\left(\nabla v\cdot A\nabla v\right)(t,x)\,\mathrm{d}x$$
$$\leq -\int_{B_R} \psi(x)v(t,x)\left(\nabla\psi\cdot A\nabla v\right)(t,x)\,\mathrm{d}x + \frac{p}{4}\int_{B_R} f(t,x)\psi^2(x)\widetilde{u}^{p-1}(t,x)\,\mathrm{d}x\,. \quad (7.7)$$

Using Schwarz' inequality and the inequality $ab \leq \frac{1}{4\varepsilon}a^2 + \varepsilon b^2$ for $\varepsilon > 0$, we estimate the first integrand on the right-hand side by

$$\begin{aligned} |v\psi\,\nabla\psi\cdot A\nabla v| &\leq \left(v^2\left(\nabla\psi\cdot A\nabla\psi\right)\psi^2\left(\nabla v\cdot A\nabla v\right)\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4\varepsilon}v^2\;\nabla\psi\cdot A\nabla\psi + \varepsilon\psi^2\;\nabla v\cdot A\nabla v. \end{aligned} \quad (7.8)$$

Choose $\varepsilon = \frac{1}{2}\left(1-\frac{1}{p}\right) > 0$. Then (7.7) and (7.8) yield

$$\frac{1}{4}\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + \varepsilon\int_{B_R} \psi^2(x)\left(\nabla v\cdot A\nabla v\right)(t,x)\,\mathrm{d}x$$
$$\leq \frac{1}{4\varepsilon}\int_{B_R} v^2(t,x)(\nabla\psi\cdot A\nabla\psi)(t,x)\,\mathrm{d}x + \frac{p}{4}\int_{B_R} f(t,x)\psi^2(x)u^{p-1}(t,x)\,\mathrm{d}x\,. \quad (7.9)$$

Using (4.3) and $\|f/\widetilde{u}\|_{L^\infty(Q)} \leq 1$ we obtain

$$\begin{aligned} &\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + 4\varepsilon\lambda\int_{B_R} \psi^2(x)\left|\nabla v(t,x)\right|^2\,\mathrm{d}x \\ &\qquad \leq \int_{B_R} v^2(t,x)\left(\frac{\Lambda}{\varepsilon}\left|\nabla\psi(x)\right|^2 + |p|\;\psi^2(x)\right)\mathrm{d}x. \end{aligned} \quad (7.10)$$

Now define[2] a piecewise differentiable function $\chi_\ominus\colon \mathbb{R} \to [0,1]$ by

$$\chi_\ominus(t) = \left(\frac{t+R^2}{R^2-r^2} \wedge 1 \right) \vee 0.$$

Multiplying (7.10) with $\chi_\ominus^2$ implies

$$\begin{aligned}\int\limits_{B_R} \partial_t(\chi_\ominus(t)\psi(x)v(t,x))^2\,\mathrm{d}x + 4\varepsilon\lambda\,\chi_\ominus^2(t) \int\limits_{B_R} \psi^2(x)\,|\nabla v(t,x)|^2\,\mathrm{d}x \\ \leq \chi_\ominus^2(t) \int\limits_{B_R} v^2(t,x) \left[\frac{\Lambda}{\varepsilon} |\nabla\psi(x)|^2 + |p|\ \psi^2(x) \right] \mathrm{d}x \\ +2\chi_\ominus(t)\big|\chi_\ominus'(t)\big| \int\limits_{B_R} \psi^2(x)v^2(t,x)\,\mathrm{d}x. \quad (7.11)\end{aligned}$$

Integrating (7.11) from $-R^2$ to some $t \in (-r^2, 0) = I_\ominus(r)$ yields

$$\begin{aligned}\int_{B_R} (\chi_\ominus(t)\psi(x)v(t,x))^2\,\mathrm{d}x + 4\varepsilon\lambda \int_{-R^2}^t \chi_\ominus^2(s) \int_{B(R)} \psi^2(x)\,|\nabla v(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_{-R^2}^t \chi_\ominus^2(s) \int_{B_R} v^2(s,x) \left(\frac{\Lambda}{\varepsilon} |\nabla\psi(x)|^2 + |p|\ \psi^2(x) \right) \mathrm{d}x\,\mathrm{d}s \quad (7.12) \\ + \int_{-R^2}^t \int_{B_R} 2\chi_\ominus(s)\,\big|\chi_\ominus'(s)\big|\,\psi^2(x)v^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\ .\end{aligned}$$

