Working Papers

Center for Mathematical Economics

464

November 2013

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http://www.imw.uni-bielefeld.de/wp/2013.php
ISSN: 0931-6558

Coherent Price Systems and Uncertainty-Neutral Valuation

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This Version: November 25, 2013

Abstract

We consider fundamental questions of arbitrage pricing arising when the uncertainty model incorporates volatility uncertainty. The resulting ambiguity motivates a new principle of preference-free valuation.

By establishing a microeconomic foundation of sublinear price systems, the principle of ambiguity-neutral valuation imposes the novel concept of equivalent symmetric martingale measures. Such measures exist when the asset price with uncertain volatility is driven by Peng's G-Brownian motion.

1 Introduction

A fundamental assumption behind models in Finance refers to the modeling of uncertainty via a single probability measure. Instead, we allow for a set of probability measures $\mathcal{P}$, such that we can guarantee awareness of potential model misspecification.[1] We investigate the implications of a related and reasonable arbitrage concept. In this context, we suggest a *fair* pricing principle associated with an appropriate martingale concept. The multiple prior setting influences the price system in terms of the simultaneous control of different null sets. This motivates a pricing theory of possible means.[2]

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[1] The distinction between measurable and unmeasurable uncertainty drawn by Knight (1921) serves as a starting point for modeling the uncertainty in the economy. Keynes (1937) later argued that single prior models cannot represent irreducible uncertainty.

[2] This was originally discussed by de Finetti and Obry (1933).

The pricing of derivatives via arbitrage arguments is fundamental. Before stating an arbitrage concept, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed such that marketed claims or tradeable assets with trading strategies can be defined. The implicit assumption is that the probabilities are known exactly. The Fundamental Theorem of Asset Pricing (FTAP) then asserts equivalence between the absence of $\mathbb{P}$-arbitrage in the market model and the existence of a consistent linear price extension such that the market model can price all contingent claims. The equivalent martingale measure is then an alternative description of this extension via the Riesz representation theorem.

In contrast to this standard setup, we introduce an uncertainty model described as a set of possibly mutually singular probability measures or priors.[3] Our leading motivation is a general form of *volatility uncertainty*. This perspective deviates from models with term structures of volatilities, including stochastic volatility models such as Heston (1993). As argued in Carr and Lee (2009), we question this confidence and avoid formulating the volatility process of a continuous-time asset price via another process whose law of motion is exactly known. Instead, the legitimacy of the probability law still depends on an infinite repetition of variable observations, as highlighted by Kolmogoroff (1933). We include this residual uncertainty by giving no concrete model for the stochastics of the volatility process and instead fix a confidence interval for the volatility variable.[4]

A coherent valuation principle changes considerably when the uncertainty is enlarged by the possibility of different probabilistic scenarios having different null sets. In order to illustrate this point, we consider for a moment the uncertainty given by one probability model, i.e. $\mathcal{P} = \{\mathbb{P}\}$. An arbitrage refers to a claim X with zero cost, a $\mathbb{P}$-almost surely positive and with a positive probability a strictly positive payoff. Formally, this can be written as $\pi(X) \leq 0$,

$$\mathbb{P}(X \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(X > 0) > 0.$$

The situation changes in the case of an uncertainty model described by a set of mutually singular priors $\mathcal{P}$. The second and third condition should be carefully reformulated, because every prior $P \in \mathcal{P}$ could be the correct market description. We rewrite an arbitrage as $\pi(X) \leq 0$,

$$\text{for all } \mathbb{P} \in \mathcal{P} \quad \mathbb{P}(X \geq 0) = 1 \quad \text{and} \quad \mathbb{P}'(X > 0) > 0 \text{ for some } \mathbb{P}' \in \mathcal{P}.$$

In accepting this new $\mathcal{P}$-arbitrage notion as a weak dominance principle, we might ask for the structure of the related objects.[5] Suppose we apply the same idea of linear and coherent extensions to the present multiple prior uncertainty model.

[3] Two priors are mutually singular if they live on two disjoint supports.

[4] For further motivation to consider volatility uncertainty, we refer to Subsection 1.1 of Epstein and Ji (2013). Very recent developments in stochastic analysis have established a complete theory to model volatility uncertainty in continuous time. A major objective refers to the sublinear expectation operator introduced by Peng (2006).

[5] See Remark 3.14 in Vorbrink (2010) for a discussion of this arbitrage definition and its implication in the G-framework.

Coherence corresponds to a strictly positive and continuous price systems on the space of claims L which is consistent with the given data of a possibly incomplete market. Marketed claims $M \subset L$ can be traded frictionless and are priced by a linear functional $\pi : M \to \mathbb{R}$.

Another important aspect focuses on the order structure for contingent claims and the underlying topology of similarity for L. This comprises the basis of any financial model that asks for coherent pricing. The representation of linear and continuous price systems[6] indicates inconsistencies between positive linear price systems and the concept of $\mathcal{P}$-arbitrage. As is usual, the easy part of establishing an FTAP is deducing an arbitrage-free market model from the existence of an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P} \in \mathcal{P}$. When seeking a modified FTAP, the following question (and answer) serves to clarify the issue:

> Is the existence of a measure $\mathbb{Q}$ equivalent to some $\mathbb{P} \in \mathcal{P}$ such that prices of all traded assets are $\mathbb{Q}$-martingales (and therefore) a sufficient condition to prevent a $\mathcal{P}$-arbitrage opportunity?

A short argument gives us a negative answer: Let $X \in M$ be a marketed claim with price $0 = \pi(X)$. We have $E^{\mathbb{Q}}[X] = 0$ since $\mathbb{Q}$ is related to a consistent price system. Suppose $X \in M$ is a $\mathcal{P}$-arbitrage with $\mathbb{P}'(X > 0) > 0$. The point is now,

> with $\mathcal{P} = \{\mathbb{P}\}$ we would observe a contradiction since $\mathbb{Q} \sim \mathbb{P}$ implies $E^{\mathbb{Q}}[X] > 0$. But $X \in M$ may be such that $\mathbb{P}'(X > 0) > 0$ with $\mathbb{P}' \in \mathcal{P}$ being mutually singular to $\mathbb{Q} \sim \mathbb{P} \in \mathcal{P}$.

This indicates that our *robust* arbitrage notion is, in general, not consistent with a linear theory of valuation. In other words, a single pricing measure $\mathbb{Q}$ is not able to contain all the information about what is possible under $\mathcal{P}$. Similarly, the concept of "no empty promises" in Willard and Dybvig (1999) refers to the possible ignorance of payoffs in states with zero probability.

Since our goal is to suggest a modified framework for a coherent pricing principle, the concept of marketed claim is reformulated by a prior-dependent notion of possible marketed spaces $M_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$. As discussed in Example 3 below, such a step is necessary to address the prior dependency of the asset span $M_{\mathbb{P}}$. The likeness of marketed spaces depends on the similarity of the priors in question. Hence, the possibility of different priors creates a friction caused by the new uncertainty.

A New Commodity-Price Duality

The very basic principle of uncertainty is the assumption of different possible future states of the world Ω .[7] In the most general framework, we assume a weakly

[6] We discuss the precise description in Section 2.2.

[7] In order to tackle the mutually singular priors, we need some structure in the state space. See Bion-Nadal and Kervarec (2010) for a discussion of different state spaces. In the most abstract setting, the states of the world $\omega \in \Omega$ build a complete separable metric space, also known as a Polish space. The state space contains all realizable paths of security prices. For the greater part of the paper, we assume $\Omega = C([0,T];\mathbb{R})$, the Banach space of continuous functions between $[0,T]$ and $\mathbb{R}$, equipped with the supremum norm.

compact set of priors $\mathcal{P}$.[8] This encourages us to consider the sublinear expectation operator

$$\mathcal{E}^{\mathcal{P}}(X) = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[X].$$

In our economy, the Banach space of contingent claims $L^2(\mathcal{P})$ consists of all random variables with a finite variance for all $\mathbb{P} \in \mathcal{P}$. The primitives are prior-dependent representative agent economies given by preference relations in $\mathbb{A}(\mathbb{P})$, being convex, continuous, strictly monotone and rational.

In the single prior setting, the expectation under an equivalent martingale measure $\mathbb{Q}$ refers to a normalized, linear and continuous price system in the sense of Arrow-Debreu. The present topological dual space of $L^2(\mathcal{P})$, a first candidate for the space of price systems, does not consist of elements which can be merely represented by a state price density ψ. Rather, in the present volatility uncertainty framework, it is represented by the pairs $(\mathbb{P}, \psi) \in \cup_{\mathbb{P}\in\mathcal{P}}\{\mathbb{P}\} \times L^2(\mathbb{P})$. As explained before, such linear valuations are inconsistent with the fine and robust arbitrage we are interested in. Loosely speaking, such price systems only see the null sets of a particular $\mathbb{P}$ and are blind for the null sets of any mutually singular prior $\mathbb{P}' \in \mathcal{P}$. We consider the space of nonlinear price functionals $L^2(\mathcal{P})^{\circledast}$ built upon this dual space. Proposition 1 lists important properties and indicates a possible axiomatic approach to the price systems inspired by the coherent risk measures of Artzner, Delbaen, Eber, and Heath (1999).

Sublinear prices are constructed by the price systems of partial equilibria, which consist of prior-dependent linear price functionals $\pi_{\mathbb{P}}$ restricted to the prior-dependent marketed spaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. These spaces are joined to a product of marketed spaces. The consolidation operation Γ transforms the extended product of price systems $\{\pi_{\mathbb{P}}\}_{P\in\mathbb{P}}$ to *one* coherent element in the price space $L^2(\mathcal{P})^{\circledast}_{+}$. Scenario-based viability can then model a preference-free valuation concept in terms of consolidation of possibilities.

The first main result, Theorem 1, gives an equivalence between our notion of scenario-based viable price systems, and the extension of sublinear functionals. The present viability concept, corresponding to a no trade equilibrium, is based on sublinear prices so that every price functional act linearly under unambiguous contingent claims.

Risk- and Ambiguity-Neutral Valuation

In the second part, we consider the dynamic framework on a time interval $[0, T]$ with an augmented filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ modeling the arrival of new information. Its special feature is its reliance on the initial σ-algebra, which does not contain all null sets. Built upon this information structure, we introduce a dynamic updating principle based on a sequence of conditional sublinear expectations $\mathcal{E}_t(\cdot) = \mathbb{E}^{\mathcal{P}}[\cdot|\mathcal{F}_t]$, $t \in [0, T]$. These operators are well defined under every $\mathbb{P} \in \mathcal{P}$ and satisfy the Law of Iterated Expectation.

[8]If one accepts a deterministic upper bound on the volatility, i.e. the derivative of every possible quadratic variation, then the (relatively) weak compactness of $\mathcal{P}$ is a sufficient condition.

With the conditional sublinear expectation, a martingale theory is available which represents a possibilistic model of a fair game against nature.[9] In this fashion, the multiple prior framework forces us to generalize the concept of equivalent martingale measures. Instead of considering *one* probability measure representing the risk-neutral world, we suggest that the appropriate concept is a set of priors $\mathcal{Q}$. The relation to the statistical set of priors $\mathcal{P}$ is induced through a prior-dependent family of state price densities $\psi_{\mathbb{P}} \in L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. This creates a new sublinear expectation, $\mathcal{E}^{\mathcal{Q}}$, generated by $\mathcal{Q}$. For this rationale, the uncertain asset price (S_t) becomes under $\mathcal{E}^{\mathcal{Q}}$ mean unambiguous, i.e. $E^{\mathbb{Q}}[S_T] = E^{\mathbb{Q}'}[S_T]$, for all $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}$.

The essential renewal is to consider $\mathcal{Q}$ as the appropriate uncertainty-neutral world. At this stage, ambiguity neutrality as a part of uncertainty neutrality comes into play. The central idea follows the same lines as in the classical risk-neutral valuation. Preferences on ambiguity become neutral when we move to the uncertainty neutral world $\mathcal{Q}$.[10] And it is exactly this kind of neutrality which corresponds to the notion of symmetric martingales, i.e (S_t) and $(-S_t)$ are $\mathcal{E}^{\mathcal{Q}}$-martingales. This reasoning motivates the modification of the martingale concept, now based on the idea of a fair game under $\mathcal{Q}$. As such, the condition that the price process S is a symmetric martingale motivates qualifying the valuation principle as *uncertainty neutral*.

The principal idea of our modified notion of $\mathcal{P}$-arbitrage was introduced by Vorbrink (2010) for the G-expectation framework (see also Section 3 in Epstein and Ji (2013)). In Theorem 2 we show that under no $\mathcal{P}$-arbitrage there is a one-to-one correspondence between the extensions of Theorem 1 and (special) *equivalent symmetric martingale measure sets* $\mathcal{Q}$, called EsMM-sets. We thus establish an asset pricing theory based on a (discounted) nonlinear expectation payoff.

Having established the relation between these concepts, we continue in the same fashion as in the classical literature with a single prior. We consider a special class of asset prices driven by G-Brownian motion, related to a G-expectation E_G. This is a zero-mean and stationary process with novel $N(0, [\underline{\sigma}, \overline{\sigma}])$-normally distributed independent increments. Such a normally distributed random variable is the outcome of a robust central limit theorem under the sublinear G-expectation. Moreover, in this uncertainty setup, independence of random variables is no longer a symmetric property.[11] This process can be regarded as a canonical generalization of the standard Brownian motion, in which the quadratic variation (or volatility) may move almost arbitrarily in a positive interval. The related heat equation is now a fully nonlinear PDE, see Peng (2006).

We consider a Black-Scholes like market under volatility uncertainty driven by

[9]More precisely, a whole hierarchy of different fairness degrees is possible.

[10]This symmetry of priors is essential for creating a process via a conditional expectation which satisfies the classical martingale representation property, see Appendix B.2.

[11]In the mathematical literature, the starting point for consideration is a sublinear expectation space, consisting of the triple $(\Omega; \mathcal{H}; \mathcal{E})$, where $\mathcal{H}$ is a given space of random variables. If the sublinear expectation space can be represented via the supremum of a set of priors, see Denis, Hu, and Peng (2011), one can take $(\Omega, \mathcal{B}(\Omega), \mathcal{P})$ as the associated *uncertainty space* or Dynkin space, see Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011).

a G-Brownian motion B^G. The uncertain asset price process (S_t) is modeled as a stochastic differential equation[12]

$$dS_t = \mu(t, S_t)d\langle B^G\rangle_t + V(t, S_t)dB_t^G, \quad S_0 = 1.$$

Intuitively, the increment dS_t is divided into the locally certain part[13] and the locally risky and ambiguous part $V(t, S_t)dB_t^G$. An interpretation of this G-Itô differential representation reads as follows:

$$\frac{d}{dr}\mathrm{var}_r^{\mathbb{P}}(S_t)\bigg|_{r=t} \in V(t, S_t)\cdot[\underline{\sigma}, \overline{\sigma}], \qquad \mathbb{P}\in\mathcal{P},$$

where $\mathrm{var}_r^{\mathbb{P}}(S_t)$ is the $(\mathcal{F}_t, \mathbb{P})$-conditional variance. In abuse of notation we could write this issue as $\mathrm{var}_t(dS_t) = V(t, S_t)^2 d\langle B^G\rangle_t$, $\mathcal{P}$-quasi surely.

In this mutually singular prior setting, the (more evolved) martingale representation property, related to a conditional sublinear expectation, is not equivalent to the completeness of the model because the volatility uncertainty is encoded in the integrator of the price process. For the state price density process we introduce an exponential martingale $(\mathtt{E}_t)_{t\in[0,T]}$[14] under G-Brownian motion and apply a new Girsanov type theorem under E_G. For every contingent claim $X \in L^2(\mathcal{P})$, this yields following robust pricing formula

$$\Psi(X) = \mathcal{E}^{\mathcal{Q}}(X) = E_G[\mathtt{E}_T X].$$

Related Literature

We embed the present paper into the existing literature. In Harrison and Kreps (1979), the arbitrage pricing principle provides an economic foundation by relating the notion of equivalent martingale measures with a linear equilibrium price system.[15] Risk-neutral pricing, as a precursor, was discovered by Cox and Ross (1976). Harrison and Pliska (1981), as well as Kreps (1981) and Yan (1980), continued laying the foundation of arbitrage free pricing. Later, Dalang, Morton, and Willinger (1990) presented a fundamental theorem of asset pricing for finite discrete time. In a general semimartingale framework, the notion of no free lunch with vanishing risk Delbaen and Schachermayer (1994) ensured the existence of an equivalent martingale measure in the given (continuous-time) financial market. All these considerations have in common that the uncertainty of the model is given by a single probability measure.

