Coherence in generic representation theory*

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Abstract

Let $\mathcal{F}_q$ be the category of functors from finite dimensional vector spaces over a finite field to vector spaces over the same field. Here q denotes the cardinality of the finite field $\mathbb{F}_q$. Initial Motivation for this paper is to show that the category $\mathcal{F}_q$ is coherent. While the Artinian conjecture would imply the subcategory of finitely generated functors in $\mathcal{F}_q$ is abelian, coherence implies that the category of finitely presented functors in $\mathcal{F}_q$ is abelian. Therefore, coherence is somewhat weaker. We document approaches of ways how to show that the category $\mathcal{F}_q$ is coherent.
A second subject of this paper is the following. The category $\mathcal{F}_q$ admits an exact endo-functor Δ. We prove that this endo-functor is diagonalizable on the Grothendiek-group of indecomposable projective functors. To achieve this goal we will use computational methods and properties of the dimension of indecomposable projective functors. A second step will then be to show that the previously computed dimension function is sufficient to compute diagonalizability.

Zusammenfassung

Wir bezeichnen mit $\mathcal{F}_q$ die Kategorie von Funktoren, die von der Kategorie endlich dimensionaler Vektorräume über einem endlichen Körper in die Kategorie von Vektoräumen über diesem Körper abbilden. Dabei ist q die Mächtigkeit des endlichen Körpers $\mathbb{F}_q$. Die initiale Motivation für diese Arbeit ist zu zeigen, dass die Kategorie $\mathcal{F}_q$ kohärent ist. Während die Artinsche Vermutung besagt, dass die Unterkategorie von $\mathcal{F}_q$, die endlich erzeugte Funktoren enthählt abelsch ist, so besagt Kohärenz, dass die Unterkategorie der endlich präsentierten Funktoren abelsch ist. Daher ist Kohärenz etwas schwächer. Wir dokumentieren eine Reihe von Ansätzen mit denen gezeigt werden sollte, dass die Kategorie $\mathcal{F}_q$ kohärent ist.
Ein weiterer Schwerpunkt dieser Arbeit ist der folgende. Die Kategorie $\mathcal{F}_q$ besitzt einen exakten Endo-Funktor Δ. Wir beweisen, dass dieser Endo-Funktor auf der Grothendiek-Gruppe der unzerlegbaren projektiven Funktoren diagonalisierbar ist. Um dieses Ziel zu erreichen werden wir Eigenschaften der Dimensionsfunktionen unzerlegbar projektiver Funktoren berechnen und benutzen. Ein zweiter Schritt wird dann darin bestehen zu zeigen, dass es für die Diagonalisierbarkeit hinreichend ist die Dimensionsfunktionen zu kennen.

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Contents

1 Introduction and notation

1.1 The main results

Theoretical basis for this work is the remarkable three paper series by N. Kuhn [Kuh93], [Kuh94], [Kuh95]. He considers functor categories to study representations of the general linear group $\mathrm{GL}(V)$, V a vector space, which is a rater classical problem in representation theory. The approach chosen by Kuhn allows to connect representations of the general linear group to unstable modules over the Steenrod algebra. The latter is a problem arising in algebraic topology. The study of representations of $\mathrm{GL}_n(\mathbb{F}_q)$ in terms of functor categories allows to define the setting of generic representation theory, which we will explain a little later.

In this paper we want to consider same setting as N. Kuhn. We start in the category of finite dimensional vector spaces over a finite field $\mathbb{F}_q$. Our object of interest is the category of functor starting in finite dimensional vector spaces with the target category of $\mathbb{F}_q$ vector spaces that are not necessary finite dimensional. We call this category $\mathcal{F}_q$. When we evaluate a functor $F \in \mathcal{F}_q$ at a vector space $V \in \operatorname{mod} \mathbb{F}_q$ the result $F(V)$ gives naturally rise to a representation of $\mathrm{GL}(V)$. Variation of V generically yields representations for all general linear groups $\mathrm{GL}(V)$. Therefore, we call this setting generic representation theory.

The general concept of functors from an additive or $\mathbb{K}$-linear category $\mathcal{C}$, for a field $\mathbb{K}$, to the category of abelian groups or vector spaces is very classical. A lot dates back to the 1960s and the work of M. Auslander. A good reference, where many details are summed up, is the work [AR74] by M. Auslander and I. Reiten. When we consider categories of such functors, the morphisms are given by natural transformations.

A class of often used functors are the standard projective functors. By the Yoneda embedding we have a standard projective P_V for every finite dimensional vector space V. A functor F is called finitely generated if it is a quotient of a finite direct sum of functors P_V. More structure of those functors will be introduced a little later.

Furthermore, the category $\mathcal{F}_q$ admits a discrete derivation functor $\Delta : \mathcal{F}_q \to \mathcal{F}_q$ that sends a functor F to the quotient $\Delta F := F(\mathbb{F}_q \oplus -)/F$. In his paper [Kuh94] N. Kuhn states that ΔF is projective if F is. He provides a formula how to compute ΔF in this case.

By using the formula for ΔP, provided by N. Kuhn, we can calculate the function $\phi(P, n) = \dim_{\mathbb{F}_q} P(\mathbb{F}_q^n)$, as a function of n, explicitly for any finitely generated projective functor P. This yields our first main result.

Proposition 1.1.1. *For an indecomposable projective functor P it holds that*

$$\phi(P,n) = \sum_{i=0}^{s} a_i q^{in}.$$

Where $a_i \in \mathbb{Q}$ and q is the cardinality of the finite field we are working over.

During discussions with L. Schwartz the author was introduced to the question if knowledge of this form for the dimension function $\phi(P,n)$ could be used to obtain eigenfunctors for the derivation functor Δ. An eigenfunctor would be a functor F such that $\Delta(F) \cong F^{\oplus m}$. Known eigenfunctors are the standard projective functors P_V which yield $\Delta(P_V) \cong P_V^{\oplus q^{\dim V}-1}$.
The Grothendiek group $K_0(\mathcal{F}_q)$ is generated by the isomorphism classes of indecomposable projective functors. If there would be enough eigenfunctors given by direct sums of projective functors, it would be possible to give a generating set of $K_0(\mathcal{F}_q)$, that consists only of eigenfunctors. We can deduce the following result.

Theorem 1.1.2. *The functor $\Delta : \mathcal{F}_q \to \mathcal{F}_q$ is diagonalizable on $K_0(\mathcal{F}_q)$ and the eigenvalues are $q^k - 1$ for $k \in \mathbb{N}_0$.*

A second subject of this paper and also the initial start of this project is the following conjecture posed by L. Schwartz.

Conjecture 1.1.3 (Artinian Conjecture due to Lionel Schwartz)**.** *The functor I_V is artinian for all finite dimensional vector spaces V.*

In this paper we use the dual approach to show that all projective functors P_V are noetherian. The point from which to start is the question wether the functors P_V are at least coherent. While noetherian would imply that any subfunctor of P_V would be finitely generated, coherence only implies that all subfunctors which are kernels of maps $f : P_V \to P_W$ are finitely generated.
The idea we start with is the following generalization of the form of a dimension function in proposition 1.1.1.

Definition 1.1.4. Let F be a functor in $\mathcal{F}_q$. We say that $\phi(F,n)$ is of closed form if

$$\phi(F,n) = \sum_{i=0}^{s} p_i(n) q^{in},$$

where $p_i(n)$ are polynomials and q is the cardinality of the field.

We conjecture the following.

Conjecture 1.1.5. *Let F be a finitely presented functor, then the function $\phi(F,n)$ is of closed form.*

Classes of functors such that $\phi(F, n)$ is of closed form are projective functors and functors of finite composition length. Our hope is that knowledge of the existence of the closed form will provide a way of understanding the structure of kernels of maps $f : P_V \to P_W$.
We shall however already remark that it is not possible to show coherence. But it is possible to show some hints that it should at all be possible to show coherence or even the Artinian conjecture.

Lemma 1.1.6. *It holds that* $\operatorname{rad}^\infty P = 0$ *for all indecomposable projectives.*

Lemma 1.1.7. *Let* F *be finitely generated, then* $\operatorname{rad}^r F/\operatorname{rad}^{r+1} F$ *is a functor of finite composition length.*

For a subfunctor G of P_V we can also show:

Lemma 1.1.8. *Let* $G \subset P_V$ *for some finite dimensional* V*. For all* $U \in \operatorname{mod} \mathbb{F}_q$ *we can find finite dimensional vector space* L *and a map* $g : P_L \to P_V$ *such that* g_U *induces an epimorphism* $P_L(U) \twoheadrightarrow G(U)$*.*

Lemma 1.1.9. *Let* F *be finitely generated via*

$$\bigoplus_{i=1}^{m} P_{V_i} \xrightarrow{\chi} F \to 0.$$

With the previous lemma we can always find P_W *and* f *such that* $\chi \circ f = 0$*. For fixed* $\mathbb{F}_q^n$ *this can be chosen such that the sequence*

$$0 \to \operatorname{Im} f_{\mathbb{F}_q^n} \to \bigoplus_{i=1}^{m} P_{V_i}(\mathbb{F}_q^n) \to F(\mathbb{F}_q^n) \to 0$$

is exact. Then it holds that this sequence is also exact for $\mathbb{F}_q^{n-k}$*.*

Unfortunately we cannot say much what happens in the case $\mathbb{F}_q^{n+k}$. But we document some attempts on promising leads. The closed form, that we conjecture to be exist for all finitely generated functors, will be of help here.
In case when the map $f : P_V \to P_W$ is rather simple, the so-called Kronecker case as it is connected to the representation theory of the Kronecker quiver, we can show that $\ker f$ is finitely generated. So far we did not say anything about how such a map $f : P_V \to P_W$ looks like. It turns out that f can be written as a formal linear combination of linear maps. For this reason the solution of wether $\ker f$ is finitely generated or not is somewhat connected to the representation theory of quivers. As mentioned before, the case of the Kronecker quiver can be solved completely.

Proposition 1.1.10. *Let $f : P_V \to P_W$ be a representable morphism such that $f = \lambda[f_1] + \mu[f_2]$, then* $\ker f$ *is finitely generated.*

The last result of this paper is the beginning of the so-called extension quiver of the category $\mathcal{F}_q$. Due to computational restrictions only could the first few steps be computed.

1.2 Outline

The thesis is organized as follows. In Section 2 we introduce the background of the problem we want to address. We will introduce notation and discuss a few examples. Equipped with this knowledge we are able to turn our attention to finitely generated projective functors. Section 3 aims at a full description of their dimension functions. Afterwards, we can prove that the endo functor $\Delta : \mathcal{F}_q \to \mathcal{F}_q$ is diagonalizable on the Grothendiek group of finitely generated projectives.
Section 4 introduces a concept based on representations of quivers of how to view the category $\mathcal{F}_q$ as a category of modules. This is afterwards used to show that the kernel of a representable map in the so called Kronecker case is finitely generated.
The following Section 5 deals with attempts to prove that the category $\mathcal{F}_q$ is coherent. Though not complete or even remotely satisfying answer to wether $\mathcal{F}_q$ is coherent can be given, the applied methods might be enlightening for future work.
In Section 6 the partial results of Section 5 will be addressed in a more general setting, aiming to get closer to the Artinian Conjecture.
The final Sections 6 and 7 deal with algorithmic approaches to a better understanding of the functors in our category $\mathcal{F}_q$. In Section 6 we use the Meataxe algorithm to compute the beginning of the extension quiver of the category $\mathcal{F}_q$.
Section 7 deals with an algorithm developed by the author that can compute and test wether a subfunctor of a given functor $F \in \mathcal{F}_q$ is finitely generated.

2 Generic representation theory

We fix a prime power $q = p^s$ and denote by $\mathbb{F}_q$ the field of characteristic p and q elements, which is unique up to isomorphism. We want to study the category $\mathcal{F}_q = \mathrm{Func}(\mathrm{mod}\,\mathbb{F}_q, \mathrm{Mod}\,\mathbb{F}_q)$, the category of all covariant functors that take finite dimensional vector spaces over a finite field to vector spaces over that field. For all $V \in \mathrm{mod}\,\mathbb{F}_q$ and $F \in \mathcal{F}_q$, $F(V)$ is naturally a representation of $\mathrm{GL}(V)$ and of the semi-group (with composition) $\mathrm{End}(V)$. Considering all $F(V)$ together we call this setting generic representation theory.
Instead of looking at all the functors, it is enough that if we $\mathbb{F}_q$-linearize the category we start with and then only consider $\mathbb{F}_q$-linear functors. We need to observe that the $\mathbb{F}_q$-linearization $\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]$ of $\mathrm{mod}\,\mathbb{F}_q$ is not additive and therefore

the functors $F : \mathbb{F}_q[\operatorname{mod}\mathbb{F}_q] \to \operatorname{Mod}\mathbb{F}_q$ do not preserve direct sums. F being $\mathbb{F}_q$-linear means that the maps

$$F_{V,W} : \operatorname{Hom}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]}(V, W) \to \operatorname{Hom}_{\mathbb{F}_q}(F(V), F(W))$$

are $\mathbb{F}_q$-linear. The category of such functors will be denoted by

$$\operatorname{Lin}(\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q], \operatorname{Mod}\mathbb{F}_q).$$

It is a general result that $\mathcal{F}_q$ and $\operatorname{Lin}(\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q], \operatorname{Mod}\mathbb{F}_q)$ are equivalent. Most of the time we implicitly work with $\operatorname{Lin}(\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q], \operatorname{Mod}\mathbb{F}_q)$.
Now what does the category $\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]$ look like? The objects in $\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]$ are the same as in $\operatorname{mod}\mathbb{F}_q$ but the morphism spaces are defined as follows:

$$\operatorname{Hom}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]}(V, W) := \left\{ \sum_{i=1}^{m} \lambda_i [f_i] \,\middle|\, \lambda_i \in \mathbb{F}_q,\ f_i : V \to W \text{ a linear map} \right\}$$

This is again a vector space. In fact it is isomorphic to the vector space $\mathbb{F}_q[\operatorname{Hom}_{\mathbb{F}_q}(V, W)]$ which has a basis that is given by $\operatorname{Hom}_{\mathbb{F}_q}(V, W)$ as a set.
In particular, it holds $\dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]}(\mathbb{F}_q^s, \mathbb{F}_q^t) = q^{st}$.
Throughout this paper we abbreviate $\operatorname{Hom}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]}(V, W)$ by (V, W). Note that $(\mathbb{F}_q, -) \oplus (\mathbb{F}_q, -) \ncong (\mathbb{F}_q^2, -)$ because their dimensions will differ if we evaluate them at an arbitrary $\mathbb{F}_q^t$.
For $V \in \operatorname{mod}\mathbb{F}_q$ the representable functors $P_V = (V, -)$ are projective objects in $\mathcal{F}_q$ by Yoneda's lemma. We call a projective functor P in $\mathcal{F}_q$ standard if it is isomorphic to a finite direct sum $\bigoplus_{i \in I}(V_i, -)$ and the $(V_i, -)$ will come from vector spaces V_i in $\operatorname{mod}\mathbb{F}_q$.
The Yoneda-embedding $V \mapsto (V, -)$ yields an embedding $(\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q])^{op} \hookrightarrow \mathcal{F}_q$. A projective is finitely generated if and only if it is a direct summand of a standard projective $\bigoplus_{i=1}^{n}(V_i, -)$ and the V_i are finite dimensional for all i.
The duality functor D is defined as follows: $DF(V) = F(V^*)^*$ where V^* denotes the vector space dual of V. This is sometimes called the Kuhn-dual. It is contravariant but sends covariant functors to covariant functors. If P is projective, it holds that DP is injective.
Before we begin with more concrete work, we should make a few remarks.

Remark 2.0.1. The category $\mathcal{F}_q$ is abelian since the target category of the functors is abelian.

Let us now start to consider special classes of functors in $\mathcal{F}_q$. We want to start with simples and indecomposable projectives. Luckily those two are closely related but we need some definitions until we can get to them. We follow the way of [HK88].

Definition 2.0.2. Let $\mathbb{F}_q$ be the field of characteristic p and $q = p^s$ elements. Then $\mathrm{GL}_n(\mathbb{F}_q)$ is the group of invertible $n \times n$-matrices with entries in $\mathbb{F}_q$ and $M_n(\mathbb{F}_q)$ is the semi-group of all $n \times n$-matrices with entries in $\mathbb{F}_q$.
By $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$ we denote the group algebra of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{F}_q$. Analogously we define the semi group algebra $\mathbb{F}_q[M_n(\mathbb{F}_q)]$.

Definition 2.0.3 ([HK88])**.** A simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-module is called singular if it is induced by a simple $\mathbb{F}_q[M_{n-1}(\mathbb{F}_q)]$-module. The module where every matrix acts as the identity we call the trivial module.

Definition 2.0.4 ([HK88])**.** Let $\det : \mathbb{F}_q[M_n(\mathbb{F}_q)] \to \mathbb{F}_q$ be the determinant representation.

Definition 2.0.5 ([HK88])**.** If N is an $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-module, let $\mathrm{Res}^{M_n}_{\mathrm{GL}_n}$ denote N restricted to $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$. We note that $\mathrm{Res}^{M_n}_{\mathrm{GL}_n}((\det)^{q-1})$ is the trivial module, although $(\det)^{q-1}$ is not.

Theorem 2.0.6 ([Kuh94], Theorem 5.17)**.**

1. *$\{N \otimes (\det)^j \,|\, N$ is singular and $0 \le j \le q-1\}$ is the set of simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-modules.*
2. *$\{\mathrm{Res}^{M_n}_{\mathrm{GL}_n}(N \otimes (\det)^j) \,|\, N$ is singular and $1 \le j \le q-1\}$ is the set of simple $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$-modules.*

Definition 2.0.7 ([Kuh94], Notation/Definition 5.18)**.** Let $n \ge 0$ and let $\Omega(q,n)$ be the set of $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $0 \le \lambda_i \le q-1$.
For each $\lambda \in \Omega(q,n)$ we get a simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-module as follows: For $n = 0$ let M_0 be the trivial module. For $n > 0$ and $\lambda = (\lambda', \lambda_n)$ with $\lambda' \in \Omega(q, n-1)$ we let

$$M_\lambda = c^n_{n-1}(M_{\lambda'}) \otimes (\det)^{\lambda_n}.$$

Where $c^n_{n-1} : \mathrm{mod}\, \mathbb{F}_q[M_{n-1}(\mathbb{F}_q)] \to \mathrm{mod}\, \mathbb{F}_q[M_n(\mathbb{F}_q)]$ is the induction functor.
It is $M_\lambda \ncong M_\mu$ for $\lambda \ne \mu$. We set $d_\lambda = \dim_{\mathbb{F}_q} M_\lambda$.

Corollary 2.0.8 ([Kuh94], Corollary 5.19)**.** *$\{M_\lambda \,|\, \lambda \in \Omega(q,n)\}$ is a complete set of simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ modules. Furthermore $\mathrm{Res}^{M_n}_{\mathrm{GL}_n}(M_\lambda)$ is a simple $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$ module and we have the identification*

$$\mathrm{Res}^{M_n}_{\mathrm{GL}_n}(M_{(\lambda',0)}) \cong \mathrm{Res}^{M_n}_{\mathrm{GL}_n}(M_{(\lambda',q-1)}).$$

Definition 2.0.9 ([Kuh94], Notation/Definition 5.20)**.** Let Ω be the set of sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers, such that $\lambda_n = 0$ for large n. For $\lambda \in \Omega$, we let $n(\lambda) = \max\{n | \lambda_n \ne 0\}$. Let $\Omega(q)$ be the subset of sequences with $\lambda_i < q$, the so called q-restricted weights.

For each $\lambda \in \Omega(q)$ there is an indecomposable projective a with simple top. The characterization of these simples is the subject of the following corollary due to Kuhn.

Corollary 2.0.10 ([Kuh94], Corollary 5.21)**.** *1. $\{F_\lambda,\ \lambda \in \Omega(q)\}$ is a complete set of simple functors in $\mathcal{F}_q$.*

2. $F_\lambda(\mathbb{F}_q^n)$ is nonzero exactly when $n \geq n(\lambda)$, and in that case is the simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$- (or $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$-) module $M_{\pi_n(\lambda)}$, where $\pi_n : \Omega(q) \to \Omega(q,n)$ is the projection onto the first n coordinates.

With this at hand we can now look further.

Proposition 2.0.11 ([Kuh94], Proposition 3.4)**.** *For $P_{\mathbb{F}_q^n} = (\mathbb{F}_q^n, -)$ it holds $P_{\mathbb{F}_q^n} = \bigoplus_\lambda d_\lambda P_\lambda$ where P_λ is the projective cover of F_λ.*

Remark 2.0.12. The result uses $\mathbb{F}_q \cong \mathrm{End}_{\mathbb{F}_q[M_n(\mathbb{F}_q)]}(M_\lambda)$.

Remark 2.0.13 ([Kuh94], Remarks 3.5)**.** 1. The M_λ remain simple when restricted to $\mathrm{GL}_n(\mathbb{F}_q)$, so that the numbers d_λ are the dimensions of simple $\mathrm{GL}_n(\mathbb{F}_q)$-modules.

2. Since $P_{\mathbb{F}_q^{n-1}}$ is a direct summand of $P_{\mathbb{F}_q^n}$, it is quite easy to deduce that the 'new' λ correspond to the simple $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$-modules.

3. Applying the duality functor to the decomposition yields $DP_{\mathbb{F}_q^n} = I_{\mathbb{F}_q^n} \cong \bigoplus_\lambda d_\lambda I_\lambda$, where I_λ has a simple socle DF_λ. We have $DF_\lambda \cong F_\lambda$.

A categorical property of $\mathcal{F}_q$ that should not stay unmentioned is the tensor product. For two functors $F, G \in \mathcal{F}_q$ the tensor product $F \otimes G$ is defined point wise via: $(F \otimes G)(V) = F(V) \otimes G(V)$. It yields the following nice isomorphism: $P_V \otimes P_W \cong P_{V \oplus W}$, where V and W are vector spaces and the direct sum is in the category of vector spaces.
A look at the Yoneda-lemma yields:

$$\mathrm{Hom}_{\mathcal{F}_q}(P_V, F) \cong F(V)$$

A corollary of this is $\mathrm{End}_{\mathcal{F}_q}(P_V) = (V, V)$. We remark that $[0] \in (V, V)$ is a non-zero idempotent.
We now turn to the problem we want to address.

2.1 The Artinian conjecture

Conjecture 2.1.1. *(Lionel Schwartz) The functor I_V is artinian for all finite dimensional vector spaces V.*

Definition 2.1.2. A functor $F \in \mathcal{F}_q$ is called finitely generated if it is a quotient of a functor $P = \bigoplus_{i=1}^{m}(V_i, -)$. Finitely co-generated functors are dually defined. The full subcategory of finitely generated functors in $\mathcal{F}_q$ is denoted by $\mathcal{F}_{q_{fg}}$.
The full subcategory of finitely presented functors is denoted by $\mathcal{F}_{q_{fp}}$.

Remark 2.1.3. In his papers [Kuh93], [Kuh94] N. Kuhn calls a finitely generated projective also p-small.

The following proposition shows equivalent formulations to this conjecture.

Proposition 2.1.4 ([Kuh94], Proposition 3.13)**.** *The following statements are equivalent:*

1. *Every finitely co-generated F has a resolution by finitely co-generated injectives.*
2. *Every finitely generated F has a resolution by finitely generated projectives.*
3. *Every quotient of a finitely co-generated object is again finitely co-generated.*
4. *Every subobject of a finitely generated object is again finitely generated.*
5. *Every finitely co-generated F is artinian.*
6. *Every finitely generated F is noetherian.*
7. *I_V is artinian for all V.*
8. *P_V is noetherian for all V.*
9. *In $\mathcal{F}_q$, every direct sum of injectives is again injective.*

The formulation of the conjecture that is used in this paper is the second one. So we try to construct resolutions for finitely generated functors F. Before we can start, we will need some definitions

Definition 2.1.5. Let $F \in \mathcal{F}_q$, then define $\Delta F = \operatorname{coker} F(\iota)$ where $\iota : V \to V \oplus \mathbb{F}_q$ is the canonical inclusion. This is a functor, well defined and since $F(\iota)$ is a split monomorphism (we make this explicit in a bit), we have $\Delta F \oplus F = F(\mathbb{F}_q \oplus -)$.
$\Delta^n F$ is then defined by induction.

Remark 2.1.6. We have that $F(\iota)$ is a split monomorphism where $\iota : V \to V \oplus \mathbb{F}_q$ is the canonical inclusion. Let $\mathrm{id} : \mathcal{F}_q \to \mathcal{F}_q$ be the identity functor and $\Sigma : \mathcal{F}_q \to \mathcal{F}_q$ the functor be defined via $\Sigma(F) = F(\mathbb{F}_q \oplus -)$. Then $\iota : \mathrm{id} \to \Sigma$ is a natural transformation. We also have a natural transformation $\pi : \Sigma \to \iota$ defined in the obvious way. It is not hard to see that $\iota \circ \pi$ is the identity transformation on id. Therefore, $F(\iota) \circ F(\pi)$ is also the identity on $F(\mathrm{id}) \cong \mathrm{id}$ making $F(\iota)$ a split monomorphism.

Remark 2.1.7. The endofunctor Δ is exact. Therefore it commutes with kernels, images and cokernels.

Definition 2.1.8. A functor $F \in \mathcal{F}_q$ is called polynomial if $\Delta^n F = 0$ for some $n \in \mathbb{N}_0$.

For the following sections the next theorem will be useful. It also yields cases of functors that admit resolutions by finitely generated projectives.

Theorem 2.1.9 ([Kuh98], Proposition 4.4)**.** *For $F \in \mathcal{F}_q$ the following are equivalent:*

1. *F is of finite length.*
2. *F takes finite dimensional values and is polynomial.*
3. *The function $n \mapsto \dim_{\mathbb{F}_q} F(\mathbb{F}_q^n)$ is a polynomial function in n with coefficients in $\mathbb{Q}$.*

Example 2.1.10. Examples of finite length functors are the n-fold tensor products T^n and symmetric powers S^n, S_n. The exterior powers Λ^n are simple for all n. These functors are defined in the following way:

$$\begin{array}{rccl} T^n : & \mathrm{mod}-\mathbb{F}_q & \to & \mathrm{mod}-\mathbb{F}_q \\ & V & \mapsto & T^n(V) := V^{\otimes n} \\ S^n : & \mathrm{mod}-\mathbb{F}_q & \to & \mathrm{mod}-\mathbb{F}_q \\ & V & \mapsto & S^n(V) := (V^{\otimes n})^{\mathcal{S}_n} \\ S_n : & \mathrm{mod}-\mathbb{F}_q & \to & \mathrm{mod}-\mathbb{F}_q \\ & V & \mapsto & S_n(V) := V^{\otimes n} / \left\langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \right\rangle \\ \Lambda^n : & \mathrm{mod}-\mathbb{F}_q & \to & \mathrm{mod}-\mathbb{F}_q \\ & V & \mapsto & \Lambda^n(V) := V^{\otimes n} / \langle v \otimes v \rangle \end{array}$$

Where $\mathcal{S}_n$ is the symmetric group on n letters and $\sigma \in \mathcal{S}_n$. On morphisms these functors are defined in the obvious way.
For more details on this we refer the reader to the work of Kuhn.

The third equivalent statement of the last theorem has an interesting characterization, and will also come in handy later. In this context there are two cases where L. Schwartz managed to prove projective resolutions to exist.

Theorem 2.1.11 ([Sch94], Theorem 5.3.8 and [FLS94], Section 10)**.** *Let $F \in \mathcal{F}_q$. If there exists an n such that $\Delta^n F$ is a finitely generated projective, then F has a resolution by finitely generated projectives.*

Since we can also think of the 0-functor as a finitely generated projective, there is the following corollary.

Corollary 2.1.12 ([Kuh94], Theorem 3.9)**.** *Every finite length functor F has a resolution by finitely generated projectives.*

Definition 2.1.13. We call a simple functor F_λ a simple Steinberg functor if

$$F_\lambda(\mathbb{F}_q^{n(\lambda)}) \cong P_\lambda(\mathbb{F}_q^{n(\lambda)}).$$

Where P_λ is the projective cover of F_λ.