Now we use the facts that

$$|\nabla\psi|^2 \leq \frac{1}{(R-r)^2}, \qquad \big|\chi_\ominus'\big| \leq \frac{1}{R^2-r^2}$$

$$\psi = 1 \text{ on } B_r, \quad \psi \leq 1 \text{ on } B_R, \qquad \chi_\ominus = 1 \text{ on } I_\ominus(r), \quad \chi_\ominus \leq 1 \text{ on } I_\ominus(R).$$

to establish

$$\begin{aligned}\sup_{t\in I_\ominus(r)} \int_{B(r)} v^2(t,x)\,\mathrm{d}x + \varepsilon\lambda \int_{Q_\ominus(r)} |\nabla v(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{1}{4} \left(\frac{\Lambda}{\varepsilon} \frac{1}{(R-r)^2} + |p| + \frac{1}{R^2-r^2} \right) \int_{Q_\ominus(R)} v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{c_1}{(R-r)^2} \left(\frac{\Lambda}{\varepsilon} + |p| + 1 \right) \int_{Q_\ominus(R)} v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\,. \quad (7.13)\end{aligned}$$

This $L^\infty(L^2)\cap L^2(H^1)$-estimate is sometimes called a Caccioppoli-type estimate. We will use these inequalities in the next step to control the constant in the parabolic Sobolev embedding.

[2]cf. Figure 6.2 on page 81

II: Now we first apply Hölder's inequality with exponents[3] $\theta = \frac{d}{d-2}$, $\theta' = \frac{d}{2}$ to the integrand $v^{2\kappa}$ (remember $\kappa = 1 + \frac{1}{\theta'}$) and then we make use of Sobolev's inequality in Proposition 5.6:

$$\begin{aligned}
\int_{Q(r)} v^{2\kappa}(t,x)\,\mathrm{d}x\,\mathrm{d}t &= \int_{Q(r)} v^2(t,x) v^{4/d}(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{I(r)} \left(\int_{B(r)} v^{2\theta}(t,x)\,\mathrm{d}x\right)^{1/\theta} \left(\int_{B(r)} v^2(t,x)\,\mathrm{d}x\right)^{1/\theta'} \mathrm{d}t \\
&\le c_2 \sup_{t\in I(r)} \left(\int_{B(r)} v^2(t,x)\,\mathrm{d}x\right)^{1/\theta'} \int_{Q(r)} \left(r^{-2}v^2(t,x) + |\nabla v(t,x)|^2\right) \mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{7.14}$$

with a constant c_2 depending on d. Using (7.13) twice and $r \ge \frac{1}{2}$ yields

$$\begin{aligned}
\int_{Q(r)} v^{2\kappa}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le c_2 &\left[\frac{c_1}{(R-r)^2}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right)\right]^{\kappa-1} \times \\
&\times \left[\frac{c_1}{(R-r)^2}\frac{1}{\lambda\varepsilon}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right) + 1\right] \left(\int_{Q(R)} v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\kappa}.
\end{aligned} \tag{7.15}$$

Observe that with $\sigma = \Lambda + \lambda^{-1}$

$$\begin{aligned}
&\left[\frac{c_1}{(R-r)^2}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right)\right]^{\kappa-1} \left[\frac{c_1}{(R-r)^2}\frac{1}{\lambda\varepsilon}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right) + 1\right] \\
&\quad = \frac{c_1^{\kappa}}{(R-r)^{2\kappa}}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right)^{\kappa-1}\left[\frac{\Lambda\lambda^{-1}}{\varepsilon^2} + \frac{\lambda^{-1}}{\varepsilon}(1+|p|) + c_1^{-1}(R-r)^2\right] \\
&\quad \le \frac{c_3}{(R-r)^{2\kappa}}\left(\frac{\sigma}{\varepsilon} + |p| + 1\right)^{\kappa-1}\left(\frac{\sigma}{\varepsilon} + |p| + 1\right)^{2}.
\end{aligned}$$

Together with (7.15) this proves (7.1) in part (i) by resubstituting $v^2 = \widetilde{u}^p$.

To prove part (ii) take $p \in (0,1)$. If $f = 0$ a.e. on Q we set $\widetilde{u} = u + \varepsilon$, $\varepsilon > 0$, and pass to the limit $\varepsilon \to 0+$ in the end. We need this assumption to guarantee the boundedness of the test function.

Next, observe that (7.6) remains valid in this case. For $p \in (0,1)$, inequality (7.6) is equivalent to

$$\begin{aligned}
-\frac{1}{4}\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x - \left(1 - \frac{1}{p}\right)\int_{B_R} \psi^2(x)\left(\nabla v \cdot A\nabla v\right)(t,x)\,\mathrm{d}x \\
\le \int_{B_R} \psi(x)v(t,x)\left(\nabla\psi \cdot A\nabla v\right)(t,x)\,\mathrm{d}x - \frac{p}{4}\int_{B_R} f(t,x)\psi^2(x)\widetilde{u}^{p-1}(t,x)\,\mathrm{d}x\,.
\end{aligned} \tag{7.16}$$

[3]In the case $d = 1, 2$ these exponents should be adopted to (5.19b)

Choose $\varepsilon = \frac{1}{2}\left(\frac{1}{p} - 1\right) = \frac{1}{2}\left|1 - \frac{1}{p}\right|$ in (7.8). Then

$$\begin{aligned} -\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + 4\varepsilon\lambda \int_{B_R} \psi^2(x)\left|\nabla v(t,x)\right|^2\,\mathrm{d}x \\ \leq \int_{B_R} v^2(t,x)\left(\frac{\Lambda}{\varepsilon}\left|\nabla\psi(x)\right|^2 + p\,\psi^2(x)\right)\mathrm{d}x. \end{aligned} \tag{7.17}$$

