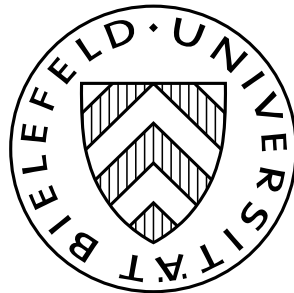# Generalized Kuhn-Tucker Conditions for N-Firm Stochastic Irreversible Investment under Limited Resources

Maria B. Chiarolla, Giorgio Ferrari and Frank Riedel

Generalized Kuhn-Tucker Conditions for N-Firm Stochastic Irreversible Investment under Limited Resources[*]

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Abstract. In this paper we study a continuous time, optimal stochastic investment problem under limited resources in a market with N firms. The investment processes are subject to a time-dependent stochastic constraint. Rather than using a dynamic programming approach, we exploit the concavity of the profit functional to derive some necessary and sufficient first order conditions for the corresponding *Social Planner* optimal policy. Our conditions are a stochastic infinite-dimensional generalization of the Kuhn-Tucker Theorem. As a subproduct we obtain an enlightening interpretation of the first order conditions for a single firm in Bank [5].

In the infinite-horizon case, with operating profit functions of Cobb-Douglas type, our method allows the explicit calculation of the optimal policy in terms of the 'base capacity' process, i.e. the unique solution of the Bank and El Karoui representation problem [4].

Keywords: stochastic irreversible investment, optimal stopping, the Bank and El Karoui Representation Theorem, base capacity, Lagrange multiplier optional measure.

MSC2010 subsject classification: 91B70, 93E20, 60G40, 60H25.

JEL classification: C02, E22, D92, G31.

1 Introduction

In the last years the theory of irreversible investment under uncertainty has received much attention in Economics as well as in Mathematics (see, for example, the extensive review in Dixit and Pindyck [15]). From the mathematical point of view, optimal irreversible investment problems under uncertainty are singular stochastic control problems. In fact, the economic constraint that does not allow disinvestment may be modeled as a 'monotone follower' problem;

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that is, a problem in which investment strategies are given by nondecreasing stochastic processes, not necessarily absolutely continuous with respect to the Lebesgue measure as functions of time. The pioneering papers by Karatzas [20], Karatzas and Shreve [21], El Karoui and Karatzas [16] (among others) started the application of 'monotone follower' problems to Economics. These Authors studied the problem of optimally minimizing a convex cost (or optimally maximizing a concave profit) functional when the capacity is a Brownian motion tracked by a nondecreasing process, i.e. the monotone follower. They showed that any such control problem is connected to a suitable optimal stopping problem whose value function v is the derivative of the value function V of the control problem. Moreover, the optimal control ν_* defines an optimal stopping time τ^* by the simple formula $\tau^* := \inf\{t \in [0, T) : \nu_*(t) > 0\} \wedge T$. Later on, this kind of link has been established also for more complicated dynamics of the controlled diffusion; that is the case, for example, of a Geometric Brownian motion (Baldursson and Karatzas [1]), or of a quite general controlled Ito diffusion (see Boetius and Kohlmann [8], Chiarolla and Haussmann [12], Chiarolla and Ferrari [13], among others). More recently, Boetius [9], Chiarolla and Haussmann [10] and [11], and Karatzas and Wang [24] showed that such connection holds in the case of bounded variation stochastic control problems as well; the value function of the control problem V satisfies $\frac{\partial}{\partial x}V = v$, where v is the saddle point of a suitable Dynkin game, that is a zero-sum optimal stopping game.

In the last decade several papers handled singular stochastic control problems of the monotone follower type by deriving first order conditions for optimality. That is the case, for instance, of Bank and Riedel [2] in which the Authors studied an intertemporal utility maximization problem with Hindy, Huang and Kreps preferences, of Bank and Riedel [3] in which the optimal dynamic choice of durable and perishable goods is analyzed, or of Riedel and Su [27] in which a very general irreversible investment problem with unlimited resources is treated. In these papers the optimal consumption, or investment policy, is constructed as the running supremum of a desirable value. Such level of satisfaction is the unique optional solution of a stochastic backward equation in the spirit of Bank-El Karoui (cf. [4], Theorem 3) and may be represented in terms of the value functions of a family of standard optimal stopping problems.

The link between irreversible investment problems and optimal stopping is also relevant in Economics. In fact a firm operating in a market with uncertainty not only has to decide how to invest but also when to invest. The optimal timing problem is then related to option theory, since it may be viewed as a 'real option', an option whose strike price is the cost of investment. It follows that exercising a real option means to invest properly at an optimal time.

The investment problem becomes even harder if one takes into account the fact that the available resources may be limited. The problem turns into a 'finite fuel' singular stochastic control problem since the total amount of effort (fuel) available to the controller (for example, the firm's manager) is limited. The mathematical literature on this field started in 1967 with Bather and Chernoff [6] in the context of controlling the motion of a spaceship. Finite fuel monotone follower problems were then studied by Benes, Shepp and Witsenhausen in 1980 [7]. In 1985 Chow, Menaldi and Robin [14] and Karatzas [22] used a PDE approach and purely probabilistic arguments, respectively, to show that the optimal policy of a 'monotone follower' problem with constant finite fuel is 'follow the unconstrained optimal policy until there is some fuel to spend'. Much more difficult is the case of finite fuel given by a time-dependent process, either deterministic or stochastic.

In 2005 Bank [5], without relying on any Markovian assumption, generalized the optimal

policy proposed by Karatzas [22] to the case of a stochastic, increasing, adapted finite fuel process $\theta(t)$. The Author characterized the optimal policy of a cost minimization problem as the unique process satisfying some first order conditions for optimality (cf. [5], Theorem 1), 'the optimal control should be exercised only when its impact on future costs is maximal; on the other hand, when the cost functional's subgradient tends to decrease, then all the available fuel must be used'. More in detail, if $\mathbb{S}(\nu)$ is the Snell envelope of the total cost functional's subgradient $\nabla_\nu \mathcal{C}(\nu)$ (i.e., $\mathbb{S}(\nu)(t) := \operatorname{ess\,inf}_{t \le \tau \le T} \mathbb{E}\{\nabla_\nu \mathcal{C}(\nu)(\tau)|\mathcal{F}_t\}$), and $\mathcal{M}(\nu) + A(\nu)$ is its Doob-Meyer decomposition, then Bank [5] proved that ν_* is optimal if and only if

$$
\begin{array}{ll}
\text{(i)} & \nu_* \text{ is flat off } \{\nabla_\nu \mathcal{C}(\nu_*) = \mathbb{S}(\nu_*)\}, \\
\text{(ii)} & A(\nu_*) \text{ is flat off } \{\nu_* = \theta\}.
\end{array}
\tag{1.1}
$$

Moreover, the Author constructed the optimal control ν_* in terms of the 'base capacity' process, i.e. a desirable value of capacity. Mathematically such process is the unique optional solution of the Bank-El Karoui representation problem [4].

In this paper we generalize Bank's single firm problem to the case of a Social Planner in a market with N firms in which the total investment is bounded by a stochastic, time-dependent, increasing, adapted finite fuel $\theta(t)$; that is, the case $\sum_{i=1}^N \nu^{(i)}(t) \le \theta(t)$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. The Social Planner's objective is to pursue a vector $\underline{\nu}_* \in \mathbb{R}_+^N$ of efficient irreversible investment processes that maximize the aggregate expected profit, net of investment cost, i.e.

$$
\sup_{\sum_{i=1}^N \nu^{(i)} \le \theta} \sum_{i=1}^N \mathbb{E}\left\{ \int_0^T e^{-\delta(t)} R^{(i)}(X(t), \nu^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \right\}.
\tag{1.2}
$$

Here the operating profit function $R^{(i)}$ of firm i, $i = 1, 2, ..., N$, depends directly on the cumulative control exercised since we do not allow for dynamics of the productive capacity. As in Kobila [25], and Riedel and Su [27], the uncertain status of the economy is modeled by an exogenous economic shock $\{X(t), t \in [0,T]\}$. Although our finite fuel θ is increasing as in Bank [5], his results cannot be directly applied to each firm since for each i the investment bound $\theta - \sum_{j \ne i} \nu^{(j)}$ is not an increasing process. To overcome this difficulty we develop a new approach based on a stochastic generalization of the classical Kuhn-Tucker method. That is accomplished as follows. By applying a version of Komlòs' theorem for optional random measures (cf. Kabanov [19], Lemma 3.5) we prove existence and uniqueness of optimal irreversible investment policies. Then we use the concavity of the profit functional to characterize the optimal Social Planner policy as the unique solution of some stochastic Kuhn-Tucker conditions. The Lagrange multiplier takes the form of a nonnegative optional random measure on $[0,T]$ whose support is the set of times for which the constraint is binding, i.e. when all the fuel is spent. Hence, as a subproduct we obtain an enlightening interpretation of the first order conditions that Bank [5] proved for a single firm optimal investment problem. In fact, we show that process $A(\nu_*)$ in (1.1) is equal to the Lagrange multiplier of our control problem. As expected in optimization under inequality constraints, our Lagrange multiplier λ grows only when the resource constraint is binding (see our equation (3.9) below). This condition corresponds exactly to Bank's (1.1)-(ii).

When the N firms have operating profit functions of Cobb-Douglas type, with a different parameter for each of them, our generalized stochastic Kuhn-Tucker approach allows for the explicit calculation of the Social Planner optimal investment strategy. Such optimal policy is

given in terms of the 'base capacity' process, i.e. the unique solution of the Bank-El Karoui Representation Problem [4]. Finally, when the finite fuel is constant, we recover two classical monotone follower problems for which we are able to identify the compensator part in the Doob-Meyer decomposition of the profit (cost) functional's supergradient (subgradient) as the Lagrange multiplier of the optimal investment problem.

The paper is organized as follows. In Section 2 we set the model. In Section 3 we introduce the generalized stochastic Kuhn-Tucker conditions for the Social Planner problem. Finally, in Section 4 we solve an N-firm Social Planner optimization problem and we test our approach on some 'finite-fuel' problems from the literature (cf. [5] and [22], among others).

2   The Model

We consider a market with N firms on a time horizon $T \leq +\infty$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual conditions. The cumulative irreversible investment of firm i, $i = 1, 2, ..., N$, denoted by $\nu^{(i)}(t)$, is an adapted process, nondecreasing, left-continuous, finite a.s. s.t. $\nu^{(i)}(0) = y^{(i)} > 0$.

