Stochastic nonlinear Schrödinger equation

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Abstract

This thesis is devoted to the study of stochastic nonlinear Schrödinger equations (abbreviated as SNLS) with linear multiplicative noise in two aspects: the well-posedness in $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ and the noise effects on blowup phenomena in the non-conservative case.

1. The well-posedness in $L^2(\mathbb{R}^d)$.

The first fundamental question when dealing with SNLS is the well-posedness problem. In the first chapter, we prove the global well-posedness results in $L^2(\mathbb{R}^d)$ with the subcritical exponents of the nonlinear term, and we also obtain the local existence, uniqueness and blowup alternative in the critical case.

Our approach is different from the standard literature on stochastic nonlinear Schroödinger equations. By a rescaling transformation we reduce the stochastic equation to a random nonlinear Schroödinger equation with lower order terms and treat the resulting equation by a fixed point argument, based on generalizations of Strichartz estimates proved by J. Marzuola, J. Metcalfe and D. Tataru in 2008. This approach allows to improve earlier well-posedness results obtained in the conservative case by a direct approach to the stochastic Schroödinger equation. In contrast to the latter, we obtain the global well-posedness in the full range $(1, 1 + 4/d)$ of admissible exponents in the non-linear part (where d is the dimension of the underlying Euclidean space), i.e. in exactly the same range as in the deterministic case.

2. The well-posedness in $H^1(\mathbb{R}^d)$.

In the second chapter, we study the well-posedness for SNLS in the energy space $H^1(\mathbb{R}^d)$. The main motivation comes from the physical significance of the energy space $H^1(\mathbb{R}^d)$, and this work develops the preliminary results and machinery for the blowup analysis in the chapter later on. We consider here both focusing and defocusing nonlinearities and obtain the global well-posedness, including also the continuous

dependence on the initial data, with the subcritical exponents α satisfying

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1, 2; \\ 1 < \alpha < 1 + \frac{4}{d-2}, & \text{if } d \geq 3, \end{cases} \tag{0.0.1}$$

in the defocusing case $\lambda = -1$, and

$$1 < \alpha < 1 + \frac{4}{d}. \tag{0.0.2}$$

in the focusing case $\lambda = 1$, i.e. in exactly the same ranges as in the deterministic case. We also prove the local existence, uniqueness and blowup alternative in the energy-critical case where $\alpha = 1 + \frac{4}{d-2}$ with $d \geq 3$.

This work improves the earlier results obtained in the conservative case by A. de Bouard and A. Debussche in 2003, where the global existence and uniqueness were restricted to the case $\alpha < 1 + \frac{2}{d-1}$ if $d \geq 6$. Moreover, the mass-critical value $\alpha = 1 + \frac{4}{d}$ in the focusing case and the energy-critical value $\alpha = 1 + \frac{4}{d-2}$ with $d \geq 3$ obtained here allow to study in the stochastic case the blowup phenomena, which are extensively studied in the deterministic case. As a matter of fact, this is also one of our main motivations to study well-posedness in the H^1 context and leads to the work in the next chapter about the noise effects on blowup phenomena in the non-conservative case.

3. The noise effects on blowup in the non-conservative case.

In this chapter we focus on the noise effects on blowup phenomena in the non-conservative focusing mass-critical/supercritical case, i.e. $\lambda = 1$, $\alpha \in [1 + \frac{4}{d}, \infty)$ with $d = 1, 2$ and $\alpha \in [1 + \frac{4}{d}, 1 + \frac{4}{d-2})$ with $d \geq 3$. Our main motivation comes from the wide interests in the field of stochastic partial differential equations (SPDE) to investigate the effects (e.g. uniqueness, blow-up) of noise on the deterministic equations. We prove here that adding a large space-independent noise one can, with high probability, prevent the blowup on the whole time interval $[0, \infty)$. Furthermore, for more generally space-dependent noise, the explosion can also be prevented with high probability on a given bounded interval $[0, T]$, $0 < T < \infty$.

These noise effects are quite different than those in the conservative case obtained by A. de Bouard and A. Debussche in 2005, where spatial smooth noise can cause blow-up immediately with positive probability for any smooth initial data.

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Bielefeld, 04 March 2014 Deng Zhang

Contents

Chapter 0

Introduction

Stochastic nonlinear Schrödinger equation

This thesis is devoted to the study of stochastic nonlinear Schrödinger equation (abbreviated as SNLS) with linear multiplicative noise

$$\begin{aligned} idX(t,\xi) &= \Delta X(t,\xi)dt + \lambda|X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\ &\quad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \qquad (0.0.1) \\ X(0) &= x. \end{aligned}$$

Here X is a complex valued function on $[0,T]\times\mathbb{R}^d$, $\lambda=-1$ (defocusing) or $\lambda=1$ (focusing) and $\alpha>1$. $W(t,\xi)$ is the colored Wiener process

$$W(t,\xi)=\sum_{j=1}^{N}\mu_j e_j(\xi)\beta_j(t),\quad t\geq 0,\quad \xi\in\mathbb{R}^d,$$

where we assume $N<\infty$ for simplicity, $\mu_j\in\mathbb{C}$, e_j are real-valued functions on $\mathbb{R}^d$, and $\beta_j(t)$, $1\leq j\leq N$, are independent real Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$.

For physical reasons (see below), we choose μ of the form $\mu(\xi)=\frac{1}{2}\sum\limits_{j=1}^{N}|\mu_j|^2 e_j^2(\xi)$, $\xi\in\mathbb{R}^d$, in order to make $|X(t)|_2^2$ a martingale from which one can define the "physical probability law".

In the case without noise effect, i.e. $\mu_j=0,\quad 1\leq j\leq N$, (0.0.1) is the classical nonlinear Schrödinger equation (NLS). This equation is one of the basic models for nonlinear wave and has widely physical applications, e.g. nonlinear optics, plasma physics and quantum field theory, etc. We refer the reader to [84] for more details. From the mathematical point of view, On one hand, NLS possesses both conservation of mass and conservation of Hamiltonian, that is

$$|X(t)|_2^2 = |x|_2^2,$$

$$\frac{1}{2}|\nabla X(t)|_2^2 - \frac{\lambda}{\alpha+1}|X(t)|_{L^{\alpha+1}}^{\alpha+1} = \frac{1}{2}|\nabla x|_2^2 - \frac{\lambda}{\alpha+1}|x|_{L^{\alpha+1}}^{\alpha+1}.$$

Hence, it is appropriate to study the well-posedness in the space $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$. On the other hand, NLS is a kind of dispersive equation. The linear part possesses dispersive properties, e.g. the Strichartz estimates and local smoothing estimates. These dispersive properties allow us in suitable Banach spaces to apply Banach's fixed point theorem and to control the nonlinearity for α in the subcritical and critical ranges. Therefore, one can obtain the local well-posedness of (0.0.1) in the subcritical and critical cases, based on the dispersive properties. Then the global existence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$) in the subcritical case follows from the conservation of mass (resp. Hamiltonian). However, the global existence in the critical case is much more delicate. In this case, the nonlinearity competes with the kinetic energy, solutions may blow up in finite time and more interesting phenomena will appear, e.g. L^2-mass concentration, self-similarity, etc. We refer the reader to [46, 22, 58] for more sophisticated studies on this argumentss.

In the stochastic case with noise effects, this equation was earlier studied in the conservative case with a purely imaginary noise, i.e., $Re\mu_j = 0, 1 \leq j \leq N$, and in this case the L^2 norm of the solution is still conserved $|X(t)|_2 = |x|_2$. This case was proposed in [2] as a model for the propagation of nonlinear dispersive waves in nonlinear or random media (see [3, 75]). The global existence and uniqueness results in L^2 and H^1 spaces were first obtained by de A. de Bouard and A. Debussche [10, 12] in the subcritical case, and they also studied the influence of noise on the blowup phenomena in the focusing mass-supercritical case [13] (see also [11] for the additive noise case). Moreover, we refer the interested reader to [14, 26, 27] for numerical simulations in the focusing mass-critical case, where the numerical tests suggest that the spatially smooth noise is able to delay the blowup and the white noise can even prevent the explosion.

The general stochastic equation (0.0.1) with $\mu_j \in \mathbb{C}$, including the two previous cases, was first studied in our recent paper [7]. In this case one of the main feature is that $|X(t)|_2^2$ is no longer independent of time, but a general martingale (see (1.3.79) in Chapter 1).

Now, let us first briefly present the physical meaning of SNLS (0.0.1). $X = X(t,\xi,\omega)$, $\xi \in \mathbb{R}^d$, $t \geq 0$, $\omega \in \Omega$, represents the quantum state at time t, while the stochastic perturbation $iXdW$ represents a stochastic continuous measurement via the quantum observables $\mu_j e_j$. Equation (0.0.1) can be also derived by a Schrödinger equation with the potential XV, where the random field V fluctuates rapidly and so can be approximated by Gaussian white noise W. A better insight in equation (0.0.1) can be gained from the analysis in [8], [9]. Then, an (at this stage) heuristic application of Itô's formula implies that

$$|X(t)|_{L^2}^2 = |x|_{L^2} + 2\sum_{j=1}^{N} \mathrm{Re}(\mu_j) \int_0^t \langle X(s), X(s)e_j\rangle_{L^2} d\beta_j(s),\ t \geq 0. \tag{0.0.2}$$

Applying Itô's formula to $\log |X(t)|_{L^2}^2$, we see that

$$|X(t)|_{L^2}^2 = |x|_2^2 \exp\left\{\sum_{j=1}^{N}\left[\int_0^t v_j(s)d\beta_j(s) - \frac{1}{2}\int_0^t v_j^2(s)ds\right]\right\},$$

where $v_j(t) = 2\mathrm{Re}\langle X(t), \mu_j e_j X(t)\rangle_{L^2} |X(t)|_{L^2}^{-2}$. Clearly, by (0.0.2), $t \to |X(t)|_{L^2}^2$ is a continuous martingale and so, if $|x|_{L^2} = 1$,

$$\widehat{\mathbb{P}}_x^T(F) = \int_F |X(T,\omega)|_{L^2}^2 d\mathbb{P}(d\omega),\ F \in \mathcal{F}_T,$$

defines a probability law on $(\Omega, \mathcal{F}_T)$ (the physical probability law) and, under this law by Girsanov's theorem the continuous process

$$\widetilde{\beta}_j(t) = \beta_j(t) - \int_0^t v_j(s)ds,\ t \in [0,T],\ j = 1, ..., N, \tag{0.0.3}$$

are independent Gaussian processes with respect to the filtration $(\mathcal{F}_t)$ (Theorem 2.14 in [5]). Here $\widehat{P}_x^T$ is the physical probability law of the events occurring in time $[0,T]$, while $\widehat{\psi}(t,\omega) = X(t,\omega)|X(t,\omega)|_2^{-1}$ is the state of the quantum system conditioned by observation of $s \to \beta_j(s,\omega)$, $0 \leq s < t$.

In the stochastic conservative case mentioned above, we have $v_j(t) = 0$, $|X(t)|_{L^2} = |x|_{L^2}$, $\forall t$ and $\widehat{P}_x^T = \mathbb{P}|_{\mathcal{F}_T}$. Then, by (0.0.3), $\widetilde{\beta}_j = \beta_j$, $\forall j$, and so, in this case, the randomness is independent of the quantum system, and the measurement does not provide any information on the quantum system.

In this thesis, we will focus on two aspects of SNLS (0.0.1): the well-posedness problems in $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ and the noise effects on blowup in the non-conservative case. The content of the thesis is briefly presented below.

Chapter 1 is concerned with the well-posedness of (0.0.1) in $L^2(\mathbb{R}^d)$, which is the fundamental problem when dealing with SNLS. Unlike in the standard literature

[10], we apply a new method to study the well-posedness of (0.0.1). We obtain the sharp global well-posedness in the subcritical case, improving the earlier results in [10]. Moreover, the local existence, uniqueness and blowup alternative in the critical case are also given.

Chapter 2 is a natural continuation of Chapter 1. In this chapter we treat the well-posedness in $H^1(\mathbb{R}^d)$. The main motivation comes from the physical significance of the energy space $H^1(\mathbb{R}^d)$, but the results claimed here will be also used for the blow-up analysis in the next chapter. We study here both defocusing and focusing cases and prove the global well-posedness of (0.0.1) in the subcritical case. We also show the local existence, uniqueness and blowup alternative in the critical case. This work improves the earlier results in [12]. Moreover, the sharper results obtained here will be also used to study the noise effects on the formation of singularities in Chapter 3.

Chapter 3 is devoted to the study of noise effects on the blowup phenomena in the non-conservative focusing mass-critical/supercritical cases. Our main motivation comes from the wide interests in the field of stochastic partial differentia equations (SPDE) for investigation of the effects (e.g. uniqueness, blow-up) of noise on the deterministic equations. It is already known that, the multiplicative noise term in deterministic evolution equations has a dissipativity effect. We show here that adding a large space-independent noise one can, with high probability, prevent the blowup on the whole time interval $[0,\infty)$. Furthermore, for more generally space-dependent noise, the explosion can also be prevented on the bounded time interval $[0,T]$, $0<T<\infty$.

These phenomena are different than those in the deterministic case, where there exist solutions that blow up in finite time (see e.g. [84, 22, 58]). The noise effects in the non-conservative case are also quite different than those in the stochastic conservative focusing mass-supercritical case, where the spatial smooth noise can cause blowup immediately with positive probability for any smooth initial data (see [12]).

We continue with a more detailed presentation of each of these three chapters.

Chapter 1. The well-posedness in $L^2(\mathbb{R}^d)$.

In this chapter we study the well-posedness of SNLS (0.0.1) in $L^2(\mathbb{R}^d)$, which is the starting point of the following works in the next chapters.

In the deterministic case, the global well-posedness in the subcritical case $1<\alpha<1+\frac{4}{d}$ was first obtained by Y. Tsutsumi [90], based on the regularization procedure with the H^1 well-posedness results and the dispersive effect of the free Schrödinger group $e^{it\Delta}$ expressed by the Strichartz estimates. Simplified fixed point arguments were presented by T. Kato [46] (see also [90]). Later on, the local well-posedness in the critical case $\alpha = 1 + \frac{4}{d}$ was proved by T. Cazenave and F. B. Weissler [19]. A more comprehensive review of the basic results can be found in [84, 22, 58].

In the stochastic setting, the global existence and uniqueness in the conservative subcritical case were first obtained by A. de Bouard and A. Debussche [10]. In this article, the authors started with the mild equation of stochastic equation (0.0.1) and applied the Burkholder inequalities, based on the γ-radonifying operators, to estimate the Banach space valued stochastic integrals. However, this approach leads to a restrictive condition on α: $1 < \alpha < 1 + \frac{2}{d-1}$ if $d \geq 3$ (see also the comments in [12]), hence one can not study the blow-up phenomena in the L^2-critical case where $\alpha = 1 + \frac{4}{d}$.

Unlike in the previous work, here we present a new approach to study the well-posedness of (0.0.1), even for the more general case $\mu \in \mathbb{C}$ as in the physical context [8, 9]. We prove the global well-posedness, including also the continuous dependence on the initial data, in the subcritical case with the exponents α of the nonlinearity as the same as in the deterministic case, i.e., $1 < \alpha < 1 + \frac{4}{d}$. Moreover, we also show the local existence, uniqueness and blowup alternative in the critical case $\alpha = 1 + \frac{4}{d}$. These sharper well-posedness results obtained here improve the work in [10] and enable one to study the blow-up phenomena in the L^2-critical case where $\alpha = 1 + \frac{4}{d}$, $d \geq 1$.

The main results.

Theorem 0.0.1. *Assume* $(H1)$ *(see Section 1.1 below). Let* $1 < \alpha < 1 + \frac{4}{d}$, $1 \leq d < \infty$*. Then, for each* $x \in L^2$ *and* $0 < T < \infty$*, there exists a unique strong solution* (X, T) *of* (0.0.1) *(see Definition 1.1.1 in Section 1.1), which satisfies*

$$X \in L^2(\Omega; C([0,T]; L^2)) \tag{0.0.4}$$

$$X \in L^\gamma(0,T; L^\rho), \ \mathbb{P} - a.s., \tag{0.0.5}$$

where (ρ, γ) *is any Strichartz pair.*

Moreover, for $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, the map* $x \to X(\cdot, x, \omega)$ *is continuous from* L^2 *to* $C([0,T]; L^2) \cap L^\gamma(0,T; L^\rho)$*, and* $t \to |X(t)|_2^2$ *is a continuous martingale with the representation*

$$|X(t)|_2^2 = |x|_2^2 + 2\sum_{j=1}^{N} \mathrm{Re}(\mu_j) \int_0^t \int_{\mathbb{R}^d} e_j |X(s)|^2 d\xi \, d\beta_j(s), \ t \in [0,T]. \tag{0.0.6}$$

In the critical case we have the following local existence, uniqueness and blowup alternative results.

Theorem 0.0.2. *Assume* $(H1)$ *and let* $\alpha = 1 + \frac{4}{d}$. *Then, for each* $x \in L^2$ *and* $0 < T < \infty$, *there exists a maximal strong solution* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *of* (0.0.1) *(see Definition 1.1.1 in Section 1.1). In particular, uniqueness holds for* (0.0.1). X *also satisfies* $\mathbb{P}$-*a.s for every* $n \geq 1$

$$X|_{[0,\tau_n]} \in C([0,\tau_n]; L^2) \cap L^{2+\frac{4}{d}}(0,\tau_n; L^{2+\frac{4}{d}}). \tag{0.0.7}$$

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-*a.e* ω, *if* $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, *then*

$$\|X(t)(\omega)\|_{L^{2+\frac{4}{d}}(0,\tau^*(x)(\omega);L^{2+\frac{4}{d}})} = \infty. \tag{0.0.8}$$

The strategy of the proof.

The main approach here is based on the rescaling transformation

$$X = e^W y. \tag{0.0.9}$$

The advantage of this transformation is that it reduces the stochastic equation (0.0.1) to a Schrödinger equation with random coefficients, to which one can apply deterministic methods pathwisely. More precisely, we have

$$\begin{aligned} &\frac{\partial y}{\partial t} = A(t)y - \lambda i e^{(\alpha-1)ReW}|y|^{\alpha-1}y, \\ &y(0) = x. \end{aligned} \tag{0.0.10}$$

Here $A(t)y := -i(\Delta y + b\cdot\nabla y + cy)$, $b(t,\xi) = 2\nabla W(t,\xi)$, $c(t,\xi) = \sum_{j=1}^{d}(\partial_j W(t,\xi))^2 + \Delta W(t,\xi) - i(\mu + \widetilde{\mu})$ and $\widetilde{\mu} = \frac{1}{2}\sum_{j=1}^{N}\mu_j^2 e_j^2$.

The explicit definitions of solutions and the equivalence between two equations (0.0.1) and (0.0.10) will be made clear in Section 1.1.

In this way, the well-posedness problems of (0.0.1) is reduced to those of (0.0.10). As in the deterministic case, the well-posedness of (0.0.10) relies crucially on the dispersive property, particularly the Strichartz estimates, of the linear part $\frac{\partial y}{\partial t} = A(t)y$. Thanks to the Strichartz estimates established in [59] for the lower order perturbations of the Laplacian, we can obtain for $\mathbb{P}$-a.e. $\omega \in \Omega$ the local existence, uniqueness and blowup alternative of (0.0.10) in the exactly the same range as in the deterministic case.

To get the global well-posedness, we need certain a priori estimates. In the L^2 case, our a priori estimate comes from the analysis of the mass $|X(t)|_2^2$. Although $|X(t)|_2^2$ is no longer conserved as in the deterministic and stochastic conservative cases, its martingale property is enough for us to obtain the desired a priori estimates and hence to establish the global well-posedness of (0.0.10) in the subcritical case. Once we prove the well-posedness results of the random equation (0.0.10), we obtain the corresponding well-posedness of SNLS (0.0.1) by the rescaling approach.

Chapter 1 is based on our recent paper [7] in joint with Prof. Viorel Barbu and Prof. Michael Röckner. The structure is as follows. In Section 1.1 we set up the preliminaries, including the definition of solutions and the rescaling transformation. Then in Section 1.2 we establish the local existence, uniqueness and blowup alternative results in both subcritical and critical cases. Section 1.3 is devoted to the global well-posedness in the subcritical case. Some comments on the relevant results are also included in Section 1.4.

Chapter 2. The well-posedness in $H^1(\mathbb{R}^d)$.

The aim of this chapter is to establish the well-posedness of SNLS (0.0.1) in the energy space $H^1(\mathbb{R}^d)$, which is a natural continuation of Chapter 1 and also develops the preliminary results and machinery for the future study of blow-up phenomena in the focusing mass-critical/supercritical cases.

In the deterministic case, the global well-posedness in the subcritical case was studied in a series works by J. Ginibre and G. Velo [35, 36, 38] (see also [39] for a compactness method to prove the existence). Later on, simplified fixed point arguments were presented by T. Kato [45, 46], based on the Strichartz estimates. Moreover, the local well-posedness in the critical case was obtained by T. Cazenave and F. B. Weissler [19]. For more systematic discussions, see e.g., [22, 58].

It should be mentioned that, unlike what happens in the L^2 case, the exponents for the global well-posedness in defocusing and focusing cases are different. More precisely, NLS is globally well posed in the defocusing case ($\lambda = -1$) for the exponents α satisfying

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1, 2; \\ 1 < \alpha < 1 + \frac{4}{d-2}, & \text{if } d \geq 3, \end{cases} \tag{0.0.11}$$

while in the focusing case ($\lambda = 1$) for α satisfying $1 < \alpha < 1 + \frac{4}{d}$. This difference comes from the signs in the associated Hamiltonian of (0.0.1), that is,

$$H(X) := \frac{1}{2}|\nabla X|_2^2 - \frac{\lambda}{\alpha+1}|X|_{L^{\alpha+1}}^{\alpha+1}.$$

To get the global well-posedness, one needs a priori estimates of $|\nabla X(t)|_2^2$ from $H(X(t))$. In the defocusing case ($\lambda = -1$), the conservation of the mass $|X(t)|_2^2$ and the Hamiltonian $H(X(t))$ give us directly the uniform bound of $|X(t)|_{H^1}^2$. While in the focusing case ($\lambda = 1$), $H(X(t))$ can be negative and one shall further use the Gagliardo-Nirenberg inequality to dominate the potential energy $|X(t)|_{L^{\alpha+1}}^{\alpha+1}$ by the kinetic energy $|\nabla X(t)|_2^2$ but for the restrictive exponents $1 < \alpha < 1 + \frac{4}{d}$. We note that, $\alpha = 1 + \frac{4}{d}$ is indeed a sharp value for the global well-posedness in the focusing case, in the sense that in this case there exist solutions that can blow up in finite time.

Now, in the stochastic setting, the global existence and uniqueness in the subcritical case were first obtained in [12] in the conservative case. In fact, the authors proved in [12] the local existence and uniqueness with α satisfying

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1 \text{ or } 2; \\ 1 < \alpha < 5, & \text{if } d = 3; \\ 2 \leq \alpha < 1 + \frac{4}{d-2}, & \text{if } d = 4, 5; \\ \alpha < 1 + \frac{2}{d-1}, & \text{if } d \geq 6, \end{cases}$$

and obtained the global well-posedness under the further assumptions $\alpha < 1 + \frac{4}{d}$ or $\lambda = -1$. Hence in the focusing case $\lambda = 1$ with dimension $d \geq 6$, the global well-posedness is established only for the restrictive exponents $\alpha < 1 + \frac{2}{d-1}$.

Following an idea from [7], we apply the rescaling approach to study the well-posedness problems of SNLS (0.0.1), including also the non-conservative case. We obtain the global well-posedness, including also the continuous dependence on the initial data, in the subcritical case with the exponents α in the same ranges as in the deterministic case. Moreover, we also show the local existence, uniqueness and blowup alternative in the H^1-critical case where $\alpha = 1 + \frac{4}{d-2}$, $d \geq 3$.

In conclusion, this work improves the results in [12]. Moreover, as mentioned above, in the focusing case $\alpha = 1 + \frac{4}{d}$ is the mass-critical value for the deterministic solutions to blow up. Therefore, the sharp value $\alpha = 1 + \frac{4}{d}$ obtained here allows to study the noise effects on blowup phenomena. This is indeed one of our motivations to study the well-posedness in the H^1 context and leads to the study of the noise effects in the non-conservative case in Chapter 3.

The main results.

Theorem 0.0.3. *Assume* $(H2)$ *(see Section 2.1). Let* α *satisfy (0.0.11) and* $1 < \alpha < 1+\frac{4}{d}$ *in the defocusing and focusing cases respectively. Then for each* $x \in H^1$ *and* $0 < T < \infty$*, there exists a unique strong solution* (X, T) *of (0.0.1) (see Definition 2.1.1 in Section 2.1), such that*

$$X \in L^2(\Omega; C([0,T]; H^1)) \cap L^{\alpha+1}(\Omega; C([0,T]; L^{\alpha+1})), \tag{0.0.12}$$

and

$$X \in L^\gamma(0,T; W^{1,\rho}), \quad \mathbb{P} - a.s., \tag{0.0.13}$$

where (ρ, γ) *is any Strichartz pair.*

Furthermore, for $\mathbb{P} - a.e$ ω*, the map* $x \to X(\cdot, x, \omega)$ *is continuous from* H^1 *to* $C([0,T]; H^1) \cap L^\gamma(0,T; W^{1,\rho})$*.*

We also have the following local existence, uniqueness and blowup alternative results in the critical case.

Theorem 0.0.4. *Assume* $(H2)$ *and* $\alpha = 1 + \frac{4}{d-2}$*,* $d \geq 3$*. For each* $x \in H^1$ *and* $0 < T < \infty$*, there exists a maximal strong solution* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *of* (0.0.1) *(see Definition 2.1.1 in Section 2.1). In particular, uniqueness holds for* (0.0.1)*.* X *also satisfies* $\mathbb{P}$*-a.s for every* $n \geq 1$

$$X|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^\gamma(0,\tau_n; W^{1,\rho}), \tag{0.0.14}$$

where (ρ, γ) *is any Strichartz pair.*

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$*-a.e* ω*, if* $\tau_n(\omega) < \tau^*(x)(\omega)$*,* $\forall n \in \mathbb{N}$*, then*

$$\|X(\omega)\|_{L^{\frac{2(d+2)}{d-2}}(0,\tau^*(x)(\omega); L^{\frac{2(d+2)}{d-2}})} = \infty, \quad \mathbb{P} - a.s. \tag{0.0.15}$$

The strategy of the proof.

Chapter 1 is the starting point of this chapter. We use the rescaling transformation (0.0.9) to reduce SNLS (0.0.1) to the random equation (0.0.10) (the equivalence between these two equations will be presented in Section 2.1). Later on, we derive the Strichartz estimates in Sobolev spaces, with which, as well as the probabilistic localization arguments involving the stopping times and the adaptedness, we are able to establish the local existence, uniqueness and blowup alternative results of (0.0.1) pathwisely in both subcritical and critical cases.

The global well-posedness in the subcritical case lies in the a priori estimates of the energy $|X(t)|_{H^1}$, which will be derived from the analyze of the Hamiltonian $H(X(t))$. Inspired by [57], we apply here a different approach than that in Chapter 1 to derive Itô's formula for the term $|X(t)|_{L^p}^p$, $p > 2$, in $H(X(t))$. Then, using standard martingale technique we obtain the desired a priori estimate of the energy and get the global well-posedness for (0.0.10), hence also for SNLS (0.0.1) by the rescaling transformation.

Chapter 2 is organized as follows. Section 2.1 includes the definition of solutions and the equivalence theorem via the rescaling transformation. Section 2.2 are concerned with the local existence, uniqueness and blowup alternative results in both subcritical and critical cases, based on the Strichartz estimates. Section 2.3 is devoted to the global well-posedness in the subcritical case. Lastly, Section 2.4 contains some comments on relevant results in the literature.

Chapter 3. The noise effects on blowup in the non-conservative case.

This chapter focuses on the noise effects on blowup in the non-conservative focusing mass-critical/supercritical case, i.e. $\lambda = 1$, $\alpha \in [1 + \frac{4}{d}, \infty)$ with $d = 1, 2$ and $\alpha \in [1 + \frac{4}{d}, 1 + \frac{4}{d-2})$ with $d \geq 3$.

In the deterministic case, an elementary proof of the existence of blowup solutions was obtained by R. T. Glassey [40]. Later on, the threshold for solutions to blow up in the focusing mass-critical case was obtained by M. I. Weinstein [94]. One major result was obtained by F. Merle [62], stating that in the focusing mass-critical case, up to symmetries, the critical mass blowup solutions are unique.

In the stochastic conservative case, the effects of noise on blowup were first mathematically studied by A. de Bouard and A. Debussche [13] (see also [11] for the additive noise). They proved that in the conservative focusing mass-supercritical case, i.e. $\lambda = 1$, $\alpha \in (1 + \frac{4}{d}, \infty)$ if $d = 1, 2$ and $\alpha \in (\frac{7}{3}, 5)$ if $d = 3$, the spatially smooth noise will cause blowup immediately with positive probability for any smooth initial data. Moreover, the numerical simulations in [14, 26, 27] suggested that in the conservative focusing mass-critical case, i.e. $\lambda = 1$, $\alpha = 1 + \frac{4}{d}$, the spatial smooth noise has the effect to delay blowup and white noise can even prevent blowup.

In contrast to the previous case, the situations in the non-conservative case are quite different. We prove that in the non-conservative focusing mass-critical/supercritical case, adding a large space-independent noise one can, with high probability, prevent blowup on the whole time interval $[0, \infty)$. Furthermore, for the general space-dependent noise, the explosion can also be avoided with high probability on the bounded time interval $[0, T]$, where $0 < T < \infty$.

The main result.

We have the non-explosion result for the space-independent noise as follows.

Theorem 0.0.5. *Consider* (0.0.1) *in the non-conservative case with* $Re\mu_1 \neq 0$. *Let* $\lambda = 1$, $\alpha \in [1+\frac{4}{d}, \infty)$ *if* $d = 1, 2$, *and* $\alpha \in [1+\frac{4}{d}, 1+\frac{4}{d-2})$ *if* $d \geq 3$. *Assume* $(H3)$ *(see Section 3.1 in Chapter 3), but with* f_j, $1 \leq j \leq N$, *also being fixed constants and* c_k *for* $2 \leq k \leq N$ *being fixed. Then for any* $x \in H^1$

$$\mathbb{P}(X(t) \text{ does not blow up on } [0,\infty)) \to 1, \quad \text{as } c_1 \to \infty.$$

For the general space-dependent noise we have

Theorem 0.0.6. *Consider* (0.0.1) *in the non-conservative case with* $Re\mu_1 \neq 0$. *let* λ *and* α *be as in Theorem 0.0.5. Assume* $(H3)$ *with* f_j, $1 \leq j \leq N$, *and* c_k, $2 \leq k \leq N$ *being fixed. Then for any* $x \in H^1$ *and* $0 < T < \infty$

$$\mathbb{P}(X(t) \text{ does not blow up on } [0,T]) \to 1, \quad \text{as } c_1 \to \infty.$$

The strategy of the proof.

The proof is based on the observation that, after the first transformation $X = e^W y$, there appears a damped term $\widehat{\mu} := \mu + \widetilde{\mu}$ in (0.0.10), which is related with the noise and has positive real part in the non-conservative case. Thus one may expect this term to prevent the explosion.

In order to explore the noise effects, we use a second transformation

$$z = e^{\widehat{\mu}t} y,$$

to reduce (0.0.10) to the equation below

$$\begin{aligned} &\frac{\partial z(t)}{\partial t} = \widehat{A}(t)z - ie^{-(\alpha-1)(Re\widehat{\mu}t - ReW(t))}|z|^{\alpha-1}z, \\ &z(0) = x \in H^1, \end{aligned} \tag{0.0.16}$$

where $\widehat{A}(t) := -i(\Delta + \widehat{b}(t)\cdot\nabla + \widehat{c}(t))$ with $\widehat{b} = -2t\nabla\widehat{\mu} + 2\nabla W(t)$ and $\widehat{c}(t) = t^2\sum_{j=1}^{N}(\partial_j\widehat{\mu})^2 - t\Delta\widehat{\mu} - 2t\nabla W(t)\cdot\nabla\widehat{\mu} + \left[\sum_{j=1}^{N}(\partial_j W(t))^2 + \Delta W(t)\right]$.

Here, an exponential decay term $e^{-(\alpha-1)Re\widehat{\mu}t}$ appears in front of the nonlinear term in (0.0.16). This fact enables us to apply the contraction arguments, developed in Chapter 2, to obtain the solutions existing on large bounded time intervals with high probability, as long as $\widehat{\mu}$ is sufficiently large. In particular, when the noise is spatial independent, one can also obtain the non-explosion with high probability on the whole time interval $[0, \infty)$.

The structure of Chapter 3 is as follows. In Section 3.1 we set up some preliminaries, including the equation (0.0.16) after the second transformation. Then in Section 3.2 and Section 3.3 we prove the non-explosion results in the non-conservative case. Some further reviews on the blow-up works are also given in Section 3.4.

Chapter 1

The well-posedness in $L^2(\mathbb{R}^d)$

This chapter is devoted to the well-posedness problems of the stochastic nonlinear Schrödinger equation (0.0.1) in $L^2(\mathbb{R}^d)$. We first introduce basic setups in Section 1.1. Then in Section 1.2 we establish the local existence, uniqueness and blowup alternative results in both subcritical and critical cases. Section 1.3 is concerned with the global well-posedness in the subcritical case. We end this chapter with brief notes on some relevant results in Section 1.4.

1.1 Preliminaries

In this section we first introduce the stochastic nonlinear Schrödinger equation (abbreviated as SNLS) in Subsection 1.1.1. Then in Subsection 1.1.2 we present the rescaling transformation (1.1.5) to reduce SNLS to a random equation, and we also prove the equivalence between two equations via the rescaling transformation.

1.1.1 Stochastic nonlinear Schrödinger equation

Let us consider the stochastic nonlinear Schrödinger equation with linear multiplicative noise

$$\begin{aligned} &idX(t,\xi) = \Delta X(t,\xi)dt + \lambda |X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\ &\qquad\qquad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \qquad (1.1.1)\\ &X(0)=x. \end{aligned}$$

Here X is a complex valued function on $[0,T]\times\mathbb{R}^d$, $\lambda=-1$ (defocusing) or $\lambda=1$ (focusing) and $\alpha>1$. W is the colored Wiener process

$$W(t,\xi)=\sum_{j=1}^{N}\mu_j e_j(\xi)\beta_j(t), t\geq 0, \xi\in\mathbb{R}^d, \tag{1.1.2}$$

where we assume $N<\infty$ for simplicity, $\mu_j\in\mathbb{C}$, $\beta_j(t)$, $1\leq j\leq N$, are independent real Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and e_j are real-valued functions under the following spatial decay assumption

(H1) *$e_j\in C_b^\infty(\mathbb{R}^d)$ such that*

$$\lim_{|\xi|\to\infty}\zeta(\xi)(|e_j(\xi)|+|\nabla e_j(\xi)|+|\Delta e_j(\xi)|)=0,$$

where $j\in\{1,...,N\}$ and

$$\zeta(\xi)=\begin{cases}1+|\xi|^2, & \text{if } d\neq 2,\\ (1+|\xi|^2)(\ln(3+|\xi|^2))^2, & \text{if } d=2.\end{cases}$$

As a matter of fact, the assumption $\lim_{|\xi|\to\infty}\zeta(\xi)|e_j(\xi)|=0$ can be removed (see Remark 3.1.1 in Chapter 3). However, we leave it here for the sake of simplicity.

For physical reasons (see Introduction), we choose μ of the form

$$\mu(\xi)=\frac{1}{2}\sum_{j=1}^{N}|\mu_j|^2e_j^2(\xi), \xi\in\mathbb{R}^d. \tag{1.1.3}$$

In this chapter we study the well-posedness of SNLS (1.1.1) in L^2 space, and the solutions are taken in the sense below.

Definition 1.1.1. *Let $x\in L^2$, $\alpha\in(1,1+\frac{4}{d}]$, fix $T>0$.*

(i). A strong solution of (1.1.1) is a pair (X,τ) with $\tau(\leq T)$ an $(\mathscr{F}_t)$-stopping time, such that $X=(X(t))_{t\in[0,T]}$ is an L^2-valued continuous $(\mathscr{F}_t)$-adapted process, $|X|^{\alpha-1}X\in L^1(0,\tau;H^{-2}(\mathbb{R}^d))$ $\mathbb{P}-a.s$, and X satisfies $\mathbb{P}-a.s$ for $t\in[0,\tau]$

$$X(t)=x-\int_0^t(i\Delta X(s)+\mu X(s)+\lambda i|X(s)|^{\alpha-1}X(s))ds+\int_0^t X(s)dW(s), \tag{1.1.4}$$

where the stochastic integral is taken in sense of Ito and equation (1.1.4) is understood as an equation in $H^{-2}(\mathbb{R}^d)$.

(ii). We say that uniqueness holds for (1.1.1)*, if for any two strong solutions* (X_i, τ_i)*,* $i = 1, 2$*, it holds* $\mathbb{P}$*-a.s. that* $X_1 = X_2$ *on* $[0, \tau_1 \wedge \tau_2]$.

(iii). A maximal strong solution of (1.1.1) *is a pair* $((X_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$*, where* (X_n, τ_n)*,* $n \in \mathbb{N}$*, are strong solutions of* (1.1.1) *with* $(\tau_n)_{n\in\mathbb{N}}$ *a sequence of increasing stopping times and* $X_{n+1} = X_n$ *on* $[0, \tau_n]$*, and " maximal" means that given any strong solution* $(\widetilde{X}, \widetilde{\tau})$*, we have for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there exists* $n(\omega) \geq 1$ *such that* $\widetilde{\tau}(\omega) \leq \tau_{n(\omega)}(\omega)$ *and* $\widetilde{X}(\omega) = X_{n(\omega)}(\omega)$ *on* $[0, \widetilde{\tau}(\omega)]$*. In particular, uniqueness holds for* (1.1.1).

To simplify the notations, we denote the maximal strong solution $((X_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$ *by the triple* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$*, where* $X = \lim_{n\to\infty} X_n \mathbb{1}_{[0,\tau^*(x))}$ *with* $\tau^*(x) = \lim_{n\to\infty} \tau_n$.

Notice that, the pair $(X, \tau^*(x))$ *is independent of the choice of* $((X_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$.

Remark 1.1.2. *In Definition 1.1.1,* $\int_0^t X(s)dW(s)$ *is an* L^2*-valued stochastic integral. Indeed,* $\int_0^t X(s)dW(s) = \int_0^t \Phi(s)d\widetilde{W}(s)$*, where* $\widetilde{W} = (\beta_1, ..., \beta_N) \in \mathbb{R}^N$ *and* $\Phi(t) : \mathbb{R}^N \to L^2(\mathbb{R}^d)$ *is defined by*

$$\Phi(t)(v) = \sum_{j=1}^N X(t)\mu_j e_j < v, f_j >$$

for $v \in \mathbb{R}^N$*,* $t \in [0, \tau]$ *and* $\{f_j\}_{j=1}^N$ *is a natural basis in* $\mathbb{R}^N$.

Hence, $\Phi(t)$ *are Hilbert-Schmidt operators from* $\mathbb{R}^N$ *to* $L^2(\mathbb{R}^d)$*,* $t \in [0, \tau]$*, and*

$$\int_0^t \|\Phi(s)\|^2_{L_2(\mathbb{R}^N;L^2(\mathbb{R}^d))} ds = \int_0^t \sum_{j=1}^N |X(s)\mu_j e_j|_2^2 ds \leq \sum_{j=1}^N |\mu_j|^2 |e_j|^2_{L^\infty} \sup_{s\in[0,\tau]} |X(s)|_2^2 \, t < \infty,$$

which implies the claim.

1.1.2 Rescaling approach

Let us introduce the rescaling transformation

$$X(t, \xi) = e^{W(t,\xi)} y(t, \xi). \tag{1.1.5}$$

The main advantage of this transformation lies in the fact that, it reduces the stochastic equation (1.1.1) to the following equation with random coefficients, to which one can apply the deterministic methods. Precisely, applying (1.1.5) to (1.1.1) we have that

$$\frac{\partial y(t, \xi)}{\partial t} = A(t)y(t, \xi) - \lambda i e^{(\alpha-1)ReW(t,\xi)} |y(t, \xi)|^{\alpha-1} y(t, \xi), \tag{1.1.6}$$

$$y(0) = x.$$

Here

$$\begin{aligned} A(t)y(t,\xi) :=& - ie^{-W}\Delta(e^W y) - (\mu + \widetilde{\mu})y \\ =& - i(\Delta + b(t,\xi)\cdot\nabla + c(t,\xi))y(t,\xi), \end{aligned} \tag{1.1.7}$$

where

$$b(t,\xi) = 2\nabla W(t,\xi), \tag{1.1.8}$$

$$c(t,\xi) = \sum_{j=1}^{d}(\partial_j W(t,\xi))^2 + \Delta W(t,\xi) - i(\mu(\xi) + \widetilde{\mu}(\xi)) \tag{1.1.9}$$

with

$$\widetilde{\mu}(\xi) = \frac{1}{2}\sum_{j=1}^{N}\mu_j^2 e_j^2(\xi). \tag{1.1.10}$$

Similarly to Definition 1.1.1, the solutions to (1.1.6) are taken in the following sense.

Definition 1.1.3. *Let $x \in L^2$, $\alpha \in (1, 1 + \frac{4}{d}]$, fix $T > 0$.*

(i). A strong solution of (1.1.6) is a pair (y, τ) with $\tau(\leq T)$ an $(\mathscr{F}_t)$-stopping time, such that $y = (y(t))_{t\in[0,T]}$ is an L^2-valued continuous $(\mathscr{F}_t)$-adapted process, $|y|^{\alpha-1}y \in L^1(0,\tau; H^{-2}(\mathbb{R}^d))$ $\mathbb{P}-a.s$, and y satisfies $\mathbb{P}-a.s$ (1.1.6) on $[0,\tau]$ as an equation in $H^{-2}(\mathbb{R}^d)$.

(ii). We say that uniqueness holds for (1.1.6), if for any two strong solutions (y_i, τ_i), $i = 1, 2$, it holds $\mathbb{P}$-a.s. that $y_1 = y_2$ on $[0, \tau_1 \wedge \tau_2]$.

(iii). A maximal strong solution of (1.1.6) is a pair $((y_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$, where (y_n, τ_n), $n \in \mathbb{N}$, are strong solutions of (1.1.6) with $(\tau_n)_{n\in\mathbb{N}}$ a sequence of increasing stopping times and $y_{n+1} = y_n$ on $[0, \tau_n]$, and " maximal" means that given any strong solution $(\widetilde{y}, \widetilde{\tau})$, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $n(\omega) \geq 1$ such that $\widetilde{\tau}(\omega) \leq \tau_{n(\omega)}(\omega)$ and $\widetilde{y}(\omega) = y_{n(\omega)}(\omega)$ on $[0, \widetilde{\tau}(\omega)]$. In particular, uniqueness holds for (1.1.6).

For simplicity, we denote the maximal strong solution $((y_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$ by the triple $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$, where $y = \lim_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$ and $\tau^*(x) = \lim_{n\to\infty}\tau_n$. Also, (y, τ^*) is independent of the choice of $((y_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$.

The following theorem establishes the equivalence between two strong solutions of (1.1.1) and (1.1.6) respectively via the rescaling transformation (1.1.5).

Theorem 1.1.4. (i) *Let* (y, τ) *be a strong solution of* (1.1.6) *in the sense of Definition 1.1.3. Define* $X := e^W y$. *Then,* (X, τ) *is a strong solution of* (1.1.1) *in the sense of Definition 1.1.1.*

(ii) *Suppose* (X, τ) *is a strong solution of* (1.1.1) *in the sense of Definition 1.1.1. Define* $y := e^{-W} X$. *Then,* (y, τ) *is a strong solution of* (1.1.6) *in the sense of Definition 1.1.3.*

Before going to the proof of Theorem 1.1.4, a few remarks are in order concerning the formal calculation given at the beginning to link (1.1.1) and (1.1.6). In fact, it is purely heuristic since we applied the Itô product to y though it is not of bounded variation in L^2. Furthermore, taking into account that the exponential is an operator of Nemitsky type in L^2 which is not differentiable, the infinite dimensional Itô formula in L^2 is not justified, see e.g. [73]. Also, when we try to apply Itô's product rule for complex valued stochastic processes after evaluating the L^2-valued processes X, W, y at $\xi \in \mathbb{R}^d$, which by itself is delicate since L^2 consists of equivalence classes of functions, we run into problems since e.g. again $X(t,\xi)$, $y(t,\xi)$, $t \in [0,T]$, might not be semi-martingales. We refer the reader to [6], [57] for relevant treatments in this case. See also the proof of Lemma 2.3.11 in Chapter 2.

The proof we give below avoids these two problems, thanks to the hilbertian structure of $L^2(\mathbb{R}^d)$ space and the stochastic Fubini theorem. (For the stochastic calculus for complex valued processes and their products in $\mathbb{C}$, we refer the reader to [48], Section 2, as background literature in regard to this.)

Proof. We only prove (i), since (ii) can be proved analogously. Let $\varphi \in H^2(\mathbb{R}^d)$. Then, for every $t \in [0, \tau]$, we have

$$\langle \varphi, e^{W(t)} y(t) \rangle_2 = \sum_{j=1}^{\infty} \langle \overline{e^{W(t)}} \varphi, f_j \rangle_2 \langle f_j, y(t) \rangle_2,$$

where $\{f_j\}_{j=1}^{\infty}$ is an orthonormal basis in L^2; $f_j \in H^2(\mathbb{R}^d)$.