Lemma 2.1.14. *Let P_λ be an indecomposable projective. Then P_λ has only finitely many composition factors that are Steinberg.*

Proof. Let F_μ be Steinberg. Since $P_\mu(\mathbb{F}_q^{n(\mu)})$ is a simple $\mathbb{F}_q[M_{n(\mu)}(\mathbb{F}_q)]$ module the functor P_μ cannot have composition factors F_ν with $n(\nu) \leq n(\mu)$. Vice versa if P_λ is arbitrary and one of its composition factors F_μ is Steinberg then $n(\lambda) > n(\mu)$. Since P_λ can only have finitely many composition factors of any given weight and there are only finitely many weights μ with $n(\mu) < n(\lambda)$ the lemma follows. $\square$

Definition 2.1.15. We denote by Serre(P) the Serre subcategory of $\mathcal{F}_q$ which is generated by all the finitely generated projectives. Serre(I) is the analogue Serre subcategory which is co-generated by the finitely co-generated injectives.

These Serre subcategories fit into our category $\mathcal{F}_q$ as follows: Let us look at the full subcategory $\mathcal{F}_{q_{fin}}$ of $\mathcal{F}_q$ that consists of all functors $F : \operatorname{mod} \mathbb{F}_q \to \operatorname{mod} \mathbb{F}_q$. Then the Kuhn-dual D induces an autoequivalence on $\mathcal{F}_{q_{fin}}$ that restricts to an equivalence of Serre(P) and Serre(I).

Remark 2.1.16. It holds that DF is of finite length if and only if F is of finite length and that $DF_\lambda = F_\lambda$ for all simple functors F_λ.

Definition 2.1.17. The functor arising from the zero vector space, $(0, -)$, is not a functor that sends everything to zero. It rather sends every object to the one dimensional vector space $\mathbb{F}_q$ and is therefore called the constant functor. We denote it by $\mathbb{F}_q$ to stay consistent with the literature on this topic.

The functor $\mathbb{F}_q$ is projective and injective and it has another special property.

Corollary 2.1.18 ([Kuh93], Appendix B, Corollary B.9)**.** *A projective P_V cannot have a nonzero polynomial subfunctor other than $\mathbb{F}_q$.*

In the category $\mathcal{F}_q$ the standard projectives are finite direct sums of representable functors. For computations to come, this is not very useful. The following lemma will tell us that, when looking at finitely generated projectives, we can restrict ourselves always to projectives that are representable and will not loose too much information.

Lemma 2.1.19. *Let $P = \bigoplus_{i=1}^{s}(V_i, -)$ be an arbitrary standard projective functor in $\mathcal{F}_q$. Then there exists a vector space $\overline{V}$ such that P is a direct summand of $(\overline{V}, -) \oplus \mathbb{F}_q^{s-1}$. Where $\mathbb{F}_q$ is the constant functor.*

Proof. Assertion 3 of theorem 2.1.9 gives us that $n \mapsto \dim_{\mathbb{F}_q} F_\lambda(\mathbb{F}_q^n)$ is a polynomial function in n for any simple F_λ. These polynomials are not constant except for $F_0 = \mathbb{F}_q$ the constant functor.
Now it is a basic analytic fact that every non-constant polynomial $p(n)$ goes to $\pm\infty$ as n goes to $+\infty$. Since the dimension of $F_\lambda(\mathbb{F}_q^n)$ can never be negative, the dimension has to go to $+\infty$. Therefore, for any finite set of simple functors $\{F_\lambda\}_\lambda$, that does not contain the constant functor, and any constant a there is an $N \in \mathbb{N}$ such that $a \leq \min\{\dim_{\mathbb{F}_q} F_\lambda(\mathbb{F}^N)\}_\lambda$. This even holds for infinite sets, but we do not need that here.
Next step is to look at the decomposition of a standard projective $(\mathbb{F}_q^m, -)$ into indecomposable ones. Remark 2.0.13 gives $(\mathbb{F}_q^m, -) = \bigoplus_\lambda d_\lambda P_\lambda$, where $d_\lambda = \dim_{\mathbb{F}_q} F_\lambda(\mathbb{F}_q^m)$. Since we can think of any standard projective $(W, -)$ as $(\mathbb{F}_q^m, -)$ for suitable m, we can apply this to an arbitrary P in the following way:

$$P = \bigoplus_{i=1}^{s}(\mathbb{F}_q^{m_i}, -) = \bigoplus_{i=1}^{s}\bigoplus_{\lambda} d_\lambda^i P_\lambda = \bigoplus_{\lambda}\left(\sum_{i=1}^{s} d_\lambda^i\right) P_\lambda$$

Where we have $d_\lambda^i = \dim F_\lambda(\mathbb{F}_q^{m_i})$. Now we can apply the first part of the proof to find a finite dimensional W such that $\dim F_\lambda(W) \geq \sum_{i=1}^{s} \dim F_\lambda(\mathbb{F}_q^{m_i})$ for all $\lambda \neq 0$.
It is easy to see now that all indecomposable projectives in P can be embedded as direct summands in $(\mathbb{F}_q^N, -)$. The only thing we cannot embed is the constant part of P. Therefore, we have to add the missing $s - 1$ constant functors $\mathbb{F}_q$ to $(\mathbb{F}_q^N, -)$ to get an injective map. □

The above lemma only states the existence of such an embedding, but as we partially work with the computer, there is always the question of optimality. It appears not to be trivial, but is really straight forward to be proven.

Lemma 2.1.20. *Let $P = \bigoplus_{i=1}^{s}(V_i, -)$ be a standard projective functor in $\mathcal{F}_q$. Then P is a direct summand of $(\overline{V}, -) \oplus \mathbb{F}_q^{s-1}$. Where $\mathbb{F}_q$ is the constant functor and $\dim \overline{V} = \sum_{i=1}^{s} \dim V_i$.*

Proof. Let $(\iota_i, -) : (V_i, -) \to (\overline{V}, -)$ be the following representable morphism. ι_i is just a basis vector of $(\overline{V}, V_i)$ with entries a_{ij} such that

$$a_{ij} = 1 \text{ if } j = \sum_{k=1}^{i-1} \dim V_k + i, \text{ 0 else.}$$

$(\iota_i, -)$ is a split monomorphism (the reversed map is given by its transpose) and the kernel map $\sum_{i=1}^{s} \iota_i : \bigoplus_{i=1}^{s}(V_i, -) \to (\overline{V}, -)$ is generated by pairs $[0]_i \oplus (p-1)[0]_j$ for $[0]_i \in (V_i, -)$ and $[0]_j \in (V_j, -)$.
Let us now modify some of the ι_i. We set $\iota'_i = (\iota_i + (p-1)[0], [0]) : (V_i, -) \to (\overline{V}, -) \oplus \mathbb{F}_q$ for $i > 1$ and $\iota'_1 = \iota_1$. These maps are again split monomorphisms but now $\sum_{i=1}^{s} \iota'_i : \bigoplus_{i=1}^{s}(V_i, -) \to (\overline{V}, -) \oplus \mathbb{F}_q^{s-1}$ has no more kernel,therefore we get again a split monomorphism. □

Lemma 2.1.20 is equivalent to the following.

Corollary 2.1.21. *It holds that $\dim F_\lambda(-)$ is super additive, i.e. $\dim F_\lambda(V \oplus W) \geq \dim F_\lambda(V) + \dim F_\lambda(W)$.*

Though most of the time we will work with resolutions respectively will try to prove in which cases they exist, we do not want to leave the cases unmentioned where functors P_V are known to be noetherian.

Theorem 2.1.22 ([Pow98b])**.** *The functors $(\mathbb{F}_q, -)$ are noetherian for all q.*

And the only known bigger cases.

Theorem 2.1.23 ([Pow98a])**.** *The functor $(\mathbb{F}_2^2, -)$ is noetherian.*

Theorem 2.1.24 ([Dja09])**.** *The functor $(\mathbb{F}_2^3, -)$ is noetherian.*

2.2 A few properties of dimension functions

The first property that comes to us just by looking at the $\mathbb{F}_q$-linearized Hom-spaces is the following.
Fact: $\dim_{\mathbb{F}_q} P(\mathbb{F}_q^n) = \sum_{i=1}^{m} q^{n \cdot \dim_{\mathbb{F}_q} V_i}$.

Following Kuhn [Kuh93] we know that there is a scalar decomposition of every $F(V)$. This actually induces a decomposition of functors, not just vector spaces.

Theorem 2.2.1 ([Kuh93], Section 3.3)**.** *A functor $F \in \mathcal{F}_q$ has a splitting $F = F_0 \oplus \cdots \oplus F_{q-1}$ where $F_i(V) = \{x \in F(V) | F(\lambda \operatorname{id}_V)(x) = \lambda^i x \,\forall\, \lambda \in \mathbb{F}_q\}$.*

Proof. This is a finite field version of MacDonald's eigenspace (degree) composition [Mac08, appendix to chap. 1]. □

Example 2.2.2. Let $q = 2$ and $F = (\mathbb{F}_2, -)$. Then $(\mathbb{F}_2, -)$ should decompose into two direct summands. Let $x = \sum_{i=1}^{s} \mu_i [x_i] \in (\mathbb{F}_2, \mathbb{F}_2^n)$.

$$(\mathbb{F}_2, -)_0(\mathbb{F}_2^n) = \left\{ x \in (\mathbb{F}_2, \mathbb{F}_2^n) \,\middle|\, (\mathbb{F}_2, \lambda \operatorname{id}_{F_2^n})(x) = \sum_{i=1}^{s} \mu_i[\lambda x_i] = \sum_{i=1}^{s} \mu_i[x_i] = \lambda^0 x \,\forall\, \lambda \in \mathbb{F}_2 \right\}$$

In particular for $\lambda = 0$ this yields

$$x = \sum_{i=1}^{s} \mu_i[x_i] = \left(\sum_{i=1}^{s} \mu_i \right) [0] \in \mathbb{F}_2[0].$$

We conclude that $(\mathbb{F}_2, -)_0(\mathbb{F}_2^n) = \mathbb{F}_2[0]$ for any n. So $(\mathbb{F}_2, -)_0 = (0, -) = \mathbb{F}_2$ the constant functor.

$$(\mathbb{F}_2, -)_1(\mathbb{F}_2^n) = \left\{ x \in (\mathbb{F}_2, \mathbb{F}_2^n) \,\middle|\, (\mathbb{F}_2, \lambda \operatorname{id}_{F_2^n})(x) = \sum_{i=1}^{s} \mu_i[\lambda x_i] = \sum_{i=1}^{s} \mu_i[x_i] = \lambda^1 x \,\forall\, \lambda \in \mathbb{F}_2 \right\}$$

In particular for $\lambda = 0$ this yields

$$0 = 0 \cdot x = \sum_{i=1}^{s} \mu_i[0 \cdot x_i] = \left(\sum_{i=1}^{s} \mu_i \right) [0].$$

We conlude that

$$(\mathbb{F}_2, -)_1(\mathbb{F}_2^n) = \left\{ x = \sum_{i=1}^{s} \mu_i[x_i] \in (\mathbb{F}_2, \mathbb{F}_2^n) \,\middle|\, \sum_{i=1}^{s} \mu_i = 0 \right\}$$

We can now even show that $(\mathbb{F}_2, -)_1$ is the projective cover of the identity functor Λ^1. The map

$$\begin{array}{rcl} (\mathbb{F}_2, -)_1(\mathbb{F}_2^n) & \to & \Lambda^1(\mathbb{F}_2^n) = \mathbb{F}_2^n \\ \sum_{i=1}^{s} \mu_i[x_i] & \mapsto & \sum_{i=1}^{s} \mu_i x_i \end{array}$$

This map is surjective since for each $y \in \mathbb{F}_2^n$ the element $[y] + (p-1)[0]$ is a preimage in $(\mathbb{F}_2, -)_1(\mathbb{F}_2^n)$. So far this is only a surjection. The missing properties for the projective cover are content of the following example.

The functor $P(\Lambda^1)$ can be described more explicitly.

Example 2.2.3. Let $q = 2$, the functor $P(\Lambda^1)$ is a uniserial by [Kuh93, Appendix B] and it is a chain of exterior powers where the right hand side are the composition factors.

$$P(\Lambda^1) = \begin{matrix} \Lambda^1 \\ \Lambda^2 \\ \Lambda^3 \\ \vdots \end{matrix}$$

Its dual, the injective functor $I(\Lambda^1)$, which is the injective hull of the identity functor, is also uniserial, just the functors are in the reversed order.

$$I(\Lambda^1) = \begin{matrix} \vdots \\ \Lambda^3 \\ \Lambda^2 \\ \Lambda^1 \end{matrix}$$

Lemma 2.2.4. *There is a decomposition of one in* $\mathrm{End}((\mathbb{F}_q, -)) = \mathbb{F}_q[\mathbb{F}_q]$ *by q orthogonal primitive idempotents given by*

$$\begin{aligned} e_0 &= [0] \\ e_i &= (p-1)\textstyle\sum_{\lambda \in \mathbb{F}_q} \lambda^i [\lambda], \ 1 \leq i \leq q-1. \end{aligned}$$

Proof. We start by showing that these elements are indeed idempotents. It is obvious that $e_0^2 = [0]^2 = [0^2] = [0] = e_0$. For e_i we have

$$e_i^2 = \sum_{\lambda, \mu \in \mathbb{F}_q} (\lambda\mu)^i [\lambda\mu]$$

Now we do some counting for each of the $[\lambda]$. If $[\lambda] = [0]$ we have precisely $2q - 1$ summands of it, q from $[0]e_i$ and one out of every $[\lambda]e_i$. Since we work over a field of characteristic p the coefficient in the result will be $p - 1$. For $[\lambda] = [1]$ there will be $q - 1$ summands since there is precisely one multiplicative inverse of each λ. So the coefficient of $[1]$ in e_i^2 will be $p - 1$. For the other non zero $[\lambda]$ the same argument yields again a coefficient of $(p - 1)$. Therefore, $e_i^2 = e_i$.
Now we need to check that $e_i e_j = 0$. It is obvious that $e_0 e_i = q[0] = 0$. Next we check $e_0 e_i = \sum_\lambda \lambda^i [0] = 0$ since taking the i-th power is an automorphism of $\mathbb{F}_q$ and each element has an additive inverse. Therefore we have found a set of q orthogonal idempotents. Since $\dim \mathbb{F}_q[\mathbb{F}_q] = q$, they must be primitive and their sum must be $[1]$. □

Lemma 2.2.5. *The projective* $(\mathbb{F}_q, -)$ *decomposes as* $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ *modul into* $\mathbb{F}_q$ *and $q - 1$ equidimensional parts P_i with $1 \leq i \leq q - 1$ and the constant part P_0.*

Proof. Without loss of generality we look at $(\mathbb{F}_q, \mathbb{F}_q^n)$. Then we use induction on n.
The above lemma guaranties us that $1_{\mathrm{End}_{\mathcal{F}_q}((\mathbb{F}_q,-))}$ decomposes into q nontrivial idempotents. Since $\mathrm{End}_{\mathcal{F}_q}((\mathbb{F}_q,-)) = \mathbb{F}_q[\mathbb{F}_q]$, this corresponds to an idempotent decomposition of $(\mathbb{F}_q, \mathbb{F}_q^n)$. Let $[\alpha]$ be a basis vector of $(\mathbb{F}_q, \mathbb{F}_q^n)$, so α is a linear map from $\mathbb{F}_q$ to $\mathbb{F}_q^n$. Then we have

$$\begin{array}{ll} [\alpha]e_0 & = [0] \\ [\alpha]e_i & = (p-1)\sum_{\lambda\in\mathbb{F}_q} \lambda^i[\lambda\alpha],\ 1 \le i \le q-1. \end{array}$$

Therefore, $\dim(\mathbb{F}_q, \mathbb{F}_q^n)e_0 = 1 = \dim P_0(\mathbb{F}_q^n)$ the constant part. We can also easily see that $[\alpha]e_i \neq 0$ for basis vectors $[\alpha]$ and $[\alpha]e_i \neq [\beta]e_i$ for $[\alpha] \neq [\beta]$. We conclude that $\dim(\mathbb{F}_q, \mathbb{F}_q^n)e_i = \dim(\mathbb{F}_q, \mathbb{F}_q^n)e_j$ for $0 < i, j$. □

This yields a decomposition as functors and it is known [Kuh93, Appendix B] that these functors are all uniserial.

Corollary 2.2.6. *Let P_i, $1 \le i \le q-1$ be as in the previous lemma. The dimension of each of the $P_i(\mathbb{F}_q^n)$ is $\sum_{j=0}^{n-1} q^j$.*

Proof. Follows from the identity of the geometric sum

$$\frac{q^n - 1}{q - 1} = \sum_{j=0}^{n-1} q^j$$

□

Example 2.2.7. As in most cases as well here the case $q = 2$ is the one that is best investigated. Here are some examples of dimensions of certain indecomposable projectives. In the case of $(\mathbb{F}_2, -)$ we have two of them, $P_0 = \mathbb{F}_q$ the constant functor and $P_1 = P(\Lambda^1)$ the projective cover of the identity functor. The notation P_0 and P_1 uses 0 and 1 as sequences in $\Omega(q, 1)$ of definition 2.0.7:

$$\dim P_0(\mathbb{F}_2^n) = 1 \text{ this even holds for arbitrary q}$$

$$\dim P_1(\mathbb{F}_2^n) = 2^n - 1$$

For $n(\lambda) > 1$ the calculation of the closed form for the dimension of a given projective is no longer that easy. Here we need the primitive Idempotents to get to results. The new indecomposables do no longer have such a nice description as before, so we give them in weight notation.

$$\dim P_{11}(\mathbb{F}_2^n) = \frac{1}{3}2^{2n} - \frac{1}{3}$$

$$\dim P_{01}(\mathbb{F}_2^n) = \frac{1}{3}2^{2n} - 2^n + \frac{2}{3}$$

The results about the growth of the projective functors in the above lemma are promising. It is possible to utilize them in a broader sense. For this we need the following general definition.

Definition 2.2.8. Let $f : \mathbb{N} \to \mathbb{C}$ a (counting) function. We say that f is of closed form if

$$f(n) = \sum_{i=1}^{k} p_i(n)\alpha_i^{t_i n}$$

with $\alpha_i \in \mathbb{C}$ and $t_i \in \mathbb{Z}$ for all i, $p_i(n)$ complex valued polynomials.

As mentioned before, this definition is very general. To make it more accessible in our context, we need to specialize it.

Definition 2.2.9. For a functor $F \in \mathcal{F}_q$ and a non-negative integer n we define $\phi(F, n) = \dim_{\mathbb{F}_q} F(\mathbb{F}_q^n)$.
We say that $\phi(F, n)$ is of closed form if there exists a $k \in \mathbb{Z}_{\geq 0}$ and integers $t_1, \ldots t_k$ such that

$$\phi(F, n) = \sum_{i=1}^{k} p_i(n)\alpha_i^{t_i n}.$$

The $p_i(n)$ are polynomials with coefficients in $\mathbb{Q}$ for all i.
We sometimes say that F is of closed form in this case.

Remark 2.2.10. Without loss of generality, we will be able to order the polynomials in such a way that we can assume $\{t_1, \ldots, t_k\} = \{0, \ldots, \max_k(t_k)\}$. For integers j not included in the set of the t_i we then set $p_j(n) = 0$.

This is somewhat an abuse of the definition of the closed form. But in our context we always find $\alpha_i = q$.
There are some specializations of this definition of the closed form. They read as follows:

Lemma 2.2.11. *Let $0 \to H \to F \to G \to 0$ be a short exact sequence in $\mathcal{F}_q$, then we have $\phi(F, n) = \phi(H, n) + \phi(G, n)$.*

Proof. It follows from the properties of $\dim_{\mathbb{F}_q}$. □

Corollary 2.2.12. *If $\phi(F, n)$ is of closed form, then also $\phi(\Delta F, n)$ is.*

Proof. Obviously, since with $\phi(F, n)$ also $\phi(F(\mathbb{F}_q \oplus -), n) = \phi(F, n+1)$ is of closed form. Then we use the above lemma. □

Example 2.2.13. A representable projective $(V, -)$ is of closed form and not polynomial (unless $V = 0$).

Before we start doing calculations, let us pose a conjecture.

Conjecture 2.2.14. *Let F be a finitely generated functor in $\mathcal{F}_q$, then F is of closed form if and only if F is finitely presented.*

3 On the Grothendiek group $K_0(\mathcal{F}_q)$

We define $K_0(\mathcal{F}_q)$ as the the Grothendiek group of $\mathcal{F}_q$ that is generated by the isomorphism classes of indecomposable projective modules. The aim of this section is to show that the difference functor Δ is diagonalizable on the Grothendieck group $K_0(\mathcal{F}_q)$.
A first step is to start with a very general remark about how the closed form of an indecomposable projective looks like.

3.1 Δ is diagonalizable on $K_0(\mathcal{F}_q)$

When looking at example 2.2.7, we can get the impression that one invariant worth looking at could be the dimensional growth of functors. The category $\mathcal{F}_q$ does not posses a dimension function. The category $\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]$ can also not be used for that. If we evaluate a functor $F \in \mathcal{F}_q$ at an object $V \in \mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]$, we get to the category of vector spaces which has a notion of dimension.
From the start there are only two kinds of functors which have a dimension function of the desired form. These are standard projectives and finite length functors. The following lemma suggests how we can get to more types of functors.

Lemma 3.1.1. *Let $F, G \subset P = \bigoplus_{i=1}^{m}(V_i, -)$ such that $\phi(F, n)$ and $\phi(G, n)$ are of closed form, then $\phi(F \cap G, n)$ is of closed form if and only if $\phi(F + G, n)$ is of closed form.*

Proof. This is just lemma 2.2.11 applied to the exact sequence

$$0 \to F \cap G \to F \oplus G \to F + G \to 0.$$

□

But here we can also see a problem. If we do not have any idea how the dimension function of all but one indecomposable projective looks like, we have no chance to find out what dimension function of the remaining indecomposable looks like. This is somewhat disappointing, so we will have to find a way around it. We will use the following combinatorial identity.

Definition 3.1.2. A counting function $f : \mathbb{N}_0 \to \mathbb{C}$ is said to be of finite recursion type if there exists $d \in \mathbb{N}$ and complex numbers $h_1, \ldots, h_d$ with $h_d \neq 0$ s.t.

$$f(n + d) + h_1 f(n + d - 1) + \cdots + h_d f(n) = 0 \forall n \in \mathbb{N}.$$

Theorem 3.1.3 ([Aig06], Theorem 3.1)**.** *Let $h_1, \ldots, h_d$ be a fixed series of complex numbers with $d \geq 1$ and $h_d \neq 0$, $g(z) = 1 + h_1 z + \ldots + h_d z^d = (1 - \alpha_1 z)^{d_1} \cdots (1 -$*

$\alpha_k z)^{d_k}$. Then for a counting function $f : \mathbb{N}_0 \to \mathbb{C}$ the following are equivalent:
(A1) f solves the recursion $f(n+d) + h_1 f(n+d-1) + \ldots + h_d f(n) = 0 \,\forall\, n \geq 0$.
(A2) f is of closed form:

$$f(n) = \sum_{i=1}^{k} p_i(n) \alpha_i^{t_i n}$$

where $p_i(n)$ are polynomials of degree $< d_i$ and t_i positive integers for $i = 1, \ldots, k$.

For more details we refer the reader to the source.
We might ask ourselves what this has to do with our case. Let us look at the following example, that will give us a connection to finite functors. Afterwards we turn to more general cases.

Example 3.1.4. Let us look at theorem 2.1.11. If F has only finitely many composition factors then, $\Delta^r F$ becomes zero for some r. We get the following dimension formula for $\Delta^t F$:

$$\Delta F(-) = F(\mathbb{F}_q \oplus -)/F(-) \Rightarrow \phi(\Delta F, n) = \phi(F, n+1) - \phi(F, n)$$

$$\Delta^2 F(-) = \Delta F(\mathbb{F}_q \oplus -)/\Delta F(-)$$

$$\begin{aligned} \Rightarrow \phi(\Delta^2 F, n) &= (\phi(F, n+2) - \phi(F, n+1)) - (\phi(F, n+1) - \phi(F, n)) \\ &= \phi(F, n+2) - 2\phi(F, n+1) + \phi(F, n) \end{aligned}$$

By induction we can derive:

$$\phi(\Delta^t F, n) = \sum_{i=0}^{t} \binom{t}{i} (-1)^{t-i} \phi(F, n+i)$$

If $\Delta^r F = 0$ for some r, then this gives rise to a recursive formula for $\phi(F, n)$, therefore it must be of closed form by theorem 3.1.3.

The next class of functors we would like to get our hands on are indecomposable projectives. Rather quickly we arrive at closed form for this class of functors.
Let P_λ be an indecomposable projective. Then it possesses the following projective presentation by standard projectives.

$$(\mathbb{F}_q^{n(\lambda)}, -) \xrightarrow{(e,-)} (\mathbb{F}_q^{n(\lambda)}, -) \twoheadrightarrow P_\lambda \longrightarrow 0$$

We know that $(e, -)$ is an idempotent since P_λ is a direct summand of $(\mathbb{F}_q^{n(\lambda)}, -)$. By the work of Kuhn [Kuh94] we also know that ΔP is again projective if P is. But we can say even more if P is a standard projective.

Definition 3.1.5. $F \in \mathcal{F}_q$ is eigenfunctor of Δ if there exists a $\lambda \in \mathbb{Z}_{\geq 0}$ such that

$$\Delta(F) = F(\mathbb{F}_q \oplus -)/F = F^{\oplus \lambda}.$$

λ is then called an eigenvalue of Δ.

The first thing that comes to mind is the following:

Example 3.1.6.

$$\phi(\Delta(\mathbb{F}_q^m, -), n) = \phi((\mathbb{F}_q^m, -), n+1) - \phi((\mathbb{F}_q^m, -), n) = q^{m(n+1)} - q^{mn} = q^{mn}(q^m - 1)$$

So just by the attained dimensions there are some candidates. But the question remains if $\Delta(\mathbb{F}_q^m, -)$ is really $(\mathbb{F}_q^m, -)^{\oplus q^m - 1}$, in other words, if it is representable. This is the content of the following lemma.

Lemma 3.1.7. *There is a functorial equivalence* $\Delta(\mathbb{F}_q^m, -) \cong (\mathbb{F}_q^m, -)^{\oplus q^m - 1}$.

Proof. For the proof we need again to consider the endo functor Σ with $\Sigma F = F(\mathbb{F}_q \oplus -)$. For a projective this yields:

$$\Sigma(\mathbb{F}_q^m, -) : \mathbb{F}_q^n \mapsto (\mathbb{F}_q^m, \mathbb{F}_q \oplus \mathbb{F}_q^n) \cong (\mathbb{F}_q^m, \mathbb{F}_q) \otimes (\mathbb{F}_q^m, \mathbb{F}_q^n)$$

Therefore, $\Sigma(\mathbb{F}_q^m, -) \cong (\mathbb{F}_q^m, \mathbb{F}_q) \otimes (\mathbb{F}_q^m, -)$ as functors. As in the definition of Δ we can write down the inclusion $(\mathbb{F}_q^m, \iota) : (\mathbb{F}_q^m, -) \to (\mathbb{F}_q^m, \mathbb{F}_q) \otimes (\mathbb{F}_q^m, -)$ via $x \mapsto x \otimes [0]$ and $(\mathbb{F}_q^m, \pi) : (\mathbb{F}_q^m, \mathbb{F}_q) \otimes (\mathbb{F}_q^m, -) \to (\mathbb{F}_q^m, -)$ via $x \otimes y \to x$. Again we have $(\mathbb{F}_q^m, \pi) \circ (\mathbb{F}_q^m, \iota) \cong \mathrm{id}_{(\mathbb{F}_q^m, -)}$ and therefore $\Delta(\mathbb{F}_q^m, -) \cong \big((\mathbb{F}_q^m, \mathbb{F}_q)/\mathbb{F}_q\big) \otimes (\mathbb{F}_q^m, -) \cong (\mathbb{F}_q^m, -)^{\oplus q^m - 1}$ since $\dim \big((\mathbb{F}_q^m, \mathbb{F}_q)/\mathbb{F}_q\big) = q^m - 1$. □

Having this in mind, we go back to indecomposable projectives. The first thing we can observe is the following lemma.