The firms are financed entirely by equities but we focus primarily on the irreversibility of investments and do not model precisely the rest of the economy. It is reasonable to assume that the firms cannot invest in natural resources as much as they like. In fact, we assume that the total amount of natural resources available at time t is a finite quantity $\theta(t)$; that is,

$$\sum_{i=1}^{N} \nu^{(i)}(t) \leq \theta(t), \quad \mathbb{P} - \text{a.s.}, \quad \text{for} \quad t \in [0,T]. \tag{2.1}$$

The stochastic time-dependent constraint $\{\theta(t), t \in [0,T]\}$ is the cumulative amount of resources extracted up to time t. It is a nonnegative and increasing adapted process with left-continuous paths, which starts at time zero from $\theta(0) = \theta_o > 0$. We assume

$$\mathbb{E}\{\theta(T)\} < +\infty. \tag{2.2}$$

We denote by $\mathcal{S}_\theta$ the nonempty set of admissible investment plans, i.e.

$$\mathcal{S}_\theta := \{\underline{\nu} : \Omega \times [0,T] \to \mathbb{R}_+^N, \text{ nondecreasing, left-continuous, adapted process s.t.}$$

$$\nu^{(i)}(0) = y^{(i)}, \ \mathbb{P} - \text{a.s.}, \ i = 1, 2, ..., N, \text{ and } \sum_{i=1}^{N} \nu^{(i)}(t) \leq \theta(t), \ \mathbb{P} - \text{a.s.} \ \forall t \in [0,T]\}.$$

Let $\{X(t), t \in [0,T]\}$ be some exogenous real-valued state variable progressively measurable with respect to $\mathcal{F}_t$. It may be regarded as an economic shock, reflecting the changes in technological ouput, demand and macroeconomic conditions which have direct or indirect effect on the firm's profit. At the moment we do not make any Markovian assumption.

We take the capital good as numeraire, hence we express profits, costs etc. in real terms, not nominal ones. Hence the price of a unitary investment is equal to one. We take the point of view of a fictitious *Social Planner* aiming to maximize the aggregate expected profit, net of investment costs, $\mathcal{J}_{SP}(\underline{\nu})$ (see equation (2.5) below), by allocating efficiently the available

resources. We denote by $\delta(t)$ the Social Planner discount factor. $\delta(t)$ is a nonnegative, optional process, bounded uniformly in $(\omega, t) \in \Omega \times [0, T]$. Assumption (2.2) ensures

$$\mathbb{E}\left\{ \int_{[0,T)} e^{-\delta(t)} \, d\nu^{(i)}(t) \right\} < +\infty, \qquad i = 1, 2, ..., N, \tag{2.3}$$

i.e. the investment plan's expected net present value of firm i is finite.

The operating profit function of firm i is $R^{(i)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., N$. At time t, when the investment of firm i is $\nu^{(i)}(t)$, $R^{(i)}\left(X(t), \nu^{(i)}(t)\right)$ represents the amount of goods produced by firm i under the shock process $X(t)$. The Social Planner problem is

$$V_{SP} := \sup_{\underline{\nu} \in \mathcal{S}_\theta} \mathcal{J}_{SP}(\underline{\nu}), \tag{2.4}$$

where

$$\mathcal{J}_{SP}(\underline{\nu}) := \sum_{i=1}^{N} \mathcal{J}_i(\nu^{(i)}) \tag{2.5}$$

and, for $i = 1, 2, ..., N$,

$$\mathcal{J}_i(\nu^{(i)}) = \mathbb{E}\left\{ \int_0^T e^{-\delta(t)} \, R^{(i)}(X(t), \nu^{(i)}(t)) dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \right\}. \tag{2.6}$$

Notice that $\mathcal{J}_i(\nu^{(i)})$ is the expected total profit, net of investment costs, of firm i when the Social Planner picks $\underline{\nu} \in \mathcal{S}_\theta$.

The operating profit functions satisfy the following concavity and regularity assumptions.

Assumption 2.1.

1. *For every $x \in \mathbb{R}$ and $i = 1, 2, ..., N$, the mapping $y \to R^{(i)}(x, y)$ is increasing, strictly concave, with continuous decreasing partial derivative $R_y^{(i)}(x, y)$ satisfying the Inada conditions*

$$\lim_{y \to 0} R_y^{(i)}(x, y) = \infty, \qquad \lim_{y \to \infty} R_y^{(i)}(x, y) = 0.$$

2. *$R^{(i)}\left(X(\omega, t), \nu^{(i)}(\omega, t)\right)$ is $d\mathbb{P} \otimes dt$-integrable, for $i = 1, 2, ..., N$.*

3. *The process $(\omega, t) \longrightarrow \sup_{\nu^{(i)}(\omega,t):\underline{\nu} \in \mathcal{S}_\theta} R^{(i)}(X(\omega, t), \nu^{(i)}(\omega, t))$ is $d\mathbb{P} \otimes dt$-integrable, for $i = 1, 2, ..., N$.*

Under (2.2) and Assumption 2.1 the net profit $\mathcal{J}_i(\nu^{(i)})$ is well defined and finite for all admissible plans.

In the next Section we show how to handle constraint (2.1) in order to find the solution to Social Planner problem (2.4).

3 A Stochastic Kuhn-Tucker Approach

In this Section we aim to find an optimal investment plan by means of a gradient approach. As in [27], proof of Theorem 2.6, by applying a suitable version of Komlòs' Theorem for optional random measures (cf. [19], Lemma 3.5) we obtain existence and uniqueness of a solution to problem (2.4). In fact, Komlòs' Theorem states that if a sequence of random variables $(Z_n)_{n \in \mathbb{N}}$ is bounded from above in expectation, then there exists a subsequence $(Z_{n_k})_{k \in \mathbb{N}}$ which converges in the Cesàro sense to some random variable Z. In our case the limit provided by Komlòs' Theorem turns out to be the optimal investment strategy.

Theorem 3.1. *Under (2.2) and Assumption 2.1, there exists a unique optimal vector of irreversible investment plans $\underline{\nu}_* \in \mathcal{S}_\theta$ for problem (2.4).*

Proof. Let $\underline{\nu} \in \mathcal{S}_\theta$ and denote by $\mathcal{H}$ the space of optional measures on $[0, T]$. Then, the investment strategies $\nu^{(i)}$ may be regarded as elements of $\mathcal{H}$, hence $\mathcal{S}_\theta \subset \mathcal{H}^N$.

Let $(\underline{\nu}_n)_{n \in \mathbb{N}}$ be a maximizing sequence of investment plans in $\mathcal{S}_\theta$, i.e. a sequence such that $\lim_{n \to \infty} \mathcal{J}_{SP}(\underline{\nu}_n) = V_{SP}$. By (2.2) we have that the sequence $(\mathbb{E}\{\nu_n^{(i)}(T)\})_{n \in \mathbb{N}}$ is bounded for $i = 1, 2, ..., N$; in fact, $\mathbb{E}\{\nu_n^{(i)}(T)\} \leq \mathbb{E}\{\theta(T)\} < \infty$. By a version of Komlòs' Theorem for optional measures (cf. [19], Lemma 3.5), there exists a subsequence $(\hat{\underline{\nu}}_n)_{n \in \mathbb{N}}$ that converges weakly a.s. in the Cesàro sense to some random vector $\underline{\nu}_* \in \mathcal{H}^N$. That is, for $i = 1, 2, ..., N$, we have, almost surely,

$$\hat{I}_n^{(i)}(t) := \frac{1}{n} \sum_{j=0}^{n} \hat{\nu}_j^{(i)}(t) \to \nu_*^{(i)}(t), \quad \text{as} \quad n \to \infty, \tag{3.1}$$

for every point of continuity of $\nu_*^{(i)}$, $i = 1, 2, ..., N$. Notice that $\hat{\underline{\nu}}_n \in \mathcal{S}_\theta$ for all n implies that also the Cesàro sequence $\hat{\underline{I}}_n$ belongs to $\mathcal{S}_\theta$ due to the convexity of $\mathcal{S}_\theta$, hence $\sum_{i=1}^{N} \hat{I}_n^{(i)}(t) \leq \theta(t)$, for $n \in \mathbb{N}$. It follows that, almost surely,

$$\sum_{i=1}^{N} \nu_*^{(i)}(t) \leq \theta(t), \tag{3.2}$$

which means $\underline{\nu}_* \in \mathcal{S}_\theta$.

Since $(\nu_n^{(i)})_{n \in \mathbb{N}}$ is a maximizing sequence so is $(\hat{I}_n^{(i)})_{n \in \mathbb{N}}$ by concavity of the profit functional. Then Jensen inequality and the dominated convergence theorem yield

$$\mathcal{J}_{SP}(\underline{\nu}_*) \geq \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} \mathcal{J}_{SP}(\hat{\underline{\nu}}_n) = V_{SP}. \tag{3.3}$$

Finally, uniqueness follows from the strict concavity of the Social Planner profit functional. $\square$

We now aim to characterize the Social Planner optimal policy as the unique solution of a set of first order generalized stochastic Kuhn-Tucker conditions. Notice that the strict concave functionals $\mathcal{J}_i$, $i = 1, 2, ..., N$, admit the supergradient

$$\nabla_y \mathcal{J}_i(\nu^{(i)})(t) := \mathbb{E}\left\{ \int_t^T e^{-\delta(s)} R_y^{(i)}(X(s), \nu^{(i)}(s)) \, ds \,\Big|\, \mathcal{F}_t \right\} - e^{-\delta(t)} \mathbb{1}_{\{t < T\}} \tag{3.4}$$

for $t \in [0, T]$.

Remark 3.2. *The quantity $\nabla_y \mathcal{J}_i(\nu^{(i)})(t)$, $i = 1, 2, ..., N$, may be interpreted as the marginal expected profit resulting from an additional infinitesimal investment at time t when the investment plan is $\nu^{(i)}$. Mathematically, $\nabla_y \mathcal{J}_i(\nu^{(i)})$ is the Riesz representation of the profit gradient at $\nu^{(i)}$. More precisely, define $\nabla_y \mathcal{J}_i(\nu^{(i)})$ as the optional projection of the progressively measurable process*

$$\Phi_i(\omega, t) := \int_t^T e^{-\delta(\omega, s)} R_y^{(i)}(X(\omega, s), \nu^{(i)}(\omega, s)) \, ds \; - \; e^{-\delta(\omega, t)} \mathbb{1}_{\{t < T\}}, \qquad (3.5)$$

for $\omega \in \Omega$, and $t \in [0, T]$. Hence $\nabla_y \mathcal{J}_i(\nu^{(i)})$ is uniquely determined up to $\mathbb{P}$-indistinguishability and it holds

$$\mathbb{E}\left\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(t) d\nu^{(i)}(t) \right\} = \mathbb{E}\left\{ \int_{[0,T)} \Phi_i(t) d\nu^{(i)}(t) \right\}$$

for all admissible $\nu^{(i)}(t)$ (cf. Theorem 1.33 in [18]).