By Itô's formula, we have for all $\xi \in \mathbb{R}^d$, $t \in [0,T]$,

$$e^{W(t,\xi)} = 1 + \int_0^t e^{W(s,\xi)} dW(s,\xi) + \widetilde{\mu}(\xi) \int_0^t e^{W(s,\xi)} ds.$$

Fix $j \in \mathbb{N}$. Then, we have $\mathbb{P}$-a.s. for all $t \in [0,T]$,

$$\langle \overline{e^{W(t)}} \varphi, f_j \rangle_2$$

$$=\langle\varphi, f_j\rangle_2+\sum_{k=1}^N\overline{\mu}_k\int_{\mathbb{R}^d}\varphi(\xi)e_k(\xi)\bar{f}_j(\xi)d\xi\int_0^t\overline{e^{W(s,\xi)}}d\beta_k(s)+\int_0^t\langle\overline{\widetilde{\mu}}\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 ds, \tag{1.1.11}$$

where we used the deterministic Fubini theorem in the last term. While, for the stochastic term, since

$$\begin{aligned}&\mathbb{E}\int_0^T\sum_{j=1}^N\Big|\int\overline{\mu}_k\varphi e_k\bar{f}_j\overline{e^{W(s)}}d\xi\Big|^2 ds\\ \leq&\left(\sum_{j=1}^N|\mu_k|^2|e_k|^2_{L^\infty}\right)|\varphi|_2^2|f_j|_2^2\int_0^T\mathbb{E}[|e^{W(s)}|^2_{L^\infty}]ds<\infty,\end{aligned}$$

hence, we can apply stochastic Fubini's theorem and derive from (1.1.11) that

$$\begin{aligned}&\langle\overline{e^{W(t)}}\varphi, f_j\rangle_2\\ =&\langle\varphi, f_j\rangle_2+\sum_{k=1}^N\overline{\mu}_k\int_0^t\langle e_k\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 d\beta_k(s)+\int_0^t\langle\overline{\widetilde{\mu}}\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 ds,\ \ t\in[0,T].\end{aligned}$$

Now, set $J_\varepsilon=(I-\varepsilon\Delta)^{-1}$ and let $y_\varepsilon=J_\varepsilon(y)$. Then, $y_\varepsilon\in C([0,\tau],H^2(\mathbb{R}^d))$ and

$$\begin{aligned}\frac{\partial y_\varepsilon}{\partial t}=&J_\varepsilon(A(t)y)-\lambda i J_\varepsilon(e^{(\alpha-1)ReW}|y|^{\alpha-1}y),\quad t\in[0,\tau],\\ y_\varepsilon(0)=&J_\varepsilon(x)=x_\varepsilon.\end{aligned}\tag{1.1.12}$$

Since $f_j\in H^2(\mathbb{R}^d)$, for each j, $\langle f_j, y_\varepsilon(t)\rangle_2,\ t\in[0,\tau]$, is of bounded variation. Hence, we can apply the Itô product rule (for scalar valued processes) to obtain

$$\begin{aligned}\langle\overline{e^{W(t)}}\varphi, f_j\rangle_2\langle f_j, y_\varepsilon(t)\rangle_2=&\langle\varphi, f_j\rangle_2\langle f_j, x_\varepsilon\rangle_2\\ &+\int_0^t\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\langle f_j, J_\varepsilon(A(s)y(s))\rangle_2 ds\\ &+\lambda i\int_0^t\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\langle f_j, J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2 ds\\ &+\sum_{k=1}^N\overline{\mu}_k\int_0^t\langle f_j, y_\varepsilon(s)\rangle_2\langle e_k\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 d\beta_k(s)\\ &+\int_0^t\langle f_j, y_\varepsilon(s)\rangle_2\langle\overline{\widetilde{\mu}}\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 ds.\end{aligned}\tag{1.1.13}$$

(We note that, since $J_\varepsilon(Ay)\in C([0,\tau];L^2)$, the second integral in the above equality makes sense.)

It is not difficult to sum over $j \in \mathbb{N}$ and interchange the infinite sum with the integrals. Indeed, for the integrals with respect to ds, we take the third term in the right hand side of (1.1.13) for example. By Definition 1.1.3 and Assumption $(H1)$, $J_\varepsilon(e^{(\alpha-1)ReW}|y|^{\alpha-1}y) \in L^1(0,t;L^2)$ with $t \in [0,\tau]$. Then for any $n \in \mathbb{N}$

$$
\begin{aligned}
&\Big|\sum_{j=1}^{n}\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\langle f_j, J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2\Big|\\
\leq&\sqrt{\sum_{j=1}^{n}\Big|\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\Big|^2}\sqrt{\sum_{j=1}^{n}\Big|\langle f_j, J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2\Big|^2}\\
\leq&|e^{W(s)}\varphi|_2\ |J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))|_2\\
\leq&\sup_{s\in[0,t]}|e^{W(s)}|_{L^\infty}\ |\varphi|_2\ |J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))|_2 \in L^1(0,t),
\end{aligned}
$$

which implies that for $t \in [0,\tau]$

$$
\begin{aligned}
&\lambda i\sum_{j=1}^{\infty}\int_0^t\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\langle f_j, J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2 ds\\
=&\lambda i\int_0^t\sum_{j=1}^{\infty}\langle\overline{e^{W(s)}}\varphi, f_j\rangle_2\langle f_j, J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2 ds\\
=&\lambda i\int_0^t\langle\varphi, e^{W(s)}J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2 ds.
\end{aligned}
$$

Similar arguments also apply to the stochastic term in (1.1.13). In fact, set $\tau_M = \inf\{t \in [0,\tau] : |y_\varepsilon(t)|_2^2 > M\} \wedge \tau$. Then, for any $n \in \mathbb{N}$

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_M}\sum_{k=1}^{N}\Big|\sum_{j=1}^{n}\overline{\mu}_k\langle f_j, y_\varepsilon(s)\rangle_2\langle e_k\,\overline{e^{W(s)}}\varphi, f_j\rangle_2\Big|^2 ds\\
\leq&\left(\sum_{k=1}^{N}|\mu_k|^2|e_k|_{L^\infty}^2\right)|\varphi|_2^2\ \mathbb{E}\int_0^{t\wedge\tau_M}|e^{W(s)}|_{L^\infty}^2|y_\varepsilon(s)|_2^2 ds\\
\leq&\left(\sum_{k=1}^{N}|\mu_k|^2|e_k|_{L^\infty}^2\right)|\varphi|_2^2 M^2 t\ \mathbb{E}(\sup_{s\in[0,t]}|e^{W(s)}|_{L^\infty}^2) < \infty,
\end{aligned}
$$

which implies that

$$
\sum_{j=1}^{\infty}\sum_{k=1}^{N}\overline{\mu}_k\int_0^t\langle f_j, y_\varepsilon(s)\rangle_2\langle e_k\,\overline{e^{W(s)}}\varphi, f_j\rangle_2 d\beta_k(s)
$$

$$=\sum_{k=1}^{N}\int_0^t\sum_{j=1}^{\infty}\overline{\mu_k}\langle f_j, y_\varepsilon(s)\rangle_2\langle e_k\overline{e^{W(s)}}\varphi, f_j\rangle_2 d\beta_k(s)$$
$$=\sum_{k=1}^{N}\int_0^t\langle\varphi, \mu_k e_k e^{W(s)}y_\varepsilon(s)\rangle_2 d\beta_k(s) \tag{1.1.14}$$

holds on $\{t\leq\tau_M\}$. Moreover, since $\sup_{t\in[0,\tau]}|y_\varepsilon(t)|_2^2<\infty$, $\mathbb{P}$-a.s, for $\mathbb{P}$-a.e $\omega\in\Omega$, there exists $M(\omega)\in\mathbb{N}$, such that $\tau_M(\omega)=\tau(\omega)$ for all $M\geq M(\omega)$. Thus

$$\bigcup_{M\in\mathbb{N}}\{t\leq\tau_M\}=\{t\leq\tau\},$$

which implies that (1.1.14) holds on $\{t\leq\tau\}$, $\mathbb{P}$-a.s.

Therefore, summing over $j\in\mathbb{N}$ in (1.1.13) and interchanging the infinite sum with the integrals, we conclude that $\mathbb{P}$-a.s., for all $t\in[0,\tau]$,

$$\begin{aligned}&\langle\varphi, e^{W(t)}y_\varepsilon(t)\rangle_2\\ =&\langle\varphi, x_\varepsilon\rangle_2+\int_0^t\langle\varphi, e^{W(s)}J_\varepsilon(A(s)y(s))\rangle_2 ds\\ &+\lambda i\int_0^t\langle\varphi, e^{W(s)}J_\varepsilon(e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s))\rangle_2 ds\\ &+\sum_{k=1}^{N}\int_0^t\langle\varphi, \mu_k e_k e^{W(s)}y_\varepsilon(s)\rangle_2 d\beta_k(s)+\int_0^t\langle\varphi, \widetilde{\mu}e^{W(s)}y_\varepsilon(s)\rangle_2 ds.\end{aligned}$$

On the other hand, since

$$J_\varepsilon(f)\to f \;\; strongly \; in \; H^k, \;\; as \; \varepsilon\to 0, \tag{1.1.15}$$

and

$$|J_\varepsilon(f)|_{H^k}\leq|f|_{H^k}, \tag{1.1.16}$$

where $f\in H^k$ and $k=0,1,2$, we may pass to the limit $\varepsilon\to 0$ in the previous equality to obtain

$$\begin{aligned}\langle\varphi, e^{W(t)}y(t)\rangle_2=&\langle\varphi, x\rangle_2+i\int_0^t {}_{H^2}\langle\varphi, \Delta(e^{W(s)}y(s))\rangle_{H^{-2}}ds-\int_0^t\langle\varphi, \mu e^{W(s)}y(s)\rangle_2 ds\\ &+\lambda i\int_0^t {}_{H^2}\langle\varphi, e^{W(s)}e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s)\rangle_{H^{-2}}ds\end{aligned}$$

$$+\sum_{k=1}^{N}\int_0^t \langle \varphi, \mu_k e_k e^{W(s)} y(s)\rangle_2 d\beta_k(s), \quad t \in [0,\tau],$$

Therefore, set $X(t) := e^{W(t)}y(t)$, $t \in [0,T]$. The above equality implies that (X,τ) is a strong solution of (1.1.1), as claimed. In the above equality, ${}_{H^2}\langle \cdot,\cdot\rangle_{H^{-2}}$ is the pairing between H^2 and H^{-2} or, equivalently,

$${}_{H^2}\langle \varphi, \Delta(e^W y)\rangle_{H^{-2}} = \int_{\mathbb{R}^d} \Delta\varphi \overline{e^W}\,\overline{y}\, d\xi, \quad \varphi \in C_0^2(\mathbb{R}^d;\mathbb{C}).$$

This completes the proof. □

Thanks to Theorem 1.1.4, the well-posedness problem of the stochastic equation (1.1.1) is equivalent to that of the random equation (1.1.6), to which we can apply deterministic methods. This leads to the works in the next section.

1.2 Local existence, uniqueness and blowup alternative

As in the deterministic case, the local well-posedness of (1.1.6) relies on the dispersive properties of its linear part, i.e., $\frac{\partial y}{\partial t} = A(t)y$. Hence, we will first introduce the corresponding evolution operators and Strichartz estimates in Subsection 1.2.1, and we also prove the equivalence between solutions of weak and mild equations in Subsection 1.2.2. Then in Subsection 1.2.3 and Subsection 1.2.4 we establish the local existence, uniqueness and blowup alternative in the subcritical and critical cases respectively.

1.2.1 Evolution operator and Strichartz estimate

Lemma 1.2.1. *Assume $(H1)$. For $\mathbb{P}-a.e.\ \omega$, the operator $A(t)$ defined in* (1.1.7) *generates evolution operators $U(t,s) = U(t,s,\omega)$, $0 \le s \le t \le T$, in the spaces $H^k(\mathbb{R}^d)$, $k \in \mathbb{R}$. Moreover, for each $x \in H^k(\mathbb{R}^d)$, the process $[s,T] \ni t \to U(t,s)x$ is continuous and $(\mathscr{F}_t)$-adapted, hence progressively measurable with respect to the filtration $(\mathscr{F}_t)_{t\ge s}$.*

Proof of Lemma 1.2.1. This lemma is a direct consequence of the work [32]. Precisely, by assumption $(H1)$ one can check that b, c as in (1.1.8) and (1.1.9) satisfy the conditions of Theorem 1.1 in [32]. Thus for every $x \in H^k$ and $f \in L^1(0,T;H^k)$, $k \in \mathbb{R}$, there exists a unique solution $y \in C([s,T];H^k)$ in the sense of distribution to the Cauchy problem

$$\frac{dy}{dt} = A(t)y + f,\ a.e\ t \in (s,T),\qquad (1.2.17)$$
$$y(s) = x,$$

where $\frac{d}{dt}$ is taken in the sense of vectorial $H^{k-2}(\mathbb{R}^d)$-valued distributions on (s,T). This means that $y : [s,T] \to H^{k-2}(\mathbb{R}^d)$ is absolutely continuous and a.e differentiable on (s,T).

Moreover

$$|y(t)|_{H^k} \leq C(|x|_{H^k} + \int_s^t |f(r)|_{H^k} dr),\ s \leq t \leq T. \qquad (1.2.18)$$

These give us the evolution operator $U(t,s) \in L(H^k, H^k)$, $k \in \mathbb{R}$, defined by $U(t,s)x = y(t)$, which indeed solves the homogeneous equation (1.2.17) with $f \equiv 0$, $0 \leq s \leq t \leq T$.

Moreover, the continuity of $[s,T] \ni t \to U(t,s)x$ is due to the property $y \in C([s,T]; H^k)$, and the progressive measurability follows from the fact that the processes $t \to b(t)$ and $t \to c(t)$ are progressively measurable with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ and the solution depends continuously on the coefficients b and c. $\square$

Remark 1.2.2. *The estimate* (1.2.18) *is stated in* (1.5) *in [31], where the coefficients b, c of the lower order terms in A are time independent. However, as the author observed in [32], the proofs work also when they are time dependent.*

Next, we will prove the Strishartz estimates which are fundamental tools in the next two subsections.

Lemma 1.2.3. *Assume $(H1)$. Then for any $T > 0$, $u_0 \in H^1$ and $f \in L^{q_2'}(0,T; W^{1,p_2'})$, the solution of*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds, 0 \leq t \leq T, \qquad (1.2.19)$$

satisfies the estimates

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \leq C_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}). \qquad (1.2.20)$$

Here (p_1, q_1) and (p_2, q_2) are Strichartz pairs, namely

$$(p_i, q_i) \in [2,\infty] \times [2,\infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i},\ if\ d \neq 2,$$

or

$$(p_i, q_i) \in [2, \infty) \times (2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i}, \ if \ d = 2,$$

Furthermore, the process C_t, $t \geq 0$, can be taken to be $(\mathscr{F}_t)$-progressively measurable, increasing and continuous.

Proof. (1.2.20) follows from the results of [59] on Strichartz estimates for the linear Schrödinger operator with nonsmooth and asymptotically flat coefficients. We will prove more than (1.2.20) as follows.

$$\|u\|_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}} \leq C_T(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}}), \tag{1.2.21}$$

where $\widetilde{X}_{[0,T]}$ is the local smoothing space introduced in [59] up to time T (see Notation) and (q_i, p_i), $i = 1, 2$, are Strichartz pairs.

Under Assumption $(H1)$, the conditions $(1.4) - (1.6)$ in [59] on $[0, T] \times \mathbb{R}^d$ are satisfied (see below). Then, by estimate (1.24) in [59] (see Theorem 1.13 and Remark 1.17 in [59]), we have

$$\|u\|_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}} \leq C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}} + \|u\|_{L^2(0,T;L^2(|\xi|\leq 2R))}\right), \tag{1.2.22}$$

for R sufficiently large.

We are going to prove first that (1.2.21) holds for T sufficiently small. To this end, we note that

$$\begin{aligned}\|u\|^2_{L^2(0,T;L^2(|\xi|\leq 2R))} &\leq (m(B_{2R}))^{\frac{p_1-2}{p_1}} \int_0^T |u(t)|^2_{L^{p_1}} dt \\ &\leq (m(B_{2R}))^{\frac{p_1-2}{p_1}} T^{\frac{q_1-2}{q_1}} \|u\|^2_{L^{q_1}(0,T;L^{p_1})} \\ &\leq (m(B_{2R}))^{\frac{p_1-2}{p_1}} T^{\frac{q_1-2}{q_1}} \|u\|^2_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}},\end{aligned}$$

where $m(B_{2R})$ is the volume of the ball B_{2R} of radius $2R$. For simplicity, we assume that $q_1 > 2$, which is the case in the application of Lemma 1.2.3 in later subsections. Then, for

$$0 < T = \left((2C)^{-2}(m(B_{2R}))^{-\frac{p_1-2}{p_1}}\right)^{\frac{q_1}{q_1-2}}, \tag{1.2.23}$$

we get by (1.2.22) that

$$\|u\|_{L^{q_1}(0,T;L^{p_1})\cap \widetilde{X}_{[0,T]}} \leq 2C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}}\right). \tag{1.2.24}$$

For $q_1 = \infty,\ p_1 = 2$, we get in a similar way

$$\|u\|_{L^\infty(0,T;L^2)\cap \widetilde{X}_{[0,T]}} \le 2C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}}\right),$$

for $0 < T < (2C)^{-2}$. Reiterating (1.2.24) on the interval $(T, 2T)$, we get therefore

$$\begin{aligned}\|u\|_{L^{q_1}(T,2T;L^{p_1})\cap \widetilde{X}_{[T,2T]}} &\le 2C\left(|u(T)|_2 + \|f\|_{L^{q_2'}(T,2T;L^{p_2'})+\widetilde{X}'_{[T,2T]}}\right)\\ &\le 2C\left[2C(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+\widetilde{X}'_{[0,T]}}) + \|f\|_{L^{q_2'}(T,2T;L^{p_2'})+\widetilde{X}'_{[T,2T]}}\right]\\ &\le 2C\left[2C|u_0|_2 + 2(2C+1)\|f\|_{L^{q_2'}(0,2T;L^{p_2'})+\widetilde{X}'_{[0,2T]}}\right],\\ &\le 8C(C+1)\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,2T;L^{p_2'})+\widetilde{X}'_{[0,2T]}}\right).\end{aligned}$$

Hence

$$\|u\|_{L^{q_1}(0,2T;L^{p_1})\cap \widetilde{X}_{[0,2T]}} \le 16C(C+1)\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,2T;L^{p_2'})+\widetilde{X}'_{[0,2T]}}\right).$$

Then, after a finite number of steps, we get estimate (1.2.21) on an arbitrary bounded interval, as claimed.

As regards the measurability in (1.2.20), for each $t \in [0, T]$, we may take

$$\begin{aligned}C_t =& \sup\{\|U(\cdot,0)u_0\|_{L^{q_1}(0,t;L^{p_1})};\ |u_0|_2 \le 1\}\\ &+ \sup\left\{\left\|\int_0^{\cdot} U(\cdot,s)f(s)ds\right\|_{L^{q_1}(0,t;L^{p_1})};\ \|f\|_{L^{q_2'}(0,t;L^{p_2'})} = 1\right\}.\end{aligned} \tag{1.2.25}$$

Obviously, the function $t \to C_t$ is monotonically increasing, $C_0 = 0$, and it follows by (1.2.20) and standard arguments that it is continuous (indeed, one can consider countable many u_0 and f, then prove the continuity with the help of Arzela-Ascoli's theorem).

Next, we prove that the process $t \to \|U(\cdot,0)u_0\|_{L^{q_1}(0,t;L^{p_1})}$ is adapted. Indeed, choose $u_n \in H^1$, $n \in \mathbb{N}$, such that $u_n \to u_0$ in L^2. By Lemma 1.2.1, the process $t \to U(t,0)u_n$ is adapted in H^1, which implies from Sobolev's imbedding theorem that $t \to U(t,0)u_n$ is adapted in L^{p_1}. Hence $t \to \|U(\cdot,0)u_n\|_{L^{q_1}(0,t;L^{p_1})}$ is adapted. Now, by Strichartz estimate (1.2.20) we have that

$$\|U(\cdot,0)u_n - U(\cdot,0)u_0\|_{L^{q_1}(0,t;L^{p_1})} \le C_T|u_n - u_0|_2 \to 0,$$

which implies that $t \to \|U(\cdot,0)u_0\|_{L^{q_1}(0,t;L^{p_1})}$ is adapted, as claimed.

Similar arguments apply to the process $t \to \left\| \int_0^\cdot U(\cdot, s) f(s) ds \right\|_{L^{q_1}(0,t;L^{p_1})}$. In fact, let us choose $f_n \in C_c^\infty(0,T) \times C_c^\infty(\mathbb{R}^d)$, such that

$$f_n \to f, \quad in\ L^{q_2'}(0,T; L^{p_2'}).$$

Since $f_n(s) \in H^1$, $s \in [0,t]$, it follows from Lemma 1.2.1 that $t \to \int_0^t U(t,s) f_n(s) ds$ is adapted in H^1, hence also in L^{p_1}, which implies that $t \to \left\| \int_0^\cdot U(\cdot, s) f_n(s) ds \right\|_{L^{q_1}(0,t;L^{p_1})}$ is adapted. Again, using Strichartz estimate (1.2.20), we deduce that

$$\begin{aligned}&\left\| \int_0^\cdot U(\cdot, s) f_n(s) ds - \int_0^\cdot U(\cdot, s) f(s) ds \right\|_{L^{q_1}(0,t;L^{p_1})} \\ \leq & C_T \| f_n - f \|_{L^{q_2'}(0,t;L^{p_2'})} \to 0,\end{aligned}$$

yielding the adaptedness of $t \to \left\| \int_0^\cdot U(\cdot, s) f(s) ds \right\|_{L^{q_1}(0,t;L^{p_1})}$ as claimed.

Now, we notice that, since by separability the sup in (1.2.25) is a sup over countably many $u_0 \in L^2$ and $f \in L^{q_2'}(0,t;L^{p_2'})$, we conclude that $t \to C_t$ is adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. But then, as a continuous process, C_t is $(\mathscr{F}_t)$-progressively measurable, thereby completing the proof. □

In the remaining of this subsection, we show that Assumption $(H1)$ is sufficient for the coefficients b, c in (1.1.8) and (1.1.9) respectively to satisfy the conditions (1.4) – (1.6) in [59] on $[0,T] \times \mathbb{R}^d$.

Let us adapt the notations in [59] $D_t := -i\partial_t$, $D_j := -i\partial_j$, $1 \leq j \leq d$, to rewrite (1.2.19) in the weak equation form

$$D_t u = -\Delta u + \sum_{k=1}^d (D_k \widetilde{b}^k + \widetilde{b}^k D_k + \widetilde{c}) u - if \tag{1.2.26}$$

where

$$\begin{aligned}\widetilde{b}^k =& - i\partial_k W_t \\ =& - i \sum_{m=1}^N \mu_m \partial_k e_m \beta_m(t), \quad 1 \leq k \leq d,\end{aligned} \tag{1.2.27}$$

and

$$\widetilde{c} = - \sum_{k=1}^d (\partial_k W)^2 + (\mu + \widetilde{\mu}) i$$

$$= -\sum_{k=1}^{d}\left(\sum_{m=1}^{N}\mu_m\partial_k e_m\beta_m(t)\right)^2 + \frac{1}{2}i\left[\sum_{m=1}^{N}(|\mu_m|^2+\mu_m^2)e_m^2\right] \tag{1.2.28}$$

Condition (1.4) in [59] is obviously satisfied for Δ. We will prove below that under Assumption $(H1)$ the coefficients $\widetilde{b}$ and $\widetilde{c}$ satisfy

$$\sum_j \sup_{A_j} <\xi> |\widetilde{b}(t,\xi)| \le \kappa_T, \tag{1.2.29}$$

$$\sup_{[0,T]\times\mathbb{R}^d} \zeta(\xi)(|\widetilde{c}(t,\xi)| + |div\widetilde{b}(t,\xi)|) \le \kappa_T, \tag{1.2.30}$$

and

$$\limsup_{|\xi|\to\infty} \zeta(\xi)(|\widetilde{c}(t,\xi)| + |div\widetilde{b}(t,\xi)|) = 0, \tag{1.2.31}$$

which implies conditions (1.5), (1.6) in [59]. Here $<\xi>=\sqrt{1+|\xi|^2}$, κ_T is a constant depending on T, $A_j=[0,T]\times B_j$ with $B_0=\{|\xi|\le 2\}$ and $B_j=\{2^j\le|\xi|\le 2^{j+1}\}$, $j\ge 1$.

Proof. We set for simplicity $|f|_\infty = |f|_{L^\infty}$ for any $f\in L^\infty(\mathbb{R}^d)$. First, for the condition (1.2.29), we notice from (1.2.27) and Assumption $(H1)$ that

$$\begin{aligned}\sum_j \sup_{A_j} <\xi> |\widetilde{b}^k| \le& \sum_{m=1}^{N}|\mu_m||\zeta\partial_k e_m|_\infty \sup_{t\in[0,T]}|\beta_m(t)| \sum_{j\in\mathbb{N}}\sup_{D_j} <\xi>^{-\frac{1}{2}} \\ \le& 2\sum_{m=1}^{N}|\mu_m||\zeta\partial_k e_m|_\infty \sup_{t\in[0,T]}|\beta_m(t)| < \infty,\end{aligned} \tag{1.2.32}$$

where we used $\sum\limits_{j\in\mathbb{N}}\sup\limits_{D_j} <\xi>^{-\frac{1}{2}} \le \sum\limits_{j\in\mathbb{N}} 2^{-j} = 2$. This yields (1.2.29).

Next, for (1.2.30), we have from (1.2.27) that

$$div\widetilde{b} = -i\sum_{m=1}^{N}\mu_m\Delta e_m\beta_m(t),$$

then by Assumption $(H1)$

$$\sup_{[0,T]\times\mathbb{R}^d}\zeta|div\widetilde{b}| \le \sum_{m=1}^{N}|\mu_m||\zeta\Delta e_m|_\infty \sup_{t\in[0,T]}|\beta_m(t)| < \infty. \tag{1.2.33}$$

Moreover, by (1.2.28) and Assumption $(H1)$

$$\sup_{[0,t]\times\mathbb{R}^d} \zeta|\widetilde{c}| \leq \sum_{m=1}^N |\mu_m|^2 \left(\sum_{k=1}^d |\partial_k e_m|_\infty |\zeta \partial_k e_m|_\infty \sup_{t\in[0,T]} |\beta_m(t)|^2 + |e_m|_\infty |\zeta e_m|_\infty \right) < \infty. \tag{1.2.34}$$

Hence, (1.2.30) follows from (1.2.33) and (1.2.34).

(1.2.31) can be proved similarly. Indeed,

$$\begin{aligned}
&\limsup_{|\xi|\to\infty} \zeta(|\widetilde{c}| + |div\widetilde{b}|) \\
\leq& \sum_{m=1}^N \Bigg[|\mu_m||\beta_m(t)| \limsup_{|\xi|\to\infty} |\zeta \Delta e_m| \\
&+ |\mu_m|^2 \left(\sum_{k=1}^d |\partial_k e_m|_\infty |\beta_m(t)|^2 \limsup_{|\xi|\to\infty} |\zeta \partial_k e_m| + |e_m|_\infty \limsup_{|\xi|\to\infty} |\zeta e_m| \right) \Bigg] \\
=&0,
\end{aligned}$$

which yields (1.2.31) and completes the proof. □

1.2.2 Weak and mild equations

This subsection is concerned with the equivalence between solutions of weak and mild equations which will be used in Subsection 1.2.3 and Subsection 1.2.4 below. we refer to [22, 21] and [1] for the semigroup case, e.g. the free Schrödinger group $e^{it\Delta}$. Let us first present the lemma below as a preparation.

Lemma 1.2.4. *Let A be as in* (1.1.7). *Then $A \in C([0,T]; L(H^1, H^{-1}))$ and also $A \in C([0,T]; L(H^{-1}, H^{-3}))$*

Proof. For any $f \in H^1$, it follows from (1.1.7) that

$$|A(t)f|_{H^{-1}} \leq |\Delta f|_{H^{-1}} + |b\cdot\nabla f|_{H^{-1}} + |cf|_{H^{-1}}.$$

Obviously, $|\Delta\varphi|_{H^{-1}} \leq |\varphi|_{H^1}$. Moreover,

$$|b(t)\cdot\nabla f|_{H^{-1}} \leq |b(t)\cdot\nabla f|_2 \leq |b(t)|_{L^\infty}|\nabla f|_2,$$

and

$$|c(t)f|_{H^{-1}} \leq |c(t)f|_2 \leq |c(t)|_{L^\infty}|f|_2.$$

Collecting the results above, we have that

$$|A(t)f|_{H^{-1}} \leq (1 + |b(t)|_{L^\infty} + |c(t)|_{L^\infty})|f|_{H^1},$$

which implies $A \in L(H^1, H^{-1})$. Furthermore, from the expressions of b, c in (1.1.8) and (1.1.9) respectively, it follows that $b \in C([0,t]; L^\infty)$ and $c \in C([0,t]; L^\infty)$. Therefore, we deduce that $A \in C([0,T]; L(H^1, H^{-1}))$. The proof for $A \in C([0,T]; L(H^{-1}, H^{-3}))$ follows from similar arguments above, so we omit it here. $\square$.

Theorem 1.2.5. *Let $x \in L^2$ and $U(t,s)$ be the evolution operators associated with A as in (1.1.7), $0 \leq s \leq t \leq T$. Let $y \in C([0,T]; L^2)$ and g be a complex function such that $g(y) \in L^1(0,T; H^{-1})$. If y satisfies the mild equation*

$$y(t) = U(t,0)x + \int_0^t U(t,s)g(y(s))ds, \quad t \in [0,T], \quad in\ H^{-1}, \tag{1.2.35}$$

then y also satisfies the weak equation

$$y(t) = x + \int_0^t A(s)y(s)ds + \int_0^t g(y(s))ds, \quad t \in [0,T], \quad in\ H^{-2}. \tag{1.2.36}$$

Moreover, the converse is also valid.

Proof. (i). We first prove the first part. Below ${}_{-k}\langle , \rangle_k$ denotes the pair between H^{-k} and H^k, $k \geq 0$.

For any $\varphi \in H^3$, we have from (1.2.35) that

$$\begin{aligned} {}_{-1}\langle y(t), \varphi \rangle_1 &= {}_{-1}\langle U(t,0)x, \varphi \rangle_1 + {}_{-1}\langle \int_0^t U(t,s)g(y(s))ds, \varphi \rangle_1 \\ &= {}_{-1}\langle U(t,0)x, \varphi \rangle_1 + \int_0^t {}_{-1}\langle U(t,s)g(y(s)), \varphi \rangle_1 ds, \end{aligned}$$

then

$${}_{-2}\langle y(t), \varphi \rangle_2 = {}_{-2}\langle U(t,0)x, \varphi \rangle_2 + \int_0^t {}_{-3}\langle U(t,s)g(y(s)), \varphi \rangle_3 ds \tag{1.2.37}$$

Lemma 1.2.1 implies that

$$U(t,0)x = x + \int_0^t A(r)U(r,0)xdr, \ in\ H^{-2}. \tag{1.2.38}$$

Moreover, since $g(y(s)) \in H^{-1}$ for $dt - a.e\ s \in [0,t]$, it follows from Lemma 1.2.1

that

$$U(t,s)g(y(s)) = g(y(s)) + \int_s^t A(r)U(r,s)g(y(s))dr, \ in \ H^{-3}, \ dt - a.e \ s \in [0,t], \tag{1.2.39}$$

implying

$$\begin{aligned} &{}_{-3}\langle \int_0^t U(t,s)g(y(s))ds, \varphi\rangle_3 \\ =&{}_{-3}\langle \int_0^t [g(y(s)) + \int_s^t A(r)U(r,s)g(y(s))dr]ds, \varphi\rangle_3 \\ =&{}_{-3}\langle \int_0^t g(y(s))ds, \varphi\rangle_3 + {}_{-3}\langle \int_0^t (\int_s^t A(r)U(r,s)g(y(s))dr)ds, \varphi\rangle_3. \end{aligned}$$

Since $A \in C([0,T]; L(H^{-1}, H^{-3}))$ and $g(y) \in L^1(0,T;H^{-1})$, we can interchange the integrals for the second term and obtain

$$\begin{aligned} &{}_{-3}\langle \int_0^t U(t,s)g(y(s))ds, \varphi\rangle_3 \\ =&{}_{-3}\langle \int_0^t g(y(s))ds, \varphi\rangle_3 + {}_{-3}\langle \int_0^t (\int_0^r A(r)U(r,s)g(y(s))ds)dr, \varphi\rangle_3 \\ =&{}_{-3}\langle \int_0^t g(y(s))ds, \varphi\rangle_3 + {}_{-3}\langle \int_0^t A(r)(\int_0^r U(r,s)g(y(s))ds)dr, \varphi\rangle_3. \end{aligned} \tag{1.2.40}$$

Therefore, plugging (1.2.38) and (1.2.40) into (1.2.37) and taking into account (1.2.35), we have

$$\begin{aligned} {}_{-2}\langle y(t), \varphi\rangle_2 =&{}_{-2}\langle x, \varphi\rangle_2 + {}_{-2}\langle \int_0^t A(r)[U(r,0)x + \int_0^r U(r,s)g(y(s))ds]dr, \varphi\rangle_2 \\ &+ {}_{-3}\langle \int_0^t g(y(s))ds, \varphi\rangle_3 \\ =&{}_{-2}\langle x, \varphi\rangle_2 + {}_{-2}\langle \int_0^t A(r)y(r)dr, \varphi\rangle_2 + {}_{-3}\langle \int_0^t g(y(s))ds, \varphi\rangle_3, \end{aligned}$$

implying

$$y(t) = x + \int_0^t A(r)y(r)dr + \int_0^t g(y(s))ds, \tag{1.2.41}$$

as an equation in H^{-3}. But $A(t)y(t) \in H^{-2}$ and $g(y) \in L^1(0,t;H^{-1}) \subset L^1(0,t;H^{-2})$, this implies that (1.2.41) holds in H^{-2}, thereby completing the proof of the first part.

(ii). For the converse part, fix $t \in [0,T]$ and define $u(s) = U(t,s)y(s)$ for $s \in (0,t)$. Then for any $h > 0$ such that $s + h < t$

$$\begin{aligned} &u(s+h) - u(s) \\ =&U(t,s+h)[(y(s+h) - y(s)) - (U(s+h,s) - I)y(s)]. \end{aligned} \tag{1.2.42}$$

Dividing (1.2.42) by h, and using (1.2.36) and Lemma 1.2.1 to take the limit $h \to 0^+$, we obtain for dt-a.e. $s \in (0,t)$

$$\begin{aligned} (\frac{d}{ds})^+ u(s) =&U(t,s)[(A(s)y(s) + g(y(s))) - A(s)y(s)] \\ =&U(t,s)g(y(s)),\ in\ H^{-2}. \end{aligned} \tag{1.2.43}$$

Next, we will prove that $s \to u(s)$ is absolutely continuous from $(0,t)$ to H^{-2}. Indeed, (1.2.36) yields

$$\begin{aligned} |y(s+h) - y(s)|_{H^{-2}} =&\left| \int_s^{s+h} A(s)y(s)ds + \int_s^{s+h} g(y(s))ds \right|_{H^{-2}} \\ \le& h\|A\|_{C([0,T];L(L^2;H^{-2}))}\|y\|_{C([0,T];L^2)} + \int_s^{s+h} |g(y(s))|_{H^{-1}}ds. \end{aligned} \tag{1.2.44}$$

Similarly, Lemma 1.2.1 implies that

$$U(s+h,s)y(s) - y(s) = \int_s^{s+h} A(r)U(r,s)y(s)dr, \tag{1.2.45}$$

then

$$\begin{aligned} &|(U(s+h,s) - I)y(s)|_{H^{-2}} \\ \le& h\|A\|_{C([0,T];L(L^2;H^{-2}))} \sup_{r\in[s,s+h]} \|U(r,s)\|_{L(L^2,L^2)}\|y\|_{C([0,T];L^2)}. \end{aligned} \tag{1.2.46}$$

Hence, for any $\varepsilon > 0$ and for any non-intersecting intervals $\{(s_i, s_i + h_i)\}_{i=1}^m \subset (0,t)$ satisfying $\sum_{i=1}^m h_i < \delta$ with δ chosen later, plugging (1.2.44) and (1.2.46) into (1.2.42) with s_i, h_i in place of s, h, we come to

$$\begin{aligned} &\sum_{i=1}^N |u(s_i + h_i) - u(s_i)|_{H^{-2}} \\ \le& \sup_{s\in[0,t]} \|U(t,s)\|_{L(H^{-2},H^{-2})} \Bigg[\delta\|A\|_{C([0,T];L(L^2;H^{-2}))}\|y\|_{C([0,T];L^2)} + \sum_{i=1}^m \int_{s_i}^{s_i+h_i} |g(y(s))|_{H^{-1}}ds \end{aligned}$$

$$+ \delta \|A\|_{C([0,T];L(L^2;H^{-2}))} \sup_{s\le r\in[0,t]} \|U(r,s)\|_{L(L^2,L^2)} \|y\|_{C([0,T];L^2)} \Big],$$

which tends to 0 as $\delta \to 0$.

Therefore, we conclude the absolutely continuity of $u(s)$, $s \in (0,t)$, which implies the existence of $\frac{d}{ds}u(s)$ for dt-a.e $s \in (0,t)$. Then, it follows from (1.2.43) that for dt-a.e $s \in (0,t)$

$$\frac{d}{ds}u(s) = U(t,s)g(y(s)), \ \text{in } H^{-2}. \tag{1.2.47}$$

Integrating (1.2.47) over $(0,t)$ and noting that the right hand side is in H^{-1}, we consequently obtain (1.2.35) and complete the proof of Theorem 1.2.5. □

1.2.3 Subcritical case

In this subsection we establish the local existence, uniqueness and blowup alternative results in the subcritical case.

Theorem 1.2.6. *Assume (H1) and let $1 < \alpha < 1 + \frac{4}{d}$. For each $x \in L^2$ and $0 < T < \infty$, there exists a maximal strong solution $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of (1.1.1) in the sense of Definition 1.1.1. In particular, uniqueness holds for (1.1.1). X also satisfies $\mathbb{P}$-a.s for every $n \geq 1$*

$$X|_{[0,\tau_n]} \in C([0,\tau_n];L^2) \cap L^\gamma(0,\tau_n;L^\rho), \tag{1.2.48}$$

for each Strichartz pair (ρ,γ).

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-a.e ω, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t\to\tau^*(x)(\omega)} |X(t)(\omega)|_2 = \infty.$$

Remark 1.2.7. *The proof of Theorem 1.2.8 below indeed implies that, if $\sup_{t\in[0,\tau^*(x))} |X(t)| < \infty$ $\mathbb{P}-a.s$, then $\tau^*(x) = T$, $\mathbb{P}-a.s$.*

Under Theorem 1.1.4, we will prove this result from the following theorem concerning the random equation (1.1.6).

Theorem 1.2.8. *Assume (H1) and let $1 < \alpha < 1 + \frac{4}{d}$. For each $x \in L^2$ and $0 < T < \infty$, there exists a maximal strong solution $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of (1.1.6) in the sense of Definition 1.1.3. In particular, uniqueness holds for (1.1.6). y also satisfies $\mathbb{P}$-a.s for every $n \geq 1$*

$$y|_{[0,\tau_n]} \in C([0,\tau_n];L^2) \cap L^\gamma(0,\tau_n;L^\rho), \tag{1.2.49}$$

for each Strichartz pair (ρ,γ).

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-a.e ω, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t\to\tau^*(x)(\omega)} |y(t)(\omega)|_2 = \infty.$$

Before the proof we remark that, the advantage to treat (1.1.6) rather than the stochastic equation (1.1.1) lies in the fact that, for $\mathbb{P}$-a.e. $\omega \in \Omega$ we can apply standard fixed point arguments (cf. e.g [58, 22, 46]) to construct a unique local solution and then extend this local solution to a maximal interval $[0,\tau^*(x))$. However, we also emphasize the detail probabilistic proofs for the construction of L^2-valued continuous stochastic process, involving stopping times and adaptedness, which are necessary to apply Itô's formula later to derive the a priori estimates.

Proof. By Theorem 1.2.5, it is equivalent to construct the solution to (1.1.6) in the "mild" sense

$$y(t) = U(t,0)x - \lambda i \int_0^t U(t,s)(e^{(\alpha-1)ReW(s)}g(y(s)))ds,\ t \in [0,T], \tag{1.2.50}$$

where $g(y) = |y|^{\alpha-1}y$. (Note that, we will prove below that $g(y) \in L^{q'}(0,t;L^{p'})$, which implies that $g(y) \in L^1(0,t;H^{-1})$ by Sobolev's imbedding's theorem, hence we can apply Theorem 1.2.5.)

We set $\mathcal{X} = C([0,T];L^2) \cap L^q(0,T;L^{\alpha+1})$ with $q = \frac{4(\alpha+1)}{d(\alpha-1)}$, and consider the integral operator

$$F(y)(t) = U(t,0)x - \lambda i \int_0^t U(t,s)(e^{(\alpha-1)ReW(s)}g(y(s)))ds, t \in [0,T], \tag{1.2.51}$$

defined on $\mathcal{X}$.

Step 1. By Stichartz estimate (1.2.20) with the Strichartz pair $(\alpha+1,q)$, we have for $y \in \mathcal{X}$,

$$\|F(y)\|_{L^q(0,T;L^{\alpha+1})} \le C_T|x|_2 + C_T\|\, e^{(\alpha-1)ReW} g(y)\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})},$$

then using Hölder's inequality we get

$$\begin{aligned}\|e^{(\alpha-1)ReW} g(y)\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})} &\le \gamma_T\, \||y|^\alpha\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})}\\ &\le \gamma_T T^\theta \|y\|^\alpha_{L^q(0,T;L^{\alpha+1})},\end{aligned}$$

where $\gamma_T := exp((\alpha-1)\|W\|_{L^\infty(0,T;L^\infty)})$ and $\theta = 1 - \frac{d(\alpha-1)}{4} > 0$.

Therefore,

$$\|F(y)\|_{L^q(0,T;L^{\alpha+1})} \le C_T \left[|x|_2 + \gamma_T T^\theta \|y\|^\alpha_{L^q(0,T;L^{\alpha+1})} \right]. \tag{1.2.52}$$

Similarly, by (1.2.20) with the Strichartz pairs $(2,\infty)$ and $(\alpha+1,q)$

$$\|F(y)\|_{L^\infty(0,T;L^2)} \le C_T \left[|x|_2 + \gamma_T T^\theta \|y\|^\alpha_{L^q(0,T;L^{\alpha+1})} \right]. \tag{1.2.53}$$

In particular, this implies that $F(\mathcal{X}) \subset \mathcal{X}$.

We note that, in (1.2.52) and (1.2.53) the constant C_T, coming from the Strichartz estimate (1.2.20), depends on $\omega \in \Omega$. However, as mentioned in Lemma 1.2.3, the process $t \to C_t$ is $(\mathscr{F}_t)$-adapted.

Now, Fix $\omega \in \Omega$ and consider the operator F on the set

$$\mathcal{X}^\tau_{M_1} = \left\{ y \in C([0,\tau];L^2) \cap L^q(0,\tau;L^{\alpha+1});\ \sup_{0\le t\le\tau} |y(t)|_2 + \|y\|_{L^q(0,\tau;L^{\alpha+1})} \le M_1 \right\}$$

where $\tau = \tau(\omega) \in (0,T]$ and $M_1 = M_1(\omega) > 0$ are random variables.

For $y \in \mathcal{X}^\tau_{M_1}$, by estimates (1.2.52) and (1.2.53)

$$\|F(y)\|_{L^\infty(0,\tau;L^2)} + \|F(y)\|_{L^q(0,\tau;L^{\alpha+1})} \le 2C_\tau(|x|_2 + \gamma_\tau \tau^\theta M_1^\alpha). \tag{1.2.54}$$

This means that $F(\mathcal{X}^\tau_{M_1}) \subset \mathcal{X}^\tau_{M_1}$, if M_1 and τ are chosen in a such way that

$$2C_\tau(|x|_2 + \alpha\gamma_\tau \tau^\theta M_1^\alpha) \le M_1 \tag{1.2.55}$$

To this end, we choose $M_1 = 3C_\tau|x|_2$ and define the real-valued continuous $(\mathscr{F}_t)$-adapted process

$$Z^{(1)}_t := 2\cdot 3^{\alpha-1}\alpha|x|_2^{\alpha-1} C_t^\alpha \gamma_t t^\theta,\ t \in [0,T].$$

Then (1.2.55) is equivalent to $Z^{(1)}_\tau \le \frac{1}{3}\cdot$ Hence, defining the $(\mathscr{F}_t)$-stopping time

$$\tau_1 = \inf\left\{ t \in [0,T] : Z^{(1)}_t > \frac{1}{3} \right\} \wedge T,$$

we have $\tau_1 > 0$ and $Z^{(1)}_{\tau_1} \le \frac{1}{3}$, hence

$$F(\mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}) \subset \mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}.$$

Now, let us show that F is a contraction in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^{\alpha+1})$. Arguing as in the proof of (1.2.52) and (1.2.53), for $y_1, y_2 \in \mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}$ we get that

$$\begin{aligned}
&\|F(y_1) - F(y_2)\|_{L^q(0,\tau_1;L^{\alpha+1})} + \|F(y_1) - F(y_2)\|_{L^\infty(0,\tau_1;L^2)} \\
\le& 2C_{\tau_1}\gamma_{\tau_1}\||y_1|^{\alpha-1}y_1 - |y_2|^{\alpha-1}y_2\|_{L^{q'}(0,\tau_1;L^{\frac{\alpha+1}{\alpha}})} \\
\le& 2\alpha C_{\tau_1}\gamma_{\tau_1}\tau_1^\theta(\|y_1\|^{\alpha-1}_{L^q(0,\tau_1;L^{\alpha+1})} + \|y_2\|^{\alpha-1}_{L^q(0,\tau_1;L^{\alpha+1})})\|y_1 - y_2\|_{L^q(0,\tau_1;L^{\alpha+1})} \\
\le& 4\alpha C_{\tau_1}\gamma_{\tau_1}\tau_1^\theta M_1^{\alpha-1}\|y_1 - y_2\|_{L^q(0,\tau_1;L^{\alpha+1})} \\
=& 2Z^{(1)}_{\tau_1}\|y_1 - y_2\|_{L^q(0,\tau_1;L^{\alpha+1})} \\
\le& \frac{2}{3}\|y_1 - y_2\|_{L^q(0,\tau_1;L^{\alpha+1})},
\end{aligned} \tag{1.2.56}$$

by definition of τ_1.

This implies that F is a contraction on the space $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^{\alpha+1})$. Hence, by Banach's fixed point theorem, we know that there exists a unique solution $y \in C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^{\alpha+1})$ satisfying $y = F(y)$ on $[0,\tau_1]$, which implies that y solves (1.2.50) on $[0,\tau_1]$.