[12]This related stochastic calculus comprises a stochastic integral notion, a G-Itô formula and a martingale representation theorem.

[13]For this part one usually has a dt-drift as the inner clock of classical Brownian motion. Since the inner clock or quadratic variation is now given by the ambiguous $\langle B^G\rangle_t$, we relate it to the drift part.

[14]The precise PDE description of the G-expectation allows the definition of a universal density. Note that in the more general case we have a prior-dependent family of densities.

[15]The efficient market hypothesis by Fama (1970) introduces information efficiency, a concept closely related to Samuelson (1965), where the notion of a martingale reached neo-classic economics for the first time. Bachelier (1900) influenced the course of Samuelson's work.

Moving to models with multiple probability measures, the concept of pasting of probability measures models the intrinsic structure of dynamic convexity, see Riedel (2004) and Delbaen (2006). This type of time consistency is related to recursive equations, see Epstein and Schneider (2003); Chen and Epstein (2002), which can result in nonlinear expectation and generates a rational updating principle. Moreover, the backward stochastic differential equations can model drift-uncertainty, a dynamic sublinear expectation, see Peng (1997). However, in these models of uncertainty, all priors are related to a reference probability measure, i.e. all priors are equivalent or absolutely continuous. Moreover, drift uncertainty does not create a significant change for a valuation principle of contingent claims.[16]

The possible insufficiency of equivalent prior models for an imprecise knowledge of the environment motivates the consideration of mutually singular priors as illustrated at the beginning of this introduction. The mathematical discussion of such frameworks can be found in Peng (2006); Nutz and Soner (2012); Bion-Nadal and Kervarec (2012). Epstein and Ji (2013) provide a discussion in economic terms. Similarly to the present paper, the volatility uncertainty is encoded in a non-deterministic quadratic variation of the underlying noise process.

Recalling Gilboa and Schmeidler (1989), this axiomatization of uncertainty aversion represents a non-linear expectation via a worst case analysis. Similarly to risk measures, see Artzner, Delbaen, Eber, and Heath (1999),[17] the related set of representing priors may be not equivalent to each other. This important change permits the application of financial markets under volatility uncertainty. We refer to Avellaneda, Levy, and Paras (1995); Denis and Martini (2006) for a pricing principle of claims via a quasi sure stochastic calculus.

Jouini and Kallal (1995) consider a non-linear pricing caused by bid-ask spreads and transaction costs, where the price system is extended to a linear functional. In Araujo, Chateauneuf, and Faro (2012), pricing rules with finitely many state are considered.[18] A price space of sublinear functionals is discussed in Aliprantis and Tourky (2002). We quote the following interpretation of the classical equilibrium concept with linear prices and its meaning (see Aliprantis, Tourky, and Yannelis (2001)):

> *A linear price system summarizes the information concerning relative scarcities and at equilibrium approximates the possibly non-linear primitive data of the economy.*

The paper is organized as follows. Section 2 introduces the primitives of the economic model and establishes the connection between our notion of viability and

[16]Cont (2006) notes that this assumption is "*actually quite restrictive: it means that all models agree on the universe of possible scenarios and only differ on their probabilities. For example, if $\mathbb{P}_0$ defines a complete market model, this hypothesis entails that there is no uncertainty on option prices!*"

[17] Markowitz (1952) postulated the importance of diversification, a fundamental principle in finance, which corresponds to sublinearity of risk measures.

[18]They establish a characterization of super-replication pricing rules via an identification of the space of frictionless claims.

extensions of price systems. Section 3 introduces the security market model associated with the marketed space. We also discuss the corresponding G-Samuelson model. Section 4 concludes and discusses the results of the paper and lists possible extensions. The first part of the appendix presents the details of the model and provides the theorem proofs. In the second part, we discuss mathematical foundations such as the space of price systems and a collection of results of stochastic analysis and G-expectations.

2 Viability and Sublinear Price Systems

We begin by recapping the case where uncertainty is given by an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as it emphasizes sensible differences with regard to the uncertainty model posited in this paper. Following, we introduce the uncertainty model as well as the related space of contingent claims. Then we discuss the space of sublinear price functionals. The last subsection introduces the economy, and Theorem 1 states an extension result.

Background: Classical Viability

Let there be two dates $t = 0, T$, claims at T are elements of the classical Hilbert lattice $L^2(\mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$. Price systems are given by linear and $L^2(\mathbb{P})$-continuous functionals. By Riesz representation theorem, elements of the related topological dual can be identified in terms of elements in $L^2(\mathbb{P})$. A strictly positive functional $\Pi : L^2(\mathbb{P}) \to \mathbb{R}$ evaluates a positive random variable X with $\mathbb{P}(X > 0) > 0$, such that $\Pi(X) > 0$.
A *price system* consists of a (closed) subspace $M \subset L^2(\mathbb{P})$ and a linear price functional $\pi : M \to \mathbb{R}$. The marketed space consists of contingent claims achievable in a frictionless manner. $\mathbb{A}(\mathbb{P})$ is the set of rational, convex, strictly monotone and $L^2(\mathbb{P})$-continuous preference relations on $\mathbb{R} \times L^2(\mathbb{P})$. The consistency condition for an economic equilibrium is given by the concept of viability. A price system is *viable* if there exists a preference relation $\succsim \in \mathbb{A}(\mathbb{P})$ and a bundle $(\hat{x}, \hat{X}) \in \mathbb{R} \times M$ with

$$(\hat{x}, \hat{X}) \in B(0, 0, \pi, M) \text{ and } (\hat{x}, \hat{X}) \succsim (x, X) \text{ for all } (x, X) \in B(0, 0, \pi, M),$$

where $B(x, X, \pi, M) = \{(y, Y) \in \mathbb{R} \times M : y + \pi(Y) \leq x + \pi(X)\}$ denotes the budget set. Harrison and Kreps (1979) prove the following fundamental result:

(M, π) *is viable if and only if there is a strictly positive extension* Π *of* π *to* $L^2(\mathbb{P})$.

The proof is achieved by a Hahn-Banach argument and the usage of the properties of $\succsim$ such that Π creates a linear utility functional and hence a preference relation in $\mathbb{A}(\mathbb{P})$.

2.1 The Uncertainty Model and the Space of Claims

We begin with the underlying uncertainty model by considering possible scenarios which share neither the same probability measure nor the same null sets. Therefore it is not possible to assume the existence of a given reference probability measure when the null sets are not the same. For this reason we need a topological structure to formulate the uncertainty model.

Let Ω, the states of the world, be a complete separable metric space, $\mathcal{B}(\Omega) = \mathcal{F}$ the Borel σ-algebra of Ω and let $\mathcal{C}_b(\Omega)$ denote the set of all bounded continuous real valued functions. The uncertainty of the model is given by a weakly compact set of Borel probability measure $\mathcal{P} \subset \mathcal{M}_1(\Omega)$ on $(\Omega, \mathcal{F})$.[19] In the following example we illustrate a construction for $\mathcal{P}$, applied in the dynamic setting of Section 3.

Example 1 *We consider a time interval* $[0, T]$*, the Wiener measure* P_0 *on the state space of continuous paths* $\Omega = \{\omega : \omega \in C([0, T]; \mathbb{R}) : \omega_0 = 0\}$ *and the canonical process* $B_t(\omega) = \omega_t$*. Let* $\mathbb{F}^o = (\mathcal{F}^o_t)_{t\in[0,T]}$*,* $\mathcal{F}^o_t = \sigma(B_s, s \in [0, t])$ *be the raw filtration of* B*. The strong formulation of volatility uncertainty is based upon martingale laws with stochastic integrals:*

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}, \quad X^\alpha_t = \int_0^t \alpha_s^{1/2} dB_s,$$

where the integral is defined $\mathbb{P}_0$ *almost surely. The process* α *is* $\mathbb{F}^o$*-adapted and has a finite first moment. A set* $\mathcal{D}$ *of* α*'s builds* $\mathcal{P}$ *via the associated prior* $\mathbb{P}^\alpha$*, such that* $\{\mathbb{P}^\alpha : \alpha \in \mathcal{D}\} = \mathcal{P}$ *is weakly compact.*[20]

We describe the set of contingent claims. Following Huber and Strassen (1973), for each $\mathcal{F}$-measurable real function X such that $E^{\mathbb{P}}[X]$ exists for every $\mathbb{P} \in \mathcal{P}$, define the upper expectation operator by $\mathcal{E}^{\mathcal{P}}(X) = \sup_{P\in\mathcal{P}} E^{\mathbb{P}}[X]$.[21] We suggest the following norm for the space of contingent claims, given by the capacity norm $c_{2,\mathcal{P}}$, defined on $\mathcal{C}_b(\Omega)$ by

$$c_{2,\mathcal{P}}(X) = \mathcal{E}^{\mathcal{P}}\left(|X|^2\right)^{\frac{1}{2}}.$$

Define the completion of $\mathcal{C}_b(\Omega)$ under the so called "Lebesgue prolongation" of $c_{2,\mathcal{P}}$[22] by $\mathcal{L}^2(\mathcal{P}) = \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$, and let $L^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$ be the quotient space of $\mathcal{L}^2(\mathcal{P})$ by the $c_{2,\mathcal{P}}$ null elements $\mathcal{N}$. We do not distinguish between classes and their representatives. Two random variables $X, Y \in L^2(\mathcal{P})$ can be distinguished if there is a prior in $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(X \neq Y) > 0$.

It is possible to define an order relation $\leq$ on $L^2(\mathcal{P})$. Classical arguments prove that $(L^2(\mathcal{P}), c_{2,\mathcal{P}}, \leq)$ is a Banach lattice, see Appendix A.1 for details.

We consider the space of contingent claims $L^2(\mathcal{P})$ so that under every probability model $\mathbb{P} \in \mathcal{P}$, we can evaluate the variance of a contingent claim. Properties of random variables are required to be true $\mathcal{P}$-quasi surely, i.e. $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$. This indicates that in contrast to drift uncertainty, a related stochastic calculus cannot be based only on one probability space.

[19] As shown in Denis, Hu, and Peng (2011), the related capacity $c(\cdot) = \sup_{P\in\mathcal{P}} P(\cdot)$ is *regular* if and only if the set of priors is relatively compact. Here, regularity refers to a reasonable continuity property. In Appendix B, we recall some related notions and we give a criterion for the weak compactness of $\mathcal{P}$ when it is constructed via the quadratic variation and a canonical process.

[20] In order to define universal objects, we need the pathwise construction of stochastic integrals, (see Föllmer (1981), Karandikar (1995)).

[21] It is easily verified that $\mathcal{C}_b(\Omega) \subset \{X\ \mathcal{F}\text{-measurable} : \mathcal{E}^{\mathcal{P}}(X) < \infty\}$ holds and $\mathcal{E}^{\mathcal{P}}(\cdot)$ satisfies the property of a sublinear expectation. For details, see Appendix A.1.1, Peng (2010) and Appendix B.2.

[22] We refer to Section 2 in Feyel and de La Pradelle (1989), see also Section 48.7-8 in Choquet (1953) and Section A in Dellacherie (1972).

2.2 Scenario-Based Viable Price Systems

This subsection is divided into three parts. First, we introduce the dual space where linear and $c_{2,\mathcal{P}}$-continuous functionals are the elements. As discussed in the introduction, we allow sublinear prices as well. This forces us to extend the linear price space where we discuss two operations on the new price space and take a leaf out of Aliprantis and Tourky (2002). We integrate over the set of priors for the linear consolidation of functionals. In Proposition 1, we list standard properties of coherent price functionals. The last part in this subsection focuses on the consolidation of prior-dependent price systems.

Linear and $c_{2,\mathcal{P}}$-Continuous Price Systems on $L^2(\mathcal{P})$

We present the basis for the modified concept of viable price systems. The mutually singular uncertainty generates a different space of contingent claims. This gives us a new topological dual space $L^2(\mathcal{P})^*$. The discussion of the dual space is only the first step to get a reasonable notion of viability which accounts for the present type of uncertainty. In the second part of the Appendix, we give a result which asserts that the topological dual, the space of all linear and $c_{2,\mathcal{P}}$-continuous functionals on $L^2(\mathcal{P})$, is given by

$$L^2(\mathcal{P})^* = \left\{ E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] : \mathbb{P} \in \mathcal{P} \text{ and } \psi_{\mathbb{P}} \in L^2(\mathbb{P}) \right\}.$$

This representation delivers an appropriate form for possible price systems. The random variable $\psi_{\mathbb{P}}$ in the representation matches the classical state price density of the Riesz representation when only one prior $\{\mathbb{P}\} = \mathcal{P}$ is present. The space's description allows for an interpretation of a state price density $\psi_{\mathbb{P}}$ based on some prior $\mathbb{P} \in \mathcal{P}$. The stronger capacity norm $c_{2,\mathcal{P}}(\cdot)$ in comparison to the classical single prior $L^2(\mathbb{P})$-norm implies a richer dual space, controlled by the set of priors $\mathcal{P}$. Moreover, one element in the dual space implicitly selects a prior $\mathbb{P} \in \mathcal{P}$ and ignores all other priors. This foreshadows the insufficiency of a linear pricing principle under the present uncertainty model, as indicated in the introduction.

The Price Space of Nonlinear Expectations

In this paragraph we introduce a set of sublinear functionals defined on $L^2(\mathcal{P})$. The singular prior uncertainty of our model induces the appearance of non-linear price systems.[23] Let $k(\mathcal{P})$ be the convex hull of $\mathcal{P}$. The *coherent price space* of $L^2(\mathcal{P})$ generated by linear $c_{2,\mathcal{P}}$-continuous functionals is given by

$$L^2(\mathcal{P})^{\circledast}_{+} = \left\{ \Psi : L^2(\mathcal{P}) \to \mathbb{R} : \Psi(\cdot) = \sup_{\mathbb{P} \in \mathcal{R}} E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] \text{ with } \mathcal{R} \subset k(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})_+ \right\}.$$

Elements in $L^2(\mathcal{P})^{\circledast}_{+}$ are constructed by a set of $c_{2,\mathcal{P}}$-continuous linear functionals $\{\Pi_{\mathbb{P}} : L^2(\mathcal{P}) \to \mathbb{R}\}_{\mathbb{P} \in \mathcal{P}}$, which are consolidated by a combination of the pointwise maximum and convex combination. Strictly positive functionals in $L^2(\mathcal{P})^{\circledast}_{++}$

[23] A subcone of the super order dual is considered in Aliprantis and Tourky (2002). They introduce the lattice theoretic framework and consider the notion of a semi lattice. In Aliprantis, Florenzano, and Tourky (2005); Aliprantis, Tourky, and Yannelis (2001) general equilibrium models with a superlinear price systems are considered in order to discuss a non-linear theory of value.

satisfy additionally $\Psi(X) > 0$ for every $X \in L^2(\mathcal{P})_+$ with $\mathbb{P}(X > 0) > 0$ for some $\mathbb{P} \in \mathcal{P}$. The following example illustrates how a sublinear functional in $L^2(\mathcal{P})^{\circledast}_+$ can be constructed.

