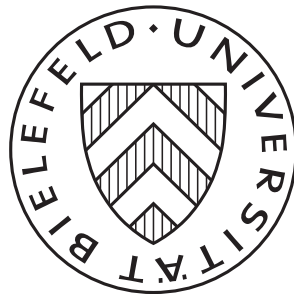# Optimal consumption and portfolio choice with ambiguity

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Optimal consumption and portfolio choice with ambiguity

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Abstract

We consider optimal consumption and portfolio choice in the presence of Knightian uncertainty in continuous-time. We embed the problem into the new framework of stochastic calculus for such settings, dealing in particular with the issue of non–equivalent multiple priors. We solve the problem completely by identifying the worst–case measure. Our setup also allows to consider interest rate uncertainty; we show that under some robust parameter constellations, the investor optimally puts all his wealth into the asset market, and does not save or borrow at all.

Key words and phrases: Robust Finance, Optimal Portfolio Choice, Knightian Uncertainty, Model Uncertainty, Ambiguity

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1 Introduction

The optimal way to invest and consume one's wealth belongs to the basic questions of finance. The standard textbook answer uses Merton's (1969) solution within the framework of the geometric Brownian motion model for risky assets. In this paper, we generalize this fundamental model to allow for Knightian uncertainty about asset and interest rate dynamics and study the consequences for ambiguity–averse investors.

In continuous time, Knightian uncertainty leads to some subtle issues. Uncertainty about volatility, as well as uncertainty about the short rate, requires the use of singular probability measures, a curious, but in an ambiguous world natural fact. The investor is cautious and presumes that nature has unpleasant surprises. In particular, the volatility of risky assets can take surprising paths, within certain limits.

Fortunately, the last years have seen the development of a new stochastic calculus[1] that extends the omnipresent Itô–calculus to such multiple prior models . The rationality of such an approach as well as the consequences for utility theory and equilibrium asset pricing have recently been discussed at length by Epstein and Ji (2011). We show here how to embed the classic Merton–Samuelson model into this new framework. The new framework has the advantage that it allows to use essentially well-known martingale arguments to establish optimality of candidate policies, just as in the classic case.

While explicit results are difficult to obtain under Knightian uncertainty in general, we are here able to solve completely the ambiguity–averse investor's optimal consumption–portfolio problem. In a first step, we derive the extension to Knightian uncertainty for the classic Hamilton–Jacobi–Bellman equation. A closer analysis of that equation leads to a conjecture for the worst–case measure. We then verify that the ambiguity–averse investor behaves as a classic expected–utility maximizer under the worst–case measure by using the new techniques. The existence of a worst case measure immediately yields a maxmin result: the value function under ambiguity is the lower envelope of the value functions under expected utility.

Ambiguity leads to different predictions for optimal portfolios and consumption plans. As in simple static models, high ambiguity about the mean return of the uncertain asset leads to non–participation in the asset market. As far as volatility is concerned, we show that our risk– and ambiguity–averse investor always uses the maximal possible volatility to determine the opti-

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[1]Denis and Martini (2006) developed such a framework for studying the pricing of options under Knightian Uncertainty; Peng (2007) develops the whole theory of stochastic calculus from scratch under Knightian uncertainty.

mal policy. A more surprising and, as far as we know, new result emerges when we take interest rate uncertainty into account: for robust parameter sets, the investor puts all his wealth into the asset market when interest rate uncertainty is sufficiently high, a phenomenon that we have observed in the aftermath of the recent financial crisis. When interest rates are low, and sufficiently high ambiguity is perceived by investors, they prefer to put all capital into assets only. Both saving and borrowing are considered to be too uncertain to be worthwhile activities.

The problem of Knightian or model uncertainty has recently attracted a great deal of attention, both in practice, as the sensitivity of many financial decisions with respect to questionable probabilistic assumptions became clear, and in theory, where an extensive theory of decision making and risk measurement under uncertainty has been developed. Gilboa and Schmeidler (1989) lay the foundation for a new approach to decisions under Knightian uncertainty by weakening the strong independence axiom or sure thing principle used previously by Savage (1954) and Anscombe and Aumann (1963) to justify (subjective) expected utility. The models are closely related to monetary risk measures (Artzner, Delbaen, Eber, and Heath (1999)). Subsequently, the theory has been generalized to variational preferences (Maccheroni, Marinacci, and Rustichini (2006a), Föllmer and Schied (2002)) and dynamic time-consistent models (Epstein and Schneider (2003), Maccheroni, Marinacci, and Rustichini (2006b), Riedel (2004)).

The pioneering results of Samuelson (1969) and Merton (1969) laid the foundation for a huge literature. As mean return, volatility, and interest rates are constants in the basic model, the consequences of having stochastic, time-varying dynamics for these parameters have been studied in great detail. Mean–reverting drift (or "predictable returns") , stochastic volatility models and models with stochastic term structures have been studied in detail. These models all work under the expected utility paradigm as they assume a known distribution for the parameters. In the same vein, one can also study incomplete information models where the investor updates his initial belief about some unknown parameter[2]. In contrast to these Bayesian models, we focus here on the recent Knightian approach where the investor takes a pessimistic, maxmin view of the world concerning the parameters of her model. One can also relax the time–additive structure of the intertemporal utility function as in recursive utility models Duffie and Epstein (1992), Hindy–Huang–Kreps models (Hindy and Huang (1992), Bank and Riedel

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[2]Barberis (2000) studies mean–reverting returns and estimation errors. Chacko and Viceira (2005) study stochastic volatility. See Liu (2007) for a recent general approach with stochastic interest rates and volatilities. Schroder and Skiadas (2002)

(2001)), or allow for trading constraints and transaction costs, topics that are outside the scope of this paper.

The Knightian approach is closely related to model uncertainty, or robustness considerations in the spirit of Anderson, Hansen, and Sargent (2003). For example, Trojani and Vanini (2002) and Maenhout (2004) study the robust portfolio choice problem with drift ambiguity. Drift ambiguity in continuous time is also discussed in Chen and Epstein (2002), Miao (2009) , Schied (2005), Schied (2008), Liu (2010), Liu (2011) among others. Föllmer, Schied, and Weber (2009) survey this literature. In these papers, a reference measure to which all priors are equivalent is used, in contrast to our approach. In particular, one cannot discuss volatility uncertainty within these models.

The paper is setup as follows. The next section formulates the Merton model under Knightian uncertainty within the new framework. The following section derives derives optimal consumption and portfolio rules for ambiguity-averse investors with fixed interest rates. Section 4 then generalizes to ambiguous interest rates. An appendix collects the relevant information about the new stochastic calculus.

2   The Samuelson-Merton Model under Knightian Uncertainty

The standard workhorse for asset pricing in continuous time has been proposed by Samuelson (1969) and Merton (1971); they work with a safe asset, or bond, with deterministic dynamics

$$dP_t = rP_t dt$$

for a known interest rate r and a risky asset S that satisfies

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

for a Brownian motion B and known drift and volatility parameters μ and σ.

This basic model has been extended in many forms, of course[3]. Here, we show how to treat the optimal portfolio-consumption choice problem for an investor who does not know the specific parameters nor their probability laws. We thus have Knightian uncertainty in the sense that the distribution of the unknown parameters is not known. However, the investor is willing

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[3]

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to work with (or knows) certain bounds for the relevant parameters; she is ambiguity–averse and aims to find a policy that is robust to such parameter uncertainty in the sense that it is optimal even against a malevolent nature.

As far as modeling is concerned, our new approach shows how to embed the Samuelson–Merton model in an extended stochastic calculus framework; the advantage of that model is that it allows to use the well-known Itô-calculus-type martingale arguments to solve the problem.

Technically, we will replace the standard Brownian motion B by a so–called G–Brownian motion that we denote by the same symbol B. A G–Brownian motion is a diffusion with unknown volatility process. It shares many properties with the known Brownian motion of classic calculus; its quadratic variation $\langle B \rangle$, however, is not equal to expired time t; only estimates of the form

$$\langle B \rangle_t \in \left[\underline{\sigma}^2 t, \overline{\sigma}^2 t \right]$$

for some volatility bounds $0 < \underline{\sigma} \le \overline{\sigma}$ are given. The classical model is recovered for $\underline{\sigma} = \overline{\sigma}$.

We will replace the drift term μdt by an ambiguous term db_t where b is a process of bounded variation that allows for any drift between two bounds $[\underline{\mu}, \overline{\mu}]$. Our new model for what we now call the uncertain, rather than the risky asset reads as

$$dS_t = S_t db_t + S_t dB_t \, .$$

We will write $dR_t = db_t + dB_t$ for the return process in the sequel. It exhibits mean and volatility uncertainty.

The riskless asset is standard, with price dynamics

$$dP_t = rP_t dt,$$

where r is the constant interest rate. One can allow for interest rate uncertainty as well as we show in Section 4. For the moment, the results are more transparent when we keep the idealized assumption of constant and known interest rates.

The investors beliefs are summarized then by a set of priors of the form $P^{\mu,\sigma}$ where μ and σ are (progressively measurable) stochastic processes. Under a prior $P^{\mu,\sigma}$, the uncertain asset has drift μ and volatility σ, and the short rate is equal to r. Note that for different volatility or short rate specifications, the priors are mutually singular to each other. The investor does not fix the null sets ex ante; under model uncertainty, one needs to reduce the number of "impossible" events. Only events that are null under all possible priors can be considered to be negligible. Technically, these events are called polar; if an event has probability one under all priors, we say it happens quasi-surely.

In the appendix, we describe the mathematical construction of the set of priors $\mathcal{P}$ and the corresponding sublinear and superlinear expectation for the more general multi-dimensional case (using Shige Peng's method) in more detail. It can be skipped at first reading.

2.1 Asset Prices

Let B be a G-Brownian motion with volatility bounds $[\underline{\sigma}, \overline{\sigma}]$. Let b be two maximally distributed increasing process with drift bounds $[\underline{\mu}, \overline{\mu}]$.

The uncertain asset prices evolve as

$$dS_t = S_t dR_t, S_0 = 1$$

where the return dynamics satisfy

$$dR_t = db_t + dB_t \, .$$

The locally riskless bond evolves as

$$dP_t = P_t r dt, P_0 = 1 \, .$$

2.2 Consumption and Trading Opportunities

The investor chooses a portfolio strategy π and a consumption plan c. Uncertainty reduces the set of possible consumption plans and trading strategies an investor might choose. This reflects the economic incompleteness of markets that uncertainty can bring. As in the classic case, we want to give precise meaning to the intertemporal budget constraint

$$
\begin{aligned}
dX_t &= X_t \pi_t^T dR_t + (1 - \pi_t^T 1) X_t r dt - c_t dt \\
&= r X_t (1 - \pi_t^T 1) dt + X_t \pi_t^T db_t - c_t dt + X_t \pi_t^T A dB_t \, . \quad (2.1)
\end{aligned}
$$

In order to do so, we have to introduce suitable restrictions, intuitively speaking, to make the stochastic differential equation meaningful under all priors simultaneously.

