p-adic L-functions of automorphic forms

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Contents

Introduction

Let F be a number field (with adele ring $\mathbb{A}_F$), and p a prime number. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Attached to π is the automorphic L-function $L(s, \pi)$, for $s \in \mathbb{C}$, of Jacquet-Langlands [JL]. Under certain conditions on π, we can also define a p-adic L-function $L_p(s, \pi)$ of π, with $s \in \mathbb{Z}_p$. It is related to $L(s, \pi)$ by the *interpolation property*: For every character $\chi : \mathcal{G}_p \to \mathbb{C}^*$ of finite order we have

$$L_p(0, \pi \otimes \chi) = \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_\mathfrak{p}, \chi_\mathfrak{p}) \cdot L(\tfrac{1}{2}, \pi \otimes \chi),$$

where $e(\pi_\mathfrak{p}, \chi_\mathfrak{p})$ is a certain Euler factor (see theorem 4.12 for its definition) and $\tau(\chi)$ is the Gauss sum of χ.

$L_p(s, \pi)$ was defined by Haran [Har] in the case where π has trivial central character and $\pi_\mathfrak{p}$ is a spherical principal series representation for all $\mathfrak{p}|p$. For a totally real field F, Spieß [Sp] has given a new construction of $L_p(s, \pi)$ that also allows for $\pi_\mathfrak{p}$ to be a special (Steinberg) representation for some $\mathfrak{p}|p$.

Here, we generalize Spieß' construction of $L_p(s, \pi)$ to automorphic representations π over any number field, with arbitrary central character. As in [Sp], we will assume that π is ordinary at all primes $\mathfrak{p}|p$ (cf. definition 2.5), that π_v is discrete of weight 2 at all real infinite places v, and a similar condition at the complex places.

Throughout most of this thesis, we follow [Sp]; for section 4.1, we follow Bygott [By], Ch. 4.2, who in turn follows Weil [We].

We define the p-adic L-function of π as an integral of the p-adic cyclotomic character $\mathcal{N}$ with respect to a certain measure μ_π on the Galois group $\mathcal{G}_p$ of the maximal abelian extension that is unramified outside p and ∞, specifically

$$L_p(s, \pi) := \int_{\mathcal{G}_p} \mathcal{N}(\gamma)^s \mu_\pi(d\gamma)$$

(cf. section 4.6 for details). Heuristically, μ_π is the image of $\mu_{\pi_\mathfrak{p}} \times W^p \left(\begin{smallmatrix} x^p & 0 \\ 0 & 1 \end{smallmatrix} \right) d^\times x^p$ under the reciprocity map $\mathbb{I}_F = F_p^* \times \mathbb{I}^p \to \mathcal{G}_p$ of global class field theory. Here $\mu_{\pi_p} = \prod_{\mathfrak{p}|p} \mu_{\pi_\mathfrak{p}}$ is the product of certain local distributions $\mu_{\pi_\mathfrak{p}}$ on $F_\mathfrak{p}^*$ attached to $\pi_\mathfrak{p}$, $d^\times x^p$ is the Haar measure on the group $\mathbb{I}^p = \prod'_{v\nmid p} F_v^*$ of p-ideles, and $W^p = \prod_{v\nmid p} W_v$ is a specific Whittaker function of $\pi^p := \otimes_{v\nmid p} \pi_v$.

The structure of this work is the following: In chapter 2, we describe the local distributions $\mu_{\pi_\mathfrak{p}}$ on $F_\mathfrak{p}^*$; they are the image of a Whittaker functional under a map δ on the dual of $\pi_\mathfrak{p}$. For constructing δ, we describe $\pi_\mathfrak{p}$ in terms of what we call the "Bruhat-Tits graph" of $F_\mathfrak{p}^2$: the directed graph whose vertices (resp. edges) are the lattices of $F_\mathfrak{p}^2$ (resp. inclusions between lattices). Roughly speaking, it is a covering of the (directed) Bruhat-Tits tree of $\mathrm{GL}_2(F_\mathfrak{p})$ with fibres $\cong \mathbb{Z}$. When $\pi_\mathfrak{p}$ is the Steinberg representation, $\mu_\mathfrak{p}$ can actually be extended to all of $F_\mathfrak{p}$.

In chapter 3, we attach a p-adic distribution μ_ϕ to any map $\phi(U, x^p)$ of an open compact subset $U \subseteq F_p^* := \prod_{\mathfrak{p}|p} F_\mathfrak{p}^*$ and an idele $x^p \in \mathbb{I}^p$ (satisfying certain conditions). Integrating ϕ over all the infinite places, we get a cohomology class $\kappa_\phi \in H^d(F^{*\prime}, \mathcal{D}_f(\mathbb{C}))$ (where $d = r + s - 1$ is the rank of the group of units of

F, $F^{*\prime} \cong F^*/\mu_F$ is a maximal torsion-free subgroup of F^*, and $\mathcal{D}_f(\mathbb{C})$ is a space of distributions on the finite ideles of F). We show that μ_ϕ can be described solely in terms of κ_ϕ, and μ_ϕ is a (vector-valued) p-adic measure if κ_ϕ is "integral", i.e. if it lies in the image of $H^d(F^{*\prime}, \mathcal{D}_f(R))$, for a Dedekind ring R consisting of "p-adic integers".

In chapter 4, we define a map ϕ_π by

$$\phi_\pi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}$$

($U \subseteq F_p^*$ compact open, $x^p \in \mathbb{I}^p$). ϕ_π satisfies the conditions of chapter 3, and we show that κ_{ϕ_π} is integral by "lifting" the map $\phi_\pi \mapsto \kappa_{\phi_\pi}$ to a function mapping an automorphic form to a cohomology class in $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)$, for a certain space of functions $\mathcal{A}_f$. (Here $\mathrm{GL}_2(F)^+$ is the subgroup of $M \in \mathrm{GL}_2(F)$ with totally positive determinant.) For this, we associate to each automorphic form φ a harmonic form ω_φ on a generalized upper-half space $\mathcal{H}_\infty$, which we can integrate between any two cusps in $\mathbb{P}^1(F)$.

Then we can define the p-adic L-function $L_p(s, \pi) := \int_{\mathcal{G}_p} \mathcal{N}(\gamma)^s \mu_\pi(d\gamma)$ as above, with $\mu_\pi := \mu_{\phi_\pi}$. By a result of Harder [Ha], $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)_\pi$ is one-dimensional, which implies that $L_p(s, \pi)$ has values in a one-dimensional $\mathbb{C}_p$-vector space.

Our construction has the following potential application: If E is a modular elliptic curve over F corresponding to π (i.e. the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places v of F), we define the p-adic L-function of E as $L_p(E, s) := L_p(s, \pi)$. The condition that π be ordinary at all $\mathfrak{p}|p$ means that E must have good ordinary or multiplicative reduction at all places $\mathfrak{p}|p$ of F.

The *exceptional zero conjecture* (formulated by Mazur, Tate and Teitelbaum [MTT] for $F = \mathbb{Q}$, and by Hida [Hi] for totally real F) states that

$$\mathrm{ord}_{s=0} L_p(E, s) \geq n, \tag{0.1}$$

where n is the number of $\mathfrak{p}|p$ at which E has split multiplicative reduction, and gives an explicit formula for the value of the n-th derivative $L_p^{(n)}(E, 0)$ as a multiple of $L(E, 1)$. The conjecture was proved in the case $F = \mathbb{Q}$ by Greenberg and Stevens [GS] and independently by Kato, Kurihara and Tsuji.

In [Sp], Spieß has used his new construction of $L_p(E, s) := L_p(s, \pi)$ to prove the conjecture for all totally real number fields F. Our generalization of $L_p(s, \pi)$ might therefore be well-suited for proving the conjecture for general F.

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1 Preliminaries

Let $\mathcal{X}$ be a totally disconnected locally compact topological space, R a topological Hausdorff ring. We denote by $C(\mathcal{X}, R)$ the ring of continuous maps $\mathcal{X} \to R$, and let $C_c(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)$ be the subring of compactly supported maps. When R has the discrete topology, we also write $C^0(\mathcal{X}, R) := C(\mathcal{X}, R)$, $C_c^0(\mathcal{X}, R) := C_c(\mathcal{X}, R)$.

We denote by $\mathfrak{Co}(\mathcal{X})$ the set of all compact open subsets of $\mathcal{X}$, and for an R-module M we denote by $\mathrm{Dist}(\mathcal{X}, M)$ the R-module of M-valued distributions on $\mathcal{X}$, i.e. the set of maps $\mu : \mathfrak{Co}(\mathcal{X}) \to M$ such that $\mu(\bigcup_{i=1}^n U_i) = \sum_{i=1}^n \mu(U_i)$ for any pairwise disjoint sets $U_i \in \mathfrak{Co}(\mathcal{X})$.

For an open set $H \subseteq \mathcal{X}$, we denote by $1_H \in C(\mathcal{X}, R)$ the R-valued indicator function of H on $\mathcal{X}$.

Throughout this paper, we fix a prime p and embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\overline{\mathcal{O}}$ denote the valuation ring of $\overline{\mathbb{Q}}$ with respect to the p-adic valuation induced by ι_p.

We write $G := \mathrm{GL}_2$ throughout the thesis, and let B denote the Borel subgroup of upper triangular matrices, T the maximal torus (consisting of all diagonal matrices), and Z the center of G.

For a number field F, we let $G(F)^+ \subseteq G(F)$ and $B(F)^+ \subseteq B(F)$ denote the corresponding subgroups of matrices with totally positive determinant, i.e. $\sigma(\det(g))$ is positive for each real embedding $\sigma : F \hookrightarrow \mathbb{R}$. (If F is totally complex, this is an empty condition, so we have $G(F)^+ = G(F)$, $B(F)^+ = B(F)$ in this case.) Similarly, we define $G(\mathbb{R})^+$ and $G(\mathbb{C})^+ = G(\mathbb{C})$.

1.1 p-adic measures

Definition 1.1. Let $\mathcal{X}$ be a compact totally disconnected topological space. For a distribution $\mu : \mathfrak{Co}(\mathcal{X}) \to \mathbb{C}$, consider the extension of μ to the $\mathbb{C}_p$-linear map $C^0(\mathcal{X}, \mathbb{C}_p) \to \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, $f \mapsto \int f d\mu$. If its image is a finitely-generated $\mathbb{C}_p$-vector space, μ is called a *p-adic measure.*

We denote the space of p-adic measures on $\mathcal{X}$ by $\mathrm{Dist}^b(\mathcal{X}, \mathbb{C}) \subseteq \mathrm{Dist}(\mathcal{X}, \mathbb{C})$. It is easily seen that μ is a p-adic measure if and only if the image of μ, considered as a map $C^0(\mathcal{X}, \mathbb{Z}) \to \mathbb{C}$, is contained in a finitely generated $\overline{\mathcal{O}}$-module. A p-adic measure can be integrated against any continuous function $f \in C(\mathcal{X}, \mathbb{C}_p)$.

2 Local results for representations with arbitrary central character

For this chapter, let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ its ring of integers, ϖ its uniformizer and $\mathfrak{p} = (\varpi)$ the maximal ideal. Let q be the cardinality of $\mathcal{O}_F/\mathfrak{p}$, and set $U := U^{(0)} := \mathcal{O}_F^\times$, $U^{(n)} := 1 + \mathfrak{p}^n \subseteq U$ for $n \geq 1$.

We fix an additive character $\psi : F \to \overline{\mathbb{Q}}^*$ with $\ker \psi = \mathcal{O}_F$. We let $|\cdot|$ be the absolute value on F^* (normalized by $|\varpi| = q^{-1}$), $\operatorname{ord} = \operatorname{ord}_\varpi$ the additive valuation, and dx the Haar measure on F normalized by $\int_{\mathcal{O}_F} dx = 1$. We define a (Haar) measure on F^* by $d^\times x := \frac{q}{q-1}\frac{dx}{|x|}$ (so $\int_{\mathcal{O}_F^\times} d^\times x = 1$).

2.1 Gauss sums

Recall that the *conductor* of a character $\chi : F^* \to \mathbb{C}^*$ is by definition the largest ideal $\mathfrak{p}^n$, $n \geq 0$, such that $\ker \chi \supseteq U^{(n)}$, and that χ is *unramified* if its conductor is $\mathfrak{p}^0 = \mathcal{O}_F$.

We will need the following two easy lemmas of [Sp]:

Lemma 2.1. *Let $X \subseteq \{x \in F^* | \operatorname{ord}(x) \leq -2\}$ be a compact open subset such that $aU^{(-\operatorname{ord}(a)-1)} \subseteq X$ for all $a \in X$. Then*

$$\int_X \psi(x) d^\times x = 0.$$

(cf. [Sp], lemma 3.1)

Lemma 2.2. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasicharacter of conductor $\mathfrak{p}^f$, $f \geq 1$, and let $a \in F^*$ with $\operatorname{ord}(a) \neq -f$. Then we have*

$$\int_U \psi(ax)\chi(x) d^\times x = 0.$$

(cf. [Sp], lemma 3.2)

Definition 2.3. Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$. The *Gauss sum* of χ (with respect to ψ) is defined by

$$\tau(\chi) := [U : U^{(f)}] \int_{\varpi^{-f}U} \psi(x)\chi(x) d^\times x.$$

For a locally constant function $g : F^* \to \mathbb{C}$, we define

$$\int_{F^*} g(x)dx := \lim_{n\to\infty} \int_{x \in F^*, -n \leq \operatorname{ord}(x) \leq n} g(x)dx,$$

whenever that limit exists. Then we have the following formula:

Lemma 2.4. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$. For $f = 0$, assume $|\chi(\varpi)| < q$. Then we have*

$$\int_{F^*} \chi(x)\psi(x)dx = \begin{cases} \frac{1-\chi(\varpi)^{-1}}{1-\chi(\varpi)q^{-1}} & \text{if } f = 0 \\ \tau(\chi) & \text{if } f > 0. \end{cases}$$

Proof. (cf. [Sp], lemma 3.4)
For $a \in F^*$, we have

$$\int_U \psi(ax)d^\times x = \begin{cases} 1, & \text{if } \operatorname{ord}(a) \geq 0 \\ -\frac{1}{q-1}, & \text{if } \operatorname{ord}(a) = -1 \\ 0, & \text{if } \operatorname{ord}(a) \leq -2 \quad \text{(by lemma 2.1).} \end{cases} \tag{2.1}$$

Since $d^\times x = \frac{dx}{(1-1/q)|x|}$, this implies

$$\int_{F^*} \chi(x)\psi(x)dx = \sum_{n=-\infty}^{\infty} (1-1/q)q^{-n} \int_{\varpi^n U} \chi(x)\psi(x)d^\times x.$$

For $f > 0$, all summands except the $(-f)$th are zero by lemma 2.2, thus we have

$$\int_{F^*} \chi(x)\psi(x)dx = (1-1/q)q^f \int_{\varpi^{-f}U} \chi(x)\psi(x)d^\times x = \tau(\chi)$$

by the definition of τ (since $[U : U^{(f)}] = (1-1/q)q^f$).
For $f = 0$, we have by (2.1)

$$\begin{aligned} \int_{F^*} \chi(x)\psi(x)dx &= (1-1/q)\left(-\frac{q}{(q-1)\chi(\varpi)} + \sum_{n=0}^{\infty}(\chi(\varpi)q^{-1})^n\right) \\ &= -\frac{1}{\chi(\varpi)} + \frac{1-1/q}{1-\chi(\varpi)q^{-1}} \qquad (\text{since } |\chi(\varpi)| < q) \\ &= \frac{1-\chi(\varpi)^{-1}}{1-\chi(\varpi)q^{-1}}. \end{aligned}$$

□

2.2 Tamely ramified representations of $\mathrm{GL}_2(F)$

For an ideal $\mathfrak{a} \subset \mathcal{O}_F$, let $K_0(\mathfrak{a}) \subseteq G(\mathcal{O}_F)$ be the subgroup of matrices congruent to an upper triangular matrix modulo $\mathfrak{a}$.

Let $\pi : \mathrm{GL}_2(F) \to \mathrm{GL}(V)$ be an irreducible admissible infinite-dimensional representation (where V is a $\mathbb{C}$-vector space), with central quasicharacter χ. It is well-known (e.g [Ge], Thm. 4.24) that there exists a maximal ideal $\mathfrak{c}(\pi) = \mathfrak{c} \subset \mathcal{O}_F$, the *conductor* of π, such that the space $V^{K_0(\mathfrak{c}),\chi} = \{v \in V | \pi(g)v = \chi(a)v \ \forall g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(\mathfrak{c})\}$ is non-zero (and in fact one-dimensional). A representation π is called *tamely ramified* if its conductor divides $\mathfrak{p}$.
If π is tamely ramified, then π is the spherical resp. special representation $\pi(\chi_1, \chi_2)$ (in the notation of [Ge] or [Sp]):
If the conductor is $\mathcal{O}_F$, π is (by definition) spherical and hence a principal series representation $\pi(\chi_1, \chi_2)$ for two unramified quasi-characters χ_1 and χ_2 with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$ ([Bu], Thm. 4.6.4).
If the conductor is $\mathfrak{p}$, then $\pi = \pi(\chi_1, \chi_2)$ with $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.

For $\alpha \in \mathbb{C}^*$, we define a character $\chi_\alpha : F^* \to \mathbb{C}^*$ by $\chi_\alpha(x) := \alpha^{\operatorname{ord}(x)}$.

So let now $\pi = \pi(\chi_1, \chi_2)$ be a tamely ramified irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$; in the special case, we assume χ_1 and χ_2 to be ordered such that $\chi_1 = |\cdot|\chi_2$.
Set $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. (We also write $\pi = \pi_{\alpha_1,\alpha_2}$ sometimes.) Set $a := \alpha_1 + \alpha_2$, $\nu := \alpha_1\alpha_2/q$. Define a distribution $\mu_{\alpha_1,\nu} := \mu_{\alpha_1/\nu} := \psi(x)\chi_{\alpha_1/\nu}(x)dx$ on F^*.

For later use, we will need the following condition on the α_i:

Definition 2.5. $\pi = \pi_{\alpha_1,\alpha_2}$ is called *ordinary* if a and ν both lie in $\overline{\mathcal{O}}^*$ (i.e. they are p-adic units in $\overline{\mathbb{Q}}$). Equivalently, this means that either $\alpha_1 \in \overline{\mathcal{O}}^*$ and $\alpha_2 \in q\overline{\mathcal{O}}^*$, or vice versa.

Proposition 2.6. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$; for $f = 0$, assume $|\chi(\varpi)| < |\alpha_2|$. Then the integral $\int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx)$ converges and we have*

$$\int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx) = e(\alpha_1, \alpha_2, \chi)\tau(\chi)L(\tfrac{1}{2}, \pi \otimes \chi),$$

where

$$e(\alpha_1, \alpha_2, \chi) = \begin{cases} \dfrac{(1 - \alpha_1\chi(\varpi)q^{-1})(1 - \alpha_2\chi(\varpi)^{-1}q^{-1})(1 - \alpha_2\chi(\varpi)q^{-1})}{(1 - \chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ spherical,} \\ \dfrac{(1 - \alpha_1\chi(\varpi)q^{-1})(1 - \alpha_2\chi(\varpi)^{-1}q^{-1})}{(1 - \chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ special,} \\ (\alpha_1/\nu)^{-f} = (\alpha_2/q)^f, & f > 0, \end{cases}$$

and where we assume the right-hand side to be continuously extended to the potential removable singularities at $\chi(\varpi) = q/\alpha_1$ or $= q/\alpha_2$.

Proof. Case 1: $f = 0$, π spherical
We have

$$L(s, \pi \otimes \chi) = \frac{1}{\left(1 - \alpha_1\chi(\varpi)q^{-(s+\frac{1}{2})}\right)\left(1 - \alpha_2\chi(\varpi)q^{-(s+\frac{1}{2})}\right)},$$

so

$$\begin{aligned} L(\tfrac{1}{2}, \pi \otimes \chi) \cdot \tau(\chi) \cdot e(\alpha_1, \alpha_2, \chi) &= \frac{1 - \alpha_2 q^{-1}\chi(\varpi)^{-1}}{1 - \chi(\varpi)\alpha_2^{-1}} \\ &= \frac{1 - \nu\alpha_1^{-1}\chi(\varpi)^{-1}}{1 - \alpha_1\chi(\varpi)\nu^{-1}q^{-1}} \\ &= \int_{F^*} \chi(x)\chi_{\alpha_1/\nu}(x)\psi(x)dx \\ &= \int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx) \end{aligned}$$

by lemma 2.4.

Case 2: $f = 0$, π special
Assuming $\chi_1 = |\cdot|\chi_2$, we have

$$L(s, \pi \otimes \chi) = \frac{1}{1 - \alpha_1 \chi(\varpi) q^{-(s+\frac{1}{2})}}$$

and thus

$$\begin{aligned} L(\tfrac{1}{2}, \pi \otimes \chi) \cdot \tau(\chi) \cdot e(\alpha_1, \alpha_2, \chi) &= \frac{1 - \nu\alpha_1^{-1}\chi(\varpi)^{-1}}{1 - \alpha_1 \nu^{-1} \chi(\varpi) q^{-1}} \\ &= \int_{F^*} \chi(x) \chi_{\alpha_1/\nu}(x) \psi(x) dx \\ &= \int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx). \end{aligned}$$

by lemma 2.4.

Case 3: $f > 0$
In this case, $L(s, \pi \otimes \chi) = 1$ for $s > 0$ and

$$\begin{aligned} \int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx) &= \tau(\chi \cdot \chi_{\alpha_1/\nu}) \\ &= q^{f-1}(q-1) \int_{\varpi^{-f}U} \psi(x)\chi(x)\chi_{\alpha_1/\nu}(x) d^\times x \\ &= (\alpha_1/\nu)^{-f} q^{f-1}(q-1) \int_{\varpi^{-f}U} \psi(x)\chi(x) d^\times x \\ &= e(\alpha_1, \alpha_2, \chi) \cdot \tau(\chi) \cdot L(\tfrac{1}{2}, \pi \otimes \chi). \end{aligned}$$

□

2.3 The Bruhat-Tits graph $\tilde{\mathcal{T}}$

Let $\tilde{\mathcal{V}}$ denote the set of lattices (i.e. submodules isomorphic to $\mathcal{O}_F^2$) in F^2, and let $\tilde{\mathcal{E}}$ be the set of all inclusion maps between two lattices; for such a map $e : v_1 \hookrightarrow v_2$ in $\tilde{\mathcal{E}}$, we define $o(e) := v_1$, $t(e) := v_2$. Then the pair $\tilde{\mathcal{T}} := (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ is naturally a directed graph, connected, with no directed cycles (specifically, $\tilde{\mathcal{E}}$ induces a partial ordering on $\tilde{\mathcal{V}}$). For each $v \in \tilde{\mathcal{V}}$, there are exactly $q+1$ edges beginning (resp. ending) in v, each.