Example 2 *Let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a partition of $\mathcal{P}$. And let $\mu_n : \mathcal{B}(\mathcal{M}_1(\Omega)) \to \mathbb{R}$ be a positive measure with support $\mathcal{P}_n$ and $\mu_n(\mathcal{P}_n) = 1$. The resulting prior $\mathbb{P}_n(\cdot) = \int_{\mathcal{P}_n} \mathbb{P}(\cdot)\mu_n(d\mathbb{P})$ is given by a weighting operation Γ_{μ_n}. When we apply Γ_{μ_n} to the density $\psi_{\mathbb{P}}$ we get $\bar{\psi}_n(\omega) = \int_{\mathcal{P}_n} \psi_{\mathbb{P}}(\omega)\mu_n(d\mathbb{P})$, $\omega \in \Omega$. These new prior density pairs $(\bar{\psi}_n, \mathbb{P}_n)$ can then be consolidated by the supremum operation of the expectations, i.e. $\Gamma(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}})(\cdot) = \sup_{n\in\mathbb{N}} E^{\mathbb{P}_n}\left[\bar{\psi}_n \cdot\right]$.*

For further details of Example 2, see Appendix A.1.1 and Appendix B.1.1. The following proposition discusses properties and the extreme case of functionals in the price space $L^2(\mathcal{P})^{\circledast}_+$.[24]

Proposition 1 *Functionals in $L^2(\mathcal{P})^{\circledast}_+$ satisfy 1. sub-additivity, 2. positive homogeneity, 3. constant preserving, 4. monotonicity and 5. $c_{2,\mathcal{P}}$-continuity.*[25]

Moreover, for every positive measure μ on $\mathcal{B}(\mathcal{P})$ with $\mu(\mathcal{P}) = 1$, we have the following inequality for every $X \in L^2(\mathcal{P})$

$$E^{\mathbb{P}_\mu}[\psi_\mu X] \leq \sup_{\mathbb{P}\in k(\mathcal{P})} E^{\mathbb{P}}[\psi_{\mathbb{P}} X], \quad \text{where } \mathbb{P}_\mu(\cdot) = \int_{\mathcal{P}} \mathbb{P}(\cdot)\mu(d\mathbb{P}).$$

Below, we introduce the consolidation operation Γ for the prior-dependent price systems. $\Gamma(\mathcal{P})$ refers to the set of priors in $\mathcal{P}$ which are relevant. In Example 2, we observe $\Gamma_{\mu_n}(\mathcal{P}) = \mathcal{P}_n$.

Remark 1 *Price systems in $L^2(\mathcal{P})^{\circledast}_+$ resemble the structure of ask prices. However, the related bid price can then be described by the super order dual $-L^2(\mathcal{P})^{\circledast}_-$, since $\sup(\cdot) = -\inf(-\cdot)$. From this perspective, we could also construct a fully nonlinear, monotone and positive homogeneous price systems Ψ as elements in $L^2(\mathcal{P})^{\circledast}_+ - L^2(\mathcal{P})^{\circledast}_-$. For some cover $\mathcal{P}_+ \cup \mathcal{P}_- = \mathcal{P}$ we have*

$$X \mapsto \Psi(X) = \sup_{\mathbb{P}\in\mathcal{P}_+} E^{\mathbb{P}}[\psi_{\mathbb{P}} X] + \inf_{\mathbb{P}'\in\mathcal{P}_-} E^{\mathbb{P}'}[\psi_{\mathbb{P}'} X]. \tag{1}$$

At this stage, the nonlinear price functional can be seen as a fully nonlinear expectation $\mathfrak{E}(\cdot) \leq \mathcal{E}^{\mathcal{P}}(\cdot)$, being dominated by $\mathcal{E}^{\mathcal{P}}$ on $L^2(\mathcal{P})$ (see Remark 3.1. below and Section 8 of Chapter III in Peng (2010) for more details).

Marketed Spaces and Scenario-Based Price Systems

In the spirit of Aliprantis, Florenzano, and Tourky (2005) our commodity-price duality is given by the following pairing $\langle L^2(\mathcal{P}), L^2(\mathcal{P})^{\circledast}_+\rangle$.

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[24] A full lattice-theoretical discussion of our price space $L^2(\mathcal{P})^{\circledast}_+$ lies beyond the scope of this paper. However, it is worthwhile to mention that Theorem 12 in Denis, Hu, and Peng (2011) characterizes σ-order continuity of sublinear functionals in $L^2(\mathcal{P})^{\circledast}_+$.

[25] Formally this means: $1.\Psi(X+Y) \leq \Psi(X) + \Psi(Y)$ for all $X, Y \in L^2(\mathcal{P})$, $2.\Psi(\lambda X) = \lambda\Psi(X)$ for all $\lambda \geq 0$, $X \in L^2(\mathcal{P})$, $3.\Psi(c) = c$ for all $c \in \mathbb{R}$, 4. If $X \geq Y$ then $\Psi(X) \geq \Psi(Y)$ for all $X, Y \in L^2(\mathcal{P})$ and 5. Let $(X_n)_{n\in\mathbb{N}}$ converge in $c_{2,\mathcal{P}}$ to some X, then we have $\lim_n \Psi(X_n) = \Psi(X)$.

For the single prior framework, viability and the extension of the price system are associated with each other. This structure allows only for linear prices. In our framework this corresponds to a consolidation via the Dirac measure $\delta_{\{\mathbb{P}\}}$ for some $\mathbb{P} \in \mathcal{P}$, so that $\Gamma(\mathcal{P}) = \{\mathbb{P}\}$.

We begin by introducing the marketed subspaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. The underlying idea is that any claim in $M_{\mathbb{P}}$ can be achieved, whenever $\mathbb{P} \in \mathcal{P}$ is the true probability measure. This input data resembles a partial equilibrium, depending on the prior under consideration. [26] Claims in the marketed space $M_{\mathbb{P}}$ can be bought and sold whenever the related prior governs the economy. We illustrate this in the following examples.

Example 3 *1. Let us consider the role of marketed spaces in the very simple situation when no prior dependency is present, i.e. $M_{\mathbb{P}} = M$ for every $\mathbb{P} \in \mathcal{P}$. Specifically, set*

$$M = \left\{ X \in L^2(\mathcal{P}) : E^{\mathbb{P}}[X] = const. \quad \text{for every } \mathbb{P} \in \mathcal{P} \right\}.$$

As we show in Corollary 1, this space consists of (unambiguous) contingent claims which do not depend on the prior of the corresponding linear expectation operator. It turns out that this space has a strong connection to symmetric martingales.

2. Suppose the set of priors is constructed by the procedure in Example 1. The marketed spaces differ because of the $\mathbb{P}$-dependent replication condition. Specifically, this is encoded in an equation which holds only $\mathbb{P}$-almost surely.

Let the marketed space be generated by the quadratic variation of an uncertain asset with terminal payoff $\langle B \rangle_T$ and a riskless asset with payoff 1. We have by construction $\langle B \rangle_T = \int_0^T \alpha_s ds$ $\mathbb{P}^\alpha$-a.s. The marketed space under $\mathbb{P}^\alpha$ is given by

$$M_{\mathbb{P}^\alpha} = \left\{ X \in L^2(\mathbb{P}^\alpha) : X = a + b \cdot \int_0^T \alpha_s ds \ \mathbb{P}^\alpha\text{-a.s.},\ a, b \in \mathbb{R} \right\}.$$

But $\langle B \rangle$ coincides with the $\mathbb{P}$-quadratic variation under every martingale law $\mathbb{P} \in \mathcal{P}$. Therefore a different $\hat{\alpha}$ builds a different marketed space $M_{\mathbb{P}^{\hat{\alpha}}}$. Suppose $\alpha = \hat{\alpha}$ $\mathbb{P}_0$-a.s. on $[0, s]$ for some $s \in (0, T]$ then we have $M_{\mathbb{P}^\alpha} \cap M_{\mathbb{P}^{\hat{\alpha}}}$ consists also of non trivial claims. Note, that $\mathbb{P}^\alpha$ and $\mathbb{P}^{\hat{\alpha}}$ are neither equivalent nor mutually singular.[27]

We fix linear price systems $\pi_{\mathbb{P}}$ on $M_{\mathbb{P}}$. As illustrated in Example 3, it is possible that the $\pi_{\mathbb{P}_1}, \pi_{\mathbb{P}_2} \in \{\pi_{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}}$ have a common domain, i.e $M_{\mathbb{P}_1} \cap M_{\mathbb{P}_2} \neq \{0\}$. In this case one may observe different evaluations among different priors, i.e $\pi_{\mathbb{P}_1}(X) \neq \pi_{\mathbb{P}_2}(X)$ with $X \in M_{\mathbb{P}_1} \cap M_{\mathbb{P}_2}$. To account for this possible phenomenon, we associate a

[26] One may think that a countable set of scenarios could be sufficient. As in Bion-Nadal and Kervarec (2012), the norm can be represented via different countable dense subsets of priors. However, for the marketed space we allow for a direct prior dependency of all possible scenarios $\mathcal{P}$. This implies that different choices of countable and dense scenarios can deliver different price systems (see Definition 1 below).

[27] The event $\{\omega : \langle B \rangle_r(\omega) = \int_0^r \alpha_t(\omega) dt, r \in [0, s]\}$ has positive mass under both priors, but the priors restricted to the complement are mutually singular. We refer to Example 3.7 in Epstein and Ji (2013) for a similar example.

linear price system $\pi_P : M_{\mathbb{P}} \to \mathbb{R}$ for each marketed space. In this context, we posit that coherence is based on *sublinear* price systems,[28] as illustrated in the following example (see also Heath and Ku (2006) for a discussion).

Example 4 *Let the uncertainty model consist of two priors $\mathcal{P} = \{\mathbb{P}, \mathbb{P}'\}$. If $\mathbb{P}$ is the true law, the market model is given by the set of marketed claims $M_{\mathbb{P}}$ priced by a linear functional $\pi_{\mathbb{P}}$. If $\mathbb{P}'$ is the true law, we get $M_{\mathbb{P}'}$ and $\pi_{\mathbb{P}'}$. As in Example 3.2, constructing a claim via self-financing strategies implies an equality of portfolio holdings that must be satisfied almost surely only for the particular probability measure. If the trader could choose between the sets $M_{\mathbb{P}'} + M_{\mathbb{P}}$ to create a portfolio, additivity would be a natural requirement with the consistency condition $\pi_{\mathbb{P}'} = \pi_{\mathbb{P}}$ on $M_{\mathbb{P}'} \cap M_{\mathbb{P}}$. However, the trader is neither free to choose a mixture of claims, nor may she choose a scenario, simply because of existing ignorance.*

An equality of prices at the intersection is less intuitive, since the different priors create a different price structure in each scenario. We therefore argue, that $\sup(\pi_{\mathbb{P}'}(X), \pi_{\mathbb{P}}(X))$ is a robust and reasonable price for a claim $X \in M_{\mathbb{P}'} \cap M_{\mathbb{P}}$ in our multiple prior framework. This yields to subadditivity. In contrast to the classical law of one price, linearity of the pricing functional is merely true under a fixed prior.[29]

The set $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ of linear scenario-based price functionals inherit all the information of the underlying financial market. In the single prior setting incompleteness means $M_{\mathbb{P}} \neq L^2(\mathbb{P})$.[30] $M_{\mathbb{P}} \otimes M_{\mathbb{P}'}$ refers to the Cartesian product of the relevant basis elements in $M_{\mathbb{P}}$ and $M_{\mathbb{P}'}$.

Definition 1 *Fix subspaces $\{M_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ with $M_{\mathbb{P}} \subset L^2(\mathbb{P})$ and a set $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ of linear price functionals $\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}$. A* price system *for $(\{\pi_{\mathbb{P}}\}_{P\in\mathbb{P}}, \Gamma)$ is a functional on the Cartesian product of Γ-relevant scenarios*

$$\pi(\otimes\mathcal{P}) : \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} M_{\mathbb{P}} \to \mathbb{R}$$

such that the projection to $M_{\mathbb{P}}$ is given by the restriction $\pi(\otimes\mathcal{P})_{\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}\restriction M_{\mathbb{P}}}$.

Each $\mathbb{P}$-related marketed space $M_{\mathbb{P}}$ consists of contingent claims which can be achieved frictionless, when $\mathbb{P}$ is the true law. We have a set of different price systems $\{\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}}$. When we aim to establish a meaningful consolidation

[28]This price system can be seen as an envelope of the price correspondence $\pi(X) = \{\pi_P(X) : X \in M_{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, as in Clark (1993).

[29]Sublinearity induced by market frictions is conceptually different. For instance, in Jouini and Kallal (1999) one convex set of marketed claims is equipped with a convex pricing functional, in which case, the possibility of different scenarios is not included.

[30]Note that Ω is separable by assumption, hence $L^2(\mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a separable Hilbert space for each $\mathbb{P} \in \mathcal{P}$ and admits a countable orthonormal basis. In terms of Example 2, $\mathbb{P}_0$ is the Wiener measure. In this situation, $L^2(\mathbb{P}_0)$ can be decomposed via the Wiener chaos expansion. A similar procedure could be done for the canonical process X^α related to some $\mathbb{P}^\alpha$. So we can generate an orthonormal basis for each $L^2(\mathbb{P}^\alpha)$, with $\alpha \in \mathcal{D}$. However, we take an infinite product, if $|\Gamma(\mathcal{P})| \not< \infty$, since an infinite orthonormal sum is not in general a Hilbert space.

of the scenarios we need an additional ingredient, namely Γ. This consolidation determines the operator which maps an extension of $\pi(\otimes\mathcal{P})$ into the price space $L^2(\mathcal{P})^{\circledast}_{++}$ and therefore influences the whole marketed space.

2.3 Preferences and the Economy

Having discussed the commodity price dual and the role of the consolidation of linear price systems, we introduce agents which are characterized by their preference of trades on $\mathbb{R}\times L^2(\mathbb{P})$, $\mathbb{P}\in\mathcal{P}$. There is a single consumption good, a numeraire, which agents will consume at $t=0,T$. Thus, bundles (x,X) are elements in $\mathbb{R}\times L^2(\mathbb{P})$, which are the units at time zero and time T with uncertain outcome. We call the set of rational preference relations $\succsim_{\mathbb{P}}$ on $\mathbb{R}\times L^2(\mathbb{P})$, $\mathbb{A}(\mathbb{P})$, which satisfies convexity, strict monotonicity, and $L^2(\mathbb{P})$-continuity. Let

$$B(x,X,\pi_{\mathbb{P}},M_{\mathbb{P}})=\{(y,Y)\in\mathbb{R}\times M_{\mathbb{P}}:y+\pi_{\mathbb{P}}(Y)\le x+\pi_P(X)\}$$

denote the *budget set* for a price functional $\pi_{\mathbb{P}}:M_{\mathbb{P}}\to\mathbb{R}$. We are ready to define an appropriate notion of viability. Such a minimal consistency criterion can be regarded as an inverse no trade equilibrium condition.

Definition 2 *A price system is* scenario-based viable*, if for each $\mathbb{P}\in\Gamma(\mathcal{P})$ there is a preference relation $\succsim_{\mathbb{P}}\in\mathbb{A}(\mathbb{P})$ and a bundle $(\hat{x}_{\mathbb{P}},\hat{X}_{\mathbb{P}})\in B(0,0,\pi_{\mathbb{P}},M_{\mathbb{P}})$ such that $(\hat{x}_{\mathbb{P}},\hat{X}_{\mathbb{P}})$ is $\succsim_{\mathbb{P}}$-maximal on $B(0,0,\pi_{\mathbb{P}},M_{\mathbb{P}})$.*

The conditions are necessary and sufficient for a classical economic equilibrium under each scenario $\mathbb{P}\in\Gamma(\mathcal{P})$, when we find such preference relations. Note that this definition has up to some degree the preference flavor of Bewley (2002). In the case of Example 3.1, scenario-based viability is exactly the existence of an agent with Bewley preferences and a maximal consumption bundle $(\hat{x},\hat{X})$, not depending on the prior.[31]

In the following, we relate the viability of $(\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}},\Gamma)$ with price systems in $L^2(\mathcal{P})^{\circledast}_{+}$. Let $M^{\mathcal{P}}_{\mathbb{P}}=M_{\mathbb{P}}\cap L^2(\mathcal{P})$, with $\mathbb{P}\in\mathcal{P}$.

