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On an Integral Equation for the Free Boundary of Stochastic, Irreversible Investment Problems*

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Abstract. In this paper we derive a new handy integral equation for the free boundary of infinite time horizon, continuous time, stochastic, irreversible investment problems with uncertainty modeled as a one-dimensional, regular diffusion $X^{0,x}$. The new integral equation allows to explicitly find the free boundary $b(\cdot)$ in some so far unsolved cases, as when $X^{0,x}$ is a three-dimensional Bessel process or a CEV process. Our result follows from purely probabilistic arguments. Indeed, we first show that $b(X^{0,x}(t)) = l^*(t)$, with $l^*(t)$ unique optional solution of a representation problem in the spirit of Bank-El Karoui [4]; then, thanks to such identification and the fact that l^* uniquely solves a backward stochastic equation, we find the integral problem for the free boundary.

Key words: integral equation, free boundary, irreversible investment, singular stochastic control, optimal stopping, one-dimensional diffusion, Bank and El Karoui's Representation Theorem, base capacity.

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1 Introduction

In this paper we find a new integral equation for the free boundary $b(\cdot)$ arising in infinite time horizon, continuous time, stochastic, irreversible investment problems of the form

$$\sup_{\nu} \mathbb{E}\bigg\{ \int_0^\infty e^{-rt}\, \pi(X^{0,x}(t), y + \nu(t))dt - \int_0^\infty e^{-rt} d\nu(t) \bigg\}, \tag{1.1}$$

with $X^{0,x}$ regular, one-dimensional diffusion modeling market's uncertainty. The integral problem for $b(\cdot)$ is derived by means of purely probabilistic arguments. After having completely characterized the solution of singular control problem (1.1) by some first order conditions for optimality and in terms of the *base capacity* process l^*, unique optional solution of a representation problem à la Bank-El Karoui [4], we show that $l^*(t) = b(X^{0,x}(t))$. Such identification, strong

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Markov property and a beautiful result in [15] on the joint law of a regular, one-dimensional diffusion and its running supremum both stopped at an independent exponentially distributed random time, lead to the integral equation for $b(\cdot)$

$$\psi_r(x)\int_x^{\overline{x}}\left(\int_{\underline{x}}^z \pi_c(y,b(z))\psi_r(y)m(dy)\right)\frac{s(dz)}{\psi_r^2(z)}=1. \tag{1.2}$$

Here $\pi_c(x,c)$ is the instantaneous marginal profit function, $\underline{x}$ and $\overline{x}$ the endpoints of the domain of $X^{0,x}$, r the discount factor, $\mathcal{G}$ the infinitesimal generator associated to $X^{0,x}$, $\psi_r(x)$ the increasing solution to the equation $\mathcal{G}u=ru$, and $m(dx)$ and $s(dx)$ the speed measure and the scale function measure of $X^{0,x}$, respectively. The rather simple structure of equation (1.2) allows to explicitly find the free boundary even in some non-trivial settings; that is, for example, the case of $X^{0,x}$ given by a three-dimensional Bessel process in which (for a Cobb-Douglas operating profit function)

$$b(x)=\left[\left(\frac{\alpha+\beta}{2\beta}\right)x^2\frac{\psi_r'(x)}{g(x)}\right]^{-\frac{1}{1-\beta}}, \quad x>0, \tag{1.3}$$

with $\psi_r'(x)$ the first derivative of the increasing function $\psi_r(x)=\frac{\sinh(\sqrt{2r}x)}{x}$, and $g(x):=\int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy$. Such result appears here for the first time.

The connection between irreversible investment problems under uncertainty, optimal stopping and free boundary problems is well known in the economic and mathematical literature (cf., e.g., the monography by Dixit and Pyndick [18]). From the mathematical point of view, a problem of optimal irreversible investment may be modeled as a 'monotone follower' problem; that is, a problem in which investment strategies are nondecreasing stochastic processes, not necessarily absolutely continuous with respect to the Lebesgue measure. Work on 'monotone follower' problems and their application to Economics started with the pioneering papers by Karatzas, Karatzas and Shreve, El Karoui and Karatzas (cf. [24], [25] and [19]), among others. These Authors studied the problem of optimally minimizing expected costs when the controlled diffusion is a Brownian motion tracked by a nondecreasing process, i.e. the monotone follower. They showed that one may associate to such a singular stochastic control problem a suitable optimal stopping problem whose value function v is related to the value function V of the original control problem by $v=\frac{\partial}{\partial x}V$. Moreover, the optimal stopping time τ^* is such that $\tau^*=\inf\{t\geq 0:\nu^*(t)>0\}$, with ν^* the optimal singular control. Later on, this kind of link has been established also for more complicated dynamics of the controlled diffusion; that is the case, for example, of a Geometric Brownian motion [1], or of a quite general controlled Ito diffusion (see [6] and [8], among others).

Usually (see [10] and [11], [27], [28], [31] and [32], among others) the optimal irreversible investment policy consists in waiting until the shadow value of installed capital is below the marginal cost of investment; on the other hand, the times at which the shadow value of installed capital equals the marginal cost of investment are optimal times to invest. It follows that from the mathematical point of view one must find the region in which it is profitable to invest immediately (the so called 'stopping region') and the region in which it is optimal to wait (the so called 'continuation region'). The boundary between these two regions is the free boundary of the optimal stopping problem naturally associated to the singular control one. The optimal investment is then the least effort to keep the controlled process inside the closure of the 'continuation region'; that is, in a diffusive setting, the local time of the optimal controlled diffusion at the free boundary.

In the last decade many papers addressed singular stochastic control problems by means of a first order conditions approach (cf., e.g., [2], [5], [12], [13], [32] and [34]), not necessarily relying on any Markovian or diffusive structure. The solution of the optimization problem is indeed related to that of a representation problem for optional processes (cf. [4]): the optimal policy consists in keeping the state variable always above the lower bound $l^*(t)$, unique optional solution of a stochastic backward equation à la Bank-El Karoui [4]. Clearly such policy acts like the optimal control of singular stochastic control problems as the original monotone follower problem (e.g., cf. [24] and [25]) or, more generally, irreversible investment problems (cf. [1], [11], [27] and [28], among others). Therefore, in a diffusive setting, the signal process l^* and the free boundary $b(\cdot)$ arising in singular stochastic control problems must be linked. In [12] the Authors studied a continuous time, singular stochastic irreversible investment problem over a finite time horizon and they showed that for a production capacity given by a controlled Geometric Brownian motion with deterministic, time-dependent coefficients one has $l^*(t) = b(t)$.

