ENTRANCE LAWS OF GENERALIZED ORNSTEIN-UHLENBECK PROCESSES AND ASYMPTOTIC STRONG FELLER PROPERTY FOR NON-AUTONOMOUS SEMIGROUPS

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Abstract

This thesis is concerned with Markovian two-parameter semigroups $(p_{s,t})_{s\leq t}$ and the associated *evolution systems of measures* (also called *entrance laws)* in Hilbert spaces. Evolution systems of measures corresponding to a two-parameter semigroup are an important generalization of the concept of invariant measures for a one-parameter semigroup. An evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ in a Hilbert space H satisfies the following identity:

$$\int_H p_{s,t}f(x)\nu_s(dx) = \int_H f(x)\nu_t(dx), \quad s \leq t,\ s,t \in \mathbb{R}.$$

The first part of this thesis deals with ergodic properties of Markovian two-parameter semigroups in Hilbert spaces. An important tool here, is the so-called *asymptotic strong Feller property*, which allows to prove the uniqueness of the corresponding evolution system of measures for a two-parameter semigroups $(p_{s,t})_{s\leq t}$. In the first part of this work, we give an analytical criterion for the asymptotic strong Feller property and show how it can be used to prove the uniqueness of T-periodic evolution systems of measures. As an application, we check the validity of the asymptotic strong Feller property for the Markovain semigroups associated with a quite general class of stochastic partial differential equations with time-dependent coefficients.

The second part of this thesis is devoted to the analysis of the set of all evolution systems of measures and to showing the existence of a representation for an arbitrary evolution system of measures as a convex combination of their extremal points. Then, the extremal points of this set is completely characterized for the particular case of *generalized Mehler semigroups* which are the transition semigroup of *generalized time-inhomogeneous Ornstein-Uhlenbeck processes*.

We also give an alternative proof of the uniqueness of T-periodic evolution system of measures (in the particular case of Ornstein-Uhlenbeck processes) by using the explicit form of the corresponding extremal points. This establishes the connection between the both parts of this work.

Contents

Chapter 1

Introduction

This thesis is concerned with Markovian two-parameter semigroups $(p_{s,t})_{s\leq t}$ acting on the space of all bounded Borel measurable function $f : H \to \mathbb{R}$, where H is a Hilbert space. They preserve positivity of this type of function and $p_{s,t}\mathbb{1} = \mathbb{1}$, $s \leq t$. Furthermore, $p_{s,s}$, $s \in \mathbb{R}$, is the identity operator and $p_{s,r}p_{r,t} = p_{s,t}$ for $s \leq r \leq t$. This type of semigroups is a natural generalization of Markovian one-parameter semigroups which are one of the central topics in modern stochastic analysis. An important example of Markovian semigroups can be constructed by means of stochastic partial differential equations (SPDEs) with time-dependent coefficients of the type

$$\begin{aligned} dX(t) &= \big(A(t)X(t) + F(t, X(t))\big)dt + \sigma(t, X(t))dL(t), \quad s \leq t, \\ X(s) &= x, \end{aligned} \tag{1.1}$$

on a Hilbert space H, where $A(t) : D(A) \subset H \to H$ are linear operators, F is a non-linear part and L is a Lévy noise in H. Then, under appropriate assumptions on the coefficients, the evolution semigroup associated with this equation will be given by $p_{s,t}f(x) := \mathbb{E}[f(X(s, t, x))]$, where $X(s, t, x)$ is the solution of equation (1.1).

One part of the thesis (Chapters 4 and 7) deals with ergodic properties of Markovian two-parameter semigroups in Hilbert spaces. An important tool here is the so-called *asymptotic strong Feller property*, which allows to prove uniqueness of the corresponding evolution system of measures for the semigroup $(p_{s,t})_{s\leq t}$ under consideration. In Chapter 4, we give an analytic criterion for the asymptotic strong Feller property and describe its role in proving uniqueness of evolution systems of measures for the two-parameter semigroup $(p_{s,t})_{s\leq t}$. As an application, in Chapter 7 we discuss the validity of the asymptotic strong Feller property for the Markovain semigroups associated with a quite general class of SPDEs of type (1.1).

The uniqueness problem of invariant measures has been always crucial in the study of stochastic differential equations. By definition, an invariant measure ν for a one-parameter semigroup $(p_t)_{t\geq 0}$ fulfills the following equation:

$$\int_H p_t f(x)\nu(dx) = \int_H f(x)\nu(dx), \quad t \geq 0, \tag{1.2}$$

for all bounded measurable functions $f : H \to \mathbb{R}$.
An important observation is that for any invariant measure ν we can uniquely extend $(p_t)_{t\geq 0}$ to a contractive $\mathcal{C}_0$-semigroup in all spaces $L^p(H,\nu)$, $p \geq 1$. Moreover, if the invariant measure ν is unique, then the dynamical system associated with this semigroup will be ergodic, i.e., the dynamical system has the same behavior averaged over time as averaged over the space of all system's states (phase space).
Here a natural question is under which conditions one can guarantee the uniqueness of the invariant measure. Note that uniqueness is often provided by a result due to Khasminskii and Doob (see e.g. Chapter 7 in [PZ96]) stating that, it is a consequence of the strong Feller property and irreducibility. Recall that for diffusions in $\mathbb{R}^n$ a possible way to check the regularity conditions for transition probabilities needed to show the strong Feller property is by using Hörmander's Theorem (see Theorem 7.1 in [Hör85]).
However, in infinite dimensional spaces not only these theorems are not available, but even the strong Feller property mostly fails to hold too.
Only when the forcing noise is sufficiently rough, e.g., the covariance of the noise is non-degenerate, the Bismut-Elworthy formula (see e.g. [Mal97]) allows to show the strong Feller property for a class of semilinear parabolic SPDE. However, in the case, when the noise is rather weak, the Bismut-Elworthy formula does not apply.
Therefore, it is extremely important to find a weaker property (as compared to the strong Feller property) that still implies uniqueness of the invariant measure and covers a wider class of stochastic differential equations.

This leads to the notion of asymptotic strong Feller property for the Markovain one-parameter semigroups which for the first time has been introduced in [HM06].

Now let us discuss how these considerations can be extended to the case of Markovian two-parameter semigroups.

Clearly, equation (1.2) is not applicable for a two-parameter semigroup. In fact, the additional second time parameter in the above semigroup calls for an additional time parameter in the invariant measure. Thus, what we get is a family of probability measures $(\nu_t)_{t\in\mathbb{R}}$ satisfying

$$\int_H p_{s,t} f(x)\nu_s(dx) = \int_H f(x)\nu_t(dx), \quad s \leq t,$$

for every bounded Borel measurable function $f : H \to \mathbb{R}$. We call this family *evolution system of measures* (ESM) (see e.g. [PR08]) or *entrance law* (see [Dyn71]).

The following question is fundamental here:

- How does the asymptotic strong Feller property look like for two-parameter semigroups and whether the theory of uniqueness by using this property is still true.

Asymptotic strong Feller property for the Markovian two-parameter semigroup $(p_{s,t})_{s\leq t}$ has been first introduced in [PD08] (see Definition 4.26 below). Their approach is based on the *homogenization* method.

Homogenization is a well-known technique in the theory of differential equations (see e.g. [DK74])), which reduces the problem to an autonomous case. We apply this procedure in proving uniqueness of invariant measures. For instance, for a given initial condition $x \in H$, equation (1.1) can be equivalently rewritten after homogenization as follows

$$\begin{aligned} &dZ(\tau) = \big[A(y(\tau))Z(\tau) + F(y(\tau), Z(\tau))\big]d\tau + \sigma(y(\tau), Z(\tau))dL(\tau) \\ &dy(\tau) = d\tau, \\ &Z(0) = x,\ y(0) = s. \end{aligned} \tag{1.3}$$

Then a solution $\big(y(\tau,t,x), Z(\tau,t,x)\big)$ to (1.3) is a process taking values in $\mathbb{R} \times H$.

We should point it out that the uniqueness theory of evolution systems of measures by using asymptotic strong Feller property presented in [PD08] deals with the concrete case of two-dimensional Navier-Stokes equations on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$ with Dirichlet boundary conditions and a periodic forcing term. Moreover, there exist some gaps and errors in proving the main assertions which cannot be justified (see Remark 4.24 for details). This motivated us to extend the theory to a general framework and to fill the gaps contained in the original proofs in [PD08].
To apply the homogenization method to study ergodic properties, we need to restrict our considerations to the T-periodic case. Consider a T-periodic stochastically continuous Markovian semigroup $(p_{s,t})_{s\leq t}$. Let $S_T \cong [0, T]$ be the torus of length $T > 0$. We define a Markovian semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$ acting on bounded measurable real-valued functions $f : S_T \times H \to \mathbb{R}$ as

$$\mathcal{P}_\tau f(t, x) = p_{t,t+\tau} f(t + \tau, .)(x), \quad \tau \geq 0,\ (t, x) \in S_T \times H.$$

Then $(\mathcal{P}_\tau)_{\tau\geq 0}$ is called the space-time Homogenization semigroup associated with $(p_{s,t})_{s\leq t}$. The semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$ is in fact the Markovian semigroup corresponding to equation (1.3), i.e.,

$$\mathcal{P}_\tau f(t, x) = \mathbb{E}\big[f(y(\tau, t, x), Z(\tau, t, x))\big].$$

Now, let $(\nu_t)_{t\in\mathbb{R}}$ be a T-periodic evolution system of measures associated with $(p_{s,t})_{s\le t}$. Then

$$\nu(dt, dx) := \frac{1}{T}\nu_t(dx)dt \tag{1.4}$$

will be an invariant *probability* measure for $(\mathcal{P}_\tau)_{\tau\ge 0}$ on $\big(S_T \times H, \mathscr{B}(S_T)\otimes \mathscr{B}(H)\big)$. And conversely, for every invariant measure of $(\mathcal{P}_\tau)_{\tau\ge 0}$, there exists a T-periodic evolution system of measures for $(p_{s,t})_{s\le t}$ for which (1.4) is satisfied.

It is worth noting that we suppose the T-periodicity of $(p_{s,t})_{s\le t}$ and choose $S_T \cong [0,T]$ as our time-parameter space, just because we cannot normalize Lebesgue measure on the whole of $\mathbb{R}$ in order to obtain an invariant *probability* measure for $(\mathcal{P}_\tau)_{\tau\ge 0}$.

Our main result about the uniqueness of evolution system of measures is stated in Theorem 4.33. The sufficient conditions in our theorem is ,however, slightly different from the corresponding ones in [PD08].

In general, it is a non-trivial problem to check whether the asymptotic strong Feller property of two-parameter semigroups (or even one-parameter semigroups) holds in infinite dimensions. In Theorem 4.29 below we give a sufficient condition for the asymptotic strong Feller property which extends a related result for one-parameter semigroups proven in [HM06].

- As an application, we consider a general SPDE and apply this theorem in order to examine the validity of asymptotic strong Feller property for the corresponding semigroup under appropriate conditions.

More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$. Moreover, let L be a Lévy process on a Hilbert space G with σ-algebra $\mathscr{B}(G)$ and let its characteristics be (b, Q, ν).
Let us fix $T > 0$ and consider the following SPDE in a Hilbert space H

$$\begin{aligned} dX(t) &= [A(t)X(t) + F(t, X(t))]dt + \sigma(t)dL(t), \\ X(s) &= x, \end{aligned} \tag{1.5}$$

where all the coefficients are deterministic. We suppose that the linear operators $A(t) : D(A) \subset H \to H$ generate a strong evolution operator $(U(s,t))_{s\le t}$ which is stable, i.e., there exist $\mathbb{M} > 0$ and $\omega \ge 0$ such that $\|U(s,t)\|_{\mathcal{L}(H)} \le \mathbb{M}e^{-\omega(t-s)}$ for every $s \le t$. Furthermore, we impose the Lipschitz continuity on F with respect to its second variable. Also, the Lévy intensity measure ν is assumed to have strong second moments (cf. Hypothesis 7.3 below).

Then, under these assumptions, we prove the existence and uniqueness of the corresponding mild solutions $X(s,t,x)$ to (1.5) in Theorem 7.14. Moreover, the statement of this theorem covers more general equations with time-dependent random coefficients.

As a result of applying the sufficient criteria stated in Theorem 4.33, we verify in Theorem 7.18 that the associated two-parameter semigroup is asymptotically strong Feller.

The second part of this thesis (Chapters 5 and 6) is focused on the analysis of the set of all evolution systems of measures related to a Markovian semigroup $(p_{s,t})_{s\leq t}$ and the characterization of its extremal points.

Informally speaking, ergodic measures are the simplest invariant measures of a dynamical system. It was recognized that it is possible to decompose an invariant measure into ergodic measures in such a way that the study of the former reduces to the study of the latter (in our context see Theorem 4.20).

On the other hand, it is well-known that in the one-parameter case the ergodic measures of a Markovian semigroup $(p_t)_{t\geq 0}$ exactly coincide with the extremal points in the set of all invariant measures of $(p_t)_{t\geq 0}$ (see respectively Theorem 2.70).
Therefore, the following motivating question arises:

- Does in the two-parameter case exist a representation of an arbitrary evolution system of measures in terms of a unique convex combination of their extremal points (property called *simplexity* in this thesis).

For the Markovian simgroups $(p_{s,t})_{s\leq t}$ which we consider, the answer is "YES". Additionally, knowing every extremal point of this set, along with the simplexity of the set of all ESMs, can help us to understand them completely. Therefore, these extremal points are the objects which we are mostly interested in.

On the other hand, as we know from the first part of the thesis, the sufficient conditions, which imply uniqueness of ESM, can not be easily verified, even not for T-periodic ESM. Actually, the generic case is that more than one ESM exist.

As a result, one of the main contributions of our work is:

- Providing an explicit description for the extremal points in the set of all ESMs, thus identifying more than one of them.

In fact, we extend the previously known result of Dynkin (Theorem 5.1 in [Dyn88]), which was devoted to Gaussian Ornstein-Uhlenbeck processes and we concentrate on Lévy Ornstein-Uhlenbeck processes in a Hilbert space H. In the one-parameter case, the corresponding semigroup is called *generalized Mehler semigroups*. They have been first introduced in [BRS96] and then extensively studied in [FR00], [PL07], [Knä11], [OR10] etc.
This type of processes constitutes a large class of explicit examples of Markov processes in infinite-dimensional spaces with rich mathematical structure. Those processes may have non-trivial invariant measures, which make them

better candidates for infinite-dimensional reference processes than Lévy processes.

Let us summarize the main results on this type of processes. Because of the complexity of the abstract framework which has been taken from [Dyn88], for the reader's convenience we separate our problem into the following two steps.

1. We first focus on one-parameter Mehler semigroups $(p_t)_{t\geq 0}$. The generalized one-parameter (time-homogeneous) Mehler semigroup $(p_t)_{t\geq 0}$ introduced in [FR00] is given by

$$p_t f(x) = \int_H f(T_t x + y)\mu_t(dy),$$

where $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on H and $(\mu_t)_{t\geq 0}$ is a family of probability measures defined in Proposition 5.24.

This type of one-parameter semigroups can be mainly illustrated by the following general autonomous Ornstein-Uhlenbeck equation

$$\begin{aligned} dX(t) &= AX(t)dt + dL(t) \\ X(s) &= x, \end{aligned}$$

where A is a linear operators and L is a Lévy process. Then the semigroup corresponding to this equation will be the one-parameter Mehler semigroup, such that A is the generator of $(T_t)_{t\geq 0}$ and the measures $(\mu_t)_{t\geq 0}$ are uniquely defined by the characteristics of L in Proposition 5.24.
The ESMs will be defined by

$$\int_H p_t f(x)\nu_s(dx) = \int_H f(x)\nu_{t+s}(dx), \quad s \leq t.$$

We prove that the set of all ESMs, which we denote by $\mathcal{K}(\pi)$, is a simplex. The method of proving this fact is based on the existence of a canonical isomorphism between Markovian semigroups and Markov processes, identified in [Dyn72].

In Theorem 5.34, we prove the main result about the explicit representation of extremal points in the set of all ESMs (under Assumption 5.32 below). This theorem establishes an isomorphism between the set of all so-called T-*entrance laws* $(\kappa_t)_{t\in\mathbb{R}} \subset H$, i.e., solutions of the equalities $T_t\kappa_s = \kappa_{s+t}$ and the set of all extremal points in $\mathcal{K}(\pi)$ (with finite first moment).

2. Next, we extend these results to the two-parameter case. In particular, analogously to the one-parameter semigroup case we show the simplexity (introduced before) of $\mathcal{K}(\pi)$.
The construction of the two-parameter Mehler semigroups goes as follows. We fix a strongly continuous evolution family of operators $(U_{s,t})_{s\leq t}$ on H.

Recalling the definition from [OR10], a generalized two-parameter (non-time-homogeneous) Mehler semigroup is given by

$$p_{s,t}f(x) = \int_H f(U_{s,t}x + y)\mu_{s,t}(dy),$$

where $(\mu_{s,t})_{s\leq t}$ is defined in Proposition 6.11 and evolution systems of measures $(\nu_t)_{t\in\mathbb{R}}$ satisfy

$$\int_H p_{s,t}f(x)\nu_s(dx) = \int_H f(x)\nu_t(dx), \quad s \leq t,$$

for every bounded Borel measurable function $f : H \to \mathbb{R}$.

In the next step, we prove an explicit form for the corresponding extremal points $(\nu_t)_{t\in\mathbb{R}}$, in Theorem 6.20, where we also establish an isomorphism between the set of all $(\kappa_t)_{t\in\mathbb{R}} \subset H$ with $U_{s,t}\kappa_s = \kappa_t$ and the set of all extremal points in (the subset of entrance laws with finite moment from) $\mathcal{K}(\pi)$.

This type of two-parameter semigroups can be mainly illustrated by the following general non-autonomous Ornstein-Uhlenbeck equation

$$\begin{aligned} dX(t) &= A(t)X(t)dt + \sigma(t)dL(t) \\ X(s) &= x, \end{aligned}$$

where $(U_{s,t})_{s\leq t}$ is assumed to solve the Cauchy problem $\frac{d}{dt}U(s,t) = A(t)U(s,t)$.

Using the explicit form of extremal points, we can prove the uniqueness of the T-periodic evolution system of measures (related to the non-autonomous Ornstein-Uhlenbeck process above). This is an alternative proof of the uniqueness of ESMs, without using asymptotic strong Feller property which this constitutes a counterpart to the first part of this thesis.

Let us describe the contents of this work chapter by chapter.

Chapter 2 contains supporting material from the theory of stochastic processes and operator semigroups. In Section 2.1, we clarify the required concepts of martingales and Markov processes as well as related Markovian semigroups. Different classes (e.g. nuclear and Hilbert-Schmidt) of operators in Hilbert spaces are described in Section 2.4. Gâteaux and Fréchet differentiability of functions on Banach spaces is briefly discussed in Section 2.5. In Section 2.6, an introduction to the Lévy processes is given. In this context, stochastic integration with respect to the theory of Lévy processes is of our prime interest in Section 2.7. In Section 2.8, we introduce the Wasserstein distance between probability distributions and review its properties. In

Section 2.9, we recall basic concepts of the ergodic theory to be used in the study of Markovian semigroup below.

Chapter 3 reviews two basic techniques to prove the uniqueness of invariant measures. In Section 3.1, we explain the first approach, which is by using strong Feller property. Then, in Section 3.2, we give a motivation why in infinite dimensions it is needed to find an appropriated modification of strong Feller property. And finally, in Section 3.3, we shortly recall the second technique which uses the so-called asymptotic strong Feller property.

Chapter 4 is devoted to the uniqueness problem of ESM in the case of T-periodic two-parameter semigroups. In Section 4.1, we give the necessary definitions and assumptions concerning the Markovian two-parameter semigroups and evolution systems of measures. In Section 4.2, the space-time homogenization method for semigroup is presented. Section 4.3 is devoted to showing existence of an one-to-one correspondence between T-periodic evolution systems of measures for $(p_{s,t})_{s\leq t}$ and invariant measures for the associated space-time homogeneous semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$. This section fill some gaps contained in the original proofs in [PD08]. The relation between the support of an invariant measure for $(\mathcal{P}_\tau)_{\tau\geq 0}$ and the support of its disintegration is considered in Section 3.5. It is the critical point which causes doubt to the correctness of the related result in [PD08]. And finally, in Sections 3.6 and 3.7, we prove the uniqueness criteria for evolution system of measures employing the asymptotic strong Feller property and under the refined assumptions as compared to [PD08].

Chapter 5 is devoted to developing a unified theory for the ESMs corresponding to one-parameter semigroups. In Section 5.2, we define the Markov process corresponding to a Markovian one-parameter semigroup with an ESM as the initial value. A general definition of convex measurable spaces (in the spirit of Dynkin [Dyn88]) is given in Section 5.3. A crucial issue here is to establish an isomorphism between the set of all ESMs and the set of such Markov processes, which is described in Section 5.4. As a result of this isomorphism, we get the simplexity of the set of all ESMS. In Section 5.5, we concentrate on the particular case of one-parameter Mehler semigroups. Then in the next section, we give the main result of this chapter, (Theorem 5.34), which is the explicit representation of the corresponding extremal points in the set of all ESMs. In Section 5.7, we give an illustrative example which satisfies all assumption required in Theorem 5.34. In the last section, our Mehler semigroup is defined in a nuclear space in order to weaken the assumptions imposed in the Hilbert setting of Section 5.6.

In *Chapter 6*, the results of Chapter 5 are extended to the two-parameter semigroups. In Section 6.1, in the case of two-parameter semigroups we establish an isomorphism which is similar to that in Section 5.4. Then with the help of the concept of a strong evolution family introduced in Section 6.2, we construct a two-parameter Mehler semigroup in Section 6.3. An explicit form for the extremal points in the set of all ESMs is given in Section 6.4. In

Section 6.5, as an example of such representation, we recall the classical result from [Dyn88] concerning the Gaussian case. In Section 6.6, an alternative proof of the uniqueness of T-periodic ESM for Mehler semigroups by using the results of Chapter 6 is given.

In *Chapter 7*, we show the validity of the asymptotic strong Feller property for a large class of SPDEs driven by Lévy noise. In Section 7.1, we introduce the SPDE as well as fix the assumptions on the noises and coefficients. In Section 7.2, we show the existence and uniqueness of the mild solution, even for a larger class of SPDEs under Hypothesis 7.1 and 7.3 to be imposed below. Differentiability of solution for the SPDE discussed in Section 7.3 will be a key technical issue to show the main result, Theorem 7.18, of Section 7.4. This theorem states the validity of the asymptotic strong Feller property for the semigroup associated to the considered SPDE.

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Chapter 2

Prerequisites

In this chapter, we recall the necessary fundamentals which are needed in our work. We collect some well-known definitions and facts concerning the martingales and Markov processes as well as associated semigroups. Also, we recall some basic facts from functional analysis especially the properties of Hilbert-Schmidt and trace class operators in Hilbert spaces. Also, notions of Gâteaux and Fréchet differentiability will be viewed in this chapter, to help understanding the differentiability of solutions of SPDEs in chapter 7. Furthermore, we introduce Lévy processes, which are our case in Chapters 5 and 6 and will serve as integrators for our stochastic equation in Chapter 7. Since in Chapter 4 a Wasserstein-type metrics will play a crucial role, so we present it also in our prerequisites. We finish this chapter with a short review on ergodic theory, with the emphasis placed on ergodic properties of Markovian semigroups.

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space and $\mathscr{B}(H)$ its Borel σ-algebra. Denote by $\mathcal{B}_b(H)$ the space of all bounded, Borel measurable functions $f : H \to \mathbb{R}$ and denote by $\mathcal{C}_b(H)$ the subspace of all continuous and bounded functions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with normal filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null-sets of $\mathcal{F}$.

2.1 Martingale and Markov process

Here we recall the basic definitions and properties of martingales and Markov processes. For more details on the theory of stochastic processes, we refer to [RY99], [BF75], [Bau96] and [PZ92].

The process $(X(t))_{t\in\mathbb{R}}$ is said to be adapted to the filtration $(X(t))_{t\in\mathbb{R}}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \in \mathbb{R}$.

Definition 2.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. Let $(X(t))_{t\in\mathbb{R}}$ be a family of H-valued Bochner integrable random variables on*

$(\Omega, \mathcal{F}, \mathbb{P})$ which is adapted to this filtration. The process $(X(t))_{t\in\mathbb{R}}$ is called a martingale with respect to the given filtration if for every pair $s, t \in \mathbb{R}$ with $s \leq t$, the following condition is fulfilled

$$\mathbb{E}(X(t) \mid \mathcal{F}_s) = X(s), \quad \mathbb{P} - a.s.,$$

where the concept of conditional expectation of a H-value random variable has been taken from [PZ92], p. 27.

All integrals in Definition 2.1 are understood in the Bochner sense. For the theory of Bochner integration in separable Hilbert spaces see Appendix A in [PR07].

Lemma 2.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\in\mathbb{R}}$ a filtration in $\mathcal{F}$ and Z an integrable H-valued random variable. Then*

$$Y(t) := \mathbb{E}(Z \mid \mathcal{F}_t), \quad t \in \mathbb{R},$$

defines a martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}}$.

Proof In the first place, each such $Y(t)$ is $\mathcal{F}_t$-measurable by definition of the conditional expectation, so $(Y(t))_{t\in\mathbb{R}}$ is adapted to the given filtration. Also for every $s \leq t$

$$\mathbb{E}(Y(t) \mid \mathcal{F}_s) = \mathbb{E}\big(\mathbb{E}(Z \mid \mathcal{F}_t) \mid \mathcal{F}_s\big) = \mathbb{E}(Z \mid \mathcal{F}_s) = Y(s), \quad \mathbb{P} - a.s.,$$

confirming the martingale property. ■

Theorem 2.3 (Backwards Martingale Convergence Theorem) *Let $(Y(t))_{t\in\mathbb{R}}$ be a martingale. Then without any further conditions, there exists a random variable $Y(-\infty) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\lim_{t\to-\infty} Y(t) = Y(-\infty), \quad \mathbb{P} - a.s.,$$

where the convergence is in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 2.4 *Let $(X(t))_{t\in\mathbb{R}}$ be a H-valued integrable process. Assume that, for all $t > s$, the random variable $X(t) - X(s)$ is independent of $\mathcal{F}_s$. Then the process $Y(t) := X(t) - \mathbb{E}\big[X(t)\big]$ is a martingale.*

Definition 2.5 *A family of H-valued stochastic processes*

$$\big(\Omega, \mathcal{F}, (X(t))_{t\in\mathbb{R}}, (\mathbb{P}_x)_{x\in H}\big)$$

is called a Markov process with state space $(H, \mathscr{B}(H))$, if

(i) *each $\mathbb{P}_x$ is a probability measure on $(\Omega, \mathcal{F})$ such that $x \mapsto \mathbb{P}_x(\Gamma)$ is $\mathscr{B}(H)$-measurable for all $\Gamma \in \mathcal{F}$;*

(ii) *there exists a filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}}$ *in* $\mathcal{F}$ *such that* $(X(t))_{t\in\mathbb{R}}$ *is* $\mathcal{F}_t$*-adapted and*

$$\mathbb{P}_x(X(s+t)\in B \mid \mathcal{F}_s) = \mathbb{P}_{X(s)}(X(t)\in B), \quad \mathbb{P}_x - a.e.,$$

for all $s\in\mathbb{R}$ *and* $t\geq 0$*, for every* $B\in\mathscr{B}(H)$ *and* $x\in H$*.*

2.2 Semigroups and invariant measures

For more details see e.g. [PZ96].

Definition 2.6 *A function* $o : H\times\mathscr{B}(H)\to[0,\infty)$ *is called a transition kernel on* $\big(H,\mathscr{B}(H)\big)$ *if*

(i) $x\mapsto o(x,B)$ *is measurable for every* $B\in\mathscr{B}(H)$*;*

(ii) $B\mapsto o(x,B)$ *is a* σ*-finite measure on* $\big(H,\mathscr{B}(H)\big)$ *for every* $x\in H$*.*

The transition kernel o is called to be Markovian if $o(x,H)=1$ *for all* $x\in H$*, e.g.,* $o(x,.)$ *is a probability measure on* $\big(H,\mathscr{B}(H)\big)$ *for each* $x\in H$*.*

Definition 2.7 *A family* $(p_t)_{t\geq 0}$ *of linear bounded operators on* $\mathcal{B}_b(H)$ *is called a Markovian semigroup if*

(i) $p_0=\mathbb{1}$*;*

(ii) $p_{s+t}=p_sp_t$ *for all* $t,s\geq 0$*;*

(iii) $p_s\mathbb{1}=\mathbb{1}$*;*

(iv) *for every* $t\geq 0$*,* p_tf *is non-negative whenever* f *is non-negative.*

Definition 2.8 *Any Markovian semigroup on* $\mathcal{B}_b(H)$ *defines a family of Markovian transition kernels on* $\big(H,\mathscr{B}(H)\big)$ *by*

$$\pi_t(x,B) := p_t\mathbb{1}_B(x), \quad x\in H,\ B\in\mathscr{B}(H).$$

Therefore for an arbitrary $f\in\mathcal{B}_b(H)$ *we have*

$$p_tf(x)=\int_H f(y)\pi_t(x,dy) \tag{2.1}$$

for every $x\in H$*. We call* $(\pi_t)_{t\geq 0}$ *the Markovian semigroup of transition kernels.*

Proposition 2.9 *There is a one-to-one correspondence between the Markovian semigroup* $(p_t)_{t\geq 0}$ *on* $\mathcal{B}_b(H)$ *and the Markovian semigroup of transition kernels* $(\pi_t)_{t\geq 0}$ *over* H *given by* $\pi_t(x,B):=p_t\mathbb{1}_B(x)$*.*

Proof See Lemma 2.4 in [Hai08]. ∎

Remark 2.10 *From the semigroup identity $p_{s+t} = p_s p_t$ for all $s, t \geq 0$, we get*

$$\pi_{s+t}(x, B) = \int_H \pi_t(y, B)\pi_s(x, dy) \tag{2.2}$$

for all $x \in H$ and $B \in \mathscr{B}(H)$.

Equation (2.2) is known as the *Chapman-Kolmogorov equation.* A heuristic interpretation of this equality can be given as follows:
the probability for a particle starting at time 0 in $x \in H$ to be in $B \in H$ at time $s + t$ is equal to the probability that the particle starts at time 0 in $x \in H$ and being in some infinitesimal small volume dy at time s and then starting new in $y \in H$ at time s and being in the subset B at time $s + t$ integrated over all intermediate points $y \in H$.

Definition 2.11 *The Markovian semigroup $(p_t)_{t\geq 0}$ is called*

(i) *Feller at time $t \geq 0$ if $p_t f \in \mathcal{C}_b(H)$ for any $f \in \mathcal{C}_b(H)$. It is called Feller if it is Feller at all times $t \geq 0$.*

(ii) *strong Feller at time $t > 0$ if $p_t f \in \mathcal{C}_b(H)$ for any $f \in \mathcal{B}_b(H)$. It is called Feller if it is Feller at all times $t > 0$.*

Definition 2.12 *A Markovian semigroup $(p_t)_{t\geq 0}$ is called stochastically continuous if $t \mapsto p_t f(x)$ is continuous for every $f \in \mathcal{C}_b(H)$ and $x \in H$.*

Let μ be a probability measure defined on $(H, \mathscr{B}(H))$ and $(\pi_t)_{t\geq 0}$ be a Markovian semigroup of transition kernels on $\big(H, \mathscr{B}(H)\big)$. For any $t \geq 0$, we set

$$p_t^* \mu(B) := \int_H \pi_t(x, B)\mu(dx), \quad t \geq 0, \; B \in \mathscr{B}(H).$$

Definition 2.13 *A probability measure μ on $(H, \mathscr{B}(H))$ is said to be invariant measure with respect to the Markovian semigroup $(p_t)_{t\geq 0}$ if*

$$p_t^* \mu = \mu, \quad \text{for all } t \geq 0.$$

We denote the set of all invariant probability measures of $(p_t)_{t\geq 0}$ by $\mathrm{Inv}(p)$.

Proposition 2.14 *Assume that μ is an invariant measure for $(p_t)_{t\geq 0}$. Then for all $q \geq 1$, a stochastic continuous Markovian semigroup $(p_t)_{t\geq 0}$ is uniquely extendible to a $\mathcal{C}_0$-semigroup of linear bounded operators in $L^q(H, \mu)$ that we still denote by $(p_t)_{t\geq 0}$. Moreover*

$$\|p_t\|_{\mathcal{L}(L^q(H,\mu))} \leq 1, \quad t \geq 0.$$

Proof See Theorem 5.8 in [Pra06]. ∎

2.3 Supporting material from stochastic analysis

Definitions and propositions of this section have been taken from [BF75], [PR07], [FR00] and [Li11].

Lemma 2.15 (Monotone Class Theorem) *Suppose that a linear space of bounded functions contains* $\mathbf{1}$ *and is closed under bounded convergence. If this space contains a family* $\mathcal{D}$ *which is closed under multiplication, then given space contains all bounded functions which are measurable w.r.t the* σ*-algebra generated by* $\mathcal{D}$*.*

Definition 2.16 *For a probability measure* μ *on* $(H, \mathscr{B}(H))$*, its characteristic function is defined by the formula*

$$\widehat{\mu}(a) = \int_H e^{i\langle a,x\rangle}\mu(dx), \quad a \in H.$$

Remark 2.17 *By the monotone class theorem every probability measure* μ *on* $(H, \mathscr{B}(H))$ *is determined uniquely by its characteristic function* $\widehat{\mu}$*.*

As usual, the dual space H^* of the Hilbert space H, will be canonically identified with H.

Definition 2.18 *A function* $\psi : H \to \mathbb{C}$ *is called positive definite if for any* $n \in \mathbb{N}$*,* $a_1, ..., a_n \in H$ *and* $c_1, ..., c_n \in \mathbb{C}$ *satisfying* $\sum_{i=1}^n c_i = 0$*, we have*

$$\sum_{i,j=1}^{n} \psi(a_i - a_j)c_i\bar{c_j} \geq 0;$$

respectively ψ *is negative definite if*

$$\sum_{i,j=1}^{n} \psi(a_i - a_j)c_i\bar{c_j} \leq 0.$$

2.4 Operators on Hilbert spaces

In this section we follow [PR07], [RS80] and [Con00]. Let $(G, \langle\cdot,\cdot\rangle_G)$ and $(H, \langle\cdot,\cdot\rangle_H)$ be two separable Hilbert spaces.

Definition 2.19 **i)** *The space of all bounded linear operators from* G *to* H *is denoted by* $\mathcal{L}(G, H)$*. For simplicity, we write* $\mathcal{L}(G)$ *instead of* $\mathcal{L}(G, G)$*.*

ii) *By* $T^* \in \mathcal{L}(H, G)$ *we denote the adjoint operator of* $T \in \mathcal{L}(G, H)$*.*

iii) *An operator* $T \in \mathcal{L}(G)$ *is called symmetric if* $\langle Tu, v\rangle_G = \langle u, Tv\rangle_G$ *for all* $u, v \in G$

iv) *An operator* $T \in \mathcal{L}(G)$ *is called non-negative if* $\langle Tu, u\rangle_G \geq 0$ *for all* $u \in G$*.*

2.4.1 Trace class operators

Definition 2.20 (trace class or nuclear operator) *An operator $T \in \mathcal{L}(G,H)$ is said to be nuclear if it can be represented by*

$$Tx = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_G, \quad x \in G,$$

where the series is convergent in H and $(a_j)_{j\in\mathbb{N}} \subset H$ and $(b_j)_{j\in\mathbb{N}} \subset G$ are such that $\sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_G < \infty$. The space of all trace class operators from G to H is denoted by $\mathcal{L}_1(G,H)$.

Proposition 2.21 *The space $\mathcal{L}_1(G,H)$ equipped with the norm*

$$\|T\|_{\mathcal{L}_1} := \inf\left\{ \sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_G \;\middle|\; Tx = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_G,\ x \in G\right\}$$

is a Banach space.

Proof See [PR07], Proposition B.0.2. ■

Definition 2.22 *Let $T \in \mathcal{L}(G)$ and let $\{e_k\}_{k\in\mathbb{N}}$, $k \in \mathbb{N}$, be an orthonormal basis of G. Then we define the trace of T by*

$$tr(T) := \sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle_G,$$

if this series is convergent.

One has to notice that this definition could depend on the choice of the orthonormal basis. However, note the following result concerning nuclear operators.

Remark 2.23 *If $T \in \mathcal{L}_1(G) := \mathcal{L}_1(G,G)$, then $tr(T)$ is well-defined independently of the choice of the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$. Moreover, we have that*

$$|tr(T)| \le \|T\|_{\mathcal{L}_1}.$$

Proof See [PR07] Remark B.0.4. ■

For two following facts see for example [Con00].

Proposition 2.24 *Let $T \in \mathcal{L}_1(G)$ be such that $T \ge 0$. Then for every two orthonormal bases $\{e_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ in G, we have*

$$\sum_{j\in\mathbb{N}} h_j \langle Th_j, x\rangle_G = \sum_{j\in\mathbb{N}} e_j \langle Te_j, x\rangle_G.$$

Therefore, the representation of Tx is independent of chosen orthonormal basis and thus

$$tr(T) = \|T\|_{\mathcal{L}_1}.$$

We denote the family of all non-negative symmetric operators $\mathcal{L}_1(G)$ by $\mathcal{L}_1^+(G)$.

Lemma 2.25 *$\mathcal{L}_1(G)$ is an operator ideal in $\mathcal{L}(G)$, i.e., $\mathcal{L}(G)\mathcal{L}_1(G)\mathcal{L}(G) \subset \mathcal{L}_1(G)$.*

2.4.2 Hilbert-Schmidt operators

Definition 2.26 (Hilbert-Schmidt operator) *A bounded linear operator $T : G \to H$ is called Hilbert-Schmidt if*

$$\sum_{k\in\mathbb{N}} \|Te_k\|_H^2 < \infty,$$

where $\{e_k\}_{k\in\mathbb{N}}$, $k \in \mathbb{N}$, is an orthonormal basis of G. The space of all Hilbert-Schmidt operators from G to H is denoted by $\mathcal{L}_2(G, H)$.

Remark 2.27 **i)** *The definition of the Hilbert-Schmidt operator and the quantity*

$$\|T\|^2_{\mathcal{L}_2(G,H)} := \sum_{k\in\mathbb{N}} \|Te_k\|^2$$

does not depend on the choice of the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$. Furthermore, $\|T\|_{\mathcal{L}_2(G,H)} = \|T^*\|_{\mathcal{L}_2(H,G)}$. For notation simplicity, we write $\|T\|_{\mathcal{L}_2}$ instead of $\|T\|_{\mathcal{L}_2(G,H)}$.

Proof See [PR07], Proposition B.0.7. ∎

Proposition 2.28 *For any $T, T' \in \mathcal{L}_2(G, H)$ and $\{e_k\}_{k\in\mathbb{N}}$ being an orthonormal basis of G, let us define the inner product*

$$\langle T, T'\rangle_{\mathcal{L}_2} := \sum_{k\in\mathbb{N}} \langle T'e_k, Te_k\rangle_H.$$

Then $(\mathcal{L}_2(G, H), \langle\cdot,\cdot\rangle_{\mathcal{L}_2})$ is a separable Hilbert space. If $\{f_j\}_{j\in\mathbb{N}}$, is an orthonormal basis of H, then $f_j \otimes e_k := f_j\langle e_k, \cdot\rangle_G$, $j, k \in \mathbb{N}$, is an orthonormal basis of $\mathcal{L}_2(G, H)$.

Proof See [PR07], Proposition B.0.7. ∎

Proposition 2.29 (Square root) *Let $T \in \mathcal{L}(G)$ be a non-negative and symmetric operator. Then, there exists exactly one element $T^{1/2} \in \mathcal{L}(G)$, which is non-negative and symmetric, such that*

$$T^{1/2} \circ T^{1/2} = T.$$

If $tr(T) < \infty$, then $T^{1/2} \in \mathcal{L}_2(G)$ with $\|T^{1/2}\|^{\frac{1}{2}} = tr(T)$ and $L \circ T^{1/2} \in \mathcal{L}_2(G, H)$ for all $L \in \mathcal{L}(G, H)$.

Proof See [PR07] Proposition 2.3.4. ∎

2.5 Different concepts of differentiability in Banach spaces

Let us recall some well-known facts from nonlinear analysis in Banach spaces (see e.g., [Wat84] and [Kno03]). Let E_1 and E_2 be two Banach spaces and let $F : E_1 \to E_2$.

Definition 2.30 (Directional derivatives) *F is said to be differentiable in the point $x \in E_1$ and along direction $y \in E_1$ if there exists an element $\partial F(x;y) \in E_2$ such that*

$$\partial F(x;y) = \lim_{h\to 0} \frac{F(x+hy) - F(x)}{h}.$$

Then $\partial F(x;y)$ is called the directional derivative of F in x_0 along direction y.

Definition 2.31 (Gâteaux differentiability) *F is said to be Gâteaux differentiable in $x \in E_1$ if there exist all directional derivatives $\partial F(x;y)$, $y \in E_1$, and if $\partial F(x;\cdot) \in \mathcal{L}(E_1, E_2)$. Then we write $\partial F(x)y$ instead of $\partial F(x;y)$ and $\partial F(x) \in \mathcal{L}(E_1, E_2)$ is called the Gâteaux derivative of F in x. If F is Gâteaux differentiable in each $x \in E_1$, we call F Gâteaux differentiable.*

Definition 2.32 (Fréchet differentiability) *F is said to be Fréchet differentiable in $x \in E_1$ if there exists an operator $DF(x) \in \mathcal{L}(E_1, E_2)$ such that for all $y \in E_1$*

$$F(x+y) = F(x) + DF(x)y + \mathcal{O}(x,y),$$

where

$$\lim_{\|y\|_{E_1}\to 0} \frac{\mathcal{O}(x,y)}{\|y\|_{E_1}} = 0.$$

Then $DF(x)$ is called the Fréchet derivative of F in x.
If F is Fréchet differentiable in each $x \in E_1$, we call F simply Fréchet differentiable.

Proposition 2.33 *Let $F : E_1 \to E_2$ be a Gâteaux differentiable. If the mapping $x \mapsto \partial F(x)$ is continuous from E_1 to $\mathcal{L}(E_1, E_2)$, then F is even Fréchet differentiable with $\partial F(x) = DF(x)$ for all $x \in E_1$.*

2.6 Introduction to Lévy processes

Our exposition and proofs are taken from [PZ07] and [App04].

Let $(G, \|\cdot\|_G)$ be a separable Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$.

