Asymptotic Expansions in Classical and Free Probability

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Contents

Chapter 1

Introduction

Asymptotic expansions in the Central Limit Theorem is one of the fundamental problems in probability theory. To describe the situation let $X_1, X_2, \ldots$ be independent identically distributed random variables with zero mean and unit variance and let us denote by $F_n(x)$ the distribution function of the normalized sum $S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Such sums appear in many applications, but unfortunately they cannot be computed easily. Therefore, it is natural to ask about suitable approximations for them. One possible solution of the problem is given by the Central Limit Theorem. This theorem says that $F_n(x)$ tends to the standard Gaussian law $\Phi(x)$ as $n \to \infty$ uniformly in x. This means that we can replace the distribution function of S_n by $\Phi(x)$. To do such a replacement we just need to estimate the difference between $F_n(x)$ and $\Phi(x)$. One of the basic results was obtained independently by Berry [10] and Esseen [22]. If X_1 has a finite absolute third moment β_3, then the Berry–Esseen inequality provides the estimate

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq c\beta_3/\sqrt{n}, \quad (n \geq 2).$$

The distribution function $F_n(x)$ also admits an asymptotic expansion in powers of $n^{-1/2}$. The first formal expansions of this type were obtained by Chebyshev [43] and Edgeworth [21]. Let us introduce this result. Assume that the random variable X_1 has moments of all orders. Then there is a formal expansion in a power series in $n^{-1/2}$:

$$F_n(x) = \Phi(x) + \varphi(x) \sum_{p=1}^{\infty} \frac{Q_p(x)}{n^{p/2}}, \tag{1.1}$$

where

$$Q_p(x) = -\sum H_{p+2s-1}(x) \prod_{m=1}^{p} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}}{(m+2)!} \right)^{k_m},$$

$\varphi(x)$ is the density of Gaussian law and γ_m is the cumulant of order m of X_1. In the last equality the summation on the right-hand side is carried out over all nonnegative integer solutions $(k_1, \dots, k_m)$ of the equations

$$k_1 + 2k_2 + \dots + pk_p = p \quad \text{and} \quad s = k_1 + \dots + k_p.$$

In terms of characteristic functions, (1.1) has the form

$$\int_{-\infty}^{\infty} e^{itx} dF_n(x) = e^{-t^2/2} + \sum_{m=1}^{\infty} \frac{P_m(t)}{n^{m/2}} e^{-t^2/2}, \tag{1.2}$$

where

$$\int_{-\infty}^{\infty} e^{itx} dQ_m(x) = P_m(t) e^{-t^2/2}.$$

The first asymptotic expansion with a sharp estimate for the error term was introduced by Esseen [22]. He proved that if X_1 has a non-lattice distribution and the third moment m_3 and the third absolute moment β_3 are finite, then

$$F_n(x) = \Phi(x) - \frac{m_3}{6\sqrt{n}} \Phi^{(3)}(x) + o(n^{-1/2}),$$

which holds uniformly in x.

The next important questions concern asymptotic expansions in the Local Limit Theorem and in the Functional Limit Theorem. As before we consider $X_1, X_2, \dots$ independent identically distributed random variables. The Local Limit Theorem states that if $F_n(x)$ has a bounded density $p_{n_0}(x)$ for some $n_0 \in \mathbb{N}$, then for $n \geq n_0$, $p_n(x)$ converges to the density $\varphi(x)$ as $n \to \infty$ uniformly with respect to x. The Functional Limit Theorem states that under the conditions of the previous theorem $\int f(x) dF_n(x)$ converges to $\int f(x) d\Phi(x)$ as $n \to \infty$ for every bounded measurable function f. In both cases $p_n(x)$ and $\int f(x) dF_n(x)$ admit asymptotic expansions. For more details we refer the reader to [40], [24] and for the multidimensional case to [11].

A new field for developing asymptotic expansions became free probability. This theory was initiated by Dan Voiculescu in the mid-1980's [44]. It started as a tool for solving the free group factor isomorphism problem. More precisely, if we have a free group with a given number of generators, we can consider the von Neumann algebra generated by this group, which is the simplest type II_1 factor. The isomorphism problem asks whether the von Neumann algebras are isomorphic for different number of generators. Since then, free probability has become an area by its own. The theory is a non-commutative counterpart of classical probability theory: A probability space is replaced by a non-commutative probability space, independence is replaced by freeness and a new type of highly non-linear convolution arises. Many free analogues of classical results have been proved: The Free Central Limit Theorem [44], the Free Law of Large Numbers [6], the classification of free infinitely divisible law [7] etc.

Nevertheless, there are aspects in free probability which vary from the ones in the classical case. One difference that we want to stress is the phenomenon of superconvergence. Let us consider a normalized sum of identically distributed random variables

$S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$ with zero mean and unit variance, and assume that X_1 has compact support. Then in the classical case, if $X_1, X_2, \ldots$ are independent, the probability $\{|S_n| > n/2\}$ is exponentially small, but greater then zero. By contrast, in the non-commutative case, if $X_1, X_2, \ldots$ are freely independent, the probability $\{|S_n| > n/2\}$ becomes identically zero for sufficiently large n. This type of convergence is called superconvergence. For more details see [9] and [31]. The effect of superconvergence also appears in random matrix theory [3], [28]. In this thesis we consider a related problem. Denote by μ_n the distribution of the sum S_n of free identically distributed random variables with compact support. We are looking for an interval I such that $I \subset \mathrm{supp}(\mu_n)$ and $p_{\mu_n}(x)$ is positive on I for sufficiently large n . Our result is that $I = [-2 + c(\mu)n^{-1/4}, 2 - c(\mu)n^{-1/4}]$, where $c(\mu)$ is a positive constant depending on μ (Theorem 5.10).

It should be noted that free probability is tightly connected to random matrix theory and its techniques have a great impact. For example, Speicher [41], Pastur and Vasilchuk [39] proved that if A_n and B_n are two Hermitian random matrices such that their spectral distributions converge weakly in probability to μ and ν as $n \to \infty$ and U_n is the random unitary Haar distributed matrix, then A_n and $U_n^* B_n U_n$ become asymptotically free as $n \to \infty$. Moreover, the spectral distribution of a sum $A_n + U_n^* B_n U_n$ converges weakly in probability to the free convolution $\mu \boxplus \nu$ as $n \to \infty$. Hence, one can apply R-transfoms and subordination functions (see Chapter 3) for further study of spectral limit distributions in random matrix theory. Furthermore, free probability has a number of applications in physics [42] and wireless communication [19], [34].

As we have already mentioned the first Free Central Limit Theorem was proved by Voiculescu [44]. This theorem states that the distribution of a normalized sum of free selfadjoint identically distributed random variables with compact support, zero mean and unit variance converges weakly to the standard semicircular law as the number of summands tends to infinity. This result was extended by Maassen for random variables with finite variance [35]. The Local Limit Theorem for densities was proved by Voiculescu and Bercovici [9] for bounded random variables. This result was extended to the class of Borel probability measures by Wang [47].

Asymptotic expansion in these Free Limit Theorems is a new challenging area of research. Chistyakov and Götze [15] established a free analogue of the Berry-Esseen inequality for free selfadjoint identically distributed random variables with zero mean, unit variance and bounded absolute forth moment [15]. Independently, Kargin derived the same type of inequality [30], but under a stronger condition, namely, the random variables must be of bounded support.

The first asymptotic expansion in the Free Central Limit Theorem was obtained by Chistyakov and Götze [17]. They obtained the Edgeworth type expansion for distribution functions of normalized sums of free random variables under minimal moment conditions. The expansions in the Free Local Limit Theorem for bounded random variables were obtained as well in [17], and for random variables with finite 8th moment in [18]. In order to establish these results approximation by the free Meixner distribution was applied.

In this thesis we deduce Edgeworth type expansions in free probability for free identically distributed random variables with compact support. We will focus on one unifying style for deriving Edgeworth type expansions in classical and free probability. Our technique was introduced by Götze in [25] and differs from the ones in [17] and [18]. In the sequel, we briefly sketch the main idea of the method. For the sake of simplicity assume that $X_1, X_2, \ldots$ are independent identically distributed random variables, the distribution function $F_1(x)$ of X_1 has a density and finite moments up to order $s \geq 3$. Denote by $\hat{F}_n$ the Fourier transform of S_n. Then $\hat{F}_n$ admits an asymptotic expansion up to order $O(n^{-(s-2)/2})$ as $n \to \infty$. Denote by ξ a random variable with a standard Gaussian distribution and consider the sum $\xi + \varepsilon_1 X_1 + \ldots + \varepsilon_s X_s$ with Fourier transform $\hat{F}_s(t; \varepsilon_1, \ldots, \varepsilon_s)$, where $\varepsilon_1, \ldots, \varepsilon_s$ are small arbitrary weights. Under appropriate conditions (see Chapter 1) we can obtain the Edgeworth expansion for $\hat{F}_n$ by means of derivatives with respect to $\varepsilon_1, \ldots, \varepsilon_s$ of $\hat{F}_s(t; \varepsilon_1, \ldots, \varepsilon_s)$ at $\varepsilon_1 = \ldots = \varepsilon_s = 0$. In order to establish an estimate for the error term we have to show that the Fourier transform of $\varepsilon_1 X_1 + \ldots + \varepsilon_{m+s} X_{m+s}$ has bounded derivatives up to order s on the set of weights vectors $(\varepsilon_1, \ldots, \varepsilon_{m+s}) \in E$ such that all but $c(s)$ components are equal to $m^{-1/2}$ and the remaining weights are bounded in absolute value by $n^{-1/2}$, $m \geq n$. This scheme can be applied to a wide class of functional limit theorems as well. In order to apply this method in free probability we replace (classical) random variables by non-commutative ones, the condition of independence by freeness, the Gaussian random variable ξ by a semicircle one ζ and the Fourier transforms by Cauchy transforms.

This thesis is organised as follows: In the second chapter we introduce the general scheme from [25]. The original paper was difficult to follow due to a number of misprints, just a very short explanation of the condition (2.1) and the proof of the essential Lemma 8.1 was sketched only. We fill these gaps and present the revised versions of the proofs in the Appendix. Moreover, the conditions of Theorem 2.3 were formulated not correctly, we reformulate these conditions and discuss them in Remark 2.4. Chapter 3 is dedicated to the main aspects of free probability. In Chapter 4 we formulate our central results, namely, Edgeworth type expansions for the Cauchy transforms (Theorem 4.4), densities (Corollary 4.5) and distributions (Corollary 4.6) of free identically distributed random variables with compact support. In Chapter 5 we are looking for an interval such that the density of $\zeta + \sum_{i=1}^{r} \varepsilon_i X_i$, $|\varepsilon_j| \leq n^{-1/2}$ is positive (Proposition 5.6). In the sequel we use this result to construct an analytic extension for the Cauchy transform of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ (Theorem 5.11). Our main tool is a subordination result for analytic functions. In Chapter 6 we are looking for an interval such that the density of $\frac{1}{\sqrt{m}} \sum_{i=1}^{m-r} X_i + \sum_{i=1}^{2r} \varepsilon_i X_i$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots, 2r$, $m \geq n$ is positive (Theorem 6.4). Then we construct an analytic extension for the Cauchy transform of $\frac{1}{\sqrt{m}} \sum_{i=1}^{m-r} X_i + \sum_{i=1}^{2r} \varepsilon_i X_i$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots, 2r$, $m \geq n$

(Theorem 6.5) and show that this extension is uniformly differentiable with respect to ε_{2r} (Theorem 4.1). Finally, we apply the technique described in Chapter 2 to prove the Edgeworth type expansions. The techniques of this chapter are similar to those of the previous one.

Throughout the text we denote by $c, c_1, c_2 \ldots$, $c(\mu), c_1(\mu), c_2(\mu), \ldots$ and $c(\mu, r)$, $c_1(\mu, r)$, $c_2(\mu, r)$,... positive constants, positive constants depending only on the measure μ and positive constants depending only on the measure μ and a parameter r, respectively. The constants are allowed to vary from place to place.

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Chapter 2

A general scheme for asymptotic expansions based on symmetry

This chapter is devoted to describing the common scheme which may be widely adopted to the realization of asymptotic expansions. This scheme is based on the analysis of a sequence of functions $h_n(\varepsilon_1, \ldots, \varepsilon_n)$ which build on statistics of n variables, where the parameters ε_j are real and such that $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots, n$. Moreover, the functionals $h_n(\varepsilon_1, \ldots, \varepsilon_n)$ are assumed to be smooth, symmetric, compatible and have vanishing first derivatives at zero. Then $h_n(\varepsilon_1, \ldots, \varepsilon_n)$ admits an asymptotic expansion in powers of $n^{-1/2}$. This scheme was introduced by Götze in [25]. Before we describe the general scheme, we introduce a smoothing procedure which helps us to consider random variables which may have a non-integrable characteristic function. Then we proceed to the actual scheme and show how the asymptotic expansion for characteristic functions can be obtained by this method.

All the necessary preliminaries and results about the general scheme are presented in this section. Original proofs of the results can be found in [25]. Moreover, for the sake of completeness of this thesis the proofs of the results from this chapter with some corrections are introduced in the Appendix.

2.1 Smoothing procedure

Suppose that $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero mean, unit variance and finite moment of order $s \geq 3$. Denote by F_1 the distribution function of X_1. Suppose that $T_n = T_n(X_1, \ldots, X_n)$ is a sequence of real-valued statistics of $X_1, \ldots, X_n$, and that f is a bounded measurable function on $\mathbb{R}$. We want to get an asymptotic expansion for the functional $h_n = \mathbb{E}f(T_n)$. In order to do so we need to assume that F_1 has a bounded density or that there exists $n_0 \in \mathbb{N}$ such that the n_0-fold convolution of F_1 has a bounded density. In the general case this condition may not hold. In this case F_1 can be convolved by a smooth kernel. More precisely, let K_δ be the distribution function of a random variable α which is independent of $X_1, \dots, X_n$. Let us assume that K_δ has an integrable characteristic function and compact support concentrated near zero on an interval $[-\delta, \delta]$, where $\delta = \delta(n)$. Furthermore, let $\widetilde{F}_n$ be the distribution function of T_n. Then, the convolution $\widetilde{F}_n * K_\delta$ will have an integrable characteristic function, hence it also has a density. The bounds of $|\int f d(\widetilde{F}_n - \widetilde{F}_n * K_\delta)|$ are given by a wide class of smoothing inequalities, and these values can be controlled by a proper choice of δ. The smoothing inequalities can be found, for instance, in Petrov [40] and for the multidimensional case in Bhattacharya and Ranga Rao [11].

Suppose that $T_n^\alpha = T_n^\alpha(X_1, \dots, X_n)$ is a statistic that involves a smoothing variable α and has an integrable characteristic function. Denote $h_n(n^{-1/2}, \dots, n^{-1/2}) = \mathbb{E} f T_n^\alpha$, and assume

$$h_n = h_n(n^{-1/2}, \dots, n^{-1/2}) + O(n^{-(s-2)/2}), \; s \geq 3. \tag{2.1}$$

Due to the condition (2.1) we can now consider a wide class of random variables without additional space assumptions.

2.2 Statement of the general scheme

We are going to replace the arguments $n^{-1/2}$ in $h_n(n^{-1/2}, \dots, n^{-1/2})$ by arbitrary weights $\varepsilon_j \in \mathbb{R}$ such that $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \dots, n$. Assume that the sequence of functions $h_n(\varepsilon_1, \dots, \varepsilon_n)$ satisfies the following conditions:

$$h_n(\varepsilon_1, \dots, \varepsilon_n) \text{ is symmetric in all arguments;} \tag{2.2}$$

the sequence h_n is compatible, which means

$$h_{n+1}(\varepsilon_1, \dots, \varepsilon_{j-1}, 0, \varepsilon_{j+1}, \dots \varepsilon_{n+1}) = h_n(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_{j+1}, \dots, \varepsilon_{n+1}), \; j = 1, \dots, n+1; \tag{2.3}$$

and all first derivatives vanish at zero:

$$\frac{\partial}{\partial \varepsilon_j} h_n(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)\Big|_{\varepsilon_j = 0} = 0, \quad j = 1, \dots, n. \tag{2.4}$$

We use the following notation for vectors $\underline{\varepsilon}_m := (\varepsilon_1, \dots, \varepsilon_m)$. Let us denote by $E^n_{m,s}$ $(m \geq n > s)$ the set of weight vectors $\underline{\varepsilon}_{m+s}$ where all but $2s$ components are equal to $m^{-1/2}$ and the remaining $2s$ components are bounded by $n^{-1/2}$. In fact, it is sufficient if condition (2.4) holds for $h_{m+s}(\underline{\varepsilon}_{m+s})$, $\underline{\varepsilon}_{m+s} \in E^n_{m,s}$. We also assume that $h_{m+s}(\underline{\varepsilon}_{m+s})$ is uniformly differentiable on $E^n_{m,s}$. Finally, introduce the following limit (which is shown to exist in Proposition 2.1 below):

$$h_\infty(\underline{\varepsilon}_s) := \lim_{m \to \infty} h_{m+s}(m^{-1/2}, \dots, m^{-1/2}, \underline{\varepsilon}_s), \;\; |\varepsilon_j| \leq n^{-1/2}, \;\; j = 1, \dots, s, \;\; h_\infty := h_\infty(0).$$

There notations are standard and can be found, for example, in [40]. The two main results from [25] are formulated below (the proofs can be found in Appendix). The first one is Proposition 2.1, which yields the existence of the limit $h_\infty(\varepsilon_1, \dots, \varepsilon_s)$. This result is obtained by Taylor expansions and requires no additional information from probability theory. The second result is Theorem 2.3, which gives an expansion for h_n in itself. The idea of the proof is also based on Taylor expansions.

Let $\alpha = (\alpha_1, \dots, \alpha_m)$ denote an m-dimensional multi-index and set $D^\alpha = \frac{\partial^{\alpha_1}}{\partial \varepsilon_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial \varepsilon_m^{\alpha_m}}$. In the sequel we introduce "cumulant" differential operators $\kappa_p(D)$ and Edgeworth polynomials $P_r(\kappa_.(D))$ by means of formal power series identities.

We begin with establishing "cumulant" differential operators $\kappa_p(D)$ via the formal identity

$$\sum_{p=2}^{\infty} p!^{-1} \varepsilon^p \kappa_p(D) = \ln\left(1 + \sum_{p=2}^{\infty} p!^{-1} \varepsilon^p D^p\right). \tag{2.5}$$

Applying formal power series in the formal variable ε on the right-hand side of this identity we obtain the definition of $\kappa_p(D)$. Here D^p denotes p-fold differentiation with respect to a single variable ε, and $D^{p_1} \cdots D^{p_r} = D^{(p_1, \dots, p_r)}$ denotes differentiation with respect to r different variables $\varepsilon_1, \dots, \varepsilon_r$ at the point $\underline{\varepsilon}_r = 0$. Since the operators are applied to symmetric functions at zero, $\kappa_p(D)$ is unambiguously defined by (2.5). The first cumulants are $\kappa_2(D) = D^2,\ \kappa_3(D) = D^3,\ \kappa_4(D) = D^4 - 3D^2D^2$, etc.

Then, we define Edgeworth polynomials by means of the following formal series in κ_r and a formal variable ε.

$$\sum_{r=0}^{\infty} \varepsilon^r P_r(\kappa_.) = \exp\left(\sum_{r=3}^{\infty} r!^{-1} \varepsilon^{r-2} \kappa_r\right) \tag{2.6}$$

which yields

$$P_r(\kappa_.) = \sum_{m=1}^{r} m!^{-1} \left\{ \sum_{(j_1, \dots j_m)} (j_1+2)!^{-1} \kappa_{j_1+2} \cdots (j_m+2)!^{-1} \kappa_{j_m+2} \right\}, \tag{2.7}$$

where the sum $\sum_{(j_1,\dots,j_m)}$ means summation over all m-tuples of positive integers $(j_1, \dots, j_m)$ satisfying $\sum_{q=1}^m j_q = r$ and $\kappa_. = (\kappa_3, \dots, \kappa_{r+2})$. Replacing the variables $\kappa_.$ in $P_r(\cdot)$ by the differential operators $\kappa_.(D) := (\kappa_3(D), \dots, \kappa_{r+2}(D))$ we obtain "Edgeworth" differential operators, say $P_r(\kappa_.(D))$.

Let $C_B^s(A)$ denote the space of s times partially differentiable functions on $A \subset \mathbb{R}^m$ with derivatives bounded in absolute values by $B > 0$. Finally, we define

$$d_s(h, n) := \sup\{|D^\alpha h_{m+s}(\underline{\varepsilon}_{m+s})| : |\alpha| = s,\ \underline{\varepsilon}_{m+s} \in E^n_{m,s},\ m \geq n\}. \tag{2.8}$$

The following proposition shows that $\lim_{m\to\infty} h_{m+2}(m^{-1/2}, \dots, m^{-1/2}, \underline{\varepsilon}_2)$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, 2$, $m \geq n > 3$ exists .

Proposition 2.1. *Assume* $h_m(\cdot)$, $m \geq n$, *satisfies conditions* (2.1) − (2.4) *and the condition* $d_3(h,n) < \infty$. *Then* $\lim_{m\to\infty} h_{m+2}(m^{-1/2},\dots,m^{-1/2},\underline{\varepsilon}_2) = h_\infty(\underline{\varepsilon}_2)$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1,2$, $m \geq n > 3$ *exists and*

$$|h_{n+2}(n^{-1/2},\dots,n^{-1/2},\underline{\varepsilon}_2) - h_\infty(\underline{\varepsilon}_2)| \leq c d_3(h,n) n^{-1/2},$$

where c *is an absolute constant.*

Remark 2.2. *Under the condition* $d_s(h,n) < \infty$, $n > s \geq 3$ *we can define the function* $h_\infty(\underline{\varepsilon}_{s-1})$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1,\dots,s-1$.

We note that Proposition 2.1 guarantees that the limit $h_\infty(\cdot)$ exists, but does not help to find the limit. The following theorem yields an asymptotic expansion for h_n.

Theorem 2.3. *Assume that* $h_m(\cdot)$, $m \geq n$, *fulfills conditions* (2.1) − (2.4) *together with*

$$h_{m+s}(\cdot) \in C_B^s(E_{m,s}^n), \qquad s \geq 3, \tag{2.9}$$

$$\sup_{\underline{\varepsilon}_{m+s}\in E_{m,s}^n} \left|D^\alpha h_{m+s}(\underline{\varepsilon}_{m+s})\right| \leq B, \quad s \geq 3, \tag{2.10}$$

where $\alpha = (\alpha_1,\dots,\alpha_{s-2})$ *such that* $\alpha_i \geq 2$, $i = 1,\dots,s-2$, $\sum_{i=1}^{s-2}(\alpha_i - 2) = s-2$. *Then*

$$\left| h_n(n^{-1/2},\dots,n^{-1/2}) - \sum_{r=0}^{s-3} n^{-r/2} P_r(\kappa_{\cdot}(D)) h_\infty(\varepsilon_1,\dots,\varepsilon_r)\Big|_{\underline{\varepsilon}_r=0} \right| \leq c_s B n^{-(s-2)/2}, \tag{2.11}$$

where $P_0(\kappa_{\cdot}(D)) = 1$ *and* $P_r(\kappa_{\cdot}(D))$ *are given explicitly in* (2.7), c_s *is an absolute constant.*

According to the previous theorem the expansion for h_n can be obtained by using derivatives of $h_\infty(\varepsilon_1,\dots,\varepsilon_{s-1})$:

$$\begin{aligned} h_n &= h_\infty(0) + \frac{1}{n^{1/2}}\left(\frac{1}{6}\frac{\partial^3}{\partial\varepsilon_1^3}\right) h_\infty(\varepsilon_1)\Big|_{\varepsilon_1=0} \\ &+ \frac{1}{n}\left(\frac{1}{24}\left(\frac{\partial^4}{\partial\varepsilon_1^4} - 3\frac{\partial^2}{\partial\varepsilon_1^2}\frac{\partial^2}{\partial\varepsilon_2^2}\right) + \frac{1}{72}\frac{\partial^3}{\partial\varepsilon_1^3}\frac{\partial^3}{\partial\varepsilon_2^3}\right) h_\infty(\varepsilon_1,\varepsilon_2)\Big|_{\varepsilon_1=\varepsilon_2=0} \\ &+ \frac{1}{48n^{3/2}}\left(\frac{1}{5}\left(\frac{\partial^5}{\partial\varepsilon_1^5} - 10\frac{\partial^3}{\partial\varepsilon_1^3}\frac{\partial^2}{\partial\varepsilon_2^2}\right)\right. \\ &+ \left.\frac{1}{3}\left(\frac{\partial^4}{\partial\varepsilon_1^4} - 3\frac{\partial^2}{\partial\varepsilon_1^2}\frac{\partial^2}{\partial\varepsilon_2^2}\right)\frac{\partial^3}{\partial\varepsilon_3^3} + \frac{1}{27}\frac{\partial^3}{\partial\varepsilon_1^3}\frac{\partial^3}{\partial\varepsilon_2^3}\frac{\partial^3}{\partial\varepsilon_3^3}\right) h_\infty(\varepsilon_1,\varepsilon_2,\varepsilon_3)\Big|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Remark 2.4. *Condition (2.9) guarantees the desired estimate for the remainder term in (2.11). Conditions (2.9) and (2.10) guarantee that the function*

$$g^{\alpha}_{m+s}(\underline{\varepsilon}_{m+s}) := D^{\alpha} h_{m+s}(\underline{\varepsilon}_{m+s}),$$

for $\alpha = (\alpha_1, \ldots, \alpha_r)$, *where* $r \leq s-3,\ s \geq 3$

$$\alpha_i \geq 2, \quad i = 1, \ldots, r, \quad \sum_{i=1}^{r} (\alpha_i - 2) = s - 3$$

satisfies the conditions of Proposition 2.1. In particular, due to Proposition 2.1 for each α *the functions* $g^{\alpha}_{m+s}(\underline{\varepsilon}_{m+s})$ *converge to* $g^{\alpha}_{s}(\underline{\varepsilon}_{s})$ *as* $m \to \infty$ *uniformly in* $\underline{\varepsilon}_s$, $|\varepsilon_j| \leq n^{-1/2}$, $m \geq n \geq 1$, $j = 1, \ldots, s$ *and due to Theorem 7.1 (see Auxiliary results) we conclude that* $g^{\alpha}_{s}(\underline{\varepsilon}_{s}) = D^{\alpha} h_{\infty}(\underline{\varepsilon}_{s})$, $r \leq s-3$.

Let us illustrate how one can apply this method to obtain an asymptotic expansion in the Central Limit Theorem. As a statistic, we consider normalized sums of independent identically distributed random variables $X_1, \ldots, X_n$ with distribution function F_1, zero mean, unit variance and finite moments m_3 and m_4. Suppose that F_1 has a bounded density and $\hat{F}_1$ is a corresponding characteristic function. Let S_n denote the sequence of the normalized sums. We denote by F_n and $\hat{F}_n$ the distribution and characteristic functions of S_n correspondingly. Replacing $n^{-1/2}$ in S_n by ε_j, we get

$$S_n(\underline{\varepsilon}_n) = \sum_{j=1}^{n} \varepsilon_j X_j.$$

One can see that the sequence of the corresponding characteristic function of $F_n^{(\underline{\varepsilon}_n)}$

$$\hat{F}_n^{(\underline{\varepsilon}_n)}(t) = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{it\varepsilon_j x} dF_1(x), \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \ldots, n$$

is symmetric and compatible as a function of $\underline{\varepsilon}_n$. Let us show that $\hat{F}_{m+s}^{(\underline{\varepsilon}_{m+s})}$ is uniformly differentiable on the set $E^n_{m,s}$:

$$\hat{F}_{m+s}^{(\underline{\varepsilon}_{m+s})}(t) = \prod_{j=1}^{m+s} \int_{\mathbb{R}} e^{it\varepsilon_j x} dF_1(x), \quad \underline{\varepsilon}_{m+s} \in E^n_{m,s}.$$

Due to the assumption of the integrability of $\hat{F}_1$, it is easy to see that $\hat{F}_{m+s}^{(\underline{\varepsilon}_{m+s})}$ has bounded derivatives D^{α} (α is defined in Theorem 2.3), since we assumed that m_4 is finite. Since X_1 has zero mean it is easy to see that the first derivatives vanish, indeed

$$\frac{\partial}{\partial \varepsilon_j} \hat{F}_m^{(\underline{\varepsilon}_m)}(t)\Big|_{\varepsilon_j = 0} = it \int_{\mathbb{R}} x e^{it\varepsilon_j x} dF_1(x)\Big|_{\varepsilon_j = 0} \prod_{k=1}^{m}{}^{*} \left[\int_{\mathbb{R}} e^{it\varepsilon_k x} dF_1(x) \right] = 0, \quad j = 1, \ldots, m,$$

where $\prod^*$ denotes multiplication over k from 1 to m without j. Due to the Central Limit Theorem we have $F_n \to \Phi$ uniformly as $n \to \infty$, where Φ is the distribution function of the standard normal law. This means that $h_\infty = \hat{\Phi}$, where $\hat{\Phi}$ is the Fourier transform of Φ. Now we can derive the second term in the asymptotic expansion (2.12). For that assume that m_3 is the third moment of F_1, then

$$\begin{aligned}\frac{\partial^3}{\partial \varepsilon^3} h_\infty(\varepsilon)\Big|_{\varepsilon=0} &= \frac{\partial^3}{\partial \varepsilon^3} \int_{\mathbb{R}} e^{it\varepsilon y} dF_1(y)\Big|_{\varepsilon=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) \\ &= -it^3 \int_{\mathbb{R}} y^3 e^{it\varepsilon y} dF_1(y)\Big|_{\varepsilon=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) = -m_3 \hat{\Phi}^{(3)}(t).\end{aligned}$$

The next term consists of three parts:

$$\frac{\partial^4}{\partial \varepsilon^4} h_\infty(\varepsilon)\Big|_{\varepsilon=0} = t^4 \int_{\mathbb{R}} y^4 e^{it\varepsilon y} dF_1(y)\Big|_{\varepsilon=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) = m_4 \hat{\Phi}^{(4)}(t);$$

$$\begin{aligned}&\frac{\partial^2}{\partial \varepsilon_1^2} \frac{\partial^2}{\partial \varepsilon_2^2} h_\infty(\varepsilon_1, \varepsilon_2)\Big|_{\varepsilon_1=\varepsilon_2=0} \\ &= \frac{\partial^2}{\partial \varepsilon_1^2} \int_{\mathbb{R}} e^{it\varepsilon_1 y} dF_1(y)\Big|_{\varepsilon_1=0} \frac{\partial^2}{\partial \varepsilon_2^2} \int_{\mathbb{R}} e^{it\varepsilon_2 y} dF_1(y)\Big|_{\varepsilon_2=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) \\ &= t^4 \int_{\mathbb{R}} y^2 e^{it\varepsilon_1 y} dF_1(y)\Big|_{\varepsilon_1=0} \int_{\mathbb{R}} y^2 e^{it\varepsilon_2 y} dF_1(y)\Big|_{\varepsilon_2=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) = \hat{\Phi}^{(4)}(t);\end{aligned}$$

$$\begin{aligned}&\frac{\partial^3}{\partial \varepsilon_1^3} \frac{\partial^3}{\partial \varepsilon_3^2} h_\infty(\varepsilon_1, \varepsilon_2)\Big|_{\varepsilon_2=0} \\ &= -\, t^6 \int_{\mathbb{R}} y^3 e^{it\varepsilon_1 y} dF_1(y)\Big|_{\varepsilon_1=0} \int_{\mathbb{R}} y^3 e^{it\varepsilon_2 y} dF_1(y)\Big|_{\varepsilon_2=0} \int_{\mathbb{R}} e^{itx} d\Phi(x) = m_3^2 \hat{\Phi}^{(6)}(t).\end{aligned}$$

We plug the derivatives into (2.12) and obtain the Edgeworth expansion for characteristic functions

$$\begin{aligned}\hat{F}_n(t) &= \hat{\Phi}(t) - \frac{1}{n^{1/2}} \left(\frac{1}{6} m_3 \hat{\Phi}^{(3)}(t) \right) \\ &+ \frac{1}{n} \left(\frac{1}{24} m_4 \hat{\Phi}^{(4)}(t) + \frac{1}{72} m_3^2 \hat{\Phi}^{(6)}(t) \right) + O(n^{-3/2}).\end{aligned}$$

This expansion was originally proved by Cramer in [20]. If we assume that for some $n_0 \in \mathbb{N}$ the distribution F_n has a density, then due to the Fourier inversion formula one can obtain an expansion for F_n:

$$\begin{aligned}F_n(x) &= \Phi(x) - \frac{1}{n^{1/2}} \left(\frac{1}{6} m_3 \Phi^{(3)}(x) \right) \\ &+ \frac{1}{n} \left(\frac{1}{24} m_4 \Phi^{(4)}(x) + \frac{1}{72} m_3^2 \Phi^{(6)}(x) \right) + O(n^{-3/2}). \qquad (2.12)\end{aligned}$$

Let us denote by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{d^m} e^{-x^2/2}$ the Hermite polynomials, i.e. $H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = x^2 - 1,\ H_3(x) = x^3 - 3x,\ H_4(x) = x^4 - 6x^2 + 3,\ H_5(x) = x^5 - 10x^3 + 15x$ etc. It is easy to see that

$$\Phi^{(k)}(x) = \frac{(-1)^{k-1}}{\sqrt{2\pi}} H_{k-1}(x) e^{-x^2/2}.$$

Therefore, (2.12) can be rewritten in terms of the Hermite polynomials

$$\begin{aligned} F_n(x) &= \Phi(x) - \frac{1}{n^{1/2}} \left(\frac{1}{6} m_3 H_2(x) \varphi(x) \right) \\ &+ \frac{1}{n} \left(\frac{1}{24} m_4 H_3(x) \varphi(x) + \frac{1}{72} m_3^2 H_5(x) \varphi(x) \right) + O(n^{-3/2}), \end{aligned}$$

where φ is the density of the standard normal law. For more details see [24] and [40]. Further examples can be found in [25].