3.1 Generalized Stochastic Kuhn-Tucker Conditions

Let $\mathcal{B}[0, T]$ denote the Borel σ-algebra on $[0, T]$. Recall that if $\beta(t)$ is a right-continuous, adapted and nondecreasing process, then the bracket operator

$$\langle \alpha, \beta \rangle = \mathbb{E}\left\{ \int_{[0,T)} \alpha(t) \, d\beta(t) \right\} \qquad (3.6)$$

is well defined (possibly infinite) for all processes $\alpha(t)$ which are nonnegative and $\mathcal{F}_T \otimes \mathcal{B}[0, T]$-measurable. Notice that the bracket is preserved when we pass from α to its optional projection $\alpha^{(o)}$ (cf. [18], Theorem 1.33); that is

$$\langle \alpha, \beta \rangle = \langle \alpha^{(o)}, \beta \rangle. \qquad (3.7)$$

Since the constraint is $\theta(t) - \sum_{i=1}^N \nu^{(i)}(t) \geq 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$ (cf. (2.1)), we define the *Lagrangian functional* of problem (2.4) as

$$
\begin{aligned}
\mathcal{L}^\theta(\underline{\nu}, \lambda) &= \mathcal{J}_{SP}(\underline{\nu}) + \langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda \rangle \\
&= \sum_{i=1}^N \mathbb{E}\left\{ \int_0^T e^{-\delta(t)} R^{(i)}(X(t), \nu^{(i)}(t)) dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \right\} \qquad (3.8) \\
&\quad + \mathbb{E}\left\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^N \nu^{(i)}(t)] d\lambda(t) \right\},
\end{aligned}
$$

where $d\lambda(\omega, t)$ is a nonnegative optional measure, which may be interpreted as the Lagrange multiplier of Social Planner problem (2.4). By using Fubini's Theorem we write the bracket $\langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda \rangle$ in a more convenient form, that is

$$
\begin{aligned}
\langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda \rangle &= \mathbb{E}\left\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^N \nu^{(i)}(t)] d\lambda(t) \right\} \\
&= \mathbb{E}\left\{ \int_{[0,T)} \left[\int_{[0,t)} (d\theta(s) - \sum_{i=1}^N d\nu^{(i)}(s)) \right] d\lambda(t) \right\} + K \, \mathbb{E}\left\{ \int_{[0,T)} d\lambda(t) \right\}
\end{aligned}
$$

$$= \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s)\Big](d\theta(t) - \sum_{i=1}^N d\nu^{(i)}(t))\Big\} + K\,\mathbb{E}\Big\{ \int_{[0,T)} d\lambda(t)\Big\},$$

where $K := \theta_o - \sum_{i=1}^N y^{(i)} = \theta(0) - \sum_{i=1}^N \nu^{(i)}(0)$. Hence

$$\mathcal{L}^\theta(\underline{\nu}, \lambda) = \mathcal{J}_{SP}(\underline{\nu}) + \langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda\rangle$$

$$= \sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T e^{-\delta(t)} R^{(i)}(X(t), \nu^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t)\Big\}$$

$$+ \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s)\Big](d\theta(t) - \sum_{i=1}^N d\nu^{(i)}(t))\Big\} + K\,\mathbb{E}\Big\{ \int_{[0,T)} d\lambda(t)\Big\}.$$

We now obtain stochastic Kuhn-Tucker conditions for optimality with a stochastic Lagrange multiplier process that takes care of our dynamic resource constraint. A similar approach may be found in [2] for an intertemporal utility maximization problem under a constant badget constraint, with Hindy, Huang and Kreps preferences.

Theorem 3.3. *Under (2.2) and Assumption 2.1, an admissible investment vector $\underline{\nu}_*$ is the unique solution of the Social Planner problem (2.4) if there exists a nonnegative Lagrange multiplier measure $d\lambda(\omega, t)$ such that $\mathbb{E}\{\int_{[0,T)} d\lambda(t)\} < \infty$, and the following generalized stochastic Kuhn-Tucker conditions hold true for $i = 1, 2, ..., N$*

$$\begin{cases} \nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) \leq \mathbb{E}\Big\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t\Big\}, \quad \mathbb{P} - a.s., \; \forall t \in [0, T), \\[2mm] \int_{[0,T)} \Big[\nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) - \mathbb{E}\Big\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t\Big\}\Big] d\nu_*^{(i)}(t) = 0, \quad \mathbb{P} - a.s., \\[2mm] \mathbb{E}\Big\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^N \nu_*^{(i)}(t)] d\lambda(t)\Big\} = 0. \end{cases} \quad (3.9)$$

Proof. Let $\underline{\nu}_*$ satisfy the first order Kuhn-Tucker conditions (3.9) and let $\underline{\nu}$ be an arbitrary admissible plan. By concavity of $R^{(i)}(x, \cdot)$, $i = 1, 2, ..., N$, and Fubini's Theorem we have

$$\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) = \sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T e^{-\delta(t)} [R^{(i)}(X(t), \nu_*^{(i)}(t)) - R^{(i)}(X(t), \nu^{(i)}(t))]dt$$

$$- \int_{[0,T)} e^{-\delta(t)} d(\nu_*^{(i)}(t) - \nu^{(i)}(t))\Big\}$$

$$\geq \sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T e^{-\delta t} R_y^{(i)}(X(t), \nu_*^{(i)}(t))\,(\nu_*^{(i)}(t) - \nu^{(i)}(t))\,dt$$

$$- \int_{[0,T)} e^{-\delta(t)} d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\}$$

$$= \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \int_s^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu_*^{(i)}(t)) \, dt \, d(\nu_*^{(i)}(s) - \nu^{(i)}(s)) \tag{3.10}$$

$$- \int_{[0,T)} e^{-\delta(s)} d(\nu_*^{(i)}(s) - \nu^{(i)}(s)) \Big\}$$

$$= \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) \, d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\}.$$

Now (3.9) implies

$$\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) \geq \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) \, d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\}$$

$$\geq \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \mathbb{E} \Big\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t \Big\} d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\} \tag{3.11}$$

$$= \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\}$$

and the nonnegativity of $d\lambda(t)$, the admissibility of $\underline{\nu}$, and another application of Fubini's Theorem give

$$\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) \geq \sum_{i=1}^{N} \mathbb{E} \Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \Big\}$$

$$= \mathbb{E} \Big\{ \int_{[0,T)} \sum_{i=1}^{N} [\nu_*^{(i)}(t) - \nu^{(i)}(t)] d\lambda(t) \Big\}$$

$$= \mathbb{E} \Big\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^{N} \nu^{(i)}(t)] d\lambda(t) \Big\} \geq 0,$$

where the last line follows from (3.9), third condition. $\qquad\square$

Conditions (3.9) are also necessary for optimality under the assumption that

$$\omega \to \theta(\omega, T) \int_0^T R^{(i)}(X(\omega, t), \theta(\omega, T)) dt \quad \text{is} \quad d\mathbb{P} - \text{integrable}, \quad i = 1, 2, ..., N. \tag{3.12}$$

The proof is based on arguments similar to those used in the finite-dimensional Kuhn-Tucker Theorem. Denote by $\mathcal{T}$ the set of all stopping times in $[0, T]$, $\mathbb{P}$-a.s., and notice that

$$\nabla_y \mathcal{J}_i(\nu_*^{(i)})(\tau) \leq \mathbb{E} \Big\{ \int_{[\tau,T)} d\lambda(s) \Big| \mathcal{F}_\tau \Big\},$$

for every $i = 1, 2, ..., N$ and for all $\tau \in \mathcal{T}$. In fact, if not, then there would exist some $\overline{\tau} \in \mathcal{T}$ such that $\nabla_y \mathcal{J}_i(\nu_*^{(i)})(\overline{\tau}) > \mathbb{E}\{\int_{[\overline{\tau}, T)} d\lambda(s) | \mathcal{F}_{\overline{\tau}}\}$ which, togheter with the continuity of $R_y^{(i)}$ and the linearity of investment costs, would imply that a sufficiently small extra investment at $\overline{\tau}$ is profitable and hence contradict the optimality of $\nu_*^{(i)}$, $i = 1, 2, ..., N$.

In the next Lemma we show that under (3.12) the optimal policy $\underline{\nu}_*$ solves the linearized problem

$$\sup_{\underline{\nu} \in \mathcal{S}_\theta} \sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \Phi_i^*(s) d\nu^{(i)}(s) \right\} \tag{3.13}$$

where Φ_i^* is the progressively measurable process associated to $\nabla_y \mathcal{J}_i(\nu_*^{(i)})$, $i = 1, 2, ..., N$, and defined in (3.5). Solutions of the linear problem will then be characterized by some 'flat-off conditions' in the second Lemma.

Lemma 3.4. *Let $\underline{\nu}_*$ be optimal for problem (2.4) and assume (3.12). Then it solves (3.13).*

Proof. Let $\underline{\nu}$ be an admissible plan. For $i = 1, 2, ..., N$ and $\epsilon \in (0, 1)$, set $\nu_\epsilon^{(i)} = \epsilon \nu^{(i)} + (1 - \epsilon) \nu_*^{(i)}$ and let Φ_i^ϵ be the progressively measurable process defined in (3.5) associated to $\nabla_{\nu_i} \mathcal{J}_i(\nu_\epsilon^{(i)})$. Then $\lim_{\epsilon \to 0} \nu_\epsilon^{(i)}(t) = \nu_*^{(i)}(t)$, $\mathbb{P}$-a.s., as well as $\lim_{\epsilon \to 0} \Phi_i^\epsilon(t) = \Phi_i^*(t)$, $\mathbb{P}$-a.s., by continuity of $R_y^{(i)}$. Optimality of $\underline{\nu}_*$, concavity of $y \to R^{(i)}(X(t), y)$ and Fubini's Theorem, imply

$$
\begin{aligned}
0 &\geq \frac{1}{\epsilon} [\mathcal{J}_{SP}(\underline{\nu}_\epsilon) - \mathcal{J}_{SP}(\underline{\nu}_*)] \\
&= \frac{1}{\epsilon} \sum_{i=1}^{N} \mathbb{E}\left\{ \int_0^T e^{-\delta(t)} [R^{(i)}(X(t), \nu_\epsilon^{(i)}(t)) - R^{(i)}(X(t), \nu_*^{(i)}(t))] dt \right. \\
&\qquad\qquad \left. - \epsilon \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \right\} \\
&\geq \sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \Phi_i^\epsilon(t) d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \right\},
\end{aligned}
\tag{3.14}
$$

since $\epsilon(\nu^{(i)} - \nu_*^{(i)}) = \nu_\epsilon^{(i)} - \nu_*^{(i)}$.