Definition 2.34 *An G-valued stochastic process $L = (L_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a Lévy process if:*

(L1) *L has independent increments, i.e., for any $0 \leq t_0 < t_1 < ... < t_n$ the random variables $L(t_1) - L(t_0)$, $L(t_2) - L(t_1), ..., L(t_n) - L(t_{n-1})$ are independent;*

(L2) *the law of $L(t) - L(s)$ depends only on the difference $t - s$;*

(L3) $L(0) = 0$, *a.s.;*

(L4) *the process L is stochastically continuous, i.e., for all $t \geq 0$ and $\varepsilon > 0$ it holds*

$$\lim_{s\to t} \mathbb{P}(\|L(s) - L(t)\|_G > \varepsilon) = 0.$$

Remark 2.35 *Every Lévy process has a càdlàg modification, i.e., it is right-continuous with left limits everywhere.*

Since the Lévy process is right continuous and has left limits, the only discontinuities that can occur are of jump type.

Definition 2.36 *Let $\Delta L(t) := L(t) - L(t_-)$ be the jump of L at time t. We define*

$$N(t, B) := card\ \{0 \leq s \leq t \mid \Delta L(s) \in B\}, \quad B \in \mathscr{B}(G \setminus \{0\}),$$

where $\mathscr{B}(G \setminus \{0\}) := \{B \in H \mid B \in \mathscr{B}(H) \text{ and } 0 \notin B\}$.

A set function $N : \mathbb{N}_+ \times \mathscr{B}(G \setminus \{0\}) \times \Omega \to \mathbb{R}_+$ is called the Poisson random measure corresponding to L.

Definition 2.37 *We define $\nu(B) := \mathbb{E}\big[N(1, B)\big]$ for every $B \in \mathscr{B}(G \setminus \{0\})$. It is called the intensity measure and it is a Lévy measure on G in the sense of the following definition.*

Definition 2.38 *A (possibly infinite) measure ν on $(G \setminus \{0\}, \mathscr{B}(G \setminus \{0\}))$ is called a Lévy measure if*

$$\int_{G\setminus\{0\}} \big(\|x\|^2 \wedge 1\big)\nu(dx) < \infty.$$

(An alternative convention is to define the Lévy measure on the whole of G via the assignment $\mu(\{0\}) = 0$.)

A set function $\tilde{N} : \mathbb{N}_+ \times \mathscr{B}(G \setminus \{0\}) \times \Omega \to \mathbb{R}_+$ with

$$\tilde{N}\big(t, B\big) := N\big(t, B\big) - t\nu(B), \quad t \geq 0,\ B \in \mathscr{B}(G \setminus \{0\}), \tag{2.3}$$

is called the compensated Poisson random measure (See Section 4.3 in [PZ07]).

Theorem 2.39 (Lévy-Itô's decomposition) *If L is a G-valued Lévy process, then there exist a drift vector $b \in G$, a Q-Brownian motion B_Q on G such that B_Q is independent of $N(t,B)$ for any $B \in \mathscr{B}(G \setminus \{0\})$, and we have:*

$$L(t) = bt + B_Q(t) + \int_{\|x\|<1} x(\underbrace{N(t,dx) - t\nu(dx)}_{\tilde{N}(t,dx)}) + \int_{\|x\|\geq 1} xN(t,dx), \quad (2.4)$$

where $N(t,\cdot)$ is the Poisson random measure associated with L and ν is the corresponding Lévy measure.

Proof (cf. [AR05]) ■

There exists a special form of the Lévy-Itô's decomposition to be used in the subsequent chapters.

Proposition 2.40 *If the intensity measure of a Lévy process $(L(t))_{t\geq 0}$ additionally obeys*

$$\int_G \|x\|^2 \ \nu(dx) < \infty,$$

then the Lévy-Itô decomposition can be written as

$$L(t) = mt + B_Q(t) + \int_G x\tilde{N}(t,dx)$$

with a drift vector $m \in G$ given by $m = b + \int_{\|x\|\geq 1} x\nu(dx)$

Proposition 2.41 (Lévy-Khinchine representation) *If L is a Lévy process, then the Fourier transform of L is equal to*

$$\mathbb{E}[e^{i\langle x, L(t)\rangle}] = e^{t\lambda(x)}, \quad x \in G, \quad (2.5)$$

where $\lambda : G \to \mathbb{C}$ is measurable and takes the following form

$$\lambda(a) = i\langle a,b\rangle - \frac{1}{2}\langle a, Qa\rangle + \int_G \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a,x\rangle}{1+\|x\|^2}\right)\nu(dx), \quad (2.6)$$

where $b \in G$, $Q \in \mathcal{L}_1^+(G)$ and ν is the same Lévy measure in (2.4). The functional λ is called the Lévy symbol of the process $(L(t))_{t\geq 0}$.

Definition 2.42 *Since $L(t)$ is completely characterized by its Fourier transform (2.5) (or by its Lévy-Itô's decomposition (2.4)), we will say that it is a Lévy process with characteristics $[b, Q, \nu]$.*

Definition 2.43 *The family of measures $(\mu_t)_{t\geq 0}$ on $(G, \mathscr{B}(G))$ satisfying the following conditions:*

(i) $\mu_0 = \delta_0$ *(δ_0 is the δ-measure places at 0) and* $\mu_{s+t} = \mu_t * \mu_s$ *(convolution of the measures) for all* $t, s \geq 0$*,*

(ii) $\mu_t\big(\{x \mid \|x\|_G < r\}\big) \to 0$ *as* $t \to 0$ *for every* $r > 0$ *or equivalently* μ_t *converges weakly to* δ_0 *as* $t \to 0$*,*

is called a convolution semigroup of measures or infinitely divisible family.

Actually the Lévy-Khinchine representation holds not only for Lévy processes but for any infinitely divisible random variable (Chapter VI in [Par67]).

Theorem 2.44 (Lévy-Khinchin Formula) *A function* $\psi : G \to \mathbb{C}$ *is the characteristic function of an infinitely divisible distribution* μ *on* G *if and only if it is of the form*

$$\psi(a) = \exp\left\{ i\langle a, b\rangle - \frac{1}{2}\langle a, Qa\rangle + \int_G \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1 + \|x\|^2} \right) \nu(dx) \right\},$$

where $b \in G$*,* $Q \in \mathcal{L}_1^+(G)$ *and* ν *is a* σ*-finite Lévy measure.*

2.7 Stochastic integration with respect to Lévy measure

Via the Lévy-Itô's decomposition, we will define stochastic integrals with respect to a Lévy process. As a result of Proposition 2.40, for this type of square integrable process, two terms which need a proper analysis are the Itô-integral with respect to the Browning motion and the stochastic integral with respect to the compensated Poisson measure

Let G, H be two separable Hilbert spaces and we fix $T > 0$. Set $\Omega_T := [0, T] \times \Omega$.

Definition 2.45 *A subset* $B_T \subset \Omega_T$ *of the form* $B_T = (s, t) \times \Gamma$ *where* $\Gamma \in \mathcal{F}_s$*,* $0 \leq s < t \leq T$*, or* $\{0\} \times \Gamma$*,* $\Gamma \in \mathcal{F}_0$*, is called a predictable rectangle.*

Let $\mathcal{R}_T$ *be the* σ*-algebra generated by the family of predictable rectangles.* $\mathcal{R}_T$ *is called the* σ*-algebra of predictable sets; a stochastic process* $(X(t))_{t\in[0,T]}$ *measurable with respect to* $\mathcal{R}_T$ *is called predictable.*

2.7.1 Integration respect to Brownian motion

We will refer to [PR07] for more details.

Definition 2.46 *A* G*-valued stochastic process* B_Q *adapted to* $(\mathcal{F}_t)_{t\geq 0}$ *is a* Q*-Brownian motion if*

(B1) $B_Q(0) = 0$ *a.s. ;*

(B2) B_Q *has increments independent of the past, i.e.,* $B_Q(t) - B_Q(s)$ *is independent of* $\mathcal{F}_s$ *for every* $s \leq t$*;*

(B3) *B_Q has stationary Gaussian increments, i.e.,* $\mathbb{P}\circ\big(B_Q(t)-B_Q(s)\big)^{-1} = N(0,(t-s)Q)$ *for all* $0 \le s < t < \infty$ *where* $N(0,(t-s)Q)$ *is normal Gaussian distribution with mean zero and variance operator* $(t-s)Q$ *where* Q *is a non-negative, symmetric trace class operator in* G.

(B4) *B_Q has $\mathbb{P}$-a.s. continuous trajectories,*

Proposition 2.47 *The Q-Brownian motion $B_Q(t)$, $t \in [0,T]$, is a continuous square-integrable martingale. Moreover,* $\mathbb{E}[\|B_Q(t)\|^2] = t \cdot tr(Q) < \infty$.

For the given G-valued non-negative, symmetric trace class operator Q, we introduce the subspace $G_0 := Q^{\frac{1}{2}}(G) \subset G$ with the inner product given by

$$\langle u_0, v_0\rangle_0 := \langle Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0\rangle_G,$$

$u_0, v_0 \in G_0$, where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Then $(G_0, \langle\ ,\ \rangle_0)$ is again a separable Hilbert space.

Let us construct the stochastic integral with respect to Brownian motion.

For each $t \in [0,T]$ and a simple process of the form $X = \sum_{i=1}^n X_i \mathbb{1}_{(t_i,t_{i+1}]}$ where $0 = t_1 < ... < t_{n+1} = T$ and $X_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}; H)$, $1 \le i \le n$, the stochastic integral is defined by

$$\int_0^t X(s)dB_Q(s) := \sum_{i=1}^n X_i\big(B_Q(t_{i+1}\wedge t) - B_Q(t_i \wedge t)\big).$$

Furthermore, we have the Itô's isometry

$$\mathbb{E}\big[\|\int_0^t X(s)dB_Q(s)\|^2\big] = \mathbb{E}\big[\int_0^t \|X(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2} ds\big] = \mathbb{E}\big[\int_0^t \|X(s)\|^2_{\mathcal{L}_2^0} ds\big].$$

By the Itô's isometry, the notion of stochastic integrals can be extended to a larger class of integrands.

Definition 2.48 *Let $\mathcal{N}_B^2(T)$ be the space of all mappings X on $\Omega_T \times H$ taking values in $\mathcal{L}_2(G_0,H)$ (the space of all Hilbert-Schmidt operator from G_0 to H), such that X is predictable, i.e.,* $\mathcal{R}_T/\mathscr{B}\big(\mathcal{L}_2^0 := \mathcal{L}_2(G_0,H)\big)$*-measurable, and we have*

$$\|X\|_{\mathcal{N}_B^2(T)} := \Big(\mathbb{E}\big[\int_0^T \|X(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2} ds\big]\Big)^{\frac{1}{2}} < \infty.$$

Proposition 2.49 (Itô's Isometry) *For any $X \in \mathcal{N}_B^2(T)$ the stochastic integral*

$$\int_0^t X(s)dB_Q(s),\ t \in [0,T],$$

is well-defined and obeys

$$\mathbb{E}\big[\|\int_0^t X(s)dB_Q(s)\|^2\big] = \mathbb{E}\big[\int_0^t \|X(s)Q^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 ds\big]$$

Moreover the process $(\int_0^t X(s)dB_Q(s))_{t\in[0,T]}$ *is a continuous square-integrable martingale with respect to* $(\mathcal{F}_t)_{t\in[0,T]}$.

2.7.2 Integration respect to the compensated Poisson measure

We follow the general scheme in [Kno03] and [AR05]. We recall the σ-algebra $\mathscr{B}(G\setminus\{0\}) := \{B \in \mathscr{B}(G) | \ 0 \notin B\}$ on the set $G \setminus \{0\}$.

Definition 2.50 *An* H*-valued process* $X(t) : \Omega \times G \to H$, $t \in [0,T]$, *is said to be elementary if there exists a partition* $0 = t_1 < ... < t_{n+1} = T$ *and* $B_1, ..., B_n$ *in* $\mathscr{B}(G\setminus\{0\})$ *such that* $\nu(B_i) < \infty$ *for each* $1 \le i \le n$, *pairwise disjoint, such that*

$$X = \sum_{i=1}^n X_i \mathbb{1}_{(t_i,t_{i+1}]\times B_i},$$

where $X_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}; H)$, $1 \le i \le n$. *The linear space of all elementary processes is denoted by* $\mathcal{E}$.

For $X \in \mathcal{E}$ and $t \in [0,T]$, we define the stochastic integral by

$$\begin{aligned} Int(X)(t) &:= \int_0^t \int_{G\setminus\{0\}} X(s,x)\tilde{N}(ds,dx) \\ &:= \sum_{i=1}^n X_i \big[\tilde{N}(t_{i+1}\wedge t, B_i) - \tilde{N}(t_i \wedge t, B_i)\big] \end{aligned}$$

Then $Int(X)$ is $\mathbb{P}$-a.s. well-defined and Int is linear in $X \in \mathcal{E}$.

Proposition 2.51 *If* $X \in \mathcal{E}$, *then* $\mathrm{Int}(X)$ *is left-continuous square-integrable* $\mathcal{F}_t$*-martingale for* $t \in [0,T]$ *and*

$$\mathbb{E}\big[\| \ \mathrm{Int}(X)(t) \ \|_H^2\big] = \mathbb{E}\big[\int_0^t \int_{G\setminus\{0\}} \|X(s,x)\|_H^2 \nu(dx)ds\big].$$

Now we extend the notion of this integral:

Proposition 2.52 *Let* $\mathcal{N}_\nu^2(T)$ *be the space of all predictable* $X : \Omega_T \times G \to H$ *such that* X *is* $\big(\mathcal{R}_T \times \mathscr{B}(G)\big)/\mathcal{B}(H)$*-measurable and has the finite norm*

$$\|X\|_{\mathcal{N}_\nu^2(T)} := \Big(\int_0^T \int_{G\setminus\{0\}} \|X(s,x)\|_H^2 \nu(dx)ds\Big)^{\frac{1}{2}} < \infty.$$

Then

$$\overline{\mathcal{E}}^{\|\cdot\|_{\mathcal{N}_\nu^2(T)}} = \mathcal{N}_\nu^2(T).$$

And the statement of Proposition 2.51 extends to every $X \in \mathcal{N}_\nu^2(T)$.

2.8 Wasserstein metric

Let $\mathcal{X}$ be a Polish (i.e., separable, completely metrizable) space. Wasserstein-type metrics are used to evaluate distances between probability distributions on $\mathcal{X}$. For an extended account on this topic see [Dud89], [Rac91], [AGS08], [Vil07] and [AGS08].

Definition 2.53 *A continuous function $d : \mathcal{X}^2 \to \mathbb{R}_+$ is called pseudo-metric on $\mathcal{X}$, if*

- *$d(x,x) = 0$ for all $x \in \mathcal{X}$;*
- *$d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in \mathcal{X}$.*

Note that in this definition, from $d(x,x) = 0$ one can not conclude $x = 0$.

Definition 2.54 *A sequence $(d_n)_{n\in\mathbb{N}}$ of pseudo-metric is called increasing if d_{n+1} is larger than d_n for all $n \in \mathbb{N}$, i.e.,*

$$d_n(x,y) \le d_{n+1}(x,y), \quad \text{for all } (x,y) \in \mathcal{X}^2 \text{ and } n \in \mathbb{N}.$$

Definition 2.55 *An increasing sequence $(d_n)_{n\in\mathbb{N}}$ of pseudo-metrics on the Polish space $\mathcal{X}$ is called totally separating system of pseudo-metrics for $\mathcal{X}$ if*

$$\lim_{n\to\infty} d_n(x,y) = 1, \quad \text{for all } (x,y) \in \mathcal{X}^2,\ x \neq y.$$

The above terminology is justified by the observation that the sequence $(d_n)_{n\ge1}$ converges pointwisely to the trivial metric

$$d_{TV}(x,y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases}$$

which totally separates all the points of $\mathcal{X}$ and therefore loses completely all information about the topology of $\mathcal{X}$.

As we will see below, the totally separating system $(d_n)_{n\in\mathbb{N}}$ provides a way of approximating the total variation distance between two probability measures by a sequence of Wasserstein distances.

Definition 2.56 *Given a pseudo-metric d on $\mathcal{X}$ and let us denote by $L_d(\mathcal{X})$ the set of all Lipschitz real-valued function f on $\mathcal{X}$ with the following seminorm*

$$\|f\|_{Lip_d} := \sup_{\substack{x,y\in\mathcal{X}\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)}.$$

Then we define a pseudo-metric on the space of probability measures on $\mathcal{X}$ via

$$\|\mu_1 - \mu_2\|_d := \sup_{\|f\|_{Lip_d}=1} \left(\int_{\mathcal{X}} f(x)\mu_1(dx) - \int_{\mathcal{X}} f(x)\mu_2(dx) \right)$$

Definition 2.57 *Let us given μ_1 and μ_2, two positive finite Borel measures on $\mathcal{X}$ with equal mass, i.e., $\mu_1(\mathcal{X}) = \mu_2(\mathcal{X}) < \infty$. We denote by $\mathcal{C}(\mu_1,\mu_2)$ the set of all measures on $\mathcal{X}^2$ with marginals μ_1 and μ_2. Given a (pseudo-) metric d on $\mathcal{X}$, we define the corresponding Wasserstein distance between μ_1 and μ_2 as*

$$W_d(\mu_1,\mu_2) = \inf_{\mu\in\mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{X}^2} d(x_1,x_2)\mu(dx_1,dx_2).$$

Remark 2.58 *When $d \leq 1$ and if μ_1,μ_2 are probability measures, we can prove that always $W_d(\mu_1,\mu_2) \leq 1$ because*

$$\begin{aligned} W_d(\mu_1,\mu_2) &= \inf_{\mu\in\mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{X}^2} d(x_1,x_2)\mu(dx_1,dx_2) \\ &\leq \inf_{\mu\in\mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{X}^2} 1\ \mu(dx_1,dx_2) \\ &= \mu_1(\mathcal{X}).\mu_2(\mathcal{X}) = 1. \end{aligned}$$

We will need this property later in proving Theorem 4.32.

Lemma 2.59 *Let d be a semi-continuous pseudo-metric on $\mathcal{X}$. Then by the Kantorovich-Rubinstein duality theorem (see, e.g. [Rac91]) for every positive measures μ_1 and μ_2 with $\mu_1(\mathcal{X}) = \mu_2(\mathcal{X}) < \infty$*

$$\|\mu_1 - \mu_2\|_d = W_d(\mu_1,\mu_2).$$

The next result is crucial to the approach taken in this work (see Lemma 3.4 in [HM06])

Proposition 2.60 *Let $(d_n)_{n\in\mathbb{N}}$ be a bounded and increasing sequence of continuous pseudo-metrics on a Polish space $\mathcal{X}$. Define a new pseudo-metric*

$$d(x,y) := \lim_{n\to\infty} d_n(x,y)$$

for all $(x, y) \in \mathcal{X}^2$. *Let* μ_1 *and* μ_2 *be two probability measures on* $\big(\mathcal{X}, \mathscr{B}(\mathcal{X})\big)$. *Then*

$$\lim_{n\to\infty} W_{d_n}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{TV}.$$

Corollary 2.61 *Let* $\mathcal{X}$ *be a Polish space and* $\{d_n\}_{n\in\mathbb{N}}$ *be a totally separating system of pseudo-metrics for* $\mathcal{X}$. *Then*

$$\|\mu_1 - \mu_2\|_{TV} = \lim_{n\to\infty} \|\mu_1 - \mu_2\|_{d_n} = \lim_{n\to\infty} W_{d_n}(\mu_1, \mu_2)$$

for any two positive measures μ_1 *and* μ_2 *with equal mass on* $\mathcal{X}$.

An important fact about the Wasserstein distance is the following:

Theorem 2.62 *If* d *is a bounded metric that generates the topology of* $\mathcal{X}$, *then the corresponding Wasserstein metric generates the topology of weak convergence on the space of probability measures on* $\mathcal{X}$.

Proof See Theorem 11.3.3 in [Dud89]. ∎

2.9 Ergodic theory

For a general account on the ergodic theory of Markov processes, see [PZ96].

A dynamical system on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(\vartheta_t)_{t\in\mathbb{R}}$ of measurable maps $\vartheta : \Omega \to \Omega$ such that

- $\vartheta_t \circ \vartheta_s = \vartheta_{s+t}$ for every $s, t \in \mathbb{R}$;
- $(\vartheta_t)_{t\in\mathbb{R}}$ preserves measure $\mathbb{P}$, i.e.,

$$\mathbb{P}(\vartheta_t \Gamma) = \mathbb{P}(\Gamma), \qquad \text{for all } \Gamma \in \mathcal{F},\ t \in \mathbb{R}.$$

We will denote this dynamical system by $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$.

Definition 2.63 *A dynamical system* $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ *is said to be continuous if the induced group of linear transformations on* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *defined by*

$$U_t\xi(\omega) := \xi(\vartheta_t\omega), \quad \xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}),\ t \in \mathbb{R},$$

is continuous for every $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.64 *Let* $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ *be a continuous dynamical system. A set* $\Gamma \in \mathcal{F}$ *is said to be invariant with respect to* $\mathcal{S}$ *if*

$$\mathbb{P}(\vartheta_t \Gamma \cap \Gamma) = \mathbb{P}(\Gamma) = \mathbb{P}(\vartheta_t \Gamma), \qquad \textit{for all } t \in \mathbb{R}.$$

Definition 2.65 *A dynamical system $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is called ergodic if*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}(\vartheta_{-t}\Gamma_1 \cap \Gamma_2)dt = \mathbb{P}(\Gamma_1)\mathbb{P}(\Gamma_2), \quad \textit{for all } \Gamma_1, \Gamma_2 \in \mathcal{F}.$$

Proposition 2.66 *Let $\mathcal{S}$ be a continuous dynamical system. Then $\mathcal{S}$ is ergodic if and only if for any invariant set Γ either $\mathbb{P}(\Gamma) = 0$ or $\mathbb{P}(\Gamma) = 1$.*

Proof refer to Theorem 1.2.4 in [PZ96]. ■

Now, with a given Markovian semigroup $(p_t)_{t\geq 0}$ on $\mathcal{B}_b(H)$ having an invariant probability measure μ on $(H, \mathscr{B}(H))$ (cf. Definitions 2.7 and 2.13) we will associate, in a unique way, a dynamical system $\mathcal{S}^\mu = (\Omega, \mathcal{F}, \mathbb{P}_\mu, (\vartheta_t)_{t\in\mathbb{R}})$. We just sketch the proof; for more details see [Hai08].

Let $\Omega = H^{\mathbb{R}}$ and $\mathcal{F} = (\mathscr{B}(H))^{\otimes\mathbb{R}}$. Consider a set of bounded cylinder functions $\varphi : H^{\mathbb{R}} \to \mathbb{R}$ which can be represented as $\varphi(x) = \tilde{\varphi}(x_1, ..., x_n)$. For every such φ with n-tuple of times $t_1 < ... < t_n$, we define

$$\begin{aligned}\mathbb{P}_\mu(\varphi) &= \mathbb{P}_\mu^{t_1,\dots,t_n}(\tilde{\varphi}) \\ &= \int_H \dots \int_H \tilde{\varphi}(x_1, ..., x_n) p_{t_n - t_{n-1}}(x_{n-1}, dx_n) ... p_{t_2 - t_1}(x_1, dx_2)\mu(dx_1).\end{aligned}$$

It is straightforward to check that this family of specifications is consistent and therefore, by Kolmogorov's extension theorem, there exists a unique measure $\mathbb{P}_\mu$ on $H^{\mathbb{R}}$ such that above equality holds.
We introduce a group of shift operators $\vartheta_t : \Omega \to \Omega$ by

$$(\vartheta_t\omega)(s) := w(s+t), \quad s, t \in \mathbb{R}.$$

Since μ is invariant, the transformations $(\vartheta_t)_{t\in\mathbb{R}}$ preserve the measure $\mathbb{P}_\mu$. As a result, $\mathcal{S}_\mu$ is a dynamical system.

Definition 2.67 *Let μ be an invariant measure with respect to the semigroup $(p_t)_{t\geq 0}$. Then μ is called ergodic if the corresponding dynamical system $\mathcal{S}^\mu$ is ergodic.*

Let us recall the following important property of invariant measures, cf. Proposition 2.14:
Assume that μ is an invariant Borel probability measure for the (stochastically continuous) Markovian semigroup $(p_t)_{t\geq 0}$. Then for all $t \geq 0$ and $q \geq 1$, $(p_t)_{t\geq 0}$ is uniquely extendible to a contractive $\mathcal{C}_0$-semigroup of linear bounded operator on $L^q(H, \mu)$ that we still denote by $(p_t)_{t\geq 0}$.

Theorem 2.68 *Let $(p_t)_{t\geq 0}$ be a stochastically continuous Markovian semigroup with an invariant measure μ. Then following conditions are equivalent:*

(i) *μ is ergodic;*

(ii) *If $f \in L^2(H, \mu)$ and we have*

$$p_t f = f, \quad \mu - a.s., \text{ for all } t > 0,$$

then f is μ-a.s. constant;

(iii) *If for a set $B \in \mathscr{B}(H)$*

$$p_t \mathbb{1}_B = \mathbb{1}_B, \quad \mu - a.s., \text{ for all } t \geq 0,$$

then either $\mu(B) = 0$ or $\mu(B) = 1$;

(iv) *For an arbitrary $f \in L^2(H, \mu)$*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T p_t f dt = \int_H f d\mu. \tag{2.7}$$

(the convergence and Bochner integrals are understood in the sense of $L^2(H, \mu)$.)

Proof For all details see the proof of Theorem 3.2.4 in [PZ96]. However, in the proof in [PZ96], there are missing arguments which allow to show that (*i*) implies (*iv*). Indeed, they are only able to prove the weak convergence in (2.7) and the rest of the arguments for the strong convergence does not work.
But by the *von Neumann ergodic theorem* (see, for instance, Section 5.4 in [Pra06]) the limit $\frac{1}{T} \int_0^T p_s f ds$ does exist in $L^2(H, \mu)$. Therefore by the consistency of the weak and strong limits we get the strong convergence in (2.7). ∎

Lemma 2.69 *Let μ be an ergodic measure and ν an invariant measure of $(p_t)_{t \geq 0}$ such that ν is absolutely continuous with respect to μ with a (μ-a.s.) bounded $\rho := \frac{d\nu}{d\mu}$. Then $\mu = \nu$.*

Proof Fix any $B \in \mathscr{B}(H)$. From the ergodicity of μ, by Theorem 2.68 it follows that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T p_s \mathbb{1}_B ds = \mu(B), \quad \text{in } L^2(H, \mu).$$

Therefore there exists a sequence $(T_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} T_n = \infty$, such that

$$\lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} p_s \mathbb{1}_B ds = \mu(B), \quad \mu - a.s.. \tag{2.8}$$

Since $\nu \ll \mu$, identity (2.8) holds also ν-a.s. Indeed, for any $F_n \to F$, $n \to \infty$, in $L^2(H,\mu)$, we can write

$$\int_H G\ F_n\ \rho d\mu = \int_H \underbrace{G\rho}_{\in L^2(H,\mu)}\ F_n\ d\mu$$
$$\xrightarrow{n\to\infty} \int_H G\rho\ F\ d\mu = \int_H G\ F\ \rho d\mu = \int_H G\ F\ d\nu$$

for every $G \in L^2(H,\rho\mu)$.
Integrating with respect to ν now yields for all $n \in \mathbb{N}$

$$\int_H \left(\frac{1}{T_n}\int_0^{T_n} p_t \mathbb{1}_B(x)dt\right)\nu(dx) = \frac{1}{T_n}\int_0^{T_n} \big(\int_H p_t\mathbb{1}_B(x)\nu(dx)\big)dt = \nu(B)$$

according to the invariance of ν.
Hence, letting $n \to \infty$ we get by (2.8)

$$\nu(B) = \lim_{n\to\infty}\int_H \left(\frac{1}{T_n}\int_0^{T_n} p_t\mathbb{1}_B(x)dt\right)\nu(dx) = \int_H \mu(B)\nu(dx) = \mu(B)$$

Now the assertion follows due to the arbitrary choice of $B \in \mathscr{B}(H)$. ■

Theorem 2.70 *The set of all ergodic Borel probability measures for* $(p_t)_{t\geq 0}$ *coincides the set of all extremal points of* $\mathrm{Inv}(p)$.

PROOF Let μ be an ergodic measure. Assume by contradiction that μ is not an extremal point of *Inv*(p). Then there exist $\mu_1, \mu_2 \in$ *Inv*(p) with $\mu_1 \neq \mu_2$ and $\alpha \in (0,1)$ such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$. Hence $\mu_1 \ll \mu$ and $\mu_2 \ll \mu$, this is a contradiction according to Lemma 2.69. Therefore μ is an extremal point of *Inv*(p).
Conversely, let μ be an extremal point of *Inv*(p) and assume that μ is not ergodic. Then there exists a set $B \in \mathscr{B}(H)$ such that $0 < \mu(B) < 1$ and

$$p_t\mathbb{1}_B = \mathbb{1}_B, \quad \mu - a.s., \text{ for all } t \geq 0. \tag{2.9}$$

Define measures μ_1 and μ_2 by the formula

$$\mu_1(D) := \frac{1}{\mu(B)}\mu(D\cap B),\ \mu_2(D) := \frac{1}{\mu_(B^c)}\mu(D\cap B^c), \quad D \in \mathscr{B}(H).$$

We should show that μ_1 and μ_2 are invariant measures for $(p_t)_{t\geq 0}$. We check this for μ_1 and for μ_2 will be the same.

$$\begin{aligned} p_t^*\mu_1(D) &= \int_H \pi_t(x,D)\mu_1(dx) = \frac{1}{\mu(B)}\int_H \pi_t(x,D)\mu(dx) \\ &= \frac{1}{\mu(B)}\int_H \pi_t(x,D\cap B)\mu(dx) + \frac{1}{\mu(B)}\int_H \pi_t(x,D\cap B^c)\mu(dx). \end{aligned}$$

It follows from (2.9) that

$$\pi_t(x, D \cap B) \leq \pi_t(x, B) = 0, \quad \mu - a.e., \ x \in B^c$$

and

$$\pi_t(x, D \cap B^c) \leq \pi_t(x, B^c) = 0, \quad \mu - a.e., \ x \in B.$$

Therefore

$$\begin{aligned} p_t^* \mu_1(D) &= \frac{1}{\mu(B)} \int_H \pi_t(x, D \cap B) \mu(dx) \\ &= \frac{1}{\mu(B)} \mu(D \cap B) = \mu_1(D) \end{aligned}$$

so that μ_1 is invariant for $(p_t)_{t \geq 0}$. Furthermore, since obviously $\mu_1 \neq \mu_2$ and

$$\mu = \mu(B)\mu_1 + (1 - \mu(B))\mu_2,$$

μ could not be extremal which contradicts to the assumption above. ∎

Chapter 3

Uniqueness of Invariant Measures

This chapter is a slightly modified version of [Hai08] and Chapters 5 and 7 in [Pra06]. Furthermore, we additionally point out to [Sch09] which has been reviewed these results.

We present two techniques of proving uniqueness of invariant measures. The first approach is based on the so-called strong Feller property (see [PZ96] Theorem 4.2.1) and the second one is employs the so-called asymptotic strong Feller property initially developed in [HM06].
We stress that the original papers were concerned with the autonomous case, therefore the associated dynamics are governed by are one-parameter semigroups.
For the convenience of the reader, in this chapter we point out the main issues of the above techniques applied to the one-parameter semigroups. This can be seen as a necessary preparation for the further constructions of Chapter 4 dealing with the two-parameter semigroups.

Recall that H is a real separable Hilbert space. By $\mathcal{B}_b(H)$ (resp. $\mathcal{C}_b(H)$) we denote the space of all bounded, Borel measurable (resp. continuous) functions $f : H \to \mathbb{R}$.

3.1 Uniqueness by using strong Feller property

In this section, our aim is to define the strong Feller property for the Markovian semigroup. Thereafter the method of proving the uniqueness of invariant measure by using the strong Feller property will be presented.

Definition 3.1 *A Markovian semigroup $p = (p_t)_{t\geq 0}$ is called strong Feller at time $t > 0$ if $p_t f \in \mathcal{C}_b(H)$ for all $f \in \mathcal{B}_b(H)$. It is called strong Feller if it is strong Feller at all times $t > 0$.*

There is a sufficient condition to check the strong Feller property for the semigroup $(p_t)_{t\geq 0}$.

Proposition 3.2 *Let H be a separable Hilbert space and $(p_t)_{t\geq 0}$ a Markovian semigroup on $\mathcal{B}_b(H)$. If for any function $f \in \mathcal{B}_b(H)$*

$$|p_t f(x) - p_t f(y)| \leq C(\|x\| \vee \|y\|) \cdot \|f\|_\infty \cdot \|x - y\|$$

for all $x, y \in H$ and $t > 0$, where $C : \mathbb{R}_+ \to \mathbb{R}$ is a fixed non-decreasing function, then $(p_t)_{t\geq 0}$ is strong Feller.

Definition 3.3 *The Markovian semigroup $(p_t)_{t\geq 0}$ is called irreducible at time $t > 0$ if $\pi_t(x, B(x_0, \delta)) > 0$, for all $\delta > 0$ and $x, x_0 \in H$ and respectively is called irreducible if it is irreducible at all times $t > 0$.*

Definition 3.4 *The Markovian semigroup $(p_t)_{t\geq 0}$ is called regular at time $t > 0$ if all probability measures $\pi_t(x, .)$, $x \in H$, are mutually equivalent. It is called regular if it is regular at all times $t > 0$.*

The following uniqueness criteria is well-known, see [PZ96], Theorem 4.2.1.

Theorem 3.5 *Let $(p_t)_{t\geq 0}$ be a Markovian semigroup which is strong Feller and irreducible. Then there is at most one invariant Borel probability measure for $(p_t)_{t\geq 0}$.*

3.2 Motivation to introduce the asymptotic strong Feller property

One of the main features of the strong Feller property is given by the following proposition.

Proposition 3.6 *If a Markovian semigroup $(p_t)_{t\geq 0}$ over H has the strong Feller property, then the topological supports of any two mutually singular invariant measures are disjoint.*

Proof See Theorem 7.7 in [Hai08]. ■

Recall that the support of any probability measure μ on $(H, \mathscr{B}(H))$, is defined by

$$supp(\mu) := \{x \in H \mid \mu(B(x, r)) > 0, \quad \text{for all } r > 0\}.$$

On the other hand, there is a weaker formulation of Definition 3.1 where we can introduce the notion of being strong Feller at some $x \in H$ (but not on the whole H).

Definition 3.7 *The Markovian semigroup* $(p_t)_{t\geq 0}$ *is called strong Feller at* $x \in H$ *if the function* $x \mapsto p_t f(x)$ *is continuous at* x *for all* $f \in \mathcal{B}_b(H)$ *and all* $t > 0$.

Definition 3.1 then means that Definition 3.7 holds for all $x \in H$.

As a result, the theorem about uniqueness of invariant measures can be stated as follows:

Theorem 3.8 *If the semigroup* $(p_t)_{t\geq 0}$ *is strong Feller (at every point in* H*) and there exists some point* $x \in H$ *such that* $x \in \operatorname{supp}(\mu)$ *for every invariant Borel probability measure* μ *for* $(p_t)_{t\geq 0}$*, then there exists at most one invariant measure* μ.

Definition 3.9 *Let* m *be a finite signed measure on* $(H, \mathscr{B}(H))$ *with Jordan decomposition* $m = m^+ - m^-$. *Then the total variation norm of* m *is given by*

$$\|m\|_{TV} := \frac{1}{2}\big(m^+(H) + m^-(H)\big).$$

We have the following characterization of the strong Feller property by the associated transition kernels $(\pi_t)_{t\geq 0}$.

Theorem 3.10 *Let* H *be a Polish space and* $(p_t)_{t\geq 0}$ *be a Markovian semigroup on* $B_b(H)$. *Then* $(p_t)_{t\geq 0}$ *is strong Feller if and only if for all* $t \geq 0$ *the transition probabilities* $\pi_t(x, \cdot)$ *are continuous in the parameter* x *with respect to the total variation norm* $\| \,.\, \|_{TV}$.

In applications, the strong Feller property often fails to hold for infinite dimension stochastic PDEs, simply because it is typically so far that any two measures in such spaces be mutually singular. It would therefore be extremely convenient to have a weaker property that still allows to get a statement similar to Proposition 3.6.

This is the idea of asymptotic strong Feller property which, instead of prescribing a smoothing property at a fixed time $t > 0$ (c.f. Theorem 3.10), prescribes some kind of smoothing property at time ∞. In fact, it is expected that such an asymptotic smoothing property will be sufficient to conclude that the topological supports of distinct ergodic invariant measures are disjoint.

Now, as we have mentioned in Section 2.8, we have by Proposition 2.60 that

$$\lim_{n\to\infty} W_{d_n}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{TV}.$$

Furthermore, by Theorem 2.62, if d is a bounded metric that generates the topology of H, then the corresponding Wasserstein metric generates the topology of weak convergence on the space of probability measures on H. Therefore as a result, we can use the Wasserstein metrics W_d to define a new weaker property as compared to the strong Feller property.

3.3 Uniqueness by using asymptotic strong Feller property

Let $(p_t)_{t\geq 0}$ is a Markovian one-parameter semigroup with the transition kernels $(\pi_t)_{t\geq 0}$.

Definition 3.11 *(See [HM06]) A Markovian semigroup $(p_t)_{t\geq 0}$ on $\mathcal{B}_b(H)$ is called asymptotically strong Feller at a given point $x \in H$ if there exists a totally separating system of continuous pseudo-metrics $(d_n)_{n\in\mathbb{N}}$ on H and a non-decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$, such that we have*

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} \sup_{y\in B(x,\gamma)} W_{d_n}\big(\pi_{t_n}(x,.), \pi_{t_n}(y,.)\big) = 0, \tag{3.1}$$

where $B(x,\gamma)$ denotes the open ball of radius $\gamma > 0$ centered at $x \in H$. Here W_d is the Wasserstein metric defined in Section 1.7.
The semigroup $(p_t)_{t\geq 0}$ is called asymptotically strong Feller, if it is asymptotically strong Feller at every $x \in H$.

Remark 3.12 *For Polish (e.g., Hilbert) spaces, the above definition is equivalent to the following one ([HM06]):*

Definition 3.13 *A Markovian one-parameter semigroup $(p_t)_{t\geq 0}$ on a Hilbert space H is called asymptotically strong Feller at any $x \in H$ if there exists a totally separating system of pseudo-metrics $(d_n)_{n\in\mathbb{N}}$ for H and a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that*

$$\inf_{U\in\mathcal{U}_x} \limsup_{n\to\infty} \sup_{y\in U} W_{d_n}(\pi_{t_n}(x,.), \pi_{t_n}(y,.)) = 0,$$

in which $\mathcal{U}_x$ is the collection of all open sets $U \subseteq H$ containing x.

One can see that the asymptotic strong Feller property is a natural generalization of the strong Feller property.

Proposition 3.14 *Let H be a Hilbert space and $(p_t)_{t\geq 0}$ a Markovian semigroup on $\mathcal{B}_b(H)$ which is strong Feller. Then $(p_t)_{t\geq 0}$ is asymptotically strong Feller.*

Proof Fix an arbitrary $x \in H$ and consider a total separating system of continuous pseudo-metrics $(d_n)_{n\in\mathbb{N}}$ for H. Let us choose a constant sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \equiv t$, $n \in \mathbb{N}$ for some fixed $t > 0$. As it has been pointed in Corollary 2.61, we have

$$W_{d_n}(\pi_t(x,.), \pi_t(y,.)) \leq \|\pi_t(x,.) - \pi_t(y,.)\|_{TV}$$

for all $n \in \mathbb{N}$ and $y \in H$. Hence $\gamma \to 0$, we get

$$\begin{aligned} &\lim_{\gamma\to 0} \limsup_{n\to\infty} \sup_{y\in B(x,\gamma)} W_{d_n}(\pi_t(x,.),\pi_t(y,.)) \\ \leq &\lim_{\gamma\to 0} \sup_{y\in B(x,\gamma)} \|\pi_t(x,.)-\pi_t(y,.)\|_{TV}. \end{aligned}$$

On the other hand, since the semigroup $(p_t)_{t\geq 0}$ is strong Feller, $(\pi_t)_{t\geq 0}$ is continuous in the total variation norm by Theorem 3.10. Hence the right hand side of the above equation equals 0.
Since $x \in H$ is arbitrary, the Definition 3.11 of the asymptotic strong Feller for $(p_t)_{t\geq 0}$ is satisfied with the constant sequence $t_n \equiv t > 0$. ∎

A useful criterion (comparable to Proposition 3.2) for checking the validity of the asymptotic strong Feller property for a given Markovian semigroup is as follows:

Proposition 3.15 *Let $(t_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be two positive sequences with t_n increasing to infinity and δ_n converging to zero as $n \to \infty$. A semigroup $(p_t)_{t\geq 0}$ on a Hilbert space H is asymptotically strong Feller if, for all Fréchet differentiable $f : H \to \mathbb{R}$ with $\|f\|_\infty < \infty$ and $\|\nabla f\|_\infty < \infty$,*

$$|p_t f(x) - p_t f(y)| \leq C(\|x\| \vee \|y\|) \cdot (\|f\|_\infty + \delta_n \cdot \|\nabla f\|_\infty) \cdot \|x-y\|$$

for all $n \in \mathbb{N}$ and $x, y \in H$, where $C : \mathbb{R}+ \to \mathbb{R}$ is a fixed non-decreasing function.

As an important consequence of the asymptotic strong Feller property, one does have the following analogue of Proposition 3.6:

Proposition 3.16 *Let $(p_t)_{t\geq 0}$ a Markovian semigroup on $\mathcal{B}_b(H)$ and consider two different ergodic Borel probability measures for $(p_t)_{t\geq 0}$. If $(p_t)_{t\geq 0}$ is asymptotically strong Feller, then the intersection of the topological support of this two ergodic measures is empty.*

The above proposition, leads to the following uniqueness criteria for invariant measures.

Theorem 3.17 *Let $(p_t)_{t\geq 0}$ be an asymptotically strong Feller Markovian semigroup on $\mathcal{B}_b(H)$ and assume that there exists a point $x \in H$ such that $x \in \text{supp}(\mu)$ for every invariant Borel probability measure μ for $(p_t)_{t\geq 0}$. Then there exists at most one invariant Borel probability measure μ.*

In the next chapter, we will extend these results to the case of two-parameter semigroups.