In this paper we aim to understand the meaning of process l^* for the whole class of infinite time horizon, irreversible investment problems of type (1.1). By means of a first order conditions approach we first find the optimal investment policy in terms of the 'base capacity' process $l^*(t)$ (cf. [32], Definition 3.2), unique solution of a representation problem in the spirit of Bank-El Karoui [4]. That completely solves control problem (1.1). Invest just enough to keep the production capacity above $l^*(t)$ turns out to be the optimal investment startegy at time t. The base capacity process defines therefore a desirable value of capacity that the controller aims to maintain. We show indeed that $l^*(t) = b(X^{0,x}(t))$, where $b(\cdot)$ is the free boundary of the optimal stopping problem

$$\inf_{\tau \geq 0} \mathbb{E}\left\{ \int_0^\tau e^{-rs} \pi_c(X^{0,x}(s), y) ds + e^{-r\tau} \right\} \tag{1.4}$$

associated to (1.1) (cf., e.g., [1], Lemma 2). Such identification, together with the fact that l^* uniquely solves a backward stochastic equation (see (3.3) below), yields a new integral equation for the free boundary (cf. (1.2) and also our Theorem 3.8 below). That equation does not rely on Ito's formula and does not require any smooth-fit property or a priori continuity of $b(\cdot)$ to be applied. In this sense it distinguishes from that of Pedersen and Peskir [29] (used in the context of stochastic, irreversible investment problems in [11]) which is instead based on a local time space calculus for semimartingales on continuous surfaces [30]. Moreover, our result differs also from that of Federico and Pham [20] obtained via a viscosity solution approach for nondegenerate diffusions and a quadratic cost functional.

The paper is organized as follows. Section 2 introduces the optimal control problem. In Section 3 we find the optimal investment strategy, we identify the link between the 'base capacity' process and the free boundary and we derive the integral equation for the latter one. Finally, in Section 4 we discuss some relevant examples, as the case in which the economic shock $X^{0,x}$ is a Geometric Brownian motion, a three-dimensional Bessel process or a CEV process.

2 The Optimal Investment Problem

On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\{\mathcal{F}_t, t \geq 0\}$ the filtration generated by an exogenous Brownian motion $\{W(t), t \geq 0\}$ and augmented by $\mathbb{P}$-null sets, consider the optimal irreversible investment problem of a firm. The uncertain status of the economy is represented by the one-dimensional, time-homogeneous diffusion $\{X^{0,x}(t), t \geq 0\}$ with state space $\mathcal{I} \subseteq \mathbb{R}$, unique pathwise solution of the stochastic differential equation

$$\begin{cases} dX^{0,x}(t) = \mu(X^{0,x}(t))dt + \sigma(X^{0,x}(t))dW(t) \\ X^{0,x}(0) = x, \end{cases} \tag{2.1}$$

for some Borel functions $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto (0, +\infty)$ such that

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < +\infty, \quad \text{for some } \epsilon > 0, \tag{2.2}$$

for every $x \in \text{int}(\mathcal{I})$. Local integrability condition (2.2) implies that the diffusion process $X^{0,x}$ is regular in $\mathcal{I}$, i.e. $X^{0,x}$ reaches y with positive probability starting at x, for any x and y in $\mathcal{I}$. Hence the state space $\mathcal{I}$ cannot be decomposed into smaller sets from which $X^{0,x}$ could not exit (see, e.g., [33], Chapter VII). We shall denote by $m(dx)$, $s(dx)$, $\mathcal{G}$ and $\mathbb{P}_x$ the speed measure, the scale function measure, the infinitesimal generator and the probability distribution of $X^{0,x}$, respectively. Notice that, under (2.2), $m(dx)$ and $s(dx)$ are well defined, and there always exist two linearly independent, positive solutions of the ordinary differential equation $\mathcal{G}u = \beta u$, $\beta > 0$ (cf. [21]). These functions are uniquely defined up to multiplication, if one of them is required to be strictly increasing and the other to be strictly decreasing. Finally, throughout this paper we assume that $\mathcal{I}$ is an interval with endpoints $-\infty \leq \underline{x} < \overline{x} \leq +\infty$.

The firm's manager aims to increase the production capacity $C^{y,\nu}(t)$ by optimally choosing an irreversible investment plan $\nu \in \mathcal{S}_o$, where

$$\mathcal{S}_o := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, \text{ nondecreasing, left-continuous, adapted s.t. } \nu(0) = 0,\ \mathbb{P} - \text{a.s.}\}$$

is the non empty, convex set of irreversible investment processes. We suppose that

$$C^{y,\nu}(t) = y + \nu(t), \qquad C^{y,\nu}(0) = y \geq 0, \tag{2.3}$$

that the firm makes profit at rate $\pi(x, c)$ when its own capacity is c and the status of economy is x, and that the firm's manager discounts revenues and costs at constant rate $r \geq 0$. As for the operating profit function $\pi : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ we make the following

Assumption 2.1.

1. *The mapping $c \mapsto \pi(x, c)$ is strictly increasing and strictly concave with continuous derivative $\pi_c(x, c) := \frac{\partial}{\partial c}\pi(x, c)$ satisfying the Inada conditions*

$$\lim_{c\to 0} \pi_c(x, c) = \infty, \qquad \lim_{c\to\infty} \pi_c(x, c) = 0.$$

2. *The process $(\omega, t) \mapsto \pi(X^{0,x}(\omega, t), C^{y,\nu}(\omega, t))$ is $\mathbb{P} \otimes e^{-rt}dt$ integrable for any $\nu \in \mathcal{S}_o$.*

The optimal investment problem is then

$$V(x,y) := \sup_{\nu \in \mathcal{S}_o} \mathcal{J}_{x,y}(\nu), \tag{2.4}$$

where the profit functional $\mathcal{J}_{x,y}(\nu)$, net of investment costs, is defined as

$$\mathcal{J}_{x,y}(\nu) = \mathbb{E}\bigg\{ \int_0^\infty e^{-rt}\, \pi(X^{0,x}(t), C^{y,\nu}(t))dt - \int_0^\infty e^{-rt} d\nu(t) \bigg\}. \tag{2.5}$$

Since $\pi(x,\cdot)$ is strictly concave, $\mathcal{S}_o$ is convex and $C^{y,\nu}$ is affine in ν, then, if an optimal solution ν^* to (2.4) does exist, it is unique. Under further minor assumptions the existence of a solution to (2.4) is a well known result (see, e.g., [32], Theorem 2.3, for an existence proof in a not necessarily Markovian framework).

3 The Optimal Solution and the Integral Equation for the Free Boundary

A problem similar to (2.4) (with depreciation in the capacity dynamics) has been completely solved by Riedel and Su in [32], or (in the case of a time-dependent, stochastic finite fuel) by Bank in [5]. By means of a first order conditions approach and without relying on any Markovian or diffusive assumption, these Authors show that it is optimal to keep the production capacity always above a desirable lower value of capacity, the *base capacity* process (see [32], Definition 3.1), which is the unique optional solution of a stochastic backward equation in the spirit of Bank-El Karoui [4]. In this Section we aim to understand the meaning of the base capacity process l^* in our setting.