In order to give the definitions of a consumption plan and portfolio choices precisely, we introduce spaces of random variables and stochastic processes, which are different from the classical case, technically speaking, because of the nonequivalence of the priors.

We denote by $\Omega = C_0^{2d}(\mathbb{R}^+)$ the space of all $\mathbb{R}^{2d}$-valued continuous path $(\omega_t)_{t \in \mathbb{R}^+}$ with $w_0 = 0$, equipped with the topology generated by the uniform convergence on compacts.

We let $L^2(\Omega)$ be the completion of the set of all bounded and continuous functions on Ω under the norm $\parallel \xi \parallel = \hat{\mathbb{E}}[|\xi|^2]^{\frac{1}{2}} := \sup_{P \in \mathcal{P}} E_P[|\xi|^2]^{\frac{1}{2}}$. For $t \in [0, T]$, we define the following space

$$L_{ip}(\Omega_t) = \Big\{ \varphi(\omega_{t_1} \cdots, \omega_{t_m}) \mid m \in \mathbb{N}, \ t_1, \cdots, t_m \in [0, t],$$
$$\text{for all bounded function } \varphi \Big\}.$$

For $p \geq 1$, we now consider the process η of the following form:

$$\eta = \sum_{j=0}^{n-1} \xi_j 1_{[t_j, t_{j+1})},$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, and $\xi_j \in L_{ip}(\Omega_{t_j}), j = 0, \cdots, n-1$, We denote the set of the above processes $M^{p,0}$. And the norm in $M^{p,0}$ is defined by

$$\parallel \eta \parallel_p = \left(\hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^p dt \Big] \right)^{\frac{1}{p}} = \left(\hat{\mathbb{E}}\Big[\sum_{j=0}^{n-1} |\xi_{t_j}|^p (t_{j+1} - t_j) \Big] \right)^{\frac{1}{p}}.$$

Finally, we denote by M^p the completion of $M^{p,0}$ under the above norm.

The investor chooses a consumption plan c, a nonnegative stochastic process such that $c \in M^1$. Also the investor can choose the fraction of wealth π_t^i invested in the i risky asset, and the fraction of wealth $1 - \sum_{i=1}^d \pi_t^i$ invested in the riskless asset.

The wealth of the investor with some initial endowment $x_0 > 0$ and portfolio–consumption policy (π, c) is given by

$$dX_t = X_t \pi_t^T dR_t + (1 - \pi_t^T 1)X_t r dt - c_t dt$$
$$= r X_t (1 - \pi_t^T 1)dt + X_t \pi_t^T db_t - c_t dt + X_t \pi_t^T A dB_t, \qquad (2.2)$$

where $\pi = (\pi^1, \cdots, \pi^d)^T$ is the trading strategy, and $1 = (1, \cdots, 1)^T$. The consumption and portfolio processes pair (π, c) is admissible if $X_t \geq 0, t \in [0, T]$, $c \in M^1$ and $\pi \in M^2$.

We denote by Π the set of all such admissible π taking value in $\mathcal{B} = (-\infty, +\infty)$. Also we denote by C the set of all such admissible c.

2.3 Utility

The investor is ambiguity–averse and maximizes the minimal expected utility over his set of priors. For any random variable X on $(\Omega, \mathcal{F}_T)$, we denote by

$$\mathbb{E}X := \inf_{P \in \mathcal{P}} E_P X$$

the lowest expected value of an uncertain outcome X.

The investor's utility of consuming $c \in M^1$ and bequesting a terminal wealth X_T is

$$U(c, X) = \underline{\mathbb{E}}[\int_0^T u(s, c_s)ds + \Phi(T, X_T)],$$

the utility function u and the bequest function Φ are strictly increasing, concave and differentiable with respect to c and X, respectively. We further suppose that u and Φ are $C^{1,3}$ and $C^{1,2}$, respectively. Furthermore, we suppose that the marginal utility is infinite at zero:

$$\lim_{c \to 0} \frac{\partial}{\partial c} u(t, x) = \infty.$$

We define the value function:

$$V(x_0) = \sup_{(\pi, c) \in \Pi \times C} U(c, X),$$

that is the indirect utility function defined over portfolio and initial wealth. In the appendix, we prove that $V(x_0)$ is increasing and concave in x.

3 Optimal consumption and portfolio choice with ambiguity

3.1 The Robust Dynamic Programming Principle

We quickly recap the classical dynamic programming approach put forward by Merton in one dimension. When $v(t, x)$ denotes the value function at time t with wealth x, the dynamic programming principle states, informally, that

$$v(t, X_t) \simeq \max_{\pi, c} u(t, c_t) + E[v(t + \Delta t, X_{t+\Delta t} | \mathcal{F}_t]$$

which, according to the usual rules of Itô calculus, leads to the typical Hamilton-Jacobi–Bellman equation

$$\sup_{\pi, c} \left\{ u(t, c) - cv_x(t, x) + \pi x(\mu - r)v_x(t, x) + \frac{1}{2}\pi^2 x^2 \sigma^2 v_{xx}(t, x) \right\} = 0.$$

It reflects the usual martingale principle: for all admissible policies (π, c) with wealth process X, the sum of indirect utility and past consumption utility

$$v(t, X_t) + \int_0^t u(s, c_s)ds$$

is a supermartingale, and a martingale for the optimal policy.

We should thus expect a similar equation here, with the caveat that nature (or our cautiousness) chooses the worst parameters for drift μ and volatility σ. For given portfolio-consumption policy (π, c), nature will thus minimize our utility, which leads locally to the *uncertain HJB equation*

$$\sup_{\pi,c} \inf_{(\mu,\sigma)\in\Theta} \left\{ u(t,c) - cv_x(t,x) + \pi x(\mu - r)v_x(t,x) \right.$$
$$\left. + \frac{1}{2}\pi^2 x^2 \sigma^2 v_{xx}(t,x) \right\} = 0. \tag{3.1}$$

In fact, if we can find a suitable smooth function that solves this adjusted HJB equation, we have solved our problem. The following verification theorem states this fact in more detail.

Theorem 3.1 *Let* $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^+)$ *be a solution of the following equation*

$$\sup_{(\pi,c)\in\mathcal{B}\times\mathcal{A}} \left\{ u(t,c) + \varphi_t(t,x) + xr\varphi_x(t,x)(1 - \pi^T 1) - c\varphi_x(t,x)] \right.$$
$$\left. + \inf_{(q,Q)\in\Theta} \left\{ \varphi_x(t,x)x\langle \pi, q\rangle + \frac{1}{2}x^2\varphi_{xx}(t,x)\langle A^T\pi\pi^T A, QQ^T\rangle \right\} \right\} = 0, \tag{3.2}$$

with boundary condition

$$\varphi(T,x) = \Phi(T,x).$$

Then we have

$$V(x_0) = \varphi(0, x_0) = \sup_{(\pi,c)\in\Pi\times C} U(c, X).$$

The above theorem delivers the uncertain HJB equation for optimal consumption and optimal choice with very general specifications of drift and volatility ambiguity. In our canonical case, these uncertainties are easily separated from each other (and are, in this sense, "independent"), but there are many more interesting possible specifications for a dependence between drift and volatility uncertainty. For example, Epstein and Ji (2013) consider

$$\Theta = \left\{ (\mu, \sigma^2) : \mu = \mu_{min} + z, \sigma^2 = \sigma_{min}^2 + \alpha z, 0 \le z \le \bar{z} \right\},$$

where $\mu_{min}, \sigma_{min}^2, \alpha > 0$ and $\bar{z}$ are fixed and deterministic parameters.

3.2 The Worst–Case Measure in the Canonical Model

In this section, we will completely solve the ambiguity–averse investor's choice problem by reducing it to a suitable classical expected utility maximizer's problem. To do so, we will analyze the HJB equation in order to guess the worst–case prior. We will then verify that the value function under the worst–case prior solves our uncertain HJB equation as well.

In the canonical model, nature's minimization problem can be explicitly solved. Let us have a look at the uncertain HJB equation again. First of all, as in the classical case, it is natural to expect that indirect utility is increasing, or $\varphi_x > 0$, and (differentiably strictly) concave, or $\varphi_{xx} < 0$.

In the canonical model, for the drift, we obtain then simply

$$\inf_{\mu \in [\underline{\mu}, \overline{\mu}]} \left\{ \varphi_x(t,x) x \langle \pi, \mu \rangle \right\}$$
$$= \varphi_x(t,x) x \sum_{i=1}^{d} \pi^i \underline{\mu}^i 1_{\{\pi^i > 0\}} + \varphi_x(t,x) x \sum_{i=1}^{d} \pi^i \overline{\mu}^i 1_{\{\pi^i \leq 0\}}.$$

Clearly, nature decides for a low drift if we are long, and for a high drift if we are short.

For the volatilities, as the value function is concave, we end up maximizing the potential volatility of returns. For volatility, we immediately conclude that the nature always chooses its maximal possible value. Risk– and ambiguity–averse investors use a cautious estimate for volatility, or add an ambiguity premium to their estimate.

Having solved nature's choice, we are left with a simple maximization problem for consumption and portfolio weights, yet with a kink in the linear part at zero, when we change from short to long position. We thus need to maximize

$$u(t,c) + \varphi_t(t,x) + \varphi_x(t,x) x r - \varphi_x(t,x) c + \varphi_x(t,x) x \sum_{i=1}^{d} \pi^i(\underline{\mu}^i - r) 1_{\{\pi^i > 0\}}$$

$$+ \varphi_x(t,x) x \sum_{i=1}^{d} \pi^i(\overline{\mu}^i - r) 1_{\{\pi^i \leq 0\}} + \frac{1}{2} x^2 \varphi_{xx}(t,x) \sum_{i=1}^{d} (\pi^i)^2 (\overline{\sigma}^i)^2$$

over π and c.

The solution for the portfolio depends on the relation of the riskless interest rate with respect to the bounds for the drift. Let us write

$$a(t,x) = -\frac{x \varphi_{xx}(t,x)}{\varphi_x(t,x)}$$

for the agent's indirect relative risk aversion. The optimal portfolio choice anticipates the worst case scenarios. The investor evaluates the best long position and the best short position, and then make a choice of the better one. The optimal portfolio is

$$\hat{\pi}^i = \frac{\underline{\mu}^i - r}{a(t,x)(\overline{\sigma}^i)^2}, i = 1, \cdots, d,$$

if the lowest possible drift is above the riskless rate, $\underline{\mu}^i > r$, this is a long position. And the optimal portfolio choice is

$$\hat{\pi}^i = \frac{\overline{\mu}^i - r}{a(t,x)(\overline{\sigma}^i)^2}$$

if in contrast $\overline{\mu}^i < r$, and this is a short position. The important case is the middle one, when ambiguity allows for lower or higher drift than the interest rate; in this case, the optimal portfolio does not invest into the uncertain asset,
$$\hat{\pi}^i = 0 \,.$$

The long position is evaluated by the lowest premium, and the short position is evaluated by the highest premium. If the investor buys risky assets with the lowest premium, or sell the risky assets with highest premium, the the return of wealth is strictly lower than the riskless rate, which results in the nonparticipation of the risky asset market.