Chapter 4

Uniqueness of Evolution System of Measures

In many important classes of stochastic differential equations, coefficients are time dependent. So, their solutions are inhomogeneous Markov processes and therefore the associated evolution operators constitute a two-parameter semigroup.
Concerning the uniqueness of evolution system of measures for such two-parameter semigroups, we should mention that this is a nontrivial problem. There is a theory proving uniqueness of a T-periodic evolution system of measures when the semigroup is T-periodic irreducible at one point and asymptotically strong Feller. This theory, being a generalization of the recent results of Hairer and Matingly for one-parameter semigroups, was developed in the particular case of the 2D Navier Stokes equation with time-periodic coefficients by Da Prato and Debussche and presented in [PD08].
This chapter is an extension of the above mentioned works to an arbitrary T-periodic semigroup, without any implicit relation to a concrete SDE generating such a semigroup. Also, in [PD08] there are some gaps and drawback in proving the main theorem, which will be filled out and corrected in our work.

4.1 Definitions

Let us use the notation $\mathscr{P}(H)$ for the set of all probability measures on $(H, \mathscr{B}(H))$. We introduce the topology of weak convergence for measures from $\mathscr{P}(H)$. This is the weakest topology such that the mapping

$$\mathscr{P}(H) \ni \mu \mapsto \int f \; d\mu(x)$$

is continuous for any $f \in \mathcal{C}_b(H)$. Note that $\mathscr{P}(H)$ with the weak topology is a Polish space, i.e., there exists a metric on $\mathscr{P}(H)$ which is consistent with this topology and $\mathscr{P}(H)$ with this metric becomes a complete separable metric space.

The generalization of Definition 2.7 for two-parameter semigroups is given as follow:

Definition 4.1 *A family $(p_{s,t})_{s\le t}$ of linear bounded operators on $\mathcal{B}_b(H)$ is called a Markovian two-parameter semigroup (or Markovian evolution family) if:*

(i) $p_{s,s} = \mathbb{1}$ *for all* $s \in \mathbb{R}$, *where* $\mathbb{1}$ *is the identity operator in* H,

(ii) $p_{s,t} = p_{s,r}p_{r,t}$ *for all* $s \le r \le t$,

(iii) $p_{s,t}\mathbb{1} = \mathbb{1}$.

(iv) *For every* $s \le t$, $p_{s,t}f$ *is positive whenever* f *is positive.*

Remark 4.2 *A typical example of two-parameter semigroups $(p_{s,t})_{s\le t}$ arises in the theory of infinite dimensional stochastic differential equation with coefficients depending on time. Namely, we define*

$$p_{s,t}f(x) = \mathbb{E}\big[f(X(s,t,x))\big], \quad f \in \mathcal{C}_b(H),$$

where $X(s,t,x)$ is the solution starting at time $s \in \mathbb{R}$ and at point x (Some classes of such semigroups will be studied in the last chapter of this thesis). Then, in this case, an evolution system of measures (μ_t) indexed by $t \in \mathbb{R}$ is a measure-valued solution of the dual Kolmogorov equation.

Definition 4.3 *Any Markovian two-parameter semigroup $(p_{s,t})_{s\le t}$ on $\mathcal{B}_b(H)$ (uniquely) defines a family of transition probability kernels on $\big(H, \mathscr{B}(H)\big)$ by*

$$\pi_{s,t}(x, B) := p_{s,t}\mathbb{1}_B(x), \quad x \in H,\ B \in \mathscr{B}(H),$$

and therefore for any $f \in \mathcal{B}_b(H)$ we have

$$p_{s,t}f(x) = \int_H f(y)\pi_{s,t}(x, dy), \quad x \in H. \tag{4.1}$$

We call $\pi = (\pi_{s,t})_{s\le t}$, which is indexed by $s,t \in \mathbb{R}$, $s \le t$, the Markovian two-parameter semigroup of transition kernels.

Remark 4.4 *From the identity $p_{s,t} = p_{s,r}p_{r,t}$ for all $s \le r \le t$, we get*

$$\pi_{s,t}(x, B) = \int_H \pi_{r,t}(y, B)\pi_{s,r}(x, dy) \tag{4.2}$$

for all $x \in H$ and $B \in \mathscr{B}(H)$. This is the so-called Kolmogorov-Chapman equality similar to equation (2.2), but now it relates to the two-parameter

case. Therefore the heuristic interpretation of equality (4.2) *which can be given as follows:*
the probability for a particle starting at time $s \in \mathbb{R}$ in $x \in H$ to be in $B \in H$ at time $t \geq s$ is equal to the probability that the particle starts at time s in $x \in H$ and being in some infinitesimal small volume dy at time r and then starting new in $y \in H$ at time $r \in [s,t]$ and being in the subset B at time t integrated over all intermediate points $y \in H$.

Definition 4.5 *The semigroup $p = (p_{s,t})_{s \leq t}$ is called forward continuous if for all $f \in \mathcal{C}_b(H)$, $x \in H$ and $s \leq t$*

$$\lim_{\substack{r \to t \\ r \geqslant s}} p_{s,r} f(x) = p_{s,t} f(x).$$

Definition 4.5 just means, for each $x \in H$, the continuity of the map

$$[s, +\infty) \ni t \mapsto \pi_{s,t}(x, \cdot) \in \mathscr{P}(H)$$

in the topology of weak convergence of measures on $(H, \mathscr{B}(H))$.

Definition 4.6 *A family $(p^*_{s,t})_{s \leq t}$ of operators on $\mathscr{P}(H)$ is called the semigroup of transposed operators associated with $(p_{s,t})_{s \leq t}$ if*

$$p^*_{s,t} \mu_s(B) = \int_H \pi_{s,t}(x, B) \mu_s(dx), \quad s \leq t, \; B \in \mathscr{B}(H),$$

where clearly $p^*_{r,t} p^*_{s,r} = p^*_{s,t}$.

Let us consider a Markovian two-parameter semigroup $(p_{s,t})_{s \leq t}$ with the corresponding Markovian two-parameter semigroup of transition kernels $(\pi_{s,t})_{s \leq t}$. Now we will give a natural generalization of the notion of invariant measures for two-parameter semigroups.

Definition 4.7 *Consider a mapping*

$$\mathbb{R} \ni t \mapsto \mu_t \in \mathscr{P}(H)$$

such that for every $f \in \mathcal{B}_b(H)$

$$\int_H p_{s,t} f(x) \mu_s(dx) = \int_H f(x) \mu_t(dx), \quad s \leq t; \tag{4.3}$$

or equivalently

$$p^*_{s,t} \mu_s = \mu_t.$$

Then every such $(\mu_t)_{t \in \mathbb{R}}$ will be called an evolution system of measures (e.g. [PR08], [KLL10]).

Remark 4.8 *If* $(\pi_{s,t})_{s\le t}$ *are transition kernels of* $(p_{s,t})_{s\le t}$*, then one can rewrite equation* (4.3) *as*

$$\int_H \pi_{s,t}(x,B)\mu_s(dx) = \mu_t(B), \quad s\le t,\ B\in\mathscr{B}(H).$$

From this point of view, in the literature $(\mu_t)_{t\in\mathbb{R}}$ *is called the* π*-entrance law. We denote the set of all* π*-entrance laws (or evolution systems of measures) by* $\mathcal{K}(\pi)$*.*

Remark 4.9 *If* $(p_{s,t})_{s\le t}$ *is forward continuous, then obviously* $t\mapsto\mu_t$ *is continuous in the topology of weak convergence of measures from* $\mathscr{P}(H)$*, i.e.,*

$$\mathbb{R}\ni t\mapsto\int_H f(x)\mu_t(dx)$$

is continuous for every $f\in\mathcal{C}_b(H)$*.*

Hypothesis: In what follows we will always assume that $t\mapsto\mu_t$ **is weakly continuous.**

Definition 4.10 *Let for a given* $T>0$*, the two-parameter semigroup* $(p_{s,t})_{s\le t}$ *obeys*

$$p_{s+T,t+T} = p_{s,t}.$$

Then, we call it a T*-periodic two-parameter semigroup.*

Definition 4.11 *Given* $T>0$*, the family* $(\mu_t)_{t\in\mathbb{R}}$ *is called* T*-periodic if*

$$\mu_{t+T}=\mu_t, \quad t\in\mathbb{R}.$$

When we are dealing with a T-periodic semigroup $p=(p_{s,t})_{s\le t}$, it seems natural to restrict our considerations only to T-periodic entrance laws $(\mu_t)_{t\in\mathbb{R}}$ (for a further motivation see e.g. [PD08]).
In the case of dissipative stochastic equations (cf. [PR06], [PL07]), one has under reasonable assumptions that

$$\lim_{s\to-\infty} p_{s,t}f(x) = \int_H f(y)\mu_t(dy), \quad x\in H,\ f\in\mathcal{C}_b(H),$$

and therefore the unique evolution system of measures corresponding to this T-periodic semigroup $(p_{s,t})_{s\le t}$ ought to be T-periodic.

4.2 Space-time homogenization semigroup

Let us fix some $T > 0$ and identify the end points of the time interval $[0, T]$. Then we get a circle (torus) S_T of length T. Then $S_T \cong [0, T]$ with $t + kT = t \ (mod\ T)$ for all $k \in \mathbb{Z}$.

We introduce the space $\mathcal{B}_b(S_T \times H)$ consisting of all Borel measurable bounded functions $f : S_T \times H \to \mathbb{R}$. Obviously, $\mathcal{B}_b(S_T \times H)$ is isomorphic to $\mathcal{B}_b^{per}(\mathbb{R} \times H)$ being the space of all Borel measurable and T-periodic bounded functions $f : \mathbb{R} \times H \to H$ such that $f(t + T, .) = f(t, .)$ for each $t \in \mathbb{R}$. In what follows, we will not distinguish between functions $f \in \mathcal{B}_b(S_T \times H)$ and the functions from $\mathcal{B}_b^{per}(\mathbb{R} \times H)$.

Note that $\mathcal{B}_b(S_T \times H)$ is a Banach space with the norm

$$\|f\| := \sup_{(t,x) \in S_T \times H} |f(t, x)|.$$

In this space we define the family of operators $(\mathcal{P}_\tau)_{\tau \geq 0}$ acting by

$$\mathcal{P}_\tau f(t, x) = p_{t,t+\tau} f(t + \tau, .)(x), \quad \tau \geq 0, \ (t, x) \in \mathbb{R} \times H. \tag{4.4}$$

It is called the *space-time homogenization semigroup* associated with $(p_{s,t})_{s \leq t}$ (see [PL07], [Knä11]).

Note that we need to assume that $(p_{s,t})_{s \leq t}$ is also T-periodic and we will keep this assumption everywhere we use the homogenization semigroup.

Lemma 4.12 *$\mathcal{P}_\tau(\mathcal{B}_b(S_T \times H)) \subset \mathcal{B}_b(S_T \times H)$ for all $\tau \geq 0$.*

Proof Since for any $f \in \mathcal{B}_b(S_T \times H) \cong \mathcal{B}_b^{per}(\mathbb{R} \times H)$ we have $f(t + T, \cdot) = f(t, \cdot)$, therefore for every $x \in H$ and $\tau \geq 0, \ t \in \mathbb{R}$

$$\mathcal{P}_\tau f(t + T, x) = p_{t+T,t+T+\tau} f(t + T + \tau, \cdot)(x) = p_{t,t+\tau} f(t + \tau, \cdot)(x) = \mathcal{P}_\tau f(t, x),$$

where the second equality was obtained from the T-periodicity of $(p_{s,t})_{s \leq t}$ and $f \in \mathcal{B}_b^{per}(\mathbb{R} \times H)$. ∎

Lemma 4.13 *$(\mathcal{P}_\tau)_{\tau \geq 0}$ is a Markovian semigroup (in the sense of Definition 2.7) in $\mathcal{B}_b(S_T \times H)$.*

Proof Since $(p_{s,t})_{s \leq t}$ is an Markovian evolution family, for every $\eta, \tau \geq 0$ and $f \in \mathcal{B}_b(S_T \times H)$ we have

$$\begin{aligned}
\mathcal{P}_\eta \mathcal{P}_\tau f(s, x) &= p_{s,s+\eta} \mathcal{P}_\tau f(s + \eta, .)(x) \\
&= p_{s,s+\eta} p_{s+\eta,s+\eta+\tau} f(s + \eta + \tau, .)(x) \\
&= p_{s,s+\eta+\tau} f(s + \eta + \tau, .)(x) \\
&= \mathcal{P}_{\eta+\tau} f(s, x)
\end{aligned}$$

for all $s \in \mathbb{R}$ and $x \in H$.
It remains to check the Markovian property $\mathcal{P}_\tau \mathbb{1} = \mathbb{1}$ for every $\tau \geq 0$. Indeed,

$$\begin{aligned}(\mathcal{P}_\tau \mathbb{1})(t,x) &= p_{t,t+\tau}\mathbb{1}(t,.)(x)\\ &= \int_H \mathbb{1}(t,.)(y)\pi_{t,t+\tau}(x,dy) = 1\end{aligned}$$

for any $x \in H$ and $t \in \mathbb{R}$. ■

Remark 4.14 *Since the semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$ is defined by $(p_{s,t})_{s\leq t}$, we can think about its transition kernels and how they are related with $(\pi_{s,t})_{s\leq t}$. Indeed, for any $f \in \mathcal{B}_b(S_T \times H)$*

$$\begin{aligned}\mathcal{P}_\tau f(s,x) &= p_{s,s+\tau} f(s+\tau,\cdot)(x)\\ &= \int_H f(s+\tau,\cdot)(y)\pi_{s,s+\tau}(x,dy)\\ &= \int_{S_T}\int_H f(t,y)\delta_{s+\tau}(t)\pi_{s,s+\tau}(x,dy)dt.\end{aligned}$$

So P_τ obeys a transition kernel, denoted by S_τ, which acts by

$$S_\tau((s,x),dt\ dy) = \delta_{s+\tau}(dt)\pi_{s,s+\tau}(x,dy), \quad (s,x) \in \mathbb{R}\times H,\ \tau \geq 0,$$

where $\delta_\tau(dt)$ is the δ-measure on S_T placed at τ.

4.3 Relation between T-periodic evolution systems of measures for $(p_{s,t})_{s\leq t}$ and invariant measures for the corresponding space-time homogeneous semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$

Our next aim is to show that there exists a *one-to-one correspondence* between T-periodic evolution system of measures associated with T-periodic Markovian two-parameter semigroup $(p_{s,t})_{s\leq t}$ and invariant measures for $(\mathcal{P}_\tau)_{\tau\geq 0}$.

Let us start with looking for candidates to be invariant measures for semigroup $(\mathcal{P}_\tau)_{\tau\geq 0}$,

Definition 4.15 *Let $(\mu_t)_{t\in\mathbb{R}}$ be a T-periodic evolution system of measures for $(p_{s,t})_{s\leq t}$, which is equivalent to say that $(\mu_t)_{t\in S_T}$ is an evolution system of measures for $(p_{s,t})_{s\leq t}$. We define a probability measure m_μ on $\big(S_T \times H, \mathscr{B}(S_T)\otimes\mathscr{B}(H)\big)$ by*

$$m_\mu(dt,dx) := \frac{1}{T}\ \mu_t(dx)dt.$$

Lemma 4.16 *The probability measure $m_\mu = \frac{1}{T}\mu_t(dx)dt$ is invariant for $(\mathcal{P}_\tau)_{\tau\ge 0}$ for $\tau \ge 0$.*

PROOF Let $f \in \mathcal{B}_b\big(S_T \times H\big)$. We have by the above definitions

$$\begin{aligned}\int_{S_T\times H} \mathcal{P}_\tau f(t,x) m_\mu(dt,dx) &= \frac{1}{T}\int_0^T \int_H p_{t,t+\tau} f(t+\tau,.)(x)\mu_t(dx)dt \\ &= \frac{1}{T}\int_0^T \int_H f(t+\tau,.)(x)\mu_{t+\tau}(dx)dt.\end{aligned}$$

Now by the T-periodicity of both f and $(\mu_t)_{t\in\mathbb{R}}$ we have

$$\begin{aligned}&\frac{1}{T}\int_0^T \int_H f(t+\tau,.)(x)\mu_{t+\tau}(dx)dt \\ &= \frac{1}{T}\int_0^T \int_H f(t,.)(x)\mu_t(dx)dt \\ &= \int_{S_T\times H} f(t,x) m_\mu(dt,dx),\end{aligned}$$

which implies the required invariance of m_μ. ∎

Definition 4.17 *The set of all invariant measures of $(\mathcal{P}_\tau)_{\tau\ge 0}$ will be denote by* $\mathrm{Inv}(\mathcal{P})$*. Equivalently, each $m \in \mathrm{Inv}(\mathcal{P})$ obeys*

$$\mathcal{P}_\tau^* m = m, \quad \tau \ge 0,$$

where $(\mathcal{P}_\tau^*)_{\tau\ge 0}$ is the transposed semigroup on $\mathscr{P}(S_T \times H)$ defined by

$$\mathcal{P}_\tau^* m(I \times B) = \int_{S_T\times H} S_\tau\big((s,x), I \times B\big) m(ds,dx),$$

for every $I \in \mathscr{B}(S_T)$ and $B \in \mathscr{B}(H)$.

It is important to know how each element of *Inv*$(\mathcal{P})$ looks like.

Proposition 4.18 *If $m \in \mathrm{Inv}(\mathcal{P})$, then*

$$m(dt\ dx) = \frac{1}{T} m_t(dx)dt$$

where $(m_t)_{t\in\mathbb{R}}$ is a T-periodic weakly continuous evolution system of measures for the forward continuous $(p_{s,t})_{s\le t}$, i.e., $(m_t)_{t\in\mathbb{R}} \in \mathcal{K}(\pi)$.

PROOF Consider the projection map

$$\theta_1(t,x) : S_T \times H \to S_T, \quad \theta_1(t,x) := t \in S_T.$$

Then the image measure of m under θ_1 is defined by

$$\lambda(dt) := m \circ \theta_1^{-1}(dt).$$

By the disintegration theorem ([Rip76], [Dud89]), there is a family of probability kernels $\big(m_t(dx)\big)_{t\in S_T}$ on $\mathscr{B}(H)$ such that

$$m(dt, dx) = m_t(dx)\lambda(dt).$$

Note that the disintegration $(m_t)_{t\in S_T}$ is defined λ-almost surely.

In the next step, we prove that λ is a Lebesgue measure. By definition, we have

$$\int_{S_T\times H} \mathcal{P}_\tau f(t,x)m(dt\ dx) \ = \int_{S_T\times H} f(t,x)m(dt\ dx)$$

for every $\mathcal{B}_b(S_T \times H)$.
On the other hand, by choosing $f(t,x) := u(t)$ independent of x, we get

$$\mathcal{P}_\tau u(t) = u(t+\tau), \quad \tau > 0,\ t \in S_T,$$

so that

$$\int_0^T u(t+\tau)\lambda(dt) = \int_0^T u(t)\lambda(dt).$$

Therefore, λ is invariant w.r.t translations of the torus $S_T \cong [0,T]$, so it coincides with the Lebesgue measure on $[0,T]$ multiplied by a positive constant. So we have

$$m(dt\ dx) = cm_t(dx)dt,$$

and since m is a probability measure, it should hold $c = 1/T$.
Clearly, $(m_t)_{t\in S_T}$ can be extended to almost every $t \in \mathbb{R}$ by T-periodicity (for which we keep the same notation).

It remains to show that there exists a version of $(m_t)_{t\in S_T}$ such that $p^*_{s,t}m_s = m_t$ for all $s \le t$. Since $m \in Inv(\mathcal{P}_\tau)$, for every $f \in \mathcal{B}_b(H)$ and every $u \in \mathcal{B}_b(S_T)$ we have for any $\tau \ge 0$

$$\int_{S_T\times H} \mathcal{P}_\tau\big(u(t)f(x)\big)m(dt,dx) = \int_{S_T\times H} u(t)f(x)m(dt,dx)$$

and thus

$$\int_0^T u(t+\tau)\int_H p_{t,t+\tau}f(x)m_t(dx)dt = \int_0^T u(t)\int_H f(x)m_t(dx)dt.$$

So we obtain

$$\int_0^T u(t+\tau)\int_H p_{t,t+\tau}f(x)m_t(dx)dt = \int_0^T u(t+\tau)\int_H f(x)m_{t+\tau}(dx)dt.$$

Therefore

$$\int_H p_{t,t+\tau}f(x)m_t(dx) = \int_H f(x)m_{t+\tau}(dx), \tag{4.5}$$

for almost all $t \in S_T$ (depending on f and τ).

Next we are going to show that it is possible to construct a continuous disintegration of m. To this end, for a given $f \in \mathcal{C}_b(H)$ let us consider the maps

$$\mathbb{R} \times \mathbb{R}_+ \ni (t,\tau) \mapsto \int_H p_{t,t+\tau}f(x)m_t(dx) \tag{4.6}$$

and

$$\mathbb{R} \times \mathbb{R}_+ \ni (t,\tau) \mapsto \int_H f(x)m_{t+\tau}(dt).$$

By construction, for each fixed $\tau \geq 0$

$$\mathbb{R} \ni t \mapsto \int_H p_{t,t+\tau}f(x)m_t(dx)$$

and

$$\mathbb{R} \ni t \mapsto \int_H f(x)m_{t+\tau}(dt)$$

are measurable.
On the other hand, we claim that these mappings are stochastically continuous at every fixed t . We will prove this for $\tau \mapsto \int_H p_{t,t+\tau}f(x)m_t(dx)$ and the same also will hold for $\tau \mapsto \int_H f(x)m_{t+\tau}(dt)$. Indeed, it would suffice to show that for any sequence $\tau_n \geq 0$, $\lim_{n\to\infty}\tau_n = \tau$,

$$\lim_{n\to\infty}\int_0^T \left| \int_H \big(p_{t,t+\tau} - p_{t,t+\tau_n}\big) f(x)m_t(dx)\right| dt = 0. \tag{4.7}$$

But $\tau \mapsto p_{t,t+\tau}$ is forward continuous and

$$\big|\big(p_{t,t+\tau} - p_{t,t+\tau_n}\big)f\big| \leq 2\|f\|_\infty.$$

So via Lebesgue's convergence theorem the claim is proved.
Therefore by Proposition 3.2 in [PZ92] both maps in (4.6)

$$(t,\tau) \mapsto \int_H p_{t,t+\tau}f(x)m_t(dx)$$

and

$$(t,\tau) \mapsto \int_H f(x)m_{t+\tau}(dt)$$

have measurable modifications. In what follows, we will work with these measurable modifications (keeping the same notation for them). Now we can rewrite (4.5) as

$$\int_0^T \left[\int_H \Big(p_{t,t+\tau} f(x) m_t(dx) - f(x) m_{t+\tau}(dx) \Big) \right] dt = 0, \quad \tau \geq 0,$$

which obviously implies

$$\int_0^\infty \int_0^T \left[\int_H \Big(p_{t,t+\tau} f(x) m_t(dx) - f(x) m_{t+\tau}(dx) \Big) \right] dt \; d\tau = 0.$$

Using Fubini's theorem yields

$$\int_0^T \int_0^\infty \int_H \Big(p_{t,t+\tau} f(x) m_t(dx) - f(x) m_{t+\tau}(dx) \Big) d\tau \; dt = 0$$

and hence

$$\int_0^\infty \int_H \Big(p_{t,t+\tau} f(x) m_t(dx) - f(x) m_{t+\tau}(dx) \Big) d\tau = 0, \quad t \in [0,T] \; (a.a.).$$

Therefore, for almost every $t \in [0,T]$ (depending on f and τ)

$$\int_H p_{t,t+\tau} f(x) m_t(dx) = \int_H f(x) m_{t+\tau}(dx), \quad \tau \geq 0 \; (a.a).$$

Actually, the above equation holds for each $\tau \in \Xi_{t,f} \in \mathscr{B}(\mathbb{R}_+)$, where the set $\Xi_{t,f}$ has the full measure. Since $\mathscr{P}(H)$ (endowed with the weak topology) is a Polish space, by taking a proper countable set $(f_N)_{N\in\mathbb{N}} \subset \mathcal{C}_b(H)$ (see Section 15.7 of [Kal83]) we conclude that for almost all $t \in [0,T]$

$$p^*_{t,t+\tau} m_t = m_{t+\tau}, \quad \tau \geq 0 \; (a.a), \tag{4.8}$$

in which $\Xi_t = \bigcap_N \Xi_{t,f_N} \in \mathscr{B}(\mathbb{R}_+)$ has again the full measure. By T-periodicity we can extend (4.8) to almost all $t \in \mathbb{R}$. Indeed, for every $k \in \mathbb{Z}$

$$p^*_{kT+t,kT+(t+\tau)} m_t = p^*_{t,t+\tau} m_{kT+t} = m_{t+\tau} = m_{kT+t+\tau}, \quad \tau \geq 0 \; (a.a).$$

Let us choose a sequence $t_n \to -\infty$, $n \in \mathbb{N}$, such that

$$p^*_{t_n,t_n+\tau} m_{t_n} = m_{t_n+\tau}, \quad \tau \in \bigcap_{n\in\mathbb{N}} \Xi_{t_n} =: \Xi.$$

Clearly Ξ also has the full measure.
For $n \in \mathbb{N}$ and all $\xi \geq t_n$, we define

$$\widetilde{m}^n_\xi := p^*_{t_n,\xi} m_{t_n}.$$

Then we have $\widetilde{m}^n_\xi = m_\xi$ almost surely for all $\xi \ge t_n$, because

$$\widetilde{m}^n_\xi = p^*_{t_n,\xi} m_{t_n} = p^*_{t_n,t_n+(\xi-t_n)} m_{t_n} = m_{t_n+(\xi-t_n)} = m_\xi.$$

Therefore $\widetilde{m}^n_\xi = \widetilde{m}^{n+1}_\xi$ almost surely for $\xi \ge t_n$.

On the other hand, for each n, $\widetilde{m}^n_\xi$ is continuous in ξ w.r.t. the topology of weak convergence. Indeed, from Definition 4.6 of the transposed operators we have for every $f \in \mathcal{B}_b(H)$

$$\begin{aligned}
\int_H f(y)\widetilde{m}^n_\xi(dy) &= \int_H f(y)(p^*_{t_n,\xi} m_{t_n})(dy)\\
&= \int_H f(y)\int_H \pi_{t_n,\xi}(x,dy) m_{t_n}(dx)\\
&= \int_H \left(\int_H f(y)\pi_{t_n,\xi}(x,dy)\right) m_{t_n}(dx)\\
&= \int_H p_{t_n,\xi} f(x) m_{t_n}(dx).
\end{aligned}$$

So, the continuity of $\widetilde{m}^n_\xi$ means that for any $f \in \mathcal{B}_b(H)$ and any sequence $\xi_k \to \xi$ as $k \to \infty$, such that $t_n \le \xi_k$, $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \int_{t_n}^\infty \left| \int_H \big(p_{t_n,\xi} - p_{t_n,\xi_k}\big) f(x) m_{t_n}(dx) \right| dt = 0.$$

And via the argument similar to that used in the proof of (4.7), we get the claim.

Thus we conclude that $\widetilde{m}^n_\xi = \widetilde{m}^{n+1}_\xi$ for all $\xi \ge t_n$. So, we can define the family $(\widetilde{m}_\xi)_{\xi\in\mathbb{R}}$ by

$$\widetilde{m}_\xi := \widetilde{m}^n_\xi, \qquad \text{for all } \xi \ge t_n.$$

Obviously $\widetilde{m}_\xi = m_\xi$ almost surely, so that $(\widetilde{m}_\xi)_{\xi\in\mathbb{R}}$ is a continuous disintegration of m.

Finally, we check that $(\widetilde{m}_\xi)_{\xi\in\mathbb{R}}$ is an evolution system of measures. For every $t \in \mathbb{R}$ we choose some t_{n_0} from the sequence $(t_n)_{n\ge 1}$ such that $t_{n_0} \le t$. Then we have

$$\begin{aligned}
p^*_{t,t+\tau}\widetilde{m}_t &= p^*_{t,t+\tau} p^*_{t_n,t} m_{t_n}\\
&= \big(p_{t_n,t} p_{t,t+\tau}\big)^* m_{t_n}\\
&= p^*_{t_n,t+\tau} m_{t_n}\\
&= \widetilde{m}_{t+\tau}, \quad \tau \ge 0.
\end{aligned}$$

Thus $(\widetilde{m}_\xi)_{\xi\in\mathbb{R}}$ is a continuous T-periodic evolution system of measures for $(p_{s,t})_{s\le t}$. ∎

4.4 On the property of ergodic measures

Now we put a closer look at ergodic invariant measures.

Theorem 4.19 *Let m_1 and m_2, $m_1 \neq m_2$, be two ergodic invariant probability measures for $(\mathcal{P}_\tau)_{\tau\geq 0}$. Then m_1 and m_2 are singular.*

Proof Let $I \times B \in \mathcal{B}\big(S_T \times H\big)$ such that $m_1(I \times B) \neq m_2(I \times B)$. Since m_1 and m_2 are ergodic, there exist a sequence $(T_n)_{n\in\mathbb{N}}$, $T_n \to \infty$, and sets $J_1 \times M,\ J_2 \times N \in \mathcal{B}\big(S_T \times H\big)$ with $m_1(J_1 \times M) = m_2(J_2 \times N) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \mathcal{P}_\tau \mathbb{1}_{I\times B}(t,x)d\tau = \int_{S_T\times H} \mathbb{1}_{I\times B}(t,x)m_1(dt,dx) = m_1(I\times B)$$

for all $(t,x) \in J_1 \times M$ and

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \mathcal{P}_\tau \mathbb{1}_{I\times B}(t,x)d\tau = \int_{S_T\times H} \mathbb{1}_{I\times B}(t,x)m_2(dt,dx) = m_2(I\times B)$$

for all $(t,x) \in J_2 \times N$.
Since $m_1(I \times B) \neq m_2(I \times B)$, this implies $(J_1 \times M) \cap (J_2 \times N) = \emptyset$ and so m_1 and m_2 are singular. ∎

The following theorem gives a presentation of the set of all invariant measure for a given Markovian semigroup.

Theorem 4.20 *For every $m \in \mathrm{Inv}(\mathcal{P})$ there exists a probability measure ρ_m on the set of all ergodic measures for $(\mathcal{P}_\tau)_{\tau\geq 0}$ such that*

$$m(B) = \int_{\mathrm{Inv}(\mathcal{P}_\tau)} \tilde{m}(B)\ d\rho_m(\tilde{m}), \quad B \in \mathscr{B}(H)\otimes\mathscr{B}(S_T). \tag{4.9}$$

Proof See Theorem 5.1 in [Hai08]. ∎

Remark 4.21 *Since we need to extend this fact to the case of two-parameter semigroups, in the next chapter we will give a alternative proof to this theorem. There we will also give a precise meanings how to understand the integral in* (4.9).

4.5 Connection between the support of an invariant measure and the support of its disintegration

The support ($supp(m)$) of a probability measure m on a Polish space is defined as the intersection of all closed subsets having probability 1.
The following Lemma (which proof is based on the so-called strong Lindelöf) is a general fact valid for measures on Banach spaces.

Lemma 4.22 *Let μ be a probability measure on $(H, \mathscr{B}(H))$. Then*

$$\mathrm{supp}(\mu) = \{x \in H \mid \mu(B(x,r) > 0 \quad \textit{for all } r > 0\}.$$

Proof See Theorem 2.1 in [Par67]. ∎

The next lemma describes the support of the invariant measure $m \in Inv(\mathcal{P})$ in terms of the support of its disintegration $(m_t)_{t\in\mathbb{R}}$.

Lemma 4.23 *Let m be a element of* $\mathrm{Inv}(\mathcal{P})$ *then*

$$a) \ \ \mathrm{supp}(m) \subseteq \overline{\{(t,x) \in S_T \times H : \ x \in \mathrm{supp}(m_t)\}};$$
$$b) \ \ \{(t,x) \in S_T \times H : \ x \in \mathrm{supp}(m_t)\} \subseteq \mathrm{supp}(m),$$

where $(m_t)_{t\in S_T}$ is the (continuous) disintegration of m.

Proof Set $A := \{(t,x) \in S_T \times H : x \in supp(m_t)\}$ so that

$$supp(m_t) = A^t := \{x \in H : \ (t,x) \in A\}.$$

Since by the disintegration of m in Proposition 4.18 we have

$$m(A) \geq \frac{1}{T}\int_0^T m_t(A^t)dt = 1,$$

this yields that $supp(m) \subset \overline{A}$.
Now we prove that $A \subset supp(m)$. Suppose $(t_0, x_0) \notin supp(m)$, then from Lemma 4.22 there exist $\varepsilon > 0$ such that

$$m\big(B\big((t_0,x_0),\varepsilon\big)\big) = m\big((t_0-\varepsilon, t_0+\varepsilon) \times B(x_0,\varepsilon)\big) = 0$$

implying

$$\begin{aligned} &\int_{(t_0-\varepsilon,t_0+\varepsilon)\times B(x_0,\varepsilon)} m(dt,dx) = 0 \\ \Longrightarrow &\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B(x_0,\varepsilon)} m_t(dx)dt = 0 \\ \Longrightarrow &\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_H \mathbb{1}_{B(x_0,\varepsilon)} m_t(dx)dt = 0 \end{aligned}$$

As a result we have

$$\Longrightarrow \int_H \mathbb{1}_{B(x_0,\varepsilon)} m_t(dx) = 0, \quad a.s. \ t \in (t_0-\varepsilon, t_0+\varepsilon). \tag{4.10}$$

Let $f : H \to \mathbb{R}$ be a continuous function such that

$$f(x) = \begin{cases} 1, & if \ \ x \in B(x_0, \varepsilon/2), \\ 0, & if \ \ x \in B(x_0,\varepsilon)^c, \end{cases}$$

and $0 \le f(x) \le 1$ for all $x \in H$. Therefore, by (4.10)

$$\int_H f(x) m_t(dx) = 0, \quad a.s.\ t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Now from the continuity of $t \to \int_H f(x) m_t(dx)$ we can deduce that $\int_H f(x) m_{t_0}(dx) = 0$. Hence $m_{t_0}(B(x_0, \varepsilon/2)) = 0$, which implies $x_0 \in supp(m_{t_0})$. ■

Remark 4.24 *In Lemma 5.4 of [PD08] it has been claimed that for every* $m \in \mathrm{Inv}(\mathcal{P})$

$$\mathrm{supp}(m) = \{(t, x) \in \mathbb{R} \times H : \ x \in \mathrm{supp}(m_t)\},$$

where $(m_t)_{t \in \mathbb{R}}$ *is the (continuous) disintegration of* m*. But it seems to be wrong at least for the following reasons.*

(1) *In general, we can not claim that the set* A *defined in Lemma 4.23 is a Borel and even a closed set. If we assume that* A *is closed then Lemma 4.23 yields that* $A = \mathrm{supp}(m)$*. This issue was over looked in the [PD08].*

(2) *Furthermore, in general the support of the measure* m *can not be represented as a product of the supports of its disintegrations. The assumption that* $(m_t)_{t \in \mathbb{R}}$ *is weakly continuous is not helpful in this case. We can reference to Proposition 5.1.8 in [AGS08] which states in following.*

Proposition 4.25 *If* $(m_n)_{n \in \mathbb{N}} \subset \mathscr{P}(X)$ *(where* X *is a separable metric space) is weakly convergent to* $m \in \mathscr{P}(X)$*, then*

$$\mathrm{supp}(m) \subset \liminf_{n \to \infty} \ \mathrm{supp}(m_n),$$

i.e., for each $x \in \mathrm{supp}(m)$ *there exists a sequence* $\{x_{n_j}\}_{j \ge 1}$ *with* $x_{n_j} \in \mathrm{supp}(m_n)$ *such that* $\lim_{n_j \to \infty} x_{n_j} = x$.

As a simple counterexample let us define $(m_t)_{t \ge 1}$ *so that*

$$m_t(\{x\}) = \begin{cases} \frac{t-1}{t}, & if \ \ x = 0 \\ \frac{1}{t}, & if \ \ x = 1 \end{cases}$$

Then each m_t *can be considered as a probability measure on* $\mathbb{R}$*. Obviously,* $\mathrm{supp}(m_t) = \{0, 1\} \subset \mathbb{R}$ *for every* $t > 1$ *and* $\mathrm{supp}(m_1) = 1$*. We observe that:*

(1) *A is not closed. Indeed, for every* $n \ge 1$, $\{0\} \in \mathrm{supp}(m_n)$, $(n, \{0\}) \in A$ *and* $(n, \{0\}) \to (1, \{0\})$*, but* $\{0\} \notin \mathrm{supp}(m_1)$ *and* $(1, \{0\}) \notin A$*. Thus* A *is not closed in* $[1, \infty) \times \mathbb{R}$.

(2) *Despite of the weak convergence* $m_t \to m_1$ *as* $t \to 1$*, we have*

$$\mathrm{supp}(m_t) \nrightarrow \mathrm{supp}(m_1).$$

4.6 Sufficient condition for the asymptotic strong Feller property of $(p_{s,t})_{s\leq t}$

Let us first define the asymptotic strong Feller property for semigroup $(p_{s,t})_{s\leq t}$. Note that in this section we do not need to assume the T-periodicity of $(p_{s,t})_{s\leq t}$.
We adapt to our setting Definition 5.2 in [PD08].

Definition 4.26 *The semigroup $(p_{s,s+r})_{r\geq 0}$ on $\mathcal{B}_b(H)$ is called asymptotically strong Feller at $x_0 \in H$ and $s_0 \in \mathbb{R}$, if there exists a totally separating system of continuous pseudo-metrics $(d_n)_{n\in\mathbb{N}}$ on H and a non-decreasing sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$, such that*

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} \sup_{x\in B(x_0,\gamma)} \sup_{s\in[s_0-\gamma,s_0+\gamma]} W_{d_n}(\pi_{s,s+r_n}(x_0,.), \pi_{s,s+r_n}(x,.)) = 0, \tag{4.11}$$

where π is the transition kernel of p and $B(x_0,\gamma)$ denotes the open ball of radius $\gamma > 0$ centered at $x_0 \in H$.
Respectively, $(p_{s,t})_{s\leq t}$ is called asymptotically strong Feller, if it is asymptotically strong Feller at every $x \in H$ and $s \in \mathbb{R}$.

Recall that for $f \in \mathcal{C}_b(H)$,

$$\|f\|_\infty := \sup_{x\in H} \|f(x)\|.$$

We need some lemmas for preparation to set the main result.

Lemma 4.27 *Let $f \in \mathcal{C}_b^1(H)$. Then*

$$\|\nabla f\|_\infty = \|f\|_{Lip_d}$$

with $d(x,y) := \|x-y\|$ for all $x,y \in H$.

Proof See Lemma 7.1.5 in [PZ96]. ∎

Lemma 4.28 *Let d be a pseudo-metric on H and $f \in L_d(H)$ (see Definition 2.56). Then there exists a sequence $(f_m)_{m\in\mathbb{N}} \subset \mathcal{C}_b^\infty(H)$ such that:*

1. *$f_m \to f$ pointwisely as $m \to \infty$;*
2. *$\|f_m\|_\infty \leq \|f\|_\infty$ for all $m \in \mathbb{N}$;*
3. *$\|f_m\|_{Lip_d} \leq \|f\|_{Lip_d}$ for all $m \in \mathbb{N}$.*

Proof See [PZ96] and [Cer99]. ∎

Here, we present the sufficient criteria (comparable to Proposition 3.15) that guarantees the asymptotic strong Feller property for the two-parameter semigroups.

Theorem 4.29 *Let H be a separable Hilbert space. If for every two positive sequences $(r_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ such that $r_n \le r_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_n = 0$, we have*

$$|p_{s,s+r_n} f(x) - p_{s,s+r_n} f(y)| \le C_s(\|x\| \vee \|y\|)(\|f\|_\infty + \delta_n \|\nabla f\|_\infty) \cdot \|x - y\|$$

for every $f \in \mathcal{C}_b^1(H)$ and all $x, y \in H$, $s \in \mathbb{R}$, $n \in \mathbb{N}$ where $C_s : \mathbb{R}_+ \to \mathbb{R}$ are non-decreasing functions, then the semigroup $(p_{s,t})_{s\le t}$ is asymptotically strong Feller.

Proof For $\varepsilon > 0$ define on H the metric

$$\begin{aligned} d_\varepsilon : H \times H &\to \mathbb{R}_+ \\ (x_1, x_2) &\to d_\varepsilon(x_1, x_2) := 1 \wedge \frac{1}{\varepsilon} \cdot \|x_1 - x_2\|. \end{aligned}$$

This is a metric on H, because for any $x, y, z \in H$

$$\begin{aligned} d_\varepsilon(x,y) + d_\varepsilon(y,z) &= \big(1 \wedge \frac{1}{\varepsilon}\|x-y\|\big) + \big(1 \wedge \frac{1}{\varepsilon}\|y-z\|\big) \\ &\ge 1 \wedge \frac{1}{\varepsilon}\big(\|x-y\| + \|y-z\|\big) \\ &\ge 1 \wedge \frac{1}{\varepsilon}\|x-z\|\big) \\ &= d_\varepsilon(x,z), \end{aligned}$$

and the non-negativity, coincidence axiom and symmetry follows immediately from the metric properties of d.

Also $(d_{\delta_n})_{n\in\mathbb{N}}$, as $\lim_{n\to\infty} \delta_n = 0$, is a totally separating system of continuous metrics for H, since:

- $\delta_n \ge \delta_{n+1} > 0$, for all $n \in \mathbb{N}$, supplies that for all $(x_1, x_2) \in H^2$

$$d_{\delta_n}(x_1,x_2) = 1 \wedge \frac{1}{\delta_n}\Big(\|x_1 - x_2\|\Big) \le 1 \wedge \frac{1}{\delta_{n+1}}\|x_1 - x_2\| = d_{\delta_{n+1}}(x_1,x_2);$$

- $\lim_{n\to\infty} d_{\delta_n}(x_1,x_2) = \lim_{n\to\infty} \Big(1 \wedge \frac{1}{\delta_n} \cdot \|x_1 - x_2\|\Big) = 1$, for all $x_1 \neq x_2$.