Chapter 3

Preliminaries

In this chapter we present the main concepts of free probability. We describe the construction of a non-commutative probability space for random variables with bounded support and unbounded ones as well. For both cases we introduce the notions of a non-commutative probability space, freeness and free convolution. For the case of Borel probability measures on $\mathbb{R}$ with compact support we refer the reader to Voiculescu, Dykema and Nica [46], Nica and Speicher [38] and Hiai and Petz [27]. The first extension of free probability beyond measures with compact support was done by Maassen [35], but he restricted himself to measures with finite variance. Bercovici and Voiculescu [8] finally introduced the extension to the class of Borel probability measures on $\mathbb{R}$ without any further restrictions.

3.1 Non-commutative probability spaces

Measures with compact support. In classical probability theory one can define a probability space by specifying a triple (Ω, Σ, P), where Ω is a set, Σ is a σ-algebra and $P : \Sigma \to [0, 1]$ is a probability measure. Let $\mathcal{A} = L^\infty(\Omega, P)$ be the space of essentially bounded measurable functions $f : \Omega \to \mathbb{C}$ and let φ be the linear functional

$$\varphi : L^\infty(\Omega, P) \to \mathbb{C}, \quad \varphi(f) = \int_\Omega f dP.$$

It is well known that $L^\infty(\Omega, P)$ is a von Neumann algebra. In this context, a commutative probability space can be described as a commutative algebra of random variables and a functional on this algebra defined by the expectation of these random variables. Replacing a commutative algebra by a non-commutative one we arrive to the following definition of a non-commutative probability space:

Definition 3.1. A non-commutative probability space *is a pair* $(\mathcal{A}, \varphi)$, *where* $\mathcal{A}$ *is a unital algebra over* $\mathbb{C}$ *and* φ *is a unital linear functional*

$$\varphi : \mathcal{A} \to \mathbb{C}, \qquad \varphi(1_\mathcal{A}) = 1.$$

An element $a \in \mathcal{A}$ is called a non-commutative random variable.

Next, we need to specify some additional properties of the functional φ.

Definition 3.2. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, where $\mathcal{A}$ is a $*$-algebra. We say that*

(1) *φ is* faithful *if $\varphi(aa^*) = 0$ implies that $a = 0$ for all $a \in \mathcal{A}$;*

(2) *φ is a* state *if $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$;*

(3) *φ is a* normal state, *if $\sup_i \varphi(a_i) = \varphi(\sup_i a_i)$ for every monotone, increasing, bounded net $\{a_i\} \subset \mathcal{A}$;*

(4) *φ is a* trace *if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.*

Let us denote a trace by τ. Then $(\mathcal{A}, \tau)$ is called tracial.

Definition 3.3. *A pair $(\mathcal{A}, \varphi)$ is called a C^*-*probability space *if $\mathcal{A}$ is a C^*-algebra and φ is a state.*

Example 3.4. Let us consider the algebra of $n \times n$ matrices $\mathcal{A} = M_n(\mathbb{C})$. We can define a state φ by

$$\varphi(X) = n^{-1} Tr(X).$$

Then $(\mathcal{A}, \varphi)$ is a non-commutative probability space.

One of the basic concepts in probability theory is the distribution of a random variable. In free probability the distribution of a non-commutative random variable is given as a collection of moments.

Definition 3.5. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The* distribution *of $a \in \mathcal{A}$ is the linear functional μ_a on $\mathbb{C}[X]$ (the algebra of complex polynomials in one variable), defined by $\mu_a(P) = \varphi(P(a))$, $P \in \mathbb{C}[X]$.*

It is a well known result that for a selfadjoint random variable $a = a^*$ in a C^*-probability space, μ_a extends to a compactly supported probability measure on the real line. More precisely, there exists a unique probability measure μ_a such that

$$\int_{\mathbb{R}} P(t) d\mu_a(t) = \varphi(P(a)) \quad \text{for all} \quad P \in \mathbb{C}[X].$$

Now we would like to discuss the concept of the joint distribution. Actually the word "free" in the name "free probability" is justified by the concept of the free product. It comes from the theory of free algebras and we would like to introduce a construction of the free product for unital algebras in the following definition.

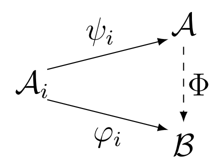

Figure 3.1

Definition 3.6. *If $(\mathcal{A}_i)_{i\in I}$ is a family of unital algebras, then their* unital algebra free product $*_{i\in I}\mathcal{A}_i$ *is the unique unital algebra $\mathcal{A}$ together with unital homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that given any unital algebra $\mathcal{B}$ and unital homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{B}$ there exists a unique unital homomorphism $\Phi = *_{i\in I}\psi_i : \mathcal{A} \to \mathcal{B}$ making the diagram in Figure 3.1 commute.*

In the next definition we need the unital algebra of non-commutative polynomials in $|I|$ variables, which we denote by $\mathbb{C}\langle X_i | i \in I\rangle = *_{i\in I}\mathbb{C}\langle X_i\rangle$. It is the linear span of 1 and non-commutative monomials of the form $X_{i_1}^{k_1} \dots X_{i_n}^{k_n}$ with $i_1 \neq i_2 \neq \dots \neq i_n$ and $k_j \geq 1$ for all j.

Definition 3.7. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $(a_i)_{i\in I}$ be random variables in $\mathcal{A}$. Their* joint distribution *is the linear functional $\mu : \mathbb{C}\langle X_i | i \in I\rangle \to \mathbb{C}$ defined by $\mu(P) = \varphi(h(P))$, where $P \in \mathbb{C}\langle X_i | i \in I\rangle$ and $h : \mathbb{C}\langle X_i | i \in I\rangle \to \mathcal{A}$ is the unique unital algebra homomorphism such that $h(X_i) = a_i$.*

Free independence. The important step from non-commutative probability to free probability is the notion of freeness or free independence. In classical probability independence is a special relation between two or more random variables which gives us a rule to calculate a joint distribution of these random variables. More precisely, if a family of random variables is independent, then the joint distribution of the family is completely determined by the knowledge of the individual distributions of the variables. In the non-commutative context the notion of classical independence has the following extension.

Definition 3.8. *In a non-commutative probability space $(\mathcal{A}, \varphi)$, a family of subalgebras $\mathcal{A}_i \subseteq \mathcal{A}$, $i \in I$, is* independent *if the algebras commute with each other (i.e. $[\mathcal{A}_i, \mathcal{A}_j] = 0$ if $i \neq j$) and $\varphi(a_1, \dots, a_n) = \varphi(a_1)\varphi(a_2)\dots\varphi(a_n)$ whenever $a_k \in \mathcal{A}_{i_k}$ and $k \neq l$ implies $i_k \neq i_l$.*

The notion of free independence is the completely non-commutative counterpart of classical one.

Definition 3.9. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. A family $(\mathcal{A}_i)_{i\in I}$ of unital subalgebras of $\mathcal{A}$ is called* free *if $\varphi(a_1a_2\dots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$, $i_1 \neq i_2 \neq \dots \neq i_n$ and $\varphi(a_j) = 0$, $j = 1, \dots, n$. A family of random variables $(a_i)_{i\in I}$ is called* free *if the unital subalgebras generated by these variables are free.*

Remark 3.10. *If $(\mathcal{A},\varphi)$ is a C^*-probability space and $(\mathcal{A}_i)_{i\in I}$ are $*$-algebras, then algebras $(\mathcal{A}_i)_{i\in I}$ are free implies corresponding C^*-algebras $(C^*(\mathcal{A}_i))_{i\in I}$ are free.*

Example 3.11. Let us consider two free selfadjoint random variables, i.e. we consider $a,\ b \in (\mathcal{A},\varphi)$. We would like to compute $\varphi(ab)$. In the beginning we center our variables $(a-\varphi(a)1_{\mathcal{A}})$, $(b-\varphi(b)1_{\mathcal{A}})$. Due to the definition of freeness we have

$$\begin{aligned} 0 &= \varphi((a-\varphi(a)1_{\mathcal{A}})(b-\varphi(b)1_{\mathcal{A}})) \\ &= \varphi(ab)-\varphi(a1_{\mathcal{A}})\varphi(b)-\varphi(a)\varphi(b1_{\mathcal{A}})+\varphi(a)\varphi(b)\varphi(1_{\mathcal{A}}) \\ &= \varphi(ab)-\varphi(a)\varphi(b). \end{aligned}$$

We thus see that $\varphi(ab)$ depends only on $\varphi(a)$ and $\varphi(b)$.

Below, we introduce a general result from [38] about the joint distribution of free random variables, which is uniquely determined by the distribution of each random variable.

Theorem 3.12. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let the unital subalgebras $(\mathcal{A}_i)_{i\in I}$ be freely independent. Denote by $\mathcal{B}$ the algebra which is generated by all $\mathcal{A}_i$, $\mathcal{B} := \mathrm{alg}(\mathcal{A}_i | i \in I)$. Then $\varphi|_{\mathcal{B}}$ is uniquely determined by $\varphi|_{\mathcal{A}_i}$ for all $i\in I$ and by the free independence condition.*

Next, we introduce a free product construction of C^*-algebras and C^*-probability spaces.

Definition 3.13. *Let $(\mathcal{A}_i)_{i\in I}$ be a family of unital C^*-algebras. Then, their* unital C^*-algebra free product *$*_{i\in I}\mathcal{A}_i$ is the unique unital algebra $\mathcal{A}$ together with unital $*$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that given any unital C^*-algebra $\mathcal{B}$ and unital $*$-homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{B}$ there exists a unique unital $*$-homomorphism $\Phi = *_{i\in I}\psi_i : \mathcal{A} \to \mathcal{B}$ making the diagram in Figure 3.2 commute.*

Definition 3.14. *Let $(\mathcal{A}_i,\tau_i)_{i\in I}$ be a family of tracial C^*-probability spaces. A C^*-probability space $(\mathcal{A},\tau)$ is called the* free product of C^*-probability spaces *if:*

(1) *the algebras $\mathcal{A}_i$ can be regarded as free subalgebras of $\mathcal{A}$;*

(2) $\tau|_{\mathcal{A}_i} = \tau_i$;

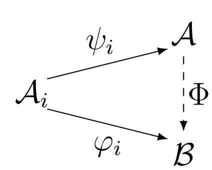

Figure 3.2

(3) *$\mathcal{A}$ is generated as a C^*-algebra by its subalgebras $\mathcal{A}_i$.*

This construction shows that for given compactly supported measures $\mu_1, \ldots, \mu_n$ it is possible to introduce free random variables $a_1, \ldots, a_n$ from some non-commutative probability space such that $\mu_{a_i} = \mu_i$ for $i = 1, \ldots, n$. In particular, we can work with measures without using random variables.

As in the classical case, we consider the distribution of the sum of free random variables. Due to Theorem 3.12 it follows that the distribution of the sum of free random variables only depends on the individual distributions of the summands. Thus we can define the sum of free random variables.

Definition 3.15. *Let μ, ν be two compactly supported measures on the real line. The* free additive convolution *(or* free convolution *in the sequel) of μ and ν is denoted by $\mu \boxplus \nu$ and defined as the distribution of $a_\mu + a_\nu$, where a_μ and a_ν are free random variables with distributions μ and ν respectively.*

We want to remark that in the case of compactly supported measures the free convolution is also a measure of compact support.

Measures with unbounded support. The extension of free probability to measures with unbounded supports is done in the context of finite von Neumann algebras. Below we review the basic concepts. For more details we refer to [8]. Let us denote by $\mathcal{M}$ the class of all Borel probability measures on $\mathbb{R}$.

Definition 3.16. *A non-commutative probability space $(\mathcal{A}, \tau)$ is called a* tracial W^*-probability space *if $\mathcal{A}$ is a finite von Neumann algebra and τ is a normal faithful tracial state.*

In the sequel we will omit the word "tracial" and speak of a W^*-probability space. Unbounded random variables should be considered as unbounded operators affiliated with some von Neumann algebra. More precisely:

Definition 3.17. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let $\mathcal{A}$ act on a Hilbert space $\mathcal{H}$. A selfadjoint operator X on $\mathcal{H}$ is said to be* affiliated *with $\mathcal{A}$ if all the spectral projections of X belong to $\mathcal{A}$. A closed, densely defined operator T on $\mathcal{H}$ is* affiliated *with $\mathcal{A}$ if its polar decomposition $T = UX$ has the property that $U \in \mathcal{A}$ and X is affiliated with $\mathcal{A}$. We will denote by $\tilde{\mathcal{A}}$ the set of all operators on $\mathcal{H}$ which are affiliated with $\mathcal{A}$, and by $\tilde{\mathcal{A}}_{sa}$ the set of selfadjoint operators. The elements of $\tilde{\mathcal{A}}_{sa}$ are* unbounded random variables*.*

An important result by Murray and von Neumann [37] asserts that $\tilde{\mathcal{A}}$ is an algebra, namely, if $X, Y \in \tilde{\mathcal{A}}$ then $X+Y$ and XY are densely defined, closable and their closures are in $\tilde{\mathcal{A}}$. The next step is introducing the distribution of an unbounded selfadjoint random variable.

Definition 3.18. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and $T \in \tilde{\mathcal{A}}_{sa}$. The* distribution *$\mu_T$ of T is the unique probability measure on $\mathbb{R}$ satisfying the equality $\tau(u(T)) = \int_{\mathbb{R}} u(x) d\mu_T(x)$ for every bounded Borel function u on $\mathbb{R}$.*

In other words, in the case of unbounded random variables it is not enough to define the distribution on polynomials but we must consider bounded Borel functions. Next, we would like to define a notion of freeness for unbounded random variables. Let $(\mathcal{A}, \tau)$ be a W^*-probability space and $\omega \subset \tilde{\mathcal{A}}_{sa}$ a collection of random variables. We will denote by $C^*(\omega)$ the C^*-algebra generated by elements of the form $u(T)$ with $T \in \omega$ and u a bounded continuous function on $\mathbb{R}$.

Definition 3.19. *A family $\{\omega_i\}_{i \in I}$ of subsets of $\tilde{\mathcal{A}}_{sa}$ is said to be* free *if $\{C^*(\omega_i)\}_{i \in I}$ is a free family of subalgebras of $\mathcal{A}$. A family $\{T_i\}_{i \in I}$ of elements of $\tilde{\mathcal{A}}_{sa}$ is said to be* free *if the family of singletons $\{T_i\}_{i \in I}$ is free.*

One can define a free product of W^*-probability spaces in the same way as it was done for C^*-probability spaces. Due to this construction it is possible to prove that for measures $\mu_1, \ldots, \mu_n \in \mathcal{M}$ there exist free random variables $a_1, \ldots, a_n$ affilated with some W^*-probability space such that $\mu_{a_i} = \mu_i$ for $i = 1, \ldots, n$.

The definition of freeness is similar to the one from the previous paragraph.

Bercovici and Voiculescu showed in [8] that also for free unbounded selfadjoint random variables their joint distribution is uniquely determined by the distribution of each random variable. By this result, the distribution of the sum only depends on the distribution of the summand and we can introduce the definition of the free convolution for unbounded free random variables:

Definition 3.20. *Let μ, ν be two Borel probability measures on the real line. The* free additive convolution *(or* free convolution *in the sequel) of μ and ν is denoted by $\mu \boxplus \nu$ and defined as the distribution of $a_\mu + a_\nu$, where a_μ and a_ν are free random variables with distributions μ and ν respectively.*

3.2 Free convolution

In this section we introduce a number of approaches which one can apply to compute the free convolution of free random variables.

Measures with bounded support. Let us first consider the case when measure μ has a compact support contained in $[-L, L]$. This implies that the measure has moments of all orders. Denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ the complex upper half-plane and by $\mathbb{C}^- = \{z \in \mathbb{C} : \Im z < 0\}$ the complex lower half-plane. The Cauchy transform of a measure $\mu \in \mathcal{M}$ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z - x}, \quad z \in \mathbb{C}^+,$$

which is an analytic function on the upper half-plane. A measure is uniquely determined by its Cauchy transform. For instance, the measure can be recovered from its Cauchy transform by the Stieltjes inversion formula:

$$d\mu(x) = -\frac{1}{\pi} \lim_{y \downarrow 0} \Im G_\mu(x + iy)dx. \tag{3.1}$$

The Cauchy transform has the following power series expansion at $z = \infty$

$$G_\mu(z) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}, \tag{3.2}$$

where m_k are the moments of the measure μ. Moreover, $|m_k| \leq L^k$. It is easy to see that $G_\mu(z) = \frac{1}{z}(1 + o(1))$ at $z = \infty$. The series (3.2) is univalent for large z ($|z| > L$) and we can define its functional inverse $K_\mu(z)$ such that $K_\mu(G_\mu(z)) = z$, which converges in a neighbourhood of zero. Let us introduce the function

$$R_\mu(z) = K_\mu(z) - \frac{1}{z}. \tag{3.3}$$

This function is called the R-transform and can be expressed as formal power series by

$$R_\mu(z) = \sum_{l=0}^{\infty} \kappa_{l+1} z^l,$$

where the coefficients κ_k are called the free cumulants of a corresponding measure. There is a combinatorial approach to free probability where free cumulants are defined by non-crossing partitions, for more details see [38]. Let us note that the three first free cumulants coincide with the classical ones, thus we will call the second free cumulant variance as well. The first free cumulant is equal to the first moment $\kappa_1 = m_1$. In the case when $m_1 = 0$ and $m_2 = 1$ we note that $\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = m_3$, $\kappa_4 = m_4 - 2$, $\kappa_5 = m_5 - 5m_3$. For cumulants of higher order the following inequalities have been established in [30]:

$$|\kappa_l| \leq \frac{2L}{l-1}(4L)^{l-1}, \quad l \geq 2. \tag{3.4}$$

A remarkable property of the R-transform was proved by Voiculescu [44]. It states that the R-transform linearises the free convolution, more precisely, for two given probability measures μ_1 and μ_2 with compact support the R-transform of the free convolution $\mu_1 \boxplus \mu_2$ is given by the formula

$$R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z), \tag{3.5}$$

on the common domain of these functions. According to this property we can compute the Cauchy transform of $\mu_1 \boxplus \mu_2$ and find the measure $\mu_1 \boxplus \mu_2$ by applying the Stieltjes inversion formula. We want to stress that the R-transform is a free counterpart of the logarithm of the Fourier transform, since the first one linearises the free convolution and the second linearises the classical one. Moreover, (3.5) implies that the free convolution is commutative and associative.

Next, we note some scaling properties of the Cauchy transform and the R-transform. We denote by $D_t\mu$ the dilation of a measure μ by the factor t:

$$D_t\mu(A) = \mu(t^{-1}A), \qquad (A \subset \mathbb{R} \quad \text{measurable}).$$

Then the Cauchy transform and the R-transform of the rescaled measure $D_t\mu$ are

$$G_{D_t\mu}(z) = t^{-1}G_\mu(t^{-1}z) \quad \text{and} \quad R_{D_t\mu}(z) = tR_\mu(tz). \tag{3.6}$$

Measures with unbounded support. In this paragraph we introduce an approach for the calculation of the free convolution for measures with unbounded support. Here, the basic problem is that measures are not necessarily uniquely defined by the moments (even if all moments are bounded). Therefore, it is more convenient to study special classes of analytic functions instead of power series. More precisely, instead of the Cauchy transform we will deal with the reciprocal Cauchy transform

$$F_\mu(z) = 1/G_\mu(z), \quad z \in \mathbb{C}^+,$$

which is an analytic self-mapping of $\mathbb{C}^+$. The class of reciprocal Cauchy transforms can be described as a subclass of the Nevanlinna functions.

Definition 3.21. *The* Nevanlinna class *is the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$ with the integral representation*

$$f(z) = a + bz + \int \frac{1+tz}{t-z}\rho(dt), \quad z \in \mathbb{C}^+, \tag{3.7}$$

where $b \geq 0$, $a \in \mathbb{R}$ and ρ is a nonnegative finite measure.

From the integral representation (3.7) it follows that $f(z) = (b + o(1))z$ for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z| \to \infty$ (or $z \to \infty$ nontangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the domain

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z,\ \Im z > \beta\}$$

for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$. For more details about Nevanlinna functions we refer to [1], Section 3 and [2], Section 6.

The class of the reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ is a subclass of the Nevanlinna functions such that $f(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$. We will denote this class by $\mathcal{F}$. It is easy to see that the reciprocal Cauchy transform F_μ admits the representation (3.7) with $b = 1$. The functions $f \in \mathcal{F}$ satisfy the inequality

$$\Im f(z) \geq \Im z, \quad z \in \mathbb{C}^+. \tag{3.8}$$

Moreover, F_μ has certain invertability properties on $\Gamma_{\alpha,\beta}$, which allow us to define the function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$. This function is called the Voiculescu transform of μ. It has the same remarkable property as the R-transform, namely, the Voiculescu transform linearises free convolution so that the equality

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z), \quad \mu_1, \mu_2 \in \mathcal{M}, \tag{3.9}$$

holds on a domain $\Gamma_{\alpha,\beta}$ such that these three function are defined.

Analytic approach to the definition of free convolution. Voiculescu showed in [45] that for compactly supported measures μ_1, μ_2, the Cauchy transform $G_{\mu_1 \boxplus \mu_2}$ of $\mu_1 \boxplus \mu_2$ is subordinated to G_{μ_j}, $j = 1, 2$, in the sense that

$$G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z)),$$

where $Z_1,\ Z_2 : \mathbb{C}^+ \to \mathbb{C}^+$ belong to $\mathcal{F}$. Biane extended this result for $\mu_1,\ \mu_2 \in \mathcal{M}$ in [13].

Chistyakov and Götze [14], [16], Bercovici and Belinschi [5], Belinschi [4] proved, using complex analytic methods, that for $\mu_1,\ \mu_2 \in \mathcal{M}$ the subordination functions $Z_1,\ Z_2 \in \mathcal{F}$ satisfy the following equations for $z \in \mathbb{C}^+$:

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)); \tag{3.10}$$

$$F_{\mu_1 \boxplus \mu_2}(z) = F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{3.11}$$

The main advantage of these equations is the fact that one can compute the free convolution without inverting the Cauchy transforms (or the reciprocal Cauchy transforms). Let us consider the result from [16] which concerns n equal measures $\mu_1 = \ldots = \mu_n$. Then there exists a unique function $Z \in \mathcal{F}$ such that

$$z = nZ(z) - (n-1)F_{\mu_1}(Z(z)) \tag{3.12}$$

$$F_{\mu_1 \boxplus \ldots \boxplus \mu_n}(z) = F_{\mu_1}(Z(z)), \quad z \in \mathbb{C}^+. \tag{3.13}$$

3.3 Free convolution with a semicircle law

The semicircle law plays a key role in free probability. This law is the free counterpart of the Gaussian one. The centered semicircle distribution of variance t is denoted by ω_t and has the density

$$p_{\omega_t}(x) = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}, \quad x \in \mathbb{R},$$

where $a_+ := \max\{a, 0\}$. We denote by ω the standard semicircle law that has zero mean, unit variance and the density

$$p_\omega(x) = \frac{1}{2\pi}\sqrt{(4 - x^2)_+}, \quad x \in \mathbb{R}.$$

The Cauchy transform of ω_t is given by

$$G_{\omega_t}(z) = \frac{z - \sqrt{z^2 - 4t}}{2t}, \qquad z \in \mathbb{C}^+.$$

The function $\sqrt{z^2 - 4t}$ is double-valued and has branch points at $z = \pm 2\sqrt{t}$. We can define two single-valued analytic branches on the complex plane cut along the segment $-2\sqrt{t} \le x \le 2\sqrt{t}$ of the real axis. Since the Cauchy transform has asymptotic behaviour $1/z$ at infinity, we can choose a branch such that $\sqrt{-1} = i$ on $\mathbb{C}^+$. The Cauchy transform $G_{\omega_t}(z)$ has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$ which acts on $\mathbb{R}$ by

$$\begin{cases} (x - i\sqrt{4t - x^2})/2t, & \text{if } |x| \le 2\sqrt{t}; \\ (x - \sqrt{x^2 - 4t})/2t, & \text{if } |x| > 2\sqrt{t}. \end{cases} \tag{3.14}$$

We see that for each $\delta > 0$, the function G_{ω_t} can be continued analytically to the domain $K = \{x + iy : x \in (-2\sqrt{t}, 2\sqrt{t}), |y| < \delta\}$ and beyond to the whole Riemann surface [1]. This analytic continuation is again denoted by G_{ω_t}. It has the explicit formula $G_{\omega_t}(z) = (z - i\sqrt{4t - z^2})/2t$, where the branch of the square root on $\mathbb{C}^+$ is chosen such that $\sqrt{-1} = i$. The function G_{ω_t} satisfies the functional equation

$$G_\omega(z) + F_\omega(z) = z, \quad z \in \mathbb{C}^+ \cup K. \tag{3.15}$$

One can compute the R-transform of the semicircle law

$$R_{\omega_t}(z) = tz.$$

The properties of free convolution by semicircular distributions have been studied in [12], [45], [47]. We review some of these results. Fix $t > 0$ and a measure $\nu \in \mathcal{M}$.

[1] Here the Riemann surface consists of two complex planes with cuts along the intervals $[-2\sqrt{t}, 2\sqrt{t}]$ glued to each other in the following way: the upper edge of one cut is glued to the lower edge of another cut.

As it was shown by Biane [12], the Cauchy transform $G_{\omega_t \boxplus \nu}$ has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$ and the measure $\omega_t \boxplus \nu$ has a density p_t which can be described as follows.

Define the function $v_t : \mathbb{R} \to [0, +\infty)$ by

$$v_t(u) = \inf \left\{ v \geq 0 : \int_{-\infty}^{\infty} \frac{1}{(u-x)^2 + v^2} d\nu(x) \leq \frac{1}{t} \right\}.$$

It was shown in [12], Lemma 2, that the function v_t is continuous on $\mathbb{R}$, analytic on the set $\{u \in \mathbb{R} : v_t(u) > 0\}$ and that for all $u \in \mathbb{R}$, we have the bound

$$\int_{-\infty}^{\infty} \frac{1}{(u-x)^2 + v_t(u)^2} d\nu(x) \leq \frac{1}{t} \tag{3.16}$$

with equality if $v_t(u) > 0$. Let us now introduce the function

$$\psi_t(u) = u + t \int_{-\infty}^{\infty} \frac{(u-x)}{(u-x)^2 + v_t(u)^2} d\nu(x), \quad u \in \mathbb{R}.$$

In [12], Biane proved the following result:

Theorem 3.22. *The function $\psi_t : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and at the point $\psi_t(u)$ the measure $\omega_t \boxplus \nu$ has a density given by*

$$p_t(\psi_t(u)) = \frac{v_t(u)}{\pi t},$$

and its Cauchy transform is given by

$$G_{\omega_t \boxplus \nu}(\psi_t(u)) = \frac{1}{t}(\psi_t(u) - u - i v_t(u)).$$

Remark 3.23. *The density p_t is analytic on the set $\{x \in \mathbb{R} \mid p_t(x) > 0\}$ (see Corollary 4 in [12]). Biane also proved that $G_{\omega_t \boxplus \nu}(z)$ has an analytic extension to wherever v_t is positive.*

We note the following estimate in [12]:

$$|G_{\omega_t \boxplus \nu}(z)| \leq \frac{1}{\sqrt{t}}, \qquad z \in \mathbb{C}^+ \cup \mathbb{R}. \tag{3.17}$$

Throughout the text we use the notation

$$\mu_n := \underbrace{D_{1/\sqrt{n}}\mu \boxplus \ldots \boxplus D_{1/\sqrt{n}}\mu}_{n \text{ times}}, \quad n \in \mathbb{N}.$$

The next lemma gives an integral representation of the reciprocal Cauchy transforms F_μ and F_{μ_n}, where $\mu \in \mathcal{M}$.

Theorem 3.24. *Suppose that $\mu \in \mathcal{M}$ and has zero mean and unit variance. Then there exists a unique probability measure $\nu \in \mathcal{M}$ such that*

$$F_\mu(z) = z - G_\nu(z), \quad z \in \mathbb{C}^+, \tag{3.18}$$

and, for every $n \geq 2$,

$$F_{\mu_n}(z) = z - G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(z), \quad z \in \mathbb{C}^+ \cup \mathbb{R}, \tag{3.19}$$

where $t = \frac{n-1}{n}$.

The proof of the representation (3.18) can be found in [2], a proof of (3.19) is provided in [47]. Furthermore, we observe that the sequence $D_{1/\sqrt{n}}\nu$ converges weakly to δ_0. The next result describes the behavior of $G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}$.

Theorem 3.25 (Wang, [47])**.** *For each $\delta \in (0,1)$ there exists $N = N(\delta) > 0$ such that, for all $n \geq N$, the function $G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(z)$ can be continued analytically to a neighbourhood of the interval $[-2+\delta, 2-\delta]$. Furthermore, this analytic continuation never vanishes on $[-2+\delta, 2-\delta]$.*

3.4 Limit theorems

The free analogue of the Central Limit Theorem was first proved by Voiculescu [44] for compactly supported measures. This result was extended by Maassen [35] to measures with finite variance.

Theorem 3.26 (Maassen, [35])**.** *Let $\mu \in \mathcal{M}$ be a probability measure on $\mathbb{R}$ with zero mean and unit variance. Then weak-$\lim_{n\to\infty} \mu_n = \omega$, where ω is semicircle with zero mean and unit variance.*

Another important result is the Local Limit Theorem for densities. The free counterpart of this classical result was first proved by Bercovici and Voiculescu in [9] for compactly supported measures. Later this result was extended for measures with finite variance by Wang [47]. The following theorem describes the behavior of G_{μ_n} and $d\mu_n/dx$ for large n.

Theorem 3.27 (Wang, [47])**.** *Suppose that $\mu \in \mathcal{M}$ has zero mean and unit variance. Then:*

(1) *the measure μ_n is Lebesgue absolutely continuous for sufficiently large n;*

(2) *for each small $\delta > 0$ there exist $\eta > 0$ and $N > 0$ such that the funcion G_{μ_n} has an analytic continuation h_n to $K = \{x+iy : x \in [-2+\delta, 2-\delta], |y| < \eta\}$ whenever $n \geq N$. Moreover, $h_n(z) \to (z - i\sqrt{4-z^2})/2$ uniformly on K as $n \to \infty$;*

(3) *the density $d\mu_n/dx$ is continuous for sufficiently large n and $d\mu_n/dx \to d\omega/dx$ uniformly on $\mathbb{R}$ as $n \to \infty$.*

Berry–Esseen type inequality. The Berry–Esseen type approximation in the Free Central Limit Theorem was proved by Chistyakov and Götze [15], [17]. Assume μ has zero mean, unit variance and finite third absolute moment β_3, then there exists an absolute constant $c > 0$ such that

$$\sup_{x\in\mathbb{R}} |\mu_n(-\infty, x)) - \omega((-\infty, x))| \leq \frac{c\beta_3}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

For more details see [17]. It was also proved that the bound is sharp. In the case of non-identically distributed free random variables the analogue of the Berry–Esseen inequality was established in [15]. Let us consider a sequence of measures $\{\mu_j\}_{j=1}^{\infty} \in \mathcal{M}$ such that $m_1(\mu_j) := \int_{\mathbb{R}} x d\mu_j(x) = 0$ and $\beta_3(\mu_j) := \int_{\mathbb{R}} |x|^3 d\mu_j(x) < \infty$ for all $j = 1, 2, \ldots$. Denote

$$A_n := \sum_{k=1}^{n} \beta_3(\mu_k), \quad B_n^2 := \sum_{k=1}^{n} \int_{\mathbb{R}} x^2 d\mu_j(x), \quad L_n := \frac{A_n}{B_n^3}.$$

Write $\mu_{nk}((-\infty, x)) := \mu_k((-\infty, B_n x))$, $x \in \mathbb{R}$, $k = 1, \ldots, n$ and $\mu^{(n)} := \mu_{n1} \boxplus \ldots \boxplus \mu_{nn}$ as well. Then there exists an absolute constant $c > 0$ such that

$$\sup_{x\in\mathbb{R}} |\mu^{(n)}(-\infty, x)) - \omega((-\infty, x))| \leq c L_n^{1/2}.$$

Kargin proved the Berry–Esseen type inequality [30] under the condition that μ has compact support $[-L, L]$, zero mean and unit variance. Under these conditions there exists an absolute constant $c > 0$ such that

$$\sup_{x\in\mathbb{R}} |\mu_n(-\infty, x)) - \omega((-\infty, x))| \leq \frac{cL^3}{\sqrt{n}}.$$

Edgeworth type expansion. Assume that $\mu \in \mathcal{M}$ has compact support. In [17] Chistyakov and Götze obtained an analogue of the formal extension (1.2), namely, a formal power expansion for the Cauchy transform of μ_n. First, denote by $U_n(x)$ the Chebyshev polynomial of the second kind of degree n, which is defined by the recurrence relation

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x). \end{aligned} \tag{3.20}$$

The formal expansion has the form

$$G_{\mu_n}(z) = G_\omega(z) + \sum_{k=1}^{\infty} \frac{B_k(G_\omega(z))}{n^{k/2}}, \tag{3.21}$$

where

$$B_k(z) = \sum_{(p,m)} c_{p,m} \frac{z^p}{(1/z - z)^m} \tag{3.22}$$

with real coefficients $c_{p,m}$ which depend on the free cumulants $\kappa_3, \dots, \kappa_{k+2}$ and do not depend on n. The summation on the right-hand side of (3.22) is taken over a finite set of non-negative integer pairs (p, m). The coefficients $c_{p,m}$ can be calculated explicitly. For the cases $k = 1, 2$ we have

$$B_1(z) = \kappa_3 \frac{z^3}{1/z - z}$$

and

$$B_2(z) = (\kappa_4 - \kappa_3^2) \frac{z^4}{1/z - z} + \kappa_3^2 \left(\frac{z^5}{(1/z - z)^2} + \frac{z^2}{(1/z - z)^3} \right).$$

This formal expansion follows from property (3.9) of the Voiculescu transform and equation (3.12).

The analogue of the Edgeworth expansion in the Free Central Limit Theorem was also established by Chistyakov and Götze [17], [18]. They approximated μ_n by the free centered Meixner measure $\mu_{a,b,d}$, where $a \in \mathbb{R}$, $b, d < 1$. The free Meixner measure has the following reciprocal Cauchy transform

$$F_{\mu_{a,b,d}}(z) = a + \frac{1}{2} \left((1 + b)(z - a) + \sqrt{(1 - b)^2 (z - a)^2 - 4(1 - d)} \right),$$

where the branch of the square root is determined by the condition $\Im F_{\mu_{a,b,d}}(z) > 0$, $z \in \mathbb{C}^+$. We notice that if $a = b = d = 0$, then $\mu_{a,b,d}$ is the standard semicircular distribution, if $b = d = 0$, $a \neq 0$, then $\mu_{a,b,d}$ is the Marchenko-Pastur distribution.