Moreover, there exists a sequence $u_{1,m} \in \mathcal{X},\ m \in \mathbb{N}$, such that $u_{1,m+1} = F(u_{1,m})$, $m \ge 1$, $u_{1,1}(t) = U(t,0)x$, $t \in [0,T]$, and $\lim_{m\to\infty} u_{1,m}|_{[0,\tau_1]} = y$ in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^{\alpha+1})$. Define $y_1(t) := y(t \wedge \tau_1)$, $t \in [0,T]$. Then

$$y_1 = \lim_{m\to\infty} u_{1,m}(\cdot \wedge \tau_1) \text{ in } C([0,T];L^2).$$

Since, each $u_{1,m}$ is $(\mathscr{F}_t)$-adapted in L^2, so is y_1.

Therefore, we conclude that (y_1,τ_1) is a strong solution of (1.1.6), such that $y_1(t) = y_1(t \wedge \tau_1)$, $t \in [0,T]$, and $y_1|_{[0,\tau_1]} \in C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$.

Step 2. We shall use an induction argument to extend (y_1,τ_1) to a new strong solution (y_{n+1},τ_{n+1}) with $\tau_{n+1} \ge \tau_1$. Suppose that at the n-th step we have a strong solution (y_n,τ_n) of (1.1.6), such that $\tau_n \ge \tau_{n-1}$, $y_n(t) = y_n(t \wedge \tau_n)$, $t \in [0,T]$, and and $y_n|_{[0,\tau_n]} \in C([0,\tau_n];L^2) \cap L^q(0,\tau_n;L^p)$.

Define the integral operator F_n on $\mathcal{X}$

$$F_n(z)(t) = U(\tau_n{+}t,\tau_n)y_n(\tau_n) - \lambda i\int_0^t U(\tau_n{+}t,\tau_n{+}s)(e^{(\alpha-1)ReW(\tau_n+s)}g(z(s)))ds, \tag{1.2.57}$$

where $t \in [0,T]$. Set

$$\mathcal{X}^{\sigma_n}_{M_{n+1}} = \left\{z \in C([0,\sigma_n];L^2) \cap L^q(0,\sigma_n;L^{\alpha+1});\ \sup_{0\le t\le\sigma_n}|z(t)|_2 + \|z\|_{L^q(0,\sigma_n;L^{\alpha+1})} \le M_{n+1}\right\}$$

where $\sigma_n = \sigma_n(\omega)$ and $M_{n+1} = M_{n+1}(\omega)$ are random variables.

Similarly to (1.2.54), we have for every $z \in \mathcal{X}^{\sigma_n}_{M_{n+1}}$,

$$\begin{aligned}&\|F_n(z)\|_{L^\infty(0,\sigma_n;L^2)} + \|F_n(z)\|_{L^q(0,\sigma_n;L^{\alpha+1})}\\ \le& 2C_{\tau_n+\sigma_n}(|y_n(\tau_n)|_2 + \gamma_{\tau_n+\sigma_n}\sigma_n^\theta M_{n+1}^\alpha),\end{aligned} \tag{1.2.58}$$

which implies that $F_n(\mathcal{X}^{\sigma_n}_{M_{n+1}}) \subset \mathcal{X}^{\sigma_n}_{M_{n+1}}$ and F_n is a contraction in $\mathcal{X}^{\sigma_n}_{M_{n+1}}$, if we take $M_{n+1} = 3C_{\tau_n+\sigma_n}|y_n(\tau_n)|_2$ and choose σ_n such that

$$2C_{\tau_n+\sigma_n}(|y_n(\tau_n)|_2 + \alpha\gamma_{\tau_n+\sigma_n}\sigma_n^\theta M_{n+1}^\alpha) \le M_{n+1},$$

i.e.,

$$2\cdot 3^{\alpha-1}\alpha|y_n(\tau_n)|_2^{\alpha-1}C^\alpha_{\tau_n+\sigma_n}\gamma_{\tau_n+\sigma_n}\sigma_n^\theta \le \frac{1}{3}\cdot \tag{1.2.59}$$

So, similarly as above, we define the real-valued continuous $(\mathscr{F}_{\tau_n+t})$-adapted process

$$Z_t^{(n)} := 2\cdot 3^{\alpha-1}\alpha|y_n(\tau_n)|_2^{\alpha-1}C^\alpha_{\tau_n+t}\gamma_{\tau_n+t}t^\theta, \quad t\in[0,T],$$

and

$$\sigma_n := \inf\left\{t\in[0,T-\tau_n] : Z_t^{(n)} > \frac{1}{3}\right\}\wedge(T-\tau_n).$$

Then $\sigma_n > 0$ and $Z^{(n)}_{\sigma_n} \le \frac{1}{3}$, i.e., (1.2.59) holds.

Set $\tau_{n+1} := \tau_n + \sigma_n$. Then τ_{n+1} is an $(\mathscr{F}_t)$-stopping time. Indeed, for $t\in[0,T]$,

$$\{\tau_n+\sigma_n<t\} = \bigcup_{\substack{q_1,q_2\in Q_+\\ q_1+q_2<t}}\{\tau_n<q_1,\sigma_n<q_2\},$$

where Q_+ denotes the nonnegative rational numbers. But, by induction, τ_n is an $(\mathscr{F}_t)$-stopping time and

$$\begin{aligned}\{\tau_n<q_1,\sigma_n<q_2\} &= \bigcup_{\substack{q\in Q_+\\ q<q_2}}\left\{\tau_n+q_2<q_1+q_2, Z_q^{(n)}>\frac{1}{3}\right\}\\ &\in \mathscr{F}_{(\tau_n+q_2)\wedge(q_1+q_2)} \subset \mathscr{F}_{q_1+q_2}\subset\mathscr{F}_t,\end{aligned}$$

since $\left\{Z_q^{(n)} > \frac{1}{3}\right\} \in \mathscr{F}_{\tau_n+q}\subset\mathscr{F}_{\tau_n+q_2}$. Since $(\mathscr{F}_t)$ is right-continuous, τ_{n+1} is thus an $(\mathscr{F}_t)$-stopping time.

Analogously to Step 1, one now shows that, by Banach's fixed point theorem, there exists a unique $z_{n+1} \in \mathcal{X}^{\sigma_n}_{M_{n+1}}$, satisfying $z_{n+1} = F_n(z_{n+1})$. We define

$$y_{n+1}(t) = \begin{cases} y_n(t), & t \in [0, \tau_n], \\ z_{n+1}((t-\tau_n) \wedge \sigma_n), & t \in (\tau_n, T]. \end{cases} \tag{1.2.60}$$

It follows from the definitions of F in Step 1 and F_n that $y_{n+1} = F(y_{n+1})$ on $[0, \tau_{n+1}]$, which implies that y_{n+1} solves (1.2.50) on $[0, \tau_{n+1}]$.

In order to prove that y_{n+1} is adapted to $(\mathscr{F}_t)$ in L^2, we first note that, by $z_{n+1} = F_n(z_{n+1})$ and Banach's fixed point theorem, there exists a sequence $\{v_{n+1,m}\}_{m\ge 1}$, adapted to $(\mathcal{F}_{\tau_n+t})$, satisfying $v_{n+1,m+1} = F_n(v_{n+1,m})$ for $m \ge 1$, $v_{n+1,1}(t) = U(\tau_n + t, \tau_n)y_n(\tau_n)$, $t \in [0, T]$, and $z_{n+1} = \lim_{m\to\infty} v_{n+1,m}|_{[0,t]}$ in $C([0,t]; L^2) \cap L^q(0, t; L^{\alpha+1}))$, $t \in [0, \sigma_n]$. Define

$$u_{n+1,m}(t) = \begin{cases} y_n(t), & t \in [0, \tau_n], \\ v_{n+1,m}(t-\tau_n), & t \in (\tau_n, \infty). \end{cases}$$

Then, $y_{n+1} = \lim_{m\to\infty} u^{\tau_{n+1}}_{n+1,m}$, in $C([0,T]; L^2)$.

Now, we show that $u_{n+1,m}$ is adapted to $(\mathscr{F}_t)$ in L^2. In fact, let f_j, $j \in \mathbb{N}$, be an orthonormal basis of L^2. We have, for each $a > 0$, $\{|\langle u_{n+1,m}(t), f_j\rangle_2| < a\} = J_{1,a} \cup J_{2,a}$, where $J_{1,a} = \{|\langle y_n(t), f_j\rangle_2| < a,\ t \le \tau_n\}$ and $J_{2,a} = \{|\langle v_{n+1,m}(t-\tau_n), f_j\rangle_2| < a,\ \tau_n < t\}$. Since y_n is adapted to $(\mathscr{F}_t)$ and τ_n is an $(\mathscr{F}_t)$-stopping time, it follows that $J_{1,a} \in \mathscr{F}_t$.

By the continuity of $t \mapsto |\langle v_{n+1,m}(t-\tau_n), f_j\rangle_2|$ we see that

$$J_{2,a} = \bigcup_{\substack{q\in Q\\ q<a}} \bigcup_{h\in\mathbb{N}} \bigcap_{s\in Q} J_{q,h,s},$$

where $J_{q,h,s} = \left\{|\langle v_{n+1,m}(s), f_j\rangle_2| < q,\ t-\tau_n-\frac{1}{h} < s < t-\tau_n, \tau_n < t\right\}$. Taking into account that $\{|\langle v_{n+1,m}(s), f_j\rangle_2| < q\} \in \mathscr{F}_{\tau_n+s}$ and $\tau_n + s < t$, we have $J_{q,h,s} \in \mathscr{F}_t$, which implies that $J_{2,a} \in \mathscr{F}_t$.

Collecting the above results, we obtain that, for any $j \in \mathbb{N}$ and $a > 0$, $\{|\langle u_{n+1,m}(t), f_j\rangle_2| < a\} \in \mathscr{F}_t$. This is enough to imply that $u_{n+1,m}$ is adapted to $(\mathscr{F}_t)$ in L^2. Hence, as the limit of $u^{\tau_{n+1}}_{n+1,m}$, y_{n+1} is also adapted to $(\mathscr{F}_t)$ in L^2.

Therefore, we conclude that (y_{n+1}, τ_{n+1}) is a new strong solution of (1.1.6) with $\tau_{n+1} \ge \tau_n$, such that $y_{n+1}(t) = y_{n+1}(t \wedge \tau_{n+1})$, $t \in [0, T]$, and $y_{n+1}|_{[0,\tau_{n+1}]} \in C([0, \tau_{n+1}]; L^2) \cap L^q(0, \tau_{n+1}; L^p)$.

Step 3. By an induction argument, we finally have a sequence of strong solutions (y_n, τ_n), $n \in \mathbb{N}$, with τ_n increasing $(\mathscr{F}_t)$-stopping times and $y_{n+1} = y_n$ on

$[0, \tau_n]$. Defining $\tau^*(x) = \lim\limits_{n\to\infty} \tau_n$ and $y = \lim\limits_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$, we obtain the triple $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$.

(1.2.49) follows from the fact that $y_n|_{[0,\tau_n]} \in C([0,\tau_n]; L^2) \cap L^q(0,\tau_n; L^p)$ and the Strichartz estimate (1.2.20). Indeed, for any Strichartz pair (ρ, γ)

$$\begin{aligned}\|y\|_{L^\gamma(0,\tau_n;L^\rho)} =&\|F(y_n)\|_{L^\gamma(0,\tau_n;L^\rho)} \\ \leq& C_{\tau_n}[|x|_2 + \|\lambda e^{(\alpha-1)ReW} g(y_n)\|_{L^{q'}(0,\tau_n;L^{\frac{\alpha+1}{\alpha}})}] \\ \leq& C_{\tau_n}[|x|_2 + \gamma_{\tau_n}\tau_n^\theta \|y_n\|^\alpha_{L^q(0,\tau_n;L^{\alpha+1})}] < \infty, \quad \mathbb{P}-a.s.\end{aligned}$$

In order to prove that $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ is indeed the maximal strong solution in the sense of Definition 1.1.3, let us first show the uniqueness and blowup alternative given below.

As regards the uniqueness, for another two strong solutions $(\widetilde{y}_i, \sigma_i)$, $i = 1, 2$, as in (1.2.56) we have for any $t, s > 0, s + t < \sigma_1 \wedge \sigma_2$

$$\begin{aligned}&\|\widetilde{y}_1 - \widetilde{y}_2\|_{L^\infty(s,s+t;L^2)} + \|\widetilde{y}_1 - \widetilde{y}_2\|_{L^q(s,s+t;L^{\alpha+1})} \\ \leq& 4\alpha C_T \gamma_T t^\theta M^{\alpha-1} \|\widetilde{y}_1 - \widetilde{y}_2\|_{L^q(s,s+t;L^{\alpha+1})}\end{aligned}$$

with $M = \|\widetilde{y}_1\|_{L^q(0,s+t;L^{\alpha+1})} + \|\widetilde{y}_2\|_{L^q(0,s+t;L^{\alpha+1})} < \infty$ a.s. Thus a properly chosen small t yields $\widetilde{y}_1 = \widetilde{y}_2$ on $[s, s+t]$, implying the uniqueness on $[0, \sigma_1 \wedge \sigma_2)$. Hence $\widetilde{y}_1 = \widetilde{y}_2$ on $[0, \sigma_1 \wedge \sigma_2]$ by the continuity in L^2.

For the proof of the blowup alternative, set $M^* := \sup\limits_{t\in[0,\tau^*(x))} |y(t)|_{L^2}$. Suppose that $\mathbb{P}(M^* < \infty; \tau_n < \tau^*(x), \forall n \in \mathbb{N}) > 0$. Define the real-valued continuous process

$$Z_t := 2 \cdot 3^{\alpha-1}\alpha(M^*)^{\alpha-1} C^\alpha_{T+t}\gamma_{T+t}t^\theta, \ t \in [0, T],$$

and

$$\sigma := \inf\left\{t \in [0, T] : Z_t > \frac{1}{6}\right\} \wedge T.$$

By the assumption above, we have $\sigma_n(\omega) < T - \tau_n(\omega)$ for $\omega \in \{M^* < \infty; \tau_n < \tau^*(x), \forall n \in \mathbb{N}\}$, hence $\sigma_n(\omega) = \inf\{t \in [0, T - \sigma_n(\omega)] : Z^{(n)}_t(\omega) > \frac{1}{3}\}$. On the other hand, since for every $n \geq 1$, $|y(\tau_n)|_2 \leq M^*$, $C_{\tau_n+t} \leq C_{T+t}$ and $\gamma_{\tau_n+t} \leq \gamma_{T+t}$, it follows that $Z_t \geq Z^{(n)}_t$. Hence $\sigma_n(\omega) > \sigma(\omega) > 0$, $\tau_{n+1}(\omega) = \tau_n(\omega) + \sigma_n(\omega) > \tau_n(\omega) + \sigma(\omega)$, which implies that $\tau_{n+1}(\omega) > \tau_1(\omega) + n\sigma(\omega)$, $n \geq 1$. Thus, after finitely many steps, $\tau_n(\omega)$ will exceed T, contradicting the fact that $\tau_n(\omega) \leq T$. Therefore, we conclude the blow-up alternative and finish the proof of Theorem 1.2.8.

Now we show that $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ is the maximal strong solution of (1.1.6). Suppose not, then by the definition and uniqueness there exists a strong solution $(\widetilde{y}, \widetilde{\tau})$ of (1.1.6), such that $\mathbb{P}(\widetilde{\tau} > \tau_n, \forall n \in \mathbb{N}) > 0$. For $\omega \in \{\widetilde{\tau} > \tau_n, \forall n \in \mathbb{N}\}$, it follows that $\sup_{t\in[0,\tau^*(x)(\omega))} |y(\omega)|_2 < \infty$. Hence, by the blowup alternative, there exists $n_0(\omega) \in \mathbb{N}$ such that $\tau_{n_0(\omega)} = \tau^*(x)(\omega)$. As $\tau_{n_0(\omega)} < \widetilde{\tau}(\omega) \leq T$, by the construction procedure in Step 2 one is able to obtain a new solution to (1.1.1) on a larger interval $[0, \tau_{n_0+1}(\omega)]$ containing $[0, \tau_{n_0}(\omega)](= [0, \tau^*(x)(\omega)])$, yielding a contradiction. Therefore, we complete the proof of Theorem 1.2.8. □

Remark 1.2.9. *In theorem 1.2.8, Assumption $(H1)$ on the spatial regularity and decay is due to Lemma 1.2.1 and Lemma 1.2.3, which allow to construct local solutions pathwisely. This in some sense indicates that spatially smoother noise helps to obtain the local well-posedness.*

It shall be mentioned that, the authors in [10] imposed a different regularity assumption on the noise. Precisely, let

$$W(t, \xi) = \sum_{k=0}^{\infty} \beta_k(t) \phi e_k(\xi), \tag{1.2.61}$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d, \mathbb{R})$ (real valued space integrable functions in $\mathbb{R}^d$). Then it was required in [10] that $\phi \in L_2(L^2(\mathbb{R}^d, \mathbb{R}), L^2(\mathbb{R}^d, \mathbb{R})) \cap R(L^2(\mathbb{R}^d, \mathbb{R}), L^{2+\delta}(\mathbb{R}^d))$ for some $\delta > 2(d-1)$. Here $R(L^2(\mathbb{R}^d, \mathbb{R}), L^{2+\delta}(\mathbb{R}^d))$ denotes the radonifying operators from $L^2(\mathbb{R}^d, \mathbb{R})$ to $L^{2+\delta}(\mathbb{R}^d)$, which enables one to control Banach space valued stochastic integrals by the Burkholder type inequality. However, this technical treatment is not applicable to solve (1.1.1) *pathwisely and leads to a restrictive condition on $\alpha : 1 < \alpha < 1 + \frac{2}{d-1}$ if $d \geq 3$. We refer the reader to [10] for more details.*

1.2.4 Critical case

This subsection is concerned with the L^2-critical case $\alpha = 1 + \frac{4}{d}$. Let us first state the local existence, uniqueness and blowup alternative results for the random equation (1.1.6).

Theorem 1.2.10. *Assume $(H1)$ and let $\alpha = 1 + \frac{4}{d}$. Then, for each $x \in L^2$ and $0 < T < \infty$, there exists a maximal strong solution $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of* (1.1.6) *in the sense of Definition 1.1.3. In particular, uniqueness holds for* (1.1.6). *y also satisfies $\mathbb{P}$-a.s for every $n \geq 1$*

$$y|_{[0,\tau_n]} \in C([0, \tau_n]; L^2) \cap L^{2+\frac{4}{d}}(0, \tau_n; L^{2+\frac{4}{d}}). \tag{1.2.62}$$

Moreover, we have the blowup alternative, namely, for $\mathbb{P}-a.e\ \omega$, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\|y(\omega)\|_{L^{2+\frac{4}{d}}(0,\tau^*(x)(\omega);L^{2+\frac{4}{d}})} = \infty.$$

Proof. The proofs below follow the lines in the proof of Theorem 1.2.8, and some similar arguments will be omitted.

Consider the integral operator F as in (1.2.51) on the set (see, e.g., [58], p. 97)

$$G^{\tau_1}_{\widetilde{M}_1} = \{y \in C([0,\tau_1];L^2) \cap L^p(0,\tau_1;L^p);\ \sup_{0\le t\le\tau_1} |y(t) - U(t,0)x|_2 + \|y\|_{L^p(0,\tau_1;L^p)} \le \widetilde{M}_1\},$$

where $p = 2 + \frac{4}{d}$. We have for $y \in G^{\tau_1}_{\widetilde{M}_1}$

$$\sup_{0\le t\le\tau_1} |F(y)(t) - U(t,0)x|_2 + \|F(y)\|_{L^p(0,\tau_1;L^p)} \le \varepsilon_1(\tau_1) + 2C_{\tau_1}\gamma_{\tau_1}\widetilde{M}_1^{\alpha},$$

where $\varepsilon_1(t) := \|U(\cdot,0)x\|_{L^p(0,t;L^p)}$ and $\gamma_t = exp((\alpha-1)\|W\|_{L^\infty(0,t;L^\infty)})$.

Moreover, for $y_1, y_2 \in G^{\tau_1}_{\widetilde{M}_1}$

$$\begin{aligned}&\|F(y_1) - F(y_2)\|_{L^\infty(0,\tau_1;L^2)} + \|F(y_1) - F(y_2)\|_{L^p(0,\tau_1;L^p)}\\ \le& 4\alpha C_{\tau_1}\gamma_{\tau_1}\widetilde{M}_1^{\alpha-1}\|y_1 - y_2\|_{L^p(0,\tau_1;L^p)}.\end{aligned}$$

Notice that $\varepsilon_1(t) \to 0$ as $t \to 0$, and $t \to \varepsilon_1(t)$ is $(\mathscr{F}_t)$-adapted (see the proof of Lemma 1.2.3). Define the continuous $(\mathscr{F}_t)$-adapted process $\widetilde{Z}^{(1)}_t = 2^{2-\alpha}3^{\alpha-1}\alpha C_t\gamma_t\varepsilon_1^{\alpha-1}(t)$, $t \in [0,T]$, set the $(\mathscr{F}_t)$-stopping time

$$\tau_1 = \inf\{t \in [0,T]; \widetilde{Z}^{(1)}_t > \frac{1}{3}\} \wedge T$$

and let $\widetilde{M}_1 = \frac{3}{2}\varepsilon_1(\tau_1)$, then F is a contraction map in $G^{\tau_1}_{\widetilde{M}_1}$. Hence, as in Step 1 in the proof of Theorem 1.2.8, we obtain a strong solution (y_1,τ_1) of (1.1.6) such that $y_1(t) = y_1(t\wedge\tau_1)$, $t \in [0,T]$, and $y_1|_{[0,\tau_1]} \in C([0,\tau_1];L^2) \cap L^{2+\frac{4}{d}}(0,\tau_1;L^{2+\frac{4}{d}})$.

Now, suppose that at the n^{th} step we have a strong solution (y_n,τ_n) of (1.1.6) with $\tau_n \ge \tau_{n-1}$, such that $y_n(t) = y_n(t\wedge\tau_n)$, $t \in [0,T]$, and $y_n|_{[0,\tau_n]} \in C([0,\tau_n];L^2)\cap L^{2+\frac{4}{d}}(0,\tau_n;L^{2+\frac{4}{d}})$.

Consider the operator F_n as in (1.2.57) and set

$$\begin{aligned}G^{\sigma_n}_{\widetilde{M}_{n+1}} =&\{z \in C([0,\sigma_n];L^2) \cap L^p(0,\sigma_n;L^p);\\ &\sup_{0\le t\le\sigma_n} |z(t) - U(\tau_n+t,\tau_n)y_n(\tau_n)|_2 + \|z\|_{L^p(0,\sigma_n;L^p)} \le \widetilde{M}_{n+1}\}.\end{aligned}$$

We have for $z \in G^{\sigma_n}_{\widetilde{M}_{n+1}}$

$$\begin{aligned}&\sup_{0\leq t\leq\sigma_n} |F_n(z)(t) - U(\tau_n + t, \tau_n)y_n(\tau_n)|_2 + \|F_n(z)\|_{L^p(0,\sigma_n;L^p)}\\ \leq&\varepsilon_{n+1}(\sigma_n) + 2C_{\tau_n+\sigma_n}\gamma_{\tau_n+\sigma_n}\widetilde{M}^{\alpha}_{n+1},\end{aligned}$$

where $\varepsilon_{n+1}(t) = \|U(\tau_n + \cdot, \tau_n)y_n(\tau_n)\|_{L^p(0,t;L^p)}$ is $(\mathscr{F}_{\tau_n+t})$-adapted and tends to 0 as $t \to 0$.

Moreover, for $z_1, z_2 \in G^{\sigma_n}_{\widetilde{M}_{n+1}}$

$$\begin{aligned}&\|F_n(z_1) - F_n(z_2)\|_{L^\infty(0,\sigma_n;L^2)} + \|F_n(z_1) - F_n(z_2)\|_{L^p(0,\sigma_n;L^p)}\\ \leq&4\alpha C_{\tau_n+\sigma_n}\gamma_{\tau_n+\sigma_n}\widetilde{M}^{\alpha-1}_{n+1}\|z_1 - z_2\|_{L^p(0,\tau_1;L^p)}.\end{aligned}$$

Similarly, in order to let F_n be a contraction map in $G^{\sigma_n}_{\widetilde{M}_{n+1}}$, we define the continuous $(\mathscr{F}_{\tau_n+t})$-adapted process $\widetilde{Z}^{(n)}_t = 2^{2-\alpha}3^{\alpha-1}\alpha C_{\tau_n+t}\gamma_{\tau_n+t}\varepsilon^{\alpha-1}_{n+1}(t)$, $t \in [0, T]$, set

$$\sigma_n = \inf\{t \in [0, T - \tau_n]; \widetilde{Z}^{(n)}_t > \frac{1}{3}\} \wedge (T - \tau_n),$$

and $\widetilde{M}_{n+1} = \frac{3}{2}\varepsilon_{n+1}(\sigma_n)$. Then Banach's fixed point theorem implies that there exists $z_{n+1} \in G^{\sigma_n}_{\widetilde{M}_{n+1}}$ such that $F_n(z_{n+1}) = z_{n+1}$. Hence, letting $\tau_{n+1} = \tau_n + \sigma_n$, defining y_{n+1} as in (1.2.60), and using the arguments as in Step 2 in the proof of Theorem 1.2.8 we deduce that (y_{n+1}, τ_{n+1}) is a new strong solution of (1.1.6), such that $y_{n+1}(t) = y_{n+1}(t \wedge \tau_{n+1})$, $t \in [0, T]$, and $y_{n+1}|_{[0,\tau_{n+1}]} \in C([0, \tau_{n+1}]; L^2) \cap L^{2+\frac{4}{d}}(0, \tau_{n+1}; L^{2+\frac{4}{d}})$.

Therefore, by induction arguments we finally construct strong solutions $(y_n, \tau_n)_{n\in\mathbb{N}}$, with τ_n increasing stopping times and $y_{n+1} = y_n$ on $[0, \tau_n]$, which give us the triple $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ defined by $\tau^*(x) = \lim_{n\to\infty} \tau_n$ and $y = \lim_{n\to\infty} y_n\mathbb{1}_{[0,\tau^*(x))}$, and (1.2.62) follows immediately. Moreover, in order to prove that $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ is the maximal strong solution in the sense of Definition 1.1.3, we only need to show the uniqueness and blowup alternative.

As regards the uniqueness, given another two solutions $(\widetilde{y}_i, \sigma_i)$, $i = 1, 2$, we define $\varsigma = \sup\{t \in [0, \sigma_1 \wedge \sigma_2] : \widetilde{y}_1 = \widetilde{y}_2 \ on \ [0, t]\}$. Suppose that $\mathbb{P}(\varsigma < \sigma_1 \wedge \sigma_2) > 0$. For $\omega \in \{\varsigma < \sigma_1 \wedge \sigma_2\}$, by the continuity in L^2, $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega)]$. Then for $t > 0$, $\varsigma(\omega) + t < \sigma_1(x)(\omega) \wedge \sigma_2(\omega)$, we have

$$\|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^p(\varsigma(\omega),\varsigma(\omega)+t;L^p)} \leq \alpha C_{\varsigma(\omega)+t}\ \gamma_{\varsigma(\omega)+t}\widetilde{M}(t)\|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^p(\varsigma(\omega),\varsigma(\omega)+t;L^p)},$$

where $\widetilde{M}(t) = \|\widetilde{y}_1(\omega)\|^{\alpha-1}_{L^p(\varsigma(\omega),\varsigma(\omega)+t;L^p)} + \|\widetilde{y}_2(\omega)\|^{\alpha-1}_{L^p(\varsigma(\omega),\varsigma(\omega)+t;L^p)} \to 0$ as $t \to 0$. Hence, a sufficient small t yields $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[\varsigma(\omega), \varsigma(\omega) + t]$ and then $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega) + t]$, contradicting the definition of ς.

For the blowup alternative, suppose that $\mathbb{P}(\|y\|_{L^p(0,\tau^*(x);L^p)} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}) > 0$. For $\omega \in \{\|y\|_{L^p(0,\tau^*(x);L^p)} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}\}$, by definition $\sigma_n(\omega) = \inf\{t \in [0, T - \tau_n(\omega)]; \widetilde{Z}_t^{(n)}(\omega) > \frac{1}{3}\}$. Moreover, $\widetilde{Z}_{\sigma_n(\omega)}^{(n)}(\omega) = \frac{1}{3}$, since $\widetilde{Z}_t^{(n)}(\omega)$ is continuous. On the other hand, for every $n \in \mathbb{N}$ and $t \in [0, \sigma_n]$, by the definition of F_n

$$\begin{aligned}
\varepsilon_{n+1}(t) =& \|U(\tau_n + \cdot, \tau_n) y(\tau_n)\|_{L^p(0,t;L^p)} \\
\leq& \|F_n(z_{n+1})\|_{L^p(0,t;L^p)} + \|\int_0^{\cdot} U(\tau_n + \cdot, \tau_n + s) e^{(\alpha-1)ReW(\tau_n+s)} g(z_{n+1}(s)) ds\|_{L^p(0,t;L^p)} \\
\leq& \|z_{n+1}\|_{L^p(0,t;L^p)} + C_{\tau_{n+1}} \gamma_{\tau_{n+1}} \|z_{n+1}\|^\alpha_{L^p(0,t;L^p)} \\
\leq& \widetilde{M_n^*} + C_T \gamma_T (\widetilde{M_n^*})^\alpha.
\end{aligned} \tag{1.2.63}$$

Since $\widetilde{M_n^*}(\omega) := \|y(\omega)\|_{L^p(\tau_n(\omega), \tau^*(x)(\omega); L^p)} \to 0$, as $n \to \infty$, there exists n large enough such that

$$\widetilde{Z}^{(n)}(\omega) := 2^{2-\alpha} 3^{\alpha-1} \alpha C_T(\omega) \gamma_T(\omega) [\widetilde{M_n^*}(\omega) + C_T(\omega)\gamma_T(\omega)(\widetilde{M_n^*})^\alpha(\omega)]^{\alpha-1} < \frac{1}{6}.$$

But, this implies from (1.2.63) and the definition of $\widetilde{Z}_t^{(n)}$ that for all $t \in [0, \sigma_n(\omega)]$, $\frac{1}{6} > \widetilde{Z}^{(n)}(\omega) > \widetilde{Z}_t^{(n)}(\omega)$, which is a contradiction, since $\widetilde{Z}_{\sigma_n(\omega)}^{(n)}(\omega) = \frac{1}{3}$.

Therefore, we obtain the blowup alternative and complete the proof of Theorem 1.2.10 □

From Theorem 1.2.10 and Theorem 1.1.4, we have the corresponding results for SNLS (1.1.1).

Theorem 1.2.11. *Assume* (H1) *and let* $\alpha = 1 + \frac{4}{d}$. *Then, for each* $x \in L^2$ *and* $0 < T < \infty$, *there exists a maximal strong solution* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *of* (1.1.1) *in the sense of Definition 1.1.1. In particular, uniqueness holds for* (1.1.1). X *also satisfies* $\mathbb{P}$*-a.s for every* $n \geq 1$

$$X|_{[0,\tau_n]} \in C([0, \tau_n]; L^2) \cap L^{2+\frac{4}{d}}(0, \tau_n; L^{2+\frac{4}{d}}). \tag{1.2.64}$$

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$*-a.e* ω, *if* $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, *then*

$$\|X(t)(\omega)\|_{L^{2+\frac{4}{d}}(0,\tau^*(x)(\omega);L^{2+\frac{4}{d}})} = \infty. \tag{1.2.65}$$

Remark 1.2.12. *Similarly to Remark 1.2.7, the proof of Theorem 1.2.10 indeed implies that if* $\|X(t)(\omega)\|_{L^{2+\frac{4}{d}}(0,\tau^*(x)(\omega);L^{2+\frac{4}{d}})} < \infty$ *$\mathbb{P}$-a.s, then* $\tau^*(x) = T$*, $\mathbb{P}$-a.s. But, unlike in the subcritical case, a priori estimate of mass does not imply the global well-posedness in the critical case. In the deterministic case, it was once conjectured that in the defocusing case one has the finite bound of* $\|X(t)\|_{L^{2+\frac{4}{d}}(0,\tau^*(x);L^{2+\frac{4}{d}})}$. *We refer to the interested reader to Section 1.4 for further reviews.*

1.3 Global well-posedness

This section is devoted to the global well-posedness in the subcritical case. We will first derive in Subsection 1.3.1 a priori estimate of the mass, with which we then obtain the global well-posedness in Subsection 1.3.2.

1.3.1 A priori estimate of the mass

From the blowup alternative in Theorem 1.2.6, the proof for the global existence in the subcritical case lies in the a priori estimate of the mass, i.e $\sup\limits_{0\le t<\tau^*(x)} |X(t)|_2 < \infty$, $\mathbb{P}$-a.s., which will be obtained in Theorem 1.3.1 below thanks to the martingale property of $|X(t)|_2^2$.

Theorem 1.3.1. *Let* $x \in L^2$, $\alpha \in (1, 1 + \frac{4}{d}]$ *and* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *be the maximal strong solution of* (1.1.1) *from Theorem 1.2.6 and Theorem 1.2.11 respectively. We have $\mathbb{P}$-a.s for* $0 \le t < \tau^*(x)$

$$|X(t)|_2^2 = |x|_2^2 + 2\sum_{k=1}^{N}\int_0^t\int_{\mathbb{R}^d} Re(\mu_k)e_k|X(s)|^2 d\xi d\beta_k(s), \tag{1.3.66}$$

Moreover

$$\mathbb{E}\left[\sup_{0\le t<\tau^*(x)} |X(t)|_2^2\right] \le \widetilde{C}(T) < \infty. \tag{1.3.67}$$

Remark 1.3.2. *In the deterministic and stochastic conservative cases, the mass is conserved* $|X(t)|_2^2 = |x|_2^2$*, since* $Re\mu_j = 0$, $1 \le j \le N$*. While in the stochastic non-conservative case where* $Re\mu_{j_0} \ne 0$ *for some* $j_0 \in \{1, ..., N\}$*, the mass is no longer a constant, but a martingale depending on time.*

Proof of Theorem 1.3.1. Let $\{f_j\}_{j\ge1} \subset H^2(\mathbb{R}^d)$ be an orthonormal basis in L^2, set $J_\varepsilon = (I - \varepsilon\Delta)^{-1}$ and $h_\varepsilon = J_\varepsilon h$ for any $h \in H^{-2}$. We also set $\phi_k = \mu_k e_k$, $1 \le k \le N$, for simplicity.

From (1.1.4) it follows that $\mathbb{P}$-a.s.

$$X_\varepsilon(t) = x_\varepsilon - \int_0^t [i\Delta X_\varepsilon(s) + (\mu X)_\varepsilon(s) + \lambda i g_\varepsilon(s)]\,ds + \sum_{k=1}^N \int_0^t (X\phi_k)_\varepsilon(s)d\beta_k(s), \qquad t \in [0, \tau_n], \tag{1.3.68}$$

where $g_\varepsilon(s) = J_\varepsilon[|X(s)|^{\alpha-1}X(s)] \in L^2$, $0 \le s < \tau^*(x)$.

Then for every f_j

$$\langle f_j, X_\varepsilon(t)\rangle_2 = \langle f_j, x_\varepsilon\rangle_2 - \langle f_j, \int_0^t [i\Delta X_\varepsilon(s) + (\mu X)_\varepsilon(s) + \lambda i g_\varepsilon(s)]ds\rangle_2 + \langle f_j, \sum_{k=1}^N \int_0^t (X\phi_k)_\varepsilon(s)d\beta_k(s)\rangle_2, \quad t \in [0, \tau_n]. \tag{1.3.69}$$

From (1.2.48) and (1.2.64), it is not difficult to interchange the integrals for the drift term. While for the stochastic integral in (1.3.69), as in the proof of Theorem 1.1.4, we set $\sigma_{n,m} = \inf\{t \in [0, \tau_n] : |X(t)|_2 > m\} \wedge \tau_n$, then by (1.1.16)

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} \sum_{k=1}^N |\langle f_j, (X\phi_k)_\varepsilon(s)\rangle_2|^2 ds \\
\le&\Big(\sum_{k=1}^N |\phi_k|_\infty^2\Big)|f_j|_2^2\,\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} |X(s)|_2^2 ds \\
\le&\Big(\sum_{k=1}^N |\phi_k|_\infty^2\Big)|f_j|_2^2\; m^2 t < \infty.
\end{aligned}$$

Hence, by stochastic Fubini's theorem,

$$\langle f_j, \sum_{k=1}^N \int_0^t (X\phi_k)_\varepsilon(s)d\beta_k(s)\rangle_2 = \sum_{k=1}^N \int_0^t \langle f_j, (X\phi_k)_\varepsilon(s)\rangle_2 d\beta_k(s) \tag{1.3.70}$$

holds on $\{t \le \sigma_{n,m}\}$. But by (1.2.48) and (1.2.64), for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $m(\omega) \in \mathbb{N}$ such that for $m \ge m(\omega)$, $\sigma_{n,m} = \tau_n$. Thus

$$\bigcup_{m\in\mathbb{N}} \{t \le \sigma_{n,m}\} = \{t \le \tau_n\}, \tag{1.3.71}$$

which implies (1.3.70) holds on $\{t \le \tau_n\}$.

Now, we conclude that $\mathbb{P}$-a.s.

$$\begin{aligned}\langle f_j, X_\varepsilon(t)\rangle_2 =&\langle f_j, x_\varepsilon\rangle_2 - \int_0^t \langle f_j, i\Delta X_\varepsilon(s) + (\mu X)_\varepsilon(s) + \lambda i g_\varepsilon(s)\rangle_2 ds \\ &+ \sum_{k=1}^N \int_0^t \langle f_j, (X\phi_k)_\varepsilon(s)\rangle_2 d\beta_k(s), \quad t \in [0, \tau_n]. \end{aligned} \tag{1.3.72}$$

Applying the Itô product rule yields

$$\begin{aligned}|\langle f_j, X_\varepsilon(t)\rangle_2|_2^2 =&|\langle f_j, x_\varepsilon\rangle_2|_2^2 + 2Re\int_0^t \langle X_\varepsilon(s), f_j\rangle_2 d\langle f_j, X_\varepsilon(s)\rangle_2 \\ &+ \langle\langle X_\varepsilon(t), f_j\rangle_2, \langle f_j, X_\varepsilon(t)\rangle_2\rangle \\ =&|\langle f_j, x_\varepsilon\rangle_2|_2^2 + 2Re\int_0^t \langle X_\varepsilon(s), f_j\rangle_2 \langle f_j, -i\Delta X_\varepsilon(s)\rangle_2 ds \\ &+ 2Re\int_0^t \langle X_\varepsilon(s), f_j\rangle_2 \langle f_j, -(\mu X)_\varepsilon(s)\rangle_2 ds \\ &+ 2Re\int_0^t \langle X_\varepsilon(s), f_j\rangle_2 \langle f_j, -\lambda i g_\varepsilon(s)\rangle_2 ds \\ &+ \sum_{k=1}^N \int_0^t |\langle f_j, (X\phi_k)_\varepsilon(s)\rangle_2|^2 ds \\ &+ 2Re\sum_{k=1}^N \int_0^t \langle X_\varepsilon(s), f_j\rangle_2 \langle f_j, (X\phi_k)_\varepsilon(s)\rangle_2 d\beta_k(s), \ t \in [0, \tau_n].\end{aligned}$$

Summing over j and interchanging the infinite sum with the integrals, which can be justified as in the proof of Theorem 1.1.4, we derive that

$$\begin{aligned}|X_\varepsilon(t)|_2^2 =&|x_\varepsilon|_2^2 + 2Re\int_0^t \langle X_\varepsilon(s), -(\mu X)_\varepsilon(s)\rangle_2 ds \\ &+ 2Re\int_0^t \langle X_\varepsilon(s), -\lambda i g_\varepsilon(s)\rangle_2 ds + \sum_{k=1}^N \int_0^t |(X\phi_k)_\varepsilon(s)|_2^2 ds \\ &+ 2Re\sum_{k=1}^N \int_0^t \langle X_\varepsilon(s), (X\phi_k)_\varepsilon(s)\rangle_2 d\beta_k(s), \quad t \in [0, \tau_n]. \end{aligned} \tag{1.3.73}$$

Therefore, since for $f \in L^p$, $p \in (1, \infty)$, as $\varepsilon \to 0$

$$|J_\varepsilon(f)|_{L^p} \le |f|_{L^p} \ \ and \ \ J_\varepsilon(f) \to f, \ in \ L^p,$$

we can pass to the limit $\varepsilon \to 0$ in (1.3.73). Indeed, we take the stochastic integral in

(1.3.73) for example. Since as $\varepsilon \to 0$, $\langle X_\varepsilon(s), (X\phi_k)_\varepsilon(s)\rangle_2 \to \langle X(s), X(s)\phi_k\rangle_2$, and

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} \sum_{k=1}^N \left(Re\langle X_\varepsilon(s), (X\phi_k)_\varepsilon(s)\rangle_2\right)^2 ds \\
\leq &(\sum_{k=1}^N |\phi_k|^2_{L^\infty})\mathbb{E}\int_0^{t\wedge\sigma_{n,m}} |X(s)|_2^4 ds \\
\leq &(\sum_{k=1}^N |\phi_k|^2_{L^\infty}) m^4 t < \infty,
\end{aligned}$$

hence

$$2Re\sum_{k=1}^N \int_0^t \langle X_\varepsilon(s), (X\phi_k)_\varepsilon(s)\rangle_2 d\beta_k(s) \to 2Re\sum_{k=1}^N \int_0^t \langle X(s), X(s)\phi_k\rangle_2 d\beta_k(s), \tag{1.3.74}$$

in probability on $\{t \leq \sigma_{n,m}\}$, which implies by (1.3.71) that (1.3.74) holds on $\{t \leq \tau_n\}$.

After taking the limit in (1.3.73) we notice that the second term cancels with the fourth term and the third term tends to 0. Consequently we obtain $\mathbb{P}$-a.s. (1.3.66) on $\{t \leq \tau_n\}$, which implies that (1.3.66) holds on $\{t \leq \tau^*(x)\}$ as $\tau_n \to \tau^*(x)$, $\mathbb{P}$-a.s.

Now, in order to get a priori estimate (1.3.67), taking into account that $\sum_{j=1}^N |\mu_j|^2 |e_j|^2_{L^\infty} < \infty$, by the Burkholder–Davis–Gundy and Young's inequality, we have for $t \in [0, T]$ and all $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} \left|\sum_{j=1}^N \int_0^s \int_{\mathbb{R}^d} \mathrm{Re}(\mu_j) e_j |X(r)|^2 d\xi\, d\beta_j(r)\right|\right] \\
\leq &C\mathbb{E}\left[\int_0^{t\wedge\tau_n} \sum_{j=1}^N \left(\int_{\mathbb{R}^d} \mathrm{Re}(\mu_j) e_j |X(s)|^2 d\xi\right)^2 ds\right]^{\frac{1}{2}} \\
\leq &C\mathbb{E}\left[\int_0^{t\wedge\tau_n} |X(s)|_2^4 ds\right]^{\frac{1}{2}} \\
\leq &C\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} |X(s)|_2 \left(\int_0^{t\wedge\tau_n} |X(s)|_2^2 ds\right)^{\frac{1}{2}}\right] \\
\leq &C\sqrt{\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]} |X(s)|_2^2}\sqrt{\mathbb{E}\int_0^{t\wedge\tau_n} |X(s)|_2^2 ds}
\end{aligned}$$

$$\leq \frac{1}{4}\,\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}|X(s)|_2^2 + C\int_0^t \mathbb{E}\left(\sup_{r\in[0,s\wedge\tau_n]}|X(r)|_2^2\right)ds,$$

where C is a constant independent of n and may change from line to line. Together with (1.3.66), this yields

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]}|X(s)|_2^2\right] \leq 2|x|_2^2 + 4C\int_0^t \mathbb{E}\left(\sup_{r\in[0,s\wedge\tau_n]}|X(r)|_2^2\right)ds,$$

which implies

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_n]}|X(t)|_2^2\right] \leq \widetilde{C}(T),$$

where $\widetilde{C}(T)$ is independent of n.

Finally, taking $n \uparrow \infty$ and applying Fatou's lemma, we obtain (1.3.67), as claimed. □

1.3.2 Subcritical case

We will show the global well-posedness for SNLS (1.1.1) in the subcritical case $\alpha \in (1, 1+\frac{4}{d})$ (see Theorem 1.3.4 below). As before, we first present the global well-posedness for the random equation (1.1.6) as follows.

Theorem 1.3.3. *Assume (H1). Let $1 < \alpha < 1+\frac{4}{d}$. For each $x \in L^2$ and $0 < T < \infty$ there exists a unique strong solution (y, T) of (1.1.6) in the sense of Definition 1.1.3, which satisfies*

$$e^W y \in L^2(\Omega; C([0,T]; L^2)) \tag{1.3.75}$$

$$y \in L^\gamma(0,T; L^\rho),\ \mathbb{P}-a.s., \tag{1.3.76}$$

where (ρ,γ) is any Strichartz pair.

Moreover, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the mapping $x \to y(\cdot, x, \omega)$ is continuous from L^2 to $C([0,T]; L^2) \cap L^\gamma(0,T; L^\rho)$,

Proof. Let $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal solution of (1.1.6) from Theorem 1.2.8. We also recall $(y_n)_{n\in\mathbb{N}}$ in the proof of Theorem 1.2.8. Note that, by Theorem 1.3.1, we have $\mathbb{P}-a.s.$

$$\sup_{0\leq s<\tau^*(x)}|e^{W(s)}y(s)|_2^2 < \infty,$$

which yields $\sup_{0\leq t<\tau^*(x)}|y(t)|_2^2 < \infty$, $\mathbb{P}$-a.s, since $\|W\|_{L^\infty(0,T;L^\infty)} < \infty$. Then it follows from the blowup alternative in Theorem 1.2.8 that $\tau^*(x) = T$ $\mathbb{P}$-a.s. (see also

Remark 1.2.7). Therefore, we can modify the definition of y in Theorem 1.2.8 by $y := \lim_{n\to\infty} y_n$ and deduce that (y,T) is the desired unique strong solution of (1.1.6) in the sense of Definition 1.1.3. Moreover, (1.3.75), (1.3.76) follow from (1.3.67) and (1.2.49) respectively.

It remains to prove the continuous dependence with respect to the initial data $x \in L^2$. Suppose that $x_m \to x$ in L^2 as $m \to \infty$. We have $\mathbb{P}$-a.s. for every $m \geq 1$ a unique strong solution (y_m, T) of equation (1.1.6) with the initial data x_m.

