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Competitive Outcomes and the Core of TU Market Games*

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Abstract

We investigate the relationship between certain subsets of the core for TU market games and competitive payoff vectors of certain markets linked to that game. This can be considered as the case in between the two extreme cases of Shapley and Shubik (1975). They remark already that their result can be extended to any closed convex subset of the core, but they omit the details of the proof which we present here. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

Keywords and Phrases: Market Games, Competitive Payoffs, Core

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1 Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik (1969). They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market is said to represent a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market that represents a game, then this game is called a market game. Shapley and Shubik (1969) prove the identity of the class of totally balanced TU games with the class of TU market games. Furthermore, Shapley and Shubik (1975) show that starting with a TU market game every payoff vector in the core of that game is competitive in a certain market, called direct market, and that for any given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector. Moreover, they claim that an analogous result holds also for closed convex subsets of the core. Shapley and Shubik (1975) give a hint how this can be shown but they omit the details of the proof. By following this remark of Shapley and Shubik (1975) we give a detailed proof how their two main results can be extended to any closed convex subset of the core. This more general case is in particular interesting, as the two theorems of Shapley and Shubik (1975) are included as special cases.

Similarly to the approach of Shapley and Shubik (1969, 1975), Inoue (2010) uses coalition production economies as in Sun et al. (2008) instead of markets. Inoue (2010) shows that every TU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy whose set of competitive payoff vectors coincides with the core of the balanced cover of the original TU game.

A different extension of Shapley and Shubik (1969, 1975) is Garratt and Qin (2000). They consider time-constrained market games, where the agents are supposed to supply one unit of time to the market. Their main result is that a TU game is a time-constrained market game if and only if it is superadditive. This result of Garratt and Qin (2000) was again extended by Bejan and Gómez (2011) introducing additionally location and free disposal constraints. They show that in this sense the entire class of TU games can be considered as market games.

For NTU market games Brangewitz and Gamp (2011) extend the NTU analogue to Shapley and Shubik (1975), namely Qin (1993), to closed subsets of the inner core. Hereby, the techniques used to show the results in the TU and the NTU case are notably different.

2 TU market games

In this section we state the main definitions and results on TU market games. The following introduction for TU market games is mainly based on Shapley and Shubik (1969) and Shapley and Shubik (1975).

Let $N = \{1, 2..., n\}$ be a set of players. The set of all non-empty coalitions is given by $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$. Thus, a coalition is a non-empty subset of players. A *cooperative game with transferable utility (TU)* is given by the pair (N, v) where N is the player set and $v : \mathcal{N} \to \mathbb{R}$ is the *characteristic* or *coalitional function.*[1] A *subgame* (T, v_T) of a TU game (N, v) is a subset of players $T \in \mathcal{N}$ and the characteristic function v_T with $v_T(S) = v(S)$ for $S \subseteq T$, $S \neq \emptyset$. A *payoff vector* for a TU game (N, v) is a vector $x \in \mathbb{R}^n$. The payoff of a coalition $S \in \mathcal{N}$ is given by $x(S) = \sum_{i \in S} x_i$. The *core* $C(v)$ of a TU game (N, v) is the set of payoff vectors where the value $v(N)$, the grand coalition N can achieve, is distributed and no coalition can improve upon,

$$C(v) = \{x \in \mathbb{R}^n |\, x(N) = v(N),\, x(S) \geq v(S) \quad \forall\, S \in \mathcal{N}\}.$$

Given a set of players $N = \{1, 2..., n\}$, a family $\mathcal{B} \subseteq \mathcal{N}$ is a *balanced family* if there exist weights $\{\gamma_S\}_{S \in \mathcal{B}}$, with $\gamma_S \geq 0$, such that for all $i \in N$ we have

$$\sum_{S \in \mathcal{B},\, S \ni i} \gamma_S = 1.$$

The weights γ_S do not depend on the individual players but on the coalition $S \in \mathcal{N}$. The above condition can be as well written as

$$\sum_{S \in \mathcal{N}} \gamma_S e^S = e^N$$

where $e^S \in \mathbb{R}^n$ is the vector with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Let the set of weights be denoted by $\Gamma(e^N)$. The balancing weights can be interpreted as the intensity with which player i participates in a coalition or the fraction of time he spends to be in this coalition.