Now, for every Fréchet differentiable function $f : H \to \mathbb{R}$, we have by our assumption and Lemma 4.27

$$\begin{aligned}
&\left| \int_H f(x)\big[\pi_{s,s+r_n}(x_1,dy)-\pi_{s,s+r_n}(x_2,dy)\big]\right| \\
&= |p_{s,s+r_n}f(x_1)-p_{s,s+r_n}f(x_2)| \\
&\leq C_s(\|x_1\|\vee\|x_2\|).(\|f\|_\infty+\delta_n\cdot\|\nabla f\|_\infty)\cdot\|x_1-x_2\| \\
&\leq C_s(\|x_1\|\vee\|x_2\|).(\|f\|_\infty+\delta_n\cdot\|f\|_{Lip_d})\cdot\|x_1-x_2\| \\
&\leq C_s(\|x_1\|\vee\|x_2\|).(\|f\|_\infty+\tfrac{\delta_n}{\varepsilon}\cdot\|f\|_{Lip_{d_\varepsilon}})\cdot\|x_1-x_2\|,
\end{aligned} \tag{4.12}$$

The last inequality is true because

$$\|f\|_{Lip_d}=\frac{1}{\varepsilon}\cdot\sup_{\substack{x,y\in H\\ x\neq y}}\frac{|f(x)-f(y)|}{\varepsilon^{-1}\cdot d(x,y)}\leq\frac{1}{\varepsilon}\cdot\sup_{\substack{x,y\in H\\ x\neq y}}\frac{|f(x)-f(y)|}{\underbrace{1\wedge\varepsilon^{-1}\cdot d(x,y)}_{=d_\varepsilon(x,y)}}=\frac{1}{\varepsilon}\cdot\|f\|_{Lip_{d_\varepsilon}}.$$

Observe that, by Lemma 4.28 for every d_ε-Lipschitz continuous $f:H\to\mathbb{R}$ with $\|f\|_{Lip_{d_\varepsilon}}\leq 1$, there exists a sequence $(f_m)_{m\in\mathbb{N}}$ of Fréchet differential functions $f_m:H\to\mathbb{R}$ such that $f_m\to f$ pointwisely as $m\to\infty$ and $\|f_m\|_{Lip_{d_\varepsilon}}\leq\|f\|_{Lip_{d_\varepsilon}}\leq 1$ for all $m\in\mathbb{N}$.
Therefore by Lebesgue's dominated convergence theorem, we have the result similar to (4.12) also for Lipschitz continuous functions f. Indeed, for any $x_1,x_2\in H$

$$\begin{aligned}
&\left|\int_H f(x)\big(\pi_{s,s+r_n}(x_1,dy)-\pi_{s,s+r_n}(x_2,dy)\big)\right| \\
=&\lim_{m\to\infty}\left|\int_H f_m(x)\big(\pi_{s,s+r_n}(x_1,dy)-\pi_{s,s+r_n}(x_2,dy)\big)\right| \\
\leq&\lim_{m\to\infty}C_s(\|x_1\|\vee\|x_2\|).\left(\|f_m\|_\infty+\frac{\delta_n}{\varepsilon}.\|f_m\|_{Lip_{d_\varepsilon}}\right).\|x_1-x_2\| \\
\leq&\, C_s(\|x_1\|\vee\|x_2\|).\left(\|f\|_\infty+\frac{\delta_n}{\varepsilon}.\|f\|_{Lip_{d_\varepsilon}}\right).\|x_1-x_2\|.
\end{aligned}$$

Since in the definition of Wasserstein metric, it suffices to consider Lipschitz functions f such that $f(0)=0$. So, without losing generality, from $\|f\|_{Lip_{d_\varepsilon}}\leq 1$ we can assume $\|f\|_\infty\leq 1$. Then again from lemma 4.28 we have $\|f_m\|_\infty\leq\|f\|_\infty\leq 1$. Thus we have

$$\begin{aligned}
\|\pi_{s,s+r_n}(x_1,.)-\pi_{s,s+r_n}(x_2,.)\|_{d_\varepsilon}&=\left|\int_H f(x)\big(\pi_{s,s+r_n}(x_1,dy)-\pi_{s,s+r_n}(x_2,dy)\big)\right| \\
&\leq C_s(\|x_1\|\vee\|x_2\|).(1+\frac{\delta_n}{\varepsilon}).\|x_1-x_2\|.
\end{aligned}$$

But applying Lemma 2.59 yields

$$W_{d_\varepsilon}(\pi_{s,s+r_n}(x_1,.),\pi_{s,s+r_n}(x_2,.)) \le C_s(\|x_1\| \vee \|x_2\|).(1+\frac{\delta_n}{\varepsilon}).\|x_1-x_2\|$$

Choosing $\varepsilon = a_n = \sqrt{\delta_n}$, we obtain

$$W_{d_{a_n}}(\pi_{s,s+r_n}(x_1,.),\pi_{s,s+r_n}(x_2,.)) \le C_s(\|x_1\| \vee \|x_2\|).(1+a_n).\|x_1-x_2\|,$$

for all $n \in \mathbb{N}$, which in turn implies that $(p_{s,t})_{s\le t}$ is asymptotically strong Feller, since $a_n \to 0$ for $n \to \infty$. ■

4.7 Uniqueness of evolution system of measures

Lemma 4.30 *For every two mutually singular measures μ and ν of* $\mathrm{Inv}(\mathcal{P})$*, their corresponding disintegrations $(\mu_t)_{t\in\mathbb{R}}$ and $(\nu_t)_{t\in\mathbb{R}}$ are also almost surely singular.*

Proof See Lemma 5.5 in [PD08]. ■

Lemma 4.31 *Let $d \ge 1$ be a pseudo-metric in H. Let μ and ν be two ergodic invariant measures for $(\mathcal{P}_\tau)_{\tau\ge 0}$ with disintegrations $(\mu_t)_{t\in\mathbb{R}}$ and $(\nu_t)_{t\in\mathbb{R}}$, respectively, and $B \in \mathscr{B}(H)$. Then we have*

$$W_d(\mu_{s+r},\nu_{s+r}) \le 1 - \mu_s(B)\wedge\nu_s(B)\Big(1-\max_{y,z\in B} W_d(\pi_{s,s+r}(y,.),\pi_{s,s+r}(z,.))\Big).$$

Proof See Lemma 5.6 in [PD08]. ■

As we have discussed in Remark 4.24, there is a serious gap in the original proof in [PD08]. Therefore we put here the complete statement and the proof of the main results.

Theorem 4.32 *Let $(p_{s,s+r})_{r\ge 0}$ be a Markovian semigroup on $\mathcal{C}_b(H)$ with its associated $(\mathcal{P}_\tau)_{\tau\ge 0}$ and let μ, ν be two ergodic invariant measures for $(\mathcal{P}_\tau)_{\tau\ge 0}$. If $(p_{s,s+r})_{r\ge 0}$ is asymptotically strong Feller at the given $(s_0,x_0) \in \mathbb{R}\times H$, then $x_0 \notin \mathrm{supp}(\mu_{s_0}) \cap \mathrm{supp}(\nu_{s_0})$ where $(\mu_t)_{t\in\mathbb{R}}$ and $(\nu_t)_{t\in\mathbb{R}}$ are continuous disintegration of μ and ν.*

Proof Every two ergodic invariant probability measures of any one-parameter semigroup are singular. Thus μ and ν are singular. Therefore for their total variation of their difference, we obtain

$$\begin{aligned}\|\mu-\nu\|_{TV} &= \tfrac{1}{2}\cdot\big((\mu-\nu)^+(S_T\times H)+(\mu-\nu)^-(S_T\times H)\big)\\ &= \tfrac{1}{2}\cdot\big(\mu(S_T\times H)+\nu(S_T\times H)\big)=1.\end{aligned} \tag{4.13}$$

Assume now by contradiction that $x_0 \in supp(\mu_{s_0}) \cap supp(\nu_{s_0})$, respectively. Therefore

$$\min\left\{\mu_{s_0}(B(x_0,\delta/2)),\nu_{s_0}(B(x_0,\delta/2))\right\} > 0$$

Now we use the continuity of

$$s \mapsto \int_H f(x)\mu_s(dx), \quad s \mapsto \int_H f(x)\nu_s(dx), \quad f \in \mathcal{C}_b(H).$$

Let $f = \mathbb{1}$ on $B(x_0,\delta/2)$ then we see that there exists $\tilde{\delta} > 0$ such that

$$\min\left\{\mu_s(B(x_0,\delta)),\nu_s(B(x_0,\delta))\right\} > 0$$

for any $|s - s_0| < \tilde{\delta}$.

On the other hand, by the definition of the asymptotic strong Feller property for $(p_{s,t})_{s\leq t}$, there exists $\delta > 0$, $N \in \mathbb{N}$ such that

$$W_{d_n}\left(\pi_{s,s+r_n}(y,.),\pi_{s,s+r_n}(z,.)\right) \leq \frac{1}{2}, \quad n \geq N,$$

for any $y, z \in B(x_0,\delta)$ and $|s - s_0| \leq \delta$.

Taking $B = B(x_0,\delta)$ and $d = d_n$ in Lemma 4.31, we get then

$$\begin{aligned} W_{d_n}(\mu_{s+r_n},\nu_{s+r_n}) &\leq 1 - \mu_s(B) \wedge \nu_s(B)\left(1 - \frac{1}{2}\right) \\ &= 1 - \frac{1}{2}\min\{\mu_s(B),\nu_s(B)\} \end{aligned}$$

for every $|s - s_0| < \tilde{\delta} \wedge \delta$. So

$$\frac{1}{T}\int_0^T W_d(\mu_{s+r_n},\nu_{s+r_n})ds \leq 1 - \frac{1}{T}\int_{|s-s_0|<\tilde{\delta}\wedge\delta} \mu_s(B) \wedge \nu_s(B)ds < 1$$

for every $n \geq N$.

But Lemma 4.31

$$\begin{aligned} \frac{1}{T}\int_0^T \|\mu_s - \nu_s\|_{TV}ds &= \lim_{n\to\infty}\frac{1}{T}\int_0^T W_{d_n}(\mu_s,\nu_s)ds \\ &= \lim_{n\to\infty}\frac{1}{T}\int_0^T W_{d_n}(\mu_{s+r_n},\nu_{s+r_n})ds < 1 \end{aligned}$$

which is in contradiction to the fact that

$$\frac{1}{T}\int_0^T \|\mu_s - \nu_s\|_{TV}ds = 1$$

by equation (4.13). ■

Theorem 4.33 *Let* $(p_{s,t})_{s\le t}$ *be an asymptotically strong Feller evolution family on* $\mathcal{B}_b(H)$*, and suppose there is a point* $(s_0, x_0) \in \mathbb{R} \times H$ *such that* $x_0 \in \operatorname{supp}(m_{t_0})$ *for every* T*-periodic evolution system of measures* $(m_t)_{t\in\mathbb{R}}$ *in* $\mathcal{K}(\pi)$*. Then there exists at most one* T*-periodic evolution system of measures* $(m_t)_{t\in\mathbb{R}}$*.*

PROOF Suppose, there is more than one invariant measure for $(\mathcal{P}_\tau)_{\tau\ge 0}$. Now *Inv*($\mathcal{P}$) is simplex i.e., for every $m \in Inv(\mathcal{P})$ there exists a probability measure ρ_m on the set of all ergodic measures for $(\mathcal{P}_\tau)_{\tau\ge 0}$ such that

$$m(B) = \int_{Inv(\mathcal{P}_\tau)} \tilde{m}(B)\ d\rho_m(\tilde{m}), \quad B \in \mathscr{B}(H) \otimes \mathscr{B}(S_T).$$

Therefore we have at least two extremal points of *Inv*($\mathcal{P}$) which we denote by μ and ν. Now we know the set of all ergodic Borel probability measures for any one-parameter semigroup is coincides the set of all extremal points in the set of all its corresponding invariant measures, so these two extremal points are ergodic measures for $(\mathcal{P}_\tau)_{\tau\ge 0}$. Let us denote the corresponding T-periodic continuous disintegrations by $(\mu_t)_{t\in\mathbb{R}}$ and $(\nu_t)_{t\in\mathbb{R}}$.
Since $(p_{s,t})_{s\le t}$ is asymptotically strong Feller, it follows from the previous theorem $x \notin supp(\mu_t) \cap supp(\nu_t)$ for any $t \in \mathbb{R}$ and $x \in H$, i.e., $supp(\mu_t) \cap supp(\nu_t) = \emptyset$ for all $t \in \mathbb{R}$.

On the other hand, if we have a $x_0 \in supp(m_{s_0})$ for every T-periodic evolution system of measures $(m_t)_{t\in\mathbb{R}}$ then $(s_0, x_0) \in \{(s, x) \in S_T \times H : x \in supp(m_t)\}$. But from Lemma 4.23, $(s_0, x_0) \in supp(m)$ for every $m \in Inv(\mathcal{P})$.

In conclusion, it should be at most one invariant measure and by the one-to-one corresponding between members of $\mathcal{K}(\pi)$ and *Inv*($\mathcal{P}$) there exists also at most one T-periodic evolution family of measures for $(p_{s,t})_{s\le t}$. ■

Remark 4.34 *The conditions of our Theorem 4.33 is a stronger than the similar ones in Proposition 5.7 in [PD08] because* $x_0 \in \operatorname{supp}(s_0)$ *means* $(s_0, x_0) \in \{(s, x) \in S_T \times H : x \in \operatorname{supp}(m_t)\}$*. But by Lemma 4.23 we get* $(s_0, x_0) \in \operatorname{supp}(m)$*. As we mentioned above, we had to impose this assumption because Lemma 5.4 in [PD08] is incorrect and we have to substitute it by Lemma 4.23.*

Chapter 5

Extremal π-Entrance Laws for One-Parameter Mehler Semigroups

A standard result in convex analysis is that any point in a convex set in $\mathbb{R}^n$ can be represented as a convex combination of the extremal points.
In the ergodic theory, this fact is well-known as "Ergodic Decomposition theorem" (see Theorem 5.2.16 in [DS89]). This chapter is devoted to the possible decomposition of an evolution system of measures with the help of their extremal points.

Furthermore, in Theorem 5.34, we give an explicit formula for the extremal points in the set of evolution systems of measures for the particular case of Mehler semigroup.

Finally, we discuss this ergodic representation on nuclear spaces (instead of Hilbert spaces) and show how our assumptions can be refined in this framework.

5.1 Definitions

Recall that H is assumed to be a real separable Hilbert space with inner product and norm respectively by $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$, and let $\mathscr{B}(H)$ be the Borel field on H. As usual, we identify H with its dual space H^*. If it does not lead to misunderstanding, we will denote the inner product and norm of H respectively by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$.

Remark 5.1 *Suppose $(\pi_t)_{t\geq 0}$ is a Markovian one-parameter semigroup of transition kernels on $(H, \mathscr{B}(H))$. Then, as follows from Definition 4.7 and Remark 4.8, every π-entrance law $\nu = (\nu_s)_{s\in\mathbb{R}} \subset \mathscr{P}(H)$ associated with*

$(\pi_t)_{t\geq 0}$ *satisfies the following identity*

$$\int_H \pi_t(x,B)\nu_s(dx) = \nu_{t+s}(B), \quad t \geq 0,\ s \in \mathbb{R},\ B \in \mathscr{B}(H).$$

Lemma 5.2 *The set of all probability π-entrance laws (which we denote by $\mathcal{K}(\pi)$, see the previous chapter) is convex.*

Proof Suppose $\nu = (\nu_s)_{s\in\mathbb{R}}$, $\eta = (\eta_s)_{s\in\mathbb{R}}$ are elements of $\mathcal{K}(\pi)$ and $\alpha \in [0,1]$. Then for any $B \in \mathscr{B}(H)$ we have

$$\begin{aligned}
&\int_H \pi_t(x,B)(\alpha\nu_s + (1-\alpha)\eta_s)(dx) \\
&= \alpha \int_H \pi_t(x,B)\nu_s(dx) + (1-\alpha)\int_H \pi_t(x,B)\eta_s(dx) \\
&= \alpha\nu_{s+t}(B) + (1-\alpha)\eta_{s+t}(B) \\
&= (\alpha\nu_{s+t} + (1-\alpha)\eta_{s+t})(B)
\end{aligned}$$

for all $t \geq 0$ and $s \in \mathbb{R}$. Therefore

$$\alpha\nu + (1-\alpha)\eta = \Big(\alpha\nu_s + (1-\alpha)\eta_s\Big)_{s\in\mathbb{R}} \in \mathcal{K}(\pi). \qquad \blacksquare$$

Definition 5.3 *An element $\nu = (\nu_t)_{t\in\mathbb{R}} \in \mathcal{K}(\pi)$ is called extremal if every $\widetilde{\nu} = (\widetilde{\nu}_t)_{t\in\mathbb{R}} \in \mathcal{K}(\pi)$ dominated by ν (which means that $\widetilde{\nu}_t$ is absolutely continuous with respect to ν_t for every $t \in \mathbb{R}$) is equal to ν, i.e., $\widetilde{\nu}_t(B) = \nu_t(B)$ for every $B \in \mathscr{B}(H)$ and every $t \in \mathbb{R}$.*
We denote the set of all extremal points of $\mathcal{K}(\pi)$ by $\mathcal{K}_e(\pi)$.

5.2 Markov processes associated with one-parameter semigroups

We refer for more details concerning the material of this section to [SV06], [RY99] and [Röc11].

Definition 5.4 *We say that a Markov process $\big(\Omega, \mathcal{F}, (X_s)_{s\in\mathbb{R}}, \mathbb{P}\big)$ corresponds to the one-parameter semigroup of transition kernels $(\pi_t)_{t\geq 0}$ and write $\mathbb{P} \in M(\pi)$ if for any $t \geq 0$ and $s \in \mathbb{R}$*

$$\mathbb{P}(X_{t+s} \in B \mid \mathcal{F}_s) = \pi_t(X_s, B), \quad \mathbb{P}-a.s., \tag{5.1}$$

where $B \in \mathscr{B}(H)$ and $\mathcal{F}_s = \sigma(X_r \mid r \leq s)$.

The following theorem shows that for every Markovian one-parameter semigroup of transition kernels and every π-entrance law of this semigroup, there exists a unique corresponding Markov process.

Proposition 5.5 *Given a Markovian semigroup* $(\pi_t)_{t\geq 0}$ *over* $\big(H, \mathscr{B}(H)\big)$ *and a* π*-entrance law* $\nu := (\nu_s)_{s\in\mathbb{R}}$ *on* $\mathscr{B}(H)$*, there exists a Markov process* $\big(\Omega, \mathcal{F}, (X_s)_{s\in\mathbb{R}}, \mathbb{P}_\nu\big)$ *in the sense of Definition 5.4 with the state space* $\big(H, \mathscr{B}(H)\big)$ *such that* $\Omega = H^{\mathbb{R}}$ *and* $\mathcal{F} := \sigma(X_s \mid s \in \mathbb{R})$*. Furthermore,* ν_t *is the law of* X_t *for any* $t \in \mathbb{R}$*.*

Proof Put $X_s(\omega) := \omega(s)$ for every $s \in \mathbb{R}$ and $\omega \in H^{\mathbb{R}}$ and set $\mathcal{F}_s = \sigma(X_r \mid r \leq s)$. In more words, $\mathcal{F}_s$ is the σ-algebra generated by the cylinder sets $\{\omega \in \Omega \mid X_r(\omega) \in B\}$ with all possible $B \in \mathscr{B}(H)$ and $r \leq s$.

The first aim is to show the existence of a measure $\mathbb{P}_\nu$ on $H^{\mathbb{R}}$.
For any n-tuple of times $-\infty < t_1 < ... < t_n < +\infty$, we define its finite dimensional distributions $\mathbb{P}_{t_1,...,t_n}$ by

$$\begin{aligned}\mathbb{P}_{t_1,...,t_n}\big[X_{t_1} \in dx_1, ..., X_{t_n} \in dx_n\big] :\\ = \pi_{t_n - t_{n-1}}(x_{n-1}, dx_n)...\pi_{t_2-t_1}(x_1, dx_2)\nu_{t_1}(dx_1).\end{aligned} \tag{5.2}$$

Since $(\pi_t)_{t\geq 0}$ satisfies the Chapman-Kolmogorov equation and $(\nu_s)_{s\in\mathbb{R}}$ is a π-entrance law, definition (5.2) is independent of choosing t_i. Furthermore, $\{\mathbb{P}_{t_1,...,t_n}\}$ is consistent in the sense that if $\{s_1, ..., s_{n-1}\}$ is obtained from $\{t_1, ..., t_n\}$ by deleting the kth element t_k, $1 \leq k \leq n$ then $\mathbb{P}_{s_1,...,s_{n-1}}$ coincides with the marginal distribution of $\mathbb{P}_{t_1,...,t_n}$ obtained by removing the kth coordinate.
Therefore, by Kolmogorov's extension theorem, there exists a unique measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ such that equality (5.2) is true.

In the next step, we construct a Markov process

$$\Big(H^{\mathbb{R}_+}, \tilde{\mathcal{F}} = \sigma((X_t) \mid t \geq 0), (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in H}\Big)$$

in the sense of Definition 2.5 corresponding to the transition kernels $(\pi_t)_{t\geq 0}$ and initial distributions δ_x for $x \in H$ at $t = 0$.
For every given $x \in H$, let us define

$$\begin{aligned}&\mathbb{E}_x\big[f(X_{t_1}, ..., X_{t_n})\big] :=\\ &\int_H ... \int_H f(x_1, ..., x_n)\pi_{t_n-t_{n-1}}(x_{n-1}, dx_n)...\pi_{t_2-t_1}(x_1, dx_2)\pi_{t_1}(x, dx_1)\end{aligned}$$

for any n-tuple of times $0 \leq t_1 < ... < t_n < \infty$ and $f : H^n \to \mathbb{R}$ bounded and $\big(\mathscr{B}(H)\big)^{\bigotimes n}$-measurable. Equivalently, each measure $\mathbb{P}_x$ is defined by Kolmogorov's theorem via its family of finite dimensional distributions

$$\begin{aligned}\mathbb{P}_{t_1,...,t_n}[X_{t_1} \in dX_1, ..., X_{t_n} \in dX_n]\\ := \pi_{t_n-t_{n-1}}(x_{n-1}, dx_n)...\pi_{t_2-t_1}(x_1, dx_2)\pi_{t_1}(x, dx_1).\end{aligned}$$

First of all, we should prove that $x \to \mathbb{P}_x(\Gamma)$ is H-measurable for all $\Gamma \in \tilde{\mathcal{F}}$. Let us consider cylinder sets $\Gamma \in \tilde{\mathcal{F}}$ of the form

$$\Gamma = \{\omega \mid X_{t_1}(\omega) \in B_1, ..., X_{t_n}(\omega) \in B_n\}$$

with arbitrary $0 \leq t_1 < ... < t_n < \infty$ and $B_i \in \mathscr{B}(H)$, $1 \leq i \leq n \in \mathbb{N}$. Note that such sets generate $\tilde{\mathcal{F}}$. Letting $f(x_1, ..., x_n) = \mathbb{1}_{B_1}(x_1)...\mathbb{1}_{B_n}(x_n)$, we get

$$\begin{aligned}&\mathbb{P}_x\big[X_{t_1} \in B_1, ..., X_{t_n} \in B_n\big] = \mathbb{E}_x[f(X_{t_1}, ..., X_{t_n})]\\ =& \int_H \pi_{t_1}(x, dx_1) \int_H \pi_{t_2-t_1}(x_1, dx_2) ... \int_H \pi_{t_n-t_{n-1}}(x_{n-1}, dx_n) f(x_1, ..., x_n),\end{aligned}$$

so that the measurability of $x \mapsto \mathbb{P}_x(\Gamma)$ is clear from the measurability of $(\pi_t)_{t\geq 0}$. By the monotone class argument, the required measurability can be proved for arbitrary $\Gamma \in \tilde{\mathcal{F}}$.
In order to check the Markov property for $(\mathbb{P}_x)_{x\in H}$, see Definition 2.5, we show a stronger fact

$$\mathbb{E}_x[f(X_{t_1+s}, ..., X_{t_n+s}) \mid \mathbb{1}_\Delta] = \mathbb{E}_x\big[\mathbb{E}_{X_s}[f(X_{t_1}, ..., X_{t_n})] \mid \mathbb{1}_\Delta\big]$$

holding for all $x \in H$, $s \in \mathbb{R}_+$, $0 \leq t_1 < ... < t_n < \infty$ and $\Delta \in \mathcal{F}_s$.
By the monotone class theorem (applied to Δ), this follows from

$$\begin{aligned}&\mathbb{E}_x\big[f(X_{t_1+s}, ..., X_{t_n+s}) g(X_{s_0}, ..., X_{s_m})\big]\\ =& \mathbb{E}_x\big[\mathbb{E}_{X_s}[f(X_{t_1}, ..., X_{t_n})] g(X_{s_0}, ..., X_{s_m})\big]\end{aligned}$$

for all $\big(\mathscr{B}(H)\big)^{\bigotimes(m+1)}$-measurable bounded $g : H^{m+1} \to \mathbb{R}$ and $0 \leq s_0 < s_1 < ... < s_m = s$. So, the left hand side is equal to

$$\begin{aligned}&\int_H \pi_{s_0}(x, dx_0) ... \int_H \pi_{s_m-s_{m-1}}(x_{m-1}, dx_m)\\ &\times \underbrace{\int_H \pi_{t_1}(x_m, dy_1) ... \int_H \pi_{t_n-t_{n-1}}(y_{n-1}, dy_n) f(y_1, ..., y_n)}_{=\mathbb{E}_{x_m}[f(X_{t_1}, ..., X_{t_n})]} g(x_0, x_1, ..., x_m)\\ &= \mathbb{E}_x[\mathbb{E}_{X_{s_m}}[f(X_{t_1}, ..., X_{t_n}) g(X_{s_0}, ..., X_{s_m})]].\end{aligned}$$

Now by previous steps, we can prove that

$$\mathbb{P}_\nu[X_{t+s} \in B \mid \mathcal{F}_s] = \mathbb{P}_{X_s}[X_t \in B] = \pi_t(X_s, B), \quad \mathbb{P}_\nu - a.s., \tag{5.3}$$

for all $t \geq 0,\ s \in \mathbb{R}$ and $B \in \mathscr{B}(H)$.
One can prove this claim similarly to the proof of the Markov property in previous step. It means that is enough to show

$$\begin{aligned}&\mathbb{E}_\nu\big[f(X_{t_1+s}, ..., X_{t_n+s}) g(X_{s_0}, ..., X_{s_m})\big]\\ =\quad& \mathbb{E}_\nu\big[\mathbb{E}_{X_s}[f(X_{t_1}, ..., X_{t_n})] g(X_{s_0}, ..., X_{s_m})\big]\end{aligned} \tag{5.4}$$

for all $0 \leq t_1 < ... < t_n < \infty$ and $-\infty < s_0 < ... < s_m = s$. The expectation on the both sides of equation (5.4) can be rewritten in terms of transition probabilities analogously to previous step.

So $\mathbb{P}_\nu$ belongs to $M(\pi)$, which ends the proof. ∎

Remark 5.6 *Conversely, if X is a Markov process with values in H, then there exist an entrance law $\nu = (\nu_t)_{t\in\mathbb{R}}$ and a Markovian transition kernel $(\pi_t)_{t\geq 0}$ such that equation (5.1) holds and the law of X_t is ν_t. Indeed, we define the kernels $\pi_t(x, dy)$ as the regular version of the conditional probabilities*

$$\pi_t(x, dy) = \mathbb{P}\{X_{t+s} \mid \ X_s = x\}, \quad B \in \mathscr{B}(H).$$

5.3 Convex measurable space

The following definitions and facts are taken from [Dyn72] and [Dyn78] and will be used later.

Consider a set $\mathcal{M}$ of non-negative functions $m : \mathcal{W} \to \mathbb{R}_+$ defined on some abstract space $\mathcal{W}$. On $\mathcal{M}$ we introduce a σ-algebra $\mathfrak{M}$, which is the smallest σ-algebra generated by all mappings

$$\mathcal{M} \ni m \mapsto m(w), \quad w \in \mathcal{W}.$$

Then $(\mathcal{M}, \mathfrak{M})$ is a measurable space.

Definition 5.7 *A function $m_\rho : \mathcal{W} \to \mathbb{R}_+$ is called the barycentre of a given probability ρ on $(\mathcal{M}, \mathfrak{M})$ if*

$$m_\rho(w) = \int_{\mathcal{M}} m(w)\ d\rho(m), \quad w \in \mathcal{W}.$$

Definition 5.8 *We say that $m \in \mathcal{M}$ is an extremal point of $\mathcal{M}$ if m is not barycentre of any measure ρ except the measure concentrated on m.*
The set of all extremal point of $\mathcal{M}$ will be denoted by $\mathcal{M}_e$.

Definition 5.9 *We say that a convex structure is introduced into measurable space $(\mathcal{M}, \mathfrak{M})$ if for every probability measure ρ on $\mathfrak{M}$, there is an associated $m_\rho \in \mathcal{M}$.*
A space $\big(\mathcal{M}, \mathfrak{M}\big)$ provided with such a structure will be called a convex measurable space.

Definition 5.10 *A convex measurable space $\mathcal{M}$ is called a simplex if the set of its extremal points $\mathcal{M}_e$ is $\mathfrak{M}$-measurable and each $m \in \mathcal{M}$ is a barycentre of one and only one probability measure ρ concentrated on $\mathcal{M}_e$.*

Definition 5.11 *Let* $(\mathcal{M}, \mathfrak{M})$ *and* $(\mathcal{M}', \mathfrak{M}')$ *be convex measurable spaces. A one-to-one mapping* ϕ *between* $\mathcal{M}$ *and* $\mathcal{M}'$ *is called an isomorphism if it preserves measurability of sets and if* $\phi(m_\rho) = m_{\rho'}$ *where* $\rho'(U) := \rho[\phi^{-1}(U)]$, $U \in \mathfrak{M}'$. *Then it is clear that under this isomorphism extremal points go to extremal points and a space isomorphic to a simplex is itself a simplex.*

Now let assume until the end of this section that $\mathcal{M}$ be a subset of probability measures on a measurable space $(\Omega, \mathcal{F})$. This is a particular case of the previous situation since each measure $m \in \mathcal{M}$ is a non-negative function $\mathcal{F} \ni \Gamma \mapsto m(\Gamma) \geq 0$.

Remark 5.12 *We define a natural* σ*-algebra on* $\mathcal{M}$ *as*

$$\mathfrak{M} := \sigma\{\Phi_\Gamma \mid \Gamma \in \mathcal{F}\}$$

where $\Phi_\Gamma : \mathcal{M} \to \mathbb{R}$ *with* $\Phi_\Gamma(m) := m(\Gamma)$.
For a probability distribution ρ *on* $(\mathcal{M}, \mathfrak{M})$, *we define as before*

$$\left(\int_{\mathcal{M}} m \; d\rho(m)\right)(\Gamma) := \int_{\mathcal{M}} m(\Gamma) \; d\rho(m), \quad \Gamma \in \mathcal{F}.$$

Definition 5.13 *For any probability distribution* ρ *on* $(\mathcal{M}, \mathfrak{M})$, *if the measure* m_ρ *given by*

$$m_\rho(\Gamma) = \int_{\mathcal{M}} m(\Gamma)\rho(dm), \quad \Gamma \in \mathcal{F}, \tag{5.5}$$

belongs to $\mathcal{M}$, *then the convex structure defined by* (5.5) *on* $\mathcal{M}$ *will be called the natural convex structure.*

Note that if $\mathcal{M}$ is a simplex, then the formula

$$m_\rho(\Gamma) = \int_{\mathcal{M}_e} m(\Gamma)\rho(dm)$$

establishes a one-to-one correspondence between $\mathcal{M}$ and the set of all probability distributions on $\mathcal{M}_e$.

Definition 5.14 *Given a measure* $m \in \mathcal{M}$, *two sets* $\Gamma_1, \Gamma_2 \in \mathcal{F}$ *are called* m*-equivalent if* $\mathbb{1}_{\Gamma_1} = \mathbb{1}_{\Gamma_2}$, $m - a.s.$. *Two* σ*-algebra* $\mathcal{F}^1, \mathcal{F}^2 \subset \mathcal{F}$ *are* $\mathcal{M}$*-equivalent if, for each* $m \in \mathcal{M}$, *every* $\Gamma_1 \in \mathcal{F}_1$ *is* m*-equivalent to some* $\Gamma_2 \in \mathcal{F}_2$ *and vice verse.*

Definition 5.15 *A* σ*-algebra* $\mathcal{F}^0 \subset \mathcal{F}$ *is called sufficient for* $\mathcal{M}$ *if all measures* $m \in \mathcal{M}$ *have a common conditional distribution relative to* $\mathcal{F}^0$; *in other words, if for each* $\omega \in \Omega$ *there exists a probability measure* δ_ω *on* $\mathcal{F}^0$ *such that, for each* Γ, $\delta_\omega(\Gamma)$ *is* $\mathcal{F}$*-measurable and*

$$m(\Gamma \mid \mathcal{F}^0) = \delta_\omega(\Gamma) \quad m - a.s.,$$

for all $m \in \mathcal{M}$.
A sufficient σ-algebra is called $\mathcal{H}$-sufficient if in addition,

$$\delta_\omega \in \mathcal{M}, \quad m - a.s..$$

Theorem 5.16 *(cf. Theorem 3.1 in [Dyn78]) Let $\mathcal{M}$ be a separable class on $(\Omega, \mathcal{F})$, i.e., $\mathcal{F}$ contains a countable family which separate the measures in $\mathcal{M}$. Assume $\mathcal{F}^0$ be an $\mathcal{H}$-sufficient σ-algebra for $\mathcal{M}$. Then the set $\mathcal{M}_e$ is measurable and each $m \in \mathcal{M}$ is a barycentre of one and only one probability measure ρ_m concentrated on $\mathcal{M}_e$. If $\mathcal{M}$ is convex, it is a simplex.*

Theorem 5.17 *(cf. Theorem 3.2 in [Dyn78]) Let a separable class $\mathcal{M}$ have an $\mathcal{H}$-sufficient σ-algebra and let $\check{\mathcal{F}}$ be the class of all sets $\Gamma \in \mathcal{F}$ with the following property:*

$$m(\Gamma) = 0 \quad or \quad m(\Gamma) = 1, \quad \text{for all } m \in \mathcal{M}_e.$$

Then a σ-algebra $\mathcal{F}^0$ is $\mathcal{H}$-sufficient for $\mathcal{M}$ if and only if it is $\mathcal{M}$-equivalent to $\check{\mathcal{F}}$.

5.4 Correspondence between Markov processes and π-entrance laws

We introduce a measurable structure on $\mathcal{K}(\pi)$. Each $\nu \in \mathcal{K}(\pi)$ can be considered as a non-negative mapping

$$\mathbb{R} \times \mathscr{B}(H) \ni (t, B) \mapsto F_{t,B}(\nu) := \nu_t(B).$$

The natural σ-algebra on $\mathcal{K}(\pi)$ is the minimal σ-algebra generated by the family of mapping

$$\nu \mapsto F_{t,B}(\nu), \quad t \geq 0,\ B \in \mathscr{B}(H).$$

We denote this σ-algebra by $\mathscr{K}$.

Lemma 5.18 *The formula*

$$\nu_s(B) = \int_{\mathcal{K}(\pi)} \widetilde{\nu}_s(B) \rho(d\widetilde{\nu}), \quad B \in \mathscr{B}(H), \tag{5.6}$$

associates with each probability measure ρ on $(\mathcal{K}(\pi), \mathscr{K})$ an element $\nu = (\nu_s)_{s\in\mathbb{R}}$ of $\mathcal{K}(\pi)$.

Proof Obviously for each $s \in \mathbb{R}$, ν_s is a probability measure. Similarly to (4.1) one can define $\widetilde{\nu}_s(f) = \int_H f(x)\widetilde{\nu}_s(dx)$ for any non-negative Borel measurable function f. For any such f, by using the monotone class argument we can conclude that $\mathcal{K}(\pi) \ni \widetilde{\nu} \mapsto \widetilde{\nu}(f) \in \mathbb{R}$ is measurable. Then by (5.6) one can rewrite

$$\nu_s(\mathbb{1}_B) = \int_{\mathcal{K}(\pi)} \widetilde{\nu}_s(\mathbb{1}_B)\rho(d\widetilde{\nu}), \quad B \in \mathscr{B}(H),$$

which implies (by the same argument) that

$$\nu_s(f) = \int_{\mathcal{K}(\pi)} \widetilde{\nu}_s(f)\rho(d\widetilde{\nu}).$$

In particular, for $f(x) = \pi_{t-s}(x, B)$ we have from Definition 4.6 and the definition of evolution system of measures that

$$\begin{aligned} p^*_{t-s}\nu_s(B) :&= \int_H \pi_{t-s}(x,B)\nu_s(dx) \\ &= \int_{\mathcal{K}(\pi)} \int_H \pi_{t-s}(x,B)\widetilde{\nu}_s(dx)\rho(d\widetilde{\nu}) \\ &= \int_{\mathcal{K}(\pi)} \nu_t(B)\rho(d\widetilde{\nu}) \\ &= \nu_t(B), \quad B \in \mathscr{B}(H), \end{aligned}$$

which shows that $\nu \in \mathcal{K}(\pi)$. ∎

Therefore there is a natural convex measurable structure on $\mathcal{K}(\pi)$.

Since $M(\pi)$ is a subset of probability measures on $\big(\Omega = H^{\mathbb{R}}, \mathcal{F} = \sigma(X_s \mid s \in \mathbb{R})\big)$, we can introduce a measurable structure on $M(\pi)$ via the arguments in Remark 5.12. Furthermore, for every probability distribution on $M(\pi)$, its barycentre also belongs to $M(\pi)$. This can be shows by the arguments similar to that used in the proof of Lemma 5.18.

Now we are in a position to prove the existence of an isomorphism between $M(\pi)$ and $\mathcal{K}(\pi)$. The idea for the proof of the following lemma is taken from Theorem 3.1 in [Dyn72]:

Lemma 5.19 *The correspondence $\nu \to \mathbb{P}_\nu$ which is defined by (the proof of) Proposition 5.5 is an isomorphism of the convex measurable spaces $\mathcal{K}(\pi)$ and $M(\pi)$ in the sense of Definition 5.9.*

Proof By Proposition 5.5, $\mathbb{P}_\nu \in M(\pi)$ for all $\nu \in \mathcal{K}(\pi)$.
Now let $\mathbb{P}$ be any element of $M(\pi)$. Define $\nu = (\nu_s)_{s\in\mathbb{R}}$ by

$$\nu_s(B) = \mathbb{P}\{X_s \in B\}, \quad s \in \mathbb{R},\ B \in \mathscr{B}(H). \tag{5.7}$$

Then formula (5.1) implies that for any $-\infty < s_1 < ... < s_n < \infty$ and $B_1, ..., B_n \in \mathscr{B}(H)$

$$\mathbb{P}\{X_{s_1} \in B_1, ..., X_{s_n} \in B_n\} = \int_{B_1} ... \int_{B_n} \nu_{s_1}(dx_1)\pi_{s_2-s_1}(x_1, dx_2)...\pi_{s_n-s_{n-1}}(x_{n-1}, dx_n).$$

Hence $\mathbb{P} = \mathbb{P}_\nu$.
Furthermore, for every $B \in \mathscr{B}(H)$

$$\nu_{s+t}(B) = \mathbb{P}(X_{s+t} \in B), \quad s \in \mathbb{R},\ t \geq 0, \tag{5.8}$$

and so

$$\int_H \pi_t(x, B)\nu_s(dx) = \mathbb{P}(X_s \in H, X_{s+t} \in B) = \nu_{s+t}(B), \quad s \in \mathbb{R},\ t \geq 0$$

Hence it is clear that ν is a π-entrance law.
On the other hand, $\mathbb{P}\{X_s \in B\} = \nu_s(B)$. Thus, no element $\mathbb{P}$ of $M(\pi)$ can have two different inverse images in $\mathcal{K}(\pi)$.

We have proved that the mapping $\nu \leftrightarrow \mathbb{P}_\nu$ defines a one-to-one correspondence between $\mathcal{K}(\pi)$ and $M(\pi)$ and that the inverse mapping is given by (5.7).
The inverse mapping (5.7) obviously preserves the convex and measurable structures.

To prove that the mapping $\nu \to \mathbb{P}_\nu$ has the same properties it is sufficient to check that for any measurable function $f \geq 0$:
a) $\mathbb{E}_\nu(f)$ is measurable with respect to ν;
b) if ν is the barycentre of a probability distribution ρ, then

$$\mathbb{E}_\nu(f) = \int_{\mathcal{K}(\pi)} \mathbb{E}_{\widetilde{\nu}}(f)\rho(d\widetilde{\nu}).$$

By the monotone class theorem, it is sufficient to prove both these assertions when $f = \mathbb{1}_{B_1}(X_{s_1})...\mathbb{1}_{B_n}(X_{s_n})$ for $-\infty < s_1 < ... < s_n < \infty$ and $B_1, ..., B_n \in \mathscr{B}(H)$.
But in this case

$$\mathbb{E}_\nu(f) = \nu_{s_1}(\Phi),$$

where

$$\Phi(x_1) = \mathbb{1}_{B_1}(x_1) \int_{B_2} ... \int_{B_n} \pi_{s_2-s_1}(x_1, dx_2)...\pi_{s_n-s_{n-1}}(x_{n-1}, dx_n)$$

so that our assertion is obvious. ∎

In this position, we are able to prove one of the crucial result in our work. Let us recall that $\mathcal{F}_s$ is the σ-algebra in Ω generated by X_r, $r \in [-\infty, s]$. Then by $\mathcal{F}_\infty$ we denote the intersection of all $\mathcal{F}_s$ taken over all $s > -\infty$.

Theorem 5.20 *$\mathcal{K}(\pi)$ is simplex, i.e., each $\nu \in \mathcal{K}(\pi)$ can be uniquely represented as*

$$\nu_t = \int_{\mathcal{K}_e(\pi)} \widetilde{\nu}_t \rho(d\widetilde{\nu}), \quad t \in \mathbb{R}, \tag{5.9}$$

where ρ is a probability measure on $\mathcal{K}_e(\pi)$.

Proof Since we know from Theorem 9.1 in [Dyn78] that $\mathcal{F}_\infty$ is $\mathcal{H}$-sufficient for $M(\pi)$ and $M(\pi)$ is convex measurable space, so $M(\pi)$ is simplex. On the other hand, it was proved in Lemma 5.19 that $\nu \to \mathbb{P}_\nu$ is an isomorphism of convex measurable space $\mathcal{K}(\pi)$ and $M(\pi)$.
Consequently $\mathcal{K}(\pi)$ is also simplex. ∎

5.5 Construction of Mehler semigroups

We start with two known facts from [Par67] which play an important role in the subsequent considerations.

Theorem 5.21 *Let $K : H^2 \to \mathbb{C}$ be of the form*

$$K(x, a) = e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1 + \|x\|^2}, \quad (x, a) \in H^2,$$

and $\phi : H \to \mathbb{C}$ be a function of the form

$$\phi(a) = e^{\int_H K(x,a)dF(x)}, \quad a \in H,$$

where F is a σ-finite measure on $(H, \mathscr{B}(H))$ which has finite mass outside every neighborhood of the origin and for which

$$\int_H \left(1 \wedge \|x\|^2\right) dF(x) < \infty.$$

Then ϕ is the characteristic function of an infinitely divisible distribution in the sense Definition 2.43.