Since $|x_m|_2 \leq |x|_2 + 1$ for all $m \geq m_1$ with m_1 large enough, we can modify the stopping time $\tau_1(\leq T)$ in Step 1 in the proof of Theorem 1.2.8 such that

$$\tau_1 = \inf\{t \in [0,T], 2 \cdot 3^{\alpha-1}\alpha(|x|_2+1)^{\alpha-1}C_t^\alpha \gamma_t t^\theta > \frac{1}{3}\} \wedge T,$$

which is independent for all $m \geq m_1$. Hence, using similar contraction arguments as in Step 1 in the proof of Theorem 1.2.8 and the uniqueness, we deduce that

$$\widetilde{M}_1 := \sup_{m\geq m_1} [\|y_m\|_{L^\infty(0,\tau_1;L^2)} + \|y_m\|_{L^q(0,\tau_1;L^{\alpha+1})}] \leq 3C_{\tau_1}(|x|_2+1).$$

Note that for $m \geq m_1$

$$\begin{aligned}&\|y_m - y\|_{L^\infty(0,\tau_1;L^2)} + \|y_m - y\|_{L^q(0,\tau_1;L^{\alpha+1})} \\ \leq& 2C_T|x_m - x|_2 + 4\alpha C_{\tau_1}\gamma_{\tau_1}\tau_1^\theta \widetilde{M}_1^{\alpha-1}\|y_m - y\|_{L^q(0,\tau_1;L^{\alpha+1})},\end{aligned}$$

where $\theta = 1 - \frac{d(\alpha-1)}{4} > 0$. By the choice of τ_1 and the bound on $\widetilde{M}_1$, we have

$$4\alpha C_{\tau_1}\gamma_{\tau_1}\tau_1^\theta \widetilde{M}_1^{\alpha-1} \leq \frac{2}{3},$$

hence

$$\|y_m - y\|_{L^\infty(0,\tau_1;L^2)} + \frac{1}{3}\|y_m - y\|_{L^q(0,\tau_1;L^{\alpha+1})} \leq 2C_T|x_m - x|_2 \to 0, \ \text{as } m \to \infty.$$

Moreover, using Lemma 1.2.3 with the Strichartz pairs (ρ,γ) and $(\alpha+1,q)$, we have

$$\begin{aligned}\|y_m - y\|_{L^\gamma(0,\tau_1;L^\rho)} =& \|F(y_m) - F(y)\|_{L^\gamma(0,\tau_1;L^\rho)} \\ \leq& C_{\tau_1}|x_m - x|_2 + C_{\tau_1}\gamma_{\tau_1}\||y_m|^{\alpha-1}y_m - |y|^{\alpha-1}y\|_{L^{q'}(0,\tau_1;L^{\frac{\alpha+1}{\alpha}})} \\ \leq& C_{\tau_1}|x_m - x|_2 + 2\alpha C_{\tau_1}\gamma_{\tau_1}\widetilde{M}_1^{\alpha-1}\tau_1^\theta\|y_m - y\|_{L^q(0,\tau_1;L^{\alpha+1})} \to 0.\end{aligned}$$

Thus we obtain the continuous dependence on the interval $[0,\tau_1]$. Now, since $y_m(\tau_1) \to y(\tau_1)$, using similar arguments as above we can extend the above results to $[0,\tau_2]$ with τ_2 depending on $|y(\tau_1)|_2$ and $\tau_1 \le \tau_2 \le T$. Reiterating the arguments, we then have an increasing sequence of stopping times τ_n depending on $|y(\tau_{n-1})|_2$, such that the continuous dependence holds on $[0,\tau_n]$, $n \in \mathbb{N}$. Since $\sup_{t\in[0,T]} |y(t)|_2 < \infty$, $\mathbb{P}$-a.s, as in the proof for the blowup alternative in Theorem 1.2.8, we deduce that for $\mathbb{P}$-a.e. ω there exists $n(\omega) < \infty$ such that $\tau_{n(\omega)}(\omega) = T$. Therefore, we obtain the continuous dependence on $[0,T]$ and consequently complete the proof of Theorem 1.3.3. □

As a consequence of Theorem 1.3.3, Theorem 1.1.4 and Theorem 1.3.1, we obtain the global well-posedness of SNLS (1.1.1) in the subcritical case.

Theorem 1.3.4. *Assume $(H1)$. Let $1 < \alpha < 1 + \frac{4}{d}$, $1 \le d < \infty$. Then, for each $x \in L^2$ and $0 < T < \infty$, there exists a unique strong solution (X,T) of (1.1.1) in the sense of Definition 1.1.1, which satisfies*

$$X \in L^2(\Omega; C([0,T]; L^2)) \tag{1.3.77}$$

$$X \in L^\gamma(0,T; L^\rho),\ \mathbb{P}-a.s., \tag{1.3.78}$$

where (ρ,γ) is any Strichartz pair.

Moreover, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $x \to X(\cdot,x,\omega)$ is continuous from L^2 to $C([0,T];L^2) \cap L^\gamma(0,T;L^\rho)$, and $t \to |X(t)|_2^2$ is a continuous martingale with the representation

$$|X(t)|_2^2 = |x|_2^2 + 2\sum_{k=1}^{N} \int_0^t \int_{\mathbb{R}^d} \mathrm{Re}(\mu_k) e_k |X(s)|^2 d\xi\, d\beta_k(s),\ t \in [0,T]. \tag{1.3.79}$$

Remark 1.3.5. *Theorem 1.3.4 implies that, given the initial data $x \in L^2$, SNLS (1.1.1) generates a global stochastic flow in $L^2(\mathbb{R}^d)$ in the subcritical case $\alpha \in (1, \frac{4}{d})$, $d \ge 1$.*

Remark 1.3.6. *For the case $N = \infty$, besides the assumption $(H1)$ we need the further assumption $(\widetilde{H1})$ below*

$(\widetilde{H1})$

$$\sum_{k=1}^{\infty} |\mu_k|^2 |e_k|_{L^\infty}^2 < \infty \tag{1.3.80}$$

and for any multi-index γ, $|\gamma| \geq 0$

$$\sum_{k=1}^{\infty} |\mu_k||\partial^\gamma e_k|_{L^\infty} < \infty. \tag{1.3.81}$$

Here (1.3.80) *suffices to the justifications of Fubini's theorem, summation and taking limits in the previous proofs in this chapter. While* (1.3.81) *is assumed for the smoothness of b, c in* (1.1.8) *and* (1.1.9) *respectively, so as to satisfy the conditions in [32] and [59]. Hence, under Assumptions* $(H1)$ *and* $(\widetilde{H}1)$ *the results in this chapter remain valid for the case $N = \infty$.*

1.4 Notes

For the theory of stochastic partial differential equations (SPDE), we refer the reader to the standard references [73], [56] and [24].

In the finite dimension case, the rescaling transformation (1.1.5) is well-known to reduce SDE to random ODE, see e.g. p.79 in [71]. While in the infinite dimension case, the equivalence between SPDE and random PDE related by the rescaling transformation is more delicate. A first rigorous proof of this equivalence was given in [4] for the study of stochastic porous media equations (see also [5] and [6]). The rescaling transformation was also used by A. de Bouard and R. Fukuizumi for stochastic nonlinear Schrödinger equations (however, only for one-dimension purely imaginary noise) in the paper [15], of which we were not aware. But the equivalence is not proved there (but only justified by informal computations).

The Strichartz estimate is one of the most stable ways of measuring dispersion. For the free Schrödinger group $e^{it\Delta}$, this estimate is first obtained in [83] as a Fourier restriction theorem. Later on, it was generalized to the homogeneous case by J. Ginibre and G. Velo [39] and to the inhomogeneous case by K. Yajima [96], T. Cazenave and F. B. Weissler [18]. The endpoint estimates were established by M. Keel and T. Tao [47]. A comprehensive review of these basic results is presented in [46, 84, 22, 58]. Since Strichartz estimate is an essential tool for the well-posedness of nonlinear Schrödinger equation, the extension of such estimates to more general Schrödinger operators is extensively studied in the literature. For the Strichartz estimates and decay estimates in the case $-\Delta + V$, we refer the reader to [44, 80, 77]. See also [33] for Strichartz estimates in the case $-\Delta + i(A.\nabla + \nabla.A) + V$. For the variable coefficients case, including the lower order perturbations of the Laplacian, see [78, 82] for the local in time Strichartz estimates. We also refer the interested reader to [88, 59] for the global in time Strichartz estimates and local smoothing estimates. The latter estimates are very useful in establishing the local well-posedness for quasilinear Schrödinger equations (see [60, 61] and [58]). We also refer the reader to [42] for the Strichartz estimate in the flat radial torus $\mathbb{T}^3$.

In the L^2-subcritical case, the global well-posedness of NLS in $L^2(\mathbb{R}^d)$ was first obtained by Y. Tsutsumi [90]. The proofs presented in this chapter adapt the simplified fixed point arguments in [46] and [58]. The interested reader are also referred to [22] for more general nonlinear terms, including the non-local nonlinearity. For the stochastic case, the global existence and uniqueness results were first proved in the conservative case by A. de Bouard and A. Debussche [10] under the restrictive condition $1 < \alpha < 1 + \frac{2}{d-1}$ if $d \geq 3$. For the general case and shaper global well-posedness result, we refer the reader to our recent paper [7]. See also Remark 1.2.9 for the discussion on spatial regularity assumptions on the noise.

In the L^2-critical case, the local well-posedness can be found in [19], including also the global well-posedness with small initial data. While, the case for the large initial data is much more difficult. It was conjectured that, in the defocusing case ($\lambda = -1$) NLS is globally well posed and solutions obey global spacetime bounds in (1.2.65), in particular, scattering holds. But, in the focusing case ($\lambda = 1$), the same conclusions hold for initial data with mass less than a threshold, characterized by the ground state. In the defocusing case ($\lambda = -1$), this conjecture has been affirmed in [86, 87, 51] for radial data when $d \geq 2$ and in [28, 29, 30] for non-radial data and all dimensions. In the focusing case ($\lambda = 1$), see [51, 55, 28, 29, 30]. We also refer the reader to [53] for detail presentations and the references therein.

Chapter 2

The well-posedness in $H^1(\mathbb{R}^d)$

In this chapter, we continue to study the well-posedness of the stochastic nonlinear Schrödinger equation (0.0.1) in the energy space $H^1(\mathbb{R}^d)$. As regards the structure of Chapter 2, we first present preliminaries concerning the definition of solutions and the rescaling transformation approach in Section 2.1. Then in Section 2.2 we establish the local existence, uniqueness and blowup alternative results in both subcritical and critical cases. Later on, in Section 2.3 we obtain the global well-posedness in the subcritical case. Some comments on relevant results in the literature are also given in Section 2.4.

2.1 Preliminaries

We come back to the stochastic nonlinear Schrödinger equaiton (SNLS)

$$\begin{aligned}
&idX(t,\xi) = \Delta X(t,\xi)dt + \lambda|X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\
&\qquad\qquad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \qquad (2.1.1)\\
&X(0) = x,
\end{aligned}$$

where $\lambda = -1$ (defocusing) or $\lambda = 1$ (focusing), $\alpha > 1$. $W(t,\xi)$ and $\mu(\xi)$ are as in (1.1.2) and (1.1.3) with $N < \infty$ for simplicity. While, due to the technical reasons in deriving the Schtricahrtz estimates in Sobolev spaces (see Subsection 2.2.1), we now work under the following spatial decay assumptions of $\{e_j\}_{j=1}^N$ in the colored Brownian motion $W(t,\xi)$

(H2) $e_j \in C_b^\infty(\mathbb{R}^d)$ *such that*

$$\lim_{|\xi|\to\infty} \zeta(\xi)|\partial^\gamma e_j(\xi)| = 0,$$

where γ is multi-index such that $|\gamma| \leq 3$, $1 \leq j \leq N$ and $\zeta(\xi)$ is as in Assumption $(H1)$.

As in Chapter 1, the assumption $\lim_{|\xi|\to\infty} \zeta(\xi)|e_j(\xi)| = 0$ can be removed (see Remark 3.1.1 in Chapter 3).

In this chapter, the well-posedness of (2.1.1) is studied in the context of energy space $H^1(\mathbb{R}^d)$, and the solutions of (2.1.1) are taken analogously as in Definition 1.1.1.

Definition 2.1.1. *Let $x \in H^1$, $T > 0$ and α satisfy*

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1, 2; \\ 1 < \alpha \leq 1 + \frac{4}{d-2}, & \text{if } d \geq 3. \end{cases} \tag{2.1.2}$$

(i). A strong solution of (2.1.1) is a pair (X, τ) with $\tau(\leq T)$ an $(\mathscr{F}_t)$-stopping time, such that $X = (X(t))_{t\in[0,T]}$ is an H^1-valued continuous $(\mathscr{F}_t)$-adapted process, $|X|^{\alpha-1}X \in L^1(0, \tau; H^{-1}(\mathbb{R}^d))$ $\mathbb{P}-a.s$, and X satisfies $\mathbb{P}-a.s$ for $t \in [0, \tau]$

$$X(t) = x - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i|X(s)|^{\alpha-1}X(s))ds + \int_0^t X(s)dW(s), \tag{2.1.3}$$

as an equation in $H^{-1}(\mathbb{R}^d)$.

(ii). We say that uniqueness holds for (2.1.1), if for any two strong solutions (X_i, τ_i), $i = 1, 2$, it holds $\mathbb{P}$-a.s. that $X_1 = X_2$ on $[0, \tau_1 \wedge \tau_2]$.

(iii). A maximal strong solution of (2.1.1) is a pair $((X_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$, where (X_n, τ_n), $n \in \mathbb{N}$, are strong solutions of (2.1.1) with $(\tau_n)_{n\in\mathbb{N}}$ a sequence of increasing stopping times and $X_{n+1} = X_n$ on $[0, \tau_n]$, and " maximal" means that given any strong solution $(\widetilde{X}, \widetilde{\tau})$, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $n(\omega) \geq 1$ such that $\widetilde{\tau}(\omega) \leq \tau_{n(\omega)}(\omega)$ and $\widetilde{X}(\omega) = X_{n(\omega)}(\omega)$ on $[0, \widetilde{\tau}(\omega)]$. In particular, uniqueness holds for (2.1.1).

For simplicity, we denote the maximal strong solution by the triple $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$, where $X = \lim_{n\to\infty} X_n \mathbb{1}_{[0,\tau^*(x))}$ with $\tau^*(x) = \lim_{n\to\infty} \tau_n$.

Notice that, the pair $(X, \tau^*(x))$ is independent of the choice of $((X_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}})$.

As in Remark 1.1.2, $\int_0^t X(s)dW(s)$ in Definition 2.1.1 is an H^1-valued stochastic integral.

Again, we apply the rescaling transformation

$$X(t, \xi) = e^{W(t,\xi)}y(t, \xi) \tag{2.1.4}$$

to reduce the stochastic equation (2.1.1) to the random equation

$$\frac{\partial y(t,\xi)}{\partial t} = A(t)y(t,\xi) - \lambda i e^{(\alpha-1)ReW(t,\xi)}|y(t,\xi)|^{\alpha-1}y(t,\xi), \tag{2.1.5}$$
$$y(0) = x,$$

where A is as in (1.1.7). The solutions of (2.1.5) are taken in the following sense.

Definition 2.1.2. *Let x, α, T be as in Definition 2.1.1.*

(i). A strong solution of (2.1.5) is a pair (y,τ) with $\tau(\leq T)$ an $(\mathscr{F}_t)$-stopping time, such that $y = (y(t))_{t\in[0,T]}$ is an H^1-valued continuous $(\mathscr{F}_t)$-adapted process, $|y|^{\alpha-1}y \in L^1(0,\tau;H^{-1}(\mathbb{R}^d))$ $\mathbb{P}-a.s$, and y satisfies $\mathbb{P}$-a.s for $t\in[0,\tau]$

$$y(t) = x + \int_0^t A(s)y(s)ds - \int_0^t \lambda i e^{(\alpha-1)ReW(s)}|y(s)|^{\alpha-1}y(s)ds, \tag{2.1.6}$$

as an equation in $H^{-1}(\mathbb{R}^d)$.

(ii). The uniqueness of (2.1.5) means that, given any two strong solutions (y_i,τ_i), $i=1,2$, it holds $\mathbb{P}$-a.s. that $y_1 = y_2$ on $[0,\tau_1\wedge\tau_2]$.

(iii). A maximal strong solution of (2.1.5) is a pair $((y_n)_{n\in\mathbb{N}},(\tau_n)_{n\in\mathbb{N}})$, where (y_n,τ_n), $n\in\mathbb{N}$, are strong solutions to (2.1.5) with $(\tau_n)_{n\in\mathbb{N}}$ a sequence of increasing stopping times and $y_{n+1} = y_n$ on $[0,\tau_n]$, and " maximal" means that given any strong solution $(\widetilde{y},\widetilde{\tau})$, we have for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists $n(\omega)\geq 1$ such that $\widetilde{\tau}(\omega)\leq\tau_{n(\omega)}(\omega)$ and $\widetilde{y}(\omega) = y_{n(\omega)}(\omega)$ on $[0,\widetilde{\tau}(\omega)]$.

For simplicity, $(y,(\tau_n)_{n\in\mathbb{N}},\tau^*(x))$ denotes the maximal strong solution $((y_n)_{n\in\mathbb{N}},(\tau_n)_{n\in\mathbb{N}})$, where $y := \lim_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$ and $\tau^*(x) := \lim_{n\to\infty}\tau_n$ are independent of the choice of $((y_n)_{n\in\mathbb{N}},(\tau_n)_{n\in\mathbb{N}})$.

As in Theorem 1.1.4, we also have in the H^1 case the equivalence between two strong solutions of (2.1.1) and (2.1.5) respectively via the rescaling transformation (1.1.5).

Theorem 2.1.3. (i) *Let (y,τ) be a strong solution of (2.1.5) in the sense of Definition 2.1.2. Set $X := e^W y$. Then (X,τ) is a strong solution of (2.1.1) in the sense of Definition 2.1.1.*

(ii) *Suppose (X,τ) be a strong solution of (2.1.1) in the sense of Definition 2.1.1. Define $y := e^{-W}X$. Then (y,τ) is a strong solution of (2.1.5) in the sense of Definition 2.1.2.*

Proof. This theorem follows immediately from Theorem 1.1.4 in the L^2 case. In fact, consider the case (i). Since $x\in H^1\subset L^2$ and y satisfies (2.1.6) in $H^{-1}\subset H^{-2}$, Theorem 1.1.4 implies that (X, τ) is a strong solution of (2.1.1) in the sense of Definition 1.1.1, in particularly, X solves (2.1.3) in H^{-2}. But, as $y \in C([0,T]; H^1)$ and $e^W \in C([0,T]; W^{1,\infty})$, we deduce that $X \in C([0,T]; H^1)$. Hence, the right hand side of (2.1.3) is in H^{-1}, which implies that (X, τ) is a strong solution of (2.1.1) in the sense of Definition 2.1.1, thereby completing the proof of (i). The proof for (ii) follows analogously. Therefore, we finish the proof of Theorem 2.1.3. □

By Theorem 2.1.3, we will focus on the H^1 well-posedness problem of the time dependent random equation (2.1.5) in the next section.

2.2 Local existence, uniqueness and blowup alternative

In this section, we will establish the local existence, uniqueness and blowup alternative in both subcritical and critical cases in Subsection 2.2.2 and Subsection 2.2.3 respectively. Before that, in Subsection 2.2.1 we first derive the Strichartz estimates in Sobolev spaces, and we also show the equivalence between solutions of weak and mild equations used in this chapter.

2.2.1 Strichartz estimate, weak and mild equations

Unlike the free Schrödinger group $e^{it\Delta}$, the evolution operator $U(t,s)$ from Lemma 1.2.1 does not commute with the gradient operator (see (2.2.7) and (2.2.10) below), hence we will use Proposition 2.3(a) in [59] to control the lower order term, which leads to the further spatial decay assumptions on $\{e_j\}_{j=1}^N$ in $(H2)$ in Section 2.1 (see Section 2.3.3 for the proof.)

Lemma 2.2.1. *Assume $(H2)$. For any $T > 0$, $u_0 \in H^1$ and $f \in L^{q_2'}(0,T; W^{1,p_2'})$, the solution of*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds, 0 \leq t \leq T, \tag{2.2.7}$$

satisfies the estimates

$$\|u\|_{L^{q_1}(0,T;W^{1,p_1})} \leq C_T(|u_0|_{H^1} + \|f\|_{L^{q_2'}(0,T;W^{1,p_2'})}). \tag{2.2.8}$$

Here (p_1, q_1) and (p_2, q_2) are Strichartz pairs as in Lemma 1.2.3.

Furthermore, the process C_t, $t \geq 0$, can be taken to be $(\mathscr{F}_t)$-progressively measurable, increasing and continuous.

Proof of Lemma 2.2.1. Since the proof relies on Theorem 1.13 and Proposition 2.3 (a) in [59], we shall adapt the notations there $D_t := -i\partial_t$, $D_j := -i\partial_j$, $1 \le j \le d$, to rewrite (2.2.7) in the weak equation form

$$D_t u = (D_j a^{jk} D_k + D_j \widetilde{b^j} + \widetilde{b^j} D_j + \widetilde{c})u - if$$

with $a^{jk} = \delta_{jk}$, $\widetilde{b^j} = -i\partial_j W_t$ and $\widetilde{c} = -\sum_{j=1}^{d} (\partial_j W)^2 + (\mu + \widetilde{\mu})i$, $1 \le j, k \le d$.

Direct computations show

$$\begin{aligned}
D_t \nabla u =& \nabla D_t u \\
=& \nabla(D_j a^{jk} D_k u) + \nabla(D_j \widetilde{b^j} u) + \nabla(\widetilde{b^j} D_j u) + \nabla(\widetilde{c} u) - i\nabla f \\
=& D_j \nabla(a^{jk} D_k u) + D_j \nabla(\widetilde{b^j} u) + \nabla(\widetilde{b^j} D_j u) + \nabla(\widetilde{c} u) - i\nabla f \\
=& (D_j \nabla a^{jk} D_k u + D_j a^{jk} D_k \nabla u) + (D_j \nabla \widetilde{b^j} u + D_j \widetilde{b^j} \nabla u) \\
&+ (\nabla \widetilde{b^j} D_j u + \widetilde{b^j} D_j \nabla u) + (\nabla \widetilde{c} u + \widetilde{c} \nabla u) - i\nabla f \\
=& (D_j a^{jk} D_k + D_j \widetilde{b^j} + \widetilde{b^j} D_j + \widetilde{c}) \nabla u \\
&+ (D_j \nabla a^{jk} D_k u + D_j \nabla \widetilde{b^j} + \nabla \widetilde{b^j} D_j + \nabla \widetilde{c}) u - i\nabla f,
\end{aligned}$$

as $a^{jk} = \delta_{jk}$, it follows that

$$\begin{aligned}
D_t \nabla u =& (-\Delta + D_j \widetilde{b^j} + \widetilde{b^j} D_j + \widetilde{c}) \nabla u \\
&+ (D_j \nabla \widetilde{b^j} + \nabla \widetilde{b^j} D_j + \nabla \widetilde{c}) u - i\nabla f,
\end{aligned} \tag{2.2.9}$$

or equivalently

$$\nabla u(t) = U(t,0)\nabla u_0 + \int_0^t U(t,s) \left[i(D_j \nabla \widetilde{b^j}(s) + \nabla \widetilde{b^j}(s) D_j + \nabla \widetilde{c}(s)) u + \nabla f(s) \right] ds. \tag{2.2.10}$$

We regard (2.2.10) as the equation for the unknown ∇u and treat the lower order term $(D_j \nabla \widetilde{b^j} + \nabla \widetilde{b^j} D_j + \nabla \widetilde{c}) u$ as the same role of ∇f. Hence applying (1.2.21) to (2.2.10) and then using Proposition 2.3 (a) to control this lower order term, we have that

$$\begin{aligned}
&\|\nabla u\|_{L^{q_1}(0,T;L^{p_1}) \cap \widetilde{X}_{[0,T]}} \\
\le& C_T \left[|\nabla u_0|_2 + \|i(D_j \nabla \widetilde{b^j} + \nabla \widetilde{b^j} D_j + \nabla \widetilde{c}) u + \nabla f\|_{L^{q_2'}(0,T;L^{p_2'}) + \widetilde{X}'_{[0,T]}} \right] \\
\le& C_T \left[|\nabla u_0|_2 + \|i(D_j \nabla \widetilde{b^j} + \nabla \widetilde{b^j} D_j + \nabla \widetilde{c}) u\|_{\widetilde{X}'_{[0,T]}} + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})} \right]
\end{aligned}$$

$$
\begin{aligned}
\leq & C_T \left[|\nabla u_0|_2 + \kappa_T \|u\|_{\widetilde{X}_{[0,T]}} + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})} \right] \\
\leq & C_T \left[|\nabla u_0|_2 + C_T \kappa_T (|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}) + \|\nabla f\|_{L^{q_2'}(0,T;L^{p_2'})} \right] \\
= & C_T (C_T \kappa_T + 1) \left[|u_0|_{H^1} + \|f\|_{L^{q_2'}(0,T;W^{1,p_2'})} \right],
\end{aligned} \tag{2.2.11}
$$

where we have used again (1.2.21) to control $\|u\|_{\widetilde{X}_{[0,T]}}$ in the last two inequality. This together with (1.2.20) yields the estimate (2.2.8).

Now, set

$$
\begin{aligned}
C_t = & \sup\{\|U(\cdot,0)u_0\|_{L^{q_1}(0,t;W^{1,p_1})}; |u_0|_{H^1} \leq 1\} \\
& + \sup\left\{ \left\| \int_0^{\cdot} U(\cdot,s)f(s)ds \right\|_{L^{q_1}(0,t;W^{1,p_1})}; \|f\|_{L^{q_2'}(0,t;W^{1,p_2'})} = 1 \right\}.
\end{aligned} \tag{2.2.12}
$$

Then the asserted properties of the process C_t, $t \geq 0$ follow analogously as in the proof of Lemma 1.2.3. Consequently, we complete the proof of Lemma 2.2.1. □

In the end of this Subsection, we show the equivalence between solutions of weak and mild equations in the H^1 case.

Theorem 2.2.2. *Let $x \in H^1$ and $U(t,s)$ be the evolution operators associated with A as in* (1.1.7)*, $0 \leq s \leq t \leq T$. Let $y \in C([0,T];H^1)$ and g be a complex function such that $g(y) \in L^1(0,T;H^{-1})$. If y satisfies the mild equation*

$$
y(t) = U(t,0)x + \int_0^t U(t,s)g(y(s))ds, \quad t \in [0,T], \quad \textit{in } H^{-1}, \tag{2.2.13}
$$

then y also satisfies the weak equation

$$
y(t) = x + \int_0^t A(s)y(s)ds + \int_0^t g(y(s))ds, \quad t \in [0,T], \quad \textit{in } H^{-1}. \tag{2.2.14}
$$

Moreover, the converse is also valid.

Proof. The proofs follow the similar arguments as in the proof of Theorem 2.1.3. In fact, Let y satisfy the mild equation (2.2.13). Since $H^{-1} \subset H^{-2}$, it follows from Theorem 1.2.5 that y satisfies the weak equation (2.2.14) in the H^{-2} sense. But, the solution y here is in $C([0,T];H^1)$, which implies that y indeed satisfies the (2.2.14) in the H^{-1} sense, thereby completing the first part. The converse part can be proved analogously. □

2.2.2 Subcritical case

The aim of this subsection is to establish the local existence, uniqueness and blowup alternative of SNLS (2.1.1) in the subcritical case, i.e.

$$\begin{cases} 1 < \alpha < \infty, & \text{if } d = 1, 2; \\ 1 < \alpha < 1 + \frac{4}{d-2}, & \text{if } d \geq 3. \end{cases} \tag{2.2.15}$$

.

Theorem 2.2.3. *Assume (H2) and let α satisfy (2.2.15). For each $x \in H^1$ and $0 < T < \infty$, there exists a maximal strong solution $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of (2.1.1) in the sense of Definition 2.1.1. In particular, uniqueness holds for (2.1.1). X also satisfies $\mathbb{P}$-a.s for every $n \geq 1$*

$$X|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^\gamma(0,\tau_n; W^{1,\rho}), \tag{2.2.16}$$

for each Strichartz pair (ρ, γ).

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-a.e ω, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t\to\tau^*(x)(\omega)} |X(t)(\omega)|_{H^1} = \infty.$$

Remark 2.2.4. *As in Remark 1.2.7, we have $\tau^*(x) = T$, $\mathbb{P}$-a.s, provided $\sup\limits_{t\in[0,\tau^*(x))} |X(t)|_{H^1} < \infty$, $\mathbb{P}$-a.s.*

By the equivalence between (2.1.1) and (2.1.5) in Theorem 2.1.3, this theorem follows from Theorem 2.2.5 below.

Theorem 2.2.5. *Assume (H2) and let α satisfy (2.2.15). For each $x \in H^1$ and $0 < T < \infty$, there exists a maximal strong solution $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of (2.1.5) in the sense of Definition 2.1.2. In particular, uniqueness holds for (2.1.5). y also satisfies $\mathbb{P}$-a.s for every $n \geq 1$*

$$y|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^\gamma(0,\tau_n; W^{1,\rho}), \tag{2.2.17}$$

for each Strichartz pair (ρ, γ).

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-a.e ω, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t\to\tau^*(x)(\omega)} |y(t)(\omega)|_{H^1} = \infty.$$

Proof. By Theorem 2.2.2, it is equivalent to solve the weak equation (2.1.6) in the mild sense, namely

$$y = U(t,0)x - \lambda i \int_0^t U(t,s)e^{(\alpha-1)ReW(s)}g(y(s))ds, \tag{2.2.18}$$

where $g(y) = |y|^{\alpha-1}y$. (Note that, the fact that $g(y) \in L^1(0,t;H^{-1})$ will follow from $g(y) \in L^{q'}(0,t;L^{p'})$ below and the Sobolev's imbedding theorem, hence Theorem 2.2.2 is applicable here.)

The following fixed point arguments are standard in the deterministic case (see e.g. [46] and [58]). Moreover, since the proofs of constructing stochastic process, especially the stopping times and adaptedness, are analogous to those in Theorem 1.2.8, we will only sketch it here.

Let us first consider the case $d \geq 3$. Choose the Strichartz pair $(p,q) = (\frac{d(\alpha+1)}{d+\alpha-1}, \frac{4(\alpha+1)}{(d-2)(\alpha-1)})$, set $\mathcal{X} = C([0,T];L^2)\cap L^q(0,T;L^p)$, $\mathcal{Y} = C([0,T];H^1)\cap L^q(0,T;W^{1,p})$, and consider the integral operator

$$F(y)(t) = U(t,0)x - \lambda i \int_0^t U(t,s)(e^{(\alpha-1)ReW(s)}g(y(s)))ds,\ t \in [0,T], \tag{2.2.19}$$

defined for $y \in \mathcal{Y}$.

We claim that

$$F(\mathcal{Y}) \subseteq \mathcal{Y}. \tag{2.2.20}$$

In fact, by Strichartz estimates in Lemma 2.2.1

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \leq C_T \left[|x|_{H^1} + \|e^{(\alpha-1)ReW}g(y)\|_{L^{q'}(0,T;W^{1,p'})}\right]. \tag{2.2.21}$$

To estimate the right-hand side, we have that

$$\begin{aligned} &\|e^{(\alpha-1)ReW}g(y)\|_{L^{q'}(0,T;W^{1,p'})} \\ \leq &D_1(T)\left(\||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} + \||y|^{\alpha-1}|\nabla y|\|_{L^{q'}(0,T;L^{p'})}\right), \end{aligned} \tag{2.2.22}$$

where in the last inequality we have used $|\nabla g(y)| \leq \alpha|y|^{\alpha-1}|\nabla y|$, $|\nabla(e^{(\alpha-1)ReW}g(y))| \leq |e^{(\alpha-1)W}|\,[(\alpha-1)|\nabla W||g(y)| + |\nabla g(y)|]$ and $D_1(T) := \alpha(|\nabla W|_{L^\infty(0,T;L^\infty)}+2)e^{(\alpha-1)|W|_{L^\infty(0,T;L^\infty)}}$.

With our choice of (p,q), it is easy to verify that $(\frac{1}{p'}, \frac{\alpha}{q}) = (\alpha-1)(\frac{1}{(\alpha-1)l}, \frac{1}{q})+(\frac{1}{p}, \frac{1}{q})$, where $\frac{1}{l} = \frac{1}{p'} - \frac{1}{p}$, satisfying $\frac{1}{(\alpha-1)l} = \frac{1}{p} - \frac{1}{d}$. Hence, from Hölder's inequality and Sobolev's imbedding $|y|_{L^{(\alpha-1)l}} \leq D|y|_{W^{1,p}}$ it follows that

$$\||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} \leq T^\theta \||y|^{\alpha-1}y\|_{L^{\frac{q}{\alpha}}(0,T;L^{p'})}$$

$$\begin{aligned}&\le T^\theta \|y\|^{\alpha-1}_{L^q(0,T;L^{(\alpha-1)l})}\|y\|_{L^q(0,T;L^p)}\\ &\le D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^q(0,T;W^{1,p})}\|y\|_{L^q(0,T;L^p)},\end{aligned} \tag{2.2.23}$$

with $\theta = \frac{1}{q'} - \frac{\alpha}{q} > 0$, and also

$$\||y|^{\alpha-1}|\nabla y|\|_{L^{q'}(0,T;L^{p'})} \le D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^q(0,T;W^{1,p})}\|\nabla y\|_{L^q(0,T;L^p)}. \tag{2.2.24}$$

Thus, taking (2.2.23), (2.2.24) into (2.2.22) and (2.2.21) yields that for $y \in \mathcal{Y}$

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \le C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^\alpha_{L^q(0,T;W^{1,p})}\right], \tag{2.2.25}$$

with $D_2(T) = D_1(T)D^{\alpha-1}$. Similarly,

$$\|F(y)\|_{L^\infty(0,T;H^1)} \le C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^\alpha_{L^q(0,T;W^{1,p})}\right]. \tag{2.2.26}$$

Hence (2.2.25) and (2.2.26) yield (2.2.20) as claimed.

We now start to construct a maximal strong solution of (2.1.5) by analogous arguments as in the proof of Theorem 1.2.8.

Step 1. Fix $\omega \in \Omega$ and consider F on the set

$$\mathcal{Y}^{\tau_1}_{M_1} = \{y \in C([0,\tau_1]; H^1) \cap L^q(0,\tau_1; W^{1,p});\ \sup_{0\le t\le \tau_1} |y(t)|_{H^1} + \|y\|_{L^q(0,\tau_1;W^{1,p})} \le M_1\},$$

where $\tau_1 = \tau_1(\omega) \in (0,T]$ and $M_1 = M_1(\omega) > 0$ are random variables.

For $y \in \mathcal{Y}^{\tau_1}_{M_1}$ by estimates (2.2.25) and (2.2.26)

$$\|F(y)\|_{L^\infty(0,\tau_1;H^1)} + \|F(y)\|_{L^q(0,\tau_1;W^{1,p})} \le 2C_{\tau_1}\left[|x|_{H_1} + D_2(\tau_1)\tau_1^\theta M_1^\alpha\right],$$

In order to obtain $F(\mathcal{Y}^{\tau_1}_{M_1}) \subset \mathcal{Y}^{\tau_1}_{M_1}$, we shall choose M_1 and τ_1 in such a way that

$$2C_{\tau_1}\left[|x|_{H_1} + D_2(\tau_1)\tau_1^\theta M_1^\alpha\right] \le M_1.$$

To this end, we define the real-valued continuous $(\mathscr{F}_t)$-adapted process

$$Z^{(1)}_t = 2\cdot 3^{\alpha-1}|x|^{\alpha-1}_{H^1} C^\alpha_t D_2(t)t^\theta,\ t \in [0,T],$$

choose the $(\mathscr{F}_t)$-stopping time

$$\tau_1 = \inf\left\{t \in [0,T], Z^{(1)}_t > \frac{1}{3}\right\} \wedge T$$

and set $M_1 = 3C_{\tau_1}|x|_{H^1}$. Then it follows that $Z^{(1)}_{\tau_1} \leq \frac{1}{3}$ and $F(\mathcal{Y}^{\tau_1}_{M_1}) \subset \mathcal{Y}^{\tau_1}_{M_1}$.

Moreover, the estimates as in the proof of (2.2.25) show that for $y_1, y_2 \in \mathcal{Y}^{\tau_1}_{M_1}$

$$\begin{aligned}
&\|F(y_1) - F(y_2)\|_{L^\infty(0,\tau_1;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,\tau_1;L^p)} \\
\leq& 2C_{\tau_1}\|\lambda e^{(\alpha-1)ReW}(g(y_1) - g(y_2))\|_{L^{q'}(0,\tau_1;L^{p'})} \\
\leq& C_{\tau_1}D_1(\tau_1)\|(|y_1|^{\alpha-1} + |y_2|^{\alpha-1})|y_1 - y_2|\|_{L^{q'}(0,\tau_1;L^{p'})} \\
\leq& C_{\tau_1}D_1(\tau_1)\tau_1^\theta D^{\alpha-1}\left(\|y_1\|^{\alpha-1}_{L^q(0,\tau_1;W^{1,p})} + \|y_2\|^{\alpha-1}_{L^q(0,\tau_1;W^{1,p})}\right)\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)} \\
\leq& 2C_{\tau_1}D_2(\tau_1)\tau_1^\theta M_1^{\alpha-1}\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)} \\
\leq& \frac{1}{3}\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)},
\end{aligned} \tag{2.2.27}$$

which implies that F is a contraction in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$.

Now, by Banach's fixed point theorem there exists a sequence $u_{1,m} \in \mathcal{Y}^{\tau_1}_{M_1}$, $m \in \mathbb{N}$, and $y \in C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$, such that $y = F(y)$, $u_{1,m+1} = F(u_{1,m})$, $m \geq 1$, $u_{1,1}(t) = U(t,0)x$, $t \in [0,T]$, and $u_{1,m}|_{[0,\tau_1]}$ converges to y in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$. As $u_{1,m}$, $m \in \mathbb{N}$, are bounded in $\mathcal{Y}^{\tau_1}_{M_1}$, from the Banach-Steinhaus theorem we deduce that $y \in \mathcal{Y}^{\tau_1}_{M_1}$ and $u_{1,m}|_{[0,\tau_1]}$ converges weakly to y in $C([0,\tau_1];H^1) \cap L^q(0,\tau_1;W^{1,p})$. Hence, defining $y_1(t) := y(t \wedge \tau_1)$, $t \in [0,T]$, we have $u_{1,m}(t \wedge \tau_1)$ converges weakly to $y_1(t)$ in H^1 for $t \in [0,T]$. This implies the $(\mathscr{F}_t)$-adaptedness of y_1, since each $u_{1,m}$ is $(\mathscr{F}_t)$-adapted in H^1.

Therefore, (y_1,τ_1) is a strong solution of (2.1.5), such that $y_1(t) = y_1(t \wedge \tau_1)$, $t \in [0,T]$, and $y_1|_{[0,\tau_1]} \in C([0,\tau_1];H^1) \cap L^q(0,\tau_1;W^{1,p})$.

Step 2. Suppose that at the n^{th} step we have a strong solution (y_n,τ_n) of (2.1.5), such that $\tau_n \geq \tau_{n-1}$, $y_n(t) = y_n(t \wedge \tau_n)$, $t \in [0,T]$, and $y_n|_{[0,\tau_n]} \in C([0,\tau_n];H^1) \cap L^q(0,\tau_n;W^{1,p})$.

Set

$$\mathcal{Y}^{\sigma_n}_{M_{n+1}} = \{z \in C([0,\sigma_n];H^1) \cap L^q(0,\sigma_n;W^{1,p});\ \sup_{0\leq t\leq\sigma_n}|z(t)|_{H^1} + \|z\|_{L^q(0,\sigma_n;W^{1,p})} \leq M_{n+1}\},$$

and define the integral operator F_n on $\mathcal{Y}$ by

$$\begin{aligned}
F_n(z)(t) =& U(\tau_n + t,\tau_n)y_n(\tau_n) \\
&- \lambda i \int_0^t U(\tau_n + t,\tau_n + s)\left(e^{(\alpha-1)ReW(\tau_n+s)}g(z(s))\right)ds, \quad t \in [0,T],
\end{aligned} \tag{2.2.28}$$

for $z \in \mathcal{Y}$.

Analogous calculations as in Step 1 show that for $z \in \mathcal{Y}^{\sigma_n}_{M_{n+1}}$

$$\begin{aligned}&\|F_n(z)\|_{L^\infty(0,\sigma_n;H^1)} + \|F_n(z)\|_{L^q(0,\sigma_n;W^{1,p})}\\ \le& 2C_{\tau_n+\sigma_n}\left[|y_n(\tau_n)|_{H_1} + D_2(\tau_n+\sigma_n)\sigma_n^\theta M_{n+1}^\alpha\right],\end{aligned}$$

and for $z_1, z_2 \in \mathcal{Y}^{\sigma_n}_{M_{n+1}}$

$$\begin{aligned}&\|F(z_1)-F(z_2)\|_{L^\infty(0,\sigma_n;L^2)} + \|F(z_1)-F(z_2)\|_{L^q(0,\sigma_n;L^p)}\\ \le& 2C_{\tau_n+\sigma_n}D_2(\tau_n+\sigma_n)\sigma_n^\theta M_{n+1}^{\alpha-1}\|z_1-z_2\|_{L^q(0,\sigma_n;L^p)}.\end{aligned}$$

Similarly, we define the continuous $(\mathscr{F}_{\tau_n+t})$-adapted process

$$Z_t^{(n)} := 2\cdot 3^{\alpha-1}|y_n(\tau_n)|_{H_1}^{\alpha-1}C^\alpha_{\tau_n+t}D_2(\tau_n+t)t^\theta,\ t\in[0,T],$$

set

$$\sigma_n = \inf\left\{t\in[0,T-\tau_n] : Z_t^{(n)} > \frac{1}{3}\right\}\wedge(T-\tau_n)$$

and choose $M_{n+1} = 3C_{\tau_n+\sigma_n}|y_n(\tau_n)|_{H^1}$. It follows that $F_n(\mathcal{Y}^{\sigma_n}_{M_{n+1}}) \subset \mathcal{Y}^{\sigma_n}_{M_{n+1}}$ and F_n is a contraction in $C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)$. Hence, because $\mathcal{Y}^{\sigma_n}_{M_{n+1}}$ is a complete metric space in $C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)$, Banach's fixed point theorem implies that there is a unique $z_{n+1}\in\mathcal{Y}^{\sigma_n}_{M_{n+1}}$ such that $z_{n+1} = F_n(z_{n+1})$ on $[0,\sigma_n]$.

Then, set $\tau_{n+1} = \tau_n + \sigma_n$ and define

$$y_{n+1}(t) = \begin{cases} y_n(t), & t\in[0,\tau_n];\\ z_{n+1}((t-\tau_n)\wedge\sigma_n), & t\in(\tau_n,T].\end{cases} \tag{2.2.29}$$

It follows from the definitions of F and F_n that $y_{n+1} = F(y_{n+1})$ on $[0,\tau_{n+1}]$. Moreover, using the arguments as in Step 2 in the proof of Theorem 1.2.8, we deduce that τ_{n+1} is an $(\mathscr{F}_t)$-stopping time and y_{n+1} is adapted to $(\mathscr{F}_t)$ in H^1. Hence, (y_{n+1},τ_{n+1}) is a strong solution of (2.1.5), such that $y_{n+1}(t) = y_{n+1}(t\wedge\tau_{n+1})$, $t\in[0,T]$, and $y_{n+1}|_{[0,\tau_{n+1}]}\in C([0,\tau_{n+1}];H^1)\cap L^q(0,\tau_{n+1};W^{1,p})$.

Step 3. By an induction argument, we finally construct a sequence of strong solutions (y_n,τ_n), $n\in\mathbb{N}$, where τ_n are increasing stopping times and $y_{n+1} = y_n$ on $[0,\tau_n]$. Hence we obtain the triple $(y,(\tau_n)_{n\in\mathbb{N}},\tau^*(x))$, where $y := \lim\limits_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$ with $\tau^*(x) := \lim\limits_{n\to\infty}\tau_n$.

The integrability (2.2.17) follows from the fact that $y_n|_{[0,\tau_n]}\in C([0,\tau_n];H^1)\cap L^q(0,\tau_n;W^{1,p})$ and the Strichartz estimate (2.2.8). In fact, for any Strichartz pair

(ρ,γ),

$$\begin{aligned}\|y\|_{L^\gamma(0,\tau_n;W^{1,\rho})} =&\|F(y_n)\|_{L^\gamma(0,\tau_n;W^{1,\rho})}\\ \leq& C_{\tau_n}[|x|_{H^1} + \|\lambda e^{(\alpha-1)ReW} g(y_n)\|_{L^{q'}(0,\tau_n;W^{1,p'})}]\\ \leq& C_{\tau_n}[|x|_{H^1} + D_2(\tau_n)\tau_n^\theta \|y_n\|^\alpha_{L^q(0,\tau_n;W^{1,p})}] < \infty, \quad \mathbb{P}-a.s.\end{aligned}$$

Moreover, as in the proof of Theorem 1.2.8, the maximality of $(y,(\tau_n)_{n\in\mathbb{N}},\tau^*(x))$ follows from the uniqueness and blowup alternative given below.

To prove the uniqueness, for another two strong solutions $(\widetilde{y}_i,\sigma_i)$, $i=1,2$, as in (2.2.27) we have for any $t,s>0, s+t<\sigma_1\wedge\sigma_2$

$$\begin{aligned}&\|\widetilde{y}_1-\widetilde{y}_2\|_{L^\infty(s,s+t;L^2)} + \|\widetilde{y}_1-\widetilde{y}_2\|_{L^q(s,s+t;L^p)}\\ \leq& 2C_T D_2(T)t^\theta M^{\alpha-1}\|\widetilde{y}_1-\widetilde{y}_2\|_{L^q(s,s+t;L^p)}\end{aligned}$$

with $M = \|\widetilde{y}_1\|_{L^q(0,s+t;W^{1,p})} + \|\widetilde{y}_2\|_{L^q(0,s+t;W^{1,p})} < \infty$ a.s. Thus a properly chosen small t yields $\widetilde{y}_1 = \widetilde{y}_2$ on $[s,s+t]$, which implies $\widetilde{y}_1 = \widetilde{y}_2$ on $[0,\sigma_1\wedge\sigma_2)$, hence $\widetilde{y}_1=\widetilde{y}_2$ on $[0,\sigma_1\wedge\sigma_2]$ by the continuity of $\widetilde{y}_i$ in H^1, $i=1,2$.