We now introduce some further notations. Assume $\mu \in \mathcal{M}$ has zero mean and unit variance. Denote by β_q the qth absolute moment of μ, and assume that $\beta_q < \infty$ for some $q \geq 2$. Moreover, denote

$$a_n := \frac{m_3}{\sqrt{n}}, \quad b_n := \frac{m_4 - m_3^2 - 1}{n}, \quad d_n := \frac{m_4 - m_3^2}{n}, \quad n \in \mathbb{N}.$$

Introduce the Lyapunov fractions

$$L_{qn} := \frac{\beta_q}{n^{(q-2)/2}} \quad \text{and let} \quad \rho_q(\mu_t) := \int_{|x|>t} |x|^q d\mu(x), \quad t > 0.$$

Write

$$q_1 := \min\{q, 3\}, \quad q_2 := \min\{q, 4\}, \quad q_3 := \min\{q, 5\}.$$

For $n \in \mathbb{N}$, set

$$\eta_{qs}(n) := \inf_{0<\varepsilon\le 10^{-1/2}} g_{qns}(\varepsilon), \quad \text{where} \quad g_{qns}(\varepsilon) := \varepsilon^{s+2-q_s} + \frac{\rho_{q_s}(\mu, \varepsilon\sqrt{n})}{\beta_{q_s}}\varepsilon^{-q_s}$$

provided that $\beta_q < \infty$, $q \ge s+1$, for $s = 1, 2, 3$, respectively. Given that, for some $q \ge 3$ and $n \in \mathbb{N}$ we have the estimate

$$\sup_{x\in\mathbb{R}} |\mu_n((-\infty, x)) - \mu_{a_n,0,0}((-\infty, x))| \le c \begin{cases} \eta_{q2}(n)L_{qn} + L_{3n}^2, & 3 \le q < 4 \\ L_{4n}, & q \ge 4. \end{cases} \tag{3.23}$$

Moreover, the following expansion holds:

$$\mu_n((-\infty, x)) = \omega((-\infty, x)) - \frac{1}{3}U_2(x/2)p_\omega(x) + \rho_{n1}(x), \quad x \in \mathbb{R},$$

where the remainder term $\rho_{n1}(x)$ admits the bound

$$|\rho_{n1}(x)| \le c \begin{cases} \eta_{q2}(n)L_{qn} + L_{3n}^2 + |a_n|^{3/2}, & 3 \le q < 4 \\ L_{4n} + |a_n|^{3/2}, & q \ge 4 \end{cases}$$

for $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Before formulating the next result, denote by κ a signed measure with density

$$p_\kappa(x) := \frac{1}{2\pi}(x^2 - 1)\sqrt{(4 - x^2)_+}, \quad x \in \mathbb{R},$$

and denote by κ_n, $n \in \mathbb{N}$, the signed measure $\kappa_n := \mu_{a_n,b_n,d_n} + \frac{1}{n}\kappa * \delta_{a_n}$, where δ_{a_n} is a Dirac measure concentrated at the point a_n. Assume that $\beta_q < \infty$ with some $q \ge 4$, then, for $n \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} |\mu_n((-\infty, x)) - \kappa((-\infty, x))| \le c \begin{cases} \eta_{q3}(n)L_{qn} + L_{4n}^{3/2}, & 4 \le q < 5 \\ L_{5n}, & q \ge 5. \end{cases}$$

Moreover, the expansion

$$\begin{aligned} \mu_n((-\infty, x + a_n)) &= \omega((-\infty, x)) \\ &+ \left(\frac{a_n^2}{2}U_1(\frac{x}{2}) + \frac{a_n}{3}(3 - U_2(\frac{x}{2})) - \frac{b_n - a_n^2 - 1/n}{4}U_3(\frac{x}{2})\right) p_\omega(x) + \rho_{2n}(x), \end{aligned} \tag{3.24}$$

holds for $x \in \mathbb{R}$, $n \in \mathbb{N}$, where

$$|\rho_{n2}(x)| \le c \begin{cases} \eta_{q3}(n)L_{qn} + L_{4n}^{3/2}, & 4 \le q < 5 \\ L_{5n}, & q \ge 5. \end{cases} \tag{3.25}$$

If $m_3 = 0$ this formula has the simple form

$$\mu_n((-\infty, x)) = \omega((-\infty, x)) - \frac{m_4 - 2}{4n} U_3(\frac{x}{2}) p_\omega(x) + \rho_{n3}(x), \tag{3.26}$$

where $\rho_{3n}(x)$ admits the bound (3.25). If μ is not a Dirac measure, then for sufficiently large n the measure μ_n is Lebesgue absolutely continuous. Denote by p_{μ_n} the density of μ_n. Assume that μ has compact support, then for $n \geq c_1(\mu)$, $p_{\mu_n}(x)$ admits the expansion

$$\begin{aligned} p_n(x + a_n) &= \left(1 + \frac{d_n}{2} - a_n^2 - \frac{1}{n} - a_n x - \left(b_n - a_n^2 - \frac{1}{n}\right) x^2\right) p_\omega(E_n x) \\ &+ \frac{c\theta}{n^{3/2}\sqrt{4 - (E_n x)^2}} \end{aligned} \tag{3.27}$$

for $x \in [-2/E_n + h, 2/E_n - h]$, where $E_n := (1 - b_n)/\sqrt{1 - d_n}$ and $h = \frac{c_2(\mu)}{n^{3/2}}$ and $|\theta| \leq 1$.

Chapter 4

Main results: Edgeworth expansion in free probability

In this chapter the main results are introduced. We formulate Edgeworth type expansions for the Cauchy transform (Theorem 4.4) as well as for the density (Corollary 4.5) and distribution (Corollary 4.6) of normalized sums of free random variables. Moreover, we state results which guarantee that the conditions of Theorem 2.3 are satisfied in the free context (Theorem 4.1, Theorem 4.2, Theorem 4.3). A couple of examples are introduced in the second part of this chapter. By agreement *we use the same notations for the Cauchy transforms and their extensions.*

4.1 Results

The fact that the general procedure as described in Chapter 2 is independent of classical probability theory allows us to apply the scheme to objects that have a different nature than classical random variables. The main aim of this work is to apply the scheme to free random variables.

In free probability a sequence of measures μ_n is absolutely continuous with respect to the Lebesgue measure for sufficiently large n, if μ is not a point mass. This means that, in contrast to the classical situation, we do not need to introduce a smoothing variable in order to get an asymptotic expansion. Therefore, in free probability condition (2.1) in the scheme is redundant.

Assume that μ is a compactly supported measure with zero mean and unit variance. Let us introduce a sequence of measures

$$\mu_n^{(\varepsilon_n)} := D_{\varepsilon_1}\mu \boxplus \ldots \boxplus D_{\varepsilon_n}\mu, \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \ldots, n.$$

Due to (3.5) and (3.6) we define

$$G^{(-1)}_{\mu_n^{(\varepsilon_n)}}(z) = \sum_{j=1}^{n} \varepsilon_j R_\mu(\varepsilon_j z) + \frac{1}{z}, \quad |\varepsilon_j| \le n^{-1/2}, \ \ j = 1, \dots, n,$$

wherever the power series $\sum_{j=1}^n \varepsilon_j R_\mu(\varepsilon_j z)$ converges in z. If this series converges on the appropriate domain then we can define the analytic continuation of $G_{\mu_n^{(\varepsilon_n)}}(z)$ to some rectangle around $(-2, 2)$. It is obvious that such a continuation is compatible and symmetric in ε_j, $j = 1, \dots, n$. Hence, we choose the sequence of extensions of the Cauchy transforms as the sequence of functionals $h_n(\underline{\varepsilon}_n)$, i.e.

$$h_n(\underline{\varepsilon}_n) := G_{\mu_n^{(\varepsilon_n)}}, \quad |\varepsilon_j| \le n^{-1/2}, \ \ j = 1, \dots, n,$$

respectively, $h_n := G_{\mu_n}$. Moreover, we define the following measure

$$\tilde{\mu}_{m+r} := D_{\varepsilon_1}\mu \boxplus \dots \boxplus D_{\varepsilon_{m+r}}\mu, \quad \underline{\varepsilon}_{m+r} \in E^n_{m,r},$$

where $E^n_{m,r}$ is the set of weight vectors $(\varepsilon_1, \dots, \varepsilon_{m+r})$ where all but $2r$ components are equal to $m^{-1/2}$ and the remaining $2r$ components are bounded by $n^{-1/2}$, $m \ge n$. Then

$$h_{m+r}(\underline{\varepsilon}_{m+r}) := G_{\tilde{\mu}_{m+r}}, \quad \underline{\varepsilon}_{m+r} \in E^n_{m,r},$$

where $G_{\tilde{\mu}_{m+r}}$ is a corresponding analytic continuation of the Cauchy transform.

We would like to extend the Cauchy transforms to the set

$$K'' := \{x + iy : x \in [-2 + 5\delta, \ 2 - 5\delta], \ |y| < \delta\sqrt{\delta}/2\}, \quad \delta \in (0, 1/10).$$

Theorem 5.11 guarantees that $G_{\mu_n}(z)$ has an analytic continuation to K'' for $n \ge c(\mu)\delta^{-4}$. This result is similar to Theorem 3.27, but the domain of convergence is different. Theorem 6.5 guarantees that $G_{\tilde{\mu}_{m+r}}(z)$ has an analytic continuation to K'', $m \ge n \ge c(\mu, r)\delta^{-4}$. Corollary 6.6 shows that $G_{\mu_m \boxplus \mu_r^{(\varepsilon_r)}}(z)$ also has an analytic continuation to K'', $m \ge n \ge c(\mu, r)\delta^{-4}$ and this continuation converges to $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$ as $m \to \infty$ uniformly on K'', where $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$ is a corresponding continuation (see Theorem 5.8). Hence, we can define $h_\infty(\underline{\varepsilon}_r)$:

$$h_\infty(\underline{\varepsilon}_r) := G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}, \quad |\varepsilon_j| \le n^{-1/2}, \ \ j = 1, \dots, r.$$

Below, for analytic continuations of the Cauchy transforms we do not use special notations, we just specify the domain, for instance, $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$, $z \in \mathbb{C}^+$ is the Cauchy transform and $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$, $z \in K''$ is the corresponding extension to K''.

Next, we formulate results which guarantee that conditions (2.4), (2.9) and (2.10) hold. The following theorems show that $G_{\tilde{\mu}_{m+r}}(z)$, $z \in K''$ belongs to $C^\infty(E^n_{m,r})$) for sufficiently large m, n.

Theorem 4.1. *Assume $\mu \in \mathcal{M}$ has compact support contained in $[-L, L]$, zero mean and unit variance. Then for each $\delta \in (0, 1/10)$, $n \geq c(\mu, r)\delta^{-4}$ and each $z \in K''$ the extension $G_{\tilde{\mu}_{m+r}}$ is in $C^\infty(E^n_{m,r})$.*

The following theorem shows that the first derivatives of $G_{\tilde{\mu}_{m+r}}(z)$, $z \in K''$, $m \geq n \geq c(\mu, r)\delta^{-4}$ with respect to ε_j vanish at $\varepsilon_j = 0$, $j = 1, \ldots, 2r$.

Theorem 4.2. *Under the assumptions of Theorem 4.1 we have for each $\delta \in (0, 1/10)$, $m \geq n \geq c(\mu, r)\delta^{-4}$ and each $z \in K''$*

$$\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)\Big|_{\varepsilon_j = 0} = 0, \quad j = 1, \ldots, 2r.$$

The next theorem shows that the derivatives of $G_{\tilde{\mu}_{m+r}}(z)$ with respect to $\underline{\varepsilon}_{2r}$ are uniformly bounded on $E^n_{m,r} \times K''$, $m \geq n \geq c(\mu, r)\delta^{-4}$.

Theorem 4.3. *Under the assumptions of Theorem 4.1 for each $\delta \in (0, 1/10)$, $m \geq n \geq c(\mu, r)\delta^{-4}$ and $r \geq 1$ the following bound holds:*

$$\sup_{z \in K''} \sup_{\underline{\varepsilon}_{m+r} \in E^n_{m,r}} \left| D^\alpha G_{\tilde{\mu}_{m+r}}(z) \right| \leq c, \quad |\alpha| \leq r.$$

Finally, the general scheme from Chapter 2 is applicable to the extension of the Cauchy transform of μ_n and we get an expansion as described in the following theorem.

Theorem 4.4. *Under assumptions of Theorem 4.1 the extension of the Cauchy transform G_{μ_n} admits the expansion*

$$\begin{aligned} G_{\mu_n}(z) &= G_\omega(z) + \frac{\kappa_3 G^4_\omega(z)}{(1 - G^2_\omega(z))\sqrt{n}} \\ &+ \left((\kappa_4 - \kappa_3^2) \frac{G^5_\omega(z)}{1 - G^2_\omega(z)} + \kappa_3^2 \Big(\frac{G^7_\omega(z)}{(1 - G^2_\omega(z))^2} + \frac{G^5_\omega(z)}{(1 - G^2_\omega(z))^3} \Big) \right) \frac{1}{n} \\ &+ \left(\frac{\kappa_5 G^6_\omega(z)}{(1 - G^2_\omega(z))} + \frac{\kappa_3^3 G^{10}_\omega(z) \left(5G^4_\omega(z) - 15G^2_\omega(z) + 12\right)}{\left(1 - G^2_\omega(z)\right)^5} \right. \\ &- \left. \frac{\kappa_3 \kappa_4 G^8_\omega(z) \left(5G^2_\omega(z) - 7\right)}{\left(1 - G^2_\omega(z)\right)^3} \right) \frac{1}{n^{3/2}} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

for $z \in K''$, $n \geq c(\mu)\delta^{-4}$.

One can see that the coefficients of this expansion coincide with the coefficients in the formal expansion (3.21). Due to the Stieltjes inversion formula we also obtain an expansion for the densities.

Corollary 4.5. *Under the assumptions of Theorem 4.1 the density p_{μ_n} admits the expansion*

$$
\begin{aligned}
p_{\mu_n}(x) &= p_\omega(x) + \frac{\kappa_3 x\,(x^2-3)\,p_\omega(x)}{(4-x^2)\sqrt{n}} \\
&+ \frac{(\kappa_3^2\,(2x^6-15x^4+30x^2-10) - \kappa_4\,(x^6-8x^4+18x^2-8))\,p_\omega(x)}{(4-x^2)^2\,n} \\
&+ \Bigg(\frac{\kappa_3^3 x(5x^8-60x^6+252x^4-420x^2+210))}{(4-x^2)^3} \\
&+ \frac{\kappa_3\kappa_4 x(5x^6-42x^4+105x^2-70)}{(4-x^2)^2} + \frac{\kappa_5 x(x^4-5x^2+5)}{(4-x^2)}\Bigg)\frac{p_\omega(x)}{n^{3/2}} + O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

for $x \in [-2+5\delta, 2-5\delta]$, $n \geq c(\mu)\delta^{-4}$.

In contrast to expansion (3.27), our expansion does not use an n-dependent shift of the point x. The expansion for μ_n can be obtained by integrating the density expansion from Corollary 4.5.

Corollary 4.6. *Under the assumptions of Theorem 4.1 the distribution μ_n admits the expansion*

$$
\begin{aligned}
\mu_n((a,b)) &= \Bigg[\omega((-\infty,x)) - \kappa_3 U_2\left(\frac{x}{2}\right)\frac{p_\omega(x)}{3\sqrt{n}} \\
&+ \left(-\kappa_4 U_3\left(\frac{x}{2}\right) + 2\kappa_3^2\left(U_3\left(\frac{x}{2}\right) + U_1\left(\frac{x}{2}\right) - \frac{U_1\left(\frac{x}{2}\right)}{4-x^2}\right)\right)\frac{p_\omega(x)}{4n} \\
&+ \Bigg(\frac{\kappa_5}{5}U_4\left(\frac{x}{2}\right) - \frac{\kappa_3\kappa_4}{4-x^2}\left(U_6\left(\frac{x}{2}\right) - U_4\left(\frac{x}{2}\right)\right) \\
&- \frac{\kappa_3^3}{3(4-x^2)^2}\left(3U_8\left(\frac{x}{2}\right) - 7U_6\left(\frac{x}{2}\right) + 4U_4\left(\frac{x}{2}\right)\right)\Bigg)\frac{p_\omega(x)}{n^{3/2}}\Bigg]\Bigg|_a^b + O\left(\frac{1}{n^2}\right) \quad (4.1)
\end{aligned}
$$

with $(a,b) \subset [-2+5\delta, 2-5\delta]$, $n \geq c(\mu)\delta^{-4}$, $U_n(x)$ are Chebychev polynomials (3.20).

Remark 4.7. *Assume that $\kappa_3 = 0$, then*

$$
\mu_n((a,b)) = \left[\omega((-\infty,x)) - \frac{\kappa_4}{4n}U_3\left(\frac{x}{2}\right)p_\omega(x) + \frac{\kappa_5}{5n^{3/2}}U_4\left(\frac{x}{2}\right)p_\omega(x)\right]\Bigg|_a^b + O\left(\frac{1}{n^2}\right)
$$

with $(a,b) \subset [-2+5\delta, 2-5\delta]$, $n \geq c(\mu)\delta^{-4}$. Two first terms in this expansion coincide with the terms in (3.26).

In contrast to the expansions (3.24) and (3.26), our expansions for the measure μ_n are local and hold under stronger assumptions, namely, we require that μ is compactly supported.

Let us show that three first terms in expansion (4.1) coincides with (3.24). We replace x on the right-hand side in (4.1) by $y := (x + \frac{\kappa_3}{\sqrt{n}})$ and expand by Taylor series:

$$\begin{aligned}
\omega((-\infty, y)) &= \omega((-\infty, x)) + \frac{m_3\sqrt{4-x^2}}{\sqrt{n}} - \frac{x m_3^2}{2n\sqrt{4-x^2}} + O(n^{-3/2});\\
\kappa_3 U_2\left(\frac{y}{2}\right)\frac{p_\omega(y)}{3\sqrt{n}} &= \frac{m_3}{3\sqrt{n}}\left(x^2-1\right)\sqrt{4-x^2} + \frac{m_3^2 x\left(3-x^2\right)}{n\sqrt{4-x^2}} + O(n^{-3/2});\\
\kappa_4 U_3\left(\frac{y}{2}\right)\frac{p_\omega(y)}{4n} &= \frac{m_4-2}{4n}\left(x(x^2-2)\sqrt{4-x^2}\right) + O(n^{-3/2});
\end{aligned}$$

$$\kappa_3^2\left(U_3\left(\frac{y}{2}\right) + U_1\left(\frac{y}{2}\right) - \frac{U_1\left(\frac{y}{2}\right)}{4-(y)^2}\right)\frac{p_\omega(y)}{2n} = \frac{m_3^2}{2n}\frac{x\left(5-5x^2+x^4\right)}{2\sqrt{4-x^2}} + O(n^{-3/2}).$$

Finally, we compute

$$\begin{aligned}
&\mu_n((a+\frac{\kappa_3}{\sqrt{n}}, b+\frac{\kappa_3}{\sqrt{n}})) = \omega((a,b))\\
&+ \left[\frac{m_3}{3\sqrt{n}}\left(3-U_2\left(\frac{x}{2}\right)\right) + \frac{m_3^2}{2n}\left(U_3\left(\frac{x}{2}\right) - U_1\left(\frac{x}{2}\right)\right) - \frac{m_4-2}{4n}U_3\left(\frac{x}{2}\right)\right]p_\omega(x)\Big|_a^b\\
&+ O(n^{-3/2})
\end{aligned}$$

with $(a,b) \subset [-2+5\delta, 2-5\delta]$, $n \geq c(\mu)\delta^{-4}$. It is easy to see that this expansion coincides with (3.24) on $[-2+5\delta, 2-5\delta]$.

In Chapter 5 we show that the measure $\omega\boxplus\mu_r^{(\varepsilon_r)}$ has a positive density on $[-2+\delta, 2-\delta]$ for each $\delta \in (0, 1/10)$, $n \geq c(\mu,r)\delta^{-4}$ (Corollary 5.7). Theorem 5.8 provides us with analytic continuation of $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ to $K := \{x+iy : x \in (-2+2\delta, 2-2\delta); |y| \leq \delta\sqrt{\delta}\}$, $n \geq c(\mu,r)\delta^{-4}$ and a uniform bound for this continuation. Theorem 5.10 guarantees that the density p_{μ_n} is positive and analytic on $[-2+\delta, 2-\delta]$, for sufficiently large n. Then we construct the analytic continuation for G_{μ_n} on K, $n \geq c(\mu)\delta^{-4}$ with a uniform bound which we need in the sequel (Theorem 5.11).

Chapter 6 is devoted to the proofs of Theorem 4.1 which is a non-trivial result by itself. Theorem 4.2 and Theorem 4.3 are also proved in Chapter 6. We focus on the measure $\tilde{\mu}_{m+r}$. We show that this measure has a density which is analytic on $[-2+2\delta, 2-2\delta]$, $n \geq c(\mu,r)\delta^{-4}$ (Theorem 6.4). Finally, we construct the analytic continuation

$$G_{\tilde{\mu}_{m+r}}(z) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad z \in K', \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \dots, r$$

where $K' := \{x+iy : x \in (-2+4\delta, 2-4\delta),\ |y| < \delta\sqrt{\delta}/2\}$, $|\tilde{l}_n(z)| \leq \frac{c(r)}{\delta n}$, $z \in K'$, $m \geq n \geq c(\mu,r)\delta^{-4}$ (Theorem 6.5). Then we prove Theorem 4.1, Theorem 4.2 and Theorem 4.3. Finally, we prove Theorem 4.4, Corollary 4.5 and Corollary 4.6.

4.2 Examples

In this section we consider asymptotic expansions for free convolutions of the free Poisson law and the Arcsine law.

Example 4.8 (Free Poisson law)**.** Let us consider the free Poisson law with density

$$p_\mu(x) = \frac{1}{2\pi(x+1)}\sqrt{4(x+1)-(x+1)^2}, \quad -1 \le x \le 3,$$

which has moments $m_1 = 0$, $m_2 = 1$, $m_3 = 1$, $m_4 = 3$, $m_5 = 6$, and cumulants $\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = 1$, $\kappa_4 = 1$, $\kappa_5 = 3$. The density of $p_{\mu_n}(x)$ is given by

$$p_{\mu_n}(x) = \frac{\sqrt{(4n-1)+2\sqrt{n}x-nx^2}}{2\pi\left(\sqrt{n}+x\right)}, \quad -2+n^{-1/2} \le x \le 2-n^{-1/2}.$$

We consider $p_{\mu_{10}}(x)$ and $p_{\mu_{100}}(x)$,

$$p_{\mu_{10}}(x) = \frac{\sqrt{39+2\sqrt{10}x-10x^2}}{2\pi\left(\sqrt{10}+x\right)}, \quad -2+1/\sqrt{10} \le x \le 2+1/\sqrt{10};$$

$$p_{\mu_{100}}(x) = \frac{\sqrt{399+20x-100x^2}}{2\pi(10+x)}, \quad -2+1/10 \le x \le 2-1/10.$$

In Figure 4.1, one can see graphics of the densities and the approximations of the densities based on Corollary 4.5.

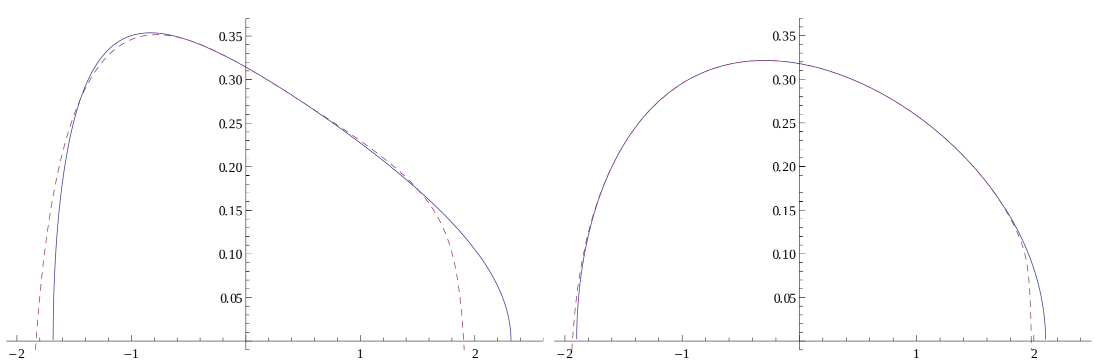

Figure 4.1: Comparison of the asymptotic expansion, shown as the dashed line, and the exact result, shown as the solid line, for free Poisson, $n = 10$ (on the left) and $n = 100$ (on the right).

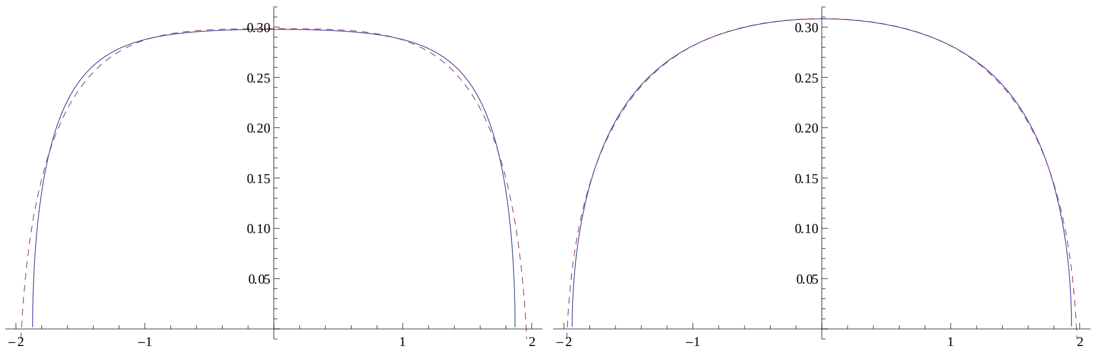

Figure 4.2: Comparison of the asymptotic expansion, shown as the dashed line, and the exact result, shown as the solid line, for arcsine, $n = 4$ (on the left) and $n = 8$ (on the right).

Example 4.9 (Arcsine law)**.** In this example we consider the arcsine law, which has the density

$$p_\mu(x) = \frac{1}{\pi\sqrt{2-x^2}}, \quad -\sqrt{2} \le x \le \sqrt{2}.$$

This law is symmetric and all odd moments vanish. Therefore, in this case the approximations will be better than in the non-symmetric case. The first even moments are $m_2 = 1$, $m_4 = 3/2$ and the cumulants are $\kappa_2 = 1$, $\kappa_4 = -1/2$. We consider the convolutions of 4 and 8 measures:

$$p_{\mu_4}(x) = \frac{4\sqrt{3.5-x^2}}{\pi\,(8-x^2)}, \quad -\sqrt{3.5} \le x \le \sqrt{3.5};$$

$$p_{\mu_8}(x) = \frac{8\sqrt{3.75-x^2}}{\pi\,(16-x^2)}, \quad -\sqrt{3.75} \le x \le \sqrt{3.75}.$$

Figure 4.2 illustrates the densities and their approximations.

Chapter 5

Convolution with the semicircle law

In this chapter we construct analytic extensions for $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ and G_{μ_n} to $K := \{x+iy : x \in (-2+2\delta,\ 2-2\delta),\ |y| < \delta\sqrt{\delta}\}$ for $\delta \in (0, 1/10)$ and $n \geq c(\mu)\delta^{-4}$. We start with finding an interval such that $p_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(x)$ is positive (Theorem 5.7). In order to do so we use an idea that was introduced by Kargin in [32] and which is based on the Newton-Kantorovich Theorem [29]. We proceed by constructing an analytic continuation of the Cauchy transform of $\omega \boxplus \mu_r^{(\varepsilon_r)}$ to the domain K (Theorem 5.8). Such a type of extension was introduced by Wang in [47] and has the form

$$G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) = G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z), \quad z \in K.$$

In this chapter we introduce a uniform estimate for $|\tilde{l}_{(\varepsilon_r)}(z)|$ on K. Then we construct an analytic continuation of G_{μ_n} to K for $\delta \in (0, 1/10)$ and $n \geq N$, where $N := c(\mu)\delta^{-4}$. First of all we deduce an interval on which $p_{\mu_n}(x)$ is positive (Theorem 5.10). Then we proceed with analytic continuation for G_{μ_n} in Theorem 5.11, which is similar to the result from [47], but we give a different estimate for $|l_n(z)|$, $z \in K$, $n \geq N$. In the original result by Wang, the assumption that μ is of compact support is relaxed.

5.1 Positive density for $\omega \boxplus \mu_r^{(\varepsilon_r)}$

The fact that the measure $\omega_t \boxplus \lambda$ has a density, where λ is an arbitrary measure on $\mathbb{R}$, was proved by Biane in [12]. Our aim is to find an interval such that the density of $\omega_t \boxplus \lambda$ is positive for a measure λ which is concentrated in a small neighborhood of zero. The main idea is based on the Newton-Kantorovich Theorem (see Theorem 7.4) and was described by Kargin in [32]. Let us consider a pair of measures ν_1 and ν_2. We can rewrite equations (3.10) and (3.11) as a system

$$\begin{cases} (z - Z_1(z) - Z_2(z))^{-1} + G_{\nu_1}(Z_1(z)) = 0 \\ (z - Z_1(z) - Z_2(z))^{-1} + G_{\nu_2}(Z_2(z)) = 0, \end{cases} \tag{5.1}$$

where G_{ν_1} and G_{ν_2} are the Cauchy transforms of ν_1 and ν_2, correspondingly. Choose another pair of measures μ_1 and μ_2 such that the Levy distance between ν_j and μ_j is sufficiently small for $j = 1, 2$. Then we can define subordination functions for the couple (μ_1, μ_2) as a solution of (5.1), where G_{ν_1} and G_{ν_2} are replaced by the Cauchy transforms of μ_1 and μ_2 correspondingly. Denote these subordination functions by t_1^0 and t_2^0. According to the Newton-Kantorovich Theorem (for a proof see [32]) one can show that the subordination functions Z_j and t_j^0, $j = 1, 2$ are sufficiently close to each other. We can choose μ_1 and μ_2 to be equal, so that $t_1^0 = t_2^0$. Such a choice essentially simplifies the structure of equations (3.10) and (3.11).

We need the Levy distance and some results about its properties for further estimations.

Definition 5.1. *Let $Q_1(x)$ and $Q_2(x)$ be the cumulative distribution functions of the two measures μ_1 and μ_2 respectively. The Levy distance between these measures is defined by the formula*

$$d_L(\mu_1, \mu_2) = \inf\{s \geq 0 : Q_2(x - s) - s \leq Q_1(x) \leq Q_2(x + s) + s \text{ for all } x \in \mathbb{R}\}.$$

The Levy distance is a metric on the space of measures and weak convergence is equivalent to convergence with respect to this metric. We also need the following result by Voiculescu and Bercovici [8] about the Levy distance.

Theorem 5.2. *If $\mu_1,\ \mu_2,\ \nu_1,$ and $\nu_2 \in \mathcal{M}$, then*

$$d_L(\mu_1 \boxplus \nu_1, \mu_2 \boxplus \nu_2) \leq d_L(\mu_1, \mu_2) + d_L(\nu_1, \nu_2).$$

Finally, let us prove some further results about the Levy distance.

Lemma 5.3. *Suppose $\mu,\ \nu \in \mathcal{M}$ are measures with compact support, zero mean and unit variance, moreover, let μ be supported on an interval $[-L, L]$. Then*

(1) $d_L(\nu, \nu \boxplus \mu^{(\varepsilon_r)}) \leq L \sum_{i=1}^{r} \varepsilon_i$;

(2) $d_L(D_{\varepsilon_1}\mu, D_{\varepsilon_2}\mu) \leq L|\varepsilon_1 - \varepsilon_2|$.