Proof See Theorem 4.8, Chapter VI in [Par67]. ∎

Definition 5.22 *Given a separable Hilbert space H. Let $\mathcal{L}_1(H)$ denote the family of all trace class linear operators on H. The class of sets*

$$\{a \in H \mid (Sa, a) < 1\}$$

indexed by all $S \in \mathcal{L}_1(H)$, defines a system of neighborhoods at the origin for the Sazonov topology on H.
By definition, Sazonov's topology is stronger than the topology induced by the inner product on H.

A Bochner-type theorem in Hilbert spaces, which is called *Minlos-Sazonov theorem*, states the following:

Theorem 5.23 *In order that a complex-valued function $\phi(a)$, $a \in H$, to be the characteristic function of a probability measure on $(H, \mathscr{B}(H))$ the following three conditions are necessary and sufficient:*
(i) $\phi(0) = 1$,
(ii) ϕ *is positive definite on* H,
(iii) ϕ *is Sazonov continuous on* H.

Proof See Theorem 2.4 Chapter VI in [Par67]. For the definition of property *(ii)* see the Introduction. ■

We are going to define a one-parameter Mehler semigroup with parameters A and λ:

Proposition 5.24 *Assume that A is the generator of a $\mathcal{C}_0$-semigroup $(T_t)_{t\geq 0}$ on H and the Lévy symbol $\lambda : H \to \mathbb{C}$ is a negative-definite Sazonov-continuous function with $\lambda(0) = 0$. Then there exists $(\mu_t)_{t\geq 0} \subset \mathscr{P}(H)$ whose characteristic function are of the form*

$$\widehat{\mu}_t(a) = \int_H e^{i\langle a,x\rangle} \mu_t(dx) = e^{-\int_0^t \lambda(T_s^* a) ds}, \quad a \in H,$$

where T^* is the adjoint operator of T.

Proof By the same argument as in [FR00], our assumptions on λ imply that, for every $t > 0$, the function $\exp(-t\lambda)$ are positive-definite and Sazonov continuous. Therefore, by Minlos-Sazonov theorem, they are characteristic functions of probability measures on H. Then $\exp(-\lambda)$ is obviously the characteristic function of an infinitely divisible probability measure on H. Now by Lévy-Khinchin Theorem, λ can be written in the form

$$\lambda(a) = -i\langle a, b\rangle + \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1 + \|x\|^2} \right) M(dx) \tag{5.10}$$

where $b \in H$, $R : H \to H$ is a symmetric non-negative trace class operator and M is a Lévy measure on $(H, \mathscr{B}(H))$. Now let us consider the extended form of $e^{-\int_0^t \lambda(T_s^* a)ds}$ (see [FR00], [Wie11])

$$\begin{aligned} &\exp\Big(-\textstyle\int_0^t \lambda(T_s^* a) ds \Big) \\ &\quad = \exp\Big(i\langle a, b_t\rangle - \tfrac{1}{2}\langle R_t a, a\rangle + \textstyle\int_H \big(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a,x\rangle}{1+\|x\|^2}\big) M_t(dx) \Big), \end{aligned} \tag{5.11}$$

where

$$R_t = \int_0^t T_s R T_s^* ds$$
$$b_t = \int_0^t T_s b \, ds + \int_0^t \int_H T_s x \Big(\frac{1}{1 + \|T_s x\|^2} - \frac{1}{1 + \|x\|^2} \Big) M(dx) ds,$$

and the measures M_t are defined by

$$M_t(B) := \int_0^t M\big((T_s)^{-1}(B \setminus \{0\})\big) ds, \qquad B \in \mathscr{B}(H).$$

Here $(T_s)^{-1}(B \setminus \{0\}) := \Big\{ y \in H \mid T_s y \in B \setminus \{0\} \Big\}$. In other words,

$$M_t(B) = \int_0^t \Big(\int_H (\mathbb{1}_{B \setminus \{0\}})(T_s y) M(dy) \Big) ds,$$

where the integrand $s \mapsto \int_H \mathbb{1}_{B\setminus\{0\}}(T_s x) M(dx)$ is obviously measurable. From the Lévy-Khinchin formula (Theorem 2.44) and Bochner-type theorem one can see that

$$e^{-\int_0^t \lambda(T_s^* a) ds} \tag{5.12}$$

is positive definite. So, if $H \ni a \mapsto \int_0^t \lambda(T_s^* a) ds$ is continuous in Sazonov's topology, then by Theorem 5.23 there is a probability measure μ_t on H such that $\widehat{\mu_t}(a) = e^{-\int_0^t \lambda(T_s^* a) ds}$. Thus, it remains to prove that $a \mapsto \int_0^t \lambda(T_s^* a) ds$ is a Sazonov continuous functional. Let us discuss the continuity of every part in the right side of (5.11) separately.

About the first part, i.e.,

$$a \mapsto \exp\big(- i \langle a, b_t \rangle \big), \tag{5.13}$$

note that b is just a vector in H and for every fixed t, $b_t \in H$ is correctly defined via Bochner integrals. Obviously, $H \ni a \mapsto \langle a, b_t \rangle b_t \in H$ defines a trace class linear operator in H and therefore (5.13) is Sazonov continuous.

Now concerning the second part,

$$a \mapsto \exp\big(- \frac{1}{2} \langle R_t a, a \rangle \big). \tag{5.14}$$

We know that R is a non-negative trace class operator. Since $\mathcal{L}_1(H)$ is an operator ideal in $\mathcal{L}(H)$, so $T_s R T_s^*$ is also trace class operator for every s. Meanwhile, every $T_s R T_s^*$ is non-negative due to the same property of R. The Bochner integral in the definition of R_t is taken in the Banach separable space $\mathcal{L}_1(H)$ of all trace class operators in H. This definition is correct since

$$\int_0^T \|T_s R T_s^*\|_{\mathcal{L}_1(H)} ds \le T \mathbb{M}_T \cdot \sup_{0 \le s \le T} \|R\|_{\mathcal{L}_1(H)} < \infty,$$

where concerning $\mathbb{M}_T$, one can say that since $(T_t)_{t\geq 0}$ is a strongly continuous semigroup, by Hida-Yoshida Theorem (see [Paz83]) we have

$$\mathbb{M}_T := \sup_{0\leq t\leq T} \|T_t\|_{\mathcal{L}(H)} < \infty.$$

Then R_t is a non-negative trace class operator with

$$\|R_t\|_{\mathcal{L}_1(H)} \leq \int_0^T \|T_s R T_s^*\|_{\mathcal{L}_1(H)} ds < \infty$$

and

$$\langle R_t a, a\rangle = \int_0^t \langle R T_s^* a, T_s^* a\rangle \; ds \geq 0, \quad a \in H.$$

Therefore (5.14) is Sazonov continuous, too.

The last part is

$$a \mapsto \exp\Big(\int_H \big[e^{i\langle a,x\rangle} - 1 - \frac{i\langle a,x\rangle}{1+\|x\|^2}\big] M_t(dx)\Big). \tag{5.15}$$

Note that

$$\begin{aligned}
\int_H \big(1\wedge \|x\|^2\big) M_t(dx) &= \int_0^t \int_H \Big(1\wedge \|T_s x\|^2\Big) M(dx) ds \\
&\leq \int_0^T \int_H \big(1\wedge \mathbb{M}_T \cdot \|x\|^2\big) M(dx) ds \\
&\leq T \max\{1, \mathbb{M}_T^2\} \int_H \big(1\wedge \|x\|^2\big) M(dx) < \infty.
\end{aligned}$$

Therefore by Theorem 5.21 we observe that (5.15) is a characteristic function of a probability measure on H, and finally Theorem 5.23 shows Sazonov continuity of the last term.
In conclusion, (5.12) is Sazonov continuous and the proof of the proposition is complete. ■

Definition 5.25 *Let $\pi_t(x,dy)$ be the translation of μ_t by $T_t x$, i.e.,*

$$\pi_t(x,dy) := \mu_t(dy - T_t x), \quad t \geq 0,\ x \in H.$$

In other words, for all $f \in \mathcal{B}_b(H)$, we have

$$\int_H f(y)\pi_t(x,dy) = \int_H f\big(T_t x + y\big)\mu_t(dy), \quad x \in H.$$

Then $(p_t)_{t\geq 0}$ is given by

$$p_t f(x) = \int_H f(y)\pi_t(x,dy) = \int_H f(T_t x + y)\mu_t(dy), \quad t \geq 0$$

is called the generalized one-parameter Mehler semigroup.

In fact, based on Proposition 2.2 in [BRS96], the semigroup property of $(p_t)_{t\geq 0}$ exists if and only if $\mu_{s+t} = \mu_s * (\mu_t \circ {T_s}^{-1})$ or equivalently,

$$\widehat{\mu}_{s+t}(a) = \widehat{\mu}_s(a)\widehat{\mu}_t(T_s^* a), \quad a \in H,\ t, s \geq 0$$

and we have

$$\begin{aligned}
\widehat{\mu}_s(a)\widehat{\mu}_t(T_s^* a) \quad &= e^{-\int_0^s \lambda\left(T_r^* a\right)dr} e^{-\int_0^t \lambda\left(T_r^* T_s^* a\right)dr} \\
&= e^{-\int_0^s \lambda\left(T_r^* a\right)dr} e^{-\int_s^{s+t} \lambda\left(T_{r-s}^* T_s^* a\right)dr} \\
&= e^{-\int_0^s \lambda\left(T_r^* a\right)dr} e^{-\int_s^{s+t} \lambda\left((T_s T_{r-s})^* a\right)dr} \qquad (5.16) \\
&= e^{-\int_0^{s+t} \lambda\left(T_r^* a\right)dr} \\
&= \widehat{\mu}_{s+t}(a),
\end{aligned}$$

for all $t, s \geq 0$ and $a \in H$. Therefore $(p_t)_{t\geq 0}$ is a semigroup and by the property (5.16) of μ, $(p_t)_{t\geq 0}$ is also a Markovian semigroup.

Remark 5.26 *The typical examples of these semigroups are the corresponding semigroup to the autonomous Ornstein-Uhlenbeck process of type*

$$\begin{aligned} dX(t) &= AX(t) + dL(t), \\ X(0) &= x \end{aligned}$$

Here, A is the mentioned linear operator in the representation of Mehler semigroup and L in a Lévy process with Lévy symbol λ (see e.g. [App06] or [Sto05]).
Mehler semigroups were first introduced by Bogachev, Röckner, and Schmuland [BRS96] within an axiomatic approach to transition semigroups of Ornstein-Uhlenbeck processes with Brownian motion and have been extended to the non-Gaussian case in [FR00].

Lemma 5.27 *The characteristic function of $(\pi_t)_{t\geq 0}$ is given by*

$$\int e^{i\langle a,y\rangle}\pi_t(x, dy) = e^{i\langle a,T_t x\rangle - \int_0^t \lambda(T_s^* a)dy}, \quad a \in H. \qquad (5.17)$$

PROOF For any $a \in H$ and $s, t \geq 0$ we have

$$\begin{aligned}
\widehat{\pi_t(x,.)}(a) &= \int_H e^{i\langle a,y\rangle} \pi_t(x, dy) \\
&= \int_H e^{i\langle a,y\rangle} \mu_t(dy - T_t x) \\
&= \int_H e^{i\langle a,y+T_t x\rangle} \mu_t(dy) \\
&= e^{i\langle a,T_t x\rangle} \int_H e^{i\langle a,y\rangle} \mu_t(dy) \\
&= e^{i\langle a,T_t x\rangle - \int_0^t \lambda(T_s^* a)dy}.
\end{aligned}$$

This end the proof. ∎

5.6 Structure of extremal π-entrance laws

Let $\nu = (\nu_s)_{s\in\mathbb{R}} \in \mathcal{K}(\pi)$ for which

$$\int_H |\langle a, x\rangle| \nu_s(dx) < \infty, \quad a \in H, \ s \in \mathbb{R}. \tag{5.18}$$

Since every ν_s is a probability measure, therefore for any $t \in \mathbb{R}$, the linear functional $a \mapsto \int_H \langle a, x\rangle \nu_s(dx)$ is continuous on H (see Proposition 07 in [PR07]) and therefore by the Riesz representation theorem there exists $\kappa_s \in H$ such that

$$\int_H \langle a, x\rangle \nu_s(dx) = \langle a, \kappa_s\rangle, \quad a \in H, \ s \in \mathbb{R}. \tag{5.19}$$

We denote the set of all $\nu \in \mathcal{K}(\pi)$ which satisfy (5.18) by $\mathcal{K}^1(\pi)$.

Definition 5.28 *For each $\nu \in \mathcal{K}^1(\pi)$, the family $(\kappa_s)_{s\in\mathbb{R}}$ obtained from (5.19) is called the projection of ν and will be denoted by $\kappa = \mathbf{p}(\nu)$.*

Definition 5.29 *Consider the semigroup $(T_t)_{t\geq 0}$ on H. A family $(\kappa_s)_{s\in\mathbb{R}} \subset H$, is called a T-entrance law if $T_t\kappa_s = \kappa_{s+t}$ for all $s \in \mathbb{R}$ and $t \geq 0$. The set of all such laws is denoted by $\mathcal{K}(T)$.*

Lemma 5.30 *Let $(p_t)_{t\geq 0}$ be a Mehler semigroup. If*

$$\int \langle a, y\rangle \pi_t(x, dy) = \langle a, T_t x\rangle, \tag{5.20}$$

then for each $\nu \in \mathcal{K}^1(\pi)$, $\kappa = \mathbf{p}(\nu)$ is a T-entrance law.

PROOF If we show that $\int_H \langle a, x\rangle \nu_{s+t}(dx) = \langle a, T_t\kappa_s\rangle$ for every $t \geq 0$ and $s \in \mathbb{R}$, the assertion is complete. By the definition of $\nu = (\nu_t)_{t\in\mathbb{R}}$

$$\int_H \langle a, x\rangle \nu_{s+t}(dx) = \int_H \left(\int_H \langle a, x\rangle \pi_t(y, dx) \right) \nu_s(dy).$$

Now if (5.20) is satisfied, then

$$\begin{aligned}\int_H \Big(\int_H \langle a,x\rangle \pi_t(y,dx)\Big)\nu_s(dy) &= \int_H \langle a, T_t y\rangle \nu_s(dy)\\ &= \int_H \langle T_t^* a, y\rangle \nu_s(dy)\\ &= \langle T_t^* a, \kappa_s\rangle = \langle a, T_t \kappa_s\rangle,\end{aligned}$$

which was needed to show. ∎

But, in order to have equality (5.20), we need to impose some conditions to our structure. The next proposition concerns this point.

Proposition 5.31 *Suppose that in the representation of* λ *in* (5.10)*, we have that* $b = 0$ *and* M *is symmetric, i.e.,* $M(dx) = M(-dx)$*. Also assume that*

$$\int_H |\langle a,y\rangle| \mu_t(dy) < \infty$$

for all $a \in H$ *and* $t \geq 0$*. Then*

$$\int_H \langle a,y\rangle \pi_t(x,dy) = \langle a, T_t x\rangle. \tag{5.21}$$

Proof **i)** We claim that M is symmetric and $b = 0$ if and only if the characteristic function of $(\mu_t)_{t\geq 0}$ is real valued, i.e., $\widehat{\mu}_t = \overline{\widehat{\mu}_t}$.
Proof of i) Since

$$\overline{\lambda}(a) = i\langle a,b\rangle + \frac{1}{2}\langle a, Ra\rangle - \int_H \big(e^{-i\langle a,x\rangle} - 1 + \frac{i\langle a,x\rangle}{1+\|x\|^2}\big)M(dx),$$

so that

$$\lambda = \overline{\lambda} \Longleftrightarrow M(dx) = M(-dx)$$

and $b = 0$.
On the other hand, $\lambda = \overline{\lambda} \Longrightarrow \lambda_t = \overline{\lambda_t}$ where $\lambda_t(a) = \int_0^t \lambda(T_s^* a)ds$.
So we conclude that $\widehat{\mu}_t = \overline{\widehat{\mu}_t}$.

ii) We claim if $\widehat{\mu}_t = \overline{\widehat{\mu}_t}$ then $\int \langle a,y\rangle \mu_t(dy) = 0$ for $a \in H$.
Proof of ii) Indeed,

$$\widehat{\mu}_t = \overline{\widehat{\mu}_t} \Longleftrightarrow \int_H e^{i\langle a,x\rangle}\mu_t(dx) = \int_H e^{-i\langle a,x\rangle}\mu_t(dx) = \int_H e^{i\langle -a,x\rangle}\mu_t(dx),$$

so that $\widehat{\mu}_t(a) = \widehat{\mu}_t(-a)$ for all $a \in H$.
Now let us substitute a by ua, $u \in \mathbb{R}$, then by Lebesgue's convergence

theorem we can calculate that

$$\begin{aligned}\frac{d}{du}\Big(\widehat{\mu}_t(ua)\Big)\Big|_{u=0} &= \frac{d}{du}\Big(\int_H e^{iu\langle a,x\rangle}\mu_t(dx)\Big)\Big|_{u=0}\\ &= \Big(\int_H i\langle a,x\rangle e^{iu\langle a,x\rangle}\mu_t(dx)\Big)\Big|_{u=0}\\ &= \int_H i\langle a,x\rangle\mu_t(dx).\end{aligned}$$

It is obvious that

$$\widehat{\mu}_t(a) = \widehat{\mu}_t(-a) \Rightarrow \frac{d}{du}\widehat{\mu}_t(ua)\Big|_{u=0} = \frac{d}{du}\widehat{\mu}_t(-ua)\Big|_{u=0},$$

which implies

$$\int i\langle a,x\rangle\mu_t(dx) = -\int i\langle a,x\rangle\mu_t(dx)$$

and hence

$$\int i\langle a,x\rangle\mu_t(dx) = 0.$$

iii) Now we can complete the proof of the proposition. We have defined $\pi_t(x,dy) = \mu_t(\cdot + T_tx)$, hence

$$\begin{aligned}\int_H \langle a,y\rangle\pi_t(x,dy) &= \int_H \langle a,y\rangle\mu_t(T_tx - dy)\\ &= \int_H \langle a, T_tx + y\rangle\mu_t(dy)\\ &= \langle a, T_tx\rangle + \int_H \langle a,y\rangle\mu_t(dy)\end{aligned}$$

and by applying **i)** and **ii)** the proof is complete. ■

Now we are able to present the main result of this paper. But first, the following assumption should be imposed to our setting which plays a technical role in the progress of the proof of the main theorem.

Assumption 5.32 *Let π be a Mehler semigroup with parameters R, M and A in the sense of Proposition* 5.24 *(see* (5.10)*). We assume that*

$$H \ni a \mapsto \int_{-\infty}^{t} \lambda(T^*_{t-s}a)ds, \quad t \geq 0,$$

is well-defined and Sazonov continuous.

Remark 5.33 *If instead of the Hilbert space H, we will use a nuclear space as a state space for our semigroup, then Sazonov continuity in Assumption 5.32 will be reduced to the continuity of $a \mapsto \int_{-\infty}^t \lambda(T^*_{t-s}a)ds$ in the corresponding projective topology. We will discuss it in the last section of this chapter.*

The following theorem, which is our main result in this chapter establishes the relation between $\mathcal{K}^1_e(\pi)$ and $\mathcal{K}(T)$. Furthermore, it gives an explicit representation for the characteristic function of every element in $\mathcal{K}^1_e(\pi)$ in terms of the elements in $\mathcal{K}(T)$. In the case of Ornstein-Uhlenbeck processes with Brownian motion this relationship has been already indicated in [Dyn88]).

Theorem 5.34 *Let $(p_t)_{t\geq 0}$ be a Mehler semigroup on $\mathscr{P}(H)$ with the following Fourier transform for corresponding $(\pi_t)_{t\geq 0}$*

$$\int_H e^{i\langle a,y\rangle}\pi_t(x,dy) = e^{i\langle a,T_t x\rangle - \int_0^t \lambda(T^*_s a)ds},$$

where $T=(T_t)_{t\in\mathbb{R}}$ is a $\mathcal{C}_0$-semigroup of linear operators on H and λ has the representation (cf. (5.10)*)*

$$\lambda(a) = \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a,x\rangle}{1+\|x\|^2}\right)M(dx),\ \ a\in H,$$

with a symmetric Lévy measure M on $(H,\mathscr{B}(H))$. Then under Assumption 5.32, for every $\kappa\in\mathcal{K}(T)$ there exists a unique extremal probability π-entrance law ν^κ such that

$$\int_H e^{i\langle a,y\rangle}\nu^\kappa_t(dy) = e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(T^*_{t-s}a)ds}. \tag{5.22}$$

Moreover, formula (5.22) *establishes a one-to-one correspondence between $\mathcal{K}(T)$ and the set of all extremal elements of $\mathcal{K}^1(\pi)$.*

Proof Three following Claims (i), (ii) and (iii) will give us a complete proof of the theorem:
Claim (i): *For every $\kappa\in\mathcal{K}(T)$, there exists $\nu^\kappa\in\mathcal{K}^1(\pi)$ which satisfies equation* (5.22)*.*
Proof of Claim (i): By Assumption 5.32 $a\mapsto\int_{-\infty}^t \lambda(T^*_{t-s}a)ds$ is Sazonov continuous. Also, note that

$$\int_{-\infty}^t \lambda(T^*_{t-s}a)ds \ = \lim_{r\to+\infty}\int_{-r}^t \lambda(T^*_{t-s}a)ds = \lim_{r\to+\infty}\int_0^{t+r}\lambda(T^*_s a)ds$$

is negative definite from the Lévy-Khinchin formula. Thus,

$$e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(T^*_{t-s}a)ds},\quad a\in H,$$

is Sazonov continuous and positive definite, and therefore by the Minlos-Sazonov theorem it is a characteristic function of a certain probability measure ν_t^κ on $(H, \mathscr{B}(H))$. Now we will prove that $(\nu_t^\kappa)_{t\in\mathbb{R}}$ belongs to $\mathcal{K}^1(\pi)$. Indeed, for every $t \geq 0$, $r \in \mathbb{R}$ and $a \in H$

$$
\begin{aligned}
\widehat{(p_t^*\nu_r^\kappa)}(a) &= \int_H \left(\int_H e^{i\langle a,y\rangle} \pi_t(x,dy) \right) \nu_r^\kappa(dx) \\
&= \int_H e^{i\langle a,T_t x\rangle - \int_0^t \lambda(T_s^* a)ds} \nu_r^\kappa(dx). \\
&= \int_H e^{i\langle T_t^* a,x\rangle} \nu_r^\kappa(dx).\ e^{-\int_0^t \lambda(T_s^* a)ds} \\
&= e^{i\langle T_t^* a,\kappa_r\rangle - \int_{-\infty}^r \lambda\left(T_{r-s}^*(T_t^* a)\right)ds}.\ e^{-\int_0^t \lambda(T_s^* a)ds} \\
&= e^{i\langle a,T_t\kappa_r\rangle - \int_{-\infty}^r \lambda((T_tT_{r-s})^* a)ds - \int_0^t \lambda(T_s^* a)ds} \\
&= e^{i\langle a,\kappa_{r+t}\rangle - \int_{-\infty}^r \lambda(T_{r+t-s}^* a)ds - \int_r^{r+t} \lambda(T_{r+t-s}^* a)ds} \\
&= e^{i\langle a,\kappa_{r+t}\rangle - \int_{-\infty}^{r+t} \lambda(T_{r+t-s}^* a)ds} \\
&= \widehat{\nu_{r+t}^\kappa}(a),
\end{aligned}
$$

which implies $(p_t^*\nu_r^\kappa) = \nu_{r+t}^\kappa$.

Claim (ii): *If ν is an extremal element of $\mathcal{K}^1(\pi)$, then the relation* (5.22) *holds with $\kappa = \mathbf{p}(\nu)$.*

Let first recall from [Dyn78] the following

Proposition 5.35 *Let $\nu \in \mathcal{K}_e(\pi)$ and let $(X_t, \mathbb{P}_\nu)$ be the corresponding Markov process on the time interval $(-\infty, +\infty)$ which we have constructed in Proposition 5.5. Then $\mathbb{P}_\nu$ is trivial.*

Proof From Theorem 9.1 in [Dyn78], we know that $\mathcal{F}_\infty$ is a $\mathcal{H}$-sufficient σ-algebra for $M(\pi)$, so Theorem 5.17 implies that $\mathcal{F}_\infty$ is $M(\pi)$-equivalent to $\check{\mathcal{F}}$ in Theorem 5.17 and hence $\mathbb{P}_\nu$ is trivial on $\mathcal{F}_\infty$. ∎

And we need the following remark to prove the claim.

Remark 5.36 *Let f be a non-negative measurable function on H such that $\mathbb{E}_\nu f(X_t) < \infty$. Since $\mathbb{P}_\nu$ is trivial on $\mathcal{F}_\infty$, we have*

$$\begin{aligned}\mathbb{E}_\nu f(X_t) &= \mathbb{E}_\nu\{f(X_t)|\mathcal{F}_\infty\}\\ &= \lim_{s\to-\infty} \mathbb{E}_\nu\{f(X_t)|\mathcal{F}_s\}\\ &= \lim_{s\to-\infty} \mathbb{E}_{X_s} f(X_{t-s}), \quad \mathbb{P}_\nu - a.s.,\end{aligned}$$

where in the the second line we applied the Backwards Martingale convergence theorem (cf. Theorem 2.3). We can use this theorem because according to Lemma 2.2 the process $\mathbb{E}_\nu\{f(X_t)|\mathcal{F}_s\}$ is a martingale. And finally, in the third line we used the Markov property of our process.

Proof of Claim (ii): Indeed, by (5.17) and Remark (5.36) we have the following identities, $\mathbb{P}_\nu - a.s.$,

$$\begin{aligned}\mathbb{E}_\nu e^{i\langle a,X_t\rangle} &= \lim_{r\to-\infty} \mathbb{E}_{X_r} e^{i\langle a,X_{t-r}\rangle}\\ &= \lim_{r\to-\infty} e^{i\langle T^*_{t-r}a,X_r\rangle - \int_0^{t-r}\lambda(T^*_s a)ds}\\ &= \lim_{r\to-\infty} e^{i\langle T^*_{t-r}a,X_r\rangle - \int_r^{t}\lambda(T^*_{t-s} a)ds}.\end{aligned}$$

Then equation (5.21), Remark 5.36, Lemma 5.30 and Remark 5.36 give us

$$\langle T^*_{t-r}a, X_r\rangle = \mathbb{E}_{X_r}\langle a, X_{t-r}\rangle \xrightarrow{r\to-\infty} \mathbb{E}_\nu\langle a, X_t\rangle = \langle a, \kappa_t\rangle, \quad \mathbb{P}_\nu - a.s..$$

Claim (iii): *If κ and ν^κ are related by (5.22), then ν^κ is extremal.*

The following notion plays an important role in the proof of Claim (iii).

Definition 5.37 *$\nu \in \mathcal{K}^1(\pi)$ is called the lifting of $\varrho \in \mathcal{K}(T)$ and will be denoted by $\nu = \mathbf{l}(\varrho)$, if for each $t \in \mathbb{R}$, $a \in H$*

$$\widehat{\nu_t}(a) = \lim_{r\to-\infty} \widehat{\pi_{t-r}(\varrho_r, .)}(a).$$

Proof of Claim (iii): We will divide the proof of Claim (iii) into five steps.
Step1: The π-entrance law ν^κ is the lifting of κ.
Proof: By the definition of ν^κ_t we have

$$\begin{aligned}\widehat{\nu^\kappa_t}(a) &= e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(T^*_{t-s}a)ds} = \lim_{r\to-\infty} e^{i\langle a,\kappa_t\rangle - \int_r^t \lambda(T^*_{t-s}a)ds}\\ &= \lim_{r\to-\infty} e^{i\langle a,\kappa_t\rangle - \int_0^{t-r} \lambda(T^*_s a)ds} = \lim_{r\to-\infty} e^{i\langle a,T_{t-r}\kappa_r\rangle - \int_0^{t-r} \lambda(T^*_s a)ds}\\ &= \lim_{r\to-\infty} \widehat{\pi_{t-r}(\kappa_r, .)}(a).\end{aligned}$$

Step2: Every $\nu \in \mathcal{K}_e^1(\pi)$ is the lifting of its projection, i.e.,

$$\mathbf{l}\big(\mathbf{p}(\nu)\big) = \nu. \tag{5.23}$$

Proof: We need to prove that for each $t \in \mathbb{R}$ and $a \in H$

$$\widehat{\nu_t}(a) = \lim_{r \to -\infty} \pi_{t-r}\widehat{(\mathbf{p}(\nu_r)}, .)(a).$$

But since $\nu \in \mathcal{K}_e^1(\pi)$ we know from the Claim (ii) that

$$\widehat{\nu_t}(a) = e^{i\langle a, \mathbf{p}(\nu_t)\rangle - \int_{-\infty}^t \lambda(T_{t-s}^* a) ds}.$$

On the other hand

$$\begin{aligned}\pi_{t-r}\widehat{(\mathbf{p}(\nu_r)}, .)(a) &= e^{i\langle a, T_{t-r}\mathbf{p}(\nu_r)\rangle - \int_0^{t-r} \lambda(T_s^* a) ds} \\ &= e^{i\langle a, \mathbf{p}(\nu_t)\rangle - \int_r^t \lambda(T_{t-s}^* a) ds}.\end{aligned}$$

Hence clearly we have the required property.
Step3: There is the following representation

$$\mathbf{l}(\kappa) = \int_{\mathcal{K}_e^1(\pi)} \mathbf{l}(\widetilde{\varrho}_t)\ \xi(d\mathbf{l}(\widetilde{\varrho})), \tag{5.24}$$

where $\widetilde{\varrho} = \mathbf{p}(\widetilde{\nu})$ and η is the image of ξ under the projection $\mathbf{p}$.
Proof: Since $\mathcal{K}^1(\pi)$ is a simplex by Theorem 5.20, we have for each $t \in \mathbb{R}$

$$\nu_t^\kappa = \int_{\mathcal{K}_e^1(\pi)} \widetilde{\nu}_t \xi(d\widetilde{\nu})$$

with some probability distribution ξ on $\mathcal{K}^1(\pi)$ (which is indeed supported by $\mathcal{K}_e^1(\pi)$). By *Step1* we have that $\nu^\kappa = \mathbf{l}(\kappa)$ and from *Step2* we know that $\widetilde{\nu} = \mathbf{l}(\mathbf{p}(\widetilde{\nu}))$ for every $\widetilde{\nu} \in \mathcal{K}_e^1(\pi)$, which yields $\widetilde{\nu} = \mathbf{l}(\widetilde{\varrho})$. Now let us denote by η the image of ξ under the projection $\mathbf{p}$. Then

$$\mathbf{l}(\kappa) = \int_{\mathcal{K}_e^1(\pi)} \mathbf{l}(\widetilde{\varrho}_t)\ \xi(d\mathbf{l}(\widetilde{\varrho}))$$

and formally we can write for each $t \in \mathbb{R}$

$$\kappa_t = \int_{\mathcal{K}(T)} \widetilde{\varrho}_t\ \eta(d\widetilde{\varrho}).$$

Step4: We claim that for every $t \in \mathbb{R}$

$$e^{i\langle a, \kappa_t\rangle} = \int_{\mathcal{K}(T)} e^{i\langle a, \widetilde{\varrho}_t\rangle} \eta(d\widetilde{\varrho}), \quad a \in H,$$

with $\kappa = \int_{\mathcal{K}(T)} \widetilde{\varrho}\, \eta(d\widetilde{\varrho})$.
Proof: From *Step3* we have $\mathbf{l}(\kappa) = \int_{\mathcal{K}(T)} \mathbf{l}(\widetilde{\varrho})\, \eta(d\widetilde{\varrho})$, therefore

$$\begin{aligned}\int_H e^{i\langle x,a\rangle}\mathbf{l}(\kappa)(dx) &= \int_H e^{i\langle x,a\rangle} \int_{\mathcal{K}(T)} \mathbf{l}(\widetilde{\varrho})\eta(d\widetilde{\varrho})\ (dx) \\ &= \int_{\mathcal{K}(T)} \int_H e^{i\langle x,a\rangle}\mathbf{l}(\widetilde{\varrho})(dx)\ \eta(d\widetilde{\varrho}), \quad a \in H\end{aligned}$$

and hence

$$\widehat{\mathbf{l}(\kappa_t)} = \int_{\mathcal{K}(T)} \widehat{\mathbf{l}(\widetilde{\varrho}_t)}\ \eta(d\widetilde{\varrho}).$$

Finally, from the definition of lifting we get

$$e^{i\langle a,\kappa_t\rangle} = \int_{\mathcal{K}(T)} e^{i\langle a,\widetilde{\varrho}_t\rangle}\eta(d\widetilde{\varrho}). \tag{5.25}$$

Step5: η is concentrated at a singleton, so that ν^κ is an extremal point.
Proof: It is easy to see that

$$e^{i\langle a,\kappa_t\rangle} = \int e^{i\langle a,\widetilde{\varrho}_t\rangle}\varepsilon_{\kappa_t}(d\widetilde{\varrho}),$$

since the barycentre for a measure on the unit circle can lie on this circle only if the measure is concentrated at one point (Here we are speaking about the probability law of $e^{i\langle a,\widetilde{\varrho}_t\rangle}$). Therefore, clearly from equation (5.25) and recent equality it holds $\eta = \varepsilon_\kappa$, which means that η is concentrated at a singleton.
Now the theorem follows from **Claim i**, **Claim ii** and **Claim iii**. ∎

5.7 An example: stable measures

In the Hilbert space H, let us consider the semigroup $T_t = e^{-t}\mathbb{1}$, $t \geq 0$, where $\mathbb{1}$ is the identity operator in H. Suppose that $\lambda : H \to \mathbb{C}$ has the same representation as in (5.10) (so it is negative definite and Sazonov continuous) and satisfies

$$\lambda(\gamma a) = \gamma^p \lambda(a), \quad a \in H,\ \gamma > 0$$

for some real number $p \in (0, 2]$. Since

$$\int_{-\infty}^t \lambda(T^*_{t-s}a)ds = \int_0^\infty \lambda(T^*_s a)ds = \frac{1}{p}\lambda(a),$$

the integral $\int_{-\infty}^t \lambda(T^*_{t-s}a)ds$ appearing in Assumption 5.32 is well-defined and Sazonov continuous, so that Assumption 5.32 is satisfied. Hence our

main Theorem 5.34 applies to this case.
Note that now

$$p_t f(x) = \int_H f(e^{-t}x + y)\mu_t(dy), \quad x \in H,$$

where $(\mu_t)_{t\geq 0}$ is defined via its Fourier transform

$$\widehat{\mu_t}(a) = \exp\left\{ -\frac{1}{p}\lambda\left[(1-e^{-pt})^{1/p}a\right]\right\}, \quad a \in H.$$

In the case $p = 2$, the measures $(\mu_t)_{t\geq 0}$ are Gaussian and the presentation for $(p_t)_{t\geq 0}$ is given by the classical Mehler formula. For $0 < p < 2$, $(\mu_t)_{t\geq 0}$ are (strictly) p-stable measure (see e.g. [Lin86], Chapter 6).

5.8 Mehler semigroups in nuclear spaces

For more details on topological properties of nuclear space see, e.g., [HKPS93] and [BK88].

Definition 5.38 *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Suppose that a sequence of Hilbert spaces $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is given. Assume that the set $S = \bigcap_{i\in\mathbb{N}} \mathcal{S}_i$ is dense in each $\mathcal{S}_i$ and that the family $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is directed by embedding, i.e.,*

$$\forall i_1, i_2 \in \mathbb{N},\ i_1 \leq i_2 : \quad \mathcal{S}_{i_2} \hookrightarrow \mathcal{S}_{i_1},$$

where all the embeddings are continuous.

Also, assume that for any $i \in \mathbb{N}$, one can find $i^{'} \in \mathbb{N}$ such that $\mathcal{S}_{i'}$ is embedded into $\mathcal{S}_i$ with a Hilbert-Schmidt embedding operator $\mathcal{O}_{i,i'} : \mathcal{S}_{i'} \hookrightarrow \mathcal{S}_i$ and also every $\mathcal{S}_i$ is continuously dense embedded in $\mathcal{S}_0 \equiv H$.

Then $\mathcal{S}$ is called a (Fréchet) nuclear space and we write $\mathcal{S} = pr \lim_{i\in\mathbb{N}} \mathcal{S}_{+i}$.

On $\mathcal{S}$ we introduce a projective limit topology τ_p with respect to the families of Hilbert spaces $(\mathcal{S}_i)_{i\in\mathbb{N}}$ and natural embeddings $\mathcal{O}_i : \mathcal{S} \hookrightarrow \mathcal{S}_i$, i.e., the weakest topology on $\mathcal{S}$ for which all the mappings $\mathcal{O}_i$, $i \in \mathbb{N}$, are continuous. An open neighborhood base at zero of this topology is given by the sets $U_{\varepsilon,i} = \{x \in \mathcal{S},\ \|x\|_i < \varepsilon\}$ with all possible choices of $\varepsilon > 0$ and $i \in \mathbb{N}$ where $\|\cdot\|_i$ is the corresponding norm of $\mathcal{S}_i$.

A sequence $\{x_n, n \in \mathbb{N}\}$ in $\mathcal{S}$ converges to $x \in \mathcal{S}$ with respect to τ_p if and only if it converges to x in every space $\mathcal{S}_i$. It turns out (cf. [GV64]) that $(\mathcal{S}, \tau_p)$ is metrizable and complete (i.e., Polish).

Definition 5.39 *Let us denote the dual space of $\mathcal{S}_i$, $i \in \mathbb{N}$ (with respect to pairing in H), by $\mathcal{S}_{-i}$ and set $\mathcal{S}_0 := H$. Then we obtain a triple*

$$\mathcal{S}_i \subset H \subset \mathcal{S}_{-i}.$$

Note that $\mathcal{S}_{i'} \hookrightarrow \mathcal{S}_i$ implies $\mathcal{S}_{-i} \hookrightarrow \mathcal{S}_{-i'}$. Moreover, if we have the Hilbert-Schmidt embedding operator $\mathcal{S}_{i'} \hookrightarrow \mathcal{S}_i$, then the embedding $\mathcal{S}_{-i} \hookrightarrow \mathcal{S}_{-i'}$ is also Hilbert-Schmidt with the same norm.
We define $\mathcal{S}^* := \bigcup_{i\in\mathbb{N}} \mathcal{S}_{-i}$ and equip it with following σ-algebra

$$\mathscr{B}(\mathcal{S}^*) := \sigma\Big(\bigcup_{i=1}^{\infty} \sigma(\mathcal{S}_{-i})\Big).$$

Then we have $\mathcal{S} \subset H \subset \mathcal{S}^*$, which will be called a Gelfand triple.

The following two propositions will be important for us.

Theorem 5.40 (Bochner-Minlos Theorem): *A function $\Psi : \mathcal{S} \to \mathbb{C}$ is the characteristic functional of a probability measure ν on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$, i.e.,*

$$\Psi(a) = \int_{\mathcal{S}^*} e^{i\langle a,x\rangle} d\nu(x), \quad a \in \mathcal{S},$$

if and only if
(i) *$\Psi(0) = 1$,*
(ii) *Ψ is continuous on $\mathcal{S}$,*
(iii) *Ψ is positive definite.*
Moreover such measure ν is uniquely determined by Ψ.

Theorem 5.41 (Lévy Continuity Theorem): *Let $\Psi : \mathcal{S} \to \mathbb{C}$ be continuous and let ν_n, $n \in \mathbb{N}$, be a sequence of probability measures on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$. If*

$$\widehat{\nu}_n(a) = \int_{\mathcal{S}^*} e^{i\langle a,x\rangle} d\nu_n(x) \xrightarrow{n\to\infty} \Psi(a), \quad a \in \mathcal{S},$$

then there exists a probability measure ν on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$ such that $\widehat{\nu} = \Psi$.

In this section, we concentrate on the following Gelfand triple $\mathcal{S} \subset H \subset \mathcal{S}^*$ (cf. [Kol88]).
Let us given a non-negative self-adjoint operator A in H. Moreover, we suppose that A has a purely discrete spectrum $\{\alpha_k\}_{k=1}^{\infty}$, i.e., $Ae_k = \alpha_k e_k$ with $\alpha_k > 0$, where $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal system in H. Furthermore, we suppose that $\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty$, i.e., the inverse operator A^{-1} has finite trace.

Define the Hilbert space $\mathcal{S}_i := \{x \in H \mid \|x\|_i^2 = \sum_{k=1}^{\infty} {\alpha_k}^i \langle x, e_k\rangle_H^2 \ < \infty\}$ which will be equipped with the inner product

$$\langle x, y\rangle_i := \sum_{k}^{\infty} {\alpha_k}^i \langle x, e_k\rangle_H \langle y, e_k\rangle_H, \quad x, y \in \mathcal{S}_i.$$

Claim 1: Denote $e_k^i := \frac{e_k}{(\sqrt{\alpha_k})^i}$, then $\{e_k^i\}_{k\in\mathbb{N}}$ is an $\mathcal{ONB}$ for $\mathcal{S}_i$.
Proof: Obviously, each $e_k^i \in \mathcal{S}_i$ and

$$\begin{aligned}\left\langle (\frac{1}{\sqrt{\alpha_k}})^i e_k, (\frac{1}{\sqrt{\alpha_\ell}})^i e_\ell \right\rangle_i : &= {\alpha_k}^{-\frac{i}{2}} {\alpha_\ell}^{-\frac{i}{2}} \langle e_k, e_\ell\rangle_i \\ &= {\alpha_k}^{-\frac{i}{2}} {\alpha_\ell}^{-\frac{i}{2}} \sum_{j=1}^{\infty} {\alpha_j}^i \langle e_k, e_j\rangle_H \langle e_\ell, e_j\rangle_H \\ &= \begin{cases} 0, & if \ \ k \neq \ell, \\ 1, & if \ \ k = \ell, \end{cases} \\ &= \delta_{k\ell}.\end{aligned}$$

Claim 2: $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i-1}$ (continuously embedded) for any $i \in \mathbb{N}$.
Proof: It is enough to prove that $\|x\|_{i-1}^2 \le \|x\|_i^2$ for any $x \in \mathcal{S}_i$. Since $\mathcal{S}$ is dense in $\mathcal{S}_i$ and $\|x\|_{i-1}^2 \le \|x\|_i^2$ for each $x \in \mathcal{S}$ by the definition, so our claim is true.