When we compare the formulas for optimal portfolios to the classic Merton solution, we come to the following conjecture: the investor behaves as if the lowest possible drift $\underline{\mu}^i$ was the real one if $\underline{\mu}^i > r$. If the interest rate belongs to the interval of possible drifts, then he behaves as if the drift was equal to the riskless rate (in this case, a standard risk-averse expected utility maximizer does not invest in the risky asset). Let us thus define the *worst case parameters* as follows: the worst case volatility is the highest possible volatility, $\sigma^* = [\overline{\sigma}^1, \cdots, \overline{\sigma}^d]$. The worst drift depends on the relation of the interest rate r with respect to $[\underline{\mu}, \overline{\mu}]$:

$$\hat{\mu}^i = \begin{cases} \underline{\mu}^i, & \text{if } \underline{\mu}^i > r; \\ r, & \text{if } \underline{\mu}^i \leq r \leq \overline{\mu}^i; \\ \overline{\mu}^i, & \text{else.} \end{cases}$$

We let $\mu^* = [\hat{\mu}^1, \cdots, \hat{\mu}^d]$. Let the the probability measure $P^* = P^{\mu^*, \sigma^*}$ the *worst case prior*. $\varphi(0, x_0)$ is the value function of an expected utility maximizer using the worst-case prior.

Theorem 3.2 *The ambiguity–averse investor chooses the same optimal policy as an expected utility-maximizer with worst case prior P^*. In particular, the value function φ of an expected utility maximizer with prior P^* solves the uncertain HJB equation (3.1),*

$$V(x_0) \;=\; \varphi(0, x_0) = \sup_{(\pi, c) \in \Pi \times C} U(c, X),$$

the optimal consumption rule is

$$\hat{c} = v(\varphi_x(t, x)),$$

where v is the inverse of u_c, and

(i) if $r \leq \inf_i \underline{\mu}^i$, then the optimal portfolio choice is $\hat{\mu}^i = \underline{\mu}^i, i = 1, \cdots, d$, and

$$\hat{\pi}^i = -\frac{\varphi_x(t, x)}{\varphi_{xx}(t, x)x} \frac{\underline{\mu}^i - r}{(\overline{\sigma}^i)^2}.$$

(ii) if $\sup_i \overline{\mu}^i \leq r$, then the optimal portfolio choice is $\hat{\mu}^i = \overline{\mu}^i, i = 1, \cdots, d$, and

$$\hat{\pi}^i = -\frac{\varphi_x(t, x)}{\varphi_{xx}(t, x)x} \frac{\overline{\mu}^i - r}{(\overline{\sigma}^i)^2}.$$

(iii) if $\inf_i \underline{\mu}^i < r < \sup_i \overline{\mu}^i$, then the optimal portfolio choice is

$$\hat{\pi}^i = -\frac{\varphi_x(t, x)}{\varphi_{xx}(t, x)x} \frac{\underline{\mu}^i - r}{(\overline{\sigma}^i)^2} 1_{\{r \leq \underline{\mu}^i\}} - \frac{\varphi_x(t, x)}{\varphi_{xx}(t, x)x} \frac{\overline{\mu}^i - r}{(\overline{\sigma}^i)^2} 1_{\{\overline{\mu}^i \leq r\}}.$$

The previous theorem allows to draw several interesting conclusions. First of all, as we have identified a worst–case measure, we have proved a minmax theorem.

Corollary 3.3 *We have the following minmax theorem: Let $\phi(P, x)$ denote the value function of an expected utility maximizer with belief P and initial capital x. Let $v(x)$ be the ambiguity–averse investor's indirect utility function. Then*

$$v(x) = \min_{P \in \mathcal{P}} \phi(P, x)$$

or

$$\max_{(\pi, c)} \min_{P \in \mathcal{P}} E^P[\int_0^T u(s, c_s)ds + \Phi(T, X_T)] = \min_{P \in \mathcal{P}} \max_{(\pi, c)} E^P[\int_0^T u(s, c_s)ds + \Phi(T, X_T)].$$

The fact that the ambiguity–averse investor behaves as an expected utility maximizer under the worst–case measure P^* does not imply that their demand functions are indistinguishable. Note that the worst–case measure is frequently the one where the drift is equal to the interest rate. Under such a belief, the expected utility maximizer does not invest at all into the risky asset, a result that is by now well established.

3.3 Explicit solution for CRRA Utility

In this subsection, we give give explicit solutions for Constant Relative Risk Aversion (CRRA) Utility, i.e.,

$$u(t,c) = \frac{c^{1-\alpha}}{1-\alpha}, \Phi(T,x) = \frac{Kx^{1-\alpha}}{1-\alpha}, \alpha \neq 1.$$

We have the following results. For the proof, see the appendix.

Proposition 3.4 (i) If $r \leq \inf_i \underline{\mu}^i$, then the optimal consumption and portfolio rules are given by the following

$$\hat{c} = \left[K^{\alpha^{-1}} e^{\beta \alpha^{-1}(T-t)} + \alpha \beta^{-1}(e^{\beta \alpha^{-1}(T-t)} - 1) \right]^{-1} x,$$

and

$$\hat{\pi}^i = \frac{1}{\alpha} \frac{\underline{\mu}^i - r}{(\overline{\sigma}^i)^2}, \ i = 1, \cdots, d,$$

where

$$\beta = \left[r + \sum_{i=1}^d \frac{(\underline{\mu}^i - r)^2}{2\alpha(\overline{\sigma}^i)^2} \right](1 - \alpha).$$

(ii) if $\sup_i \overline{\mu}^i \leq r$, then the optimal consumption and portfolio rules are given by the following

$$\hat{c} = \left[K^{\alpha^{-1}} e^{\beta \alpha^{-1}(T-t)} + \alpha \beta^{-1}(e^{\beta \alpha^{-1}(T-t)} - 1) \right]^{-1} x,$$

and

$$\hat{\pi}^i = \frac{1}{\alpha} \frac{\overline{\mu}^i - r}{(\overline{\sigma}^i)^2}, \ i = 1, \cdots, d,$$

where

$$\beta = \left[r + \sum_{i=1}^d \frac{(\overline{\mu}^i - r)^2}{2\alpha(\overline{\sigma}^i)^2} \right](1 - \alpha).$$

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(iii) if $\inf_i \underline{\mu}^i < r < \sup_i \overline{\mu}^i$, *then the optimal consumption and portfolio rules are given by the following*

$$\hat{c} = \left[K^{\alpha^{-1}} e^{\beta \alpha^{-1}(T-t)} + \alpha \beta^{-1} (e^{\beta \alpha^{-1}(T-t)} - 1) \right]^{-1} x,$$

and

$$\hat{\pi}^i = \frac{1}{\alpha} \frac{\underline{\mu}^i - r}{(\overline{\sigma}^i)^2} 1_{\{r \leq \underline{\mu}^i\}} + \frac{1}{\alpha} \frac{\overline{\mu}^i - r}{(\overline{\sigma}^i)^2} 1_{\{\overline{\mu}^i \leq r\}},$$

where

$$\beta = \left[r + \sum_{i=1}^{d} \frac{(\underline{\mu}^i - r)^2}{2\alpha(\overline{\sigma}^i)^2} \right] (1 - \alpha) 1_{\{r \leq \underline{\mu}^i\}} + \left[r + \sum_{i=1}^{d} \frac{(\overline{\mu}^i - r)^2}{2\alpha(\overline{\sigma}^i)^2} \right] (1 - \alpha) 1_{\{\overline{\mu}^i \leq r\}}.$$

In particular, if $\underline{\mu}^i < r < \overline{\mu}^i$, *then the optimal portfolio choice is*

$$\hat{\pi}^i = 0.$$

3.4 Comparative Statics

From the above results in the above subsection we obtain immediately following comparative statics.

Proposition 3.5 *Let drift ambiguity be given by* $[\mu_0 - \kappa, \mu_0 + \kappa]$ *for some* $\kappa > 0$. *As* κ *increases, asset holdings decrease. After some critical level of ambiguity* κ^*, *refrains from trading the asset altogether.*

We have here the well–known phenomenon that high uncertainty about mean returns keeps ambiguity–averse investors away form the asset market, as Dow and Werlang (1992) first pointed out.

Proposition 3.6 *The exposure of investors decreases with ambiguity: for parameter sets* $\Theta \subset \hat{\Theta}$, *let* π *and* $\hat{\pi}$ *denote the optimal portfolio choices. Then* $\|\pi\| \geq \|\hat{\pi}\|$.

This result follows from the fact that the investor always works with the maximal volatility. If the asset is profitable, he uses the minimal mean excess return, and if she is going short, she uses the maximal mean return in computing the portfolio. The absolute amount of assets held optimally thus decreases with ambiguity.

4 Interest rate uncertainty

A fixed and known interest rate, or in other words, a flat term structure, is not a reasonable assumption for long-term investors. Investors face considerable uncertainty about the short rate; on the one hand, stochastic demand for credit leads to short term variability, on the other hand, the short rate is partially determined by central bank policies. The latter are known to be quite ambiguous, and sometimes deliberately so as central bankers have strong incentives to conceal their real objectives. We thus also allow for interest rate uncertainty.

Introducing Knightain uncertainty about the short rate *requires* the use of singular measures: if we model the possibility that the bond dynamics satisfy

$$dP_t = r_t P_t dt$$

under one measure, and

$$dP_t = \hat{r}_t P_t dt$$

under another, for a different short rate $\hat{r}$, then these measures need to be singular to each other. This is in contrast to the well–studied models of drift ambiguity for the uncertain asset where the presence of the noise term allows to work with equivalent probability measures.

But we are already used to work with singular measures, and so our framework can be extended to cover such Knightian uncertainty about the short rate as well. We model the ambiguity about the short rate via an interval $[\underline{r}, \overline{r}]$, similar to the ambiguity about drift and volatility.

Interest rate uncertainty leads to some new phenomena. The most interesting case arises, in our view, when the uncertain asset is profitable, but interest rate uncertainty is high. Then it is optimal not to participate in the market for bonds at all and to put all capital into the uncertain asset. We thus obtain non–participation in the credit market; a phenomenon that we have seen during the financial crisis as well. Of course, we do not model or explain the origin of such uncertainty here, but we show that interest rate uncertainty can play an important role in asset decisions.

Let us now come to the formal model. In a first step, as in Section 2, we construct a set of priors. For $\theta = (\mu, \sigma)$ and r, which are an $\mathcal{F}$-progressively measurable processes with values in $\Theta = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$ and $[\underline{r}, \overline{r}]$, respectively, we consider stochastic differential equation

$$dX_t^\theta = \mu_t dt + \sigma_t dB_t, X_0 = 0,$$

and

$$dY_t^r = r_t dt, \ Y_0 = 0,$$

under our reference measure P_0.

We let $P^{r,\theta}$ be the distribution of (X^θ, Y^r), i.e.

$$P^{r,\theta}(A) = P_0((X^\theta, Y^r) \in A),$$

for all $A \in \mathcal{F}_T$.