Proof. First, we prove inequality (1). From Theorem 5.2, we get

$$d_L(\nu, \nu \boxplus \mu_r^{(\varepsilon_r)}) \leq d_L(\delta_0, \mu_r^{(\varepsilon_r)}),$$

where δ_0 is a delta function concentrated at zero and

$$d_L(\delta_0, \mu_r^{(\varepsilon_r)}) \leq \sum_{i=1}^{r} d_L(\delta_0, D_{\varepsilon_i}\mu).$$

We know that $\text{supp}(\mu) \subset [-L, L]$, hence $\text{supp}(D_{\varepsilon_i}\mu) \subset [-\varepsilon_i L, \varepsilon_i L]$, $i = 1, \dots, r$. We conclude that $d_L(\delta_0, D_{\varepsilon_i}\mu) \leq \varepsilon_i L$, $i = 1, \dots, r$, and

$$d_L(\delta_0, \mu_r^{(\varepsilon_r)}) \leq L \sum_{i=1}^{r} \varepsilon_i.$$

Finally, we arrive at

$$d_L(\nu, \nu \boxplus \mu_r^{(\varepsilon_r)}) \leq L \sum_{i=1}^{r} \varepsilon_i.$$

Now, we prove inequality (2). Let $Q(x)$ be the distribution function of μ, then

$$\begin{aligned} &d_L(D_{\varepsilon_1}\mu, D_{\varepsilon_2}\mu) \\ &\quad = \inf\{s \geq 0 : Q((x-s)/\varepsilon_1) - s \leq Q(x/\varepsilon_2) \leq Q((x+s)/\varepsilon_1) + s,\ x \in \mathbb{R}\} \\ &\quad \leq \inf\{s \geq 0 : Q((x-s)/\varepsilon_1) \leq Q(x/\varepsilon_2) \leq Q((x+s)/\varepsilon_1),\ x \in \mathbb{R}\}. \end{aligned}$$

We have to consider the two situations $\varepsilon_1 > \varepsilon_2$ and $\varepsilon_1 < \varepsilon_2$ (the case $\varepsilon_1 = \varepsilon_2$ is trivial). Let $\underline{\varepsilon_1 > \varepsilon_2}$. Since a distribution function does not decrease, we get

$$\begin{aligned} &\inf\{s \geq 0 : Q\left(\frac{x-s}{\varepsilon_1}\right) \leq Q\left(\frac{x}{\varepsilon_2}\right) \leq Q\left(\frac{x+s}{\varepsilon_1}\right),\ x \in \mathbb{R}\} \\ &= \max\left\{\inf\{s \geq 0 : Q\left(\frac{x-s}{\varepsilon_1}\right) \leq Q\left(\frac{x}{\varepsilon_2}\right) \leq Q\left(\frac{x+s}{\varepsilon_1}\right),\ \varepsilon_1 L \leq |x|\},\right. \\ &\qquad \inf\{s \geq 0 : Q\left(\frac{x-s}{\varepsilon_1}\right) \leq Q\left(\frac{x}{\varepsilon_2}\right) \leq Q\left(\frac{x+s}{\varepsilon_1}\right),\ \varepsilon_2 L \leq |x| \leq \varepsilon_1 L\}, \\ &\qquad \left.\inf\{s \geq 0 : Q\left(\frac{x-s}{\varepsilon_1}\right) \leq Q\left(\frac{x}{\varepsilon_2}\right) \leq Q\left(\frac{x+s}{\varepsilon_1}\right),\ |x| \leq \varepsilon_2 L\}\right\}. \end{aligned} \tag{5.2}$$

We note that the first infimum in (5.2) is equal to zero. For the second term in (5.2), we consider $x \geq 0$ (remember that μ has zero mean). But then the left inequality is trivial and we only need to consider the right inequality which holds if s satisfies the inequality

$$x/\varepsilon_2 \leq (x+s)/\varepsilon_1.$$

Hence, $(\varepsilon_1 - \varepsilon_2)x \leq \varepsilon_2 s$ must hold for $x \in [\varepsilon_2 L, \varepsilon_1 L]$. To prove this, we consider the difference $Q((x+s)/\varepsilon_1) - Q(x/\varepsilon_2)$. Since we have $Q(x/\varepsilon_2) = 1$ for all $x \geq \varepsilon_2 L$, we can take s such that $Q((\varepsilon_2 L + s)/\varepsilon_1) = 1$, which implies $Q((x+s)/\varepsilon_1) = 1$ for all $x \geq \varepsilon_2 L$. We see that we can set $s = L(\varepsilon_1 - \varepsilon_2)$. For $x < 0$ the same arguments show that $s = L(\varepsilon_1 - \varepsilon_2)$.

For the third infimum in (5.2) we consider $x \geq 0$ and the right inequality. If we set $x = \varepsilon_2 y$, where $y \in [0, L]$, then $Q((x+s)/\varepsilon_1) = Q((\varepsilon_2 y + s)/\varepsilon_1)$. We need s such that $(\varepsilon_2 y + s)/\varepsilon_1 = y$, hence $s = (\varepsilon_1 - \varepsilon_2)y$, and we conclude that $s = (\varepsilon_1 - \varepsilon_2)L$. For negative x the same arguments show that we can take $s = (\varepsilon_1 - \varepsilon_2)L$ and

$$d_L(D_{\varepsilon_1}\mu, D_{\varepsilon_2}\mu) \le (\varepsilon_1 - \varepsilon_2)L.$$

Assume $\underline{\varepsilon_2 > \varepsilon_1}$. This case can be proved in the same way as the previous one and we obtain that $s = (\varepsilon_2 - \varepsilon_1)L$.

From these two cases we finally conclude that

$$d_L(D_{\varepsilon_1}\mu, D_{\varepsilon_2}\mu) \le |\varepsilon_1 - \varepsilon_2|L.$$

The lemma is proved. □

In the sequel we need the following estimates for G_ω and F_ω. An estimate of this type for G_ω can be found in [47].

Lemma 5.4. *For each $\delta \in (0, 1/10)$ we define the set*

$$K_\delta = \{x + iy : x \in [-2 + \delta,\ 2 - \delta], |y| \le 2\delta\sqrt{\delta}\}.$$

Then, we have $G_\omega(K_\delta) \subset D_{\theta,1.4} = \{z \in \mathbb{C}^- : \arg z \in (-\pi + \theta, -\theta); |z| < 1.4\}$, where the angle $\theta = \theta(\delta)$ is chosen in such a way that $2\sin\theta = \sqrt{\frac{\delta}{4}\left(1 - \frac{\delta}{4}\right)}$. Moreover, $|F_\omega(z)| < 1.4$, $z \in K_\delta$.

Proof. Figure 5.1 illustrates the sets K_δ and $D_{\theta,1.4}$.

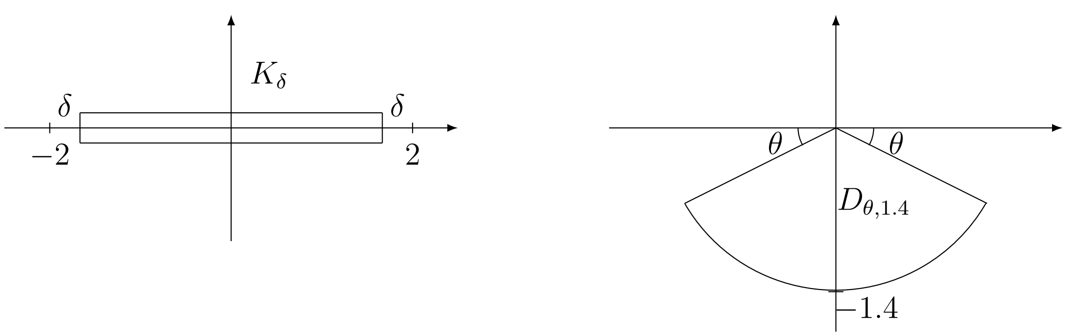

Figure 5.1

In the beginning we show that $G_\omega(K_\delta) \subseteq D_{\theta,1.4}$, where G_ω is an analytic extension of the Cauchy transform of ω on K_δ. Let $z_0 \in K_\delta$ be given, and write $G_\omega(z_0) = Re^{i\psi}$. In order to prove $G_\omega(z_0) \in D_{\theta,1.4}$ we need to verify that $|\sin\psi| > \sin\theta$ and $R < 1.4$. From the functional equation (3.15) we have

$$\left(R + \frac{1}{R}\right)\cos\psi + i\left(R - \frac{1}{R}\right)\sin\psi = z_0.$$

By the fact that $|\Re z_0| \le 2 - \delta$, we get

$$2|\cos\psi| \le \left(R + \frac{1}{R}\right)|\cos\psi| \le 2 - \delta.$$

This implies $|\cos\psi| \le 1 - \delta/2$, hence

$$|\sin\psi| \;=\; \sqrt{1-\cos^2\psi} \ge \sqrt{1-(1-\delta/2)^2} = \sqrt{\delta/4\,(1-\delta/4)} > \sin\theta.$$

We obtain the desired result $|\sin\psi| > \sin\theta$.

In order to estimate R we consider the imaginary part of z_0

$$2\delta\sqrt{\delta} > |\Im z_0| = |\sin\psi|\left|R - \frac{1}{R}\right| > \frac{|R^2-1|}{R}\frac{\sqrt{\delta}}{2}. \tag{5.3}$$

If $R > 1$, we get the inequality $R^2 - 4\delta R - 1 < 0$. Therefore, R must be bounded from above by the intercept of the positive x-axis and the parabola $y = R^2 - 4\delta R - 1$. The roots of the equation $R^2 - 4\delta R - 1 = 0$ are

$$R = 2\delta \pm \sqrt{4\delta^2 + 1}.$$

Because of the choice of δ we have $2\delta + \sqrt{4\delta^2+1} < 1.22$. This implies $R < 1.4$.

In order to estimate $|F_\omega(z)|$ from above on K, we need inequality (5.3) for $R < 1$, hence $R^2 + 4\delta R - 1 > 0$. Therefore, R must be bounded from below by the intercept of the positive x-axis and the parabola $y = R^2 + 4\delta R - 1$. This means that $R \ge -2\delta + \sqrt{4\delta^2+1}$, and by the choice of δ, we conclude $R > 0.81$ and $1/R < 1.22$. □

The following inequalities are due to Kargin [32].

Lemma 5.5. *Let $d_L(\mu_1, \mu_2) \le s$ and $z = x + iy$, where $y > 0$. Then*

1. $|G_{\mu_1}(z) - G_{\mu_2}(z)| < \widetilde{c} s y^{-1} \max\{1,\ y^{-1}\}$, *where $\widetilde{c} > 0$ is a numerical constant;*
2. $|\frac{d^r}{dz^r}(G_{\mu_2}(z) - G_{\mu_1}(z))| < \widetilde{c}_r s y^{-1-r} \max\{1,\ y^{-1}\}$, *where $\widetilde{c}_r > 0$ are numerical constants.*

Consider a pair of measures (ν_1, ν_2) and introduce a function $F(t) : \mathbb{C}^2 \to \mathbb{C}^2$ by the formula

$$F(t) = \begin{pmatrix} (z - t_1 - t_2)^{-1} + G_{\nu_1}(t_1) \\ (z - t_1 - t_2)^{-1} + G_{\nu_2}(t_2) \end{pmatrix}. \tag{5.4}$$

The equation $F(t) = 0$ has a unique solution, say $Z = (Z_1(z), Z_2(z))$, where $Z_1(z)$ and $Z_2(z)$ are subordination functions. Let (μ_1, μ_2) be another pair of measures. Assume $t^0 = (t_1^0, t_2^0) = (t_1^0(z), t_2^0(z))$ solves the system of equations

$$\begin{cases} (z - t_1^0 - t_2^0)^{-1} + G_{\mu_1}(t_1^0) = 0 \\ (z - t_1^0 - t_2^0)^{-1} + G_{\mu_2}(t_2^0) = 0. \end{cases}$$

Then $F(t^0)$ has the form

$$F(t^0) = \begin{pmatrix} (z - t_1^0 - t_2^0)^{-1} + G_{\nu_1}(t_1^0) \\ (z - t_1^0 - t_2^0)^{-1} + G_{\nu_2}(t_2^0) \end{pmatrix}.$$

The derivative of F with respect to t at t_0 is

$$F'(t^0) = \begin{pmatrix} (z - t_1^0 - t_2^0)^{-2} + G'_{\nu_1}(t_1^0) & (z - t_1^0 - t_2^0)^{-2} \\ (z - t_1^0 - t_2^0)^{-2} & (z - t_1^0 - t_2^0)^{-2} + G'_{\nu_2}(t_2^0) \end{pmatrix}.$$

Obviously

$$G_{\mu_j}(t_j^0) = -\frac{1}{(z - t_1^0 - t_2^0)}, \quad G^2_{\mu_j}(t_j^0) = \frac{1}{(z - t_1^0 - t_2^0)^2}, \quad j = 1, 2. \tag{5.5}$$

Taking into account (5.5), $F(t^0)$ and $F'(t^0)$ can be rewritten as

$$F(t^0) = \begin{pmatrix} G_{\nu_1}(t_1^0) - G_{\mu_1}(t_1^0) \\ G_{\nu_2}(t_2^0) - G_{\mu_2}(t_2^0) \end{pmatrix},$$

$$F'(t^0) = \begin{pmatrix} G'_{\nu_1}(t_1^0) + G^2_{\mu_1}(t_1^0) & G^2_{\mu_1}(t_1^0) \\ G^2_{\mu_2}(t_2^0) & G'_{\nu_2}(t_2^0) + G^2_{\mu_2}(t_2^0) \end{pmatrix}.$$

The inverse matrix of $F'(t^0)$ is

$$[F'(t^0)]^{-1} = \frac{1}{\det[F'(t^0)]} \begin{pmatrix} G'_{\nu_2}(t_2^0) + G^2_{\mu_2}(t_2^0) & -G^2_{\mu_1}(t_1^0) \\ -G^2_{\mu_2}(t_2^0) & G'_{\nu_1}(t_1^0) + G^2_{\mu_1}(t_1^0) \end{pmatrix}, \tag{5.6}$$

where

$$\det[F'(t^0)] = (G'_{\nu_2}(t_2^0) + G^2_{\mu_2}(t_2^0))(G'_{\nu_1}(t_1^0) + G^2_{\mu_1}(t_1^0)) - G^2_{\mu_1}(t_1^0)G^2_{\mu_2}(t_2^0). \tag{5.7}$$

After simple computations, we obtain

$$[F'(t^0)]^{-1}F(t^0) = \frac{1}{\det[F'(t^0)]} \tag{5.8}$$
$$\times \begin{pmatrix} (G'_{\nu_2}(t_2^0) + G^2_{\mu_2}(t_2^0))(G_{\nu_1}(t_1^0) - G_{\mu_1}(t_1^0)) - G^2_{\mu_1}(t_1^0)(G_{\nu_2}(t_2^0) - G_{\mu_2}(t_2^0)) \\ (G'_{\nu_1}(t_1^0) + G^2_{\mu_1}(t_1^0))(G_{\nu_2}(t_2^0) - G_{\mu_2}(t_2^0)) - G^2_{\mu_2}(t_2^0)(G_{\nu_1}(t_1^0) - G_{\mu_1}(t_1^0)) \end{pmatrix}.$$

The second derivative of F with respect to t at t_0 is

$$F''(t^0) = \begin{pmatrix} (G''_{\nu_1}(t_1^0) - 2G^3_{\mu_1}(t_1^0)) & 2G^3_{\mu_2}(t_2^0) & 2G^3_{\mu_1}(t_1^0) & 2G^3_{\mu_2}(t_2^0) \\ 2G^3_{\mu_1}(t_1^0) & 2G^3_{\mu_2}(t_2^0) & 2G^3_{\mu_1}(t_1^0) & (G''_{\nu_2}(t_2^0) - 2G^3_{\mu_2}(t_2^0)) \end{pmatrix}. \tag{5.9}$$

The next result is about the behaviour of the measure $\omega \boxplus \lambda$ with arbitrary $\lambda \in \mathcal{M}$. The main work on this question was done by Biane [12]. Our aim was to find an interval on which the density $p_{\omega\boxplus\lambda}$ is positive.

Proposition 5.6. *Let $\lambda \in \mathcal{M}$ and there exists $c(\lambda)$ such that $d_L(\omega, \omega_t \boxplus \lambda) \le c(\lambda)\delta^2$, $t > 0$. Then for each $\delta \in (0, 1/10)$ the density $p_{\omega_t \boxplus \lambda}(x)$ is positive and analytic on $[-2+\delta, 2-\delta]$.*

Proof. The measure $\omega_t \boxplus \lambda$ always has a density. We would like find an interval where the density is positive. For that, define a subordination function $Z_{\omega_{1/2}}(z)$ which solves the equations

$$z = 2Z_{\omega_{1/2}}(z) - F_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) \quad \text{and} \quad F_\omega(z) = F_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)).$$

It easy to calculate that

$$Z_{\omega_{1/2}}(z) = \frac{3z + \sqrt{z^2-4}}{4},$$

and an analytic continuation of $Z_{\omega_{1/2}}$ to $(-2, 2)$ is given by

$$Z_{\omega_{1/2}}(z) = \frac{3z + i\sqrt{4-z^2}}{4}.$$

It easy to see that the following inequality holds:

$$\lim_{y \downarrow 0} \Im Z_{\omega_{1/2}}(x+iy) > \sqrt{\delta}/3, \quad x \in [-2+\delta, 2-\delta].$$

We set in (5.4) $\nu_1 = \omega_{t/2}$, $\nu_2 = \omega_{t/2} \boxplus \lambda$. On an 2-dimensional complex space $\mathbb{C}^2$ we choose the norm:

$$\|(z_1, z_2)\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

Now we apply the Newton-Kantorovich Theorem (see Theorem 7.4) to the equation $F(t) = 0$ for $z \in M := \{x + iy : x \in [-2+\delta, 2-\delta],\ 0 < y < \delta\sqrt{\delta}\}$. In formulas (5.6), (5.8) and (5.9) we set $\mu_1 = \mu_2 = \omega_{1/2}$ and $t_1^0 = t_2^0 = Z_{\omega_{1/2}}$. Since $|Z_{\omega_{1/2}}(z)| < 2$, $z \in M$, we choose the branch of $G_{\omega_{1/2}}$ such that $G_{\omega_{1/2}}(z) = z - i\sqrt{2-z^2}$, $|z| < 2$. Let s be such that $d_L(\omega, \omega_t \boxplus \lambda) \le s$.

1. First, we estimate $\|[F'(t^0)]^{-1}\|$. We have computed $\det[F'(t^0)]$ above. Moreover, due to Lemma 5.5 we have $G'_{\nu_j}(t_j^0) = G'_{\omega_{1/2}}(t_j^0) + f_j(t_j^0)$, where $|f_j(t_j^0)| \le \tilde{c}_1 s \delta^{-3/2}$ on M, $j = 1, 2$. Hence,

$$\begin{aligned}
&\det[F'(t^0)] \\
&= (G^2_{\omega_{1/2}}(t_2^0) + G'_{\omega_{1/2}}(t_2^0) + f_2(t_2^0))(G^2_{\omega_{1/2}}(t_1^0) + G'_{\omega_{1/2}}(t_1^0) + f_1(t_1^0)) - G^2_{\omega_{1/2}}(t_1^0)G^2_{\omega_{1/2}}(t_2^0) \\
&= g(z) + (f_1(t_1^0) + f_2(t_1^0))(G'_{\omega_{1/2}}(t_1^0) + G^2_{\omega_{1/2}}(t_1^0)) + f_1(t_1^0)f_2(t_1^0),
\end{aligned}$$

where

$$g(z) = \Big(G^2_\omega(z) + G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))\Big)^2 - \ G^4_\omega(z).$$

We find that

$$G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) = 1 + \frac{iZ_{\omega_{1/2}}(z)}{\sqrt{2 - Z_{\omega_{1/2}}(z)}} = 1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}}. \tag{5.10}$$

Finally, we obtain

$$\begin{aligned} g(z) = &\left(1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}} + \frac{1}{4}\left(z - i\sqrt{4-z^2}\right)^2\right)^2 - \frac{1}{16}\left(z - i\sqrt{4-z^2}\right)^4 \\ = &\left(1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}}\right) \\ \times &\left(1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}} + \frac{1}{2}\left(z - i\sqrt{4-z^2}\right)^2\right). \end{aligned} \tag{5.11}$$

The function $36 - 10z^2 - 6iz\sqrt{4-z^2}$ has zeros at $\pm 3/\sqrt{2}$, hence $g(z)$ is analytic on M and continuous up to the boundary. Let us check that $g(z)$ does not vanish on M. The first multiplier in (5.11) has no zeros on M:

$$\begin{aligned} 1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}} &= 0; \\ 36 - 10z^2 - 6iz\sqrt{4-z^2} &= 4 - 10z^2 - 6iz\sqrt{4-z^2}; \\ 36 &\neq 4. \end{aligned} \tag{5.12}$$

Let us compute the zeros of the second multiplier in (5.11):

$$\begin{aligned} 1 + \frac{3iz - \sqrt{4-z^2}}{\sqrt{36 - 10z^2 - 6iz\sqrt{4-z^2}}} + \frac{1}{2}\left(z - i\sqrt{4-z^2}\right)^2 &= 0; \\ 4 - 21z^2 + 9z^4 - z^6 + iz(9 - 7z^2 + z^4)\sqrt{4-z^2} &= 0; \\ 4 + 39z^2 - 18z^4 + 2z^6 &= 0. \end{aligned}$$

The last equation has the solutions: ± 2, $\pm\sqrt{\frac{5}{2} + \frac{3\sqrt{3}}{2}}$ and $\pm i\sqrt{\frac{3\sqrt{3}}{2} - \frac{5}{2}}$. The solutions of the initial equation are ± 2 and due to the choice of the branch of a square root we conclude that -2 is the only zero of $g(z)$ and $g(z)$ does not vanish on M. First of all we estimate $|g(z)|$ on an interval $[-2+\delta, 2-\delta]$. Let us denote

$$g_1(x) := \frac{3ix - \sqrt{4-x^2}}{\sqrt{36 - 10x^2 - 6ix\sqrt{4-x^2}}}.$$

We compute

$$\sqrt{(36 - 10x^2 - 6ix\sqrt{4-x^2})(36 - 10x^2 + 6ix\sqrt{4-x^2})} = 4(9 - 2x^2)$$

and

$$36-10x^2+6ix\sqrt{4-x^2} \;=\; 9(4-x^2)-x^2+6ix\sqrt{4-x^2}=(3\sqrt{4-x^2}+ix)^2.$$

Due to computations above we find

$$g_1(x)=\frac{(3ix-\sqrt{4-x^2})(3\sqrt{4-x^2}+ix)}{4(9-2x^2)}=\frac{2ix\sqrt{4-x^2}-3}{9-2x^2}$$

and hence

$$g(x)=\frac{6-2x^2+2ix\sqrt{4-x^2}}{9-2x^2}\left(\frac{6-2x^2+2ix\sqrt{4-x^2}}{9-2x^2}+x^2-2-2ix\sqrt{4-x^2}\right).$$

The following estimates hold

$$\left|\frac{6-2x^2+2ix\sqrt{4-x^2}}{9-2x^2}\right|=\frac{2}{\sqrt{9-2x^2}}\geq\frac{2}{3}$$

and

$$\left|\frac{6-2x^2+2ix\sqrt{4-x^2}}{9-2x^2}+x^2-2-2ix\sqrt{4-x^2}\right|=\frac{2\sqrt{4-x^2}}{\sqrt{9-2x^2}}\geq c\sqrt{\delta}$$

for $x\in[-2+\delta,2-\delta]$. We conclude that $|g(x)|\geq c_1\sqrt{\delta},\quad x\in[-2+\delta,2-\delta]$.

In order to estimate $|g(z)|$ on M we expand $g(x+iy)$ with respect to y at zero:

$$g(x+iy)=g(x)+R(x,y),\quad x\in[-2+\delta,2-\delta],\quad 0<y<\delta\sqrt{\delta},$$

where $R(x,y)$ is a remainder term such that

$$|R(x,y)|\leq\max_{\substack{x\in[-2+\delta,2-\delta]\\ 0<y<\delta\sqrt{\delta}}}|g'(x+iy)|\delta\sqrt{\delta}.$$

We find that $g'(z)=g_2(z)/g_3(z)$, where

$$\begin{aligned} g_2(z) &= -1488\\ &+\; 4\Bigg(2z^6-28z^4+186z^2-\left(z^4-9z^2-9\right)\sqrt{(4-z^2)(36-10z^2-6iz\sqrt{4-z^2})}\\ &-\; 2iz\left(z^4-12z^2+7\right)\sqrt{4-z^2}-iz\left(z^4-11z^2+39\right)\sqrt{36-10z^2-6iz\sqrt{4-z^2}}\Bigg),\end{aligned}$$

$$g_3(z)=i\left(18-5z^2-3iz\sqrt{4-z^2}\right)^2\sqrt{4-z^2}.$$

We conclude that $|g'(z)| \le c/\sqrt{\delta}$, $z \in M$. Hence $|R(x,y)| \le c\delta$ and $|g(z)| \ge c_1\sqrt{\delta}$, $z \in M$. Then $|\det[F'(t^0)]| \ge ||g(z)| - c_2 s\delta^{-3/2}| \ge \sqrt{\delta}(c_1 - c_2 s\delta^{-2})$ and for $s \le c_3\delta^2$ we have $\|[F'(t^0)]^{-1}\| \le c_4\delta^{-1/2} =: \beta_0$, $z \in M$.

2. We now estimate $\|[F'(t^0)]^{-1}F(t^0)\|$. Due to Lemma 5.5 we arrive at

$$\|[F'(t^0)]^{-1}F(t^0)\| \;<\; cs\delta^{-3/2} =: \eta_0, \quad z \in M.$$

3. Last, we estimate $\|F''(t^*)\|$, where $t^* = (t_1^*(z), t_2^*(z))$ such that $\|t^* - t^0\| \le 2\eta_0$. It is very important that $2\eta_0 < \sqrt{\delta}/3$, since this condition guarantees $\Im t_j^*(z) > 0$, $z \in M$, $j = 1, 2$, and therefore, we obtain $cs \le \delta^2$. Note $|G_{\omega_{1/2}}(z)| \le \sqrt{2}$ for $z \in \mathbb{C}^+ \cup \mathbb{R}$.

$$\|F''(t^0)\| \;\le\; \max\{|G''_{\nu_j}(Z_{\omega_{1/2}}) - 2G^3_{\omega_{1/2}}(Z_{\omega_{1/2}})|,\; 2|G^3_{\omega_{1/2}}(Z_{\omega_{1/2}})|,\; j = 1, 2\}.$$

Due to Lemma 5.5 we have

$$G''_{\nu_j}(Z_{\omega_{1/2}}) = G''_{\omega_{1/2}}(Z_{\omega_{1/2}}) + f(Z_{\omega_{1/2}}), \quad j = 1, 2,$$

where $|f(Z_{\omega_{1/2}})| \le \tilde{c}_2 s\delta^{-2}$ on M. Let us estimate $G''_{\omega_{1/2}}(Z_{\omega_{1/2}})$ on M. We find that

$$\begin{aligned} G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) &= 2i(2 - Z^2_{\omega_{1/2}}(z))^{-3/2} = 2i\left(2 + \frac{1}{16}(\sqrt{4 - z^2} - 3iz)^2\right)^{-3/2} \\ &= \frac{i}{8\sqrt{2}}(18 - 5z^2 - 3iz\sqrt{4 - z^2})^{-3/2} = \frac{i}{8\sqrt{2}}(z^2 - 9/2)^{-3/2}. \end{aligned}$$

Then the estimate $|G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))| \le c$ holds on M. Since we choose $s\delta^{-2} \le c$ we conclude that

$$\|F''(t^0)\| \le c_1 =: K_0.$$

The function $(z - t_1^*(z) - t_2^*(z))^{-3}$ is continuous for $z \in M$ because $\Im t_j^*(z) > 0$, $j = 1, 2$. It follows that the estimate for the second derivative holds for t^* such that $\|t^* - t^0\| < 2\eta_0$, $z \in M$.

The Newton-Kantorovich Theorem (see Theorem 7.4 in Auxiliary results) teaches us that if β_0, η_0 and K_0 satisfy the inequality $h_0 := \beta_0\eta_0K_0 \le 1/2$, then the equation $F(t) = 0$ has the solution $(Z_1(z), Z_2(z))$ for $z \in M$. It follows that parameters s and δ should satisfy the inequality:

$$c_1 s\delta^{-2} \le 1/2.$$

If we choose $h_0 = 1/2$, we get $\delta^2 = c_2 s$ from which we conclude

$$|Z_{\omega_{1/2}}(z) - Z_j(z)| \le \frac{1 - \sqrt{1 - 2h_0}}{h_0}\eta_0 = \frac{1}{\beta_0 K_0} = c_3\sqrt{\delta} = c_4 s^{1/4}, \quad j = 1, 2.$$

If we set $h_0 < 1/2$, then δ satisfies the inequality $c_1 s < \delta^2$ and the following estimate holds

$$|Z_{\omega_{1/2}}(z) - Z_j(z)| \le 2\eta_0 \le c_5 s\delta^{-3/2} \le c_6 s^{1/4}, \quad j = 1, 2.$$

In both cases we get a term of order $s^{1/4}$. Finally, we compute an estimate for the Cauchy transform

$$\left|\frac{1}{z-2Z_{\omega_{1/2}}(z)}-\frac{1}{z-Z_1(z)-Z_2(z)}\right|\le\frac{2|G_\omega^2(z)|cs\delta^{-3/2}}{|1-2cs\delta^{-3/2}|G_\omega^2(z)||},$$

$$|G_\omega(z)-G_{\omega_t\boxplus\lambda}(z)|<c_6s\delta^{-3/2}=c_7s^{1/4},\quad z\in M.$$

The limits $G_\omega(x):=\lim_{y\downarrow 0}G_\omega(x+iy)$ and $G_{\omega_t\boxplus\lambda}(x):=\lim_{y\downarrow 0}G_{\omega_t\boxplus\lambda}(x+iy)$ exist and the estimate

$$|G_\omega(x)-G_{\omega_t\boxplus\lambda}(x)|\le c_7s^{1/4},\quad x\in[-2+\delta,2-\delta]$$

holds. Hence we have the following estimate for densities

$$|p_\omega(x)-p_{\omega_t\boxplus\lambda}(x)|\le c_8s^{1/4},\quad x\in[2-\delta,2+\delta].$$

It follows that the optimal choice of $s=s(\delta)$ is such that $\delta\ge cs^{1/4}$. It easy to see $p_\omega(x)>\sqrt{\delta}/\pi$ on $[-2+\delta,\ 2-\delta]$. If we assume also $c_8s^{1/4}<\sqrt{\delta}/2\pi$, then $p_{\omega_t\boxplus\lambda}(x)>0$ on $[-2+\delta,\ 2-\delta]$. Analyticity follows from Remark 3.23. □

Corollary 5.7. *For each $\delta\in(1,1/10)$ and $n\ge c(\mu,r)\delta^{-4}$*

(1) *the density $p_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(x)$ is positive on $[-2+\delta,2-\delta]$;*

(2) *the function $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ has an analytic continuation to $[-2+\delta,2-\delta]$ and the imaginary part of this continuation does not vanish.*

Proof. According to Lemma 5.3 (1) we have

$$d_L(\omega,\omega\boxplus\mu_r^{(\varepsilon_r)})\le L\sum_{i=1}^{r}\varepsilon_i\le Lr/\sqrt{n}.$$

Applying Proposition 5.6 we complete the proof. □

5.2 Analytic continuation for $G_{\omega\boxplus\mu_r^{(\varepsilon r)}}$

Below we prove Theorem 5.8 which shows that the Cauchy transform $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ has an analytic continuation on

$$K:=\{x+iy:x\in(-2+2\delta,2-2\delta);|y|<\delta\sqrt{\delta}\}.$$

The idea of the proof is due to Wang [47].

Theorem 5.8. *Let μ be a compactly supported measure on $\mathbb{R}$ with supp$(\mu) \subset [-L, L]$, zero mean and unit variance. For every $\delta \in (0, 1/10)$ and $n \geq N^*(:= c(\mu, r)\delta^{-4})$ the Cauchy transform $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}$ has an analytic continuation on K such that*

$$G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) = G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z), \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \dots, r, \tag{5.13}$$

where $|\tilde{l}_{(\varepsilon_r)}(z)| \leq \frac{c(r)}{\sqrt{\delta}} \sum_{j=1}^r |\varepsilon_j|^2$ on K.

Proof. The inverse function of $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}$ can be expressed as

$$G^{(-1)}_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(w) = R_{\mu_r^{(\varepsilon_r)}}(w) + R_\omega(w) + \frac{1}{w} = \sum_{j=1}^r R_{D_{\varepsilon_j}\mu}(w) + w + \frac{1}{w},$$

wherever the series $\sum_{j=1}^r R_{D_{\varepsilon_j}\mu}(w)$ converges. Because of the rescaling property of the R-transform we have $R_{D_{\varepsilon_j}\mu}(w) = \varepsilon_j R_\mu(\varepsilon_j w)$. If we denote by κ_l the cumulants of μ, then for $w \in D_{\theta,1.4}$ (see Lemma 5.4) we have

$$\begin{aligned}
\left| \sum_{j=1}^r \varepsilon_j R_\mu(\varepsilon_j w) \right| &\leq \left| \sum_{j=1}^r \varepsilon_j \sum_{l=1}^\infty \kappa_{l+1} (\varepsilon_j w)^l \right| \\
&\leq \sum_{j=1}^r |\varepsilon_j|^2 |w| + \sum_{j=1}^r |\varepsilon_j| \sum_{l=2}^\infty |\kappa_{l+1}| (|\varepsilon_j||w|)^l \\
&\leq \sum_{j=1}^r |\varepsilon_j|^2 |w| + \sum_{j=1}^r |\varepsilon_j| \frac{32 L^3 |\varepsilon_j|^2 |w|^2}{1 - 4L|\varepsilon_j||w|}.
\end{aligned}$$

We can choose n such that

$$\frac{32 L^3 |\varepsilon_j||w|}{1 - 4L|\varepsilon_j||w|} \leq 1, \quad w \in D_{\theta,1.4},$$

which leads to the estimate

$$\left| \sum_{j=1}^r \varepsilon_j R_\mu(\varepsilon_j w) \right| \leq \sum_{j=1}^r 2|\varepsilon_j|^2 |w| \leq \frac{3r}{n}, \quad w \in D_{\theta,1.4}.$$

Due to Lemma 5.4 we know $G_\omega(K_\delta) \subset D_{\theta,1.4}$. Thus we replace w by G_ω and take the functional equation (3.15) into account to get

$$f_{(\varepsilon_r)}(z) := G^{(-1)}_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(G_\omega(z)) = z + g_{(\varepsilon_r)}(z), \quad z \in K_\delta, \tag{5.14}$$

where the power series in z

$$g_{(\varepsilon_r)}(z) = \sum_{j=1}^r \varepsilon_j R_\mu(\varepsilon_j G_\omega(z))$$

converges uniformly on K_δ to zero as $n \to \infty$ and the estimate

$$|g_{(\underline{\varepsilon}_r)}(z)| \le 3r \sum_{j=1}^{r} |\varepsilon_j|^2 \le c(r)\delta^4$$

holds uniformly on K_δ and $n \ge N^*$. The uniform bound of $g_{(\underline{\varepsilon}_r)}$ and (5.14) imply that the rectangle K is contained in the set $f_{(\underline{\varepsilon}_r)}(K_\delta)$. Rouché's Theorem (Theorem 7.2) implies that each function $f_{(\underline{\varepsilon}_r)}$ has an analytic inverse $f^{(-1)}_{(\underline{\varepsilon}_r)}$ defined on K. Due to (5.14) it follows

$$z = f_{(\underline{\varepsilon}_r)}\left(f^{(-1)}_{(\underline{\varepsilon}_r)}(z)\right) = f^{(-1)}_{(\underline{\varepsilon}_r)}(z) + g_{(\underline{\varepsilon}_r)}\left(f^{(-1)}_{(\underline{\varepsilon}_r)}(z)\right)$$

$$f^{(-1)}_{(\underline{\varepsilon}_r)}(z) = z - \widetilde{g}_{(\underline{\varepsilon}_r)}(z), \quad z \in K,$$

where $\widetilde{g}_{(\underline{\varepsilon}_r)}(z) = -g_{(\underline{\varepsilon}_r)}\left(f^{(-1)}_{(\underline{\varepsilon}_r)}(z)\right)$, $f^{(-1)}_{(\underline{\varepsilon}_r)}(z) \in K_\delta$ for $z \in K$, hence

$$|\widetilde{g}_{(\underline{\varepsilon}_r)}(z)| \le 3r \sum_{j=1}^{r} |\varepsilon_j|^2,$$

for $z \in K$ and $n \ge N^*$.