In order to prove that

$$\sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \Phi_i^*(t) d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \right\} \leq 0$$

we need to apply Fatou's Lemma to conclude (by (3.14))

$$\sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \Phi_i^*(t) d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \right\} \leq \liminf_{\epsilon \to 0} \sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \Phi_i^\epsilon(t) d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \right\} \leq 0.$$

To check the hypothesis of Fatou's Lemma, we must find $d\mathbb{P}$-integrable random variables, $G_i(\omega)$, $i = 1, 2, ..., N$, such that

$$I_i^\epsilon(\omega) := \int_{[0,T)} \Phi_i^\epsilon(\omega, t) d(\nu^{(i)}(\omega, t) - \nu_*^{(i)}(\omega, t)) \geq G_i(\omega), \quad \omega \in \Omega, \ \epsilon \in (0, 1).$$

We write I_i^ϵ as

$$I_i^\epsilon = \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu_\epsilon^{(i)}(t))(\nu^{(i)}(t) - \nu_*^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \quad (3.15)$$

by Fubini's Theorem. Then, from concavity of $R^{(i)}(x, \cdot)$ and

$$\nu_\epsilon^{(i)}(t) \begin{cases} \leq \nu^{(i)}(t), & \text{on } \{t : \nu^{(i)}(t) - \nu_*^{(i)}(t) \geq 0\}, \\ \\ > \nu^{(i)}(t), & \text{on } \{t : \nu^{(i)}(t) - \nu_*^{(i)}(t) < 0\}. \end{cases} \quad (3.16)$$

we obtain

$$\begin{aligned} I_i^\epsilon \;\geq\; & \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))\,\mathbb{1}_{\{\nu^{(i)} \geq \nu_*^{(i)}\}}(t)dt \\ & + \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))\,\mathbb{1}_{\{\nu^{(i)} < \nu_*^{(i)}\}}(t)dt \\ & - \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \\ =\; & \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))dt \\ & - \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \\ =\; & \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(t)\, d(\nu^{(i)}(t) - \nu_*^{(i)}(t)). \end{aligned}$$

Hence we define

$$G_i(\omega) := \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(\omega, t) d(\nu^{(i)}(\omega, t) - \nu_*^{(i)}(\omega, t)), \quad \omega \in \Omega, \; i = 1, 2, ..., N. \quad (3.17)$$

Now (2.2), Assumption 2.1 and condition (3.12), imply the integrability of $G_i(\omega)$ since $|G_i(\omega)| \leq C[\theta(\omega, T) + (1 + \theta(\omega, T)) \int_0^T R^{(i)}(X(\omega, t), \theta(\omega, T))\, dt]$, $\omega \in \Omega$, with C a constant. $\qquad \square$

Lemma 3.5. *Let* f_i, $i = 1, 2, ..., N$ *be optional processes and define*

$$\mu(s) := \max \left\{ f_1^+(s), f_2^+(s), ..., f_N^+(s) \right\}. \quad (3.18)$$

Then every solution $\hat{\underline{\nu}}$ *to the linear optimization problem*

$$\sup_{\underline{\nu} \in \mathcal{S}_\theta} \sum_{i=1}^N \mathbb{E}\left\{ \int_{[0,T)} f_i(s)\, d\nu^{(i)}(s) \right\} \quad (3.19)$$

satisfies the 'flat-off conditions'

$$\mathbb{E}\left\{ \int_{[0,T)} (f_i(s) - \mu(s))\, d\hat{\nu}^{(i)}(s) \right\} = 0, \qquad i = 1, 2, ..., N. \quad (3.20)$$

Proof. Obviously

$$\sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} f_i(s)\, d\nu^{(i)}(s) \right\} \leq \sum_{i=1}^{N} \mathbb{E}\left\{ \int_{[0,T)} \mu(s)\, d\nu^{(i)}(s) \right\}. \tag{3.21}$$

The equality holds if and only if $\underline{\nu}$ satisfies (3.20). In fact (3.20) implies the equality. Conversely, if equality holds in (3.21), then $\sum_{i=1}^{N} \mathbb{E}\{\int_{[0,T)}(f_i(s) - \mu(s))\, d\nu^{(i)}(s)\} = 0$. Hence (3.20) follows from the fact that the integrands are nonpositive. $\qquad\square$

Remark 3.6. *We point out that our stochastic Kuhn-Tucker approach may be generalized to the case of investment processes also bounded from below by a stochastic process. In that case the Lagrangian functional is defined in terms of two Lagrange multipliers, $d\lambda_1(\omega, t)$ and $d\lambda_2(\omega, t)$.*

4   Applications of the Kuhn-Tucker Conditions

In this Section we test our approach on some 'finite-fuel' problems from the literature (cf. [5] and [22], among others) and we solve a N-firms Social Planner optimization problem. In the following examples we assume $\delta(t) = \delta t$, with $\delta > 0$, and $T = +\infty$.

4.1   The Finite Fuel Monotone Follower of Bank [5]

In the setting of Section 2, under (2.2) and Assumption 2.1, we take $N = 1$ and $T = +\infty$. We set $\nu := \nu^{(1)}$, $y := y^{(1)}$, $R := R^{(1)}$ and $\mathcal{J} := \mathcal{J}_1$. Notice that with

$$c(\omega, t, \nu(\omega, t)) := -e^{-\delta t} R(X(\omega, t), \nu(\omega, t)),$$

and instantaneous cost of investment

$$k(\omega, t) := -e^{-\delta t},$$

we recover Bank's model [5]. Recall that Bank's optimal investment (cf. [5], Theorem 2) was given by

$$\nu_*(t) := \sup_{0 \leq s < t} (l(s) \wedge \theta(s)) \vee y \tag{4.1}$$

in terms of the 'base capacity' process $l(t)$ (cf. [27] for this definition) which solves uniquely the stochastic backward equation (cf. [4], Theorem 3)

$$\mathbb{E}\left\{ \int_{\tau}^{\infty} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u))\, ds \,\Big|\, \mathcal{F}_\tau \right\} = e^{-\delta \tau}, \quad \forall \tau \in \mathcal{T}. \tag{4.2}$$

When $l(t)$ is a continuous process we show the optimality of $\nu_*(t)$ by means of our Generalized Kuhn-Tucker conditions; as a subproduct we obtain an enlightening interpretation of the first order conditions stated in [5], Theorem 1, for a single firm optimal investment problem. Notice that continuity of $l(t)$ is guaranteed when the shock process $X(t)$ is continuous as well, as in the case of a diffusion (cf. [27], Theorem 6.5).

Recall that the supergradient of the net profit functional is the unique optional process given by

$$\nabla_y \mathcal{J}(\nu)(t) := \mathbb{E}\left\{ \int_t^\infty e^{-\delta s} R_y\left(X(s), \nu(s)\right) ds \,\Big|\, \mathcal{F}_t \right\} - e^{-\delta t}. \tag{4.3}$$

By Theorem 3.3 an investment plan $\nu_*(t)$ is optimal if

$$\nabla_y \mathcal{J}(\nu_*)(t) \leq \mathbb{E}\left\{ \int_t^\infty d\lambda(s) \,\Big|\, \mathcal{F}_t \right\}, \quad \mathbb{P} - \text{a.s.,} \ t \geq 0, \tag{4.4}$$

$$\int_0^\infty \left[\nabla_y \mathcal{J}(\nu_*)(t) - \mathbb{E}\left\{ \int_t^\infty d\lambda(s) \,\Big|\, \mathcal{F}_t \right\} \right] d\nu_*(t) = 0, \quad \mathbb{P} - \text{a.s.,} \tag{4.5}$$

$$\nu_*(t) \leq \theta(t), \quad \mathbb{P} - \text{a.s.,} \ \forall t \geq 0, \tag{4.6}$$

$$\mathbb{E}\left\{ \int_0^\infty \left(\theta(t) - \nu_*(t) \right) d\lambda(t) \right\} = 0, \tag{4.7}$$

for some nonnegative optional random measure $d\lambda(\omega, t)$ such that $\mathbb{E}\{\int_0^\infty d\lambda(s)\} < +\infty$.

Lemma 4.1. *For almost every $\omega \in \Omega$ one has $[R_y\left(X(\omega, t), \theta(\omega, t)\right) - \delta]\, \mathbb{1}_{\{\nu_*(\omega, \cdot) = \theta(\omega, \cdot)\}}(t) \geq 0$.*

Proof. Fix $t \geq 0$. Then, for any stopping time $\tau_1 \geq t$ a.s., equation (4.2) and the decreasing property of R_y in its second argument imply that

$$e^{-\delta t} \leq \mathbb{E}\left\{ \int_t^{\tau_1} e^{-\delta s} R_y(X(s), \sup_{t \leq u < s} l(u)) ds \,\Big|\, \mathcal{F}_t \right\} + \mathbb{E}\left\{ e^{-\delta \tau_1} \,\Big|\, \mathcal{F}_t \right\} \quad \text{a.s.,}$$

hence

$$\mathbb{E}\left\{ \int_t^{\tau_1} e^{-\delta s} R_y(X(s), l(t)) ds \,\Big|\, \mathcal{F}_t \right\} \geq \mathbb{E}\left\{ e^{-\delta t} - e^{-\delta \tau_1} \,\Big|\, \mathcal{F}_t \right\} \quad \text{a.s.}$$

In particular, for $\epsilon > 0$, define $\tau_1(\epsilon) := \inf\{s \geq t : R_y\left(X(s), l(t)\right) > R_y\left(X(t), l(t)\right) + \epsilon\}$ to obtain

$$\mathbb{E}\left\{ \int_t^{\tau_1(\epsilon)} e^{-\delta s} R_y\left(X(s), l(t)\right) ds \,\Big|\, \mathcal{F}_t \right\} \leq \frac{1}{\delta} \left(R_y\left(X(t), l(t)\right) + \epsilon \right) \mathbb{E}\left\{ e^{-\delta t} - e^{-\delta \tau_1(\epsilon)} \,\Big|\, \mathcal{F}_t \right\} \quad \text{a.s.;}$$

that is, $R_y\left(X(t), l(t)\right) + \epsilon \geq \delta$ a.s. for all $\epsilon > 0$. It follows $R_y\left(X(t), l(t)\right) \geq \delta$ a.s., and hence $[R_y\left(X(t), \theta(t)\right) - \delta]\, \mathbb{1}_{\{l(\cdot) \geq \theta(\cdot)\}}(t) \geq 0$ a.s. for all $t \geq 0$, which provides the result since $\nu_*(\omega, t) = \theta(\omega, t)$ is equivalent to $l(\omega, t) \geq \theta(\omega, t)$ for almost every $\omega \in \Omega$. $\qquad \square$