Let $\mathcal{P}_0$ be consist of all probability measures $P^{r,\theta}$ constructed in this way. Our set of priors $\mathcal{P}$ is the closure of $\mathcal{P}_0$ under the topology of weak convergence. The set of priors leads naturally to a sublinear expectation:

$$\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot].$$

The sublinear function $G : \mathbb{R}^3 \times \to \mathbb{R} :$

$$G(p_1, p_2, p_3) = \sup_{(r,\mu,\sigma) \in [\underline{r},\overline{r}] \times [\underline{\mu},\overline{\mu}] \times [\underline{\sigma},\overline{\sigma}]} \left\{ rp_1 + \mu p_2 + \frac{1}{2}\sigma^2 p_3 \right\}.$$

As introduced in the Section 2, let $(B, b, \hat{b})$ be a pair of random vectors under $\hat{\mathbb{E}}$ such that B is a G-Brownian motion and $(b, \hat{b})$ is a G-distributed process, which has mean uncertainty. B is a G-normal distributed process, which just has volatility uncertainty.

In a financial market, we consider the optimal consumption and portfolio choice, not only with ambiguity about returns and volatility, but also with interest rate uncertainty. The price of the riskless asset is now defined by

$$dP_t = P_t d\hat{b}_t,$$

where $\hat{b}$ is a G-distributed process, and $\hat{b}_t$ has mean uncertainty $[\underline{r}t, \overline{r}t]$.

In this section, we just consider one risky asset in the financial market. The classical "risky" assets where we assume that expected return and volatility are unknown. The uncertain asset prices evolve as

$$dS_t = S_t dR_t$$

and we model the return dynamics via

$$dR_t = db_t + dB_t.$$

As we did in Section 2, we can define the consumption and trading opportunities, and utility in the same way.

We consider the case corresponding to the Constant Relative Risk Aversion (CRRA) Utility, i.e.,

$$u(t, c) = \frac{c^{1-\alpha}}{1-\alpha}, \Phi(T, x) = \frac{Kx^{1-\alpha}}{1-\alpha}, \alpha > 0, \alpha \neq 1.$$

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The following three potentially optimal portfolio shares play a role as candidate optimal policies in the following. First,

$$\pi^1 = \frac{\overline{\mu} - r}{\alpha \overline{\sigma}^2}$$

corresponds to the case when maximal drift and maximal interest rate are the worst case parameters. Similarly, we define

$$\pi^2 = \frac{\underline{\mu} - r}{\alpha \overline{\sigma}^2}$$

and

$$\pi^3 = \frac{\underline{\mu} - \overline{r}}{\alpha \overline{\sigma}^2} \, .$$

Note that

$$\pi^1 \geq \pi^2 \geq \pi^3.$$

We have the following main result in this section.

Theorem 4.1 *The optimal consumption–investment policies under interest rate uncertainty can be divided into five cases:*

1. *If $\pi^1 \geq 0$,*

    (a) *and $\pi^2 \leq 0$, non–participation in the asset market, or $\pi^* = 0$, is optimal,*

    (b) *and $0 < \pi^2 < 1$, the investor goes long and saves, i.e. $\pi^* = \pi^2$,*

    (c) *and for $\pi^2 \geq 1$, we have*

        i. *in case $\pi^3 < 1$, the investor puts all capital in the uncertain asset and does not participate in the credit market, or $\pi^* = 1$,*

        ii. *in case $\pi^3 \geq 1$, the investor goes long and borrows (leveraged consumption) and $\pi^* = \pi^3$.*

2. *if $\pi^1 < 0$, the investor goes short and saves (leveraged consumption) and $\pi^* = \pi^1$.*

For the above theorem, see the following figures.
If $\underline{\mu} \leq \underline{r}$, then

$$\underline{\mu} \xrightarrow{\quad \pi^* = 0 \quad} \overline{\mu} \xrightarrow[\underline{r}]{\quad \pi^* = \frac{\overline{\mu} - r}{\alpha \overline{\sigma}^2} \quad} \, .$$

If $\underline{\mu} \geq \underline{r}$, then

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$$\underline{r} - \alpha\overline{\sigma}^2 \xrightarrow{\ \pi^* = \frac{\mu - r}{\alpha\overline{\sigma}^2}\ } \underline{r} \xrightarrow{\ \pi^* = 1\ } \overline{r} \xrightarrow[\underline{\mu} - \alpha\overline{\sigma}^2]{\ \pi^* = \frac{\mu - \overline{r}}{\alpha\overline{\sigma}^2}\ } \ .$$

The details of the proof are in the appendix. Our model allows to quantify under what conditions non-participation in the credit market is optimal for ambiguity-averse investors. This occurs when the highest possible drift exceeds the highest possible interest rate, and investment in the uncertain asset is thus potentially profitable, but when the lowest possible drift, adjusted by a mean-variance term involving risk aversion, $\underline{\mu} - \alpha\overline{\sigma}^2$, belongs to the interval of possible interest rates $[\underline{r}, \overline{r}]$.

Appendix

A G-Brownian motion

Peng (2007) introduced the theory of G-Brownian motion. For the convenience of the readers, we recall the basic definitions and some results of the theory of G-Brownian motion.

Let Ω be a given nonempty set and $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $x_1, \cdots, x_n \in \mathcal{H}$, then $\varphi(x_1, \cdots, x_n) \in \mathcal{H}$, for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$. Here $C_{l,lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|, \text{ for all } x, y \in \mathbb{R}^n,$$

for some $C > 0$ and $n \in \mathbb{N}$, both depending on φ. The space $\mathcal{H}$ is considered as a set of random variables.

Definition A.1 A sublinear expectation $\hat{\mathbb{E}}$ *on* $\mathcal{H}$ *is a functional* $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ *satisfying the following properties: for all* $X, Y \in \mathcal{H}$, *we have*

(i) **Monotonicity:** *If* $X \geq Y$, *then* $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

(ii) **Preservation of constants:** $\hat{\mathbb{E}}[c] = c$, *for all* $c \in \mathbb{R}$.

(iii) **Subadditivity:** $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$.

(iv) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, *for all* $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a sublinear expectation space.*

Remark A.2 *The sublinear expectation space can be regarded as a generalization of the classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the linear expectation associated with $\mathbb{P}$.*

Definition A.3 *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n), Y_i \in \mathcal{H}$, is said to be independent under $\hat{\mathbb{E}}$ of another random vector $X = (X_1, \cdots, X_m), X_i \in \mathcal{H}$, denoted by $X \perp Y$, if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^{m+n})$ we have*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Definition A.4 *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, X and Y are called identically distributed, and denoted by $X \overset{d}{=} Y$, if for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$ we have*

$$\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)].$$

Definition A.5 (G-distribution) *A pair of random variables (X, η) in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-distributed, if for all $a, b \geq 0$,*

$$(aX + b\bar{X}, a^2\eta + b^2\bar{\eta}) \overset{d}{=} \sqrt{a^2 X + b^2}\bar{X} + (a^2 + b^2)\eta,$$

where $(\bar{X}, \bar{\eta})$ is an independent copy of (X, η), i.e., $(\bar{X}, \bar{\eta}) \overset{d}{=} (X, \eta)$ and $(\bar{X}, \bar{\eta}) \perp (X, \eta)$.

If (X, η) is d dimensional G-distributed, for $\varphi \in C_{l,lip}(\mathbb{R}^d)$, let us define

$$u(t, x, y) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\xi, y + t\eta)], \quad (t, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

is the solution of the following parabolic partial differential equation:

$$\begin{cases} \partial_t u(t, x) = G(D_y u(t, x, y), D^2_{xx} u(t, x, y)), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \varphi(x). \end{cases}$$

Here G is the following sublinear function:

$$G(p, A) = \hat{\mathbb{E}}[\frac{1}{2}\langle AX, X \rangle + \langle p, \eta \rangle], \quad (p, A) \in \mathbb{R}^d \times \mathbb{S}_d,$$

where $\mathbb{S}_d$ is the collection of $d \times d$ symmetric matrices. There exists a bounded and closed subset Θ of $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ such that

$$G(p, A) = \sup_{(q, Q) \in \Theta} \left\{ \langle p, q \rangle + \frac{1}{2} tr(AQQ^T) \right\}, \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}_d.$$

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Let $\Omega = C_0^{2d}(\mathbb{R}^+)$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i}\left[(\max_{t \in [0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1\right], \ \omega^1, \omega^2 \in \Omega.$$

For each $t \in [0, +\infty)$, we set $\omega_t = \{\omega_{\cdot \wedge t}, \omega \in \Omega\}$. We will consider the canonical process $\hat{B}_t(\omega) = (B_t, b_t)(\omega) = \omega_t, t \in [0, +\infty), \omega \in \Omega$.

For each $T > 0$, we consider the following space of random variables:

$$L_{ip}(\Omega_T) := \Big\{ \varphi(\omega_{t_1} \cdots, \omega_{t_m}) \mid t_1, \cdots, t_m \in [0, T],$$
$$\varphi \in C_{l,lip}(\mathbb{R}^{d \times m}), \ m \geq 1 \Big\}.$$

Obviously, it holds that $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$, for all $t \leq T < \infty$. We further define

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

For each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(\hat{B}_{t_1} - \hat{B}_{t_0}, \hat{B}_{t_2} - \hat{B}_{t_1}, \cdots, \hat{B}_{t_m} - \hat{B}_{t_{m-1}})$$

for some $m \geq 1, \varphi \in C_{l,lip}(\mathbb{R}^{2d \times m})$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_m < \infty$, we set

$$\hat{\mathbb{E}}[\varphi(\hat{B}_{t_1} - \hat{B}_{t_0}, \hat{B}_{t_2} - \hat{B}_{t_1}, \cdots, \hat{B}_{t_m} - \hat{B}_{t_{m-1}})]$$
$$= \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, (t_1 - t_0)\eta_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m, (t_m - t_{m-1})\eta_m)],$$

where $\{(\xi_1, \eta_1), \cdots, (\xi_m, \eta_m)\}$ is a random vector in a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ such that (ξ_i, η_i) is G-distributed (ξ_{i+1}, η_{i+1}) is independent of $\{(\xi_1, \eta_1), \cdots, (\xi_i, \eta_i)\}$, for every $i = 1, 2, \cdots, m - 1$.

The related conditional expectation of $X = \varphi(\hat{B}_{t_1} - \hat{B}_{t_0}, \hat{B}_{t_2} - \hat{B}_{t_1}, \cdots, \hat{B}_{t_m} - \hat{B}_{t_{m-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X | \Omega_{t_j}] = \hat{\mathbb{E}}[\varphi(\hat{B}_{t_1} - \hat{B}_{t_0}, \hat{B}_{t_2} - \hat{B}_{t_1}, \cdots, \hat{B}_{t_m} - \hat{B}_{t_{m-1}}) | \Omega_{t_j}]$$
$$= \psi(\hat{B}_{t_1} - \hat{B}_{t_0}, \hat{B}_{t_2} - \hat{B}_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x_1, x_2, \cdots, x_j)$$
$$= \widetilde{\mathbb{E}}[\varphi(x_1, x_2, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, (t_{j+1} - t_j)\eta_{j+1} \cdots,$$
$$\sqrt{t_m - t_{m-1}}\xi_m, (t_m - t_{m-1})\eta_m)],$$

for $(x_1, x_2, \cdots, x_j) \in \mathbb{R}^j, 0 \le j \le m$.