By Corollary 5.7 the function $G_{\omega \boxplus \mu_r^{(\underline{\varepsilon}_r)}}$ has an analytic continuation to the interval $(-2+\delta, 2-\delta)$ for $n \ge N^*$. The composition $G_\omega^{(-1)} \circ G_{\omega \boxplus \mu_r^{(\underline{\varepsilon}_r)}}$ is defined and analytic in a neighbourhood of the interval $(-2+\delta, 2-\delta)$ and hence, it coincides with the function $f^{(-1)}_{(\underline{\varepsilon}_r)}$ on $(-2+2\delta,\ 2-2\delta)$. We conclude

$$G_\omega^{(-1)}(G_{\omega \boxplus \mu_r^{(\underline{\varepsilon}_r)}}(z)) = f^{(-1)}_{(\underline{\varepsilon}_r)}(z) = z + \widetilde{g}_{(\underline{\varepsilon}_r)}(z), \quad z \in K, \quad n \ge N^*. \tag{5.15}$$

Let us estimate $|G'_\omega(z)|$ on K. It is easy to see

$$|G'_\omega(z)| = \left|\frac{1}{2} + \frac{iz}{2\sqrt{4-z^2}}\right| \le \left|\frac{1}{2} + \frac{i(2-i\delta\sqrt{\delta})}{4\sqrt{2\delta}}\right| \le \frac{1}{2\sqrt{\delta}}, \quad z \in K. \tag{5.16}$$

Applying G_ω on (5.15), we get

$$G_{\omega \boxplus \mu_r^{(\underline{\varepsilon}_r)}}(z) = G_\omega(z + \widetilde{g}_{(\underline{\varepsilon}_r)}(z)) = G_\omega(z) + \tilde{l}_{(\underline{\varepsilon}_r)}(z), \quad z \in K, \quad n \ge N^*,$$

where

$$|\tilde{l}_{(\underline{\varepsilon}_r)}(z)| \le \sup_{z \in K} |G'_\omega(z)||\widetilde{g}_{(\underline{\varepsilon}_r)}(z)| \le \frac{c(r)}{\sqrt{\delta}} \sum_{j=1}^{r} |\varepsilon_j|^2, \quad z \in K, \quad n \ge N^*. \tag{5.17}$$

Finally, $G_{\omega \boxplus \mu_r^{(\underline{\varepsilon}_r)}}(z) = G_\omega(z) + \tilde{l}_{(\underline{\varepsilon}_r)}(z)$, $z \in K$, $n \ge N^*$ and $\tilde{l}_{(\underline{\varepsilon}_r)}(z) \to 0$ uniformly on K as $n \to \infty$. The theorem is proved. □

Corollary 5.9. *Under the assumptions of the previous theorem we have*

$$G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + F_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) = z + q_n(z), \quad |\varepsilon_j| \le n^{-1/2}, \quad j = 1, \dots, r, \tag{5.18}$$

where $|q_n(z)| < 2.7|\tilde{l}_{(\varepsilon_r)}(z)|$, *for* $z \in K$, $n \ge N$.

Proof. In order to prove equation (5.18) we apply the representation (5.13) to $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ and $F_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$. Summing up we obtain

$$\begin{aligned} G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + F_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) &= G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z) + \frac{1}{G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z)} \\ &= G_\omega(z) + F_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z) - \frac{F_\omega(z)\tilde{l}_{(\varepsilon_r)}(z)}{G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z)} \\ &= z + \tilde{l}_{(\varepsilon_r)}(z) - \frac{F_\omega(z)\tilde{l}_{(\varepsilon_r)}(z)}{G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z)}, \end{aligned}$$

for $z \in K$, $n \ge N$. Let us denote $q_n(z) := \tilde{l}_{(\varepsilon_r)}(z)(1 - \frac{F_\omega(z)}{G_\omega(z)+l_n(z)})$. With that, we get the bound

$$\left|\frac{F_\omega(z)}{G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z)}\right| = \left|\frac{F_\omega^2(z)}{1 + \tilde{l}_{(\varepsilon_r)}(z)F_\omega(z)}\right| \le \frac{|F_\omega^2(z)|}{|1 + \tilde{l}_{(\varepsilon_r)}(z)F_\omega(z)|} \le \frac{|F_\omega^2(z)|}{\left|1 - |\tilde{l}_{(\varepsilon_r)}(z)||F_\omega(z)|\right|}$$

for $z \in K$, $n \ge N^*$. Due to Lemma 5.4, we have $|F_\omega(z)| \le 1.22$. Therefore, we obtain

$$\frac{|F_\omega^2(z)|}{\left|1 - |\tilde{l}_{(\varepsilon_r)}(z)||F_\omega(z)|\right|} \le \frac{1.5}{\left|1 - 1.22|\tilde{l}_{(\varepsilon_r)}(z)|\right|}, \quad z \in K, \quad n \ge N.$$

Due to (5.17) and the choice of n we get $\frac{1.5}{\left|1-1.22|\tilde{l}_{(\varepsilon_r)}(z)|\right|} < 1.7$ and

$$|q_n(z)| < |\tilde{l}_{(\varepsilon_r)}(z)|(1 + 1.7) = 2.7|\tilde{l}_{(\varepsilon_r)}(z)|, \quad \text{for} \ \ z \in K, \ \ n \ge N. \tag{5.19}$$

The corollary is proved. □

5.3 Analytic continuations for G_{μ_n}

Theorem 5.10. *Assume* μ *is compactly supported on* $[-L, L]$ *with zero mean and unit variance. For each* $\delta \in (0, 1/10)$ *and* n *such that* $n \ge N$

(1) *the density* $p_{\mu_n}(x)$ *is positive on* $[-2 + \delta, 2 - \delta]$*;*

(2) *the Cauchy transform G_{μ_n} has an analytic continuation to the interval $[-2+\delta, 2-\delta]$ and the imaginary part of this analytic continuation never vanishes on $[-2+\delta, 2-\delta]$.*

Proof. We will prove this result using the representations (3.18) and (3.19), where $t := (n-1)/n$. Due to Remark 3.23 it is sufficient to prove that G_{μ_n} has a positive density on $[-2+\delta, 2-\delta]$ for $n \geq c(\mu)\delta^{-4}$ (here we write $c(\mu)$ because ν is defined by μ, see (3.18)). According to Proposition 5.6 we know that $G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}$ has an analytic extension to $[-2+\delta, 2-\delta]$ which does not vanish for $\delta \geq cs^{1/2}$, where $s := d_L(\omega, \omega_t \boxplus D_{1/\sqrt{n}}\nu)$. It is known by Theorem 5.2 that

$$d_L(\omega, \omega_t \boxplus D_{1/\sqrt{n}}\nu) \leq d_L(\omega, \omega_t) + d_L(\delta_0, D_{1/\sqrt{n}}\nu).$$

Furthermore, it was shown that $d_L(\omega, \omega_t) \leq 2(1-\sqrt{(n-1)/n}) \leq 3/\sqrt{n}$. Due to (3.18), the measure ν has compact support such that $\operatorname{supp}(\nu) \subset [-L, L]$, and it follows that

$$d_L(\delta_0, D_{1/\sqrt{n}}\nu) \leq L/\sqrt{n}.$$

Finally, we obtain $s \leq (L+3)/\sqrt{n}$. Due to Proposition 5.6 n and δ must satisfy the inequality $n \geq c(\mu)\delta^{-4}$. By (3.19) we conclude

$$G_{\mu_n}(x) = \frac{1}{x - G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(x)}, \quad x \in \mathbb{R}.$$

We have shown $G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(x)$ is analytic and $\Im G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(x) \neq 0$ for all $x \in [-2+\delta, 2-\delta]$ and $n \geq c(\mu)\delta^{-4}$. Hence, $\Im(x - G_{\omega_t \boxplus D_{1/\sqrt{n}}\nu}(x)) \neq 0$ for all $x \in [-2+\delta, 2-\delta]$ and we conclude $G_{\mu_n}(x)$ is analytic on $[-2+\delta, 2-\delta]$ and has non-zero imaginary part for $n \geq c(\mu)\delta^{-4}$. □

Theorem 5.11. *Assume μ is compactly supported on $[-L, L]$ with zero mean and unit variance. For each $\delta \in (0, 1/10)$ and n such that $n \geq N$, the Cauchy transform G_{μ_n} has the analytic extension*

$$G_{\mu_n}(z) = G_\omega(z) + l_n(z), \quad z \in K, \tag{5.20}$$

where $K := \{x + iy : x \in (-2+2\delta, 2-2\delta),\ |y| < \delta\sqrt{\delta}\}$, and $|l_n(z)| \leq \frac{L^3}{\sqrt{\delta n}}$ on K.

Proof. First, we define sets

$$K_\delta = \{x + iy : x \in (-2+\delta,\ 2-\delta), |y| < 2\delta\sqrt{\delta}\}$$

and

$$D_{\theta,1.4} = \{z \in \mathbb{C}^- : \arg z \in (-\pi+\theta, -\theta);\ |z| < 1.4\},$$

where the angle $\theta = \theta(\delta)$ is chosen in such a way that $2\sin\theta = \sqrt{\frac{\delta}{4}\left(1-\frac{\delta}{4}\right)}$. Figure 5.1 illustrates these sets. In Lemma 5.4 we showed $G_\omega(K_\delta) \subset D_{\theta,1.4}$. Therefore, if we assume $n \geq N$, then G_{μ_n} has an analytic continuation to the interval $(-2+2\delta, 2-2\delta)$. Moreover, assume $c(\mu) \geq 6L^3$. Additivity of the R-transform shows

$$G_{\mu_n}^{(-1)}(w) = R_{\mu_n}(w) + \frac{1}{w} = nR_{D_{1/\sqrt{n}}\mu}(w) + \frac{1}{w} = \sqrt{n}R_\mu(w/\sqrt{n}) + \frac{1}{w}.$$

Since μ has compact support, the R-transform has a series expansion $R_\mu(w) = \sum_{l=0}^{\infty} \kappa_{l+1} w^l$, where $\kappa_1 = m_1,\ \kappa_2 = m_2$ and due to our assumptions $\kappa_1 = 0,\ \kappa_2 = 1$. Therefore,

$$G_{\mu_n}^{(-1)}(w) = w + \frac{1}{w} + \sqrt{n}\sum_{l=2}^{\infty} \kappa_{l+1}\left(\frac{w}{\sqrt{n}}\right)^l.$$

The function $G_{\mu_n}^{(-1)}$ is defined on a domain where the series $\sqrt{n}\sum_{l=2}^{\infty}\kappa_{l+1}\left(\frac{w}{\sqrt{n}}\right)^l$ converges. Note $|\kappa_3| = |m_3| \leq L^3$, then the estimates for the cumulants (3.4) show

$$\begin{aligned}\left|\sqrt{n}\sum_{l=2}^{\infty}\kappa_{l+1}\left(\frac{w}{\sqrt{n}}\right)^l\right| &\leq \frac{L^3|w|^2}{\sqrt{n}}\left(1+\sum_{l=3}^{\infty}\frac{128}{l}\left(\frac{4L|w|}{\sqrt{n}}\right)^{l-2}\right)\\ &\leq \frac{L^3|w|^2}{\sqrt{n}}\left(1+\frac{512L|w|}{3\sqrt{n}-12L|w|}\right).\end{aligned}$$

Since $c(\mu) \geq 6L^3$, we have $6L^3/\sqrt{n} < \delta^2$ and

$$\frac{L^3|w|^2}{\sqrt{n}}\left(1+\frac{512L|w|}{3\sqrt{n}-12L|w|}\right) \leq \frac{L^3|w|^2}{\sqrt{n}}(1+0.5) \leq \frac{1.5L^3}{\sqrt{n}},$$

for $w \in D_{\theta,1.4}$ and $n \geq N$. Replacing w by G_ω and taking the functional equation (3.15) into account we obtain

$$f_n(z) := G_{\mu_n}^{(-1)}(G_\omega(z)) = z + g_n(z), \quad z \in K_\delta, \ \ n \geq N^*, \tag{5.21}$$

where the sequence

$$g_n(z) = \sqrt{n}\sum_{l=2}^{\infty}\kappa_{l+1}\left(\frac{G_\omega(z)}{\sqrt{n}}\right)^l$$

converges uniformly on K_δ to zero as $n \to \infty$ and the bound

$$|g_n(z)| < \frac{1.5L^3}{\sqrt{n}} < \delta^2$$

is uniform in $z \in K_\delta$ and $n \geq N$. The uniform bound of g_n and (5.21) imply the rectangle $K = \{x + iy : x \in (-2+2\delta,\ 2-2\delta), |y| < \delta\sqrt{\delta}\}$ is contained in the set $f_n(K_\delta)$. Now Rouché's theorem (Theorem 7.2) shows each function f_n has an analytic inverse $f_n^{(-1)}$ defined on K, and due to (5.21) we have

$$z = f_n(f_n^{(-1)}(z)) = f_n^{(-1)}(z) + g_n(f_n^{(-1)}(z)),$$

hence $f_n^{(-1)}(z) = z - \widetilde{g_n}(z)$, $z \in K$, where $\widetilde{g_n}(z) = -g_n(f_n^{(-1)}(z))$, $f_n^{(-1)}(z) \in K_\delta$ for $z \in K$. Finally, we conclude that $|\widetilde{g_n}(z)| \leq 1.5L^3/\sqrt{n}$, for $z \in K$, $n \geq N$.

Theorem 5.10 implies that the function G_{μ_n} has an analytic continuation to the interval $(-2+2\delta,\ 2-2\delta)$ for $n \geq N$. The composition $G_\omega^{(-1)} \circ G_{\mu_n}$ is defined and analytic in a neighbourhood of the interval $(-2+2\delta, 2-2\delta)$ and, hence, it coincides with the function $f_n^{(-1)}$ on $(-2+2\delta,\ 2-2\delta)$. We conclude

$$G_\omega^{(-1)}(G_{\mu_n}(z)) = f_n^{(-1)}(z) = z + \widetilde{g_n}(z), \quad z \in K, \quad n \geq N. \tag{5.22}$$

In (5.16) we showed $|G'_\omega(z)| \leq \frac{1}{2\sqrt{\delta}}$. Applying G_ω on (5.22), we get

$$G_{\mu_n}(z) = G_\omega(z + \widetilde{g_n}(z)) = G_\omega(z) + l_n(z), \quad z \in K, \quad n \geq N,$$

where

$$|l_n(z)| \leq \sup_{z \in K} |G'_\omega(z)||\widetilde{g_n}(z)| \leq \frac{L^3}{\sqrt{\delta n}} \leq \delta^{3/2}, \quad z \in K, \quad n \geq N. \tag{5.23}$$

Finally, $G_{\mu_n}(z) = G_\omega(z) + l_n(z)$, $z \in K$, $n \geq N$ and $l_n(z) \to 0$ uniformly on K as $n \to \infty$. □

Corollary 5.12. *Under the assumptions of the previous theorem we have*

$$G_{\mu_n}(z) + F_{\mu_n}(z) = z + p_n(z), \tag{5.24}$$

where $|p_n(z)| < 2.7|l_n(z)|$ *for* $z \in K$, $n \geq N$.

Proof. In order to prove equation (5.24) we apply the representation (5.20) of Theorem 5.11 to G_{μ_n} and F_{μ_n}. Summing up we obtain

$$\begin{aligned} G_{\mu_n}(z) + F_{\mu_n}(z) &= G_\omega(z) + l_n(z) + \frac{1}{G_\omega(z) + l_n(z)} \\ &= G_\omega(z) + F_\omega(z) + l_n(z) - \frac{F_\omega(z) l_n(z)}{G_\omega(z) + l_n(z)} \\ &= z + l_n(z) - \frac{F_\omega(z) l_n(z)}{G_\omega(z) + l_n(z)}, \end{aligned}$$

for $z \in K$, $n \geq N$. Let us denote $p_n(z) := l_n(z)(1 - \frac{F_\omega(z)}{G_\omega(z)+l_n(z)})$. With that, we get the bound

$$\left|\frac{F_\omega(z)}{G_\omega(z)+l_n(z)}\right| = \left|\frac{F_\omega^2(z)}{1+l_n(z)F_\omega(z)}\right| \leq \frac{|F_\omega^2(z)|}{|1+l_n(z)F_\omega(z)|} \leq \frac{|F_\omega^2(z)|}{|1-|l_n(z)||F_\omega(z)||}$$

for $z \in K$, $n \geq N^*$. Due to Lemma 5.4, we have $|F_\omega(z)| \leq 1.22$. Therefore, we obtain

$$\frac{|F_\omega^2(z)|}{|1-|l_n(z)||F_\omega(z)||} \leq \frac{1.5}{|1-1.22|l_n(z)||}, \quad z \in K, \quad n \geq N.$$

Due to (5.23) and the choice of δ we get $\frac{1.5}{|1-1.22|l_n(z)||} < 1.7$ and we conclude

$$|p_n(z)| < |l_n(z)|(1+1.7) = 2.7|l_n(z)|, \quad \text{for} \;\; z \in K, \;\; n \geq N. \tag{5.25}$$

The corollary is proved. □

Chapter 6

Proofs of main results

In this chapter we deduce an analogue of the Edgeworth expansion in free probability. We develop the expansion terms based on $h_\infty(\underset{\sim}{\varepsilon}_r)(:= G_{\omega \boxplus \mu_r^{(\varepsilon_r)}})$ as well as the error terms according to the procedure described in Chapter 2. In order to obtain the expansion we have to construct an analytic extension for the Cauchy transform $G_{\tilde{\mu}_{m+r}}$ to $K'' := \{x + iy : x \in [-2 + 5\delta, 2 - 5\delta],\ |y| < \delta\sqrt{\delta}/2\}$, $\delta \in (0, 1/10)$, $m \geq n \geq c(\mu, r)\delta^{-4}$ and check whether this extension is uniformly differentiable with respect to $\underset{\sim}{\varepsilon}_r$ ($|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \dots, r$). Finally, we prove Theorem 4.1, Theorem 4.2 and Theorem 4.3. and deduce local expansions for the extension of the Cauchy transform G_{μ_n} (Theorem 4.4), the density p_{μ_n} (Corollary 4.5) and the distribution μ_n (Corollary 4.6).

6.1 Analytic continuations for $G_{\tilde{\mu}_{m+r}}$

Below we consider a measure $\tilde{\mu}_{m+r}$, with $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \dots, 2r$, $m \geq n$. We show that the measure has a density $p_{\tilde{\mu}_{m+r}}(x)$ for sufficiently large n and the density is analytic on $[-2+2\delta,\ 2-2\delta]$. As before we consider the following system of equations:

$$\begin{cases} (z - Z_1(z) - Z_2(z))^{-1} + G_{\nu_1}(Z_1(z)) = 0 \\ (z - Z_1(z) - Z_2(z))^{-1} + G_{\nu_2}(Z_2(z)) = 0, \end{cases} \tag{6.1}$$

where G_{ν_1} and G_{ν_1} are the Cauchy transforms of ν_1 and ν_2.

The next result is due to Belinschi [4] (see Theorem 3.3 and Theorem 4.1).

Theorem 6.1. *Let $\nu_1,\ \nu_2$ be two Borel probability measures on $\mathbb{R}$, neither of them a point mass. The following hold:*

(1) *The subordination functions from (6.1) have limits* $\lim_{y \downarrow 0} Z_j(x + iy)$, $j = 1, 2$, $x \in \mathbb{R}$.

(2) *The absolutely continuous part of $\nu_1 \boxplus \nu_2$ is always nonzero, and its density is analytic wherever positive and finite, and $F_{\nu_1 \boxplus \nu_2}$ extends analytically in a neighbourhood of every point where the density is positive and finite.*

Let us introduce the notation

$$\mu^{r\boxplus} := \underbrace{\mu \boxplus \ldots \boxplus \mu}_{r\ times}.$$

We set $\nu_1 := D_{1/\sqrt{m}}\mu^{[(m-r)/2]\boxplus} \boxplus \mu^{(\varepsilon_r)}$ and $\nu_2 := D_{1/\sqrt{m}}\mu^{(m-r)-[(m-r)/2]\boxplus} \boxplus \mu^{(\varepsilon_{r+1},\ldots,\varepsilon_{2r})}$. Due to Lemma 5.3 the following inequality holds:

$$d_L\left(\mu_1, \nu_1\right) \leq L\left([(m-r)/2]\left(\frac{1}{\sqrt{2[m/2]}} - \frac{1}{\sqrt{m}}\right) + \sum_{j=1}^{r} \varepsilon_j + \frac{r}{\sqrt{2[m/2]}}\right).$$

Let us estimate $[m/2]\left(\frac{1}{\sqrt{2[m/2]}} - \frac{1}{\sqrt{m}}\right)$. If $m = 2k$ this is trivial, if $m = 2k+1$ we have

$$k\left(\frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2k+1}}\right) \leq \frac{1}{2\sqrt{2k+1}},$$

thus

$$d_L\left(\mu_1, \nu_1\right) \leq L\left(\frac{r+1}{\sqrt{m}} + \sum_{j=1}^{r} \varepsilon_j\right). \tag{6.2}$$

In the same way we obtain

$$d_L\left(\mu_2, \nu_2\right) \leq L\left(\frac{r+2}{\sqrt{m}} + \sum_{j=r+1}^{2r} \varepsilon_j\right). \tag{6.3}$$

Recall some notations. Define $F(t) : \mathbb{C}^2 \to \mathbb{C}^2$ by the formula

$$F(t) = \begin{pmatrix} (z - t_1 - t_2)^{-1} + G_{\nu_1}(t_1) \\ (z - t_1 - t_2)^{-1} + G_{\nu_2}(t_2) \end{pmatrix}. \tag{6.4}$$

The equation $F(t) = 0$ has a unique solution, say $Z = (Z_1(z), Z_2(z))$. Consider an approximation solution $t^0 = (t_1^0(z), t_2^0(z))$ of $F(t) = 0$ which is defined by the measures $\mu_1 = \mu_2 := D_{1/\sqrt{2[m/2]}}\mu^{[m/2]\boxplus}$ and solves the system:

$$\begin{cases} (z - t_1^0 - t_2^0)^{-1} + G_{\mu_1}(t_1^0) = 0 \\ (z - t_1^0 - t_2^0)^{-1} + G_{\mu_2}(t_2^0) = 0. \end{cases} \tag{6.5}$$

Note that $t_1^0 = t_2^0$. Next, we shall estimate $\lim_{y\downarrow 0} \Im t_1^0(x+iy)$ from below for $x \in (-2, 2)$.

Lemma 6.2. *For every $\delta \in (0, 1/10)$ and $m \geq N$ the following estimate holds*

$$\lim_{y\downarrow 0} \Im t_1^0(x+iy) \geq \sqrt{\delta}/2, \quad x \in [-2+2\delta, 2-2\delta].$$

Proof. Let us define measure $\mu_{2[m/2]} = \mu_1 \boxplus \mu_2$. Due to the subordination equation we find

$$t_1^0(z) = \frac{z + F_{\mu_{2[m/2]}}(z)}{2}.$$

In Corollary 5.12 we showed that $F_{\mu_{2[m/2]}}(z) = z + p_{2[m/2]} - G_{2[m/2]}(z)$, $z \in K$, $m \geq N$. It is easy to see that

$$t_1^0(z) = z + \frac{p_{2[m/2]}(z) - G_{\mu_{2[m/2]}}(z)}{2}.$$

Thus, we have the estimate

$$\begin{aligned}
\Im t_1^0(z) &= \Im z + \frac{1}{2}\Im(p_{2[m/2]}(z) - G_{\mu_{2[m/2]}}(z)) \\
&= \Im z + \frac{1}{2}\Im(p_{2[m/2]}(z) - G_\omega(z) - l_{2[m/2]}(z)) \\
&\geq \Im(z - \frac{1}{2}G_\omega(z)) - \frac{1}{2}(|p_{2[m/2]}(z)| + |l_{2[m/2]}(z)|),
\end{aligned}$$

for $z \in K$. In addition, due to Theorem 5.11 and Corollary 5.12, the following estimate holds:

$$\frac{1}{2}(|p_{2[m/2]}(z)| + |l_{2[m/2]}(z)|) < \frac{L^3}{\sqrt{\delta m}}, \quad z \in K, \quad m \geq N.$$

On the other hand

$$\Im(z - \frac{1}{2}G_\omega(z)) \geq -\Im G_\omega(x) \geq \sqrt{\delta}/2, \quad z \in \mathbb{C}^+ \cup K.$$

Finally, we conclude

$$\Im t_1^0(z) \geq \sqrt{\delta(1-\delta)} + \frac{L^3}{\delta\sqrt{m}} \geq \sqrt{\delta}/2, \quad z \in K,$$

due to the choice of m. □

The following lemma provides estimates for $\|[F'(t^0)]^{-1}\|$, $\|[F'(t^0)]^{-1}F(t^0)\|$ and $\|[F''(t^*)]\|$ where t^* such that $\|t^0 - t^*\| \leq 2\|[F'(t^0)]^{-1}F(t^0)\|$. (See formulas (5.6), (5.8) and (5.9)).

Lemma 6.3. *Assume that $F(t)$ is defined by (6.4), $t^0 = (t_1^0, t_2^0)$ solves (6.5) and $d_L(\mu_1, \nu_1) \le s$, $d_L(\mu_1, \nu_2) \le s$. For every $\delta \in (0, 1/10)$ and m such that $m \ge N^*(:= c(\mu, r)\delta^{-4})$ the following estimates hold:*

(1) $\|[F'(t^0)]^{-1}\| \le \beta_0$, *where β_0 is a numerical constant;*

(2) $\|[F'(t^0)]^{-1} F(t^0)\| \le \eta_0 = c_1 s/\delta$;

(3) $\|F''(t)\| \le K_0 = c_5 s/\delta^2$ *for all t such that $\|t - t^0\| < 2\eta_0$,*

for all $z \in M := \{x + iy : x \in (-2 + 2\delta, 2 - 2\delta) : 0 < y < \delta\sqrt{\delta}/2\}$.

Proof. (1) Due to Lemma 6.2 and Lemma 5.5 we get the estimate

$$G'_{\nu_j}(t_j^0) = G'_{\mu_j}(t_j^0) + f_j(t_j^0), \quad |f_j(t_j^0)| \le \widetilde{c}_1 s \delta^{-3/2}, \quad j = 1, 2 \quad z \in M,$$

where from (6.2), (6.3) we conclude that $s = L\left(\frac{r+2}{\sqrt{m}} + \sum_{j=1}^{2r} \varepsilon_j\right)$.

Let us estimate $|\det[F'(t^0)]|$. Due to (5.11) we have

$$\begin{aligned}
\det[F'(t^0)] &= (G'_{\nu_2}(t_1^0) + G^2_{\mu_1}(t_1^0))(G'_{\nu_1}(t_1^0) + G^2_{\mu_1}(t_1^0)) - G^4_{\mu_1}(t_1^0) \\
&= (G'_{\mu_1}(t_1^0) + G^2_{\mu_1}(t_1^0))^2 - G^4_{\mu_1}(t_1^0) \\
&+ (f_1(t_1^0) + f_2(t_1^0))(G'_{\mu_1}(t_1^0) + G^2_{\mu_1}(t_1^0)) + f_1(t_1^0) f_2(t_1^0).
\end{aligned}$$

We can find a derivative of G_{μ_1} in the following way

$$G'_{\mu_1}(z) = -G^2_{\mu_1}(z) F'_{\mu_1}(z), \quad z \in \mathbb{C}^+. \tag{6.6}$$

Due to representation (3.19) we have

$$F_{\mu_{2[m/2]}}(z) = z - G_{\omega_t \boxplus \nu_{2[m/2]}}(z), \quad z \in \mathbb{C}^+.$$

After rescaling we obtain

$$F_{\mu_1}(z) = z - \frac{1}{2} G_{D_{1/\sqrt{2}}\omega_t \boxplus \nu_{2[m/2]}}(z), \quad z \in \mathbb{C}^+.$$

We differentiate the last formula and arrive at

$$F'_{\mu_1}(z) = 1 - \frac{1}{2} G'_{D_{1/\sqrt{2}}\omega_t \boxplus \nu_{2[m/2]}}(z), \quad z \in \mathbb{C}^+. \tag{6.7}$$

Due to Theorem 5.11 we have

$$F_{\mu_m}(z) = \frac{1}{G_\omega(z) + l_m(z)} = F_\omega(z) + f_3(z), \quad |f_3(z)| \le c(\mu)(\delta m)^{-1/2}, \quad z \in M, \ m \ge N.$$

Let us define $t_1^0(z)$ via $Z_{\omega_{1/2}}(z)$ for $z \in M$, $m \geq N$

$$t_1^0(z) = (F_{\mu_{2[m/2]}}(z) + z)/2 = (F_\omega(z) + z + f_3(z))/2 = Z_{\omega_{1/2}}(z) + f_3(z)/2. \tag{6.8}$$

Due to Lemma 5.5 we have

$$G'_{D_{1/\sqrt{2}}\omega_t \boxplus \nu_{2[m/2]}}(t_1^0(z)) = G'_{\omega_{1/2}}(t_1^0(z)) + f_4(t_1^0(z)), \tag{6.9}$$

where $|f_4(t_1^0(z))| \leq \tilde{c}_1 \delta^{-3/2} m^{-1/2}$, $z \in M$. Due to (6.8) we obtain

$$G'_{\omega_{1/2}}(Z_{\mu_1}(z)) = G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z) + f_3(z)/2) = G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) + f_5(z),$$

where $|f_5(z)| \leq \sup_{z\in M} |G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) f_3(z)/2|$. Let us estimate $G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))$:

$$\begin{aligned} G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) &= \frac{2i}{(2 - Z^2_{\omega_{1/2}}(z))^{3/2}} = \frac{2i}{(2 + (-3iz + \sqrt{4-z^2})^2/16)^{3/2}} \\ &= \frac{128i}{(3\sqrt{4-z^2} - iz)^3} \end{aligned}$$

and

$$\left| \frac{1}{3\sqrt{4-z^2} - iz} \right| = \left| \frac{3\sqrt{4-z^2} + iz}{4(9 - 2z^2)} \right| \leq \frac{8}{4|9 - 2z^2|} \leq 2, \quad z \in M.$$

We conclude that $|G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))| \leq 2048$, $z \in M$ and the estimate holds

$$|f_5(z)| \leq cc_1(\mu)(\delta m)^{-1/2}, \quad z \in M, \quad m \geq N.$$

Finally, by (6.6), (6.7) and (6.9) we obtain

$$G'_{\mu_1}(t_1^0(z)) = -G^2_{\mu_1}(t_1^0(z))(1 - G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))/2) + f_6(z), \tag{6.10}$$

where $f_6(z) := -G^2_{\mu_1}(t_1^0(z))(f_4(z) + f_5(z))$, $z \in M$. Applying (6.10) we obtain

$$\begin{aligned} &\left(G'_{\mu_1}(t_1^0(z)) + G^2_{\mu_1}(t_1^0(z))\right)^2 - G^4_{\mu_1}(t_1^0(z)) \\ &\qquad = G^4_{\mu_1}(t_1^0(z)) \left(G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))/2 - 1\right) \left(G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))/2 + 1\right) \\ &\qquad + f_6(z) G^4_{\mu_1}(t_1^0(z))(2G'_{\mu_1}(t_1^0(z)) + 2 + f_6(z)). \end{aligned}$$

Let us estimate $\left|G^4_{\mu_1}(t_1^0(z)) g(z)\right|$, where

$$g(z) := \left(G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))/2 - 1\right) \left(G'_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))/2 + 1\right)$$

from below on M. Due to (5.10) we have

$$g(z)=\frac{1}{4}\left(\frac{3iz-\sqrt{4-z^2}}{\sqrt{36-10z^2-6iz\sqrt{4-z^2}}}-1\right)\left(\frac{3iz-\sqrt{4-z^2}}{\sqrt{36-10z^2-6iz\sqrt{4-z^2}}}+3\right). \quad (6.11)$$

We showed in the proof of Proposition 5.6 that $36-10z^2-6iz\sqrt{4-z^2}$ is non-zero on M. Therefore, we conclude that $g(z)$ is analytic on M. The first multiplier in (6.11) has no zeros on M due to computations similar to (5.12). The second multiplier has zero at -2 due to the choice of the branch of a square root. Then we see that $g(z)$ has no zeroes on M. Let us estimate $|g(x)|$ on $[-2+2\delta, 2-2\delta]$. Due to computations in Proposition 5.6 it is easy to see that $|g(x)|\geq c\sqrt{\delta}$, $x\in[-2+2\delta,2-2\delta]$. In order to estimate $|g(z)|$ on M we expand $g(x+iy)$ with respect to y at zero:

$$g(x+iy)=g(x)+R(x,y),\ x\in[-2+2\delta,2-2\delta],\quad 0<y<\delta\sqrt{\delta},$$

where $R(x,y)$ is a remainder term such that

$$|R(x,y)|\leq \max_{\substack{x\in[-2+2\delta,2-2\delta]\\ 0<y<\delta\sqrt{\delta}}}|g'(x+iy)|\delta\sqrt{\delta}.$$

We find that

$$g'(z)=\frac{16\left(iz-3\sqrt{4-z^2}\right)\left(\sqrt{4-z^2}-2\sqrt{2}\sqrt{18-5z^2-3iz\sqrt{4-z^2}}-3iz\right)}{\sqrt{4-z^2}\left(18i-5iz^2+3z\sqrt{4-z^2}\right)^2}.$$

We conclude that $|g'(z)|\leq\frac{c}{\sqrt{\delta}}$, $x\in[-2+2\delta,2-2\delta]$, then $|g(z)|\geq c\sqrt{\delta}$, $\ z\in M$. Finally, $\det[F'(t^0)]\geq c\sqrt{\delta}-c_1 s\delta^{-3/2}$ and $\det[F'(t^0)]^{-1}\leq c(\mu,r)\delta^{-1/2}$. Due to the choice of n, all entries of $[F'(t^0)]^{-1}$ are bounded, therefore the norm of $[F'(t^0)]^{-1}$ is also bounded at $t^0=(t_1^0(z),t_2^0(z))$ for all $z\in M$, $n\geq N$

(2) The inverse matrix of $F'(t^0)$ has been computed in (5.6). Due to Lemma 6.2 and Lemma 5.5 we obtain the estimate

$$|G_{\nu_j}(t_j^0)-G_{\mu_j}(t_j^0)|\leq\widetilde{c}s\delta^{-1},\quad j=1,2,\quad z\in M,\ m\geq n\geq N^*.$$

Then the following estimate for the first component of $[F'(t^0)]^{-1}F(t^0)$ holds:

$$\begin{aligned}&|(G'_{\nu_2}(t_2^0)+2G^2_{\mu_2}(t_2^0))(G_{\nu_1}(t_1^0)-G_{\mu_1}(t_1^0))-G^2_{\mu_1}(t_1^0)(G_{\nu_2}(t_2^0)-G_{\mu_2}(t_2^0))|\\ &\leq\ cs\delta^{-1}|(G'_{\nu_2}(t_2^0)+2G^2_{\mu_2}(t_2^0))-G^2_{\mu_1}(t_1^0)|\leq c_1 s\delta^{-3/2}=:\eta_0,\quad z\in M,\ m\geq n\geq N.\end{aligned}$$

We conclude that the first component of $[F'(t^0)]^{-1}F(t^0)$ is bounded at t^0 for $z\in M$. The same estimate can be obtained for the second component of $[F'(t^0)]^{-1}F(t^0)$. We conclude that $\|[F'(t^0)]^{-1}F(t^0)\|\leq c_1 s/\delta^{3/2}$ for all $z\in M$, $m\geq n\geq N$.