Finally, the proof for the blowup alternative follows the analogous arguments as those of Theorem 1.2.8. Suppose that $\mathbb{P}(M^*<\infty;\tau_n<\tau^*(x),\forall n\in\mathbb{N})>0$, where $M^* := \sup_{t\in[0,\tau^*(x))} |y(t)|_{H^1}$. Let us define the real-valued continuous process

$$Z_t := 2\cdot 3^{\alpha-1}(M^*)^{\alpha-1}C^\alpha_{T+t}D_2(T+t)t^\theta,\ t\in[0,T],$$

and

$$\sigma := \inf\left\{t\in[0,T] : Z_t > \frac{1}{6}\right\}\wedge T.$$

For $\omega\in\{M^*<\infty;\tau_n<\tau^*(x),\forall n\in\mathbb{N}\}$, since $\tau_n(\omega)<T$, $\forall n\in\mathbb{N}$, by the definition of σ_n in Step 2, we have

$$\sigma_n(\omega) = \inf\left\{t\in[0,T-\tau_n(\omega)] : Z_t^{(n)}(\omega) > \frac{1}{3}\right\}.$$

Moreover, since, for every $n\geq 1$, $|y(\tau_n(\omega))|_{H^1}\leq M^*$, $C_{\tau_n(\omega)+t}\leq C_{T+t}$ and $D_2(\tau_n(\omega)+t)\leq D_2(T+t)$, it follows that $Z_t(\omega)\geq Z_t^{(n)}(\omega)$, then $\sigma_n(\omega)>\sigma(\omega)>0$. Hence $\tau_{n+1}(\omega)=\tau_n(\omega)+\sigma_n(\omega)>\tau_n(\omega)+\sigma(\omega)$, which implies $\tau_{n+1}(\omega)>\tau_1(\omega)+n\sigma(\omega)$ for every $n\geq1$, contradicting the fact that $\tau_n(\omega)\leq T$. Therefore, we conclude the blow-up alternative and finish the proof of Theorem 2.2.5 for the case $d\geq 3$.

For the case $d = 2$, we modify the Strichartz pair (p, q) by $p = \alpha + 1$ and $q = \frac{4(\alpha+1)}{d(\alpha-1)}$. Then $(\frac{1}{p'}, \frac{1}{q}) = (\alpha - 1)(\frac{1}{p}, 0) + (\frac{1}{p}, \frac{1}{q})$ and $2 < p < \infty$. Hölder's inequality and Sobolev's imbedding $|y|_{L^p} \leq D|y|_{H^1}$ give

$$\begin{aligned}\||y|^{\alpha-1}y\|_{L^{q'}(0,T;L^{p'})} \leq & T^\theta \||y|^{\alpha-1}y\|_{L^q(0,T;L^{p'})} \\ \leq & T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;L^p)} \|y\|_{L^q(0,T;L^p)} \\ \leq & D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|y\|_{L^q(0,T;L^p)} \end{aligned} \tag{2.2.30}$$

where $\theta = 1 - \frac{2}{q} > 0$, and

$$\||y|^{\alpha-1}\nabla y\|_{L^{q'}(0,T;L^{p'})} \leq D^{\alpha-1}T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|\nabla y\|_{L^q(0,T;L^p)}, \tag{2.2.31}$$

Hence, the estimates (2.2.25) and (2.2.26) are accordingly modified by

$$\|F(y)\|_{L^q(0,T;W^{1,p})} \leq C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|y\|_{L^q(0,T;W^{1,p})}\right], \tag{2.2.32}$$

and

$$\|F(y)\|_{L^\infty(0,T;H^1)} \leq C_T \left[|x|_{H_1} + D_2(T)T^\theta \|y\|^{\alpha-1}_{L^\infty(0,T;H^1)} \|y\|_{L^q(0,T;W^{1,p})}\right]. \tag{2.2.33}$$

Similarly to (2.2.27)

$$\begin{aligned} & \|F(y_1) - F(y_2)\|_{L^\infty(0,T;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,T;L^p)} \\ \leq & 2C_T \|\lambda e^{(\alpha-1)ReW}(g(y_1) - g(y_2))\|_{L^{q'}(0,T;L^{p'})} \\ \leq & C_T D_2(T)T^\theta \left(\|y_1\|^{\alpha-1}_{L^\infty(0,T;H^1)} + \|y_2\|^{\alpha-1}_{L^\infty(0,T;H^1)}\right) \|y_1 - y_2\|_{L^q(0,T;L^p)}. \end{aligned} \tag{2.2.34}$$

Therefore, similar arguments as those after (2.2.25) and (2.2.26) yield the asserted results in the case $d = 2$.

The remaining case $d = 1$ is much simpler due to the Sobolev embedding $|y|_{L^\infty} \leq D|y|_{H^1}$. In this case for $y \in C([0, T]; H^1)$

$$\begin{aligned}\|F(y)\|_{L^\infty(0,T;H^1)} \leq & C_T \left[|x|_{H^1} + \|\lambda e^{(\alpha-1)ReW} g(y)\|_{L^1(0,T;H^1)}\right] \\ \leq & C_T \left[|x|_{H^1} + D_1(T)(\||y|^{\alpha-1}y\|_{L^1(0,T;L^2)} + \||y|^{\alpha-1}\nabla y\|_{L^1(0,T;L^2)})\right] \\ \leq & C_T \left[|x|_{H^1} + D_1(T)T\|y\|^{\alpha-1}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \|y\|_{L^\infty(0,T;H^1)}\right] \\ \leq & C_T \left[|x|_{H^1} + D_2(T)T\|y\|^{\alpha}_{L^\infty(0,T;H^1)}\right], \end{aligned}$$

and for $y_1, y_2 \in C([0,T]; H^1)$

$$\begin{aligned}&\|F(y_1) - F(y_2)\|_{L^\infty(0,T;L^2)}\\ \leq & C_T D_2(T) T (\|y_1\|^{\alpha-1}_{L^\infty(0,T;H^1)} + \|y_2\|^{\alpha-1}_{L^\infty(0,T;H^1)}) \|y_1 - y_2\|_{L^\infty(0,T;L^2)}.\end{aligned}$$

Therefore, the new spaces $\mathcal{X} = C([0,T]; L^2)$ and $\mathcal{Y} = C([0,T]; H^1)$ are enough for us to apply the above fixed point arguments to conclude the desired results. This completes the proof of Theorem 2.2.5. □

Remark 2.2.6. *As in remark 1.2.9, the spatial assumption* $(H2)$ *on the noise is due to Lemma 2.2.1. In [12] the authors assumed differently for the noise as in* (1.2.61) *that* $\phi \in L_2(L^2(\mathbb{R}^d, \mathbb{R}), H^1(\mathbb{R}^d, \mathbb{R})) \cap R(L^2(\mathbb{R}^d, \mathbb{R}), W^{1,k}(\mathbb{R}^d))$ *with* $k > 2d$. *Here* $R(L^2(\mathbb{R}^d, \mathbb{R}), W^{1,k}(\mathbb{R}^d))$ *means the radonifying operators from* $L^2(\mathbb{R}^d, \mathbb{R})$ *to* $W^{1,k}(\mathbb{R}^d)$ *and allows to apply the theory established for the Banach space valued stochastic integrals. However, as in [10], a restrictive condition was imposed on the exponent* α, *i.e* $\alpha < 1 + \frac{2}{d-1}$ *if* $d \geq 6$.

2.2.3 Critical case

In this subsection we consider the critical case $\alpha = 1 + \frac{4}{d-2}$ with $d \geq 3$.

Theorem 2.2.7. *Assume* $(H2)$ *and* $\alpha = 1 + \frac{4}{d-2}$, $d \geq 3$. *For each* $x \in H^1$ *and* $0 < T < \infty$, *there exists a maximal strong solution* $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *of* (2.1.1) *in the sense of Definition 2.1.1. In particular, uniqueness holds for* (2.1.1). X *also satisfies* $\mathbb{P}$-*a.s for every* $n \geq 1$

$$X|_{[0,\tau_n]} \in C([0,\tau_n]; H^1) \cap L^\gamma(0,\tau_n; W^{1,\rho}), \tag{2.2.35}$$

where (ρ,γ) *is any Strichartz pair.*

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$-*a.e* ω, *if* $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, *then*

$$\|X(\omega)\|_{L^{\frac{2(d+2)}{d-2}}(0,\tau^*(x)(\omega); L^{\frac{2(d+2)}{d-2}})} = \infty, \quad \mathbb{P} - a.s. \tag{2.2.36}$$

Following the lines in Subsection 2.2.2, Theorem 2.2.7 follows from Theorem 2.2.8 below owing to the equivalence between (2.1.1) and (2.1.5) in Theorem 2.1.3.

Theorem 2.2.8. *Consider the situation of Theorem 2.2.7. For each* $x \in H^1$ *and* $0 < T < \infty$, *there exists a maximal strong solution* $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ *of* (2.1.5) *in the sense of Definition 2.1.2. In particular, uniqueness holds for* (2.1.5). y *also satisfies* $\mathbb{P}$-*a.s for every* $n \geq 1$

$$y|_{[0,\tau_n]} \in C([0,\tau_n];H^1) \cap L^\gamma(0,\tau_n;W^{1,\rho}), \tag{2.2.37}$$

where (ρ,γ) *is any Strichartz pair.*

Moreover, we have the blowup alternative, namely, for $\mathbb{P}$*-a.e* ω*, if* $\tau_n(\omega) < \tau^*(x)(\omega)$*,* $\forall n \in \mathbb{N}$*, then*

$$\|y(\omega)\|_{L^{\frac{2(d+2)}{d-2}}(0,\tau^*(x)(\omega);L^{\frac{2(d+2)}{d-2}})} = \infty, \quad \mathbb{P}-a.s. \tag{2.2.38}$$

Proof of Theorem 2.2.8. Since the arguments are similar to the previous subcritical case in Theorem 2.2.5, we shall only give a sketch of it. At the first step, let us consider the operator F as in (2.2.19) on the space $G^{\tau_1}_{\widetilde{M_1}} = \{y \in C([0,\tau_1];L^2)\cap L^q(0,\tau_1;L^p) : \|y - U(\cdot,0)x\|_{L^\infty(0,\tau_1;H^1)} + \|y\|_{L^q(0,\tau_1;W^{1,p})} \le \widetilde{M_1}\}$ with the Strichartz pair $(p,q) = (\frac{2d^2}{d^2-2d+4}, \frac{2d}{d-2})$. As in (2.2.25) and (2.2.26), we observe that for $y \in G^{\tau_1}_{\widetilde{M_1}}$

$$\|F(y) - U(\cdot,0)x\|_{L^\infty(0,\tau_1;H^1)} + \|F(y)\|_{L^q(0,\tau_1;W^{1,p})} \le \epsilon_1(\tau_1) + 2C_{\tau_1}D_2(\tau_1)\widetilde{M}_1^\alpha,$$

where $\epsilon_1(t) := \|U(\cdot,0)x\|_{L^q(0,t;W^{1,p})}$ is $(\mathscr{F}_t)$-adapted. By Theorem 2.2.1, $\epsilon_1(t) = \|\mathbb{1}_{(0,t)}(\cdot)U(\cdot,0)x\|_{L^q(0,T;W^{1,p})} \le C_T|x|_{H^1} < \infty$, and $\mathbb{1}_{(0,t)}(\cdot)U(\cdot,0)x \to 0$, as $t \to 0^+$. This implies $\epsilon(t) \to 0$, as $t \to 0^+$.

Furthermore, as in (2.2.27)

$$\begin{aligned}&\|F(y_1) - F(y_2)\|_{L^\infty(0,\tau_1;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,\tau_1;L^p)}\\ \le& 2C_{\tau_1}D_2(\tau_1)\widetilde{M}_1^{\alpha-1}\|y_1 - y_2\|_{L^q(0,\tau_1;L^p)}.\end{aligned}$$

Taking into account the above two estimates, we define the continuous $(\mathscr{F}_t)$-adapted process

$$\widetilde{Z}^{(1)}_t := 2^\alpha C_t D_2(t)\epsilon_1^{\alpha-1}(t), \ t \in [0,T],$$

the $(\mathscr{F}_t)$-stopping time

$$\tau_1 = \inf\left\{t \in [0,T], \widetilde{Z}^{(1)}_t > \frac{1}{2}\right\} \wedge T$$

and take $\widetilde{M_1} = 2\epsilon_1(\tau_1)$. It follows that $F(G^{\tau_1}_{\widetilde{M_1}}) \subset G^{\tau_1}_{\widetilde{M_1}}$ and F is a contraction in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^p)$. Using the arguments as in Step 1 in the proof of Theorem 2.2.5, we obtain a strong solution (y_1,τ_1), such that $y_1(t) = y_1(t\wedge\tau_1)$, $t \in [0,T]$, and $y_1|_{[0,\tau_1]} \in C([0,\tau_1];H^1) \cap L^q(0,\tau_1;W^{1,p})$.

Suppose that at the n^{th} step we have a strong solution (y_n,τ_n) with $\tau_n \ge \tau_{n-1}$, such that $y_n(t) = y_n(t\wedge\tau_n)$, $t \in [0,T]$, and $y_n|_{[0,\tau_n]} \in C([0,\tau_n]; H^1)\cap L^q(0,\tau_n; W^{1,p})$.

Consider the operator F_n as in (2.2.28) and set $G^{\sigma_n}_{\widetilde{M}_{n+1}} = \{z \in C([0,\sigma_n]; L^2) \cap L^q(0,\sigma_n; L^p) : \|z - U(\tau_n + \cdot, \tau_n)y_n(\tau_n)\|_{L^\infty(0,\sigma_n;H^1)} + \|z\|_{L^q(0,\sigma_n;W^{1,p})} \le \widetilde{M}_{n+1}\}$, then for $z \in G^{\sigma_n}_{\widetilde{M}_{n+1}}$

$$\begin{aligned}&\|F_n(z) - U(\tau_n + \cdot, \tau_n)y_n(\tau_n)\|_{L^\infty(0,\sigma_n;H^1)} + \|F_n(z)\|_{L^q(0,\sigma_n;W^{1,p})}\\ \le&\epsilon_{n+1}(\sigma_n) + 2C_{\tau_n+\sigma_n}D_2(\tau_n+\sigma_n)\widetilde{M}^{\alpha}_{n+1},\end{aligned}$$

and for $z_1, z_2 \in G^{\sigma_n}_{\widetilde{M}_{n+1}}$

$$\begin{aligned}&\|F(z_1) - F(z_2)\|_{L^\infty(0,\sigma_n;L^2)} + \|F(y_1) - F(y_2)\|_{L^q(0,\sigma_n;L^p)}\\ \le&2C_{\tau_n+\sigma_n}D_2(\tau_n+\sigma_n)\widetilde{M}^{\alpha-1}_{n+1}\|z_1 - z_2\|_{L^q(0,\sigma_n;L^p)},\end{aligned}$$

where $\epsilon_{n+1}(t) := \|U(\tau_n + \cdot, \tau_n)y_n(\tau_n)\|_{L^q(0,t;W^{1,p})}$ is $(\mathscr{F}_{\tau_n+t})$-adapted and tends to 0 as $t \to 0$.

Hence, define the continuous $(\mathscr{F}_{\tau_n+t})$-adapted process

$$\widetilde{Z}^{(n)}_t := 2^\alpha C_{\tau_n+t}D_2(\tau_n+t)\epsilon^{\alpha-1}_{n+1}(t), \quad t \in [0,T],$$

set

$$\sigma_n = \inf\left\{t \in [0, T-\tau_n], \widetilde{Z}^{(n)}_t > \frac{1}{2}\right\} \wedge (T-\tau_n)$$

and let $\widetilde{M}_{n+1} = 2\epsilon_{n+1}(\sigma_n)$. We have that $F_n(G^{\sigma_n}_{\widetilde{M}_{n+1}}) \subset G^{\sigma_n}_{\widetilde{M}_{n+1}}$ and F_n is a contraction in $C([0,\sigma_n]; L^2) \cap L^q(0,\sigma_n; L^p)$. Then using similar arguments as in Step 2 in the proof of Theorem 2.2.5, we obtain a strong solution (y_{n+1}, τ_{n+1}) with $\tau_{n+1} := \tau_n + \sigma_n$, such that $y_{n+1}(t) = y_{n+1}(t \wedge \tau_{n+1})$, $t \in [0,T]$, and $y_{n+1}|_{[0,\tau_{n+1}]} \in C([0,\tau_{n+1}]; H^1) \cap L^q(0,\tau_{n+1}; W^{1,p})$.

Therefore, by induction arguments we obtain a sequence of strong solutions $(y_n, \tau_n)_{n\in\mathbb{N}}$ with τ_n increasing stopping times and $y_{n+1} = y_n$ on $[0,\tau_n]$, then we define the triple $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ by $y = \lim\limits_{n\to\infty} y_n \mathbb{1}_{[0,\tau^*(x))}$ with $\tau^*(x) := \lim\limits_{n\to\infty}\tau_n$. (2.2.37) follows from the fact that and $y_n|_{[0,\tau_n]} \in C([0,\tau_n]; L^2) \cap L^q(0,\tau_n; W^{1,p})$ and the Strichartz estimate (2.2.8). Moreover, the maximality of $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ follows from the uniqueness and blowup alternative below.

The proof for the uniqueness is analogous to that in the L^2-critical case. For any two strong solutions $(\widetilde{y}_i, \sigma_i)$, $i = 1, 2$, define $\varsigma = \sup\{t \in [0, \sigma_1 \wedge \sigma_2] : \widetilde{y}_1 = \widetilde{y}_2 \textit{ on } [0,t]\}$. Suppose that $\mathbb{P}(\varsigma < \sigma_1 \wedge \sigma_2) > 0$. For $\omega \in \{\varsigma < \sigma_1 \wedge \sigma_2\}$, we have

$\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega)]$ by the continuity in H^1, and for $t \in [0, \sigma_1 \wedge \sigma_2(\omega) - \varsigma(\omega))$

$$\begin{aligned}&\|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^q(\varsigma(\omega),\varsigma(\omega)+t;L^p)}\\ \le& 2C_{\varsigma(\omega)+t} D_2(\varsigma(\omega)+t)\widetilde{M}(t)\|\widetilde{y}_1(\omega) - \widetilde{y}_2(\omega)\|_{L^q(\varsigma(\omega),\varsigma(\omega)+t;L^p)},\end{aligned}$$

where $\widetilde{M}(t) := \|\widetilde{y}_1(\omega)\|^{\alpha-1}_{L^q(\varsigma(\omega),\varsigma(\omega)+t;W^{1,p})} + \|\widetilde{y}_2(\omega)\|^{\alpha-1}_{L^q(\varsigma(\omega),\varsigma(\omega)+t;W^{1,p})} \to 0$ as $t \to 0$. Therefore, with t small enough we deduce that $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[\varsigma(\omega), \varsigma(\omega)+t]$, which implies $\widetilde{y}_1(\omega) = \widetilde{y}_2(\omega)$ on $[0, \varsigma(\omega)+t]$ and yields a contradiction.

It remains to prove the blowup alternative. We will adapt the augments as in [22] and [19]. Set $q_1 = \frac{2(d+2)}{d-2}$. Besides the Strichartz pair $(p,q) = (\frac{2d^2}{d^2-2d+4}, \frac{2d}{d-2})$, let us choose another Strichartz pair $(p_2, p_2) = (2+\frac{4}{d}, 2+\frac{4}{d})$. Then $\frac{1}{p_2'} = \frac{\alpha-1}{q_1} + \frac{1}{p_2}$.

Suppose that $\mathbb{P}(\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}) > 0$. For $\omega \in \{\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty;\ \tau_n < \tau^*(x), \forall n \in \mathbb{N}\}$, we have $\sigma_n(\omega) = \inf\{t \in [0, T - \tau_n(\omega)]; \widetilde{Z}^{(n)}_t(\omega) > \frac{1}{2}\}$ and $\widetilde{Z}^{(n)}_{\sigma_n(\omega)}(\omega) = \frac{1}{2}$. For convenience, we omit the dependence of ω below.

From the definition F_n and the construction of y, one can verify that for every $n \ge 1$ and $t \in [0, \tau^*(x) - \tau_n)$

$$y(\tau_n + t) = U(\tau_n + t, \tau_n)y(\tau_n) - \lambda i \int_{\tau_n}^{\tau_n+t} U(\tau_n+t, s)e^{(\alpha-1)ReW(s)} g(y(s))ds.$$

Then by Lemma 2.2.1 and Höder's inequality, for every $n \ge 1$ and $t \in [\tau^*(x) - \tau_n)$

$$\begin{aligned}\|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})} \le& C_T|y(\tau_n)|_{H^1} + C_T\|e^{(\alpha-1)ReW(s)}g(y(s))\|_{L^{p_2'}(\tau_n,\tau_n+t;W^{1,p_2'})}\\ \le& C_T|y(\tau_n)|_{H^1} + C_T D_1(T)\|y\|^{\alpha-1}_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})}\|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})},\end{aligned}$$

where $D_1(T)$ is defined as in the proof of Theorem 2.2.5.

Since $\|y\|_{L^{q_1}(0,\tau^*(x);L^{q_1})} < \infty$ and $\tau_n \to \tau^*(x)$, we have $\|y\|_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})} \to 0$ as $n \to \infty$. Hence, choosing n large enough, such that $C_T D_2(T)\|y\|^{\alpha-1}_{L^{q_1}(\tau_n,\tau^*(x);L^{q_1})} < \frac{1}{2}$, we have for $t \in [\tau^*(x) - \tau_n)$

$$\|y\|_{L^{p_2}(\tau_n,\tau_n+t;W^{1,p_2})} \le 2C_T|y(\tau_n)|_{H^1},$$

yielding

$$\|y\|_{L^{p_2}(0,\tau^*(x);W^{1,p_2})} < \infty.$$

Therefore

$$\|y\|_{L^q(0,\tau^*(x);W^{1,p})} \le C_T|x|_{H^1} + C_T\|e^{(\alpha-1)ReW(s)}g(y(s))\|_{L^{p_2'}(0,\tau^*(x);W^{1,p_2'})}$$

$$\leq C_T|x|_{H^1} + C_T D_1(T)\|y\|^{\alpha-1}_{L^{q_1}(0,\tau^*(x);L^{q_1})}\|y\|_{L^{p_2}(0,\tau^*(x);W^{1,p_2})} < \infty.$$

Now, as in the proof of Theorem 1.2.10, we note that for every $n \geq 1$ and $t \in [0, \sigma_n]$

$$\varepsilon_{n+1}(t) = \|U(\tau_n + \cdot, \tau_n)y(\tau_n)\|_{L^q(0,t;W^{1,p})} \leq \widetilde{M}^*_n + C_T D_2(T)(\widetilde{M}^*_n)^\alpha,$$

where $\widetilde{M}^*_n(\omega) := \|y(\omega)\|_{L^q(\tau_n(\omega),\tau^*(x)(\omega);W^{1,p})} \to 0$, as $n \to \infty$. Then choose n large enough such that

$$\widetilde{Z}^{(n)}(\omega) := 2^\alpha C_T(\omega)D_2(T)(\omega)[\widetilde{M}^*_n(\omega) + C_T(\omega)D_2(T)(\omega)(\widetilde{M}^*_n)^\alpha(\omega)]^{\alpha-1} < \frac{1}{6}.$$

But this implies $\frac{1}{6} > \widetilde{Z}^{(n)}(\omega) > \widetilde{Z}^{(n)}_t(\omega)$ for any $t \in [0, \sigma_n(\omega)]$, in particular, $\frac{1}{6} > \widetilde{Z}^{(n)}(\omega) > \widetilde{Z}^{(n)}_{\sigma_n(\omega)}(\omega) = \frac{1}{2}$, yieding a contradiction. Therefore, we finish the proof of Theorem 2.2.8. □

Remark 2.2.9. *In the critical case, a bound on the solution in $H^1(\mathbb{R}^d)$ can not give us the global existence, which is different from the subcritical case (see Theorem 2.2.3). In the last decade, it was extensively studied to derive a finite bound in* (2.2.36) *for NLS in the defocusing case ($\lambda = -1$). We refer to interested reader to Section 2.4 for further reviews and references therein.*

Remark 2.2.10. *Comparing the ranges of exponents α for the local well-posedness results in Section 1.2 and Section 2.2, we notice that smoother initial data allow to obtain local well-posedness for more general nonlinearity.*

In the deterministic case, the authors in [20] proved that given the initial data $x \in H^s$, where s is an integral and $0 \leq s < \frac{d}{2}$, (2.1.1) *is locally wellposed for $1 < \alpha \leq 1 + \frac{4}{n-2s}$ (if s is not an even integer, suppose also $s < \alpha$). The case when s is not an integer was also studied there in the context of Besov's spaces.*

2.3 Global well-posedness

This section is devoted to the global well-posedness of SNLS (2.1.1) in the subcritical case. We will first derive a priori estimates of the energy in Subsection 2.3.1, then we present the global well-posedness in Subsection 2.3.2. Some technical proofs are put in Subsection 2.3.3.

2.3.1 A priori estimate of the energy

As we saw in Theorem 2.2.3, the key ingredient to obtain the global solution in the subcritical case is an a priori bound on the energy $\sup_{0\le t<\tau^*(x)} |X(t)|_{H^1}$, which will be obtained in Lemma 2.3.5 and Theorem 2.3.6 below. Like in the deterministic case, we obtain such estimate from the Hamiltonian H defined on H^1 and for α satisfying (2.1.2)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 d\xi - \frac{\lambda}{\alpha+1}\int |u|^{\alpha+1} d\xi, \quad u \in H^1. \tag{2.3.39}$$

Note that, the condition on the exponents α of the nonlinearity ensures from Sobolev' imbedding theorem that the Hamiltonian H is well-defined.

Let us start with an evolution formula for $H(X(t))$ and some technical lemmas. Some details of the proofs are postponed to Subsection 2.3.3.

Theorem 2.3.1. *Let α satisfy* (2.2.15) *and $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal strong solution of* (2.1.1) *from Theorem 2.2.3. It holds $\mathbb{P}$-a.s. for $0 \le t < \tau^*(x)$,*

$$\begin{aligned}
&H(X(t)) \\
=&H(x) + \int_0^t Re\langle -\nabla(\mu X(s)), \nabla X(s)\rangle_2 ds + \frac{1}{2}\sum_{j=1}^N \int_0^t |\nabla(X(s)\phi_j)|_2^2 ds \\
&- \frac{1}{2}\lambda(\alpha-1)\sum_{j=1}^N \int_0^t \int (Re\phi_j)^2 |X(s)|^{\alpha+1} d\xi ds \\
&+ \sum_{j=1}^N \int_0^t Re\langle \nabla(\phi_j X(s)), \nabla X(s)\rangle_2 d\beta_j(s) \\
&- \lambda \sum_{j=1}^N \int_0^t \int Re\phi_j |X(s)|^{\alpha+1} d\xi d\beta_j(s),
\end{aligned}$$

where $\phi_j = \mu_j e_j$, $j = 1, ..., N$.

Remark 2.3.2. *In the deterministic case $\mu_j = 0$, $1 \le j \le N$, the Hamiltonian* (2.3.39) *is conserved, i.e $H(X(t)) = H(x)$.*

In the stochastic conservative case $\mu_j = -i\widetilde{\mu}_j$, $\widetilde{\mu}_j \in \mathbb{R}$, $1 \le j \le N$, notice that for $1 \le j \le N$, $Re\phi_j = 0$, $-Re\langle\nabla(\mu X), \nabla X\rangle_2 + \frac{1}{2}\sum_{j=1}^N |\nabla(X\phi_j)|_2^2 = \frac{1}{2}\sum_{j=1}^N |\nabla\widetilde{\phi}_j X|_2^2$ with $\widetilde{\phi}_j = \widetilde{\mu}_j e_j$, and $Re < \nabla(\phi_j X), \nabla X >_2 = -Im < \nabla X, \nabla\widetilde{\phi}_j X >_2$. Then it follows from Theorem 2.3.1 that

$$H(X(t))$$

$$=H(x)+\frac{1}{2}\sum_{j=1}^{N}\int_0^t|\nabla\widetilde{\phi_j}X(s)|_2^2ds-\sum_{j=1}^{N}Im\int_0^t<\nabla X(s),\nabla\widetilde{\phi_j}X(s)>_2 d\beta_j(s),$$

which coincides with (4.26) *in [12].*

Proof of Theorem 2.3.1. This formula follows heuristically by applying Itô's formula to the integrands in $H(X(t))$ with the variable ξ fixed and then integrating over $\mathbb{R}^d$. To prove it rigourously, we introduce the operators Θ_m, $m\in\mathbb{N}$, used in [12] and defined for any $f\in\mathcal{S}$ by

$$\Theta_m f:=\left(\theta(\frac{|\cdot|}{m})\right)^{\vee}*f\ \ (=m^d\theta^{\vee}(m\cdot)*f),$$

where $\theta\in C_c^{\infty}$ is real-valued, nonnegative and $\theta(x)=1$ for $|x|\le 1$, $\theta(x)=0$ for $|x|>2$.

By the Hausdorff-Young inequality and $\int\theta^{\vee}d\xi=1$, for any $p\in[1,\infty)$

$$\|\Theta_m\|_{L^p\to L^p}\le C, \tag{2.3.40}$$

where $C=C(p)$ is independent of m, and

$$\Theta_m f\to f\ \ in\ L^p,\ \ as\ m\to\infty. \tag{2.3.41}$$

Moreover, for any $f\in L^{\frac{\alpha+1}{\alpha}}$

$$\Theta_m f\in L^{\alpha+1}, \tag{2.3.42}$$

$$Re\int if(\xi)\overline{\Theta_m f(\xi)}d\xi=0. \tag{2.3.43}$$

(See Subsection 2.3.3 for the proof.)

Consider the approximating equation

$$\begin{aligned}&idX_m=\Delta X_m dt-i\mu X_m dt+\lambda\Theta_m(g(X_m))dt+iX_m dW,\ \ t\in(0,T),\\&X_m(0)=x,\end{aligned} \tag{2.3.44}$$

where $g(X_m)=|X_m|^{\alpha-1}X_m$. Since the bound in (2.3.40) is independent of m, the arguments in the proof of Theorem 2.2.5 show that there exists a maximal strong solution $(X_m,(\tau_n)_{n\in\mathbb{N}},\tau^*(x))$ of (2.3.44), with τ_n, $n\in\mathbb{N}$, independent of m, and it holds that $\mathbb{P}-a.s.$

$$R(t):=\sup_{m\ge1}(\|X_m\|_{C([0,t];H^1)}+\|X_m\|_{L^q(0,t;W^{1,\alpha+1})})<\infty,\ \ t<\tau^*(x), \tag{2.3.45}$$

where $q = \frac{4(\alpha+1)}{d(\alpha-1)}$.

Moreover, it follows from Lemma 2.3.11 and Lemma 2.3.12 in Subsection 2.3.3 that

$$\begin{aligned}
&H(X_m(t)) \\
=&H(x) + \int_0^t Re\langle -\nabla(\mu X_m), \nabla(X_m)\rangle_2 dt + \frac{1}{2}\sum_{j=1}^N \int_0^t |\nabla(X_m(s)\phi_j)|_2^2 ds \\
&- \frac{1}{2}\lambda(\alpha-1)\sum_{j=1}^N \int_0^t \int (Re\phi_j)^2 |X_m(s)|^{\alpha+1} d\xi ds \\
&- \lambda \int_0^t Re \int i\nabla[(\Theta_m - 1)g(X_m)]\nabla\overline{X_m} d\xi ds \qquad (2.3.46)\\
&+ \sum_{j=1}^N \int_0^t Re\langle \nabla(\phi_j X_m(s)), \nabla X_m(s)\rangle_2 d\beta_j(s) \\
&- \lambda \sum_{j=1}^N \int_0^t \int Re\phi_j |X_m(s)|^{\alpha+1} d\xi d\beta_j(s).
\end{aligned}$$

We also have $\mathbb{P} - a.s.$ for $t < \tau^*(x)$

$$X_m \to X, \ in \ L^\infty(0,t;H^1) \cap L^q(0,t;W^{1,\alpha+1}), \qquad (2.3.47)$$

it particular,

$$X_m \to X, \quad \nabla X_m \to \nabla X, \quad in \ measure \ dt \times d\xi \qquad (2.3.48)$$

(see Subsection 2.3.3 for its proof).

Now, to take the limit in (2.3.46), let us take the fifth term for example. We will show that $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\lambda \int_0^t Re \int i\nabla[(\Theta_m - 1)g(X_m)]\nabla\overline{X_m} d\xi ds \to 0, \ as \ m \to \infty. \qquad (2.3.49)$$

Indeed, by (2.3.48), Lemma 2.3.10 and (2.3.47), it suffices to prove that $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\nabla[(\Theta_m - 1)g(X_m)] \to 0, \ in \ L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}}). \qquad (2.3.50)$$

Notice that, by (2.3.40)

$$\|\nabla[(\Theta_m - 1)g(X_m)]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}$$

$$\leq \|(\Theta_m - 1)(\nabla g(X_m) - \nabla g(X))\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} + \|(\Theta_m - 1)\nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}$$
$$\leq C\|\nabla g(X_m) - \nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} + \|(\Theta_m - 1)\nabla g(X)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})},$$

where C is independent of m. Using the arguments after (2.3.112) we deduce that the first term tends to 0. Moreover, the second term also converges to 0, due to (2.3.41) and (2.3.40). Therefore, we obtain (2.3.49), as claimed.

One easily verifies that we can also take the limit for the remaining terms in (2.3.46) using (2.3.47). Consequently we complete the proof of Theorem 2.3.1. □

We next state some technical lemmas before the proof of a priori estimates in Theorem 2.3.6.

Lemma 2.3.3. *For $1 < \alpha < 1 + \frac{4}{d}$, $d \geq 1$,*

$$|X|_{L^{\alpha+1}}^{\alpha+1} \leq C|X|_2^{\beta}|\nabla X|_2^{\gamma}, \tag{2.3.51}$$

where $\beta = (1-\theta)(\alpha+1)$ and $\gamma = \theta(\alpha+1) \in (0,2)$ with $\theta = \frac{d(\alpha-1)}{2(\alpha+1)} \in (0,1)$.

Moreover, we have

$$|X|_{L^{\alpha+1}}^{\alpha+1} \leq C_\epsilon |X|_2^{\beta p} + \epsilon|\nabla X|_2^2, \tag{2.3.52}$$

where $\beta p > 2$ and $p > 1$.

Proof. (2.3.51) is the well-known Gagliardo-Nirenberg inequality, (2.3.52) follows immediately from (2.3.51) and Young's inequality $ab \leq C_\epsilon a^p + \epsilon b^q$, $\frac{1}{p} + \frac{1}{q} = 1$ by choosing $\gamma q = 2$. □

Lemma 2.3.4. *Let $Y \geq 0$ be real-valued progressively measurable process, we have*

$$\mathbb{E}\left(\int_0^t Y(s)^2 ds\right)^{\frac{1}{2}} \leq \epsilon\mathbb{E}\sup_{s\leq t} Y(s) + C_\epsilon \int_0^t \mathbb{E}\sup_{r\leq s} Y(r) ds.$$

Proof. This lemma follows from

$$\mathbb{E}\left(\int_0^t Y(s)^2 ds\right)^{\frac{1}{2}} \leq \mathbb{E}\left[\sup_{s\leq t} Y(s)^{\frac{1}{2}}\left(\int_0^t Y(s) ds\right)^{\frac{1}{2}}\right] \leq \sqrt{\mathbb{E}\sup_{s\leq t} Y(s)}\sqrt{\mathbb{E}\int_0^t Y(s) ds},$$

and the inequality $\sqrt{ab} \leq \epsilon a + C_\epsilon b$. □

Unlike in the deterministic and stochastic conservative cases, $|X(t)|_2^2$ is no longer independent of t, but a general martingale (see (1.3.79)). When we use Lemma 2.3.3 to control $|X(t)|_{L^{\alpha+1}}^{\alpha+1}$, we shall need the following lemma to bound the p-power of $|X(t)|_2$. The proof of this lemma is postponed to Subsection 2.3.3.

Lemma 2.3.5. *Take $p \geq 2$. Let $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal strong solution of* (2.1.1) *from Theorem 2.2.3. Then there exists $\widetilde{C}(T) < \infty$ such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*(x))} |X(t)|_2^p \leq \widetilde{C}(T) < \infty.$$

With the above preliminaries, we are now ready to prove the following main a priori estimate in this subsection.

Theorem 2.3.6. *Let α satisfy* (2.2.15) *in the defocusing case and $1 < \alpha < 1 + \frac{4}{d}$ in the focusing case. Let $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal strong solution of* (2.1.1) *from Theorem 2.2.3. There exists $\widetilde{C}(T) < \infty$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,\tau^*(x))} \left(|\nabla X(t)|_2^2 + |X(t)|_{L^{\alpha+1}}^{\alpha+1}\right)\right] \leq \widetilde{C}(T) < \infty. \tag{2.3.53}$$

Proof. (i) First assume that $\lambda = 1$ (focusing). From the definition of H in (2.3.39) and Theorem 2.3.1, it follows that $\mathbb{P}$-a.s. for every $n \geq 1$ and $t \in [0, T]$

$$\begin{aligned}
&\frac{1}{2}|\nabla X(t\wedge\tau_n)|_2^2 \\
=&H(X(t\wedge\tau_n)) + \frac{1}{\alpha+1}|X(t\wedge\tau_n)|_{L^{\alpha+1}}^{\alpha+1} \\
=&H(x) + \frac{1}{\alpha+1}|X(t\wedge\tau_n)|_{L^{\alpha+1}}^{\alpha+1} \\
&+ \int_0^{t\wedge\tau_n} \left[Re\langle -\nabla(\mu X(s)), \nabla X(s)\rangle_2 + \frac{1}{2}\sum_{j=1}^N |\nabla(X(s)\phi_j)|_2^2 \right] ds \\
&- \frac{1}{2}(\alpha-1)\sum_{j=1}^N \int_0^{t\wedge\tau_n} \int (Re\phi_j)^2 |X(s)|^{\alpha+1} d\xi ds \\
&+ \sum_{j=1}^N \int_0^{t\wedge\tau_n} Re\langle \nabla(\phi_j X(s)), \nabla X(s)\rangle_2 d\beta_j(s) \\
&- \sum_{j=1}^N \int_0^{t\wedge\tau_n} \int Re\phi_j |X(s)|^{\alpha+1} d\xi d\beta_j(s) \\
=&H(x) + \frac{1}{\alpha+1}|X(t\wedge\tau_n)|_{L^{\alpha+1}}^{\alpha+1} + J_1(t\wedge\tau_n) + J_2(t\wedge\tau_n) + J_3(t\wedge\tau_n) + J_4(t\wedge\tau_n),
\end{aligned} \tag{2.3.54}$$

where ϕ_j, $1 \leq j \leq N$, are defined as in Theorem 2.3.1.

To estimate the second term in (2.3.54), we note that, from (2.3.52) and Lemma

2.3.5 it follows that

$$\begin{aligned}\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{\alpha+1}\le&\frac{1}{\alpha+1}C_\epsilon\mathbb{E}\sup_{s\le t\wedge\tau_n}|X(s)|_2^{\beta p}+\epsilon\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|\nabla X(s)|_2^2\\ \le&\frac{1}{\alpha+1}C_\epsilon\widetilde{C}(T)+\epsilon\frac{1}{\alpha+1}\mathbb{E}\sup_{s\le t\wedge\tau_n}|\nabla X(s)|_2^2.\end{aligned}\tag{2.3.55}$$

For $J_1(t\wedge\tau_n)$, since $|\phi_j|_\infty<\infty$ and $|\nabla\phi_j|_\infty<\infty$, $1\le j\le N$, we have that

$$\begin{aligned}J_1(t)=&\int_0^t\Big[Re\langle-\nabla\mu X(s)-\mu\nabla X(s),\nabla X(s)\rangle_2\\ &+\frac{1}{2}\sum_{j=1}^N|\nabla X(s)\phi_j+X(s)\nabla\phi_j|_2^2\Big]ds\\ \le&C\int_0^t|\nabla X(s)|_2^2+|X(s)|_2^2ds.\end{aligned}$$

where C depends on $|\phi_j|_\infty$ and $|\nabla\phi_j|_\infty$, $1\le j\le N$. Hence by Lemma 2.3.5

$$\begin{aligned}\mathbb{E}\sup_{s\le t\wedge\tau_n}|J_1(s)|\le&C\mathbb{E}\sup_{s\le t\wedge\tau_n}\int_0^s|X(r)|_2^2+|\nabla X(r)|_2^2dr\\ \le&C\widetilde{C}(T)t+C\int_0^t\mathbb{E}\sup_{r\le s\wedge\tau_n}|\nabla X(r)|_2^2ds.\end{aligned}\tag{2.3.56}$$

Moreover, since

$$\mathbb{E}\sup_{s\le t\wedge\tau_n}|J_2(s)|\le(\alpha-1)|\mu|_{L^\infty}\int_0^t\mathbb{E}\sup_{r\le s\wedge\tau_n}|X(r)|_{L^{\alpha+1}}^{\alpha+1}ds,\tag{2.3.57}$$

using the estimate (2.3.55) we have that

$$\begin{aligned}\mathbb{E}\sup_{s\le t\wedge\tau_n}|J_2(s)|\le&(\alpha-1)|\mu|_{L^\infty}C_\epsilon\widetilde{C}(T)t\\ &+\epsilon(\alpha-1)|\mu|_{L^\infty}\int_0^t\mathbb{E}\sup_{r\le s\wedge\tau_n}|\nabla X(r)|_2^2ds.\end{aligned}\tag{2.3.58}$$

To estimate J_3, since $|\nabla\phi_j|_\infty<\infty$, $|\phi_j|_\infty<\infty$, $1\le j\le N$, the Burkholder-Davis-Gundy inequality shows that

$$\mathbb{E}\sup_{s\le t\wedge\tau_n}|J_3(s)|\le C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\sum_{j=1}^N\left(Re\langle\nabla(\phi_jX(s)),\nabla X(s)\rangle_2\right)^2ds\right]^{\frac{1}{2}}$$

$$
\begin{aligned}
&= C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\sum_{j=1}^N\left(Re\langle\nabla\phi_j X(s)+\phi_j\nabla X(s),\nabla X(s)\rangle_2\right)^2 ds\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left(\int_0^{t\wedge\tau_n}|X(s)|_2^4+|\nabla X(s)|_2^4 ds\right)^{\frac{1}{2}},\\
&\leq C\mathbb{E}\left(\int_0^{t\wedge\tau_n}|X(s)|_2^4 ds\right)^{\frac{1}{2}}+C\mathbb{E}\left(\int_0^{t\wedge\tau_n}|\nabla X(s)|_2^4 ds\right)^{\frac{1}{2}},
\end{aligned}
$$

Then, applying Lemma 2.3.4 with Y replaced by $|X(s)|_2^2$ and $|\nabla X(s)|_2^2$ respectively and using Lemma 2.3.5 yield

$$
\begin{aligned}
&\mathbb{E}\sup_{s\leq t\wedge\tau_n}|J_3(s)|\\
\leq&\epsilon C\widetilde{C}(T)+CC_\epsilon\widetilde{C}(T)t+\epsilon C\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2\\
&+CC_\epsilon\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|\nabla X(r)|_2^2 ds. \qquad (2.3.59)
\end{aligned}
$$

For the remaining term J_4, it follows similarly from the Burkholder-Davis-Gundy inequality and Lemma 2.3.4 with Y replaced by $|X|_{L^{\alpha+1}}^{\alpha+1}$ that

$$
\begin{aligned}
\mathbb{E}\sup_{s\leq t\wedge\tau_n}|J_4(s)|\leq&C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\sum_{j=1}^N\left(\int Re\phi_j|X(s)|^{\alpha+1}d\xi\right)^2 ds\right]^{\frac{1}{2}}\\
\leq&C\mathbb{E}\left(\int_0^{t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{2(\alpha+1)}ds\right)^{\frac{1}{2}}\\
\leq&\epsilon C\mathbb{E}\sup_{s\leq t\wedge\tau_n}|X(s)|_{L^{\alpha+1}}^{\alpha+1}+CC_\epsilon\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|X(r)|_{L^{\alpha+1}}^{\alpha+1}ds. \qquad (2.3.60)
\end{aligned}
$$

Then from (2.3.55) it follows that

$$
\begin{aligned}
\mathbb{E}\sup_{s\leq t\wedge\tau_n}|J_4(s)|\leq&CC_\epsilon(\epsilon\widetilde{C}(T)+C_\epsilon\widetilde{C}(T)t)+\epsilon^2C\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2\\
&+\epsilon CC_\epsilon\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|\nabla X(r)|_2^2 ds. \qquad (2.3.61)
\end{aligned}
$$

Taking (2.3.55)-(2.3.61) into (2.3.54) and summing up the like terms, we conclude that

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2\leq&C_1(T)+\epsilon C_2(T)\mathbb{E}\sup_{s\leq t\wedge\tau_n}|\nabla X(s)|_2^2\\
&+C_3(T)\int_0^t\mathbb{E}\sup_{r\leq s\wedge\tau_n}|\nabla X(r)|_2^2 ds,
\end{aligned}
$$

where the constants $C_k(T)$, $1 \leq k \leq 3$, depend on T, $H(x)$, α, $|\phi_j|_\infty$, $|\nabla\phi_j|_\infty$, $1 \leq j \leq N$, and $\mathbb{E} \sup\limits_{t\in[0,\tau^*(x))} |X(t)|_2^p$ with $p \geq 2$.

Therefore, choosing a sufficiently small ϵ and using Gronwall's lemma, we come to

$$\mathbb{E} \sup_{t\in[0,\tau_n]} |\nabla X(t)|_2^2 \leq \widetilde{C}(T) < \infty.$$

Then taking $n \to \infty$ and appylying Fatou's lemma give us

$$\mathbb{E} \sup_{t\in[0,\tau^*(x))} |\nabla X(t)|_2^2 \leq \widetilde{C}(T) < \infty,$$

which yields (2.3.53) by (2.3.52) and Lemma 2.3.5.

(ii) In the defocusing case $\lambda = -1$, the fact that $\frac{1}{2}|\nabla X(t)|_2^2 \leq H(X(t))$ allows to estimate $|X|_{L^{\alpha+1}}^{\alpha+1}$ more directly without using Lemma 2.3.3. Therefore the calculations in the previous case can be much simplified and the condition on α is less restrictive than the focusing case.