For $p \ge 1$, $\|X\|_p = \hat{\mathbb{E}}^{\frac{1}{p}}[|X|^p]$, $X \in L_{ip}(\Omega)$, defines a norm on $L_{ip}^0(\Omega)$. Let $L^p(\Omega)$ (resp. $L^p(\Omega_t)$) be the completion of $L_{ip}(\Omega)$ (resp. $L_{ip}(\Omega_t)$) under the norm $\| \cdot \|_p$. Then the space $(L^p(\Omega), \| \cdot \|_p)$ is a Banach space and the operators $\hat{\mathbb{E}}[\cdot]$ (resp. $\hat{\mathbb{E}}[\cdot|\Omega_t]$) can be continuously extended to the Banach space $L^p(\Omega)$ (resp. $L^p(\Omega_t)$). Moreover, we have $L^p(\Omega_t) \subseteq L^p(\Omega_T) \subset L^p(\Omega)$, for all $0 \le t \le T < \infty$.

Definition A.6 (G-normal distribution) *Let Γ be a given non-empty, bounded and closed subset of $\mathbb{R}^{d \times d}$. A random vector ξ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is said to be G-normal distributed, denoted by $\xi \sim \mathcal{N}(0, \Gamma)$, if for each $\varphi \in C_{l,lip}(\mathbb{R}^d)$, the following function defined by*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,$$

is the unique viscosity solution of the following parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = G(D^2 u), & (t, x) \in [0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \varphi(x). \end{cases} \tag{A.1}$$

Here $D^2 u$ is the Hessian matrix of u, i.e., $D^2 u = (\partial^2_{x^i x^j} u)^d_{i,j=1}$, and

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} tr(\gamma \gamma^T A), \quad A \in \mathbb{S}_d.$$

Example A.7 *In one dimensional case, i.e., $d = 1$, we take $\Gamma = [\underline{\sigma}^2, \overline{\sigma}^2]$, where $\underline{\sigma}$ and $\overline{\sigma}$ are constants with $0 \le \underline{\sigma} \le \overline{\sigma}$. Then equation (A.1) has the following form*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}[\overline{\sigma}^2 (\partial^2_{xx} u)^+ - \underline{\sigma}^2 (\partial^2_{xx} u)^-], & (t, x) \in [0, \infty) \times \mathbb{R}, \\ u(0, x) = \varphi(x). \end{cases}$$

If $\underline{\sigma} = \overline{\sigma}$, the G-normal distribution is the classical normal distribution.

Example A.8 *In multidimensional case, we consider one typical case when*

$$\Gamma = \left\{ diag[\gamma^1, \cdots, \gamma^d], \gamma^i \in [(\underline{\sigma}^i)^2, (\overline{\sigma}^i)^2], i = 1, \cdots, d \right\},$$

where $\underline{\sigma}^i$ and $\overline{\sigma}^i$ are constants with $0 \le \underline{\sigma}^i \le \overline{\sigma}^i$. Then equation (A.1) has the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^d [(\overline{\sigma}^i)^2 (\partial^2_{x^i x^i} u)^+ - (\underline{\sigma}^i)^2 (\partial^2_{x^i x^i} u)^-], \\ u(0, x) = \varphi(x). \end{cases}$$

Definition A.9 *A process* $B = \{B_t, t \geq 0\}$ *in a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a G-Brownian motion, if the following properties are satisfied:*

(i) $B_0 = 0$;

(ii) *for each* $t, s \geq 0$, *the difference* $B_{t+s} - B_t$ *is* $\mathcal{N}(0, \Gamma s)$-*distributed and is independent of* $(B_{t_1}, \cdots, B_{t_n})$, *for all* $n \in \mathbb{N}$ *and* $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

Definition A.10 (Maximal distribution) *Let* Λ *be a given non-empty, bounded and closed subset of* $\mathbb{R}^d$. *A random vector* ξ *in a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is said to be Maximal distributed, denoted by* $\xi \sim \mathcal{N}(\Lambda, \{0\})$, *if for each* $\varphi \in C_{l,lip}(\mathbb{R}^d)$, *the following function defined by*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + t\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,$$

is the unique viscosity solution of the following parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = g(Du), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \varphi(x), \end{cases}$$

where $Du = (\partial_{x^i} u)_{i=1}^d$, *and*

$$g(p) = \frac{1}{2} \sup_{q \in \Lambda} \langle p, q \rangle, \quad p \in \mathbb{R}^d.$$

Proposition A.11 *If* $b_t \sim \mathcal{N}([\underline{\mu} t, \overline{\mu} t], \{0\})$, *where* $\underline{\mu}$ *and* $\overline{\mu}$ *are constants with* $\underline{\mu} \leq \overline{\mu}$, *then for* $\varphi \in C_{l,lip}(\mathbb{R})$

$$\hat{\mathbb{E}}[\varphi(b_t)] = \sup_{v \in [\underline{\mu} t, \overline{\mu} t]} \varphi(vt).$$

Proposition A.12 (Itô's formula) *Let* $b_t \sim \mathcal{N}([\underline{\mu} t, \overline{\mu} t], \{0\})$ *and* $B_t \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$ *where* $\underline{\mu}$ *and* $\overline{\mu}$ *are constants with* $\underline{\mu} \leq \overline{\mu}$, *and* $\underline{\sigma}$ *and* $\overline{\sigma}$ *are constants with* $\underline{\sigma} \leq \overline{\sigma}$. *Then for* $\varphi \in C^2(\mathbb{R})$ *and*

$$X_t = X_0 + \int_0^t \alpha_s db_s + \int_0^t \beta_s dB_s, \text{ for all } t \in [0, T],$$

where α *in* M^1 *and* $\beta \in M^2$, *we have*

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \partial_x \varphi(X_u) \beta_u dB_u + \int_0^t \partial_x \varphi(X_u) \alpha_u db_u$$
$$+ \int_0^t \frac{1}{2} \partial_{xx}^2 \varphi(X_u) \beta_u^2 d\langle B \rangle_u, \quad 0 \leq t \leq T.$$

B Construction of the Set of Priors

In an ambiguous world, the investor is uncertain about the law that governs the price dynamics of risky assets. We thus do not fix a probability measure ex ante. We set up the canonical model in such a continuous–time Knightian setting first.

As we want to study the standard Samuelson–Merton consumption–portfolio problem in the Knightian case, we look at asset prices with continuous sample paths. We let $C([0,T])$ be the set of all continuous paths with values in $\mathbb{R}^d$ over the finite time horizon $[0,T]$ endowed with the sup norm. Our state space is

$$\Omega_0 = \left\{ \omega : \omega \in C([0,T]), \omega_0 = 0 \right\}.$$

The coordinate process $B = (B_t)_{t \geq 0}$ is $B_t(\omega) = \omega_t$.

As in the classic case, the coordinate process $B_t(\omega) = \omega(t)$ will play the role of noise, but here it will be uncertain, rather than probabilistic noise; ambiguous, or, as Peng calls it, G–Brownian motion. In order to model such ambiguous Brownian motion, we construct a set of priors. We take as a starting point the classic Wiener measure P_0 under which B is a standard Brownian motion. Note that P_0 does not reflect the investor's view of the world; it merely plays the role of a construction tool for the set of priors.

Let $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ denote the filtration generated by B, completed by all P_0-null sets.

In the continuous–time diffusion framework, essentially two parameter processes describe all uncertainty, drift and volatility. We thus model ambiguity with the help of a convex and compact subset $\Theta \subset \mathbb{R}^d \times \mathbb{R}^{d \times d}$. The investor is not sure about the exact value or distribution of the drift process $\mu = (\mu_t)$ with values in $\mathbb{R}^d$ nor about the exact value or distribution of the volatility process $\sigma = (\sigma_t)$ with values in $\mathbb{R}^{d \times d}$.

For every "hypothesis" $\theta = (\mu, \sigma)$, an $\mathcal{F}$-progressively measurable process with values in Θ, the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dB_t, X_0 = 0$$

has a unique solution X^θ under our reference measure P_0. We let P^θ be the distribution of X^θ, i.e.

$$P^\theta(A) = P_0(X^\theta \in A)$$

for all $A \in \mathcal{F}_T$.

Let $\mathcal{P}_0$ be consist of all probability measures P^θ constructed in this way. Our set of priors $\mathcal{P}$ is the closure of $\mathcal{P}_0$ under the topology of weak convergence. Ambiguous volatility gives rise to nonequivalent priors. For example,

let $P^{\overline{\sigma}}$ and $P^{\underline{\sigma}}$ be the distribution of the processes $(\overline{\sigma}B_t)_{t\geq 0}$ and $(\underline{\sigma}B_t)_{t\geq 0}$, respectively. Then $P^{\overline{\sigma}}$ and $P^{\underline{\sigma}}$ are mutually singular, i.e.,

$$P^{\overline{\sigma}}(\langle B\rangle_T = \overline{\sigma}^2 T) = P^{\underline{\sigma}}(\langle B\rangle_T = \underline{\sigma}^2 T) = 1,$$

where the quadratic variation process of B is defined as follows, for $0 = t_1 \leq \cdots < t_m = T$ and $\Delta t_k = t_{k+1} - t_k$,

$$\langle B\rangle_T = \lim_{\Delta t_k \to 0} \sum_{k=1}^{m-1} |B_{t_{k+1}} - B_{t_k}|^2.$$

The preceding construction is the canonical continuous-time model for a world in which investors face ambiguity about drift and volatility.

The set of priors leads naturally to a sublinear expectation:

$$\hat{\mathbb{E}}[\cdot] = \sup_{P\in\mathcal{P}} E_P[\cdot].$$

One advantage of our continuous-time uncertainty model is the fact that one can describe uncertainty by a quadratic real function. The sublinear function $G : \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$:

$$G(p, A) = \sup_{(q,Q)\in\Theta} \left\{\langle p, q\rangle + \frac{1}{2}tr(AQQ^T)\right\}, \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}_d,$$

where $\mathbb{S}_d$ is the collection of $d \times d$ symmetric matrices, will describe locally, at the level of parameters, uncertainty of drift and volatility in our model.

Let (B, b) be a pair of random vectors under $\hat{\mathbb{E}}$ such that B is a G-Brownian motion and b is a G-distributed process, which just has mean uncertainty. B is a G-normal distributed process, which just has volatility uncertainty. To see this, we consider $d = 1$ and $\Theta = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$. Then the process b has mean uncertainty with parameters $[\underline{\mu}, \overline{\mu}]$, i.e.,

$$\hat{\mathbb{E}}[b_t] = \overline{\mu}t, \text{ and } -\hat{\mathbb{E}}[-b_t] = \underline{\mu}t.$$

And the process B does not have mean uncertainty, i.e.,

$$\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0,$$

but has volatility uncertainty with parameters $[\underline{\sigma}^2, \overline{\sigma}^2]$, i.e.,

$$\hat{\mathbb{E}}[B_t^2] = \overline{\sigma}^2 t, \text{ and } -\hat{\mathbb{E}}[-B_t^2] = \underline{\sigma}^2 t.$$

The preceding construction is the canonical model for a world in which investors face ambiguity about drift and volatility, but do know certain bounds on these processes.