Theorem 1 *A price system $(\{\pi_{\mathbb{P}}\}_{P\in\mathcal{P}},\Gamma)$ is scenario-based viable if and only if there is an $\Psi\in L^2(\mathcal{P})^{\circledast}_{++}$ such that $\pi_{\mathbb{P}\restriction M^{\mathcal{P}}_{\mathbb{P}}}\le\Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}}$ for each $\mathbb{P}\in\Gamma(\mathcal{P})$.*

This characterization of scenario-based viability takes scenario-based marketed spaces $\{M_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ as given. Moreover, the consolidation operator Γ is a given characteristic of the coherent price system. With this in mind, one should think that in a general equilibrium system the locally given prices $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ are be part of it.

[31] The fundamental theorem of asset pricing in Dybvig and Ross (2003) contains a third equivalent statement, the existence of an agent (preferring more than less) being in an optimal state. The adequate concept of strict monotone preferences is subtle and important when the uncertainty is given by a set of mutually singular priors. For instance, the classical strict monotonicity ($X\ge Y$ and $X\ne Y$ implies $X\succ Y$) seems to be too strong. For instance, maxmin preferences of Gilboa and Schmeidler (1989) do not satisfy this monotonicity under the $\mathcal{P}$.

The extension we perceive can be seen as a regulated and coherent price system for every claim in $L^2(\mathcal{P})$.

In comparison to the single prior case, the structure of incompleteness depends on the set of relevant priors $\Gamma(\mathcal{P})$. As described in Example 3.2, this is a natural situation. As such, prior-dependent prices $\pi_{\mathbb{P}}$ are also plausible. The expected payoff as a pricing principle depends on the prior under consideration, as well. In this way, the concept of scenario-based prices accounts for every Γ-relevant price system simultaneously.

As indicated in Example 3.1, there is a closed subspace of unambiguous claims where the valuation is unique. In Section 3, we use the related symmetry property for the introduction of a reasonable martingale notion. Let $\mathcal{R} \subset \mathcal{P}$ and define the *$\mathcal{R}$-marketed space* by

$$\mathbb{M}(\mathcal{R}) = \left\{ X \in L^2(\mathcal{P}) : E^{\mathbb{P}}[X] \text{ is constant for all } \mathbb{P} \in \mathcal{R} \right\}.$$

Only the continent claims in $\mathbb{M}(\mathcal{R})$ reduce the valuation to a linear pricing, if $\Gamma(\mathcal{P}) = \mathcal{R}$.[32] Claims in $\mathbb{M}(\mathcal{R})$ are unambiguous. This can also be formulated as a property of events $\mathcal{U}(\mathcal{R}) = \{A \in \mathcal{F} : P(A) \text{ is constant for all } \mathbb{P} \in \mathcal{R}\}$.[33] From Theorem 1 we have the following corollary.

Corollary 1 *Every Ψ in Theorem 1 is linear and $c_{2,\mathcal{P}}$-continuous on $\mathbb{M}(\Gamma(\mathcal{P}))$.*

We have two operations which constitute the distillation of uncertainty. This consolidation can be seen as a characterization of the Walrasian auctioneer, in which case diversification should be encouraged. But this refers to the sublinearity of Ψ.

Remark 2 *One may ask which Γ is appropriate. Such a question is related to the concept of mechanism design. The market planner can choose a consolidation that influences the indirect utility of a reported preference relation. However, the full discussion of these issues lies beyond the scope of this paper.*[34]

3 Asset Markets and Symmetric Martingales

We extend the primitives with trading dates and trading strategies. A time interval is considered where the market consists of a riskless security and a security under volatility uncertainty. Within the financial market model, we discuss the modified notions of arbitrage and equivalent martingale measures. Theorem 2 associates scenario-based viability with equivalent symmetric martingale measure sets. The last section considers the so called G-framework. Here, the uncertain security

[32] Or unless Γ is given a priori by a linear pricing, e.g. $\Gamma = \delta_{\{\mathbb{P}\}}$ for some $\mathbb{P} \in \mathcal{P}$.

[33] Note, that for the single prior case every closed subspace of $L^2(\mathbb{P})$ can be identified with a sub σ-algebra in terms of a projection via the conditional expectation operator. Although $\mathcal{U}$ is not a σ-algebra, but a Dynkin System, it identifies in a similar way a certain subspace. See also Epstein and Zhang (2001) for a definition of unambiguous events and an axiomatization of preferences on this domain.

[34] A starting point could be Lopomo, Rigotti, and Shannon (2009), who consider a mechanism design problem under Knightian uncertainty.

process is driven by a G-Itô process, which shows that the concept of symmetric martingale measure sets is far from empty.

Background: Risk-neutral asset pricing with one prior

In order to introduce dynamics and trading dates, we fix a time interval $[0, T]$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Fix an $\mathbb{F}$-adapted risky asset price $(S_t) \in L^2(\mathbb{P}\otimes dt)$ and a riskless bond $S^0 \equiv 1$. We next review some terminology.

The portfolio process of a strategy $\eta = (\eta^0, \eta^1)$ is called X^η. Simple self-financing strategies are piecewise constant $\mathbb{F}$-adapted processes η such that $dX^\eta = \eta dS$, which we call $\mathcal{A}(\mathbb{P})$. A $\mathbb{P}$-arbitrage in $\mathcal{A}(\mathbb{P})$ is a strategy (with zero initial capital) such that $X_T^\eta \geq 0$ and $\mathbb{P}\left(X_T^\eta > 0\right) > 0$.

A claim is marketed, i.e. $X \in M$, if there is a $\eta \in \mathcal{A}(\mathbb{P})$ such that $X = \eta_T S_T$ $\mathbb{P}$-a.s., then we have the (by the law of one price) $\pi(X) = \eta_0 S_0$. An equivalent martingale measure (EMM) $\mathbb{Q}$ must satisfy that S is a $\mathbb{Q}$-martingale and $d\mathbb{Q} = \psi d\mathbb{P}$, where $\psi \in L^2(\mathbb{P})_{++}$ is a Radon Nykodym-Density with respect to $\mathbb{P}$. Theorem 2 of Harrison and Kreps (1979) states the following:

Under no $\mathbb{P}$-arbitrage, there is a one to one correspondence between the continuous linear and strictly positive extension of $\pi : M \to \mathbb{R}$ to $L^2(\mathbb{P})$ and a EMM $\mathbb{Q}$. The relation is given by $\mathbb{Q}(B) = \Pi(1_B)$ and $\Pi(X) = E^{\mathbb{Q}}[X]$, $B \in \mathcal{F}_T$ and $X \in L^2(\mathbb{P})$.

This result can be seen as a preliminary version of the first fundamental theorem of asset pricing.

3.1 Volatility Uncertainty, Dynamics and Arbitrage

We specify the mathematical framework and the modified notions, such as arbitrage. The present uncertainty model $(\Omega, \mathcal{F}, \mathcal{P})$ is based on the explicit formulation of volatility uncertainty. Afterwards, we introduce the notion of a martingale with respect to a conditional sublinear expectation, the financial market and the robust arbitrage concept.

3.1.1 Dynamics and Martingales under Sublinear Expectation

The principle idea is to transfer the results from Section 2 into a dynamic setup. The specification in Example 1 of Section 2.1 serves as our uncertainty model. We can directly observe the sense in which the quadratic variation creates volatility uncertainty. We introduce the sublinear expectation $\mathcal{E} : L^2(\mathcal{P}) \to \mathbb{R}$ given by the supremum of expectations of $\mathcal{P} = \{\mathbb{P}^\alpha : \alpha \in \mathcal{D}\}$. It is possible to work within the larger space $\hat{L}^2(\mathcal{P})$. An explicit representation of $\hat{L}^2(\mathcal{P})$ is given in Appendix A.1. Moreover, we assume that $\mathcal{P}$ is stable under pasting (see Appendix A.2. for details).

As we aim to equip the financial market with the dynamics of a sublinear conditional expectation, we introduce the information structure of the financial market given by an augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The setting is based on the dynamic sublinear expectation terminology as instantiated by Nutz and Soner (2012).

We give a generalization of Peng's G-expectation as an example, satisfying the weak compactness of $\mathcal{P}$ when the sublinear expectation is represented in terms of a supremum of linear expectations. In Section 3.3 and in Appendix B.2, we consider the G-expectation as an important special case. That said, a possible association of results in Section 2 depends heavily on the weak compactness of the generated set of priors $\mathcal{P}$.

Example 5 *Suppose a trader is confronted with a pool of models describing volatility, such as the stochastic volatility model in Heston (1993). After a statistical analysis of the data, two models remain plausible $\mathbb{P}^{\alpha}$ and $\mathbb{P}^{\hat{\alpha}}$. Nevertheless, the implications for the trading decision deviate considerably. Even the asset span on its own depends on each scenario (see Example 3). A mixture of both models does not change this uncertain situation at all. In order to address the possibilistic issue, let us define the universal extreme cases $\underline{\sigma}_t = \inf(\alpha_t, \hat{\alpha}_t)$ and $\overline{\sigma}_t = \sup(\alpha_t, \hat{\alpha}_t)$. When thinking about a reasonable* uncertainty management, *no scenario between $\underline{\sigma}$ and $\overline{\sigma}$ should be ignored. The uncertainty model which accounts for all these cases is given by*

$$\mathcal{P} = \{\mathbb{P}^{\alpha} : \alpha_t \in [\underline{\sigma}_t, \overline{\sigma}_t] \quad \mathbb{P}_0 \otimes dt \, a.e.\}.$$

A related construction of a sublinear conditional expectation is achieved in Nutz (2012), where the deterministic bounds of the G-expectation are replaced by path dependent bounds.[35]

In the following, we introduce an appropriate concept for the dynamics of the continuous-time multiple-prior uncertainty model. The associated objectives are trading dates, the information structure and the price process (as the carrier of the uncertainty). In order to introduce the price process $S = (S_t)_{t\in[0,T]}$ of an uncertain and long lived security, we have to impose further primitives. Define the time-depending set of priors

$$\mathcal{P}(t, \mathbb{P})^o = \{\mathbb{P}' \in \mathcal{P} : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_t^o\}.$$

This set of priors consists of all extensions $\mathbb{P} : \mathcal{F}_t^o \to [0, 1]$ from $\mathcal{F}_t^o$ to $\mathcal{F}$ in $\mathcal{P}$. In other words, $\mathcal{P}(t, \mathbb{P})^o$ contains exactly all probability measures in $\mathcal{P}$ defined on $\mathcal{F}$ that agree with $\mathbb{P}$ in the events up to time t. Fix a contingent claim $X \in L^2(\mathcal{P})$. In Nutz and Soner (2012), the unique existence of a sublinear expectation $(\mathcal{E}_t^{\mathcal{P}}(X))_{t\in[0,T]}$ is provided by the following construction[36]

$$\mathcal{E}_t^{\mathcal{P}}(X)(\omega) = \sup_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})^o} E^{\mathbb{P}'}[X|\mathcal{F}_t](\omega) \quad \mathbb{P}\text{-a.s.} \quad \text{for all } \mathbb{P} \in \mathcal{P}$$

The conditional expectation operator satisfies the Law of Iterated Expectation, i.e. $\mathcal{E}_s^{\mathcal{P}}(\mathcal{E}_t^{\mathcal{P}}) = \mathcal{E}_s^{\mathcal{P}}$ with $s \leq t$. We can define a martingale similarly to the single prior

[35]This framework is also included in Epstein and Ji (2013). In this setting, drift and volatility uncertainty are considered simultaneously. Drift uncertainty or κ-ambiguity are well known terms in financial economics. A coherent and well-developed theory, known as g-expectation, is available under a Brownian filtration.

[36]$\mathbb{P}$ess sup denotes the essential supremum under $\mathbb{P}$. Representations of such martingales can be formulated via a 2BSDE. This concept is introduced in Cheridito, Soner, Touzi, and Victoir (2007), see also Soner, Touzi, and Zhang (2012).

setting.[37] The nonlinearity implies that if a process $X = (X_t)_{t\in[0,T]}$ is a martingale under $\mathcal{E}_t^{\mathcal{P}}(\cdot)$ then $-X$ is not necessarily a martingale.

Definition 3 *An $\mathbb{F}$-adapted process $X = (X_t)$ is a* $\mathcal{P}$-martingale *if*

$$\mathcal{E}_s^{\mathcal{P}}(X_t) = X_s \quad \mathcal{P}\text{-q.s.,} \quad \text{for all } s \le t.$$

We call X a symmetric $\mathcal{P}$-martingale *if X and $-X$ are both $\mathcal{P}$-martingales.*

In the next subsection we discuss the martingale property of asset prices processes under a modified sublinear expectation. As we will see, the space $\mathbb{M}(\mathcal{P})$ is closely related to symmetric martingales. Conceptually, the symmetry refers to a generalized Put-Call parity and formalizes the uncertainty-neutral valuation in terms of martingales.

3.1.2 The Primitives of the Financial Market and Arbitrage

For the sake of simplicity, we assume that the riskless asset is $S_t^0 = 1$, for every $t \in [0,T]$, i.e. the interest rate is zero. We call the related abstract financial market $\mathcal{M}(1,S)$ on the filtered space uncertainty space $(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{F})$, whenever the price process of the uncertain asset (S_t) satisfies $S_t \in L^2(\mathcal{P})$ for every $t \in [0,T]$ and $\mathbb{F}$-adaptedness.

A *simple trading strategy*[38] is an $\mathbb{F}$-adapted stochastic process (η_t) in $L^2(\mathcal{P}) \times L^2(\mathcal{P})$ when there is a finite sequence of dates $0 < t_0 \le \cdots \le t_N = T$ such that $\eta = (\eta^0, \eta^1)$ can be written with $\eta^i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathcal{P})$ as $\eta_t = \sum_{i=0}^{N-1} 1_{[t_{i+1},t_i)}(t)\eta^i$. The fraction invested in the riskless asset is denoted by η_t^0, $t \in [0,T]$. A trading strategy is *self-financing* if $\eta^0_{t_{n-1}} S^0_{t_n} + \eta^1_{t_{n-1}} S_{t_n} = \eta^0_{t_n} S^0_{t_n} + \eta^1_{t_n} S_{t_n}$ $\mathcal{P}$-q.s. and for every $n \le N$. The value of the portfolio satisfies $X_t^\eta \in L^2(\mathcal{P})$ for every $t \in [0,T]$.

The set of simple self-financing trading strategies is denoted by $\mathcal{A}$. This financial market $\mathcal{M}(1,S)$ with trading strategies in $\mathcal{A}$ is called $\mathcal{M}(1,S,\mathcal{A})$.

It is well known that a necessary condition for equilibrium is the absence of arbitrage. Therefore, with regard to the equilibrium consistency condition of the last section, we introduce arbitrage in the financial market of securities. The modeled uncertainty of the financial market motivates us to consider a stronger and robust notion of absence of arbitrage.

Definition 4 *Let $\mathcal{R} \subset \mathcal{P}$. We say there is an* $\mathcal{R}$-arbitrage opportunity *in $\mathcal{M}(1,S,\mathcal{A})$ if there exists an admissible pair $\eta \in \mathcal{A}$ such that $\eta_0 S_0 \le 0$,*

$$\eta_T S_T \ge 0 \;\; \mathcal{R}\text{-q.s., and } \;\; \hat{\mathbb{P}}\left(\eta_T S_T > 0\right) > 0 \text{ for at least one } \hat{\mathbb{P}} \in \mathcal{R}.$$

The choice of the definition is based on the following observation. This arbitrage strategy is riskless for each $\mathbb{P} \in \mathcal{R}$ and if the prior $\hat{\mathbb{P}}$ constitutes the market one would gain a profit with a strictly positive probability. With this in mind, the $\mathcal{P}$-arbitrage

[37] For the multiple prior case with mutually equivalent priors we refer to Riedel (2009).