More precisely, taking (2.3.56), (2.3.57), (2.3.59) and (2.3.60) into Theorem 2.3.1 and summing up the like terms, we derive that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E} \sup_{s\leq t\wedge\tau_n} |\nabla X(s)|_2^2 + \frac{1}{\alpha+1}\mathbb{E} \sup_{s\leq t\wedge\tau_n} |X(s)|_{L^{\alpha+1}}^{\alpha+1} \\
\leq& C_1(T) + \epsilon C_2(T)\ \mathbb{E} \sup_{s\leq t\wedge\tau_n} (|\nabla X(s)|_2^2 + |X(s)|_{L^{\alpha+1}}^{\alpha+1}) \\
&+ C_3(T) \int_0^t \mathbb{E} \sup_{r\leq s\wedge\tau_n} (|\nabla X(r)|_2^2 + |X(r)|_{L^{\alpha+1}}^{\alpha+1}) ds,
\end{aligned}$$

where the constants $C_k(T)$, $1 \leq k \leq 3$, depend on T, $H(x)$, α, $|\phi_j|_\infty$, $|\nabla\phi_j|_\infty$, $1 \leq j \leq N$, and $\mathbb{E} \sup\limits_{t\in[0,\tau^*(x))} |X(t)|_2^2$.

Therefore, similar arguments as in the end of the previous case yield (2.3.53). This completes the proof of Theorem 2.3.6. □

2.3.2 Subcritical case

Let us first prove the global well-posedness in the subcritical case for the random equation (2.1.5).

Theorem 2.3.7. *Assume $(H2)$. Let α satisfy (2.2.15) and $1 < \alpha < 1 + \frac{4}{d}$ in the defocusing and focusing cases respectively. For each $x \in H^1$ and $0 < T < \infty$, there exists a unique strong solution (y, T) of (2.1.5) in the sense of Definition 2.1.2, such* *that*

$$e^W y \in L^2(\Omega; C([0,T]; H^1)) \cap L^{\alpha+1}(\Omega; C([0,T]; L^{\alpha+1})), \tag{2.3.62}$$

and

$$y \in L^\gamma(0,T; W^{1,\rho}), \quad \mathbb{P} - a.s., \tag{2.3.63}$$

where (ρ, γ) *is any Strichartz pair.*

Moreover, for $\mathbb{P}$*-a.e* ω*, the map* $x \to y(\cdot, x, \omega)$ *is continuous from* H^1 *to* $C([0,T]; H^1) \cap L^\gamma(0,T; W^{1,\rho})$.

Proof. Let $(y, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal solution of (2.1.5) from Theorem 2.2.5. We also recall $(y_n)_{n\in\mathbb{N}}$ in the proof of Theorem 2.2.5. By Lemma 2.3.5 and Theorem 2.3.6

$$\sup_{0\le t<\tau^*(x)} \left(|X(t)|_2^2 + |\nabla(X(t))|_2^2 \right) < \infty, \quad \mathbb{P} - a.s.$$

Since $\|e^{-W}\|_{L^\infty(0,T;W^{1,\infty})} < \infty$, $\mathbb{P}$-a.s, direct computations show that

$$\sup_{0\le t<\tau^*(x)} \left(|y(t)|_2^2 + |\nabla y(t)|_2^2 \right) < \infty, \quad \mathbb{P} - a.s. \tag{2.3.64}$$

Therefore, arguing as in the proof of Theorem 1.3.3 and modifying the definition of y by $y := \lim\limits_{n\to\infty} y_n$, we conclude that that (y, T) is the desired unique strong solution of (2.1.5) in the sense of Definition 2.1.2. Moreover, (2.3.62) follows from Lemma 2.3.5 and Theorem 2.3.6, and (2.3.63) follows from (2.2.17).

We are left to prove the continuous dependence. This proof is analogous to that of (2.3.47), hence we only sketch it.

Suppose that $x_m \to x$ in H^1 and let (y_m, T) be the unique strong solutions of (2.1.5) corresponding to the initial data x_m, $m \ge 1$. Choose the Strichartz pair $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$. As in the proof of Theorem 1.3.3, we can choose a uniform $\tau_1 (\le T)$ such that for all $m \ge m_1$ with m_1 large enough

$$\widetilde{R} := \sup_{m\ge m_1} \left(\|y_m\|_{L^\infty(0,\tau_1;H^1)} + \|y_m\|_{L^q(0,\tau_1;W^{1,p})} \right) < \infty, \quad \mathbb{P} - a.s.$$

We will first prove the continuous dependence on the interval $[0, \tau_1]$. Analogous calculations as in (2.2.27) show that

$$\begin{aligned}&\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^p)}\\&\le 2C_T|x_m - x|_2 + 2C_T D_2(T)\widetilde{R}^{\alpha-1}t^\theta\|y_m - y\|_{L^q(0,t;L^p)},\end{aligned} \tag{2.3.65}$$

where $\theta = 1 - \frac{2}{q} > 0$. Then taking t small enough and independent of m yields that

$$\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^p)} \to 0,\ \ as\ m \to \infty. \tag{2.3.66}$$

To obtain

$$\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,p})} \to 0, \tag{2.3.67}$$

we notice from (2.2.10) that for $m \ge m_1$

$$\begin{aligned}\nabla(y_m - y) =&U(t,0)\nabla(x_m - x) + \int_0^t U(t,s)\Big\{i(D_j\nabla\widetilde{b^j} + \nabla\widetilde{b^j}D_j + \nabla\widetilde{c})(y_m - y)\\&- \lambda i\nabla\left[e^{(\alpha-1)ReW(s)}\left(g(y_m(s)) - g(y(s))\right)\right]\Big\}ds.\end{aligned} \tag{2.3.68}$$

Similarly to (2.3.111) we have that

$$\begin{aligned}&\|i(D_j\nabla\widetilde{b^j} + \nabla\widetilde{b^j}D_j + \nabla\widetilde{c})(y_m - y)\|_{\widetilde{X}'_{[0,t]}}\\ \le&C(T)|x_m - x|_2 + C(T)t^\theta\|y_m - y\|_{L^q(0,t;L^p)},\end{aligned} \tag{2.3.69}$$

where $C(T)$ depends on κ_T, C_T, $\|e^W\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}$.

From (2.3.69), as in (2.3.112), we derive that

$$\begin{aligned}&\|\nabla y_m - \nabla y\|_{L^\infty(0,t;L^2)} + \|\nabla y_m - \nabla y\|_{L^q(0,t;L^p)}\\ \le&C(T)|x_m - x|_{H^1} + C(T)t^\theta\|y_m - y\|_{L^q(0,t;L^p)}\\&+ C(T)\|\nabla g(y_m) - \nabla g(y)\|_{L^{q'}(0,t;L^{p'})}.\end{aligned} \tag{2.3.70}$$

Collecting (2.3.65), (2.3.70), and choosing t small enough and independent of m, we come to

$$\begin{aligned}&\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,p})}\\ \le&C(T)|x_m - x|_{H^1} + C(T)\|\nabla g(y_m) - \nabla g(y)\|_{L^{q'}(0,t;L^{p'})}.\end{aligned} \tag{2.3.71}$$

Therefore, applying similar arguments as we do after (2.3.111) to control the last term, we finally deduce that (2.3.67) holds for t small enough and independent of m.

Reiterating this procedure in finite steps we obtain the continuous dependence on the interval $[0,\tau_1]$. Consequently, as the arguments in the proof of Theorem 1.3.3, this suffices to obtain the continuous dependence on $[0,T]$, thereby completing the proof of Theorem 2.3.7. □

As a consequence of Theorem 2.3.7, Theorem 2.1.3 and the boundedness in space and time of $W(\omega)$ and $\nabla W(\omega)$ for $\mathbb{P}$-a.e $\omega\in\Omega$, we obtain the global well-posedness for SNLS (2.1.1) in the subcritical case.

Theorem 2.3.8. *Assume* $(H2)$. *Let* α *satisfy (2.2.15) and* $1<\alpha<1+\frac{4}{d}$ *in the defocusing and focusing cases respectively. Then for each* $x\in H^1$ *and* $0<T<\infty$, *there exists a unique strong solution* (X,T) *of (2.1.1) in the sense of Definition 2.1.1, such that*

$$X\in L^2(\Omega;C([0,T];H^1))\cap L^{\alpha+1}(\Omega;C([0,T];L^{\alpha+1})), \tag{2.3.72}$$

and

$$X\in L^{\gamma}(0,T;W^{1,\rho}),\quad \mathbb{P}-a.s., \tag{2.3.73}$$

where (ρ,γ) *is any Strichartz pair.*

Furthermore, for $\mathbb{P}-a.e\ \omega$, *the map* $x\to X(\cdot,x,\omega)$ *is continuous from* H^1 *to* $C([0,T];H^1)\cap L^{\gamma}(0,T;W^{1,\rho})$.

Remark 2.3.9. *Theorem 2.3.8 implies that SNLS* (2.1.1) *generates a global stochastic flow in* $H^1(\mathbb{R}^d)$ *in the subcritical case when* α *satisfies* (2.2.15) *and* $1<\alpha<1+\frac{4}{d}$ *in the defocusing and focusing cases respectively.*

2.3.3 Appendix

This subsection contains some details of the proofs used in the previous subsections. Let us first show that Assumption $(H2)$ allows to use Proposition 2.3(a) in [59].

Proof. Recalling the coefficients $\widetilde{b}$ and $\widetilde{c}$ in (1.2.27) and (1.2.28) respectively, we need to prove (1.2.29)-(1.2.31) with $\partial_l\widetilde{b}$ and $\partial_l\widetilde{c}$ replacing $\widetilde{b}$ and $\widetilde{c}$ respectively, $1\leq l\leq d$. For simplicity set $|f|_\infty=|f|_{L^\infty}$ for any $f\in L^\infty(\mathbb{R}^d)$.

From (1.2.27) and (1.2.28), we have that

$$\partial_l\widetilde{b}^k=-\,i\sum_{m=1}^{N}\mu_m\partial_{lk}e_m\beta_m(t),\quad 1\leq k,l\leq d, \tag{2.3.74}$$

$$div\partial_l\widetilde{b} = -i\sum_{m=1}^{N}\mu_m\Delta\partial_l e_m\beta_m(t), \quad 1\le l\le d, \tag{2.3.75}$$

and

$$\begin{aligned}\partial_l\widetilde{c} =& -2\sum_{k=1}^{d}\left(\sum_{m=1}^{N}\mu_m\partial_k e_m\beta_m(t)\right)\left(\sum_{m=1}^{N}\mu_m\partial_{kl}e_m\beta_m(t)\right)\\ &+ i\left[\sum_{m=1}^{N}(|\mu_m|^2+\mu_m^2)e_m\partial_l e_m\right].\end{aligned} \tag{2.3.76}$$

Now, as in (1.2.32), we have from (2.3.74) and Assumption $(H2)$ that for $1\le l,k\le d$

$$\sum_j \sup_{A_j} <\xi> |\partial_l\widetilde{b}^k| \le 2\sum_{m=1}^{N}|\mu_m||\zeta\partial_{lk}e_m|_\infty \sup_{t\in[0,T]}|\beta_m(t)|<\infty,$$

which yields (1.2.29) for $\partial_l\widetilde{b}$, $1\le l\le d$.

Moreover, by (2.3.75) and Assumption $(H2)$, we have for $1\le l\le d$

$$\sup_{[0,T]\times\mathbb{R}^d}\zeta|div\partial_l\widetilde{b}| \le \sum_{m=1}^{N}|\mu_m||\zeta\Delta\partial_l e_m|_\infty\sup_{t\in[0,T]}|\beta_m(t)|<\infty$$

and

$$\limsup_{|\xi|\to\infty}\zeta|div\partial_l\widetilde{b}| \le \sum_{m=1}^{N}|\mu_m||\beta_m(t)|\limsup_{|\xi|\to\infty}|\zeta\Delta\partial_l e_m|_\infty = 0,$$

which yield (1.2.30) and (1.2.31) for $\partial_l\widetilde{b}$, $1\le l\le d$.

Similar argument can be applied to $\partial_l\widetilde{c}$. Indeed, by (2.3.76) and Assumption $(H2)$, we have that for $1\le l\le d$

$$\begin{aligned}\sup_{[0,T]\times\mathbb{R}^d}\zeta|\partial_l\widetilde{c}| \le& 2\sum_{k=1}^{d}\left(\sum_{m=1}^{N}|\mu_m||\zeta\partial_k e_m|_\infty\sup_{t\in[0,T]}|\beta_m(t)|\right)\left(\sum_{m=1}^{N}|\mu_m||\partial_{kl}e_m|_\infty\sup_{t\in[0,T]}|\beta_m(t)|\right)\\ &+2\left(\sum_{m=1}^{N}|\mu_m|^2|e_m|_\infty|\zeta\partial_l e_m|_\infty\right)<\infty,\end{aligned}$$

and

$$\limsup_{|\xi|\to\infty} \zeta|\partial_l \widetilde{c}| \leq 2\sum_{k=1}^{d}\left(\sum_{m=1}^{N}|\mu_m| \limsup_{|\xi|\to\infty}|\zeta\partial_k e_m||\beta_m(t)|\right)\left(\sum_{m=1}^{N}|\mu_m||\partial_{kl}e_m|_\infty|\beta_m(t)|\right)$$
$$+ 2\left(\sum_{m=1}^{N}|\mu_m|^2|e_m|_\infty \limsup_{|\xi|\to\infty}|\zeta\partial_l e_m|\right) = 0,$$

which yield (1.2.30) and (1.2.31) for $\partial_l\widetilde{c}$, $1 \leq l \leq d$, and complete the proof. □

Proof of (2.3.40)**.** By Hausdorf-Young's inequality, for $p \in [1, \infty)$

$$|\Theta_m f|_{L^p} = |m^d\theta^\vee(m\cdot) * f|_{L^p}$$
$$\leq |m^d\theta^\vee(m\cdot)|_{L^1}|f|_{L^p} \leq |\theta^\vee|_{L^1}|f|_{L^p}.$$

Since $\theta \in C_c^\infty \subset \mathcal{S}$, $\theta^\vee \in \mathcal{S} \subset L^1$, thus $|\Theta_m|_{L^p\to L^p} \leq |\theta^\vee|_{L^1} < \infty$, implying the result. □

Proof of (2.3.42)**.** The argument is similar as the previous proof. Hausdorf-Young's inequality shows that

$$|\Theta_m f|_{L^{\alpha+1}} = |(\theta(\frac{|\cdot|}{m}))^\vee * f|_{L^{\alpha+1}}$$
$$\leq |(\theta(\frac{|\cdot|}{m}))^\vee|_{L^{\frac{\alpha+1}{2}}}|f|_{L^{\frac{\alpha+1}{\alpha}}}.$$

Since $\theta(\frac{|\cdot|}{m}) \in C_c^\infty \subset \mathcal{S}$, we have $(\theta(\frac{|\cdot|}{m}))^\vee \in \mathcal{S} \subset L^{\frac{\alpha+1}{2}}$ implying $|\Theta_m f|_{L^{\alpha+1}} < \infty$. □

Proof of (2.3.43)**.** It follows from Fourier's inversion formula and Fubini's theorem that for $f \in L^{\frac{\alpha+1}{\alpha}} \cap L^1$

$$Re\int if(\xi)\overline{\Theta_m(f)(\xi)}d\xi$$
$$= \frac{1}{(2\pi)^d}Re\int if(\xi)d\xi\overline{\int \widehat{\Theta_m(f)}(\eta)e^{i\xi\cdot\eta}d\eta}$$
$$= \frac{1}{(2\pi)^d}Re\int if(\xi)d\xi\int \theta(\frac{|\eta|}{m})\overline{\widehat{f}}(\eta)e^{-i\xi\cdot\eta}d\eta$$
$$= \frac{1}{(2\pi)^d}Re\int i\theta(\frac{|\eta|}{m})\overline{\widehat{f}}(\eta)d\eta\int f(\xi)e^{-i\xi\cdot\eta}d\xi$$
$$= \frac{1}{(2\pi)^d}Re\int i\theta(\frac{|\eta|}{m})\overline{\widehat{f}}(\eta)\widehat{f}(\eta)d\eta$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^d} Re \int i\theta(\frac{|\eta|}{m})|\widehat{f}(\eta)|^2 d\eta \\
&= 0.
\end{aligned}
$$

For the general case $f \in L^{\frac{\alpha+1}{\alpha}}$, we can take smooth cut-off function χ_n, which is supported on $[-n-1, n+1]$ and equals to 1 on $[-n, n]$. Since $f_n := f\chi_n \in L^{\frac{\alpha+1}{\alpha}} \cap L^1$, it follows from the above result that $Re \int i f_n(\xi)\overline{\Theta_m(f_n)(\xi)} d\xi$, which yields (2.3.43) due to (2.3.42) and the fact that $f_n \to f$ in $L^{\frac{\alpha+1}{\alpha}}$. □

We start to prove Itô's formula for $|X_m(t)|^{\alpha+1}_{L^{\alpha+1}}$. We first notice that Theorem 2.1 in [57] can not apply directly here, since we do not have $X \in L^{\alpha+1}(0,t;W^{1,\alpha+1})$ and $|X|^{\alpha-1}X \in L^{\alpha+1}(0,t;L^{\alpha+1})$ from Theorem 2.2.3. However, in the approximating equation (2.3.44) we have by (2.3.42) and (2.3.43) for the nonlinearity that $\Theta_m(g(X_m)) \in L^{\alpha+1}$ and $Re \int i g(X_m)\Theta_m(g(X_m)) d\xi = 0$, which allow to use the technique in [57] to obtain the Ito formula for $|X(t)|^{\alpha+1}_{L^{\alpha+1}}$.

We adapt the same notation as in [57] and set $h^\varepsilon = h * \psi_\varepsilon$ for any locally integrable function h mollified by ψ_ε, where $\psi_\varepsilon = \varepsilon^{-d}\psi(\frac{x}{\varepsilon})$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ is a real-valued nonnegative function with unit integral. Recall that $|h^\varepsilon|_{L^p} \le |h|_{L^p}$ and if $h \in L^p$, then $h^\varepsilon \to h$ in L^p as $\varepsilon \to 0$, $p > 1$, which will be used in the later estimates. We also need the following lemma, which is a modification of Corollary 3.2 in [57], to justify some limit procedure later.

Lemma 2.3.10. *Let $(\mathbb{E}, \mathscr{E}, \mathfrak{M})$ be a measure space, $u_n, u \in L^r(\mathbb{E})$, $v_n, v \in L^s(\mathbb{E})$ with $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$, $r, s \in (1, \infty)$. Assume that $u_n \to u$, $v_n \to v$ in measure, and $|u_n|_{L^r} \to |u|_{L^r}$, $|v_n|_{L^s} \to |v|_{L^s}$. Then*

$$u_n v_n \to uv, \ \text{in } L^p.$$

Proof. From the above assumptions it follows that $\{u_n\}$, $\{v_n\}$ are weakly compact in L^r and L^s respectively. Hence, $u_n \to u$, $v_n \to v$ weakly in L^r and L^s respectively. Since L^r and L^s are uniformly convex and $|u_n|_{L^r} \to |u|_{L^r}$, $|v_n|_{L^s} \to |v|_{L^s}$, we have

$$u_n \to u \text{ in } L^r, \quad v_n \to v \text{ in } L^s,$$

and the later clearly implies the desired conclusion. □

Lemma 2.3.11. *Let $(X_m, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ be the maximal strong solution of (2.3.44) with τ_n, $n \in \mathbb{N}$, independent of m, and α satisfies (2.2.15). Set $p = \alpha + 1$. We have*

$\mathbb{P}$-*a.s. for* $t < \tau^*(x)$

$$\begin{aligned}|X_m(t)|^p_{L^p} =&|x|^p_{L^p} - p\int_0^t Re\int i\nabla g(X_m)(s)\nabla\overline{X_m}(s)d\xi ds\\&+\frac{1}{2}p(p-2)\sum_{j=1}^N\int_0^t\int (Re\phi_j)^2|X_m(s)|^p d\xi ds\\&+p\sum_{j=1}^N\int_0^t\int Re\phi_j|X_m(s)|^p d\xi d\beta_j(s),\end{aligned} \tag{2.3.77}$$

where $g(X_m) = |X_m|^{p-2}X_m$ *and* $\phi_j = \mu_j e_j$, $1 \le j \le N$.

Proof. From (2.3.44) we have that $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$X_m(t) = x + \int_0^t [-i\Delta X_m(s) - \mu X_m(s) - \lambda i g_m(s)]\, ds + \int_0^t X_m(s)\phi_j d\beta_j(s), \tag{2.3.78}$$

where $g_m(s) = \Theta_m(g(X_m(s)))$, the above equation is taken in the H^{-1} sense and we have used the summation convention over repeated indices for simplicity.

Taking convolution of both sides of the (2.3.78) with the mollifies ψ_ε, we claim that there exists $\widetilde{\Omega}$ of full probability, such that on $\widetilde{\Omega}$ for every $\xi \in \mathbb{R}^d$

$$\begin{aligned}(X_m(t))^\varepsilon(\xi) = x^\varepsilon(\xi) + &\int_0^t [-i\Delta(X_m(s))^\varepsilon(\xi) - (\mu X_m(s))^\varepsilon(\xi) - \lambda i(g_m(s))^\varepsilon(\xi)]\, ds\\&+\int_0^t (X_m(s)\phi_j)^\varepsilon(\xi)d\beta_j(s), \quad t < \tau^*(x).\end{aligned} \tag{2.3.79}$$

Indeed, fix $\xi \in \mathbb{R}^d$, since $\Delta X_m \in C([0,t];H^{-1})$ for $t < \tau^*(x)$, $\int_0^t -i\Delta X_m(s)ds$ is Bochner integrable in H^{-1}. Moreover, $v \to {}_{H^1}\langle\psi_\varepsilon(\xi - \cdot), \overline{v(\cdot)}\rangle_{H^{-1}}$ is a continues linear functional in H^{-1}, hence

$$\begin{aligned}\left(\psi_\varepsilon * \int_0^t -i\Delta X_m(s)ds\right)(\xi) =&\langle\psi_\varepsilon(\xi - \cdot), \overline{\int_0^t -i\Delta X_m(s)ds}\rangle\\=&\int_0^t \langle\psi_\varepsilon(\xi - \cdot), \overline{-i\Delta X_m(s)}\rangle ds\\=&\int_0^t -i\left(\psi_\varepsilon * \Delta X_m(s)\right)(\xi)ds\\=&\int_0^t -i(\Delta X_m(s))^\varepsilon(\xi)ds\\=&\int_0^t -i\Delta(X_m(s))^\varepsilon(\xi)ds\end{aligned}$$

where $\langle\ ,\ \rangle$ is the dual pair between H^1 and H^{-1}. Similar arguments can also be applied to the remaining two drift terms. Moreover, for the last stochastic term, since $\int_0^t X_m(s)\phi_j d\beta_j(s)$ is an H^1-valued stochastic integral and $v \to \langle \psi_\varepsilon(\xi-\cdot), \overline{v(\cdot)}\rangle_2$ is a continues linear functional in L^2, by Lemma 2.4.1 in [73]

$$\begin{aligned}\left(\psi_\varepsilon * \int_0^t X_m(s)\phi_j d\beta_j(s)\right)(\xi) =&\langle \psi_\varepsilon(\xi-\cdot), \overline{\int_0^t X_m(s)\phi_j d\beta_j(s)}\rangle_2 \\ =&\int_0^t \langle \psi_\varepsilon(\xi-\cdot), \overline{X_m(s)\phi_j}\rangle_2 d\beta_j(s) \\ =&\int_0^t (X_m(s)\phi_j)^\varepsilon(\xi) d\beta_j(s).\end{aligned}$$

Therefore, for any fix $\xi \in \mathbb{R}^d$, there exists $\Omega_\xi \in \mathcal{F}$ with $\mathbb{P}(\Omega_\xi) = 1$, such that (2.3.79) holds on Ω_ξ.

In order to find a uniform $\widetilde{\Omega}$ independent of ξ, we need in (2.3.79) the continuity with respect to ξ. Let us check this for the stochastic integral term in (2.3.79). Set $\sigma_{n,l} = \inf\{s \in [0,\tau_n] : |X_m(s)|_{H^1} > l\} \wedge \tau_n$. Since $\xi \to (X_m(s)\phi_j)^\varepsilon(\xi)$ is continuous and

$$\begin{aligned}&\mathbb{E}\Big|\sum_{j=1}^N \int_0^{t\wedge\sigma_{n,l}} (X_m(s)\phi_j)^\varepsilon(\xi) d\beta_j(s)\Big|^2 \\ =&\mathbb{E}\int_0^{t\wedge\sigma_{n,l}} \sum_{j=1}^N |(X_m(s)\phi_j)^\varepsilon(\xi)|^2 ds \\ \le&\mathbb{E}\int_0^{t\wedge\sigma_{n,l}} (\sum_{j=1}^N |\phi_j|_{L^\infty}^2)|\psi_\varepsilon|_2^2 |X_m(s)|_2^2 ds \\ \le&(\sum_{j=1}^N |\phi_j|_{L^\infty}^2)|\psi_\varepsilon|_2^2 l^2 t < \infty,\end{aligned}$$

it follows that $\xi \to \int_0^t (X_m(s)\phi_j)^\varepsilon(\xi) d\beta_j(s)$ is continuous on $\{t \le \sigma_{n,l}\}$. But $\sup_{t\in[0,\tau_n]} |X_m(t)|_{H^1} < \infty$, $\mathbb{P}$-a.s, for $\mathbb{P}$-a.e $\omega \in \Omega$ there exists $l(\omega) \in \mathbb{N}$ such that $\sigma_{n,l}(\omega) = \tau_n(\omega)$ for all $l \ge l(\omega)$. Then

$$\bigcup_{l\in\mathbb{N}} \{t \le \sigma_{n,l}\} = \{t \le \tau_n\}, \tag{2.3.80}$$

implying that $\xi \to \int_0^t (X_m(s)\phi_j)^\varepsilon(\xi) d\beta_j(s)$ is continuous on $\{t \le \tau_n\}$ hence on

$\{t \le \tau^*(x)\}$. One can also check the continuity in ξ for the drift terms in (2.3.79). Therefore, we conclude the claim.

Now, we set for simplicity $X_m^\varepsilon(t) = (X_m(t))^\varepsilon(\xi)$ and correspondingly for the other arguments. Then by Itô's formula we have $\mathbb{P}$-a.s.

$$\begin{aligned}
|X_m^\varepsilon(t)|^p =&|x^\varepsilon|^p - p\int_0^t Re(ig(\overline{X_m^\varepsilon})(s)\Delta X_m^\varepsilon(s))ds \\
&- p\int_0^t Re(g(\overline{X_m^\varepsilon})(s)(\mu X_m)^\varepsilon(s))ds \\
&- \lambda p\int_0^t Re(ig(\overline{X_m^\varepsilon})(s)g_m^\varepsilon(s))ds \\
&+ \frac{p}{2}\int_0^t |X_m^\varepsilon(s)|^{p-2}|(X_m\phi_j)^\varepsilon(s)|^2 ds \\
&+ \frac{1}{2}p(p-2)\int_0^t |X_m^\varepsilon(s)|^{p-4}[Re(\overline{X_m^\varepsilon}(s)(X_m\phi_j)^\varepsilon(s))]^2 ds \\
&+ p\int_0^t Re(g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s))d\beta_j(s), \quad t < \tau^*(x),
\end{aligned} \tag{2.3.81}$$

where $g(\overline{X_m^\varepsilon}) = |X_m^\varepsilon|^{p-2}\overline{X_m^\varepsilon}$.

We next integrate this equality over $\mathbb{R}^d$ and the integrability property (2.3.45) allows us to interchange integrals by deterministic and stochastic Fubini's theorems.

For example, for the second term in the right hand side of (2.3.81), notice that $\Delta X_m^\varepsilon \in C([0,t];H^1)$ and by Sobolev's imbedding theorem

$$\|\Delta X_m^\varepsilon\|_{L^q(0,t;L^{\alpha+1})} \le Dt^{\frac{1}{q}}\|\Delta X_m^\varepsilon\|_{L^\infty(0,t;H^1)}.$$

Hence

$$\begin{aligned}
&\int_0^t\int |Re(ig(\overline{X_m^\varepsilon})\Delta X_m^\varepsilon)|d\xi ds \\
\le&\|g(\overline{X_m^\varepsilon})\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}\|\Delta X_m^\varepsilon\|_{L^q(0,t;L^{\alpha+1})} \\
\le&Dt^{\theta+\frac{1}{q}}\|X_m^\varepsilon\|^\alpha_{L^q(0,t;L^{\alpha+1})}\|\Delta X_m^\varepsilon\|_{L^\infty(0,t;H^1)} < \infty
\end{aligned}$$

with $\theta = 1 - \frac{2}{q} > 0$, implying that

$$\begin{aligned}
&-p\int\int_0^t Re(ig(\overline{X_m^\varepsilon})\Delta X_m^\varepsilon)d\xi ds \\
=&-p\int_0^t\int Re(ig(\overline{X_m^\varepsilon})\Delta X_m^\varepsilon)d\xi ds
\end{aligned}$$

$$=-p\int_0^t Re\int i\nabla g(X_m^\varepsilon)\nabla\overline{X_m^\varepsilon}d\xi ds. \tag{2.3.82}$$

For the fourth term concerning g_m^ε in the right hand side of (2.3.81), by (2.3.42) and Sobolev's imbedding theorem

$$\begin{aligned}&\lambda p\int_0^t\int\left|Re(ig(\overline{X_m^\varepsilon})(s)g_m^\varepsilon(s))\right|d\xi ds\\ \leq&C\int_0^t|g(\overline{X_m^\varepsilon})(s)|_{L^{p'}}|g_m^\varepsilon(s)|_{L^p}ds\\ \leq&C\int_0^t|X_m(s)|_{L^p}^{p-1}|g(X_m)(s)|_{L^{p'}}ds\\ \leq&C\int_0^t|X_m(s)|_{L^p}^{2(p-1)}ds\\ \leq&Ct\sup_{s\in[0,t]}|X_m(s)|_{H^1}^{2(p-1)}<\infty\end{aligned} \tag{2.3.83}$$

with C depending on m, hence

$$-\lambda p\int\int_0^t Re(ig(\overline{X_m^\varepsilon})(s)g_m^\varepsilon(s))dsd\xi=-\lambda p\int_0^t Re\int ig(\overline{X_m^\varepsilon})(s)g_m^\varepsilon(s)d\xi ds. \tag{2.3.84}$$

Moreover, for the last stochastic integrals in the right hand side of (2.3.81), since $|\phi_j|_{L^\infty}<\infty$, $1\leq j\leq N$

$$\begin{aligned}&\int_0^t\left(\int|g(\overline{X_m^\varepsilon})(s)||(X_m\phi_j)^\varepsilon(s)|d\xi\right)^2ds\\ \leq&C\int_0^t\left[|g(\overline{X_m^\varepsilon})(s)|_{L^{p'}}|(X_m\phi_j)^\varepsilon(s)|_{L^p}\right]^2ds\\ \leq&C\int_0^t|X_m(s)|_{L^p}^{2p}ds\\ \leq&Ct\sup_{s\in[0,t]}|X_m(s)|_{H^1}^{2p}<\infty,\end{aligned} \tag{2.3.85}$$

thus by stochastic Fubini's theorem, it follows that

$$\begin{aligned}&p\int\int_0^t Re(g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s))d\beta_j(s)d\xi\\ =&p\int_0^t Re\int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s)d\xi d\beta_j(s).\end{aligned} \tag{2.3.86}$$

The remaining integrals in (2.3.81) can be treated similarly. Therefore, we obtain that

$$\begin{aligned}|X_m^\varepsilon(t)|_{L^p}^p =&|x^\varepsilon|_{L^p}^p - p\int_0^t Re\int i\nabla g(X_m^\varepsilon)(s)\nabla\overline{X_m^\varepsilon}(s)d\xi ds\\ &- p\int_0^t Re\int (\mu X_m)^\varepsilon(s)g(\overline{X_m^\varepsilon})(s)d\xi ds\\ &- \lambda p\int_0^t Re\int ig(\overline{X_m^\varepsilon})(s)g_m^\varepsilon(s)d\xi ds\\ &+\frac{p}{2}\int_0^t\int |X_m^\varepsilon(s)|^{p-2}|(X_m\phi_j)^\varepsilon(s)|^2 d\xi ds\\ &+\frac{1}{2}p(p-2)\int_0^t\int |X_m^\varepsilon(s)|^{p-4}[Re(\overline{X_m^\varepsilon}(s)(X_m\phi_j)^\varepsilon(s))]^2 d\xi ds\\ &+p\int_0^t Re\int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s)d\xi d\beta_j(s)\\ =&|x^\varepsilon|_{L^p}^p + K_1+K_2+K_3+K_4+K_5+K_6. \end{aligned} \tag{2.3.87}$$

We want to pass to the limit $\varepsilon\to 0$ in (2.3.87). Below we mainly show the asymptotics for K_1, K_3 and K_6.

First, we notice that as $\varepsilon\to 0^+$

$$X_m^\varepsilon \to X_m, \quad in\ L^q(0,t;W^{1,p}), \tag{2.3.88}$$

and in particular,

$$X_m^\varepsilon \to X_m,\ \nabla X_m^\varepsilon\to\nabla X_m \quad in\ measure\ dt\times d\xi. \tag{2.3.89}$$

Indeed, (2.3.88) follows from Lebesgue's dominated theorem, due to the fact that

$$X_m^\varepsilon(s)\to X_m(s), \quad in\ W^{1,p}, \quad dt-a.e.\ s\in[0,t],$$

and by (2.3.45)

$$\|X_m^\varepsilon(s)\|_{W^{1,p}} \le \|X_m(s)\|_{W^{1,p}}\in L^q(0,t), \quad dt-a.e.\ s\in[0,t].$$

Now, in order to take the limit in K_1, by Lemma 2.3.10 and (2.3.89) we need only to show as $\varepsilon\to 0$

$$\|\nabla X_m^\varepsilon\|_{L^q(0,t;L^p)}\to\|\nabla X_m\|_{L^q(0,t;L^p)}, \tag{2.3.90}$$

and

$$\|\nabla g(X_m^\varepsilon)\|_{L^{q'}(0,t;L^{p'})} \to \|\nabla g(X_m)\|_{L^{q'}(0,t;L^{p'})}. \tag{2.3.91}$$

(2.3.90) follows directly from (2.3.88). For (2.3.91), direct calculations show that

$$\nabla g(X_m^\varepsilon) = \frac{p-2}{2}|X_m^\varepsilon|^{p-4}(X_m^\varepsilon)^2\nabla\overline{X_m^\varepsilon} + \frac{p}{2}|X_m^\varepsilon|^{p-2}\nabla X_m^\varepsilon. \tag{2.3.92}$$

To treat the first term in the right hand side above, observe that for $dt-a.e$ $s \in [0,t]$ as $\varepsilon \to 0$

$$\begin{aligned}&||X_m^\varepsilon|^{p-4}(s)(X_m^\varepsilon)^2(s)|_{L^{\frac{p}{p-2}}} = ||X_m^\varepsilon|^{p-2}(s)|_{L^{\frac{p}{p-2}}} = |X_m^\varepsilon(s)|_{L^p}^{p-2}\\ \to&|X_m(s)|_{L^p}^{p-2} = ||X_m|^{p-4}(s)(X_m)^2(s)|_{L^{\frac{p}{p-2}}},\end{aligned}$$

and

$$|\nabla\overline{X_m^\varepsilon}(s)|_{L^p} \to |\nabla\overline{X_m}(s)|_{L^p},$$

thus by Lemma 2.3.10 and (2.3.89), as $\varepsilon \to 0$

$$\frac{p-2}{2}|X_m^\varepsilon|^{p-4}(s)(X_m^\varepsilon)^2(s)\nabla\overline{X_m^\varepsilon}(s) \to \frac{p-2}{2}|X_m|^{p-4}(s)(X_m)^2(s)\nabla\overline{X_m}(s),\ in\ L^{p'}.$$

Similar results hold also for the second term in the right hand side of (2.3.92). Thus for $dt-a.e$ $s \in [0,t]$ as $\varepsilon \to 0$

$$\nabla g(X_m^\varepsilon)(s) \to \nabla g(X_m)(s),\ in\ L^{p'}. \tag{2.3.93}$$

Moreover

$$|\nabla g(X_m^\varepsilon)(s)|_{L^{p'}} \le (p-1)|X_m(s)|_{L^p}^{p-2}|\nabla X_m(s)|_{L^p} \in L^{q'}(0,t). \tag{2.3.94}$$

Therefore, it follows from (2.3.93), (2.3.94) and Lebesgue's convergence theorem that (2.3.91) holds, which implies that for K_1

$$\begin{aligned}&\lim_{\varepsilon\to 0} -p\int_0^t Re\int i\nabla g(X_m^\varepsilon)(s)\nabla\overline{X_m^\varepsilon}(s)d\xi ds\\ =&-p\int_0^t Re\int i\nabla g(X_m)(s)\nabla\overline{X_m}(s)d\xi ds.\end{aligned}$$

For the term K_3 concerning g_m^ε, first observe that as $\varepsilon \to 0$

$$|g_m^\varepsilon(s) - g_m(s)|_{L^p} \to 0,\quad |g(X_m^\varepsilon)(s) - g(X_m)(s)|_{L^{p'}} \to 0,\quad s \in [0,t],$$

thus as $\varepsilon \to 0$

$$Re \int ig(\overline{X_m^\varepsilon})(s) g_m^\varepsilon(s) d\xi \to Re \int ig(\overline{X_m})(s) g_m(s) d\xi, \quad s \in [0,t],$$

Moreover, as in estimate (2.3.83)

$$\big| Re \int ig(\overline{X_m^\varepsilon})(s) g_m^\varepsilon(s) d\xi \big| \le C \sup_{s\in[0,t]} |X_m(s)|_{H^1}^{2(p-1)} < \infty.$$

Therefore, it follows from Lebesgue's dominated convergence theorem and (2.3.43) that

$$\begin{aligned}\lim_{\varepsilon\to 0} K_3 =& \lim_{\varepsilon\to 0} -\lambda p \int_0^t Re \int ig(\overline{X_m}^\varepsilon)(s) g_m^\varepsilon(s) d\xi ds \\ =& -\lambda p \int_0^t Re \int ig(\overline{X_m})(s) g_m(s) d\xi ds \\ =& 0.\end{aligned}$$

Finally, as regards the last stochastic term K_6, we will first prove that for $\sigma_{n,l}$ defined above, as $\varepsilon \to 0$

$$\mathbb{E} \int_0^{t\wedge\sigma_{n,l}} Re \left[\int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s) d\xi - \int g(\overline{X_m})(s)(X_m\phi_j)(s) d\xi \right]^2 ds \to 0, \tag{2.3.95}$$

In fact, using similar arguments as above, we have

$$Re \int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s) d\xi - Re \int g(\overline{X_m})(s)(X_m\phi_j)(s) d\xi \to 0, \quad s \in [0, t\wedge\sigma_{n,l}]. \tag{2.3.96}$$

Furthermore, as in the estimate (2.3.85)

$$\big| \int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s) d\xi \big|^2 \le C \sup_{s\in[0,t\wedge\sigma_{n,l}]} |X_m(s)|_{H^1}^{2p} < Cl^{2p}, \ s \in [0, t\wedge\sigma_{n,l}]. \tag{2.3.97}$$

Therefore, (2.3.95) follows from the Lebesgue dominated convergence theorem, hence

$$p \int_0^t Re \int g(\overline{X_m^\varepsilon})(s)(X_m\phi_j)^\varepsilon(s) d\xi d\beta_j(s) \to p \int_0^t \int Re\phi_j |X_m|^p(s) d\xi d\beta_j(s) \tag{2.3.98}$$

in $\mathbb{P}$-measure on $\{t \le \sigma_{n,l}\}$ as $\varepsilon \to 0$, which implies by (2.3.80) that (2.3.98) holds on $\{t \le \tau_n\}$. Therefore, as $\tau_n \to \tau^*(x)$ $\mathbb{P}$-a.s, we conclude that (2.3.98) holds $\mathbb{P}$-a.s. for $t < \tau^*(x)$.

Similar results also hold for the remaining integrals in (2.3.87). Therefore, we may pass to the limit $\varepsilon \to 0$ in (2.3.87) and observe that K_2 and K_4 are canceled after taking the limit. We finally conclude the desired formula (2.3.77) . □

We next prove the Ito formula for $|\nabla X_m|_2^2$.

Lemma 2.3.12. *Assume the conditions in Lemma 2.3.11 to hold. We have $\mathbb{P}$-a.s. for $t < \tau^*(x)$*

$$\begin{aligned}|\nabla X_m(t)|_2^2 =& |\nabla x|_2^2 + 2\int_0^t Re\langle -\nabla(\mu X_m)(s), \nabla X_m(s)\rangle_2 ds + \sum_{j=1}^N \int_0^t |\nabla(X_m(s)\phi_j)|_2^2 ds \\ &- 2\lambda \int_0^t Re \int i\nabla g_m(s)\nabla\overline{X_m}(s)d\xi ds \\ &+ 2\sum_{j=1}^N \int_0^t Re\langle \nabla(\phi_j X_m(s)), \nabla X_m(s)\rangle_2 d\beta_j(s). \end{aligned} \tag{2.3.99}$$

Proof. We follow the ideas as in the proof of Theorem 1.3.1 in Section 1.3.1 to derive (2.3.99). Let $\{f_k | k \in \mathbb{N}\} \subset H^2$ be an orthonormal basis of L^2, set $J_\varepsilon = (I - \varepsilon\Delta)^{-1}$ and $h_\varepsilon := J_\varepsilon(h) \in H^1$ for any $h \in H^{-1}$. Then we have from equation (2.3.44) that $\mathbb{P}$-a.s. for $t \in (0, \tau^*(x))$

$$\begin{aligned} &idX_{m,\varepsilon} = \Delta X_{m,\varepsilon}dt - i(\mu X_m)_\varepsilon dt + \lambda g_{m,\varepsilon}dt + i(X_m\phi_j)_\varepsilon d\beta_j, \\ &X_{m,\varepsilon}(0) = x_\varepsilon, \end{aligned} \tag{2.3.100}$$

where $g_{m,\varepsilon} = [\Theta_m(g(X_m))]_\varepsilon$ and we used the summation convention.

Noticing that $\partial_l f_k \in H^1$ for each f_k, $1 \le l \le d$, $k \in \mathbb{N}$, it follows from (2.3.100) and Fubini's theorem that $\mathbb{P}$-a.s. for $t \in (0, \tau^*(x))$

$$\begin{aligned} &\langle X_{m,\varepsilon}(t), \partial_l f_k\rangle_2 \\ =& \langle x_\varepsilon, \partial_l f_k\rangle_2 + \int_0^t \langle -i\Delta X_{m,\varepsilon}(s), \partial_l f_k\rangle_2 ds + \int_0^t \langle -(\mu X_m)_\varepsilon(s), \partial_l f_k\rangle_2 ds \\ &+ \int_0^t \langle -\lambda i g_{m,\varepsilon}(s), \partial_l f_k\rangle_2 ds + \int_0^t \langle (X_m(s)\phi_j)_\varepsilon, \partial_l f_k\rangle_2 d\beta_j. \end{aligned}$$

Applying Itô's product rule and integrating by parts, we deduce that

$$\begin{aligned} &|\langle X_{m,\varepsilon}(t), \partial_l f_k\rangle_2|^2 \\ =& |\langle x_\varepsilon, \partial_l f_k\rangle_2|^2 \\ &+ 2Re\int_0^t \overline{\langle X_{m,\varepsilon}(s), \partial_l f_k\rangle_2} d\langle X_{m,\varepsilon}(s), \partial_l f_k\rangle_2 + \langle\langle X_{m,\varepsilon}(t), \partial_l f_k\rangle_2, \overline{\langle X_{m,\varepsilon}(t), \partial_l f_k\rangle_2}\rangle \end{aligned}$$

$$
\begin{aligned}
=&|\langle \partial_l x_\varepsilon, f_k\rangle_2|^2 \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\varepsilon}(s), f_k\rangle_2}\langle -i\partial_l \Delta X_{m,\varepsilon}(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\varepsilon}(s), f_k\rangle_2}\langle -\partial_l (\mu X_m)_\varepsilon(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\varepsilon}(s), f_k\rangle_2}\langle -\lambda i\partial_l g_{m,\varepsilon}(s), f_k\rangle_2 ds \\
&+ 2Re\int_0^t \overline{\langle \partial_l X_{m,\varepsilon}(s), f_k\rangle_2}\langle \partial_l (X_m(s)\phi_j)_\varepsilon, f_k\rangle_2 d\beta_j(s) \\
&+ \int_0^t |\langle \partial_l (X_m(s)\phi_j)_\varepsilon, f_k\rangle_2|^2 ds, \quad t < \tau^*(x), \quad \mathbb{P}-a.s.
\end{aligned}
$$

Notice that $\Delta X_{m,\varepsilon}$ and $g_{m,\varepsilon}$ are in H^1, thus the above integrals make sense. This is the reason that we introduce the operator J_ε.

Now summing over $k \in \mathbb{N}$ and interchanging the infinite sum with integrals (which can be justified as in the proof of Theorem 1.1.4), we obtain $\mathbb{P}$-a.s. for all $t \in (0, \tau^*(x))$

$$
\begin{aligned}
&|\partial_l X_{m,\varepsilon}(t)|_2^2 \\
=&\sum_{k=1}^\infty |\langle X_{m,\varepsilon}(t), \partial_l f_k\rangle_2|^2 \\
=&|\partial_l x_\varepsilon|_2^2 + 2\int_0^t Re\langle i\Delta X_{m,\varepsilon}(s), \partial_l^2 X_{m,\varepsilon}(s)\rangle_2 ds \\
&+ 2\int_0^t Re\langle -\partial_l(\mu X_m)_\varepsilon(s), \partial_l X_{m,\varepsilon}(s)\rangle_2 ds + \int_0^t |\partial_l (X_m(s)\phi_j)_\varepsilon|_2^2 ds \\
&- 2\lambda\int_0^t Re\langle i\partial_l g_{m,\varepsilon}(s), \partial_l X_{m,\varepsilon}(s)\rangle_2 ds + 2\int_0^t Re\langle \partial_l (X_m(s)\phi_j)_\varepsilon, \partial_l X_{m,\varepsilon}(s)\rangle_2 d\beta_j(s).
\end{aligned}
$$

Finally, summing over $l : 1 \le l \le d$, and using (1.1.15), (1.1.16) for $k = -1, 0, 1$, we can pass to the limit $\varepsilon \to 0$ in the above equality and conclude the evolution formula (2.3.99). □

Proof of (2.3.47)**.** By the rescaling transformation $X_m = e^W y_m$, it suffices to prove that $\mathbb{P}$-a.s.

$$
y_m \to y, \quad in\ L^\infty(0,t;H^1) \cap L^q(0,t;W^{1,\alpha+1}), \quad t < \tau^*(x),
$$

where $q = \frac{4(\alpha+1)}{d(\alpha-1)}$.