Recall that the *Bruhat-Tits tree* $\mathcal{T} = (\mathcal{V}, \vec{\mathcal{E}})$ of $G(F)$ is the directed graph whose vertices are homothety classes of lattices of F^2 (i.e. $\mathcal{V} = \tilde{\mathcal{V}}/\sim$, where $v \sim \varpi^i v$ for all $i \in \mathbb{Z}$), and the directed edges $\bar{e} \in \vec{\mathcal{E}}$ are homothety classes of inclusions of lattices. We can define maps $o, t : \vec{\mathcal{E}} \to \mathcal{V}$ analogously. For each edge $\bar{e} \in \vec{\mathcal{E}}$, there is an opposite edge $\bar{e}' \in \vec{\mathcal{E}}$ with $o(\bar{e}') = t(\bar{e})$, $t(\bar{e}') = o(\bar{e})$; and the undirected graph underlying $\mathcal{T}$ is simply connected. We have a natural "projection map" $\pi : \tilde{\mathcal{T}} \to \mathcal{T}$, mapping each lattice and each homomorphism to its homothety class. Choosing a (set-theoretic) section $s : \mathcal{V} \to \tilde{\mathcal{V}}$, we get a bijection $\mathcal{V} \times \mathbb{Z} \xrightarrow{\cong} \tilde{\mathcal{V}}$ via $(v, i) \mapsto \varpi^i s(v)$.

The group $G(F)$ operates on $\tilde{\mathcal{V}}$ via its standard action on F^2, i.e. $gv = \{gx | x \in v\}$ for $g \in G(F)$, and on $\tilde{\mathcal{E}}$ by mapping $e : v_1 \to v_2$ to the inclusion map $ge : gv_1 \to gv_2$. The stabilizer of the standard vertex $v_0 := \mathcal{O}_F^2$ is $G(\mathcal{O}_F)$.

For a directed edge $\overline{e} \in \vec{\mathcal{E}}$ of the Bruhat-Tits tree $\mathcal{T}$, we define $U(\overline{e})$ to be the set of ends of $\overline{e}$ (cf. [Se1]/[Sp]); it is a compact open subset of $\mathbb{P}^1(F)$, and we have $gU(\overline{e}) = U(g\overline{e})$ for all $g \in G(F)$.

For $n \in \mathbb{Z}$, we set $v_n := \mathcal{O}_F \oplus \mathfrak{p}^n \in \tilde{\mathcal{V}}$, and denote by e_n the edge from v_{n+1} to v_n; the "decreasing" sequence $(\pi(e_{-n}))_{n \in \mathbb{Z}}$ is the geodesic from ∞ to 0. (The geodesic from 0 to ∞ traverses the $\pi(v_n)$ in the natural order of $n \in \mathbb{Z}$.) We have $U(\pi(e_n)) = \mathfrak{p}^{-n}$ for each n.

Now (following [BL] and [Sp]), we can define a "height" function $h : \mathcal{V} \to \mathbb{Z}$ as follows: The geodesic ray from $v \in \mathcal{V}$ to ∞ must contain some $\pi(v_n)$ ($n \in \mathbb{Z}$), since it has non-empty intersection with $A := \{\pi(v_n) | n \in \mathbb{Z}\}$; we define $h(v) := n - d(v, \pi(v_n))$ for any such v_n; this is easily seen to be well-defined, and we have $h(\pi(v_n)) = n$ for all $n \in \mathbb{Z}$. We have the following lemma of [Sp]:

Lemma 2.7. *(a) For all $\overline{e} \in \mathcal{E}$, we have*

$$h(t(\overline{e})) = \begin{cases} h(o(\overline{e})) + 1 & \text{if } \infty \in U(\overline{e}), \\ h(o(\overline{e})) - 1 & \text{otherwise.} \end{cases}$$

(b) For $a \in F^*$, $b \in F$, $\overline{v} \in \mathcal{V}$ we have

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \overline{v}\right) = h(\overline{v}) - \operatorname{ord}_{\varpi}(a).$$

Proof. (cf. [Sp], Lemma 3.6)
(a) is clear from the definition of h. For (b) we can assume $\overline{v} = \pi(v_0) =: \overline{v_0}$ since $B'(F) := \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) | a \in F^*, b \in F\}$ operates transitively on $\mathcal{V}$. Put $\overline{e} := \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \pi(e_0)$; since $U(\overline{e}) = a\mathcal{O}_F + b$ does not contain ∞, we have

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) = h(t(\overline{e})) = h(o(\overline{e})) - 1 = h\left(\begin{pmatrix} a\varpi^{-1} & b \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) - 1.$$

If $b \neq 0$, we can iterate this n times such that $\operatorname{ord}(a\varpi^{-n}) \geq \operatorname{ord} b$ and get

$$\begin{aligned} h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) &= h\left(\begin{pmatrix} a\varpi^{-n} & b \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) - n = h\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) - n \\ &= h\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \overline{v_0}\right) = h(\pi(v_{-\operatorname{ord}(a)})) = -\operatorname{ord}(a). \end{aligned}$$

□

2.4 Hecke structure of $\tilde{\mathcal{T}}$

Let R be a ring, M an R-module. We let $C(\tilde{\mathcal{V}}, M)$ be the R-module of maps $\phi : \tilde{\mathcal{V}} \to M$, and $C(\tilde{\mathcal{E}}, M)$ the R-module of maps $\tilde{\mathcal{E}} \to M$. Both are $G(F)$-modules via $(g\phi)(v) := \phi(g^{-1}v)$, $(gc)(e) := c(g^{-1}e)$.

We let $\mathcal{C}_c(\tilde{\mathcal{V}}, M) \subseteq C(\tilde{\mathcal{V}}, M)$ and $\mathcal{C}_c(\tilde{\mathcal{E}}, M) \subseteq C(\tilde{\mathcal{E}}, M)$ be the ($G(F)$-stable) submodules of maps with compact support, i.e. maps that are zero outside a finite set. We get pairings

$$\langle \text{-}\,,\text{-}\rangle : C_c(\tilde{\mathcal{V}}, R) \times C(\tilde{\mathcal{V}}, M) \to M, \quad \langle \phi_1, \phi_2 \rangle := \sum_{v\in\tilde{\mathcal{V}}} \phi_1(v)\phi_2(v) \tag{2.2}$$

and

$$\langle \text{-}\,,\text{-}\rangle : C_c(\tilde{\mathcal{E}}, R) \times C(\tilde{\mathcal{E}}, M) \to M, \quad \langle c_1, c_2 \rangle := \sum_{e\in\tilde{\mathcal{E}}} c_1(v)c_2(v). \tag{2.3}$$

We define Hecke operators $T, \mathcal{R} : \mathcal{C}(\tilde{\mathcal{V}}, M) \to \mathcal{C}(\tilde{\mathcal{V}}, M)$ by

$$T\phi(v) = \sum_{t(e)=v} \phi(o(e)) \quad \text{and} \quad \mathcal{R}\phi := \varpi\phi \text{ (i.e. } \mathcal{R}\phi(v) = \phi(\varpi^{-1}v))$$

for all $v \in \tilde{\mathcal{V}}$. These restrict to operators on $C_c(\tilde{\mathcal{V}}, R)$, which we sometimes denote by T_c and $\mathcal{R}_c$ for emphasis. With respect to (2.2), T_c is adjoint to $T\mathcal{R}$, and $\mathcal{R}_c$ is adjoint to its inverse operator $\mathcal{R}^{-1} : \mathcal{C}_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R)$.

T and $\mathcal{R}$ obviously commute, and we have the following Hecke structure theorem on compactly supported functions on $\tilde{\mathcal{V}}$ (an analogue of [BL], Thm. 10):

Theorem 2.8. *$C_c(\tilde{\mathcal{V}}, R)$ is a free $R[T, \mathcal{R}^{\pm 1}]$-module (where $R[T, \mathcal{R}^{\pm 1}]$ is the ring of Laurent series in $\mathcal{R}$ over the polynomial ring $R[T]$, with $\mathcal{R}$ and T commuting).*

Proof. Fix a vertex $v_0 \in \tilde{\mathcal{V}}$. For each $n \geq 0$, let C_n be the set of vertices $v \in \tilde{\mathcal{V}}$ such that there is a directed path of length n from v_0 to v in $\tilde{\mathcal{V}}$, and such that $d(\pi(v_0), \pi(v)) = n$ in the Bruhat-Tits tree $\mathcal{T}$. So $C_0 = \{v_0\}$, and C_n is a lift of the "circle of radius n around v_0" in $\mathcal{T}$, in the parlance of [BL].

One easily sees that $\bigcup_{n=0}^{\infty} C_n$ is a complete set of representatives for the projection map $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$; specifically, for $n > 1$ and a given $v \in C_{n-1}$, C_n contains exactly q elements adjacent to v in $\tilde{\mathcal{V}}$; and we can write $\tilde{\mathcal{V}}$ as a disjoint union $\bigcup_{j\in\mathbb{Z}} \bigcup_{n=0}^{\infty} \mathcal{R}^j(C_n)$.

We further define $V_0 := \{v_0\}$ and choose subsets $V_n \subseteq C_n$ as follows: We let V_1 be any subset of cardinality q. For $n > 1$, we choose $q-1$ out of the q elements of C_n adjacent to v', for every $v' \in C_{n-1}$, and let V_n be the union of these elements for all $v' \in C_{n-1}$. Finally, we set

$$H_{n,j} := \{\phi \in C_c(\tilde{\mathcal{V}}, R) |\, \mathrm{Supp}(\phi) \subseteq \bigcup_{i=0}^{n} \mathcal{R}^j(C_i)\} \quad \text{for each } n \geq 0, j \in \mathbb{Z},$$

$H_n := \bigcup_{j\in\mathbb{Z}} H_{n,j}$, and $H_{-1} := H_{-1,j} := \{0\}$. (For ease of notation, we identify $v \in \tilde{\mathcal{V}}$ with its indicator function $1_{\{v\}} \in C_c(\tilde{\mathcal{V}}, R)$ in this proof.)

Define $T' : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R)$ by

$$T'(\phi)(v) := \sum_{\substack{t(e)=(v),\\ o(e)\in\mathcal{R}^j(C_n)}} \phi(o(e)) \quad \text{for each } v \in \mathcal{R}^j(C_{n-1}), j \in \mathbb{Z};$$

T' can be seen as the "restriction to one layer" $\bigcup_{n=0}^{\infty} \mathcal{R}^j(C_n)$ of T. We have $T'(v) \equiv T(v) \mod H_{n-1}$ for each $v \in H_n$, since the "missing summand" of T' lies in H_{n-1}.

We claim that for each $n \geq 0$, the set $X_{n,j} := \bigcup_{i=0}^{n} \mathcal{R}^j T^{n-i}(V_i)$ is an R-basis for $H_{n,j}/H_{n-1,j}$. By the above congruence, we can replace T by T' in the definition of $X_{n,j}$.

The claim is clear for $n = 0$. So let $n \geq 1$, and assume the claim to be true for all $n' \leq n$. For each $v \in C_{n-1}$, the q points in C_n adjacent to v are generated by the $q-1$ of these points lying in V_n, plus $T'v$ (which just sums up these q points). By induction hypothesis, v is generated by $X_{n-1,0}$, and thus (taking the union over all v), C_n is generated by $T'(X_{n-1,0}) \cup V_n = X_{n,0}$. Since the cardinality of $X_{n,0}$ equals the R-rank of $H_{n,0}/H_{n-1,0}$ (both are equal to $(q+1)q^{n-1}$), $X_{n,0}$ is in fact an R-basis.

Analoguously, we see that $H_{n,j}/H_{n-1,j}$ has $\mathcal{R}^j(X_{n,0}) = X_{n,j}$ as a basis, for each $j \in \mathbb{Z}$.

From the claim, it follows that $\bigcup_{j\in\mathbb{Z}} X_{n,j}$ is an R-basis of H_n/H_{n-1} for each n, and that $V := \bigcup_{n=0}^{\infty} V_n$ is an $R[T, \mathcal{R}^{\pm 1}]$-basis of $C_c(\tilde{\mathcal{V}}, R)$. □

For $a \in R$ and $\nu \in R^*$, we let $\tilde{\mathcal{B}}_{a,\nu}(F, R)$ be the "common cokernel" of $T - a$ and $\mathcal{R} - \nu$ in $C_c(\tilde{\mathcal{V}}, R)$, namely $\tilde{\mathcal{B}}_{a,\nu}(F, R) := C_c(\tilde{\mathcal{V}}, R)/(\mathrm{Im}(T - a) + \mathrm{Im}(\mathcal{R} - \nu))$; dually, we define $\tilde{\mathcal{B}}^{a,\nu}(F, M) := \ker(T - a) \cap \ker(\mathcal{R} - \nu) \subseteq C(\tilde{\mathcal{V}}, M)$.

For a lattice $v \in \tilde{\mathcal{V}}$, we define a valuation ord_v on F as follows: For $w \in F^2$, the set $\{x \in F | xw \in v\}$ is some fractional ideal $\varpi^m \mathcal{O}_F \subseteq F$ $(m \in \mathbb{Z})$; we set $\mathrm{ord}_v(w) := m$. This map can also be given explicitly as follows: Let λ_1, λ_2 be a basis of v. We can write any $w \in F^2$ as $w = x_1\lambda_1 + x_2\lambda_2$; then we have $\mathrm{ord}_v(w) = \min\{\mathrm{ord}_\varpi(x_1), \mathrm{ord}_\varpi(x_2)\}$. This gives a "valuation" map on F^2, as one easily checks. We restrict it to $F \cong F \times \{0\} \hookrightarrow F^2$ to get a valuation ord_v on F, and consider especially the value at $e_1 := (1, 0)$.

Lemma 2.9. *Let $\alpha, \nu \in R^*$, and put $a := \alpha + q\nu/\alpha$. Define a map $\varrho = \varrho_{\alpha,\nu} : \tilde{\mathcal{V}} \to R$ by $\varrho(v) := \alpha^{h(\pi(v))}\nu^{-\mathrm{ord}_v(e_1)}$. Then $\varrho \in \tilde{\mathcal{B}}^{a,\nu}(F, R)$.*

Proof. One easily sees that $\left(v \mapsto \nu^{-\mathrm{ord}_v(e_1)}\right) \in \ker(\mathcal{R} - \nu)$. It remains to show that $\varrho \in \ker(T - a)$:

We have the Iwasawa decomposition $G(F) = B(F)G(\mathcal{O}_F) = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}Z(F)G(\mathcal{O}_F)$; thus every vertex in $\tilde{\mathcal{V}}$ can be written as $\varpi^i v$ with $v = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) v_0$, with $i \in \mathbb{Z}$, $a \in F^*, b \in F$.

Now the lattice $v = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) v_0$ is generated by the vectors $\lambda_1 = \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$ and $\lambda_2 = \left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right) \in \mathcal{O}_F^2$, so $e_1 = a^{-1}\lambda_1$ and thus $\mathrm{ord}_v(e_1) = \mathrm{ord}_\varpi(a^{-1}) = -\mathrm{ord}_\varpi(a)$. The $q+1$ neighbouring vertices v' for which there exists an $e \in \tilde{\mathcal{E}}$ with $o(e) = v', t(e) = v$ are given by $N_i v$, $i \in \{\infty\} \cup \mathcal{O}_F/\mathfrak{p}$, with $N_\infty := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, and $N_i := \left(\begin{smallmatrix} \varpi & i \\ 0 & 1 \end{smallmatrix}\right)$ where $i \in \mathcal{O}_F$ runs through a complete set of representatives $\mod \varpi$. By lemma 2.7, $h(\pi(N_\infty v)) = h(\pi(v)) + 1$ and $h(\pi(N_i v)) = h(\pi(v)) - 1$ for $i \neq \infty$. By considering the basis $\{N_i\lambda_1, N_i\lambda_2\}$ of $N_i v$ for each N_i, we see that $\mathrm{ord}_{N_\infty v}(e_1) = \mathrm{ord}_v(e_1)$ and $\mathrm{ord}_{N_i v}(e_1) = \mathrm{ord}_v(e_1) - 1$ for $i \neq \infty$. Thus we have

$$(T\varrho)(v) = \sum_{t(e)=v} \alpha^{h(\pi(o(e)))}\nu^{-\operatorname{ord}_{o(e)}(e_1)} = \alpha^{h(\pi(v))+1}\nu^{-\operatorname{ord}_v e_1} + q\cdot\alpha^{h(\pi(v))-1}\nu^{1-\operatorname{ord}_v(e_1)}$$
$$= (\alpha + q\alpha^{-1}\nu)\alpha^{h(\pi(v))}\nu^{-\operatorname{ord}_v e_1} = a\varrho(v),$$

and also $(T\varrho)(\varpi^i v) = (T\mathcal{R}^{-i}\varrho)(v) = \mathcal{R}^{-i}(a\varrho)(v) = a\varrho(\varpi^i v)$ for a general $\varpi^i v \in \tilde{\mathcal{V}}$, which shows that $\varrho \in \ker(T-a)$. □

If $a^2 \neq \nu(q+1)^2$ (we will call this the "spherical case"[i]), we put $\mathcal{B}_{a,\nu}(F,R) := \tilde{\mathcal{B}}_{a,\nu}(F,R)$ and $\mathcal{B}^{a,\nu}(F,M) := \tilde{\mathcal{B}}^{a,\nu}(F,M)$.

In the "special case" $a^2 = \nu(q+1)^2$, we need to assume that the polynomial $X^2 - a\nu X + q\nu^{-1} \in R[X]$ has a zero $\alpha' \in R$. Then the map $\varrho := \varrho_{\alpha',\nu} \in C(\tilde{\mathcal{V}}, R)$ defined as above lies in $\tilde{\mathcal{B}}^{a\nu,\nu^{-1}}(F,R) = \ker(T\mathcal{R}-a) \cap \ker(\mathcal{R}^{-1}-\nu)$ by Lemma 2.9, since $a\nu = \alpha' + q\nu^{-1}/\alpha'$. In other words, the kernel of the map $\langle\cdot,\varrho\rangle : C_c(\tilde{\mathcal{V}}, R) \to R$ contains $\operatorname{Im}(T-a) + \operatorname{Im}(\mathcal{R}-\nu)$; and we define

$$\mathcal{B}_{a,\nu}(F,R) := \ker(\langle\cdot,\varrho\rangle) / (\operatorname{Im}(T-a) + \operatorname{Im}(\mathcal{R}-\nu))$$

to be the quotient; evidently, it is an R-submodule of codimension 1 of $\tilde{\mathcal{B}}_{a,\nu}(F,R)$. Dually, $T-a$ and $\mathcal{R}-\nu$ both map the submodule $\varrho M = \{\varrho\cdot m, m \in M\}$ of $C(\tilde{\mathcal{V}}, M)$ to zero and thus induce endomorphisms on $C(\tilde{\mathcal{V}}, M)/\varrho M$; we define $\mathcal{B}^{a,\nu}(F,M)$ to be the intersection of their kernels.

In the special case, since $\nu = \alpha^2$, Lemma 2.9 states that $\varrho(gv_0) = \chi_\alpha(ad)\varrho(v_0) = \chi_\alpha(\det g)\varrho(v_0)$ for all $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(F)$, and thus for all $g \in G(F)$ by the Iwasawa decomposition, since $G(\mathcal{O}_F)$ fixes v_0 and lies in the kernel of $\chi_\alpha \circ \det$. By the multiplicity of det, we have $(g^{-1}\varrho)(v) = \varrho(gv) = \chi_\alpha(\det g)\varrho(v)$ for all $g \in G(F)$, $v \in \tilde{\mathcal{V}}$. So $\phi \in \ker\langle\cdot,\varrho\rangle$ implies $\langle g\phi, \varrho\rangle = \langle\phi, g^{-1}\varrho\rangle = \chi_\alpha(\det g)\langle\phi,\varrho\rangle = 0$, i.e. $\ker\langle\cdot,\varrho\rangle$ and thus $\mathcal{B}_{a,\nu}(F,R)$ are $G(F)$-modules.

By the adjointness properties of the Hecke operators T and $\mathcal{R}$, we have pairings $\operatorname{coker}(T_c - a) \times \ker(T\mathcal{R}-a) \to M$ and $\operatorname{coker}(\mathcal{R}_c - \nu) \times \ker(\mathcal{R}^{-1}-\nu) \to M$, which "combine" to give a pairing

$$\langle \text{-}, \text{-} \rangle : \mathcal{B}_{a,\nu}(F,R) \times \mathcal{B}^{a\nu,\nu^{-1}}(F,M) \to M$$

(since $\ker(T\mathcal{R}-a) \cap \ker(\mathcal{R}^{-1}-\nu) = \ker(T-a\nu) \cap \ker(\mathcal{R}-\nu^{-1})$), and a corresponding isomorphism $\mathcal{B}^{a\nu,\nu^{-1}}(F,M) \xrightarrow{\cong} \operatorname{Hom}(\mathcal{B}_{a,\nu}(F,R), M)$.

Definition 2.10. Let G be a totally disconnected locally compact group, $H \subseteq G$ an open subgroup. For a smooth $R[H]$-module M, we define the *(compactly) induced*

[i]We use this term since these pairs of a, ν will later be seen to correspond to a spherical representation of $\mathrm{GL}_2(F)$. The case $a^2 = \nu(q+1)^2$ means that there exists an $\alpha \in R^*$ with $a = \alpha(q+1)$, $\nu = \alpha^2$, which will correspond to a special representation.

G-representation of M, denoted $\operatorname{Ind}_H^G M$, to be the space of maps $f : G \to M$ such that $f(hg) = f(g)$ for all $g \in G, h \in H$, and such that f has compact support modulo H. We let G act on $\operatorname{Ind}_H^G M$ via $g \cdot f(x) := f(xg)$. (We can also write $\operatorname{Ind}_H^G M = R[G] \otimes_{R[H]} M$, cf. [Br], III.5.)

We further define $\operatorname{Coind}_H^G M := \operatorname{Hom}_{R[H]}(R[G], M)$. Finally, for an $R[G]$-module N, we write $\operatorname{res}_H^G N$ for its underlying $R[H]$-module ("restriction of scalars").