Claim 3: $\mathbb{I}_i : \mathcal{S}_i \to \mathcal{S}_{i-1}$, with identity operator $\mathbb{I}_i$ is a Hilbert-Schmidt operator for any $i \in \mathbb{N}$.
Proof:

$$\begin{aligned}\sum_{k=1}^{\infty} \|\mathbb{I}_i e_k^i\|_{i-1}^2 &= \sum_{k=1}^{\infty} \|(\frac{e_k}{(\sqrt{\alpha_k})^i})\|_{i-1}^2 \\ &= \sum_{k=1}^{\infty} \alpha_k^{i-1} \left\langle (\frac{1}{\sqrt{\alpha_k}})^i e_k, e_k \right\rangle^2 \\ &= \sum_{k=1}^{\infty} \frac{\alpha_k^{i-1}}{\alpha_k^i} \\ &= \sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty.\end{aligned}$$

Obviously, $\mathcal{S}$ is a dense subset of H, since $\{e_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$ is linearly dense in H. So, each $\mathcal{S}_i$ can be seen as a completion of $\mathcal{S}$ in the corresponding norm $\|\cdot\|_i$.

Denoting by $\mathcal{S}_{-i}$ (with norm $\|\cdot\|_{-i}$) the dual of $\mathcal{S}_i$ w.r.t $H = \mathcal{S}_0$, we have the chain

$$\bigcap_i \mathcal{S}_i \subset ... \subset \mathcal{S}_i \subset ... \subset \mathcal{S}_1 \subset H \subset \mathcal{S}_{-1} \subset ... \subset \mathcal{S}_{-i} \subset ... \subset \bigcup_i \mathcal{S}_{-i}. \quad (5.26)$$

Setting $\mathcal{S} := \bigcap_i \mathcal{S}_i$ and $\mathcal{S}^* := \bigcup_i \mathcal{S}_{-i}$, we get the Gelfand triple $\mathcal{S} \subset H \subset \mathcal{S}^*$.

Remark 5.42 *An important feature of this construction is that the semigroup $T_t = e^{-tA}$, $t \geq 0$, maps continuously $\mathcal{S}$ into $\mathcal{S}$. Indeed we can show that for each $i \in \mathbb{N}$*

$$e^{-tA} : \mathcal{S}_i \to \mathcal{S}_i,$$

is continuous and we have

$$\|e^{-sA}a\|_i^2 = \sum_{k=1}^{\infty} {\alpha_k}^i \langle e^{-sA}a, e_k\rangle^{\,2} < \infty,$$

but

$$\langle e^{-sA}a, e_k\rangle^{\,2} = \langle a, e^{-sA}e_k\rangle^{\,2} = \langle a, e^{-s\alpha_k}e_k\rangle_H^2 = e^{-2s\alpha_k}\langle a, e_k\rangle_H.$$

Note that $e^{-2s\alpha_k} \leq 1$, since $\alpha_k \geq 0$ for all $k \in \mathbb{N}$. Now $a \in \mathcal{S}$, therefore $a \in \mathcal{S}_i$ for any $i \in \mathbb{N}$ and we obtain

$$\|e^{-sA}a\|_i^2 = \sum_{k=1}^{\infty} {\alpha_k}^i \langle e^{-sA}a, e_k\rangle^{\,2} \leq \sum_{k=1}^{\infty} {\alpha_k}^i \langle a, e_k\rangle^{\,2} = \|a\|_i^2 < \infty.$$

Now since the operators e^{-tA}, $t \geq 0$, are self-adjoint on H, we can extend this semigroup to the bounded continuous operators $e^{-tA} : \mathcal{S}^* \to \mathcal{S}^*$ by the duality

$$\langle e^{-tA}x, a\rangle := \langle x, e^{-tA}a\rangle, \quad x \in \mathcal{S}^*,\ a \in \mathcal{S}.$$

Also, we can consider symmetric non-negative linear continuous operators $R : \mathcal{S} \to \mathcal{S}^*$ obeying

$$\begin{aligned} \langle Ra_1, a_1\rangle &\geq 0 \\ \langle Ra_1, a_2\rangle &= \langle a_1, Ra_2\rangle \end{aligned}$$

for any $a_1,\ a_2 \in \mathcal{S}$.
The continuity of $R : \mathcal{S} \to \mathcal{S}^*$ means that for each $i_1 \in \mathbb{N}$ there exists $i_2 \in \mathbb{N}$ such that $R : \mathcal{S}_{i_1} \to \mathcal{S}_{-i_2}$ is continuous. We denote the class of such continuous operators by $\mathcal{L}(\mathcal{S}, \mathcal{S}^*)$.

The following analogue of Proposition 5.24 takes place in the nuclear case.

Proposition 5.43 *Assume that $A \geq 0$ is a self-adjoint infinitesimal generator of the $\mathcal{C}_0$-semigroup $T_t = e^{-tA}$, $t \geq 0$, on $\mathcal{S}$ and $R : \mathcal{S} \to \mathcal{S}^*$ is a symmetric non-negative linear continuous operator. Let $\lambda : H \to \mathbb{C}$ is a function with the below representation*

$$\lambda(a) = \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1 + \|x\|^2}\right) M(dx)$$

where $R : \mathcal{S} \to \mathcal{S}^*$ is a symmetric non-negative linear continuous operator and M is a symmetric Lévy measure on H. Then there exists a family of measures $(\mu_t)_{t\geq 0}$ on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$ whose characteristic function is of the form

$$\widehat{\mu}_t(a) = \int_{\mathcal{S}^*} e^{i\langle a,x\rangle} \mu_t(dx) = e^{-\int_0^t \lambda(e^{-sA}a)ds}, \quad a \in \mathcal{S}.$$

Proof : Comparing the result in Hilbert space, since in the nuclear space similar result to the Lévy-Khinchin formula does not exist, we need to apply another arguments. Consider the extended form of $\exp\big(\int_0^t \lambda(e^{-sA}a)ds\big)$

$$\begin{aligned}\exp&\left(-\int_0^t \lambda(e^{-sA}a)ds\right)\\ &= \exp\left(i\langle a, b_t\rangle - \tfrac{1}{2}\langle R_t a, a\rangle + \int_H \left(e^{i\langle a,x\rangle} - 1 - \tfrac{i\langle a,x\rangle}{1+\|x\|^2}\right) M_t(dx)\right),\end{aligned} \tag{5.27}$$

where

$$\begin{aligned}R_t &= \int_0^t e^{-sA} R e^{-sA} ds \in \mathcal{L}(\mathcal{S}, \mathcal{S}^*),\\ b_t &= \int_0^t \int_H e^{-sA} x \left(\frac{1}{1 + \|e^{-sA}x\|^2} - \frac{1}{1 + \|x\|^2}\right) M(dx)ds \in \mathcal{S}^*,\end{aligned}$$

and

$$M_t(B) := \int_0^t M\Big((e^{-sA})^{-1}(B \setminus \{0\})\Big)ds, \quad B \in \mathscr{B}(H).$$

We need to check that $a \to \int_0^t \lambda(e^{-sA}a)ds$ is a positive-definite and continuous functional on $\mathcal{S}$. Then by the Bochner-Milnos theorem there exists a probability measure μ_t on $\mathcal{S}^*$ such that $\widehat{\mu_t}(a) = e^{-\int_0^t \lambda(e^{-sA}a)ds}$.
Let us to examine of each part in the right side of (5.27) separately.

About the first part, i.e.,

$$\mathcal{S} \ni a \mapsto \exp\big(-i\langle a, b_t\rangle\big)$$

we observe the following.
Since $b \in \mathcal{S}_{-i}$ for some $i \in \mathbb{N}$ and e^{-tA}, $t \geq 0$, maps continuously $\mathcal{S}_{-i}$ to some $\mathcal{S}_{-i'}$ $(i' \geq i)$, we can define $b_t \in \mathcal{S}_{-i'}$ via Bochner integrals in $\mathcal{S}_{-i'}$. Now, for every fix t, b_t is a vector in $\mathcal{S}_{-i'}$, so the first part is clearly continuous.

Concerning the second part,

$$\mathcal{S} \ni a \mapsto \exp\big(-\frac{1}{2}\langle R_t a, a\rangle\big), \tag{5.28}$$

we first need to define the integral $R_t = \int_0^t e^{-sA}Re^{-sA}ds$. By the definition of $\mathcal{S}$, $a \in \mathcal{S}_{i_0}$ for every fixed $i_0 \in \mathbb{N}$, so that we have

$$\begin{aligned} \forall i_1 \in \mathbb{N} \quad \exists i_0 \in \mathbb{N}, &\quad \text{such that} \quad e^{-sA} : \mathcal{S}_{i_0} \hookrightarrow \mathcal{S}_{i_1}, \\ \exists i_2 \in \mathbb{N}, &\quad \text{such that} \quad R : \mathcal{S}_{i_1} \hookrightarrow \mathcal{S}_{-i_2}, \\ \exists i_3 \in \mathbb{N}, &\quad \text{such that} \quad e^{-sA} : \mathcal{S}_{-i_2} \hookrightarrow \mathcal{S}_{-i_3}, \end{aligned}$$

so $e^{-tA}Re^{-tA} \in \mathcal{L}(\mathcal{S}_{i_0}, \mathcal{S}_{-i_3})$. Since $\mathcal{S}$ is nuclear, this implies that for some $i_4 \in \mathbb{N}$ we have $e^{-sA}Re^{-sA} \in \mathcal{L}_1(\mathcal{S}_{i_0}, \mathcal{S}_{-i_4})$ i.e., a trace class operator. Also $e^{-sA}Re^{-sA}$ is positive, i.e., $\langle e^{-sA}Re^{-sA}, a\rangle \geq 0$, for any $a \in \mathcal{S}$. Hence the Bochner integral can be taken in the separable Banach space $\mathcal{L}_1(\mathcal{S}_{i_0}, \mathcal{S}_{-i_4})$ of all trace class operators similar to Proposition 5.24. So $R_t \in \mathcal{L}_1(\mathcal{S}_{i_0}, \mathcal{S}_{-i_3})$ and therefore $R_t \in \mathcal{L}(\mathcal{S}, \mathcal{S}^*)$. Obviously, the result does not depend on the concrete choice of the spaces $\mathcal{S}_{i_0}, \mathcal{S}_{i_1}, ..., \mathcal{S}_{i_4}$.
Equivalently, we can define the integral (5.28) in the weak sense, so that

$$\langle R_t a, a\rangle := \int_0^t \langle e^{-sA}Re^{-sA}a, a\rangle ds, \quad a \in \mathcal{S}.$$

The last part is

$$\mathcal{S} \ni a \mapsto \exp\bigg(\int_H \big(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1+\|x\|^2}\big)M_t(dx)\bigg). \tag{5.29}$$

The integral over M lives in H, thus the Lévy-Khinchin formula and Milnos-Sazonov theorem gives the positive definiteness and continuity of the third part in (5.27).
And proof of the proposition is complete. ■

Now we are in a position to present a similar result to Theorem 5.34, but now related to the nuclear case.
First, we need to define the concept of $\mathcal{K}(\pi)$ and $\mathcal{K}(T)$ in the nuclear spaces.

Definition 5.44 *Let $\mathscr{P}(\mathcal{S}^*)$ be the set of all probability measures on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$, then we define similarly to the Hilbert space, $\mathcal{K}(\pi)$ to be the set of all π-entrance law $(\nu_t)_{t\in\mathbb{R}} \subset \mathscr{P}(\mathcal{S}^*)$ associated with Markovian semigroup $(\pi_t)_{t\geq 0}$ on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$ which satisfying following identity*

$$\int_{\mathcal{S}^*} \pi_t(x, B)\nu_s(dx) = \nu_{t+s}(B), \quad t \geq 0,\ s \in \mathbb{R}, \quad B \in \mathscr{B}(\mathcal{S}^*),$$

for each $x \in \mathcal{S}$. Similarly to the Hilbert space state space, $\mathcal{K}^1(\pi)$ is the notation for the set of all probability π-entrance laws with finite weak first moments.

Definition 5.45 *Consider the semigroup $(T_t = e^{-tA})_{t\geq 0}$ to $\mathcal{S}^*$ (continuous extended form of $(e^{-tA})_{t\geq 0}$ on $\mathcal{S}$). A family $(\kappa_s)_{s\in\mathbb{R}} \subset \mathcal{S}^*$, is called a T-entrance law if $T_t\kappa_s = \kappa_{s+t}$ for all $s \in \mathbb{R}$ and $t \geq 0$. The set of all such laws is denoted by $\mathcal{K}(T)$.*

We define similarly to the Hilbert space case, $\pi_t(x, dy)$ to be the translation of μ_t by $e^{-tA}x$, i.e.,

$$\pi_t(x, dy) := \mu_t(dy - e^{-tA}x), \quad t \geq 0,\ x \in \mathcal{S}^*.$$

Theorem 5.46 *Let π be a Mehler semigroup on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$ with the corresponding Fourier transform*

$$\int_{\mathcal{S}^*} e^{i\langle a,y\rangle}\pi_t(x, dy) = e^{i\langle a,T_t x\rangle - \int_0^t \lambda(T_s a)ds},$$

where $T = (T_t)_{t\in\mathbb{R}}$ is a family of $\mathcal{C}_0$-semigroups generated by a self-adjoint $A \geq 0$ with positive discrete spectrum $(\alpha_k)_{k\geq 0}$ such that $\sum_k \alpha_k^{-1} < \infty$. Let λ have a representation

$$\lambda(a) = \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1 + \|x\|^2}\right)M(dx), \quad a \in \mathcal{S},$$

with parameters R and M in the sense of Proposition 5.43. In addition assume that M is a symmetric Lévy measure on $(H, \mathscr{B}(H))$ and that also

$$\mathcal{S} \ni a \mapsto \int_{-\infty}^t \lambda(T_{t-s}a)ds, \quad t \geq 0,$$

is well-defined and continuous.
Then for every $\kappa \in \mathcal{K}(T)$ there exists a unique extremal probability π-entrance law ν^κ such that

$$\int_{\mathcal{S}^*} e^{i\langle a,y\rangle}\nu_t^\kappa(dy) = e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(T_{t-s}a)ds}, \quad a \in \mathcal{S}, \tag{5.30}$$

in which $(\kappa_t)_{t\in\mathbb{R}} \subset \mathcal{S}^*$.
Moreover, formula (5.30) establishes a one-to-one correspondence between $\mathcal{K}(T)$ and the set of all extremal elements of $\mathcal{K}^1(\pi)$.

Proof The proof of theorem is almost the same Theorem 5.34. The only major difference is to show the existence of a certain probability measure ν_t^κ on $(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*))$ obeying the characteristic function (5.30), (cf. Claim i). For a fix $t \in \mathbb{R}$ by using (6.13) we have the sequence

$$\widehat{M_r}(a) = e^{i\langle a,\kappa_t\rangle - \int_r^t \lambda(T_{t-s}a)ds}$$

where $M_r(dy) := \pi_{t-r}(\kappa_r, dy)$ when $r \to -\infty$. On the other hand, by the assumptions of theorem we have continuity of the right-hand side of (5.30) and our statement follows from the Lévy continuity theorem. ∎

Remark 5.47 *Concerning the above construction of the Gelfand triple* (5.26), *we note that the following more general fact is valid (cf. Section 1, Chapter 4 in [BK88]).*
Let $A \geq 0$ *be self-adjoint operator acting in a real separable Hilbert space* H, *having a spectrum of finite multiplicity. Then one can construct a nuclear Gelfand triple* $\mathcal{S} \subset H \subset \mathcal{S}^*$ *such that:*

1. $\mathcal{S} = pr\lim_{i\in\mathbb{N}} \mathcal{S}_{+i}$ *where each* $(\mathcal{S}_i, \langle\cdot,\cdot\rangle_i)$ *is a separable Hilbert space with a Hilbert-Schmidt embeddings* $\mathcal{S}_i \hookrightarrow \mathcal{S}_0 := H$, $\mathcal{S}_{i+1} \hookrightarrow \mathcal{S}_i$, $i \in \mathbb{N}$.

2. $\mathcal{S} \subset \mathcal{D}(A)$, $A \in \mathcal{L}(\mathcal{S},\mathcal{S})$ *and* $e^{-tA} \in \mathcal{L}(\mathcal{S},\mathcal{S})$ *for all* $t \geq 0$.

In the case of general self-adjoint $A \geq 0$, *we can construct a nuclear space* $\mathcal{S} = pr\lim_{i\in\Lambda} \mathcal{S}_i$ *with a not necessarily countable index set* $\Lambda = \{i\}$.

Chapter 6

Two-Parameter Mehler Semigroups and Corresponding Extremal π-Entrance Laws

Evolution systems of measures corresponding to a two-parameter semigroup are an important generalization of the concept of invariant measures for a one-parameter semigroup. Similarly to the previous chapter, we will show that the set $\mathcal{K}(\pi)$ of all π-entrance laws associated with the two-parameter semigroup $(p_{s,t})_{s\leq t}$, is a simplex. Furthermore, we will prove an explicit representation for the extremal points of $\mathcal{K}(\pi)$ corresponding the two-parameter Mehler semigroup with Lévy processes. This is one of the main results of the whole thesis and will be stated in Theorem 6.20.
The above theorem will be illustrated by one example in Section 6.5.
Finally, in Section 6.6, we will show the uniqueness of T-periodic evolution system of measures by using the explicit representation of extremal points obtained in Section 6.4.

Let us recall again:

Definition 6.1 *Suppose $(\pi_{s,t})_{s\leq t}$ is a Markovian two-parameter semigroup of transition kernels on H. Every one-parameter family of probability measures $(\nu_s)_{s\in\mathbb{R}}$ which are connected by the relation*

$$\int_H \pi_{s,t}(x,B)\nu_s(dx) = \nu_t(B), \quad s\leq t,\ s,t\in\mathbb{R},\ B\in\mathscr{B}(H)$$

is called a π-entrance law. The set of all such $(\nu_s)_{s\in\mathbb{R}}$ is denoted by $\mathcal{K}(\pi)$.

The proof of convexity for $\mathcal{K}(\pi)$ is more or less similar to the non-autonomous case but we will put it again:

Lemma 6.2 $\mathcal{K}(\pi)$ *is convex.*

PROOF Suppose $(\nu_s)_{s\in\mathbb{R}}$, $(\eta_s)_{s\in\mathbb{R}}$ are in $\mathcal{K}(\pi)$ and $\alpha \in [0,1]$. Then for every arbitrary $B \in \mathscr{B}(H)$, we have

$$\begin{aligned}
&\int_H \pi_{s,t}(x,B)(\alpha\nu_s + (1-\alpha)\eta_s)(dx) \\
&= \alpha \int_H \pi_{s,t}(x,B)\nu_s(dx) + (1-\alpha)\int_H \pi_{s,t}(x,B)\eta_s(dx) \\
&= \alpha\nu_t(B) + (1-\alpha)\eta_t(B) \\
&= (\alpha\nu_t + (1-\alpha)\eta_t)(B)
\end{aligned}$$

for all $s \le t$ and $s,t \in \mathbb{R}$. Therefore

$$\alpha\nu + (1-\alpha)\eta = \Big(\alpha\nu_s + (1-\alpha)\eta_s\Big)_{s\in\mathbb{R}} \in \mathcal{K}(\pi).$$ ■

Note that the concept of extremal point on the set $\mathcal{K}(\pi)$ is defined exactly like as in the previous chapter (cf. Definition 5.3).

6.1 The Markov process associated with two-parameter semigroups

Definition 6.3 *We say that a Markov process $\big(\Omega, \mathcal{F}, (X_t)_{t\in\mathbb{R}}, \mathbb{P}\big)$ corresponds to the two-parameter semigroup of transition kernels $(\pi_{s,t})_{s\le t}$ and write $\mathbb{P} \in M(\pi)$ if for any $s \le t$ and $s,t \in \mathbb{R}$*

$$\mathbb{P}(X_t \mid \mathcal{F}_s) = \pi_{s,t}(X_s, B), \quad \mathbb{P} - a.s., \tag{6.1}$$

where $B \in \mathscr{B}(H)$ and $\mathcal{F}_s = \sigma(X_r \mid r \le s)$.

Here we will extend the statement of Proposition 5.5 to the case of Markovian two-parameter semigroups:

Proposition 6.4 *Given a Markovian semigroup $(\pi_{s,t})_{s\le t}$ over $\big(H, \mathscr{B}(H)\big)$ and a π-entrance law $\nu := (\nu_s)_{s\in\mathbb{R}}$ on $\mathscr{B}(H)$, then there is a Markov process $\big(\Omega, \mathcal{F}, (X_s)_{s\in\mathbb{R}}, \mathbb{P}_\nu\big)$ in the sense of Definition 6.3 with state space $\big(H, \mathscr{B}(H)\big)$, where $\Omega = H^{\mathbb{R}}$ and $\mathcal{F} := \sigma((X_s)|s \in \mathbb{R})$. Furthermore, ν_t is the law of X_t for any $t \in \mathbb{R}$.*

PROOF Put $X_s(\omega) := \omega(s)$ for every $s \in \mathbb{R}$ and every $\omega \in H^{\mathbb{R}}$ and set $\mathcal{F}_s$ to be the σ-algebra generated by $\{X_r \mid r \le s\}$.

The first is to check the existence of a measure $\mathbb{P}_\nu$ with the following finite dimensional distributions

$$\begin{aligned}
&\mathbb{P}_{t_1,\dots,t_n}\big[X_{t_1} \in dx_1, , \dots, X_{t_n} \in dx_n\big] : \\
= \quad &\pi_{t_{n-1},t_n}(x_{n-1}, dx_n) \dots \pi_{t_1,t_2}(x_1, dx_2)\nu_{t_1}(dx_1)
\end{aligned} \tag{6.2}$$

for each n-tuple of times $-\infty < t_1 < ... < t_n < +\infty$, which can be done similarly to the one-parameter semigroup case considered in Proposition 5.5.

In the next step, we construct a Markov process

$$\Big(H^{\mathbb{R}}, \mathcal{F} = \sigma((X_t) \mid t \in \mathbb{R}), (X_t)_{t\in\mathbb{R}}, (\mathbb{P}_\nu)\Big)$$

with state space $\big(H, \mathscr{B}(H)\big)$, which corresponds to the given $(\pi_{s,t})_{s\le t}$ and the initial distribution $(\nu_t)_{t\in\mathbb{R}}$ in the sense of Definition 6.3.

To this end, for any $s \in \mathbb{R}$ and $x \in H$, we define the corresponding probability measure $\mathbb{P}_{(s,x)}$ on $\big(H^{\mathbb{R}}, \mathcal{F}_{>s} := \sigma(X_r \mid r \ge s)\big)$ by following formula

$$\begin{aligned} &\mathbb{E}_{(s,x)}\big[f(X_{t_1}, ..., X_{t_n})\big] : \\ = \quad &\textstyle\int_H ... \int_H f(x_1, ..., x_n)\pi_{t_{n-1},t_n}(x_{n-1}, dx_n)...\pi_{t_1,t_2}(x_1, dx_2)\pi_{s,t_1}(x, dx_1) \end{aligned} \tag{6.3}$$

for any n-tuple of times $s < t_1 < ... < t_n < \infty$ and $f : H^n \to \mathbb{R}$ bounded and $\big(\mathscr{B}(H)\big)^{\bigotimes n}$-measurable.

Given $(\nu_t)_{t\in\mathbb{R}}$, we then set

$$\begin{aligned} &\mathbb{E}_\nu\big[f(X_{t_1}, ..., X_{t_n})\big] : \\ = &\int_H ... \int_H f(x_1, ..., x_n)\pi_{t_{n-1},t_n}(x_{n-1}, dx_n)...\pi_{t_1,t_2}(x_1, dx_2)\nu_{t_1}(dx_1) \end{aligned}$$

for any n-tuple of times $-\infty < t_1 < ... < t_n < \infty$ and $f : H^n \to \mathbb{R}$ bounded and $\big(\mathscr{B}(H)\big)^{\bigotimes n}$-measurable.

By checking the Markov property in the sense of Definition 2.5, namely showing that

$$\mathbb{E}_\nu[X_t \in B \mid \mathcal{F}_s] = \mathbb{P}_{(s,X_s)}[X_t \in B] \tag{6.4}$$

for every $s \le t$ and $B \in \mathscr{B}(H)$, our assertion will be proven because from definition (6.3) we have

$$\mathbb{P}_{(s,X_s)}[X_t \in B] = \pi_{s,t}(X_s, B).$$

Indeed, instead of (6.4) we will show a stronger fact

$$\mathbb{E}_\nu[f(X_{t_1}, ..., X_{t_n}) \mid \mathbb{1}_\Delta] = \mathbb{E}_\nu\big[\mathbb{E}_{(s,X_s)}[f(X_{t_1}, ..., X_{t_n})] \mid \mathbb{1}_\Delta\big]$$

holding for each entrance law $(\nu_t)_{t\in\mathbb{R}}$, every $s < t_1 < ... < t_n < \infty$ and $\Delta \in \mathcal{F}_s$.

By the monotone class theorem (applied to Δ), this follows from the identity

$$\begin{aligned} &\mathbb{E}_\nu\big[f(X_{t_1}, ..., X_{t_n})g(X_{s_1}, ..., X_{s_m})\big] = \\ &\mathbb{E}_\nu\big[\mathbb{E}_{(s,X_s)}[f(X_{t_1}, ..., X_{t_n})]g(X_{s_1}, ..., X_{s_m})\big] \end{aligned}$$

holding for all $\big(\mathscr{B}(H)\big)^{\bigotimes(m)}$-measurable bounded $g : H^m \to \mathbb{R}$ and $-\infty < s_1 < ... < s_m = s$. So, the left hand side is equal to

$$\begin{aligned}&\int \nu_{s_1}(dx_1)\int \pi_{s_1,s_2}(x_1,dx_2)...\int \pi_{s_{m-1},s_m}(x_{m-1},dx_m)\\ \times &\underbrace{\int \pi_{s,t_1}(x_m,dy_1)...\int \pi_{t_{n-1},t_n}(y_{n-1},dy_n)f(y_1,...,y_n)}_{=\mathbb{E}_{(s,x_m)}[f(X_{t_1},...,X_{t_n})]}g(x_1,...,x_m)\\ = &\ \mathbb{E}_\nu[\mathbb{E}_{(s,X_s)}[f(X_{t_1},...,X_{t_n})g(X_{s_0},...,X_{s_m})]].\end{aligned}$$

So $\mathbb{P}_\nu$ belongs to $M(\pi)$, which ends the proof. ∎

Remark 6.5 *Conversely, if X is a Markov process with values in H, then there exist an entrance law $\nu = (\nu_t)_{t\in\mathbb{R}}$ and a Markovian transition kernel $(\pi_{s,t})_{s\le t}$ such that equation* (6.1) *holds and the law of X_t is ν_t*

Like as in the one-parameter case, we can establish the isomorphism between $\mathcal{K}(\pi)$ and $M(\pi)$ also for the Markovian two-parameter semigroup of transition kernels $(\pi_{s,t})_{s\le t}$. The proof is quite similar to the one-parameter case dealt with in Lemma 5.19. For the convenience of the reader, we will sketch the proof here.

Lemma 6.6 *The correspondence $\nu \to \mathbb{P}_\nu$ defined by proposition 6.4 is an isomorphism of convex measurable spaces $\mathcal{K}(\pi)$ and $M(\pi)$.*

Proof Similarly to the previous chapter one can define a measurable structure for $M(\pi)$. Also, by Lemma 5.18 there is a natural convex structure associated to $\mathcal{K}(\pi)$. Via Proposition 6.4, $\mathbb{P}_\nu \in M(\pi)$ for any $\nu \in \mathcal{K}(\pi)$. Now let $\mathbb{P}$ be any element of $M(\pi)$. Define $\nu = (\nu_s)_{s\in\mathbb{R}}$ by

$$\nu_s(B) = \mathbb{P}\{X_s \in B\}, \quad s \in \mathbb{R},\ B \in \mathscr{B}(H). \tag{6.5}$$

Then formula (6.1) implies that for any $-\infty < s_1 < ... < s_n < \infty$ and $B_1, ..., B_n \in \mathscr{B}(H)$

$$\mathbb{P}\{X_{s_1} \in B_1, ..., X_{s_n} \in B_n\} = \int_{B_1}...\int_{B_n} \nu_{s_1}(dx_1)\pi_{s_1,s_2}(x_1,dx_2)...\pi_{s_{n-1},s_n}(x_{n-1},dx_n).$$

Hence $\mathbb{P} = \mathbb{P}_\nu$.
Furthermore, for every $B \in \mathscr{B}(H)$

$$\nu_t(B) = \mathbb{P}\{\mathbb{1}_B(X_t)\}, \quad t \in \mathbb{R}, \tag{6.6}$$

and thus

$$\int_H \pi_{s,t}(x,B)\nu_s(dx) = \mathbb{P}(X_s \in H, X_t \in B) = \nu_t(B), \quad s \le t$$

Hence it is clear that ν is a π-entrance law.
On the other hand, $\mathbb{P}\{X_s \in B\} = \nu_s(B)$. Thus, no element $\mathbb{P}$ of $M(\pi)$ can have two different inverse images in $\mathcal{K}(\pi)$.

We have proved that the mapping $\nu \longleftrightarrow \mathbb{P}_\nu$ defines a one-to-one correspondence between $\mathcal{K}(\pi)$ and $M(\pi)$ and that the inverse mapping given by (6.6) obviously preserves the convex and measurable structures. Proving that the mapping $\nu \to \mathbb{P}_\nu$ has the same properties can be done in the similar way as it has been done in Lemma 5.19. The previous scheme works with an obvious modification that the mapping Φ in the proof of Lemma 5.19 should be substituted by

$$\Phi(x_1) = \mathbb{1}_{B_1}(x_1) \int_{B_2} ... \int_{B_n} \pi_{s_1,s_2}(x_1, dx_2)...\pi_{s_{n-1},s_n}(x_{n-1}, dx_n).$$

And the proof is complete. ∎

Remark 6.7 *Similarly to Proposition 5.5, the measure $\mathbb{P}_\nu$ introduced above belongs to $M(\pi)$.*

6.2 Strong evolution operators

We present the definition of an strong evolution operator from [Yad86]:

Definition 6.8 *Let us define*

$$\Delta := \{(s,t) \subset \mathbb{R}^2 \mid T_* \leq s \leq t \leq T^*\}$$

where $T_* \in \mathbb{R} \cup \{-\infty\}$ and $T^* \in \mathbb{R} \cup \{+\infty\}$.
A family $U = (U(s,t))_{T_* \leq s \leq t \leq T^*} \subset \mathcal{L}(H)$ is called an strong evolution family if the following holds:

(i) *$U_{t,t} = I$ for all $t \in [T_*, T^*]$, where I denotes the identity operator in H;*

(ii) *$U_{r,t}U_{s,r} = U_{s,t}$ for $T_* \leq s \leq r \leq t \leq T^*$;*

(iii) *U is strongly continuous, i.e., the map $U_{(\cdot,\cdot)}x : \Delta \to H$ is continuous for any $x \in H$ and*

$$\mathbb{M}_T := \sup_{\Delta} \|U_{s,t}\| < \infty;$$

(iv) *For any $t \in [T_*, T^*]$, there exists a closed linear operator $A(t)$ on H such that $U_{s,t} : D(A(s)) \to D(A(t))$ for all $s < t$ and*

$$\int_s^t A(r)U_{s,r} f dr = (U_{s,t} I) f$$

for any $f \in \mathcal{D}_{s,t}(A) := \{f \in H \mid U_{s,r} f \in D(A(r)) \text{ for all } r \in [s,t]\}$.

(v) *Obviously (iv) implies that for every $f \in \mathcal{D}_{s,t}(A)$*

$$\frac{\partial}{\partial t} U_{s,t} f = A(t) U_{s,t} f$$

for Lebesgue-almost all $t \in [T_*, T^*]$, which justifies the terminology.

Analogously to the theory of one-parameter semigroups, $(A(t))_{t\in[T_*,T^*]}$ is called the generator of U.

Remark 6.9 *From* **(iii)** *in the Definition 6.8 we have*

$$t \mapsto U_{s,t}x \in H, \quad \text{for all } t \in [s, T^*],$$

is a continuous mapping for any $s \in \mathbb{R}$ and $x \in H$ and

$$s \mapsto U_{s,t}x \in H, \quad \text{for all } s \in [T_*, t],$$

is a continuous mapping for any $t \in \mathbb{R}$ and $x \in H$.

Remark 6.10 *In fact, despite an extensive literature on the solution of non-autonomous Cauchy problem (see [Paz83])*

$$X'(t) = A(t)X(t), \quad X(s) = x_s \in D(A(s)), \quad s \le t, \tag{6.7}$$

but there is by now, a basically missing an existence result analogous to the well-known Hille- Yoshida theorem giving a characterization of the infinitesimal generators of $\mathcal{C}_0$-semigroups. Since this issue is beyond the scope of this thesis, it will be assumed that we are readily given an evolution family $U = (U_{s,t})_{s\le t}$ which is properly associated with the Cauchy problem (6.7).

6.3 Two-parameter Mehler semigroup

In this section, we define a two-parameter Mehler semigroup. The idea to introduce this type of Mehler semigroup belongs to [OR10].

Proposition 6.11 *Let $(U_{s,t})_{s\le t}$ be a strong evolution family of linear operators on H, $\sigma : \mathbb{R} \to H$ be a strongly continuous and bounded in operator norm and the Lévy symbol λ is a negative-definite and Sazonov continuous function on H with $\lambda(0) = 0$. Then there exists $(\mu_{s,t})_{s\le t} \subset \mathscr{P}(H)$ whose characteristic functions are of the form*

$$\widehat{\mu}_{s,t}(a) = \int_H e^{i\langle a,x\rangle} \mu_{s,t}(dx) = e^{-\int_s^t \lambda(\sigma^*(r)U^*_{r,t}a)dr}, \quad a \in H, \tag{6.8}$$

where U^* is the adjoint operator of U.

Proof In fact, by applying the same argument as in Chapter 5 concerning the property of λ and using Minlos-Sazonov theorem and Lévy-Khinchin Formula, we conclude that $\exp(-\lambda)$ is the characteristic function of an infinitely divisible probability measure on H with

$$\lambda(a) = -i\langle a, b\rangle + \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1+\|x\|^2}\right)M(dx) \qquad (6.9)$$

where $b \in H$, $R \in \mathcal{L}_1^+(H)$ and M is a Lévy measure on H.
Now, if we consider the extended form of $e^{-\int_s^t \lambda(\sigma(r)^* U_{r,t}^* a)dr}$

$$\exp\Bigg(- \int_s^t \lambda(\sigma^*(r)U_{r,t}^* a)dr\Bigg)$$

$$= \exp\Bigg(\int_s^t i\langle a, U_{r,t}\sigma(r)b\rangle dr - \int_s^t \tfrac{1}{2}\langle \sigma^*(r)U_{r,t}^* a, R\sigma^*(r)U_{r,t}^* a\rangle dr$$
$$+ \int_s^t \int_H \left(e^{i\langle a, U_{r,t}\sigma(r)x\rangle} - 1 - \tfrac{i\langle a, U_{r,t}\sigma(r)x\rangle}{1+\|x\|^2}\right) M(dx)dr\Bigg)$$

$$= \exp\Bigg(i\langle a, b_{s,t}\rangle - \tfrac{1}{2}\langle R_{s,t}a, a\rangle + \int_H \left(e^{i\langle a,x\rangle} - 1 - \tfrac{i\langle a,x\rangle}{1+\|x\|^2}\right) M_{s,t}(dx)\Bigg), \qquad (6.10)$$

where

$$R_{s,t} = \int_s^t U_{r,t}\sigma(r)R\sigma^*(r)U_{r,t}^* dr$$
$$b_{s,t} = \int_s^t U_{r,t}\sigma(r)b dr + \int_s^t \int_H U_{r,t}\sigma(r)x\left(\frac{1}{1+\|U_{r,t}\sigma(r)x\|^2} - \frac{1}{1+\|x\|^2}\right)M(dx)dr,$$

and the measures $M_{s,t}$ are defined by

$$M_{s,t}(B) := \int_s^t M\Big((U_{r,t}\sigma(r))^{-1}(B\setminus\{0\})\Big)dr, \quad B \in \mathscr{B}(H).$$

Here

$$(U_{r,t}\sigma(r))^{-1}(B\setminus\{0\}) := \Big\{ y \in H \mid U_{r,t}\sigma(r)y \in B\setminus\{0\}\Big\}.$$

From the Lévy-Khinchin formula, if one prove that for every fixed s and t, $b_{s,t}$ is well-defined in H, $R_{s,t} \in \mathcal{L}_1^+(H)$ and $M_{s,t}$ is a Lévy measure, then there is a probability measure $\mu_{s,t}$ on H such that

$$\widehat{\mu_{s,t}}(a) = e^{-\int_s^t \lambda(\sigma^*(r)U_{r,t}^* a)dr}. \qquad (6.11)$$

Let us discuss the required properties of every part separately.
note that b is just a vector in H and for every fixed s and t, $b_{s,t} \in H$ is

correctly defined via Bochner integrals.

Concerning $(R_{s,t})_{s\le t}$, since by our assumption $(U_{r,t})_{r\le t}$ is a strong evolution operators, strongly continuity of $(U_{r,t})_{r\le t}$ give us

$$\mathbb{M}_T := \sup_{T_*\le r\le t\le T^*} \|U_{r,t}\|_{\mathcal{L}(H)} < \infty.$$

Now, note that the strong measurability of σ is equivalent to measurability of $r \mapsto \sigma(r)e_k \in H$ for every $\mathcal{ONB}$, $\{e_k\}_{k\in\mathbb{N}}$ on H and furthermore the measurability of $r \mapsto \sigma^*(r)R\sigma(r)e_k \in H$. Since $\mathcal{L}_1(H)$ is ideal in $\mathcal{L}(H)$ so we get the measurability of $\sigma^*(r)R\sigma(r)$ in the separable Banach space $\mathcal{L}_1(H)$. Then $t \mapsto U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}$ is also strongly measurable, i.e.,

$$t \mapsto U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}x \in \mathcal{L}^+(H),$$

and we are able to define $R_{s,t}$ as the Bochner integral of $U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}$. We know that R is a non-negative trace class operator. Since $\mathcal{L}_1(H)$ is an operator ideal in $\mathcal{L}(H)$, so $U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}$ is also trace class operator for every r and t. Meanwhile, every $U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}$ is non-negative due to the same property of R. Furthermore, it is also symmetric because

$$\begin{aligned}\langle U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}a, a\rangle &= \langle R\sigma^*(r)U^*_{r,t}a, \sigma^*(r)U^*_{r,t}a\rangle \\ = \langle \sigma^*(r)U^*_{r,t}a, R\sigma^*(r)U^*_{r,t}a\rangle &= \langle a, U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}a\rangle.\end{aligned}$$

Next, the Bochner integral in the definition of $R_{s,t}$ is taken in the Banach separable space $\mathcal{L}_1(H)$ of all trace class operators in H. This definition is correct since

$$\int_{T_*}^{T^*} \|U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}\|_{\mathcal{L}_1(H)}dr \le (T^* - T_*)\mathbb{M}_T \cdot \sup_{T_*\le r\le t\le T^*} \|R\|_{\mathcal{L}_1(H)} < \infty.$$

Then $R_{s,t}$ is a symmetric non-negative trace class operator with

$$\|R_{s,t}\|_{\mathcal{L}_1(H)} \le \int_{T_*}^{T^*} \|U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}\|_{\mathcal{L}_1(H)}dr < \infty$$

and

$$\langle R_{s,t}a, a\rangle = \int_s^t \langle \sigma(r)R\sigma^*(r)U^*_{r,t}a, U^*_{r,t}a\rangle dr \ge 0, \quad a \in H$$

and

$$\begin{aligned}\langle R_{s,t}a, a\rangle &= \int_s^t \langle U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}a, a\rangle dr \\ &= \int_s^t \langle a, U_{r,t}\sigma(r)R\sigma^*(r)U^*_{r,t}a\rangle dr = \langle a, R_{s,t}a\rangle\end{aligned}$$

The last part is $(M_{s,t})_{s\leq t}$. Note that clearly $M_{s,t}(0)=0$ and also

$$\begin{aligned}\int_H \big(1\wedge\|x\|^2\big)M_{s,t}(dx) &= \int_s^t\int_H \Big(1\wedge\|U_{r,t}\sigma(r)x\|^2\Big)M(dx)dr\\ &\leq \int_{T_*}^{T^*}\int_H \big(1\wedge \mathbb{M}_T\cdot\|x\|^2\big)M(dx)dr\\ &\leq \max\{1,\mathbb{M}_T^2\}(T^*-T_*)\int_H\big(1\wedge\|x\|^2\big)M(dx)<\infty.\end{aligned}$$

In conclusion, similarly to one-parameter case, (6.11) is Sazonov continuous and the proof of the proposition is complete. ∎

Definition 6.12 *Let $\pi_{s,t}(x,dy)$ be the translation of $\mu_{s,t}(dy)$ by $U_{s,t}x$, i.e.,*

$$\pi_{s,t}(x,dy)=\mu_{s,t}(dy-U_{s,t}x),\quad s\leq t,\ x\in H.$$

In other words, for all $f\in\mathcal{B}_b(H)$, we have

$$\int_H f(y)\pi_{s,t}(x,dy)=\int_H f\big(U_{s,t}x+y\big)\mu_{s,t}(dy),\quad x\in H.$$

Following Definition 4.3, the semigroup $(p_{s,t})_{s\leq t}$ is given by

$$p_{s,t}f(x)=\int_H f(y)\pi_{s,t}(x,dy)=\int_H f(U_{s,t}x+y)\mu_{s,t}(dy),\quad s\leq t$$

is called the generalized two-parameter Mehler semigroup.

In fact, the semigroup property can be concluded in following.

Proposition 6.13 *$(p_{s,t})_{s\leq t}$ is a semigroup on $\mathcal{B}_b(H)$ if and only if for all $s\leq r\leq t$*

$$\mu_{s,t}=(\mu_{s,r}\circ U_{r,t}^{-1})*\mu_{r,t} \tag{6.12}$$

(where $*$ is convolution operator) or equivalently, in terms of the characteristic functions,

$$\widehat{\mu}_{s,t}(a)=\widehat{\mu}_{r,t}(a)\widehat{\mu}_{s,r}(U_{r,t}^*a),\quad a\in H,\ s\leq r\leq t.$$

(Note that this proposition can be found in [OR10], Proposition 2.2.)