We choose a slightly different time-dependent test function in this case, namely $\chi_\oplus(t) = \left(\frac{R^2 - t}{R^2 - r^2} \wedge 1\right) \vee 0$. We multiply the latter inequality with $\chi_\oplus^2$ and integrate from some $t \in I_\oplus(r) = (0, r^2)$ to R^2, which yields

$$\begin{aligned} \sup_{t\in I_\oplus(r)} \int_{B(r)} v^2(t,x)\,\mathrm{d}x + \varepsilon\lambda \int_{Q_\oplus(r)} \left|\nabla v(t,x)\right|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{1}{4}\left(\frac{\Lambda}{\varepsilon}\frac{1}{(R-r)^2} + |p| + \frac{1}{R^2 - r^2}\right)\int_{Q_\oplus(R)} v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{c_1}{(R-r)^2}\left(\frac{\Lambda}{\varepsilon} + |p| + 1\right)\int_{Q_\oplus(R)} v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

Starting from this inequality the assertion (7.2) can be established in the same way as in Step II above – of course with $Q_\ominus$ and $I_\ominus$ replaced by $Q_\oplus$ and $I_\oplus$, respectively.

In order to prove part (iii) take $p \in (1, \infty)$. If $f = 0$ a.e. on Q we set $\widetilde{u} = u + \varepsilon$, $\varepsilon > 0$, and pass to the limit $\varepsilon \to 0+$ in the end. Since u is a subsolution we obtain

$$\begin{aligned} \frac{1}{p}\int_{B_R} \psi^2(x)\partial_t v^2(t,x)\,\mathrm{d}x + \frac{4(p-1)}{p^2}\int_{B_R} \psi^2(x)\left(\nabla v \cdot A\nabla v\right)(t,x)\,\mathrm{d}x \\ \leq -\frac{4}{p}\int_{B_R} \psi(x)v(t,x)\left(\nabla\psi \cdot A\nabla v\right)(t,x)\,\mathrm{d}x + \int_{B_R} f(t,x)\psi^2(x)\widetilde{u}^{-p-1}\,\mathrm{d}x\,, \end{aligned} \tag{7.18}$$

which is equivalent to (7.9). The same steps as in the proof of part (i) prove (7.3).

The proof of Proposition 7.1 is complete. □

7.2 Estimates for the infimum of a supersolution and the supremum of a solution

Theorem 7.3. *Let* $\frac{1}{2} \leq r < R \leq 1$, $\sigma = \lambda^{-1} + \Lambda$ *and* $0 < p \leq \sigma^{-1}$.

(i) There is a constant $C = C(d, \Lambda, \lambda) > 0$ *such that for every nonnegative* supersolution u *of* (PE$_2$) *in* $Q = I \times \Omega$, $Q \Supset Q_\ominus(R)$, *with* $u \geq \varepsilon > 0$ *in* Q *the following estimate holds:*

$$\sup_{Q_\ominus(r)} \widetilde{u}^{-1} \leq \left(\frac{C}{(R-r)^{d+2}}\right)^{1/p}\left(\int_{Q_\ominus(R)} \widetilde{u}^{-p}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}, \tag{7.19}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$.

(ii) There is a constant $C' = C'(d,\Lambda,\lambda) > 0$ *such that for every nonnegative* solution u *of* (PE_2) *in* $Q = I \times \Omega$, $Q \Supset Q(R)$, *the following estimate holds:*

$$\sup_{Q(r)} \widetilde{u} \le \left(\frac{C'}{(R-r)^{d+2}}\right)^{1/p} \left(\int_{Q(R)} \widetilde{u}^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}, \tag{7.20}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$.

Remark 7.4. The proof below establishes the inequalities (7.19) and (7.20) in the case $d \ge 3$. The assertion is still true if $d = 1, 2$. The proof of this fact can be found in Remark 6.5 just by setting $\alpha = 2$ therein. ♦

Proof. In view of Lemma 5.11 choose $\kappa = 1 + \frac{d}{2}$, $p_0 = \sigma^{-1}$. Furthermore, for $p \in (0, \sigma^{-1}]$ and $m \in \mathbb{N}_0$ set $p_m = p\kappa^m$ as well as $\varepsilon_m = \frac{1}{2}\left(1 + \frac{1}{p_m}\right)$ and choose the sequence of radii defined by

$$r_m = r + \frac{R-r}{2^m}.$$

Note that $r_0 = R$, $r_m > r$ and $\lim_{m\to\infty} r_m = r$.

ad (i): We consider the increasing sequence of subsets $(Q_\ominus(r), 0 < r \le 1)$ of $\mathbb{R}^{d+1}$.