Following [5], [13] or [32] (among others), we start by deriving first order conditions for optimality and by finding the solution to (2.4) in terms of a base capacity process. Then, as a main new result, we identify the link between l^* and the free boundary of the optimal stopping problem naturally associated to the original singular control one (cf. (2.4)) and we determine an integral equation for the latter one.

Let $\mathcal{T}$ denote the set of all $\mathcal{F}_t$-stopping times $\tau \geq 0$ a.s. and notice that we may associate to $\mathcal{J}_{x,y}(\nu)$ its supergradient as the unique optional process defined by

$$\nabla \mathcal{J}_{x,y}(\nu)(\tau) := \mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs} \pi_c(X^{0,x}(s), C^{y,\nu}(s))ds \Big| \mathcal{F}_\tau \bigg\} - e^{-r\tau}, \tag{3.1}$$

for any $\tau \in \mathcal{T}$.

Theorem 3.1. *Under Assumption 2.1, a process $\nu^*(t) \in \mathcal{S}_o$ is the unique optimal investment strategy for problem (2.4) if and only if the following first order conditions for optimality*

$$\begin{cases} \nabla \mathcal{J}_{x,y}(\nu^*)(\tau) \leq 0, \\ \mathbb{E}\bigg\{ \displaystyle\int_0^\infty \nabla \mathcal{J}_{x,y}(\nu^*)(t) d\nu^*(t) \bigg\} = 0, \end{cases} \tag{3.2}$$

hold true for any $\tau \in \mathcal{T}$.

Proof. Sufficiency follows from concavity of $\pi(x,\cdot)$ (see, e.g., [5]), whereas for necessity see [34], Proposition 3.2. $\square$

Even if first order conditions (3.2) completely characterize the optimal investment plan ν^*, they are not binding at any time and thus they cannot be directly applied to determine ν^*. Nevertheless, the optimal control may be obtained in terms of the solution of a suitable Bank-El Karoui's representation problem [4] directly related to (3.2).

For a fixed $T \leq +\infty$, the Bank-El Karoui Representation Theorem (cf. [4], Theorem 3) states that, given

- an optional process $Y = \{Y(t), t \in [0,T]\}$ of class (D), lower-semicontinuous in expectation with $Y(T) = 0$,
- a nonnegative optional random Borel measure $\mu(\omega, dt)$,
- $f(\omega,t,y) : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ such that $f(\omega,t,\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is continuous in y, strictly decreasing from $+\infty$ to $-\infty$, and the stochastic process $f(\cdot,\cdot,y) : \Omega \times [0,T] \mapsto \mathbb{R}$ is progressively measurable and integrable with respect to $d\mathbb{P} \otimes \mu(\omega,dt)$,

then there exists a unique optional process $\xi = \{\xi(t), t \in [0,T]\}$ such that for all $\tau \in \mathcal{T}$

$$f(t, \sup_{\tau \leq u \leq t} \xi(u))\mathbb{1}_{[\tau,T)}(t) \in \mathbf{L}^1\left(d\mathbb{P} \otimes \mu(\omega,dt)\right)$$

and

$$\mathbb{E}\bigg\{ \int_\tau^T f(s, \sup_{\tau \leq u \leq s} \xi(u))\, \mu(ds) \,\bigg|\, \mathcal{F}_\tau \bigg\} = Y(\tau).$$

Proposition 3.2. *Under Assumption 2.1 there exists a unique strictly positive optional solution l^* to the backward stochastic equation*

$$\mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs}\pi_c(X^{0,x}(s), \sup_{\tau \leq u \leq s} l^*(u))ds \bigg| \mathcal{F}_\tau \bigg\} = e^{-r\tau}, \quad \tau \in \mathcal{T}. \tag{3.3}$$

Moreover, the process l^* has upper semi right-continuous paths, i.e. $l^*(t) \geq \limsup_{s \downarrow t} l^*(s)$.

Proof. We apply the Bank-El Karoui Representation Theorem with $T = +\infty$ to

$$Y(\omega,t) := e^{-rt}, \qquad \mu(\omega,dt) := e^{-rt}dt \tag{3.4}$$

and

$$f(\omega,t,y) := \begin{cases} \pi_c\left(X(\omega,t), -\frac{1}{y}\right), & \text{for } y < 0, \\ \\ -y\,, & \text{for } y \geq 0. \end{cases} \tag{3.5}$$

Then there exists a unique optional process ξ^* such that, for all $\tau \in \mathcal{T}$

$$\mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs} f(s, \sup_{\tau \leq u \leq s} \xi^*(u))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} = e^{-r\tau}. \tag{3.6}$$

If now ξ^* has upper semi right-continuous paths and it is strictly negative, then the strictly positive, upper semi right-continuous process $l^*(t) = -\frac{1}{\xi^*(t)}$ solves

$$\begin{aligned} e^{-r\tau} &= \mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs}\pi_c\Big(X^{0,x}(s), \frac{1}{-\sup_{\tau\le u\le s}(-\frac{1}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \bigg\} \\ &= \mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs}\pi_c\Big(X^{0,x}(s), \frac{1}{\inf_{\tau\le u\le s}(\frac{1}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \bigg\} \\ &= \mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs}\pi_c(X^{0,x}(s), \sup_{\tau\le u\le s} l^*(u))\, ds \,\Big|\, \mathcal{F}_\tau \bigg\}, \end{aligned}$$

thanks to (3.5) and (3.6).

To conlude the proof, we must show that $\xi^*(t)$ is indeed upper semi right-continuous and strictly negative. We start by proving upper semi right-continuity of ξ^* following the ideas in [7], Theorem 1. By [16], Proposition 2, it suffices to show that $\lim_{n\to\infty} \xi^*(\tau_n) \le \xi^*(\tau)$, for any sequence of stopping times $\{\tau_n\}_{n\ge1}$ such that $\tau_n \downarrow \tau$ and for which there exists a.s. $\zeta := \lim_{n\to\infty} \xi^*(\tau_n)$. For Y, μ and f as in (3.4) and (3.5), set

$$\Xi^l(t) := \operatorname*{ess\,inf}_{\tau\ge t} \mathbb{E}\bigg\{ \int_t^\tau f(s,l)\mu(ds) + Y(\tau) \Big| \mathcal{F}_t \bigg\}, \qquad l \in \mathbb{R},\ t \ge 0,$$