Lemma 3.1.8. *If* $(e, -)$ *is idempotent, then so is* $\Delta(e, -)$*. It is also representable again.*

Proof. Let us again look at the projective presentation of P_λ.

$$(\mathbb{F}_q^{n(\lambda)}, -) \xrightarrow{(e,-)} (\mathbb{F}_q^{n(\lambda)}, -) \twoheadrightarrow P_\lambda \longrightarrow 0$$

If we apply the exact functor Δ to this sequence, we get the following.

$$(\mathbb{F}_q^{n(\lambda)}, -)^{\oplus q^{m-1}-1} \xrightarrow{\Delta(e,-)} (\mathbb{F}_q^{n(\lambda)}, -)^{\oplus q^{m-1}-1} \twoheadrightarrow \Delta P_\lambda \longrightarrow 0$$

Since $(\mathbb{F}_q^{n(\lambda)}, -)^{\oplus q^{m-1}-1}$ is again representable, the same must hold for $\Delta([e], -)$ and since ΔP_λ is projective again, it also should be an idempotent. □

Corollary 3.1.9. $\Delta(e, -)$ *decomposes into primitive idempotents.*

Proof. Easy to see, since ΔP_λ decomposes into indecomposable projectives. $\square$

Now we would like to say something about ΔP_λ explicitly in order to be able to derive a conclusion for $\phi(P_\lambda, n)$. Section 6 of [Kuh94] deals explicitly with this. We need to borrow a few definitions and lemmata from it.

Definition 3.1.10 ([Kuh94], Definition 6.14)**.** Let $\lambda, \mu, \nu \in \Omega(q)$ then $a^\nu_{\lambda,\mu}$ and $a^\nu_{\lambda,\mu}$ are defined as follows:

$$F_\lambda \otimes F_\mu = \sum_\nu a^\nu_{\lambda,\mu} F_\nu$$

$$P_\lambda \otimes P_\mu = \sum_\nu b^\nu_{\lambda,\mu} P_\nu$$

Lemma 3.1.11 ([Kuh94], Lemma 6.15)**.** *It holds that*

$$\Delta P_\nu = \sum_{\lambda,\mu} a^\nu_{\lambda,\mu} P_\lambda \otimes P_\mu$$

and

$$\Delta F_\nu = \sum_{\lambda,\mu} b^\nu_{\lambda,\mu} F_\lambda \otimes F_\mu.$$

Definition 3.1.12 ([Kuh94], Notation/Definition 6.16)**.** Let $\lambda, \mu \in \Omega$ then

- $\lambda + \mu \in \Omega$ is defined by $(\lambda + \mu)_i = \lambda_i + \mu_i$.
- $\lambda \cdot \mu$ is defined recursively on $n(\lambda) + n(\mu)$ as follows. Let $d = \min\{\lambda_{n(\lambda)}, \mu_{n(\mu)}\}$. Let λ' equal λ except that $\lambda_{n(\lambda')}$ is replaced by $\lambda_{n(\lambda)} - d$. Similarly, μ' is defined. Then we define $(\lambda \cdot \mu)_i = \begin{cases} d, & \text{if } i = n(\lambda) + n(\mu), \\ (\lambda' \cdot \mu')_i, & \text{if } i < n(\lambda) + n(\mu), \\ 0, & \text{if } i > n(\lambda) + n(\mu). \end{cases}$
- $\lambda \leq \mu$ if, for all $k \geq 1$,

$$\sum_{i=1}^{k} i\lambda_i + \sum_{i=k+1}^{\infty} k\lambda_i \leq \sum_{i=1}^{k} i\mu_i + \sum_{i=k+1}^{\infty} k\mu_i$$

It is not hard to check that if $\lambda, \mu \in \Omega(q)$, then so is $\lambda \cdot \mu$ and that if $\lambda \leq \lambda'$ and $\mu \leq \mu'$, then both $\lambda + \mu \leq \lambda' + \mu'$ and $\lambda \cdot \mu \leq \lambda' \cdot \mu'$.

Theorem 3.1.13 ([Kuh94], Theorem 6.17)**.** *Let* $\lambda, \mu, \nu \in \Omega(q)$*.*

- *If* $a^{\nu}_{\lambda,\mu} \neq 0$, *then* $\nu \leq \lambda + \mu$ *and* $n(\nu) \geq \max\{n(\lambda), n(\mu)\}$.

- *If* $\lambda + \mu \in \Omega(q)$, *then* $a^{\lambda+\mu}_{\lambda,\mu} = 1$.

Theorem 3.1.14 ([Kuh94], Theorem 6.18)**.** *Let* $\lambda, \mu, \nu \in \Omega(q)$.

- *If* $b^{\nu}_{\lambda,\mu} \neq 0$, *then* $\nu \geq \lambda \cdot \mu$ *and* $n(\nu) \leq n(\lambda) + n(\mu)$.

- $b^{\lambda\cdot\mu}_{\lambda,\mu} = 1$.

With this at hand we can prove the following lemma.

Lemma 3.1.15. *It holds that*

$$\Delta P_\lambda = P_\lambda^{\oplus a} \oplus \bigoplus_{\mu,\ n(\mu)<n(\lambda)} m_\mu P_\mu.$$

Proof. From the above theorems from [Kuh94] we know the following

$$\Delta P_\nu = \sum_{\lambda,\mu} a^{\nu}_{\lambda,\mu} P_\lambda \otimes P_\mu = \sum_{\lambda,\mu} a^{\nu}_{\lambda,\mu} \sum_{\rho} b^{\rho}_{\lambda,\mu} P_\rho.$$

So we should look for which combinations of λ, μ, ρ the product $a^{\nu}_{\lambda,\mu} b^{\rho}_{\lambda,\mu} \neq 0$. We already know that this has to be zero if $n(\rho) > n(\nu)$.
So we want to look at $n(\rho) = n(\nu)$. Let $\nu \geq \rho$ and choose $\lambda = \nu$ and $\mu = 0$ then $a^{\nu}_{\lambda,\mu} = a^{\nu}_{\nu,0} = 1$ and $b^{\nu}_{\lambda,\mu} = b^{\nu}_{\nu,0} = 1$. But $\nu \geq \rho \geq \lambda \cdot \mu = \nu$ yields $\rho = \nu$.
Let now $\rho \geq \nu$. Then $\rho = \nu + \sigma$ and $n(\sigma) < n(\rho)$. To have $a^{\nu}_{\lambda,\mu} \neq 0$, we need to have $\lambda + \mu \geq \nu$ and $n(\nu) \geq \max\{n(\lambda), n(\mu)\}$, but to have $b^{\rho}_{\lambda,\mu} \neq 0$ we need $\rho \geq \lambda \cdot \mu$ and $n(\rho) \leq n(\lambda) + n(\mu)$. If $\lambda + \mu \geq \rho$, we have that $\lambda \cdot \mu \geq \rho \cdot 0 = \rho$ and equality can hold if and only if one of them is zero. If $\rho \geq \lambda + \mu \geq \nu$, we first assume that $\rho = \nu + (1)$. Then $\nu \cdot (1)$ is defined on entries $(\nu \cdot (1))_{i+1} = \rho_i$ and $(\nu \cdot (1))_2 = \min\{1, \nu_1\}$ and $(\nu \cdot (1))_1 = \rho_1 - (\nu \cdot (1))_2$. By the definition of $\lambda \geq \mu$ with $k \geq 3$ this yields that we still have $\lambda \cdot \mu \geq \rho$ and therefore $b^{\rho}_{\lambda,\mu} = 0$ in this case. It is not hard to see that this still holds true if $\rho = \nu + (0, \dots, 0, k)$. Let $\rho = \nu + \sigma_1 + \cdots + \sigma_s$ where $\sigma_i = (0, \dots, 0, k_i)$. Then we have $\rho \geq (\nu + \sigma_1 + \cdots + \sigma_{s-1}) \cdot \sigma_s \geq (\nu + \sigma_1 + \cdots + \sigma_{s-2}) \cdot \sigma_{s-1} < \cdots$ unless $\sigma_i = 0$ for any i.
Therefore, either $a^{\nu}_{\lambda,\mu} = 0$ or $b^{\rho}_{\lambda,\mu} = 0$ unless $\rho = \nu$.
We did not say anything so far about terms of lower order but we do not have to. □

Corollary 3.1.16. *It holds that* $\Delta P_\lambda = P_0 \oplus P_\lambda^{\oplus q-1}$ *if* $n(\lambda) = 1$.

Proof. The components that can occur follow from the above theorem. The multiplicity of P_λ follows from $\Delta(\mathbb{F}_q, -) = (\mathbb{F}_q, -)^{\oplus q-1}$. □

Proposition 3.1.17. *It holds that* $\Delta P_\lambda = P_\lambda^{\oplus q^{n(\lambda)}-1} \oplus \bigoplus_{\mu,\ n(\mu)<n(\lambda)} m_\mu P_\mu$.

Proof. Follows by induction on $n(\lambda)$ from the above corollary. □

Now we can proof that $\phi(P_\lambda, n)$ is of closed form and describe how the $p_i(n)$ and the α_i look like.

Lemma 3.1.18. **??** *The dimension function* $\phi(P_\lambda, n)$ *is of closed form and in* $\phi(P_\lambda, n)$ *all the* $p_i(n)$ *are constant and* $\alpha_i = q^i$ *for all* P_λ.

Proof. By the above corollary we have $\phi(P_\lambda, n+1) - q^{n(\lambda)}\phi(P_\lambda, n) = \sum_\mu \phi(P_\mu, n)$. By induction on $n(\lambda)$ it follows immediately that $\phi(P_\lambda, n)$ is of closed form and that $p_{n(\lambda)}$ is a constant. Since from this construction we can also follow that the length of a recursion that $\phi(P_\lambda, n)$ fulfills is at most of length $n(\lambda)+1$, we have that all the other $p_i(n)$ must be constants as well.
Induction now yields $\alpha_i = q^i$. □

With this identities at our disposal, it is relatively easy to show that Δ is diagonalizable.

Definition 3.1.19. Let $K_0(\mathcal{F}_q)_d := \operatorname{span}_{\mathbb{Z}}\{[P_\lambda] \mid n(\lambda) \leq d\} \subset K_0(\mathcal{F}_q)$ be the subgroup generated by the isomorphism classes of indecomposable direct summands of $(\mathbb{F}_q^d, -)$.

Remark 3.1.20. It is obvious that we have $K_0(\mathcal{F}_q)_d \subset K_0(\mathcal{F}_q)_{d+1}$ and $K_0(\mathcal{F}_q) = \sum_{d\in\mathbb{N}} K_0(\mathcal{F}_q)_d$.

As a corollary of the above calculations we can obtain the following proposition.

Proposition 3.1.21. *It holds that*

1. $\Delta(K_0(\mathcal{F}_q)_d) \subset K_0(\mathcal{F}_q)_d$.
2. Δ *is diagonalizable on* $K_0(\mathcal{F}_q)_d$ *with minimal polynomial*

$$m_{\Delta|_{K_0(\mathcal{F}_q)_d}}(T) = \prod_{k=0}^{d}(T - (q^k - 1))$$

Proof. The first assertion follows directly from proposition 3.1.17.
For the second assertion a second look at propostion 3.1.17 yields $(\Delta - (q^d - 1))(K_0(\mathcal{F}_q)_d) \subset K_0(\mathcal{F}_q)_{d-1}$. This implies $m_{\Delta|_{K_0(\mathcal{F}_q)_d}}(\Delta) = 0$ on $K_0(\mathcal{F}_q)_d$. Therefore the minimal polynomial of Δ on $K_0(\mathcal{F}_q)_d$ must divide $m_{\Delta|_{K_0(\mathcal{F}_q)_d}}(T)$. If we consider $[P] := \bigoplus_{k=0}^{d}[(\mathbb{F}_q^k, -)]$, which is an element in $K_0(\mathcal{F}_q)_d$, we see that $m_{\Delta|_{K_0(\mathcal{F}_q)_d}}(T)$ is in fact the polynomial of minimal degree that lets Δ vanish on $K_0(\mathcal{F}_q)_d$. Since all zeros of $m_{\Delta|_{K_0(\mathcal{F}_q)_d}}(T)$ are pairwise different, this implies that we can diagonalize Δ. □

Remark 3.1.22. The above proposition yields diagonalizablility of Δ only on the Grothendiek group. However it is possible for small dimensions to actually compute a diagonal action of Δ on finite direct sums of indecomposable projective functors.

3.2 On $n \mapsto \dim P_\lambda(\mathbb{F}_q^n)$

The aim of this subsection is to provide more detailed calculations of the coefficients a_i in the closed form of an indecomposable projective.

Lemma 3.2.1. *Let P_λ be an indecomposable projective and $\phi(P_\lambda, n) = \sum_{i=0}^{n(\lambda)} a_i q^{ni}$. Then $a_i \in \mathbb{Q}\, \forall\, i$.*

Proof. Let $a_{n(\lambda)}$ be non-zero. We need to look at the $(n(\lambda)+1) \times (n(\lambda)+1)$ system of linear equations in $(n(\lambda)+1)$ indeterminats a_i. We define $b_k := \phi(P_\lambda, k)$.

$$\begin{array}{llllll} a_{n(\lambda)} & +a_{n(\lambda)-1} & \cdots & +a_0 & = & b_0 \\ a_{n(\lambda)}q^{n(\lambda)} & +a_{n(\lambda)-1}q^{n(\lambda)-1} & \cdots & +a_0 q^0 & = & b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(\lambda)}q^{n(\lambda)n} & +a_{n(\lambda)-1}q^{(n(\lambda)-1)n} & \cdots & +a_0 q^{0n} & = & b_{n(\lambda)} \end{array}$$

Since the b_k are integers, as they are the dimensions of $P_\lambda(\mathbb{F}_q^k)$, and the determinant of the Vandermonde matrix

$$V_{n(\lambda)} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)} & q^{n(\lambda)-1} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ q^{n(\lambda)n} & q^{(n(\lambda)-1)n} & \cdots & 1 \end{pmatrix}$$

is both non-zero and rational, we see that also all the coefficients a_i must be rational. ☐

Corollary 3.2.2. *For $\phi(P_\lambda, n) = \sum_{i=0}^{n(\lambda)} a_i q^{ni}$ it holds that $a_i \in \mathbb{Z}[\det V_{n(\lambda)}^{-1}]$.*

Remark 3.2.3. The entries of the general inverse Vandermonde matrix can be computed explicitly for $V_{n(\lambda)}^{-1}$. They are:

$$(V_{n(\lambda)}^{-1})_{ij} = \frac{\displaystyle\sum_{\substack{0 \le m_0 < \ldots < m_{n(\lambda)-i} \le n(\lambda) \\ m_0, \ldots, m_{n(\lambda)-i} \neq j}} (-1)^{i-1} q^{m_0} \cdots q^{m_{n(\lambda)-i}}}{q^j \displaystyle\prod_{\substack{0 \le m \le n(\lambda) \\ m \neq j}} (q^m - q^j)}$$

Corollary 3.2.4. *Let P_λ be an indecomposable projective functor and $\phi(P_\lambda, n) = \sum_{i=0}^{n(\lambda)} a_i q^{ni}$. Let us further assume $i < j \in \{0, \dots, n(\lambda)\}$ such that $a_i a_j \neq 0$ and $a_k = 0$ for $i < k < j$. Then $a_i a_j < 0$.*

Proof. Since we can also use the adjoint matrix to compute the inverse of $V_{n(\lambda)}$, this follows directly from the computation. □

For most of the P_λ an explicit calculation of the coefficients is rather hard since we cannot say anything about the values of $\phi(P_\lambda, n)$ for $n \leq n(\lambda)$. If P_λ is the projective cover of a simple Steinberg functor, the situation is different.

Remark 3.2.5. For a simple Steinberg functor F_λ it holds that $\phi(F_\lambda, n) = 0$ for $n < n(\lambda)$ and $\phi(F_\lambda, n(\lambda)) = q^{(n(\lambda)-1)n(\lambda)/2}$. We further have $\phi(F_\lambda, n) = \phi(P_\lambda, n)$ for $n \leq n(\lambda)$.

Lemma 3.2.6. *Let P_λ be the projective cover of a simple Steinberg functor F_λ then it holds that*

$$a_0 = (\dim F_\lambda(\mathbb{F}_q^{n(\lambda)}))^2 \det(V_{n(\lambda)-1}) / \det(V_{n(\lambda)})$$

and

$$a_{n(\lambda)} = (-1)^{n(\lambda)+1} \dim F_\lambda(\mathbb{F}_q^{n(\lambda)}) \det(V_{n(\lambda)-1}) / \det(V_{n(\lambda)})$$

Proof. We know that the Vandermonde matrix $V_{n(\lambda)}$ is invertible. Therefore, we can solve the system of linear equations $V_{n(\lambda)} a = b$ by multiplying with the inverse of $V_{n(\lambda)}$. For the the entry w_{ij} in $V_{n(\lambda)}^{-1}$ it holds that

$$w_{ij} = (-1)^{i+j} \det(V_{n(\lambda)})_{ji} / \det V_{n(\lambda)},$$

where $(V_{n(\lambda)})_{ji}$ is the matrix resulting form $(V_{n(\lambda)})_{ji}$ by eliminating the j-th row and the i-th column. So if the only non-zero entry in b is the $(n(\lambda)+1)$st, we only need to look at the $(n(\lambda)+1)$st column of the inverse. This yields:

$$a_0 = (-1)^{2n(\lambda)} \frac{b_{n(\lambda)+1}}{\det V_{n(\lambda)}} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)} & q^{n(\lambda)-1} & \cdots & q^1 \\ \vdots & \vdots & \vdots & \vdots \\ q^{n(\lambda)(n(\lambda)-1)} & q^{(n(\lambda)-1)(n(\lambda)-1)} & \cdots & q^{1(n(\lambda)-1)} \end{pmatrix}$$

We can simplify this in the following way

$$a_0 = \frac{b_{n(\lambda)+1} \cdot \prod_{i=1}^{n(\lambda)-1} q^i}{\det V_{n(\lambda)}} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)-1} & q^{n(\lambda)-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ q^{(n(\lambda)-1)(n(\lambda)-1)} & q^{(n(\lambda)-2)(n(\lambda)-1)} & \cdots & 1 \end{pmatrix}$$

Since $\prod_{i=1}^{n(\lambda)-1} q^i = b_{n(\lambda)+1} = \phi(F_\lambda, n(\lambda))$, we have

$$a_0 = (\dim F_\lambda(\mathbb{F}_q^{n(\lambda)}))^2 \det(V_{n(\lambda)-1})/\det(V_{n(\lambda)})$$

as claimed. For $a_{n(\lambda)}$ we have

$$a_{n(\lambda)} = (-1)^{n(\lambda)+1}\frac{b_{n(\lambda)+1}}{\det V_{n(\lambda)}} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)-1} & q^{n(\lambda)-1} & \cdots & q^0 \\ \vdots & \vdots & \vdots & \vdots \\ q^{n(\lambda)(n(\lambda)-1)} & q^{(n(\lambda)-1)(n(\lambda)-1)} & \cdots & q^{0(n(\lambda)-1)} \end{pmatrix}.$$

From identity this we can see directly

$$a_{n(\lambda)} = (-1)^{n(\lambda)+1} \dim F_\lambda(\mathbb{F}_q^{n(\lambda)}) \det(V_{n(\lambda)-1})/\det(V_{n(\lambda)})$$

□

Next, we need to say something about the other a_i.

Lemma 3.2.7. *Let $i \leq \frac{n(\lambda)}{2}$ then $a_{n(\lambda)-i} = -a'_{n(\lambda)-i}\frac{q^{i-1}}{q^i-1}$ and $a_i = a'_{i-1}\frac{1}{q^i-1}$. Where a'_i are the coefficients in $\phi(P_\mu, n)$ with P_μ the projective cover of the Steinberg functor with $n(\mu) = n(\lambda) - 1$.*

Proof. Let $i \leq \frac{n(\lambda)}{2}$, then we have by the same argumentation as in the above lemma:

$$a_i = (-1)^{n(\lambda)+i}\frac{b_{n(\lambda)+1}}{\det V_{n(\lambda)}} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)} & q^{n(\lambda)-1} & \cdots & q^0 \\ \vdots & \vdots & \vdots & \vdots \\ q^{n(\lambda)(n(\lambda)-1)} & q^{(n(\lambda)-1)(n(\lambda)-1)} & \cdots & q^{0(n(\lambda)-1)} \end{pmatrix}$$

as well as

$$a'_{i-1} = (-1)^{n(\lambda)+i}\frac{b_{n(\lambda)}}{\det V_{n(\lambda)-1}} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)-1} & q^{n(\lambda)} & \cdots & q^0 \\ \vdots & \vdots & \vdots & \vdots \\ q^{(n(\lambda)-1)(n(\lambda)-2)} & q^{(n(\lambda)-1)(n(\lambda)-2)} & \cdots & q^0 \end{pmatrix}$$

Now we set $(q^i - 1)a_i = a'_{i-1}$. After some eliminations we get

$$\frac{(q^i-1)q^{n(\lambda)-1}}{\prod_{j=0}^{n(\lambda)-1}(q^j - q^{n(\lambda)})} \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)} & q^{n(\lambda)-1} & \cdots & q^0 \\ \vdots & \vdots & \vdots & \vdots \\ q^{n(\lambda)(n(\lambda)-1)} & q^{(n(\lambda)-1)(n(\lambda)-1)} & \cdots & q^{0(n(\lambda)-1)} \end{pmatrix} =$$

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{n(\lambda)-1} & q^{n(\lambda)-1} & \cdots & q^0 \\ \vdots & \vdots & \vdots & \vdots \\ q^{(n(\lambda)-1)(n(\lambda)-2)} & q^{(n(\lambda)-1)(n(\lambda)-2)} & \cdots & q^{0(n(\lambda)-2)} \end{pmatrix}$$

Now both matrices are again of Vandermonde type so their determinants can be computed explicitly. This yields:

$$(q^i - 1)q^{n(\lambda)-1} / \prod_{j=0}^{n(\lambda)-1} (q^j - q^{n(\lambda)}) \cdot \prod_{0 \le k < j \le n(\lambda)-1} (x_k - x_j) = \prod_{0 \le k < j \le n(\lambda)-1} (y_k - y_j)$$

Where $x_j = q^j$ for $j < i$ and $x_j = q^{j+1}$ for $j \geq i$ and $y_j = q^j$ for $j < i - 1$ and $x_j = q^{j+1}$ for $j \geq i - 1$. With this at hand we can simplify both sides. First of all we will eliminate all the terms coming from $q^{n(\lambda)}$.

$$(q^i - 1)q^{n(\lambda)-1} / (q^i - q^{n(\lambda)}) \cdot \prod_{0 \le k < j \le n(\lambda)-2} (x_k - x_j) = \prod_{0 \le k < j \le n(\lambda)-1} (y_k - y_j)$$

Next we eliminate all the factors that occur on both sides.

$$(q^i - 1)q^{n(\lambda)-1} / (q^i - q^{n(\lambda)}) \cdot \prod_{j=i}^{n(\lambda)-2} (q^{i-1} - q^j) \cdot \prod_{j=0}^{i-2} (q^j - q^{i-1}) =$$

$$\prod_{j=i+1}^{n(\lambda)-2} (q^i - q^j) \cdot \prod_{j=0}^{i-1} (q^j - q^i)$$

Now we can factor out on the right hand side and eliminate even more.

$$(q^i - 1)q^{n(\lambda)-1} / (q^i - q^{n(\lambda)}) \cdot (q^{i-1} - q^{n(\lambda)-2}) = q^{n(\lambda)-i-1} q^{i-1} (q^i - 1)$$

And we can finally see that our claim $a_i = a'_{i-1} \frac{1}{q^i - 1}$ holds.
The second claim, $a_{n(\lambda)-i} = -a'_{n(\lambda)-i} \frac{q^{i-1}}{q^i - 1}$, is proven similarly. The proof also yields that if $n(\lambda)$ is even, both values for $a_{\frac{n(\lambda)}{2}}$ coincide. □

This recursive definition has a nice consequence.

Corollary 3.2.8. *For coefficients a_i and a'_i as in the above lemma it holds that $|a'_i| \geq |a_i|$.*

Proof. This follows right from the recursive definition of the a_i. □

So far we did only mention the case where P_λ is the projective cover of a Steinberg functor. In the more general case we can obtain the following lemma.

Proposition 3.2.9. *Let P_μ be an indecomposable projective which is not the projective cover of a Steinberg functor. Let us further assume $\phi(P_\mu, n) = \sum_{i=0}^{n(\mu)} c_i q^{in}$. Then $|c_i| < |a_i|$ with a_i the coefficients in the closed form of the projective cover P_λ of a Steinberg functor.*

Proof. Though we do not use induction directly, we prove the lemma by starting to look at the case where $\phi(P_\mu, n) = 0$ for $n < n(\mu) - 1$ and afterwards by looking at the case for $n(\mu) - k$ for bigger k.
Let $b_{n(\mu)} := \phi(P_\mu, n(\mu))$ and $b_{n(\mu)-1} := \phi(P_\mu, n(\mu) - 1)$. We further assume that $V_{n(\mu)}^{-1} = (w_{ij})_{i,j}$, then we have

$$c_i = w_{i,n(\mu)-1} b_{n(\mu)-1} + w_{i,n(\mu)} b_{n(\mu)}.$$

We can use the following simplifications that will make the right hand side bigger:

$$w_{i,n(\mu)-1} < -(q^{n(\mu)} + 1) w_{i,n(\mu)-1}$$

This follows from the definition of the general element of the inverse of the Vandermonde matrix. We can further use

$$q^{n(\mu)-1} b_{n(\mu)-1} < b_{n(\mu)} < q^{n(\mu)} b_{n(\mu)-1}.$$

So we can follow

$$c_i = w_{i,n(\mu)-1} b_{n(\mu)-1} + w_{i,n(\mu)} b_{n(\mu)} < w_{i,n(\mu)} (b_{n(\mu)} - (q^{n(\mu)} - 1) b_{n(\mu)-1}) <$$
$$w_{i,n(\mu)} (q^{n(\mu)} b_{n(\mu)-1} - (q^{n(\mu)} - 1) b_{n(\mu)-1}) = w_{i,n(\mu)} b_{n(\mu)-1} < w_{i,n(\mu)} b_{n(\mu)}$$

From here it follows that

$$|c_i| = |w_{i,n(\mu)} b_{n(\mu)}| < |w_{i,n(\mu)} q^{\frac{n(\mu)}{2}(n(\mu)-1)}| = |a_i|.$$

Now we look at the general case. We use the same simplifications. If follows:

$$\sum_{j=0}^{k} w_{i,n(\mu)-j} b_{n(\mu)-j} < \sum_{j=0}^{k} (-1)^{k-j} \left(\prod_{l=0}^{j} q^{l(n(\mu)-1)+1} \right) w_{i,n(\mu)} q^{n(\mu)(k-j)} b_{n(\mu)-k} <$$
$$\sum_{j=0}^{k} (-1)^{k-j} \left(\prod_{l=0}^{j} q^{(l+k-j)n(\mu)-l+1} \right) w_{i,n(\mu)} b_{n(\mu)-k} <$$
$$w_{i,n(\mu)} b_{n(\mu)-k} q^{k(n(\mu)-1)} < w_{i,n(\mu)} b_{n(\mu)}$$

Since the all the summands in the alternating sum are limited by $q^{k(n(\mu)-1)}$, this again yields

$$|c_i| = |w_{i,n(\mu)} b_{n(\mu)}| < |w_{i,n(\mu)} q^{\frac{n(\mu)}{2}(n(\mu)-1)}| = |a_i|.$$

□

The insight of this proposition can be used to determine when instead of diagonalizability on $K_0(\mathcal{F}_q)$ we can actually diagonalize Δ on $\mathcal{F}_q$.