In Proposition 7.1(i) we established for all $j \in \mathbb{N}_0$

$$\left(\int_{Q_\ominus(r_{j+1})} \widetilde{u}^{-p_{j+1}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p_{j+1}} \le A_j(p)^{1/p_j} \left(\int_{Q_\ominus(r_j)} \widetilde{u}^{-p_j}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p_j},$$

where

$$A_j(p) = \frac{c_1}{(r_j - r_{j+1})^2}\left(\frac{\sigma}{\varepsilon_j} + p_j + 1\right)^{\frac{\kappa+1}{\kappa}} \quad \text{for some constant } c_1 = c_1(d) \ge 1.$$

Thus, condition (5.22) of Lemma 5.11 is satisfied for the function $\widetilde{u}^{-1}$. In order to verify condition (5.23) observe that $(r_j - r_{j+1})^{-2} = 2^{2j+2}(R-r)^{-2}$. Additionally, by the property $p \le \sigma^{-1}$,

$$\left(\frac{\sigma}{\varepsilon_j} + p_j + 1\right)^{\frac{\kappa+1}{\kappa}} \le \left(\frac{2\sigma p_j}{1+p_j} + p_j + 1\right)^2 \le \left(2\kappa^j + p\kappa^j + 1\right)^2 \le 16\kappa^{2j}.$$

Therefore, since $\displaystyle\sum_{j=0}^{\infty} \frac{2j}{\kappa^j} < \infty$, there is a constant $c_2 = c_2(d,\lambda,\Lambda) \ge 1$ such that

$$\prod_{j=0}^{\infty} A_j(p)^{1/\kappa^j} \le c_2\,(R-r)^{-2\sum_{j=0}^{\infty}\kappa^{-j}} = c_2 (R-r)^{-2\frac{1+2/d}{2/d}} = \frac{c_2}{(R-r)^{d+2}}.$$

Taking $M = c_2(R-r)^{-d-2}$ in Lemma 5.11 implies

$$\sup_{Q_\ominus(r)} \widetilde{u}^{-1} \le \left(\frac{c_2}{(R-r)^{d+2}}\right)^{1/p} \left(\int_{Q_\ominus(R)} \widetilde{u}^{-p}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \qquad \text{for all } p \in (0, \sigma^{-1}],$$

which had to be shown in part (i).

ad (ii): We consider the increasing sequence of subsets $(Q(r), 0 < r \le 1)$ of $\mathbb{R}^{d+1}$. Corollary 7.2 states that for all $j \in \mathbb{N}_0$

$$\left(\int_{Q(r_{j+1})} \widetilde{u}^{p_{j+1}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p_{j+1}} \le A_j(p)^{1/p_j} \left(\int_{Q(r_j)} \widetilde{u}^{p_j}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p_j},$$

where

$$A_j(p) = \frac{c_3\, 2^{2j+2}}{(R-r)^2}\left(\frac{\sigma}{\varepsilon_j} + p_j + 1\right)^{\frac{\kappa+1}{\kappa}} \quad \text{for some constant } c_3(d,\lambda,\Lambda) \ge 1.$$

Assume $p = \frac{1}{2}\kappa^{-\nu}(\kappa+1)$ for some $\nu \in \mathbb{N}$. For p of this form we have

$$|p_m - 1| = \frac{1}{2}\left|\kappa^{m-\nu}(\kappa+1) - 2\right| \ge \frac{1}{2}(1-\kappa^{-1}) \quad \text{for all } m,\nu \in \mathbb{N}. \tag{7.21}$$

This inequality can be verified as follows: The continuous, strictly monotone increasing function $x \mapsto \kappa^x(\kappa+1) - 2$, $x \in \mathbb{R}$, is zero if and only if $x = \frac{\log(\frac{2}{\kappa+1})}{\log \kappa} \in (-1,0)$. Now

$$\left|\kappa^{-1}(\kappa+1) - 2\right| = 1 - \kappa^{-1} \quad \text{and} \quad \left|\kappa^{0}(\kappa+1) - 2\right| = \kappa - 1 \ge 1 - \kappa^{-1}.$$

This shows inequality (7.21). The fact $p \in (0,\sigma^{-1}]$ and (7.21) imply

$$\frac{\sigma}{\varepsilon_m} + p_m = \frac{2\sigma p_m}{|p_m - 1|} + p_m \le \frac{2\kappa^m}{|p_m - 1|} + p_m \le \frac{4\kappa^m}{1-\kappa^{-1}} + p\kappa^m \le \frac{5\kappa^m}{1-\kappa^{-1}}$$

and

$$A_j(p) \le \frac{c_3\, 2^{2j+2}}{(R-r)^2}\left(\frac{5\kappa^j}{1-\kappa^{-1}} + 1\right)^2 \le \frac{c_4^j}{(R-r)^2}$$

for some constant $c_4 = c_4(d,\lambda,\Lambda) \ge 1$. By the same computations as in the proof of part (i) we establish (7.20) for the specific choice of p mentioned a few lines above.