By Theorem 2.8, $T_c - a$ (as well as $\mathcal{R}_c - \nu$) is injective, and the induced map

$$\mathcal{R}_c - \nu : \operatorname{coker}(T_c - a) = C_c(\tilde{\mathcal{V}}, R)/\operatorname{Im}(T_c - a) \to \operatorname{coker}(T_c - a)$$

(of $R[T, \mathcal{R}^{\pm 1}]/(T - a) = R[\mathcal{R}^{\pm 1}]$-modules) is also injective. Now since $G(F)$ acts transitively on $\tilde{\mathcal{V}}$, with the stabilizer of $v_0 := \mathcal{O}_F^2$ being $K := G(\mathcal{O}_F)$, we have an isomorphism $C_c(\tilde{\mathcal{V}}, R) \cong \operatorname{Ind}_K^{G(F)} R$. Thus we have exact sequences

$$0 \to \operatorname{Ind}_K^{G(F)} R \xrightarrow{T-a} \operatorname{Ind}_K^{G(F)} R \to \operatorname{coker}(T_c - a) \to 0 \tag{2.4}$$

and (for a, ν in the spherical case)

$$0 \to \operatorname{coker}(T_c - a) \xrightarrow{\mathcal{R}-\nu} \operatorname{coker}(T_c - a) \to \mathcal{B}_{a,\nu}(F, R) \to 0, \tag{2.5}$$

with all entries being free R-modules. Applying $\operatorname{Hom}_R(\cdot, M)$ to them, we get:

Lemma 2.11. *We have exact sequences of R-modules*

$$0 \to \ker(T\mathcal{R} - a) \to \operatorname{Coind}_K^{G(F)} M \xrightarrow{T-a} \operatorname{Coind}_K^{G(F)} M \to 0$$

and, if $\mathcal{B}_{a,\nu}(F, M)$ is spherical (i.e. $a^2 \neq \nu(q+1)^2$),

$$0 \to \mathcal{B}^{a\nu,\nu^{-1}}(F, M) \to \ker(T\mathcal{R} - a) \xrightarrow{\mathcal{R}-\nu} \ker(T\mathcal{R} - a) \to 0.$$

For the special case, we have to work a bit more to get similar exact sequences:

By [Sp], eq. (22), for the representation $St^-(F, R) := \mathcal{B}_{-(q+1),1}(F, R)$ (i.e. $\nu = 1$, $\alpha = -1$) with trivial central character, we have an exact sequence of G-modules

$$0 \to \operatorname{Ind}_{KZ}^G R \to \operatorname{Ind}_{K'Z}^G R \to St^-(F, R) \to 0, \tag{2.6}$$

where $K' = \langle W \rangle K_0(\mathfrak{p})$ is the subgroup of KZ generated by $W := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ and the subgroup $K_0(\mathfrak{p}) \subseteq K$ of matrices that are upper-triangular modulo $\mathfrak{p}$. (Since $W^2 \in Z$, $K_0(\mathfrak{p})Z$ is a subgroup of K' of order 2.) Now (π, V) can be written as $\pi = \chi \otimes St^-$ for some character $\chi = \chi_Z$ (cf. the proof of lemma 2.14 below), and we have an obvious G-isomorphism

$$(\pi, V) \cong (\pi \otimes (\chi \circ \det), V \otimes_R R(\chi \circ \det)),$$

where $R(\chi \circ \det)$ is the ring R with G-module structure given via $gr = \chi(\det(g))r$ for $g \in G, r \in R$. Tensoring (2.6) with $R(\chi \circ \det)$ over R gives an exact sequence of G-modules

$$0 \to \operatorname{Ind}_{KZ}^G \chi \to \operatorname{Ind}_{K'Z}^G \chi \to V \to 0. \tag{2.7}$$

It is easily seen that $R(\chi\circ\det)$ fits into another exact sequence of G-modules

$$0\to \operatorname{Ind}_H^G R \xrightarrow{\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)-\chi(\varpi)\,\mathrm{id}} \operatorname{Ind}_H^G R \xrightarrow{\psi} R(\chi\circ\det)\to 0,$$

where $H := \{g\in G|\det g\in \mathcal{O}_F^\times\}$ is a normal subgroup containing K, $\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)(f)(g) := f(\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}g)$ for $f\in \operatorname{Ind}_H^G R = \{f: G\to R| f(Hg) = f(g)\text{ for all } g\in G\}$, $g\in G$, is the natural operation of G, and where ψ is the G-equivariant map defined by $1_U\mapsto 1$.

Now since $H\subseteq G$ is a normal subgroup, we have $\operatorname{Ind}_H^G R\cong R[G/H]$ as G-modules (in fact $G/H\cong\mathbb{Z}$ as an abstract group). Let $X\subseteq G$ be a subgroup such that the natural inclusion $X/(X\cap H)\hookrightarrow G/H$ has finite cokernel; let g_iH, $i=1,\dots n$ be a set of representatives of that cokernel. Then we have a (non-canonical) X-isomorphism $\bigoplus_{i=0}^n \operatorname{Ind}_{X\cap H}^X \to \operatorname{Ind}_H^G R$ defined via $(1_{(X\cap H)x})_i\mapsto 1_{Hxg_i}$ for each $i=1,\dots,n$ (cf. [Br], III (5.4)).

Using this isomorphism and the "tensor identity" $\operatorname{Ind}_H^G M\otimes N\cong \operatorname{Ind}_H^G(M\otimes \operatorname{res}_H^G N)$ for any groups $H\subseteq G$, H-module M and G-module N ([Br] III.5, Ex. 2), we have

$$\begin{aligned}\operatorname{Ind}_{KZ}^G R\otimes_R \operatorname{Ind}_H^G R &\cong \operatorname{Ind}_{KZ}^G(\operatorname{res}_{KZ}^G(\operatorname{Ind}_H^G R))\\ &= \operatorname{Ind}_{KZ}^G((\operatorname{Ind}_{KZ\cap H}^{KZ} R)^2)\\ &= (\operatorname{Ind}_{KZ}^G(\operatorname{Ind}_K^{KZ} R))^2 = (\operatorname{Ind}_K^G R)^2\end{aligned}$$

(since $KZ/KZ\cap H\hookrightarrow G/H$ has index 2), and similarly

$$\begin{aligned}\operatorname{Ind}_{K'Z}^G R\otimes_R \operatorname{Ind}_H^G R &\cong \operatorname{Ind}_{K'Z}^G(\operatorname{res}_{K'Z}^G(\operatorname{Ind}_H^G R))\\ &\cong \operatorname{Ind}_{K'Z}^G((\operatorname{Ind}_{K'Z\cap H}^{K'Z} R)^2)\\ &\cong (\operatorname{Ind}_{K'}^G R)^2\end{aligned}$$

and thus, we can resolve the first and second term of (2.7) into exact sequences

$$\begin{gathered}0\to(\operatorname{Ind}_K^G R)^2\to(\operatorname{Ind}_K^G R)^2\to \operatorname{Ind}_{KZ}^G\chi\to 0,\\ 0\to(\operatorname{Ind}_{K'}^G R)^2\to(\operatorname{Ind}_{K'}^G R)^2\to \operatorname{Ind}_{\langle W\rangle K_0(\mathfrak{p})Z}^G\chi\to 0.\end{gathered}$$

Dualizing (2.7) and these by taking $\operatorname{Hom}(\cdot,M)$ for an R-module M, we get a "resolution" of $\mathcal{B}^{a\nu,\nu^{-1}}(F,M)$ in terms of coinduced modules:

Lemma 2.12. *We have exact sequences*

$$\begin{aligned}0\to&\mathcal{B}^{a\nu,\nu^{-1}}(F,M)\to \operatorname{Coind}_{K'Z}^G M(\chi)\to \operatorname{Coind}_{KZ}^G M(\chi)\to 0,\\ 0\to&\operatorname{Coind}_{KZ}^G M(\chi)\to(\operatorname{Coind}_K^G R)^2\to(\operatorname{Coind}_K^G R)^2\to 0,\\ 0\to&\operatorname{Coind}_{K'Z}^G M(\chi)\to(\operatorname{Coind}_{K'}^G R)^2\to(\operatorname{Coind}_{K'}^G R)^2\to 0\end{aligned}$$

for all special $\mathcal{B}_{a,\nu}(F,R)$ (i.e. $a^2=\nu(q+1)^2$), where $\chi=\chi_Z$ is the central character.

It is easily seen that the above arguments could be modified to get a similar set of exact sequences in the spherical case as well (replacing K' by K everywhere), in addition to that given in lemma 2.11; but we will not need this.

2.5 Distributions on $\tilde{\mathcal{T}}$

For $\varrho \in C(\tilde{\mathcal{V}}, R)$ we define R-linear maps

$$\tilde{\delta}_\varrho : C(\tilde{\mathcal{E}}, M) \to C(\tilde{\mathcal{V}}, M), \quad \tilde{\delta}_\varrho(c)(v) := \sum_{v=t(e)} \varrho(o(e))c(e) - \sum_{v=o(e)} \varrho(t(e))c(e),$$

$$\tilde{\delta}^\varrho : C(\tilde{\mathcal{V}}, M) \to C(\tilde{\mathcal{E}}, M), \quad \tilde{\delta}^\varrho(\phi)(e) := \varrho(o(e))\phi(t(e)) - \varrho(t(e))\phi(o(e)).$$

One easily checks that these are adjoint with respect to (2.2) and (2.3), i.e. we have $\langle \tilde{\delta}_\varrho(c), \phi \rangle = \langle c, \tilde{\delta}^\varrho(\phi) \rangle$ for all $c \in C_c(\tilde{\mathcal{E}}, R)$, $\phi \in C(\tilde{\mathcal{V}}, M)$. We denote the maps corresponding to $\varrho \equiv 1$ by $\delta := \tilde{\delta}_1$, $\delta^* := \tilde{\delta}^1$.

For each ϱ, the map $\tilde{\delta}_\varrho$ fits into an exact sequence

$$C_c(\tilde{\mathcal{E}}, R) \xrightarrow{\tilde{\delta}_\varrho} C_c(\tilde{\mathcal{V}}, R) \xrightarrow{\langle \cdot, \varrho \rangle} R \to 0$$

but it is not injective in general: e.g. for $\varrho \equiv 1$, the map $\tilde{\mathcal{E}} \to R$ symbolized by

$$\begin{array}{ccc} \cdot & \xrightarrow{-1} & \cdot \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle -1} \\ \cdot & \xrightarrow{1} & \cdot \end{array}$$

(and zero outside the square) lies in $\ker \delta$.

The restriction $\delta^*|_{C_c(\tilde{\mathcal{V}}, R)}$ to compactly supported maps is injective since $\tilde{\mathcal{T}}$ has no directed circles, and we have a surjective map

$$\operatorname{coker}\left(\delta^* : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{E}}, R)\right) \to C^0(\mathbb{P}^1(F), R)/R, \quad c \mapsto \sum_{e \in \tilde{\mathcal{E}}} c(e) 1_{U(\pi(e))}$$

(which corresponds to an isomorphism of the similar map on the Bruhat-Tits tree $\mathcal{T}$). Its kernel is generated by the functions $1_{\{e\}} - 1_{\{e'\}}$ for $e, e' \in \tilde{\mathcal{E}}$ with $\pi(e) = \pi(e')$.

For $\varrho_1, \varrho_2 \in C(\tilde{\mathcal{V}}, R)$ and $\phi \in C(\tilde{\mathcal{V}}, M)$ it is easily checked that

$$\left(\tilde{\delta}_{\varrho_1} \circ \tilde{\delta}^{\varrho_2}\right)(\phi) = (T + T\mathcal{R})(\varrho_1 \cdot \varrho_2) \cdot \phi - \varrho_2 \cdot (T + T\mathcal{R})(\varrho_1 \cdot \phi).$$

For $a' \in R$ and $\varrho \in \ker(T + T\mathcal{R} - a')$, applying this equality for $\varrho_1 = \varrho$ and $\varrho_2 = 1$ shows that $\tilde{\delta}_\varrho$ maps $\operatorname{Im} \delta^*$ into $\operatorname{Im}(T + T\mathcal{R} - a')$, so we get an R-linear map

$$\tilde{\delta}_\varrho : \operatorname{coker}\left(\delta^* : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{E}}, R)\right) \to \operatorname{coker}(T_c + T_c\mathcal{R}_c - a').$$

Let now again $\alpha, \nu \in R^*$, and $a := \alpha + q\nu/\alpha$. We let $\varrho := \varrho_{\alpha,\nu} \in \tilde{\mathcal{B}}^{a,\nu}(F, R)$ as defined in lemma 2.9, and write $\tilde{\delta}_{\alpha,\nu} := \tilde{\delta}_\varrho$. Since $\tilde{\delta}_{\alpha,\nu}$ maps $1_{\{e\}} - 1_{\{\varpi e\}}$ into $\operatorname{Im}(R - \nu)$, it induces a map

$$\tilde{\delta}_{\alpha,\nu} : C^0(\mathbb{P}^1(F), R)/R \to \mathcal{B}_{a,\nu}(F, R)$$

(same name by abuse of notation) via the commutative diagram

$$\begin{array}{ccc} \operatorname{coker}\delta^* & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & \operatorname{coker}(T_c + T_c\mathcal{R}_c - a') \\ \downarrow & & \downarrow {\scriptstyle \bmod (\mathcal{R}-\nu)} \\ C^0(\mathbb{P}^1(F), R)/R & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & \mathcal{B}_{a,\nu}(F,R) \end{array}$$

with $a' := a(1+\nu)$, since $\varrho \in \mathcal{B}^{a,\nu}(F,R) \subseteq \ker(T + T\mathcal{R} - a')$.

Lemma 2.13. *We have* $\varrho\,(gv) = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$, *and thus*

$$\tilde{\delta}_{\alpha,\nu}(gf) = \chi_\alpha(d/a')\chi_\nu(a')g\tilde{\delta}_{\alpha,\nu}(f),$$

for all $v \in \tilde{\mathcal{V}}$, $f \in C^0(\mathbb{P}^1(F), R)/R$ *and* $g = \begin{pmatrix} a' & b \\ 0 & d \end{pmatrix} \in B(F)$.

Proof. (a) Using lemma 2.7(b) and the fact that $\operatorname{ord}_{gv}(e_1) = -\operatorname{ord}_\varpi(a') + \operatorname{ord}_v(e_1)$, we have

$$\varrho\left(\begin{pmatrix} a' & b \\ 0 & d \end{pmatrix} v\right) = \alpha^{h(v)-\operatorname{ord}_\varpi(a'/d)}\nu^{\operatorname{ord}_\varpi(a')-\operatorname{ord}_v(e_1)} = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$$

for all $v \in \tilde{\mathcal{V}}$. For f and g as in the assertion, we thus have

$$\begin{aligned} \tilde{\delta}_{\alpha,\nu}(gf)(v) &= \sum_{v=t(e)} \varrho(o(e))f(g^{-1}e) - \sum_{v=o(e)} \varrho(t(e))f(g^{-1}e) \\ &= \sum_{g^{-1}v=t(e)} \varrho(o(ge))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(ge))f(e) \\ &= \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)\left(\sum_{g^{-1}v=t(e)} \varrho(o(e))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(e))f(e)\right) \\ &= \chi_\alpha(d/a')\chi_\nu(a')g\tilde{\delta}_{\alpha,\nu}(f)(v). \end{aligned}$$

□

We define a function $\delta_{\alpha,\nu} : C_c(F^*, R) \to \mathcal{B}_{a,\nu}(F,R)$ as follows: For $f \in C_c(F^*, R)$, we let $\psi_0(f) \in C_c(\mathbb{P}^1(F), R)$ be the extension of $x \mapsto \chi_\alpha(x)\chi_\nu(x)^{-1}f(x)$ by zero to $\mathbb{P}^1(F)$. We set $\delta_{\alpha,\nu} := \tilde{\delta}_{\alpha,\nu} \circ \psi_0$. If $\alpha = \nu$, we can define $\delta_{\alpha,\nu}$ on all functions in $C_c(F,R)$.

We let F^* operate on $C_c(F,R)$ by $(tf)(x) := f(t^{-1}x)$; this induces an action of the group $T^1(F) := \{\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) | t \in F^*\}$, which we identify with F^* in the obvious way. With respect to it, we have

$$\psi_0(tf)(x) = \chi_\alpha(t)\chi_\nu(t)^{-1}t\psi_0(f)(x)$$

and

$$\tilde{\delta}_{\alpha,\nu}(tf) = \chi_\alpha^{-1}(t)\chi_\nu(t)t\tilde{\delta}_{\alpha,\nu}(f),$$

so $\delta_{\alpha,\nu}$ is $T^1(F)$-equivariant.

For an R-module M, we define an F^*-action on $\mathrm{Dist}(F^*, M)$ by $\int f d(t\mu) := t\int (t^{-1}f) d\mu$. Let $H \subseteq G(F)$ be a subgroup, and M an $R[H]$-module. We define an H-action on $\mathcal{B}^{a\nu,\nu^{-1}}(F, M)$ by requiring $\langle \phi, h\lambda\rangle = h \cdot \langle h^{-1}\phi, \lambda\rangle$ for all $\phi \in \mathcal{B}_{a,\nu}(F, M)$, $\lambda \in \mathcal{B}^{a\nu,\nu^{-1}}(F, M)$, $h \in H$. With respect to these two actions, we get a $T^1(F) \cap H$-equivariant mapping

$$\delta^{\alpha,\nu} : \mathcal{B}^{a\nu,\nu^{-1}}(F, M) \to \mathrm{Dist}(F^*, M), \quad \delta^{\alpha,\nu}(\lambda) := \langle \delta_{\alpha,\nu}(\cdot), \lambda\rangle$$

dual to $\delta_{\alpha,\nu}$.

2.6 Local distributions

Now consider the case $R = \mathbb{C}$. Let $\chi_1, \chi_2 : F^* \to \mathbb{C}^*$ be two unramified characters. We consider (χ_1, χ_2) as a character on the torus $T(F)$ of $\mathrm{GL}_2(F)$, which induces a character χ on $B(F)$ by

$$\chi \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} := \chi_1(t_1)\chi_2(t_2).$$

Put $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. Set $\nu := \chi_1(\varpi)\chi_2(\varpi) = \alpha_1\alpha_2 q^{-1} \in \mathbb{C}^*$, and $a := \alpha_1 + \alpha_2 = \alpha_i + q\nu/\alpha_i$ for either i. When a and ν are given by the α_i like this, we will often write $\mathcal{B}_{\alpha_1,\alpha_2}(F, R) := \mathcal{B}_{a,\nu}(F, R)$ and $\mathcal{B}^{\alpha_1,\alpha_2}(F, M) := \mathcal{B}^{a\nu,\nu^{-1}}(F, M)$ (!) for its dual.

In the special case $a^2 = \nu(q+1)^2$, we assume the χ_i to be sorted such that $\chi_1 = |\cdot|\chi_2$ (not vice versa).

Let $\mathcal{B}(\chi_1, \chi_2)$ denote the space of continuous maps $\phi : G(F) \to \mathbb{C}$ such that

$$\phi\left(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} g\right) = \chi_{\alpha_1}(t_1)\chi_{\alpha_2}(t_2)|t_1|\phi(g) \tag{2.8}$$

for all $t_1, t_2 \in F^*$, $u \in F$, $g \in G(F)$. $G(F)$ operates canonically on $\mathcal{B}(\chi_1, \chi_2)$ by right translation (cf. [Bu], Ch. 4.5). If $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, $\mathcal{B}(\chi_1, \chi_2)$ is a model of the spherical representation $\pi(\chi_1, \chi_2)$; if $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, the special representation $\pi(\chi_1, \chi_2)$ can be given as an irreducible subquotient of codimension 1 of $\mathcal{B}(\chi_1, \chi_2)$.[ii]

Lemma 2.14. *We have a G-equivariant isomorphism $\tilde{\mathcal{B}}_{a,\nu}(F, \mathbb{C}) \cong \mathcal{B}(\chi_1, \chi_2)$. It induces an isomorphism $\mathcal{B}_{a,\nu}(F, \mathbb{C}) \cong \pi(\chi_1, \chi_2)$ both for spherical and special representations.*

Proof. We choose a "central" unramified character $\chi_Z : F^* \to \mathbb{C}$ satisfying $\chi_Z^2(\varpi) = \nu$; then we have $\chi_1 = \chi_Z\chi_0{}^{-1}$, $\chi_2 = \chi_Z\chi_0$ for some unramified character χ_0. We set $a' := \sqrt{q}\,(\chi_0(\varpi)^{-1} + \chi_0(\varpi))$, which satisfies $a = \chi_Z(\varpi)a'$.

For a representation (π, V) of $G(F)$, by [Bu], Ex. 4.5.9, we can define another representation $\chi_Z \otimes \pi$ on V via

$$(g, v) \mapsto \chi_Z(\det(g))\pi(g)v \quad \text{for all } g \in G(F), v \in V,$$

[ii]Note that [Bu] denotes this special representation by $\sigma(\chi_1, \chi_2)$, not by $\pi(\chi_1, \chi_2)$.

and with this definition we have $\mathcal{B}(\chi_1,\chi_2)\cong\chi_Z\otimes\mathcal{B}(\chi_0^{-1},\chi_0)$. Since $\mathcal{B}(\chi_0^{-1},\chi_0)$ has trivial central character, [BL], Thm. 20 (as quoted in [Sp]) states that $\mathcal{B}(\chi_0^{-1},\chi_0)\cong\mathcal{B}_{a',1}(F,\mathbb{C})\cong\operatorname{Ind}_{KZ}^{G(F)}R/\operatorname{Im}(T-a')$.

Define a G-linear map $\phi:\operatorname{Ind}_K^G R\to\chi_Z\otimes\operatorname{Ind}_{KZ}^G R$ by $1_K\mapsto(\chi_Z\circ\det)\cdot 1_{KZ}$. Since 1_K (resp. $(\chi_Z\circ\det)\cdot 1_{KZ}$) generates $\operatorname{Ind}_K^G R$ (resp. $\chi_Z\otimes\operatorname{Ind}_{KZ}^G R$) as a $\mathbb{C}[G]$-module, ϕ is well-defined and surjective.

ϕ maps $\mathcal{R}1_K=\left(\begin{smallmatrix}\varpi&0\\0&\varpi\end{smallmatrix}\right)1_K$ to

$$\left(\begin{smallmatrix}\varpi&0\\0&\varpi\end{smallmatrix}\right)((\chi_Z\circ\det)\cdot 1_{KZ})=\chi_Z(\varpi)^2\cdot((\chi_Z\circ\det)\cdot 1_{KZ})=\nu\cdot\phi(1_K).$$

Thus $\operatorname{Im}(\mathcal{R}-\nu)\subseteq\ker\phi$, and in fact the two are equal, since the preimage of the space of functions of support in a coset KZg ($g\in G(F)$) under ϕ is exactly the space generated by the 1_{Kzg}, $z\in Z(F)=Z(\mathcal{O}_F)\{\left(\begin{smallmatrix}\varpi&0\\0&\varpi\end{smallmatrix}\right)\}^{\mathbb{Z}}$.

Furthermore, ϕ maps $T1_K=\sum_{i\in\mathcal{O}_F/(\varpi)\cup\{\infty\}}N_i1_K$ (with the N_i as in Lemma 2.9) to

$$\sum_i\chi_Z(\det(N_i))\cdot((\chi_Z\circ\det)\cdot N_i1_{KZ})=\chi_Z(\varpi)\cdot(\chi_Z\circ\det)T1_{KZ}$$

(since $\det(N_i)=\varpi$ for all i),and thus $\operatorname{Im}(T-a)$ is mapped to $\operatorname{Im}\big(\chi_Z(\varpi)T-a\big)=\operatorname{Im}\big(\chi_Z(\varpi)(T-a')\big)=\operatorname{Im}(T-a')$.