Proof First we prove that $(p_{s,t})_{s\leq t}$ is a semigroup if we have (6.12). Let $f\in\mathcal{B}_b(H)$ and $s\leq r\leq t$. Then we calculate for all $x\in H$

$$\begin{aligned}p_{s,r}(p_{r,t}f)(x) &= (\mu_{s,r}*p_{r,t}f)(U_{s,r}x)\\ &= (\mu_{s,r}*(\mu_{r,t}*f)(U_{r,t}\cdot))(U_{s,r}x)\\ &= \big(((\mu_{s,r}\circ U_{r,t}^{-1})*\mu_{r,t})*f\big)(U_{s,t}x).\end{aligned}$$

Now using (6.12) in the right-hand side we get that

$$p_{s,r}(p_{r,t}f)(x) = (\mu_{s,t} * f)(U_{s,t}x) = p_{s,t}f(x).$$

Repeating the above arguments in the inverse direction, we show that (6.12) is also necessary for each semigroup $(p_{s,t})_{s\leq t}$. ■

On the other hand, $(\mu_{s,t})_{s\leq t}$ satisfies the following relation

$$\begin{aligned}
\widehat{\mu}_{s,r}(U^*_{r,t}a)\widehat{\mu}_{r,t}(a) &= e^{-\int_s^r \lambda\left(\sigma^*(\ell)U^*_{\ell,r}(U^*_{r,t}a)\right)d\ell}\; e^{-\int_r^t \lambda\left(\sigma^*(\ell)U^*_{\ell,t}a\right)d\ell} \\
&= e^{-\int_s^r \lambda\left(\sigma^*(\ell)(U_{r,t}U_{\ell,r})^*a\right)d\ell}\; e^{-\int_r^t \lambda\left(\sigma^*(\ell)U^*_{\ell,t}a\right)d\ell} \\
&= e^{-\int_s^r \lambda\left(\sigma^*(\ell)U^*_{\ell,t}a\right)d\ell}\; e^{-\int_r^t \lambda\left(\sigma^*(\ell)U^*_{\ell,t}a\right)d\ell} \\
&= e^{-\int_s^t \lambda\left(\sigma^*(\ell)U^*_{\ell,t}a\right)d\ell} \\
&= \widehat{\mu}_{s,t}(a).
\end{aligned}$$

Therefore $(p_{s,t})_{s\leq t}$ is a semigroup. Moreover by the property of $(\mu_{s,t})_{s\leq t}$, we get that $(p_{s,t})_{s\leq t}$ is clearly Markovian semigroup.

Lemma 6.14 *The characteristic function of $(\pi_{s,t})_{s\leq t}$ is given by*

$$\int e^{i\langle a,y\rangle}\pi_{s,t}(x,dy) = e^{i\langle a,U_{s,t}x\rangle - \int_s^t \lambda_r(U^*_{r,t}a)dr}. \tag{6.13}$$

Proof

$$\begin{aligned}
\widehat{\pi_{s,t}(x,.)}(a) &= \int e^{i\langle a,y\rangle}\pi_{s,t}(x,dy) \\
&= \int e^{i\langle a,y\rangle}\mu_{s,t}(dy - U_{s,t}x) \\
&= \int e^{i\langle a,y+U_{s,t}x\rangle}\mu_{s,t}(dy) \\
&= e^{i\langle a,U_{s,t}x\rangle}\int e^{i\langle a,y\rangle}\mu_{s,t}(dy) \\
&= e^{i\langle a,U_{s,t}x\rangle - \int_s^t \lambda_r(U^*_{r,t}a)dr}.
\end{aligned}$$

■

6.4 Extremal π-entrance laws

The set $\mathcal{K}^1(\pi)$ will be defined in the whole section the same Definition 5.18 in previous chapter, i.e.,

Definition 6.15 *By $\mathcal{K}^1(\pi)$ we denote the set of all $\nu \in \mathcal{K}(\pi)$ for which*

$$\int_H |\langle a, x\rangle| \nu_t(dx) < \infty, \quad a \in H,\ t \in \mathbb{R}.$$

Similarly to Chapter 5, we will have the family $(\kappa_t)_{t\in\mathbb{R}} \subset H$ such that

$$\int_H \langle a, x\rangle \nu_t(dx) = \langle a, \kappa_t\rangle, \quad a \in H,\ t \in \mathbb{R} \tag{6.14}$$

and we call it the projection of ν and denote it by $\kappa = \mathbf{p}(\nu)$.

Definition 6.16 *Consider the semigroup $(U_{s,t})_{s\le t}$ which has been introduced in Section 6.2. A family $(\kappa_t)_{t\in\mathbb{R}} \subset H$, is a U-entrance law, if $U_{s,t}\kappa_s = \kappa_t$ for all $s, t \in \mathbb{R}$ and $s \le t$. The set of all such laws is denoted by $\mathcal{K}(U)$.*

Lemma 6.17 *For any Mehler semigroup $(p_{s,t})_{s\le t}$ if*

$$\int \langle a, y\rangle \pi_{s,t}(x, dy) = \langle a, U_{s,t}x\rangle, \tag{6.15}$$

then for each $\nu \in \mathcal{K}^1(\pi)$, $\kappa = \mathbf{p}(\nu)$ is a U-entrance law.

Proof If we show that $\int_H \langle a, x\rangle \nu_t(dx) = \langle a, U_{s,t}\kappa_s\rangle$ for every $s \le t$, the assertion is complete. By the definition of $\nu = (\nu_t)_{t\in\mathbb{R}}$

$$\int_H \langle a, x\rangle \nu_t(dx) = \int_H \left(\int_H \langle a, x\rangle \pi_{s,t}(y, dx)\right) \nu_s(dy).$$

Now if (6.15) is satisfied, then

$$\begin{aligned}\int_H \left(\int_H \langle a, x\rangle \pi_{s,t}(y, dx)\right) \nu_s(dy) &= \int_H \langle a, U_{s,t}y\rangle \nu_s(dy)\\ &= \int_H \langle U^*_{s,t}a, y\rangle \nu_s(dy)\\ &= \langle U^*_{s,t}a, \kappa_s\rangle = \langle a, U_{s,t}\kappa_s\rangle,\end{aligned}$$

which was needed to show. ∎

But, the similar proposition to Proposition 5.31 related to the first moment of $\pi_{s,t}(x, \cdot)$ which should be imposed to our structure:

Proposition 6.18 *Suppose that in the representation of λ in* (6.9)*, we have $b = 0$ and M is symmetric. Also, assume that*

$$\int_H |\langle a, y\rangle| \mu_{s,t}(dy) < \infty, \quad a \in H,\ s \le t$$

for the $(\mu_{s,t})_{s\le t}$ obtained by (6.8)*. Then*

$$\int \langle a, y\rangle \pi_{s,t}(x, dy) = \langle a, U_{s,t}x\rangle. \tag{6.16}$$

Now we are able to present the main result of this paper. But first, the following assumption should be imposed to our setting which plays a technical role in the progress of the proof of the main theorem.

Assumption 6.19 *We assume that*

$$H \ni a \mapsto \int_{-\infty}^{t} \lambda(\sigma^*(r)U_{r,t}^* a)dr, \quad t \in \mathbb{R} \tag{6.17}$$

is well-defined and Sazonov continuous.

Our main result in this chapter will be presented in the following Theorem. It establishes the relation between $\mathcal{K}_e^1(\pi)$ and $\mathcal{K}(U)$. Furthermore, it gives an explicit representation for the characteristic function of every element in $\mathcal{K}_e^1(\pi)$ in terms of the elements in $\mathcal{K}(U)$.

Theorem 6.20 *Let $(\pi_{s,t})_{s\leq t}$ be the family of transition kernels of a Mehler semigroup on H with corresponding Fourier transform*

$$\int e^{i\langle a,y\rangle}\pi_{s,t}(x,dy) = e^{i\langle a,U_{s,t}x\rangle - \int_s^t \lambda(\sigma^*(r)U_{r,t}^* a)dr},$$

where $U = (U_{s,t})_{s\leq t}$ is a strong evolution family on H, $\sigma : \mathbb{R} \to \mathcal{L}(H)$ is strongly measurable and bounded and λ is a negative-definite and Sazonov continuous function on H which satisfies the following representation

$$\lambda(a) = \frac{1}{2}\langle a, Ra\rangle - \int_H \left(e^{i\langle a,x\rangle} - 1 - \frac{i\langle a, x\rangle}{1+\|x\|^2}\right)M(dx), \quad a \in H,$$

where $R \in \mathcal{L}_1^+(H)$ and M is a symmetric Lévy measures. Then under Assumption 6.19, for every $\kappa \in \mathcal{K}(U)$ there exists a unique extremal probability π-entrance law ν^κ such that

$$\int_H e^{i\langle a,y\rangle}\nu_t^\kappa(dy) = e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(\sigma^*(r)U_{r,t}^* a)dr}. \tag{6.18}$$

Moreover, formula (6.18) *establishes a one-to-one correspondence between $\mathcal{K}(U)$ and the set of all extremal elements of $\mathcal{K}^1(\pi)$.*

Proof Three following claims (i), (ii) and (iii) give us a complete proof of the theorem:

Claim (i) For every $\kappa \in \mathcal{K}(U)$, there exists $\nu^\kappa \in \mathcal{K}(\pi)$ which satisfies equation (6.18).

Proof of Claim (i): Since mapping (6.17) by Assumption 6.19 is Sazonov continuous and meanwhile

$$\exp\left(\int_{-\infty}^t \lambda(\sigma^*(r)U_{r,t}^* a)dr\right) = \lim_{s\to-\infty} \exp\left(\int_s^t \lambda(\sigma^*(r)U_{r,t}^* a)dr\right)$$

is positive definite. Thus one have that

$$e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^{t} \lambda(\sigma^*(r)U^*_{r,t}a)dr}$$

is Sazonov continuous and positive definite. Thus, it is a characteristic function of a measure ν_t^κ on $(H, \mathscr{B}(H))$ by using Minlos-Sazonov Theorem. Now we will prove that $(\nu_t^\kappa)_{t\in\mathbb{R}}$ belongs to $\mathcal{K}(\pi)$.

$$\begin{aligned}
\widehat{(p^*_{s,t}\ \nu_s^\kappa)}(a) &= \int_H \left(\int_H e^{i\langle a,y\rangle}\pi_{s,t}(x,dy)\right)\nu_s^\kappa(dx) \\
&= \int_H e^{i\langle a,U_{s,t}x\rangle - \int_s^t \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr}\nu_s^\kappa(dx). \\
&= \int_H e^{i\langle U^*_{s,t}a,x\rangle}\nu_s^\kappa(dx).\ e^{-\int_s^t \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr} \\
&= e^{i\langle U^*_{s,t}a,\kappa_s\rangle - \int_{-\infty}^{s} \lambda\left(\sigma^*(r)U^*_{r,s}(U^*_{s,t}a)\right)dr}.\ e^{-\int_s^t \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr} \\
&= e^{i\langle a,U_{s,t}\kappa_s\rangle - \int_{-\infty}^{s} \lambda\left(\sigma^*(r)\ (U_{s,t}U_{r,s})^*a\right)dr - \int_s^t \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr} \\
&= e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^{s} \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr - \int_s^t \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr} \\
&= e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^{t} \lambda\left(\sigma^*(r)U^*_{r,t}a\right)dr} \\
&= \widehat{\nu_t^\kappa}(a)
\end{aligned}$$

which implies the claim.

Claim (ii) If ν is an extremal element of $\mathcal{K}^1(\pi)$, then the relation (6.18) holds with $\kappa = \mathbf{p}(\nu)$.
Proof of Claim (ii): For every positive measurable function f on H which $\mathbb{E}_\nu f(X_t) < \infty$, since $\mathbb{P}_\nu$ is trivial on $\mathcal{F}_\infty$ so we have

$$\begin{aligned}
\mathbb{E}_\nu f(X_t) &= \mathbb{E}_\nu\{f(X_t)|\mathcal{F}_\infty\} \\
&= \lim_{s\to-\infty}\mathbb{E}_\nu\{f(X_t)|\mathcal{F}_s\} \qquad (6.19) \\
&= \lim_{s\to-\infty}\mathbb{E}_{(s,X_s)}f(X_t), \quad \mathbb{P}_\nu - a.s.
\end{aligned}$$

where in the the second line we applied the Backwards Martingale convergence theorem. We can use this theorem because the process $\mathbb{E}_\nu\{f(X_t)|\mathcal{F}_s\}$ is a martingale. And finally, in the third line we used the Markov property of our process. Indeed by (6.19) and (6.13),

$$\begin{aligned}\mathbb{E}_\nu e^{i\langle a,X_t\rangle} &= \lim_{s\to-\infty}\mathbb{E}_{(s,X_s)}e^{i\langle a,X_t\rangle}\\ &= \lim_{s\to-\infty} e^{i\langle U^*_{s,t}a,X_s\rangle-\int_s^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}, \quad \mathbb{P}_\nu - a.s..\end{aligned}$$

Then (6.15), (6.19) and equality (6.14) and Lemma 6.17 give us

$$\langle U^*_{s,t}a, X_s\rangle = \mathbb{E}_{(s,X_s)}\langle a, X_t\rangle \xrightarrow{s\to-\infty} \mathbb{E}_\nu\langle a, X_t\rangle = \langle a,\kappa_t\rangle, \quad \mathbb{P}_\nu - a.s..$$

Claim (iii) If κ and ν^κ are connected by (6.18), then ν^κ is extremal.
Let us first define the concept of Lifting for the elements of $\mathcal{K}(U)$.

$\nu\in\mathcal{K}(\pi)$ is called the *lifting of* $\varrho\in\mathcal{K}(U)$ denoted by $\nu=\mathbf{l}(\varrho)$ if

$$\widehat{\nu_t}(a) = \lim_{s\to-\infty}\pi_{s,t}\widehat{(\varrho_s,.)}(a).$$

Proof of (iii): We will divide the proof of Claim (iii) to five steps.
Step1: The π-entrance law ν^κ is the lifting of κ.
Proof: By the definition of ν^κ_t we get

$$\begin{aligned}\widehat{\nu^\kappa_t}(a) &= e^{i\langle a,\kappa_t\rangle-\int_{-\infty}^t\lambda(\sigma^*(r)U^*_{r,t}a)dr} = \lim_{s\to-\infty} e^{i\langle a,\kappa_t\rangle-\int_s^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}\\ &= \lim_{s\to-\infty} e^{i\langle a,U_{s,t}\kappa_s\rangle-\int_s^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}\\ &= \lim_{s\to-\infty}\pi_{s,t}\widehat{(\kappa_s,.)}(a).\end{aligned}$$

step2: Every $\nu\in\mathcal{K}^1_e(\pi)$ is the lifting of its projection, i.e.,

$$\mathbf{l}\big(\mathbf{p}(\nu)\big)=\nu. \tag{6.20}$$

Proof: We need to prove that for each $t\in\mathbb{R}$ and $a\in H$

$$\widehat{\nu_t}(a) = \lim_{s\to-\infty}\pi_{s,t}\widehat{(\mathbf{p}(\nu_s),.)}(a).$$

But since $\nu\in\mathcal{K}^1_e(\pi)$ we know from the Claim (ii) that

$$\widehat{\nu_t}(a) = e^{i\langle a,\mathbf{p}(\nu_t)\rangle-\int_{-\infty}^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}.$$

On the other hand

$$\begin{aligned}\pi_{s,t}\widehat{(\mathbf{p}(\nu_s),.)}(a) &= e^{i\langle a,U_{s,t}\mathbf{p}(\nu_s)\rangle-\int_s^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}\\ &= e^{i\langle a,\mathbf{p}(\nu_t)\rangle-\int_s^t\lambda(\sigma^*(r)U^*_{r,t}a)dr}.\end{aligned}$$

Hence clearly we have the required property.
Whereas the proofs of *step3*, *step4* and *step5* are implicitly the same ones in Theorem 5.34, we will only put them without their proofs and we refer for more details to proof of Theorem 5.34.

Step3: There is the following representation

$$\mathbf{l}(\kappa) = \int_{\mathcal{K}_e^1(\pi)} \mathbf{l}(\widetilde{\varrho}_t)\ \xi(d\mathbf{l}(\widetilde{\varrho})),$$

where $\widetilde{\varrho} = \mathbf{p}(\widetilde{\nu})$ and η is the image of ξ under the projection $\mathbf{p}$.

Step4: We claim that for every $t \in \mathbb{R}$

$$e^{i\langle a,\kappa_t\rangle} = \int_{\mathcal{K}(U)} e^{i\langle a,\widetilde{\varrho}_t\rangle}\eta(d\widetilde{\varrho}), \quad a \in H,$$

with $\kappa = \int_{\mathcal{K}(U)} \widetilde{\varrho}\ \eta(d\widetilde{\varrho})$.

Step5: η is concentrated at a singleton, so that ν^κ is an extremal point.

In conclusion, the theorem follows from **Claim i**, **Claim ii** and **Claim iii**. ■

6.5 An example: Gaussian Ornstein-Uhlenbeck process

In this section we will give an example for the representation of extremal points in the concrete case of Ornstein-Uhlenbeck processes driven by Brownian motion.
Let $\mathcal{S} \subset H \subset \mathcal{S}^*$ be a Gelfand triple.
Consider a family $(\mu_{s,t})_{s\le t}$ of centered Gaussian probability measures on $\big(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*)\big)$. The measures $\mu_{s,t}$, $s \le t$, are determined by the Fourier transform, which is given by Proposition 6.11 in the spacial case $b = 0$ and $M = 0$. Therefore, for every $s \le t$, we have

$$\widehat{\mu}_{s,t}(a) = \int_H e^{i\langle a,x\rangle}\mu_{s,t}(dx) = e^{-\int_s^t \lambda(\sigma^*(r)U_{r,t}^*a)dr}, \quad a \in \mathcal{S},$$

where $\lambda(a) = \frac{1}{2}\langle a, Ra\rangle$. Recall that, $(U_{s,t})_{s\le t}$ is a strong evolution family of the self-adjoint operators and R is a symmetric non-negative trace class operator on H.

Set $Q_t(a) = 2\lambda(\sigma^*(r))(a)$. Then we have Theorem 5.1 in [Dyn78] that states:

Theorem 6.21 *Let $(\pi_{s,t})_{s\le t}$ be the above introduced two-parameter Mehler semigroup on $\big(\mathcal{S}^*, \mathscr{B}(\mathcal{S}^*)\big)$. The set $\mathcal{K}^1(\pi)$ is empty unless*

$$a \mapsto \phi(a) = \int_{-\infty}^t Q_s(U_{s,t}a)ds$$

is a continuous functional on $\mathcal{S}$. If this condition is satisfied, then for every $\kappa \in \mathcal{K}(U)$ there exist a unique extremal π-entrance law ν^κ such that

$$\int_H e^{i\langle a,y\rangle}\nu_t^\kappa(dy) = e^{i\langle a,\kappa_t\rangle - \frac{1}{2}\int_{-\infty}^t Q_r(U_{r,t}a)dr}.$$

Moreover, this formula establishes a one-to-one correspondence between $\mathcal{K}(U)$ and the set of all extremal elements of $\mathcal{K}^1(\pi)$.

As one can see, this example is covered by our Theorem 6.20 completely.

6.6 Uniqueness of extremal points of $\mathcal{K}^1(\pi)$ associated with the T-periodic two-parameter semigroup

In this section, we will show how the explicit representation (6.18) of extremal points obtained in Theorem 6.20 can be used to prove uniqueness of the evolution system of measures. More precisely, in Theorem 6.22, we show that the extremal point is unique in the T-periodic case. This can be seen as the alternative approach for proving uniqueness, which does not require assuming the (asymptotic) strong Feller property for the semigroup. In this sense, Chapter 6 can serve as a counterpart to the ergodicity technique developed in Chapter 4.

Theorem 6.22 *Let in the representation of generalized two-parameter semigroup in Proposition 6.11, $(U_{s,t})_{s\le t}$ as well as $\sigma(r) \in \mathcal{L}(H)$ are T-periodic and the Lévy symbol of L has $b = 0$ and symmetric M. Furthermore, assume that there exist $\omega > 0$ such that U is contractive, i.e., $\|U(s,t)\|_{\mathcal{L}(H)} \le e^{-\omega(t-s)}$ for every $s \le t$. Then we have a unique T-periodic π-entrance law.*

Proof From Theorem 6.20, we know that every extremal point $(\nu_t)_{t\in\mathbb{R}}$ obeys the identity

$$\int_H e^{i\langle a,y\rangle}\nu_t(dy) = e^{i\langle a,\kappa_t\rangle - \int_{-\infty}^t \lambda(\sigma^*(r)U_{r,t}^*a)dr}.$$

Moreover, due to the periodicity of U and σ we have

$$\begin{aligned}\int_H e^{i\langle a,y\rangle}\nu_{t+T}(dy) &= e^{i\langle a,\kappa_{t+T}\rangle - \int_{-\infty}^{t+T} \lambda(\sigma^*(r)U_{r,t+T}^*a)dr}\\ &= e^{i\langle a,\kappa_{t+T}\rangle - \int_{-\infty}^{t} \lambda(\sigma^*(r+T)U_{r+T,t+T}^*a)dr}\\ &= e^{i\langle a,\kappa_{t+T}\rangle - \int_{-\infty}^{t} \lambda(\sigma^*(r)U_{r,t}^*a)dr}.\end{aligned}$$

On the other hand, if $(\nu_t)_{t\in\mathbb{R}}$ is T-periodic, then $\nu_t = \nu_{t+T}$ for every $t \in \mathbb{R}$. Therefore

$$\begin{aligned}\widehat{\nu}_t = \widehat{\nu}_{t+T} &\Longrightarrow e^{i\langle a,\kappa_t\rangle} = e^{i\langle a,\kappa_{t+T}\rangle}\\ &\Longrightarrow \kappa_t = \kappa_{t+T}\\ &\Longrightarrow \kappa_t = U_{t,t+T}\kappa_t.\end{aligned}$$

Now under the contraction assumption imposed on $(U_{s,t})_{s\le t}$, we have $\kappa_t = 0$ for all $t \in \mathbb{R}$. As a result, there exists only one extremal point $(\nu_t)_{t\in\mathbb{R}}$ with the characteristic function

$$\int_H e^{i\langle a,y\rangle}\nu_t(dy) = e^{-\int_{-\infty}^t \lambda(\sigma^*(r)U^*_{r,t}a)dr}. \qquad \blacksquare$$

Now we want to justify our result by two examples in which we have T-periodic semigroup.

1. We start from the following finite dimensional result, which is taken from [PL07]. Let us consider the stochastic differential equation in $\mathbb{R}^n$

$$\begin{aligned}dX(t) &= [A(t)X(t) + F(t)]dt + \sigma(t)dW(t)\\ X(r) &= x,\end{aligned} \tag{6.21}$$

where $W(t)$ is a standard n-dimensional Brownian motion and $x \in \mathbb{R}^n$. Also, we assume that $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n)$, $F : \mathbb{R} \to \mathbb{R}^n$ and $\sigma : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n)$ are continuous and T-periodic, for some $T > 0$. Problem (6.21) under the imposed assumptions has obviously a unique mild solution $X(r,t,x)$.

Let $(p_{r,t})_{r\le t}$ be the associated semigroup to the mild solution of equation (6.21) given by

$$p_{r,t}f(x) := \mathbb{E}[f(X(r,t,x))] = \int_{\mathbb{R}^n} f(x)\mathcal{N}_{g(r,t),Q(r,t)}(dy), \quad f \in \mathcal{C}_b(\mathbb{R}^n),\ r \le t,$$

where the Gaussian measure $\mathcal{N}_{g(r,t),Q(r,t)}$ is the law of $X(r,t,x)$. Its mean and covariance are defined respectively by

$$m(r,t) := U_{r,t}x + g(r,t), \quad Q(r,t) = \int_r^t U_{s,t}\sigma(s)\sigma^*(s)U^*_{s,t}ds \in \mathcal{L}(\mathbb{R}^n),$$

where

$$g(r,t) := \int_r^t U_{s,t}F(s)ds \in \mathbb{R}^n.$$

Proposition 6.23 *The family of Gaussian measures (=normal laws) $(\nu_t)_{t\geq 0}$ defined by*

$$\nu_t = \mathcal{N}_{g(-\infty,t),Q(-\infty,t)} \tag{6.22}$$

constitutes a T-periodic evolution system of measures associated with $(p_{r,t})_{r\leq t}$.

Conversely, if an evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ is T-periodic, then they are the measures defined by (6.22).

Proof See Proposition 3.1 in [PL07]. ∎

Observe that

$$\widehat{\mathcal{N}}_{m,Q}(a) = e^{i\langle m,a\rangle - \frac{1}{2}\langle Qa,a\rangle}, \quad a \in \mathbb{R}^n,$$

which can be rewritten in our framework (dealing with λ) as

$$\widehat{\nu}_t(a) = e^{i\langle a,g(-\infty,t)\rangle - \int_{-\infty}^t \lambda\left(\sigma^*(s)U_{s,t}^* a\right)ds}.$$

If we assume that $F = 0$ and $\sigma = \mathbb{1}$, then we will get (6.18) with $\lambda(a) = \frac{1}{2}\langle a, Qa\rangle$ and $\kappa = 0$. So, this result is completely covered by our Theorem 6.20.

2. The previous example has been generalized in [Knä11] to infinite dimensional framework with Lévy noise. Let us consider the following SDE

$$\begin{aligned} dX(t) &= [A(t)X(t) + F(t)]dt + \sigma(t)dL(t) \\ X(r) &= x \end{aligned} \tag{6.23}$$

on a Hilbert space H, where $A(t) : D(A) \subset H \to H$ are linear operators which generate a strong evolution family $U = (U(r,t))_{0\leq r\leq t\leq T}$ in H, all coefficients are T-periodic, and L is an H-valued Lévy process. Furthermore, $\sigma : \mathbb{R} \to \mathcal{L}(H)$ is strongly continuous and bounded in operator norm and $F : \mathbb{R} \to H$ is uniformly Hölder continuous.

Let $p_{r,t}f(x) := \mathbb{E}[f(X(r,t,x))]$ for $r \leq t$, where $X(r,t,x)$ is the unique mild solution of (6.23) and $f \in \mathcal{B}_b(H)$. Then, in Section 4 of [Knä11], it is proved that $(p_{r,t})_{r\leq t}$ is a Markovian semigroup.
Then the measures $\mu_{r,t}$, $r \leq t$, defined in Proposition 6.11 are in fact the distribution of the stochastic convolution $\int_r^t U_{s,t}\sigma(s)dL(s)$.
Furthermore, $\mu_{r,t}$, $r \leq t$, are also T-periodic and therefore λ_t introduced in (6.9) is also T-periodic.

In [Knä11], it is considered a special case when λ is time-independent. Theorem 4.11 states the following uniqueness result.

Theorem 6.24 *Let us assume that* $\int_{\|x\|>1} \|x\| \nu(dx) < \infty$ *for the Lévy intensity measure* ν *and* U *is a family of contraction strong evolution operators. Then the functions*

$$\widehat{\nu}_t(a) = e^{i\langle a, \int_{-\infty}^t U_{s,t} F(s) ds\rangle - \int_{-\infty}^t \lambda\left(\sigma^*(r) U_{s,t}^* a\right) ds}$$

are Fourier transforms of an T*-periodic evolution system of measures associated to* $(p_{r,t})_{r \leq t}$.
Any other T*-periodic evolution system of measures coincides with the above.*

In particular, if we assume that $F = 0$ and $\sigma = \mathbb{1}$, then

$$\widehat{\nu}_t(a) = e^{\int_{-\infty}^t \lambda\left(U_{s,t}^* a\right) ds}.$$

As we see, the above special case is completely covered by our general Theorem 6.20 when $\kappa = 0$.

Chapter 7

Applications to Stochastic Differential Equations

This Chapter is devoted to concrete applications of our general results obtained in Chapter 4. Mainly, we want to check the sufficient condition to have the unique evolution system of measures. We will discuss the validity of asymptotic strong Feller property for the semigroups associated with a quite general SPDEs with additive jump noise. In Section 7.2, we first prove the existence and uniqueness of mild solutions to a large class of SPDEs driven by Lévy noise. This result is of an essential interest in its own and will be used later to construct the corresponded Markovian semigroup. The main result of this chapter is Theorem 7.18, which states the asymptotic strong Feller property for the two-parameter semigroup associated with the SPDE (7.12).

7.1 Non-autonomous SPDE

We are given real separable Hilbert spaces G and H. If this does not lead to misunderstanding, we denote the norm in G resp. H just by $\|\cdot\|$. Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

We consider a general non-autonomous stochastic differential equation on the time interval $t \in [r, T]$, $-\infty < T \leq \infty$,

$$\begin{aligned} dX(t) &= [A(t)X(t) + F(t, X(t))]dt + \sigma(t, X(t))dL(t) \\ X(r) &= x \end{aligned} \tag{7.1}$$

where L is a G-valued Lévy process with characteristics (b, Q, ν).
Without lose of generality, one can concentrate on the case $r = 0$ as the

initial starting time. So we have for $t \in [0,T]$

$$\begin{aligned} dX(t) &= [A(t)X(t) + F(t,X(t))]dt + \sigma(t,X(t))dL(t), \\ X(0) &= x. \end{aligned} \tag{7.2}$$

Let recall that $\Omega_T := \Omega \times [0,T]$ and $\mathcal{R}_T$ is the σ-algebra of predictable subsets of Ω_T (cf. Definition 2.45). We impose the following conditions on the coefficients of equation (7.2)

Hypothesis 7.1 **(i)** *$x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ is a given initial condition*

(ii) *$(A(t))_{t\in[0,T]}$ generates a strong evolution family $U = (U(s,t))_{0\le s\le t\le T}$ in H, as we have introduced in Section 6.2.*

(iii) *F is a measurable function from $\big(\Omega_T \times H, \mathcal{R}_T \otimes \mathscr{B}(H)\big)$ to $(H, \mathscr{B}(H))$.*

(iv) *σ is a strongly measurable mapping from $\Omega_T \times H$ to $\mathcal{L}(G,H)$, i.e., $\sigma(x)$ is $\big(\mathcal{R}_T \otimes \mathscr{B}(H)\big)/\mathscr{B}(H)$-measurable for each $x \in G$.*

We also assume that

- there exist $\mathbb{M} > 0$ and $\omega > 0$ such that

$$\|U(s,t)\|_{\mathcal{L}(H)} \le \mathbb{M}e^{-\omega(t-s)}, \quad s \le t. \tag{7.3}$$

Hence we can set

$$\mathbb{M}_T := \sup_{0\le s\le t\le T} \|U(s,t)\|_{\mathcal{L}(H)} < \infty.$$

Definition 7.2 *A process $Y : \Omega_T \to H$ is called H-predictable if it is $\mathcal{R}_T/\mathscr{B}(H)$-measurable. We define the Banach space*

$$\mathscr{H}^2(T,H) := \{Y(t),\ t \in [0,T] \mid \ Y \text{ is an } H\text{-predictable processs.t.} \sup_{t\in[0,T]} \mathbb{E}[\|Y(t)\|^2] < \infty\}$$

which is equipped with the norm

$$\|Y\|_{\mathscr{H}^2} := \sup_{t\in[0,T]} (\mathbb{E}[\|Y(t)\|^2])^{\frac{1}{2}}.$$

Hypothesis 7.3 **(i)** *F is Lipschitz continuous, i.e., there exists a constant $Lip_F > 0$ such that for all $t \ge 0$, $\omega \in \Omega$ and $x,y \in H$*

$$\|F(t,\omega,x) - F(t,\omega,y)\|_H \le Lip_F \cdot \|x-y\|.$$

(ii) *σ is Lipschitz continuous, i.e., there exists a constant $Lip_\sigma > 0$ such that all $t \ge 0$, $\omega \in \Omega$ and $x,y \in H$*

$$\|\sigma(t,\omega,x) - \sigma(t,\omega,y)\|_{\mathcal{L}(G,H)} \le Lip_\sigma \cdot \|x-y\|.$$

(iii) *There is a constant $C > 0$ such that*

$$\sup_{(s,\omega)} \|F(s,\omega,0)\|_H \leq C, \quad \sup_{(s,\omega)} \|\sigma(s,\omega,0)\|_{\mathcal{L}(G,H)} \leq C.$$

(iv) *L fulfills the following condition on its intensity measure ν*

$$\int_{\{\|x\|\geq 1\}} \|x\|^2 \nu(dx) < \infty.$$

Remark 7.4 *For the measure ν, we obtain $\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) < \infty$, because*

$$\begin{aligned} \int_{\|x\|\leq 1} \|x\|^2 \nu(dx) &= \limsup_{n\to\infty} \int_{\frac{1}{n}<\|x\|\leq 1} \|x\|^2 \nu(dx) \\ &\limsup_{n\to\infty} Var\left[\int_{\frac{1}{n}<\|x\|\leq 1} xN(1,dx) \right] = Var[L_j^1] < \infty \end{aligned}$$

Remark 7.5 *If*

$$\int_{\{\|x\|\geq 1\}} \|x\|^2 \nu(dx) < \infty,$$

then by Remark 7.4 we have

$$\int_{G\backslash\{0\}} \|x\|^2 \nu(dx) < \infty. \tag{7.4}$$

Conventionally, we will extend ν and $\tilde{N}(t,.)$ to $(G, \mathscr{B}(G))$ by assigning $\tilde{N}(t,\{0\}) = \nu(\{0\}) = 0$ for all $t \in [0,T]$.
Let us set $C_\nu := \int_G \|x\|^2 \ \nu(dx)$.
Actually, (7.4) *is equivalent to claiming that the Lévy process L on a Hilbert space G is square integrable.*

Remark 7.6 *The Lipschitz constants Lip_F and Lip_σ can be chosen in such a way that*

$$\begin{aligned} \|F(s,\omega,x)\|_H &\leq Lip_F(1+\|x\|_H) \\ \|\sigma(s,\omega,x)\|_{\mathcal{L}} &\leq Lip_\sigma(1+\|x\|_H) \end{aligned}$$

for all $s \in [0,T]$, $\omega \in \Omega$ and $x \in H$. Indeed, we have

$$\begin{aligned} \|F(s,\omega,x)\|_H &\leq \|F(s,\omega,x) - F(s,\omega,0)\|_H + \|F(s,\omega,0)\|_H \\ &\leq Lip_F\|x\| + \sup_{(s,\omega)} \|F(s,\omega,0)\|_H \\ &\leq (Lip_F \vee C)(1+\|x\|). \end{aligned}$$

The same also holds for σ.

Remark 7.7 *In Subsection 2.7.1 we defined the Hilbert space* $G_0 := Q^{\frac{1}{2}}(G)$ *for the non-negative symmetric trace class operator* Q*. Then we set* $\mathcal{L}_2^0 := \mathcal{L}_2(G_0 = Q^{\frac{1}{2}}(G), H))$*, which is the space of all Hilbert-Schmidt operator from* G_0 *to* H*. Then any operator* $A \in \mathcal{L}(G, H)$ *certainly belongs to* $\mathcal{L}_2(G_0, H)$ *and obeys* $\|A\|_{\mathcal{L}_2^0} \leq \|A\|_{\mathcal{L}(G,H)} \cdot \; [tr(Q)]^{\frac{1}{2}}$*. Obviously, condition* (**ii**) *in Hypothesis 7.3 implies that* σ *is Lipschitz continuous w.r.t.* x *in the space* $\mathcal{L}_2(G_0, H)$*, i.e.,*

$$\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|_{\mathcal{L}_2^0} \leq Lip_\sigma \cdot [tr(Q)]^{\frac{1}{2}} \cdot \|x - y\|. \tag{7.5}$$

For convenience, set $\widetilde{Lip_\sigma} := Lip_\sigma \cdot (tr(Q))^{\frac{1}{2}}$*.*
Similarly, Condition (**iii**) *in Hypothesis 7.3, implies that*

$$\sup_{(s,\omega)} \|\sigma(s, \omega, 0)\|_{\mathcal{L}_2^0} \leq C \cdot [tr(Q)]^{\frac{1}{2}}. \tag{7.6}$$

Respectively, we set $\tilde{C} := C \cdot (tr(Q))^{\frac{1}{2}}$*.*
Then the Lipschitz continuity condition (7.5) *and boundedness condition* (7.6) *guarantee the solvability of SDE* (7.2) *w.r.t the Wiener noise* B_Q*. Also, the strong measurability of* σ *stated in Hypothesis* 7.1 (**iv**) *implies the* $\mathcal{L}_2(G_0, H))$*-measurability of* σ*.*

Definition 7.8 *An* H*-valued predictable process* X_t*,* $t \in [0, T]$*, is called a mild solution of equation* (7.2) *if the following identity holds* $\mathbb{P}$*-a.s.*

$$X(t) = U(0, t)x + \int_0^t U(s, t)F(s, X(s))ds + \int_0^t U(s, t)\sigma(s, X(s))dL(s)$$

for each $t \in [0, T]$*.*
Here, the first integral is a Bochner-type integral which will be discussed in Lemma 7.9 and Lemma 7.11 and second one is a stochastic convolution-type integral which will be considered during the proof of Theorem 7.14.

Note that in the general case with the initial condition x at starting time r we say $(X_t)_{t\in[r,T]}$ is a mild solution of equation (7.1) if $\mathbb{P}$-a.s.

$$X(t) = U(r, t)x + \int_r^t U(s, t)F(s, X(s))ds + \int_r^t U(s, t)\sigma(s, X(s))dL(s).$$

7.2 Mild solution to non-autonomous SDE with corresponding Lévy process

In this section we show existence of the unique mild solution to the Cauchy problem (7.2) under appropriate conditions. The existence and uniqueness of Mild solution to an equation similar to (7.2) but with Brownian motion has been obtained in [Ver10].

Before proving the main assertion, we need some preparation (similar results were proven in the case of one-parameter semigroup in [FK01] and [Kno03]).

Lemma 7.9 *If $Y : \Omega_T \to H$ is $\mathcal{R}_T$-measurable, then the mapping*

$$\begin{aligned} \tilde{Y} : \Omega_T &\to H \\ (r,\omega) &\to \mathbb{1}_{[0,t[}(r)U(r,t)Y(r,\omega) \end{aligned}$$

is also $\mathcal{R}_T$-measurable for each fixed $t \in [0,T]$.

Proof We will do the proof in two steps.

Step 1: Let $Y = \sum_{k=1}^n x_k \mathbb{1}_{A_k}$ where $n \in \mathbb{N}$, $x_k \in H$, $1 \le k \le n$ and $A_k \in \mathcal{R}_T$, $1 \le k \le n$, is a disjoint covering of Ω_T. Then

$$\begin{aligned} \tilde{Y} : \Omega_T &\to H \\ (r,\omega) &\to \mathbb{1}_{[0,t[}(r)U(r,t)Y(r,\omega) = \mathbb{1}_{[0,t[}(r)\sum_{k=1}^n U(r,t)x_k \mathbb{1}_{A_k(r,\omega)}. \end{aligned}$$

Now for $B \in \mathscr{B}(H)$

$$\tilde{Y}^{-1}(B) = \bigcup_{k=1}^n \underbrace{\Big(\underbrace{\{r \in [0,T] \mid \mathbb{1}_{[0,t[}(r)U(r,t)x_k \in B\}}_{\in \mathscr{B}([0,T])} \times \Omega\Big) \cap A_k}_{\in \mathcal{R}_T}$$

since U is strongly continuous (See Definition 6.8, part (iii)). So $\tilde{Y}$ is $\mathcal{R}_T$-measurable.

Step 2: Let Y be an arbitrary predictable process.
From Lemma 1.1 in [PZ92], we know that there exists a sequence Y_n, $n \in \mathbb{N}$, of simple H-valued predictable process such that Y_n goes pointwisely and monotonely to Y, i.e.,

$$Y_n(r,\omega) \xrightarrow{n\to\infty} Y(r,\omega), \quad \text{for all } (r,\omega) \in \Omega_T.$$

Furthermore, $U(s,t) \in \mathcal{L}(H)$ for all $s,t \in [0,T]$, so that

$$\tilde{Y}(r,\omega) := \mathbb{1}_{[0,t[}(r)U(r,t)Y(r,\omega) = \lim_{n\to\infty} \mathbb{1}_{[0,t[}(r)U(r,t)Y_n(r,\omega).$$

Hence the predictability of $\tilde{Y}$ follows from the predictability of $\tilde{Y}_n$ in Step1. And the proof of lemma is complete. ■

Lemma 7.10 *If Y is a predictable H-valued process and $\sigma \in \mathcal{L}_2^0 = \mathcal{L}_2(G_0 = Q^{\frac{1}{2}}(G), H)$ then the mapping*

$$(r, \omega) \mapsto \mathbb{1}_{[0,t[}(r) U(r,t) \sigma(r, Y(r,\omega))$$

is $\mathcal{R}_T/\mathscr{B}(\mathcal{L}_2^0)$-measurable

Proof Arguments follow directly from the previous lemma. Let $\tilde{h}_j := \sqrt{\lambda_j} h_j$, where $(\lambda_j)_{j\in\mathbb{N}}$, $\lambda_j \geq 0$, are eigenvalues of Q and $(h_j)_{j\in\mathbb{N}}$ are eigenvectors of Q. Then the vectors $(e_k \otimes \tilde{h}_j = e_k\langle \tilde{h}_j, \cdot\rangle_G)_{k,j\in\mathbb{N}}$ constitute an orthonormal bases of $\mathcal{L}_2(Q^{\frac{1}{2}}(G), H)$, where $(e_k)_{k\in\mathbb{N}}$ and $(\tilde{h}_j)_{j\in\mathbb{N}}$ are orthonormal bases in H and G_0, respectively. So,

$$\begin{aligned}(s,\omega) &\mapsto \langle e_k \otimes \tilde{h}_j, \mathbb{1}_{[0,t[}(s) U(s,t)\sigma(s, Y(s,\omega))\rangle_{\mathcal{L}_2^0} \\ &= \langle e_k, \mathbb{1}_{[0,t[}(s) U(s,t)\sigma(s, Y(s,\omega))\sqrt{\lambda_j} h_j\rangle_H\end{aligned}$$

is predictable by Lemma 7.9. Hence, we conclude that

$$(s,\omega) \mapsto \mathbb{1}_{[0,t[}(s) U(s,t)\sigma(s, Y(s,\omega))$$

is predictable. ■

Lemma 7.11 *Let Φ be a predictable H-valued process which is $\mathbb{P}$-a.s. Bochner integrable. Then the process given by*

$$\int_0^t U(s,t)\Phi(s) ds, \quad t \in [0,T],$$

is $\mathbb{P}$-a.s. continuous in H and $\mathcal{F}_t$-adapted. This especially implies that it is predictable.

Proof From Lemma 7.9 we have that the integrand process $\mathbb{1}_{[0,t[}(s)U(s,t)\Phi(s)$, $s \in [0,T]$, is predictable and obeys in addition

$$\|\mathbb{1}_{[0,t[}(s) U(s,t)\Phi(s)\| \leq \mathbb{M}_T \|\Phi(s)\|, \quad s \in [0,T].$$

Hence the Bochner integrals $\int_0^t U(s,t)\Phi(s)ds$, $t \in [0,T]$, are well-defined $\mathbb{P}$-a.s.