Theorem 4.2. *If the base capacity process $l(t)$ has continuous paths, then $\nu_*(t)$ (cf. (4.1)) is optimal and the Lagrange multiplier $d\lambda(t)$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. It suffices to check the Generalized Kuhn-Tucker conditions (4.4) - (4.7) for $\nu_*(t)$. Obviously $\nu_*(t)$ satisfies (4.6). Recall that the available resources process $\theta(t)$ is increasing and left-continuous. To show (4.4) and (4.5), fix $\tau \in \mathcal{T}$, set $\tau_0 := \tau$, and recursively define

$$\begin{cases} \tau_{2n} := \inf\{s > \tau_{2n-1} : l(s) \leq \theta(s+)\} \\ \tau_{2n+1} := \inf\{s > \tau_{2n} : l(s) > \theta(s)\} \end{cases} \tag{4.8}$$

with the convention $\inf\{\emptyset\} = +\infty$. Notice that time τ_{2n+1}, $n \geq 0$, is a time of increase for $l(t)$. Then

$$\nu_*(s) = \theta(s) \qquad \text{for } s \in (\tau_{2n+1}, \tau_{2n+2}],$$

and

$$\nu_*(s) = \sup_{\tau_{2n} \leq u < s} l(u) \qquad \text{for } s \in (\tau_{2n}, \tau_{2n+1}],$$

by the continuity of $l(t)$. Moreover we have $l(s) \leq \theta(s)$ for $s \in (\tau, \tau_1]$, hence $\sup_{\tau \leq u < s}(l(u) \wedge \theta(u)) = \sup_{\tau \leq u < s} l(u)$.

Recalling (4.1) and the previous considerations we have

$$\mathbb{E}\left\{\int_\tau^\infty e^{-\delta s} R_y(X(s), \nu_*(s)) \, ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$= \mathbb{E}\left\{\int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \nu_*(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$+ \sum_{n=1}^\infty \mathbb{E}\left\{\int_{\tau_n}^{\tau_{n+1}} e^{-\delta s} R_y(X(s), \nu_*(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$\leq \mathbb{E}\left\{\int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u) \wedge \theta(u)) ds \,\Big|\, \mathcal{F}_\tau\right\} \qquad (4.9)$$

$$+ \sum_{n=1}^\infty \mathbb{E}\left\{\int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$+ \sum_{n=1}^\infty \mathbb{E}\left\{\int_{\tau_{2n}}^{\tau_{2n+1}} e^{-\delta s} R_y(X(s), \sup_{\tau_{2n} \leq u < s} l(u)) ds \,\Big|\, \mathcal{F}_\tau\right\},$$

where the equality holds if and only if τ is a point of increase for ν_*. By definition of τ_1, from (4.9) we get

$$\mathbb{E}\left\{\int_\tau^\infty e^{-\delta s} R_y(X(s), \nu_*(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$\leq \mathbb{E}\left\{\int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u)) ds \,\Big|\, \mathcal{F}_\tau\right\} \qquad (4.10)$$

$$+ \sum_{n=1}^\infty \mathbb{E}\left\{\int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}$$

$$+ \sum_{n=1}^\infty \mathbb{E}\left\{\int_{\tau_{2n}}^{\tau_{2n+1}} e^{-\delta s} R_y(X(s), \sup_{\tau_{2n} \leq u < s} l(u)) ds \,\Big|\, \mathcal{F}_\tau\right\}.$$

Since τ_1 and all odd indexed stopping times are times of increase for the process $l(t)$, hence $\sup_{\tau \leq u < s} l(u) = \sup_{\tau_1 \leq u < s} l(u)$ for $s > \tau_1$, and $\sup_{\tau_{2n} \leq u < s} l(u) = \sup_{\tau_{2n+1} \leq u < s} l(u)$ for $s > \tau_{2n+1}$. Therefore, from (4.10) the stochastic backward equation (4.2) implies

$$\mathbb{E}\left\{\int_\tau^\infty e^{-\delta s} R_y(X(s), \nu_*(s)) ds \,\Big|\, \mathcal{F}_\tau\right\} = \mathbb{E}\left\{e^{-\delta \tau} - e^{-\delta \tau_1} \,\Big|\, \mathcal{F}_\tau\right\}$$

$$+ \sum_{n=1}^{\infty} \mathbb{E}\left\{ \int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \,\Big|\, \mathcal{F}_\tau \right\}$$

$$+ \sum_{n=1}^{\infty} \mathbb{E}\left\{ e^{-\delta \tau_{2n}} - e^{-\delta \tau_{2n+1}} \,\Big|\, \mathcal{F}_\tau \right\}$$

$$= e^{-\delta \tau} + \sum_{n=1}^{\infty} \mathbb{E}\left\{ \int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} [R_y(X(s), \theta(s)) - \delta] ds \,\Big|\, \mathcal{F}_\tau \right\}$$

$$= e^{-\delta \tau} + \mathbb{E}\left\{ \int_{\tau}^{\infty} e^{-\delta s} [R_y(X(s), \theta(s)) - \delta] \mathbb{1}_{\{\nu_* = \theta\}}(s) ds \,\Big|\, \mathcal{F}_\tau \right\}.$$

Notice that the process $e^{-\delta t}[R_y(X(t), \theta(t)) - \delta] \mathbb{1}_{\{\nu_* = \theta\}}(t)$ is nonnegative by Lemma 4.1 and it is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$−measurable. Hence, we set

$$d\lambda(t) := e^{-\delta t}[R_y(X(t), \theta(t)) - \delta] \mathbb{1}_{\{\nu_* = \theta\}}(t) dt \qquad (4.11)$$

and we show that it is the optional measure Lagrange multiplier. Let us start by showing that $d\lambda(t)$ is an optional random measure on $\mathbb{R}_+$. That is, the continuous, increasing process

$$\Lambda(t) := \int_{[0,t)} d\lambda(s) \qquad (4.12)$$

is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Assumption 2.1 and concavity of R in the second argument, imply that

$$\begin{aligned}
\mathbb{E}\{\Lambda(t)\} &= \mathbb{E}\left\{ \int_0^t e^{-\delta s} [R_y(X(s), \theta(s)) - \delta] \mathbb{1}_{\{\nu_* = \theta\}}(s) \, ds \right\} \\
&\leq \mathbb{E}\left\{ \int_0^t e^{-\delta s} R_y(X(s), \theta_o) \mathbb{1}_{\{\nu_* = \theta\}}(s) \, ds \right\} \\
&\leq \mathbb{E}\left\{ \int_0^{\infty} e^{-\delta s} R_y(X(s), \theta_o) \, ds \right\} \qquad (4.13) \\
&\leq \frac{1}{\theta_o} \mathbb{E}\left\{ \int_0^{\infty} \sup_{\nu(s) \in \mathcal{S}_\theta} R(X(s), \nu(s)) \, ds \right\} < +\infty.
\end{aligned}$$

Hence $\Lambda(t)$ is $d\mathbb{P}$-integrable and $e^{-\delta t}[R_y(X(t), \theta(t)) - \delta] \mathbb{1}_{\{\nu_* = \theta\}}(t)$ is $d\mathbb{P} \otimes dt$-integrable on $\Omega \times \mathbb{R}_+$. Therefore, by Fubini's Theorem, the application $\omega \to \Lambda(\omega, t)$ is $\mathcal{F}_t$-measurable and hence Λ is adapted. Then it is predictable since it is continuous.

It follows that (4.4) and (4.5) hold and hence the process (4.1) is optimal by Theorem 3.3. $\square$

Remark 4.3. *The usual interpretation of the Lagrange multiplier as the shadow price of the value function may be heuristically shown as follows. After an integration by parts on the cost term, we may write the value function as*

$$V(\theta) = \mathbb{E}\left\{ \int_0^{\infty} e^{-\delta t} \Big[R(X(t), \sup_{0 \leq s < t} (l(s) \wedge \theta(s))) - \delta \sup_{0 \leq s < t} (l(s) \wedge \theta(s)) \Big] dt \right\}.$$

Now, if $\nu_*(t) = \sup_{0 \leq s < t}(l(s) \wedge \theta(s))$, then $\mathbb{1}_{\{\nu_* = \theta\}}$ is the derivative (in some sense) of ν_* with respect to θ. We thus expect that the 'derivative' of V with respect to the constraint θ is

$e^{-\delta t}[R_y(X(t), \theta(t)) - \delta]\mathbb{1}_{\{\nu_* = \theta\}}(t)$, *which is exactly the density of the Lagrange multiplier in the case of a continuous 'base capacity' $l(t)$.*

Proposition 4.4. *The process $\mathcal{G}(t) := \mathbb{E}\{\Lambda(\infty)\,|\mathcal{F}_t\}$ is a uniformly integrable martingale.*

Proof. By Assumption 2.1 the random variable $\Lambda(\infty)$ (cf. (4.12)) is $d\mathbb{P}$-integrable. Hence the process $\mathcal{G}(t)$ is a uniformly integrable martingale. $\qquad\square$

Proposition 4.5. *The process*

$$\mathcal{U}(t) := \mathbb{E}\left\{\int_t^\infty d\lambda(s)\,\Big|\mathcal{F}_t\right\} = \mathbb{E}\{\Lambda(\infty)\,|\mathcal{F}_t\} - \Lambda(t) = \mathcal{G}(t) - \Lambda(t) \tag{4.14}$$

is a supermartingale of class (D) and $\mathcal{G}(t) - \Lambda(t)$ is its unique Doob-Meyer decomposition.