From (2.3.44) and Theorem 2.1.3 with the nonlinear term $|X|^{\alpha-1}X$ replaced by $\Theta_m[g(X_m)]$, it follows that $(y_m, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ with τ_n independent of m is the maximal strong solution of the random equation below

$$\begin{aligned} &dy_m = -ie^{-W}\Delta(e^W y_m)dt - (\mu + \widetilde{\mu})y_m dt - \lambda i e^{-W}\Theta_m(g(e^W y_m))dt, \\ &y_m(0) = x, \end{aligned} \tag{2.3.101}$$

or equivalently

$$y_m = U(t,0)x - \lambda i \int_0^t U(t,s)e^{-W(s)}\Theta_m(g(e^{W(s)}y_m(s)))ds. \tag{2.3.102}$$

By (2.3.45), (2.2.17) and $\|W\|_{L^\infty(0,T;W^{1,\infty})} < \infty$, we have $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\begin{aligned} \widetilde{R}(t) := &\sup_{m\geq 1}(\|y_m\|_{C([0,t];H^1)} + \|y_m\|_{L^q(0,t;W^{1,\alpha+1})}) \\ &+ (\|y\|_{C([0,t];H^1)} + \|y\|_{L^q(0,t;W^{1,\alpha+1})}) < \infty. \end{aligned} \tag{2.3.103}$$

Moreover, taking into account (2.2.18) and (2.3.102), we have

$$y_m - y = -\lambda i \int_0^t U(t,s)e^{-W(s)}\left[\Theta_m(g(e^{W(s)}y_m(s))) - g(e^{W(s)}y(s))\right] ds. \tag{2.3.104}$$

Let us first show that there exists t small enough and independent of m, such that

$$\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} \to 0, \ \textit{as } m \to \infty. \tag{2.3.105}$$

and particularly

$$y_m \to y \ \ \textit{in measure } dt \times d\xi. \tag{2.3.106}$$

Indeed, applying Strichartz estimate (1.2.20) to (2.3.104) we have

$$\begin{aligned} &\|y_m - y\|_{L^\infty(0,t;L^2)} + \|y_m - y\|_{L^q(0,t;L^{\alpha+1})} \\ \leq &C_T\|e^{-W}\|_{L^\infty(0,T;L^\infty)}\|\Theta_m(g(e^W y_m)) - g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ \leq &C(T)\|\Theta_m[g(e^W y_m) - g(e^W y)]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ &+ C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}, \end{aligned} \tag{2.3.107}$$

where $C(T)$ depends on C_T and $\|W\|_{L^\infty(0,T;L^\infty)}$.

It follows from (2.3.40) and (2.2.34) that

$$\|\Theta_m[g(e^W y_m) - g(e^W y)]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \leq C(T)t^\theta \|y_m - y\|_{L^q(0,t;L^{\alpha+1})}, \tag{2.3.108}$$

where $C(T)$ depends on C_T, $\|W\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}(t^*)$ with any fixed $t^* \in (t, \tau^*(x))$, and $\theta = 1 - \frac{2}{q} > 0$. Choosing t small enough, plugging (2.3.108) into (2.3.107) and then using (2.3.41) we consequently obtain (2.3.105).

We next prove that for t sufficiently small and independent of m

$$\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,\alpha+1})} \to 0, \ as\ m \to \infty. \tag{2.3.109}$$

Indeed, we notice from (2.2.10) that

$$\begin{aligned}\nabla(y_m - y) = \int_0^t U(t,s)\Big\{ & i(D_j \nabla \widetilde{b}^j + \nabla \widetilde{b}^j D_j + \nabla \widetilde{c})(y_m - y) \\ & - \lambda i \nabla \left[e^{-W}\left(\Theta_m(g(e^W y_m)) - g(e^W y)\right)\right]\Big\} ds. \end{aligned} \tag{2.3.110}$$

Using Proposition 2.3(a) in [59], applying the estimate (1.2.21) to (2.3.104), and then using(2.3.107) and (2.3.108), we derive that

$$\begin{aligned}&\|i(D_j\nabla\widetilde{b}^j + \nabla\widetilde{b}^j D_j + \nabla\widetilde{c})(y_m - y)\|_{\widetilde{X}'_{[0,t]}} \\ \leq & \kappa_T \|y_m - y\|_{\widetilde{X}_{[0,t]}} \\ \leq & \kappa_T C_T \|e^{-W}\left(\Theta_m(g(e^W y_m)) - g(e^W y)\right)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ \leq & C(T)t^\theta\|y_m - y\|_{L^q(0,t;L^{\alpha+1})} + C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}, \end{aligned} \tag{2.3.111}$$

where $\widetilde{X}_{[0,t]}$ is the local smoothing space defined in [59], $C(T)$ depends on κ_T, C_T, $\|W\|_{L^\infty(0,T;L^\infty)}$ and $\widetilde{R}(t^*) < \infty$, $\mathbb{P}$-a.s, and $\theta = 1 - \frac{2}{q} > 0$.

Thus, applying the estimate (1.2.21) to (2.3.110), using (2.3.111) and similar estimates as in (2.3.107), we have for $m \geq 1$

$$\begin{aligned}&\|\nabla y_m - \nabla y\|_{L^\infty(0,t;L^2)} + \|\nabla y_m - \nabla y\|_{L^q(0,t;L^{\alpha+1})} \\ \leq & 2C_T\|i(D_j\nabla\widetilde{b}^j + \nabla\widetilde{b}^j D_j + \nabla\widetilde{c})(y_m - y)\|_{\widetilde{X}'_{[0,t]}} \\ & + 2C_T\|\lambda i\nabla\left[e^{-W}\left(\Theta_m(g(e^W y_m)) - g(e^W y)\right)\right]\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ \leq & C(T)t^\theta\|y_m - y\|_{L^q(0,t;L^{\alpha+1})} + C(T)\|\nabla g(e^W y_m) - \nabla g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \\ & + C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \end{aligned}$$

$$+ C(T)\|(\Theta_m - 1)\nabla g(e^W y)\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}, \tag{2.3.112}$$

where $C(T)$ is independent of t and m.

Let us estimate the second term in the right hand side of (2.3.112). Since

$$\nabla g(y) = F_1(y)\nabla y + F_2(y)\nabla \overline{y}$$

with $F_1(y) = \frac{\alpha+1}{2}|y|^{\alpha-1}$ and $F_2(y) = \frac{\alpha-1}{2}|y|^{\alpha-3}y^2$, we have

$$\begin{aligned}
&\nabla g(e^W y_m) - \nabla g(e^W y)\\
=&F_1(e^W y_m)[\nabla(e^W y_m) - \nabla(e^W y)] + [F_1(e^W y_m) - F_1(e^W y)]\nabla(e^W y)\\
&+ F_2(e^W y_m)[\nabla(\overline{e^W y_m}) - \nabla(\overline{e^W y})] + [F_2(e^W y_m) - F_2(e^W y)]\nabla(\overline{e^W y})\\
=&I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.3.113}$$

Since $|I_1| + |I_3| \leq \alpha|e^W y_m|^{\alpha-1}|\nabla(e^W y_m - e^W y)|$, (2.2.31) yields

$$\|I_1 + I_3\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \leq C(T)t^\theta \|y_m - y\|_{L^q(0,t;W^{1,\alpha+1})} \tag{2.3.114}$$

with $C(T)$ depending on $\widetilde{R}(t^*)$ and $\|e^W\|_{L^\infty(0,T;W^{1,\infty})}$.

Thus taking (2.3.113) and (2.3.114) into (2.3.112), together with (2.3.107), we derive that

$$\begin{aligned}
&\|y_m - y\|_{L^\infty(0,t;H^1)} + \|y_m - y\|_{L^q(0,t;W^{1,\alpha+1})}\\
\leq&C(T)t^\theta\|y_m - y\|_{L^q(0,t;W^{1,\alpha+1})} + C(T)\|(\Theta_m - 1)g(e^W y)\|_{L^{q'}(0,t;W^{1,\frac{\alpha+1}{\alpha}})}\\
&+ C(T)\|I_2 + I_4\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})}
\end{aligned} \tag{2.3.115}$$

Therefore, choosing t small enough and independent of m, taking the first term to the left and then applying (2.3.41) to the second term in the right hand side of (2.3.115), we deduce that (2.3.109) holds once we prove that

$$\|I_2 + I_4\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \to 0, \ \textit{as } m \to \infty. \tag{2.3.116}$$

To prove (2.3.116), by (2.3.105) we have for dt-a.e. $s \in [0,t]$, as $m \to \infty$

$$e^{W(s)}y_m(s) \to e^{W(s)}y(s), \quad \textit{in } L^{\alpha+1},$$

which yields that

$$\begin{aligned}
&|F_1(e^{W(s)}y_m(s))|_{L^{\frac{\alpha+1}{\alpha-1}}} = \frac{\alpha+1}{2}|e^{W(s)}y_m(s)|^{\alpha-1}_{L^{\alpha+1}} \\
\to&\frac{\alpha+1}{2}|e^{W(s)}y(s)|^{\alpha-1}_{L^{\alpha+1}} = |F_1(e^{W(s)}y(s))|_{L^{\frac{\alpha+1}{\alpha-1}}}.
\end{aligned}$$

Then by Lemma 2.3.10 and (2.3.106), for dt-a.e. $s \in [0,t]$

$$F_1(e^{W(s)}y_m(s)) \to F_1(e^{W(s)}y(s)), \quad in\ L^{\frac{\alpha+1}{\alpha-1}},$$

hence

$$F_1(e^{W(s)}y_m(s))\nabla(e^{W(s)}y(s)) \to F_1(e^{W(s)}y(s))\nabla(e^{W(s)}y(s)), \quad in\ L^{\frac{\alpha+1}{\alpha}}.$$

Moreover, by (2.2.31), for dt-a.e. $s \in [0,t]$

$$\begin{aligned}
&|F_1(e^{W(s)}y_m(s))\nabla(e^{W(s)}y(s)) - F_1(e^{W(s)}y(s))\nabla(e^{W(s)}y(s))|_{L^{\frac{\alpha+1}{\alpha}}} \\
\le&C(T)(|y_m(s)|^{\alpha-1}_{L^{\alpha+1}} + |y(s)|^{\alpha-1}_{L^{\alpha+1}})\|y(s)\|_{W^{1,\alpha+1}} \\
\le&C(T)D^{\alpha-1}(\|y_m\|^{\alpha-1}_{L^\infty(0,t;H^1)} + \|y\|^{\alpha-1}_{L^\infty(0,t;H^1)})\|y(s)\|_{W^{1,\alpha+1}} \\
\le&C(T)D^{\alpha-1}\widetilde{R}(t^*)\|y(s)\|_{W^{1,\alpha+1}} \ \in L^{q'}(0,t),
\end{aligned}$$

where $C(T)$ depends on $\|e^W\|_{L^\infty(0,T;W^{1,\infty})} < \infty$, $\mathbb{P}$-a.s. Therefore, from Lebesgue's dominated convergence theorem it follows that

$$\|I_2\|_{L^{q'}(0,t;L^{\frac{\alpha+1}{\alpha}})} \to 0.$$

The proof for I_4 is similar and consequently we complete the proof of (2.3.47) for t sufficiently small and independent of m. Reiterating this procedure with the estimates as above we conclude (2.3.47) for any $t < \tau^*(x)$. □

Proof of Lemma 2.3.5. As in the proof of Theorem 1.3.1, we have $\mathbb{P}$-a.s.

$$|X(t)|_2^2 = |x|_2^2 + 2\sum_{j=1}^N \int_0^t Re\mu_j < X(s), X(s)e_j >_2 d\beta_j(s), \quad t < \tau^*(x), \quad (2.3.117)$$

where $\tau^*(x)$ is as in Theorem 2.2.3.

Then, applying Itô's formula to $|X(t)|_2^p$ shows

$$|X(t)|_2^p = |x|_2^p + p\int_0^t |X(s)|_2^{p-2}\sum_{j=1}^N Re\mu_j\langle X(s), X(s)e_j\rangle_2 d\beta_j(s)$$

$$+\frac{1}{2}p(p-2)\int_0^t |X(s)|_2^{p-4}\sum_{j=1}^N (Re\mu_j)^2\langle X(s),X(s)e_j\rangle_2^2 ds, \quad t<\tau^*(x).$$

Hence, by Burkholder-Davis-Gundy's inequality and Lemma 2.3.4 with Y replaced by $|X|_2^{2p}$, we derive that for every τ_n, $n\in\mathbb{N}$

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}|X(s)|_2^p \le& |x|_2^p + C\mathbb{E}\left[\int_0^{t\wedge\tau_n} p^2|X(s)|_2^{2p-4}\sum_{j=1}^N (Re\mu_j)^2|\langle X(s),X(s)e_j\rangle_2|^2 ds\right]^{\frac{1}{2}}\\
&+\frac{1}{2}p(p-2)\sum_{j=1}^N (Re\mu_j)^2\ \mathbb{E}\int_0^{t\wedge\tau_n}|X(s)|_2^{p-4}|\langle X(s),X(s)e_j\rangle_2|^2 ds\\
\le& |x|_2^p + \sqrt{2|\mu|_\infty}pC\mathbb{E}\left[\int_0^{t\wedge\tau_n}|X(s)|_2^{2p}ds\right]^{\frac{1}{2}}\\
&+2p(p-2)|\mu|_\infty\mathbb{E}\int_0^{t\wedge\tau_n}|X(s)|_2^p ds\\
\le& |x|_2^p + \epsilon\sqrt{2|\mu|_\infty}pC\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}|X(s)|_2^p\\
&+C_\epsilon\sqrt{2|\mu|_\infty}pC\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\tau_n]}|X(r)|_2^p ds\\
&+2p(p-2)|\mu|_\infty\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\tau_n]}|X(r)|_2^p ds.
\end{aligned}$$

Choosing ϵ small enough and applying Gronwall's inequality, we have that

$$\mathbb{E}\left[\sup_{t\in[0,\tau_n]}|X(t)|_2^p\right]\le \widetilde{C}(T)<\infty,$$

which yields Lemma 2.3.5 by taking $n\to\infty$ and applying Fatou's lemma. □

Remark 2.3.13. *Similarly to Remark 1.3.6, the results in this chapter still hold under Assumption $(H2)$ and Assumption $(\widetilde{H2})$ below*

$(\widetilde{H2})$

$$\sum_{k=1}^\infty |\mu_k|^2(|e_k|_{L^\infty}^2+|\nabla e_k|_{L^\infty}^2)<\infty \tag{2.3.118}$$

and for any multi-index γ, $|\gamma| \geq 0$

$$\sum_{k=1}^{\infty} |\mu_k||\partial^\gamma e_k|_{L^\infty} < \infty. \tag{2.3.119}$$

2.4 Notes

In the subcritical case, the global well-posedness of NLS goes back to the papers by J. Ginibre and G. Velo [35, 36, 37]. The fixed point arguments in Theorem 2.2.5 benefit from [46] and [58]. We also refer the reader to [39, 22] for a compactness argument. This method allows to obtain the existence result in general domains in $\mathbb{R}^d$, where the Strichartz estimates do not hold, while it can not give us the uniqueness result.

In the stochastic case, the global existence and uniqueness were first obtained by A. de Bouard and A. Debussche [12] in the conservative case. The proofs there follow the direct approach as in [10], which leads to the restrictive condition on $\alpha : \alpha < 1 + \frac{2}{d-1}$ if $d \geq 6$.

In the critical case, the local well-posedness of NLS was studied in [19], which also included the global well-posedness for small initial data. The extension to general H^s space can be found in [20]. See also Remark 2.2.10 for the smoothness effect of initial data on the well-posedness results.

For the global well-posedness of NLS with large initial data in the critical case, two aspects are extensively studied.

One aspect is concerned with the focusing mass-critical case, i.e $\lambda = 1$, $\alpha = 1 + \frac{4}{d}$. M. I. Weinstein [94] found a threshold Q, arising from the optimal constant in Gagliardo-Nirenberg's inequality, and proved the global well-posedness for initial data x with $|x|_2 < |Q|_2$. This work inspired numerous works focusing on the blowup phenomena in the case when $|x|_2 \geq |Q|_2$ (see Section 3.4 in the next chapter for more details).

Another aspect is concerned with the energy-critical case $\alpha = 1 + \frac{4}{d-2}$ $(d \geq 3)$. As in the L^2-critical case in Notes 1.4 it was conjectured that, in the defocusing case $(\lambda = -1)$ NLS is globally well posed and solutions obey global spacetime bounds in (2.2.36), in particular, scattering holds. While in the focusing case $(\lambda = 1)$, the same results hold for initial data less than a threshold which is characterized by the ground state. The first major step was obtained by J. Bourgain [17] for the defocusing case $(\lambda = -1)$ in three and four dimensions with radial data. The case for higher dimensions with radial data was proved by T. Tao [85]. For the defocusing non-radial case, see [23] in three dimension and [79, 92, 93] in higher dimensions.

In the focusing ($\lambda = 1$) radial data case, the existence of minimal blowup solutions was proved by C. E. Kenig and F. Merle [49] in dimensions $3, 4, 5$, based on the concentration compactness arguments. For all dimensions see [54]. For non-radial data in dimensions five and higher see [52]. We also refer the interested reader to [53] for comprehensive reviews on this conjecture.

Chapter 3

The noise effects on blowup in the non-conservative case

This chapter is devoted to the study of noise effects on the blow-up phenomena in the non-conservative focusing mass-critical/supercritical cases. We first introduce some preliminaries in Section 3.1. Then in Section 3.2 we prove the non-explosion results in the non-conservative case, and some technical proofs are included in Section 3.3. Finally, some reviews of relevant results are contained in Section 3.4.

3.1 Preliminaries.

Consider the stochastic nonlinear Schrödinger equation

$$\begin{aligned} & idX(t,\xi) = \Delta X(t,\xi)dt + \lambda |X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\ & \qquad\qquad - i\mu(\xi)X(t,\xi)dt + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \qquad (3.1.1) \\ & X(0) = x \in H^1. \end{aligned}$$

In this chapter, we will study the non-conservative focusing mass-critical/supercritical case: $\lambda = 1$ and α satifies

$$\begin{cases} \alpha \in [1+\frac{4}{d},\infty), & \text{if } d=1,2; \\ \alpha \in [1+\frac{4}{d}, 1+\frac{4}{d-2}), & d\geq 3, \end{cases} \qquad (3.1.2)$$

and $W(t,\xi)$ is defined as in (1.1.2), i.e.,

$$W(t,\xi) = \sum_{j=1}^{N} \mu_j e_j(\xi)\beta_j(t), t\geq 0, \xi\in\mathbb{R}^d, \qquad (3.1.3)$$

where we assume $N < \infty$, $\beta_j(t)$, $1 \le j \le N$, are independent real Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\ge 0}$.

Moreover, "non-conservative" means that

$$\exists j_0 : 1 \le j_0 \le N,\ \textit{such that } Re(\mu_{j_0}) \neq 0. \tag{3.1.4}$$

We may assume that $Re\mu_1 \neq 0$ without loss of generality.

(Note that, in the non-conservative case, the mass $|X(t)|_2^2$ is a general martingale depending on time, on longer conserved.)

The real-valued functions e_j, $1 \le j \le N$, in the colored Brownian motion W are under the following assumption.

(H3) *$e_j = f_j + c_j$, $1 \le j \le N$, where c_j are real constants and f_j are real-valued functions, such that $f_j \in C_b^\infty$ and*

$$\lim_{|\xi|\to\infty} \zeta(\xi) \sum_{1\le|\gamma|\le 3} |\partial^\gamma f_j(\xi)| = 0,$$

where γ is multi-index and ζ is as in Assumption $(H1)$ in Chapter 1.

In the previous two chapters, we have applied the rescaling transformation (1.1.5) to (3.1.1) and obtained the random equation

$$\begin{aligned} \frac{\partial y(t,\xi)}{\partial t} &= A(t)y(t,\xi) - ie^{(\alpha-1)ReW(t,\xi)}|y(t,\xi)|^{\alpha-1}y(t,\xi), \\ y(0) &= x, \end{aligned} \tag{3.1.5}$$

with $A(t)$ as in (1.1.7), i.e.

$$A(t) = -i(\Delta + b(t)\cdot\nabla + c(t)), \tag{3.1.6}$$

$$b(t) = 2\nabla W(t), \tag{3.1.7}$$

$$c(t) = \sum_{j=1}^{d} (\partial_j W(t))^2 + \Delta W(t) - i\widehat{\mu}, \tag{3.1.8}$$

and

$$\widehat{\mu} := \mu + \widetilde{\mu} \tag{3.1.9}$$

with μ, $\widetilde{\mu}$ as in (1.1.3) and (1.1.10) respectively.

Observe that the real part of the damped term $\widehat{\mu}$ is positive in the non-conservative case. Indeed,

$$Re\widehat{\mu} = \sum_{j=1}^{N}(Re\mu_j)^2 e_j^2 \geq (Re\mu_1)^2 c_1^2 > 0. \tag{3.1.10}$$

Therefore, in order to explore the noise effects, we apply to (3.1.5) a second transformation

$$z(t,\xi) = e^{\widehat{\mu}t}y(t,\xi), \tag{3.1.11}$$

and derive that

$$\begin{aligned}&\frac{\partial z(t)}{\partial t} = \widehat{A}(t)z(t) - ie^{-(\alpha-1)(Re\widehat{\mu}t - ReW(t))}|z(t)|^{\alpha-1}z(t),\\ &z(0) = x \in H^1,\end{aligned} \tag{3.1.12}$$

where

$$\widehat{A}(t) = -i(\Delta + \widehat{b}(t)\cdot\nabla + \widehat{c}(t)) \tag{3.1.13}$$

with

$$\widehat{b(t)} = -2t\nabla\widehat{\mu} + 2\nabla W(t), \tag{3.1.14}$$

and

$$\begin{aligned}\widehat{c(t)} =& t^2\sum_{j=1}^{N}(\partial_j\widehat{\mu})^2 - t\Delta\widehat{\mu} - 2t\nabla W(t)\cdot\nabla\widehat{\mu}\\ &+\left[\sum_{j=1}^{N}(\partial_j W(t))^2 + \Delta W(t)\right].\end{aligned} \tag{3.1.15}$$

We notice that there appears here an exponential decay term $e^{-(\alpha-1)Re\widehat{\mu}t}$ in front of the nonlinear term in (3.1.12), hence we may expect that the blow-up can be prevented provided μ is sufficiently large (or in other sense, the noise is sufficiently large). This is indeed the case that we will prove in the next section. For this purpose, let us rewrite equation (3.1.12) in the mild form

$$z(t) = V(t,0)x + \int_0^t (-i)V(t,s)\left[h(s)|z(s)|^{\alpha-1}z(s)\right]ds, \tag{3.1.16}$$

where

$$h(s) := e^{-(\alpha-1)(Re\widehat{\mu}s - ReW(s))} \tag{3.1.17}$$

and $V(t,s)$ is the evolution operator generated by $\widehat{A}(t)$, that is, $V(t,s) = z(t)$, $s \leq t \leq T$, where

$$\frac{dz(t)}{dt} = \widehat{A}(t)z(t), \quad a.e\ t \in (s,T), \tag{3.1.18}$$
$$z(s) = x \in H^1.$$

(The existence of the evolution operator $V(t,s)$ follows similarly as in Lemma 1.2.1.)

Remark 3.1.1. *In Chapter 1 and Chapter 2, the local existence and uniqueness are established under the assumption that e_j satisfy the further decay assumption $\lim\limits_{|\xi|\to 0} \zeta(\xi)|e_j(\xi)| = 0$. Indeed, we can remove this restriction, due to the fact that $\widehat{b}, \widehat{c}$ only involves the gradient terms of $\widehat{\mu}$ and $W(t)$ (see Section 3.3 for the proof). Hence we are allowed to take c_1 very large to prevent blow-up.*

Moreover, under Assumption $(H3)$ *one can check from [59] that Strichartz estimates* (2.2.8) *also hold for $V(t,s)$.*

3.2 The non-explosion results

Let us first consider the case when the noise $W(t)$ is independent of the space variable.

Theorem 3.2.1. *Consider* (3.1.1) *in the non-conservative case* (3.1.4)*. Let $\lambda = 1$ and α satisfy* (3.1.2)*. Assume* $(H3)$*, but with f_j, $1 \leq j \leq N$, also being fixed constants and c_k for $2 \leq k \leq N$ being fixed. Then for any $x \in H^1$,*

$$\mathbb{P}(X(t) \text{ does not blow up on } [0,\infty)) \to 1, \quad \text{as } c_1 \to \infty.$$

(where we recall that by assumption (3.1.4) *we have $Re\mu_1 \neq 0$.)*

Similar phenomena happen for the deterministic damped nonlinear Schrödinger equation ([72]),

$$i\partial_t u + \Delta u + |u|^{\alpha-1}u + iau = 0, \quad a > 0. \tag{3.2.19}$$

Notice that, this equation is analogous to (3.1.5) in the case where the noise $W(t)$ is space-independent and $\mu_k \in \mathbb{R}$, $1 \leq k \leq N$, namely

$$i\partial_t y - \Delta y - e^{(\alpha-1)ReW(t)}|y|^{\alpha-1}y + i\widehat{\mu}y = 0, \quad \widehat{\mu} > 0.$$

(This similarity indicates that the multiplicative noise term has a dissipativity effect in the non-conservative case.)

Under the assumption that a is large enough, the authors obtained the global well-posedness for (3.2.19), based on the decay estimate of $e^{it\Delta}$ (see Lemma 4 in [72]).

The proof we present below is quite different than that in [72]. It is based on the contraction mapping arguments as in Chapter 2 without use of the decay estimate. The adavantage of this proof is that it can also be applied to the case when the noise is space-dependent (see Theorem 3.2.2 below).

Proof. By the transformations (1.1.5) and (3.1.11), it is equivalent to prove this theorem for the random equation (3.1.12).

Let us first observe in equation (3.1.18) that $\widehat{b} = \widehat{c} = 0$, hence $V(t,s) = e^{-i(t-s)\Delta}$ and the Strichartz coefficient $C_t \equiv C$ is independent of t.

Choose the Strichartz pair $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$, Set

$$\mathcal{Z}_M^\tau = \{u \in C(0,\tau;L^2) \cap L^q(0,\tau;L^p) : \|u\|_{L^\infty(0,\tau;H^1)} + \|u\|_{L^q(0,\tau;W^{1,p})} \le M\}, \tag{3.2.20}$$

and define the integral operator G on $\mathcal{Z}_M^\tau$ by

$$G(u)(t) = V(t,0)x + \int_0^t (-i)V(t,s)\left[h(s)|u(s)|^{\alpha-1}u(s)\right] ds, \tag{3.2.21}$$

for $u \in \mathcal{Z}_M^\tau$.

We first claim that, for $u \in \mathcal{Z}_M^\tau$

$$\begin{aligned} &\|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \\ \le& 2C|x|_{H^1} + 2C\|h|u|^{\alpha-1}u\|_{L^{q'}(0,\tau;W^{1,p'})} \\ \le& 2C|x|_{H^1} + 2CD_3(\tau)\|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)}\|u\|_{L^q(0,\tau;W^{1,p})} \\ \le& 2C|x|_{H^1} + 2CD_3(\tau)M^\alpha, \end{aligned} \tag{3.2.22}$$

where

$$D_3(t) = \alpha D^{\alpha-1}\|h\|_{L^v(0,t)} \tag{3.2.23}$$

with $v > 1$ satisfying $\frac{1}{v} = 1 - \frac{2}{q} > 0$.

Indeed, as in (2.2.30), Hölder's inequality and Sobolev's imbedding theorem yield

$$\begin{aligned}\|h|u|^{\alpha-1}u\|_{L^{q'}(0,\tau;L^{p'})} \leq& |h|_{L^v(0,\tau)}\||u|^{\alpha-1}u\|_{L^q(0,\tau;L^{p'})}\\ \leq& D^{\alpha-1}|h|_{L^v(0,\tau)}\|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)}\|u\|_{L^q(0,\tau;L^p)},\end{aligned} \tag{3.2.24}$$

and similarly to (2.2.31)

$$\begin{aligned}\|h\nabla(|u|^{\alpha-1}u)\|_{L^{q'}(0,\tau;L^{p'})} \leq& \alpha\|h|u|^{\alpha-1}|\nabla u|\|_{L^{q'}(0,\tau;L^{p'})}\\ \leq& \alpha D^{\alpha-1}|h|_{L^v(0,\tau)}\|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)}\|\nabla u\|_{L^q(0,\tau;L^p)}.\end{aligned} \tag{3.2.25}$$

Hence (3.2.22) follows from (3.2.24) and (3.2.25), as claimed.

Moreover, similarly to (2.2.27), we have that for $u_1, u_2 \in \mathcal{Z}^\tau_M$

$$\begin{aligned}&\|G(u_1) - G(u_2)\|_{L^\infty(0,\tau;L^2)} + \|G(u_1) - G(u_2)\|_{L^q(0,\tau;L^p)}\\ \leq& 4CD_3(\tau)M^{\alpha-1}\|u_1 - u_2\|_{L^q(0,\tau;L^p)}.\end{aligned} \tag{3.2.26}$$

Let $M = 3C|x|_{H^1}$ and choose the $(\mathscr{F}_t)$-stopping time $\tau = \tau(c_1)$ defined by

$$\tau = \inf\left\{t > 0 : 2 \cdot 3^\alpha |x|^{\alpha-1}_{H^1} C^\alpha D_3(t) > 1\right\}. \tag{3.2.27}$$

Then, using similar arguments as in the proof of Theorem 2.2.5, we obtain a strong solution (z, τ) of (3.1.12).

We next show that $\mathbb{P}(\tau = \infty) \to 1$, as $c_1 \to \infty$. Since the definition of τ involves the term $D_3(t)$, from (3.2.23) we need to estimate $\|h\|_{L^v(0,\infty)}$.

Set $\phi_k = \mu_k e_k$, $1 \leq k \leq N$. By the scaling property of Brownian motion, i.e. $\mathbb{P} \circ [Re\phi_k\ \beta_k(\cdot)]^{-1} = \mathbb{P} \circ [\beta_k((Re\phi_k)^2\cdot)]^{-1}$, we have for any $c \geq 0$

$$\begin{aligned}&\mathbb{P}(\|h\|^v_{L^v(0,\infty)} \geq c)\\ =&\mathbb{P}\left(\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s - Re\phi_k\beta_k(s)]}ds \geq c\right)\\ =&\mathbb{P}\left(\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s - \beta_k((Re\phi_k)^2 s)]}ds \geq c\right),\end{aligned} \tag{3.2.28}$$

Moreover, from the law of the iterated logarithm of Brownian motion, it follows that

$$\widetilde{C}_1 := \int_0^\infty e^{-(\alpha-1)v[s-\beta_1(s)]}ds < \infty,\ a.s, \tag{3.2.29}$$

and

$$\widetilde{C} := \max_{2\le k\le N} \sup_{s\ge 0} e^{-(\alpha-1)v[(Re\phi_k)^2 s-\beta_k((Re\phi_k)^2 s)]} < \infty,\ a.s. \tag{3.2.30}$$

(We may also take $\widetilde{C} > 1$.)

Hence $\mathbb{P}$-a.s.

$$\begin{aligned}
&\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v[(Re\phi_k)^2 s-\beta_k((Re\phi_k)^2 s)]}ds\\
\le&\widetilde{C}^N \int_0^\infty e^{-(\alpha-1)v[(Re\phi_1)^2 s-\beta_1((Re\phi_1)^2 s)]}ds\\
\le&\frac{1}{(Re\phi_1)^2}\widetilde{C}^N\widetilde{C}_1.
\end{aligned} \tag{3.2.31}$$

Taking (3.2.31) into (3.2.28), we conclude that, for any fixed $c \ge 0$

$$\mathbb{P}(\|h\|^v_{L^v(0,\infty)} \ge c) \le \mathbb{P}\left(\widetilde{C}^N\widetilde{C}_1 \ge c(Re\phi_1)^2\right) \to 0, \quad as\ c_1 \to \infty, \tag{3.2.32}$$

where the last convergence is due to the fact that $\widetilde{C}^N\widetilde{C}_1 < \infty,\ a.s.$ and $Re^2\phi_1 \ge Re^2\mu_1 c_1^2 \to \infty$ as $c_1 \to \infty$.

Now, choose $c = \left[4\cdot 3^\alpha \alpha |x|_{H^1}^{\alpha-1} C^\alpha D^{\alpha-1}\right]^{-v} > 0$. By the definition of τ in (3.2.27) and (3.2.32), we derive that

$$\begin{aligned}
&\mathbb{P}(\tau = \infty)\\
=&\mathbb{P}\left(2\cdot 3^\alpha |x|_{H^1}^{\alpha-1} C^\alpha D_3(t) < 1,\ \forall t \in [0,\infty)\right)\\
\ge&\mathbb{P}\left(2\cdot 3^\alpha\alpha |x|_{H^1}^{\alpha-1} C^\alpha D^{\alpha-1}\|h\|_{L^v(0,\infty)} \le \frac{1}{2}\right)\\
\ge&1-\mathbb{P}\left(\|h\|^v_{L^v(0,\infty)} \ge c\right)\\
\to&1, \quad as\ c_1 \to \infty,
\end{aligned}$$

which completes the proof. □

Next, we consider the general case when the noise $W(t)$ is space-dependent.

Theorem 3.2.2. *Consider* (3.1.1) *in the non-conservative case* (3.1.4). *let* $\lambda = 1$, α *satisfy* (3.1.2). *Assume* $(H3)$ *with* f_j, $1 \le j \le N$, *and* c_k, $2 \le k \le N$ *being fixed. Then for any* $x \in H^1$ *and* $0 < T < \infty$

$$\mathbb{P}(X(t)\ \textit{does not blow up}\ \ on\ [0,T]) \to 1, \quad as\ c_1 \to \infty.$$

In this case, we have in $\widehat{A}(t)$ (see (3.1.13)) the additional lower order terms, for which we are not sure whether the decay estimate as in [72] still hold. The proof presented below is based on the contraction mapping arguments as in Theorem 3.2.1 as well as the Strichartz estimates (2.2.8) established for the lower order perturbations of the Laplacian. Since we just have local in time Strichartz estimate, that is, the Strichartz coefficient C_T in (2.2.8) depends on time and goes to ∞ as $T \to \infty$, we focus on the finite time interval in Theorem 3.2.2.

Proof of Theorem 3.2.2. As in the proof of Theorem 3.2.1, it is equivalent to prove this theorem for the random equation (3.1.12).

Choose the Strichartz pair $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$ and set $\mathcal{Z}_M^\tau$, G as in (3.2.20) and (3.2.21) repectively

Similarly to (3.2.22), for $u \in \mathcal{Z}_M^\tau$

$$\begin{aligned}
&\|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \\
\leq& 2C_\tau |x|_{H^1} + 2C_\tau \|h|u|^{\alpha-1}u\|_{L^{q'}(0,\tau;W^{1,p'})} \\
\leq& 2C_\tau |x|_{H^1} + 2C_\tau D_4(\tau) \|u\|^{\alpha-1}_{L^\infty(0,\tau;H^1)} \|u\|_{L^q(0,\tau;W^{1,p})} \\
\leq& 2C_\tau |x|_{H^1} + 2C_\tau D_4(\tau) M^\alpha,
\end{aligned} \tag{3.2.33}$$

where

$$D_4(t) = \alpha D^{\alpha-1} \|h\|_{L^v(0,t;W^{1,\infty})}. \tag{3.2.34}$$

with $v > 1$ satisfying $\frac{1}{v} = 1 - \frac{2}{q} > 0$.

Moreover, for $u_1, u_2 \in \mathcal{Z}_M^\tau$

$$\begin{aligned}
&\|G(u_1) - G(u_2)\|_{L^\infty(0,\tau;L^2)} + \|G(u_1) - G(u_2)\|_{L^q(0,\tau;L^p)} \\
\leq& 4C_\tau D_4(\tau) M^{\alpha-1} \|u_1 - u_2\|_{L^q(0,\tau;L^p)}.
\end{aligned} \tag{3.2.35}$$

Set $M = 3C_\tau |x|_{H^1}$ and choose the $(\mathscr{F}_t)$-stopping time $\tau = \tau(c_1)$ defined by

$$\tau = \inf\{t \in [0, T], 2 \cdot 3^\alpha |x|_{H^1}^{\alpha-1} C_t^\alpha D_4(t) > 1\} \wedge T. \tag{3.2.36}$$

It follows from (3.2.33), (3.2.35) that $G(\mathcal{Z}_M^\tau) \subset \mathcal{Z}_M^\tau$ and G is a contraction on $C([0, \tau]; L^2) \cap L^q(0, \tau; L^p)$. Therefore, using the arguments as in Step 1 in the proof of Theorem 2.2.5, we obtain a strong solution (z, τ) to (3.1.12).

We next prove that $\mathbb{P}(\tau = T) \to 1$, as $c_1 \to \infty$. Taking into account (3.2.36) and (3.2.34), we need to estimate $\|h\|_{L^v(0,t;W^{1,\infty})}$. For simplicity, we set $|f|_\infty = |f|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R}^d)$ and $\phi_k = \mu_k e_k$, $1 \le k \le N$.

Let us first consider the norm $\|h\|_{L^v(0,t;L^\infty)}$. From the expression of h in (3.1.17) and (3.1.10), it follows that

$$\begin{aligned}|h(t)|_{L^\infty} =& \left|e^{-(\alpha-1)\sum_{k=1}^{N}[Re^2\phi_k t - Re\phi_k\beta_k(t)]}\right|_{L^\infty} \\ \le& e^{-(\alpha-1)\sum_{k=1}^{N}[\frac{(Re\mu_1)^2c_1^2}{N}t - |Re\phi_k|_\infty|\beta_k(t)|]}. \end{aligned} \tag{3.2.37}$$

Analogously to (3.2.30)

$$\widetilde{C} := \max_{2\le k\le N}\sup_{t\ge0} e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t - |\beta_k(|Re\phi_k|_\infty^2 t)|]} < \infty, \quad a.s. \tag{3.2.38}$$

Here, we also take $\widetilde{C} > 1$.

Moreover, choosing c_1 large enough such that $c_1 > |f_1|_{L^\infty}$, we have

$$\begin{aligned}&\int_0^T e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t - |\beta_1(|Re\phi_1|_\infty^2 t)|]}dt \\ =&\frac{1}{|Re\phi_1|_\infty^2}\int_0^{|Re\phi_1|_\infty^2 T} e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N|Re\phi_1|_\infty^2}t - |\beta_1(t)|]}dt \\ \le&\frac{1}{|Re\phi_1|_\infty^2}\int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t - |\beta_1(t)|]}dt \\ \le&\frac{1}{|Re\phi_1|_\infty^2}\widetilde{C}_1, \end{aligned} \tag{3.2.39}$$

where $\widetilde{C}_1 := \int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t - |\beta_1(t)|]}dt < \infty$, $\mathbb{P}$-a.s.

Thus, as in (3.2.28), it follows from (3.2.37)-(3.2.39) and the scaling property of β_k, $1 \le k \le N$, that for any $c > 0$ fixed

$$\begin{aligned}&\mathbb{P}\left(C_T^{\alpha v}\|h\|_{L^v(0,T;L^\infty)}^v \ge c\right) \\ \le&\mathbb{P}\left(C_T^{\alpha v}\int_0^T\prod_{k=1}^N e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t - |Re\phi_k|_\infty|\beta_k(t)|]}dt \ge c\right) \\ \le&\mathbb{P}\left(C_T^{\alpha v}\int_0^T\prod_{k=1}^N e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t - |\beta_k(|Re\phi_k|_\infty^2 t)|]}dt \ge c\right) \\ \le&\mathbb{P}\left(C_T^{\alpha v}\widetilde{C}^N\ \widetilde{C}_1 \ge |Re\phi_1|_\infty^2 c\right) \\ \to&0, \quad as\ c_1 \to \infty, \quad \mathbb{P}-a.s., \end{aligned} \tag{3.2.40}$$

where C_T is the Strichartz coefficient and the last convergence is due to the fact that $C_T^{\alpha v}\widetilde{C}^N\ \widetilde{C}_1 < \infty$, $\mathbb{P}$-a.s.

Similar arguments can also be applied to the norm $\|\nabla h\|_{L^v(0,t;L^\infty)}$. Indeed, from (3.1.17) and (3.1.10)

$$\begin{aligned}\nabla h(t) =& h(t)\left[-(\alpha-1)\sum_{k=1}^{N}\left(2Re\phi_k\nabla(Re\phi_k)t-\nabla(Re\phi_k)\beta_k(t)\right)\right]\\ =& h(t)\left[-(\alpha-1)\sum_{k=1}^{N}\left(2Re\phi_k(Re\mu_k\nabla f_k)t-Re\mu_k\nabla f_k\beta_k(t)\right)\right],\end{aligned}$$

then

$$|\nabla h(t)|_\infty \le (\alpha-1)|h(t)|_\infty\sum_{k=1}^{N}\left(2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty t+|Re\mu_k\nabla f_k|_\infty|\beta_k(t)|\right).$$

Hence, for any $c>0$ fixed

$$\begin{aligned}&\mathbb{P}(C_T^{\alpha v}\|\nabla h\|^v_{L^v(0,T;L^\infty)}\ge c)\\ \le&\mathbb{P}\big(C_T^{\alpha v}\int_0^T(\alpha-1)|h(t)|^v_{L^\infty}\\ &\qquad\left[\sum_{k=1}^{N}2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty t+|Re\mu_k\nabla f_k|_\infty|\beta_k(t)|\right]^v dt\ge c\big)\\ \le&\mathbb{P}\Bigg(C_T^{\alpha v}\int_0^T(\alpha-1)\left[\prod_{k=1}^{N}e^{-(\alpha-1)v[\frac{(Re\mu_1)^2c_1^2}{N}t-|\beta_k(|Re\phi_k|^2_{L^\infty}t)|]}\right]\\ &\qquad\left[\left(2\sum_{k=1}^{N}|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty t+|Re\mu_k\nabla f_k|_\infty|\beta_k(t)|\right)\right]^v dt\ge c\Bigg)\\ \le&\mathbb{P}\Bigg(C_T^{\alpha v}\widetilde{C}^N\frac{1}{|Re\phi_1|^2_\infty}\int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t-|\beta_1(t)|]}\\ &\left[\sum_{k=1}^{N}\frac{2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|^2_\infty}t+|Re\mu_k\nabla f_k|_\infty\left|\beta_k\left(\frac{t}{|Re\phi_1|^2_\infty}\right)\right|\right]^v dt\ge\frac{c}{\alpha-1}\Bigg).\end{aligned}$$

Choosing c_1 large enough, such that $\sum_{k=1}^{N}\frac{2|Re\phi_k|_\infty|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|^2_\infty}<1$ and $\frac{|Re\mu_k\nabla f_k|_\infty}{|Re\phi_1|_\infty}<1$, we have

$$\begin{aligned}&\mathbb{P}(C_T^{\alpha v}\|\nabla h\|^v_{L^v(0,T;L^\infty)}\ge c)\\ \le&\mathbb{P}(C_T^{\alpha v}\widetilde{C}^N C_1'\ge\frac{c}{\alpha-1}|Re\phi_1|^2_\infty)\end{aligned}$$

$$\rightarrow 0, \quad as \ c_1 \rightarrow \infty, \tag{3.2.41}$$

where C_T is the Stichartz coefficient and $C_1' := \int_0^\infty e^{-(\alpha-1)v[\frac{1}{4N}t-|\beta_1(t)|]} \left[t + \sum_{k=1}^{N} \beta_k(t) \right]^v dt < \infty$, $\mathbb{P}$-a.s.

Now, we turn back to the definition of τ in (3.2.36). Choosing $c = [4{\cdot}3^\alpha \alpha D^{\alpha-1}|x|_{H^1}^{\alpha-1}]^{-v} > 0$, we deduce from (3.2.40) and (3.2.41) that

$$\begin{aligned}
&\mathbb{P}(\tau = T) \\
\geq &\mathbb{P}(2 \cdot 3^\alpha |x|_{H^1}^{\alpha-1} C_t^\alpha D_1(t) < 1, \forall t \in [0,T]) \\
\geq &\mathbb{P}(2 \cdot 3^\alpha \alpha D^{\alpha-1} |x|_{H^1}^{\alpha-1} C_T^\alpha \|h\|_{L^v(0,T,W^{1,\infty})} < \frac{1}{2}) \\
\geq &1 - \mathbb{P}(C_T^{\alpha v} \|h\|_{L^v(0,T,W^{1,\infty})}^v \geq c) \\
\geq &1 - \mathbb{P}(C_T^{\alpha v} \|h\|_{L^v(0,T,L^\infty)}^v \geq \frac{1}{2}c) - \mathbb{P}(C_T^{\alpha v} \|\nabla h\|_{L^v(0,T,L^\infty)}^v \geq \frac{1}{2}c) \\
\rightarrow &1, \quad as \ c_1 \rightarrow \infty.
\end{aligned}$$

Therefore, we complete the proof of Theorem 3.2.2. □

One may ask further whether the result in Theorem 3.2.1 (also in Theorem 3.2.2) holds with probability 1. This is not generally valid and we have the following result.

Theorem 3.2.3. *Assume the condition in Theorem 3.2.2 to hold. Furthermore, assume* $\mu_k \in \mathbb{R}$, $1 \leq k \leq N$. *Let* $x \in \sum := \{u \in H^1, \int |\xi|^2 |u(\xi)|^2 d\xi < \infty.\}$ *with* $H(x) < 0$, *where* H *is the Hamiltonian defined as in* (2.3.39).

Then there exists $\epsilon_0 > 0$, *such that for* $0 < \epsilon < \epsilon_0$ *and* $0 \leq \sum_{1 \leq k \leq N} |\nabla f_k|_{L^\infty} < \epsilon$, *the solution to (3.1.1) blows up in finite time with positive probability.*

In particular, in the case where f_j, $1 \leq j \leq N$, *are fixed constants, the solution to (3.1.1) blows up in finite time with positive probability.*

Proof of Theorem 3.2.3. The proof follows from the classical virial analysis (see e.g [58]). We remark that, unlike in the deterministic case, there will appear a positive drift term involving a monomial at^3 in the estimate of the variance evolution formula (see (3.2.44) below), this is the reason why we impose the smallness condition on $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ to control this term.