[38] As mentioned in Harrison and Pliska (1981) simple strategies rule out the introduction of doubling strategies and hence a notion of admissibility.

notion can be seen as a weak arbitrage opportunity with the corresponding cone $L^2(\mathcal{P})_+\setminus\{0\}$. Alternatively, we could argue that absence of $\mathcal{R}$-arbitrage is consistent with a weak-dominance principle based on $\mathcal{R}$.

To connect the prior-dependent marketed spaces of Definition 1, we say that a claim $X \in L^2(\mathbb{P})$ is *marketed* in $\mathcal{M}(1, S, \mathcal{A})$ at time zero under $\mathbb{P} \in \mathcal{P}$ if there is an $\eta \in \mathcal{A}$ such that $X = \eta_T S_T$ holds only $\mathbb{P}$-almost surely. In this case we say η hedges X and lies in $M_{\mathbb{P}}$. $\eta_0 S_0 = \pi_{\mathbb{P}}(X)$ is the price of X in $\mathcal{M}(1, S, \mathcal{A})$ under $\mathbb{P} \in \mathcal{P}$.

With Example 3 and 4 in mind, fix the marketed spaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. The price of a marketed claim under the prior $\mathbb{P}$ should be well defined. Let $\eta, \eta' \in \mathcal{A}(\mathbb{P})$ generating the same claim $X \in M_{\mathbb{P}}$, i.e. $\eta_T S_T = \eta'_T S_T$ $\mathbb{P}$-a.s. We have $\eta_0 S_0 = \eta'_0 S_0 = \pi_{\mathbb{P}}(X)$ under absence of $\mathbb{P}$-arbitrage. Note, that this may not be true under no $\hat{\mathbb{P}}$-arbitrage, with $\mathbb{P} \neq \hat{\mathbb{P}} \in \mathcal{P}$. This is related to the law of one price under a fixed prior. Now, similarly to the single prior case, we define viability in a financial market. We say that a financial market $\mathcal{M}(1, S, \mathcal{A})$ is *viable* if it is $\Gamma(\mathcal{P})$-arbitrage free and the associated price system $(\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}, \Gamma)$ is scenario-based viable.

3.2 Equivalent Symmetric Martingale Measure Sets

In Section 2 we introduced the price space of sublinear functionals generated by a set of linear $c_{2,\mathcal{P}}$-continuous functionals. The extension of the price functional is strongly related to the involved linear functionals which constitutes the price systems locally. In this fashion, we introduce a modified notion of fair pricing. In essence, we associate a risk-neutral prior to each local and linear extension of a price system. Here, the term local refers to a fixed prior, so that at this stage no volatility uncertainty is present.

In our uncertainty model, the price of a claim equals the (discounted) value under a specific sublinear expectation. Exploration of available information, when multiple priors are present, changes the view of a rational expectation.

In economic terms, the notion of symmetric martingales eliminates preferences for ambiguity in the valuation. This is the base to introduce the following rational pricing principle in terms of sublinear expectations with a symmetry condition.

Definition 5 *A set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ is called* equivalent symmetric martingale measure set (EsMM-set) *if the given conditions hold:*

1. *For every $\mathbb{Q} \in \mathcal{Q}$ there is a $\mathbb{P} \in k(\mathcal{P})$ such that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent to each other, so that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{P})$.*

2. *The uncertain asset (S_t) is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale, where $\mathcal{E}^{\mathcal{Q}}$ is the conditional sublinear expectation under $\mathcal{Q}$.*

The first condition formulates a direct relationship between an element $\mathbb{Q}$ in the EsMM-set $\mathcal{Q}$ and the primitive priors $\mathbb{P} \in \mathcal{P}$. The square integrability is a technical condition that guarantees the association to the equilibrium theory of Section 2. The second is the accurately adjusted martingale condition. The idea of a fair gamble should reflect the neutrality of preferences for risk and ambiguity. Under the new sublinear expectation, the asset price and hence the portfolio process, are symmetric martingales. This implies, as discussed in the introduction, that the value of the claim does not depend on the prior. The valuation is mean unambiguous, i.e. preferences for ambiguity under $\mathcal{Q}$ are neutral. One can think of the ambiguity-neutral part in the valuation in terms of maxmin preferences from Gilboa and Schmeidler (1989).[39] In this situation, the expected utility is under every prior $\mathbb{Q} \in \mathcal{Q}$ the same. Similarly to pricing under risk, where risk preferences do not matter, analogous reasoning should be true concerning preferences for ambiguity. As such, saying everyone is uncertainty neutral immediately leads one to come up with the uncertainty-neutral expectation $\mathcal{E}^{\mathcal{Q}}$.

The case of only one prior is related to the well-known risk-neutral valuation principle. Under volatility uncertainty, this principle needs a new requirement due to the more complex uncertainty model. In this sense the symmetry condition encodes ambiguity neutrality as part of *uncertainty neutrality*.

Remark 3 *1. In the light of Remark 1, let us mention that Definition 3 and 5 can be generalized to the notion nonlinear conditional expectations* $(\mathfrak{E}_t)$ *satisfying the Law of Iterated Expectation, see Section 9 in Chapter III of Peng (2010). The definition of a* $\mathfrak{E}$*-martingale is straightforward.*

Concerning the definition of an EsMM-set, the object $\mathcal{Q}$ *would refer to the set of priors representing* $\mathfrak{E}$*. In Remark 1, a possible construction is illustrated.*

A further weakening of the symmetric martingale property is possible. Instead of that we could merely require the $\mathfrak{E}$*-martingale property of* (S_t)*.*

2. Note that in the case of a single prior framework, i.e. $\mathcal{P} = \{\mathbb{P}\}$*, the notion of EsMM-sets is reduced to accommodate EMM's. In this regard we can think of canonical generalization. On the other hand, classical EMM's and a linear price theory are still present. Every single-valued EsMM-set* $\{\mathbb{Q}\}$ *can be seen as an EMM under* $\mathbb{P} \in \mathcal{P}$*. Here, the consolidation is given by* $\Gamma = \delta_{\mathbb{P}}$ *and we have* $\Gamma(\mathcal{P}) = \{\mathbb{P}\}$*. In this situation,* Γ *reveals the ignorance of every other possible prior* $\mathbb{P}' \in \mathcal{P}$*.*

The following result justifies the discussion involving uncertainty neutrality and the symmetry condition for martingales. The one to one mapping of Theorem 2 and the choice of the price space fall into place. In this manner we show that the existence of an $\mathcal{R}$-arbitrage in $\mathcal{M}(1, S, \mathcal{A})$ with $\Gamma(\mathcal{P}) = \mathcal{R}$ is inconsistent with an economic equilibrium for agents in $\mathbb{A}(\mathbb{P})$, with $\mathbb{P} \in \mathcal{R}$. We fix an associated price system using the procedure described at the end of Subsection 3.1.

Theorem 2 *Suppose the financial market model* $\mathcal{M}(1, S, \mathcal{A})$ *does not allow any* $\mathcal{P}$*-arbitrage opportunity. Then there is a bijection between coherent price systems* $\Psi : L^2(\mathcal{P}) \to \mathbb{R}$ *in* $L^2(\mathcal{P})^{\circledast}_{++}$ *of Theorem 1 and EsMM-sets, satisfying stability under*

[39]However, the same argument is applicable to the α-MEU preferences of Ghirardato, Maccheroni, and Marinacci (2004), the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005) and variational preferences of Maccheroni, Marinacci, and Rustichini (2006).

pasting of the induced set $\Gamma(\mathcal{P})$.[40] *The relationship is given by* $\Psi(X) = \mathcal{E}^{\mathcal{Q}}(X)$, *where*

$$\mathcal{Q} = \left\{\mathbb{Q} \in \mathcal{M}_1(\Omega) : \tfrac{d\mathbb{Q}}{d\mathbb{P}} = \psi_{\mathbb{P}}, \mathbb{P} \in \Gamma(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})_{++}\right\}$$

is the associated EsMM-set.

Let $\mathcal{R} \subset \mathcal{P}$ and $\mathtt{M}(\mathcal{R})$ be the set of all EsMM-sets $\mathcal{Q}$ such that the related consolidation Γ satisfies $\Gamma(\mathcal{P}) = \mathcal{R}$. Theorem 2 can be seen as the formulation of a one-to-one mapping between a subset of

$$L^2(\mathcal{P})^{\circledast}_{++} \text{ and } \bigcup_{\mathcal{R}\subset k(\mathcal{P})} \mathtt{M}(\mathcal{R}).$$

There is a hierarchy of sublinear expectations, related to the chosen consolidation operator Γ and the EsMM-sets, which are ordered by the inclusion relation. We illustrate the relationship between Γ and an EsMM-set in the following example.

Example 6 *For the sake of simplicity, let us assume that* $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4\}$, *so that any pasting property is ignored. Starting with the sublinear price system, we have four price functionals* $\pi_1, \pi_2, \pi_3, \pi_4$ *and the consolidation operator* Γ. *Let us assume that* $\Gamma = (+, \wedge)$ *and* $\lambda \in (0, 1)$. *This gives us* $\lambda\pi_1 + (1-\lambda)\pi_2 = \pi^\lambda$ *and* $\Gamma(\pi_1, \pi_2, \pi_3, \pi_4) = \pi^\lambda \wedge \pi_3$. *The resulting EsMM-set is given by* $\mathcal{Q} = \{\psi^\lambda \times \mathbb{P}^\lambda, \psi_3 \times \mathbb{P}_3\} \in \mathtt{M}(\mathcal{P} \setminus \{\mathbb{P}_4\})$, *where* $\mathbb{P}^\lambda = \lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$, $\psi^\lambda = \lambda\psi_1 + (1-\lambda)\psi_2$ *and* $E^{\mathbb{P}^\lambda}[\psi^\lambda] = 1 = E^{\mathbb{P}_3}[\psi_3]$.

We close this consideration with some results analogous to those of the single prior setting where we combine Theorem 2 and Theorem 1.

Corollary 2 *Let* $\mathcal{R} = \Gamma(\mathcal{P}) \subset \mathcal{P}$ *be stable under pasting and given.*

1. $\mathcal{M}(1, S, \mathcal{A})$ *is viable if and only if there is an EsMM-set.*
2. *Market completeness, i.e* $M_{\mathbb{P}} = L^2(\mathbb{P})$ *for each* $\mathbb{P} \in \mathcal{R}$, *is equivalent to the existence of exactly one EsMM-set in* $\mathtt{M}(\mathcal{R})$.
3. *If* $\mathtt{M}(\mathcal{R})$ *is nonempty, then there exists no* $\mathcal{R}$*-arbitrage.*
4. *If there is a strategy* $\eta \in \mathcal{A}$ *with* $\eta_0 S_0 \leq 0$, $\eta_T S_T \geq 0$ $\mathcal{R}$*-q.s. and* $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) > 0$, *for some* $\mathcal{Q} \in \mathtt{M}(\mathcal{R})$, *then there is an* $\mathcal{R}$*-arbitrage opportunity.*

The result does not depend on the preference of the agent. The expected return under the sublinear expectation $\mathcal{E}^{\mathcal{Q}}$ equals the riskless asset. Hence, the value of a claim can be considered as the expected value in the uncertainty-neutral world.[41]

3.3 A Special Case: G-Expectation

Now, we select a stronger calculus to model the asset prices as a stochastic differential equation driven by a G-Brownian motion.[42] In this situation the volatility of

[40] See Definition 6 in Appendix A for this important concept. In essence, this condition is needed to define a conditional sublinear expectation based on $\mathcal{Q}$, satisfying the iterated law of conditional expectation.

[41] However, the sublinear expectation depends on Γ.

[42] An illustration of the concept in a discrete time framework is achievable, via the application of the results in Cohen, Ji, and Peng (2011).

the process concentrates the uncertainty in terms of the derivative of the quadratic variation. The quadratic variation of a G-Brownian motion creates volatility uncertainty. Again, we review the related result of the single prior framework.

Background: Itô processes in the single prior framework
We specify the asset price in terms of an Itô process $dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t$, $S_0 = 1$, driven by a Brownian motion $B = (B_t)_{t\in[0,T]}$ on the given filtered probability space, $\mu(\cdot, x), \sigma(\cdot, x)$, with $x \in \mathbb{R}_+$ are adapted processes such that (S_t) is a well defined process taking values in $\mathbb{R}_+$. The filtration is generated by B. The interest rate is $r = 0$. Let E^θ be the exponential martingale, given by $d\mathrm{E}^\theta_t = \mathrm{E}^\theta_t \theta_t dB_t$, $\mathrm{E}^\theta_0 = 1$, with a consistent kernel θ we can apply Girsanov theorem. The following result is from Harrison and Kreps (1979):

The set of equivalent martingale measures is not empty if and only if $\rho = \mathrm{E}^\theta_T \in L^2(\mathbb{P})$, $\theta \in L^2(\mathbb{P} \otimes dt)$ and $S^* = \int \sigma dB$ is a $\mathbb{P}$-martingale.

ρ can be interpreted as a state price density. The associated market price of risk $\theta_t = \frac{\mu_t - r}{\sigma_t}$ is the Girsanov or pricing kernel of the state price density.

3.3.1 Security prices as G-Itô processes and sublinear valuation

An important special case is an uncertainty model given by the G-expectation $E_G : L^2_G(\Omega) \to \mathbb{R}$,[43] where $L^2_G(\Omega) = \hat{L}^2(\mathcal{P})$ is described at the beginning of Appendix A.1. More precisely, the uncertainty model is induced by the following sublinear expectation space $(C([0,T];\mathbb{R}), L^2_G(\Omega), E_G)$ as given.

We select the next rational base, namely an interval $[\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}_{++}$, instead of a constant volatility σ. As indicated in the introduction, volatility uncertainty refers to the awareness that every adapted process (σ_t) taking values in $[\underline{\sigma}, \overline{\sigma}]$ may constitute one possible prior or scenario. We introduce an asset price process driven by a G-Brownian motion $(B^G_t)_{t\in[0,T]}$. In Appendix B.2 we present a small primer of the applied results.

Under the objective description of the real world, given by $\mathcal{P}$ and induced by $[\underline{\sigma}, \overline{\sigma}]$, the asset price is driven by the following G-stochastic differential equation

$$dS_t = \mu(t, S_t)d\langle B^G\rangle_t + V(t, S_t)dB^G_t, \quad S_0 = 1.$$

Let $\mu : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $V : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}_+$ be processes such that a unique solution exists.[44] Moreover, let $V(\cdot, x)$ be a strictly positive process for each $x \in \mathbb{R}_+$. The riskless asset has interest rate zero.

The Girsanov theorem for G-Brownian motion is precisely what is needed to verify the symmetric $\mathcal{E}^{\mathcal{Q}}$-martingales property of the price processes S under some sublinear expectation given by an EsMM-set $\mathcal{Q}$. The present uncertainty model enables us to apply the necessary stochastic calculus. As such, we model the financial market in the G-expectation setting, introduced in Peng (2006) and Peng (2010).

[43]It is shown in Theorem 52 by Denis, Hu, and Peng (2011), that this sublinear expectation can be represented by a weakly compact set, when the domain is in $L^2_G(\Omega)$.

[44]We refer to Chapter 5 in Peng (2010) for existence results of G-SDE's.

Central results, such as a martingale representation and a well behaved underlying topology are desired for the foundational grounding of asset pricing. The second condition of Definition 5 highlights how a Girsanov transformation adapts to a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale and thus guarantees the existence of nontrivial EsMM-sets.[45] For this purpose we define the related sublinear expectation generated by an EsMM-set, $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \psi d\mathbb{P},\ \mathbb{P} \in \mathcal{P}\}$:

$$\sup_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}[X] = \mathcal{E}^{\mathcal{Q}}(X) = E_G[\psi X], \quad X \in L^2_G(\Omega).$$

Note that we consider an aggregated family of state price densities $\psi \in L^2_G(\Omega)$ which is defined $\mathcal{P}$-q.s. This means that the density is now a uniform object under our uncertainty model, i.e. $\psi = \psi_{\mathbb{P}}$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$ (see also Remark 4 below). Theorem 3 justifies the choice of this shifted sublinear expectation when the asset price is restrained to a symmetric martingale for an uncertainty-neutral expectation.