A TU game (N, v) is *balanced* if for every balanced family $\mathcal{B}$ with weights $\{\gamma_S\}_{S \in \mathcal{B}}$ we have

$$\sum_{S \in \mathcal{B}} \gamma_S v(S) \leq v(N).$$

[1]Shapley and Shubik (1969) define the characteristic function as well for the empty set with $v(\emptyset) = 0$. Others, for example Billera and Bixby (1974), exclude the empty set from this definition.

A TU game (N, v) is *totally balanced* if all its subgames are balanced. The *totally balanced cover* of a TU game (N, v) is the smallest TU game $(N, \bar{v})$ that is totally balanced and contains the game (N, v).

Shapley and Shubik (1969) recall the following result of Shapley (1965):

Theorem (Shapley and Shubik (1969))**.** *A game has a non-empty core if and only if it is balanced.*

In oder to define a TU market game we first need to introduce the notion of a market. For the TU case it is sufficient to consider markets without production.

Definition (market)**.** Let $N = \{1, 2..., n\}$ be the set of agents (or players). A *market* is given by $\mathcal{E} = (X^i, \omega^i, u^i)_{i\in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$, $\ell \in \mathbb{N}$ is the number of commodities,
- $\omega^i \in X^i$ is the initial endowment vector,
- $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in the case with non-transferable utility (NTU) usually markets with production are considered, see for example Billera and Bixby (1974) or Qin (1993).

Let $S \in \mathcal{N}$ be a coalition. The set of *feasible S-allocations* is given by

$$F(S) = \left\{ (x^i)_{i\in S} \,\middle|\, x^i \in X^i \text{ for all } i \in S, \sum_{i\in S} x^i = \sum_{i\in S} \omega^i \right\}.$$

Elements of $F(S)$ are often denoted for short by x^S. The feasible S-allocations are those allocations the coalition S can achieve by redistributing their initial endowments within the coalition.

Now we define a TU market game in the following way:

Definition (TU market game)**.** A TU game (N, v) that is representable by a market is a *TU market game.* This means there exists a market $\mathcal{E}$ such that $(N, v_{\mathcal{E}}) = (N, v)$ with

$$v_{\mathcal{E}}(S) = \max_{x^S\in F(S)} \sum_{i\in S} u^i(x^i) \quad \text{for all } S \in \mathcal{N}.$$

For a TU market game there exists a market such that the value a coalition S can reach according to the coalitional function coincides with the joint utility that is generated by feasible S-allocations in the market.

Given a TU game we can generate a market from this game in a "natural" way. Shapley and Shubik (1969) call this market a direct market.

Definition (direct market)**.** A TU game (N, v) generates a *direct market* $\mathcal{D}_v = (X^i, \omega^i, u^i)_{i\in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+$,
- the initial endowment $\omega^i = e^{\{i\}}$ with $e^{\{i\}}_i = 1$ and $e^{\{i\}}_j = 0$ for $j \neq i$,
- the utility function $u^i(x) = \max\left\{ \sum_{S\in\mathcal{N}} \gamma_S v(S) \middle| \gamma_S \geq 0 \,\forall\, S \in \mathcal{N},\, \sum_{S\in\mathcal{N}} \gamma_S e^S = x \right\}$.

The utility function $u^i(\cdot)$ of the direct market $\mathcal{D}_v$ is identical for every individual $i \in N$ and is homogeneous of degree 1, concave and continuous. Note that in a direct market every consumer owns initially his own (private) good or interpreted differently every player "is" himself a good. Using the direct market $\mathcal{D}_v$, Shapley and Shubik (1969) obtain the following characterization of TU market games.