and recall that $\xi^*(t) = \sup\{l \in \mathbb{R} : \Xi^l(t) = Y(t)\}$ (cf. [4]). Now, given $\epsilon > 0$, for such sequence of stopping times we have

$$\Xi^{\zeta-\epsilon}(\tau) = \lim_{n\to\infty} \Xi^{\zeta-\epsilon}(\tau_n) = Y(\tau),$$

where we have used right-continuity of $t \mapsto \Xi^l(t)$, the fact that $l \mapsto \Xi^l(t)$ is a continuous, decreasing mapping (cf. [4], Lemma 4.12) and the threshold representation of ξ^*. Hence $\zeta - \epsilon \le \xi^*(\tau)$ and upper semi right-continuity of ξ^* follows by arbitraryness of ϵ. Finally, we now show that ξ^* is strictly negative. Define

$$\sigma := \inf\{t \ge 0 : \xi^*(t) \ge 0\},$$

then for $\omega \in \{\omega : \sigma(\omega) < +\infty\}$, the upper semi right-continuity of ξ^* implies $\xi^*(\sigma) \ge 0$ and therefore $\sup_{\sigma\le u\le s} \xi^*(u) \ge 0$ for all $s \ge \sigma$. Therefore, (3.6) with $\tau = \sigma$, i.e.

$$e^{-r\sigma} = -\mathbb{E}\bigg\{ \int_\sigma^\infty e^{-rs} \sup_{\sigma\le u\le s} \xi^*(u)\, ds \,\Big|\, \mathcal{F}_\sigma \bigg\}, \tag{3.7}$$

is not possible for $\omega \in \{\omega : \sigma(\omega) < +\infty\}$ since the right-hand side of (3.7) is nonpositive, whereas the left-hand side is always strictly positive. It follows that $\sigma = +\infty$ a.s. and hence $\xi^*(t) < 0$ for all $t \ge 0$ a.s. □

Proposition 3.3. *Under Assumption 2.1, the unique optimal irreversible investment process for problem (2.4) is given by*

$$\nu^*(t) = (\sup_{0\le s\le t} l^*(s) - y) \vee 0, \tag{3.8}$$

where $l^*(t)$ is the unique optional upper semi right-continuous solution to (3.3).

Proof. See, e.g., [32], Theorem 3.2. □

In the literature on stochastic, irreversible investment problems (cf. [1], [10], [11] and [12], among others), or more generally on singular stochastic control problems of monotone follower type (see, e.g., [5], [19], [25]), it is well known that to a monotone control problem one may associate a suitable optimal stopping problem whose optimal solution, τ^*, is related to the optimal control, ν^*, by the simple relation $\tau^* = \inf\{t \geq 0 : \nu^*(t) > 0\}$. Economically, it means that a firm's manager has to decide how to optimally invest or, equivalently, when to profitably exercise the investment option. Indeed, if we introduce the level passage times $\tau^\nu(q) := \inf\{t \geq 0 : \nu(t) > q\}$, $q \geq 0$, then for any $\nu \in \mathcal{S}_o$ and $y \geq 0$ we may write (cf., e.g., [1], Lemma 2)

$$\begin{aligned}
\mathcal{J}_{x,y}(\nu) - \mathcal{J}_{x,y}(0) &= \int_y^\infty \mathbb{E}\bigg\{ \int_{\tau^\nu(z-y)}^\infty e^{-rs}\pi_c(X^{0,x}(s), z)ds - e^{-r\tau^\nu(z-y)} \bigg\} dz \\
&\leq \int_y^\infty \sup_{\tau\geq 0} \mathbb{E}\bigg\{ \int_\tau^\infty e^{-rs}\pi_c(X^{0,x}(s), z)ds - e^{-r\tau} \bigg\} dz \\
&= \int_y^\infty \mathbb{E}\bigg\{ \int_0^\infty e^{-rs}\pi_c(X^{0,x}(s), z)ds \bigg\} dz \\
&\qquad - \int_y^\infty \inf_{\tau\geq 0} \mathbb{E}\bigg\{ \int_0^\tau e^{-rs}\pi_c(X^{0,x}(s), z)ds + e^{-r\tau} \bigg\} dz.
\end{aligned}$$

Therefore, if a process $\nu^* \in \mathcal{S}_o$ is such that its level passage times are optimal for the previous optimal stopping problems, then ν^* must be optimal for problem (2.4). Hence

$$v(x,y) := \inf_{\tau\geq 0} \mathbb{E}\bigg\{ \int_0^\tau e^{-rs}\pi_c(X^{0,x}(s), y)ds + e^{-r\tau} \bigg\} \tag{3.9}$$

is the optimal timing problem naturally associated to optimal investment problem (2.4). Notice that $v(x,y) \leq 1$, for all $x \in \mathcal{I}$ and $y > 0$, and that the mapping $y \mapsto v(x,y)$ is strictly decreasing for any $x \in \mathcal{I}$, being $\pi(x,\cdot)$ strictly concave. We may now define the continuation region

$$\mathcal{C} := \{(x,y) \in \mathcal{I} \times (0,\infty) : v(x,y) < 1\} \tag{3.10}$$

and the stopping region

$$\mathcal{S} := \{(x,y) \in \mathcal{I} \times (0,\infty) : v(x,y) = 1\}. \tag{3.11}$$

Intuitively $\mathcal{S}$ is the region in which it is optimal to invest immediately, whereas $\mathcal{C}$ is the region in which it is profitable to delay the investment option. The decreasing property of $y \mapsto v(x,y)$ implies that $\mathcal{S}$ is below $\mathcal{C}$ and therefore that

$$b(x) := \sup\{y > 0 \,:\, v(x,y) = 1\}, \qquad x \in \mathcal{I}, \tag{3.12}$$

is the boundary between these two regions, i.e. the free boundary.

The next Theorem gives us a new representation for the base capacity l^* in our setting.

Theorem 3.4. *Under Assumption 2.1 one has*

$$l^*(t) = \sup\{y > 0 \,:\, v(X^{0,x}(t), y) = 1\}, \tag{3.13}$$

with $l^*(t)$ the unique optional solution to (3.3) and $v(x, y)$ being defined as in (3.9).