(3) In order to estimate $F''(t^0)$ we need to estimate $G''_{\nu_j}(t^0_j)$, $j=1,2$ on M. By (6.6) we have

$$G''_{\mu_1}(z) = -(G^2_{\mu_1}(z)F'_{\mu_1}(z))' = -2G_{\mu_1}(z)G'_{\mu_1}(z)F'_{\mu_1}(z) - G^2_{\mu_1}(z)F''_{\mu_1}(z)$$
$$F''_{\mu_1}(t^0_1(z)) = -G''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) + G'''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))f_3(z)/2 + f_7(z),$$

where by Lemma 5.5 $|f_7(z)| \leq \tilde{c}_2\delta^{-2}m^{-1/2}$, $z \in M$. We compute the third derivative of $G_{\omega_{1/2}}$:

$$G'''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z)) = \frac{6iZ_{\omega_{1/2}}(z)}{(2 - Z_{\omega_{1/2}}(z)^2)^{5/2}} = \frac{3(3z + i\sqrt{4-z^2})}{2(3\sqrt{4-z^2} - iz)^5}$$

and conclude that $|G'''_{\omega_{1/2}}(Z_{\omega_{1/2}}(z))| \leq c$ on M. Finally, $|G''_{\mu_1}(z)| \leq c_2$ on M, $m \geq n \geq N^*$. Let us consider t^* such that $\|t^* - t^0\| < 2\eta_0$ with parameter $\eta_0 < c_4\sqrt{\delta}$ due to the choice of m and n. Since it is known that $\Im t^0_j(z) \geq \sqrt{\delta}/2$, $j=1,2$, $z \in M$, we can find c_4 such that $\Im t^*_j(z) > 0$ for $z \in M$, $j=1,2$. The Cauchy transform is analytic on the upper half plane. Therefore, we have $F''(t^*) < c_4\delta^{-2}m^{-1/2}$ for t^* such that $\|t^* - t^0\| < 2\eta_0$ and $z \in M$, $m \geq n \geq N^*$. With regard to the Newton-Kantorovich theorem the parameters β_0, η_0 and K_0 have to fulfill the inequality

$$h_0 = \beta_0\eta_0K_0 \leq \frac{1}{2},$$

which means that we need to choose m such that this inequality is satisfied and we find out that $m \geq c(\mu, r)\delta^{-4} =: N^*$. □

In the following theorem the measure $\tilde{\mu}_{m+r}$ is approximated by $\mu_{2[m/2]}$ in order to show that $\tilde{\mu}_{m+r}$ is absolutely continuous with respect to the Lebesgue measure in the interior of $(-2, 2)$ for sufficiently large m.

Theorem 6.4. *For every $\delta \in (0, 1/10)$ and m, n such that $m \geq n \geq N^*$, the measure $\tilde{\mu}_{m+r}$ is absolutely continuous with the density $p_{\tilde{\mu}_{m+r}}$, which is positive and analytic on $[-2+2\delta,\ 2-2\delta]$. Moreover, the Cauchy transform $G_{\tilde{\mu}_{m+r}}(z)$ can be continued analytically to a neighborhood of the interval $[-2+2\delta,\ 2-2\delta]$ and this continuation never vanishes on $[-2+2\delta,\ 2-2\delta]$.*

Proof. Recall that $\mu_1 = \mu_2 := D_{1/\sqrt{2[m/2]}}\mu^{[m/2]\boxplus}$, $\nu_1 := D_{1/\sqrt{m}}\mu^{[(m-r)/2]\boxplus} \boxplus \mu^{(\varepsilon_r)}$, $\nu_2 := D_{1/\sqrt{m}}\mu^{(m-r)-[(m-r)/2]\boxplus} \boxplus \mu^{(\varepsilon_{r+1},\ldots,\varepsilon_{2r})}$ and $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots 2r$. From the previous lemma it follows that the solution $Z = (Z_1, Z_2)$ of $F(t) = 0$ lies in the ball $B_0 := \{t \in X :\ \|t - t^0\| \leq \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0 = c_8\eta_0\}$ and

$$|Z_j(z) - t^0_j(z)| < c_8\eta_0 = \frac{c_9L}{\delta^{3/2}}\left(\frac{r+2}{\sqrt{m}} + \sum_{i=1}^{2r}\varepsilon_i\right), \quad j = 1, 2, \quad z \in M, \quad m \geq n \geq N^*.$$

Due to Theorem 6.1 the limits $Z_j(x) := \lim_{y\downarrow 0} Z_j(x+iy)$, $j = 1,2$ exist for $x \in [-2+2\delta, 2-2\delta]$, $m \geq n \geq N$. Thus it follows the limit $\lim_{y\downarrow 0} G_{\tilde{\mu}_{m+r}}(x+iy)$ exists for $x \in [-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$ and the bound

$$|G_{\tilde{\mu}_{m+r}}(x) - G_{\mu_{2[m/2]}}(x)| \leq \frac{c_{10}L}{\delta^{3/2}} \left(\frac{r+2}{\sqrt{m}} + \sum_{j=1}^{2r} \varepsilon_j \right)$$

holds uniformly on $[-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$. This implies that the measure $\tilde{\mu}_{m+r}$ is absolutely continuous on $[-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$ with density $p_{\tilde{\mu}_{m+r}}$ such that

$$|p_{\tilde{\mu}_{m+r}}(x) - p_{\mu_{2[m/2]}}(x)| \leq \frac{c_{11}L}{\delta^{3/2}} \left(\frac{r+2}{\sqrt{m}} + \sum_{j=1}^{2r} \varepsilon_j \right) \tag{6.12}$$

for $x \in [-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$.

Let us show that $p_{\tilde{\mu}_{m+r}}$ is analytic on $[-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$. Due to Theorem 6.1 it is sufficiently to show that $p_{\tilde{\mu}_{m+r}}$ is positive on $[-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$, then it follows $p_{\tilde{\mu}_{m+r}}$ is analytic and $G_{\tilde{\mu}_{m+r}}$ has an analytic continuation to $[-2+2\delta, 2-2\delta]$, $m \geq n \geq N^*$. If the inequality holds for some $x_0 \in [-2+2\delta, 2-2\delta]$

$$p_\omega(x_0) - \frac{|l_m(x_0)|}{\pi} - \frac{c_{11}L}{\delta^{3/2}} \left(\frac{r+2}{\sqrt{m}} + \sum_{j=1}^{2r} \varepsilon_j \right) > 0, \tag{6.13}$$

then Theorem 5.11 and inequality (6.12) show that $p_{\tilde{\mu}_{m+r}}(x_0) > 0$. From (6.13) it follows that

$$\sqrt{4 - x_0^2} \;\; > \;\; \frac{2\pi L^3}{\sqrt{\delta m}} + \frac{2\pi c_1 L}{\delta^{3/2}} \left(\frac{r+2}{\sqrt{m}} + \sum_{j=1}^{2r} \varepsilon_j \right) \geq \frac{c_1(\mu, r)}{\delta^{3/2}\sqrt{m}}, \quad m \geq n \geq N^*.$$

Hence, let us find all x_0 such that the inequality $\sqrt{4 - x_0^2} > \frac{c_1(\mu,r)}{\delta^{3/2}\sqrt{n}}$ holds. We have

$$x_0^2 < 4 - \frac{c_1^2(\mu, r)}{\delta^3 n}, \quad -2 + \frac{c_1^2(\mu, r)}{4\delta^3 n} < x_0 < 2 - \frac{c_1^2(\mu, r)}{4\delta^3 n}.$$

Therefore, we can choose $n \geq c(\mu, r)\delta^{-4}$ such that $[-2+2\delta,\ 2-2\delta] \subset [-2+\frac{c_1^2(\mu,r)}{4\delta^3 n},\ 2-\frac{c_1^2(\mu,r)}{4\delta^3 n}]$. Finally, $p_{\tilde{\mu}_{m+r}}(x) > 0$ for all $x \in [-2+2\delta,\ 2-2\delta]$, $m \geq n \geq N^*$. Theorem 6.1 concludes the proof. $\square$

Next we show that $G_{\tilde{\mu}_{m+r}}$ has the analytic continuation

$$G_{\tilde{\mu}_{m+r}}(z) = G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \dots, r, \quad z \in K'', \quad m \geq n \geq N^*.$$

The proof works much in the same way as the proof of Theorem 5.11. In the end of the section we show that all first derivatives $\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)$, $j = 1, \dots, 2r$ vanish at zero.

Theorem 6.5. *For each $\delta \in (0, 1/10)$ and m, n such that $m \geq n \geq N^*$ the Cauchy transform $G_{\tilde{\mu}_{m+r}}$ has the analytic extension*

$$G_{\tilde{\mu}_{m+r}}(z) = G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad z \in K', \quad |\varepsilon_j| \leq n^{-1/2}, \quad j = 1, \dots, r$$

where $K' := \{x + iy : x \in (-2 + 4\delta, 2 - 4\delta),\ |y| < \delta\sqrt{\delta}/2\}$ and $|\tilde{l}_n(z)| \leq \frac{c(r)}{\delta n}$ on K'.

Proof. Let $\delta \in (0,\ 1/10)$. Define the sets

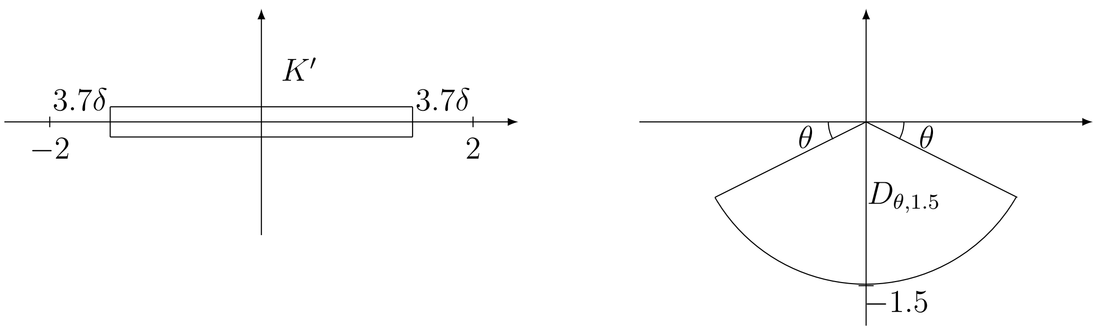

Figure 6.1

$$\hat{K} = \{x + iy : x \in (-2 + 3.7\delta,\ 2 - 3.7\delta), |y| < \delta\sqrt{\delta}\}$$

and

$$D_{\theta,1.5} = \{z \in \mathbb{C}^- : \arg z \in (-\pi + \theta, -\theta); |z| < 1.5\},$$

where the angle $\theta = \theta(\delta)$ is chosen such that $2\sin\theta = \sqrt{\frac{\delta}{4}\left(1 - \frac{\delta}{4}\right)}$. Figure 6.1 illustrates these sets. First of all, we show that $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(\hat{K}) \subset D_{\theta,1.5}$, where $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}$ is the analytic extension from Theorem 5.8. We fix $z_0 \in \hat{K}$ with $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z_0) = Re^{i\psi}$. Let us estimate θ. Due to the functional equation from Corollary 5.9 we obtain

$$\left(R + \frac{1}{R}\right)\cos\psi + i\left(R - \frac{1}{R}\right)\sin\psi = z_0 + q_n(z_0).$$

By the fact that $|\Re z_0| \leq 2 - 4\delta$ and estimate (5.25) we get

$$2|\cos\psi| \leq \left(R + \frac{1}{R}\right)|\cos\psi| \leq 2 - 3.7\delta + |q_n(z_0)| \leq 2 - \delta.$$

This inequality coincides with one from the proof of Lemma 5.4 and we have the same estimate for θ:

$$|\sin\psi| = \sqrt{\delta/4\,(1 - \delta/4)} > \sin\theta.$$

We now estimate R by using an inequality from Theorem 5.8.

$$|G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)| = |G_\omega(z) + \tilde{l}_{(\varepsilon_r)}(z)| \leq |G_\omega(z)| + |\tilde{l}_{(\varepsilon_r)}(z)| \leq 1.4 + \frac{c(r)}{\sqrt{\delta} n} < 1.5 \qquad (6.14)$$

for $z \in K := \{x + iy : x \in (-2 + 2\delta, 2 - 2\delta),\ |y| < \delta\sqrt{\delta}\}$ and $n \geq N^*$. Thus we conclude that $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(\hat{K}) \subset D_{\theta,1.5}$.

The additivity of the R-transform shows

$$\begin{aligned} G^{(-1)}_{\tilde{\mu}_{m+r}}(w) &= \frac{1}{w} + R_{\mu_m}(w) + R_{\mu_{2r}^{(\varepsilon_{2r})}}(w) - \frac{r}{\sqrt{m}} R_\mu(w/\sqrt{m}) \\ &= G^{(-1)}_{\mu_m}(w) + \sum_{j=1}^{2r} \varepsilon_j R_\mu(\varepsilon_j w) - \frac{r}{\sqrt{m}} R_\mu(w/\sqrt{m}). \qquad (6.15) \end{aligned}$$

We want to estimate $\left|\sum_{j=1}^{2r} \varepsilon_j R_\mu(\varepsilon_j w)\right|$ on $D_{\theta,1.5}$. Since μ has compact support, the R-transform has the series expansion $R_\mu(w) = \sum_{l=1}^{\infty} \kappa_{l+1} w^l$. Thus,

$$\begin{aligned} \left|\sum_{j=1}^{2r} \varepsilon_j R_\mu(\varepsilon_j w)\right| &\leq \left|\sum_{j=1}^{2r} \varepsilon_j \sum_{l=1}^{\infty} \kappa_{l+1} (\varepsilon_j w)^l\right| \\ &\leq \sum_{j=1}^{2r} |\varepsilon_j|^2 |w| + \sum_{j=1}^{2r} |\varepsilon_j| \sum_{l=2}^{\infty} |\kappa_{l+1}| (|\varepsilon_j||w|)^l \\ &\leq \sum_{j=1}^{2r} |\varepsilon_j|^2 \left(|w| + \frac{32L^3 |\varepsilon_j||w|^2}{1 - 4L|\varepsilon_j||w|}\right). \end{aligned}$$

Notice that $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots, 2r$. Therefore, the following inequality holds:

$$\frac{32L^3|\varepsilon_j||w|^2}{1 - 4L|\varepsilon_j||w|} \leq \frac{32L^3|w|^2}{\sqrt{n} - 4L|w|}, \quad j = 1, \ldots, 2r, \quad n \geq N^*.$$

On the other hand we have

$$\begin{aligned} \left|\frac{r}{\sqrt{m}} R_\mu(w/\sqrt{m})\right| &= \left|\frac{r}{\sqrt{m}} \sum_{l=1}^{\infty} \kappa_{l+1} (w/\sqrt{m})^l\right| \\ &\leq \frac{r}{\sqrt{m}} \sum_{l=1}^{\infty} |\kappa_{l+1}||w|^l / m^{l/2} \leq \frac{r}{m} \left(|w| + \frac{32L^3|w|^2}{\sqrt{m} - 4L|w|}\right) \end{aligned}$$

Let us choose N^* such that for all $m \geq n \geq N^*$ the estimates hold:

$$\frac{32L^3|w|^2}{\sqrt{m} - 4L|w|} < \frac{32L^3|w|^2}{\sqrt{n} - 4L|w|} < \frac{1}{2}, \quad r \leq n^{1/4}/10,$$

then

$$\left|\sum_{j=1}^{2r}\varepsilon_j R_\mu(\varepsilon_j w) - \frac{r}{\sqrt{m}}R_\mu(w/\sqrt{m})\right| \;<\; (|w|+0.5)\left(\sum_{j=1}^{2r}|\varepsilon_j|^2+\frac{r}{m}\right)$$
$$< \; 2\left(\sum_{j=1}^{2r}|\varepsilon_j|^2+\frac{r}{m}\right).$$

Replacing w by $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ in (6.15) gives

$$f_{(\underline{\varepsilon}_{2r})}(z) := G^{(-1)}_{\tilde{\mu}_{m+r}}(G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z)) = z + g_{(\underline{\varepsilon}_{2r})}(z), \quad z\in K', \quad m\ge n\ge N^*, \tag{6.16}$$

where the series

$$g_{(\underline{\varepsilon}_{2r})}(z) = \sum_{j=1}^{2r}\varepsilon_j R_\mu(\varepsilon_j w) - \frac{r}{\sqrt{m}}R_\mu(w/\sqrt{m})$$

is uniformly bounded on $\hat{K}$ and $m\ge n\ge N^*$ since

$$|g_{(\underline{\varepsilon}_{2r})}(z)| < 2\left(\sum_{j=1}^{2r}|\varepsilon_j|^2+\frac{r}{m}\right)\le\delta^4.$$

The uniform bound of $g_{(\underline{\varepsilon}_{2r})}$ and (6.16) imply that the rectangle K' is contained in the set $f_{(\underline{\varepsilon}_{2r})}(\hat{K})$. Now by Rouché's theorem (Theorem 7.2) each function $f_{(\underline{\varepsilon}_{2r})}$ has an analytic inverse $f^{(-1)}_{(\underline{\varepsilon}_{2r})}$ defined on K', namely

$$f^{(-1)}_{(\underline{\varepsilon}_{2r})}(z) = z - \widetilde{g}_{(\underline{\varepsilon}_{2r})}(z), \quad z\in K',$$

where $|\widetilde{g}_{(\underline{\varepsilon}_{2r})}(z)|\le 2(\sum_{j=1}^{2r}|\varepsilon_j|^2 + r/m)$, for $z\in K'$ and $m\ge n\ge N^*$.

By Theorem 6.4 the function $G_{\tilde{\mu}_{m+r}}$ has an analytic continuation to the interval $[-2+2\delta, 2-2\delta]$ for $m\ge n\ge N^*$. The composition $G^{(-1)}_{\omega\boxplus\mu_r^{(\varepsilon_r)}}\circ G_{\tilde{\mu}_{m+r}}$ is defined and analytic in the neighbourhood of the interval $[-2+2\delta, 2-2\delta]$ and hence, it coincides with the function $f^{(-1)}_{(\underline{\varepsilon}_{2r})}$ on the interval $(-2+3.7\delta,\ 2-3.7\delta)$. We conclude that

$$G^{(-1)}_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(G_{\tilde{\mu}_{m+r}}(z)) = f^{(-1)}_{(\underline{\varepsilon}_{2r})}(z) = z + \widetilde{g}_{(\underline{\varepsilon}_{2r})}(z), \quad z\in K', \quad m\ge n\ge N^*. \tag{6.17}$$

By the Cauchy inequalities (see Theorem 7.3), we have $|G'_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z)|\le 1.5/\delta$ on K'. Applying $G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}$ on (6.17), we get

$$G_{\tilde{\mu}_{m+r}}(z) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z+\widetilde{g}_{(\underline{\varepsilon}_{2r})}(z)) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad z\in K', \quad m\ge n\ge N^*,$$

where

$$|\tilde{l}_n(z)| \le \sup_{z\in K} |G'_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z)||\widetilde{g}_{(\underline{\varepsilon}_{2r})}(z)| \le \frac{3}{\delta}\left(\sum_{j=1}^{2r} |\varepsilon_j|^2 + \frac{r}{m}\right), \quad z \in K'.$$

Finally, $G_{\tilde{\mu}_{m+r}}(z) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z)$ with $|\tilde{l}_n(z)| \le \frac{3}{\delta}\left(\sum_{j=1}^{2r} |\varepsilon_j|^2 + r/m\right)$, $z \in K'$, $m \ge n \ge N^*$. □

Corollary 6.6. *For each $\delta \in (0, 1/10)$ and m, n such that $m \ge n \ge N^*$, the Cauchy transform $G_{\mu_m\boxplus\mu_r^{(\varepsilon_r)}}$ has the following analytic extension*

$$G_{\mu_m\boxplus\mu_r^{(\varepsilon_r)}}(z) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad z \in K', \quad |\varepsilon_j| \le n^{-1/2}, \quad j = 1, \dots, r$$

where $|\tilde{l}_n(z)| \le \frac{c(r)}{\delta n}$ uniformly on on K'.

Proof. The corollary follows immediately from Theorem 6.5 after replacing in $\tilde{\mu}_{m+r}$ r weights by zeros. □

6.2 Proofs of Theorem 4.1, Theorem 4.2 and Theorem 4.3

Let us remind

$$K'' := \{x + iy : x \in [-2 + 5\delta, 2 - 5\delta],\ |y| < \delta\sqrt{\delta}/2\}.$$

Now we have all we need to proof Theorem 4.1.

Proof of Theorem 4.1. Let us define the set

$$U_0 := \{\underline{\eta}_{2r} \in \mathbb{C}^{2r} :\ |\eta_j| \le 1/\sqrt{n},\ j = 1, \dots, 2r\}$$

and the function

$$G^{(-1)}(\underline{\eta}_{2r}, w) = w + \frac{1}{w} + \sqrt{m-r}\sum_{l=2}^{\infty} \kappa_{l+1}\left(\frac{w}{\sqrt{m-r}}\right)^l + \sum_{j=1}^{2r} \eta_j \sum_{l=1}^{\infty} \kappa_{l+1}(\eta_j w)^l,$$

where $w \in D_{\theta,1.5}$, $\underline{\eta}_{2r} \in U_0$, such that

$$G^{(-1)}(\underline{\eta}_{2r}, w)\Big|_{\underline{\eta}_{2r}=\underline{\varepsilon}_{2r}} = G^{(-1)}_{\tilde{\mu}_{m+r}}(w).$$

$G^{(-1)}(\underline{\eta}_{2r}, w)$ is analytic on $U_0 \times D_{\theta,1.5}$. Consider the function

$$F(\underline{\eta}_{2r}, z, w) = w + \frac{1}{w} + \sqrt{m-r} \sum_{l=2}^{\infty} \kappa_{l+1} \left(\frac{w}{\sqrt{m-r}} \right)^l + \sum_{j=1}^{2r} \eta_j \sum_{l=1}^{\infty} \kappa_{l+1} (\eta_j w)^l - z,$$

$w \in D_{\theta,1.5}$, $z \in G^{(-1)}(\underline{\eta}_{2r}, D_{\theta,1.5})$ and $\underline{\eta}_{2r} \in U_0$. This function is analytic on $U_0 \times G^{(-1)}(\underline{\eta}_{2r}, D_{\theta,1.5}) \times D_{\theta,1.5}$. For fixed $\underline{\varepsilon}_{2r}^0 \in \mathbb{R}^{2r} \cap U_0$, $w_0 \in D_{\theta,1.5}$ and fixed $z_0 = G^{(-1)}(\underline{\varepsilon}_{2r}^0, w_0) \in G^{(-1)}(\underline{\varepsilon}_{2r}^0, D_{\theta,1.5})$ we have

$$F(\underline{\varepsilon}_{2r}^0, z_0, w_0) = w_0 + \frac{1}{w_0} + \sqrt{m-r} \sum_{l=2}^{\infty} \kappa_{l+1} \left(\frac{w_0}{\sqrt{m-r}} \right)^l + \sum_{j=1}^{2r} \varepsilon_j^0 \sum_{l=1}^{\infty} \kappa_{l+1} (\varepsilon_j^0 w_0)^l - z_0 = 0,$$

and

$$\frac{\partial}{\partial w} F(\underline{\varepsilon}_{2r}^0, z_0, w_0) = 1 - \frac{1}{w_0^2} + \sum_{l=2}^{\infty} l\kappa_{l+1} \left(\frac{w_0}{\sqrt{m-r}} \right)^{l-1} + \sum_{j=1}^{2r} (\varepsilon_j^0)^2 \sum_{l=1}^{\infty} l\kappa_{l+1} (\varepsilon_j^0 w_0)^{l-1}.$$

The following estimates hold: $|w_0^2 - 1| > \sin^2\theta > \delta/16$ on $D_{\theta,1.5}$ and

$$\left| \sum_{l=2}^{\infty} l\kappa_{l+1} \left(\frac{w_0}{\sqrt{m-r}} \right)^{l-1} + \sum_{j=1}^{2r} (\varepsilon_j^0)^2 \sum_{l=1}^{\infty} l\kappa_{l+1} (\varepsilon_j^0 w_0)^{l-1} \right| \le \frac{c}{\sqrt{n}} \le c_1 \delta^2,$$

hence

$$\left| \frac{\partial}{\partial w} F(\underline{\varepsilon}_{2r}^0, z_0, w_0) \right| > c_2 \delta > 0.$$

Due to the Implicit Function Theorem (see Theorem 7.5) there is a neighbourhood $U = \tilde{U}_0 \times U_{z_0} \times U_{w_0} \subset U_0 \times G^{(-1)}(\underline{\varepsilon}_{2r}^0, D_{\theta,1.5}) \times D_{\theta,1.5}$ and an analytic function

$$g : \tilde{U}_0 \times U_{z_0} \to U_{w_0}; \quad g = G(\underline{\eta}_{2r}, z, \underline{\varepsilon}_{2r}^0, z_0), \quad \underline{\eta}_{2r} \in \tilde{U}_0, \quad z \in U_{z_0}.$$

Moreover, $G(\underline{\eta}_{2r}, z, \underline{\varepsilon}_{2r}^0, z_0)\Big|_{\underline{\eta}_{2r} = \underline{\varepsilon}_{2r}} = G_{\tilde{\mu}_{m+r}}(z)$, for $z_0 \in K' \subset G^{(-1)}(\underline{\varepsilon}_{2r}^0, D_{\theta,1.5})$. Note, that for $z_0^1 \ne z_0^2$, $z \in U_{z_0^1} \cap U_{z_0^2}$ and $\underline{\varepsilon}_{2r}^{0,1} \ne \underline{\varepsilon}_{2r}^{0,2}$, $\underline{\eta}_{2r} \in U_{\underline{\varepsilon}_{2r}^{0,1}} \cap U_{\underline{\varepsilon}_{2r}^{0,2}}$ the functions $G(\underline{\eta}_{2r}, z, \underline{\varepsilon}_{2r}^{0,1}, z_0^1)$ and $G(\underline{\eta}_{2r}, z, \underline{\varepsilon}_{2r}^{0,2}, z_0^2)$ do not necessary coincide, however

$$G(\underline{\varepsilon}_{2r}, z, \underline{\varepsilon}_{2r}^{0,1}, z_0^1) = G(\underline{\varepsilon}_{2r}, z, \underline{\varepsilon}_{2r}^{0,2}, z_0^2) = G_{\tilde{\mu}_{m+r}}(z), \quad z_0^1, z_0^2 \in K',$$

since $G_{\tilde{\mu}_{m+r}}(z)$ is uniquely defined for $z \in K'$ by Theorem 6.6. We conclude that $G_{\tilde{\mu}_{m+r}}(z) \in C^\infty(E_{m,r}^n)$, $z \in K''$, $m \ge n \ge N^*$. $\square$

Proof of Theorem 4.2. We introduce the notation:

$$\tilde{\mu}_{m-r} := \underbrace{D_{m^{-1/2}}\mu \boxplus \ldots \boxplus D_{m^{-1/2}}\mu}_{m-r \ times}.$$

Let us calculate $\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)$ at $\varepsilon_j = 0$, $j = 1, \ldots, 2r$ for $z \in K''$, $m \geq n \geq N^*$. For this purpose, we differentiate the equation

$$z = R_{\tilde{\mu}_{m+r}}(G_{\tilde{\mu}_{m+r}}(z)) + \frac{1}{G_{\tilde{\mu}_{m+r}}(z)},$$

and arrive at

$$\begin{aligned} 0 &= \Big[R'_{\tilde{\mu}_{m-r}}(G_{\tilde{\mu}_{m+r}}(z)) \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z) + R_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \varepsilon_j R'_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z))(G_{\tilde{\mu}_{m+r}}(z) + \varepsilon_j \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \sum_{i=1}^{2r}{}^* \varepsilon_i^2 R'_\mu(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z)) \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z) - \frac{\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)}{G^2_{\tilde{\mu}_{m+r}}(z)} \Big] \Big|_{\varepsilon_j = 0}, \end{aligned} \tag{6.18}$$

where $\sum_{i=1}^{2r}{}^*$ means summation over all $i \neq j$. After simple computations we get

$$\begin{aligned} 0 &= R'_{\tilde{\mu}_{m-r}}(G_{\tilde{\mu}_{m+r}}(z)) \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)\Big|_{\varepsilon_j = 0} - \frac{\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)}{G^2_{\tilde{\mu}_{m+r}}(z)}\Big|_{\varepsilon_j = 0} \\ &+ \sum_{i=1}^{2r}{}^* \varepsilon_i^2 R'_\mu(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z)) \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)\Big|_{\varepsilon_j = 0}, \end{aligned}$$

By the definition of the R-transform and taking into account that μ has zero mean and unit variance we obtain

$$R'_{\tilde{\mu}_{m-r}}(z) = \frac{m-r}{m} R'_\mu(z/\sqrt{m}) = (1 + r/m) \left(1 + \sum_{l=2}^{\infty} l \kappa_{l+1} \left(\frac{z}{\sqrt{m}} \right)^{l-1} \right).$$

Finally, we have the following equation for $\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)$:

$$\begin{aligned} &\left[(1 + r/m) \left(1 + \sum_{l=2}^{\infty} l \kappa_{l+1} \left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m}} \right)^{l-1} \right) G^2_{\tilde{\mu}_{m+r}}(z) - 1 \right. \\ &+ \left. G^2_{\tilde{\mu}_{m+r}}(z) \sum_{i=1}^{2r}{}^* \varepsilon_i^2 \sum_{l=2}^{\infty} l \kappa_{l+1} \left(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z) \right)^{l-1}) \right] \frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z) \Big|_{\varepsilon_j = 0} = 0. \end{aligned} \tag{6.19}$$

Due to the representations

$$G_{\tilde{\mu}_{m+r}}(z) = G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) + \tilde{l}_n(z), \quad z \in K'', \ \ m \geq n \geq N^*$$
$$G_{\omega\boxplus\mu_r^{(\varepsilon_r)}}(z) = G_\omega(z) + \tilde{l}_{(\underline{\varepsilon}_r)}(z), \quad z \in K'', \ \ m \geq n \geq N^*,$$

we have

$$G_{\tilde{\mu}_{m+r}}(z) = G_\omega(z) + \tilde{L}_n(z), \quad z \in K'', \ \ m \geq n \geq N^* \tag{6.20}$$

where

$$\tilde{L}_n(z) = \tilde{l}_n(z) + \tilde{l}_{(\underline{\varepsilon}_r)}(z). \tag{6.21}$$

According to Theorem 5.8 and Theorem 6.5 we have the bound

$$|\tilde{L}_n(z)| \leq \frac{c(\mu, r)}{\sqrt{\delta n}} \quad z \in K'', \ \ m \geq n \geq N^*. \tag{6.22}$$

Thus, we can rewrite equation (6.19) in the following way

$$(G_\omega^2(z) - 1 + f_m^{(\underline{\varepsilon}_{2r})}(z))\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)\bigg|_{\varepsilon_j=0} = 0,$$

where

$$\begin{aligned} f_m^{(\underline{\varepsilon}_{2r})}(z) &:= 2G_\omega(z)\tilde{L}_n(z) + \tilde{L}_n^2(z) + \frac{2r}{m}\left(1 + G^2_{\tilde{\mu}_{m+r}}(z)\right) \\ &+ \ G^2_{\tilde{\mu}_{m+r}}(z) \sum_{i=1}^{2r}{}^* \varepsilon_i^2 \sum_{l=2}^{\infty} l\kappa_{l+1}\left(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z)\right)^{l-1} \\ &+ \ G^2_{\tilde{\mu}_{m+r}}(z)\left(1 + \frac{2r}{m}\right)\sum_{l=2}^{\infty} l\kappa_{l+1}\left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m}}\right)^{l-1}. \end{aligned}$$

Finally, we can find such N^* that for all $m \geq n \geq N^*$

$$|G_\omega^2(z) - 1| > |f_m^{(\underline{\varepsilon}_{2r})}(z)|, \quad z \in \partial K'',$$

see (6.27) later. By Rouché's theorem we conclude that $G_\omega^2(z) - 1 + f_m^{(\underline{\varepsilon}_{2r})}(z)$ has no roots on K'', $m \geq n \geq N^*$, thus $\frac{\partial}{\partial \varepsilon_j} G_{\tilde{\mu}_{m+r}}(z)\Big|_{\varepsilon_j=0} = 0$, $z \in K''$, $m \geq n \geq N^*$. The theorem is proved. □

Proof of Theorem 4.3. By Theorem 4.1 $G_{\tilde{\mu}_{m+r}}$ is infinitely differentiable with respect to $\underline{\varepsilon}_{2r}$. In order to prove that derivatives of $G_{\tilde{\mu}_{m+r}}$ are bounded we apply induction on the order of derivatives l, starting with $l = 1$. The extension $G_{\tilde{\mu}_{m+r}}(z)$ solves the equation

$$z = R_{\tilde{\mu}_{m+r}}(G_{\tilde{\mu}_{m+r}}(z)) + \frac{1}{G_{\tilde{\mu}_{m+r}}(z)}, \quad z \in K'', \quad m \geq n \geq N^*. \tag{6.23}$$

Let us recall the equation for the first derivative of $G_{\tilde{\mu}_{m+r}}(z)$:

$$\begin{aligned} 0 &= \Big[R'_{\tilde{\mu}_{m-r}}(G_{\tilde{\mu}_{m+r}}(z))\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z) + R_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \varepsilon_j R'_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z))(G_{\tilde{\mu}_{m+r}}(z) + \varepsilon_j\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \sum_{i=1}^{2r}{}^{*}\varepsilon_i^2 R'_\mu(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z))\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z)\frac{\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z)}{G^2_{\tilde{\mu}_{m+r}}(z)}\Big], \quad j=1,\dots,2r, \end{aligned}$$

where $\sum_{i=1}^{2r}{}^{*}$ means summation without component j. It follows that $\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z) = \frac{f(\underline{\varepsilon}_{2r},z)}{g(\underline{\varepsilon}_{2r},z)}$, where

$$\begin{aligned} f(\underline{\varepsilon}_{2r},z) &:= R_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z)) + \varepsilon_j R'_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z))G_{\tilde{\mu}_{m+r}}(z), \\ g(\underline{\varepsilon}_{2r},z) &:= \big[R'_{\tilde{\mu}_{m-r}}(G_{\tilde{\mu}_{m+r}}(z)) + \varepsilon_j^2 R'_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \sum_{i=1}^{2r}{}^{*}\varepsilon_i^2 R'_\mu(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z)) - \frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)}\Big], \quad j=1,\dots,2r. \end{aligned}$$

It is easy to see $|f(\underline{\varepsilon}_{2r},z)| \le c|\varepsilon_j| \le c/\sqrt{n}$, $z\in K''$, $m\ge n\ge N^*$ and

$$\begin{aligned} |g(\underline{\varepsilon}_{2r},z)| &= \big|\big[R'_{\tilde{\mu}_{m-r}}(G_{\tilde{\mu}_{m+r}}(z)) + \varepsilon_j^2 R'_\mu(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z)) \\ &+ \sum_{i=1}^{2r}{}^{*}\varepsilon_i^2 R'_\mu(\varepsilon_i G_{\tilde{\mu}_{m+r}}(z)) - \frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)}\Big]\Big| \ge \left|\left|1-\frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)}\right| - \frac{c_1(\mu)}{\sqrt{n}}\right|. \end{aligned}$$

Due to (6.20) and (6.21) we have

$$\left|1-\frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)}\right| = \left|\frac{G^2_\omega(z) - 1 + \tilde{L}_n(z)(G_\omega(z)+\tilde{L}_n(z))}{(G_\omega(z)+\tilde{L}_n(z))^2}\right|.$$

The domain $D_{\theta,1.4}$ was described in Lemma 5.4. It was shown that $G_\omega(K'')\subset D_{\theta,1.4}$ for $2\sin\theta = \sqrt{\frac{\delta}{4}\left(1-\frac{\delta}{4}\right)}$, hence, $|G^2_\omega(z)-1| \ge |\cos^2\theta - 1|$. Due to (6.22) and the choice of n we have

$$\left|1-\frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)}\right| \ge c_3(\mu,r)\delta \tag{6.24}$$

and $|g(\underline{\varepsilon}_{2r},z)| \ge |c_3(\mu,r)\delta - c_2(\mu,r)\delta^2| > 0$, $z\in K''$, $m\ge n\ge N^*$. We conclude $\frac{\partial}{\partial\varepsilon_j}G_{\tilde{\mu}_{m+r}}(z)$ is bounded for $z\in K''$, $|\varepsilon_j|\le n^{-1/2}$, $j=1,\dots,2r$, $m\ge n\ge N^*$.