Putting everything together, we thus have G-isomorphisms

$$\begin{aligned}C_c(\tilde{\mathcal{V}},\mathbb{C})/\big(\operatorname{Im}(T-a)+\operatorname{Im}(\mathcal{R}-\nu)\big)&\cong\operatorname{Ind}_K^G R/\big(\operatorname{Im}(T-a)+\operatorname{Im}(\mathcal{R}-\nu)\big)\\&\cong\chi_Z\otimes\big(\operatorname{Ind}_{KZ}^G R/\operatorname{Im}(T-a')\big)\quad(\text{via }\phi)\\&\cong\chi_Z\otimes\mathcal{B}(\chi_0^{-1},\chi_0)\cong\mathcal{B}(\chi_1,\chi_2).\end{aligned}$$

Thus, $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is isomorphic to the spherical principal series representation $\pi(\chi_1,\chi_2)$ for $a^2\neq\nu(q+1)^2$.

In the special case, $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is a G-invariant subspace of $\tilde{\mathcal{B}}_{a,\nu}(F,\mathbb{C})$ of codimension 1, so it must be mapped under the isomorphism to the unique G-invariant subspace of $\mathcal{B}(\chi_1,\chi_2)$ of codimension 1 (in fact, the unique infinite-dimensional irreducible G-invariant subspace, by [Bu], Thm. 4.5.1), which is the special representation $\pi(\chi_1,\chi_2)$. □

By [Bu], section 4.4, there exists thus for all pairs a,ν a *Whittaker functional* λ on $\mathcal{B}_{a,\nu}(F,\mathbb{C})$, i.e. a nontrivial linear map $\lambda:\mathcal{B}_{a,\nu}(F,\mathbb{C})\to\mathbb{C}$ such that $\lambda(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\phi)=\psi(x)\lambda(\phi)$. It is unique up to scalar multiples.

From it, we furthermore get a *Whittaker model* $\mathcal{W}_{a,\nu}$ of $\mathcal{B}_{a,\nu}(F,\mathbb{C})$:

$$\mathcal{W}_{a,\nu}:=\{W_\xi:GL_2(F)\to\mathbb{C}\,|\,\xi\in\mathcal{B}_{a,\nu}(F,\mathbb{C})\},$$

where $W_\xi(g):=\lambda(g\cdot\xi)$ for all $g\in GL_2(F)$. (see e.g. [Bu], Ch. 3, eq. (5.6).)

Now write $\alpha:=\alpha_1$ for short. Recall the distribution $\mu_{\alpha,\nu}=\psi(x)\chi_{\alpha/\nu}(x)dx\in\operatorname{Dist}(F^*,\mathbb{C})$. For $\alpha=\nu$, it extends to a distribution on F.

Proposition 2.15. *(a) There exists a unique Whittaker functional $\lambda=\lambda_{a,\nu}$ on $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ such that $\delta^{\alpha,\nu}(\lambda)=\mu_{\alpha,\nu}$.*

(b) For every $f \in C_c(F^*, \mathbb{C})$, there exists $W = W_f \in \mathcal{W}_{a,\nu}$ such that

$$\int_{F^*} (af)(x)\mu_{\alpha,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\alpha = \nu$, then for every $f \in C_c(F, \mathbb{C})$, there exists $W_f \in \mathcal{W}_{a,\nu}$ such that

$$\int_{F} (af)(x)\mu_{\alpha,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Let $H \subseteq U = \mathcal{O}_F^\times$ be an open subgroup, and put $W_H := W_{1_H}$. For every $f \in C_c^0(F^*, \mathbb{C})^H$ we have

$$\int_{F^*} f(x)\mu_{\alpha,\nu}(dx) = [U : H] \int_{F^*} f(x) W_H \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x.$$

Proof. (a) (cf. [Sp], prop. 3.10 for the first part) We let the additive group F act on $C_c(F, \mathbb{C})$ by $(x \cdot f)(y) := f(y - x)$, and on $C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C}$ by $x\phi := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \phi$. Thus the functional

$$\Lambda : C_c(F, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_F f(x)\psi(x)dx$$

satisfies $\Lambda(xf) = \psi(x)\Lambda(f)$ for all $x \in F$ and all $f \in C_c(F, \mathbb{C})$, and there is an F-equivariant isomorphism

$$C^0(\mathbb{P}^1(F, \mathbb{C})/\mathbb{C} \to C_c(F, \mathbb{C}), \quad \phi \mapsto f(x) := \phi(x) - \phi(\infty).$$

Thus the composite

$$St(F, \mathbb{C}) := C^0(\mathbb{P}^1(F, \mathbb{C})/\mathbb{C} \xrightarrow{\cong} C_c(F, \mathbb{C}) \xrightarrow{\Lambda} \mathbb{C} \tag{2.9}$$

is a Whittaker functional of the Steinberg representation.

Let now $\lambda : \mathcal{B}_{a,\nu}(F, \mathbb{C}) \to \mathbb{C}$ be a Whittaker functional of $\mathcal{B}_{a,\nu}(F, \mathbb{C})$. By lemma 2.13,

$$(\lambda \circ \tilde{\delta}_{\alpha,\nu})(u\phi) = \lambda(u\tilde{\delta}_{\alpha,\nu}(\phi)) = \psi(x)\lambda(\tilde{\delta}_{\alpha,\nu}(\phi)),$$

so $\lambda \circ \tilde{\delta}_{\alpha,\nu}(\phi)$ is a Whittaker functional if it is not zero.
To describe the image of $\tilde{\delta}_{\alpha,\nu}$, consider the commutative diagram

$$\begin{array}{ccccccc}
C_c(\tilde{\mathcal{E}}, R) & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & C_c(\tilde{\mathcal{V}}, R) & & & & \\
\downarrow{\scriptstyle (2.10)} & & \downarrow{\scriptstyle \phi \mapsto \phi \cdot \varrho} & & & & \\
C_c(\tilde{\mathcal{E}}, R) & \xrightarrow{\delta} & C_c(\tilde{\mathcal{V}}, R) & \xrightarrow{\langle \cdot, 1 \rangle} & R & \longrightarrow & 0
\end{array}$$

where the vertical maps are defined by

$$C_c(\tilde{\mathcal{E}}, R) \to C_c(\tilde{\mathcal{E}}, R), \quad c \mapsto \big(e \mapsto c(e)\varrho(o(e))\varrho(t(e))\big) \tag{2.10}$$

resp. by mapping ϕ to $v \mapsto \phi(v)\varrho(v)$; both are obviously isomorphisms.

Since the lower row is exact, we have $\operatorname{Im}\delta = \ker\langle\cdot,1\rangle =: C_c^0(\tilde{\mathcal{V}},R)$ and thus $\operatorname{Im}\tilde{\delta}_{\alpha,\nu} = \varrho^{-1}\cdot C_c^0(\tilde{\mathcal{V}},R)$.

Since $\lambda \neq 0$ and $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is generated by (the equivalence classes of) the $1_{\{v\}}$, $v \in \tilde{\mathcal{V}}$, there exists a $v \in \tilde{\mathcal{V}}$ such that $\lambda(1_{\{v\}}) \neq 0$. Let ϕ be this $1_{\{v\}}$, and let $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in B(F)$ such that $x \notin \ker\psi = \mathcal{O}_F$. Then

$$\varrho\cdot(u\phi-\phi) = \varrho\cdot(1_{\{u^{-1}v\}} - 1_{\{v\}}) = \varrho(v)(1_{\{u^{-1}v\}} - 1_{\{v\}}) \in C_c^0(\tilde{\mathcal{V}},R)$$

by lemma 2.13, so $0 \neq u\phi - \phi \in \operatorname{Im}\tilde{\delta}_{\alpha,\nu}$, but $\lambda(u\phi-\phi) = \psi(x)\lambda(\phi) - \lambda(\phi) \neq 0$.

So $\lambda \circ \tilde{\delta}_{\alpha,\nu} \neq 0$ is indeed a Whittaker functional. By replacing λ by a scalar multiple, we can assume $\lambda\circ\tilde{\delta}_{\alpha,\nu} = (2.9)$.

Considering λ as an element of $\mathcal{B}^{a\nu,\nu^{-1}}(F,\mathbb{C}) \cong \operatorname{Hom}(\mathcal{B}_{a,\nu}(F,\mathbb{C}),\mathbb{C})$, we have

$$\begin{aligned}
\delta^{\alpha,\nu}(\lambda)(f) &= \langle\delta_{\alpha,\nu}(f),\lambda\rangle \\
&= \Lambda(\chi_\alpha\chi_\nu^{-1}f) \\
&= \int_{F^*}\chi_\alpha(x)\chi_\nu^{-1}(x)f(x)\psi(x)dx \\
&= \mu_{\alpha,\nu}(f).
\end{aligned}$$

(b) For given f, set $W_f(g) := \lambda(g\cdot\delta_{\alpha,\nu}(f))$. Then $W_f \in \mathcal{W}_{a,\nu}$, and for all $a \in F^*$ we have:

$$\begin{aligned}
W_f\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \lambda\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha,\nu}(f)\right) \\
&= \lambda(\delta_{\alpha,\nu}(af)) \qquad \text{(by the } T^1(F)\text{-invariance of } \delta_{\alpha,\nu}) \\
&= \int_{F^*}(af)(x)\mu_{\alpha,\nu}(dx).
\end{aligned}$$

(c) Without loss of generality we can assume $f = 1_{aH}$ for some $a \in F^*$.

We have

$$\begin{aligned}
\int_{F^*} 1_{aH}(x)\mu_{\alpha,\nu}(dx) &= \int_{F^*} 1_H(a^{-1}x)\mu_{\alpha,\nu}(dx) \\
&= \int_{F^*}(a\cdot 1_H)(x)\mu_{\alpha,\nu}(dx) \\
&= W_H\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ by (b),}
\end{aligned}$$

and since the left-hand side is invariant under replacing a by ah (for $h \in H$), the right-hand side also is, so we can integrate this constant function over H:

$$
\begin{aligned}
&= [U:H]\int_H W_H\begin{pmatrix} ax & 0\\ 0 & 1\end{pmatrix} d^\times x\\
&= [U:H]\int_{F^*} 1_H(x) W_H\begin{pmatrix} ax & 0\\ 0 & 1\end{pmatrix} d^\times x\\
&= [U:H]\int_{F^*} 1_H(a^{-1}x) W_H\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} d^\times x\\
&= [U:H]\int_{F^*} 1_{aH}(x) W_H\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} d^\times x.
\end{aligned}
$$

□

2.7 Semi-local theory

We can generalize many of the previous constructions to the semi-local case, considering all primes $\mathfrak{p}|p$ at once.

So let $F_1, \ldots, F_m$ be finite extensions of $\mathbb{Q}_p$, and for each i, let q_i be the number of elements of the residue field of F_i. We put $\underline{F} := F_1 \times \cdots \times F_m$.

Let R again be a ring, and $a_i \in R, \nu_i \in R^*$ for each $i \in \{1, \ldots, m\}$. Put $\underline{a} := (a_1, \ldots, a_m)$, $\underline{\nu} := (\nu_1, \ldots, \nu_m)$. We define $\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R)$ as the tensor product

$$\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R) := \bigotimes_{i=1}^m \mathcal{B}_{a_i,\nu_i}(F_i, R).$$

For an R-module M, we define $\mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(\underline{F}, M) := \mathrm{Hom}_R(\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R), M)$; let

$$\langle\cdot,\cdot\rangle : \mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R) \times \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(\underline{F}, M) \to M \tag{2.11}$$

denote the evaluation pairing.

We have an obvious isomorphism

$$\bigotimes_{i=1}^m C_c^0(F_i^*, R) \to C_c^0(\underline{F}^*, R), \quad \bigotimes_i f_i \mapsto \left((x_i)_{i=1,\ldots,m} \mapsto \prod_{i=1}^m f_i(x_i)\right). \tag{2.12}$$

Now when we have $\alpha_{i,1}, \alpha_{i,2} \in R^*$ such that $a_i = \alpha_{i,1} + \alpha_{i,2}$ and $\nu_i = \alpha_{i,1}\alpha_{i,2}q_i^{-1}$, we can define the $T^1(\underline{F})$-equivariant map

$$\delta_{\underline{\alpha}_{1,2}} := \delta_{\underline{\alpha_1},\underline{\nu}} : C_c^0(\underline{F}, R) \to \mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R)$$

as the inverse of (2.12) composed with $\bigotimes_{i=1}^m \delta_{\alpha_{i,1},\nu_i}$.

Again, we will often write $\mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F, R) := \mathcal{B}_{\underline{a\nu},\underline{\nu}^{-1}}(F, R)$ and $\mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F, M) := \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$.

If $H \subseteq G(F)$ is a subgroup, and M an $R[H]$-module, we define an H-action on $\mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$ by requiring $\langle\phi, h\lambda\rangle = h \cdot \langle h^{-1}\phi, \lambda\rangle$ for all $\phi \in \mathcal{B}_{\underline{a},\underline{\nu}}(F, M)$, $\lambda \in \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$, $h \in H$, and get a $T^1(\underline{F}) \cap H$-equivariant mapping

$$\delta^{\underline{\alpha_1},\underline{\alpha_2}} : \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M) \to \mathrm{Dist}(\underline{F}^*, M), \quad \delta^{\underline{\alpha_1},\underline{\alpha_2}}(\lambda) := \langle\delta_{\underline{\alpha_1},\underline{\alpha_2}}(\cdot), \lambda\rangle.$$

Finally, we have a homomorphism

$$
\begin{aligned}
\bigotimes_{i=1}^{m} \mathcal{B}^{a_i\nu_i,\nu_i^{-1}}(F_i,R) &\xrightarrow{\cong} \bigotimes_{i=1}^{m} \mathrm{Hom}_R(\mathcal{B}_{a_i\nu_i,\nu_i^{-1}}(F_i,R),R) \\
&\to \mathrm{Hom}(\mathcal{B}_{a_1,\nu_1}(F_1,R),\mathrm{Hom}(\mathcal{B}_{a_2,\nu_2}(F_2,R),\mathrm{Hom}(\ldots,R))...) \\
&\xrightarrow{\cong} \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F,R).
\end{aligned}
\tag{2.13}
$$

where the second map is given by $\otimes_i f_i \mapsto (x_1 \mapsto (x_2 \mapsto (\ldots \mapsto \prod_i f_i(x_i))...),$ and the last map by iterating the adjunction formula of the tensor product.

3 Cohomology classes and global measures

3.1 Definitions

From now on, let F denote a number field, with ring of integers $\mathcal{O}_F$. For each finite prime v, let $U_v := \mathcal{O}_v^*$. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adeles of F, and $\mathbb{I} = \mathbb{I}_F$ the group of ideles of F. For a finite subset S of the set of places of F, we denote by $\mathbb{A}^S := \{x \in \mathbb{A}_F \mid x_v = 0 \ \forall v \in S\}$ the S-adeles and by $\mathbb{I}^S$ the S-ideles, and put $F_S := \prod_{v\in S} F_v$, $U_S := \prod_{v\in S} U_v$, $U^S := \prod_{v\notin S} U_v$ (if S contains all infinite places of F), and similarly for other global groups.

For ℓ a prime number or ∞, we write S_ℓ for the set of places of F above ℓ, and abbreviate the above notations to $\mathbb{A}^\ell := \mathbb{A}^{S_\ell}$, $\mathbb{A}^{p,\infty} := \mathbb{A}^{S_p \cup S_\infty}$, and similarly write $\mathbb{I}^p$, $\mathbb{I}^\infty$, F_p, F_∞, U^∞, U_p, $U^{p,\infty}$, $\mathbb{I}_\infty$ etc.

Let F have r real embeddings and s pairs of complex embeddings. Set $d := r+s-1$. Let $\{\sigma_0, \ldots, \sigma_{r-1}, \sigma_r, \ldots, \sigma_d\}$ be a set of representatives of these embeddings (i.e. for $i \geq r$, choose one from each pair of complex embeddings), and denote by $\infty_0, \ldots, \infty_d$ the corresponding archimedian primes of F. We let $S_\infty^0 := \{\infty_1, \ldots, \infty_d\} \subseteq S_\infty$.

We fix an additive character $\psi : \mathbb{A} \to \mathbb{C}^*$ which is trivial on F, and let ψ_v denote the restriction of ψ to $F_v \hookrightarrow \mathbb{A}$, for all primes v; we assume that $\ker(\psi_v) = \mathcal{O}_{F_v}$ for all $\mathfrak{p}|p$.

For each place v, let dx_v denote the associated self-dual Haar measure on F_v, and $dx := \prod_v dx_v$ the associated Haar measure on $\mathbb{A}_F$. We define Haar measures $d^\times x_v$ on F_v^* by $d^\times x_v := c_v \frac{dx_v}{|x_v|_v}$, where $c_v = (1 - \frac{1}{q_v})^{-1}$ for v finite, $c_v = 1$ for $v|\infty$.

For $v|\infty$ complex, we use the decomposition $\mathbb{C}^* = \mathbb{R}_+^* \times S^1$ (with $S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$) to write $d^\times x_v = d^\times r_v \, d\vartheta_v$ for variables r_v, ϑ_v with $r_v \in \mathbb{R}_+^*$, $\vartheta_v \in S^1$.

Let $S_1 \subseteq S_p$ be a set of primes of F lying above p, $S_2 := S_p - S_1$. Let R be a topological Hausdorff ring.

Definition 3.1. We define the module of continuous functions

$$\mathcal{C}(S_1, R) := C(F_{S_1} \times F_{S_2}^* \times \mathbb{I}^{p,\infty}/U^{p,\infty}, R);$$

and let $\mathcal{C}_c(S_1, R)$ be the submodule of all compactly supported $f \in \mathcal{C}(S_1, R)$. We write $\mathcal{C}^0(S_1, R)$, $\mathcal{C}_c^0(S_1, R)$ when R is assumed to have the discrete topology.

Definition 3.2. For an R-module M, let $\mathcal{D}_f(S_1, M)$ denote the R-module of maps

$$\phi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}_F^{p,\infty} \to M$$

that are $U^{p,\infty}$-invariant and such that $\phi(\cdot, x^{p,\infty})$ is a distribution for each $x^{p,\infty} \in \mathbb{I}_F^{p,\infty}$.

Since $\mathbb{I}_F^{p,\infty}/U^{p,\infty}$ is a discrete topological group, $\mathcal{D}_f(S_1, M)$ naturally identifies with the space of M-valued distributions on $F_{S_1} \times F_{S_2}^* \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$. So there exists a canonical R-bilinear map

$$\mathcal{D}_f(S_1, M) \times \mathcal{C}_c^0(S_1, R) \to M, \quad (\phi, f) \mapsto \int f \, d\phi, \tag{3.1}$$

which is easily seen to induce an isomorphism $\mathcal{D}_f(S_1, M) \cong \mathrm{Hom}_R(\mathcal{C}_c^0(S_1, R), M)$.

For a subgroup $E \subseteq F^*$ and an $R[E]$-module M, we let E operate on $\mathcal{D}_f(S_1, M)$ and $\mathcal{C}_c^0(S_1, R)$ by $(a\phi)(U, x^{p,\infty}) := a\phi(a^{-1}U, a^{-1}x^{p,\infty})$ and $(af)(x^\infty) := f(a^{-1}x^\infty)$ for $a \in E$, $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, $x^\cdot \in \mathbb{I}_F^\cdot$; thus we have $\int (af)\ d(a\phi) = a \int f\ d\phi$ for all a, f, ϕ.

When $M = V$ is a finite-dimensional vector space over a p-adic field, we write $\mathcal{D}_f^b(S_1, V)$ for the subset of $\phi \in \mathcal{D}_f(S_1, V)$ such that ϕ is even a measure on $F_{S_1} \times F_{S_2} \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$.

Definition 3.3. For a $\mathbb{C}$-vector space V, define $\mathcal{D}(S_1, V)$ to be the set of all maps $\phi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^p \to V$ such that:

(i) ϕ is invariant under $F^\times$ and $U^{p,\infty}$.

(ii) For $x^p \in \mathbb{I}^p$, $\phi(\cdot, x^p)$ is a distribution of F_p.

(iii) For all $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, the map $\phi_U : \mathbb{I} = F_p^\times \times \mathbb{I}^p \to V, (x_p, x^p) \mapsto \phi(x_p U, x^p)$ is smooth, and rapidly decreasing as $|x| \to \infty$ and $|x| \to 0$.

We will need a variant of this last set: Let $\mathcal{D}'(S_1, V)$ be the set of all maps $\phi \in \mathcal{D}(S_1, V)$ that are "$(S^1)^s$-invariant", i.e. such that for all complex primes ∞_j of F and all $\zeta \in S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$, we have

$$\phi(U, x^{p,\infty_j}, \zeta x_{\infty_j}) = \phi(U, x^{p,\infty_j}, x_{\infty_j}) \text{ for all } x^p = (x^{p,\infty_j}, x_{\infty_j}) \in \mathbb{I}^p.$$

There is an obvious surjective map

$$\mathcal{D}(S_1, V) \to \mathcal{D}'(S_1, V), \quad \phi \mapsto \left((U, x) \mapsto \int_{(S^1)^s} \phi(U, x) d\vartheta_r \cdots d\vartheta_{r+s-1} \right)$$

given by integrating over $(S^1)^s \subseteq (\mathbb{C}^*)^s \hookrightarrow \mathbb{I}_\infty$.

Let $F^{*\prime} \subseteq F^*$ be a maximal torsion-free subgroup (so that $F/F^{*\prime} \cong \mu_F$, the roots of unity of F). If F has at least one real embedding, we specifically choose $F^{*\prime}$ to be the set F_+^* of all totally positive elements of F (i.e. positive with respect to every real embedding of F). For totally complex F, there is no such natural subgroup available, so we just choose $F^{*\prime}$ freely. We set

$$E' := F^{*\prime} \cap O_F^\times \subseteq O_F^\times,$$

so E' is a torsion-free $\mathbb{Z}$-module of rank d. E' operates freely and discretely on the space

$$\mathbb{R}_0^{d+1} := \left\{ (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} | \sum_{i=0}^{d} x_i = 0 \right\}$$

via the embedding

$$\begin{array}{ccc} E' & \hookrightarrow & \mathbb{R}_0^{d+1} \\ a & \mapsto & (\log |\sigma_i(a)|)_{i \in S_\infty} \end{array}$$

(cf. proof of Dirichlet's unit theorem, e.g. in [Neu], Ch. 1), and the quotient $\mathbb{R}_0^{d+1}/E'$ is compact. We choose the orientation on $\mathbb{R}_0^{d+1}$ induced by the natural orientation on $\mathbb{R}^d$ via the isomorphism $\mathbb{R}^d \cong \mathbb{R}_0^{d+1}$, $(x_1, \ldots, x_d) \mapsto (-\sum_{i=1}^d x_i, x_1, \ldots, x_d)$. So $\mathbb{R}_0^{d+1}/E'$ becomes an oriented compact d-dimensional manifold.