Theorem (Shapley and Shubik (1969))**.** *A game is a market game if and only if it is totally balanced.*

This means that in order to consider TU market games it is sufficient to consider just those TU games that are totally balanced. To obtain the above result Shapley and Shubik (1969) start by looking at an arbitrary TU game and its direct market. Hereafter, they consider the TU game of the direct market and show that it is equal to the totally balanced cover of the TU game they started with.

In a second paper Shapley and Shubik (1975) investigate the relationship between competitive payoffs, that arise from a competitive solution in the market, and the core of TU market games.

Definition (competitive solution)**.** A *competitive solution* is an ordered pair $(p^*, (x^{*i})_{i\in N})$, where p^* is an arbitrary n-vector of prices and x^{*N} is a feasible N-allocation, such that

$$u^i(x^{*i}) - p^* \cdot x^{*i} = \max_{x^i\in\mathbb{R}^l_+} [u^i(x^i) - p \cdot x^i] \quad \text{for all } i \in N.$$

We are in a setting with transferable utility. Thus, there is implicitly the additional commodity money, that makes the transfer of utility possible. Suppose ξ^i_0 are the initial money holdings of agent i. Then his "true" maximization problem is

$$\max_{x^i\in\mathbb{R}^l_+} [u^i(x^i) + \xi^i_0 - p \cdot \left(x^i - \omega^i\right)].$$

Since the solution of the maximization problem is independent of the initial money holdings and the initial endowment, it is equivalent to solve the in the definition above stated maximization problem.

Definition (competitive payoff vector)**.** A vector α^* is a *competitive payoff vector* if it arises from a competitive solution $(p^*, (x^{*i})_{i\in N})$ such that

$$\alpha^{*i} = u^i(x^{*i}) - p^* \cdot (x^{*i} - \omega^i).$$

Shapley and Shubik (1975) show the following two relationships between the core and competitive payoff vectors.

Theorem (1, Shapley and Shubik (1975))**.** *Every payoff vector in the core of a TU market game is competitive in the direct market of that game.*

Theorem (2, Shapley and Shubik (1975))**.** *Among the markets that generate a given totally balanced TU game, there exists a market having any given core point as its unique competitive payoff vector.*

These two theorems represent the two extreme cases where on the one hand the whole core equals the set of competitive payoff vectors of the direct market and one the other hand a given core point is the unique competitive payoff vector of a certain other market. The main ideas to prove the above two theorems are the following: For the first result Shapley and Shubik (1975) use the direct market to show that its competitive payoff vectors coincide with the core of the TU market game. To prove the second theorem they introduce a second game with a modified coalitional function for the grand coalition N. Afterwards they look at the direct market of the original game with a modified utility function depending on a given core point. Finally they show that this market represents the original TU game and has a given core point as its unique competitive payoff vector.

3 Results on TU market games

Shapley and Shubik (1975) already remark that for TU market games a extension of their proof for their second theorem leads to the following result.

Theorem. *Let (N, v) be a totally balanced TU game and let A be a closed, convex subset of the core. Then there exists a market such that this market represents the game (N, v) and such that the set of competitive payoff vectors of this market is the set A.*

Shapley and Shubik (1975) omit the details of the proof. We elaborate on them here. They remark that it is enough to change the definition of the utility function.

In the following we first define the according market and show afterwards in two steps that this market satisfies the properties we require.