Assume that we have proved that $D^\alpha_{\underline{\varepsilon}_{2r}}G_{\tilde{\mu}_{m+r}}(z)$, $|\alpha|\le r-1$ are bounded for $z\in K''$, $|\varepsilon_j|\le n^{-1/2}$, $j=1,\dots,2r$, $m\ge n\ge N^*$. In order to show that $D^\alpha_{\underline{\varepsilon}_{2r}}G_{\tilde{\mu}_{m+r}}(z)$

are bonded for $|\alpha| = s$ we differentiate equation (6.23):

$$\begin{aligned} R_{\tilde{\mu}_{m+r}}(G_{\tilde{\mu}_{m+r}}(z)) &= G_{\tilde{\mu}_{m+r}}(z) + \sqrt{m-r}\sum_{l=2}^{\infty}\kappa_{l+1}\left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m-r}}\right)^{l} \\ &+ \sum_{j=1}^{2r}\varepsilon_j\sum_{l=1}^{\infty}\kappa_{l+1}(\varepsilon_j G_{\tilde{\mu}_{m+r}}(z))^{l} + \frac{1}{G_{\tilde{\mu}_{m+r}}(z)}. \end{aligned}$$

Let us consider the sum $\sum_{l=1}^{\infty}\kappa_{l+1}(\varepsilon_j)^{l+1}(G_{\tilde{\mu}_{m+r}}(z))^{l}$, which is a function of $u_1(\varepsilon_j) = \varepsilon_j$ and $u_2(\underline{\varepsilon}_{2r}) = G_{\tilde{\mu}_{m+r}}(z)$. Thus, we introduce a function

$$F_j(\underline{u}_2(\underline{\varepsilon}_{2r})) = \sum_{l=1}^{\infty}\kappa_{l+1}(\varepsilon_j)^{l+1}(G_{\tilde{\mu}_{m+r}}(z))^{l}, \quad j = 1, \dots, 2r.$$

We apply Faà di Bruno formula (see Theorem 7.6) to find derivatives of $F_j(\underline{u}_2(\underline{\varepsilon}_{2r}))$. Let us introduce the following notations: $\underline{m}_2 = (m_1, m_2)$, $\underline{\psi}_2 = (\psi_1, \psi_2)$, $\underline{n}_{2r} = (n_1, \dots, n_{2r})$, $\underline{p}_{2r} = (p_1, \dots, p_{2r})$ with $\sum_{i=1}^{2r} p_i = r$,

$$\begin{aligned} A(\underline{p}_{2r}) &= (\{0,\dots,p_1\}\times\dots,\times\{0,\dots,p_{2r}\})\setminus(\{0\}\times\dots\times\{0\}), \\ C(r,r) &= \{(m_1,m_2)\in A(r,r) : m_1+m_2\le r\}. \end{aligned}$$

We define ψ_i as a map $\psi_i : A(\underline{p}_{2r}) \to \{0, 1, \dots, m_i\}$, $i = 1, 2$ and

$$\begin{aligned} V(\underline{m}_2) &= \Bigg\{\underline{\psi}_2 : \sum_{\underline{n}_{2r}\in A(\underline{p}_{2r})}\psi_i(\underline{n}_{2r}) = m_i, \quad (i = 1, 2); \\ &\qquad \sum_{\underline{n}_{2r}\in A(\underline{p}_{2r})} n_l(\psi_1(\underline{n}_{2r}) + \psi_2(\underline{n}_{2r})) = p_l, \quad (1\le l\le 2r)\Bigg\}. \end{aligned}$$

We define the functions:

$$B_{\psi_1}(\varepsilon_j) = 1, \quad B_{\psi_2}(u_2(\underline{\varepsilon}_{2r})) = \prod_{\underline{n}_{2r}\in A(\underline{p}_{2r})}\left\{\frac{1}{\psi_2(\underline{n}_{2r})!}\left(\frac{D_{\underline{\varepsilon}_{2r}}^{\underline{n}_{2r}}u_2(\underline{\varepsilon}_{2r})}{\prod_{l=1}^{2r} n_l!}\right)^{\psi_i(\underline{n}_{2r})}\right\}.$$

Faà di Bruno formula gives

$$\frac{\partial^s F_j(\varepsilon_j, G_{\tilde{\mu}_{m+r}})}{\partial\varepsilon_1^{p_1}\dots\partial\varepsilon_{2r}^{p_{2r}}} = \left(\prod_{j=1}^{2r} p_j!\right)\sum_{m_1+m_2\le r} D_{\underline{u}_2}^{\underline{m}_2}F_j(\varepsilon_j, G_{\tilde{\mu}_{m+r}})\sum_{\underline{\psi}_2\in V(\underline{m}_2)} B_{\psi_2}(G_{\tilde{\mu}_{m+r}}).$$

Let us note that the derivative of the order r ($n_i = p_i,\ i = 1, \ldots, 2r$) appears, when $m_1 = 0,\ m_2 = 1$, and $\psi_1 = 0,\ \psi_2 = 1$, thus

$$\begin{aligned}
\frac{\partial^r F_j(\varepsilon_j, G_{\tilde{\mu}_{m+r}})}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}} &= \left(\prod_{j=1}^{2r} p_j!\right) \sum_{\substack{m_1+m_2\le r\\(m_1,m_2)\neq(0,1)}} D_{\underline{u}_2}^{\underline{m}_2} F_j(\varepsilon_j, G_{\tilde{\mu}_{m+r}}) \sum_{\underline{\psi}_2\in V(\underline{m}_2)} B_{\psi_2}(G_{\tilde{\mu}_{m+r}}) \\
&+ \left(\prod_{j=1}^{2r} p_j!\right) \frac{\partial}{\partial u_2} F_j(\varepsilon_j, G_{\tilde{\mu}_{m+r}}) \sum_{\substack{\underline{\psi}_2\in V((0,1))\\(\psi_1,\psi_2)\neq(0,1)}} B_{\psi_2}(G_{\tilde{\mu}_{m+r}}) \\
&+ \sum_{l=1}^{\infty} l\kappa_{l+1}\varepsilon_j^{l+1} G_{\tilde{\mu}_{m+r}}^{l-1} \frac{\partial^r G_{\tilde{\mu}_{m+r}}}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}}.
\end{aligned}$$

We consider the sum $\sum_{l=2}^{\infty}\kappa_{l+1}\left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m-r}}\right)^l$ as a function $F(u_1(\underline{\varepsilon}_{2r}))$, where $u_1(\underline{\varepsilon}_{2r}) = G_{\tilde{\mu}_{m+r}}(z)$ and by Faà di Bruno formula we have

$$\begin{aligned}
\frac{\partial^r F(G_{\tilde{\mu}_{m+r}})}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}} &= \left(\prod_{j=1}^{2r} p_j!\right) \sum_{m_1\le r} \frac{d^{m_1}}{du_1^{m_1}} F(G_{\tilde{\mu}_{m+r}}) \sum_{\psi_1\in V(m_1)} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) \\
&= \left(\prod_{j=1}^{2r} p_j!\right) \sum_{\substack{m_1\le r\\m_1\neq 1}} \frac{d^{m_1}}{du_1^{m_1}} F(G_{\tilde{\mu}_{m+r}}) \sum_{\substack{\psi_1\in V(m_1)\\\psi_1\neq 1}} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) \\
&+ \left(\prod_{j=1}^{2r} p_j!\right) \frac{d}{du_1} F(G_{\tilde{\mu}_{m+r}}) \sum_{\substack{\psi_1\in V(m_1)\\\psi\neq 1}} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) \\
&+ \sum_{l=2}^{\infty} l\kappa_{l+1} \frac{G_{\tilde{\mu}_{m+r}}^{l-1}}{(m-r)^{l/2}} \frac{\partial^s G_{\tilde{\mu}_{m+r}}}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}}.
\end{aligned}$$

In the same way we find the derivatives for $1/G_{\tilde{\mu}_{m+r}}$:

$$\begin{aligned}
\frac{\partial^r G_{\tilde{\mu}_{m+r}}^{-1}}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}} &= \left(\prod_{j=1}^{2r} p_j!\right) \sum_{m_1\le s} \frac{d^{m_1}}{du_1^{m_1}} G_{\tilde{\mu}_{m+r}}^{-1} \sum_{\psi_1\in V(m_1)} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) \\
&= \left(\prod_{j=1}^{2r} p_j!\right) \sum_{\substack{m_1\le r\\m_1\neq 1}} \frac{d^{m_1}}{du_1^{m_1}} G_{\tilde{\mu}_{m+r}}^{-1} \sum_{\substack{\psi_1\in V(m_1)\\\psi_1\neq 1}} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) \\
&- \left(\prod_{j=1}^{2r} p_j!\right) \frac{1}{G_{\tilde{\mu}_{m+r}}^2} \sum_{\substack{\psi_1\in V(m_1)\\\psi\neq 1}} B_{\psi_1}(G_{\tilde{\mu}_{m+r}}) - \frac{1}{G_{\tilde{\mu}_{m+r}}^2} \frac{\partial^r G_{\tilde{\mu}_{m+r}}}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}}.
\end{aligned}$$

We find that

$$\frac{\partial^r G_{\tilde{\mu}_{m+r}}(z)}{\partial\varepsilon_1^{p_1}\ldots\partial\varepsilon_{2r}^{p_{2r}}} = \frac{f(\underline{\varepsilon}_{2r}, z)}{g(\underline{\varepsilon}_{2r}, z)},$$

where $f(z, \underline{\varepsilon}_{2r})$ depends on the derivatives of order $r-1$, thus, due to the induction assumption, $|f(z, \underline{\varepsilon}_{2r})|$ is bounded for $z \in K''$ and

$$g(\underline{\varepsilon}_{2r}, z) = 1 - \frac{1}{G^2_{\tilde{\mu}_{m+r}}(z)} + \sum_{j=1}^{2r} \sum_{l=1}^{\infty} l\kappa_{l+1} \varepsilon_j^{l+1} G^{l-1}_{\tilde{\mu}_{m+r}}(z) + \sum_{l=2}^{\infty} l\kappa_{l+1} \left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m-r}} \right)^{l-1}.$$

The following bounds hold:

$$\left| \sum_{l=1}^{\infty} l\kappa_{l+1} \varepsilon_j^{l+1} G^{l-1}_{\tilde{\mu}_{m+r}}(z) \right| \leq \frac{c_4(\mu, r)}{n}; \quad \left| \sum_{l=2}^{\infty} l\kappa_{l+1} \left(\frac{G_{\tilde{\mu}_{m+r}}(z)}{\sqrt{m-r}} \right)^{l-1} \right| \leq \frac{c_5(\mu, r)}{\sqrt{n}}. \quad (6.25)$$

Due to (6.24), (6.25) and the choice of n we have $|g(\underline{\varepsilon}_{2r}, z)| \geq c_6(\mu, r)\delta$.

Finally, we conclude that $\frac{\partial^r G_{\tilde{\mu}_{m+r}}(z)}{\partial \varepsilon_1^{p_1} \ldots \partial \varepsilon_{2r}^{p_{2r}}}$ is bounded for $z \in K''$, $|\varepsilon_j| \leq n^{-1/2}$, $j = 1, \ldots, 2r$, $m \geq n \geq N^*$. □

6.3 Proofs of Theorem 4.4, Corollary 4.5 and Corollary 4.6

We start this section with computing derivatives of $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}$. Let us remind

$$K'' := \{x + iy : x \in [-2 + 5\delta,\ 2 - 5\delta],\ |y| < \delta\sqrt{\delta}/2\}.$$

The extension $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$ is defined by (see (3.2))

$$z = \sum_{i=1}^{r} R_{D_{\varepsilon_i}\mu}(G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)) + G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) + \frac{1}{G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)}.$$

Taking into account the rescaling property of the R-transform we arrive at

$$z = \sum_{i=1}^{r} \varepsilon_i R_\mu(\varepsilon_i G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)) + G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) + \frac{1}{G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)}.$$

We set

$$F(\underline{\varepsilon}_r, z, G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)) := \sum_{i=1}^{r} \varepsilon_i R_\mu(\varepsilon_i G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)) + G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z) + \frac{1}{G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)} - z.$$

With that, we can find the derivatives of $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$ as solutions of the equations

$$D^\alpha F(\underline{\varepsilon}_r, z, G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)) \Big|_{\varepsilon_1 = \ldots = \varepsilon_r = 0} = 0, \quad |\alpha| \leq r.$$

Let us compute the first derivative of $G_{\omega\boxplus\mu_1^\varepsilon}(z)$ at $\varepsilon=0$, $z\in K''$:

$$\frac{\partial}{\partial\varepsilon}F(\varepsilon,z,G_{\omega\boxplus\mu_1^\varepsilon}(z))\Big|_{\varepsilon=0}=0,$$

hence

$$\Big[R_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))+\varepsilon R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\left(G_{\omega\boxplus\mu_1^\varepsilon}(z)+\varepsilon\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right) \\ +\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)-\frac{\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)}{G^2_{\omega\boxplus\mu_1^\varepsilon}(z)}\Big]\Big|_{\varepsilon=0}=0. \tag{6.26}$$

Thus, we have the following equation at $\varepsilon=0$:

$$\left(1-\frac{1}{G^2_\omega(z)}\right)\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0}=0$$

Due to Lemma 5.4, $G_\omega(K'')\subset D_{\theta,1.4}$, where $2\sin\theta=\sqrt{\frac{\delta}{4}\left(1-\frac{\delta}{4}\right)}$, hence $|G^2_\omega(z)|\le 1-\delta/16$ and

$$|G^2_\omega(z)-1|\ge\delta/16>0,\quad z\in K''. \tag{6.27}$$

Thus, we have $\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0}=0$. From (6.26) it follows that

$$\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)=\frac{R_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))+\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))}{\frac{1}{G^2_{\omega\boxplus\mu_1^\varepsilon}(z)}-\varepsilon^2R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))-1}.$$

Let us denote

$$\begin{aligned} g(\varepsilon):&=R_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))+\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z));\\ f(\varepsilon):&=\frac{1}{G^2_{\omega\boxplus\mu_1^\varepsilon}(z)}-\varepsilon^2R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))-1.\end{aligned}$$

We have

$$\frac{\partial^3}{\partial\varepsilon^3}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0}=\left[\frac{2g(\varepsilon)(f'(\varepsilon))^2}{f^3(\varepsilon)}-\frac{2f'(\varepsilon)g'(\varepsilon)}{f^2(\varepsilon)}-\frac{g(\varepsilon)f''(\varepsilon)}{f^2(\varepsilon)}+\frac{g''(\varepsilon)}{f(\varepsilon)}\right]\Big|_{\varepsilon=0}.$$

It is easy to see that $g(\varepsilon)\Big|_{\varepsilon=0}=0$ and

$$g'(\varepsilon)\Big|_{\varepsilon=0} \\ =\left(G_{\omega\boxplus\mu_1^\varepsilon}(z)+\varepsilon\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right)\left(2R'_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))+\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)R''_\mu(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\right)\Big|_{\varepsilon=0}=0.$$

Finally, we see that $\frac{\partial^3}{\partial\varepsilon^3}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0} = \frac{g''(\varepsilon)}{f(\varepsilon)}\Big|_{\varepsilon=0}$.

Let us compute $g''(\varepsilon)$:

$$\begin{aligned} g''(\varepsilon) &= 2\varepsilon R'(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\frac{\partial^2}{\partial\varepsilon^2}G_{\omega\boxplus\mu_1^\varepsilon}(z) \\ &+ \varepsilon^2\left(\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right)^2\left(3R''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)) + \varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)R'''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\right) \\ &+ G_{\omega\boxplus\mu_1^\varepsilon}(z)\left(3G_{\omega\boxplus\mu_1^\varepsilon}(z)R''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)) + \varepsilon^2\left(\frac{\partial^2}{\partial\varepsilon^2}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right)R''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\right. \\ &+ \left.\varepsilon G^2_{\omega\boxplus\mu_1^\varepsilon}(z)R'''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\right) + 4\left(\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right)R'(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)) \\ &+ 2\varepsilon\left(\frac{\partial}{\partial\varepsilon}G_{\omega\boxplus\mu_1^\varepsilon}(z)\right)G_{\omega\boxplus\mu_1^\varepsilon}(z)\left(4R''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)) + \varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z)R'''(\varepsilon G_{\omega\boxplus\mu_1^\varepsilon}(z))\right). \end{aligned}$$

We conclude $g''(\varepsilon)\Big|_{\varepsilon=0} = 3G_\omega^2(z)R''(0)$, where $R''(0) = 2\kappa_3$. Due to all these computations we conclude that

$$\frac{\partial^3}{\partial\varepsilon^3}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0} = \frac{6\kappa_3 G_\omega^4(z)}{1-G_\omega^2(z)}.$$

We carry on this scheme and compute all necessary derivatives:

$$\begin{aligned} \frac{\partial^4}{\partial\varepsilon^4}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0} &= \frac{4G_\omega^5(z)\left(12-6G_\omega^2(z)+6\kappa_4\left(G_\omega^2(z)-1\right)^2\right)}{\left(1-G_\omega^2(z)\right)^3}; \\ \frac{\partial^5}{\partial\varepsilon^5}G_{\omega\boxplus\mu_1^\varepsilon}(z)\Big|_{\varepsilon=0} &= \frac{5G_\omega^6(z)\left(\kappa_3(120-72G_\omega^2(z))+48\kappa_5\left(G_\omega^2(z)-1\right)^2\right)}{\left(1-G_\omega^2(z)\right)^3}; \\ \frac{\partial^4}{\partial\varepsilon_1^2\partial\varepsilon_2^2}G_{\omega\boxplus\mu_2^{(\varepsilon_2)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=0} &= \frac{8G_\omega^5(z)\left(2-G_\omega^2(z)\right)}{\left(1-G_\omega^2(z)\right)^3}; \\ \frac{\partial^5}{\partial\varepsilon_1^2\partial\varepsilon_2^3}G_{\omega\boxplus\mu_2^{(\varepsilon_2)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=0} &= \frac{12\kappa_3G_\omega^6(z)\left(5-3G_\omega^2(z)\right)}{\left(1-G_\omega^2(z)\right)^3}; \\ \frac{\partial^6}{\partial\varepsilon_1^3\partial\varepsilon_2^3}G_{\omega\boxplus\mu_2^{(\varepsilon_2)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=0} &= \frac{72\kappa_3^2G_\omega^7(z)\left(3-2G_\omega^2(z)\right)}{\left(1-G_\omega^2(z)\right)^3}; \end{aligned}$$

$$\begin{aligned} &\frac{\partial^7}{\partial\varepsilon_1^3\partial\varepsilon_2^4}G_{\omega\boxplus\mu_2^{(\varepsilon_2)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=0} \\ &= -\frac{144\kappa_3G_\omega^8(z)\left(5\kappa_4G_\omega^6(z)-G_\omega^4(z)\left(17\kappa_4+6\right)+G_\omega^2(z)\left(21+19\kappa_4\right)-7\left(\kappa_4+3\right)\right)}{\left(1-G_\omega^2(z)\right)^5}; \end{aligned}$$

$$\frac{\partial^7}{\partial \varepsilon_1^3 \partial \varepsilon_2^2 \partial \varepsilon_3^2} G_{\omega \boxplus \mu_3^{(\varepsilon_3)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0} = \frac{144\kappa_3 G_\omega^8(z)\,(7 - 7G_\omega^2(z) + 2G_\omega^4(z))}{(1 - G_\omega^2(z))^5};$$

$$\frac{\partial^9}{\partial \varepsilon_1^3 \partial \varepsilon_2^3 \partial \varepsilon_3^3} G_{\omega \boxplus \mu_3^{(\varepsilon_3)}}(z)\Big|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0} = \frac{1296\kappa_3^3 G_\omega^{10}(z)\,(12 - 15G_\omega^2(z) + 5G_\omega^4(z))}{(1 - G_\omega^2(z))^5}.$$

These symbolic computations can be done, for example, in Mathematica.

Proof of Theorem 4.4. In order to compute the expansion for G_{μ_n} we apply Theorem 2.3. Recall that

$$h_\infty(\underline{\varepsilon}_r) := G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z), \quad z \in K'', \quad |\varepsilon_j| \le n^{-1/2}, \quad j = 1, \dots, r.$$

The extension $G_{\tilde{\mu}_{m+r}}$ is symmetric and compatible, thus conditions (2.2), (2.3) hold. Due to Theorem 4.1 the extension $G_{\tilde{\mu}_{m+r}}$ is infinitely differentiable with respect to $\underline{\varepsilon}_{r+q}$, $z \in K''$, $m \ge n \ge N^*$. Moreover, Theorem 4.3 guarantees that conditions (2.9) and (2.10) hold, namely, $G_{\tilde{\mu}_{m+r}}$ has derivatives uniformly bounded in absolute value up to order $s \ge 1$ for $z \in K''$, $m \ge n \ge N^*$. Theorem 4.2 shows that condition (2.4) holds. Therefore, we can deduce the expansion terms and estimates for the error term based on (2.11). In order to get the expansion for $G_{\mu_n}(z)$, $z \in K''$, $n \ge N$ we need to compute the derivatives of $G_{\omega \boxplus \mu_r^{(\varepsilon_r)}}(z)$, $z \in K''$ at zero and plug them into (2.12). We found all derivatives in the beginning of this section, plugging them into (2.11) we obtain:

$$\begin{aligned} G_{\mu_n}(z) &= G_\omega(z) + \frac{\kappa_3 G_\omega^4(z)}{(1 - G_\omega^2(z))\sqrt{n}} \\ &+ \left((\kappa_4 - \kappa_3^2)\frac{G_\omega(z)^5}{1 - G_\omega^2(z)} + \kappa_3^2\Big(\frac{G_\omega(z)^7}{(1 - G_\omega(z)^2)^2} + \frac{G_\omega(z)^5}{(1 - G_\omega(z)^2)^3}\Big)\right)\frac{1}{n} \\ &- \left(\frac{\kappa_5 G_\omega^6(z)}{(G_\omega^2(z) - 1)} + \frac{\kappa_3^3 G_\omega^{10}(z)\,(5G_\omega^4(z) - 15G_\omega^2(z) + 12)}{(G_\omega^2(z) - 1)^5} \right. \\ &- \left. \frac{\kappa_3\kappa_4 G_\omega^8(z)\,(5G_\omega^2(z) - 7)}{(G_\omega^2(z) - 1)^3} \right) \frac{1}{n^{3/2}} + O\left(\frac{1}{n^2}\right) \end{aligned} \tag{6.28}$$

for $z \in K''$, $n \ge c(\mu)\delta^{-4}$. □

Proof of Corollary 4.5. In order to get the expansion for the densities we have to substitute the extension $G_\omega(z)$ by expression (3.14) on the left-hand side of (6.28) and after this, due to Stieltjes inversion formula (3.1), we take imaginary part. These symbolic computations can be done, for example, in Mathematica. Finally, we obtain the desired expansion for densities. □

Proof of Corollary 4.6. We integrate the expansion for densities and obtain the desired expansion for distributions. □

Chapter 7

Auxiliary results

Theorem 7.1 ([48]). *Consider vector spaces X, Y over $\mathbb{R}$ and a sequence $\{f_n\}_n$ of functions $f_n : A \to Y$, $A \subset X$. If all functions f_n are differentiable on A and the sequence $\{f_n'\}_n$ converges uniformly on A, and if the sequence $\{f_n\}_n$ converges at one point $x_0 \in A$, then $\{f_n\}_n$ converges to f uniformly on A. Moreover, f is differentiable and $f'(x) = \lim_{n\to\infty} f_n'(x)$, $x \in A$.*

Theorem 7.2 (Rouché's theorem, [26]). *Let the functions f and g be analytic in the simply connected region D, let Γ be a Jordan curve in D, and let $|f(z)| > |g(z)|$ for all $z \in \Gamma$. Then the functions $f + g$ and f have the same number of zeros in the interior of Γ.*

Theorem 7.3 (Cauchy inequalities, [36]). *Let $f(z)$ be an analytic function on a domain G, and suppose G contains the circle $\gamma_\rho : |z - z_0| = \rho$ and its interior $I(\gamma_\rho)$. Then*

$$|f^{(n)}(z_0)| \le n!\frac{M(\rho)}{\rho^n} \quad (n = 0, 1, \ldots),$$

where $M(\rho) = \max_{z\in\gamma_\rho} |f(z)|$.

Theorem 7.4 (Newton-Kantorovich, [29]). *Consider vector spaces X, Y over $\mathbb{C}$ and a functional equation $F(x) = 0$, where $F : X \to Y$. Assume that the conditions hold:*

(1) *F is differentiable at $x_0 \in X$, $\|F'(x_0)^{-1}\|_Y \le \beta_0$.*

(2) *x_0 solves approximately $F(x) = 0$ with the estimate $\|F'(x_0)^{-1}F(x_0)\|_Y \le \eta_0$.*

(3) *The second derivative $F''(x)$ is bounded in B_0 (see below): $\|F''(x)\|_Y \le K_0$.*

(4) *β_0, η_0, K_0 satisfy the inequality $h_0 = \beta_0\eta_0K_0 \le \frac{1}{2}$.*

Then there is a unique root x^* of F in $B_0 := \{x \in X : \ \|x - x_0\|_X \le \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0\}$.

Theorem 7.5 (Implicit function theorem, [23]). *Let $B \subset \mathbb{C}^{r+1} \times \mathbb{C}$ be an open set, $F : B \to \mathbb{C}$ an analytic mapping, and $(z_0, w_0) \in B$ a point with $F(z_0, w_0) = 0$ and*

$$\det\left(\frac{\partial F}{\partial z_{r+2}}(z_0, w_0)\right) \neq 0.$$

Then there is an open neighborhood $U = U' \times U'' \subset B$ and an analytic map $g : U' \to U''$ such that $\{(z, w) \in U' \times U'' : F(z, w) = 0\} = \{(z, g(z)) : z \in U'\}$.

Multivariate Faà di Bruno formula. The multivariate Faà di Bruno formula is a formula for pth derivatives $G^{(p)}(\underline{z}_N)$ of composite function $G(\underline{z}_N) = F(\underline{u}_M(\underline{z}_N))$. Let us introduce the following notations: $\underline{m}_M = (m_1, \ldots, m_M)$, $\underline{\psi}_M = (\psi_1, \ldots, \psi_M)$, $\underline{n}_N = (n_1, \ldots, n_N)$, $\underline{p}_N = (p_1, \ldots, p_N)$ with $\sum_{i=1}^N p_i = p$,

$$\begin{aligned} A(\underline{p}_N) &= (\{0, \ldots, , p_1\} \times \ldots, \times \{0, \ldots, , p_N\}) \backslash (\{0\} \times \ldots \times \{0\}), \\ C(p, \ldots, p) &= \left\{ \underline{m}_M \in A(p, \ldots, p) : \sum_{l=1}^M m_l \leq p \right\}. \end{aligned}$$

We define ψ_i as a map $\psi_i : A(\underline{p}_N) \to \{0, 1, \ldots, m_i\}$ and

$$\begin{aligned} V(\underline{m}_M) = \Bigg\{ \underline{\psi}_M : & \sum_{\underline{n}_N \in A(\underline{p}_N)} \psi_i(\underline{n}_N) = m_i, \quad (1 \leq i \leq M); \\ & \sum_{\underline{n}_N \in A(\underline{p}_N)} n_l \sum_{j=1}^M \psi_j(\underline{n}_N) = p_l, \quad (1 \leq l \leq N) \Bigg\}. \end{aligned}$$

Theorem 7.6 (Multivariate Faà di Bruno formula, [33]). *Suppose $F(\underline{u}_M)$ is $C^{(p+1)}$ and $u_i(\underline{z}_N)$ $(i = 1, \ldots, M)$ have continuous derivatives to order $(p_1 + 1, \ldots, p_N + 1)$ on appropriate domains. Define*

$$B_{\psi_i}(u_i(\underline{z}_N)) = \prod_{\underline{n}_N \in A(\underline{p}_N)} \left\{ \frac{1}{\psi_i(\underline{n}_N)!} \left(\frac{D_{\underline{z}_N}^{\underline{n}_N} u_i(\underline{z}_N)}{\prod_{l=1}^N n_l!} \right)^{\psi_i(\underline{n}_N)} \right\} \quad (1 \leq i \leq M).$$

If $G(z) = F(\underline{u}_M(\underline{z}_N))$ and $\underline{p}_N \neq \underline{0}$, then

$$\frac{\partial^p G(\underline{z}_N)}{\partial z_1^{p_1} \ldots \partial z_N^{p_N}} = \left(\prod_{j=1}^N p_j!\right) \sum_{\underline{m}_M \in C(p, \ldots, p)} D_{\underline{u}_M}^{\underline{m}_M} F \sum_{\underline{\psi}_M \in V(\underline{m}_M)} \prod_{l=1}^M B_{\psi_l}(u_l(\underline{z}_N)).$$

Chapter 8

Appendix

Below the proofs of some results from Chapter 2 are presented.