Proof. Recall that process ξ^* of (3.6) admits the representation (cf. [4], formula (23) on page 1049)

$$\xi^*(t) = \sup\left\{l < 0 \,:\, \operatorname*{ess\,inf}_{\tau\geq t} \mathbb{E}\left\{\int_t^\tau e^{-rs}\pi_c(X^{0,x}(s), -\frac{1}{l})ds + e^{-r\tau}\Big|\mathcal{F}_t\right\} = e^{-rt}\right\}. \tag{3.14}$$

To take care of the previous conditional expectation, let now $(\Omega, \mathbb{P})$ be the canonical probability space where $\mathbb{P}$ is the measure induced by $X^{0,x}$ on $C(\mathbb{R}_+)$, the space of continuous functions on $\mathbb{R}_+$. Moreover, let $\theta_t : \Omega \mapsto \Omega$, $t \geq 0$, be the shift operator (cf., e.g., [26], page 77) such that if X is the coordinate mapping process $X(\omega, t) = \omega(t)$, then $\omega(s) \circ \theta_t = \omega(s + t)$, $s \geq 0$. Hence, for any $\tau \in \mathcal{T}$,

$$\begin{aligned}
&\mathbb{E}\left\{\int_t^\tau e^{-rs}\pi_c(X^{0,x}(s), -\frac{1}{l})ds + e^{-r\tau}\Big|\mathcal{F}_t\right\}\\
&= e^{-rt}\mathbb{E}\left\{\int_0^{\tau-t} e^{-ru}\pi_c(X^{0,x}(u+t), -\frac{1}{l})du + e^{-r(\tau-t)}\Big|\mathcal{F}_t\right\}\\
&= e^{-rt}\mathbb{E}\left\{\left(\int_0^{\tau} e^{-ru}\pi_c(X^{0,x}(u), -\frac{1}{l})du + e^{-r\tau}\right)\circ\theta_t\,\Big|\mathcal{F}_t\right\}\\
&= e^{-rt}\mathbb{E}\left\{\int_0^{\tau} e^{-ru}\pi_c(X^{0,z}(u), -\frac{1}{l})du + e^{-r\tau}\right\}_{z=X^{0,x}(t)},
\end{aligned}$$

by Markov property, and therefore

$$\begin{aligned}
\xi^*(t) &= \sup\left\{l < 0 \,:\, \operatorname*{ess\,inf}_{\tau\geq t} \mathbb{E}\left\{\int_t^\tau e^{-rs}\pi_c(X^{0,x}(s), -\frac{1}{l})ds + e^{-r\tau}\Big|\mathcal{F}_t\right\} = e^{-rt}\right\}\\
&= \sup\left\{l < 0 \,:\, v(X^{0,x}(t), -\frac{1}{l}) = 1\right\},
\end{aligned}$$

with v as in (3.9).

Finally, since $l^*(t) = -\frac{1}{\xi^*(t)}$ (cf. proof of Proposition 3.2), we may write for $y > 0$

$$\begin{aligned}
l^*(t) &= -\frac{1}{\sup\left\{l < 0 \,:\, v(X^{0,x}(t), -\frac{1}{l}) = 1\right\}} = \frac{1}{-\sup\left\{-\frac{1}{y} < 0 \,:\, v(X^{0,x}(t), y) = 1\right\}}\\
&= \frac{1}{\inf\left\{\frac{1}{y} > 0 \,:\, v(X^{0,x}(t), y) = 1\right\}} = \sup\left\{y > 0 \,:\, v(X^{0,x}(t), y) = 1\right\}. \qquad (3.15)
\end{aligned}$$

□

Theorem 3.5. *Under Assumption 2.1 one has*

$$l^*(t) = b(X^{0,x}(t)). \tag{3.16}$$

Proof. Theorem 3.4 and (3.12) immediately yield the result. □

Theorem 3.5 clarifies why in the literature (cf. [2], [13] or [32], among others) one usually refers to l^* as a 'desirable value of capacity' that the controller aims to maintain in a 'minimal way'. Indeed, as in the classical monotone follower problems (see, e.g., [19] and [25]), the optimal investment policy ν^* (cf. Proposition 3.3) is the solution of a Skorohod problem being the least effort needed to reflect the production capacity at the moving (random) boundary $l^*(t) = b(X^{0,x}(t))$; that is,

$$\nu^*(t) = \sup_{0\leq s\leq t} (b(X^{0,x}(s)) - y) \vee 0.$$

In Theorem 3.8 below, we shall show that relation (3.16) allows to find by (3.3), and by exploiting purely probabilistic arguments, an integral equation for the free boundary. For that we also need to prove nondecreasing property of $b(\cdot)$ which is a direct consequence of the following result.

Proposition 3.6. *If $x \mapsto \pi_c(x, c)$ is a nondecreasing mapping, then, under Assumption 2.1, $x \mapsto v(x, y)$ is nondecreasing for any $y > 0$.*

Proof. For $y > 0$, take $x_1 > x_2$, $x_1, x_2 \in \mathcal{I}$, let τ^* be optimal for (x_1, y) and $\theta \in \mathcal{T}$ be a generic stopping time. Then

$$v(x_1, y) - v(x_2, y) \geq \mathbb{E}\bigg\{ \int_0^{\tau^*} e^{-rs}\pi_c(X^{0,x_1}(s), y)ds + e^{-r\tau^*} - \int_0^{\theta} \pi_c(X^{0,x_2}(s), y)ds - e^{-r\theta} \bigg\},$$

for any $\theta \in \mathcal{T}$. Take now $\theta \equiv \tau^*$ to obtain

$$v(x_1, y) - v(x_2, y) \geq \mathbb{E}\bigg\{ \int_0^{\tau^*} e^{-rs}[\pi_c(X^{0,x_1}(s), y) - \pi_c(X^{0,x_2}(s), y)]ds \bigg\} \geq 0,$$

being $x \mapsto X^{0,x}(t)$ a.s. increasing for any $t \geq 0$. □

Corollary 3.7. *Assume that $x \mapsto \pi_c(x, c)$ is a nondecreasing mapping. Then, under Assumption 2.1 the free boundary $b(x)$ between the continuation region and the stopping region is nondecreasing for any $x \in \mathcal{I}$.*

Proof. Use the result of Proposition 3.6 and arguments similar to those in [22], proof of Proposition 2.2. □

We may now state the main result of this paper.