Let (N, v) be a totally balanced TU game with $N = \{1, ..., n\}$ the set of players and the coalitional function v. Let $\mathcal{D}_v$ be its direct market as defined before. For $d \in \mathbb{R}_{++}$ define the TU game (N, v_d) by

$$v_d(S) = v(S) \quad \text{for all } S \subset N$$

and

$$v_d(N) = v(N) + d.$$

Since $d > 0$ the game (N, v_d) is totally balanced. Analogously let $\mathcal{D}_{v_d}$ be the direct market of (N, v_d). Let $(u_d^i)_{i \in N}$ denote the utility functions of $\mathcal{D}_{v_d}$, i.e.

$$u_d^i(x) = \max \left\{ \sum_{S \in \mathcal{N}} \gamma_S v_d(S) \middle| \gamma_S \geq 0 \, \forall \, S \in \mathcal{N}, \sum_{S \in \mathcal{N}} \gamma_S e^S = x \right\}.$$

As the utility functions u_d^i in the direct market $\mathcal{D}_{v_d}$ are identical for every individual $i \in N$, we write for short u_d.

Let A be a any non-empty closed convex subset of the core. For $\alpha \in A$ let $u_{d,\alpha}$ be defined as

$$u_{d,\alpha}(x) = \min(u_d(x), \alpha \cdot x).$$

Then define the function $u_{d,A}$ by

$$u_{d,A}(x) = \min_{\alpha \in A} u_{d,\alpha}(x).$$

Since $u_{d,A}$ is continuous and concave we can define a market by

$$\mathcal{E}_{v_d} = \left(\mathbb{R}_+^n, e^{\{i\}}, u_{d,A}^i \right)_{i \in N}.$$

with $u_{d,A}^i = u_{d,A}$ for all $i \in N$. It is easy to see that $u_{d,A}$ is homogeneous of degree 1.

Next, we show first that the market game of this market is (N, v) and second that the set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ is exactly the set A.

Proposition 1. *The market $\mathcal{E}_{v_d}$ represents the game (N, v).*

Proof. Recall that for the market $\mathcal{E}_{v_d}$ the set

$$F(S) = \left\{ x^S \in \mathbb{R}_+^{n \cdot S} | \sum_{i \in S} x^i = \sum_{i \in S} e^{\{i\}} \right\}$$

is the set of feasible allocations for a coalition $S \in \mathcal{N}$.

Looking at the market game generated by the market $\mathcal{E}_{v_d}$ we obtain

$$\begin{aligned}
v_{\mathcal{E}_{v_d}}(S) &= \max_{x^S \in F(S)} \sum_{i \in S} u^i_{d,A}(x^i) \\
&= |S| \max_{x^S \in F(S)} \sum_{i \in S} \frac{1}{|S|} u_{d,A}(x^i) \\
&\overset{(1)}{=} |S| \max_{x^S \in F(S)} u_{d,A} \left(\frac{e^S}{|S|} \right) \\
&= |S| u_{d,A} \left(\frac{e^S}{|S|} \right) \\
&\overset{(2)}{=} u_{d,A}(e^S) \\
&= \min_{\alpha \in A} u_{d,\alpha}(e^S) \\
&= \min_{\alpha \in A} (\min(u_d(e^S), \alpha \cdot e^S)) \\
&\overset{(3)}{=} \min_{\alpha \in A} (\min(v_d(S), \alpha \cdot e^S)) \\
&= \min_{\alpha \in A} (v_d(S), \alpha \cdot e^S) \\
&\overset{(4)}{=} v(S)
\end{aligned}$$

The detailed arguments are the following:

(1) First observe that $\sum_{i \in S} \frac{1}{|S|} u_{d,A}(x^i) \leq u_{d,A} \left(\sum_{i \in S} \frac{x^i}{|S|} \right) = u_{d,A} \left(\frac{e^S}{|S|} \right)$ from the concavity of $u_{d,A}$ and the market clearing condition. We take the maximum on both sides over the feasible S-allocations $F(S)$ and we observe that $\bar{x}^i = \frac{1}{|S|} e^S$ for all $i \in S$ is a feasible S-allocation. Therefore, we obtain that setting $\left(\bar{x}^i \right)_{i \in S}$ maximizes the expression on the left side and hence we get equality.

(2) The equality follows from the homogeneity of degree 1 of $u_{d,A}$.