For general p' we may find p such that $p \le p' < p\kappa$ and (7.20) holds for this p. By Hölder's inequality we obtain

$$\begin{aligned}\left(\frac{C}{(R-r)^{d+2}}\int_{Q(R)} \widetilde{u}^p\right)^{1/p} &\le \left(\frac{C\,|Q(R)|}{(R-r)^{d+2}}\right)^{1/p}\left(\frac{1}{|Q(R)|}\int_{Q(R)} \widetilde{u}^{p'}\right)^{1/p'} \\ &\le \left(\frac{C\,|Q(R)|}{(R-r)^{d+2}}\right)^{1/p'}\left(\frac{1}{|Q(R)|}\int_{Q(R)} \widetilde{u}^{p'}\right)^{1/p'}\end{aligned}$$

$$= \left(\frac{C}{(R-r)^{d+2}}\right)^{1/p'} \left(\int_{Q(R)} \widetilde{u}^{p'}\right)^{1/p'},$$

where we have used $\frac{|Q(R)|}{(R-r)^{d+2}} = \frac{R^{d+2}}{(R-r)^{d+2}} > 1$ in the last inequality. This finishes the proof of Theorem 7.3. □

7.3 An estimate for the L^1-norm of a supersolution

Of course, one could also prove a result in the spirit of Theorem 6.6 for nonnegative supersolutions of the second order equation. However, in order to prove a strong Harnack inequality for solutions to the second order equation, we need an estimate for the supremum of a solution, which we established in Theorem 7.3(ii). This is much stronger and explains why we discount a possible analog of Theorem 6.6 in this context.

7.4 An inequality for $\log u$

The following result is stated in [Mos71] and was proved in [Mos64].

Proposition 7.5. *There is $C = C(d) > 0$ such that for every supersolution u of* (PE$_2$) *in $Q = (-1,1) \times B_2(0)$ which satisfies $u \geq \varepsilon > 0$ in $(-1,1) \times B_2(0)$, there is a constant $a = a(\widetilde{u}) \in \mathbb{R}$ such that the following inequalities hold simultaneously:*

$$\forall s > 0\colon (\,\mathrm{d}t \otimes \,\mathrm{d}x)\,(Q_{\oplus}(1) \cap \{\log \widetilde{u} < -s - a\}) \leq \frac{C\sigma}{s}, \tag{7.22a}$$

$$\forall s > 0\colon (\,\mathrm{d}t \otimes \,\mathrm{d}x)\,(Q_{\ominus}(1) \cap \{\log \widetilde{u} > s - a\}) \leq \frac{C\sigma}{s}, \tag{7.22b}$$

where $\widetilde{u} = u + \|f\|_{L^\infty(\Omega)}$ and $\sigma = \lambda^{-1} + \Lambda$.

Proof. We just need to prove an inequality analogous to (6.22) in the proof of Proposition 6.9. As is explained there the arguments in the proofs of the local and nonlocal case coincide. We follow the lines of [Mos64, p. 121].

Let u be a supersolution such that $u \geq \varepsilon > 0$ on $(-1,1) \times B_2(0)$ and $\widetilde{u} = u + \|f\|_{L^\infty(\Omega)}$. We set

$$v(t,x) = -\log \widetilde{u}(t,x), \qquad \phi(t,x) = \psi^2(x)\widetilde{u}^{-1}(t,x),$$

where $\psi^2(x) = \Psi(x) = (\frac{3}{2} - |x|) \wedge 1$ as in Lemma 5.8. Note that

$$\nabla v = \frac{-\nabla \widetilde{u}}{\widetilde{u}} \quad \text{and} \quad \partial_t v = \frac{-\partial_t \widetilde{u}}{\widetilde{u}}.$$

We apply the test function ϕ in (5.9): For a.e. $t \in (-1,1)$

$$\int_{B_{3/2}} \psi^2 \widetilde{u}^{-1} \partial_t u \,\mathrm{d}x$$

$$+\int_{B_{3/2}} \left[\widetilde{u}^{-1}\left(\nabla(\psi^2)\cdot A\nabla\widetilde{u}\right) + \psi^2\left(\nabla(\widetilde{u}^{-1})\cdot A\nabla\widetilde{u}\right)\right]\,\mathrm{d}x \geq \int_{B_{3/2}} \psi^2 f \widetilde{u}^{-1}\,\mathrm{d}x\,.$$

In terms of v this inequality reads

$$\int_{B_{3/2}} \psi^2 \partial_t v\,\mathrm{d}x + \int_{B_{3/2}} \left[2\psi\left(\nabla\psi\cdot A\nabla v\right) + \psi^2\left(\nabla v\cdot A\nabla v\right)\right]\,\mathrm{d}x \;\leq\; -\int_{B_{3/2}} \psi^2 f \widetilde{u}^{-1}\,\mathrm{d}x\,.$$

Using Schwarz' inequality (cf. (7.8))

$$|\psi\left\langle \nabla\psi, A\nabla v\right\rangle| \leq \left\langle \nabla\psi, A\nabla\psi\right\rangle + \frac{1}{4}\psi^2\left\langle \nabla v, A\nabla v\right\rangle,$$