Proof. Recall (cf. proof of Theorem 4.2) that the process $\Lambda(t)$ of (4.12) is increasing, adapted, continuous and integrable. Then $\mathcal{U}(t)$ is $d\mathbb{P}$-integrable. Moreover, being $d\lambda$ nonnegative, for $s \leq t$ we have $\mathbb{E}\{\mathcal{U}(t)\,|\mathcal{F}_s\} \leq \mathcal{U}(s)$, i.e. $\mathcal{U}(t)$ is a supermartingale. Assumption 2.1 guarantees that it belongs to class (D). Hence $\mathcal{G}(t) - \Lambda(t)$ is the unique Doob-Meyer decomposition of the supermartingale $\mathcal{U}(t)$ and therefore the process $\Lambda(t)$ is the compensator of $\mathcal{U}(t)$. $\qquad\square$

If $\mathbb{S}(\nu)$ is the Snell envelope of the supergradient $\nabla_y \mathcal{J}(\nu)$, i.e.

$$\mathbb{S}(\nu)(t) = \operatorname*{ess\,sup}_{t \leq \tau \leq +\infty} \mathbb{E}\left\{\nabla_y \mathcal{J}(\nu)(\tau)|\mathcal{F}_t\right\}, \tag{4.15}$$

then [5], Theorem 1, claims that the optimal investment plan ν_* is characterized by the following conditions

$$\begin{cases} \nu_* \text{ is flat off } \{\nabla_y \mathcal{J}(\nu_*) = \mathbb{S}(\nu_*)\} \\ A(\nu_*) \text{ is flat off } \{\nu_* = \theta\}, \end{cases} \tag{4.16}$$

where $A(\nu_*)$ is the predictable increasing process in the Doob-Meyer decomposition of the supermartingale $\mathbb{S}(\nu_*)$. Moreover $\mathbb{S}(\nu_*)(t) = \mathbb{E}\{A(\infty) - A(t)\,|\mathcal{F}_t\}$ since $\nabla_y \mathcal{J}(\nu_*)(\infty) = 0$. If (3.12) holds, then (4.16), (4.4) and (4.5) imply that

$$\mathcal{U}(t) \equiv \mathbb{S}(\nu_*)(t) \tag{4.17}$$

at times of investment (when $A(\nu_*)$ and $d\lambda$ are not flat). This argument allows an enlightening interpretation of the increasing, predictable, integrable process $\Lambda(t)$. In fact at times of investment

$$\mathcal{G}(t) - \Lambda(t) = \mathbb{E}\left\{\int_t^\infty d\lambda(s)\,\Big|\mathcal{F}_t\right\} \equiv \mathbb{S}(\nu_*)(t) = \mathcal{M}(\nu_*)(t) - A(\nu_*)(t), \tag{4.18}$$

where $\mathcal{M}(\nu_*)$ is the martingale process in the unique Doob-Meyer decomposition of $\mathbb{S}(\nu_*)$. By uniqueness

$$\mathbb{E}\left\{\int_0^\infty d\lambda(s)\,\Big|\mathcal{F}_t\right\} \equiv \mathcal{M}(\nu_*)(t) \quad \text{and} \quad \Lambda(t) \equiv A(\nu_*)(t), \tag{4.19}$$

hence

$$dA(\nu_*)(t) \equiv d\lambda(t), \ \ \mathbb{P} - \text{a.s.}, \ \forall t \geq 0. \tag{4.20}$$

Therefore the second first order condition of (4.16) coincides with the Kuhn-Tucker condition (4.7); that is the Lagrange multiplier acts only when the constraint is binding.

When $l(t)$ is continuous, the explicit form of the Lagrange multiplier is known (cf. (4.11)), hence the compensator $A(\nu_*)(t)$ is known as well. It follows that its paths are absolutely continuous with respect to the Lebesgue measure and the Radon-Nykodym derivative of $dA(\nu_*)(t)$ is $e^{-\delta t}[R_y(X(t), \theta(t)) - \delta]\mathbb{1}_{\{\nu_* = \theta\}}(t)$.

4.2 N Firms: Finite Fuel and Operating Profit of Cobb-Douglas Type

In the setting of Section 2, with $T = +\infty$, we consider the Social Planner optimal investment problem (2.4) for a market with N firms endowed with operating profit functions of Cobb-Douglas type, i.e. $R^{(i)}(x, y) = \frac{x^{\alpha_i} y^{1-\alpha_i}}{1-\alpha_i}$ with $\alpha_i \in (0,1)$, $i = 1, 2, ..., N$.

Suppose that the economic shock process $X(t)$ is given by $X(t) = \exp\{Y(t)\}$ for some Levy process $Y(t)$ such that $Y(0) = 0$ and with finite Laplace transform. Then (cf. [27], Proposition 7.1)

$$l^{(i)}(t) = k_i X(t), \qquad i = 1, 2, ..., N, \tag{4.21}$$

with

$$k_i = \left(\mathbb{E}\left\{\int_0^{+\infty} e^{-\delta t} e^{\alpha_i \inf_{0 \le u < t} Y(u)} dt\right\}\right)^{\frac{1}{\alpha_i}}, \qquad i = 1, 2, ..., N,$$

is the unique optional solution of the stochastic backward equation

$$\mathbb{E}\left\{\int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \sup_{\tau \le u < s} l^{(i)}(u)) ds \,\Big|\, \mathcal{F}_\tau\right\} = e^{-\delta \tau}, \qquad \forall \tau \in \mathcal{T}. \tag{4.22}$$

Define the optional process

$$\beta_i(t) := \frac{l^{(i)}(t)}{\sum_{j=1}^N l^{(j)}(t)}. \tag{4.23}$$

Here $\beta_i(t)$ may be thought as the fraction of desirable investment of the i-th firm. By (4.21), for $t \ge 0$ and $i = 1, 2, ..., N$, we have that $\beta_i(t)$ is constant in time; in fact $\beta_i(t) = \frac{k_i}{\sum_{j=1}^N k_j} =: \beta_i$.

Fix $\tau \in \mathcal{T}$ and introduce the random times

$$\begin{cases} \sigma_1(\tau) = \inf\{s \ge \tau : \sum_{i=1}^N l^{(i)}(s) > \theta(s)\} \\[2mm] \sigma_2(\tau) = \inf\{s \ge \tau : l^{(i)}(s) > \beta_i \theta(s), \ \forall i = 1, 2, ..., N\}. \end{cases} \tag{4.24}$$

Lemma 4.6. *For all $\tau \in \mathcal{T}$ we have $\sigma_1(\tau) = \sigma_2(\tau)$ $\mathbb{P}$-almost surely.*

Proof. Notice that (4.21) implies $\sigma_1(\tau) = \inf\{s \ge \tau : X(s) > \frac{\theta(s)}{\sum_{i=1}^N k_i}\} = \inf\{s \ge \tau : k_i X(s) > \beta_i \theta(s), \ \forall i = 1, 2, ..., N\} = \sigma_2(\tau)$. $\qquad\square$

Remark 4.7. *If $\tau \in \mathcal{T}$ is a time of investment for all firms, that is $d\nu_*^{(i)}(\tau) > 0$ for all i, then the first Kuhn-Tucker condition in (3.9) guarantees that*

$$\mathbb{E}\left\{\int_\tau^{+\infty} e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s)) ds \,\Big|\, \mathcal{F}_\tau\right\} = \mathbb{E}\left\{\int_\tau^{+\infty} e^{-\delta s} R_y^{(j)}(X(s), \nu_*^{(j)}(s)) ds \,\Big|\, \mathcal{F}_\tau\right\}.$$

Notice that if X is continuous, then $l^{(i)}$ is continuous too due to (4.21).

Theorem 4.8. *If the shock process $X(t)$ is continuous then the process $\underline{\nu}_*$ with components*

$$\nu_*^{(i)}(t) = \sup_{0 \le u < t} (l^{(i)}(u) \wedge \beta_i \theta(u)) \vee y^{(i)}, \qquad i = 1, 2, ..., N, \tag{4.25}$$

is optimal for problem (2.4). Moreover, the Lagrange multiplier $d\lambda(t)$ associated to (2.4) is absolutely continuous with respect to the Lebesgue measure.

Proof. Let us check that $\nu_*^{(i)}(t)$ satisfies the first order conditions of Theorem 3.3. Obviously $\sum_{i=1}^{N} \nu_*^{(i)}(t) \le \theta(t)$ a.s. for all $t \ge 0$.

The arguments of the proof are similar to those in the proof of Theorem 4.2. Fix $\tau \in \mathcal{T}$, set $\tau_0 := \tau$ and define the sequence of stopping times τ_n as in (4.8) but with $\sum_{i=1}^{N} l^{(i)}$ instead of l; that is,

$$\begin{cases} \tau_{2n+1} := \inf\{s > \tau_{2n} : \sum_{i=1}^{N} l^{(i)}(s) > \theta(s)\} \\ \tau_{2n+2} := \inf\{s > \tau_{2n+1} : \sum_{i=1}^{N} l^{(i)}(s) \le \theta(s+)\}. \end{cases} \tag{4.26}$$

Notice that the continuity of $l^{(i)}$ implies

$$\nu_*^{(i)}(s) = \sup_{\tau_{2n} \le u < s} l^{(i)}(u) \qquad \text{for } s \in (\tau_{2n}, \tau_{2n+1}].$$

Also $\tau_{2n+1} = \sigma_1(\tau_{2n}) = \sigma_2(\tau_{2n})$ by Lemma 4.6, hence τ_{2n+1} is a time of increase for all $l^{(i)}$. It follows

$$\nu_*^{(i)}(s) = \beta_i \theta(s) \qquad \text{for } s \in (\tau_{2n+1}, \tau_{2n+2}].$$

Fix $i = 1, 2, ..., N$, and consider $\mathbb{E}\{\int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s))ds \,|\mathcal{F}_\tau\}$. Split the integral into two integrals $\int_\tau^{\tau_1}$ and $\int_{\tau_1}^\infty$. Since τ_1 is a time of increase for every $l^{(i)}$, Remark (4.7) holds and we may write

$$\mathbb{E}\left\{ \int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s))ds \,\Big|\mathcal{F}_\tau \right\} = \mathbb{E}\left\{ \int_\tau^{\tau_1} e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s))ds \,\Big|\mathcal{F}_\tau \right\} \tag{4.27}$$

$$+ \mathbb{E}\left\{ \int_{\tau_1}^\infty e^{-\delta s} \beta_i R_y^{(i)}(X(s), \nu_*^{(i)}(s))ds \,\Big|\mathcal{F}_\tau \right\} + \mathbb{E}\left\{ \int_{\tau_1}^\infty e^{-\delta s} \sum_{j \ne i} \beta_j R_y^{(j)}(X(s), \nu_*^{(i)}(s))ds \,\Big|\mathcal{F}_\tau \right\}$$

since $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_1}$. Now, as in the proof of Theorem 4.2, we use the stopping times τ_n to split the last two integrals above and by the backward equation (4.22) corresponding to $l^{(i)}(t)$ we may write

$$\mathbb{E}\left\{ \int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s))ds \,\Big|\mathcal{F}_\tau \right\} \tag{4.28}$$

$$\le e^{-\delta \tau} + \mathbb{E}\left\{ \int_\tau^\infty e^{-\delta s} \Big[\sum_{i=1}^{N} \beta_i R_y^{(i)}(X(s), \beta_i \theta(s)) - \delta \Big] \mathbb{1}_{\{\sum_{i=1}^{N} \nu_*^{(i)} = \theta\}}(s)ds \,\Big|\mathcal{F}_\tau \right\},$$

with equality if and only if $d\nu_*^{(i)}(\tau) > 0$. Hence

$$\rho(t) := e^{-\delta t} \Big[\sum_{i=1}^{N} \beta_i(R_y^{(i)}(X(t), \beta_i \theta(t)) - \delta) \Big] \mathbb{1}_{\{\sum_{i=1}^{N} \nu_*^{(i)} = \theta\}}(t)$$

is nonnegative by Lemma 4.1. We may now define the Lagrange multiplier for the N-firms Social Planner problem by $d\lambda(t) := \rho(t)dt$ since such $d\lambda$ is a nonnegative optional measure as in the proof of Theorem 4.2. □