8.1 Proof of Proposition 2.1

Proof of Proposition 2.1. As before, we denote $\underline{\varepsilon}_m := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m$, where if not specified otherwise $\varepsilon_1 = \ldots = \varepsilon_m = m^{-1/2}$. Let us denote $\underline{\sigma}_2 := (\sigma_1, \sigma_2) \in \mathbb{R}^2$ such that $|\sigma_j| \le n^{-1/2}$, $j = 1, 2$, $m \ge n > 3$. We will identify $(\underline{\varepsilon}_m, \underline{\sigma}_2)$, and $(\underline{\varepsilon}_m, 0, \underline{\sigma}_2) \in \mathbb{R}^{m+3}$. In particular, notice that

$$h_{m+3}(\underline{\varepsilon}_m, 0, \underline{\sigma}_2) = h_{m+2}(\underline{\varepsilon}_m, \underline{\sigma}_2).$$

We will also use the following notation

$$h_m(\underline{\varepsilon}_{m-k}) := h_m(\underline{\varepsilon}_{m-k}, \underbrace{0, \ldots, 0}_{k}), \quad m \ge k > 0.$$

Now we expand the function $h_{m+3}(\underline{\varepsilon}_{m+1}, \underline{\sigma}_2)$ at the point $(\underline{\varepsilon}_m, \underline{\sigma}_2)$ and get

$$\begin{aligned} & h_{m+3}(\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) \\ = \quad & h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) + \sum_{|\alpha| \le 2} \alpha!^{-1} D^\alpha h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)((\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) - (\underline{\varepsilon}_m, \underline{\sigma}_2))^\alpha + R_3(m), \end{aligned} \tag{8.1}$$

where $R_3(m)$ is a remainder in the Lagrange form:

$$R_3(m) = \frac{1}{3!} \left(t_1 \frac{\partial}{\partial \varepsilon_1} + \ldots + t_{m+1} \frac{\partial}{\partial \varepsilon_{m+1}} \right)^3 h_{m+1}(\underline{\varepsilon}_{m+1} - \theta \underline{t}_{m+1}), \tag{8.2}$$

where $t_j = m^{-1/2} - (m+1)^{-1/2}$, $j = 1, \ldots, m$, $t_{m+1} = m^{-1/2}$ and $0 < \theta < 1$. We can deduce the estimate for $R_3(m)$ from $|m^{-1/2} - (m+1)^{-1/2}| \le cm^{-3/2}$ and counting number of terms in (8.2):

$$|R_3(m)| \le cd_3(h, n) m^{-3/2}, \quad m \ge n > s. \tag{8.3}$$

We rewrite (8.1) in the following way:

$$\begin{aligned} & h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) - h_{m+3}(\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) \\ & \quad = \; -\sum_{|\alpha|\le 2} \alpha!^{-1} D^\alpha h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)((\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) - (\underline{\varepsilon}_m, \underline{\sigma}_2))^\alpha - R_3(m). \end{aligned} \tag{8.4}$$

The next step is expanding the derivatives on the right-hand side and making use of condition (2.4). We start with the second mixed derivatives in (8.4)

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon_j}\frac{\partial}{\partial \varepsilon_k} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) \\ & \quad = \frac{\partial}{\partial \varepsilon_j}\frac{\partial}{\partial \varepsilon_k} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_j=\varepsilon_k=0} + O(d_3(h,n)m^{-1/2}) = O(d_3(h,n)m^{-1/2}), \quad j \neq k. \end{aligned}$$

The other derivatives in (8.4) have the expansions

$$\begin{aligned} \frac{\partial}{\partial \varepsilon_j} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) &= \frac{\partial^2}{\partial \varepsilon_j^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_j=0} m^{-1/2} + O(d_3(h,n)m^{-1}), \\ \frac{\partial^2}{\partial \varepsilon_j^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) &= \frac{\partial^2}{\partial \varepsilon_j^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_j=0} + O(d_3(h,n)m^{-1/2}). \end{aligned}$$

Replacing the derivatives in (2.4) by their expansions we obtain

$$\begin{aligned} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) \; &- \; h_{m+3}(\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) \\ &= \sum_{j=1}^m \frac{\partial^2}{\partial \varepsilon_j^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_j=0} \left[m^{-1/2}\left(m^{-1/2} - (m+1)^{-1/2}\right)\right. \\ &- \left.\frac{1}{2}\left(m^{-1/2} - (m+1)^{-1/2}\right)^2\right] \\ &- \frac{1}{2}(m+1)^{-1} \frac{\partial^2}{\partial \varepsilon_{m+1}^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_{m+1}=0} + O\left(d_3(h,n)m^{-3/2}\right) \\ &= \sum_{j=1}^m \frac{\partial^2}{\partial \varepsilon_j^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_j=0} \left[\frac{1}{2}(m^{-1} - (m+1)^{-1})\right] \\ &- \frac{1}{2}(m+1)^{-1} \frac{\partial^2}{\partial \varepsilon_{m+1}^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_{m+1}=0} + O\left(d_3(h,n)m^{-3/2}\right). \end{aligned}$$

Since the function $h_{m+3}(\cdot)$ is symmetric we arrive at

$$\begin{aligned} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2) \; - \; h_{m+3}(\underline{\varepsilon}_{m+1}, \underline{\sigma}_2) \; &= \; \frac{1}{2}(m+1)^{-1} \frac{\partial^2}{\partial \varepsilon_1^2} h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_1=0} \\ - \; \frac{1}{2}(m+1)^{-1} \frac{\partial^2}{\partial \varepsilon_{m+1}^2} & h_{m+3}(\underline{\varepsilon}_m, \underline{\sigma}_2)\Big|_{\varepsilon_{m+1}=0} + O\left(d_3(h,n)m^{-3/2}\right). \end{aligned} \tag{8.5}$$

In order to eliminate zero at the $(m+1)$st place of $\frac{\partial^2}{\partial\varepsilon_j^2}h_{m+3}(\underline{\varepsilon}_m,\underline{\sigma}_2)\Big|_{\varepsilon_j=0}$, $j=1,\dots,m$ we apply the Taylor series in the following way:

$$\frac{\partial^2}{\partial\varepsilon_j^2}h_{m+3}(\underline{\varepsilon}_m,\underline{\sigma}_2)\Big|_{\varepsilon_j=0}=\frac{\partial^2}{\partial\varepsilon_j^2}h_{m+3}(\underline{\varepsilon}_m,\underline{\sigma}_2)\Big|_{\varepsilon_j=0,\ \varepsilon_{m+1}=m^{-1/2}}+O\left(d_3(h,n)m^{-1/2}\right). \quad (8.6)$$

Pluging (8.6) into (8.5) and using the symmetry condition we conclude

$$h_{m+3}(\underline{\varepsilon}_m,\underline{\sigma}_2)-h_{m+3}(\underline{\varepsilon}_{m+1},\underline{\sigma}_2)=O\left(d_3(h,n)m^{-3/2}\right).$$

It is easy to see that

$$h_{m+k+2}(\underline{\varepsilon}_{m+k},\underline{\sigma}_2)-h_{m+k+3}(\underline{\varepsilon}_{m+k+1},\underline{\sigma}_2)=O\left(d_3(h,n)(m+k)^{-3/2}\right).$$

Summing up these differences for $r\geq m$, we obtain

$$\sum_{k=0}^{r-1}(h_{m+k+2}(\underline{\varepsilon}_{m+k},\underline{\sigma}_2)-h_{m+k+3}(\underline{\varepsilon}_{m+k+1},\underline{\sigma}_2))=O\left(d_3(h,n)\right)\sum_{k=0}^{r-1}(m+k)^{-3/2}.$$

Hence,

$$h_{m+2}(\underline{\varepsilon}_m,\underline{\sigma}_2)-h_{m+r+2}(\underline{\varepsilon}_{m+r},\underline{\sigma}_2)=O\left(d_3(h,n)\right)\sum_{k=0}^{r-1}(m+k)^{-3/2}. \quad (8.7)$$

Finally, (8.7) shows that $h_{m+2}(\underline{\varepsilon}_m,\underline{\sigma}_2)$, $m=n,n+1,\dots$ is a Cauchy sequence in m with a limit which we denote by $h_\infty(\underline{\sigma}_2)$, $|\sigma_j|\leq n^{-1/2}$, $j=1,2$. Taking $m=n$ and letting $r\to\infty$ in (8.7) we obtain

$$h_{n+2}(n^{-1/2},\dots,n^{-1/2},\underline{\sigma}_2)-h_\infty(\underline{\sigma}_2)=O\left(d_3(h,n)n^{-1/2}\right),$$

which proves the proposition. □

8.2 Proof of Theorem 2.3

The following lemma describes the procedure of eliminating zeros like the one that is used in (8.6). The lemma shows that additional variables can be introduced (according to the compatibility property of h_m). Then we can differentiate with respect to the additional variables at zero instead of differentiating with respect to ε_j, $j=1,...,m+1$.

Lemma 8.1. *Suppose that conditions* (2.2) – (2.4) *hold. Then*

$$\sum_{j=1}^{k}\frac{\partial^j}{\partial\varepsilon^j}h_{m+1}(\varepsilon,\varepsilon_2,\dots,\varepsilon_{m+1})\Big|_{\varepsilon=0}j!^{-1}(\eta^j-\varepsilon^j) \quad (8.8)$$

$$=\sum_{r=1}^{k}\widetilde{P}_r\left((\eta^{\cdot}-\varepsilon^{\cdot})\kappa_{\cdot}(D)\right)h_{m+1+r}(\lambda_1,\dots,\lambda_r,\varepsilon,\varepsilon_2,\dots,\varepsilon_{m+1})\Big|_{\lambda_1=\ldots=\lambda_r=0}+O(m^{-k/2}),$$

where the differential operators $\widetilde{P}_r$ *and* κ_p *are defined in* (8.9) *below and* (2.5)*, and*

$$(\eta^{\cdot} - \varepsilon^{\cdot})\kappa_{\cdot}(D) := ((\eta^p - \varepsilon^p)\kappa_p(D), \quad p = 1, \ldots, r).$$

Proof. The differential operators $\widetilde{P}_r(\tau_{\cdot}\kappa_{\cdot})$ are polynomials in the cumulant operators κ_p (see (2.5)) multiplied by formal variables τ_p, $p = 1, \ldots, r$. These polynomials are defined by the formal power series in τ_p

$$\sum_{j=0}^{\infty} \widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot}(D))\mu^j = \exp\left(\sum_{j=2}^{\infty} j!^{-1}\tau_j\kappa_j(D)\mu^j\right). \tag{8.9}$$

When $\tau_j = \tau^j, \ j \geq 1$, then due to (2.5) we have

$$\sum_{j=0}^{\infty} \widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot}(D)) = 1 + \sum_{j=2}^{\infty} j!^{-1}\tau_j D^j.$$

Hence, $\widetilde{P}_0(\tau_{\cdot}\kappa_{\cdot}(D)) = 1$, $\widetilde{P}_1(\tau_{\cdot}\kappa_{\cdot}(D)) = 0$ and $\widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot}(D)) = j!^{-1}\tau_j D^j$, $j \geq 2$, which means that the differential operators $\widetilde{P}_r$ are nothing else than derivatives of order r multiplied by $r!^{-1}$ and the corresponding power of the formal variable τ_r. It easy to see that $\widetilde{P}_r$ gives the rth term in the Taylor expansion so that we can write

$$h_m(\underline{\varepsilon}_m) = \sum_{j=0}^{\infty} \widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot}(D))h_m(\underline{\varepsilon}_m)\Big|_{\varepsilon_i=0}, \quad i = 1, \ldots, m.$$

Notice that $\widetilde{P}_r$ depends on the cumulant differential operators $\kappa_{\cdot}(D)$. These operators consist of derivatives with respect to multi-variables, for instance $\kappa_4(D) = D^4 - 3D^2D^2$. Here D^2D^2 denotes differentiation with respect to two different variables (we do not need to specify the variables because of the symmetry condition). Therefore, we introduce additional variables, say λ, and write

$$h_m(\underline{\varepsilon}_m) = \sum_{j=0}^{\infty} \widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot}(D))h_{m+j}(\lambda_1, \ldots, \lambda_j, \underline{\varepsilon}_m)\Big|_{\lambda_1=\ldots=\lambda_j=\varepsilon_i=0}, \quad i = 1, \ldots, m.$$

The advantage of the operators $\widetilde{P}_r$ is that they are defined by exponents which can be easily reordered by the properties of exponential functions. Due to (8.9) and the multiplication theorem for exponential functions we obtain

$$\sum_{j+l=r} \widetilde{P}_j(\tau_{\cdot}\kappa_{\cdot})\widetilde{P}_l(\tau'_{\cdot}\kappa_{\cdot}) = \widetilde{P}_r((\tau_{\cdot} + \tau'_{\cdot})\kappa_{\cdot}) \quad (\tau_{\cdot} = (\tau_1, \ldots, \tau_r)).$$

In order to prove the theorem we start from the right-hand side of (8.8):

$$\begin{aligned}
& \sum_{r=1}^{k} \widetilde{P}_r((\eta^{\cdot} - \varepsilon^{\cdot})\kappa_{.}(D)) h_{m+1+r}(\lambda_1, \ldots \lambda_r, \varepsilon, \varepsilon_2, \ldots, \varepsilon_{m+1})\Big|_{\lambda_1=\ldots=\lambda_r=0} \\
= & \sum_{r=1}^{k} \widetilde{P}_r((\eta^{\cdot} - \varepsilon^{\cdot})\kappa_{.}(D)) \sum_{l=0}^{k-r} \widetilde{P}_l(\varepsilon^{\cdot}\kappa_{.}(D)) \\
\times & \; h_{m+1+l+r}(\lambda_1, \ldots, \lambda_{l+r}, \varepsilon, \varepsilon_2, \ldots, \varepsilon_{m+1})\Big|_{\lambda_1=\ldots=\lambda_{l+r}=\varepsilon=0} + O(m^{-k/2}) \\
= & \sum_{j=1}^{k} \sum_{\substack{l+r=j \\ r\geq 1}} \widetilde{P}_r((\eta^{\cdot} - \varepsilon^{\cdot})\kappa_{.}(D)) \widetilde{P}_l(\varepsilon^{\cdot}\kappa_{.}(D)) \\
\times & \; h_{m+1+j}(\lambda_1, \ldots, \lambda_j, \varepsilon, \varepsilon_2, \ldots, \varepsilon_{m+1})\Big|_{\lambda_1=\ldots=\lambda_j=\varepsilon=0} + O(m^{-k/2}) \\
= & \sum_{j=1}^{k} \left(\widetilde{P}_j(\eta^{\cdot}\kappa_{.}(D) - \widetilde{P}_j(\varepsilon^{\cdot}\kappa_{.}(D)\right) h_{m+1+j}(\lambda_1, \ldots, \lambda_j, \varepsilon, \varepsilon_2, \ldots, \varepsilon_{m+1})\Big|_{\lambda_1=\ldots=\lambda_j=\varepsilon=0} \\
+ & \; O(m^{-k/2}) \\
= & \sum_{j=1}^{k} \frac{\partial^j}{\partial \varepsilon^j} h_{m+1}(\varepsilon, \varepsilon_2, \ldots, \varepsilon_{m+1})\Big|_{\varepsilon=0} j!^{-1}(\eta^j - \varepsilon^j) + O(m^{-k/2}).
\end{aligned}$$

The last expression coincides with the left-hand side in (8.8), thus the theorem is proved. □

Proof of Theorem 2.3. The theorem will be proved by induction on the length of the expansion, starting with $s = 3$. The case $s = 3$ was shown in Proposition 2.1. Assume that $m \geq n$ $(n \geq 1)$. We start with the expansion

$$\begin{aligned}
& h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \\
& \quad = \; - \sum_{0<|\alpha|<s} \alpha!^{-1} D^{\alpha} h_{m+1}(\underline{\varepsilon}_m)(\underline{\varepsilon}_{m+1} - \underline{\varepsilon}_m)^{\alpha} + R_s(m),
\end{aligned} \tag{8.10}$$

where

$$|R_s(m)| \leq C d_s(h, n) m^{-s/2}. \tag{8.11}$$

The last inequality is similar to inequality (8.3) in the proof of Proposition 2.1.

In order to apply condition (2.4) on the first derivatives we expand $D^{\alpha} h_{m+1}(\underline{\varepsilon}_m)$, $\alpha = (\alpha_{j_1}, \ldots, \alpha_{j_p})$, $1 \leq j_1 < \ldots < j_p \leq m+1$, around $\varepsilon_{j_r} = 0$, $r = 1, \ldots, p$. This yields

$$D^{\alpha} h_{m+1}(\underline{\varepsilon}_m) \; = \sum_{0<|\alpha|+|\beta|<s} D^{\alpha+\beta} h_{m+1}(\underline{\varepsilon}_m^*) \underline{\varepsilon}_m^{\beta} \beta!^{-1} + \widetilde{R}_s(m), \tag{8.12}$$

where $\widetilde{R}_s(m)$ satisfies inequality (8.11), $\underline{\varepsilon}_m^*$ is equal to $\underline{\varepsilon}_m$ except for the components $\varepsilon_{j_1}, \dots, \varepsilon_{j_p}$, which are zero, and β is a vector of partial derivatives in the components $j_1, \dots, j_p$. Rewrite the derivatives in (8.10) by their expressions from (8.12)

$$\begin{aligned} & h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \\ &= - \sum_{0<|\alpha|+|\beta|<s} \alpha!^{-1} \beta!^{-1} D^{\alpha+\beta} h_{m+1}(\underline{\varepsilon}_m^*)(\underline{\varepsilon}_{m+1} - \underline{\varepsilon}_m)^{\alpha} \underline{\varepsilon}_m^{\beta} + \widetilde{\widetilde{R}}_s(m), \end{aligned}$$

where $\widetilde{\widetilde{R}}_s(m)$ denotes a remainder term satisfying (8.3).

Let $\varepsilon_{m,j} = m^{-1/2}$ and $\varepsilon_{m+1,j} = (m+1)^{-1/2}$, $j = 1, \dots m+1$, but $\varepsilon_{m,m+1} = 0$. Using the following relation (see Lemma 8.2 below)

$$\sum_{\substack{j+k=r \\ j\geq 1}} j!^{-1} k!^{-1} (\varepsilon - \eta)^j \eta^k = r!^{-1} (\varepsilon^r - \eta^r), \quad r \geq 1,\ k \geq 0,$$

we then obtain

$$h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) = -\sum_{0<|\gamma|<s} \gamma!^{-1} D^{\gamma} h_{m+1}(\underline{\varepsilon}_m^*) \prod_{j=1}^{m+1}{}^{*} (\varepsilon_{m+1,j}^{\gamma_j} - \varepsilon_{m,j}^{\gamma_j}) + \widetilde{\widetilde{R}}_s(m), \quad (8.13)$$

where $\gamma = (\gamma_1, \dots, \gamma_{m+1})$, $\prod^*$ denotes multiplication over all $\gamma_j > 0$, $j = 1, \dots, m+1$.

The next step is replacing $\underline{\varepsilon}_m^*$ by $\underline{\varepsilon}_m$ in (8.13). For this purpose we apply Lemma 8.1 to each partial derivative $\gamma_j > 0$. More precisely, we will take further derivatives with respect to additional variables at zero and make use of the symmetry condition. Introduce the notation

$$\Delta_{m,j}^{\cdot} := (\varepsilon_{m+1,j}^p - \varepsilon_{m,j}^p, p = 1, \dots, s-1).$$

Applying Lemma 8.1 to the derivatives in (8.13) we arrive at

$$\begin{aligned} & h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \quad (8.14) \\ &= -\sum_{k=1}^{m+1} \sum_{(r)}{}^{*} \widetilde{P}_{r_1}\left(\Delta_{m,j_1}^{\cdot} \kappa_{\cdot}(D)\right) \dots \widetilde{P}_{r_k}\left(\Delta_{m,j_k}^{\cdot} \kappa_{\cdot}(D)\right) h_{m+1+r}(\underline{\varepsilon}_m, 0, \dots, 0) + R_{1,s}(m), \end{aligned}$$

where $\sum_{(r)}^*$ means summation over all combinations of $r_1, \dots, r_k \geq 1$, $k = 1, \dots, m+1$, such that $r = r_1 + \dots + r_k < s$ and all ordered k-tuples $(j_1, \dots, j_k)$ of indices $1 \leq j_r \leq m+1$ without repetition. Note that the derivatives on the right-hand side of (8.14) define due to conditions (2.9) and (2.10). The remainder term $R_{1,s}(m)$ satisfies (8.3). It easy to see that such a procedure changes nothing for the $(m+1)$st component because the derivatives $\frac{\partial^j}{\partial \varepsilon_{m+1}^j} h_{m+1}(\underline{\varepsilon}_m)$ are extended at the same point $\underline{\varepsilon}_m$. Relation

(8.14) serves as the induction step in the induction on the length of the expansion, say l.

Assume that conditions (2.2) - (2.4) and (2.9) - (2.10) hold with $(s+q)$ instead of s. Assume we have already proved that for $l = 3, \ldots, s-1$, $m \geq n$, and $|\alpha| \leq s+q$ we have

$$D^\alpha h_{m+r}(m^{-1/2}, \ldots, m^{-1/2}, \varepsilon_1, \ldots, \varepsilon_r)\Big|_{\varepsilon_1=\ldots=\varepsilon_r=0} \tag{8.15}$$
$$= \sum_{j=0}^{l-3} m^{-j/2} P_j(\kappa_{\cdot}(D)) D^\alpha h_\infty(\lambda_1, \ldots, \lambda_j, \varepsilon_1, \ldots, \varepsilon_r)\Big|_{\lambda_1=\ldots=\lambda_j=\varepsilon_1=\ldots=\varepsilon_r=0} + R_{2,l}(m),$$

where $R_{2,l}(m)$ satisfies

$$|R_{2,l}(m)| \leq c(s) B m^{-(l-2)/2}. \tag{8.16}$$

The case $l = 3$ follows from Proposition 2.1, where

$$h_m(\cdot) = D^\alpha h_{m+r}(\cdot, \varepsilon_1, \ldots, \varepsilon_r)\Big|_{\varepsilon_1=\ldots=\varepsilon_r=0},$$

which satisfies conditions $(2.2)-(2.4)$ and (2.8) with $q = 3$.

In order to prove (8.15) for $l = s$, observe that (8.14) starts with $m+1$ terms of order $O(m^{-3/2})$. The induction assumption (8.15) with $|\alpha| = 0$ applied to the terms of (8.14) yields

$$h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \tag{8.17}$$
$$= -\sum_{k=1}^{m+1} \sum_{(r)}{}^{**} \widetilde{P}_{r_1}\left(\Delta^{\cdot}_{m,j_1}\kappa_{\cdot}(D)\right) \ldots \widetilde{P}_{r_k}\left(\Delta^{\cdot}_{m,j_k}\kappa_{\cdot}(D)\right) m^{-r_0/2} P_{r_0}(k_{\cdot}(D))$$
$$\times \; h_\infty(\lambda_1, \ldots, \lambda_{r_0}, \varepsilon_1, \ldots, \varepsilon_r)\Big|_{\underline{\lambda}=\underline{\varepsilon}=0} + R_{3,s}(m),$$

where $R_{3,s}(m)$ satisfies (8.16) with $l = s+2$, and $\sum_{(r)}^{**}$ denotes summation over all indices $r_1, \ldots, r_k \geq 1$, $r_0 \geq 0$ such that $r_0 + \ldots + r_k \leq s$ and all ordered k-tuples $(j_1, \ldots, j_k)$ of indices without repetition.

By definition (8.9) of $\widetilde{P}_r$, the following formal identity holds:

$$\sum_{j=1}^{\infty} \widetilde{P}_j((\eta^{\cdot} - \varepsilon^{\cdot})\kappa_{\cdot}(D)) = \exp\left(\sum_{j=2}^{\infty} j!^{-1}(\eta^j - \varepsilon^j)\kappa_j(D)\right) - 1. \tag{8.18}$$

In order to apply this identity to (8.17) we need to change the order of summation in (8.17) in the following way

$$h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \tag{8.19}$$
$$= -\sum_{r_0=0}^{s-4} m^{-r_0/2} P_{r_0}(\kappa_{\cdot}(D)) \sum_{k=1}^{m+1} \sum_{(j)}{}^{*} \left[\prod_{l=1}^{s-r_0} \left\{\sum_{v_l=1}^{\infty} \widetilde{P}_{v_l}\left(\Delta^{\cdot}_{m,j_l}\kappa_{\cdot}(D)\right)\right\}\right]_{s-r_0} h_\infty + R_{3,s}(m),$$

where $h_\infty := h_\infty(\lambda_1, \ldots, \lambda_{r_0}, \varepsilon_1, \ldots, \varepsilon_r)\Big|_{\underline{\lambda}=\underline{\varepsilon}=0}$, $[\ ]_r$ denotes all terms of the enclosed formal power series which are proportional to monomials $\Delta_{m,j_1}^{p_1} \ldots \Delta_{m,j_k}^{p_k}$ with $p_1+\ldots+p_k \le r$, $k \le m+1$, and $\sum_{(j)}^*$ denotes summation over all ordered k-tuples $(j_1, \ldots, j_k)$ without repetition of the indices. Applying (8.18) to (8.19), we get

$$\begin{aligned} &h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \\ &= -\sum_{r_0=0}^{s-4} m^{-r_0/2} P_{r_0}(\kappa_.(D)) \left[\sum_{k=1}^{m+1} \sum_{(j)}^* \prod_{l=1}^{s-r_0} \left\{\exp\left[\sum_{p=2}^{\infty} \Delta_{m,j_l}^p p!^{-1} \kappa_p\right] - 1\right\}\right]_{s-r_0} h_\infty \\ &+ R_{3,s}(m). \end{aligned} \tag{8.20}$$

The identity $\sum_{k=1}^{m+1} \sum_{(j)}^* \prod_{r=1}^{k} (e_{j_r} - 1) = \prod_{l=1}^{m+1} e_{j_l} - 1$ together with the symmetry condition of $h_m(\cdot)$, $m \ge 1$, shows that (8.20) is equal to

$$\begin{aligned} &h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \\ &= -\sum_{r_0=0}^{s-4} m^{-r_0/2} P_{r_0}(\kappa_.(D)) \left[\exp\left[\sum_{p=2}^{\infty} \left(\sum_{k=1}^{m+1} \Delta_{m,k}^p\right) p!^{-1} \kappa_p\right] - 1\right]_{s-r_0} h_\infty + R_{4,s}(m). \end{aligned} \tag{8.21}$$

It is easy to see that

$$\sum_{k=1}^{m+1} \Delta_{m,k}^2 = \sum_{k=1}^{m+1} \frac{1}{m+1} - \sum_{k=1}^{m} \frac{1}{m} = 0 \tag{8.22}$$

("equality of variances") and

$$\sum_{k=1}^{m+1} \Delta_{m,k}^p = O(m^{-p/2}), \quad p \ge 3. \tag{8.23}$$

Due to relation (8.22) the terms for $p = 2$ in (8.20) cancel.

By the definition of P_r and $\widetilde{P}_r$ (see (2.6) and (8.9)) it follows that

$$\sum_{r=1}^{l} [P_r(\tau_.\kappa_.)]_l = \sum_{r=3}^{l} \widetilde{P}_r(\tau_.\kappa_.), \tag{8.24}$$

where, according to the definitions, on the left-hand side $\tau_. = (\tau_3, \ldots, \tau_{r+2})$ and on the right-hand side $\tau_. = (\tau_3, \ldots, \tau_r)$, and $[\ ]_l$ denotes the sum of all monomials $\tau_3^{p_3} \ldots \tau_{r+2}^{p_{r+2}}$ in $P_r(\tau_.\kappa_.)$ such that $3p_3 + \ldots + (r+2)p_{r+2} \le l$, $l \ge 3$. The summation on the right-hand side starts at 3, because the first derivatives are equal to zero and the second cancel by (8.22).

Applying (8.18) and (8.24) we turn to P_r in (8.21) and get

$$m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))\left[\exp\left(\sum_{p=3}^{\infty}\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{p}\right)p!^{-1}\kappa_p(D)\right)-1\right]_{s-r_0}h_\infty \tag{8.25}$$

$$= m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))\sum_{r=3}^{s-r_0}\widetilde{P}_r\left(\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\right)\kappa_\cdot(D)\right)h_\infty$$

$$= m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))\sum_{r=1}^{s-r_0}\left[P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)\right]_{s-r_0}h_\infty.$$

Finally, (8.23) together with condition (2.10) shows that

$$m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)h_\infty \tag{8.26}$$

$$= m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))\left[P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)\right]_{s-r_0}h_\infty + R_{5,s}(m),$$

where

$$|R_{5,s}(m)| \le Bm^{-s/2} \quad \text{for every } m \ge n. \tag{8.27}$$

Note that by definition (2.7), the partial derivatives $D^{(\alpha_1,\ldots,\alpha_p)}$ of h_∞ on the right-hand side of (8.26) are such that $\alpha_j \ge 2,\ j = 1,\ldots,p,\ p \le k$, and $\sum_{j=1}^{p}(\alpha_j-2) \le s-3$. Relations (8.23), (8.25) and (8.26) show that (8.21) is equal to

$$-\sum_{r_0=0}^{s-4}m^{-r_0/2}P_{r_0}(\kappa_\cdot(D))\sum_{r=1}^{s-r_0}P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)h_\infty + R_{6,s}(m), \tag{8.28}$$

where $R_{6,s}(m)$ satisfies (8.27). Changing the order of summation and applying the relation

$$m^{-r_0/2}P_{r_0}(\kappa_\cdot(D)) = P_{r_0}\left(\sum_{j=1}^{m}\varepsilon_{m,j}^{\cdot}\kappa_\cdot(D)\right),$$

we obtain that (8.28) is equal to

$$-\sum_{l=1}^{s-3}\sum_{\substack{r_0+r=l\\ r\ge 1}}\left[P_{r_0}\left(\sum_{j=1}^{m}\varepsilon_{m,j}^{\cdot}\kappa_\cdot(D)\right)P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)\right]h_\infty + R_{6,s}(m)$$

$$= -\sum_{l=0}^{s-3}\sum_{r_0+r=l}\left[P_{r_0}\left(\sum_{j=1}^{m}\varepsilon_{m,j}^{\cdot}\kappa_\cdot(D)\right)P_r\left(\sum_{k=1}^{m+1}\Delta_{m,k}^{\cdot}\kappa_\cdot(D)\right)\right]h_\infty$$

$$- \sum_{r_0=0}^{s-3}P_{r_0}\left(\sum_{j=1}^{m}\varepsilon_{m,j}^{\cdot}\kappa_\cdot(D)\right)h_\infty + R_{6,s}(m).$$

By the multiplication theorem for exponential functions

$$\sum_{r+q=k} P_r(\tau_{\cdot}\kappa_{\cdot}(D))P_q(\tau'_{\cdot}\kappa_{\cdot}(D)) = P_k((\tau_{\cdot}+\tau'_{\cdot})\kappa_{\cdot}(D)), \quad q, r, k \geq 0,$$

we obtain

$$\begin{aligned}
&h_{m+1}(\underline{\varepsilon}_m) - h_{m+1}(\underline{\varepsilon}_{m+1}) \\
&= -\sum_{l=0}^{s-3}\left[P_l\left(\sum_{j=1}^{m}\varepsilon^{\cdot}_{m,j}\kappa_{\cdot}(D) + \sum_{j=1}^{m+1}\Delta^{\cdot}_{m,j}\kappa_{\cdot}(D)\right) - P_l\left(\sum_{j=1}^{m}\varepsilon^{\cdot}_{m,j}\kappa_{\cdot}(D)\right)\right]h_\infty + R_{6,s}(m) \\
&= -\sum_{l=0}^{s-3}\left[P_l\left(\sum_{j=1}^{m+1}\varepsilon^{\cdot}_{m+1,j}\kappa_{\cdot}(D)\right) - P_l\left(\sum_{j=1}^{m}\varepsilon^{\cdot}_{m,j}\kappa_{\cdot}(D)\right)\right]h_\infty + R_{6,s}(m) \\
&= -\sum_{l=1}^{s-3}\left[\left\{P_l\left(\sum_{j=1}^{m+1}\varepsilon^{\cdot}_{m+1,j}\kappa_{\cdot}(D)\right)+1\right\} - \left\{P_l\left(\sum_{j=1}^{m}\varepsilon^{\cdot}_{m,j}\kappa_{\cdot}(D)\right)+1\right\}\right]h_\infty + R_{6,s}(m) \\
&= -\sum_{l=1}^{s-3}\left[P_l\left(\sum_{j=1}^{m+1}\varepsilon^{\cdot}_{m+1,j}\kappa_{\cdot}(D)\right) - P_l\left(\sum_{j=1}^{m}\varepsilon^{\cdot}_{m,j}\kappa_{\cdot}(D)\right)\right]h_\infty + R_{6,s}(m).
\end{aligned}$$

This implies

$$\begin{aligned}
h_m(\underline{\varepsilon}_m) - h_\infty(0) &= \sum_{k=m}^{\infty}\left[h_k(\underline{\varepsilon}_k) - h_{k+1}(\underline{\varepsilon}_{k+1})\right] \\
&= \sum_{k=m}^{\infty}\left[\sum_{l=1}^{s-3}(k^{-l/2} - (k+1)^{-l/2})P_l(\kappa_{\cdot}(D))h_\infty + R_{6,s}(k)\right] \\
&= \sum_{l=1}^{s-3} m^{-l/2}P_l(\kappa_{\cdot}(D))h_\infty + R_{7,s}(m),
\end{aligned}$$

with $|R_{7,s}(m)| \leq c(s)Bm^{-(s-2)/2}$, where $c(s) > 0$ is a constant depending on s. This proves (8.15) for $l = s$ and $|\alpha| = 0$. The case $|\alpha| > 0$ can be proved similarly. Hence, the induction is completed and the theorem is proved. □

Lemma 8.2.

$$\sum_{\substack{j+k=r \\ j\geq 1}} j!^{-1}k!^{-1}(\varepsilon-\eta)^j\eta^k = r!^{-1}(\varepsilon^r - \eta^r), \quad r \geq 1,\ k \geq 0.$$

Proof. In fact, this relation is similar to Lemma 8.1. In order to prove this one we need to multiply both sides by a formal variable μ in the power r and sum up with respect to r from 0 to ∞. Doing so, we obtain

$$\sum_{r=0}^{\infty}\left[\sum_{\substack{j+k=r\\ j\geq 1}} j!^{-1}k!^{-1}(\varepsilon-\eta)^j\eta^k\right]\mu^r=\sum_{r=0}^{\infty}\left[r!^{-1}(\varepsilon^r-\eta^r)\right]\mu^r.$$

On the left-hand side there is no summand when $j=0$ and $k=r$. Thus, we add and subtract this term to get

$$\sum_{r=0}^{\infty}\left[\sum_{\substack{j+k=r\\ j,k\geq 0}} j!^{-1}k!^{-1}(\varepsilon-\eta)^j\eta^k - r!^{-1}\eta^r\right]\mu^r=\sum_{r=0}^{\infty}\left[r!^{-1}(\varepsilon^r-\eta^r)\right]\mu^r.$$

The right-hand side is equal to $e^{\varepsilon\mu}-e^{\eta\mu}$. If we change the indices of summation on the left-hand side, we obtain

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} j!^{-1}k!^{-1}(\varepsilon-\eta)^j\eta^k\mu^{j+k}-\sum_{r=0}^{\infty} r!^{-1}\eta^r\mu^r=e^{(\varepsilon-\eta)\mu}e^{\eta\mu}-e^{\eta\mu}=e^{\varepsilon\mu}-e^{\eta\mu}.$$

The lemma is proved. $\square$

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