$\left\|f\widetilde{u}^{-1}\right\|_{L^\infty(B_2(0))} \leq 1$ and (4.3), we get

$$\int_{B_{3/2}} \psi^2 \partial_t v\,\mathrm{d}x + \frac{\lambda}{2}\int_{B_{3/2}} \psi^2\left|\nabla v\right|^2\,\mathrm{d}x \leq 2\Lambda\int_{B_{3/2}} \left|\nabla\psi\right|^2\,\mathrm{d}x + \int_{B_{3/2}} \psi^2\,\mathrm{d}x\,. \tag{7.23}$$

Lemma 5.8 applied to the second term on the left-hand side implies

$$\int_{B_{3/2}} \psi^2 \partial_t v\,\mathrm{d}x + c_1\lambda\int_{B_{3/2}} \psi^2\left(v(t,x) - V(t)\right)^2\,\mathrm{d}x \leq 2\Lambda\int_{B_{3/2}} \left|\nabla\psi\right|^2\,\mathrm{d}x + \int_{B_{3/2}} \psi^2\,\mathrm{d}x\,.$$

where

$$V(t) = \frac{\displaystyle\int_{B_2} v(t,x)\psi^2(x)\,\mathrm{d}x}{\displaystyle\int_{B_2} \psi^2(x)\,\mathrm{d}x}\,.$$

Due to $\sigma = \lambda^{-1} + \Lambda$ we may write this inequality in the form

$$\int_{B_{3/2}} \psi^2 \partial_t v\,\mathrm{d}x + c_1\sigma^{-1}\int_{B_{3/2}} \psi^2\left(v(t,x) - V(t)\right)^2\,\mathrm{d}x \leq c_2\sigma\int_{B_{3/2}} \left|\nabla\psi\right|^2\,\mathrm{d}x + \left|B_{3/2}\right|\,, \tag{7.24}$$

which is now the desired analogous inequality to (6.22). The same arguments as in the proof of Proposition 6.9 lead to (7.22a) and (7.22b). □

7.5 Strong Harnack inequality for solutions

Theorem 7.6 (Strong Harnack inequality for solutions to second order equation)**.** *There is a constant $C = C(d,\lambda,\Lambda) > 0$ such that for every nonnegative solution u of* (PE_2) *on $Q = (-1,1)\times B_2(0)$ the following inequality holds:*

$$\sup_{U_\ominus} u \leq C\left(\inf_{U_\oplus} u + \|f\|_{L^\infty(Q)}\right), \tag{7.25}$$

where $U_\oplus = (3/4,1)\times B_{1/2}(0)$ and $U_\ominus(-1,-3/4)\times B_{1/2}(0)$.

The main difference between the following proof and the foregoing proof is that – due to Theorem 7.3 – there is an L^∞-estimate for both $\widetilde{u}$ and $\widetilde{u}^{-1}$. This means that we are able to apply Lemma 5.13 with $p_0 = \infty$ for both w and $\widehat{w}$. Let us provide the details:

Proof. Let u as in the assumption and define $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$. If $f = 0$ a.e. on Q we set $\widetilde{u} = u + \varepsilon$ and pass to the limit $\varepsilon \to 0+$ in the end.

Furthermore, set $w = e^{-a}\widetilde{u}^{-1}$ and $\widehat{w} = w^{-1} = e^a\widetilde{u}$, where $a = a(\widetilde{u})$ is chosen according to Proposition 7.5, i.e. there is $c_1 > 0$ such that for every $s > 0$

$$|Q_\oplus(1) \cap \{\log w > s\}| \le \frac{c_1\sigma}{s}, \quad \text{and} \quad |Q_\ominus(1) \cap \{\log \widehat{w} > s\}| \le \frac{c_1\sigma}{s}. \tag{7.26}$$

Apply Lemma 5.13 to w and a family of domains $\mathcal{U} = (U(r))_{\theta \le r \le 1}$ – and to $\widehat{w}$ and a family of domains $\widehat{\mathcal{U}} = (\widehat{U}(r))_{\widehat{\theta} \le r \le 1}$. Define the families $\mathcal{U}$ and $\widehat{\mathcal{U}}$ by

$$\begin{aligned} U(1) &= Q_\oplus(1), & \theta &= \frac{1}{2}, & U(r) &= \big(1 - r^2, 1\big) \times B_r, \\ \widehat{U}(1) &= Q_\ominus(1), & \widehat{\theta} &= \frac{1}{2} & \widehat{U}(r) &= \big(-1, -1 + r^2\big) \times B_r \end{aligned}$$

By virtue of (7.26) we see that condition (5.31) is satisfied for both w and $\widehat{w}$.