Let us introduce the universal state price density $\mathtt{E}^\theta$, with $\psi = \mathtt{E}^\theta_T$ $\mathcal{P}$-q.s., being a symmetric martingale of exponential type under the G-expectation, with an integrable pricing kernel $(\theta_t)_{t\in[0,T]}$ (or market price of uncertainty)

$$d\mathtt{E}^\theta_t = \mathtt{E}^\theta_t \theta(t, S_t) dB^G_t, \ \ \mathtt{E}^\theta_0 = 1.$$

Applying the results in Appendix B.2 allows us to write $\mathtt{E}^\theta$ explicitly as

$$\mathtt{E}^\theta_t = \exp\bigg(-\frac{1}{2}\int_0^t \theta(r, S_r)^2 d\langle B^G\rangle_r - \int_0^t \theta(r, S_r) dB^G_r \bigg), \quad t \in [0, T].$$

Let the pricing kernel solve $V(t, S_t)\theta(t, S_t) = \mu(t, S_t)$ $\mathcal{P}$-quasi surely, for every $t \in [0, T]$. Before we formulate the last result we define $S^*_t = S^*_0 + \int_0^t V(r, S^*_r) dB^G_r$, $t \in [0, T]$ and assume that a unique solution on $(\Omega, L^2_G(\Omega), E_G)$ exists for some state-dependent process V, see Peng (2010).

Theorem 3 *The set of EsMM-sets contains a* $\mathcal{Q} \in \mathtt{M}(\mathcal{P})$ *if and only if* S^* *is an* E_G*-martingale and*

$$E_G\bigg[\exp\bigg(\delta \cdot \int_0^T \theta(r, S_r)^2 d\langle B^G\rangle_r\bigg)\bigg] < \infty, \ \text{for some } \delta > \frac{1}{2}.$$

With Theorem 2 in mind we can associate the concept of scenario-based viability. Let $X \in L^2_G(\Omega)$ be a contingent claim such that it is priced by $\mathcal{P}$-arbitrage then the fair value is given by $\Psi(X) = E_G[\mathtt{E}^\theta_T X]$, whenever Γ consists only of a consolidation via the maximum operation.

Moreover, one can define a new $\hat{G}$-expectation related to a volatility uncertainty of a closed subinterval $[\hat{\sigma}_1, \hat{\sigma}_2] \subset [\underline{\sigma}, \overline{\sigma}]$. We can identify a consolidation operator by $\Gamma_{\hat{G}}(\mathcal{P}) = \{\mathbb{P}^\alpha : \alpha \in [\hat{\sigma}_1, \hat{\sigma}_2]\}$. In this case Theorem 3 can be reformulated in terms of the existence of an EsMM-set $\mathcal{Q}_{\hat{G}} \in \mathtt{M}(\Gamma_{\hat{G}}(\mathcal{P}))$.

[45] Trivial EsMM-sets consist of mutually equivalent priors, associated to a single $\mathbb{P} \in \mathcal{P}$.

Remark 4 *The more precise calculus of the G-expectation is based on an analytic description of nonlinear partial differential equations. This allows us to create a uniform state price density process in terms of an exponential martingale, based on a G-martingale representation theorem (see Appendix B 3). With this in mind, a more elaborated notion of EsMM-sets can be formulated by requiring that the family of densities* $\{\psi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ *create a uniform process as a symmetric martingale under the sublinear expectation* $\mathcal{E}^{\mathcal{P}} = E_G$.

Remark 5 *Under the assumption of G-Brownian motion, Epstein and Ji (2013) obtain an analogous state price density process. Without applying a Girsanov-type theorem, they use a density process in the important case* $\mu(t, S_t) = \mu_t S_t$ *and* $V(t, S_t) = V_t S_t$. *In comparison to our flexible local functional form, their asset price is apparently governed by* $dS_t = S_t(b_t dt + V_t dB_t^G)$.[46] *In this case, the relationship between the asset price processes, with the special pricing kernel* $\left(\frac{\mu_t}{V_t}\right) = (\theta_t) \in M_G^2(0, T)$, *is given by*

$$\int_0^t \mu_s d\langle B^G\rangle_s = \int_0^t \mu_s \hat{a}_s ds = \int_0^t b_s ds, \qquad \text{where } \hat{a}_s = \frac{d}{ds}\langle B^G\rangle_s.$$

Extensions to continuous trading strategies seem to be next natural step. Nevertheless, an admissibility condition should be requested, in order to exclude doubling strategies. Considering markets with more than one uncertain security requires a multidimensional Girsanov theorem.[47] We close this section with an example of the connection between super-replication of claims and EsMM-sets.

Example 7 *Under one prior* $\mathcal{P} = \{\mathbb{P}\}$, *Delbaen (1992) obtained the super-replication price in terms of martingale measures in* $\mathtt{M}(\{\mathbb{P}\})$:

$$\Lambda^{\mathbb{P}}(X) = \inf\{y \geq 0 | \exists\ \theta \in \mathcal{A}(\mathbb{P}) : y + \theta_T S_T \geq X\ \mathbb{P}\text{-}a.s.\} = \sup_{\mathbb{Q}\in\mathtt{M}(\{\mathbb{P}\})} E^{\mathbb{Q}}[X]$$

When the uncertainty is given by a set of mutually singular priors, a super-replication price can be derived, see Denis and Martini (2006), in terms of a set of martingale laws $\mathcal{M}$ *such that* $\Lambda^{\mathcal{P}}(X) = \sup_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[X]$. *In the G-framework with simple trading strategies, this set* $\mathcal{M}$ *is an EsMM-set. When applying our theory to this problem, we get*

$$\begin{aligned}\Lambda^{\mathcal{P}}(X) &= \inf\{y \geq 0 | \exists\ \theta \in \mathcal{A} :\ y + \theta_T S_T \geq X \quad \mathcal{P}\text{-}q.s.\} \\ &= \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathtt{M}(\{\mathbb{P}\})} E^{\mathbb{Q}}[X] \\ &= E_G[\mathtt{E}_T X],\end{aligned}$$

upon applying our Theorem 3 as well as Theorem 3.6 from Vorbrink (2010). This is associated to the maximal EsMM-set in $\mathtt{M}(\mathcal{P})$. *However, with Proposition 1 every EsMM-set delivers a price below this super-hedging price.*

[46]This allows us to model local volatility structures and volatility uncertainty at the same time.
[47]See Osuka (2011).

4 Discussion and Conclusion

We present a framework and a theory of derivative security pricing where the uncertainty model is given by a set of mutually singular probability measures which incorporates volatility uncertainty. The notion of equivalent martingale measures changes, and the related linear expectation principle becomes a nolinear theory of valuation. The associated arbitrage principle should consider all remaining uncertainty in the consolidation.

The results of this paper establishes preliminary version of the fundamental theorem of asset pricing (FTAP) under mutually singular uncertainty. In Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998), a definitive FTAP is achieved for the single prior uncertainty model in great generality. The notion of arbitrage is in principle a separation property of convex sets in a topological space. In this regard, the choice of the underlying topological structure is essential for observing a FTAP. For instance, Levental and Skorohod (1995), establish a FTAP with an *approximate arbitrage* based on a different notion of convergence.

In our setting, two aspects must be kept in mind for deriving a FTAP with mutually singular uncertainty. Firstly, the spaces of claims and portfolio processes are based on a capacity norm, and thus forces one to argue for the quasi sure analysis, a fact implied in our definition of arbitrage (see Definition 4). A corresponding notion of free lunch with vanishing *uncertainty* will have to incorporate this more sensitive notion of random variables.

Secondly, the sublinear structure of the price system allows for a nonlinear separation of convex sets. With one prior, the equivalent martingale measure separates achievable claims with arbitrage strategies. In our small meshed structure of random variables this separation is guided by the consolidation operator Γ. The preference-free pricing principle gives us a valuation via expected payoffs of different adjusted priors. In comparison to the preference and distribution free results in a perfectly competitive market, see Ross (1976), the implicit assumption is the common knowledge of uncertainty, described by a single probability measure. The design of uncertainty prescribes the consequences for pricing without a consumption-based utility gradient approach.

The valuation of claims, determined by $\mathcal{P}$-arbitrage, contains a new object Γ, which may inspire skepticism. However, note that the consolidation operator Γ could be seen as a tool to regulate financial markets. The valuation of claims in the balance sheet of a bank should depend on Γ. For instance, this may affect fluctuations of opinion in the market as a consequence of uncertainty. In Remark 2 of Section 2 we describe how a good consolidation may be found via consideration of mechanism design. Such considerations may provide a base for the choice of the valuation principle under multiple priors.[48]

Nonlinear Expectations and Market Efficiency

[48] As a first heuristic, it is possible that utilitarian (convex combination) and Rawlsian (supremum operation) welfare functions may constitute a principle of fair pricing. Here, the prior is chosen behind the "veil of ignorance". See also Section 4 of Wilson (1996).

In Remark 1 of Section 2 and Remark 3.1 of Section 3 we indicate how a fully nonlinear price system can be accomplished. In fact this is an approach hinting at a positive theory of nonlinear expectations, where the observable aggregated market sentiment could be captured by the partition of optimists and pessimists.

Such an attempt is a possible starting point to measure the degree of market efficiency. In fact, if markets are efficient in the weak form, deflated asset prices would by symmetric martingales and reveal all information. An *approximately efficient market* could be detected by observing the martingale property under a nonlinear conditional expectation. In this case the market prices can be regarded as the *best linear* approximation of the nonlinear market expectation of the economy.

Preferences and Asset pricing

The uncertainty model in this paper is closely related to Epstein and Wang (1994) and especially to Epstein and Ji (2013) as they consider equilibria with linear prices in their economy. This leads to an indeterminacy in terms of a continuum of linear price systems. The relationship between uncertainty and indeterminacy is caused by the constraint to pick one *effective* prior. The Lucas critique[49] applies insofar as it describes the unsuitable usage of a pessimistic investor to fix an effective prior in reduced form.

Our valuation principle is based on a *preference-free* approach. We value contingent claims in terms of mean unambiguous asset price processes. In other words, the pricing measures of the uncertainty neutral world yield expectations of the security price that do *not* merely depend on the chosen "risk-neutral" prior. Nevertheless, the idea of a risk-neutral valuation principle is not appropriate, as different mutually singular priors deliver different expectations, that cannot be related via a single density.

From this point of view, we disarrange the indeterminacy of linear prices, and allow for the appearance of a planner to configure the sublinearity. In this sense, the regulator as a policy maker is now able to confront unmeasurable sudden fluctuations in volatility. A single prior, as a part of the equilibrium output, can create an invisible threat of convention, which may be used to create the illusion of security when faced with an uncertain future. In a model with mutually singular priors, the focus on a single prior creates a hazard. Events with a positive probability under an ignored prior may be a null set under an effective prior in a consumption-based approach.

Sublinear prices and regulation via consolidation

In this context, sublinearity is associated with the principle of diversification. In these terms, an equilibrium with a sublinear price system covers the concept of Walrasian prices which decentralize with the coincidental awareness of different scenarios. A priori, an instructed Walrasian auctioneer has no knowledge of which prior $\mathbb{P}$ in $\mathcal{P}$ occurs. The degree of discrimination is related to the intensity of nonlinearity. Note that this is a normative category and opens the door to an economic foundation of regulation. Each prior is a probabilistic scenario. The

[49]See Section 3.2 in Epstein and Schneider (2010).

auctioneer *consolidates* the price for each possible scenario into one certain and robust valuation. This is also true for an agent in the model, hence the auctioneer should be able to *discriminate under-diversification* in terms of ignorance of priors in this uncertainty model. Further, a von Neumann-Morgenstern utility assumption results in an overconfidence of certainty in the associated agent.

Since we want to generalize fundamental theorems of asset pricing, we are concerned with the relationship between equivalent martingale measures, viable price systems and arbitrage. In this setting, these concepts must be recast because of the multiple prior uncertainty. In contrast, with one prior an equivalent martingale measure is associated with a linear price system. The underlying neoclassical equilibrium concept is a fully positive theory. In the multiple prior setting such a price extension can be regarded as a diversification-neutral valuation principle. Here, diversification is focused on a given set of priors $\mathcal{P}$. Should the unlucky situation arise that an unconsidered prior governs the market, it is the task of the regulator to robustify these possibilities via an appropriate price system. For instance, uniting two valuations of contingent claims cannot be worse than adding the uncertain outcomes separately. This is the diversification principle under $\mathcal{P}$.

Recalling the quotation of Aliprantis, Tourky, and Yannelis (2001) in the introduction, the degree of sublinearity in our approximation is regulated by the type of consolidation of scenario-dependent linear price systems.

A Details and Proofs

A.1 Section 2

Let $L^0(\Omega)$ denote the space of all measurable real-valued functions on Ω. $\hat{L}^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$ be the quotient of $\mathcal{L}^2(\mathcal{P})$, the closure of $\mathcal{C}_b(\Omega)$ by $c_{2,\mathcal{P}}$ in $L^0(\Omega)$. $\mathcal{N}$ denotes the ideal of $c_{2,\mathcal{P}}$ in $L^0(\Omega)$ null elements. Such null elements are characterized by random variables which are $\mathcal{P}$-polar. $\mathcal{P}$-polar sets evaluated under every prior are zero or one, although, the value may differ between different priors. A property holds *quasi-surely* (q.s.) if it holds outside a polar set. Furthermore, the space $\hat{L}^2(\mathcal{P})$ is characterized by

$$\hat{L}^2(\mathcal{P}) = \left\{ X \in L^0(\Omega) : X \text{ has a q.c. version, } \lim_{n\to\infty} \mathcal{E}^{\mathcal{P}}(|X|^2 1_{\{|X|>n\}}) = 0 \right\},$$

A mapping $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous if $\forall \varepsilon > 0$ there exists an open set O with $\sup_{P\in\mathcal{P}} P(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. We say that $X : \Omega \to \mathbb{R}$ has a *quasi-continuous version* (q.c.) if there exists a quasi–continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s. The mathematical framework provided enables the analysis of stochastic processes for several mutually singular probability measures simultaneously. All equations are understood in the quasi-sure sense. This means that a property holds almost-surely for all scenarios $P \in \mathcal{P}$. Since, for all $X, Y \in L^2(\mathcal{P})$ with $|X| \leq |Y|$ imply $c_{2,\mathcal{P}}(X) \leq c_{2,\mathcal{P}}(Y)$, we have that $L^2(\mathcal{P})$ is a Banach lattice.[50]

[50] This is of interest for existence results from general equilibrium theory.

In the following we discuss the different operations for consolidation. Let $\Pi_{\mathbb{P}} = E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot] \in L^2(\mathcal{P})^*$, with $\mathbb{P} \in \mathcal{P}$. Let μ be a measure on the Borel measurable space $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$ with $\mu(\mathcal{P}) = 1$ and full support on $\mathcal{P}$. In this context we can consider the additive case in $L^2(\mathcal{P})^{\circledast}_{+}$, where a new prior is generated:[51]

$$\Gamma_\mu : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^* \to L^2(\mathcal{P})^{\circledast}_{+}, \quad \Gamma_\mu(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \int_{\mathcal{P}} \psi_{\mathbb{P}} \cdot \mu(d\mathbb{P}) = E^{\mathbb{P}_\mu}[\psi_{\mathbb{P}_\mu}\cdot],$$

where ψ_{P_μ} is constructed as in Example 2. We can consider the Dirac measure $\delta_{\mathbb{P}}$ as an example for μ. The related consideration of only one special prior in $\mathcal{P}$ is in essence the uncertainty model in Harrison and Kreps (1979). The operation in question is given by $(\Pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}} \mapsto E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$. The second operation in $L^2(\mathcal{P})^{\circledast}_{+}$ is a point-wise maximum:

$$\Gamma_{\sup} : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^* \to L^2(\mathcal{P})^{\circledast}_{+}, \quad \Gamma_{\sup}(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot].$$

This is an extreme form of consolidation and can be considered as the highest awareness of all priors. Note that combinations between the maximum and an addition operation are possible as indicated in Example 2 and Proposition 1.