Corollary 3.2.10. *Let P_d be the full subcategory of $\mathcal{F}_q$ that consists of direct sums of indecomposable projective functors P_λ with $n(\lambda) < d$. Then Δ is diagonalizable on P_d if and only if the a direct sum of the projective cover of a Steinberg functor and functors of lower weight is an eigenfunctor in P_d.*

Proof. Let P_μ be the projective cover of a non-Steinberg simple functor. Let further P_λ be a projective cover of a Steinberg functor such that $n(\lambda) < n(\mu)$. Since $|c_i| < |a_i|$ for any non-Steinberg projective we can always find $x, y \in \mathbb{N}$ such that in $\phi(P_\mu^{\oplus x} \oplus P_\lambda^{\oplus y}, n) = \sum_{i=1}^{n(\mu)} \hat{c}_i q^{in}$ we have $\hat{c}_j < 0$ for $0 \leq j < n(\mu)$ the largest index such that $\hat{c}_j \neq 0$. Now we can iterate as we would do in the Steinberg case. The converse is obviously true. □

The following example will shed some light on how to construct such eigenfunctors in $\mathcal{F}_q$.

Example 3.2.11. Let $K_0(\mathcal{F}_q)_d$ be the subgroup generated by the isomorphism classes of indecomposable projectives P_λ such that $n(\lambda) \leq d$.
If $d = 0$ the only functor we can look at is $\mathbb{F}_q$, the constant functor. We have $\Delta F_q = 0 = \mathbb{F}_q^{\oplus(q^0-1)}$ as we claimed.
If $d > 0$, we look at P_λ the projective cover of one of the Steinberg functors for $n(\lambda) = d$. From the proofs of the above lemmata we can observe that if $\phi(P_\lambda, n) = \sum_{i=0}^{n(\lambda)} a_i q^{in}$ is the closed form of its dimension function, then $a_i/a_{n(\lambda)} \in \mathbb{Z}$, which is just a consequence of the construction of the inverse via the adjoint matrix.
We now want to study $\phi(P_\lambda^{\oplus a_{n(\lambda)}^{-1}} \oplus P_\mu^{\oplus((q-1)a_{n(\lambda)})^{-1}}, n) = \sum_{i=0}^{n(\lambda)} \hat{a}_i q^{in}$ with P_μ the projective cover of a Steinberg functor with $n(\mu) = n(\lambda) - 1$. We have $\hat{a}_{n(\lambda)} = 1$ by construction. For $\hat{a}_{n(\lambda)-1}$ we can obtain

$$\hat{a}_{n(\lambda)-1} = \frac{a_{n(\lambda)-1}}{a_{n(\lambda)}} + \frac{a'_{n(\lambda)-1}}{a_{n(\lambda)}(q-1)}.$$

Substitution via the above lemma yields:

$$\hat{a}_{n(\lambda)-1} = \frac{a_{n(\lambda)-1}}{a_{n(\lambda)}} + \frac{a'_{n(\lambda)-1}}{a_{n(\lambda)}(q-1)} = -\frac{a'_{n(\lambda)-1}}{a_{n(\lambda)}(q-1)} + \frac{a'_{n(\lambda)-1}}{a_{n(\lambda)}(q-1)} = 0.$$

Calculations for $\hat{a}_{n(\lambda)-2}$ yield

$$\hat{a}_{n(\lambda)-2} = \frac{a_{n(\lambda)-2}}{a_{n(\lambda)}} + \frac{a'_{n(\lambda)-2}}{a_{n(\lambda)}(q-1)} = -\frac{qa'_{n(\lambda)-2}}{(q^2-1)a_{n(\lambda)}} + \frac{a'_{n(\lambda)-2}}{a_{n(\lambda)}(q-1)} = \frac{a'_{n(\lambda)-2}}{(q^2-1)a_{n(\lambda)}}.$$

Therefore $\hat{a}_{n(\lambda)-2}$ is negative since $a'_{n(\lambda)-2}$ has to be smaller than zero. We can also follow $\hat{a}_{n(\lambda)-2} > a_{n(\lambda)-2}$. For the general $\hat{a}_{n(\lambda)-i}$ we have

$$\hat{a}_{n(\lambda)-i} = \frac{a_{n(\lambda)-i}}{a_{n(\lambda)}} + \frac{a'_{n(\lambda)-i}}{a_{n(\lambda)}(q-1)} = -\frac{a'_{n(\lambda)-i}q^{i-1}}{a_{n(\lambda)}(q^i-1)} + \frac{a'_{n(\lambda)-i}}{a_{n(\lambda)}(q-1)} =$$

$$\frac{a'_{n(\lambda)-i}}{a_{n(\lambda)}} \frac{\sum_{j=0}^{i-2} q^j}{(q^i-1)} \Rightarrow |a_{n(\lambda)}\hat{a}_{n(\lambda)-i}| < |a'_{n(\lambda)-i}|.$$

On the other hand we get for $\hat{a}_i$

$$\hat{a}_i = \frac{a_i}{a_{n(\lambda)}} + \frac{a'_i}{a_{n(\lambda)}(q-1)} = \frac{a'_{i-1}}{a_{n(\lambda)}(q^i-1)} + \frac{a'_i}{a_{n(\lambda)}(q-1)}.$$

If $a'_i > 0$ we can now follow

$$\hat{a}_i = \frac{a'_{i-1}}{a_{n(\lambda)}(q^i-1)} + \frac{a'_i}{a_{n(\lambda)}(q-1)} < \frac{-a'_i}{a_{n(\lambda)}(q^i-1)} + \frac{a'_i}{a_{n(\lambda)}(q-1)} = \frac{a'_i}{a_{n(\lambda)}} \frac{\sum_{j=1}^{i-1} q^j}{q^i-1}.$$

For $a'_k > 0$ we can argue similarly. It follows $|a_{n(\lambda)}\hat{a}_i| < |a'_i|$ for any i. So we can continue with $P_\lambda^{\oplus a_{n(\lambda)}^{-1}} \oplus P_\mu^{\oplus((q-1)a_{n(\lambda)})^{-1}}$ to construct an eigenfunctor to the eigenvalue $q^{n(\lambda)} - 1$ as $|a_{n(\lambda)}\hat{a}_i| < |a'_i|$ for any i.
The next step is to add $(a_n(q^2-1)(q-1))^{-1}$ copies of P_ν, a projective cover of a Steinberg functor with $n(\nu) = n(\lambda) - 2$, to $P_\lambda^{\oplus a_{n(\lambda)}^{-1}} \oplus P_\mu^{\oplus((q-1)a_{n(\lambda)})^{-1}}$ to make $\hat{a}_{n(\lambda)-2}$ vanish. By similar computations as above we can follow that

$$\left|\hat{a}_i + \frac{a''_i}{a_{n(\lambda)}(q^2-1)(q-1)}\right| \leq \left|\frac{a'_i}{a_{n(\lambda)}(q-1)} + \frac{a''_i}{a_{n(\lambda)}(q^2-1)(q-1)}\right|.$$

Therefore, in the k-th step of this process we have $\hat{a}^{(k)}_{n(\lambda)-k-1} < 0$ by an induction on $n(\lambda)$ and the above corollary. We can continue and will obtain an eigenfunctor of Δ in $\mathcal{F}_q$ for each projective cover of a Steinberg functor.

4 The category $\mathcal{F}_q$ viewed as a module category

In this section we provide an idea on how to view finitely presented functors as modules over a path algebra. This result is then used to calculate kernels on one example, the Kronecker case.

4.1 Finitely presented functors and quivers

In this subsection we restrict ourselves to finitely generated functors. The benefit is that we can describe them more explicitly as modules over a ring with several objects as defined in the previous subsection.

Definition 4.1.1. Let γ be the following quiver. We start with the infinite bipartite multigraph with countably many edges between two vertices. Now we orient all edges from one set of vertices to the other.
Explicitly $\gamma = (Q_0, Q_1)$ with

1. $Q_0 = \{i \in \mathbb{N}\}\dot\cup\{j \in \mathbb{N}\} =: Q_0(s)\dot\cup Q_0(t)$
2. $Q_1 = \{i \xrightarrow{ijl} l \,|\, i, j, l \in \mathbb{N},\, i \in Q_0(s),\, j \in Q_0(t)\}$

Definition 4.1.2. Let Γ be the following quiver. We start with the infinite bipartite graph with exactly one edge between two vertices. Now we orient all edges from one set of vertices to the other.
Explicitly $\gamma = (Q_0, Q_1')$ with

1. Q_0 as before
2. $Q_1' = \{i \xrightarrow{ij} j \,|\, i, j \in \mathbb{N},\, i \in Q_0(s),\, j \in Q_0(t)\}$

Definition 4.1.3. Let Q be a quiver. We define the category $\operatorname{rep}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]} Q$ as follows. The objects are sets (V_i, V_a) such that V_i is a finite dimensional vector space and $V_a \in (V_{s(a)}, V_{t(a)})$. Morphisms are defined in the obvious way.

Definition 4.1.4. We define the following map

$$\Phi : \operatorname{rep}_{\mathbb{F}_q} \gamma \to \operatorname{rep}_{\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]} \Gamma$$

on objects

$$(V_i, V_i \xrightarrow{a_{ijl}} V_j) \mapsto (V_i, \sum_{l \in \mathbb{N}} [a_{ijl}])$$

on morphisms

$$\begin{array}{ccccccc} V_i & \xrightarrow{a_{ijl}} & V_j & & V_i & \xrightarrow{\sum[a_{ijl}]} & V_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} & \longmapsto & \downarrow{\scriptstyle [f_i]} & & \downarrow{\scriptstyle [f_j]} \\ W_i & \xrightarrow{b_{ijl}} & W_j & & W_i & \xrightarrow{\sum[b_{ijl}]} & W_j \end{array}$$

This defines a functor.

Lemma 4.1.5. *The functor Φ is faithful. If $p = q$, then Φ is essentially surjective.*

Proof. It is faithful because it is on Hom-sets a restriction of the injection

$$\bigoplus_{i\in Q_0} \mathrm{Hom}_{\mathbb{F}_q}(V_i, W_i) \longrightarrow \bigoplus_{i\in Q_0}(V_i, W_i)$$

$$f_i \longmapsto [f_i]$$

If $p = q$ it is easily seen to be essentially surjective. □

But what does this have to do with our category $\mathcal{F}_q$?

Lemma 4.1.6. *The functor* $\eta : \mathrm{rep}_{\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]}\Gamma \to \mathcal{F}_{q\,fp}$ *defined by*

$$(V_i, a_{ij}) \mapsto \mathrm{coker}\,\mathbf{a}$$

with

$$\mathbf{a} = \big((a_{ij}, -)_{j\in Q_0(t), i\in Q_0(s)}\big) : \bigoplus_{i\in Q_0(t)} (V_i, -) \to \bigoplus_{j\in Q_0(s)} (V_j, -)$$

is dense and full.

Proof. This is obviously dense since we can display every morphism and by that every finitely presented functor in that way.
To show that it is full, we just have to look at the following resolutions, where we make use of lemma 2.1.19 to simplify to one standard projective on either side.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & (W,-) & \xrightarrow{(f,-)} & (V,-) & \xrightarrow{\pi_F} & F & \longrightarrow 0 \\ & & & \downarrow & & \downarrow \theta & \\ \cdots \longrightarrow & (W',-) & \xrightarrow{(f',-)} & (V',-) & \xrightarrow{\pi_G} & G & \longrightarrow 0 \end{array}$$

If $\theta : F \to G$ is a natural transformation, we have that π_G is surjective and $\theta \circ \pi_F$ starts in a projective, therefore we must have a map from $(V, -)$ to $(V', -)$ such that the square commutes. By passing to the image of $(f, -)$ we will also get a fitting map from $(W, -)$ to $(W', -)$. But this does mean nothing else than that we are able to obtain a map of the representations from this. Therefore, the map η is full. □

Remark 4.1.7. The functor η is not faithful since the representation $V \xrightarrow{\mathrm{id}} V$ is mapped to zero for any V.
However, η preserves direct sums.

Definition 4.1.8. Let $K \subset \mathrm{rep}_{\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]}$ be the full subcategory with objects M such that $\eta(M) = 0$. By previous remark it is additive.

Corollary 4.1.9. *The functor η induces an equivalence of categories*

$$\overline{\eta} : \mathrm{rep}_{\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]}\,\Gamma/K \cong \mathcal{F}_{q\,fp}.$$

Two morphisms f_1 and f_2 are equivalent under $\overline{\eta}$ if $\eta(f_1) = \eta(f_2)$.

Let us put together what we got here.

$$\begin{array}{ccccc} K & \hookrightarrow & \mathrm{rep}_{\mathbb{F}_q[\mathrm{mod}\,\mathbb{F}_q]}\,\Gamma & \xrightarrow{\eta'} \!\!\!\!\rightarrow & \mathcal{F}_{q\,fp} \\ & & & & \big\downarrow i \\ & & & & \mathcal{F}_{q\,fg} \end{array}$$

4.2 An example: the Kronecker case

In this subsection we study kernels of representable morphisms $(f,-) : (V,-) \to (W,-)$ with $f = [f_1] + \mu[f_2]$. The associated representation of the Kronecker quiver is

$$K_f : W \underset{f_2}{\overset{f_1}{\rightrightarrows}} V$$

Proposition 4.2.1. *Let $(f,-) : (V,-) \to (W,-)$ be a representable morphism such that $f = [f_1] + \mu[f_2]$, then* $\ker(f,-)$ *is finitely generated.*

Remark 4.2.2. Without loss of generality we can assume $f_1 \neq f_2$ and $\mu \neq 0$ because kernels of maps $([g],-)$ with g linear are completely described by the rank of the matrix g.
Also, once we understand the kernel of $([f_1]+\mu[f_2],-)$, $\mu \neq 0$ we are done because this kernel is isomorphic to the kernel of $([f_2] + \mu^{-1}[f_1],-)$. This means the roles of f_1 and f_2 can be interchanged.
For now we assume that K_f is indecomposable. These are the cases we need to look at:

1. $f_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $f_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

2. $f_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$, $f_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$

3. $f_1 = \mathrm{id}$, $f_2 = B_\lambda$ with B_λ a matrix in Jordan-normal-form with eigenvalue λ.

Remark 4.2.3. If $(f, -)$ is a epimorphism, then

$$0 \to \ker(f, -) \to (V, -) \to (W, -) \to 0$$

is a split short exact sequence, so the kernel is a direct summand of $(V, -)$. The projection $(V, -) \twoheadrightarrow \ker(f, -)$ shows that $\ker(f, -)$ is finitely generated. Therefore, it is sometimes easier to look at the image in order to determine the kernel.

Now we go through each of the cases one by one to determine the kernel. In any case we can again assume by pre-compositon that we look at $f = [f_1] + \mu[f_2]$. It turns out that if $\mu \neq -1$, calculations are somewhat different than if $\mu = -1$. For all cases we assume that we look at $(f, \mathbb{F}_q^n) : (\mathbb{F}_q^{m'}, \mathbb{F}_q^n) \to (\mathbb{F}_q^m, \mathbb{F}_q^n)$.
Case 1 Here we start by looking at the image. Let $\mu \neq -1$ then $[0] \circ f = (\mu+1)[0]$ and therefore $[0]$ is in the image of f. Now we look at a general basis vector $[\alpha] \in (\mathbb{F}_q^{m+1}, \mathbb{F}_q^n)$. We have $[\alpha] \circ f = [\alpha_{m+1}] + \mu[\alpha_1]$ with $[\alpha_i] = [\alpha_1, \dots, \hat{\alpha_i}, \dots, \alpha_{m+1}] \in (\mathbb{F}_q^m, \mathbb{F}_q^n)$ is the element where the i-th column "has to take its head". Therefore, an element $[\beta]$ with only non-zero entries in the first column is mapped to $[\beta_{m+1}]+\mu[0]$. As $[0]$ is in the image, all the elements with just the first column non-zero are in the image. Now we look at the elements $[\beta]$ with just entries in the first and second column. Those are mapped to $[\beta_{m+1}]+\mu[\beta_1]$ where $[\beta_1]$ is an element with non-zero entries just in the first column. By the discussion before all elements with entries in the second column are in the image. We can iterate this process and see that all elements with non-zero entries in m columns are in the image. Therefore, $(f, -)$ is an epimorphism in this case. Since $(\mathbb{F}_q^m, -)$ is projective, the kernel of $(f, -)$ must be a direct summand of $(\mathbb{F}_q^{m+1}, -)$ and therefore finitely generated.
If $\mu = -1$, we have $[0] \circ f = 0$. But the rest of the arguments works as well. Therefore we can associate an element $[\alpha] \circ f$ to every non-zero basis vector $[\beta] \in (\mathbb{F}_q^m, \mathbb{F}_q^n)$. As f is not injective, we get that the image of f is $(\mathbb{F}_q^m, -)/\mathbb{F}_q$. The kernel is therefore isomorphic to $(\mathbb{F}_q^{m+1}, -)/(\mathbb{F}_q^m, -) \oplus \mathbb{F}_q$ which is finitely generated.
Case 2: Again we will first assume $\mu \neq -1$ and also start by looking at the image. In this case $[0]$ is in the image of $(f, \mathbb{F}_q^n)$ and a general $[\alpha]$ is mapped to $[\alpha, 0] + \mu[0, \alpha]$. Therefore, it is not hard to see that $(f, -)$ is a monomorphism and therefore has only a trivial kernel. If $\mu = -1$, the element $[0]$ and therefore the constant functor is in the kernel. In both cases we have a finitely generated kernel.
Case 3: This case is of course the most complicated one. There are several phenomena that can appear. We work through them case by case. First of all we assume that $\lambda = 0$. In this case the computation is completely analogue to case 1 and $(f, -)$ is an epimorphism, so even an isomorphism for dimension reasons.
The case for general λ is not so easy anymore. It is the content of the following lemmata.

Lemma 4.2.4. *Let $A \in \mathrm{GL}_m(\mathbb{F}_q)$ and $\mu \in \mathbb{F}_q^\times$. If $(-\mu)^{\operatorname{ord} A} = 1$, then $f = [E_m] + \mu[A]$ is not invertible.*

Proof. It holds that $(-\mu)^{\operatorname{ord} A} = 1$. We can follow that

$$\sum_{j=0}^{\operatorname{ord}(-\mu)-1} ((-\mu)^{\operatorname{ord} A})^j = \operatorname{ord}(-\mu) \cdot 1 \neq 0$$

since $p \not| \operatorname{ord}(-\mu)$ because $p \not| q-1$. By multiplication with $-\mu$ the above yields

$$a_1 := \sum_{j=0}^{\operatorname{ord} A - 1} (-\mu)^{j \operatorname{ord} A + 1} \neq 0.$$

Let $l := \operatorname{ord}(-\mu) \operatorname{ord} A$ and

$$g := \sum_{i=1}^{l} (-\mu)^i [A]^i = \sum_{i=1}^{\operatorname{ord} A} (-\mu)^{i-1} a_1 [A^i] \neq 0 \text{ since } a_1 \neq 0,$$

where the coefficient in front of $[A^i]$ is $a_i := \sum_{j \in J} (-\mu)^j$ with $J = \{1 \leq i \leq l \mid \operatorname{ord} A | (j-i)\}$. It holds that $(-\mu) a_i = a_{i+1}$ and $a_1 = \sum_{j \mid \operatorname{ord} A | (j-1)} (-\mu)^j = \sum_{j=0}^{\operatorname{ord}(-\mu)-1} (-\mu)^{j \operatorname{ord} A + 1}$. With g defined in such a way we can now look at the composition $f \circ g$. We have

$$\begin{aligned} f \circ g &= \sum_{i=1}^{l} (-\mu)^i [A^i] + \mu \sum_{i=1}^{l} (-\mu)^i [A^{i+1}] \\ &= \sum_{i=1}^{l} (-\mu)^i [A^i] - \sum_{i=1}^{l} (-\mu)^i [A^{i+1}] = g - g = 0 \end{aligned}$$

□

Remark 4.2.5. Let $B_{\lambda,m} = \lambda E_m + J_m$ with

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Now $r_m := \operatorname{ord} B_{\lambda,m}$ is the minimal r such that $\operatorname{ord} \lambda | r$ and

$$E_m - B_{\lambda,m}^r = r \lambda^{r-1} J_m + \binom{r}{2} \lambda^{r-2} J_m^2 + \ldots + \binom{r}{k} \lambda^{r-k} J_m^k = 0$$

$$\Leftrightarrow p|\binom{r}{j},\ 1 \leq j \leq k.$$

For $1 \leq n \leq m$ define

$$g_n := \sum_{j=1}^{l_n} (-\mu)^j \left[\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & B^j_{\lambda,n} \end{pmatrix} \right]$$

and

$$l_n := \operatorname{lcm}(\operatorname{ord}(-\mu), \operatorname{ord} B_{\lambda,n})$$

Lemma 4.2.6. *It holds for $f = [E_m] + \mu[B_{\lambda,m}]$ that*

$$\ker(f, -) = \sum_{n=1}^{m} \operatorname{Im}(g_n, -).$$

Proof. Let $\alpha = \sum_{j=1}^r \nu_j[\alpha_j]$, $\nu_j \neq 0$, $\alpha_i \neq \alpha_j \,\forall\, i \neq j$ be in (V, U). We have $\alpha \in \ker(f, U) \Leftrightarrow \alpha = -\mu\alpha[B_{\lambda,m}] \Leftrightarrow \pi \in \mathcal{S}_r$ such that $\nu_j = -\mu\nu_{\pi(j)}$ and $\alpha_j = \alpha_{\pi(j)}B_{\lambda,m}$. Without loss of generality $\pi = (1, \ldots, r) \in \mathcal{S}_r$. If π is not an r-cycle, we can write it as a product of disjoint cycles and replace α by a summand over the support of one of these cycles. It follows:

1. $\nu_1 = (-\mu)^r\nu_1 \Leftrightarrow \operatorname{ord}(-\mu)|r$
2. $\alpha_1 = \alpha_1 B^r_{\lambda,m} \Leftrightarrow \operatorname{ord}\lambda|r$ and $\alpha_1(E_m - B^r_{\lambda,m}) = 0$

From now on we assume that $\operatorname{ord}(-\mu)|r$ and $\operatorname{ord}\lambda|r$. We also assume that $\nu_1 = 1$.
Case 1: It holds that $p|\binom{r}{j}$, $1 \leq j \leq k = \min(m-1, r)$. Since $\operatorname{ord}(-\mu)|r$, it follows that $l_m|r$. We can now follow that $(E_m - B^r_{\lambda,m}) = 0$ and α_1 can be chosen arbitrary, i.e.

$$\alpha = \sum_{j=1}^{r} (-\mu)^j[\alpha_1 B^j_{\lambda,m}] = \frac{r}{l_m}\sum_{j=1}^{l_m} (-\mu)^j[\alpha_1 B^j_{\lambda,m}].$$

Since $\alpha \neq 0$, we get $\frac{r}{l_m} \neq 0$, i.e. not divisible by p, and we obtain all elements of $\operatorname{Im}(g_m, U)$ in this way.
Case 2: There is an $u \in \{1, \ldots, k = \min(m-1, r)\}$ such that $p|\binom{r}{j}$ for $1 \leq j \leq u-1$ and $p\not|\binom{r}{u}$. It follows that $l_u|r$. We now have

$$\ker((E_m - B^r_{\lambda,m})^t) = \ker((J^u_m)^t) = \left\langle e^t_{m-u+1}, \ldots, e^t_m \right\rangle.$$

It holds that $\alpha_1(E_m - B^r_{\lambda,m}) = 0 \Leftrightarrow \alpha_1 = (0,\dots,0,x_{m-u+1},\dots,x_m)$ with arbitrary columns x_i. Then it holds that

$$\begin{aligned}\alpha &= \sum_{j=1}^{r}(-\mu)^j[\alpha_1 B^j_{\lambda,m}] \\ &= \sum_{j=1}^{r}(-\mu)\left[x\begin{pmatrix}0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & B^j_{\lambda,u}\end{pmatrix}\right] \\ &= \frac{r}{l_u}\sum_{j=1}^{l_u}(-\mu)\left[x\begin{pmatrix}0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & B^j_{\lambda,u}\end{pmatrix}\right]\end{aligned}$$

for $x \in M_{t\times m}(\mathbb{F}_q)$. Since $\alpha \neq 0$, we get $\frac{r}{l_u} \neq 0$, i.e. not divisible by p. Clearly, we obtain all elements in $\mathrm{Im}(g_u, U)$ in this way. □

Remark 4.2.7. The algorithm presented in section 8 gives $\mathrm{Im}\left(\sum_{n=1}^{m} g_n, -\right)$ as kernel of $(f,-)$. But for our purposes the weaker version presented in the lemma is enough.

Lemma 4.2.8. *Let $(f,-) : (V,-) \to (W,-)$ be a representable morphism such that $f = [f_1] + \mu[f_2]$ and f one of the indecomposable Kronecker cases, then $\ker(f,-)$ is finitely generated.*

Proof. The lemma is the result of the above computations. □

Now we can study what happens in the case of a decomposable K_f. We could again use a case by case study, but luckily we can obtain the following nice construction.

Remark 4.2.9. Let $B_{\lambda,m}$ be a matrix in Jordan normal form and r_m as in remark 4.2.5. Now we consider a tuple of positive integers $\underline{m} = (m_1,\dots,m_s)$, $\underline{n}$ analogue. We say $\underline{n} \leq \underline{m}$ if $n_i \leq m_i$ for any i. Let us further consider a tuple $\underline{\lambda} = (\lambda_1,\dots,\lambda_s)$ of elements in $\mathbb{F}_q$. We set $B_{\underline{\lambda},\underline{m}}$ to be the diagonal block matrix with i-th block B_{λ_i,m_i}. Then for $S \subset \{1,\dots,s\}$ define

$$g_{S,\underline{n}} := \sum_{\underline{j}\leq l_{S,\underline{n}}} (-\mu)^{\sum_{k\in S} j_k}\left[\sum_{s\in S}\begin{pmatrix}0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & B^{j_s}_{\lambda_s,n_s} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0\end{pmatrix}\right]$$

and

$$l_{S,\underline{n}} \in \mathbb{N}^{|S|},\ (l_{S,\underline{n}})_k = \operatorname{lcm}(\operatorname{ord}(-\mu), \operatorname{ord} B_{\lambda_k,n_k})$$

With this definition at hand we can prove an analogue of lemma 4.2.6 with an analogue proof.