Theorem 3.8. *Assume $x \mapsto \pi_c(x, c)$ nondecreasing and let Assumption 2.1 hold. Denote by $\mathcal{G}$ the infinitesimal generator associated to $X^{0,x}$, and by $\psi_r(x)$ the increasing solution to the equation $\mathcal{G}u = ru$. Moreover, let $m(dx)$ and $s(dx)$ be the speed measure and the scale function measure, respectively, associated to the diffusion $X^{0,x}$. Then, the free boundary $b(x)$ between the continuation region and the stopping region is the unique nondecreasing solution to the integral equation*

$$\psi_r(x) \int_x^{\overline{x}} \left(\int_{\underline{x}}^{z} \pi_c(y, b(z))\psi_r(y)m(dy) \right) \frac{s(dz)}{\psi_r^2(z)} = 1. \tag{3.17}$$

Proof. Since l^* uniquely solves (3.3) and $l^*(t) = b(X^{0,x}(t))$ (cf. Theorem 3.5), then, for any $\tau \in \mathcal{T}$,

$$\begin{aligned} r &= \mathbb{E}\Big\{ \int_\tau^\infty re^{-r(s-\tau)}\pi_c(X^{0,x}(s), \sup_{\tau\le u\le s} b(X^{0,x}(u)))ds \Big| \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_0^\infty re^{-rt}\pi_c(X^{0,x}(t+\tau), b(\sup_{0\le u\le t} X^{0,x}(u+\tau)))dt \Big| \mathcal{F}_\tau \Big\}, \end{aligned} \tag{3.18}$$

where in the second equality we have used the fact that $b(\cdot)$ is nondecreasing by Corollary 3.7. Now, by strong Markov property, (3.18) amounts to find $b(\cdot)$ such that

$$\mathbb{E}_x\Big\{ \int_0^\infty re^{-rt}\pi_c(X^{0,x}(t), b(\sup_{0\le u\le t} X^{0,x}(u)))dt \Big\} = r;$$

that is, such that

$$\mathbb{E}_x\Big\{ \pi_c(X^{0,x}(\tau_r), b(M^{0,x}(\tau_r))) \Big\} = r,$$

where $M^{0,x}(t) := \sup_{0\le s\le t} X^{0,x}(s)$ and τ_r denotes an independent exponentially distributed random time with parameter r. Integral equation (3.17) now follows since for a one-dimensional regular diffusion $X^{0,x}$ (cf. [15], page 185) one has

$$\mathbb{P}_x(X^{0,x}(\tau_r) \in dy, M^{0,x}(\tau_r) \in dz) = r\frac{\psi_r(x)\psi_r(y)}{\psi_r^2(z)}m(dy)s(dz), \qquad y \le z, \;\; x \le z.$$

□

Notice that the arguments used in the proof of Theorem 3.8 resemble those of [3], proof of Lemma 3.2. However, in [3] the Authors studied a representation problem of a different form (their equation (34)) and did not point out any connection between its solution and the free boundary of the associated optimal stopping problem. Our integral equation (3.17) distinguishes also from that of Pedersen and Peskir in [29] based on a local time space calculus formula for semimartingales on continuous surfaces [30], or from the result of Federico and Pham [20] for nondegenerate diffusions and quadratic costs. Indeed, thanks to (3.16) and strong Markov property, (3.17) follows immediately from backward equation (3.3) for $l^*(t) = b(X^{0,x}(t))$, and therefore it does not require any regularity of the value function, smooth-fit property or a priori continuity of $b(\cdot)$ itself to be applied. It thus represents an extremely useful tool to determine the free boundary of the whole class of infinite time horizon, singular, stochastic irreversible investment problems of type (2.4). As we shall see in the next Section, equation (3.17) may be analitically solved in some non-trivial cases.

Remark 3.9. *The result of Theorem 3.5 still holds if one introduces depreciation in the production capacity dynamics as in [32]; that is, if*

$$C^{y,\nu}(t) = -\rho C^{y,\nu}(t)dt + d\nu(t), \qquad C^{y,\nu}(0) = y \ge 0,$$

for some $\rho > 0$. Moreover, in this case one has (cf. [32], Theorem 3.2)

$$\nu^*(t) = \int_{[0,t)} e^{-\rho s}d\overline{\nu}^*(s), \qquad \textit{with} \qquad \overline{\nu}^*(t) = \sup_{0\le s\le t}\Big(\frac{b(X^{0,x}(s)) - ye^{-\rho s}}{e^{-\rho s}}\Big) \vee 0.$$

4 Explicit Results

In this Section we aim to explicitly solve integral equation (3.17) when the economic shock $X^{0,x}$ is a Geometric Brownian motion, a three-dimensional Bessel process and a CEV (constant elasticity of volatility) process. We shall find the free boundary $b(\cdot)$ of optimal stopping problem (3.9) for Cobb-Douglas and logarithmic operating profit functions; that is, for $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ with $\alpha, \beta \in (0,1)$, and $\pi(x,c) = \alpha \ln(x) + \beta \ln(c)$, $\alpha, \beta > 0$, respectively.

To the best of our knowledge, this is the first time that the free boundary of a singular stochastic control problem of type (2.4) (and of the optimal stopping problem associated) is explicitly determined for underlying given by a three-dimensional Bessel process or by a CEV process.

4.1 Geometric Brownian Motion and Cobb-Douglas Operating Profit

Let $X^{0,x}(t) = xe^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}$, $x > 0$, with $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, and let the operating profit function be of Cobb-Douglas type; that is, $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ for $\alpha, \beta \in (0,1)$. If we denote by $\delta := \frac{\mu}{\sigma^2} - \frac{1}{2}$, then it is well known (cf., e.g., [9]) that

$$m(dx) = \frac{2}{\sigma^2} x^{2\delta-1} dx$$

and

$$s(dx) := \begin{cases} x^{-2\delta-1}dx, & \delta \neq 0, \\ \frac{1}{x}dx, & \delta = 0. \end{cases}$$

Finally, the ordinary differential equation $\mathcal{G}u = ru$, i.e. $\frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) = ru$, admits the increasing solution

$$\psi_r(x) = x^{\gamma_1},$$

where γ_1 is the positive root of the equation $\frac{1}{2}\sigma^2\gamma(\gamma-1) + \mu\gamma = r$.

Proposition 4.1. *For any $\delta \in \mathbb{R}$ and $x > 0$, one has*

$$b(x) = K_\delta x^{\frac{\alpha}{1-\beta}}, \tag{4.1}$$

with $K_\delta := \left[\sigma^2\gamma_1(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)\right]^{-\frac{1}{1-\beta}}$.

Proof. Let us start with the case $\delta \neq 0$. For any $x > 0$ by (3.17) we have

$$\int_x^\infty \left(\int_0^z y^{\alpha+\gamma_1+2\delta-1}dy\right) b^{\beta-1}(z) z^{-2\delta-1-2\gamma_1} dz = x^{-\gamma_1}\left(\frac{\alpha+\beta}{2\beta}\right)\sigma^2;$$

that is,

$$\int_x^\infty b^{\beta-1}(z) z^{\alpha-\gamma-1} dz = \sigma^2(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right) x^{-\gamma_1}.$$

Take now $b(x) = (A_\delta z)^{\frac{\alpha}{1-\beta}}$, for some constant A_δ, to obtain

$$A_\delta^{-\alpha}\int_x^\infty z^{-\gamma_1-1}dz = \frac{A_\delta^{-\alpha}}{\gamma_1} x^{-\gamma_1} = \sigma^2(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right) x^{-\gamma_1},$$

which is satisfied by $A_\delta := \Big[\sigma^2\gamma_1(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)\Big]^{-\frac{1}{\alpha}}$. Hence $b(x) = K_\delta x^{\frac{\alpha}{1-\beta}}$ with $K_\delta := A^{\frac{\alpha}{1-\beta}}$. Similar calculations also apply to the case $\delta = 0$ to have $b(x) = K_0 x^{\frac{\alpha}{1-\beta}}$. □