The domains $U(r)$ and $\widehat{U}(r)$ are obtained from $Q_\ominus(r)$ and $Q_\oplus(r)$, respectively, by shifting in time, i.e. transformations of the type $(t, x) \mapsto (t + \tau, x)$. Therefore, Theorem 7.3 implies that condition (5.30) of Lemma 5.13 is satisfied for both w and $\widehat{w}$ with $p_0 = \infty$ and arbitrary η. By Lemma 5.13, there are constants $C, \widehat{C} > 0$ depending only on d, λ and Λ such that

$$\sup_{U(\theta)} w = e^{-a} \sup_{U(\theta)} \widetilde{u}^{-1} \le C \qquad \text{and} \qquad \sup_{\widehat{U}(\widehat{\theta})} \widehat{w} = e^a \sup_{\widehat{U}(\theta)} \widehat{u} \le \widehat{C}.$$

Multiplying these two inequalities eliminates a and yields

$$\sup_{\widehat{U}(\widehat{\theta})} \widetilde{u} \le c_2 \inf_{U(\theta)} \widetilde{u}$$

for a constant $c_2 = C\widehat{C}$ that depends only on d, λ and Λ. This proves (7.25) since $U_\oplus = U(\theta)$, $U_\ominus = \widehat{U}(\widehat{\theta})$ and

$$\sup_{\widehat{U}(\widehat{\theta})} u \le \sup_{\widehat{U}(\widehat{\theta})} \widetilde{u} \le c_2 \left(\inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right). \qquad \square$$

7.6 Hölder regularity for weak solutions

Theorem 7.7 (Hölder regularity estimate)**.** *There is a constant $\beta = \beta(d, \lambda, \Lambda)$ such that for every solution u of* (PE$_2$) *in $Q = I \times \Omega$ with $f = 0$ and every $Q' \Subset Q$ the following estimate holds:*

$$\sup_{(t,x),(s,y)\in Q'} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/2}\right)^{\beta}} \leq \frac{\|u\|_{L^\infty(Q)}}{\eta^\beta}, \tag{7.27}$$

with some constant $\eta = \eta(Q, Q') > 0$.

Proof. Let v be a nonnegative solution to (PE$_2$) on $U = (-1,1) \times B_2(0)$ and recall

$$U_\oplus = (3/4, 1) \times B_{1/2}(0) \quad \text{and} \quad U_\ominus(-1, -3/4) \times B_{1/2}(0)$$

from Theorem 7.6. Write

$$\mathcal{M}_\ominus(v) = \frac{1}{|U_\ominus|} \int_{U_\ominus} v(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq \sup_{U_\ominus} v.$$

As a consequence of Harnack's inequality we obtain that there is a constant $\gamma \in (0,1)$ depending only on d, λ and Λ such that

$$\inf_{U_\oplus} v \geq \gamma\,\mathcal{M}_\ominus(v)\,. \tag{7.28}$$

Given a solution u (that may change its sign) on U of (PE$_2$) we define

$$M = \sup_U u, \quad m = \inf_U u, \quad M^\oplus = \sup_{U_\oplus} u, \quad m^\oplus = \inf_{U_\oplus} u\,.$$

Clearly, $M - u$ and $u - m$ are nonnegative solutions of (PE$_2$) and

$$\mathcal{M}_\ominus(M - u) = M - \mathcal{M}_\ominus(u), \qquad \mathcal{M}_\ominus(u - m) = \mathcal{M}_\ominus(u) - m.$$

(7.28) implies

$$M - M^\oplus \geq \gamma(M - \mathcal{M}_\ominus(u)) \quad \text{and} \quad m^\oplus - m \geq \gamma(\mathcal{M}_\ominus(u) - m),$$

which in turn implies

$$\operatorname*{osc}_U u - \operatorname*{osc}_{U_\oplus} u \geq \gamma \operatorname*{osc}_U u, \quad \text{or equivalently} \quad \operatorname*{osc}_{U_\oplus} u \leq (1-\gamma) \operatorname*{osc}_U u, \tag{7.29}$$

where $\operatorname{osc}_D v = \sup_D v - \inf_D v$ denotes the oscillation of a function v on a domain D.

The strategy now is to repeat estimate (7.29) on a sequence of nested cylindrical domains such that the oscillation reduces in each step. To this end define for $(t,x) \in \mathbb{R}^{d+1}$ a distance function

$$\rho((t,x)) = \begin{cases} \max\left(\frac{1}{2}|x|, \frac{1}{2}\sqrt{-t}\right) & \text{if } t \in (-2, 0], \\ \infty & \text{if } t \notin (-2, 0]. \end{cases}$$

Note that $\rho((t,x)) \neq \rho(-(t,x))$. We define

$$D_r((t_0,x_0)) = \Big\{(t,x) \in \mathbb{R}^{d+1} \;\mid \rho((t,x)-(t_0,x_0)) < r\Big\}, \quad D_r((0,0)) = D_r\,.$$

Observe that $D(1) = (-2,0)\times B_2(0) = U$ and $D(1/4) = U_\oplus$. By a simple scaling argument we deduce from (7.29) that a solution u on $D(1)$ satisfies

$$\operatorname*{osc}_{D(4^{-n-1})} u \leq (1-\gamma)^{n+1} \operatorname*{osc}_{D(1)} u \quad \text{for every } n \in \mathbb{N}_0\,. \tag{7.30}$$