Lemma 4.2.10. *It holds for $f = [E_{\underline{m}}] + \mu[B_{\underline{\lambda},\underline{m}}]$ that*

$$\ker(f, -) = \sum_{S\subset\{1,\dots,s\}} \sum_{\underline{n}\leq\underline{m}} \operatorname{Im}(g_{S,\underline{n}}, -).$$

Proof. Let $\alpha = \sum_{i=1}^{s}\sum_{j=1}^{r_i} \nu_{ij}[\alpha_{1,i},\dots,\alpha_{i,j},\dots,\alpha_{s,i}]$, $\nu_j \neq 0$ and the $[\alpha_{1,i},\dots,\alpha_{i,j},\dots,\alpha_{s,i}]$ pairwise different, be in (V, U). We have $\alpha \in \ker(f, U) \Leftrightarrow \alpha = -\mu\alpha[B_{\underline{\lambda},\underline{m}}] \Leftrightarrow \pi_i \in \mathcal{S}_{r_i}$ for at least one i such that $\nu_{ij} = -\mu\nu_{i\pi_i(j)}$ and $[\alpha_{1,j},\dots,\alpha_{i,j},\dots,\alpha_{s,j}] = [\alpha_{1,j},\dots,\alpha_{i,\pi_i(j)}B_{\lambda_i,m_i},\dots,\alpha_{s,j}]$. Without loss of generality $\pi_i = (1,\dots,r_i) \in \mathcal{S}_{r_i}$. If π_i is not an r_i-cycle, we can write it as a product of disjoint cycles and replace α by a summand over the support of one of these cycles. All the i as described above will be collected in S. It follows:

1. $\nu_{i1} = (-\mu)^{r_i}\nu_{i1} \Leftrightarrow \operatorname{ord}(-\mu)|r_i$
2. $\alpha_{i,1} = \alpha_{i,1}B^{r_i}_{\underline{\lambda},\underline{m}} \Leftrightarrow \alpha_{i,1}(E_{\underline{m}} - B^{r_i}_{\underline{\lambda},\underline{m}}) = 0$

for any $i \in S$. From now on we assume that $\operatorname{ord}(-\mu)|r_i$ and $\operatorname{ord}\lambda_i|r_i$ for $i \in S$. We also assume that $\nu_{i1} = 1$. For the elements $i \in S$ there are two cases that can occur, analogue to lemma 4.2.6. This shows the case for $|S| = 1$.
It remains to show how to treat the case $|S| > 1$. Let $\alpha = \sum_{i=1}^{s}\sum_{j=1}^{r_i} \nu_{ij}[\alpha_{1,i},\dots,\alpha_{i,j},\dots,\alpha_{s,i}]$, $\nu_j \neq 0$, $[\alpha_{1,i},\dots,\alpha_{i,j},\dots,\alpha_{s,i}]$ pairwise different, be in $\ker(f, U)$. Let $S \subset \{1,\dots,s\}$ and define $\underline{n}$ via:

$$\underline{n}_i := \begin{cases} m_i \text{ respectively } u_i & i \in S \\ 0 & \text{else} \end{cases}.$$

m_i and u_i are results of the two cases of lemma 4.2.6. So for $|S| > 1$ an arbitrary $\alpha \in \ker(f, U)$ must be a product of all $\alpha_i \in \ker([E_{m_i}] + \mu[B_{\lambda_i,m_i}])$ for $i \in S$. It follows that

$$\alpha = \sum_{i=1}^{|S|} \frac{r_i}{(l_{S,\underline{n}})_i} \sum_{j_1,\dots,j_{|S|}=1}^{(l_{S,\underline{n}})_1,\dots,(l_{S,\underline{n}})_{|S|}} \prod_{k=1}^{|S|}(-\mu)^{j_k} \left[x \sum_{s\in S} \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & B^{j_s}_{\lambda_s,n_s} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \right]$$

$$= \sum_{i=1}^{|S|} \frac{r_i}{(l_{S,\underline{n}})_i} \sum_{\underline{j} \leq l_{S,\underline{n}}} (-\mu)^{\sum_{k\in S} j_k} \left[x \sum_{s\in S} \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & B^{j_s}_{\lambda_s,n_s} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \right]$$

for $x \in M_{\underline{n}\times\underline{m}}(\mathbb{F}_q)$ (a block matrix with block k of size $u_k \times m_k$). Since we have $\frac{r_i}{l_{S,\underline{n}}} \neq 0$, i.e. not divisible by p, we obtain all elements in $\mathrm{Im}(g_{S,\underline{n}}, U)$ in this way.

□

Remark 4.2.11. The above lemma only deals with maps f such that $[f_1]$ has full rank and $[f_2]$ is of jordan normal form. We further only deal with endomorphisms. Dealing with endomorphisms is however not problematic since we can just embed $(V, -)$ and $(W, -)$ into the the bigger functor. We might only add finitely many indecomposable projectives to the kernel.
Therefore, the indecomposable cases 1 and 2 can be integrated in the above lemma. We just need to add the epimorphism onto the kernel to the construcion of $g_{S,\underline{n}}$ in remark 4.2.9

This concludes the Kronecker case. More complicated cases such as the 3-Kronecker case cannot be solved this way as we have no chance of determining or parameterizing all the indecomposable cases as those algebras are wild.

5 Approaches to coherence

Before we start to document our attempts, we want to recall the definition of coherence and give a reason why it is interesting to consider it. Though it is not possible to show that the category $\mathcal{F}_{q_{f.g.}}$ is in fact coherent or even abelian, we want to use this section to give some motivation why this could be true at all.

Definition 5.0.1. A functor F in the category $\mathcal{F}_q$ is finitely presented if it is the cokernel of a representable morphism. So we have a sequence

$$P_1 \xrightarrow{(f,-)} P_0 \twoheadrightarrow F \longrightarrow 0$$

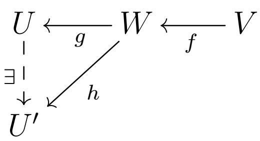
Definition 5.0.2. A functor F in the category $\mathcal{F}_q$ is coherent if it is finitely generated and its finitely generated subfunctors are again finitely presented.
The category $\mathcal{F}_q$ is coherent if all the finitely generated projectives are coherent.

Proposition 5.0.3. *The following are equivalent:*

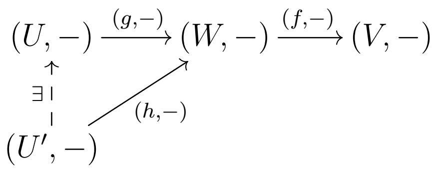
1. *The category $\mathcal{F}_q$ is coherent.*

2. *The category $\mathcal{F}_{q_{fp}}$ is abelian.*
3. *The category $\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]$ has weak cokernels*
4. *Every finitely presented functor in the category $\mathcal{F}_q$ has a resolution by finitely generated projectives.*

Proof. M. Auslander and I. Reiten claim in [AR74] that this is well known, but we will give a proof anyway.
$1 \Rightarrow 4$: Let $\mathcal{F}_q$ be coherent and F be finitely presented. Then $F = \operatorname{coker}(f, -)$ for some $(f, -) : P_1 \to P_0$. Then $\operatorname{Im}(f, -)$ is a finitely generated subfunctor of P_0. In accordance with the coherence it follows that $\operatorname{Im}(f, -)$ is again finitely presented which is equivalent to $\ker(f, -)$ being finitely generated. Since $(f, -)$ is chosen at random, we can now inductively construct a projective resolution for all finitely presented F in $\mathcal{F}_q$.
$1 \Leftarrow 4$ Let G be a finitely generated subfunctor of some P_0 then $G = \operatorname{Im}(g, -)$ for some $(g, -) : P_1 \to P_0$. Since $\operatorname{coker}(g, -)$ has a resolution by finitely generated projectives, it especially follows that $\ker(g, -)$ is finitely generated, and therefore, $G = \operatorname{Im}(g, -)$ is finitely presented.
$3 \Leftrightarrow 4$: Let $\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]$ have weak cokernels. Then for each $f : V \to W$ in $\mathbb{F}_q[\operatorname{mod}\mathbb{F}_q]$ there exists a vector space U and a homomorphism $g : W \to U$ such that each $h : W \to U'$ with $h \circ f = 0$ factors through g.



If now embed this into $\mathcal{F}_q$ via the Yoneda embedding we get:



Which is equivalent to $\ker(f, -)$ being finitely generated for all representable morphisms $(f, -)$. The last statement is furthermore equivalent to the existence of a projective resolution for each finitely generated functor in $\mathcal{F}_q$.
The converse is shown similarly.
$2 \Leftrightarrow 4$: Let $\mathcal{F}_{q_{f.p.}}$ be abelian. This implies that for each given morphism $\phi : F \to G$ for finitely presented F and G the functors $\ker \phi$ and $\operatorname{coker} \phi$ are again finitely pre-

sented. So we can obtain the following commutative diagram

$$\begin{array}{ccccccc}
P_1' & \longrightarrow & P_1 & \longrightarrow & Q_1 & \longrightarrow & Q_1' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_0' & \longrightarrow & P_0 & \xrightarrow{(f,-)} & Q_0 & \longrightarrow & Q_0' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker\phi & \longrightarrow & F & \xrightarrow{\phi} & G & \longrightarrow & \operatorname{coker}\phi
\end{array}$$

But this is equivalent to $\ker(f,-)$ being finitely generated. Since we can obtain all representable morphisms $(f,-)$ in that way, we conclude that $\ker(f,-)$ is finitely generated for all of them. This again yields projective resolutions for all finitely generated functors in $\mathcal{F}_q$.
The converse is again shown similarly.

□

5.1 The radical

In this subsection we discuss two definitions of the radical of a functor. Though it might not seem that way it will turn out that those two definitions will coincide. In our category this is one thing less to worry about. The concept we use works in very general cases. Therefore we introduce it in all generality and deduce the result for our category $\mathcal{F}_q$ afterwards.
We want to work with the following settings: Let $\mathbb{K}$ be a field and $\mathcal{C}$ a Hom-finite $\mathbb{K}$ linear category.

Definition 5.1.1. We can define the functor

$$\mathrm{Fun} : \mathcal{C} \to \mathbb{K}\text{-}\mathrm{mod}$$

as the category of $\mathbb{K}$-linear functors.

The radical of a Functor F can now be defined in the following ways.

Definition 5.1.2. Classically, we can define for a functor $F \in \mathrm{Fun}$:

$$\operatorname{rad} F = \bigcap_{G \subset F,\ G \text{ maximal}} G$$

For the second, the categorical description of the radical, we need to go into further details.

Definition 5.1.3. Let $\mathcal{C}$ be an additive category and $X, Y \in \mathcal{C}$ then

$$\operatorname{rad}_{\mathcal{C}}(X,Y) = \{\, \phi \in \operatorname{Hom}_{\mathcal{C}}(X,Y) |\, \phi \circ \psi \in J(\operatorname{End}_{\mathcal{C}}(Y)) \,\forall\, \psi \in \operatorname{Hom}_{\mathcal{C}}(Y,X)\}$$

with $J(\operatorname{End}_{\mathcal{C}}(Y))$ the Jacobson radical of the endomorphism ring.

Proposition 5.1.4 ([Kra], Proposition 1.8.1)**.** *The radical* $\operatorname{rad}_{\mathcal{C}}$ *is the unique two-sided ideal of* $\mathcal{C}$ *such that* $\operatorname{rad}_{\mathcal{C}}(X,X) = J(\operatorname{End}_{\mathcal{C}}(X))$ *for every object* $X \in \mathcal{C}$.

This enables us to state the following.

Definition 5.1.5. Let $F \in \operatorname{Fun}$ then

$$\operatorname{Rad} F(Y) = \sum_{f \in \operatorname{rad}_{\mathcal{C}}(X,Y)} \operatorname{Im}\left(F(f) : F(X) \to F(Y)\right)$$

with $\operatorname{rad}_{\mathcal{C}}(X,Y)$ being the radical in the domain category $\mathcal{C}$.
Therefore we can regard $\operatorname{Rad} F(Y)$ as precomposing $F(Y)$ with elements in $\operatorname{rad}_{\mathcal{C}}$ and will also use the notation $\operatorname{Rad} F = \operatorname{rad}_{\mathcal{C}} \cdot F$ which is borrowed from ring-theory.

Before we can prove equality, we first need to introduce further characteristics of the category $\mathcal{F}_{q_{f.g.}}$. We need the following definition and the following lemmata.

Definition 5.1.6. Let $\mathcal{C}$ be an additive category and $\operatorname{rad}_{\mathcal{C}}$ its radical. We say the quotient $\overline{\mathcal{C}} \cong \mathcal{C}/\operatorname{rad}_{\mathcal{C}}$ is semi-simple, if $\operatorname{End}_{\overline{\mathcal{C}}}(X) \cong \operatorname{End}_{\mathcal{C}}(X)/J(\operatorname{End}_{\mathcal{C}}(X))$ is semi-simple for all $X \in \mathcal{C}$.

Lemma 5.1.7. *The category* $\mathcal{C}$ *is semi-simple modulo its categorical radical.*

Proof. For all $X \in \mathcal{C}$ the endomorphism algebra $\operatorname{End}_{\mathcal{C}}(F)$ is finite dimensional as $\mathcal{C}$ is Hom-finite. Therefore $\operatorname{End}_{\mathcal{C}}(F)/J(\operatorname{End}_{\mathcal{C}}(F))$ is semi-simple. As F is chosen arbitrary, the quotient $\mathcal{C}/\operatorname{rad}_{\mathcal{C}}$ must be semi-simple as well. □

Now we want to show that $\operatorname{rad} F = \operatorname{Rad} F$ for F in our general category Fun. We will do so by proving the following three lemmata.

Lemma 5.1.8. *It holds that* $F/\operatorname{Rad} F$ *is semi simple in* Fun *and therefore,*

$$\operatorname{rad}(F/\operatorname{Rad} F) = 0 \Leftrightarrow \operatorname{rad} F \subset \operatorname{Rad} F.$$

Proof. In the previous lemma we have established that $\mathcal{C}/\operatorname{rad}_{\mathcal{C}}$ is a semi-simple category. As $\operatorname{Rad} F = \operatorname{rad}_{\mathcal{C}} \cdot F$ we have that $F/\operatorname{Rad} F$ is a functor on $\mathcal{C}/\operatorname{rad}_{\mathcal{C}}$. Since $\mathcal{C}/\operatorname{rad}_{\mathcal{C}}$ is semi-simple, the functor $F/\operatorname{Rad} F$ is a semi-simple functor on $\mathcal{C}$. □

For the proof of equality between the two radicals we need the following characterization of $\operatorname{rad} F$

Lemma 5.1.9. *Let* $F \in \text{Fun}$ *and* $\operatorname{rad} F$ *be its classic radical. Then it holds that* $\operatorname{rad}(F/\operatorname{rad} F) = 0$ *and if* $\operatorname{rad}(F/G) = 0$ *it follows that* $\operatorname{rad} F \subset G$.

Proof. The book [AF10] of Anderson and Fuller is of great help here. Proposition 9.15 in [AF10] guarantees that $\operatorname{rad}(F/\operatorname{rad} F) = 0$ for any functor $F \in \text{Fun}$. For the second assertion we take a look at Proposition 9.14 from the same book. It tells us that for any natural transformation $f : F \to H$ we have $f(\operatorname{rad} F) \subset \operatorname{rad} H$. In our situation this means for the canonical projection $\pi : F \to F/G$ we have $\pi(\operatorname{rad} F) \subset \operatorname{rad}(F/G) = 0$. Therefore, we must have $\operatorname{rad} F \subset \ker \pi = G$. □

Lemma 5.1.10. *It holds that* $\operatorname{Rad} F$ *is contained in every maximal subfunctor of* F *and therefore* $\operatorname{Rad} F \subset \operatorname{rad} F$.

Proof. Let G be a maximal subfunctor of F. Then F/G is simple and $\operatorname{rad}_{\mathcal{C}} \cdot F/G = 0$. Therefore, $\operatorname{rad}_{\mathcal{C}} \cdot F = \operatorname{Rad} F \subset G$. As G is an arbitrary maximal subfunctor of F, we have $\operatorname{Rad} F \subset \operatorname{rad} F$. □

Corollary 5.1.11. *It holds that* $\operatorname{Rad} F = \operatorname{rad} F$ *for all* $F \in \text{Fun}$.

Now we want to adapt this knowledge to the situation of the category $\mathcal{F}_q$.

Lemma 5.1.12. $\operatorname{Rad} F = \operatorname{rad} F$ *for any* $F \in \mathcal{F}_q$.

Proof. Since the domain category $\mathbb{F}_q[\operatorname{mod} \mathbb{F}_q]$ of $\mathcal{F}_q$ is Hom-finite, the quotient category $\mathbb{F}_q[\operatorname{mod} \mathbb{F}_q]/\operatorname{rad}(\mathbb{F}_q[\operatorname{mod} \mathbb{F}_q])$ is semi-simple. The only thing we have to take into the account is that $\mathbb{F}_q[\operatorname{mod} \mathbb{F}_q]$ does a priori not have split idempotents. But since in the end we work with $\mathcal{F}_q$, we can just pass to the idempotent completion without doing any harm.
Application of the above lemmata for the equality of the two definitions of the radical yields the result for $\mathcal{F}_q$. □

Now that we know what to think of when we talk about the radical of a functor, we turn back to projective functors. Corollary 2.1.12 especially implies that a finitely generated projective P must also have a finitely generated radical as simple functors are the unique tops of indecomposable projectives. This has some interesting consequences which are a subject of the next subsection.

5.2 Properties of the radical

In this subsection we will show two nice properties of the radical of a functor.

Lemma 5.2.1. *Let P_λ be an indecomposable projective, then* $\operatorname{rad} P_\lambda$ *is finitely generated.*

Proof. Since $P_\lambda / \operatorname{rad} P_\lambda = F_\lambda$ is simple and therefore polynomial, it has a resolution by finitely generated projectives. This is a consequence of [Sch94, Theorem 5.3.8] respectively [FLS94, Section 10]. □

Lemma 5.2.2. *Let F be a finitely generated functor. Then* $\operatorname{rad} F$ *is finitely generated as well.*

Proof. F being finitely generated means that it admits a projective cover P. The projective cover P is calculated by looking at $\operatorname{top} F \cong F / \operatorname{rad} F$. The cover P then will have the same top as F. We get the following commutative diagram.

$$\begin{array}{ccc} P & \twoheadrightarrow & \operatorname{top} P \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow \cong \\ F & \twoheadrightarrow & \operatorname{top} F \end{array}$$

We can complete either of the rows into a short exact sequence by mentioning the kernel on the left hand side, the kernel being nothing else but the radical.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{rad} P & \longrightarrow & P & \twoheadrightarrow & \operatorname{top} P & \longrightarrow & 0 \\ \big\downarrow p_0 & & \big\downarrow p_1 & & \big\downarrow\!\!\!\downarrow p_2 & & p_3 \big\downarrow \cong & & \\ 0 & \longrightarrow & \operatorname{rad} F & \longrightarrow & F & \twoheadrightarrow & \operatorname{top} F & \longrightarrow & 0 \end{array}$$

The map p_1 exists and is well defined because rad is a functor. Now p_0 and p_2 are epimorphisms and p_3 is a monomorphism. By the 4-lemma also p_1 has to be an epimorphism. Therefore $\operatorname{rad} F$ is finitely generated since $\operatorname{rad} P$ is. □

Lemma 5.2.3. *Let F be finitely generated with* $\operatorname{rad}^r F \neq 0$*. Then it holds*

$$\operatorname{rad}^{r+1} F \subsetneq \operatorname{rad}^r F.$$

Proof. By the Nakayama-lemma we have for any finitely generated functor G:

$$\operatorname{rad} G = G \quad \Rightarrow \quad G = 0$$

With $G = \operatorname{rad}^r F$ we get the claim of the lemma. □

Something that will not be of immediate use, but is interesting enough in itself is the infinite radical, $\operatorname{rad}^\infty = \bigcap_{i \geq 0} \operatorname{rad}^i$, of a projective. We want to show that it vanishes for all the indecomposable projectives. The proof is done in two steps. Before we start we have to recall some definitions.

Definition 5.2.4. Let $\mathcal{C}$ be a category. A non-empty subcategory $\mathcal{A}$ is called a Serre subcategory if it is closed under taking subobjects and quotients.

Definition 5.2.5. We denote by Serre(P) the Serre subcategory of $\mathcal{F}_q$ which is generated by all the finitely generated projectives. Serre(I) is the analogue Serre subcategory which is generated by the finitely co-generated injectives.

These Serre subcategories fit into our category $\mathcal{F}_q$ as follows: Let us address the full subcategory $\mathcal{F}_{q_{fin}}$ of $\mathcal{F}_q$ that consists of all functors $F : \operatorname{mod} \mathbb{F}_q \to \operatorname{mod} F_q$. Then the Kuhn-dual D induces an autoequivalence on $\mathcal{F}_{q_{fin}}$ that restricts to an equivalence of Serre(P) and Serre(I).

Lemma 5.2.6. *It holds that* $\operatorname{rad} G \subsetneq G$ *for all non-zero functors* G *in* Serre(P).

Proof. We need to start in the category Serre(I). From the work of Kuhn [Kuh93] [Appendix B] we know that all the injective functors in $\mathcal{F}_q$ and therefore all objects in Serre(I) are locally finite. That means that we can write them as a union of their finite length subfunctors. We especially have $F = \bigcup_{i\geq 0} \operatorname{soc}^i F$.
We claim that the Kuhn-dual D sends the socle of a functor in Serre(I) to the top of a functor in Serre(P), the top being defined as functor modulo its radical.
We start with a simple socle. If $F_\lambda \hookrightarrow I_\lambda$ and we apply D to that sequence, we get $P_\lambda \twoheadrightarrow F_\lambda$ since $D(I_\lambda) \cong P_\lambda$ and simples are selfdual. An arbitrary socle might be bigger, but it is most certainly semi-simple as $\operatorname{soc} F = \sum_S S$ with $S \subset F$, S simple. Since D is additive, we are done and have an equivalence $\operatorname{soc} F \cong \operatorname{top} D(F)$.
By the definition of soc we can see that it will never be 0. Therefore, the top of the dual will never be 0 as well. But this is equivalent to the claim of the lemma. □

Lemma 5.2.7. *It holds that* $\operatorname{rad}^\infty P_\lambda = 0$ *for all indecomposable projectives* P_λ.

Proof. We need to start in the category Serre(I). Again from the work of Kuhn [Kuh93][Appendix B] we know that all the injective functors in $\mathcal{F}_q$ and therefore all objects in Serre(I) are locally finite. That means we can write them as the union of their finite length subfunctors. We especially have $F = \bigcup_{i\geq 0} \operatorname{soc}^i F$.
We have the following exact sequence.

$$0 \to \operatorname{soc}^i F \to F \to F/\operatorname{soc}^i F \to 0$$

The Kuhn dual sends this to

$$0 \to \operatorname{rad}^i DF \to DF \to DF/\operatorname{rad}^i DF \to 0$$

As $F = \bigcup_{i\geq 0} \operatorname{soc}^i F$ this implies that $\operatorname{rad}^\infty DF = \bigcap_{i\geq 0} \operatorname{rad}^i DF = 0$. □

Corollary 5.2.8. *It holds that* $\operatorname{rad}^\infty G = 0$ *for all functors* G *in* Serre(P).

Lemma 5.2.9. *Let F be finitely generated, then $\operatorname{rad}^r F/\operatorname{rad}^{r+1} F$ is a polynomial functor.*

Proof. The duality functor D transfers the exact sequence

$$0 \to \operatorname{rad}^{r+1} F \to \operatorname{rad}^r F \to \operatorname{rad}^r F/\operatorname{rad}^{r+1} F \to 0$$

to the sequence

$$0 \to \operatorname{soc}^r DF \to DF/\operatorname{soc}^r DF \to DF/\operatorname{soc}^{r+1} DF \to 0.$$

Since $\operatorname{soc}^r DF$ is polynomial, so is $\operatorname{rad}^r F/\operatorname{rad}^{r+1} F$. □

5.3 Behavior of the dimension of $\operatorname{coker}(f, \mathbb{F}_q^n)$

In this subsection we attempt to describe the growth of the dimension of a representable functor in the category $\mathcal{F}_q$. We mostly use the same way as in the previous subsection and try to generalize the lemmata.
For all of this section we fix $F = \operatorname{coker}(f, -)$ with $(f, -) : (V, -) \to (W, -)$. Lemma 2.1.19 is the reason why we can restrict ourselves to this case. The general case for arbitrary representable $\mathbf{f}$ would follow immediately, but there are some difficulties which could not be overcome.
First of all we would like to have an analogue of lemma 3.1.15. We know that as $\Delta(W, -)$ and $\Delta(V, -)$ are again representable, the same must hold for $\Delta(f, -)$. But we cannot say if $\Delta(f, -) : \Delta(V, -) \to \Delta(W, -)$ will decompose into pieces of the form $\Delta(f_i, -) : (V, -) \to (W, -)$. So let us summarize what we can say so far.

Lemma 5.3.1. *Let $\mathbf{f} = ((f_1, -), \ldots, (f_t, -)) : \bigoplus_{i=1}^t (V_i, -) \to (W, -)$, $V_i, W \in \operatorname{mod} \mathbb{F}_q$ arbitrary, then*

1. $\operatorname{Im}(\mathbf{f}) = \sum_{i=1}^t \operatorname{Im}(f_i, -)$ *with* $(f_i, -) : (V_i, -) \to (W, -)$.
2. *There is a $k \in \mathbb{N}$ and a map $\mathbf{g} : \bigoplus_{i=1}^k (V, -) \to (W, -)$ such that $\operatorname{Im}(\mathbf{h}) \subset \operatorname{Im}(\mathbf{g})$ for all $\mathbf{h} : \bigoplus_{i=1}^t (V, -) \to (W, -)$ with $t > k$.*
3. *For fixed V, W there are only finitely many direct summands, depending on V and W, of which the image of $\mathbf{h}$ can consist of.*

Proof. For the first assertion we fix $\mathbb{F}_q^n$ and choose α to be an element of $(W, \mathbb{F}_q^n)$ with $(f_i, \mathbb{F}_q^n)(\beta) = \alpha$. Then $(0, \ldots, 0, \beta, 0, \ldots, 0) \in \bigoplus_{i=1}^t (V_i, -)$, with β in the i-th component, is a preimage of α under $\mathbf{f}$.
The second assertion follows from the first assertion together with the fact that (V, W) is finite as a set.
Assertion three is then a combination of the first two. □

Example 5.3.2. In the above lemma we do not necessary have that $\mathrm{Im}(\mathbf{f})$ is a direct summand of $\mathrm{Im}(\mathbf{g})$.
Let $q = 2$, $V = \mathbb{F}_2^2$ and $W = \mathbb{F}_2$. If we now choose $\mathbf{f} : (\mathbb{F}_2^2, -) \twoheadrightarrow \mathrm{rad}\, P(\Lambda^1)$ and $\mathbf{g} : (\mathbb{F}_2^2, -) \twoheadrightarrow (\mathbb{F}_2, -)$, then we have $\mathrm{Im}(\mathbf{f}) \subset \mathrm{Im}(\mathbf{g})$ but not as a direct summand; cf. example 2.2.2.

We can also dualize this construction.

Lemma 5.3.3. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_1, -) \\ (f_2, -) \\ \vdots \\ (f_m, -) \end{pmatrix} : (V, -) \to \bigoplus_{j=1}^{m} (W_j, -)$$

with $V, W_j \in \mathrm{mod}\mathbb{F}_q$ *arbitrary, then*

1. $\ker(\mathbf{f}) = \bigcap_{j=1}^{m} \ker(f_j, -)$ *with* $(f_j, -) : (V, -) \to (W_j, -)$.
2. *There is a* $k \in \mathbb{N}$ *and a map* $\mathbf{g} : (V, -) \to \bigoplus_{i=1}^{k} (W, -)$ *such that* $\mathrm{Im}(\mathbf{f}) \hookrightarrow \mathrm{Im}(\mathbf{g})$ *(as a direct summand) for all* $\mathbf{f} : (V, -) \to \bigoplus_{j=1}^{m} (W, -)$ *with* $m > k$.
3. *For fixed* V, W *there are only finitely many direct summands, depending on* V *and* W, *of which the image of* $\mathbf{f}$ *can consist of.*

Proof. For the first assertion it is not hard to see that for fixed $\mathbb{F}_q^n$ it is $\mathbf{f}(\mathbb{F}_q^n)(\alpha) = 0$ if and only if $(f_j, \mathbb{F}_q^n)(\alpha) = 0 \,\forall\, j$. Therefore, $\ker(\mathbf{f}(\mathbb{F}_q^n)) = \bigcap_{j=1}^{m} \ker(f_j, \mathbb{F}_q^n)$.
Assertion two requires us to take care of one specific detail. Let $\mathbf{g} : (V, -) \to (W, -) \oplus (W, -)$ with $(g_1, -) = (g_2, -)$. Then the composition with a projection on either component yields an isomorphism $\mathrm{Im}(\mathbf{g}) \cong \mathrm{Im}(g_i, -)$. Therefore, we can assume that $g_i \neq g_j$ for $i \neq j$. We can further assume $k = |(V, W)|$. Now we take a map of sets $\sigma : \{1, \ldots, m\} \to \{1, \ldots, k\}$ with $g_{\sigma(i)} = f_i$.
Together with the first part of the proof of assertion two, this does not only yield a direct embedding $\mathrm{Im}(\mathbf{f}) \hookrightarrow \mathrm{Im}(\mathbf{g})$ but tells us, via σ, how it looks like.
The third part, analogue to lemma 5.3.1, is a combination of part one and two. □

The next step would be to plug those two sides together, but unfortunately this is not possible. We give the conjectured proposition and show where the proof fails.