B.1 The Canonical Model

While the abstract characterization of optimal policies holds in a very general setting, we will frequently focus on the special case where ambiguity about drift is independent of ambiguity about volatility of the individual asset returns.

For given constants $\underline{\mu}^i \leq \overline{\mu}^i, \underline{\sigma}^i \leq \overline{\sigma}^i, i = 1, \cdots, d$, we consider

$$[\underline{\mu}, \overline{\mu}] = \left\{ [\mu^1, \cdots, \mu^d]^T, \mu^i \in [\underline{\mu}^i, \overline{\mu}^i], i = 1, \cdots, d \right\},$$

and

$$\Gamma = \left\{ diag[\gamma^1, \cdots, \gamma^d], \gamma^i \in [\underline{\sigma}^i, \overline{\sigma}^i], i = 1, \cdots, d \right\}.$$

In order to give an explicit solution, we consider a special case of $\Theta = [\underline{\mu}, \overline{\mu}] \times \Gamma$, and $A = diag\{1, \cdots, 1\}$. We call this the *canonical* model.

C   Proofs

C.1   Properties of $V(x_0)$

Proposition C.1 $V(x_0)$ *is increasing and concave in x.*

Proof. Just for the proof of this proposition, we denote by

$$J(\pi, c, x_0) = \underline{\mathbb{E}}[\int_0^T u(s, c_s)ds + \Phi(T, X_T)],$$

Also we denote the solution of (2.2) by X^{x_0}. For any arbitrary $0 < x \leq y$, by the Comparison theorem of stochastic differential equations driven by G-Brownian motion we have $X^x \leq X^y$. Since the utility function u and the bequest function Φ are strictly increasing, by the the monotonicity of $\underline{\mathbb{E}}$-expectation, we know that J is increasing in x. Therefore, V is increasing in x.

For any arbitrary $0 < x_1, x_2$ and $\lambda \in [0, 1]$, we denote by $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$. For any $c^1, c^2 \in C$ and $\pi^1, \pi^2 \in \Pi$, we consider

$$\begin{cases} dX_t^1 = rX_t^1(1 - (\pi_t^1)^T 1)dt + X_t^1(\pi_t^1)^T db_t - c_t^1 dt + X_t^1(\pi_t^1)^T A dB_t, \\ X_0^1 = x_1, \end{cases}$$

and

$$\begin{cases} dX_t^2 = rX_t^2(1 - (\pi_t^2)^T 1)dt + X_t^2(\pi_t^2)^T db_t - c_t^2 dt + X_t^2(\pi_t^2)^T A dB_t, \\ X_0^2 = x_2, \end{cases}$$

We denote by $X^\lambda := \lambda X^1 + (1-\lambda)X^2$, $c^\lambda := \lambda c^1 + (1-\lambda)c^2$ and

$$\pi^\lambda := \frac{\lambda \pi^1 X^1 + (1-\lambda)\pi^2 X^2}{\lambda X^1 + (1-\lambda)X^2}.$$

Then $c^\lambda \in C$, $\pi^\lambda \in \Pi$ and X^λ satisfies the following

$$\begin{cases} dX_t^\lambda = rX_t^\lambda(1-(\pi_t^\lambda)^T 1)dt + X_t^\lambda(\pi_t^\lambda)^T db_t - c_t^\lambda dt + X_t^\lambda(\pi_t^\lambda)^T A dB_t, \\ X_0^\lambda = x_\lambda, \end{cases}$$

Since the functions u and Φ are concave with respect to c and X, respectively, we have

$$\Phi(T, X_T^\lambda) \geq \lambda \Phi(T, X_T^1) + (1-\lambda)\Phi(T, X_T^2).$$

and

$$u(s, X_s^\lambda) \geq \lambda u(s, X_s^1) + (1-\lambda)u(s, X_s^2), s \in [0, T].$$

Therefore, by virtue of the positive homogeneity and subadditivity of $\hat{\mathbb{E}}$, the following holds

$$\underline{\mathbb{E}}[\int_0^T u(s, c_s^\lambda)ds + \Phi(T, X_T^\lambda)]$$
$$\geq \quad \underline{\mathbb{E}}[\int_0^T u(s, c_s^1)ds + \Phi(T, X_T^1)] + (1-\lambda)\underline{\mathbb{E}}[\int_0^T u(s, c_s^2)ds + \Phi(T, X_T^2)],$$

i.e.,

$$J(\pi^\lambda, c^\lambda, x^\lambda) \geq J(\pi^1, c^1, x_1) + J(\pi^2, c^2, x_2).$$

Consequently,

$$V(x^\lambda) \geq J(\pi^1, c^1, x_1) + J(\pi^2, c^2, x_2).$$

Since the above holds true for any $c^1, c^2 \in C$ and $\pi^1, \pi^2 \in \Pi$, it follows that

$$V(x^\lambda) \geq V(x_1) + V(x_2).$$

This means that V is concave in x. $\qquad \square$

C.2 Proof of Theorem 3.1

Proof.   For any arbitrary $(\pi, c) \in \Pi \times C$ we let X be a solution of equation (2.2) associated with (π, c). From Itô formula it follows that

$$
\begin{aligned}
\varphi(T, X_T) \;=\; & \int_0^T [\varphi_t(t, X_t) + rX_t\varphi_x(t, X_t)(1 - \pi_t^T 1) - \varphi_x(t, X_t)c_t]dt \\
& + \int_0^T \varphi_x(t, X_t)X_t\pi_t^T db_t + \int_0^T \varphi_x(t, X_t)X_t\pi_t^T AdB_t \\
& + \int_0^T \frac{1}{2}\varphi_{xx}(t, X_t)X_t^2\langle A^T\pi_t\pi_t^T A, d < B >_t\rangle + \varphi(0, x_0).
\end{aligned}
$$

Since for all $x \in \mathbb{R}$, $\varphi(T, x) = \Phi(T, x)$. Then taking expectation yields

$$
\begin{aligned}
& \mathbb{E}[\Phi(T, X_T) + \int_0^T u(t, c_t)dt] \\
= \; & \mathbb{E}[\int_0^T [\varphi_t(t, X_t) + rX_t\varphi_x(t, X_t)(1 - \pi_t^T 1) - \varphi_x(t, X_t)c_t]dt \\
& + \int_0^T \varphi_x(t, X_t)X_t\pi_t^T db_t + \int_0^T \frac{1}{2}\varphi_{xx}(t, X_t)X_t^2\langle \pi_t\pi_t^T, d < B >_t\rangle \\
& + \int_0^T U(t, c_t)dt\Big] + \varphi(0, x_0) \\
= \; & \mathbb{E}\Big[\int_0^T [\varphi_t(t, X_t) + rX_t\varphi_x(t, X_t)(1 - \pi_t^T 1) - \varphi_x(t, X_t)c_t]dt + \int_0^T u(t, c_t)dt \\
& - \int_0^T G(-\varphi_x(t, X_t)X_t\pi_t, -X_t^2\varphi_{xx}(t, X_t)A^T\pi_t\pi_t^T A)dt \\
& + \int_0^T \varphi_x(t, X_t)X_t\pi_t^T db_t + \int_0^T \frac{1}{2}\varphi_{xx}(t, X_t)X_t^2\langle \pi_t\pi_t^T, d < B >_t\rangle \\
& + \int_0^T G(-\varphi_x(t, X_t)X_t\pi_t, -X_t^2\varphi_{xx}(t, X_t)A^T\pi_t\pi_t^T A)dt\Big] + \varphi(0, x_0).
\end{aligned}
$$

By virtue of equation (3.2), we obtain that

$$
\begin{aligned}
& \mathbb{E}[\Phi(T, X_T) + \int_0^T u(t, c_t)dt] \\
\leq \; & \mathbb{E}\Big[\int_0^T \varphi_x(t, X_t)X_t\pi_t^T db_t + \int_0^T \frac{1}{2}\varphi_{xx}(t, X_t)X_t^2\langle \pi_t\pi_t^T, d < B >_t\rangle \\
& + \int_0^T G(-\varphi_x(t, X_t)X_t\pi_t, -X_t^2\varphi_{xx}(t, X_t)A^T\pi_t\pi_t^T A)dt\Big] + \varphi(0, x_0)
\end{aligned}
$$

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$$= \varphi(0, x_0).$$

For the last inequality we use the property of G-stochastic calculus,

$$\mathbb{E}\Big[\int_0^T \varphi_x(t, X_t) X_t \pi_t^T db_t + \int_0^T \frac{1}{2} \varphi_{xx}(t, X_t) X_t^2 \langle \pi_t \pi_t^T, d < B >_t \rangle$$
$$+ \int_0^T G(-\varphi_x(t, X_t) X_t \pi_t, -X_t^2 \varphi_{xx}(t, X_t) A^T \pi_t \pi_t^T A) dt \Big] = 0.$$

Consequently,

$$V(x_0) \leq \varphi(0, x_0).$$

Let $(\hat{\pi}, \hat{c})$ satisfy

$$\sup_{(\pi, c) \in \mathcal{B} \times \mathcal{A}} \Big\{ u(t, c) + \varphi_t(t, x) + xr\varphi_x(t, x)(1 - \pi^T 1) - c\varphi_x(t, x)]$$
$$+ \inf_{(q, Q) \in \Theta} \{ \varphi_x(t, x) x \langle \pi, q \rangle + \frac{1}{2} x^2 \varphi_{xx}(t, x) \langle A^T \pi \pi^T A, QQ^T \rangle \} \Big\}$$
$$= \Big\{ u(t, \hat{c}) + \varphi_t(t, x) + xr\varphi_x(t, x)(1 - (\hat{\pi})^T 1) - \hat{c}\varphi_x(t, x)]$$
$$+ \inf_{(q, Q) \in \Theta} \{ \varphi_x(t, x) x \langle \hat{\pi}, q \rangle + \frac{1}{2} x^2 \varphi_{xx}(t, x) \langle A^T \hat{\pi}(\hat{\pi})^T A, QQ^T \rangle \} \Big\},$$

and

$$\begin{cases} dX_t = [rX_t(1 - \hat{\pi}(t, X_t)^T 1)dt + X_t \hat{\pi}(t, X_t)^T db_t - \hat{c}(t, X_t)dt + X_t \hat{\pi}(t, X_t)^T AdB_t, \\ X_0 = x_0. \end{cases}$$

If $\pi = \hat{\pi}, c = \hat{c}$, then from the above proof it follows that

$$V(x_0) = \varphi(0, x_0)$$

Therefore, we have

$$V(x_0) = \varphi(0, x_0) = \sup_{(\pi, c) \in \Pi \times C} J(\pi, c, x_0) = J(\hat{\pi}, \hat{c}, x_0).$$