Proof of Proposition 1 *Since $L^2(\mathcal{P})$ is a Banach lattice, the 5th claim follows from Theorem 1 in Biagini and Frittelli (2010), whereas the other claims follow directly from the construction of the functionals in $L^2(\mathcal{P})^{\circledast}_{+}$.* ■

For the proof of Theorem 1, we define the shifted preference relationship $\succsim^0_{\mathbb{P}}$ such that every feasible net trade is worse off than $(0,0) \in B(0,0,\pi_{\mathbb{P}}, M_{\mathbb{P}})$. Obviously, an agent given by $\succsim^0_P$ does not trade. Hence, an initial endowment constitutes a no trade equilibrium.

Proof of Theorem 1 *Let the price system $(\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}, \Gamma)$ be given and we have a $\Psi \in L^2(\mathcal{P})^{\circledast}_{+}$ on $L^2(\mathcal{P})$ such that $\pi_{\mathbb{P}\restriction M^{\mathcal{P}}_{\mathbb{P}}} \leq \Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}}$ for each $\mathbb{P} \in \Gamma(\mathcal{P})$, where $M^{\mathcal{P}}_{\mathbb{P}} = M_{\mathbb{P}} \cap L^2(\mathcal{P})$. The preference relation on $\mathbb{R} \times L^2(P)$, given by*

$$(x, X) \succcurlyeq^0_{\mathbb{P}} (x', X') \quad if \;\; x + -\Pi_{\mathbb{P}}(-X) \geq x' + -\Pi_{\mathbb{P}}(-X'),$$

is in $\mathbb{A}(\mathbb{P})$. For each $\mathbb{P} \in \Gamma(\mathcal{P})$, the bundle $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) = (0,0)$ satisfies the viability condition of Definition 2, hence $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\Gamma(\mathcal{P})}$ is scenario-based viable.

In the other direction, let $\pi(\otimes\mathcal{P}) : \otimes M_{\mathbb{P}} \to \mathbb{R}$ be a price system. The preference relation $\succcurlyeq^0_{\mathbb{P}} \in \mathbb{A}(\mathbb{P})$ satisfies for each $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}})$, $\mathbb{P} \in \Gamma(\mathcal{P})$, the viability condition. We may assume for each $\mathbb{P}$, $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) = (0,0)$, since it is only a geometric deferment. Consider the following sets

$$\begin{aligned} \succ^0_{\mathcal{P}} &= \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} \{(x,X) \in \mathbb{R} \times L^2(\mathbb{P}) : (x,X) \succ_{\mathbb{P}} (0,0)\}, \\ B(\otimes\mathcal{P}) &= \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} B(0,0,\pi_{\mathbb{P}}, M_{\mathbb{P}}). \end{aligned}$$

[51] The related operation of convex functionals would corresponds to the convolution operation. Without convexity of $\mathcal{P}$, the prior $\mathbb{P}_\mu$ may only lie in the convex hull of $k(\mathcal{P})$.

We have that $B(\otimes\mathcal{P})$ and $\succ^0_{\mathcal{P}}$ are convex sets. The Riesz space product $\otimes L^2(\mathbb{P}) = \otimes_{\mathbb{P}\in\Gamma(\mathcal{P})} L^2(\mathbb{P})$ (see paragraph 352 K in Fremlin (2000)), is under the norm $c_{2,\mathcal{P}}$ again a Banach lattice (see paragraph 354 X (b) in Fremlin (2000)). By the $L^2(\mathbb{P})$-continuity of each $\succsim^0_{\mathbb{P}}$, the set $\succ^0_{\mathcal{P}}$ is $c_{2,\mathcal{P}}$-open in $\otimes L^2(\mathbb{P})$.

According to the separation theorem for a topological vector space, for each $\mathbb{P} \in \Gamma(\mathcal{P})$ there is a non zero linear and $c_{2,\mathcal{P}}$-continuous functional $\phi_{\mathbb{P}}$ on $\otimes_{\mathbb{P}\in\Gamma(\mathcal{P})}(\mathbb{R} \times L^2(\mathbb{P}))$ with

1. *$\phi_{\mathbb{P}}(x, X) \geq 0$ for all $(x, X) \in \succ^0_{\mathcal{P}}$*
2. *$\phi(x, X) \leq 0$ for all $(x, X) \in B(\otimes\mathcal{P})$*
3. *$\{(y_{\mathbb{P}}, Y_{\mathbb{P}})\}_{\mathbb{P}\in\Gamma(\mathcal{P})} = (y, Y)$ with $\mathrm{pr}_{\mathbb{R}\times L^2(\mathbb{P})}(\phi_{\mathbb{P}})(y, Y) =: \phi_{\restriction\mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) < 0$,*

since $\phi_{\mathbb{P}}$ is non-trivial. Note that condition 3. depends on the chosen $\mathbb{P}$.

Strict monotonicity of $\succsim^0_{\mathbb{P}}$ implies $(1, 0) \succ^0_{\mathbb{P}} (0, 0)$. The $L^2(\mathbb{P})$-continuity of each $\succsim^0_{\mathbb{P}}$ gives us $(1 + \varepsilon y, \varepsilon Y) \succ^0_{\mathbb{P}} (0, 0)$, for some $\varepsilon > 0$, hence

$$\begin{aligned} \phi_{\restriction\mathbb{P}}(1 + \varepsilon y_{\mathbb{P}}, \varepsilon Y_{\mathbb{P}}) &= -\phi_{\restriction\mathbb{P}}(1, 0) + \varepsilon\phi_{\restriction\mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) \leq 0 \\ \text{and} \quad \phi_{\restriction\mathbb{P}}(1, 0) &\geq -\varepsilon\phi_{\restriction\mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) > 0. \end{aligned}$$

We have $\phi_{\restriction\mathbb{P}}(1, 0) > 0$ and after a renormalization let $\phi_{\restriction\mathbb{P}}(1, 0) = 1$. Moreover, we can write $\phi_{\restriction\mathbb{P}}(x_{\mathbb{P}}, X_{\mathbb{P}}) = x_{\mathbb{P}} + \Pi_{\mathbb{P}}(X_{\mathbb{P}})$, where $\Pi_{\mathbb{P}} : L^2(\mathbb{P}) \to \mathbb{R}$ can be identified as an element in the topological dual $L^2(\mathbb{P})^*$.

We show strict positivity of $\Pi_{\mathbb{P}}$ on $L^2(\mathbb{P})$. Let $X \in L^2(\mathbb{P})_+ \setminus \{0\}$ we have $(0, X) \succ^0_{\mathbb{P}} (0, 0)$, hence $(-\varepsilon, X) \succ^0_{\mathbb{P}} (0, 0)$, and therefore $\Pi_P(X) - \varepsilon \geq 0$.

Moreover we have $L^2(\mathcal{P})$-positivity of $\Pi_{\mathbb{P}\restriction L^2(\mathcal{P})}$ on $L^2(\mathcal{P})$, i.e. $X \geq 0$ $\mathcal{P}$-q.s. implies $\Pi_{\mathbb{P}\restriction L^2(\mathcal{P})} \geq 0$. Since $L^2(\mathcal{P})$ is a Banach lattice, $\Pi_{\mathbb{P}} \in L^2(\mathcal{P})^*$ follows.

Let $X \in M^{\mathcal{P}}_{\mathbb{P}}$, since $(-\pi_{\mathbb{P}}(X), X), (\pi_{\mathbb{P}}(X), -X) \in B(0, 0, \pi_{\mathbb{P}}, M^{\mathcal{P}}_{\mathbb{P}})$ we have $0 = \phi(\pi_{\mathbb{P}}(X), X) = \pi_P(X) - \Pi_P(X)$ and $\Pi_{\mathbb{P}\restriction M^{\mathcal{P}}_{\mathbb{P}}} = \pi_{\mathbb{P}}$ follows.

$\Gamma(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\Gamma(\mathcal{P})}) = \Psi$ is by construction in $L^2(\mathcal{P})^{\circledast}_+$. The strict positivity of Ψ follows from the strict positivity of each $\Pi_{\mathbb{P}}$. $\Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$ follows from an inequality in the last part of Proposition 1 and $\Pi_{\mathbb{P}\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}}$. ■

We illustrate the construction in the following diagram:

$$\begin{array}{ccc} \{\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}} \longmapsto \pi(\otimes\mathcal{P}) : \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} M_{\mathbb{P}} \to \mathbb{R} & & \\ \quad\quad\quad\quad\quad\quad\quad\quad \downarrow {\scriptstyle Hahn\ |\ Banach} & & \\ \quad\quad\quad\quad \left\{\Pi_{\mathbb{P}} : L^2(\mathbb{P}) \to \mathbb{R}\right\}_{\mathbb{P}\in\Gamma(\mathcal{P})} \overset{\Gamma}{\longmapsto} \Psi : L^2(\mathcal{P}) \to \mathbb{R} & & \end{array}$$

Proof of Corollary 1 *By construction every functional Ψ can be represented as the supremum of priors, which are given by convex combinations. Since $X \in \mathbb{M}(\Gamma(\mathcal{P}))$, the supremum operation has no effect on X and the assertion follows.* ■

A.2 Section 3

Next, we discuss the augmentation of our information structure. The unaugmented filtration is given by $\mathbb{F}^o$. As mentioned in Subsection 3.1, the set of priors have to be stable under pasting in order to apply the framework of Nutz and Soner (2012). For the sake of completeness, we recall this notion.

Definition 6 *The set of priors is* stable under pasting *if for every* $\mathbb{P} \in \mathcal{P}$, *every* $\mathbb{F}^o$*-stopping time* τ, $B \in \mathcal{F}^o_\tau$ *and* $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\mathcal{F}^o_\tau, \mathbb{P})$, *We have* $\mathbb{P}_\tau \in \mathcal{P}$, *where*

$$\mathbb{P}_\tau(A) = E^P\big[\mathbb{P}_1(A|\mathcal{F}^o_\tau)1_B + \mathbb{P}_2(A|\mathcal{F}^o_\tau)1_{B^c}\big], \quad A \in \mathcal{F}^o_\tau.$$

In the multiple prior setting, with a given reference measure this property is equivalent to the well-known notion of *time consistency.* However, this is not true if there is no dominant prior.[52]

The usual condition of a "rich" σ-algebra at time 0 is widely used in mathematical finance. But the economic meaning is questionable. Our uncertainty model of mutually singular priors can be augmented, similarly to the classical case, using the right continuous filtration given by $\mathbb{F}^+ = \{\mathcal{F}^+_t\}_{t\in[0,T]}$ where

$$\mathcal{F}^+_t = \bigcap_{s>t} \mathcal{F}^o_t, \text{ for } t \in [0,T).$$

The second step is to augment the minimal right continuous filtration $\mathbb{F}^+$ by all polar sets of $(\mathcal{P}, \mathcal{F}^o_T)$, i.e. $\mathcal{F}_t = \mathcal{F}^+_t \vee \mathcal{N}(\mathcal{P}, \mathcal{F}^o_T)$. This augmentation is strictly smaller than the universal augmentation $\bigcap_{P\in\mathcal{P}} \overline{\mathbb{F}^o}^P$. This choice is economically reasonable since the initial σ-field does not contain all 0-1 limit events. An agent considers this exogenously specified information structure. It describes what information the agent *can* know at each date. This is the analogue to a filtration in the single prior framework satisfying the usual conditions. For the proof below, we need results from Appendix B.1.

Proof of Theorem 2 *We fix an EsMM-set* $\mathcal{Q}$. *The related consolidation* Γ *gives us the set of relevant priors* $\Gamma(\mathcal{P}) \subset \mathcal{P}$. *Let* $\psi_\mathbb{P} = \frac{d\mathbb{Q}}{d\mathbb{P}}$, *for each* $\mathbb{Q} \in \mathcal{Q}$ *and the related* $\mathbb{P} \in \mathcal{P}$. *We have* $\psi_\mathbb{P} \in L^2(\mathbb{P})$. *Let the associated strictly positive* $\Psi \in L^2(\mathcal{P})^{\circledast}_{++}$ *be given.*

Take a marketed claim $X^m \in M^\mathcal{P}_\mathbb{P}$ *with* $\mathbb{P} \in \Gamma(\mathcal{P})$ *and let* $\eta \in \mathcal{A}$ *be a self-financing trading strategy that hedges* X^m. *Since* $\eta \in \mathcal{A}$, *by the decomposition rule for conditional* $\mathcal{E}^\mathcal{Q}$*-expectation, see for instance Theorem 2.6 (iv) in* **?**, *and since* S *is a symmetric* $\mathcal{E}^\mathcal{Q}$*-martingale, the following equalities*

$$\mathcal{E}^\mathcal{Q}_t(\eta_u S_u) = \eta^+_t \mathcal{E}^\mathcal{Q}_t(S_u) + \eta^-_t \mathcal{E}^\mathcal{Q}_t(-S_u) = \eta^+_t S_t - \eta^-_t S_t = \eta_t S_t,$$

hold, where $\eta = \eta^+ - \eta^-$ *with* $\eta^+, \eta^- \geq 0$ $\mathcal{P}$*-quasi surely and* $0 \leq t \leq u \leq T$. *Therefore we achieve*

$$\Psi(X^m) = \mathcal{E}^\mathcal{Q}_0(\eta_T S_T) = \eta_0 S_0 \geq \pi_\mathbb{P}(X^m), \quad \mathbb{P} \in \Gamma(\mathcal{P}).$$

[52] Additionally, the set of priors must be chosen maximally. For further consideration, we refer the reader to Section 3 in Nutz and Soner (2012).

For the other direction, let $\Psi \in L^2(\mathcal{P})^{\circledast}_{++}$ *with* $\Psi_{\restriction M_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$*, related to a set of linear functionals* $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ *and* $\{\Pi_{\mathbb{P}}\}_{P\in\mathcal{P}}$*, such that* $\Pi_{\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}}$*. This can be inferred from* Ψ *and the construction in the proof of the second part of Theorem 1. Now, we define* $\mathcal{Q}$ *in terms of* Γ*.*

We illustrate the possible cases which can appear. For simplicity we assume $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\}$*. Let* $\mathbb{P}^{k,j} = \frac{1}{2}\mathbb{P}_k + \frac{1}{2}\mathbb{P}_j$ *and* $\psi^{k,j} = \frac{1}{2}\psi^k + \frac{1}{2}\psi^j$*, recall that we can represent each functional* $\Pi_{\mathbb{P}}(\cdot)$ *by* $E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$*. We have*

$$\frac{1}{2}(\Pi_1 + \Pi_2) \wedge \Pi_3 \text{ becomes } \left\{\psi^{1,2} \times \mathbb{P}^{1,2}, \psi_3 \times \mathbb{P}_3\right\} = \mathcal{Q}.$$

Consequently, $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \psi_{\mathbb{P}} d\mathbb{P},\ \mathbb{P} \in \Gamma(\mathcal{P}) \subset k(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})\}$*, where* $\psi_{\mathbb{P}}$*, with* $\mathbb{P} \notin \mathcal{P}$*, is constructed by the procedure of Example 2. The first condition of Definition 5 follows, since the square integrability of each* $\psi_{\mathbb{P}}$ *follows from the* $c_{2,\mathcal{P}}$*-continuity of linear functionals which generate* Ψ*.*

We prove the symmetric $\mathcal{Q}$*-martingale property of the asset price process. Let* $B \in \mathcal{F}_t$*,* $\eta \in \mathcal{A}$ *be a self-financing trading strategy and*

$$\eta^1_s = \begin{cases} 1 & s \in [t,u) \text{ and } \omega \in B \\ 0 & \text{else }, \end{cases} \qquad \eta^0_s = \begin{cases} S_t, & s \in [t,u) \text{ and } \omega \in B \\ S_u - S_t, & s \in [u,T) \text{ and } \omega \in B \\ 0 & \text{else.} \end{cases}$$

This strategy yields a portfolio value

$$\eta_T S_T = (S_u - S_t) \cdot 1_B,$$

the claim $\eta_T S_T$ *is marketed at price zero. In terms of the modified conditional sublinear expectation* $(\mathcal{E}^{\mathcal{Q}}_t)_{t\in[0,T]}$*, we have with* $t \leq u$

$$\mathcal{E}^{\mathcal{Q}}_t((S_t - S_u)1_B) = 0.$$

By Theorem 4.7 Xu and Zhang (2010), it follows that $S_t = \mathcal{E}^{\mathcal{Q}}_t(S_u)$*.*[53] *But this means that* $(S_t)_{t\in[0,T]}$ *is an* $\mathcal{E}^{\mathcal{Q}}$*-martingale. The same argumentation holds for* $-S$*, hence the asset price* S *is a symmetric* $\mathcal{E}^{\mathcal{Q}}$*-martingale.* ■

Proof of Corollary 2

1. *Suppose there is a* $\mathcal{Q} \in \mathtt{M}(\mathcal{P})$ *and let* $\eta \in \mathcal{A}$ *such that* $\eta_T S_T \geq 0$ $\mathcal{P}$*-q.s. and* $\mathbb{P}'(\eta_T S_T > 0) > 0$ *for some* $\mathbb{P}' \in \mathcal{P}$*. Since for all* $\mathbb{Q} \in \mathcal{Q}$ *there is a* $\mathbb{P} \in k(\mathcal{P})$ *such that* $\mathbb{Q} \sim \mathbb{P}$*, there is a* $\mathbb{Q}' \in \mathcal{Q}$ *with* $\mathbb{Q}'(\eta_T S_T > 0) > 0$*. Hence,* $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) > 0$ *and by Theorem 2 we observe* $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) = \eta_0 S_0$*. This implies that no* $\mathcal{P}$*-arbitrage exists.*

2. *In terms of Theorem 1, each* $P \in \mathcal{R}$ *admits exactly one extension. With Theorem 2 the result follows.*

[53]The result is proven for the G-framework. However the assertion in our setting holds true as well by an application of the martingale representation in Proposition 4.10 by Nutz and Soner (2012).