For any $u \in \sum$, define the variance

$$V(u) = \int |\xi|^2 |u(\xi)|^2 d\xi, \tag{3.2.42}$$

and the momentum

$$G(u) = Im \int \xi u(\xi) \cdot \overline{\nabla u(\xi)} d\xi. \tag{3.2.43}$$

We will prove this theorem by contradiction. Assume that the solution $X(t)$ to (3.1.1) exits globally in H^1, $\mathbb{P} - a.s.$

Let us first show that

$$\mathbb{E}V(X(t)) \leq V(x) + 4G(x)t + 8H(x)t^2 + at^3 \tag{3.2.44}$$

with

$$a = \frac{4}{3} \sum_{k=1}^{N} |\mu_k| |\nabla f_k|^2_{L^\infty} |x|^2_2.$$

Indeed, from Theorem 2.3.1 in Chapter 2, Lemma 3.3.1 and Lemma 3.3.2 in Section 3.3, we have that

$$\begin{aligned} V(X(t)) =& V(x) + 4G(x)t + 8H(x)t^2 \\ &+ 4 \sum_{k=1}^{N} \int_0^t (t-s)^2 |\nabla \phi_k X(s)|_2^2 ds \\ &- 4(\alpha - 1) \sum_{k=1}^{N} \int_0^t (t-s)^2 \int \phi_k^2 |X(s)|^{\alpha+1} d\xi ds \\ &+ \frac{16}{\alpha+1} \left[1 - \frac{d(\alpha-1)}{4} \right] \int_0^t (t-s) |X(s)|_{\alpha+1}^{\alpha+1} ds \\ &+ M_t, \end{aligned} \tag{3.2.45}$$

where $\phi_k = \mu_k e_k$, $1 \leq k \leq N$, and

$$\begin{aligned} M_t =& 8 \sum_{k=1}^{N} \int_0^t (t-s)^2 \left[Re \langle \nabla(\phi_k X(s)), \nabla X(s) \rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi \right] d\beta_k(s) \\ &- 8 \sum_{k=1}^{N} \int_0^t (t-s) Im \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s) \\ &+ 2 \sum_{k=1}^{N} \int_0^t \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \end{aligned}$$

Fix $t > 0$ and define for $r \in [0, \infty)$

$$\begin{aligned}\widetilde{M}(t,r) =& 8\sum_{k=1}^{N}\int_0^r (t-s)^2 \left[Re\langle \nabla(\phi_k X(s)), \nabla X(s)\rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi\right] d\beta_k(s) \\ &- 8\sum_{k=1}^{N}\int_0^r (t-s) Im \int \xi\cdot\nabla X(s)\overline{X(s)}\phi_k d\xi d\beta_k(s) \\ &+ 2\sum_{k=1}^{N}\int_0^r \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \end{aligned} \tag{3.2.46}$$

Set $\sigma_m = \inf\{s \in [0,t], |\nabla X_m(s)|_2^2 > m\} \wedge t$, then $\sigma_m \to t$, as $m \to \infty$. Direct computations show that $\widetilde{M}(t, \cdot \wedge \sigma_m)$ is a square integrable martingale, in particular, $\mathbb{E}[\widetilde{M}(t, t\wedge \sigma_m)] = 0$. Indeed, we take the second term in the right hand side of (3.2.46) for example. Note that

$$\begin{aligned}&\mathbb{E}\int_0^{r\wedge\sigma_m} \sum_{k=1}^{N} \left|(t-s) Im \int \xi\cdot\nabla X(s)\overline{X(s)}\phi_k d\xi\right|^2 ds \\ \leq& C\mathbb{E}\int_0^{r\wedge\sigma_m} (t-s)^2 V(X(s)) |\nabla X(s)|_2^2 ds \\ \leq& mC\mathbb{E}\sup_{s\in[0,\sigma_m]} V(X(s)) \int_0^r (t-s)^2 ds, \end{aligned} \tag{3.2.47}$$

where $C = \sum\limits_{k=1}^{N} |\phi_k|_{L^\infty}^2 < \infty$. Then, using the arguments as in the proof of (3.3.57) below, we deduce that the right hand side above is finite.

Now, since the fifth and sixth terms in the right hand side of (3.2.45) are non-positive for α satisfying (3.1.2), taking the expectation in (3.2.45), we consequently conclude that

$$\begin{aligned}\mathbb{E}V(X(\sigma_m\wedge t)) \leq& V(x) + 4G(x)(\sigma_m\wedge t) + 8H(x)(\sigma_m \wedge t)^2 \\ &+ 4\mathbb{E}\int_0^{\sigma_m\wedge t} (\sigma_m\wedge t - s)^2 \sum_{k=1}^{N} |\nabla\phi_k X(s)|_2^2 ds, \quad t < \infty.\end{aligned}$$

Then, letting $m \to \infty$, using Fatou's lemma, and noting that $\nabla\phi_k = \mu_k \nabla f_k$ and $\mathbb{E}|X(t)|_2^2 = |x|_2^2$, we finally obtain (3.2.44) as claimed.

Let $f(t)$ denote the right hand side of (3.2.44), namely,

$$f(t) = V(x) + 4G(x)t + 8H(x)t^2 + at^3.$$

We claim that if $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ is small enough, then there exists $T > 0$ such that $f(T) < 0$. Taking into account that $\mathbb{E}V(X(t)) \geq 0$ and the inequality (3.2.44), we then come to the contradiction.

It remains to prove the claim. As

$$f'(t) = 3at^2 + 16H(x)t + 4G(x),$$

for $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ small enough the discriminant is positive, that is

$$16^2(H(x))^2 - 3 \cdot 4^2 aG(x) > 0.$$

This implies that $f'(t)$ has two roots with the largest one

$$t_* = \frac{2G(x)}{-4H(x) - \sqrt{16(H(x))^2 - 3aG(x)}} > 0.$$

Hence to prove the claim is equivalent to showing that $f(t_*) < 0$. By simple computations and $f'(t_*) = 0$, it follows that

$$f(t_*) = \frac{8}{3}H(x)t_*^2 + \frac{8}{3}G(x)t_* + V(x).$$

We denote the right hand side by $g(t_*)$, where

$$g(t) = \frac{8}{3}H(x)t^2 + \frac{8}{3}G(x)t + V(x)$$

with the largest root

$$\widetilde{t}_* = \frac{-G(x) - \sqrt{(G(x))^2 - \frac{3}{2}H(x)V(x)}}{2H(x)}.$$

Observe that showing $f(t_*) < 0$ is equivalent to proving that $\widetilde{t}_* < t_*$. This is indeed the case, since $\widetilde{t}_*$ is independent of a but $t_* \to \infty$ as $a \to 0$.

Consequently we prove the claim and finish the proof of Theorem 3.2.3. □

3.3 Appendix.

Let us first show that the assumption $\lim_{|\xi|\to\infty} \zeta(\xi)|e_j(\xi)| = 0$ in Assumption $(H2)$ can be removed in the non-conservative case. Since the proofs for (1.2.29) and (1.2.30) are similar to those in Section 2.3.3 in Chapter 2, we only need to check (1.2.31). Let us take the term $t^2(\partial_j\widehat{\mu})^2$ in $\widehat{c}$ (3.1.15) for example. Since $\partial_j\widehat{\mu} = 2(Re\mu_k)^2 e_k\partial_j f_k$, by Assumption $(H3)$

$$\lim_{|\xi|\to\infty} \zeta(\xi)[t^2(\partial_j\widehat{\mu}(\xi))^2]$$
$$\leq 4(Re\mu_k)^4 t^2 |e_k|^2_{L^\infty(\mathbb{R}^d)} |\partial_j f_k|_{L^\infty(\mathbb{R}^d)} \lim_{|\xi|\to\infty} (\zeta(\xi)|\partial_j f_k(\xi)|) = 0.$$

Similar arguments can also be applied to the other terms in $\widehat{b}$, $\widehat{c}$ in (3.1.14) and (3.1.15) respectively. Therefore, Assumption $(H3)$ suffices to yield (1.2.31).

Moreover, since we also need Proposition 2.3(a) in [59] to derive the Strichartz estimates in Sobolev spaces, we shall also check (1.2.29)-(1.2.31) with $\partial_l\widehat{b}$, $\partial_l\widehat{c}$ replacing $\widehat{b}$ and $\widehat{c}$ respectively. However, the proof is similar as above, hence we omit the details here.

Now, for $x \in \sum$, we can use the arguments as in the proof of Theorem 2.2.5 to obtain the maximal strong solution $(z, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of equation (3.1.12). Then by transformations (1.1.5) and (3.1.11), we obtain the corresponding maximal strong solution $(X, (\tau_n)_{n\in\mathbb{N}}, \tau^*(x))$ of (3.1.1) in the sense of Definition 2.1.1, and X satisfies $\mathbb{P}$-a.s. for any Strichartz pair (ρ, γ)

$$X|_{[0,t]} \in C([0,t]; H^1) \cap L^\gamma(0,t; W^{1,\rho}), \quad t < \tau^*(x). \tag{3.3.48}$$

Now, we start to derive the evolution formula for the variance.

Lemma 3.3.1. *For* $x \in \sum := \{u \in H^1 : \int |\xi|^2 |u(\xi)|^2 d\xi < \infty\}$, *it holds that* $\mathbb{P}$*-a.s. for* $t < \tau^*(x)$

$$V(X(t)) = V(x) + 4\int_0^t G(X(s))ds + M_1(t), \tag{3.3.49}$$

where G *is defined as in* (3.2.43) *and*

$$M_1(t) = 2\sum_{k=1}^N \int_0^t \int |\xi|^2 |X(s)|^2 Re\phi_k d\xi d\beta_k(s)$$

with $\phi_k = \mu_k e_k$, $1 \leq k \leq N$.

Proof. (3.3.49) follows heuristically by applying Itô's formula to the integrands in $V(X(t))$ with the space variable ξ fixed and then taking the integration over $\mathbb{R}^d$. To prove it rigourously we use the techniques as in Lemma 2.3.11 in Chapter 2 and lemma 6.5.2 in [22].

Denote $\varphi^\varepsilon = \varphi * \phi_\varepsilon$ for any locally integrable function φ mollified by ϕ_ε, where $\phi_\varepsilon = \varepsilon^{-d}\phi(\frac{x}{\varepsilon})$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a real-valued nonnegative function with unit integral. Set $V_\eta(u) = \int e^{-\eta|\xi|^2}|\xi|^2|u(\xi)|^2 d\xi$ and $V(u) = \int |\xi|^2|u(\xi)|^2 d\xi$, for any $u \in \sum$.

As in (2.3.79), we have $\mathbb{P}$-a.s. for every $\xi \in \mathbb{R}^d$, $t < \tau^*(x)$

$$\begin{aligned}(X(t))^\varepsilon(\xi) = x^\varepsilon(\xi) + \int_0^t [-i\Delta(X(s))^\varepsilon(\xi) - (\mu X(s))^\varepsilon(\xi) - i(g(X(s)))^\varepsilon(\xi)]\, ds \\ + \sum_{k=1}^N \int_0^t (X(s)\phi_j)^\varepsilon(\xi) d\beta_j(s),\end{aligned} \tag{3.3.50}$$

where $g(X(s)) = |X(s)|^{\alpha-1}X(s)$. For simplicity, we set $X^\varepsilon(t) = (X(t))^\varepsilon(\xi)$ and correspondingly for the other arguments.

Applying the product rule yields $\mathbb{P}$-a.s.

$$\begin{aligned}|X^\varepsilon(t)|^2 =& |x^\varepsilon|^2 - 2Re\int_0^t \overline{X}^\varepsilon(s) i\Delta X^\varepsilon(s) ds - 2Re\int_0^t \overline{X}^\varepsilon(s)(\mu X(s))^\varepsilon ds \\ &- 2Re\int_0^t \overline{X}^\varepsilon(s) i[g(X(s))]^\varepsilon ds + \sum_{k=1}^N \int_0^t |(X(s)\phi_k)^\varepsilon|^2 ds \\ &+ 2\sum_{k=1}^N Re\int_0^t \overline{X}^\varepsilon(s)(X(s)\phi_k)^\varepsilon d\beta_k(s), \quad t < \tau^*(x).\end{aligned}$$

Integration over $\mathbb{R}^d$ with $e^{-\eta|\xi|^2}|\xi|^2$, interchanging the integrations and then integrating by parts, we have $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\begin{aligned}V_\eta(X^\varepsilon(t)) =& V_\eta(x^\varepsilon) + 4Im\int_0^t \int e^{-\eta|\xi|^2}(1-\eta|\xi|^2)X^\varepsilon(s)\xi\cdot\nabla\overline{X^\varepsilon}(s) d\xi ds \\ &- 2Re\int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\varepsilon(s)(\mu X(s))^\varepsilon d\xi ds \\ &- 2Re\int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\varepsilon(s) i[g(X(s))]^\varepsilon d\xi ds \\ &+ \sum_{k=1}^N \int_0^t \int e^{-\eta|\xi|^2}|\xi|^2|(X(s)\phi_k)^\varepsilon|^2 d\xi ds \\ &+ 2\sum_{k=1}^N Re\int_0^t \int e^{-\eta|\xi|^2}|\xi|^2\overline{X}^\varepsilon(s)(X(s)\phi_k)^\varepsilon d\xi d\beta_k(s).\end{aligned} \tag{3.3.51}$$

As $\sup_{\xi\in\mathbb{R}^d} e^{-\eta|\xi|^2}[|(1-\eta|\xi|^2)\xi| + |\xi|^2] < \infty$, similar arguments in the proof of Lemma 2.3.11 could be applied to pass to the limit $\varepsilon \to 0$ in (3.3.51). After that, in the right hand side of (3.3.51) the fourth term equals to 0 and the third term cancels with the fifth term. We thus conclude that

$$\begin{aligned} V_\eta(X(t)) =& V_\eta(x) + 4Im\int_0^t\int e^{-\eta|\xi|^2}(1-\eta|\xi|^2)X(s)\xi\cdot\nabla\overline{X}(s)d\xi ds \\ &+2\sum_{k=1}^N\int_0^t\int e^{-\eta|\xi|^2}|\xi|^2|X(s)|^2Re\phi_k d\xi d\beta_k(s),\quad t<\tau^*(x). \end{aligned} \tag{3.3.52}$$

In order to pass to the limit $\eta\to 0$, we will prove that

$$\sup_{s\in[0,\tau_n]} V(X(s))\le \widetilde{C}(n)<\infty,\quad \mathbb{P}-a.s. \tag{3.3.53}$$

Then by (3.3.52), (3.3.53), $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|e^{-\eta|\xi|^2}(1-\eta|\xi|^2)|=1$ and Lebesque's dominated theorem, we obtain (3.3.49) for $t\le\tau_n$, $n\in\mathbb{N}$. Since $\tau_n\to\tau^*(x)$, as $n\to\infty$, we consequently conclude (3.3.49) for $t<\tau^*(x)$.

In order to prove (3.3.53), for every $n\in\mathbb{N}$, set $\sigma_{n,m}=\inf\{s\in[0,\tau_n]:|\nabla X(s)|_2^2>m\}\wedge\tau_n$. By Burkholder-Davis-Gundy's inequality

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t\wedge\sigma_{n,m}]}V_\eta(X(s)) \le& 4\mathbb{E}\int_0^{t\wedge\sigma_{n,m}}\int e^{-\eta|\xi|^2}|1-\eta|\xi|^2||\overline{X}(s)\xi\cdot\nabla X(s)|d\xi ds \\ &+c\mathbb{E}\sqrt{\int_0^{t\wedge\sigma_{n,m}}\sum_{k=1}^N\left(\int e^{-\eta|\xi|^2}|\xi|^2|X(s)|^2Re\phi_k d\xi\right)^2ds} \\ =& J_1+J_2, \end{aligned} \tag{3.3.54}$$

where c is independent of n, m and η.

While, since $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|e^{-\eta|\xi|^2}(1-\eta|\xi|^2)|=1$ and $\mathbb{E}\sup_{s\in[0,\sigma_{n,m}]}|\nabla X(s)|_2^2\le m<\infty$

$$\begin{aligned} J_1 \le& 4\mathbb{E}\int_0^{t\wedge\sigma_{n,m}}\sqrt{V(X(s))}|\nabla X(s)|_2 d\xi \\ \le& 4\int_0^t\mathbb{E}\sup_{r\in[0,s\wedge\sigma_{n,m}]}V(X(r))ds+4mT. \end{aligned} \tag{3.3.55}$$

Moreover, by Lemma 2.3.4 with $V_\eta(X(s))$ replacing $Y(s)$

$$J_2\le C\mathbb{E}\sqrt{\int_0^{t\wedge\sigma_{n,m}}[V_\eta(X(s))]^2ds}$$

$$\leq \varepsilon C \mathbb{E} \sup_{s\in[0,t\wedge\sigma_{n,m}]} V_\eta(X(s)) + CC_\varepsilon \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\sigma_{n,m}]} V_\eta(X(r))ds, \tag{3.3.56}$$

where C depends on $|\phi_k|_{L^\infty}$, $1 \leq k \leq N$, and is independent of n, m and η.

Hence, plugging (3.3.55) and (3.3.56) into (3.3.54), taking ε small enough, and noting that $V_\eta(X) \leq V(X)$, we derive that

$$\mathbb{E} \sup_{s\in[0,t\wedge\sigma_{n,m}]} V_\eta(X(s)) \leq c_1 \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\sigma_{n,m}]} V(X(r))ds + c_2(m,T),$$

with c_1 and $c_2(m,T)$ independent of η. Then letting $\eta \to 0$ and using Fatou's lemma, we have

$$\mathbb{E} \sup_{s\in[0,t\wedge\sigma_{n,m}]} V(X(s)) \leq c_1 \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\sigma_{n,m}]} V(X(r))ds + c_2(m,T), \quad t\in[0,T],$$

which implies by Gronwall's lemma that

$$\mathbb{E} \sup_{t\in[0,\sigma_{n,m}]} V(X(t)) \leq C(m,T) < \infty, \tag{3.3.57}$$

hence $\sup_{t\in[0,\sigma_{n,m}]} V(X(t)) \leq \widetilde{C}(m,T) < \infty$, $\mathbb{P}$-a.s. But, since $\sup_{t\in[0,\tau_n]} |\nabla X(t)|_2^2 < \infty$, $\mathbb{P}$-a.s, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\exists m(\omega) < \infty$ such that $\sigma_{n,m(\omega)}(\omega) = \tau_n(\omega)$. Then $\mathbb{P}\left(\bigcup_{m\in\mathbb{N}} \{\sigma_{n,m} = \tau_n\}\right) = 1$. This implies (3.3.53) and completes the proof of Lemma 3.3.1. □

Next, we derive the evolution formula for the momentum.

Lemma 3.3.2. *For $x \in \sum$, it holds that $\mathbb{P}$-a.s for $t < \tau^*(x)$*

$$\begin{aligned} G(X(t)) =& G(x) + 4\int_0^t P(X(s))ds \\ &- \sum_{k=1}^N \int_0^t Im \int \xi\cdot\nabla\phi_k |X(s)|^2 \overline{\phi_k} d\xi ds + M_2(t), \end{aligned} \tag{3.3.58}$$

where

$$\begin{aligned} P(X) =& \frac{1}{2}|\nabla X|_2^2 - \frac{d(\alpha-1)}{4(\alpha+1)}|X|_{L^{\alpha+1}}^{\alpha+1} \\ =& H(X) + \frac{1}{\alpha+1}[1 - \frac{d(\alpha-1)}{4}]|X|_{L^{\alpha+1}}^{\alpha+1}, \end{aligned}$$

$\phi_k = \mu_k e_k$, $1 \leq k \leq N$, and

$$M_2(t) = d\sum_{k=1}^{N}\int_0^t \int |X(s)|^2 Im\phi_k d\xi d\beta_k(s) \\ - 2\sum_{k=1}^{N}\int_0^t Im \int \xi \cdot \nabla X(s)\overline{X(s)}\overline{\phi_k} d\xi d\beta_k(s).$$

Here, d is the dimension of the space.

Proof. (3.3.58) follows from a heuristic application of Itô's formula. To prove it rigourously, we use similar arguments as in Lemma 3.3.1. We also use summation convention in the following calculations.

Set $G_\eta(u) = Im \int e^{-\eta|\xi|^2}\xi u(\xi) \cdot \overline{\nabla u} d\xi$ for any $u \in \sum$ and consider X^ε as in (3.3.50). Below we omit the arguments ξ for the sake of simplicity.

By the product rule

$$\overline{X}^\varepsilon(t)\partial_j X^\varepsilon(t) = \overline{x}^\varepsilon \partial_j x^\varepsilon + \int_0^t \partial_j X^\varepsilon(s) d\overline{X}^\varepsilon(s) + \int_0^t \overline{X}^\varepsilon(s) d(\partial_j X^\varepsilon)(s) \\ + \int_0^t \overline{(X(s)\phi_k)^\varepsilon}\partial_j[(X(s)\phi_k)^\varepsilon]ds, \quad t < \tau^*(x), \quad \mathbb{P}-a.s.$$

Integrating over $\mathbb{R}^d$ with $e^{-\eta|\xi|^2}\xi_j$, we have $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\begin{aligned} &- G_\eta(X^\varepsilon(t)) \\ =& - G_\eta(x^\varepsilon) + Im \int e^{-\eta|\xi|^2}\xi_j \overline{X}^\varepsilon(t)\partial_j X_m^\varepsilon(t) d\xi \\ =& - G_\eta(x^\varepsilon) + Im \int \int_0^t e^{-\eta|\xi|^2}\xi_j \partial_j X^\varepsilon(s) d\overline{X}^\varepsilon(s) d\xi \\ &+ Im \int \int_0^t e^{-\eta|\xi|^2}\xi_j \overline{X}^\varepsilon(s) d(\partial_j X^\varepsilon)(s) d\xi \\ &+ Im \int_0^t \int e^{-\eta|\xi|^2}\xi_j \overline{(X(s)\phi_k)^\varepsilon}\partial_j[(X(s)\phi_k)^\varepsilon] d\xi ds, \end{aligned} \tag{3.3.59}$$

where we used Fubini's theorem for the last term.

Using integration by parts, we have that for the third term in the right hand side of (3.3.59)

$$Im \int \int_0^t e^{-\eta|\xi|^2}\xi_j \overline{X}^\varepsilon(s) d(\partial_j X^\varepsilon)(s) d\xi$$

$$
\begin{aligned}
=&- Im \int\int_0^t e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)\overline{X}^{\varepsilon}(s)dX^{\varepsilon}(s)d\xi \\
&+ Im \int\int_0^t e^{-\eta|\xi|^2}\xi_j\partial_j X^{\varepsilon}(s)d\overline{X}^{\varepsilon}(s)d\xi. \qquad (3.3.60)
\end{aligned}
$$

(This equality can be justifies rigorously as in the proof of Lemma 2.3.11. First, we obtain the equation of $\partial_j X^\varepsilon$ from (3.3.50) on a set $\widetilde{\Omega}$ with full probability and independent of $\xi \in \mathbb{R}^d$. Applying both deterministic and stochastic Fubini's theorem, we interchange the integrals. Then we use integration by parts to obtain (3.3.60). Since these technical treatments are just used in this step, we omit them here for the sake of simplicity.)

Moreover, for the fourth term in the right hand side of (3.3.59)

$$
\begin{aligned}
&Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}\partial_j[(X(s)\phi_k)^{\varepsilon}]dsd\xi \\
=&Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}[\partial_j(X(s)\phi_k)]^{\varepsilon}dsd\xi \\
=&Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}(\partial_j X(s)\phi_k)^{\varepsilon}d\xi ds \\
&+ Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}(X(s)\partial_j\phi_k)^{\varepsilon}d\xi ds. \qquad (3.3.61)
\end{aligned}
$$

Hence, plugging (3.3.60) and (3.3.61) into (3.3.59), we obtain $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$
\begin{aligned}
&- G_\eta(X^\varepsilon(t)) \\
=&- G_\eta(x^\varepsilon) - Im \int\int_0^t e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)\overline{X}^{\varepsilon}(s)dX^{\varepsilon}(s)d\xi \\
&+ 2Im \int\int_0^t e^{-\eta|\xi|^2}\xi_j\partial_j X^{\varepsilon}(s)d\overline{X}^{\varepsilon}(s)d\xi \\
&+ Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}(\partial_j X(s)\phi_k)^{\varepsilon}d\xi ds \\
&+ Im \int_0^t\int e^{-\eta|\xi|^2}\xi_j\overline{(X(s)\phi_k)^{\varepsilon}}(X(s)\partial_j\phi_k)^{\varepsilon}d\xi ds \\
=&- G_\eta(x^\varepsilon) + K_1 + K_2 + K_3 + K_4. \qquad (3.3.62)
\end{aligned}
$$

First, by (3.3.48) and (3.3.53), we can use Lebesque's dominated theorem to derive that $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$
\lim_{\eta\to 0}\lim_{\varepsilon\to 0}(K_3 + K_4)
$$

$$=2Im\int_0^t\int \xi\mu\overline{X}(s)\cdot\nabla X(s)ds+Im\int_0^t\int \xi_j\overline{\phi}_k\partial_j\phi_k|X(s)|^2ds. \tag{3.3.63}$$

For K_1, we will show that $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\varepsilon\to 0}K_1=&-d\cdot\int_0^t|\nabla X(s)|_2^2ds+d\cdot\int_0^t|X(s)|_{L^{\alpha+1}}^{\alpha+1}ds\\&-d\cdot\int_0^t\int|X(s)|^2Im\phi_kd\xi d\beta_k(s),\end{aligned} \tag{3.3.64}$$

where d is the dimension of the space.

Indeed, it follows from equation (3.3.50) and Fubini's theorem that

$$\begin{aligned}K_1=&-Im\int_0^t\int e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)\overline{X}^\varepsilon(s)\Big[(-i)\Delta X^\varepsilon(s)\\&+(-1)(\mu X(s))^\varepsilon+(-i)[g(X(s))]^\varepsilon\Big]d\xi ds\\&-Im\int_0^t\int e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)\overline{X}^\varepsilon(s)(X(s)\phi_k)^\varepsilon d\xi d\beta_k(s)\\=&K_{11}+K_{12}+K_{13}+K_{14}.\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}K_{11}=&-Im\ i\int_0^t\int e^{-\eta|\xi|^2}(-2\eta\xi)(d-2\eta|\xi|^2)\overline{X}^\varepsilon(s)\nabla X^\varepsilon(s)d\xi ds\\&-Im\ i\int_0^t\int e^{-\eta|\xi|^2}(-4\eta\xi)\overline{X}^\varepsilon(s)\nabla X^\varepsilon(s)d\xi ds\\&-\int_0^t\int e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)|\nabla X^\varepsilon(s)|^2d\xi ds.\end{aligned}$$

Since for $\eta>0$ fixed, $e^{-\eta|\xi|^2}$ is exponentially decay, we can use (3.3.48) to apply Lebesque's dominated theorem to take the limit $\varepsilon\to 0$. Then, as $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)|<\infty$, we can use (3.3.48) and (3.3.53) to pass to the limit $\eta\to 0$. Hence, we obtain $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}K_{11}=-d\cdot\int_0^t|\nabla X(s)|_2^2ds,\quad t<\tau^*(x),\quad \mathbb{P}-a.s. \tag{3.3.65}$$

Similarly

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}K_{12}=d\cdot Im\int_0^t\int\mu|X(s)|^2d\xi ds=0,\quad t<\tau^*(x),\quad \mathbb{P}-a.s. \tag{3.3.66}$$

and

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} K_{14} = -d\cdot\int_0^t\int |X(s)|^2 Im\phi_k d\xi d\beta_k(s),\quad t<\tau^*(x),\ \mathbb{P}-a.s. \tag{3.3.67}$$

We are also allowed to pass to the limits for K_{13}. Indeed, for $s<t<\tau^*(x)$

$$\begin{aligned}&\int \left|e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)\overline{X}^{\varepsilon}(s)[g(X(s))]^{\varepsilon}\right|d\xi\\ \leq& C|X^{\varepsilon}(s)|_{L^{\alpha+1}}|[g(X(s))]^{\varepsilon}|_{L^{\frac{\alpha+1}{\alpha}}}\\ \leq& C|X(s)|^{\alpha+1}_{L^{\alpha+1}}\\ \leq& CD^{\alpha+1}\sup_{s\in[0,t]}|X(s)|^{\alpha+1}_{H^1}<\infty,\end{aligned}$$

with $C=\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)|<\infty$. Thus dominated convergence theorem yields

$$\lim_{\eta\to\infty}\lim_{\varepsilon\to 0} K_{13} = d\cdot\int_0^t |X(s)|^{\alpha+1}_{L^{\alpha+1}}ds,\quad t<\tau^*(x),\quad \mathbb{P}-a.s. \tag{3.3.68}$$

Therefore, (3.3.64) follows from (3.3.65)-(3.3.68).

K_2 could be treated in a similar way, though the calculations are more complicated. We will prove that $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\varepsilon\to 0} K_2 =&(d-2)\int_0^t|\nabla X(s)|_2^2ds-\frac{2d}{\alpha+1}\int_0^t|X(s)|^{\alpha+1}_{L^{\alpha+1}}ds\\ &-2Im\int_0^t\int\xi_j\partial_jX(s)\mu\overline{X}(s)d\xi ds\\ &+2Im\int_0^t\int\xi_j\partial_jX(s)\overline{X}(s)\overline{\phi}_k d\xi d\beta_k(s),\end{aligned} \tag{3.3.69}$$

where d is the dimension of the space.

Indeed, from equation (3.3.50), (3.3.62) and Fubini's theorem it follows that

$$\begin{aligned}K_2 =&2Im\int_0^t\int e^{-\eta|\xi|^2}\xi_j\partial_jX^{\varepsilon}(s)\left[i\Delta\overline{X^{\varepsilon}}(s)+(-1)(\mu\overline{X}(s))^{\varepsilon}+i\overline{[g(X(s))]^{\varepsilon}}\right]d\xi ds\\ &+2Im\int_0^t\int e^{-\eta|\xi|^2}\xi_j\partial_jX^{\varepsilon}(s)\overline{(X(s)\phi_k)^{\varepsilon}}d\xi d\beta_k(s)\\ =&K_{21}+K_{22}+K_{23}+K_{24}.\end{aligned} \tag{3.3.70}$$

It follows easily that $\mathbb{P}$-a.s. for $t<\tau^*(x)$

$$\begin{aligned}&\lim_{\eta\to 0}\lim_{\varepsilon\to 0}(K_{22}+K_{24})\\ =&-2Im\int_0^t\int \xi_j\partial_j X(s)\mu\overline{X}(s)d\xi ds+2Im\int_0^t\int \xi_j\partial_j X(s)\overline{X}(s)\overline{\phi}_k d\xi d\beta_k(s).\end{aligned} \tag{3.3.71}$$

To pass to the limits in K_{23}, since for $\eta>0$ fixed, $\sup_{\xi\in\mathbb{R}^d}|e^{-\eta|\xi|^2}\xi_j|<\infty$, $1\le j\le N$, by (3.3.48) we can apply Lebesque's dominated theorem to take the limit $\varepsilon\to 0$ and obtain

$$\lim_{\varepsilon\to 0}K_{23}=2Im\ i\int_0^t\int e^{-\eta|\xi|^2}\xi_j\partial_j X(s)\overline{g(X(s))}d\xi ds,\quad t<\tau^*(x),\quad \mathbb{P}-a.s.$$

Using $Im\ iz=Rez$, $z\in\mathbb{C}$, and $Re[(\partial_j X)\overline{g(X)}]=\frac{1}{\alpha+1}\partial_j(|X|^{\alpha+1})$, we derive that the right hand side above is equal to

$$\begin{aligned}&2\int_0^t\int e^{-\eta|\xi|^2}\xi_j Re(\partial_j X(s)\overline{g(X(s))})d\xi ds\\ =&\frac{2}{\alpha+1}\int_0^t\int e^{-\eta|\xi|^2}\xi_j\partial_j(|X(s)|^{\alpha+1})d\xi ds\\ =&-\frac{2}{\alpha+1}\int_0^t\int e^{-\eta|\xi|^2}(d-2\eta|\xi|^2)|X(s)|^{\alpha+1}d\xi ds.\end{aligned}$$

Now, as $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}e^{-\eta|\xi|^2}|d-2\eta|\xi|^2|<\infty$ and $|X(s)|^{\alpha+1}_{L^{\alpha+1}}\le D^{\alpha+1}\sup_{s\in[0,t]}|X(s)|^{\alpha+1}_{H^1}<\infty$, $t<\tau^*(x)$, $\mathbb{P}-a.s.$, we can pass to the limit $\eta\to 0$ and obtain

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}K_{23}=-\frac{2d}{\alpha+1}\int_0^t|X(s)|^{\alpha+1}_{L^{\alpha+1}}ds,\quad t<\tau^*(x),\quad \mathbb{P}-a.s. \tag{3.3.72}$$

We are now left to prove for K_{21} that

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}K_{21}=(d-2)\int_0^t|\nabla X(s)|_2^2 ds,\quad t<\tau^*(x),\quad \mathbb{P}-a.s. \tag{3.3.73}$$

Indeed, from integration by parts it follows that

$$\begin{aligned}K_{21}=&2Im\ i\int_0^t\int\Delta(e^{-\eta|\xi|^2}\xi_j\partial_j X^\varepsilon(s))\overline{X}^\varepsilon(s)d\xi ds\\ =&2Im\ i\int_0^t\int\Big[\Delta e^{-\eta|\xi|^2}\xi_j\partial_j X^\varepsilon(s)+e^{-\eta|\xi|^2}\xi_j\partial_j\Delta X^\varepsilon(s)+2e^{-\eta|\xi|^2}\Delta X^\varepsilon(s)\end{aligned}$$

$$
\begin{aligned}
&+ 2\nabla e^{-\eta|\xi|^2}\nabla\xi_j\partial_j X^\varepsilon(s) + 2\nabla e^{-\eta|\xi|^2}\nabla\partial_j X^\varepsilon(s)\xi_j\Big]\overline{X}^\varepsilon(s)ds \\
=&K_{211} + K_{212} + K_{213} + K_{214} + K_{215}.
\end{aligned}
$$

Notice that, $\Delta e^{-\eta|\xi|^2} = 4\eta^2|\xi|^2 e^{-\eta|\xi|^2} + 2\eta e^{-\eta|\xi|^2}$, hence $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|\Delta e^{-\eta|\xi|^2}| < \infty$ and $\lim_{\eta\to 0}\Delta e^{-\eta|\xi|^2} = 0$. By (3.3.48) and (3.3.53) we can take the limits and obtain

$$
\lim_{\eta\to 0}\lim_{\varepsilon\to 0} K_{211} = 0, \quad t < \tau^*(x),\ \mathbb{P} - a.s. \tag{3.3.74}
$$

Moreover, as $\nabla e^{-\eta|\xi|^2} = -2\eta\xi e^{-\eta|\xi|^2}$, hence $\sup_{\eta>0}\sup_{\xi\in\mathbb{R}^d}|\nabla e^{-\eta|\xi|^2}| < \infty$ and $\lim_{\eta\to 0}\nabla e^{-\eta|\xi|^2} = 0$. By (3.3.48) we get

$$
\lim_{\eta\to 0}\lim_{\varepsilon\to 0} K_{214} = 0, \quad t < \tau^*(x),\ \mathbb{P} - a.s. \tag{3.3.75}
$$

Similarly, $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$
\begin{aligned}
K_{213} =& -4Im\ i\int_0^t\int \nabla(e^{-\eta|\xi|^2}\overline{X}^\varepsilon(s))\nabla X^\varepsilon(s)d\xi ds \\
=& -4Im\ i\int_0^t\int [\nabla e^{-\eta|\xi|^2}\overline{X}^\varepsilon(s)\nabla X^\varepsilon(s) + e^{-\eta|\xi|^2}|\nabla X^\varepsilon(s)|^2]d\xi ds \\
\to& -4\int_0^t |\nabla X(s)|_2^2 ds, \quad as\ \varepsilon\to 0,\ \eta\to 0,
\end{aligned} \tag{3.3.76}
$$

and

$$
K_{215} = -4Im\ i\int_0^t\int \partial_j(\nabla e^{-\eta|\xi|^2}\xi_j\overline{X}^\varepsilon(s))\nabla X^\varepsilon(s) \to 0. \tag{3.3.77}
$$

Finally, for K_{212} we notice that

$$
\begin{aligned}
K_{212} =& -2Im\ i\int_0^t\int \partial_j(e^{-\eta|\xi|^2}\xi_j\overline{X}^\varepsilon(s))\Delta X^\varepsilon(s) \\
=& -2Im\ i\int_0^t\int (\partial_j e^{-\eta|\xi|^2}\xi_j\overline{X}^\varepsilon(s)\Delta X^\varepsilon(s) + e^{-\eta|\xi|^2}d\cdot\overline{X}^\varepsilon(s)\Delta X^\varepsilon(s) \\
&+ e^{-\eta|\xi|^2}\xi_j\partial_j\overline{X}^\varepsilon(s)\Delta X^\varepsilon(s) \\
=& -2Im\ i\int_0^t\int \partial_j e^{-\eta|\xi|^2}\xi_j\overline{X}^\varepsilon(s)\Delta X^\varepsilon(s)d\xi ds - \frac{d}{2}K_{213} - K_{21} \\
=&2Im\ i\int_0^t\int \partial_k(\partial_j e^{-\eta|\xi|^2}\xi_j\overline{X}^\varepsilon(s))\partial_k X^\varepsilon(s)d\xi ds - \frac{d}{2}K_{213} - K_{21}.
\end{aligned} \tag{3.3.78}
$$

It is not difficult to check that $\mathbb{P}$-a.s. for $t < \tau^*(x)$

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} 2Im\ i \int_0^t \int \partial_k(\partial_j e^{-\eta|\xi|^2} \xi_j \overline{X}^\varepsilon(s)) \partial_k X^\varepsilon(s) d\xi ds = 0 \qquad (3.3.79)$$

Taking (3.3.74)-(3.3.79) together and passing to the limits, we consequently obtain (3.3.73).

Therefore, (3.3.69) follows from (3.3.70)-(3.3.73). Plugging (3.3.63), (3.3.64) and (3.3.69) into (3.3.62) we complete the proof of (3.3.58). □

3.4 Notes

The blowup phenomena studied here is in the H^1 context, i.e. the initial data x belongs to H^1.

In the deterministic case, it is well-known that, initial data with negative Hamiltonian can cause the solutions blow up in finite time (see [40, 22, 58]). The proof is based on the analyze of the variation.

In the deterministic focusing mass-critical case, i.e. $\lambda = 1$, $\alpha = 1 + \frac{4}{d}$, M. I. Weinstein [94] proved the global well-posedness for the initial data with $|x|_2 < |Q|_2$, where Q is the so-called ground state which appears in the sharp Gagliardo-Nirenberg inequality and satisfies the elliptic equation

$$\Delta Q - Q + |Q|^{\frac{4}{d}} Q = 0.$$

Q is indeed the threshold for blowup in this case, in the sense that there exist blow up solutions with initial data $|x|_2 = |Q|_2$. (see also [63] for a general type of nonlinear Schrödinger equations). Later on, a major progress was obtained by F. Merle [62]. He proved that, up to symmetries, the only blowup solution with $|x|_2 = |Q|_2$ is the solitary wave $X = e^{-it}Q$. The proofs were based on the local virial analysis and the variation characterization of the ground state. In the recent years, numerous works focus on the L^2 small supercritical mass case, i.e. $|Q|_2 \le |x|_2 < |Q|_2 + \varepsilon$, for some $\varepsilon > 0$ small enough. We refer the interested reader to a series of papers by F. Merle and P. Raphaël [64, 65, 66, 67, 68]. See also [58] for a brief review.

For the further phenomena related with blowup, e.g. the L^2-mass concentration phenomena and the self-similarity, see [91, 89, 70, 95]. Recently, T. Hmidi and S. Keraani [43] applied a new idea based on the profile decomposition to reprove the classical results mentioned above. Moreover, this idea was also applied in [50] to prove the existence of minimal mass blow-up solutions in the L^2 context when $x \in L^2$, and it was also used in [49] to determine the threshold W for blowup in the focusing energy-critical case where $\lambda = 1$ and $\alpha = 1 + \frac{4}{d-2}$, $d \geq 3$ (see also Section 2.4 in Chapter 2).

In the stochastic case, the blowup phenomena were first mathematically studied by A. de Bouard and A. Debussche [13] in the conservative focusing mass-supercritical case where $\lambda = 1$, $\alpha \in (1 + \frac{4}{d}, \infty)$ if $d = 1, 2$ and $\alpha \in (\frac{7}{3}, 5)$ if $d = 3$. They proved that the spatially smooth noise can cause blowup immediately with positive probability for any given smooth initial data. See also [11] for the additive noise case. In contrast, different phenomena happen in the focusing mass-critical case where $\lambda = 1$ and $\alpha = 1 + \frac{4}{d}$. The numerical simulations in [14, 26, 27] suggest that spatially smooth noise is able to delay the blowup, moreover, white noise can even prevent blowup. We refer the interested reader to [34] for a comprehensive review.

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Notations

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^d)$ is the space of all p-integrable complex valued functions with the norm $|\cdot|_{L^p}$. We use the notation $L^q(0,T;L^p)$ for all measurable functions $u : [0,T] \to L^p$ such that $t \to |u(t)|_{L^p}$ belongs to $L^q(0,T)$ and the norm denoted by

$$\|u\|_{L^q(0,T;L^p)} = \left(\int_0^T (\int_{\mathbb{R}^d} |u(t,\xi)|^p d\xi)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

$C([0,T];L^p)$ similarly denotes the continuous L^p-valued functions with the sup norm in t.

$W^{1,p} = W^{1,p}(\mathbb{R}^d)$ is the classical Sobolev space, i.e. $W^{1,p} = \{u \in L^p : \nabla u \in L^p\}$ equipped with the norm $\|u\|_{W^{1,p}} = |u|_{L^p} + |\nabla u|_{L^p}$. Here $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d})$ is the gradient operator, and we set $\partial_k = \frac{\partial}{\partial x_k}$, $1 \leq k \leq d$. We also set for multi-index $\gamma = (\gamma_1, ..., \gamma_d)$ with γ_j nonnegative integrals

$$\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_d^{\gamma_d}.$$

The order of γ is $|\gamma| = \gamma_1 + \cdots + \gamma_d$. If $|\gamma| = 0$, $\partial^\gamma f = f$. Moreover, The spaces $L^q(0,T;W^{1,p})$ and $C([0,T];W^{1,p})$ are understood similarly as above.

In the particular case $p = 2$, L^2 is the Hilbert space endowed with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^d} u(\xi)\overline{v}(\xi) d\xi;\ u, v \in L^2,$$

and we will set $|\cdot|_2 = |\cdot|_{L^2}$. For simplicity we also set $H^k = W^{k,2}$ and denote by H^{-k} the dual space of H^k, $k \in \mathbb{N}$. Their norms are denoted by $|\cdot|_{H^k}$, $k \in \mathbb{Z}$.

For two Banach spaces X and Y, $L(X,Y)$ means the bounded operators from X to Y. When X and Y are Hilbert spaces, we use the notations $L_2(X,Y)$ for the Hilbert-Schmidt operators from X to Y. For any $\Phi \in L_2(X,Y)$, $\|\Phi\|_{L_2(X,Y)} := \sum_{j=1}^{\infty} \|\Phi e_j\|_Y^2$, where $\{e_j\}_{j=1}^\infty$ is an orthonormal basis in X.

$C_c^\infty(\mathbb{R}^d)$ denotes the compactly supported smooth functions on $\mathbb{R}^d$. $\mathcal{S}$ and $\mathcal{S}'$ denote the space of rapidly decreasing functions and the tempered distributions respectively. Then for $f \in \mathcal{S}$, $\widehat{f}$ means the Fourier transformation $\widehat{f}(\eta) = \int f(\xi)e^{-i\xi\cdot\eta}d\xi$. Moreover, for $f \in \mathcal{S}'$, and $f^\vee$ denotes the inversion Fourier transform of f, $f^\vee(\xi) = \frac{1}{(2\pi)^d}\int f(\eta)e^{i\xi\cdot\eta}d\eta$.

$\widetilde{X}_{[0,T]}$ is the local smoothing space constructed in [59] up to time $T > 0$. Precisely, set $B_0 = \{|\xi| \le 2\}$, $B_j = \{2^j \le |\xi| \le 2^{j+1}\}$, $j = 1, ..., 2$, and $B_{<j} = \{|\xi| \le 2^j\}$. Let $A_j = [0,T] \times B_j$, $j \ge 0$, $A_{<j} = [0,T] \times B_{<j}$, $j \ge 1$. We consider a dyadic partition of unity of frequency, i.e. $1 = \sum\limits_{k=-\infty}^{\infty} S_k(D)$. We say a function f is localized at frequency 2^k, if $\widehat{f}$ is supported in $\{2^{k-1} < |\xi| < 2^{k+1}\}$. The functions at frequency 2^k are measured using the norm

$$\begin{aligned}
\|u\|_{X_k(T)} =& \|u\|_{L^2(A_0)} + \sup_{j>0} \| <\xi>^{-\frac{1}{2}} u\|_{L^2(A_j)}, \quad k \ge 0,\\
\|u\|_{X_k(T)} =& 2^{\frac{k}{2}}\|u\|_{L^2(A_{<-k})} + \sup_{j\ge -k} \|(|\xi| + 2^{-k})^{-\frac{1}{2}} u\|_{L^2(A_j)}, \quad k < 0,
\end{aligned}$$

where $<\xi>= \sqrt{1+|\xi|^2}$. Then the local smoothing space $\widetilde{X}_{[0,T]}$ defined by the norm

$$\begin{aligned}
\|u\|^2_{\widetilde{X}_{[0,T]}} &= \| <\xi>^{-1} u\|^2_{L^2([0,T]\times\mathbb{R}^d)} + \sum_{k=-\infty}^{\infty} 2^k \|S_k u\|^2_{X_k(T)}, \quad d \ne 2\\
\|u\|^2_{\widetilde{X}_{[0,T]}} &= \| <\xi>^{-1} (\ln(2+|\xi|))^{-1} u\|^2_{L^2([0,T]\times\mathbb{R}^d)} + \sum_{k=-\infty}^{\infty} 2^k \|S_k u\|^2_{X_k(T)}, \quad d = 2.
\end{aligned}$$

In this thesis, C and $\widetilde{C}$ will denote various constants, which may change from line to line.