Conjecture 5.3.4. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_{11}, -) & \cdots & (f_{1t}, -) \\ (f_{21}, -) & \cdots & (f_{2t}, -) \\ & \vdots & \\ (f_{m1}, -) & \cdots & (f_{mt}, -) \end{pmatrix} : \bigoplus_{i=1}^{t} (V, -) \to \bigoplus_{j=1}^{m} (W, -).$$

Then

1. $\mathrm{Im}(\mathbf{f}) = \sum_{i=1}^{t} \mathrm{Im}(\mathbf{f}_i)$ *with* $\mathbf{f}_i : (V,-) \to \bigoplus_{j=1}^{m}(W,-)$.

2. *There are* $k, m \in \mathbb{N}$ *and a map* $\mathbf{g} : \bigoplus_{i=1}^{k}(V,-) \to \bigoplus_{j=1}^{m}(W,-)$ *such that* $\mathrm{Im}(\mathbf{f}) \subset \mathrm{Im}(\mathbf{g})$ *for all* $\mathbf{f} : \bigoplus_{i=1}^{t}(V,-) \to \bigoplus_{j=1}^{s}(W,-)$ *with* $t > k$ *and* $s > m$.

3. *For fixed* V, W *there is a finite list* $\mathcal{S}$ *of subfunctors of* $(W,-)$ *that depends only on* V *and* W *such that for all* k, m *and maps* $\mathbf{g}$ *we have that* $\mathrm{Im}(\mathbf{g}) \cong \bigoplus_{S\in\mathcal{S}} S^{d_S},\ d_S \in \mathbb{N}_0$.

Motivation: The first assertion is a direct consequence of lemma 5.3.1 and true even though the other assertions are not.
From lemma 5.3.1 and its dual, we know that there can only be a finite number of $\mathbf{f}_i : (V,-) \to \bigoplus_{j=1}^{m}(W,-)$ until the image cannot grow bigger anymore as well as there are only finitely many possibilities of how $\mathbf{f}_i$ can look like.
The functors S would then be given by the finitely many indecomposable direct summands of $\mathrm{Im}(\mathbf{f}) = \sum_{i=1}^{t} \mathrm{Im}(\mathbf{f}_i)$. There are only finitely many indecomposable direct summands since the image is finitely generated. Finally, as there are only finitely many $\mathbf{f}_i$, the list of all indecomposable direct summands of all $\mathbf{f}_i$ has to be finite for each combination of t and s.
The problem arising is that the number of $\mathbf{f}_i$ depends on either t or s. If we could fix one of them as in the previous lemmata, this proof would work. But for the proof of assertions two and three we need precisely the independence of t and s.

For our purposes it would however suffice to assume a weaker version of this conjecture.

Conjecture 5.3.5. *Let*

$$(f,-) : (V,-) \to (W,-).$$

Then there is a finite list $\mathcal{S}$ *of subfunctors of* $(W,-)$ *that depends only on* $(f,-)$ *such that for all* $k \in \mathbb{N}$ $\mathrm{Im}(\Delta^k(f,-)) \cong \bigoplus_{S\in\mathcal{S}}^{d_S},\ d_S \in \mathbb{N}_0$.

For the remainder of the subsection we will assume conjecture 5.3.5. If we do so, we get the following corollary.

Corollary 5.3.6. *The cardinality of the set* $\mathcal{S}$ *of all possible direct summands of the image any representable morphism* $\Delta^k(f,-) : \Delta^k(V,-) \to \Delta^k(W,-)$ *is restricted by* $N = 2^{q^{q^{\dim V \cdot \dim W}}}$.

Proof. The number N is the cardinality of the set of all maps $\Delta^k(f,-)$ in conjecture 5.3.5. The set of all maps $(f,-)$ is the set of all subsets of (V,W) viewed as a set. □

If the previously stated conjecture 5.3.5 would be true, we could now derive an analogue of lemma 3.1.15. We should remark that since Δ is an exact functor, it commutes with images.

Lemma 5.3.7. *There is a $k \in \mathbb{N}$ such that there exists a partition $J \dot\cup I = \{0, \dots, k\}$ and positive integers m_i, $0 \le i \le k$ and $m_k \ne 0$, with*

$$\bigoplus_{j \in J} \Delta^j \mathrm{Im}(f, -)^{\oplus m_j} \cong \bigoplus_{i \in I} \Delta^i \mathrm{Im}(f, -)^{\oplus m_i}.$$

Proof. Let

$$\mathcal{S} = \{S,\, S \subset \Delta^i \mathrm{Im}(f, -) \text{ direct summand}\}/\cong.$$

Conjecture 5.3.5 gives us that this set should be finite. Therefore, the $\mathbb{Q}$ vector space $X = \langle \mathcal{S} \rangle_{\mathbb{Q}}$ is finite dimensional.
So there must be a $k \in \mathbb{N}$ such that the set $\{\Delta^i \mathrm{Im}(f, -),\, i \le k\}$, viewed as elements in X, is linear dependent in X. Therefore, there are $m_i \in \mathbb{N}$ for $0 \le i \le k$, $m_k \ne 0$, such that

$$\bigoplus_{j \in J} \Delta^j \mathrm{Im}(f, -)^{\oplus m_j} \cong \bigoplus_{i \in I} \Delta^i \mathrm{Im}(f, -)^{\oplus m_i}, \text{ with } J \dot\cup I = 0, \dots, k.$$

With out loss of generality we can assume $k \in J$. □

Assuming conjecture 5.3.5 we could further prove.

Lemma 5.3.8. *$\phi(G, n)$ is of closed form for all finitely presented functors G in $\mathcal{F}_q$.*

Proof. Let $H = \mathrm{Im}(f, -)$; H being of closed form is equivalent to a finitely presented functor being of closed form. Then by the previous lemma we have $\bigoplus_{j \in J} \Delta^j \mathrm{Im}(f, -)^{\oplus m_j} \cong \bigoplus_{i \in I} \Delta^i \mathrm{Im}(f, -)^{\oplus m_i}$. Since this sum is direct, it follows:

$$\sum_{j \in J} m_j \phi(\mathrm{Im}\Delta^j(f, -), n) = \sum_{i \in I} m_i \phi(\mathrm{Im}\Delta^i(f, -), n)$$

We will again write $\phi(\Delta^k H, n)$ as a function in $\phi(H, n)$ via

$$\phi(\Delta^k H, n) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \phi(H, n + j)$$

If we plug this in on both sides of the recursion formula for $\Delta^k \mathrm{Im}(f, -)$, we get

$$\sum_{j \in J} m_j \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \phi(H, n + l) = \sum_{i \in I} m_i \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} \phi(H, n + l).$$

By reordering we will in fact obtain a recursion formula for $\phi(H, n+k)$. The coefficient for $\phi(H, n+k)$ is not zero. In fact it is m_k. Because of theorem 3.1.3 this is equivalent to $\phi(H, n)$ being of closed form. $\square$

If we now look at the general case of an arbitrary morphism $\mathbf{f} = (f_{ij}, -)_{ij}$ between direct sums of standard projectives. We need to show this case to obtain coherence for the category $\mathcal{F}_q$. It might look like there is a lot more work to do. But that is not the case since we have lemma 2.1.19.

Corollary 5.3.9. *Let* $\mathbf{f}$ *be a matrix of representable morphisms* $Mor\mathcal{F}_q$. *Then* $\phi(\mathrm{Im}\mathbf{f}, n)$ *is of closed form if* $\phi(\mathrm{Im}(f, -), n)$ *is of closed form for all representable morphisms* $(f, -)$.

Proof. We know that $\phi(\mathrm{Im}(f, -), n)$ is of closed form whenever $(f, -) : (V, -) \to (W, -)$.
Now let $\mathbf{f}$ be arbitrary with $P_1 = \bigoplus_{i=1}^{s}(V_i, -)$ and $P_2 = \bigoplus_{j=1}^{t}(W_j, -)$. The structure of the decomposition of $(\mathbb{F}_q^m, -)$ into indecomposable projectives admits a section

$$P_1 \hookrightarrow (\overline{V}, -) \oplus \mathbb{F}_q^{\oplus(s-1)}$$

where $\mathbb{F}_q$ is the constant functor and $(\overline{V}, -)$ is a standard projective as constructed in lemma 2.1.19 ($\overline{W}$ is the analogue on the W-side). Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\ker(\mathbf{f}) & \hookrightarrow & P_1 & \xrightarrow{\mathbf{f}} & P_2 & \xrightarrow{\pi} & \mathrm{coker}(f, -) \\
{\scriptstyle\pi_{ker}}\uparrow\downarrow{\scriptstyle\iota_{ker}} & & {\scriptstyle\pi_V}\uparrow\downarrow{\scriptstyle\iota_V} & & {\scriptstyle\pi_W}\uparrow\downarrow{\scriptstyle\iota_W} & & \| \\
\ker(\overline{f}, -) & \hookrightarrow & (\overline{V}, -) \oplus \mathbb{F}_q^{\oplus(s-1)} & \xrightarrow{([\overline{f}], -)} & (\overline{W}, -) \oplus \mathbb{F}_q^{\oplus(t-1)} & \xrightarrow{\overline{\pi}} & \mathrm{coker}(\overline{f}, -)
\end{array}$$

The maps are defined as follows: $\overline{\pi} = \pi \circ \pi_W$ which has again a finitely generated kernel. Now choose $(\overline{f}, -) : (\overline{V}, -) \twoheadrightarrow \ker \pi_W$. We already derive the desired result since $(\overline{f}, -)$ is covered by lemma 5.3.8 up to the constant part. Therefore, the claim holds for the images of arbitrary matrices of representable morphisms. $\square$

If we now knew that finitely presented functors are of closed form, we would again like to know how the closed form looks like.

Lemma 5.3.10. *In* $\phi(F, n)$ *we have* $\alpha_i = q^i$ *for all* F *finitely presented. If we assume conjecture 5.3.5.*

Proof. F is a factor of a finite direct sum of indecomposable projectives and the subfunctor that is to be factored out is also finitely generated. For indecomposable projectives the this was shown in lemma **??**. $\square$

Corollary 5.3.11. *Let $(f,-),(g,-)$ be representable morphisms in $\mathcal{F}_q$ such that the composition $f \circ g$ is defined. We have that* $\ker(f,-)$ *and* $\ker(g,-)$ *are of closed form. It follows that* $\ker(f \circ g,-)$ *is also of closed form.*

Proof. Let $f = \sum_{i=1}^{s} \lambda_i[f_i]$ and $g = \sum_{j=1}^{t} \mu_i[g_i]$ then we have

$$f \circ g = \sum_{i=1}^{s}\sum_{j=1}^{t} \lambda_i\mu_j[f_i \cdot g_j] = \sum_{l=1}^{s\cdot t} \nu_l[h_l].$$

But the kernels of such morphisms are of closed form. □

So far we have said nothing about how we get to polynomials in the closed form. The following example shows how to get arbitrary high degrees of polynomials for q^0. It is also a hint why the closed form for finitely presented functors exists after all.

Example 5.3.12. These following results have been obtained by using the computer. A direct calculation by hand is rather complicated but manageable. Let $q = 2$ and

$$f = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix},$$

then $\phi(\mathrm{Im}(f,-),n) = 2^n - n$.
We calculate the dimension of the image of $(f,-)$ by taking a look at the elements of its target space $(\mathbb{F}_2,-)$. Let $[\alpha] \in (\mathbb{F}_2,\mathbb{F}_2^n)$ have at least two nonzero entries. Then we can decompose α into α_1 and α_2 both not equal to $[0]$ and wlog. α_1 with only one nonzero entry. The element $(\alpha_1,\alpha_2) \in (\mathbb{F}_2^2,\mathbb{F}_2^n)$ is then a preimage under $(f,\mathbb{F}_2^n)$ of $\alpha+\alpha_1+\alpha_2$. If $\alpha \in (\mathbb{F}_2,\mathbb{F}_2^n)$ does only have one non-zero entry, we cannot use this construction to get to a preimage. So when passing to the cokernel all α with only one non-zero entry will generate the cokernel. But these α, viewed as a subset of $\mathbb{F}_2^n$, have a name: the canonical basis $\{e_i, 1 \leq i \leq n\}$. So the set of all $[e_i]$, $1 \leq i \leq n$, generates the cokernel of $(f,\mathbb{F}_2^n)$ which is therefore at most n-dimensional.
Now we want to show that those residue classes are indeed linear independent in the cokernel. We have $\sum_{i=1}^{n} \lambda_i[e_i] = 0$ in the cokernel if $\sum_{i=1}^{n}[e_i] = \sum_{j=1}^{t} \mu_j([\alpha] + [\alpha_1] + [\alpha_2])$ but this is not possible since by the previous construction at least one of the tree terms $[\alpha],[\alpha_1],[\alpha_2]$ has to have at least two non-zero entries, unless all are $[0]$. As $\phi((\mathbb{F}_2,-),n) = 2^n$ this yields the result.
We also get

$$f = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$$

with $\phi(\mathrm{Im}(f,-),n) = 2^n - \frac{1}{2}(n^2+n)$.
The calculation is analogue for three entries not zero. Therefore, the cokernel is generated by the residue classes

$$[e_i],\ 1 \leq i \leq n \text{ and } [e_i + e_j],\ 1 \leq i \neq j \leq n.$$

Counting those elements yields a cokernel that is of dimension $\frac{1}{2}(n-1)n + n = \frac{1}{2}(n+1)n$.
In this manner we can construct functors as cokernels of arbitrary high polynomial degrees.

5.4 Conceptual approach by using the closed form

In this subsection we attempt to prove that the kernels of representable morphisms in the category $\mathcal{F}_q$ are finitely generated. This is equivalent to the existence of projective resolutions of finitely presented functors where all the projectives are finitely generated.
The approach uses the existence of the closed form for the finitely presented functors in $\mathcal{F}_q$ which we derive by assuming conjecture 5.3.5 in the previous subsection. In addition, since it is not clear if the closed form actually exists, the attempt runs into problems on its own. However there are some statements that can still be made even without assuming the closed form.

Lemma 5.4.1. *Let $(f,-)$ and $(g,-)$ be representable morphisms in $\mathcal{F}_q$ such that $(f,-) \circ (g,-) = 0$ and that $\phi((\ker(f,-),n) = \phi(\mathrm{Im}(g,-),n)$. Then $\ker(f,\mathbb{F}_q^n) = \mathrm{Im}(g,\mathbb{F}_q^n)\,\forall\, n$.*

Proof. If $\phi(\ker(f,-),n) = \phi(\mathrm{Im}(g,-),n)\,\forall\, n \in \mathbb{N}_0$, we have equivalence since $\mathrm{Im}(g,-) \subset \ker(f,-)$. □

Lemma 5.4.2. *Let $\chi : (V,-) \to F$ for some finitely generated F. For all $U \in \mathrm{mod}\,\mathbb{F}_q$ we can find a pair (L,g) such that*

$$(L,U) \xrightarrow{(g,U)} (V,U) \xrightarrow{\chi(U)} F(U)$$

is exact.

Proof. For all $U \in \mathrm{mod}\,\mathbb{F}_q$, (V,U) is a finite length $\mathrm{GL}(U)$ and $\mathrm{End}(U)$ module. Therefore, $\ker\chi(U)$ must have a finite top consisting of simple $\mathrm{End}(U)$ modules M_λ. We know that simple $\mathrm{End}(U)$ modules are in 1:1 correspondence with simple functors F_λ of restricted weight. With P_λ the projective cover of F_λ we get:

$$\begin{array}{ccc} \bigoplus M_\lambda & \overset{1:1}{\longleftrightarrow} & \bigoplus F_\lambda \\ & & \uparrow \\ & & \bigoplus P_\lambda \end{array}$$

Now [Kuh94, Lemma 3.7] guaranties that $P_\lambda(U)$ is a projective $\mathrm{End}(U)$ module. Since the M_λ are simple, $P_\lambda(U)$ must be their projective cover. This yields the following commutative diagram.

$$\begin{array}{ccc} \ker\chi(U) & \twoheadrightarrow & \bigoplus M_\lambda \\ & \nwarrow_{\varphi} & \uparrow \\ & & \bigoplus P_\lambda(U) \end{array}$$

The map φ must exist because of the universal property of projectives. It must also be surjective since $P_\lambda(U)$ is a projective cover of M_λ.
Again by [Kuh94, Lemma 3.7] we get

$$\mathrm{Hom}_{\mathcal{F}_q}(\oplus P_\lambda, \ker\chi) \cong \mathrm{Hom}_{\mathrm{End}(U)}(\oplus P_\lambda(U), \ker\chi(U)).$$

This means that φ from the above diagram must come from a φ' completing the following:

$$\begin{array}{ccc} \ker\chi & \twoheadrightarrow & \bigoplus F_\lambda \\ & \nwarrow_{\varphi'} & \uparrow \\ & & \bigoplus P_\lambda \end{array}$$

So far $\bigoplus P_\lambda$ is just an arbitrary projective but it can be easily embedded as a direct summand into a standard projective $(L, -)$. Together with the canonical inclusion of $\ker\chi$ into $(V, -)$ we can obtain

$$\begin{array}{ccc} (L,-) & \xrightarrow{(g,-)} & (V,-) \\ \downarrow & & \uparrow \\ \bigoplus P_\lambda & \xrightarrow{\varphi'} & \ker(\chi,-) \end{array}$$

where $(g, -)$ is just the composite of the other maps. It must be representable since $\mathrm{Hom}_{\mathcal{F}_q}((L, -), (V, -)) \cong (V, L)$. Because of the choice of $(g, -)$ we clearly have $\chi \circ (g, -) = 0$ and since χ is surjective, we can also derive that $\mathrm{Im}(g, U) = \ker\chi(U)$. □

The next lemma guaranties that if we find (L, g) to resolve $\ker\chi(U)$, this L will also do for $U' \subset U$.

Lemma 5.4.3. *Let F be finitely generated via*

$$\bigoplus_{i=1}^{m} (V_i, -) \xrightarrow{\chi} F \to 0.$$

By the previous lemma we can always find $(W,-)$ *and* $(f,-)$ *such that* $\chi \circ (f,-) = 0$. *For fixed* $\mathbb{F}_q^n$ *this can be chosen such that the sequence*

$$0 \to \operatorname{Im}(f, \mathbb{F}_q^n) \to \bigoplus_{i=1}^{m} (V_i, \mathbb{F}_q^n) \to F(\mathbb{F}_q^n) \to 0$$

is exact. Then it holds that this sequence is also be exact for $\mathbb{F}_q^{n-k}$.

Proof. By applying the exact functor Δ to the sequence

$$\operatorname{Im}(f,-) \to \bigoplus_{i=1}^{m} (V_i, -) \to F(-)$$

and evaluating at $\mathbb{F}_q^{n-1}$ we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Im}(f, \mathbb{F}_q^{n-1}) & \xrightarrow{a} & \bigoplus_{i=1}^{m} (V_i, \mathbb{F}_q^{n-1}) & \xrightarrow{b} & F(\mathbb{F}_q^{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Im}(f, \mathbb{F}_q^{n}) & \longrightarrow & \bigoplus_{i=1}^{m} (V_i, \mathbb{F}_q^{n}) & \longrightarrow & F(\mathbb{F}_q^{n}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Delta\operatorname{Im}(f, \mathbb{F}_q^{n-1}) & \xrightarrow{c} & \Delta \bigoplus_{i=1}^{m} (V_i, \mathbb{F}_q^{n-1}) & \xrightarrow{d} & \Delta F(\mathbb{F}_q^{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Where the middle row is exact. Now it is straight forward to show that a is injective and d is surjective. The injectivity of c follows because $c = \Delta a$ and the surjectivity of b is guaranteed by χ being an epimorphism.
It remains to show that the first and second row are exact. This can be done via counting of dimensions. It holds that:

$$\phi(\operatorname{Im}(f,-), n) + \phi(F, n) = \phi\left(\bigoplus_{i=1}^{m} (V_i, -), n\right)$$

Since the columns are exact this can be reformulated to

$$\phi(\operatorname{Im}(f,-), n-1) + \phi(\Delta\operatorname{Im}(f,-), n-1) + \phi(F, n-1) + \phi(\Delta F, n-1) =$$

$$\phi\left(\bigoplus_{i=1}^{m} (V_i, -), n-1\right) + \phi\left(\Delta \bigoplus_{i=1}^{m} (V_i, -), n-1\right)$$

Since $\chi \circ (f,-) = 0$ we get that

$$\phi(\mathrm{Im}(f,-), n-1) + \phi(F, n-1) \leq \phi\left(\bigoplus_{i=1}^{m}(V_i,-), n-1\right)$$

and

$$\phi(\Delta\mathrm{Im}(f,-), n-1) + \phi(\Delta F, n-1) \leq \phi\left(\Delta\bigoplus_{i=1}^{m}(V_i,-), n-1\right).$$

So there has to be equality in both equations.
To get the result for arbitrary $\mathbb{F}_q^{n-k}$ we iterate this process. □

Remark 5.4.4. Due to additivity this also holds for $\mathbf{f} : P_1 \to P_2$.

Assuming conjecture 5.3.5 and the result that for a finitely presented functor $\phi(F, n)$ is of closed form, we could further prove.

Lemma 5.4.5. *Let $(f,-) : (V,-) \to (W,-)$ be a representable morphism. Then there exist $m_i \in \mathbb{N}$ for $0 \leq i \leq k$ such that*

$$\bigoplus_{i\in I} m_i\Delta^i \ker(f,-) \cong \bigoplus_{j\in J} m_j\Delta^j \ker(f,-)$$

with $I\dot{\cup}J = \{0,\dots,k\}$.

Proof. By lemma 5.3.7 we have the same assertion for the image of a representable morphism. We also have that $\Delta(V,-) \cong (V,-)^{\oplus q^{\dim V}-1}$. So we can find positive integers m_i for $0 \leq i \leq k$ where $m_k \neq 0$ and $I\dot{\cup}J = \{0,\dots,k\}$ such that:

$$\bigoplus_{j\in J} m_j\Delta^j \ker(f,-) \cong \bigoplus_{i\in I} m_i\Delta^i \ker(f,-)$$

and

$$\bigoplus_{j\in J} m_j\Delta^j(V,-) \cong \bigoplus_{i\in I} m_i\Delta^i(V,-)$$

Therefore, we have the following diagram of short exact sequences with commuting squares:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\in I} m_i\Delta^i \ker(f,-) & \longrightarrow & \bigoplus_{i\in I} m_i\Delta^i(V,-) & \longrightarrow & \bigoplus_{i\in I} m_i\Delta^i\mathrm{Im}(f,-) & \longrightarrow & 0 \\
 & & \downarrow c & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \bigoplus_{j\in J} m_j\Delta^j \ker(f,-) & \longrightarrow & \bigoplus_{j\in J} m_j\Delta^j(V,-) & \longrightarrow & \bigoplus_{j\in J} m_j\Delta^j\mathrm{Im}(f,-) & \longrightarrow & 0
\end{array}$$

Since the first two maps are isomorphisms, the map c also has to be an isomorphism. □

Our hope is now that this recursive definition of the kernel of a representable morphism is able to help us to control its behavior. We would like to prove the following conjecture; a suggested proof is attached.

Conjecture 5.4.6. *Let* $\ker(f,-) \subset (V,-)$ *be a kernel of a representable morphism* $(f,-) : (V,-) \to (W,-)$. *Then* $\ker(f,-)$ *is finitely generated.*

Motivation: Following lemma 5.3.7 there exists a $k' \in \mathbb{N}$ and positive integers m_i, $0 \leq i \leq k'$ and $m_{k'} \neq 0$, such that

$$\bigoplus_{j \in J'} m'_j \Delta^j \ker(f,-) \cong \bigoplus_{i \in I'} m'_i \Delta^i \ker(f,-);$$

where $I' \dot{\cup} J' = \{0, \dots, k'\}$.
Let further $(L,-)$ be a projective standard such that $(g, \mathbb{F}_q^s) : (L, \mathbb{F}_q^s) \twoheadrightarrow \ker(f, \mathbb{F}_q^s)$ for some $s > k$. Since $\Delta(L,-) \cong (L,-)^{\oplus q^{\dim L} - 1}$, we can find positive integers m_i for $0 \leq i \leq k$ where $m_k \neq 0$ and $k = k' + 1$ as well as $I \dot{\cup} J = \{0, \dots, k\}$ such that:

$$\bigoplus_{j \in J} m_j \Delta^j \ker(f,-) \cong \bigoplus_{i \in I} m_i \Delta^i \ker(f,-)$$

and

$$\bigoplus_{j \in J} m_j \Delta^j (L,-) \cong \bigoplus_{i \in I} m_i \Delta^i (L,-)$$

Since $(g, \mathbb{F}_q^n)$ is surjective for $n \leq s$ and $\Delta F \cong F(\mathbb{F}_q \oplus -)/F$, we have that $\Delta(g, \mathbb{F}_q^n)$ is surjective for $n \leq s - k$. Therefore, we get:

$$\begin{array}{ccccc}
\bigoplus_{i \in I} m_i \Delta^i (L, \mathbb{F}_q^{s-k}) & \xrightarrow{m_i \Delta^i (g, \mathbb{F}_q^{s-k})} & \bigoplus_{i \in I} m_i \Delta^i \ker(f, \mathbb{F}_q^{s-k}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\bigoplus_{j \in J} m_j \Delta^j (L, \mathbb{F}_q^{s-k}) & \xrightarrow{m_j \Delta^j (g, \mathbb{F}_q^{s-k})} & \bigoplus_{j \in J} m_j \Delta^j \ker(f, \mathbb{F}_q^{s-k}) & \longrightarrow & 0
\end{array}$$

Without loss of generality we can assume that $k \in J$, so if we pass from $\mathbb{F}_q^{s-k}$ to $\mathbb{F}_q^{s-k+1}$, we obtain:

$$\begin{array}{ccccc}
\bigoplus_{i \in I} m_i \Delta^i (L, \mathbb{F}_q^{s-k+1}) & \xrightarrow{m_i \Delta^i (g, \mathbb{F}_q^{s-k+1})} & \bigoplus_{i \in I} m_i \Delta^i \ker(f, \mathbb{F}_q^{s-k+1}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\bigoplus_{j \in J} m_j \Delta^j (L, \mathbb{F}_q^{s-k+1}) & \xrightarrow{m_j \Delta^j (g, \mathbb{F}_q^{s-k+1})} & \bigoplus_{j \in J} m_j \Delta^j \ker(f, \mathbb{F}_q^{s-k+1}) & \longrightarrow & 0
\end{array}$$

In this case we so far know that $\bigoplus_{i \in I} m_i \Delta^i (g, \mathbb{F}_q^{s-k+1})$ because its image is fully described $(g, \mathbb{F}_q^i)$ with $i \leq s$. The commutativity can be used to obtain that

$\bigoplus_{j\in J} m_j\Delta^j(g,\mathbb{F}_q^{s-k+1})$ must be also surjective. At the level of dimensions this yields

$$\sum_{j\in J} m_j\phi(\Delta\mathrm{Im}(g,-),s-k+1)=\sum_{j\in J} m_j\phi(\Delta\ker(f,-),s-k+1).$$

The recursive definition of $\phi(\Delta^i F,n)$ (example 3.1.4) can be used to reformulate this to

$$\sum_{j\in J} a_j\phi(\mathrm{Im}(g,-),s-k+1+j)=\sum_{j\in J} a_j\phi(\ker(f,-),s-k+1+j).$$

Where $a_k=m_k\neq 0$. Since we already know that $\phi(\mathrm{Im}(g,-),s-k+1+j)=\phi(\ker(f,-),s-k+1+j)$ for $j<k$ this yields $\phi(\mathrm{Im}(g,-),s+1)=\phi(\ker(f,-),s+1)$ and therefore, $\mathrm{Im}(g,\mathbb{F}_q^{s+1})\cong\ker(f,\mathbb{F}_q^{s+1})$.
If this is true, we could iterate this process and obtain that we have an epimorphism $(L,\mathbb{F}_q^n)\twoheadrightarrow\ker(f,\mathbb{F}_q^n)$ for all $n\in\mathbb{N}$. Since by construction $(L,-)$ is finitely generated, it would appear that it holds that $\ker(f,-)$ is finitely generated as well.