(3) Using the totally balancedness of the game (N, v_d) we obtain

$$u_d(e^S) = \max \left\{ \sum_{T \in \mathcal{N}} \gamma_T v_d(T) \middle| (\gamma_T) \geq 0, \sum_{T \in \mathcal{N}} \gamma_T e^T = e^S \right\} = v_d(S).$$

(4) For $S \subset N$ this minimum is equal to $v(S)$, since α is in the core of the TU game (N, v) and therefore $\alpha \cdot e^S \geq v(S) = v_d(S)$. For $S = N$ the minimum is equal to $\alpha' \cdot e^N$ for some $\alpha' \in A$ and since α' is in the core of (N, v) we have $\alpha' \cdot e^N = v(N)$. As $d > 0$ we have $v(N) < v_d(N)$.

Thus $v_{\mathcal{E}_{v_d}} = v$ and hence the market $\mathcal{E}_{v_d}$ generates the game (N, v). □

Proposition 2. *The set of competitive payoff vectors of the market $\mathcal{E}_{v_d}$ are coincides with the set A.*

Proof. The proof is divided into five parts:

1. First, suppose $\left((x^{*i})_{i \in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then competitive payoffs are of the form $\left(p^* \cdot e^{\{i\}}\right)_{i \in N}$.

   From the definition of a competitive solution it follows that $(x^{*i})_{i \in N}$ clears the markets,
   $$\sum_{i=1}^{n} x^{*i} = \sum_{i=1}^{n} e^{\{i\}} = e^N$$
   and maximizes for each trader i his trading profit given by
   $$u_{d,A}(x^i) - p \cdot x^i.$$
   Moreover, we have from the existence of a maximum and the fact that the trading profit as a function of the consumption bundle is homogeneous of degree 1 that
   $$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} = 0.$$
   Looking at the competitive payoffs of competitive solutions we observe
   $$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

2. Second, suppose $\left((x^{*i})_{i \in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then $\left(\left(\frac{1}{n} e^N\right)_{i \in N}, p^*\right)$ is as well a competitive solution in the market $\mathcal{E}_{v_d}$. In addition the competitive payoffs coincide.

From the fact that the trading profit equals zero we obtain

$$
\begin{aligned}
u_{d,A}\left(\frac{1}{n}e^N\right) - p^* \cdot \frac{1}{n}e^N &= u_{d,A}\left(\frac{1}{n}\sum_{i=1}^{n} x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^{n} x^{*i} \\
&\overset{(1)}{=} \frac{1}{n}\sum_{i=1}^{n} u_{d,A}\left(x^{*i}\right) - p^* \cdot \frac{1}{n}\sum_{i=1}^{n} x^{*i} \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} u_{d,A}\left(x^{*i}\right) - p^* \cdot \sum_{i=1}^{n} x^{*i}\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \left(u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i}\right)\right) \\
&= 0.
\end{aligned}
$$

The detailed argument is the following:

(1) Using the concavity of $u_{d,A}$ gives us "$\geq$" and from maximality of x^{*i} we obtain the equality.

As already seen in 1., looking at the competitive payoffs of these competitive solutions we observe

$$u_{d,A}\left(x^{*i}\right) - p^* \cdot x^{*i} + p^* \cdot e^{\{i\}} = u_{d,A}\left(\frac{1}{N}e^N\right) - p^* \cdot \left(\frac{1}{N}e^N\right) + p^* \cdot e^{\{i\}} = p^* \cdot e^{\{i\}}.$$

To summarize these results mean that looking for competitive solutions and their competitive payoffs we can focus on possible equilibrium prices of the allocation $\left(\frac{1}{N}e^N\right)_{i\in N}$. Then those competitive solutions give us all possible competitive payoffs.