This complete the proof. $\quad \square$

C.3   Proof of Theorem 3.2

Proof.   From Theorem 3.1, we consider the following uncertain HJB equation:

$$\sup_{(\pi, c) \in \mathcal{B} \times \mathcal{A}} \Big\{ u(t, c) + \varphi_t(t, x) + xr\varphi_x(t, x)(1 - \pi^T 1) - c\varphi_x(t, x)]$$

$$+ \inf_{(\mu,Q)\in[\underline{\mu},\overline{\mu}]\times\Gamma} \left\{ \varphi_x(t,x)x\langle\pi,q\rangle + \frac{1}{2}x^2\varphi_{xx}(t,x)\langle\pi\pi^T, QQ^T\rangle \right\} \right\} = 0,$$

with boundary condition

$$\varphi(T,x) = \Phi(T,x).$$

The above equation can be written as follows:

$$\sup_{(\pi,c)\in\mathcal{B}\times\mathcal{A}} \left\{ u(t,c) + \varphi_t(t,x) + xr\varphi_x(t,x)(1-\pi^T 1) - c\varphi_x(t,x)] \right.$$
$$\left. + \inf_{\mu\in[\underline{\mu},\overline{\mu}]}\{\varphi_x(t,x)x\langle\pi,q\rangle\} + \frac{1}{2}x^2\inf_{Q\in\Gamma}\{\varphi_{xx}(t,x)\langle\pi\pi^T, QQ^T\rangle\} \right\} = 0,$$

If $\varphi_x(t,x) > 0$, then the optimal control $\hat{\mu}^i = \underline{\mu}^i 1_{\{\pi^i>0\}} + \overline{\mu}^i 1_{\{\pi^i\leq0\}}, i = 1,\cdots,d$, and

$$\inf_{\mu\in[\underline{\mu},\overline{\mu}]} \left\{ \varphi_x(t,x)x\langle\pi,\mu\rangle \right\}$$
$$= \varphi_x(t,x)x\sum_{i=1}^{d}\pi^i\underline{\mu}^i 1_{\{\pi^i>0\}} + \varphi_x(t,x)x\sum_{i=1}^{d}\pi^i\overline{\mu}^i 1_{\{\pi^i\leq0\}}.$$

If $\varphi_{xx}(t,x) < 0$, then

$$\inf_{Q\in\Gamma}\{\varphi_{xx}(t,x)\langle\pi\pi^T, QQ^T\rangle\} = \varphi_{xx}(t,x)\sum_{i=1}^{d}(\pi^i)^2(\overline{\sigma}^i)^2$$

Therefore, we have the following equation:

$$\sup_{(\pi,c)\in\mathcal{B}\times\mathcal{A}} \left\{ u(t,c) + \varphi_t(t,x) + \varphi_x(t,x)xr - \varphi_x(t,x)c \right.$$
$$+\varphi_x(t,x)x\sum_{i=1}^{d}\pi^i(\underline{\mu}^i - r)1_{\{\pi^i>0\}} + \varphi_x(t,x)x\sum_{i=1}^{d}\pi^i(\overline{\mu}^i - r)1_{\{\pi^i\leq0\}}$$
$$\left. +\frac{1}{2}x^2\varphi_{xx}(t,x)\sum_{i=1}^{d}(\pi^i)^2(\overline{\sigma}^i)^2 \right\} = 0. \tag{C.1}$$

From the first order condition it follows that

$$u_c(t,\hat{c}) = \varphi_x(t,x).$$

Since $u_c(t,c)$ is decreasing with respect to c, then its inverse exists and it is denoted by v. Therefore, the optimal consumption rule is the following

$$\hat{c} = v(\varphi_x(t,x)).$$

Lemma C.2 *If* $a = \frac{1}{2}\overline{\sigma}^2 x^2 \varphi_{xx}(t,x) < 0$ *and* $b = \varphi_x(t,x)x > 0$,

$$f(\pi) = a\pi^2 + b\pi(\underline{\mu} - r)1_{\{\pi>0\}} + b\pi(\overline{\mu} - r)1_{\{\pi\leq 0\}},$$

then $\sup_{\pi} f(\pi)$ *has the the following three cases.*

(i) *If* $r \leq \underline{\mu}$, *then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = -\frac{b^2(\underline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\underline{\mu} - r)^2}{2\overline{\sigma}^2},$$

where

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu} - r}{\overline{\sigma}^2}.$$

(ii) *If* $\underline{\mu} < r < \overline{\mu}$, *then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = f(0) = 0,$$

where $\hat{\pi} = 0$.

(iii) *If* $\overline{\mu} \leq r$, *then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = -\frac{b^2(\overline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\overline{\mu} - r)^2}{2\overline{\sigma}^2},$$

where

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu} - r}{\overline{\sigma}^2}.$$

Proof of Lemma C.2.
Case I: If $r \leq \underline{\mu}$, then

$$\sup_{\pi<0} f(\pi) = 0,$$

and

$$\sup_{\pi\geq 0} f(\pi) = f(\overline{\pi}) = -\frac{b^2(\underline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\underline{\mu} - r)^2}{2\overline{\sigma}^2},$$

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$$\bar{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\mu - r}{\overline{\sigma}^2}.$$

Therefore,

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = -\frac{b^2(\underline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\underline{\mu} - r)^2}{2\overline{\sigma}^2},$$

where

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu} - r}{\overline{\sigma}^2}.$$

Case II: If $\underline{\mu} < r < \overline{\mu}$, then

$$\sup_{\pi < 0} f(\pi) = f(0) = 0,$$

and

$$\sup_{\pi \geq 0} f(\pi) = f(0) = 0,$$

Therefore,

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = f(0) = 0,$$

where $\hat{\pi} = 0$.

Case III: If $\overline{\mu} \leq r$, then

$$\sup_{\pi \geq 0} f(\pi) = 0,$$

and

$$\sup_{\pi < 0} f(\pi) = f(\bar{\pi}) = -\frac{b^2(\overline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\overline{\mu} - r)^2}{2\overline{\sigma}^2},$$

where

$$\bar{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu} - r}{\overline{\sigma}^2}.$$

Therefore,

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = -\frac{b^2(\overline{\mu} - r)^2}{4a} = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\frac{(\overline{\mu} - r)^2}{2\overline{\sigma}^2},$$

where

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu}-r}{\overline{\sigma}^2}.$$

The proof is complete. $\quad\square$

We now turn back to our proof. We define

$$f(\pi^i) = \frac{1}{2}(\overline{\sigma}^i)^2 x^2 \varphi_{xx}(t,x)(\pi^i)^2 + \varphi_x(t,x)x\pi^i(\underline{\mu}^i - r)1_{\{\pi^i>0\}}$$
$$+\varphi_x(t,x)x\pi^i(\overline{\mu}^i - r)1_{\{\pi^i\leq0\}},$$

We want to get $\sum_{i=1}^{d}\sup_{\pi^i} f(\pi^i)$. We consider the following cases.

Case I: $r \leq \inf_i \underline{\mu}^i$.

From the above lemma it follows that

$$\sum_{i=1}^{d}\sup_{\pi} f(\pi^i) = \sum_{i=1}^{d} f(\hat{\pi}^i) = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\sum_{i=1}^{d}\frac{(\underline{\mu}^i - r)^2}{2(\overline{\sigma}^2)^i},$$

where

$$\hat{\pi}^i = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}^i - r}{(\overline{\sigma}^2)^i}.$$

Case II: $\sup_i \overline{\mu}^i \leq r$.

From the above lemma it follows that

$$\sum_{i=1}^{d}\sup_{\pi} f(\pi^i) = \sum_{i=1}^{d} f(\hat{\pi}^i) = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\sum_{i=1}^{d}\frac{(\overline{\mu}^i - r)^2}{2(\overline{\sigma}^2)^i},$$

where

$$\hat{\pi}^i = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu}^i - r}{(\overline{\sigma}^2)^i}.$$

Case III: $\inf_i \underline{\mu}^i < r < \sup_i \overline{\mu}^i$.

We denote by $\mathbb{A}_1 = \{i \mid \underline{\mu}^i \geq r, i = 1, \cdots, d\}, \mathbb{A}_2 = \{i \mid \overline{\mu}^i \leq r, i = 1, \cdots, d\}, \mathbb{A}_3 = \{i \mid \underline{\mu}^i < r < \overline{\mu}^i, i = 1, \cdots, d\}$. From the above lemma it follows that

$$\sum_{i=1}^{d}\sup_{\pi} f(\pi^i) = \sum_{i=1}^{d} f(\hat{\pi}^i) = -\frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\sum_{i\in\mathbb{A}_1}\frac{(\underline{\mu}^i - r)^2}{2(\overline{\sigma}^2)^i} - \frac{\varphi_x^2(t,x)}{\varphi_{xx}(t,x)}\sum_{i\in\mathbb{A}_1}\frac{(\overline{\mu}^i - r)^2}{2(\overline{\sigma}^2)^i},$$

where for $i \in \mathbb{A}_1$

$$\hat{\pi}^i = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}^i - r}{(\overline{\sigma}^2)^i};$$

for $i \in \mathbb{A}_2$

$$\hat{\pi}^i = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu}^i - r}{(\overline{\sigma}^2)^i}.$$

for $i \in \mathbb{A}_3, \hat{\pi}^i = 0$.

(C.1) is equivalent to the following equation

$$\begin{cases} u(t, v(\varphi_x(t,x))) + \varphi_t(t,x) + \varphi_x(t,x)xr - \varphi_x(t,x)v(\varphi_x(t,x)) \\ \quad -\sum_i \dfrac{\varphi_x^2(t,x)(\underline{\mu}^i - r)^2}{2(\overline{\sigma}^i)^2\varphi_{xx}(t,x)}1_{\{r \leq \underline{\mu}^i\}} - \sum_i \dfrac{\varphi_x^2(t,x)(\overline{\mu}^i - r)^2}{2(\overline{\sigma}^i)^2\varphi_{xx}(t,x)}1_{\{\overline{\mu}^i \leq r\}} = 0, \text{(C.2)} \\ \varphi(T,x) = \Phi(T,x). \end{cases}$$

C.4 Proof of Propositions 3.4

Proof of Proposition 3.4.   We just give the proof of (i), since the proof of (ii) and (iii) are similar. By the definition of u and Φ, then the equation (C.2) has the following form

$$\begin{cases} \dfrac{\varphi_x^{1-\alpha^{-1}}}{1-\alpha} + \varphi_t + \varphi_x xr - \varphi_x(t,x)v(\varphi_x) - \sum_i \dfrac{\varphi_x^2(t,x)(\underline{\mu}^i - r)^2}{2(\overline{\sigma}^i)^2\varphi_{xx}} = 0, \\ \varphi(T,x) = \dfrac{Kx^{1-\alpha}}{1-\alpha}. \end{cases} \quad \text{(C.3)}$$