Let $\mathcal{G}_p$ be the Galois group of the maximal abelian extension of F which is unramified outside p and ∞; for a $\mathbb{C}$-vector space V, let $\mathrm{Dist}(\mathcal{G}_p, V)$ be the set of V-valued distributions of $\mathcal{G}_p$. Denote by $\varrho : \mathbb{I}_F/F^* \to \mathcal{G}_p$ the projection given by global reciprocity.

3.2 Global measures

Now let $V = \mathbb{C}$, equipped with the trivial $F^{*\prime}$-action. We want to construct a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_\phi} & H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})\big) \\ & {\scriptstyle \phi \mapsto \mu_\phi} \searrow \quad \swarrow {\scriptstyle \kappa \mapsto \mu_\kappa = \kappa \cap \partial(\cdot)} & \\ & \mathrm{Dist}(\mathcal{G}_p, \mathbb{C}) & \end{array} \tag{3.2}$$

First, let R be any topological Hausdorff ring. Let $\overline{E'}$ denote the closure of E' in U_p. The projection map $\mathrm{pr} : \mathbb{I}^\infty/U^{p,\infty} \to \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty})$ induces an isomorphism

$$\mathrm{pr}^* : C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R) \to H^0(E', C_c(\mathbb{I}^\infty/U^{p,\infty}, R)),$$

and the reciprocity map induces a surjective map $\overline{\varrho} : \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}) \to \mathcal{G}_p$.

Now we can define a map

$$\varrho^\sharp : H_0(F^{*\prime}/E', C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R)) \to C(\mathcal{G}_p, R)$$
$$[f] \mapsto \Big(\overline{\varrho}(x) \mapsto \sum_{\zeta \in F^{*\prime}/E'} f(\zeta x) \text{ for } x \in \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty})\Big).$$

This is an isomorphism, with inverse map $f \mapsto [(f \circ \overline{\varrho}) \cdot 1_{\mathcal{F}}]$, where $1_{\mathcal{F}}$ is the characteristic function of a fundamental domain $\mathcal{F}$ of the action of $F^{*\prime}/E'$ on $\mathbb{I}^\infty/U^\infty$.

We get a composite map

$$\begin{aligned} C(\mathcal{G}_p, R) &\xrightarrow{(\varrho^\sharp)^{-1}} H_0\big(F^{*\prime}/E', C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R)\big) \\ &\xrightarrow{\mathrm{pr}^*} H_0\big(F^{*\prime}/E', H^0(E', C_c(\mathbb{I}^\infty/U^{p,\infty}, R))\big) \\ &\longrightarrow H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c(S_1, R))\big), \end{aligned} \tag{3.3}$$

where the last arrow is induced by the "extension by zero" from $C_c(\mathbb{I}^\infty/U^{p,\infty}, R)$ to $\mathcal{C}_c(S_1, R)$.

Now let $\eta \in H_d(E', \mathbb{Z}) \cong \mathbb{Z}$ be the generator that corresponds to the given orientation of $\mathbb{R}_0^{d+1}$. This gives us, for every R-module A, a homomorphism

$$H_0\big(F^{*\prime}/E', H^0(E', A)\big) \xrightarrow{\cap \eta} H_0\big(F^{*\prime}/E', H_d(E', A)\big)$$

Composing this with the edge morphism

$$H_0\big(F^{*\prime}/E', H_d(E', A)\big) \to H_d(F^{*\prime}, A) \tag{3.4}$$

(and setting $A = \mathcal{C}_c(S_1, R)$) gives a map

$$H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c(S_1, R))\big) \to H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big) \tag{3.5}$$

We define

$$\partial : C(\mathcal{G}_p, R) \to H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big)$$

as the composition of (3.3) with this map.

Now, letting M be an R-module equipped with the trivial $F^{*\prime}$-action, the bilinear form (3.1)

$$\begin{array}{rcl} \mathcal{D}_f(S_1, M) \times \mathcal{C}_c(S_1, R) & \to & M \\ (\phi, f) & \mapsto & \displaystyle\int f \, d\phi \end{array}$$

induces a cap product

$$\cap : H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, M)\big) \times H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big) \to H_0(F^{*\prime}, M) = M. \tag{3.6}$$

Thus for each $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, M))$, we get a distribution μ_κ on $\mathcal{G}_p$ by defining

$$\int_{\mathcal{G}_p} f(\gamma)\, \mu_\kappa(d\gamma) := \kappa \cap \partial(f) \tag{3.7}$$

for all continuous maps $f : \mathcal{G}_p \to R$.

Now let $M = V$ be a finite-dimensional vector space over a p-adic field K, and let $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V))$. We identify κ with its image in $H^d(F^{*\prime}, \mathcal{D}_f(S_1, V))$; then it is easily seen that μ_κ is also a measure, i.e. we have a map

$$H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V)) \to \mathrm{Dist}^b(\mathcal{G}_p, V). \tag{3.8}$$

Definition 3.4. The *p-adic cyclotomic character* $\mathcal{N} : \mathcal{G}_p \to \mathbb{Z}_p^*$ is defined by requiring $\gamma\zeta = \zeta^{\mathcal{N}(\gamma)}$ for $\gamma \in \mathcal{G}_p$ and all p-power roots of unity ζ. We put $\mathcal{N}(\gamma)^s := \exp_p(s \log_p(\mathcal{N}(\gamma)))$ for all $s \in \mathbb{Z}_p$.

Definition 3.5. Let K be a p-adic field, V a finite-dimensional K-vector space. We define the *p-adic L-function* of $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V))$ as

$$L_p(s, \kappa) := \int_{\mathcal{G}_p} \mathcal{N}(\gamma)^s \mu_\kappa(d\gamma)$$

for all $s \in \mathbb{Z}_p$.

Remark 3.6. Let $\Sigma := \{\pm 1\}^r$, where r is the number of real embeddings of F. The group isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}, \varepsilon \mapsto (-1)^\varepsilon$, induces a pairing

$$\langle \cdot, \cdot \rangle : \Sigma \to \{\pm 1\}, \quad \langle ((-1)^{\varepsilon_i})_i, ((-1)^{\varepsilon'_i})_i \rangle := (-1)^{\sum_i \varepsilon_i \varepsilon'_i}.$$

For a field k of characteristic zero, a $k[\Sigma]$-module V and $\underline{\mu} = (\mu_0, \ldots, \mu_{r-1}) \in \Sigma$, we put $V_{\underline{\mu}} := \{v \in V \mid \langle \underline{\mu}, \underline{\nu} \rangle v = \underline{\nu} v \ \forall \underline{\nu} \in \Sigma\}$, so that we have $V = \bigoplus_{\underline{\mu} \in \Sigma} V_{\underline{\mu}}$. We write $v_{\underline{\mu}}$ for the projection of $v \in V$ to $V_{\underline{\mu}}$, and $v_+ := v_{(1,\ldots,1)}$.

We identify Σ with $F^*/F^{*\prime}$ via the isomorphism $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}^*_+ \cong F^*/F^{*\prime}$. Then for each F^*-module M, Σ acts on $H^d(F^{*\prime}, \mathcal{D}_f(S_1, M))$ and on $H^d(F^{*\prime}, \mathcal{D}^b_f(S_1, M))$. The exact sequence $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}^*_+ = \mathbb{I}_\infty/\mathbb{I}^0_\infty \to \mathcal{G}_p \to \mathcal{G}_p^+ \to 0$ of class field theory (where $\mathbb{I}^0_\infty$ is the maximal connected subgroup of $\mathbb{I}_\infty$) yields an action of Σ on $\mathcal{G}_p$. We easily check that (3.8) is Σ-equivariant, and that the map $\gamma \mapsto \mathcal{N}(\gamma)^s$ factors over $\mathcal{G}_p \to \mathcal{G}_p^+$. Therefore we have $L_p(s, \kappa) = L_p(s, \kappa_+)$.

For $\phi \in \mathcal{D}(S_1, V)$ and $f \in C^0(\mathbb{I}/F^*, \mathbb{C})$, let

$$\int_{\mathbb{I}/F^*} f(x)\phi(d^\times x_p, x^p)\, d^\times x^p := [U_p : U] \int_{\mathbb{I}/F^*} f(x)\phi_U(x)\, d^\times x,$$

where we choose an open set $U \subseteq U_p$ such that $f(x_p u, x^p) = f(x_p, x^p)$ for all $(x_p, x^p) \in \mathbb{I}$ and $u \in U$; such a U exists by lemma 3.7 below.

Since this integral is additive in f, there exists a unique V-valued distribution μ_ϕ on $\mathcal{G}_p$ such that

$$\int_{\mathcal{G}_p} f\, d\mu_\phi = \int_{\mathbb{I}/F^*} f(\varrho(x))\phi(d^\times x_p, x^p)\, d^\times x^p \tag{3.9}$$

for all functions $f \in C^0(\mathcal{G}_p, V)$.

Lemma 3.7. *Let $F : \mathbb{I}/F^* \to X$ be a locally constant map to a set X. Then there exists an open subgroup $U \subseteq \mathbb{I}$ such that f factors over $\mathbb{I}/F^*U$.*

Proof. (cf. [Sp], lemma 4.20)
$\mathbb{I}_\infty = \prod_{v|\infty} F_v$ is connected, thus f factors over $\overline{f} : \mathbb{I}/F^*\mathbb{I}_\infty \to X$. Since $\mathbb{I}/F^*\mathbb{I}_\infty$ is profinite, $\overline{f}$ further factors over a subgroup $U' \subseteq \mathbb{I}^\infty$ of finite index, which is open. □

Let $U^0_\infty := \prod_{v \in S^0_\infty} \mathbb{R}^*_+$; the isomorphisms $U^0_\infty \cong \mathbb{R}^d$, $(r_v)_v \mapsto (\log r_v)_v$, and $\mathbb{R}^d \cong \mathbb{R}^{d+1}_0$ give it the structure of a d-dimensional oriented manifold (with the natural orientation). It has the d-form $d^\times r_1 \cdot \ldots \cdot d^\times r_d$, where (by slight abuse of notation) we choose $d^\times r_i$ on F_{∞_i} corresponding to the Haar measure $d^\times x_i$ resp. $d^\times r_i$ on $\mathbb{R}^*_+ \subseteq F^*_{\infty_i}$.

E' operates on U^0_∞ via $a \mapsto (|\sigma_i(a)|)_{i \in S^0_\infty}$, making the isomorphism $U^0_\infty \cong \mathbb{R}^{d+1}_0$ E'-equivariant.

For $\phi \in \mathcal{D}'(S_1, V)$, set

$$\int_0^\infty \phi \, d^\times r_0 \colon \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^{p,\infty_0} \to \mathbb{C}$$
$$(U, x^{p,\infty_0}) \mapsto \int_0^\infty \phi(U, r_0, x^{p,\infty_0}) \, d^\times r_0,$$

where we let $r_0 \in F_{\infty_0}$ run through the positive real line $\mathbb{R}_+^*$ in F_{∞_0}. Composing this with the projection $\mathcal{D}(S_1, V) \to \mathcal{D}'(S_1, V)$ gives us a map

$$\begin{aligned} &\mathcal{D}(S_1, V) \to H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))\big), \\ &\phi \mapsto \int_{(S^1)^s} \left(\int_0^\infty \phi \, d^\times r_0 \right) d\vartheta_r \, d\vartheta_{r+1} \ldots d\vartheta_{r+s-1} \end{aligned} \tag{3.10}$$

(where $C^\infty(U_\infty^0, V)$ denotes the space of smooth V-valued functions on U_∞^0), since one easily checks that $\int_0^\infty \phi \, d^\times r_0$ is $F^{*\prime}$-invariant.

Define the complex $C^\bullet := \mathcal{D}_f(S_1, \Omega^\bullet(U_\infty^0, V))$. By the Poincare lemma, this is a resolution of $\mathcal{D}_f(S_1, V)$. We now define the map $\phi \mapsto \kappa_\phi$ as the composition of (3.10) with the composition

$$H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))\big) \to H^0(F^{*\prime}, C^d) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, V)), \tag{3.11}$$

where the first map is induced by

$$C^\infty(U_\infty^0, V) \to \Omega^d(U_\infty^0, V), \quad f \mapsto f(r_1, \ldots, r_d) d^\times r_1 \cdot \ldots \cdot d^\times r_d, \tag{3.12}$$

and the second is an edge morphism in the spectral sequence

$$H^q(F^{*\prime}, C^p) \Rightarrow H^{p+q}(F^{*\prime}, \mathcal{D}_f(S_1, V)). \tag{3.13}$$

Specializing to $V = \mathbb{C}$, we now have:

Proposition 3.8. *The diagram* (3.2) *commutes, i.e., for each* $\phi \in \mathcal{D}(S_1, \mathbb{C})$, *we have*

$$\mu_\phi = \mu_{\kappa_\phi}.$$

Proof. (cf. [Sp], prop. 4.21)
We define a pairing

$$\langle\ ,\ \rangle \colon \mathcal{D}(S_1, \mathbb{C}) \times C^0(\mathcal{G}_p, \mathbb{C}) \to \mathbb{C}$$

as the composite of (3.10) × (3.3) with

$$\begin{aligned} &H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, \mathbb{C}))\big) \times H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c^0(S_1, \mathbb{C}))\big) \\ &\xrightarrow{\cap} H_0\big(F^{*\prime}/E', H^0(E', C^\infty(U_\infty^0, \mathbb{C}))\big) \to H_0(F^{*\prime}/E', \mathbb{C}) \cong \mathbb{C}, \end{aligned} \tag{3.14}$$

where $\cap$ is the cap product induced by (3.1), and the second map is induced by

$$H^0\big(E', \mathcal{C}^\infty(U_\infty^0, \mathbb{C})\big) \to \mathbb{C}, \quad f \mapsto \int_{U_\infty^0/E'} f(r_1, \ldots, r_d) \, d^\times r_1 \ldots d^\times r_d. \tag{3.15}$$

An easy computation shows that

$$\langle \phi, f \rangle = \int_{\mathcal{G}_p} f(\gamma)\, \mu_\phi(d\gamma) \quad \text{for all } f \in C^0(\mathcal{G}_p, \mathbb{C}).$$

So we need to show that $\kappa_\phi \cap \partial(f) = \langle \phi, f \rangle$; i.e. it suffices to show that the diagram

$$\begin{array}{ccc}
H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U^0_\infty, \mathbb{C}))\big) \times H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}^0_c(S_1, \mathbb{C}))\big) & & \\
\Big\downarrow {\scriptstyle (3.11)\times(3.5)} & \searrow^{(3.14)} & \\
 & & \mathbb{C} \\
 & \nearrow_{\cap} & \\
H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})) \times H_d\big(F^{*\prime}, \mathcal{C}^0_c(S_1, \mathbb{C})\big) & &
\end{array} \tag{3.16}$$

commutes. For this consider the commutative diagram

$$\begin{array}{ccc}
H^0(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U^0_\infty, \mathbb{C}))) \times H_0(F^{*\prime}/E', H^0(E', \mathcal{C}^0_c(S_1, \mathbb{C}))) & \xrightarrow{\cap} & H_0(F^{*\prime}/E', H^0(E', C^\infty(U^0_\infty, \mathbb{C}))) \\
\Big\downarrow {\scriptstyle \mathrm{id}\times\eta} & & \Big\downarrow {\scriptstyle \eta} \\
H^0(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U^0_\infty, \mathbb{C}))) \times H_0(F^{*\prime}/E', H_d(E', \mathcal{C}^0_c(S_1, \mathbb{C}))) & \xrightarrow{\cap} & H_0(F^{*\prime}/E', H_d(E', C^\infty(U^0_\infty, \mathbb{C}))) \\
\Big\downarrow {\scriptstyle 3\times\mathrm{id}} & & \Big\downarrow {\scriptstyle 4} \\
H^0(F^{*\prime}, \mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) \times H_0(F^{*\prime}/E', H_d(E', \mathcal{C}^0_c(S_1, \mathbb{C}))) & \xrightarrow{\cap} & H_0(F^{*\prime}/E', H_d(E', \Omega^d(U^0_\infty))) \\
\Big\downarrow {\scriptstyle \mathrm{id}\times 5} & & \Big\downarrow {\scriptstyle 6} \\
H^0(F^{*\prime}, \mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) \times H_d(F^{*\prime}, \mathcal{C}^0_c(S_1, \mathbb{C}))) & \xrightarrow{\cap} & H_d(F^{*\prime}, \Omega^d(U^0_\infty)) \\
\Big\downarrow {\scriptstyle 7\times\mathrm{id}} & & \Big\downarrow {\scriptstyle 8} \\
H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})) \times H_d(F^{*\prime}, \mathcal{C}^0_c(S_1, \mathbb{C}))) & \xrightarrow{\cap} & H_0(F^{*\prime}, \mathbb{C}) = \mathbb{C}
\end{array}$$

where the horizontal maps are cap-products induced by the pairing (3.1), η denotes cap-product with η, 3 and 4 are induced by (3.12), 5 and 6 by the edge morphism (3.4), and 7 and 8 by an edge morphism of (3.13) and a homological spectral sequence for the resolution $0 \to \mathbb{C} \to \Omega^\bullet(U^0_\infty)$, respectively.

Since the composition of the left-hand-side vertical maps is $(3.11) \times (3.5)$, we need to show that the composition of the right-hand-side vertical maps is induced by (3.15). But this follows easily from the commutativity of the diagram

$$\begin{array}{ccccc}
H^0(E', C^\infty(U^0_\infty, \mathbb{C})) & \xrightarrow{(3.12)_*} & H^0(E', \Omega^d(U^0_\infty, \mathbb{C})) & \longrightarrow & H^d(E', \mathbb{C}) \\
\Big\downarrow {\scriptstyle \cap\eta} & & \Big\downarrow {\scriptstyle \cap\eta} & & \Big\downarrow {\scriptstyle \cap\eta} \\
H_d(E', C^\infty(U^0_\infty, \mathbb{C})) & \xrightarrow{(3.12)_*} & H_d(E', \Omega^d(U^0_\infty, \mathbb{C})) & \longrightarrow & H_0(E', \mathbb{C})
\end{array}$$

since for a d-form on the d-dimensional oriented manifold $M := \mathbb{R}^{d+1}_0/E' \cong U^0_\infty/E'$, integration over M corresponds to taking the cap product with the fundamental class η of M under the canonical isomorphism $H^d_{dR}(M) \cong H^d_{sing}(M) = H^d(E', \mathbb{C})$. $\square$

3.3 Integral cohomology classes

Definition 3.9. For $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ and a subring R of $\mathbb{C}$, we denote the image of

$$H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)) \to H_0(F^{*\prime}, \mathbb{C}) = \mathbb{C}, \quad x \mapsto \kappa \cap x$$

by $L_{\kappa,R}$. ("Module of periods of R")

Lemma 3.10. *Let $R \subseteq \overline{\mathbb{Q}}$ be a Dedekind ring.*
(a)For a subring $R' \supseteq R$ of $\mathbb{C}$, we have $L_{\kappa,R'} = R' L_{\kappa,R}$.
(b) If $\kappa \neq 0$, then $L_{\kappa,R} \neq 0$.

Proof. (cf. [Sp], lemma 4.15)
(a) We have $\mathcal{C}_c^0(S_1, R') = \mathcal{C}_c^0(S_1, R) \otimes R'$, and since R' is a flat R-module, we have $H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R')) = H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)) \otimes R'$.

(b) The pairing (3.1), and thus the cap-product (3.6), is non-degenerate for $M = R = \mathbb{C}$. Thus $L_{\kappa,\mathbb{C}} \neq 0$, and (a) implies $L_{\kappa,R} \neq 0$. □

Definition 3.11. A nonzero cohomology class $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ is called *integral* if κ lies in the image of $H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ for some Dedekind ring $R \subseteq \overline{\mathcal{O}}$. If, in addition, there exists a torsion-free R-submodule $M \subseteq H^d(F^{*\prime}, \mathcal{D}_f(S_1, R))$ of rank ≤ 1 (i.e. M can be embedded into R, by the classification of finitely generated R-modules) such that κ lies in the image of $M \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$, then κ is *integral of rank* ≤ 1.

Proposition 3.12. *Let $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$. The following conditions are equivalent:*

(i) κ is integral (resp. integral of rank ≤ 1).

(ii) There exists a Dedekind ring $R \subseteq \overline{\mathcal{O}}$ such that $L_{\kappa,R}$ is a finitely generated R-module (resp. a torsion-free R-module of rank ≤ 1).

(iii) There exists a Dedekind ring $R \subseteq \overline{\mathcal{O}}$, a finitely generated R-module M (resp. a torsion-free R-module of rank ≤ 1) and an R-linear map $f : M \to \mathbb{C}$ such that κ lies in the image of the induced map $f_* : H^d(F^{*\prime}, \mathcal{D}_f(S_1, M)) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$.

Proof. (cf. [Sp], prop. 4.17)
(i) ⇒ (ii): Let R be such that κ lies in the image of $H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$. Then $\kappa = \sum_{i=1}^n x_i \kappa_i$ with $x_i \in \mathbb{C}$, $\kappa_i \in \mathrm{Im}(H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)))$ (with $n \leq 1$ if κ has rank ≤ 1) and thus $L_{\kappa,R} \subseteq \sum_{i=1}^n x_i L_{\kappa_i,R} \subseteq \sum_{i=1}^n x_i R$.

(ii) ⇒ (iii): We have a commutative diagram

$$\begin{array}{ccc} H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) & \longrightarrow & \mathrm{Hom}_R(H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)), L_{\kappa,R}) \\ \downarrow & & \downarrow \\ H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})) & \longrightarrow & \mathrm{Hom}_R(H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)), \mathbb{C}) \end{array} \tag{3.17}$$

where the horizontal maps are given by the cap-product and the vertical ones are induced by the inclusion $L_{\kappa,R} \hookrightarrow \mathbb{C}$. By the universal coefficient theorem (using the isomorphism $\mathcal{D}_f(S_1, M) \cong \mathrm{Hom}_R(\mathcal{C}_c^0(S_1, R), M)$), the lower horizontal map is an isomorphism, and the kernel and cokernel of the upper horizontal map are R-torsion; since the map $\kappa \cap \cdot$ lies in $\mathrm{Hom}_R(H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)), L_{\kappa,R})$, some multiple $a \cdot \kappa$, $a \in R^*$, must have a preimage in $H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R}))$. Thus we can choose $M = L_{\kappa,R}$ and $f : L_{\kappa,R} \to \mathbb{C}, x \mapsto a^{-1}x$ in (iii).