For Q' as in the assumption define

$$\eta(Q',Q) = \eta = \sup\big\{r \in (0,\tfrac{1}{2}] \mid \forall (t,x) \in Q' \colon D_r(t,x) \subset Q\big\}\,.$$

Fix $(t,x),(s,y) \in Q'$. Without loss of generality we may assume $t \leq s$. If $\rho((t,x)-(s,y)) < \eta$ we choose $n \in \mathbb{N}_0$ such that

$$\frac{\eta}{4^{n+1}} \leq \rho((t,x)-(s,y)) < \frac{\eta}{4^n}\,.$$

By assumption the rescaled function $\overline{u}(t,x) = u(\eta^2 t + s, \eta x + y)$ is a solution of the local equation on $D(1) = (-2,0) \times B_2(0)$ and by (7.30) we find

$$\begin{aligned}
|u(t,x)-u(s,y)| &= \big|\overline{u}(\eta^{-2}(t-s),\eta^{-1}(x-y)) - \overline{u}(0,0)\big| \\
&\leq (1-\gamma)^n \operatorname*{osc}_{D(1)} \overline{u} \\
&\leq 2(1-\gamma)^n \,\|u\|_{L^\infty(Q)} \\
&\leq 2\left(\frac{1}{4}\right)^{\beta n} \|u\|_{L^\infty(Q)} \\
&\leq \frac{8}{\eta^\beta}\,\|u\|_{L^\infty(Q)} \left(\frac{\rho((t,x)-(s,y))}{\eta}\right)^{\beta},
\end{aligned}$$

where $\beta = \dfrac{-\log(1-\gamma)}{\log 4}$.

This proves (7.27) in the case $\rho((t,x)-(s,y)) < \eta$. In the case $\rho((t,x)-(s,y)) \geq \eta$, the Hölder continuity follows directly as is explained in the last argument in the proof of Theorem 4.5 on page 102. $\square$

Notation

$A \subset B$ means that every element of A is contained in B. If A is a proper subset of B we write $A \subsetneq B$.
The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$ and the scalar product is written as

$$x \cdot y = \sum_{i=1}^{d} x_i y_i \,.$$

We use the notation $f \asymp g$ if there is a constant $C > 1$ such that $C^{-1} f \leq g \leq C\, f$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we use $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and for $u \in C^\infty(\Omega)$

$$\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d} u \,.$$

The norm of a Banach space V is denoted by $\|\cdot\|_V$. The dual space of V is denoted by V^* and for $f \in V^*$ and $v \in V$ we write

$$\langle f, v \rangle_V = f(v) \quad \text{and} \quad \|f\|_{V^*} = \sup_{v \in V \setminus \{0\}} \frac{\langle f, v \rangle_V}{\|v\|_V} \,.$$

For two Banach spaces V, W we write $V \hookrightarrow W$ if there is a continuous, injective mapping $j \colon V \to W$. If j is not specified, then $j = \mathrm{id} \colon V \to W$. We write $V \overset{d}{\hookrightarrow} W$ if $V \hookrightarrow W$ and V is dense in W.

If $B(\Omega)$ is a Banach space of functions acting on Ω and taking values in some normed space Y, then we write

$$B_{loc}(\Omega) = \{u \colon \Omega \to Y \mid u\phi \in B(\Omega) \text{ for every } \phi \in C_c^\infty(\Omega)\} \,.$$

$\mathbb{1}_A$ stands for the characteristic function of a set A.

Let $m \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^d$ open. Then

$$\begin{aligned}
C^m(\Omega) &= \Big\{ u \colon \Omega \to \mathbb{R} \;\Big|\; \partial^\alpha u \text{ is continuous for every } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m \Big\} \,, \\
C^m(\overline{\Omega}) &= \Big\{ u \in C^m(\Omega) \;\Big|\; \partial^\alpha u \text{ has a continuous extension to } \overline{\Omega} \\
&\qquad\qquad \text{for every } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m \Big\} \,, \\
C^\infty(\Omega) &= \bigcap_{m \in \mathbb{N}_0} C^m(\Omega), \qquad C^\infty(\overline{\Omega}) = \bigcap_{m \in \mathbb{N}_0} C^m(\overline{\Omega}) \,.
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^d$ be open. If $u\colon \Omega \to \mathbb{R}$ is continuous and bounded, we set

$$\|u\|_{C(\overline{\Omega})} = \sup_{x\in\Omega} |u(x)| \ .$$

Furthermore, we define for $\beta \in (0,1]$ the *Hölder seminorm*

$$[u]_{C^{0,\beta}(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\beta} \,.$$

For $k \in \mathbb{N}_0$ the Banach space $C^{k,\beta}(\overline{\Omega})$ consists of all functions $u\colon \Omega \to \mathbb{R}$ such that

$$\|u\|_{C^{k,\beta}(\overline{\Omega})} = \sum_{|\alpha|\leq k} \|\partial^\alpha\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [\partial^\alpha u]_{C^{0,\beta}(\overline{\Omega})} < \infty \,.$$

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