3. *By Theorem 2, there is a related* Ψ *in* $L^2(\mathcal{P})^{\circledast}_{+}$*, wit* $\Gamma(\mathbb{P}) = \mathcal{R}$*. Fix a costless strategy* $\eta \in \mathcal{A}$ *such that* $\eta_0 S_0 = 0$ *hence* $\Psi(\eta_T S_T) = 0$*. The viability of* Ψ *implies* $\eta_T S_T = 0$ $\mathcal{R}$*-q.s. Hence, no* $\mathcal{R}$*-arbitrage exists.*

4. *This then follows by the same argument as in Harrison and Pliska (1981) (see the Lemma on p.228), since* $\mathcal{E}^{\mathcal{Q}}$ *is strictly positive, by Theorem 2.* ■

For the proof of Theorem 3, we apply results from stochastic analysis in the G-framework. The results are collected in Appendix B.2.

Proof of Theorem 3 *Let* $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \rho d\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$ *be an EsMM-set, where the density* ρ *satisfies* $\rho \in L^2_G(\Omega)$ *and* $E_G[\rho] = -E_G[-\rho]$*. Next define the stochastic process* $(\rho_t)_{t\in[0,T]}$ *by* $\rho_t = E_G[\rho|\mathcal{F}_t]$ *resulting in a symmetric G-martingale to which we apply the martingale representation theorem for G-expectation, stated in Appendix B.2. Hence, there is a* $\gamma \in M^2_G(0,T)$ *such that we can write*

$$\rho_t = 1 + \int_0^t \gamma_s dB^G_s, \quad t \in [0,T], \quad \mathcal{P}\text{-}q.s.$$

By the G-Itô formula, stated in the Appendix B.2, we have

$$ln(\rho_t) = \int_0^t \phi_s dB^G_s + \frac{1}{2}\int_0^t \phi_s^2 d\langle B^G\rangle_s, \quad \mathcal{P}\text{-}q.s$$

for every $t \in [0,T]$ *in* $L^2_G(\Omega_t)$ *and hence*

$$\rho = \mathtt{E}^{\phi}_T = \exp\Big(-\frac{1}{2}\int_0^T \theta_s^2 d\langle B^G\rangle_s - \int_0^T \theta_s dB^G_s\Big), \quad \mathcal{P}\text{-}q.s.$$

With this representation of the density process we can apply the Girsanov theorem, stated in Appendix B.2. Set $\phi_t = \frac{\rho_t}{\gamma_t}$ *and consider the process*

$$B^{\phi}_t = B^G_t - \int_0^t \phi_s ds, \quad t \in [0,T].$$

We deduce that B^{ϕ} *is a G-Brownian motion under* $\mathcal{E}^{\phi}(\cdot) = E_G[\phi\cdot]$ *and* S *satisfies*

$$S_t = S_0 + \int_0^t V_s dB^{\phi}_s + \int_0^t (\mu_s + V_s\phi_s) d\langle B^{\phi}\rangle_s \quad t \in [0,T]$$

on $(\Omega, L^2_G(\Omega)), \mathcal{E}^{\phi})$*. Since* V *is a bounded process, the stochastic integral is a symmetric martingale under* $\mathcal{E}^{\phi}$*.* S *is a symmetric* $\mathcal{E}^{\phi}$*-martingale if and only if* $\mu_t + V_t\phi_t = 0$ $\mathcal{P}$*-q.s. We have shown that* ρ *is a simultaneous Radon-Nikodym type density of the EsMM-set* $\mathcal{Q}$*. Hence, there is a nontrivial EsMM-set in* $\mathtt{M}(\mathcal{P})$*, since* $\phi_t = \theta_t$ $\mathcal{P}$*-q.s for every* $t \in [0,T]$*.* ■

B Required results

In this Appendix we introduce the mathematical framework more carefully. We also collect all the results applied in Sections 2 and 3. Before, we state the mentioned criterion for the weak compactness of $\mathcal{P}$. Let $\sigma^1, \sigma^2 : [0,T] \to \mathbb{R}_+$ be two measures with a Holder continuous distribution function $t \mapsto \sigma^i([0,t]) = \sigma^i(t)$. As introduced in Section 2.1, a measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a martingale probability measure if the coordinate process is a martingale with regard to the canonical (raw) filtration.

Criterion for weak compactness of priors, Denis and Kervarec (2013): Let $\mathcal{P}(\sigma^1, \sigma^2)$ be the set of martingale probability measures with

$$d\sigma^1(t) \leq d\langle B\rangle_t^{\mathbb{P}} \leq d\sigma^2(t),$$

where $\langle B\rangle^P$ is the quadratic variation of B under $\mathbb{P}$. Then the set $\mathcal{P}(\sigma^1, \sigma^2)$ is weakly compact.

B.1 The sub order dual

In this subsection we discuss the mathematical preliminaries for the price space of sublinear functionals for Section 3.

The topological dual space:

1. Let $c_{2,\mathcal{P}}$ be a capacity norm, defined in Section 2.2. Every continuous linear form l on $L^2(\mathcal{P})$ admits a representation:

$$l(X) = \int X d\mu \quad \forall X \in L^2(\mathcal{P}),$$

where μ is a bounded signed measure defined on a σ-algebra containing the Borel σ-algebra of Ω. If l is a non-negative linear form, the measure μ is non-negative finite.

2. We have $L^2(\mathcal{P})^* = \left\{\mu = \int \psi_{\mathbb{P}} d\mathbb{P} : \mathbb{P} \in \mathcal{P} \text{ } and \text{ } \psi_{\mathbb{P}} \in L^2(\mathbb{P})\right\}$.

The first claim is stated in Proposition 11 of Feyel and de La Pradelle (1989). The second assertion can be proven via a modification of of Lemma I.28 and Theorem I.30 in Kervarec (2008), where the case of $L^1(\mathcal{P})^*$ is treated.

B.1.1 Semi lattices and their intrinsic structure

The space of coherent price systems $L^2(\mathcal{P})^{\circledast}_{++}$ plays a central role in Theorem 1 and 2. Every consolidation operator has a domain in $\bigotimes_{P\in\mathcal{P}} L^2(\mathcal{P})^*$ and maps to $L^2(\mathcal{P})^{\circledast}$. We begin with the most simple operation of consolidation, ignoring a subset of priors and giving a weight to the others.

Let $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$ be the positive measure μ such that $\mu(\mathcal{P}) \leq 1$. The underlying space is $\bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathcal{P})^*$, when considering simultaneously the representations of continuous and linear functionals on $L^2(\mathcal{P})$. So let $N \subset \mathcal{B}(\mathcal{P})$ be a Borel measurable set and $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$. The consolidation via convex combination is given by

$$\Gamma(\mu, N) : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathcal{P})^{\circledast} \to L^2(\mathcal{P})^*,\ \{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}} \mapsto \int_N \psi_{\mathbb{P}} d\mu(d\mathbb{P}).$$

The size of N determines the degree of ignorance, related to the exclusion of the prior in the countable reduction. A measure with a mass strictly less than one implies an ignorance. Here, a Dirac measure on $\mathcal{P}$ is a projection to one certain probability model.

Next, we consider the supremum operation of functionals. The operation of point-wise maximum preserves the convexity. We review a result which gives an iterated application of the Hahn-Banach Theorem.

Representation of sublinear functionals Frittelli (2000): Let ψ be a sublinear functional on a topological vector space V, then

$$\psi(X) = \max_{x^*\in P_\psi} x^*(X),$$

where $P_\psi = \{x^* \in X^* : x^*(X) \leq \psi(X) \text{ for all } X \in V\} \neq \emptyset$

The maximum operation can also be associated to a lattice structure. In economic terms this is related to a normative choice of the super hedging intensity. The diversification valuation operator consolidation is set to *one nonlinear* valuation functional. Note that the operation preserves monotonicity.

B.2 Stochastic analysis with G-Brownian motion

We introduce the notion of sublinear expectation for the G-Brownian motion. This includes the concept of G-expectation, the Itô calculus with G-Brownian motion and related results concerning the representation of G-expectation and (symmetric) G-martingales. For a more precise detour we refer to the Appendix of Vorbrink (2010) and to references therein. At the end of this section we present a Girsanov theorem for G-Brownian motion, which we apply in Theorem 3 of Subsection 3.3. Let $\Omega \neq \emptyset$ be a given set. Let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. Note that in our model we choose $\mathcal{C}_b(\Omega) = \mathcal{H}$ and $\Omega = \Omega_T = C_0([0,T])$.

A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying monotonicity, constant preserving, sub-additivity and positive homogeneity. The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a *sublinear expectation space*. For the construction of the G-expectation, the notion of independence and G-normal distributions we refer to Peng (2010).

A process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a G–*Brownian motion* if the following properties are satisfied:

(i) $B_0 = 0$.

(ii) For each $t, s \geq 0$: $B_{t+s} - B_t \sim B_t$ and $\hat{E}[|B_t|^3] \to 0$ as $t \to 0$.

(iii) The increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

(iv) $\hat{E}[B_t] = -\hat{E}[-B_t] = 0 \quad \forall t \geq 0$.

The following observation is important for the characterization of G–martingales. The space $C_{l,Lip}(\mathbb{R}^n)$, where $n \geq 1$ is the space of all real-valued continuous functions φ defined on $\mathbb{R}^n$ such that $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \forall x, y \in \mathbb{R}^n$. We define $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \cdots, t_n \in [0,T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}$. The Itô integral can also be defined for the following processes: Let $H_G^0(0,T)$ be the collection of processes η having the following form: For a partition $\{t_0, t_1, \cdots, t_N\}$ of $[0,T]$, $N \in \mathbb{N}$, and $\xi_i \in L_{ip}(\Omega_{t_i})$ $\forall i = 0, 1, \cdots, N-1$, let η be given by

$$\eta_t(\omega) := \sum_{0 \leq j \leq N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t) \quad \textit{for all } t \in [0,T].$$

For $\eta \in H_G^0(0,T)$ let $\|\eta\|_{M_G^2} := \left(E_G\left[\int_0^T |\eta_s|^2 ds\right]\right)^{\frac{1}{2}}$ and denote by $M_G^2(0,T)$ the completion of $H_G^0(0,T)$ under this norm. We can construct Itô's integral I on $H_G^0(0,T)$ and extend it to $M_G^2(0,T)$ continuously, by $I : M_G^2(0,T) \to L^2(\mathcal{P})$. The next result is an Itô formula. The presentation of basic notions on stochastic calculus with respect to G-Brownian motion lies beyond the scope of this appendix.

Itô-formula, Li and Peng (2011): Let $\Phi \in C^2(\mathbb{R})$ and $dX_t = \mu_t d\langle B^G \rangle_t + V_t dB_T^G$, $t \in [0,T]$, $\mu, V \in M_G^2(0,T)$ are bounded processes. Then we have for every $t \geq 0$:

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial\Phi(X_u) V_u dB_u^G + \frac{1}{2}\int_s^t \partial\Phi(X_u)\mu_u + \partial^2\Phi(X_u) V_u^2 d\langle B^G \rangle_u.$$

Next, we discuss martingales in the G-framework. In Song (2011), this identity declares that a G-martingale M can be seen as a multiple prior martingale which is a supermartingale for any $P \in \mathcal{P}$ and a martingale for an optimal measure.

Characterization for G-martingales, Soner, Touzi, and Zhang (2011): Let $x \in \mathbb{R}, z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$. Then the process

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \leq T,$$

is a G–martingale.

In particular, the nonsymmetric part $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$, $t \in [0,T]$, is a G-martingale which is different compared to classical probability theory since $\{-K_t\}_{t \in [0,T]}$ is continuous, and non-increasing with a quadratic variation equal to zero. M is a symmetric G–martingale if and only if $K \equiv 0$.

Martingale representation, Song (2011): Let $\xi \in L_G^2(\Omega_T)$. Then the G–martingale X with $X_t := E_G[\xi | \mathcal{F}_t], t \in [0,T]$, has the following unique representation

$$X_t = X_0 + \int_0^t z_s dB_s - K_t,$$

where K is a continuous, increasing process with $K_0 = 0, K_T \in L_G^\alpha(\Omega_T), z \in H_G^\alpha(0,T), \forall \alpha \in [1,2)$, and $-K$ a G–martingale. Here, $H_G^\alpha(0,T)$ is the completion of $H_G^0(0,T)$ under the norm $\|\eta\|_{H_G^\alpha} := \left(E_G \left[\int_0^T |\eta_s|^2 ds \right]^{\frac{\alpha}{2}} \right)^{\frac{1}{\alpha}}$. If ξ is bounded from above we have $z \in M_G^2(0,T)$ and $K_T \in L_G^2(\Omega_T)$, see Song (2011).

Finally we state a Girsanov type theorem with G-Brownian motion. In Subsection 3.3 we discussed some heuristics in terms of a G-Doleans Dade exponential. Define the density process by $\mathtt{E}^\theta$ as the unique solution of $d\mathtt{E}_t^\theta = \mathtt{E}_t^\theta \theta_t dB_t^G$, $\mathtt{E}_0^\theta = 1$. The proof of the Girsanov theorem is based on a Levy martingale characterization theorem for G-Brownian motion.

Girsanov for G-expectation, Xu, Shang, and Zhang (2011): Assume the following Novikov type condition: There is an $\varepsilon > \frac{1}{2}$ such that

$$E_G \left[\exp \left(\varepsilon \cdot \int_0^T \theta_s^2 d\langle B^G \rangle_s \right) \right] < \infty$$

Then $B_t^\theta = B_t^G - \int_0^t \theta_s \langle B^G \rangle_s$ is a G-Brownian motion under the sublinear expectation $\mathcal{E}^\theta(\cdot)$ given by, $\mathcal{E}^\theta(X) = E_G[\mathtt{E}_T^\theta \cdot X]$, $\mathcal{P}^\theta = \mathtt{E}_T^\theta \cdot \mathcal{P}$ with $X \in L^2(\mathcal{P}^\theta)$.

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