We suppose that $\varphi(t,x)$ has the following form

$$\varphi(t,x) = f(t)\frac{x^{1-\alpha}}{1-\alpha},$$

where $f(t)$ is a function and given later. Therefore, substituting the above form of $\varphi(t,x)$ in to (C.3), we obtain the following equation

$$\begin{cases} \alpha f(t)^{1-\alpha^{-1}} + \beta f(t) + f'(t) = 0, \\ f(T) = K, \end{cases}$$

where

$$\beta = \left[r + \sum_i \frac{(\underline{\mu}^i - r)^2}{2(\overline{\sigma}^i)^2\alpha}\right](1-\alpha).$$

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The solution of the above equation is given by

$$f(t) = \left[K^{\alpha^{-1}} e^{\beta \alpha^{-1}(T-t)} + \alpha \beta^{-1} (e^{\beta \alpha^{-1}(T-t)} - 1) \right]^{\alpha}.$$

Therefore, from Theorem 3.2 we can get the desired result. The proof is complete. $\square$

C.5 Proof of Theorem C.3

Similar to the Section 3, we have the corresponding uncertain HJB as follows.

$$\sup_{(\pi,c) \in \mathcal{B} \times \mathcal{A}} \left\{ u(t,c) + \varphi_t(t,x) - c\varphi_x(t,x) + \inf_{r \in [\underline{r}, \overline{r}]} \{ xr\varphi_x(t,x)(1-\pi) \} \right.$$

$$\left. + \inf_{(\mu,\sigma) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]} \left\{ \varphi_x(t,x)x\pi\mu + \frac{1}{2}x^2\varphi_{xx}(t,x)\pi^2\sigma^2 \right\} \right\} = 0, \qquad \text{(C.4)}$$

with boundary condition

$$\varphi(T,x) = \Phi(T,x).$$

Before giving the proof of Theorem 4.1, we give the following theorem, which will be needed in what follows.

Theorem C.3 *Let $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^+)$ be a solution of (C.4) and $\varphi_{xx} < 0$, then the optimal consumption is*

$$\hat{c} = v(\varphi_x(t,x)),$$

where v is the inverse of u_c, and

(i) if $\overline{\mu} \leq \underline{r}$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\overline{\mu} - \underline{r}}{\overline{\sigma}^2}.$$

(ii) if $\underline{\mu} < \underline{r} < \overline{\mu}$, then the optimal portfolio choice is $\hat{\pi} = 0$.

(iii) if $\underline{r} < \underline{\mu} < \overline{r}$, then the optimal portfolio choice is

$$\hat{\pi} = \left[-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\underline{\mu} - \underline{r}}{\overline{\sigma}^2} \right] \wedge 1.$$

(iv) if $\underline{\mu} \geq \overline{r}$, and if $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} < x$, then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2};$$

and if $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} > x$, then then the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2};$$

and if $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2} \leq x \leq -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2}$, then then the optimal portfolio choice is $\hat{\pi} = 1$.

Proof of Theorem C.3. From the first order condition it follows that

$$\hat{c} = v(\varphi_x(t,x)).$$

where v is the inverse of $u_c(t,c)$.

We denote by $a = \frac{1}{2}\overline{\sigma}^2 x^2 \varphi_{xx}(t,x) < 0$ and $b = \varphi_x(t,x)x > 0$. Let us consider the following functions

$$\begin{aligned}
f(\pi) &= a\pi^2 + b\pi(\underline{\mu}-\overline{r})1_{\{\pi>1\}} + b\pi(\underline{\mu}-\underline{r})1_{\{0\leq\pi\leq1\}} \\
&\quad + b\pi(\overline{\mu}-\underline{r})1_{\{\pi\leq0\}} + b\overline{r}1_{\{\pi>1\}} + b\underline{r}1_{\{\pi\leq1\}}.
\end{aligned}$$

For this, we define the following functions:

$$\begin{aligned}
f_1(\pi) &= a\pi^2 + b\pi(\underline{\mu}-\overline{r}) + b\overline{r}, \ \pi > 1, \\
f_2(\pi) &= a\pi^2 + b\pi(\underline{\mu}-\underline{r}) + b\underline{r}, \ 0 \leq \pi \leq 1, \\
f_3(\pi) &= a\pi^2 + b\pi(\overline{\mu}-\underline{r}) + b\underline{r}, \ \pi \leq 0.
\end{aligned}$$

Let us consider $\sup_{\pi} f(\pi)$ in the following cases.

Case I: If $\overline{\mu} \leq \underline{r}$, then

$$\sup_{\pi>1} f_1(\pi) = f_1(1) = a + b\underline{\mu},$$

$$\sup_{0\leq\pi\leq1} f_2(\pi) = f_2(0) = b\underline{r},$$

$$\sup_{\pi<0} f_3(\pi) = f_3(\overline{\pi}) = b\underline{r} - \frac{b^2(\overline{\mu}-\underline{r})^2}{4a},$$

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where

$$\overline{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu}-r}{\overline{\sigma}^2}.$$

Since $a < 0$ and $b > 0$, we have

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\overline{\mu}-r}{\overline{\sigma}^2}.$$

Case II: If $\underline{\mu} < \underline{r} < \overline{\mu}$, then it follows that

$$\sup_{\pi>1} f_1(\pi) = f_1(1) = a + b\underline{\mu},$$

$$\sup_{0\le\pi\le1} f_2(\pi) = f_2(0) = b\underline{r},$$

$$\sup_{\pi<0} f(\pi) = f_3(0) = b\underline{r}.$$

Since $a < 0$ and $b > 0$, we have

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = 0$.

Case III: If $\underline{r} < \underline{\mu} < \overline{r}$ then

$$\sup_{\pi>1} f_1(\pi) = f_1(1) = a + b\underline{\mu},$$

$$\sup_{0\le\pi\le1} f_2(\pi) = f_2(\overline{\pi}) > f_2(0) = f_3(0), f_2(\overline{\pi}) > f_2(1) = f_1(1)$$

where

$$\overline{\pi} = [-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-r}{\overline{\sigma}^2}] \wedge 1,$$

$$\sup_{\pi<0} f_3(\pi) = f_3(0) = b\underline{r},$$

Therefore, it follows that

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where

$$\hat{\pi} = \left[-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\mu-r}{\sigma^2}\right] \wedge 1.$$

Case IV: $\underline{\mu} \geq \overline{r}$. (a) If $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\mu-r}{\sigma^2} < x$, then

$$\sup_{\pi>1} f_1(\pi) = f_1(1) = a + b\underline{\mu},$$

$$\sup_{0\leq\pi\leq 1} f_2(\pi) = f_2(\bar{\pi}) > f_2(0) = f_3(0), f_2(\bar{\pi}) > f_2(1) = f_1(1),$$

where

$$\bar{\pi} = \frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\underline{r}}{\overline{\sigma}^2},$$

$$\sup_{\pi<0} f_3(\pi) = f_3(0) = b\underline{r}.$$

Consequently,

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

and the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\mu-r}{\overline{\sigma}^2}.$$

(b) If $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)}\frac{\mu-\overline{r}}{\overline{\sigma}^2} > x$, then

$$\sup_{\pi\geq 1} f_1(\pi) = f_1(\bar{\pi}) > f_2(0) = f_3(0), f_2(\bar{\pi}) > f_2(1) = f_1(1),$$

where

$$\bar{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\underline{\mu}-\overline{r}}{\overline{\sigma}^2},$$

$$\sup_{0\leq\pi\leq 1} f_2(\pi) = f_2(1) = f_1(1) = a + b\underline{\mu} > f_2(0) = f_3(0),$$

$$\sup_{\pi<0} f_3(\pi) = f_3(0) = b\underline{r}.$$

From the above it follows that the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x} \frac{\mu - \overline{r}}{\overline{\sigma}^2}.$$

(c) If $-\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)} \frac{\mu-\overline{r}}{\overline{\sigma}^2} \leq x \leq -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)} \frac{\mu-r}{\overline{\sigma}^2}$, then

$$\sup_{\pi\geq 1} f_1(\pi) = f_1(1),$$

$$\sup_{0\leq\pi\leq 1} f_2(\pi) = f_2(1) = f_1(1) = a + b\underline{\mu} > f_2(0) = f_3(0),$$

$$\sup_{\pi<0} f_3(\pi) = f_3(0) = b\underline{r}.$$

Therefore the optimal portfolio choice is $\hat{\pi} = 1$. The proof is complete. $\quad\square$

Proof of Theorem 4.1.

Similar to the proof of Proposition 3.4, from Theorem C.3 we can get that

(i) if $\overline{\mu} \leq \underline{r}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\overline{\mu} - r}{\alpha\overline{\sigma}^2}.$$

(ii) if $\underline{\mu} < \underline{r} < \overline{\mu}$, then the optimal portfolio choice is $\hat{\pi} = 0$.

(iii) if $\underline{r} < \underline{\mu} < \overline{r}$, then the optimal portfolio choice is

$$\hat{\pi} = \left[\frac{\underline{\mu} - \underline{r}}{\alpha\overline{\sigma}^2}\right] \wedge 1.$$

We now consider $\underline{\mu} \geq \overline{r}$ in the following cases.

Case I. Suppose $\dfrac{\underline{\mu} - \underline{r}}{\alpha\overline{\sigma}^2} < 1$, from Theorem C.3, then the equation (C.4) has the following form

$$\begin{cases} \dfrac{\varphi_x^{1-\alpha^{-1}}}{1-\alpha} + \varphi_t + \varphi_x xr - \varphi_x v(\varphi_x) - \dfrac{\varphi_x^2(\underline{\mu} - \underline{r})^2}{2\overline{\sigma}^2\varphi_{xx}} = 0, \\ \varphi(T,x) = \dfrac{Kx^{1-\alpha}}{1-\alpha}. \end{cases} \tag{C.5}$$

Similar to the proof of Proposition 3.4, the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\mu - r}{\overline{\sigma}^2} = \frac{\mu - r}{\alpha\overline{\sigma}^2}.$$

Case II. Suppose $\dfrac{\mu - \overline{r}}{\alpha\overline{\sigma}^2} > 1$, from Theorem C.3, then the equation (C.4) has the following form

$$\begin{cases} \dfrac{\varphi_x^{1-\alpha^{-1}}}{1-\alpha} + \varphi_t + \varphi_x xr - \varphi_x v(\varphi_x) - \dfrac{\varphi_x^2(\underline{\mu} - \overline{r})^2}{2\overline{\sigma}^2\varphi_{xx}} = 0, \\ \varphi(T,x) = \dfrac{Kx^{1-\alpha}}{1-\alpha}. \end{cases} \tag{C.6}$$

Similar to the proof of Proposition 3.4, the optimal portfolio choice is

$$\hat{\pi} = -\frac{\varphi_x(t,x)}{\varphi_{xx}(t,x)x}\frac{\mu - \overline{r}}{\overline{\sigma}^2} = \frac{\mu - \overline{r}}{\alpha\overline{\sigma}^2}.$$

Case II. If $\dfrac{\mu - \overline{r}}{\alpha\overline{\sigma}^2} \le 1 \le \dfrac{\mu - r}{\alpha\overline{\sigma}^2}$, then then the optimal portfolio choice is $\hat{\pi} = 1$. The proof is complete. $\square$

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