Remark 4.9. *For general operating profit functions satisfying Assumption 2.1, we expect the solution for the Social Planner problem (2.4) to be*

$$\nu_*^{(i)}(t) = \sup_{0 \leq u < t} (l^{(i)}(u) \wedge \beta_i(u)\theta(u)) \vee y^{(i)}, \qquad i = 1, 2, ..., N,$$

with

$$\beta_i(t) := \frac{l^{(i)}(t)}{\sum_{j=1}^{N} l^{(j)}(t)}.$$

4.3 Constant Finite Fuel and Quadratic Cost

Here we consider a monotone follower problem with constant finite fuel similar to those studied by Karatzas ([20], [22]), and Karatzas and Shreve [21] (among others). In particular we discuss the example (cf. [5]) of optimal cost minimization for a firm that does not incur into investment's costs and has a running cost flow given by the convex function $c(x, y) = \frac{1}{2}(x-y)^2$ of the economic shock x and the investment y. That is, we study the constrained convex minimization problem

$$\inf_{\nu \in \mathcal{S}_{\theta_o}} \mathcal{C}(\nu) := \inf_{\nu \in \mathcal{S}_{\theta_o}} \mathbb{E}\left\{ \int_0^\infty \delta e^{-\delta s} \frac{1}{2}(W(t) - \nu(t))^2 \, dt \right\} \tag{4.29}$$

where $W(t)$ is a standard Brownian motion and θ_o is the positive constant finite fuel such that $\nu(t) \leq \theta_o$, $\mathbb{P}$-a.s. for all $t \geq 0$.

We expect to find a nonpositive Lagrange multiplier. Notice that

$$\nabla_y \mathcal{C}(\nu)(t) = \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s}(\nu(s) - W(s)) \, ds \,\Big|\, \mathcal{F}_t \right\}. \tag{4.30}$$

Moreover, the backward equation

$$\mathbb{E}\left\{ \int_\tau^\infty \delta e^{-\delta s} \sup_{\tau \leq u < s} l(u) \, ds \,\Big|\, \mathcal{F}_\tau \right\} = e^{-\delta \tau} W(\tau), \quad \forall \tau \in \mathcal{T}, \tag{4.31}$$

is uniquely solved by the base capacity

$$l(s) = W(s) - c, \tag{4.32}$$

where c is the positive constant $c := \mathbb{E}\{\int_0^\infty \delta e^{-\delta s} \sup_{0 \leq u < s} W(u) \, ds\}$, by independence and time-homogeneity of Brownian increments.

From [5] we know that the optimal investment policy is

$$\nu_*(t) = \sup_{0 \leq s < t} ((W(s) - c) \wedge \theta_o) \vee \nu(0), \tag{4.33}$$

which is the well known strategy of reflecting the Brownian motion at the threshold c until all the fuel is spent (cf. [22]).

We may write the subgradient (4.30) at ν_* as

$$
\begin{aligned}
\nabla_y \mathcal{C}(\nu_*)(t) &= \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s}\left(\nu_*(s) - W(s)\right) ds \,\Big|\, \mathcal{F}_t \right\} - 0 \\
&= \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s}\left(\nu_*(s) - W(s)\right) ds \,\Big|\, \mathcal{F}_t \right\} \\
&\quad - \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s} \sup_{t \leq u < s}(W(u) - c)\, ds \,\Big|\, \mathcal{F}_t \right\} + e^{-\delta t} W(t) \\
&= \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s}\left[\nu_*(s) - \sup_{t \leq u < s}(W(u) - c)\right] ds \,\Big|\, \mathcal{F}_t \right\}
\end{aligned}
$$

where we have used (4.31) in the second equality with l given by (4.32). With this trivial trick we are in the same setting as [5], proof of Theorem 2. Hence we have that the Snell envelope of the subgradient evaluated at the optimum ν_* (cf. (4.33)) is

$$
\mathbb{S}(\nu_*)(t) = \mathbb{E}\left\{ \int_t^\infty \delta e^{-\delta s}\left[\nu_*(s) - \sup_{t \leq u < s}(W(u) - c)\right] \wedge 0\, ds \,\Big|\, \mathcal{F}_t \right\} \tag{4.34}
$$

or, equivalently,

$$
\mathbb{S}(\nu_*)(t) = \mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty \delta e^{-\delta s}\left[\theta_o - \sup_{t \leq u < s}(W(u) - c)\right] ds \,\Big|\, \mathcal{F}_t \right\}
$$

with

$$
\tau_{\theta_o}(t) := \inf\{s \geq t : W(s) - c > \theta_o\}, \tag{4.35}
$$

by means of (4.33). Notice that $\tau_{\theta_o}(t)$ is a time of increase for $W(t) - c$. Hence we have $\sup_{t \leq u < s}(W(u) - c) = \sup_{\tau_{\theta_o}(t) \leq u < s}(W(u) - c)$ for $s \in (\tau_{\theta_o}(t), +\infty]$. Therefore (4.31) implies

$$
\mathbb{S}(\nu_*)(t) = \mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty \delta e^{-\delta s} \theta_o\, ds \,\Big|\, \mathcal{F}_t \right\} - \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} W(\tau_{\theta_o}(t)) \,\Big|\, \mathcal{F}_t \right\};
$$

that is,

$$
\mathbb{S}(\nu_*)(t) = \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)}\left[\theta_o - W(\tau_{\theta_o}(t))\right] \,\Big|\, \mathcal{F}_t \right\}. \tag{4.36}
$$

Now we first find the explicit form of the Snell envelope $\mathbb{S}(\nu_*)(t)$, and then we use it to identify the compensator part of its Doob-Meyer decomposition; that is, the Lagrange multiplier of problem (4.29) (cf. (4.20)). We start by showing that $S(\nu_*)$ is an $\mathcal{F}_u$-martingale until the base capacity $l(t) = W(t) - c$ is below the finite fuel θ_o; that is, $\{\mathbb{S}(\nu_*)(u \wedge \tau_{\theta_0}(t)\}_{u \geq t}$ is an $\mathcal{F}_u$-martingale. In fact, by [26], Corollary 3.6 it suffices to prove that $\{\mathbb{S}(\nu_*)(u \wedge \tau_{\theta_0}(t)\}_{u \geq t}$ is an $\mathcal{F}_{u \wedge \tau_{\theta_o}(t)}$-martingale, and that follows by iterated conditioning and the fact that $\tau_{\theta_o}(u_1 \wedge \tau_{\theta_o}(t)) = \tau_{\theta_o}(u_2 \wedge \tau_{\theta_o}(t))$ for all $u_1, u_2 \in [t, \infty)$. Next, we define

$$
\sigma_{\theta_o}(t) := \inf\{s > t : W(s) \leq \theta_o + c\},
$$

and we show that $\mathbb{S}(\nu_*)$ is an $\mathcal{F}_u$-submartingale when the base capacity $l(t)$ is above the finite fuel θ_o; that is $\{\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t)\}_{u \geq t}$ is an $\mathcal{F}_u$-submartingale. Again, as above, it suffices to prove

that it is an an $\mathcal{F}_{u \wedge \sigma_{\theta_o}(t)}$-submartingale. In fact, from $\tau_{\theta_o}(u \wedge \sigma_{\theta_0}(t)) = u \wedge \sigma_{\theta_o}(t)$ for all $u \geq t$, the martingale property of W and $\delta e^{-\delta s} W(s) ds = -d(e^{-\delta s} W(s)) + e^{-\delta s} dW(s)$, follows that the process $\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t) + \int_t^{u \wedge \sigma_{\theta_o}(t)} \delta e^{-\delta s}(\theta_o - W(s)) ds$ is an $\mathcal{F}_{u \wedge \sigma_{\theta_o}(t)}$-martingale, hence an $\mathcal{F}_u$-martingale. Therefore $\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t))$ is an $\mathcal{F}_u$-submartingale with absolutely continuous compensator $A(\nu_*)$ given by

$$dA(\nu_*)(s) := -\delta e^{-\delta s}(\theta_o - W(s)) \, ds, \quad s \in [t, u \wedge \sigma_{\theta_o}(t)], u \geq t, \tag{4.37}$$

since $W(\cdot) < \theta_o$ on $[t, u \wedge \sigma_{\theta_o}(t)]$, $u \geq t$.

Now, as the Lagrange multiplier (4.11) acts only when $\nu_*(t) = \theta_o$, i.e. only when $l(t) > \theta_o$, we conclude that the Lagrange multiplier of problem (4.29) is

$$d\lambda(t) = \delta e^{-\delta t} [\theta_o - W(t)] \, \mathbb{1}_{\{W(\cdot) > \theta_o + c\}}(t) \, dt, \tag{4.38}$$

which, as expected, is negative and coincides with the opposite of the optional measure $dA(\nu_*)(t)$ (cf. (4.37)).

Remark 4.10. *In [7] Benes, Shepp and Witsenhausen considered a problem with the same cost functional but they allowed controls of bounded variation.*

4.4 Constant Finite Fuel and Operating Profit of Cobb-Douglas Type

We consider the maximization problem of profit, net of investment costs,

$$\sup_{\nu \in \mathcal{S}_{\theta_o}} \mathcal{J}(\nu) := \sup_{\nu \in \mathcal{S}_{\theta_o}} \mathbb{E} \left\{ \int_0^\infty e^{-\delta s} R(X(s), \nu(s)) \, ds - \int_0^\infty e^{-\delta s} d\nu(s) \right\}. \tag{4.39}$$

The finite fuel is given by the positive constant θ_o, hence the controls satisfy $0 \leq \nu(t) \leq \theta_o$ $\mathbb{P}$-a.s., for all $t \geq 0$. The economic shock process $X(t)$ is modeled by a Geometric Brownian motion

$$X(t) = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \qquad \text{with} \quad x_0 > 0. \tag{4.40}$$

The firm's operating profit function is of the Cobb-Douglas type and depends on the economic shock x and the investment policy y; i.e., $R(x, y) = \frac{1}{1-\alpha} x^\alpha y^{1-\alpha}$ with $0 < \alpha < 1$. As pointed out in [27] this construction is consistent with a competitive firm which produces at decreasing returns to scale or with a monopolist firm facing a constant elasticity demand function and constant returns to scale production. Notice that problem (4.39) has been studied in detail in [25] in the case of $\theta_o = +\infty$ by a dynamic programming approach.