To check the continuity, let us estimate for $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned} & \left\| \int_0^{t_1} U(s,t_1)\Phi(s)ds - \int_0^{t_2} U(s,t_2)\Phi(s)ds \right\| \\ \leq \quad & \int_0^{t_1} \left\|[U(s,t_1) - U(s,t_2)]\Phi(s)\right\| ds + \int_{t_1}^{t_2} \left\|U(s,t_2)\Phi(s)\right\| ds \end{aligned} \tag{7.7}$$

Concerning the first integral, let us observe that $|t_1 - t_2| \to 0$ implies

$$\|\mathbb{1}_{[0,t_1[}(s)\big[U(s,t_1) - U(s,t_2)\big]\Phi(s)\| \to 0, \quad s \in [0,T],$$

because of the strong continuity of U. Furthermore, we have the uniform bound

$$\begin{aligned}
&\|\mathbb{1}_{[0,t_1[}(s)\big[U(s,t_1) - U(s,t_2)\big]\Phi(s)\| \\
&\leq \mathbb{1}_{[0,t_1[}(s)\big[\|U(s,t_1)\|_{\mathcal{L}(H)} + \|U(s,t_2)\|_{\mathcal{L}(H)}\big]\|\Phi(s)\| \\
&\leq 2\mathbb{M}_T\|\Phi(s)\|.
\end{aligned}$$

So, one can apply Lebesgue's dominated convergence theorem to get the continuity of the first integral in the right-hand side of (7.7).
Concerning the second integral, we observe that similarly to the above

$$\int_{t_1}^{t_2} \|U(s,t_2)\Phi(s)\|ds \leq \int_{t_1}^{t_2} \mathbb{M}_T\|\Phi(s)\|ds \to 0,$$

as $|t_1 - t_2| \to 0$, and via the same argument we get the required continuity.

Finally, note that for every fixed $t \in [0,T]$

$$(s,\omega) \mapsto \mathbb{1}_{[0,t[}(s)U(s,t)\Phi(s,\omega)$$

is $\mathscr{B}([0,T])\otimes\mathcal{F}_t$-measurable because $(s,\omega) \to U(s,t)\Phi(s,\omega)$ is $\mathcal{R}_T$-measurable and

$$\big([0,t[\times\Omega\big) \cap \mathcal{R}_T \subset \mathcal{B}\big([0,T]\big) \otimes \mathcal{F}_t.$$

Hence,

$$\begin{aligned}
\omega \mapsto \left\langle \int_0^t U(s,t)\Phi(s,\omega)ds, x \right\rangle &= \int_0^t \langle U(s,t)\Phi(s,\omega), x\rangle ds \\
&= \int_0^T \langle \mathbb{1}_{[0,t[}(s)U(s,t)\Phi(s,\omega), x\rangle ds
\end{aligned}$$

is $\mathcal{F}_t$-measurable by the real Fubini theorem and therefore the Bochner integral itself is $\mathcal{F}_t$-measurable. ■

The predictability follows from the following general fact.

Lemma 7.12 *Let Φ be a process on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ taking values in a Banach space H. If Φ is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and stochastically continuous then there exists a predictable version of Φ.*

Proof ([PZ92], Proposition 3.6 (ii)) ■

Lemma 7.13 *Let $(x_{n,m})_{m\in\mathbb{N}}$, $n \in \mathbb{N}$, be sequences of real numbers such that for each $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}$ with $x_{n,m} \to x_n$ as m goes to ∞. If there exists a further sequence y_n, $n \in \mathbb{N}$, such that $|x_{n,m}| \leq y_n$ for all $m \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} y_n < \infty$, then*

$$\lim_{m\to\infty} \sum_{n\in\mathbb{N}} x_{n,m} = \sum_{n\in\mathbb{N}} x_n$$

Proof The claim is a simple consequence of Lebesgue's dominated convergence theorem. ∎

Now, we prove the main result of this section. Note that in [Knä06], the existence and uniqueness of mild solution for the autonomous version of equation (7.2) has been discussed.

Theorem 7.14 *Under Hypothesis 7.1 and 7.3 there exists a unique mild solution $X(x)$ of equation (7.2) with initial condition $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.*

Proof Let $t \in [0,T]$ and $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ be fixed. For $X \in \mathscr{H}^2(T,H)$ we define

$$\gamma(X)(t) := U(0,t)x + \int_0^t U(s,t)F(s,X(s))ds + \int_0^t U(s,t)\sigma(s,X(s))dL(s) \tag{7.8}$$

We will apply the fixed point method to find the unique X such that $\gamma(X) = X$ in $\mathscr{H}^2(T,H)$.
Thus, we show that γ is a well-defined mapping on $\mathscr{H}^2(T,H)$ and it is a strict contraction. Then by the Banach fixed-point theorem we get the existence and uniqueness of the mild solution.

Note that we can use Lévy-Itô's decomposition theorem (see Theorem 2.39) to rewrite Definition 7.8. Meanwhile, Remark 2.40 and part (iv) of Hypothesis 7.3 as well as Remark 7.5, together cause that our decomposition appear as the following identity

$$\gamma(X)(t) = U(0,t)x + I_F(t) + I_{\sigma,m}(t) + I_{\sigma,Q}(t) + I_{\sigma,\tilde{N}}(t)$$

holding $\mathbb{P}$-a.s. for each $t \in [0,T]$, where

$$\begin{aligned}
I_F(t) &= \int_0^t U(s,t)F(s,X(s))ds \\
I_{\sigma,m}(t) &= \int_0^t U(s,t)\sigma(s,X(s))mds \\
I_{\sigma,Q}(t) &= \int_0^t U(s,t)\sigma(s,X(s))dB_Q(s) \\
I_{\sigma,\tilde{N}}(t) &= \int_0^t \int_G U(s,t)\sigma(s,X(s))x\tilde{N}(ds,dx)
\end{aligned}$$

As one see clearly, $I_F(t)$ and $I_{\sigma,m}(t)$ are Bochner's integrals with respect to Lebesgue's measure, so they can be examined by the same arguments. Furthermore, $I_{\sigma,Q}(t)$ and $I_{\sigma,\tilde{N}}(t)$ are stochastic convolution-type integrals resp. Brownian motion B_Q and compensated poisson measure $\tilde{N}$ and we will evaluate them during the proof of Theorem 7.14.

Step1: The mapping $\gamma : \mathscr{H}^2(T,H) \to \mathscr{H}^2(T,H)$ is well-defined.
Let $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $X \in \mathscr{H}^2(T,H)$.

• $U(0,t)x$: is predictable because, for every fixed ω, the map $t \mapsto U(0,t)x(\omega)$ is continuous and, for every fix $t \in [0,T]$, $U(0,t)x$ is $\mathcal{F}_0$-measurable. So $(t,\omega) \mapsto U(0,t)x(\omega)$ is predictable.
Furthermore, it has finite norm in $\mathscr{H}^2(T,H)$ due to the estimate

$$\|U(0,t)x\|_{\mathscr{H}^2} = \sup_{t\in[0,T]} \big(\mathbb{E}(\|U(0,t)x\|^2)\big)^{\frac{1}{2}} \le \mathbb{M}_T\|x\| < \infty.$$

• $I_F(t)$ and $I_{\sigma,m}(t)$: we can apply Lemma 7.11 to show the existence of a predictable version. To this end, it is sufficient to check the Bochner integrability of F. Indeed,

$$\begin{aligned}
\mathbb{E}[\int_0^t \|F(s,X(s))\|ds] \;&\le \int_0^t \mathbb{E}\big[Lip_F(1+\|X(s)\|)ds\big] \\
&\le T\ Lip_F\ (1+\|X\|_{\mathscr{H}^2}) < \infty.
\end{aligned}$$

Similarly, one can analyse the term $I_{\sigma,m}$.
Next, $\|I_F(t)\|_{\mathscr{H}^2}$ can be estimated as follows:

$$\begin{aligned}
\|I_F(t)\|_{\mathscr{H}^2} &= \sup_{0\le t\le T}\left[\mathbb{E}[\|\int_0^t U(s,t)F(s,X(s))ds\|^2]\right]^{\frac{1}{2}} \\
&\le \mathbb{M}_T T^{1/2}\ Lip_F \sup_{t\in[0,T]} \big(\int_0^t \mathbb{E}(1+\|X(s)\|)^2 ds\big)^{\frac{1}{2}} \\
&\le \sqrt{2}\ \mathbb{M}_T\ T^{1/2}\ Lip_F \sup_{t\in[0,T]} \big(\int_0^t (1+\mathbb{E}\|X(s)\|^2)ds\big)^{\frac{1}{2}} \\
&\le \sqrt{2}\ \mathbb{M}_T\ T\ Lip_F\ (1+\|X\|_{\mathscr{H}^2}) < \infty,
\end{aligned}$$

where we used an elementary inequality $(a+b)^2 \le 2(a^2+b^2)$.
Also, $\|I_{\sigma,m}\|_{\mathscr{H}^2} < \infty$ can be shown similarly.

• $I_{\sigma,Q}(t)$: First we should check the well-definiteness of the integral by applying Lemma 7.10. Since the integrand $\mathbb{1}_{(0,t]}(s)U(s,t)\sigma(s,X(s))$, $s \in [0,t]$, is in $\mathcal{N}_B^2(T)$, then

$$(s,\omega) \mapsto \mathbb{1}_{(0,t]}(s)U(s,t)\sigma(s,\omega,X(s,\omega))$$

is $\mathcal{R}_T/\mathscr{B}(\mathcal{L}_2(G_0,H))$-measurable.

Moreover, by Itô's isometry (see Proposition 2.49):

$$\begin{aligned}
\|I_{\sigma,Q}(t)\|^2_{\mathcal{N}^2_B(T)} &= \mathbb{E}\big[\big\| \textstyle\int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)\big\|^2\big] \\
&= \mathbb{E}[\textstyle\int_0^t \big\|U(s,t)\sigma(s,X(s))Q^{\frac{1}{2}}\big\|^2_{\mathcal{L}_2} ds] \\
&\le \mathbb{M}_T^2\ \widetilde{Lip_\sigma}^2\ tr(Q) \textstyle\int_0^t \mathbb{E}(1+\|X(s)\|^2)ds \\
&\le \mathbb{M}_T^2\ T\ \widetilde{Lip_\sigma}^2\ tr(Q)\ 2(1+\|X\|^2_{\mathscr{H}^2}) < \infty.
\end{aligned} \tag{7.9}$$

So, for each $t \in [0,T]$, the integral $I_{\sigma,Q}(t)$ is well-defined and $\mathcal{F}_t$-adapted. Next we check whether it has the finite $\mathscr{H}^2(T,H)$-norm.
Since

$$\begin{aligned}
\|I_{\sigma,Q}(t)\|_{\mathscr{H}^2} &= \sup_{t\in[0,T]} \left(\mathbb{E}\big[\big\| \int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)\big\|^2\big]\right)^{1/2} \\
&= \sup_{t\in[0,T]} \left(\mathbb{E}\int_0^t \|U(s,t)\sigma(s,X(s))Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2} ds\right)^{1/2},
\end{aligned}$$

thus the assertion follows by estimate (7.9).

And finally, the existence of a predictable version can be obtained from Lemma 7.12. Since the required $\mathcal{F}_t$-adaptedness has been already shown, it just remains to check the stochastic continuity property. If one can prove that $t \mapsto I_{\sigma,Q}(t) \in H$ is continuous in the mean-square, then the claim is obtained.

We follow the method used in a similar situation in [FK01]. The idea is to show first that, for any fixed $\alpha > 1$, the process

$$[0,T] \ni t \mapsto \int_0^{t/\alpha} U(s,t)\sigma(s,X(s))dB_Q(s)$$

is mean-square continuous. Note that for $t \in [0,T]$

$$\begin{aligned}
\int_0^{t/\alpha} U(s,t)\sigma(s,X(s))dB_Q(s) &= \int_0^{t/\alpha} U(\alpha s,t)U(s,\alpha s)\sigma(s,X(s))dB_Q(s) \\
&= \int_0^{t/\alpha} U(\alpha s,t)\Phi_\alpha(s)dB_Q(s),
\end{aligned}$$

where

$$\Phi_\alpha(s) := \mathbb{1}_{(0,T]}(s)U(s,\alpha s)\sigma(s,X(s)), \quad s \in [0,T],$$

belongs to $\mathcal{N}^2_B(T)$. The method is based on [PZ92], Lemma 1.1.

(a) In the first step, let Φ be a simple process of the form $\Phi = \sum_{i=1}^m u_i \mathbb{1}_{A_i}$ with $u_i \in \mathcal{L}_2^0(G,H)$ and $A_i \in \mathcal{R}_T$. Now, for every $t \in [0,T]$ and Φ being a simple process, we have

$$\begin{aligned}\mathbb{E}\Big(\|\int_0^t U(s,t)\Phi(s)dB_Q(s)\|^2\Big) &\leq \mathbb{E}\Big(\int_0^t \|U(s,t)\Phi(s)Q^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 ds\Big) \\ &\leq \sum_{i=1}^m \mathbb{E}\big[\int_0^t \mathbb{1}_{A_i}(s,\cdot)\|U(s,t)u_iQ^{\frac{1}{2}}\|_{\mathcal{L}_2}^2\big] \\ &\leq tr(Q)\cdot\sum_{i=1}^m\int_0^t \|U(s,t)u_i\|_{\mathcal{L}_2^0}^2 ds.\end{aligned}$$

So,

$$\begin{aligned}&\Big(\mathbb{E}\Big[\|\int_0^{t/\alpha} U(\alpha s,t)\Phi(s)dB_Q(s) - \int_0^{r/\alpha} U(\alpha s,r)\Phi(s)dB_Q(s)\|^2\Big]\Big)^{\frac{1}{2}} \\ &\leq [\mathbb{E}\|\int_0^{r/\alpha}(U(\alpha s,t)-U(\alpha s,r))\Phi(s)dB_Q(s)\|^2]^{\frac{1}{2}} \\ &+ [\mathbb{E}\|\int_{r/\alpha}^{t/\alpha} U(\alpha s,t)\Phi(s)dB_Q(s)\|^2]^{\frac{1}{2}} \\ &\leq tr(Q)^{\frac{1}{2}}\sum_{i=1}^m\big(\int_0^{r/\alpha}\|[U(\alpha s,t)-U(\alpha s,r)]u_i\|_{\mathcal{L}_2^0}^2 ds\big)^{\frac{1}{2}} \\ &+ tr(Q)^{\frac{1}{2}}\sum_{i=1}^m\underbrace{\big(\int_{r/\alpha}^{t/\alpha}\|U(\alpha s,t)u_i\|_{\mathcal{L}_2^0}^2 ds\big)}_{\leq\frac{t-r}{\alpha}\mathbb{M}_T^2\|u_i\|_{\mathcal{L}_2^0}^2\xrightarrow{|t-r|\to 0}0}{}^{\frac{1}{2}}.\end{aligned}$$

For the first summation, by the strong continuity of the semigroup U we have for each $n \in \mathbb{N}$ and an orthonormal basis $(\tilde{e}_n)_{n\in\mathbb{N}}$ of G_0,

$$\mathbb{1}_{[0,r/\alpha)}(s)\|\big(U(\alpha s,t)-U(\alpha s,r)\big)u_i\tilde{e}_n\|^2 \xrightarrow{|t-r|\to 0} 0. \tag{7.10}$$

Note that (7.10) is obviously bounded by $4\mathbb{M}_T^2\|u_i\tilde{e}_n\|^2$. Then we get from Lemma 7.13 that

$$\begin{aligned}&\mathbb{1}_{[0,r/\alpha)}(s)\|\big(U(\alpha s,t)-U(\alpha s,r)\big)u_i\|_{\mathcal{L}_2^0}^2 \\ &=\sum_{n\in\mathbb{N}}\mathbb{1}_{[0,r/\alpha)}(s)\|\big(U(\alpha s,t)-U(\alpha s,r)\big)u_ie_n\|^2\end{aligned}$$

is pointwisely convergent to 0 as $|t-r| \to 0$. Also it is bounded by $4\mathbb{M}_T^2\|u_i\|_{\mathcal{L}_2^0}^2$. Finally Lebesgue's dominated theorem gives us the convergence of the discussed summation to 0.

All together, we have the mean-square continuity of the

$$\int_0^t U(s,t)\Phi(s)dB_Q(s)$$

when Φ is a simple process.

(b) Let Φ be an arbitrary element in $\mathcal{N}_B^2(T)$. Applying [PZ92], Lemma 1.1, there exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of simple processes such that

$$\mathbb{E}\big[\int_0^T \|\Phi(s)-\Phi_n(s)\|^2_{\mathcal{L}_2^0}ds\big] \xrightarrow{n\to\infty} 0.$$

Moreover,

$$\sup_{t\in[0,T]} \mathbb{E}\big[\big\|\int_0^{t/\alpha} U(\alpha s,t)(\Phi_n(s)-\Phi(s))dB_Q(s)\big\|^2\big]$$
$$\leq \mathbb{M}_T^2\ tr(Q)\ \mathbb{E}\big[\int_0^T \|(\Phi_n(s)-\Phi(s))\|^2_{\mathcal{L}_2^0}ds\big] \xrightarrow{n\to 0} 0.$$

It means that $\int_0^{t/\alpha} U(\alpha s,t)\Phi_n(s)dB_Q(s)$ converges in $L^2(\Omega;H)$ to

$$\int_0^{t/\alpha} U(\alpha s,t)\Phi(s)dB_Q(s)$$

uniformly in $t\in[0,T]$. So, for every process $\Phi\in\mathcal{N}_B^2(0,T)$ and particularly for Φ_α, we get the mean-square continuity of

$$\int_0^{t/\alpha} U(s,t)\sigma(s,X(s))dB_Q(s).$$

Now, we are able to prove that $\int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)$, $t\in[0,T]$, is also mean-square continuous. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\alpha_n\searrow 1$ as $n\to\infty$, then

$$\sup_{t\in[0,T]} \mathbb{E}\Bigg[\Bigg\|\int_0^{t/\alpha_n} U(s,t)\sigma(s,X(s))dB_Q(s)-\int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)\Bigg\|^2\Bigg]$$
$$\leq \mathbb{M}_T^2\ tr(Q)\ \widetilde{Lip_\sigma}^2 \underbrace{\sup_{t\leq T}\mathbb{E}\Bigg[\int_0^T \mathbb{1}_{(\frac{t}{\alpha_n},t]}(s)(1+\|X(s)\|)^2ds\Bigg]}_{\leq 2(1+\|X\|_{\mathscr{H}^2})\sup_{t\leq T}(t-\frac{t}{\alpha_n})\leq 2(1+\|X\|_{\mathscr{H}^2})T\frac{\alpha_n-1}{\alpha_n}} \xrightarrow{n\to\infty} 0.$$

This shows that

$$\int_0^{t/\alpha_n} U(s,t)\sigma(s,X(s))dB_Q(s)$$

converges uniformly in $t \in [0,T]$ to $\int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)$ as $n \to \infty$, so that $\int_0^t U(s,t)\sigma(s,X(s))dB_Q(s)$ obeys the continuity property that we are interested in. By Lemma 7.12, there exists a predictable version for it.

•$I_{\sigma,\tilde{N}}(t)$: For well-definiteness of the stochastic integral $I_{\sigma,\tilde{N}}(t)$ we need to check that for each $t \in [0,T]$ the integrand process $\mathbb{1}_{(0,t]}(s)U(s,t)\sigma(s,X(s))$, $s \in [0,t]$, is in $\mathcal{N}_\nu^2(T)$. By Lemma 7.10, we know that

$$(s,\omega) \to \mathbb{1}_{(0,t]}(s)U(s,t)\sigma(s,\omega,X(s,\omega))$$

is $\mathcal{R}_T/\mathscr{B}(\mathcal{L}(G,H))$-measurable. This means that for all $g \in G$,

$$(s,\omega) \to \mathbb{1}_{(0,t]}(s)U(s,t)\sigma(s,\omega,X(s,\omega))g$$

is $\mathcal{R}_T/\mathscr{B}(H)$-measurable. Moreover,

$$\begin{aligned}
&\|I_{\sigma,\tilde{N}}(t)\|^2_{\mathcal{N}_\nu^2(T)} \\
=\ & \mathbb{E}\big[\big\| \textstyle\int_0^t \int_G U(s,t)\sigma(s,X(s))x\tilde{N}(ds,dx)\big\|^2\big] \\
=\ & \mathbb{E}\big[\textstyle\int_0^t \int_G \big\|U(s,t)\sigma(s,X(s))x\big\|_H^2 \nu(dx)ds\big] \qquad (7.11) \\
\leq\ & \textstyle\int_0^t \mathbb{E}\big[\|U(s,t)\|^2_{\mathcal{L}(H)}\|\sigma(s,X(s))\|^2_{\mathcal{L}(G,H)}\big] \int_G \|x\|_G^2 \nu(dx)ds \\
\leq\ & \mathbb{M}_T^2\ T\ Lip_\sigma^2\ 2C_\nu(1+\|X\|^2_{\mathscr{H}^2}) < \infty.
\end{aligned}$$

Furthermore, by the construction of stochastic integrals with respect to a compensated Poisson measure $\tilde{N}$ (see Subsection 2.7.2), $I_{\sigma,\tilde{N}}(t)$ is $\mathcal{F}_t$-adapted.

Next we check, whether $I_{\sigma,\tilde{N}}(t)$ has the finite $\mathscr{H}^2(T,H)$-norm. We have

$$\begin{aligned}
\|I_{\sigma,\tilde{N}}(t)\|_{\mathscr{H}^2} &= \sup_{t\in[0,T]} \left(\mathbb{E}\big[\big\| \int_0^t \int_G U(s,t)\sigma(s,X(s))x\tilde{N}(ds,dx)\big\|^2\big]\right)^{1/2} \\
&= \sup_{t\in[0,T]} \left(\mathbb{E}\big[\int_0^t \int_G \|U(s,t)\sigma(s,X(s))x\|_H^2 \nu(dx)ds\big]\right)^{1/2},
\end{aligned}$$

which is finite due to estimates (7.11).

Next, we should prove the existence of a predictable version for adapted process $I_{\sigma,\tilde{N}}(t)$. But the mean-square continuity of $I_{\sigma,\tilde{N}}(t)$ would immediately imply the required property.

Let $\alpha > 1$ and $r \leq t$, then we have by Itô's isometry (see Proposition 2.52)

$$\begin{aligned}
&\Bigg(\mathbb{E}\Bigg[\Bigg\| \int_0^{t/\alpha}\int_G U(\alpha s,t)\Phi(s)x\tilde{N}(ds,dx) \\
&- \int_0^{r/\alpha}\int_G U(\alpha s,r)\Phi(s)x\tilde{N}(ds,dx)\Bigg\|^2\Bigg]\Bigg)^{\frac{1}{2}} \\
&\leq \Bigg(\mathbb{E}\Bigg[\Bigg\| \int_0^{r/\alpha}\int_G [U(\alpha s,t) - U(\alpha s,r)]\Phi(s)x\tilde{N}(ds,dx)\Bigg\|^2\Bigg]\Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\mathbb{E}\Bigg[\Bigg\| \int_{r/\alpha}^{t/\alpha}\int_G U(\alpha s,t)\Phi(s)x\tilde{N}(ds,dx)\Bigg\|^2\Bigg]\Bigg)^{\frac{1}{2}} \\
&\leq \Bigg(\mathbb{E}\Bigg[\int_0^{r/\alpha}\int_G \Bigg\|[U(\alpha s,t) - U(\alpha s,r)]\Phi(s)x\Bigg\|_H^2 \nu(dx)ds\Bigg]\Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\mathbb{E}\Bigg[\int_{r/\alpha}^{t/\alpha}\int_G \Bigg\|U(\alpha s,t)\Phi(s)x\Bigg\|_H^2 \nu(dx)ds\Bigg]\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

The first summand can be written as

$$\Bigg(\mathbb{E}\Bigg[\int_0^T\int_G \mathbb{1}_{(0,r/\alpha]}\Bigg\|[U(\alpha s,t) - U(\alpha s,r)]\Phi(s)x\Bigg\|_H^2 \nu(dx)ds\Bigg]\Bigg)^{\frac{1}{2}}.$$

It converges to 0 as $|t-r| \to 0$ by Lebesgue's dominated convergence theorem since the integrand converges pointwisely to 0 as $|t-r| \to 0$ and is bounded by $4\mathbb{M}_T^2\|\Phi(s)\|^2$, where

$$\mathbb{E}\Bigg[\int_0^T\int_G \|\Phi(s)x\|_H^2\nu(dx)ds\Bigg] < \infty.$$

For the second integral, we observe that

$$\begin{aligned}
&\Bigg(\mathbb{E}\Bigg[\int_{r/\alpha}^{t/\alpha}\int_G \Bigg\|U(\alpha s,t)\Phi(s)x\Bigg\|_H^2 \nu(dx)ds\Bigg]\Bigg)^{\frac{1}{2}} \\
&\leq \mathbb{M}_T\Bigg(\mathbb{E}\Bigg[\int_{r/\alpha}^{t/\alpha}\|\Phi\|_{\mathcal{L}(G,H)}^2\int_G \|x\|^2\nu(dx)ds\Bigg]\Bigg)^{\frac{1}{2}} \\
&\leq \mathbb{M}_T^2\ Lip_\sigma\ \sqrt{2}\ C_\nu^{\frac{1}{2}}\ \left(\int_0^T \mathbb{1}_{(r/\alpha,t/\alpha]}(s)\big(1+\|X(s)\|\big)^2 ds\right)^{\frac{1}{2}} \\
&\leq \mathbb{M}_T^2\ Lip_\sigma\ 2\ C_\nu^{\frac{1}{2}}\big(1+\|X\|_{\mathscr{H}^2}\big)T(\frac{t-r}{\alpha}).
\end{aligned}$$

This part also vanishes as $|t-r| \to 0$. And this ends the proof for $I_{\sigma,\tilde{N}}(t)$. As a result, we have the mean-square continuity of $I_{\sigma,Q}(t)$ and $I_{\sigma,\tilde{N}}(t)$.

Step2: The mapping $\gamma : \mathscr{H}^2(T,H) \to \mathscr{H}^2(T,H)$ is a strict contraction for small enough $T > 0$.
Let $X_1, X_2 \in \mathscr{H}^2(T,H)$ and $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$, then

$$\begin{aligned}
&\|\gamma(X_1) - \gamma(X_2)\|_{\mathscr{H}^2} \\
\leq &\sup_{t\in[0,T]} \left(\mathbb{E}[\|\int_0^t U(s,t)(F(s,X_1(s)) - F(s,X_2(s)))\|^2]\right)^{\frac{1}{2}} \\
+ &\sup_{t\in[0,T]} \left(\mathbb{E}[\|\int_0^t U(s,t)(\sigma(s,X_1(s)) - \sigma(s,X_2(s)))mds\|^2]\right)^{\frac{1}{2}} \\
+ &\sup_{t\in[0,T]} \left(\mathbb{E}[\|\int_0^t U(s,t)(\sigma(s,X_1(s)) - \sigma(s,X_2(s)))dB_Q(s)\|^2]\right)^{\frac{1}{2}} \\
+ &\sup_{t\in[0,T]} \left(\mathbb{E}[\|\int_0^t \int_G U(s,t)(\sigma(s,X_1(s)) - \sigma(s,X_2(s)))x\tilde{N}(dt,dx)\|^2]\right)^{\frac{1}{2}} \\
\leq &\mathbb{M}_T\ Lip_F\ T\ \|X_1 - X_2\|_{\mathscr{H}^2} + \mathbb{M}_T\ Lip_\sigma\ \|m\|\ T\ \|X_1 - X_2\|_{\mathscr{H}^2} \\
+ &\mathbb{M}_T\ \widetilde{Lip_\sigma}\ tr(Q)^{\frac{1}{2}}\ T^{1/2}\ \|X_1 - X_2\|_{\mathscr{H}^2} + \mathbb{M}_T\ Lip_\sigma\ T^{1/2}\ C_\nu^{\frac{1}{2}}\ \|X_1 - X_2\|_{\mathscr{H}^2}
\end{aligned}$$

Here, the last inequality follows from the same procedure which we have done to prove the finiteness of $\mathscr{H}^2$-norm for every component. For $T = T_1$ chosen sufficiently small, the contraction of γ holds in interval $[0, T_1]$. So there is a unique mild solution X in $[0, T_1]$. In order to construct a solution for a general T once we have a solution in $[0, T_1]$, we start again with the new initial value $X(T_1)$ in interval $[T_1, 2T_1]$. Since the constants involved only depend on T, the method will be followed in the same way as for $[0, T_1]$. By repeating this procedure, we can construct the solution X on the whole interval $[0, T]$.

So the proof of theorem is complete. ∎

Remark 7.15 *Actually, our method based on the approximation of convolution integral* $\int_0^t U(s,t)\sigma(s,x)dB(s)$ *by* $\int_0^{t/\alpha} U(s,t)\sigma(s,x)dB(s)$ *as* $\alpha \searrow 1$ *allows to prove the mean-square continuity of the stochastic convolution integrals even with cylindrical Wiener noise (see [FK01] and [Knä06]).*

From now on and until the end of the chapter, we assume that F, σ are deterministic (i.e., non-random) maps and σ does not also depend on X. So, the main object of our study will be the SDE of the following form:

$$\begin{aligned}
dX(t) &= [A(t)X(t) + F(t, X(t))]dt + \sigma(t)dL(t) \\
X(0) &= x
\end{aligned} \tag{7.12}$$

Concerning the differentiability of our mild solution from [MPR10] we have:

Theorem 7.16 *If F is Gâteaux differentiable such that $\partial F \in \mathcal{C}(H \times H, H)$ then the mild solution $X(s,t,x)$ is Gâteaux differentiable with respect to the initial condition x, $\mathbb{P}$-a.s. and for any direction $h \in H$. Moreover, we have $D_h X(s,t,x) = \eta^h(s,t,x)$, $\mathbb{P}-a.s.$, where $\eta^h(s,t,x)$ is the mild solution of the linear equation*

$$\tfrac{d}{dt}\eta^h(t,x) = A(t)\eta^h(t,x) + DF(t,X(t,x))\eta^h(t,x),$$

$$\eta^h(s,x) = h,$$

that is , $\eta^h(s,t,x)$ is the solution of the integral equation

$$\eta^h(s,t,x) = U_{s,t}h + \int_s^t U_{r,t}DF(r,X(r,x))\eta^h(r,x)dr, \; s \le t \tag{7.13}$$

Proof See Theorem 2.6 in [MPR10]. ■

7.3 Asymptotic strong Feller property of semigroups associated with non-autonomous SDEs

Let $(p_{s,t})_{s\le t}$ be the Markovian semigroup corresponding to the SDE (7.12). In this section we are going to prove the main result of this chapter claiming the validity of the asymptotic strong Feller property for the semigroup $(p_{s,t})_{s\le t}$.
To this end, we will crucially use dissipating assumption (7.3).

First we need a technical result about approximation of Lipschitz continuous functions.

Lemma 7.17 *Let $G : H \to H$, be measurable and Lipschitz continuous with the constant Lip_G. Then there exists a sequence of $F_{k,n} : H \to H$, $k,n \in \mathbb{N}$ which are measurable and Lipschitz with constants $Lip_{F_{k,n}} \le Lip_G$. Moreover, $x \mapsto F_{k,n}(x)$ is infinitely many times continuously differentiable and for each x*

$$\|F_{k,n}(x) - G(x)\| \to 0$$

as $n \to \infty$ and $k \to \infty$.

Proof The proof is a standard progress. First we do projection. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in H. We define $\mathbb{P}_n$ as the projector on the corresponding n-dimension subspace. Define $G_n(x) := G(\mathbb{P}_n x)$. Then we set $G_n(x) = g_n\big(\langle x, e_1\rangle_H, ..., \langle x, e_n\rangle\big)$ where $g_n : \mathbb{R}^n \to H$. Obviously, g_n, $n \in \mathbb{N}$ satisfy the Lipschitz continuity with $Lip_{g_n} = Lip_G$, $n \in \mathbb{N}$.
Next, we need regularizations for g_n. To this end, we shall use a standard convolution operator in $\mathbb{R}^n$. Fix any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{L^1} = 1$ and let $(\varphi_k(x))_{k\in\mathbb{N}}$ be the corresponding Dirac sequence constructed by

$$\varphi_k(x) := k^d\varphi(kx), \quad x \in \mathbb{R}^n.$$

Now, for any g_n, $n \in \mathbb{N}$, $(g_n * \varphi_k)(x) = \int_{\mathbb{R}^n} g_n(x-y)\varphi_k(y)dy$ holds

$$\|(g_n * \varphi_k)\|_{Lip} \leq \|g_n\|_{Lip}, \quad \|(g_n * \varphi_k)(x) - g_n(x)\| \xrightarrow{k\to\infty} 0$$

and also $D^\alpha(g_n * \varphi_k) = g_n * D^\alpha \varphi_k$ for any $\alpha \in \mathbb{Z}^+$.
Therefore it is enough to define $F_{n,k} := g_n * \varphi_k$. ∎

The following theorem is the main result of this section.

Theorem 7.18 *Assume that for equation* (7.1) *with its imposed hypothesis, F also obeys Lipschitz continuity w.r.t. the second variable with the constant Lip_F for some $\omega + \mathbb{M}\ Lip_F < 0$ where ω and $\mathbb{M}$ are the constants in* (7.3). *Then the Markovian semigroup $(p_{s,t})_{s\leq t}$ associated with equation* (7.1) *is asymptotically strong Feller.*

Proof We will divide the proof in two steps.

Step1: We prove the claim under the additional assumption that for each fixed $t \in [s,T]$, $H \ni x \mapsto F(t,x)$ is smooth as the approximations construction in Lemma 7.17. Then the Markovian semigroup $(p_{s,t})_{s\leq t}$ is asymptotically strong Feller.
Equation (7.13) (which describes the Gâteaux derivative $\eta^h(s,t,x)$ of the process $X(s,t,x)$ along direction h) with the initial starting time $s \in [0,T]$ will be in the following form

$$\eta^h(s,t,x) = U_{s,t}h + \int_s^t U_{r,t}DF(r, X(s,r,x))\eta^h(r,x)dr, \ s \leq t.$$

Thus we get

$$\|\eta^h(s,t,x)\| \leq \mathbb{M}e^{\omega(t-s)}\|h\| + \mathbb{M}\ Lip_F \int_s^t e^{\omega(t-r)}\|\eta^h(s,r,x)\|dr, \quad t \in [s,T],$$

which is equivalent to

$$e^{-\omega(t-s)}\|\eta^h(s,t,x)\| \leq \mathbb{M}\|h\| + \mathbb{M}\ Lip_F \int_s^t e^{-\omega(r-s)}\|\eta^h(s,r,x)\|dr.$$

Hence by Gronwall's lemma

$$e^{-\omega(t-s)}\|\eta^h(s,t,x)\| \leq \mathbb{M}\|h\| \cdot e^{\mathbb{M}\ Lip_F(t-s)}, \quad t \in [s,T],$$

which implies

$$\|\eta^h(s,t,x)\| \leq \mathbb{M}\|h\| \cdot e^{\mathbb{M}\ Lip_F t}e^{\omega(t-s)} = \mathbb{M}e^{(\omega+\mathbb{M}\ Lip_F)(t-s)}\|h\|.$$

So $D_hX(s,t,z) = \eta^h(s,t,x)$ is bounded by a linear operator and for its norm we have

$$\|DX(s,t,x)\|_{\mathcal{L}(H)} \leq \mathbb{M}e^{(\omega+\mathbb{M}\ Lip_F)(t-s)}, \quad \mathbb{P}-a.s., \tag{7.14}$$

for all $s \leq t$ and all $x \in H$.
Next, we want to use Theorem 4.29 for showing the asymptotic strong Feller property for $(p_{s,t})_{s\leq t}$. We have by the mean-value theorem and (7.14)

$$\begin{aligned} & |p_{s,t}f(x) - p_{s,t}f(y)| \leq \mathbb{E}[\|f(X(s,t,x)) - f(X(s,t,y))\|] \\ \leq \quad & \|\nabla f\|_\infty \cdot \mathbb{E}[\|X(s,t,x) - X(s,t,y)\|] \\ \leq \quad & \|\nabla f\|_\infty \cdot \mathbb{E}\Big[\sup_{\alpha\in[0,1]}\|DX(s,t,y+\alpha(x-y))\|_{\mathcal{L}(H)}\Big] \cdot \|x-y\| \\ \leq \quad & \|\nabla f\|_\infty \cdot \mathbb{M}e^{(\omega+\mathbb{M}\ Lip_F)(t-s)} \cdot \|x-y\| \end{aligned} \tag{7.15}$$

for all $f \in \mathcal{C}_b^1(H)$, $x, y \in H$ and $r \leq t$. Let $\{t_n\}_{n\in\mathbb{N}} \subset [s,T]$ be such that $\{t_n - s\}_{n\in\mathbb{N}}$ is non decreasing and $\lim_{n\to\infty}(t_n - s) = \infty$. Define $\delta_n := \exp\big\{(\omega + \mathbb{M}\ Lip_F)(t_n - s)\big\}$ for all $n \in \mathbb{N}$. Since $\omega < -\mathbb{M}\ Lip_F \leq 0$, we conclude that $\delta_n \to 0$ when $n \to \infty$. So the sufficient condition in Proposition 4.29 is satisfies by $C_s = \mathbb{M}$. And this end the proof of *Step1*.

Step2: We will drop the smoothness assumptions on F. So, let F be just Lipschitz continuous.
We approximate F with smooth functions $F_{n,k}$ as in Lemma 7.17. Then for every fixed $t \in [s,T]$

$$\lim_{n\to\infty}\lim_{k\to\infty} F_{n,k}(t,x) = F(t,x), \quad x \in H. \tag{7.16}$$

Furthermore, since the function $F_{n,k}$ satisfy Lipschitz continuity with $Lip_{F_{n,k}} \leq Lip_F$, thus similarly to Theorem 7.14 one can show that the Cauchy problem

$$\begin{aligned} & dX_{n,k}(t) = [AX_{n,k}(t) + F_{n,k}(t, X_{n,k})]dt + \sigma(t)dL(t), \quad t \geq s, \\ & X_{n,k}(s) = x, \end{aligned}$$

has a unique mild solution $X_{n,k}(s,t,x)$. Moreover, we have a pointwise convergence

$$\lim_{n\to\infty}\lim_{k\to\infty} X_{n,k}(s,t,x) = X(s,t,x) \tag{7.17}$$

for all $x \in H$ and $s \leq t$. Let us prove it. Indeed, We have

$$\begin{aligned} & \|X_{n,k}(s,t,x) - X(s,t,x)\| \\ \leq & \int_s^t \Big\|U(r,t)\Big(F_{n,k}(r, X_{n,k}(r)) - F(r, X(r))\Big)\Big\| dr \\ \leq & \int_s^t \Big\|U(r,t)\Big(F_{n,k}(r, X_{n,k}(r)) - F_{n,k}(r, X(r))\Big)\Big\| dr \\ + & \int_s^t \Big\|U(r,t)\Big(F_{n,k}(r, X(r)) - F(r, X(r)))\Big)\Big\| dr. \end{aligned}$$

Since $F_{n,k}$ is Lipschitz continuous with $Lip_{F_{n,k}} \leq Lip_F$ and $\|F_{n,k}(r, X(r)) - F(r, X(r))\|$ goes to zero via (7.16), we have

$$\|X_{n,k}(s,t,x) - X(s,t,x)\| \leq \mathbb{M}_T\ Lip_F \int_s^t \|X_{n,k}(s,t,x) - X(s,t,x)\| dr + \mathcal{K}_{n,k}(t),$$

where $\mathcal{K}_{n,k}(t) = \int_s^t \big\|U(r,t)\big(F_{n,k}(r, X(r)) - F(r, X(r))\big)\big\| dr$ vanishes as $n \to \infty$ and $k \to \infty$ by Lebesgue's dominated convergence. Now by Gronwall's lemma, we conclude $\|X_{n,k}(s,t,x) - X(s,t,x)\| \to 0$ pointwisely as $n \to \infty$ and $k \to \infty$.

Let us define

$$p_{s,t}^{n,k} f(x) := \mathbb{E}[f(X_{n,k}(s,t,x))], \quad n, k \in \mathbb{N},\ s \leq t,\ x \in H,$$

for all $f \in \mathcal{B}_b(H)$.
Clearly, from (7.17) and Lebesgue's dominated convergence theorem, we can check that

$$\lim_{n\to\infty} \lim_{k\to\infty} p_{s,t}^{n,k} f(x) = \lim_{n\to\infty} \lim_{k\to\infty} \mathbb{E}[f(X_{n,k}(s,t,x))] = \mathbb{E}[f(X(s,t,x))] = p_{s,t} f(x) \tag{7.18}$$

for all $f \in \mathcal{C}_b(H)$, $x \in H$, $s \leq t$.

Let us fix $n, k \in \mathbb{N}$ then according to Lemma 7.17, since $F_{n,k} \in \mathcal{C}_b^2(H, H)$, $(p_{s,t}^{n,k})_{s\leq t}$ is asymptotically strong Feller. whereas the Lipschitz constant $F_{n,k}$ is less or equal to Lipschitz constant F, we obtain from (7.15)

$$\begin{aligned} |p_{s,t}^{n,k} f(x) - p_{s,t}^{n,k} f(y)| &\leq \|\nabla f\|_\infty \cdot \mathbb{M} e^{(\omega + \mathbb{M}\ Lip_{F_{n,k}})(t_n - s)} \cdot \|x - y\| \\ &\leq \|\nabla f\|_\infty \cdot \mathbb{M}\delta_n \cdot \|x - y\| \end{aligned}$$

for all $f \in \mathcal{C}_b^1(H)$, $x, y \in H$, $n \in \mathbb{N}$ with $\delta_n = e^{(\omega + \mathbb{M}\ Lip_F)(t_n - s)}$.
Letting $n, k \to \infty$ and using the convergence, we conclude that (7.18) $(p_{s,t})_{s\leq t}$ satisfies the required inequality and so is asymptomatically strong Feller. ∎

Remark 7.19 *In conclusion, under the imposed hypothesis 7.1 and 7.3 with* $\omega + \mathbb{M}\ Lip_F < 0$*, we can show asymptotic strong Feller property for the associated semigroup* $(p_{s,t})_{s\leq t}$ *to the mild solution of SDE*

$$\begin{aligned} dX(t) &= [A(t)X(t) + F(t, X(t))]dt + \sigma(t) dL(t) \\ X(s) &= x. \end{aligned}$$

If we additionally assume that there exists a common point $(s_0, x) \in \mathbb{R} \times H$*, then we have the uniqueness of* T*-periodic evolution system of measures and therefore ergodicity of the system.*
Unfortunately, it is not clear whether one can show that there exists a point (s_0, x) *such that* x *is in the support of* μ_{s_0} *for every* $(\mu_s)_{s\in\mathbb{R}}$ *(see [PD08] in the case of Navier-Stokes equation).*

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