(iii) $\Rightarrow$ (i): Since $f(M)$ is a torsion-free finitely generated module over a Dedekind ring, it can be embedded into a free module $R^n \hookrightarrow \mathbb{C}$ (with $n \leq 1$ if M has rank ≤ 1). Then f factorizes over $M \to f(M) \hookrightarrow R^n \hookrightarrow \mathbb{C}$, and thus f_* factorizes over $H^d(F^{*\prime}, \mathcal{D}_f(S_1, R^n))$. Thus, we can assume that $M = R^n$.

Now let $x_1, \ldots, x_n \in \mathbb{C}$ be the images of the standard basis of M under f. Then we have

$$\begin{aligned}\kappa \in \mathrm{Im}(f_*) &= \sum_{i=1}^{n} x_i \,\mathrm{Im}\left(H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))\right) \\ &\subseteq \mathrm{Im}\left(H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))\right).\end{aligned}$$

□

Corollary 3.13. *Let $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ be integral and $R \subseteq \overline{\mathcal{O}}$ be as in proposition 3.9. Then*
(a) μ_κ is a p-adic measure, and
(b) the map $H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}} \to \mathcal{H}^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ is injective and κ lies in its image.

Proof. (cf. [Sp], cor. 4.18.)
The image of $C^0(\mathcal{G}_p, \overline{\mathcal{O}}) \to \mathbb{C}$, $f \mapsto \int f \mu_\kappa = \kappa \cap \partial(f)$ is contained in $L_{\kappa,\overline{\mathcal{O}}}$ since $\partial(f) \in H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, \overline{\mathcal{O}}))$. Condition (iii) in the proposition implies that $L_{\kappa,\overline{\mathcal{O}}}$ is a finitely generated $\overline{\mathcal{O}}$-module, from which (a) follows.
(b): In the proof of (ii)$\Rightarrow$ (iii) above, the right-hand vertical map in (3.17) is injective, thus the left-hand map tensored with $\overline{\mathbb{Q}}$ also is (and κ lies in its image), since the horizontal maps are isomorphisms after tensoring with $\overline{\mathbb{Q}}$. □

Remark 3.14. Let κ be integral with Dedekind ring R as above. By (b) of the corollary, we can view κ as an element of $H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}}$. Put $V_\kappa := L_{\kappa,R} \otimes_R \mathbb{C}_p$; let $\overline{\kappa}$ be the image of κ under the composition

$$H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \overline{\mathbb{Q}} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \mathbb{C}_p \to H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V_\kappa)),$$

where the second map is induced by $\mathcal{D}_f(S_1, L_{\kappa,R}) \otimes_R \mathbb{C}_p \to \mathcal{D}_f^b(S_1, V_\kappa)$. By lemma 3.10 (a), $\overline{\kappa}$ does not depend on the choice of R.

Since μ_κ is a p-adic measure, $\mu_{\overline{\kappa}}$ allows integration of all continuous functions $f \in C(\mathcal{G}_p, \mathbb{C}_p)$, and by abuse of notation, we write $L_p(s, \kappa) := \int_{\mathcal{G}_p} \mathcal{N}(\gamma)^s \mu_\kappa(d\gamma) := L_p(s, \overline{\kappa})$ (cf. remark 3.6). So $L_p(s, \kappa)$ has values in the finite-dimensional $\mathbb{C}_p$-vector space V_κ.

4 p-adic L-functions of automorphic forms

We keep the notations from chapter 3; so F is again a number field with r real embeddings and s pairs of complex embeddings.

For an ideal $0 \neq \mathfrak{m} \subseteq \mathcal{O}_F$, we let $K_0(\mathfrak{m})_v \subseteq G(\mathcal{O}_{F_v})$ be the subgroup of matrices congruent to an upper triangular matrix modulo $\mathfrak{m}$, and we set $K_0(\mathfrak{m}) := \prod_{v\nmid\infty} K_0(\mathfrak{m})_v$, $K_0(\mathfrak{m})^S := \prod_{v\nmid\infty, v\notin S} K_0(\mathfrak{m})_v$ for a finite set of primes S. For each $\mathfrak{p}|p$, let $q_\mathfrak{p} = N(\mathfrak{p})$ denote the number of elements of the residue class field of $F_\mathfrak{p}$.

We denote by $|\cdot|_\mathbb{C}$ the square of the usual absolute value on $\mathbb{C}$, i.e. $|z|_\mathbb{C} = z\overline{z}$ for all $z \in \mathbb{C}$, and write $|\cdot|_\mathbb{R}$ for the usual absolute value on $\mathbb{R}$ in context.

Definition 4.1. Let $\mathfrak{A}_0(G, \underline{2}, \chi_Z)$ denote the set of all *cuspidal automorphic representations* $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_F)$ with central character χ_Z such that $\pi_v \cong \sigma(|\cdot|_{F_v}^{1/2}, |\cdot|_{F_v}^{-1/2})$ at all archimedian primes v. Here we follow the notation of [JL]; so $\sigma(|\cdot|_{F_v}^{1/2}, |\cdot|_{F_v}^{-1/2})$ is the discrete series of weight 2, $\mathcal{D}(2)$, if v is real, and is isomorphic to the principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1(z) = z^{1/2}\overline{z}^{\,-1/2}$, $\mu_2(z) = z^{-1/2}\overline{z}^{1/2}$ if v is complex (cf. section 4.5 below).

We will only consider automorphic representations that are *p-ordinary* , i.e $\pi_\mathfrak{p}$ is ordinary (in the sense of chapter 2) for every $\mathfrak{p}|p$.

Therefore, for each $\mathfrak{p}|p$ we fix two non-zero elements $\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2} \in \overline{\mathcal{O}} \subseteq \mathbb{C}$ such that $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}$ is an ordinary, unitary representation. By the classification of unitary representations (see e.g. [Ge], Thm. 4.27), a spherical representation $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if either χ_1, χ_2 are both unitary characters (i.e. $|\alpha_{\mathfrak{p},1}| = |\alpha_{\mathfrak{p},2}| = \sqrt{q_\mathfrak{p}}$)[iii], or $\chi_{1,2} = \chi_0|\cdot|^{\pm s}$ with χ_0 unitary and $-\frac{1}{2} < s < \frac{1}{2}$. A special representation $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if the central character $\chi_1\chi_2$ is unitary. In all three cases, we have thus $\max\{|\alpha_{\mathfrak{p},1}|, |\alpha_{\mathfrak{p},2}|\} \geq \sqrt{q_\mathfrak{p}}$. Without loss of generality, we will assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$.

As in chapter 2, we define $a_\mathfrak{p} := \alpha_{\mathfrak{p},1} + \alpha_{\mathfrak{p},2}$, $\nu_\mathfrak{p} := \alpha_{\mathfrak{p},1}\alpha_{\mathfrak{p},2}/q_\mathfrak{p}$.

Let $\underline{\alpha_i} := (\alpha_{\mathfrak{p},i}, \mathfrak{p}|p)$, for $i = 1, 2$. We denote by $\mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$ the subset of all $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z)$ such that $\pi_\mathfrak{p} = \pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}$ for all $\mathfrak{p}|p$.

Let $S_1 \subseteq S_p$ be the set of places such that $\pi_\mathfrak{p}$ is the Steinberg representation (i.e. $\alpha_{\mathfrak{p},1} = \nu_\mathfrak{p} = 1$, $\alpha_{\mathfrak{p},2} = q$).[iv]

For later use we note that $\pi^\infty = \otimes_{v\nmid\infty}\pi_v$ is known to be defined over a finite extension of $\mathbb{Q}$, the smallest such field being the *field of definition* of π (cf. [Sp]).

[iii] To avoid confusion: By $|\alpha_{\mathfrak{p},i}|$ we always mean the archimedian absolute value of $\alpha_{\mathfrak{p},i} \in \mathbb{C}$; whereas in the context of the p-adic characters χ_i, $|\cdot|$ always means the p-adic absolute value, unless otherwise noted.

[iv] Note that all $\mathfrak{p}|p$ with $\alpha_{\mathfrak{p},1} = \nu_\mathfrak{p} \in \overline{\mathcal{O}}^*$, i.e. $\alpha_{\mathfrak{p},2} = q$, already lie in S_1, since $|\alpha_{\mathfrak{p},2}| < q$ in the spherical case. $L_p(s, \pi)$ should have an exceptional zero for each $\mathfrak{p} \in S_1$, according to the exceptional zero conjecture.

4.1 Upper half-space

Let $\mathcal{H}_2 := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\} \cong \mathbb{R} \times \mathbb{R}_+^*$ be the complex upper half-plane, and let $\mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_+^*$ be the 3-dimensional upper half-space. Each $\mathcal{H}_m$ is a differentiable manifold of dimension i. If we write $x = (u,t) \in \mathcal{H}_m$ with $t \in \mathbb{R}_+^*$, u in $\mathbb{R}$ or $\mathbb{C}$, respectively, it has a Riemannian metric $ds^2 = \frac{dt^2 + du\, d\overline{u}}{t}$, which induces a hyperbolic geometry on $\mathcal{H}_m$, i.e. the geodesic lines on $\mathcal{H}_m$ are given by "vertical" lines $\{u\} \times \mathbb{R}_+^*$ and half-circles with center in the line or plane $t = 0$.

We have the decomposition $\mathrm{GL}_2(\mathbb{C}) = B'_{\mathbb{C}} \cdot Z(\mathbb{C}) \cdot K_{\mathbb{C}}$, where $B'_{\mathbb{C}}$ is the subgroup of matrices $\left(\begin{smallmatrix} \mathbb{R}_+^* & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$, Z is the center, and $K_{\mathbb{C}} = \mathrm{SU}(2)$ (cf. [By], Cor. 43); and analogously $\mathrm{GL}_2(\mathbb{R})^+ = B'_{\mathbb{R}} \cdot Z(\mathbb{R}) \cdot K_{\mathbb{R}}$ with $B'_{\mathbb{R}} = \{\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) | x \in \mathbb{R}, y \in \mathbb{R}_+^*\}$ and $K_{\mathbb{R}} = \mathrm{SO}(2)$.

We can identify $B'_{\mathbb{C}}$ with $\mathcal{H}_3$ via $\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \mapsto (z,t)$, and $B'_{\mathbb{R}}$ with $\mathcal{H}_2$ via $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto x + iy$. This gives us natural projections

$$\pi_{\mathbb{R}} : \mathrm{GL}_2(\mathbb{R})^+ \twoheadrightarrow \mathrm{GL}_2(\mathbb{R})^+/\mathbb{R}^* \,\mathrm{SO}(2) \cong \mathcal{H}_2$$

and

$$\pi_{\mathbb{C}} : \mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* \,\mathrm{SU}(2) \cong \mathcal{H}_3.$$

The corresponding left actions on cosets are invariant under the Riemannian metrics on $\mathcal{H}_m$, and can be given explicitly as follows:

$\mathrm{GL}_2(\mathbb{R})^+$ operates on $\mathcal{H}_2 \subseteq \mathbb{C}$ via Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := \frac{az+b}{cz+d},$$

and $\mathrm{GL}_2(\mathbb{C})$ operates on $\mathcal{H}_3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z,t) := \left(\frac{(az+b)(\overline{cz+d}) + a\overline{c}t^2}{|cz+d|^2 + |ct|^2}, \frac{|ad-bc|t}{|cz+d|^2 + |ct|^2} \right)$$

([By], (3.12)); specifically, we have

$$\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} (0,1) = (z,t) \quad \text{ for } (z,t) \in \mathcal{H}_3.$$

A differential form ω on $\mathcal{H}_m$ is called *left-invariant* if it is invariant under the pullback L_g^* of left multiplication $L_g : x \mapsto gx$ on $\mathcal{H}_m$, for all $g \in G$. Following [By], eqs. (4.20), (4.24), we choose the following basis of left invariant differential 1-forms on $\mathcal{H}_3$:

$$\beta_0 := -\frac{dz}{t}, \quad \beta_1 := \frac{dt}{t}, \quad \beta_2 := \frac{d\overline{z}}{t},$$

and on $\mathcal{H}_2$ (writing $z = x + iy \in \mathcal{H}_2$):

$$\beta_1 := \frac{dz}{y}, \quad \beta_2 := -\frac{d\overline{z}}{y}.$$

We note that a form $f_1\beta_1 + f_2\beta_2$ is harmonic on $\mathcal{H}_2$ if and only if f_1/y and f_2/y are holomorphic functions in z ([By], lemma 60).

Let $k \in \{\mathbb{R}, \mathbb{C}\}$. The Jacobian $J(g,(0,1))$ of left multiplication by g in $(0,1) \in \mathcal{H}_m$ with respect to the basis $(\beta_i)_i$ gives rise to a representation

$$\varrho = \varrho_k : Z(k) \cdot K_k \to \mathrm{SL}_m(\mathbb{C})$$

with $\varrho|_{Z(k)}$ trivial, which on K_k is explicitly given by

$$\varrho_{\mathbb{C}} \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\overline{v} & u\overline{u} - v\overline{v} & v\overline{u} \\ \overline{v}^2 & -2\overline{uv} & \overline{u}^2 \end{pmatrix},$$

resp.

$$\varrho_{\mathbb{R}} \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} = \begin{pmatrix} e^{2i\vartheta} & 0 \\ 0 & e^{-2i\vartheta} \end{pmatrix}$$

([By], (4.27), (4.21)). In the real case, we will only consider harmonic forms on $\mathcal{H}_2$ that are multiples of β_1, thus we sometimes identify $\varrho_{\mathbb{R}}$ with its restriction $\varrho_{\mathbb{R}}^{(1)}$ to the first basis vector β_1,

$$\varrho_{\mathbb{R}}^{(1)} : \mathrm{SO}(2) \to S^1 \subseteq \mathbb{C}^*, \quad \kappa_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mapsto e^{2i\vartheta}.$$

For each i, let ω_i be the left-invariant differential 1-form on $\mathrm{GL}_2(k)$ which coincides with the pullback $(\pi_{\mathbb{C}})^*\beta_i$ at the identity. Write $\underline{\omega}$ (resp. $\underline{\beta}$) for the column vector of the ω_i (resp. β_i). Then we have the following lemma from [By]:

Lemma 4.2. *For each i, the differential ω_i on G induces β_i on $\mathcal{H}_m$, by restriction to the subgroup $B'_k \cong \mathcal{H}_m$. For a function $\phi : G \to \mathbb{C}^m$, the form $\phi \cdot \underline{\omega}$ (with $\mathbb{C}^m$ considered as a row vector, so $\cdot$ is the scalar product of vectors) induces $f \cdot \underline{\beta}$, where $f : \mathcal{H}_m \to \mathbb{C}^m$ is given by*

$$f(z,t) := \phi\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right).$$

(See [By], Lemma 57.)

To consider the infinite primes of F all at once, we define

$$\mathcal{H}_\infty := \prod_{i=0}^{d} \mathcal{H}_{m_i} = \prod_{i=0}^{r-1} \mathcal{H}_2 \times \prod_{i=r}^{d} \mathcal{H}_3$$

(where $m_i = 2$ if σ_i is a real embedding, and $= 3$ if σ_i is complex), and let $\mathcal{H}_\infty^0 := \prod_{i=1}^{d} \mathcal{H}_{m_i}$ be the product with the zeroth factor removed.[v]

For each embedding σ_i, the elements of $\mathbb{P}^1(F)$ are cusps of $\mathcal{H}_{m_i}$: for a given complex embedding $F \hookrightarrow \mathbb{C}$, we can identify F with $F \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ and define the "extended upper half-space" as $\overline{\mathcal{H}_3} := \mathcal{H}_3 \cup F \cup \{\infty\} \subseteq \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$;

[v] The choice of the 0-th factor is for convenience; we could also choose any other infinite place, whether real or complex.

similarly for a given real embedding $F \hookrightarrow \mathbb{R}$, we get the extended upper half-plane $\overline{\mathcal{H}_2} := \mathcal{H}_2 \cup F \cup \{\infty\}$. A basis of neighbourhoods of the cusp ∞ is given by the sets $\{(u,t) \in \mathcal{H}_m | t > N\}$, $N \gg 0$, and of $x \in F$ by the open half-balls in $\mathcal{H}_m$ with center $(x, 0)$.

Let $G(F)^+ \subseteq G(F)$ denote the subgroup of matrices with totally positive determinant . It acts on $\mathcal{H}^0_\infty$ by composing the embedding

$$G(F)^+ \hookrightarrow \prod_{v|\infty, v\neq v_0} G(F_v)^+, \qquad g \mapsto (\sigma_1(g), \ldots, \sigma_d(g)),$$

with the actions of $G(\mathbb{C})^+ = G(\mathbb{C})$ on $\mathcal{H}_3$ and $G(\mathbb{R})^+$ on $\mathcal{H}_2$ as defined above, and on $\Omega^d_{\text{harm}}(\mathcal{H}^0_\infty)$ by the inverse of the corresponding pullback, $\gamma \cdot \underline{\omega} := (\gamma^{-1})^* \underline{\omega}$. Both are left actions.

Denote by $S_\mathbb{C}$ (resp. $S_\mathbb{R}$) the set of complex (resp. real) archimedean primes of F. For each complex v, we write the codomain of ϱ_{F_v} as

$$\varrho_{F_v} : Z(F_v) \cdot K_{F_v} \to \mathrm{SL}_3(\mathbb{C}) =: \mathrm{SL}(V_v),$$

for a three-dimensional $\mathbb{C}$-vector space V_v. We denote the harmonic forms on $\mathrm{GL}_2(F_v)$, $\mathcal{H}_{F_v}$ defined above by $\underline{\omega_v}$, $\underline{\beta_v}$ etc.

Let $V = \bigotimes_{v \in S_\mathbb{C}} V_v \cong (\mathbb{C}^3)^{\otimes s}$, $Z_\infty = \prod_{v|\infty} Z(F_v)$, $K_\infty = \prod_{v|\infty} K_{F_v}$. We can merge the representations ϱ_{F_v} for each $v|\infty$ into a representation

$$\varrho = \varrho_\infty := \bigotimes_{v \in S_\mathbb{C}} \varrho_\mathbb{C} \otimes \bigotimes_{v \in S_\mathbb{R}} \varrho^{(1)}_\mathbb{R} : Z_\infty \cdot K_\infty \to \mathrm{SL}(V),$$

and define V-valued vectors of differential forms $\underline{\omega} := \bigotimes_{v\in S_\mathbb{C}} \underline{\omega_v} \otimes \bigotimes_{v \in S_\mathbb{R}} \omega^1_v$, $\underline{\beta} := \bigotimes_{v \in S_\mathbb{C}} \underline{\beta_v} \otimes \bigotimes_{v\in S_\mathbb{R}} (\beta_v)_1$ on $\mathrm{GL}_2(F_\infty)$ and $\mathcal{H}_\infty$, respectively.

4.2 Automorphic forms

Let $\chi_Z : \mathbb{A}^*_F / F^* \to \mathbb{C}^*$ be a Hecke character that is trivial at the archimedean places. We also denote by χ_Z the corresponding character on $Z(\mathbb{A}_F)$ under the isomorphism $\mathbb{A}^*_F \to Z(\mathbb{A}_F)$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

Definition 4.3. An *automorphic cusp form of parallel weight* $\underline{2}$ *with central character* χ_Z is a map $\phi : G(\mathbb{A}_F) \to V$ such that

(i) $\phi(z\gamma g) = \chi_Z(z)\phi(g)$ for all $g \in G(\mathbb{A})$, $z \in Z(\mathbb{A})$, $\gamma \in G(F)$.

(ii) $\phi(g k_\infty) = \phi(g)\varrho(k_\infty)$ for all $k_\infty \in K_\infty$, $g \in G(\mathbb{A})$ (considering V as a row vector).

(iii) ϕ has "moderate growth" on $B'_\mathbb{A} := \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G(\mathbb{A})\}$, i.e. $\exists C, \lambda\ \forall A \in B'_\mathbb{A}$: $\|\phi(A)\| \leq C \cdot \sup(|y|^\lambda, |y|^{-\lambda})$ (for any fixed norm $\|\cdot\|$ on V);
and $\phi|_{G(\mathbb{A}_\infty)} \cdot \underline{\omega}$ is the pullback of a harmonic form $\omega_\phi = f_\phi \cdot \underline{\beta}$ on $\mathcal{H}_\infty$.

(iv) There exists a compact open subgroup $K' \subseteq G(\mathbb{A}^\infty)$ such that $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K'$.

(v) For all $g \in G(\mathbb{A}_F)$,

$$\int_{\mathbb{A}_F/F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \, dx = 0. \qquad \textit{(“Cuspidality”)}$$

We denote by $\mathcal{A}_0(G, \mathrm{harm}, \underline{2}, \chi_Z)$ the space of all such maps ϕ.

For each $g^\infty \in \mathbb{A}_F^\infty$, let $\omega_\phi(g^\infty)$ be the restriction of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ from $G(\mathbb{A}_F^\infty)$ to $\mathcal{H}_\infty$; it is a $(d+1)$-form on $\mathcal{H}_\infty$.

We want to integrate $\omega_\phi(g^\infty)$ between two cusps of the space $\mathcal{H}_{m_0}$. (We will identify each $x \in \mathbb{P}^1(F)$ with its corresponding cusp in $\overline{\mathcal{H}_{m_0}}$ in the following.) The geodesic between the cusps $x \in F$ and ∞ in $\overline{\mathcal{H}_{m_0}}$ is the line $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ and the integral of ω_ϕ along it is finite since ϕ is uniformly rapidly decreasing:

Theorem 4.4. *(Gelfand, Piatetski-Shapiro) An automorphic cusp form ϕ is rapidly decreasing modulo the center on a fundamental domain $\mathcal{F}$ of $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)$; i.e. there exists an integer r such that for all $N \in \mathbb{N}$ there exists a $C > 0$ such that*

$$\phi(zg) \leq C|z|^r \|g\|^{-N}$$

for all $z \in Z(\mathbb{A}_F)$, $g \in \mathcal{F} \cap \mathrm{SL}_2(\mathbb{A}_F)$. Here $\|g\| := \max\{|g_{i,j}|, |(g^{-1})_{i,j}|\}_{i,j \in \{1,2\}}$.

(See [CKM], Thm. 2.2; or [Kur78], (6) for quadratic imaginary F.)

In fact, the integral of $\omega_\phi(g^\infty)$ along $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ equals the integral of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ along a path $g_t \in \mathrm{GL}_2(F_{\infty_0})$, $t \in \mathbb{R}_+^*$, where we can choose

$$g_t = \frac{1}{\sqrt{t}} \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{t}} & \frac{x}{\sqrt{t}} \\ 0 & \sqrt{t} \end{pmatrix},$$

and thus have $\|g_t\| = \sqrt{t}$ for all $t \gg 0$, $\|g_t\| = C\frac{1}{\sqrt{t}}$ for $t \ll 1$, so the integral $\int_x^\infty \omega_\phi(g^\infty) \in \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty)$ is well-defined by the theorem.