It is known (cf. [5]) that the unique optimal solution for problem (4.39) is given by

$$\nu_*(t) = \sup_{0 \leq s < t} (l(s) \wedge \theta_o) \vee \nu(0), \tag{4.41}$$

where the optional process $l(t)$ uniquely solves the stochastic backward equation (cf. [4])

$$\mathbb{E} \left\{ \int_\tau^\infty e^{-\delta s} X^\alpha(s) \left(\sup_{\tau \leq u < s} l(u) \right)^{-\alpha} ds \Big| \mathcal{F}_\tau \right\} = e^{-\delta \tau}, \quad \forall \tau \in \mathcal{T}. \tag{4.42}$$

As shown in [27], Proposition 7.1, when the shock process is of exponential Levy type, i.e. $X(t) = x_0 e^{Y(t)}$, with $Y(t)$ a Levy process such that $Y(0) = 0$, then the solution of (4.42) is given by the base capacity

$$l(t) = kX(t), \tag{4.43}$$

where $k = (\frac{1}{\delta}\mathbb{E}\{e^{\alpha\underline{Y}(\tau(\delta))}\})^{\frac{1}{\alpha}}$, $\underline{Y}(t) := \inf_{0 \le u \le t} Y(u)$ and $\tau(\delta)$ is an independent exponentially distributed time with parameter δ.

From (4.39) we have

$$\nabla_y \mathcal{J}(\nu)(t) = \mathbb{E}\left\{ \int_t^\infty e^{-\delta s} X^\alpha(s) \nu^{-\alpha}(s) ds \,\Big|\, \mathcal{F}_t \right\} - e^{-\delta t}. \tag{4.44}$$

Following [5], proof of Theorem 2, we know that the Snell envelope of supergradient (4.44) evaluated at the optimal control policy (4.41) is

$$\mathbb{S}(\nu_*)(t) \tag{4.45}$$
$$= \mathbb{E}\left\{ \int_t^\infty e^{-\delta s} \Big[X^\alpha(s) \Big(\Big(\sup_{0 \le u < s} (kX(u) \wedge \theta_o) \vee \nu(0) \Big)^{-\alpha} - \Big(\sup_{t \le u < s} kX(u) \Big)^{-\alpha} \Big) \Big]^+ ds \,\Big|\, \mathcal{F}_t \right\}.$$

Fix $t \ge 0$ and define the stopping time

$$\tau_{\theta_o}(t) := \inf\{s \ge t : kX(s) > \theta_o\}. \tag{4.46}$$

It is a time of increase for $X(t)$. Now we split the integral into $\int_t^{\tau_{\theta_o}(t)} + \int_{\tau_{\theta_o}(t)}^\infty$, then the first one vanishes due to (4.46) and we are left with

$$\mathbb{S}(\nu_*)(t) = \mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty e^{-\delta s} \Big[X^\alpha(s) \Big((\theta_o)^{-\alpha} - (\sup_{t \le u < s} kX(u))^{-\alpha} \Big) \Big] ds \,\Big|\, \mathcal{F}_t \right\}$$
$$= (\theta_o)^{-\alpha} \mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty e^{-\delta s} X^\alpha(s) \, ds \,\Big|\, \mathcal{F}_t \right\} - \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} \,\Big|\, \mathcal{F}_t \right\}$$

where we have used (4.42) to obtain the second equality.

Lemma 4.11. *Assume $\delta > \mu + \sigma^2$. Then for every $t \ge 0$, one has*

$$\mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty e^{-\delta s} X^\alpha(s) \, ds \,\Big|\, \mathcal{F}_t \right\} = \frac{1}{(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha)} \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} X^\alpha(\tau_{\theta_o}(t)) \,\Big|\, \mathcal{F}_t \right\}. \tag{4.47}$$

Proof. The proof follows from the Markov property and the Laplace transform of a Gaussian process. Independence of Brownian increments, together with $W(u + \tau_{\theta_o}(t)) - W(\tau_{\theta_o}(t)) \sim W(u)$, allow us to write

$$\mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty e^{-\delta s} X^\alpha(s) \, ds \,\Big|\, \mathcal{F}_t \right\} = \mathbb{E}\left\{ \mathbb{E}\left\{ \int_{\tau_{\theta_o}(t)}^\infty e^{-\delta s} X^\alpha(s) \, ds \,\Big|\, \mathcal{F}_{\tau_{\theta_o}(t)} \right\} \,\Big|\, \mathcal{F}_t \right\}$$
$$= \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} X^\alpha(\tau_{\theta_o}(t)) \mathbb{E}\left\{ \int_0^\infty e^{-\delta u} e^{\alpha(\mu - \frac{1}{2}\sigma^2)u + \alpha\sigma(W(u+\tau_{\theta_o}(t)) - W(\tau_{\theta_o}(t)))} \, ds \right\} \,\Big|\, \mathcal{F}_t \right\}$$
$$= \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} X^\alpha(\tau_{\theta_o}(t)) \mathbb{E}\left\{ \int_0^\infty e^{-\delta u} e^{\alpha(\mu - \frac{1}{2}\sigma^2)u + \alpha\sigma W(u)} \, ds \right\} \,\Big|\, \mathcal{F}_t \right\}$$
$$= \mathbb{E}\left\{ e^{-\delta \tau_{\theta_o}(t)} X^\alpha(\tau_{\theta_o}(t)) \int_0^\infty e^{-(\delta - \mu\alpha)u - \frac{1}{2}\sigma^2\alpha(1-\alpha)u} du \,\Big|\, \mathcal{F}_t \right\}.$$

Notice that $(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha) > 0$ by the assumption, hence (4.47) follows. $\qquad\square$

Now Lemma 4.11 and (4.47) imply

$$\mathbb{S}(\nu_*)(t) = \frac{(\theta_o)^{-\alpha}}{(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha)} \mathbb{E}\left\{ e^{-\delta\tau_{\theta_o}(t)} X^\alpha(\tau_{\theta_o}(t)) \Big| \mathcal{F}_t \right\} - \mathbb{E}\left\{ e^{-\delta\tau_{\theta_o}(t)} \Big| \mathcal{F}_t \right\}. \qquad (4.48)$$

By arguments similar to those used in Subsection 4.3 we find the explicit form of the Snell envelope $\mathbb{S}(\nu_*)(t)$ and hence we identify the compensator part of its Doob-Meyer decomposition as the Lagrange multiplier of problem (4.39). In fact, we have that $S(\nu_*)$ is an $\mathcal{F}_u$-martingale until the base capacity $l(t) = kX(t)$ is below the finite fuel θ_o, since $\tau_{\theta_o}(u_1 \wedge \tau_{\theta_o}(t)) = \tau_{\theta_o}(u_2 \wedge \tau_{\theta_o}(t))$ for all $u_1, u_2 \in [t, \infty)$. Then, if

$$\sigma_{\theta_o}(t) := \inf\{s > t : kX(s) \leq \theta_o\},$$

we show that $\mathbb{S}(\nu_*)$ is an $\mathcal{F}_u$-supermartingale when the base capacity $l(t)$ is above the finite fuel θ_o; that is $\{\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t)\}_{u \geq t}$ is an $\mathcal{F}_u$-supermartingale. It suffices to prove that it is an $\mathcal{F}_{u \wedge \sigma_{\theta_o}(t)}$-supermartingale (cf. [26], Corollary 3.6). In fact, from $\tau_{\theta_o}(u \wedge \sigma_{\theta_0}(t)) = u \wedge \sigma_{\theta_o}(t)$ for all $u \geq t$ and $d(e^{-\delta s} X^\alpha(s)) = -\delta e^{-\delta s} X^\alpha(s)ds + e^{-\delta s}dX^\alpha(s)$ follows that the process $\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t) + \int_t^{u \wedge \sigma_{\theta_o}(t)} e^{-\delta s}(X^\alpha(s)(\theta_o)^{-\alpha} - \delta)ds$ is an $\mathcal{F}_{u \wedge \sigma_{\theta_o}(t)}$-martingale, hence an $\mathcal{F}_u$-martingale.

Notice that $X^\alpha(s)(\theta_o)^{-\alpha} > \delta$ for all $s \in [t, u \wedge \sigma(t))$. In fact, we have $X^\alpha(s)(\theta_o)^{-\alpha} > k^{-\alpha}$ for $s \in [t, u \wedge \sigma(t))$ since $k = (\frac{1}{\delta}\mathbb{E}\{e^{\alpha\underline{Y}(\tau(\delta))}\})^{\frac{1}{\alpha}}$, $\underline{Y}(u) := \inf_{0 \leq s \leq u}\left[(\mu - \frac{1}{2}\sigma^2)s + \sigma W(s)\right]$, and $\mathbb{E}\{e^{\alpha\underline{Y}(\tau(\delta))}\} = \beta_-(\beta_- - \alpha)^{-1} < 1$ with β_- the negative root of $\frac{1}{2}\sigma^2 x^2 + (\mu - \frac{1}{2}\sigma^2)x - \delta = 0$ and $\alpha > 0$. Therefore, $\mathbb{S}(\nu_*)(u \wedge \sigma_{\theta_o}(t))$ is an $\mathcal{F}_u$-supermartingale with absolutely continuous compensator $A(\nu_*)$ given by

$$dA(\nu_*)(u) := e^{-\delta u}\left(X^\alpha(u)(\theta_o)^{-\alpha} - \delta\right)du. \qquad (4.49)$$

Finally, as the Lagrange multiplier optional measure $d\lambda$ (cf. (4.11)) acts only at times when $\nu_*(t) = \theta_o$ (i.e., only when $l(t) > \theta_o$), we conclude that for problem (4.39), $d\lambda$ must be

$$d\lambda(t) = e^{-\delta t}\left(X^\alpha(t)(\theta_o)^{-\alpha} - \delta\right)\mathbb{1}_{\{kX(\cdot) > \theta_o\}}(t)\,dt; \qquad (4.50)$$

that is, $d\lambda(t)$ coincides with the random measure $dA(\nu_*)(t)$ (cf. (4.49)).

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