4.2 Geometric Brownian Motion and Logarithmic Operating Profit

In the same setting of Section 4.1, assume now that $\pi(x,c) = \alpha\ln(x) + \beta\ln(c)$, $\alpha,\beta > 0$. Then

Proposition 4.2. *For any $\delta \in \mathbb{R}$ and $x > 0$ one has*

$$b(x) = \frac{2\beta}{\sigma^2\gamma_1(2\delta+\gamma_1)}. \tag{4.2}$$

Proof. For $\delta \neq 0$ and $x > 0$ we may write from (3.17)

$$x^{\gamma_1}\int_x^\infty\left(\int_0^z \frac{\beta}{b(z)}y^{\gamma_1}\frac{2}{\sigma^2}y^{2\delta-1}dy\right)\frac{z^{-2\delta-1}}{z^{2\gamma_1}}dz = 1;$$

that is,

$$\int_x^\infty\left(\int_0^z y^{\gamma_1+2\delta-1}dy\right)\frac{z^{-2\delta-1-2\gamma_1}}{b(z)}dz = \frac{\sigma^2}{2\beta}x^{-\gamma_1}.$$

By integrating one easily obtains

$$\int_x^\infty \frac{z^{-\gamma_1-1}}{b(z)}dz = \frac{\sigma^2(2\delta+\gamma_1)}{2\beta}x^{-\gamma_1},$$

which is solved by $b(x) = \frac{2\beta}{\sigma^2\gamma_1(2\delta+\gamma_1)}$. Similar arguments apply to the case $\delta = 0$ to obtain $b(x) = \frac{2\beta}{\sigma^2\gamma_1^2}$. □

4.3 Three-Dimensional Bessel Process and Logarithmic Operating Profit

Let now $X^{0,x}(t)$ be a three-dimensional Bessel process; that is, the strong solution of

$$dX^{0,x}(t) = \frac{1}{X^{0,x}(t)}dt + dW(t), \qquad X^{0,x}(0) = x > 0.$$

It is a diffusion with state space $(0,\infty)$, generator $\mathcal{G} := \frac{1}{2}\frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx}$ and scale and speed measures given by $s(dx) = x^{-2}dx$ and $m(dx) = 2x^2dx$, respectively (cf. [23], Chapter VI). Further, since $X^{0,x}(t)$ may be characterized as a killed Brownian motion at zero, conditioned never to hit zero, the three-dimensional Bessel process may be viewed as an excessive transform of a killed Brownian motion with excessive function $h(x) = x$; that is, the scale function of the Brownian motion. Therefore $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$ (cf. [23], Chapter VI or [15], Section 6.2, among others).

Moreover, assume that the operating profit function is of logarithmic type, i.e. $\pi(x,c) = \alpha\ln(x) + \beta\ln(c)$, $\alpha,\beta > 0$. Then

Proposition 4.3. *For any $x > 0$, one has*

$$b(x) = \frac{\beta}{r}. \tag{4.3}$$

Proof. In this case (3.17) becomes

$$\frac{1}{2\beta}\frac{x}{\sinh(\sqrt{2r}x)} = \int_x^\infty \left(\int_0^z y\sinh(\sqrt{2r}y)dy\right)\frac{dz}{b(z)\sinh^2(\sqrt{2r}z)} = \int_x^\infty g(z)\frac{dz}{b(z)\sinh^2(\sqrt{2r}z)},$$

with $g(x) := \int_0^x y\sinh(\sqrt{2r}y)dy = \frac{1}{2r}[\sqrt{2r}x\cosh(\sqrt{2r}x) - \sinh(\sqrt{2r}x)]$, thanks to an integration by parts. Take now $b(x) := \frac{\beta}{r}$ and the result follows since

$$\int \frac{g(x)}{\sinh^2(\sqrt{2r}z)} = -\frac{1}{2r}\frac{x}{\sinh(\sqrt{2r}x)} + \text{const.}$$

□

4.4 Three-Dimensional Bessel Process and Cobb-Douglas Operating Profit

In the same setting of Section 4.3, suppose now that the operating profit is of Cobb-Douglas type; that is, $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ for $\alpha, \beta \in (0,1)$. The following result holds.

Proposition 4.4. *For any $x > 0$ one has*

$$b(x) = \left[\left(\frac{\alpha+\beta}{2\beta}\right)x^2\frac{\psi_r'(x)}{g(x)}\right]^{-\frac{1}{1-\beta}}, \tag{4.4}$$

where $\psi_r'(x)$ denotes the first derivative of the increasing function $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$, and $g(x) := \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy$.

Proof. From integral equation (3.17) we may write

$$\begin{aligned}\left(\frac{\alpha+\beta}{2\beta}\right)\frac{x}{\sinh(\sqrt{2r}x)} &= \int_x^\infty \left(\int_0^z y^{\alpha+1}\sinh(\sqrt{2r}y)dy\right)\frac{b^{\beta-1}(z)}{\sinh^2(\sqrt{2r}z)}dz \\ &= \int_x^\infty g(z)\frac{b^{\beta-1}(z)}{\sinh^2(\sqrt{2r}z)}dz,\end{aligned}$$

with $g(x) := \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy$. By differentiating, one obtains

$$b^{\beta-1}(x) = \left(\frac{\alpha+\beta}{2\beta}\right)\frac{[x\sqrt{2r}\cosh(\sqrt{2r}x) - \sinh(\sqrt{2r}x)]}{g(x)} = \left(\frac{\alpha+\beta}{2\beta}\right)x^2\frac{\psi_r'(x)}{g(x)}, \tag{4.5}$$

i.e.

$$b(x) = \left[\left(\frac{\alpha+\beta}{2\beta}\right)x^2\frac{\psi_r'(x)}{g(x)}\right]^{-\frac{1}{1-\beta}}.$$

Notice that $b(\cdot)$ is positive since $\psi_r(\cdot)$ is increasing and $g(\cdot)$ is positive.