It may not appear obvious that this is false. The problem rests in the behavior of the maps $\Delta^i(g,-)$. Of course, those remain surjective when being evaluated at sufficiently small $\mathbb{F}_q^n$. Also, we have an $l\in\mathbb{N}$ a partition $I\dot\cup J=\{0,\dots,l\}$ and positive integers m_i for $0\le i\le l$ with $m_l\neq 0$ such that $\bigoplus_{i\in I} m_i\Delta^i\mathrm{Im}(g,-)\cong\bigoplus_{j\in J} m_j\Delta^j\mathrm{Im}(g,-)$. But we have no chance to restrict l independent of $\dim L$. Therefore, it is not known if it is possible to find a common recursion for both $\mathrm{Im}(g,-)$ and $\ker(f,-)$ such that

$$\begin{array}{ccccc}
\bigoplus_{i\in I} m_i\Delta^i\mathrm{Im}(g,\mathbb{F}_q^1) & \xrightarrow{\cong} & \bigoplus_{i\in I} m_i\Delta^i\ker(f,\mathbb{F}_q^1) & \longrightarrow & 0\\
\downarrow{\cong} & & \downarrow{\cong} & & \\
\bigoplus_{j\in J} m_j\Delta^j\mathrm{Im}(g,\mathbb{F}_q^1) & \longrightarrow & \bigoplus_{j\in J} m_j\Delta^j\ker(f,\mathbb{F}_q^1) & \longrightarrow & 0
\end{array}$$

So we can conclude that this approach cannot be used. Although it would have some very nice consequences.

Corollary 5.4.7. *Let* $\mathbf{f}:P_1\to P_2$ *be a matrix representable morphisms in* $\mathcal{F}_q$. *Then* $\ker(\mathbf{f})$ *is finitely generated.*

This result would also generalize the results of L. Schwartz which we stated in theorem 2.1.11. Firstly we can observe that functors of finite length have resolutions by finitely generated projectives. Since $\dim F(\mathbb{F}_q^n)$ is a polynomial in n, all the functions of the dimensions of images and kernels involved must be of closed form.

The more general result is that F has a resolution by finitely generated projectives if $\exists k$ s.t. $\Delta^k F$ is a finitely generated projective. If $\Delta^k F = P$ then $\dim P(\mathbb{F}_q^n)$ is of closed form. And since F is part of the following sequence

$$P = \Delta^k F \longrightarrow \Delta F^k(\mathbb{F}_q \oplus -) \longrightarrow \cdots \longrightarrow \Delta F(\mathbb{F}_q \oplus -) \longrightarrow F$$
$$\Delta F^k(\mathbb{F}_q \oplus -) \searrow \Delta^{k-1} F \nearrow \cdots$$

we see that $\dim F(\mathbb{F}_q^n)$ must also be of closed form.

5.5 Semi-direct approach

In this subsection we will try to approach coherence by using a method that relies on both, the closed form and explicitly calculated representable morphisms $(g, -)$ such that $g \circ f = 0$. We must assume that $\phi(\ker(f, -), n)$ is of closed form for any representable morphism $(f, -)$.
For all of this subsection we abbreviate $F = \ker(f, -)$ for some $(f, -) : (V, -) \to (W, -)$. We further set $G = \mathrm{Im}(g, -)$ for an approximation $(g, -) : (L, -) \to (V, -)$ such that $(f, \mathbb{F}_q^n) \circ (g, \mathbb{F}_q^n)$ is exact for $n \leq N$. The map $(g, -)$ will be obtained as in lemma 5.4.2.

Remark 5.5.1. Since $\phi(F, n)$ and $\phi(G, n)$ are of closed form, there are d, $d' \in \mathbb{N}$ and a_i, $b_i \in \mathbb{C}$ such that

$$\phi(F, n + d) = \sum_{i=0}^{d-1} a_i \phi(F, n + i) \text{ respectively } \phi(G, n + d) = \sum_{i=0}^{d'-1} b_i \phi(G, n + i).$$

For coherence we would now need that $\phi(F, n) = \phi(G, n)$ for all $n \in \mathbb{N}$. But the closed form allows us to restrict to a finite number of n, namely those smaller or equal to d. So our hope is that if

$$\phi(F, n) = \phi(G, n) \,\forall\, n \leq d$$

that we can follow $\phi(F, n) = \phi(G, n) \,\forall\, n \in \mathbb{N}$. But sadly, there is no chance to a priori determine d' or restrict it. Therefore, we are in the same situation as in the previous subsection and cannot proceed any further with this approach. But there are still some more things to say.

Remark 5.5.2. Let $H = F/G$ with $\phi(H, n) = \sum_{i=0}^{s} p_i(n) q^{in}$. Now we choose $N \in \mathbb{N}$ such that both $\phi(H, N) \neq 0$ and all $p_i(N) \neq 0$. If we use lemma 5.4.2 again to find an approximation $G' = \mathrm{Im}(g', -)$ such that $g' \circ f = 0$ and $\phi(G', n) = \phi(F, n)$ for $n \leq N$, then we get

$$G \subsetneq G' + G \subset F.$$

Lemma 5.5.3. *Let $H' = F/(G' + G)$ with G and G' as before, then it holds that $\phi(H', n) \leq \phi(H, n)$ for all n and we can further obtain from the closed forms $\phi(H, n) = \sum_{i=0}^{s} p_i(n)q^{in}$ and $\phi(H', n) = \sum_{i=0}^{s'} p'_i(n)q^{in}$ that it must hold that $s' \leq s$ and $\deg p_s \geq \deg p'_s$.*

Proof. Since $G \subsetneq G' + G$ it is obvious that $\phi(H', n) \leq \phi(H, n)$ for all n as $\phi(G' + G, n) \geq \phi(G, n)$ for all n. If we now assume that $s' > s$ then we would end up with $\phi(H', n) > \phi(H, n)$ for large enough n. The same can be obtained for the degree of the leading polynomial. □

It is however not possible to make any statement on how G' looks like. Lemma 5.4.2 barely gives us the existence of such a finitely generated subfunctor of F. But we do not know what happens to the closed form. To be more precise, we cannot prove anything better than $s' \leq s$ in terms of the leading non zero q^n term. Another hope would be that we would at least have $\deg p_s > \deg p'_s$. As the degree of any polynomial is finite, after finitely many approximations as in lemma 5.4.2 we would be able to archive $s' < s$. But all attempts to show this have also been running into difficulties and $\deg p_s \geq \deg p'_s$ is the best we could show.

Corollary 5.5.4. *Let $\phi(H, n) = \sum_{i=0}^{s} p_i(n)q^{in}$ and $\phi(H', n) = \sum_{i=0}^{s} p'_i(n)q^{in}$ with $\phi(H', n) < \phi(H, n)$ for large n then there must be at least one polynomial p'_i such that $p'_i(n) < p_i(n)$ for large n.*

The problem, that is described in the above corollary, is that if $p_i(n)' < 0$ for large enough n being smaller than another polynomial, can also mean that $\deg p'_i > \deg p_i$. The definition of the length of the recursion for $\phi(G' + G, n)$ would of course imply that this length grows. Computationally, this would take us away even further from $\phi(F, n)$.
The final approach that comes to mind is the following.

Conjecture 5.5.5. *Let $H = F/G$ with $\phi(H, n) = \sum_{i=0}^{s} p_i(n)q^{in}$. Then there is a finitely generated subfunctor $\hat{F}$ of F such that $G \subsetneq G + \hat{F} \subset F$. and $\phi((G + \hat{F})/G, n) = \sum_{i=0}^{s} \hat{p}_i(n)q^{in}$.*

Assuming this conjecture we can prove the following.

Lemma 5.5.6. *Let $H = F/G$ with $\phi(H, n) = \sum_{i=0}^{s} p_i(n)q^{in}$ then adding a finite number of functors of the form $\hat{F}$, $\hat{F}$ as described in the above conjecture, will decrease $\deg p_s(n)$ after finitely steps.*

Proof. We use induction on s. For $s = 0$ we would obviously face the problem that subtracting something positive from $\phi(H, n)$, this expression would become negative for some n. This is a contradiction since H is a functor.

For $s > 0$ we can just use the functor Δ as for some iteration of $\Delta^k H$ the functional $\phi(\Delta^k H, n)$ will be a $q^n \cdot \left(\sum_{i=0}^{s-1} \overline{p}_i q^{in}\right)$. In this case $\overline{p}_i(n)$ will still be a polynomial of the same degree as $p_i(n)$ and even $\overline{p}_i(n) = \sum_{i=0}^{k} q^i p_i(n)$ which yields that $\phi(\Delta^k H, n)$ would become negative which is a contradiction. □

If this lemma could be proven, we could follow this nice corollary from it.

Corollary 5.5.7. *Let $F \subset (V, -)$ be of closed form, then F is finitely generated.*

Proof. Let G be a finitely generated approximation at F and $\phi(F/G, n) = \sum_{i=0}^{s} p_i(n) q^{in}$. Then by the previous lemma we can find finitely many functors of the form $\hat{F}$ such that $G' = G + \sum_{i=1}^{l} \hat{F}_i$ admits $\phi(F/G', n) = \sum_{i=0}^{s} p_i'(n) q^{in}$ with $\deg p_s > \deg p_s'$. Iterating this process will yield a G'' with $\phi(F/G'', n) = \sum_{i=0}^{s'} p_i''(n) q^{in}$ with $s' < s$.
Further iteration yields a $\hat{G}$ with $\phi(F/\hat{G}, n) = 0$. Therefore, $\hat{G} = F$ and since $\hat{G}$ is finitely generated, so must F be. □

The problem here is that it is not clear how to construct the finitely generated functors $\hat{F}$. The idea is that if we have $H = F/G$ with $\phi(H, n) = \sum_{i=0}^{s} p_i(n) q^{in}$, then for large enough $\mathbb{F}_q^n$ there is a subset $\{\alpha\} \subset F(\mathbb{F}_q^n)$, $\{\alpha\} \not\subset G(\mathbb{F}_q^n)$ such that all elements $\alpha \in \{\alpha\}$ have s columns such that all entries in this columns can be chosen freely without leaving the functor F and the set $\{\alpha\}$. From such a subset we would then construct a finitely generated subfunctor $\hat{F} \subset F$. But it is not clear at all how to choose the subset that we stay in the kernel.

5.6 Inductive approach

This subsection uses the proof of L. Schwartz in [Sch94][Theorem 5.3.8]. The enveloping method used is induction as well as an explicit description of a map $\Delta(f, -)$ respectively matrices of such maps. We are only be able to prove the inductive start and step in a rather special case that will not lead us to a proof of coherence in the category $\mathcal{F}_q$.

Definition 5.6.1. Let $(f, -) : (V, -) \to (W, -)$ be a representable morphism with $f = \sum_{i=1}^{s} \lambda_i [f_i]$. Then $(f, \mathbb{F}_q \oplus -)$ is a $q^{\dim W} \times q^{\dim V}$ matrix with rows indexed by elements $w \in (W, \mathbb{F}_q)$ and columns indexed by elements $v \in (V, \mathbb{F}_q)$ such that

$$(f, \mathbb{F}_q \oplus -)_{w,v} = \sum_{i \in I_{w,v}} \lambda_i [f_i].$$

The sets $I_{w,v}$ are defined such that $i \in I_{w,v}$ if $v \circ [f_i] = w$. The entry at $0, 0$ corresponds to $(f, -)$. Furthermore, the column indexed by $v = 0$ corresponds to the direct summand $(f, -) : (V, -) \to (W, -)$ of $(f, \mathbb{F}_q \oplus -) : (V, \mathbb{F}_q \oplus -) \to (W, \mathbb{F}_q \oplus -)$.

Lemma 5.6.2. *Let $(f,-):(V,-)\to(W,-)$ be a representable morphism with $f=\sum_{i=1}^s \lambda_i[f_i]$. For the index set $I_{\binom{w_1}{w_2},\binom{v_1}{v_2}}$ needed in $\Delta^2(f,-)$ it holds that:*

$$I_{\binom{w_1}{w_2},\binom{v_1}{v_2}} = I_{w_1,v_1}\cap I_{w_2,v_2}$$

Proof. We have $i\in I_{\binom{w_1}{w_2},\binom{v_1}{v_2}}$ if $\binom{v_1}{v_2}\circ[f_i]=\binom{w_1}{w_2}$. But this holds if and only if $v_1\circ[f_i]=w_1$ and $v_2\circ[f_i]=w_2$. Therefore, the claim of the lemma holds. □

Corollary 5.6.3. *For $\Delta^k(f,-)$ the index sets $I_{w,v}$ are defined inductively by the above lemma.*

Lemma 5.6.4. *Let $(f,-):(V,-)\to(W,-)$ be a representable morphism with $f=\sum_{i=1}^s\lambda_i[f_i]$. Then there is a $k\in\mathbb{N}$ such that $|I_{w,v}|<s$, $I_{w,v}$ index sets in $\Delta^k(f,-)$, for all w,v if and only if there are no $g\in\mathbb{F}_q[\mathrm{GL}(V)]$ and $h\in\mathbb{F}_q[\mathrm{GL}(W)]$ such that in $f'=h\circ f\circ g=\sum_{j=1}^t[f'_j]$ all the $[f'_j]$ have one row in common.*

Proof. Since $[f_i]\neq[f_j]$ for all $i\neq j$, there must be a $v\in(V,\mathbb{F}_q^n)$, for some $n\in\mathbb{N}$, such that $v\circ[f_i]\neq v\circ[f_j]$ unless all $[f_i]$ have a common row, which we can find after base change as described above. Therefore, $i\in I_{w,v}$ and $j\in I_{w',v}$ for $w\neq w'$ which shows that $|I_{w,v}|<s$.
The other direction is obvious. □

The following lemma stems from the work of L. Schwartz. Though it is very important in the original appearance, it is not stated in this very manner, therefore, we restate it a little.

Lemma 5.6.5. *Let G be a functor in $\mathcal{F}_q$ then G is finitely generated if and only if ΔG is.*

Proof. This is result of the proof of Theorem 5.3.8 and Lemma 5.3.9 in [Sch94]. □

The importance of this lemma becomes clear in a moment. Let us first state a conjecture for an inductive start. We will prove it a little later.

Conjecture 5.6.6. *Let*

$$\mathbf{f}=\begin{pmatrix}(f_{11},-) & \cdots & (f_{1t},-)\\ (f_{21},-) & \cdots & (f_{2t},-)\\ & \vdots & \\ (f_{m1},-) & \cdots & (f_{mt},-)\end{pmatrix}:\bigoplus_{i=1}^t(V,-)\to\bigoplus_{j=1}^m(W,-).$$

If $(f_{ji},-)=([f_{ji}],-)$ for all j,i then $\ker(\mathbf{f})$ is finitely generated.

If we assume this conjecture, we can prove the following proposition which would be the inductive step in an inductive proof on the length of the linear combinations that each f_{ji} consists of.

Proposition 5.6.7. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_{11}, -) & \cdots & (f_{1t}, -) \\ (f_{21}, -) & \cdots & (f_{2t}, -) \\ & \vdots & \\ (f_{m1}, -) & \cdots & (f_{mt}, -) \end{pmatrix} : \bigoplus_{i=1}^{t} (V, -) \to \bigoplus_{j=1}^{m} (W, -).$$

Where $f_{ji} = \sum_{l=1}^{s_{ji}} \lambda_l^{(ij)} [f_l^{(ij)}]$ *and* $s = \max\{s_{ji}\}$. *If we know that* $\ker(\mathbf{f})$ *is finitely generated for* $\max\{s_{ji}\} < s$ *such that there is a* k *such that* $\Delta^k \mathbf{f}$ *is a matrix of map that are basis vectors, then* $\ker(\mathbf{f})$ *is finitely generated if there is a* $k \in \mathbb{N}$ *such that in* $\Delta^k \mathbf{f}$ *we have* $\max\{s_{ji}\} \leq 1$.

Proof. Let us assume that there is a $k \in \mathbb{N}$ such that in the matrix corresponding to $\Delta^k \mathbf{f}$, all components are sums of basis vectors at most length 1. The above conjecture respectively the inductive hypothesis we know that $\ker \Delta^k \mathbf{f}$ is finitely generated. From 5.6.5 we know that this is equivalent to $\ker \mathbf{f}$ being finitely generated. □

If we could prove the inductive start given by conjecture 5.6.6, we could show that $\ker \mathbf{f}$ is finitely generated for any matrix of representable morphism which are basis vectors. We now turn our attention to a proof of the inductive start.

Lemma 5.6.8. *Let*

$$\mathbf{f} = \begin{pmatrix} ([f_1], -) \\ ([f_2], -) \\ \vdots \\ ([f_m], -) \end{pmatrix} : (V, -) \to \bigoplus_{j=1}^{m} (W, -).$$

Then $\ker \mathbf{f}$ *is finitely generated by the set*

$$G = \bigcup_{J \subset \{1,\ldots,m\}} \left(\sum_{I \subset J} (p-1)^{|I|} [g + k_I^J] \right),$$

$[g]$ *arbitrary, with* $k_\emptyset^J = 0$, $k_I^J = \sum_{i \in I} k_i^J$ *such that* $k_i^J \circ f_i = 0$ *and for* $k \notin J$ *we have* $k_i^J \circ f_k = 0$.

Proof. We use induction on m. For $m = 1$ we can assume $[f_1]$ to be in rank normal form with rank r:

$$[f_1] = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The kernel and image of $([f_1], -)$ are finitely generated projectives as $[f_1]$ is an idempotent. The kernel is generated by elements of the form $[g_i, r_i] + (p-1)[g_i, 0]$ as claimed.

Let us now make the inductive step. We suppose that we know that for one m the kernel of $\mathbf{f}$ is finitely generated. We further assume the map $[f_{m+1}]$ to be in rank normal form as above. From previous computations we know that

$$\ker \mathbf{f} = \bigcap_{i=1}^{m+1} \ker([f_i], -).$$

Now take $x \in \bigcap_{i=1}^{m} \ker([f_i], \mathbb{F}_q^n)$ for some n.

$$x = \sum_{i=1}^{t} \lambda_i \sum_{J \subset \{1,\ldots,m\}} \sum_{I \subset J} (p-1)^{|I|} [g_i + k_I^{J,(i)}]$$

If $x \circ [f_{m+1}] = 0$, we know that there must also be a representation

$$x = \sum_{i=1}^{s} \mu_i ([g_i, r_i] + (p-1)[g_i, 0]).$$

We now want to show that

$$G = \bigcup_{J \subset \{1,\ldots,m+1\}} \left(\sum_{I \subset J} (p-1)^{|I|} [g + k_I^J] \right)$$

which is a candidate for a generating set of $\bigcap_{i=1}^{m+1} \ker([f_i], \mathbb{F}_q^n)$ is in fact a generating set. Suppose we have an element in this intersection of the kernels.

$$x = \sum_{i=1}^{t} \lambda_i \sum_{J \subset \{1,\ldots,m\}} \sum_{I \subset J} (p-1)^{|I|} [g_i + k_I^{J,(i)}] = \sum_{i=1}^{t'} \mu_i ([g_i] + (p-1)[g_i + k_i])$$

where we have $k_i \circ f_{m+1} = 0$. Since $x \in \bigcap_{i=1}^{m+1} \ker([f_i], \mathbb{F}_q^n)$, we have

$$0 = \sum_{i=1}^{t} \lambda_i \sum_{J \subset \{1,\dots,m\}} \sum_{I \subset J} (p-1)^{|I|} [g_i + k_I^{J,(i)}] \circ f_{m+1}.$$

We know that for every $[g_i]$ there must be elements $[g_j] = [g_i + k_I^{J,(i)}]$ (if the element $[g_j]$ is represented as $[g_j + k_I^{J,(j)}]$, we just rename it) with $k_I^{J,(i)} \circ f_{m+1} = 0$ such that $\sum_{[g_j]} \lambda_j = 0$. Now there are two cases that can occur.

Case 1: If not all of those elements come from the same summand in the representation of x as element of $\bigcap_{i=1}^{m} \ker([f_i], \mathbb{F}_q^n)$, we need to take a second summand $-\lambda_{i'} \sum_{J \subset \{1,\dots,m\}} \sum_{I \subset J} (p-1)^{|I|} [g_i' + k_I^{J,(i')}]$ into the account. We might not be able to take everything of this summand out as we may have $\lambda_{i'} \neq -\lambda_i$, but we can decrease the number of summands in the sum $\sum_{[g_j]} \lambda_j = 0$ and can therefore iterate this process. But this yields x being generated by elements in G.

Case 2: If $0 = \sum_{i=1}^{t} \lambda_i \sum_{I \subset \{1,\dots,m\}} (p-1)^{|I|} [g_i + k_I^{J,(i)}] \circ f_{m+1}$ and the elements $[g_i] \circ f_{m+1} = [g_i + k_I^{J,(i)}] \circ f_{m+1}$ such that $\sum_{[g_j]} \lambda_j = 0$ all stem from the same summand. This means that x is generated by the elements of

$$\bigcup_{J \subset \{1,\dots,m\}} \left(\sum_{I \subset J} (p-1)^{|I|} [g + k_I^J] \right) \subset G,$$

in accordance with the definition of k_I^J for $m + 1 \notin J$, and therefore by G.

□

Corollary 5.6.9. *Let $G \subset (V, -)$ where $G = \bigcap_{i=1}^{m} P_{\lambda_i}$, where $P_{\lambda_i} \subset (V, -)$ is an indecomposable projective. Then G is finitely generated.*

Proof. We can find maps $([f_i], -)$ that each have a different isomorphic copy of P_λ in their kernel. This intersection is finitely generated by the above lemma as our given G will be a direct summand of such a kernel. □

Corollary 5.6.10. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_1, -) \\ (f_2, -) \\ \vdots \\ (f_m, -) \end{pmatrix} : (V, -) \to \bigoplus_{j=1}^{m} (V, -).$$

Then $\ker \mathbf{f}$ *is finitely generated if there are invertible maps $g_j, h : V \to V$. such* *that*

$$g_j \circ f_j \circ h = \mathrm{id}_V + (p-1)\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all j. Where the matrix on the right hand side is of rank r_j.

Proof. The kernel of such a map is again a finitely generated projective again since $f_j^2 = f_j$ and therefore the kernel of $\mathbf{f}$ can be written as kernel of a map as in lemma 5.6.8. □

In the previous corollary we did require that $\mathbf{f} : (V, -) \to \bigoplus_{j=1}^{m} (V, -)$ though technically we do not need that restriction and the claim holds more generally.

Corollary 5.6.11. *Let $\mathbf{f} = ((f_1, -), \ldots, (f_t, -)) : \bigoplus_{i=1}^{t} (V, -) \to (W, -)$, then $\ker \mathbf{f}$ is finitely generated.*

Proof. We can calculate that $\ker \mathbf{f} \cong \bigoplus_{i=1}^{t} \ker([f_i], -) \oplus \bigoplus_{S \subset \{1,\ldots,t\}} \bigcap_{i \in T} \mathrm{Im}([f_i], -)$. The sum of the kernels we know to be finitely generated. For the intersection we can use lemma 5.6.8 as the intersection of the images is of the form of a kernel with respect to corollary 5.6.10. □

Corollary 5.6.12. *Let*

$$\mathbf{f} = \begin{pmatrix} ([f_{11}], -) & \cdots & ([f_{1t}], -) \\ ([f_{21}], -) & \cdots & ([f_{2t}], -) \\ & \vdots & \\ ([f_{m1}], -) & \cdots & ([f_{mt}], -) \end{pmatrix} : \bigoplus_{i=1}^{t} (V, -) \to \bigoplus_{j=1}^{m} (W, -).$$

Then $\ker \mathbf{f}$ is finitely generated.

Proof. It holds that $\ker \mathbf{f} \cong \bigcap_{i=1}^{t} \ker \mathbf{f}_j$ where $\mathbf{f}_j : \bigoplus_{i=1}^{s} (V, -) \to (V, -)$ (j-th copy). We further have $\ker \mathbf{f}_j = \bigoplus_{i=1}^{s} \ker([f_{ij}], -) \oplus \bigcap_{i=1}^{s} \mathrm{Im}([f_{ij}], -)$ which is finitely generated by corollary 5.6.11. Therefore, we have

$$\ker \mathbf{f} \cong \bigcap_{j=1}^{t} \left(\bigoplus_{i=1}^{s} \ker([f_{ij}], -) \oplus \bigcap_{i=1}^{s} \mathrm{Im}([f_{ij}], -) \right) \cong$$

$$\bigoplus_{J \dot\cup J' = \{1,\ldots,t\}} \left(\bigcap_{j \in J} \left(\bigoplus_{i=1}^{s} \ker([f_{ij}], -) \right) \cap \bigcap_{j' \in J'} \left(\bigcap_{i=1}^{s} \mathrm{Im}([f_{ij'}], -) \right) \right) \cong$$

$$\bigoplus_{k=1}^{t'}\bigcap_{l=1}^{s'} P_{\lambda_{tl}} \text{ where } \lambda_{tl} = \lambda_{t'l'} \text{ is possible.}$$

This is a finite direct sum of finite intersections of finitely generated projective functors and therefore finitely generated by corollary 5.6.9. □

Corollary 5.6.13. *Conjecture 5.6.6 holds true.*

After proving this conjecture, we can now prove a better version of proposition 5.6.7.

Proposition 5.6.14. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_{11}, -) & \cdots & (f_{1t}, -) \\ (f_{21}, -) & \cdots & (f_{2t}, -) \\ & \vdots & \\ (f_{m1}, -) & \cdots & (f_{mt}, -) \end{pmatrix} : \bigoplus_{i=1}^{t}(V, -) \to \bigoplus_{j=1}^{m}(W, -).$$

Where $f_{ji} = \sum_{l=1}^{s_{ji}} \lambda_l [f_l]$ *and* $s = \max\{s_{ji}\}$*. If there is a* $k \in \mathbb{N}$ *such that in* $\Delta^k \mathbf{f}$*, we have* $\max\{s_{ji}\} \leq 1$*, then* $\ker \mathbf{f}$ *is finitely generated.*

Proposition 5.6.15. *Let* $(f, -) : (V, -) \to (W, -)$ *be a representable morphism, with* $f = \sum_{i=1}^{s} \lambda_i [f_i]$*, then* $\ker(f, -)$ *is finitely generated if* $\ker(f', -)$ *is finitely generated where* $f' = \sum_{i=1}^{s} \lambda_i [f'_i]$ *and* $[f'_i]$ *is made from* $[f_i]$ *by removing all the rows that are common (after base change) for all* i*.*

Proof. Lemma 5.6.4 tells us precisely how the part that we cut out of every $[f_i]$ looks like to get to $[f'_i]$. The part we cut out we denote by $[m]$. Then we want to use corollary **??** as we can find a decomposition

$$(V', -) \otimes (V'', -) \xrightarrow{(f',-)\otimes([m],-)} (V, -)\ .$$

Now it is not hard to see that we also have $\text{Im}(f, -) \cong \text{Im}(f', -) \otimes \text{Im}([m], -)$ and can therefore obtain that $\ker(f, -)$ is finitely generated as it is the tensor product of two finitely generated functors. □

This proposition shows a way on how to deal with common rows in $(f, -)$. But it does not provide a way of treating maps $\mathbf{f}$ where multiple representable maps have the same starting vector space. So unfortunately, this is as close as we can get to coherence. We have however managed to obtain finitely generatedness for a large group of kernels of morphisms.
The following conjecture motivates how to proceed with the obtained information.