For any two cusps $a, b \in \mathbb{P}^1(F)$, we now define

$$\int_a^b \omega_\phi(g^\infty) := \int_a^\infty \omega_\phi(g^\infty) - \int_b^\infty \omega_\phi(g^\infty) \in \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty).$$

Since ϕ is *uniformly* rapidly decreasing ($\|g_t\|$ does not depend on x, for $t \gg 0$), this integral along the path $(a, 0) \to (a, \infty) = (b, \infty) \to (b, 0)$ in $\overline{\mathcal{H}_{m_0}}$ is the same as the limit (for $t \to \infty$) of the integral along $(a, 0) \to (a, t) \to (b, t) \to (b, 0)$; and since ω_ϕ is harmonic (and thus integration is path-independent within $\mathcal{H}_{m_0}$) the latter is in fact independent of t, so equality holds for each $t > 0$, or along any path from $(a, 0)$ to $(b, 0)$ in $\mathcal{H}_{m_0}$. Thus we have

$$\int_a^b \omega_\phi(g^\infty) + \int_b^c \omega_\phi(g^\infty) = \int_a^c \omega_\phi(g^\infty)$$

for any three cusps $a, b, c \in \mathbb{P}^1(F)$. Let $\mathrm{Div}(\mathbb{P}^1(F))$ denote the free abelian group of divisors of $\mathbb{P}^1(F)$, and let $\mathcal{M} := \mathrm{Div}_0(\mathbb{P}^1(F))$ be the subgroup of divisors of degree 0.

We can extend the definition of the integral linearly to get a homomorphism

$$\mathcal{M} \to \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty), \quad m \mapsto \int_m \omega_\phi(g^\infty).$$

For $\gamma \in G(F)^+$, $g \in G(\mathbb{A}^\infty)$, $m \in \mathcal{M}$ and $x^0_\infty \in G(F_{S^0_\infty})$, we have

$$\begin{aligned}
\gamma^* \left(\int_{\gamma m} \omega_\phi(\gamma g) \right)(x^0_\infty) &= \int_{\gamma m} \omega_\phi(\gamma g)(\gamma x^0_\infty) \\
&= \int_{\gamma m} \phi(\gamma g, \gamma x^0_\infty, *) \cdot \omega \\
&= \int_{\gamma m} \phi(g, x^0_\infty, \gamma^{-1} *) \cdot \underline{\omega} \qquad \text{(by (i) of definition 4.3)} \\
&= \int_{m} \phi(g, x^0_\infty, *) \cdot \underline{\omega} \quad \text{(since } \underline{\omega} \text{ is } G(F_\infty)\text{-left invariant)} \\
&= \int_{m} \omega_\phi(g)(x^0_\infty),
\end{aligned}$$

i.e.

$$\gamma^* \left(\int_{\gamma m} \omega_\phi(\gamma g) \right) = \int_m \omega_\phi(g). \tag{4.1}$$

Now let $\mathfrak{m}$ be an ideal of F prime to p, let χ_Z be a Hecke character of conductor dividing $\mathfrak{m}$, and $\underline{\alpha_1}, \underline{\alpha_2}$ as above.

Definition 4.5. We define $S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2})$ to be the $\mathbb{C}$-vector space of all maps

$$\Phi : G(\mathbb{A}^p) \to \mathcal{B}^{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, V) = \mathrm{Hom}(\mathcal{B}_{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, \mathbb{C}), V)$$

such that:

(a) ϕ is "almost" $K_0(\mathfrak{m})$-invariant (in the notation of [Ge]), i.e. $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A}^p)$ and $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$, and $\phi(gk) = \chi_Z(a)\phi(g)$ for all $v | \mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^p)$.

(b) For each $\psi \in \mathcal{B}_{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, \mathbb{C})$, the map

$$\langle \Phi, \psi \rangle : G(\mathbb{A}) = G(F_p) \times G(\mathbb{A}^p) \to V, \ (g_p, g^p) \mapsto \Phi(g^p)(g_p \psi)$$

lies in $\mathcal{A}_0(G, \mathrm{harm}, \underline{2}, \chi_Z)$.

Note that (a) implies that ϕ is K'-invariant for some open subgroup $K' \subseteq K_0(\mathfrak{m})^p$ of finite index ([By]/[We]).

4.3 Cohomology of $\mathrm{GL}_2(F)$

Let M be a left $G(F)$-module and N an $R[H]$-module, for a ring R and a subgroup $H \subseteq G(F)$. Let $S \subseteq S_p$ be a set of primes of F dividing p; as above, let $\chi = \chi_Z$ be a Hecke character of conductor $\mathfrak{m}$ prime to p.

Definition 4.6. For a compact open subgroup $K \subseteq K_0(\mathfrak{m})^S \subseteq G(\mathbb{A}^{S,\infty})$, we denote by $\mathcal{A}_f(K, S, M; N)$ the R-module of all maps $\Phi : G(\mathbb{A}^{S,\infty}) \times M \to N$ such that

1. $\Phi(gk, m) = \Phi(g, m)$ for all $g \in G(\mathbb{A}^{S,\infty})$, $m \in M$, $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$;
2. $\Phi(gk) = \chi_Z(a)\Phi(g)$ for all $v|\mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^{S,\infty})$, $m \in M$.

We denote by $\mathcal{A}_f(S, M; N)$ the union of the $\mathcal{A}_f(K, S, M; N)$ over all compact open subgroups K.

$\mathcal{A}_f(S, M; N)$ is a left $G(\mathbb{A}^{S,\infty})$-module via $(\gamma \cdot \Phi)(g, m) := \Phi(\gamma^{-1}g, m)$ and has a left H-operation given by $(\gamma \cdot \Phi)(g, m) := \gamma\Phi(\gamma^{-1}g, \gamma^{-1}m)$, commuting with the $G(\mathbb{A}^{S,\infty})$-operation.

In contrast to our previous notation, we consider two subsets $S_1 \subseteq S_2 \subseteq S_p$ in this section. We put $(\underline{\alpha_1}, \underline{\alpha_2})_{S_1} := \{(\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}) | \mathfrak{p} \in S_1\}$, we set

$$\mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) = \mathcal{A}_f(S_2, M; \mathcal{B}^{(\underline{\alpha_1}, \underline{\alpha_2})_{S_1}}(F_{S_1}, N));$$

we write $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) := \mathcal{A}_f(K_0(\mathfrak{m}), (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N)$. If $S_1 = S_2$, we will usually drop S_2 from all these notations.

We have a natural identification of $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N)$ with the space of maps $G(\mathbb{A}^{S,\infty}) \times M \times \mathcal{B}_{(\underline{\alpha_1}, \underline{\alpha_2})_S}(F_S, R) \to N$ that are "almost" K-invariant.

Let $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_p$ be subsets. The pairing (2.11) induces a pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \times \mathcal{B}_{(\underline{\alpha_1}, \underline{\alpha_2})_{S_0}}(F_{S_0}, R) \to \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N), \tag{4.2}$$

which, when restricting to K-invariant elements, induces an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \cong \mathcal{B}^{(\underline{\alpha_1}, \underline{\alpha_2})_{S_1 - S_0}}(F_{S_1 - S_0}, \mathcal{A}_f(\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N). \tag{4.3}$$

Putting $S_0 := S_1 - \{\mathfrak{p}\}$ for a prime $\mathfrak{p} \in S_1$, we specifically get an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \cong \mathcal{B}^{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}(F_{\mathfrak{p}}, \mathcal{A}_f(\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N).$$

Lemmas 2.11 and 2.12 now immediately imply the following:

Lemma 4.7. *Let $S \subseteq S_p$, $\mathfrak{p} \in S$, $S_0 := S - \{\mathfrak{p}\}$. Let $K \subseteq G(\mathbb{A}^{S,\infty})$ be a compact open subgroup.*
(a) If $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is spherical, we have exact sequences

$$0 \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N) \to Z \xrightarrow{\mathcal{R} - \nu_{\mathfrak{p}}} Z \to 0$$

and

$$0 \to Z \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N) \xrightarrow{T - a_{\mathfrak{p}}} \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N) \to 0$$

for a $G(\mathbb{A}^{S_0,\infty})$*-module* Z *and a compact open subgroup* $K_0 = K \times K_{\mathfrak{p}}$ *of* $G(\mathbb{A}^{S_0,\infty})$*.*

(b) If $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}$ *is special (with central character* $\chi_{\mathfrak{p}}$*) , we have exact sequences*

$$0 \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N) \to Z' \to Z \to 0$$

and

$$\begin{aligned} 0 \to &Z \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to 0, \\ 0 \to &Z' \to \mathcal{A}_f(K_0', (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to \mathcal{A}_f(K_0', (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to 0, \end{aligned}$$

with $Z := \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S, M; N(\chi_{\mathfrak{p}}))$ *and* $Z' := \mathcal{A}_f(K_0', (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S, M; N(\chi_{\mathfrak{p}}))$*, where* $K_0 = K \times K_{\mathfrak{p}}$ *and* $K_0' = K \times K_{\mathfrak{p}}'$ *are compact open subgroups of* $G(\mathbb{A}^{S_0,\infty})$*.*

Proposition 4.8. *Let* $S \subseteq S_p$ *and let* K *be a compact open subgroup of* $G(\mathbb{A}^{S,\infty})$*.*

(a) For each flat R*-module* N *(with trivial* $G(F)$*-action), the canonical map*

$$H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)) \otimes_R N \to H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; N))$$

is an isomorphism for each $q \geq 0$*.*

(b) If R *is finitely generated as a* $\mathbb{Z}$*-module, then* $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)$ *is finitely generated over* R*.*

Proof. (cf. [Sp], Prop. 5.6)
(a) The exact sequence of abelian groups $0 \to \mathcal{M} \to \mathrm{Div}(\mathbb{P}^1(F)) \cong \mathrm{Ind}_{B(F)}^{G(F)} \mathbb{Z} \to \mathbb{Z} \to 0$ induces a short exact sequence of $G(\mathbb{A}^{S,\infty})$-modules

$$\begin{aligned} 0 \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathbb{Z}; N) \to \mathrm{Coind}_{B(F)^+}^{G(F)^+} \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathbb{Z}; N) \\ \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; N) \to 0. \end{aligned} \tag{4.4}$$

Using the five-lemma on the associated diagram of long exact cohomology sequences $H^q(\cdot, R) \otimes_R N$ (which is exact due to flatness) and $H^q(\cdot, N)$, it is enough to show that (4.4) holds for $\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathbb{Z}; \cdot)$ and $\mathrm{Coind}_{B(F)^+}^{G(F)^+} \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathbb{Z}; \cdot)$ instead of $\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; \cdot)$. By lemma 4.7, it is furthermore enough to consider the case $S = \varnothing$. Since $\mathcal{A}_f(K, \mathbb{Z}; N) \cong \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N$, we thus have to show that

$$\begin{aligned} H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} R) \otimes_R N \to H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N), \\ H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} R) \otimes_R N \to H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N) \end{aligned}$$

are isomorphisms for all $q \geq 0$ and all flat R-modules N.

Since every flat module is the direct limit of free modules of finite rank, it suffices to show that $N \mapsto H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N)$ and $N \mapsto H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N)$ commute with direct limits.

For $g \in G(\mathbb{A}^\infty)$, put $\Gamma_g := G(F)^+ \cap gKg^{-1}$, By the strong approximation theorem, $G(F)^+ \backslash G(\mathbb{A}^\infty)/K$ is finite. Choosing a system of representatives $g_1, \ldots, g_n$, we have

$$H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N) = \bigoplus_{i=1}^{n} H^q(\Gamma_{g_i}, N).$$

Since the groups Γ_g are arithmetic, they are of type (VFL), and thus the functors $N \mapsto H^q(\Gamma_g, N)$ commute with direct limits by [Se2], remarque on p. 101.

Similarly, the Iwasawa decomposition $G(\mathbb{A}^\infty) = B(\mathbb{A}^\infty) \prod_{v \nmid \infty} G(\mathcal{O}_v)$ implies that $B(F)^+ \backslash G(\mathbb{A}^\infty)/K$ is finite. Therefore, the same arguments show that $N \mapsto H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbb{A}^\infty)} N)$ commutes with direct limits.

(b) This follows along the same line of reasoning as (a), since $H^q(\Gamma_g, R)$ is finitely generated over $\mathbb{Z}$ by [Se2], remarque on p. 101. $\square$

With the notation as above, we define

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; R)) := \varinjlim H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; R))$$

where the limit runs over all compact open subgroups $K \subseteq G(\mathbb{A}^{S,\infty})$; and similarly define $H^q_*(B(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)$. The proposition immediately implies

Corollary 4.9. *Let $R \to R'$ be a flat ring homomorphism. Then the canonical map*

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)) \otimes_R R' \to H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R')$$

is an isomorphism, for all $q \geq 0$.

If $R = k$ is a field of characteristic zero, $H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; R)$ is a smooth $G(\mathbb{A}^{S,\infty})$-module, and we have

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)^K = H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; k).$$

We identify $G(F)/G(F)^+$ with the group $\Sigma = \{\pm 1\}^r$ via the isomorphism

$$G(F)/G(F^+) \xrightarrow{\det} F^*/F^*_+ \cong \Sigma$$

(with all groups being trivial for $r = 0$). Then Σ acts on $H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)$ and $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)$ by conjugation.
For $\pi \in \mathfrak{A}_0(G, \underline{2})$ and $\underline{\mu} \in \Sigma$, we write $H^q_*(G(F)^+, \cdot)_{\pi,\underline{\mu}} := \mathrm{Hom}_{G(\mathbb{A}^{S,\infty})}(\pi^S, H^q_*(G(F)^+, \cdot))_{\underline{\mu}}$.

Now we can show that π occurs with multiplicity 2^r in $H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; k)$:

Proposition 4.10. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$, $S \subseteq S_p$. Let k be a field which contains the field of definition of π. Then for every $\underline{\mu} \in \Sigma$, we have*

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; k)_{\pi,\underline{\mu}} = \begin{cases} k, & \text{if } q = d; \\ 0, & \text{if } q \in \{0, \ldots, d-1\} \end{cases} \tag{4.5}$$

Proof. (cf. [Sp], prop. 5.8)
First, assume $S = \varnothing$. The sequence (4.4) induces a cohomology sequence

$$\ldots \to H^q_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k)) \to H^q_*(B(F)^+, \mathcal{A}_f(\mathbb{Z}, k)) \to H^q_*(G(F)^+, \mathcal{A}_f(\mathcal{M}, k)) \\ \to H^{q+1}_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k)) \to \ldots$$

Harder ([Ha]) has determined the action of $G(\mathbb{A}^\infty)$ on $H^q_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k))$ and $H^q_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k))$: For $q < d$, $H^q_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k))$ is a direct sum of one-dimensional representations; for $q = d$ there is a $G(\mathbb{A}^\infty)$-stable decomposition

$$H^{d+1}_*(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k)) = H^{d+1}_{\text{cusp}} \oplus H^{d+1}_{\text{res}} \oplus H^{d+1}_{\text{Eis}},$$

with the last two summands again being direct sums of one-dimensional representations, and

$$H^{d+1}_{\text{cusp}}(G(F)^+, \mathcal{A}_f(\mathbb{Z}, k))_{\pi,\underline{\mu}} \cong k$$

([Ha], 3.6.2.2); $H^q_*(B(F)^+, \mathcal{A}_f(\mathbb{Z}, k))$ always decomposes into one-dimensional $G(\mathbb{A}^\infty)$-representations. Since π^S does not map to one-dimensional representations, this proves the claim for $S = \varnothing$.

Now for $S = S_0 \cup \{\mathfrak{p}\}$ and $\pi_\mathfrak{p}$ spherical, lemma 4.7(a) and the statement for S_0 give an isomorphism

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, \mathcal{M}; k))_{\pi,\underline{\mu}} \cong H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; k))_{\pi,\underline{\mu}}$$

since the Hecke operators $T_\mathfrak{p}$, $\mathcal{R}_\mathfrak{p}$ act on the left-hand side by multiplication with $a_\mathfrak{p}$ or $\nu_\mathfrak{p}$, respectively. If $\pi_\mathfrak{p}$ is special, we can similarly deduce the statement for S from that for S_0, using the first exact sequence of lemma 4.7(b) (cf. [Sp]), since the results of [Ha] also hold when twisting k by a (central) character. □

4.4 Eichler-Shimura map

Given a subgroup $K_0(\mathfrak{m})^p \subseteq G(\mathbb{A}^{p,\infty})$ as above, there is a map

$$I_0 : S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \to H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}; \Omega^d_{\text{harm}}(\mathcal{H}^0_\infty)))$$

given by

$$I_0(\Phi) : (\psi, (g, m)) \mapsto \int_m \omega_{\langle\Phi,\psi\rangle}(1_p, g),$$

for $\psi \in \mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C}), g \in G(\mathbb{A}^{p,\infty}), m \in \mathcal{M}$, where 1_p denotes the unity element in $G(F_p)$.

This is well-defined since both sides are "almost" $K_0(\mathfrak{m})$-invariant, and the $G(F)^+$-invariance of $I_0(\Phi)$ follows from the similar invariance for differential forms, and the definition of the $G(F)^+$-operations on $\mathcal{A}_f(M, N)$, $\mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F_p, N)$ and $\Omega^d_{\text{harm}}(\mathcal{H}^0_\infty)$: For each $\psi \in \mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C}), g \in G(\mathbb{A}^{p,\infty}), m \in \mathcal{M}$, we have

$$
\begin{aligned}
(\gamma I_0(\Phi))(\psi,(g,m)) &= \gamma I_0(\Phi)(\gamma^{-1}\psi,(\gamma^{-1}g,\gamma^{-1}m)) \\
&= \gamma\cdot\int_{\gamma^{-1}m}\omega_{\langle\Phi,\gamma^{-1}\psi\rangle}(1_p,\gamma^{-1}g) \\
&= (\gamma^{-1})^*\int_{\gamma^{-1}m}\omega_{\langle\Phi,\gamma^{-1}\psi\rangle}(1_p,\gamma^{-1}g) \\
&= \int_{m}\omega_{\langle\Phi,\gamma^{-1}\psi\rangle}(\gamma 1_p,g) \qquad\qquad \text{(by (4.1))} \\
&= I_0(\Phi)(\psi,(g,m)).
\end{aligned}
$$

We have a complex $\mathcal{A}_f(m,\underline{\alpha_1},\underline{\alpha_2},\mathcal{M};\mathbb{C}) \to C^\bullet := \mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M};\Omega^\bullet_{\text{harm}}(\mathcal{H}^0_\infty))$. Therefore we get a map

$$
S_2(G,\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2}) \to H^d(G(F)^+,\mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M};\mathbb{C})) \tag{4.6}
$$

by composing I_0 with the edge morphism $H^0(G(F)^+,C^d) \to H^d(G(F)^+,\mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M};\mathbb{C}))$ of the spectral sequence

$$
H^q(G(F)^+,C^p) \implies H^{p+q}(G(F)^+,C^\bullet).
$$

Using the map $\delta^{\underline{\alpha_1},\underline{\alpha_2}} : \mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F,V) \to \mathrm{Dist}(F_p^*,V)$ from section 2.7, we next define a map

$$
\Delta_V^{\underline{\alpha_1},\underline{\alpha_2}} : S_2(G,\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2}) \to \mathcal{D}(S_1,V) \tag{4.7}
$$

by

$$
\Delta_V^{\underline{\alpha_1},\underline{\alpha_2}}(\Phi)(U,x^p) = \delta^{\underline{\alpha_1},\underline{\alpha_2}}\left(\Phi\begin{pmatrix} x^p & 0 \\ 0 & 1\end{pmatrix}\right)(U)
$$

for $U \in \mathfrak{Co}(F_{S_1}\times F_{S_2}), x^p \in \mathbb{I}^p$, and we denote by $\Delta^{\underline{\alpha_1},\underline{\alpha_2}} : S_2(G,\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2}) \to \mathcal{D}(S_1,\mathbb{C})$ its (1,...,1)th coordinate function (i.e. corresponding to the harmonic forms $\bigotimes_{v|\infty}(\omega_v)_1$, $\bigotimes_{v|\infty}(\beta_v)_1$ in section 4.1):

$$
\Delta^{\underline{\alpha_1},\underline{\alpha_2}}(\Phi)(U,x^p) = \delta^{\underline{\alpha_1},\underline{\alpha_2}}\left(\Phi\begin{pmatrix} x^p & 0 \\ 0 & 1\end{pmatrix}\right)_{(1,\ldots,1)}(U).
$$

Since for each complex prime v, $S^1 \cong \mathrm{SU}(2)\cap T(\mathbb{C})$ operates via ϱ_v on Φ, $\Delta^{\underline{\alpha_1},\underline{\alpha_2}}$ is easily seen to be S^1-invariant, i.e. it lies in $\mathcal{D}'(S_1,\mathbb{C})$.

We also have a natural (i.e. commuting with the complex maps of each $C^\bullet$) family of maps

$$
\mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M},\Omega^i_{\text{harm}}(\mathcal{H}^0_\infty)) \to \mathcal{D}_f(S_1,\Omega^i(U^0_\infty,\mathbb{C})) \tag{4.8}
$$

for all $i \geq 0$, and

$$
\mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M},\mathbb{C}) \to \mathcal{D}_f(S_1,\mathbb{C}) \tag{4.9}
$$

(the $i=-1$-th term in the complexes), by mapping $\Phi \in \mathcal{A}_f(\mathfrak{m},\underline{\alpha_1},\underline{\alpha_2},\mathcal{M},\cdot)$ first to

$$
(U,x^{p,\infty}) \mapsto \Phi\left(\begin{pmatrix} x^{p,\infty} & 0 \\ 0 & 1\end{pmatrix},\infty-0\right)(\delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U)) \in \Omega^i_{\text{harm}}(\mathcal{H}^0_\infty) \text{ resp. } \in \mathbb{C},
$$

and then for $i \geq 0$ restricting the differential forms to $\Omega^i(U^0_\infty)$ via

$$U^0_\infty = \prod_{v \in S^0_\infty} \mathbb{R}^*_+ \hookrightarrow \prod_{v \in S^0_\infty} \mathcal{H}_v = \mathcal{H}^0_\infty.$$

One easily checks that (4.8) and (4.9) are compatible with the homomorphism of "acting groups" $F^{*\prime} \hookrightarrow G(F)^+, x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so we get induced maps in cohomology

$$H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty))) \to H^0(\mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) \tag{4.10}$$

and

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C})) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})), \tag{4.11}$$

which are linked by edge morphisms of the respective spectral sequences to give a commutative diagram (given in the proof below).