To conclude the proof it suffices now to check that the mapping $x \mapsto b(x)$ is actually nondecreasing as suggested by Proposition 3.7; that is, $x \mapsto b^{\beta-1}(x)$ is nonincreasing. From (4.5) we have

$$\begin{aligned}\frac{d}{dx}b^{\beta-1}(x) &= \left(\frac{\alpha+\beta}{2\beta g^2(x)}\right)\Big[g(x)(2x\psi_r'(x) + x^2\psi_r''(x)) - g'(x)x^2\psi_r'(x)\Big] \\ &= \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\Big[2rg(x)\psi_r(x) - g'(x)\psi_r'(x)\Big],\end{aligned} \tag{4.6}$$

since $\psi_r(x)$ solves $\frac{1}{2}\psi_r''(x) + \frac{1}{x}\psi_r'(x) = r\psi_r(x)$. Recall now that $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$, $g'(x) = x^{\alpha+1}\sinh(\sqrt{2r}x)$ and notice that, by an integration by parts,

$$g(x) = \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy = \frac{1}{\sqrt{2r}}x^{\alpha+1}\cosh(\sqrt{2r}x) - \frac{\alpha+1}{\sqrt{2r}}I(x),$$

with $I(x) := \int_0^x y^{\alpha}\cosh(\sqrt{2r}y)dy$. Therefore from (4.6) we may write

$$\begin{aligned}\frac{d}{dx}b^{\beta-1}(x) &= \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\frac{\sinh(\sqrt{2r}x)}{x}\Big[-(\alpha+1)\sqrt{2r}I(x) + \sinh(\sqrt{2r}x)x^{\alpha}\Big] \\ &=: \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\frac{\sinh(\sqrt{2r}x)}{x}T(x). \end{aligned} \tag{4.7}$$

Since $T(0) = 0$ and $T'(x) = \alpha x^{\alpha-1}[\sinh(\sqrt{2r}x) - x\sqrt{2r}\cosh(\sqrt{2r}x)] = -\alpha x^{\alpha+1}\psi_r'(x) < 0$, being $x \mapsto \psi_r(x)$ increasing, it follows that $x \mapsto T(x)$ is negative for any $x > 0$. The decreasing property of $x \mapsto b^{\beta-1}(x)$ is therefore proved. ☐

4.5 CEV Process and Cobb-Douglas Operating Profit

Let now the diffusion $X^{0,x}$ be of CEV (Constant Elasticity of Variance) type; that is,

$$dX^{0,x}(t) = rX^{0,x}(t)dt + \sigma(X^{0,x})^{1-\gamma}(t)dW(t), \qquad X^{0,x}(0) = x > 0, \tag{4.8}$$

for some $r > 0$, $\sigma > 0$ and $\gamma \in (0,1)$. CEV process was introduced in the financial literature by John Cox in 1975 [14] in order to capture the stylized fact of a negative link between equity volatility and equity price (the so called 'leverage effect'). In this case we have

$$m(dx) = \frac{2}{\sigma^2 x^{2(1-\gamma)}}e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}}dx, \qquad s(dx) = e^{-\frac{r}{\gamma\sigma^2}x^{2\gamma}}dx,$$

and $\psi_r(x) = x$ (cf., e.g., [17], Section 6.2). Moreover, assume that $\pi(x,c) = \frac{x^{\alpha}c^{\beta}}{\alpha+\beta}$, for $\alpha, \beta \in (0,1)$.

Proposition 4.5. *For any $x > 0$ one has*

$$b(x) = \left[\frac{2\beta}{\sigma^2(\alpha+\beta)}g(x)e^{-\frac{r}{\gamma\sigma^2}x^{2\gamma}}\right]^{\frac{1}{1-\beta}}, \tag{4.9}$$

with $g(x) := \int_0^x y^{2\gamma+\alpha-1}e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}}dy$.

Proof. From (3.17) one has

$$\int_x^{\infty}\left(\int_0^z y^{2\gamma+\alpha-1}e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}}dy\right)\frac{b^{\beta-1}(z)}{z^2}e^{-\frac{r}{\gamma\sigma^2}z^{2\gamma}}dz = \frac{\sigma^2}{x}\left(\frac{\alpha+\beta}{2\beta}\right),$$

i.e.

$$\int_x^{\infty} g(z)\frac{b^{\beta-1}(z)}{z^2}e^{-\frac{r}{\gamma\sigma^2}z^{2\gamma}}dz = \frac{\sigma^2}{x}\left(\frac{\alpha+\beta}{2\beta}\right),$$

with $g(x) := \int_0^x y^{2\gamma+\alpha-1} e^{\frac{r}{\gamma\sigma^2} y^{2\gamma}} dy$. Take now

$$b^{\beta-1}(x) = \frac{\sigma^2}{g(x)} \left(\frac{\alpha+\beta}{2\beta}\right) e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}}$$

to obtain the sought result.

To conclude the proof we shall now show that $b(x)$ as in (4.9) is nondecreasing, or, equivalently, that $x \mapsto b^{\beta-1}(x)$ is nonincreasing. Indeed we have

$$\begin{aligned} \frac{d}{dx} b^{\beta-1}(x) &= \frac{\sigma^2}{g^2(x)} \left(\frac{\alpha+\beta}{2\beta}\right) x^{2\gamma-1} e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}} \left[\frac{2r}{\sigma^2} g(x) - x^{\alpha} e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}}\right] \\ &= -\frac{\alpha\sigma^2}{g^2(x)} \left(\frac{\alpha+\beta}{2\beta}\right) x^{2\gamma-1} e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}} \int_0^x y^{\alpha-1} e^{\frac{r}{\gamma\sigma^2} y^{2\gamma}} dy < 0, \end{aligned} \tag{4.10}$$

being $g(x) = \frac{\sigma^2}{2r}[e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}} x^{\alpha} - \alpha \int_0^x y^{\alpha-1} e^{\frac{r}{\gamma\sigma^2} y^{2\gamma}} dy]$, thanks to an integration by parts. $\square$

4.6 CEV Process and Logarithmic Operating Profit

In the same setting of Section 4.5, suppose now that the operating profit is of logarithmic type; that is $\pi(x, c) = \alpha \ln(x) + \beta \ln(c)$, for some $\alpha, \beta > 0$. Then

Proposition 4.6. *For any $x > 0$ one has*

$$b(x) = \frac{\beta\sigma^2}{r}\left(1 - e^{-\frac{r}{\gamma\sigma^2} x^{2\gamma}}\right). \tag{4.11}$$

Proof. From (3.17) we may write

$$\int_x^{\infty} \left(\int_0^z y^{2\gamma-1} e^{\frac{r}{\gamma\sigma^2} y^{2\gamma}} dy\right) \frac{1}{b(z) z^2} e^{-\frac{r}{\gamma\sigma^2} z^{2\gamma}} dz = \frac{1}{2\beta x};$$

that is,

$$\int_x^{\infty} \frac{1}{b(z) z^2} \left(1 - e^{-\frac{r}{\gamma\sigma^2} z^{2\gamma}}\right) dz = \frac{r}{x\beta\sigma^2},$$

which is obviously satisfied if $b(x)$ is as in (4.11). $\square$

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