3. Third, as in the proof of Proposition 1, equality (3)

$$u_d\left(\frac{1}{N}e^N\right) = \frac{1}{N}v_d(N) > \frac{1}{N}v(N) = u_{d,A}\left(\frac{1}{N}e^N\right)$$

and furthermore

$$u_{d,A}\left(\frac{1}{N}e^N\right) = \alpha' \cdot \left(\frac{1}{N}e^N\right)$$

for all $\alpha' \in A$. Because of the continuity of $u_d(\cdot)$ it follows for all $\alpha' \in A$ that $u_d(x) > \alpha' \cdot x$ for x in a small neighborhood of $\frac{1}{N}e^N$. Thus, in a neighborhood of $\frac{1}{N}e^N$, $u_{d,A}(x) = \min_{\alpha' \in A}\left(\alpha' \cdot x\right)$.

4. Forth, it remains to check for which prices p^* the pair $\left(\left(\frac{1}{N}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. In a first step we show that each $p^* \in A$ can be chosen as an equilibrium price vector, in a second step we show that any $p^* \notin A$ cannot be an equilibrium price vector. For the second step it is enough to concentrate on $p^* \in C(v) \setminus A$ as we have seen in 1. that the equilibrium price vector determines the competitive payoff vector, which are necessarily in the core.

   Step 1: Suppose $p^* \in A$. Then for all $x^i \in \mathbb{R}^n_+$ we have

$$\min_{\alpha' \in A} \left(\alpha' \cdot x^i\right) - p^* \cdot x^i \leq p^* \cdot x^i - p^* \cdot x^i = 0$$

   and furthermore

$$\min_{\alpha' \in A} \left(\alpha' \cdot \left(\frac{1}{N}e^N\right)\right) - p^* \cdot \left(\frac{1}{N}e^N\right) = 0.$$

   Hence, $x^i = \frac{1}{N}e^N$ maximizes the trading profit of agent i. Furthermore, the markets clear, as $\sum\limits_{i\in N} \frac{1}{N}e^N = e^N$.
   So, the pair $\left(\left(\frac{1}{N}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution.

   Step 2: Suppose $p^* \in C(v) \setminus A$. Recall that the set A is compact and convex. Hence, we can apply the separating hyperplane theorem[2] and obtain that there exists $\bar{x} \in \mathbb{R}^n_+$ such that for all $\alpha \in A$

$$\alpha \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

   Therefore we conclude that

$$\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x} > 0.$$

   Now, for sufficiently small $\varepsilon > 0$ we have that $\frac{1}{N}e^N + \varepsilon\bar{x}$ is in a neighborhood of $\frac{1}{N}e^N$ where we have $u_{d,A}(x) = \min_{\alpha' \in A}\left(\alpha' \cdot x\right)$. But

$$\min_{\alpha' \in A} \left(\alpha' \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right)\right) - p^* \cdot \left(\frac{1}{N}e^N + \varepsilon\bar{x}\right) = \varepsilon\left(\min_{\alpha' \in A} \alpha' \cdot \bar{x} - p^* \cdot \bar{x}\right) > 0.$$

   This implies that $\frac{1}{N}e^N$ does not maximize agent i's trading profit for $p^* \notin A$.

5. To summarize the line of argument:

[2] See for example Mas-Colell et al. (1995, Theorem M.G.2, p.948).

If $\left((x^{*i})_{i\in N}, p^*\right)$ is a competitive solution in the market $\mathcal{E}_{v_d}$, then by 2. we have that $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. By 4. we show that $p^* \in A$ and by 1. we know that its competitive payoff vector is equal to p^*.
On the other hand if $p^* \in A$ then by 4. we have that $\left(\left(\frac{1}{n}e^N\right)_{i\in N}, p^*\right)$ is a competitive solution. The competitive payoff vector is equal to p^*.

□

4 Concluding Remarks

Shapley and Shubik (1975) investigate the relationship between competitive payoffs of markets that represent a cooperative game and their relation to solution concepts for cooperative games. We presented the details of the proof of Shapley and Shubik (1975), that extends their two main results to closed, convex subsets of the core. This shows also the two theorems of Shapley and Shubik (1975). In a further contribution (Brangewitz and Gamp, 2011) we establish an analogue result for NTU market games.

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