Proposition 4.11. *We have a commutative diagram:*

$$\begin{array}{ccc} S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) & \xrightarrow{(4.6)} & H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C})) \\ \downarrow \Delta^{\underline{\alpha_1}, \underline{\alpha_2}} & & \downarrow (4.11) \\ \mathcal{D}'(\mathcal{G}_m, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_\phi} & H^d\big(F^{*\prime}, \mathcal{D}_f(\mathbb{C})\big) \end{array}$$

Proof. The given diagram factorizes as

$$\begin{array}{ccccc} S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) & \xrightarrow{I_0} & H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty))) & \longrightarrow & H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C})) \\ \downarrow \Delta^{\underline{\alpha_1}, \underline{\alpha_2}} & & \downarrow (4.10) & & \downarrow (4.11) \\ \mathcal{D}'(\mathcal{G}_m, \mathbb{C}) & \longrightarrow & H^0(\mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) & \longrightarrow & H^d\big(F^{*\prime}, \mathcal{D}_f(\mathbb{C})\big) \end{array}$$

The right-hand square is the naturally commutative square mentioned above; the commutativity of the left-hand square can be checked by hand:

Let $\Phi \in S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2})$. Then $I_0(\Phi)$ is the map $(\psi, (g, m)) \mapsto \int_m \omega_{\langle \Phi, \psi \rangle}(1_p, g)$, which is mapped under (4.10) to

$$\begin{aligned} (U, x^{p,\infty}) \ &\mapsto \ \int_0^\infty \omega_{\langle \Phi, \delta_{\underline{\alpha_1}, \underline{\alpha_2}}(1_U) \rangle} \left(1_p, \begin{pmatrix} x^{p,\infty} & 0 \\ 0 & 1 \end{pmatrix}\right)\Big|_{U^0_\infty} \\ &= \int_0^\infty \Phi_{(1,\ldots,1)} \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} (\delta_{\underline{\alpha_1}, \underline{\alpha_2}}(1_U)) \frac{dt_0}{t_0} \frac{dt_1}{t_1} \ldots \frac{dt_d}{t_d}; \end{aligned}$$

along the other path, Φ is mapped under $\Delta^{\underline{\alpha_1}, \underline{\alpha_2}}$ to the map

$$(U, x^p) \mapsto \delta^{\underline{\alpha_1}, \underline{\alpha_2}} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix}\right)_{(1,\ldots,1)} (U) = \Phi_{(1,\ldots,1)} \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} (\delta_{\underline{\alpha_1}, \underline{\alpha_2}}(1_U))$$

and then also to

$$(U, x^{p,\infty}) \mapsto \int_0^\infty \Phi_{(1,\ldots,1)} \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} (\delta_{\underline{\alpha_1}, \underline{\alpha_2}}(1_U))\, d^\times r_0\, d^\times r_1 \ldots d^\times r_d$$

(with $x^p = (x^{p,\infty}, r_0, r_1, \ldots, r_d)$). □

4.5 Whittaker model

We now consider an automorphic representation $\pi = \otimes_\nu \pi_\nu \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$. Denote by $\mathfrak{c}(\pi) := \prod_{v \text{ finite}} \mathfrak{c}(\pi_v)$ the conductor of π.

Let $\chi : \mathbb{I}^\infty \to \mathbb{C}^*$ be a unitary character of the finite ideles; for each finite place v, set $\chi_v = \chi|_{F_v^*}$. For each prime v of F, let $\mathcal{W}_v$ denote the Whittaker model of π_v. For each finite and each real prime, we choose $W_v \in \mathcal{W}_v$ such that the local L-factor equals the local zeta function at $g = 1$, i.e. such that

$$L(s, \pi_v \otimes \chi_v) = \int_{F_v^*} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \chi_v(x)|x|^{s-\frac{1}{2}}\, d^\times x \tag{4.12}$$

for any unramified quasi-character $\chi_v : F_v^* \to \mathbb{C}^*$ and $\mathrm{Re}(s) \gg 0$.

This is possible by [Ge], Thm. 6.12 (ii); and by loc.cit., Prop. 6.17, W_v can be chosen such that SO(2) operates on W_v via ϱ_v for real archimedean v, and is "almost" $K_0(\mathfrak{c}(\pi_v))$-invariant for finite v.

For complex primes v of F, we can also choose a W_v satisfying (4.12) and which behaves well with respect to the SU(2)-action ϱ_v, as follows:

By [Kur77], there exists a three-dimensional function

$$\underline{W_v} = (W_v^0, W_v^1, W_v^2) : G(F_v) \to \mathbb{C}^3$$

such that $W_v^i \in \mathcal{W}_v$ for all i, and such that SU(2) operates by the right via ϱ_v on $\underline{W_v}$; i.e. for all $g \in G(F_v)$ and $h = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in \mathrm{SU}(2)$, we have

$$\underline{W_v}(gh) = \underline{W_v}(g) M_3(h),$$

where

$$M_3(h) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\overline{v} & u\overline{u} - v\overline{v} & v\overline{u} \\ \overline{v}^2 & -2\overline{uv} & \overline{u}^2 \end{pmatrix}.$$

Note that W_v^1 is thus invariant under right multiplication by a diagonal matrix $\begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}$ with $u \in S^1 \subseteq \mathbb{C}$. Since π_v has trivial central character for archimedean v by our assumption, a function in $\mathcal{W}_v$ is also invariant under $Z(F_v)$. Thus we have

$$W_v^1 \left(g \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) = W_v^1(g) \quad \text{for all } g \in G(F_v),\ u \in S^1.$$

W_v^1 can be described explicitly in terms of a certain Bessel function, as follows. The modified Bessel differential equation of order $\alpha \in \mathbb{C}$ is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \alpha^2)y = 0.$$

Its solution space (on $\{\mathrm{Re}\, z > 0\}$) is two-dimensional; we are only interested in the second standard solution K_v, which is characterised by the asymptotics

$$K_v(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

(as defined in [We]; see also [DLMF], 10.25).[vi]

[vi] Note that [Kur77] uses a slightly different definition of the K_v, which is $\frac{2}{\pi}$ times our K_v.

By [Kur77], we have $W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{\pi} x^2 K_0(4\pi x)$.
(W_v^0 and W_v^2 can also be described in term of Bessel functions; they are linearly dependent and scalar multiples of $x^2 K_1(4\pi x)$.)

By [JL], Ch. 1, Thm. 6.2(vi), $\sigma(|\cdot|_{\mathbb{C}}^{1/2}, |\cdot|_{\mathbb{C}}^{-1/2}) \cong \pi(\mu_1, \mu_2)$ with

$$\mu_1(z) = z^{1/2}\overline{z}^{\,-1/2} = |z|_{\mathbb{C}}^{-1/2} z, \qquad \mu_2(z) = z^{-1/2}\overline{z}^{1/2} = |z|_{\mathbb{C}}^{-1/2}\overline{z};$$

and the L-series of the representation is the product of the L-factors of these two characters:

$$\begin{aligned} L_v(s, \pi_v) = L(s, \mu_1) L(s, \mu_2) &= 2\,(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\tfrac{1}{2}) \cdot 2\,(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\tfrac{1}{2}) \\ &= 4\,(2\pi)^{-(2s+1)}\Gamma(s+\tfrac{1}{2})^2. \end{aligned}$$

On the other hand, letting $d^\times x = \frac{dx}{|x|_{\mathbb{C}}} = \frac{dr}{r} d\vartheta$ (for $x = re^{i\vartheta}$), we have for $\mathrm{Re}(s) > -\frac{1}{2}$:

$$\begin{aligned} \int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, d^\times x &= \int_{S^1}\int_{\mathbb{R}_+} W_v^1 \begin{pmatrix} re^{i\vartheta} & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, \frac{dr}{r}\, d\vartheta \\ &= 4\int_0^\infty x^2 K_0(4\pi x) x^{2s-1}\, \frac{dx}{x} \\ &\qquad \text{(invariance under } \mathrm{SU}(2)\cdot Z(F_v) \text{ gives a constant integral w.r.t. } \vartheta) \\ &= 4\,(4\pi)^{-2s+1}\int_0^\infty K_0(x) x^{2s}\, dx \\ &= 4\,(4\pi)^{-2s+1}\, 2^{2s-1}\, \Gamma(s+\tfrac{1}{2})^2 \qquad \text{(by [DLMF] 10.43.19)} \\ &= 4\,(2\pi)^{-2s+1}\, \Gamma(s+\tfrac{1}{2})^2 \end{aligned}$$

Thus we have

$$\int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, d^\times x = (2\pi)^2\, L_v(s, \pi_v)$$

for all $\mathrm{Re}(s) > -\frac{1}{2}$.

We set $W_v := (2\pi)^{-2}\, W_v^1$; thus (4.12) holds also for complex primes.

Now that we have defined W_v for all primes v, put $W^p(g) := \prod_{v \nmid p} W_v(g_v)$ for all $g = (g_v)_v \in G(\mathbb{A}^p)$.
We will also need the vector-valued function $\underline{W^p} : G(\mathbb{A}_F) \to V$ given by

$$\underline{W^p}(g) := \prod_{v \nmid p \text{ finite or } v \text{ real}} W_v(g_v) \cdot \bigotimes_{v \text{ complex}} (2\pi)^{-2} \underline{W_v}(g_v).$$

4.6 p-adic measures of automorphic forms

Now return to our $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$. We fix an additive character $\psi : \mathbb{A} \to \mathbb{C}^*$ which is trivial on F, and let ψ_v denote the restriction of ψ to $F_v \hookrightarrow \mathbb{A}$, for all primes v. We further require that $\ker(\psi_\mathfrak{p}) = \mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p}|p$, so that we can apply the results of chapter 2.

As in chapter 2, let $\mu_{\pi_\mathfrak{p}} := \mu_{\alpha_{\mathfrak{p},1}/\nu_\mathfrak{p}} = \mu_{q_\mathfrak{p}/\alpha_{\mathfrak{p},2}}$ denote the distribution $\chi_{q_\mathfrak{p}/\alpha_{\mathfrak{p},2}}(x)\psi_\mathfrak{p}(x)dx$ on $F_\mathfrak{p}$, and let $\mu_{\pi_p} := \prod_{\mathfrak{p}|p} \mu_{\pi_\mathfrak{p}}$ be the product distribution on $F_p := \prod_{\mathfrak{p}|p} F_\mathfrak{p}$.

Define $\phi = \phi_\pi : \mathfrak{Co}(F_{S_1} \times F^*_{S_2}) \times \mathbb{I}^p \to \mathbb{C}$ by

$$\phi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}.$$

By proposition 2.15(a), we have for each $U \in \mathfrak{Co}(F_{S_1} \times F^*_{S_2})$:

$$\begin{aligned} \phi(x_p U, x^p) &= \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta x_p U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\zeta \in F^*} W_U \begin{pmatrix} \zeta x_p & 0 \\ 0 & 1 \end{pmatrix} W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\zeta \in F^*} W \begin{pmatrix} \zeta x & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $W(g) := W_U(g_p)W^p(g^p)$ lies in the global Whittaker model $\mathcal{W} = \mathcal{W}(\pi)$ for all $g = (g_p, g^p) \in G(\mathbb{A})$, putting $W_U := W_{1_U}$; so ϕ is well-defined and lies in $\mathcal{D}(S_1, \mathbb{C})$ (since W is smooth and rapidly decreasing; distribution property, F^*- and $U^{p,\infty}$-invariance being clear by the definitions of ϕ and W^p).

Let $\mu_\pi := \mu_{\phi_\pi}$ be the distribution on $\mathcal{G}_p$ corresponding to ϕ_π, as defined in (3.9), and let $\kappa_\pi := \kappa_{\phi_\pi} \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ be the cohomology class defined by (3.10) and (3.11).

Theorem 4.12. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$; we assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$.*[vii]

(a) Let $\chi : \mathcal{G}_p \to \mathbb{C}^*$ be a character of finite order with conductor $\mathfrak{f}(\chi)$. Then we have the interpolation property

$$\int_{\mathcal{G}_p} \chi(\gamma)\mu_\pi(d\gamma) = \tau(\chi) \prod_{\mathfrak{p} \in S_p} e(\pi_\mathfrak{p}, \chi_\mathfrak{p}) \cdot L(\tfrac{1}{2}, \pi \otimes \chi),$$

where

$$e(\pi_\mathfrak{p}, \chi_\mathfrak{p}) = \begin{cases} \dfrac{(1 - \alpha_{\mathfrak{p},1} x_\mathfrak{p} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p}^{-1} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p} q_\mathfrak{p}^{-1})}{(1 - x_\mathfrak{p}\alpha_{\mathfrak{p},2}^{-1})}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0 \text{ and } \pi \text{ spherical}, \\ \dfrac{(1 - \alpha_{\mathfrak{p},1} x_\mathfrak{p} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p}^{-1} q_\mathfrak{p}^{-1})}{(1 - x_\mathfrak{p}\alpha_{\mathfrak{p},2}^{-1})}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0 \text{ and } \pi \text{ special}, \\ (\alpha_{\mathfrak{p},2}/q_\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi))}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) > 0 \end{cases}$$

[vii] So we have $\chi_{\mathfrak{p},1} = |\cdot|\chi_{\mathfrak{p},2}$ for all special $\pi_\mathfrak{p}$.

and $x_{\mathfrak{p}} := \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})$.

(b) Let $U_p := \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}$, *put* $\phi_0 := (\phi_\pi)_{U_p}$. *Then*

$$\int_{\mathbb{I}/F^*} \phi_0(x) d^\times x = \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, 1) \cdot L(\tfrac{1}{2}, \pi).$$

(c) κ_π *is integral (cf. definition 3.11). For* $\underline{\mu} \in \Sigma$, *let* $\kappa_{\pi,\underline{\mu}}$ *be the projection of* κ_π *to* $H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))_{\pi,\underline{\mu}}$. *Then* $\kappa_{\pi,\underline{\mu}}$ *is integral of rank 1.*

Proof. (a) We consider χ as a character on $\mathbb{I}_F/F^*$ (which is unitary and trivial on $\mathbb{I}_\infty$), and choose a subgroup $V \subseteq U_p$ such that $\chi_p|_V = 1$ (where $\chi_p := \chi|_{F_p}$) and V is a product of subgroups $V_{\mathfrak{p}} \subseteq U_{\mathfrak{p}}$.

Let $W_V \in \mathcal{W}_p$ be the product of the $W_{V_{\mathfrak{p}}}$, as defined in prop. 2.15, set $W(g) := W^p(g^p)W_V(g_p) \in \mathcal{W}$, and let

$$\phi_V(x) := \phi(x_p V, x^p) = \sum_{\zeta \in F^*} W \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}.$$

Since π is unitary, we have $|\alpha_{\mathfrak{p},2}| \geq \sqrt{q_{\mathfrak{p}}} > 1 = |\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})|$ for all $\mathfrak{p}$, thus $e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)$ is always non-singular, and we will be able to apply proposition 2.6 locally below.

We want to show that the equality

$$[U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x)|x|^s \phi_V(x) d^\times x = N(\mathfrak{f}(\chi))^s \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \cdot L(s + \tfrac{1}{2}, \pi \otimes \chi)$$

holds for $s = 0$. Since both the left-hand side and $L(s + \frac{1}{2}, \pi \otimes \chi)$ are holomorphic in s (see [Ge], Thm. 6.18 and its proof), it suffices to show this equality for $\mathrm{Re}(s) \gg 0$.

For such s, we have

$$\begin{aligned}
[U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x)|x|^s \phi_V(x) d^\times x &= \int_{\mathbb{I}_F} \chi(x)|x|^s W \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x \qquad \text{(def. of } \phi_V\text{)} \\
&= [U_p : V] \int_{F_p^*} \chi_p(x)|x|^s W_U \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x \cdot \int_{\mathbb{I}_F^p} \chi^p(y)|y|^s W^p \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^\times y \\
&= \prod_{\mathfrak{p}|p} \int_{F_{\mathfrak{p}}^*} \chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^s \mu_{\pi_{\mathfrak{p}}}(dx) \cdot L_{S_p}(s + \tfrac{1}{2}, \pi \otimes \chi) \qquad \text{(by prop. 2.15 and (4.12))} \\
&= \prod_{\mathfrak{p}|p} \left(e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \tau(\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \right) \cdot L(s + \tfrac{1}{2}, \pi \otimes \chi) \qquad \text{(by prop. 2.6)} \\
&= N(\mathfrak{f}(\chi))^s \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \cdot L(s + \tfrac{1}{2}, \pi \otimes \chi).
\end{aligned}$$

For $s = 0$, we get the claimed statement, since by (3.9) we have

$$\int_{\mathcal{G}_p} \chi(\gamma) \mu_\pi(d\gamma) = \int_{\mathbb{I}_F/F^*} \chi(x) \phi(dx_p, x^p) d^\times x^p = [U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x) \phi_V(x) d^\times x.$$

(b) This follows immediately from (a), setting $\chi = 1$, since $\tau(1) = 1$.

(c) Let $\lambda_{\underline{\alpha_1},\underline{\alpha_2}} \in \mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C})$ be the image of $\otimes_{v|p}\lambda_{a_v,\nu_v}$ under the map (2.13). For each $\psi \in \mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C})$, define

$$\begin{aligned} \langle \Phi_\pi, \psi \rangle (g^p, g_p) &:= \sum_{\zeta \in F^*} \lambda_{\underline{\alpha_1},\underline{\alpha_2}} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g_p \cdot \psi \right) \underline{W^p} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g^p \right) \\ &=: \sum_{\zeta \in F^*} \underline{W_\psi} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

for a V-valued function $\underline{W_\psi}$ whose every coordinate function is in $\mathcal{W}(\pi)$.

This defines a map $\Phi_\pi : G(\mathbb{A}^p) \to \mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F_p, V)$. In fact, Φ_π lies in $S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2})$, where $\mathfrak{m}$ is the prime-to-p part of $\mathfrak{f}(\pi)$:

Condition (a) of definition 4.5 follows from the fact that the W_v are almost $K_0(\mathfrak{c}(\pi_v))$-invariant, for $v \nmid p, \infty$.

For condition (b), we check that $\langle \Phi_\pi, \psi \rangle$ satisfies the conditions (i)-(v) in the definition of $\mathcal{A}_0(G, \mathrm{harm}, \underline{2}, \chi)$:

Each coordinate function of $\langle \Phi_\pi, \psi \rangle$ lies in (the underlying space of) π by [Bu], Thm. 3.5.5, thus $\langle \Phi, \psi \rangle$ fulfills (i) and (v), and has moderate growth. (ii) and (iv) follow from the choice of the W_v and $\underline{W_v}$.

Now since $\pi_v \cong \sigma(|\cdot|_v^{1/2}, |\cdot|_v^{-1/2})$ for $v|\infty$, it follows from those conditions that $\langle \Phi, \psi \rangle|_{B'_{F_v}} \cdot \underline{\beta_v} = C \sum_{\zeta \in F^*} \underline{W_v} \left(\begin{smallmatrix} \zeta t & 0 \\ 0 & 1 \end{smallmatrix} \right) \cdot \underline{\beta_v}$ is harmonic for each archimedian place v of F: for real v, it is well-known that $f(z)/y$ is holomorphic for $f \in \mathcal{D}(2)$, and thus $f \cdot (\beta_v)_1$ is harmonic; for complex v, this is also true, see e.g. [Kur78], p. 546 or [We].

Now we have

$$\begin{aligned} \Delta^{\underline{\alpha_1},\underline{\alpha_2}}(\Phi_\pi)(U, x^p) &= \delta^{\underline{\alpha_1},\underline{\alpha_2}} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right)_{(1,\ldots,1)} (U) \\ &= \sum_{\zeta \in F^*} \lambda_{\underline{\alpha_1},\underline{\alpha_2}} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U) \right) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &\overset{(*)}{=} \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} = \phi_\pi(U, x^p), \end{aligned}$$

where (*) follows from the calculation (with w_0 as defined in Ch. 2)

$$\begin{aligned} \lambda_{\underline{\alpha_1},\underline{\alpha_2}} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U) \right) &= \prod_{\mathfrak{p}|p} \int_{F_\mathfrak{p}} \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}(1_U) \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \psi_\mathfrak{p}(-x) dx \\ &= \prod_{\mathfrak{p}|p} \int_{F_\mathfrak{p}} \delta_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}(1_U) \underbrace{\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)}_{= \left(\begin{smallmatrix} 0 & 1 \\ -\zeta^{-1} & -x \end{smallmatrix} \right)} \psi_\mathfrak{p}(-x) dx \\ &= \prod_{\mathfrak{p}|p} \int_{F_\mathfrak{p}} \chi_{\alpha_{\mathfrak{p},2}}(-x)\, \chi_{\alpha_{\mathfrak{p},1}}(-1)\, 1_U(-x\zeta)\, \psi_\mathfrak{p}(-x)\, dx \\ &= \int_{\zeta U} \prod_{\mathfrak{p}|p} \chi_{\alpha_{\mathfrak{p},2}}(-x) \psi_\mathfrak{p}(-x) dx = \mu_{\pi_p}(\zeta U) \end{aligned}$$

for all $\zeta \in F^*$.

Let R be the integral closure of $\mathbb{Z}[a_\mathfrak{p}, \nu_\mathfrak{p}; \mathfrak{p}|p]$ in its field of fractions; thus R is a Dedekind ring $\subseteq \overline{\mathcal{O}}$ for which $\mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F, R)$ is defined. $\mathbb{C}$ is flat as an R-module (since torsion-free modules over a Dedekind ring are flat); thus by proposition 4.8, the natural map

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, R)) \otimes \mathbb{C} \to H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C}))$$

is an isomorphism. The map (4.11) can be described as the "R-valued" map

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, R)) \to H^d\big(F^{*\prime}, \mathcal{D}_f(R)\big)$$

tensored with $\mathbb{C}$. By proposition 4.11, κ_π lies in the image of (4.11), and thus in $H^d\big(F^{*\prime}, \mathcal{D}_f(R)\big) \otimes \mathbb{C}$; i.e. it is integral.

Similarly, it follows from propositions 4.8 and 4.10 that $\kappa_{\pi,\underline{\mu}}$ is integral of rank 1. □

Corollary 4.13. *μ_π is a p-adic measure.*

Proof. By proposition 3.8, $\mu_\pi = \mu_{\phi_\pi} = \mu_{\kappa_\pi}$. Since κ_π is integral, μ_{κ_π} is a p-adic measure by corollary 3.13. □

We can now define the *p-adic L-function* of $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$ by

$$L_p(s, \pi) := L_p(s, \kappa_\pi) := L_p(s, \kappa_{\pi,+}) := \int_{\mathcal{G}_p} \mathcal{N}(\gamma)^s \mu_\pi(d\gamma)$$

for all $s \in \mathbb{Z}_p$, where $\mathcal{N}$ is the p-adic cyclotomic character (definition 3.4; cf. remark 3.14). $L_p(s, \pi)$ is a locally analytic function with values in the one-dimensional $\mathbb{C}_p$-vector space $V_{\kappa_{\pi,+}} = L_{\kappa,\overline{\mathcal{O}},+} \otimes_{\overline{\mathcal{O}}} \mathbb{C}_p$.

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