Conjecture 5.6.16. *Let*

$$\mathbf{f} = \begin{pmatrix} (f_{11}, -) & \cdots & (f_{1t}, -) \\ (f_{21}, -) & \cdots & (f_{2t}, -) \\ & \vdots & \\ (f_{m1}, -) & \cdots & (f_{mt}, -) \end{pmatrix} : \bigoplus_{i=1}^{t} (V, -) \to \bigoplus_{j=1}^{m} (W, -),$$

where $f_{ji} = \sum_{l=1}^{s_{ji}} \lambda_l [f_l]$ *and* $s = \max\{s_{ji}\}$. *If there is a* $k \in \mathbb{N}$ *such that in* $\Delta^k \mathbf{f}$ *we have* $\max\{s_{ji}\} \leq t$, *then* $\ker \mathbf{f}$ *is finitely generated. Here* t *is maximal such that we know that* $\ker(f, -)$, *for* $f = \sum_{i=1}^{t} \lambda_i [f_i]$, *is finitely generated.*

Motivation: One would again start by setting $t = 1$ and use a proof similar to lemma 5.6.8 to show that $\ker \mathbf{f}$ is finitely generated in such a case. Instead of doubling the length of a generator the length could now increase by factor l if l is the max length of a generator for one of the kernels of the f_i. Solving this part is already as hard as showing that $F \cap G$ is finitely generated for $F, G \subset (V, -)$ finitely generated subfunctors which are kernels of representable morphisms.
Lastly, we could prove an analogue of corollary 5.6.12 as the kernel of such a given $\mathbf{f}$ is just again a finite sum of finite intersections of finitely generated kernels.

Assuming the conjecture, we can prove the following.

Proposition 5.6.17. *Let* $(f, -) : (V, -) \to (W, -)$ *be a representable morphism with* $f = \sum_{i=1}^{s} \lambda_i [f_i]$ *then* $\ker(f, -)$ *is finitely generated.*

Proof. We use induction on s. For $s = 1$ the claim can be proved by a direct computation.
For $s > 1$ we can have two cases. If all $[f_i]$ have common rows (after base change) then we can use proposition 5.6.15 to arrive at an $(f', -)$ that does no longer have common rows. If there is no common row, we can just continue with $(f, -)$. In both cases $\Delta(f, -)$ and $\Delta(f', -)$ respectively allows to use the inductive hypothesis as all component maps are sums of length less than s. The kernel of $\Delta(f, -)$ and $\Delta(f', -)$ respectively is then a finite sum of finite intersections of finitely generated functors, and as stated in the above conjecture, finitely generated. From lemma 5.6.5 we know this to be equivalent to $\ker(f, -)$ and $\ker(f', -)$ respectively being finitely generated. In case we work with $(f, -)$, we are done. For $(f', -)$ we need to use proposition 5.6.15 once more. □

If the above conjecture holds, in other words if we could show that a finite intersection of two finitely generated kernels of representable morphisms is finitely generated, the above proposition would yield coherence.

Remark 5.6.18. Moreover corollary 5.6.9 gives rise to the following heuristics: Let $(f,-) : (V,-) \to (W,-)$ a representable morphism. Then $\mathrm{Im}(f,-)$ is finitely generated, as it is a finite sum of finitely generated projectives. Therefore $\ker(f,-)$ should be, analogue to the subspace case, a finite sum of finite intersections of finitely generated projectives and therefore also finitely generated. The subspace case arising from the connection to representations of certain quivers as described in subsection 4.1.

6 Approach to finitely generated functors

Finitely generated functors come with less structure than finitely presented ones. This causes some problems that we have to deal with. Again this section lists just some attempts and approaches.

Since an arbitrary subfunctor $G \subset \bigoplus_{i=1}^m (V_i,-)$ does not necessary come from a representable morphism we have little hope to be able to describe the growth of the dimension of G, e.g. in closed form explicitly.
Let us summarize what we can say so far.

Remark 6.0.1. Let G be a proper sub functor of $\bigoplus_{i=1}^m (V_i,-)$. Then it holds that

$$\dim G(\mathbb{F}_q^n) \leq \sum_{i=1}^m q^{n \dim V_i} \,\forall\, n \in \mathbb{N}.$$

6.1 Dimension formula for subfunctors of projectives

We now try to describe the dimensional growth of a non-finitely generated subfunctor of a projective functor. The desired result being that such a functor cannot exist.
Let us suppose that $G \subset \bigoplus_{i=1}^m (V_i,-)$ is not finitely generated. We get the following remark:

Remark 6.1.1. If G is not finitely generated then $\nexists N \in \mathbb{Z}_{\geq 0}$ and vector spaces $W_i,\ i \in \{1,\dots,N\}$ s.t.

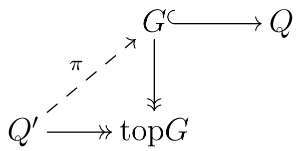
$$\bigoplus_{i=1}^N (W_i,-) \twoheadrightarrow G \to 0.$$

Definition 6.1.2. Let G be any functor and P a projective, s.t. there is a surjection $P \twoheadrightarrow G$. Then we set $\mathrm{top}G = \mathrm{top}P$. This is well defined for P projective cover of G. $\mathrm{top}P_\lambda$ is defined for all indecomposable projectives by $\mathrm{top}P_\lambda = P_\lambda / \operatorname{rad} P_\lambda$, where $\operatorname{rad} P_\lambda$ is the unique maximal submodule. Further, we have $\operatorname{rad}\left(\bigoplus_{i\in I} P_{\lambda_i}\right) = \bigoplus_{i \in I} \operatorname{rad}(P_{\lambda_i})$ for all index sets I.

If G is finitely generated, then top$G$ is obviously a semi-simple module of finite length. Though a non-finitely generated module must not necessary have an infinite top$G$, but it must in our case.

Lemma 6.1.3. *If G is a non finitely generated submodule of a finitely generated projective, then* top *must be a semi simple module of infinite length.*

Proof. Let G be a non finitely generated submodule of some $\bigoplus_{i=1}^{m}(V_i,-) = Q$.We need to show that top$G$ is in fact of infinite length. Suppose top$G$ is of finite length and Q' its projective cover.



We need to show that the map π is an epimorphism. If π is not surjective, then Imπ is a proper subfunctor of G, so we can look at the cokernel. By definition of top$G$ $G/\mathrm{Im}\pi$ can only have a trivial top. Therefore, we must have $\mathrm{rad}\,(G/\mathrm{Im}\pi) = G/\mathrm{Im}\pi$. This is equivalent to $G/\mathrm{Im}\pi$ not having any maximal submodules.
Let us now look at $\big(\mathrm{rad}^i(Q/\mathrm{Im}\pi)\big)\cap(G/\mathrm{Im}\pi)$. This is again a submodule of $G/\mathrm{Im}\pi$ and by the structure of Q there must be a minimal i such that this is a proper submodule of $\big(\mathrm{rad}^{i-1}(Q/\mathrm{Im}\pi)\big)\cap(G/\mathrm{Im}\pi)$. Then

$$\big((\mathrm{rad}^{i-1}(Q/\mathrm{Im}\pi))\cap(G/\mathrm{Im}\pi)\big)\,/\,\big((\mathrm{rad}^{i}(Q/\mathrm{Im}\pi))\cap(G/\mathrm{Im}\pi)\big)$$

is of finite length since $\mathrm{rad}^i Q/\mathrm{rad}^{i-1} Q$ is. So we can construct a maximal submodule of $G/\mathrm{Im}\pi$ from it and $(\mathrm{rad}^i(Q/\mathrm{Im}\pi))\cap(G/\mathrm{Im}\pi)$, which is a contradiction. Therefore, π must be an epimorphism and G must be finitely generated if its top is. □

Corollary 6.1.4. *Let G be a subfunctor of a finitely generated projective, then G is non finitely generated if and only if* top$G$ *is of infinite length.*

But how can such a non finitely generated subfunctor of a finitely generated projective look like? Let us restate lemma 5.2.9 here.

Lemma 6.1.5. *Let F be finitely generated, then* $\mathrm{rad}^r F/\mathrm{rad}^{r+1} F$ *is a polynomial functor.*

Corollary 6.1.6. *Let $G \subset \bigoplus_{i=1}^{m}(V_i,-)$ be a non finitely generated subfunctor. Then* $\mathrm{top}G \cap \mathrm{top}\,(\mathrm{rad}^r \bigoplus_{i=1}^{m}(V_i,-)) \neq 0$ *for infinitely many r.*

Corollary 6.1.7. *Let $G \subset \bigoplus_{i=1}^{m}(V_i, -)$ be a non finitely generated subfunctor. Then the composition*

$$G \cap \mathrm{rad}^r \bigoplus_{i=1}^{m}(V_i, -) \hookrightarrow G \twoheadrightarrow \mathrm{top}G$$

is nontrivial for infinitely many r.

With this information we can ask ourselves if on the journey through this infinitely many layers we must also come across arbitrary high weights. The following lemma tells us that this is in fact true.

Lemma 6.1.8. *Let $G \subset \bigoplus_{i=1}^{m}(V_i, -)$ be non finitely generated, then* $\mathrm{top}(G)$ *contains functors of arbitrary high weight.*

Proof. If n would be the highest weight of all the simple functors in $\mathrm{top}(G)$, then we would have

$$\sum_{i=1}^{m} q^{n \dim V_i} = \dim \bigoplus_{i=1}^{m}(V_i, \mathbb{F}_q^n) \geq \dim G(\mathbb{F}_q^n) = \infty,$$

which is a contradiction. □

Corollary 6.1.9. *Let G be a subfunctor of F, F finitely generated. Then for every $n \in \mathbb{Z}_{\geq 0}$ there exist only finitely many simple composition factors F_λ of G with $n(\lambda) = n$.*

Let us look at the following construction. We use an implicite induction on $n(\lambda)$ since we know that P_λ is noetherian for any λ with $n(\lambda) \leq 1$.
From now on we again assume that we know about the existence of the closed form for finitely presented functors.

Conjecture 6.1.10. *Let P_λ be a projective such that $n(\lambda)$ is minimal for P_λ not noetherian. Then there exists an infinitely generated $G \subset P_\lambda$ s.t.*

$$\phi(P_\lambda/G, n) = \overline{p_0}(n) \text{ a power series}$$

but no infinitely generated H s.t.

$$\phi(P_\lambda/H, n) = \overline{p_1}(n)q^{1n} + \overline{p_0}(n), \ \overline{p_i}(n) \text{ power series}$$

Motivation : G resp. H being non-finitely generated means that we can have a family $([f]_n, -)$ of representable morphisms s.t.

$$\begin{array}{ccc} (W_n, -) & \xrightarrow{([f]_n, -)} & (V, -) \\ \downarrow\!\!\downarrow & & \uparrow \\ G & \hookrightarrow & P_\lambda \end{array}$$

for $\mathbb{F}_q^k$ with $k \leq n$ and by increasing the dimension of W_n we get a new order polynomial that approximates the power series.
If the family $([f]_n, -)$ admits a power series at q^{1n} than it has one degree of freedom that is not used to create the dimension of said power series. So we can take this degree of freedom out of $([f]_n, -)$ and still have a subfunctor of a projective which is not of closed form and, therefore, non-finitely generated. But that is a contradiction to the minimality of $n(\lambda)$. Therefore, only G and not H can exist.
The problem with this construction is that it is not so clear how the taking out of a degree of freedom works.

Corollary 6.1.11. *Given G as described above:*

$$\phi(P_\lambda/G, n) = \overline{p_0}(n),\ \overline{p_0}(n) \text{ a power series}$$

Corollary 6.1.12. *With G as above, P_λ/G cannot be embedded in a projective functor.*

Proof. The function $\phi(F, n)$ must be of closed form for F finitely generated submodule of a projective. □

The following looks a bit surprising but is not arbitrary at all.

Lemma 6.1.13. *Every strictly ascending in terms the radical layer of chain of simples C inside a projective P_λ going to infinity has dimensional growth of at least q^{1n}.*

Proof. Without loss of generality we can assume that the simple functor at the top of C is the top of the projective P_λ, it is lying in. This makes C a factor functor of P_λ. The kernel of the projection map is generated by the projective covers of $\operatorname{top}(\operatorname{rad} P_\lambda)/\operatorname{top}(\operatorname{rad} C)$. Therefore, C is finitely presented and its dimension of closed form.
Since C is of infinite length the closed form cannot be just a polynomial but must have at least a term that grows with q^{1n}. □

Corollary 6.1.14. *Every infinite length functor has dimensional growth of at least q^{1n}.*

Now we want to look at $\operatorname{top}(G)$, where G is constructed as above. We order the simple functors in there by the radical layer of P_λ they fall in, so $F_\mu < F_\nu$ if $F_\mu \in \operatorname{top}(\operatorname{rad}^j P_\lambda)$, $F_\nu \in \operatorname{top}(\operatorname{rad}^i P_\lambda)$ and $j < i$. If two simples lie in the in the same radical layer, we use an arbitrary order.

Lemma 6.1.15. *Let G be as in lemma 6.1.11. There exists an infinite number of uniserial chains of simples in P_λ/G descending to infinity which have dimensional growth of at least q^{1n}.*

Proof. We start in the top of P_λ/G which is also the top of P_λ. Now we descend through each radical layer and add one simple functor to our chain C that is not in G. This is possible by corollary 6.1.6 and the fact that P_λ/G has to be indecomposable since it is only generated by one indecomposable projective with a simple top.
But going to infinity with this construction, it yields a uniserial factor functor of P_λ which is also factor functor of P_λ/G.
Now we repeat this process by considering $\hat{C}/G$, where $\hat{C}$ is defined as the kernel of the projection $P_\lambda \twoheadrightarrow C$. As we can apply corollary 6.1.6 to $\hat{C}$, we can now find another uniserial chain C', which is a factor of $\hat{C}/G$. This process may be infinitely many times continued. □

Corollary 6.1.16. *For every $b \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that* $\mathrm{top}((\mathrm{rad}^k P_\lambda) \cap G)$ *and* $\mathrm{top}(\mathrm{rad}^k P_\lambda)/\mathrm{top}((\mathrm{rad}^k P_\lambda) \cap G)$ *contain more than b simples.*

Since every polynomial or power series in a dimension formula arises from simple modules, we can similarly argue for higher q^{1n} terms that there cannot be power series, replacing polynomials in a dimension formula.

Conjecture 6.1.17. *$\phi(G, n)$ is of closed form for any subfunctor of an indecomposable projective.*

Motivation : Starting from lemma 6.1.11, we work the case for a power series at the q^{0n} term. Now we can use lemma 6.1.10 to turn this into an inductive proof on $n(\lambda)$, where λ is the weight of $P_\lambda \subset (\mathbb{F}_q^{n(\lambda)}, -)$. To explicitly make it, let $n(\lambda) = 1$. Then $(\mathbb{F}_q, -)$ is a direct sum of uniserial functors. Therefore, each factor is a direct sum of uniserial and finite length functors. So every subfunctor of such a projective has closed form dimension with a possible polynomial as coefficient for q^{0n}. Let $n(\lambda) > 1$. In this case we can apply lemma 6.1.15. It implies that

$$\phi(P_\lambda/G, n) = \overline{p_1}(n)q^{1n} + \overline{p_0}(n),\ \overline{p_i}(n) \text{ power series}$$

since the chains of simples have arbitrary high simples at their top. We can even demand that $\overline{p_0}(n) \geq 0$ for all n.
By assumption, at least one of the $\overline{p_i}(n)$ must be a proper power series. If $\overline{p_1}(n)$ would be a polynomial and $\overline{p_0}(n)$ a power series, then we could further apply lemma 6.1.15 to decrease $\overline{p_0}(n)$ and thus raising the polynomial degree of $\overline{p_1}(n)$. If the degree of the polynomial $\overline{p_1}(n)$ would not be raised and the values of $\overline{p_1}(n)$ would just get bigger, then we can assume that there is an a such that $\overline{p_1}(a) > q^{n(\lambda)a}$

which is a contradiction.
It follows that $\overline{p_1}(n)$ must be a power series in any case. But this would be a contradiction to conjecture 6.1.10. So $\overline{p_1}(n)$ should be a polynomial and by the above argument, so must $\overline{p_0}(n)$ be.
Therefore, we should always get closed form for the dimension functional for submodules G of P_λ for fixed $n(\lambda)$ given that all P_μ with $n(\mu) < n(\lambda)$ are noetherian.

Once we would be able to show that indecomposable projectives are noetherian, the rest is relatively easy. For the proof of the Artinian conjecture we need to show that all the standard projectives are noetherian but this is standard.

Lemma 6.1.18. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of modules. Then Y is noetherian if and only if X and Z are.*

Proof. Let X, Y, Z be modules and X_i, Y_i, Z_i ascending sequences of submodules such that $X_i = X \cap Y_i$. We need to look at the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X_i & \longrightarrow & Y_i & \longrightarrow & Z_i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X_{i-1} & \longrightarrow & Y_{i-1} & \longrightarrow & Z_{i-1} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0
\end{array}$$

Let us now suppose that X and Z are noetherian and Y_i is any ascending sequence in Y. Now the sequences X_i and Z_i must be stationary after finitely many steps. Therefore this will also hold for Y_i, so Y is noetherian.
The converse follows similar. We start with a sequence X_i in X, but this can be viewed as a sequence in Y, which therefore must be stationary after finitely many steps. To show that Z is also noetherian, we start with a sequence Z_i in Z. Then we look at pre-images of the Z_i in Y. This is again an ascending sequence in Y, which then must be stationary after finitely many steps. So, Z must also be noetherian. □

Conjecture 6.1.19. *Standard projectives are noetherian.*

Motivation: First of all, we can use the above lemma in our situation, since the category $\mathcal{F}_q$, that we are working in, can be viewed as a module category.
Standard projectives are a finite direct sum of indecomposable projectives. So our

method of choice is induction on the number of the indecomposable summands. But this is nothing more than lemma 6.1.18.

$$0 \to P_\lambda \to P_\lambda \oplus P_\mu \to P_\mu \to 0$$

The inductive step is performed as follows.

$$0 \to P_\lambda \to P_\lambda \oplus \bigoplus_{i=1}^{n-1} P_{\mu_i} \to \bigoplus_{i=1}^{n-1} P_{\mu_i} \to 0$$

6.2 Remarks on the Strong Artinian conjecture

The aim of this subsection is to give an approach to the Strong Artinian conjecture. This approach relies strongly on the the Artinian conjecture or better the closed form of the dimension function that we now claim to have for all subfunctors of all projectives.

Conjecture 6.2.1 (Strong Artinian conjecture (Schwartz)). $(\mathbb{F}_q^m, -)$ *is noetherian of type m for any q and m.*

We first need to explain what it means to be noetherian of type m. Afterwards we give an idea of how we can describe the type in this case.

Definition 6.2.2. Let $\mathcal{C}$ be an abelian category with a dimension function. We say that an object C is noetherian of type m if there exists a finite filtration of C such that all the composition factors are simple noetherian of type m.
An object C is simple noetherian of type m if all its subobjects are noetherian of type $m-1$.
A simple object is noetherian of type 0.

Remark 6.2.3. The noetherian type is also sometimes referred to as the dimension.

Example 6.2.4. In the category $\mathcal{F}_q$ we know that all indecomposable direct summands of $(\mathbb{F}_q, -)$ are uniserial. We now say that a subfunctor of such an indecomposable is noetherian of type 0 if its quotient is of finite composition length (that is going to be every subfunctor, but we do not mind).
In this way every indecomposable direct summand of $(\mathbb{F}_q, -)$ becomes simple noetherian of type 1. Since there are only finitely many of them, this makes $(\mathbb{F}_q, -)$ noetherian of type 1 by taking the indecomposable direct summands as composition factors of a finite composition series.

Definition 6.2.5. Call C simple noetherian of type m if all subfunctors G of C produce a factor functor that has lower order closed form, i.e. if $\phi(C, n) = \sum_{i=0}^{m} p_i^C(n) q^{in}$ then $\phi(C/G, n) = \sum_{i=0}^{m'} p_i^{C/G}(n) q^{in}$ with $m' < m$.

Motivation for the proof of the Strong Artinian conjecture. We will assume the Artinian conjecture as well as the closed form. In this case we can show the following. We need to find a finite filtration for each $(\mathbb{F}_q^m, -)$ such that each composition factor is of simple noetherian type m.
A first obvious step is to filter $(\mathbb{F}_q^m, -)$ by its indecomposable projectives. Since there are only finitely many such functors, we can restrict our attention to one of them.
Then we filter an indecomposable P_λ by some subfunctors C_j such that:

1. $C_t \subset C_{t-1} \subset \cdots \subset C_1 \subset C_0 = P_\lambda$
2. C_t is the largest subfunctor of P_λ s.t. $\phi(C_t, n) = \sum_{i=0}^{s} p_i^{C_t}(n) q^{in}$ with $s < n(\lambda)$.
3. C_j for $j < t$, s.t. C_j/C_{j+1} does not admit a subfunctor G s.t. $\phi((C_j/C_{j+1})/G, n) = \sum_{i=0}^{s} p_i^{G}(n) q^{in}$ with $s = n(\lambda)$.

Now the second point is fulfilled because P_λ is assumed to be noetherian. Especially, C_t is again finitely generated. Further C_j can be constructed in such a way that they comply with the third point. We only need finitely many of those again since P_λ is assumed to be noetherian. □

7 The extension quiver

When looking at the resolutions above, there is one naturally arising question. Since we can always find projective resolutions for simple functors, we would like to know how those simple functors stick together. A question that can be answered by looking at the extension or Gabriel quiver. This quiver was introduced by P. Gabriel in [Gab72] and [Gab].

7.1 Theoretical aspects

Definition 7.1.1. The vertices of the *extension quiver* of a $\mathbb{K}$-linear hom finite category $\mathcal{A}$ are given by the simple objects $S \in \mathrm{Obj}(\mathcal{A})$. For two simples S, S' we get d arrows $S \to S'$ iff $\dim \mathrm{Ext}^1(S, S') = d$.

Since the category $\mathcal{F}_q$ has infinitely many simple functors, we would like to know which simple functors can occur in the top of the radical or in the second term of such a resolution respectively. The following lemma helps us to compute extensions.

Lemma 7.1.2. *We have*

$$\mathrm{Ext}^1_{\mathcal{F}_q}(F_\mu, F_\lambda) = (\mathrm{rad}\, P_\mu, F_\lambda) \subset ((V, -), F_\lambda) = F_\lambda(V),$$

where $(V, -)$ *is the smallest standard projective that has* $\mathrm{rad}\, P_\mu$ *as a factor.*

Proof. The equality of the Ext and the Hom space follows from the long exact sequence of homology. $(\mathrm{rad}\, P_\mu, F_\lambda) \subset ((V, -), F_\lambda)$ is then due to the fact that every indecomposable projective has a simple top. □

As a corollary we get that this is of course 0 if $\dim V < n(\lambda)$. But since we have no chance to decide which standard projective will admit an epimorphism to $\mathrm{rad}\, P_\mu$ by just looking at F_μ, we cannot make any statement of the appearance or non appearance of arrows from here.

Remark 7.1.3. Since it is not possible to make any assertion about how far the the arrows can reach in the quiver, in terms of lengths of weights, we need to pick a suitable bound for our computations to come.
We have that $\phi(\mathrm{rad}\, P_\lambda, n)$ is of closed form with

$$\phi(\mathrm{rad}\, P_\lambda, n) = \sum_{i=0}^{n(\lambda)} a_i q^{in} - p^\lambda(n),$$

where we have $p^\lambda(n) = \phi(F_\lambda, n)$.
Now theorem 3.1.3 tells us that this closed form is determined by a recursion of length $n(\lambda) + \deg p^\lambda + 1$ and the same number of starting values. Therefore we will use the bound $n(\lambda) + \deg p^\lambda + 1$ in our computations as upper bound for the degree of simple functors in $\mathrm{top}(\mathrm{rad}\, P_\lambda)$, though this bound is not accurate.

In [Kuh94] we can even find an upper bound for $\deg(F_\lambda)$. Let $\mathcal{M}_n(q)$ the category of $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-modules. Section 3 of [Kuh94] yields an extension functor $c_n^\infty : \mathcal{M}_n(q) \to \mathcal{F}_q$ that satisfies the following:

Theorem 7.1.4 ([Kuh94], Theorem 4.8)**.**

1. $c_n^\infty(M)$ *is an extension of* M*: There are natural isomorphisms* $c_n^\infty(M)(\mathbb{F}_q^n) \cong M$ *of* $\mathbb{F}_q[M_n(\mathbb{F}_q)]$*-modules.*
2. $c_n^\infty(M)$ *is minimal with respect to 1: if* F *is any other functor satisfying* $F(\mathbb{F}_q^n) \cong M$ *as* $\mathbb{F}_q[M_n(\mathbb{F}_q)]$*-modules, then* $c_n^\infty(M)$ *is a quotient of* F*.*
3. c_n^∞ *preserves monos, epis and direct sums.*
4. *If* M *is simple* $\mathbb{F}_q[M_n(\mathbb{F}_q)]$*-module, then* $c_n^\infty(M)$ *is a simple functor.*
5. *If* M *is finitely generated, then* $c_n^\infty(M)$ *is of finite length.*

6. *$c_n^\infty(M)$ is always locally finite. In fact, $c_n^\infty(M)$ will be a polynomial functor of degree not more than $n^2(q-1)$.*

By applying 6 of the above theorem to remark 7.1.3 we get the following specialization.

Remark 7.1.5. Since $c^\infty_{n(\lambda)}(M_\lambda)(\mathbb{F}_q^n) = F_\lambda(\mathbb{F}_q^n) = 0$ if $n < n(\lambda)$ we have that $\dim F_\lambda(\mathbb{F}_q^n)$ is a polynomial of degree smaller or equal $n(\lambda)^2(q-1)$. We also know that $\operatorname{rad} P_\lambda$ is finitely generated. Therefore we get for the highest degree of a simple functor in the top of the radical of P_λ:

$$m = n(\lambda)^2(q-1) + n(\lambda) + 1$$

Also the work of Piriou and Schwartz [PS98] shows that there are no self extensions of simples and therefore no loops in the quiver.

Theorem 7.1.6 ([PS98])**.**

$$\mathrm{Ext}^1_{\mathcal{F}_q}(S,S) = 0 \,\forall\, S \in \mathcal{F}_q,\ S \textit{ simple}.$$

The work of Harris and Kuhn, [HK88], provides a classification of the simple functors in $\mathcal{F}_q$. It turns out they are induced by simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-modules. We call a simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-module singular if it is induced by a simple $\mathbb{F}_q[M_{n-1}(\mathbb{F}_q)]$-module, i.e. definition 2.0.3.

Theorem 7.1.7 ([HK88], Theorem 6.1)**.**
1. *$\{N \otimes (\det)^j \mid N$ singular and $0 \leq j \leq q-1\}$ is the set of simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-modules.*
2. *$\{\mathrm{Res}^{M_n}_{\mathrm{GL}_n}(N \otimes (\det)^j) \mid N$ singular and $1 \leq j \leq q-1\}$ is the set of simple $\mathbb{F}_q[\mathrm{GL}_n(\mathbb{F}_q)]$-modules.*

Here det is the determinant representation $\det : \mathbb{F}_q[M_n(\mathbb{F}_q)] \to \mathbb{F}_q$.

7.2 Computational aspects

How do we compute resolutions or just indecomposable projectives? Since we deal with finite fields usage of the computer comes to mind. The naive attempt is to use the regular representation of $\mathrm{End}_{\mathcal{F}_q}((\mathbb{F}_q^n, -)) = \mathbb{F}_q[M_n(\mathbb{F}_q)]$ and decompose it into indecomposables by multiplication with primitive idempotents.

Algorithm 7.2.1. Computing the regular representation of $M_n(\mathbb{F}_q)$

1. Choose a generating set for $M_n(\mathbb{F}_q)$.
2. Compute the actions of the generators on every element of $M_n(\mathbb{F}_q)$.

3. The resulting matrices give the generators for the regular representation.

Remark 7.2.2. It is a simple exercise in linear algebra to check that a generating set of $M_n(\mathbb{F}_q)$ consists exactly of 3 elements: a nilpotent matrix of rank $n-1$, an n-cycle and an unipotent matrix with only the entry at (1,2) not zero.

Remark 7.2.3. Unfortunately the dimension of the regular representation of $M_n(\mathbb{F}_q)$ is q^{n^2}, so even for small q this will become very big, even for considerably small n. For example the dimension of the regular representation of $M_4(\mathbb{F}_2)$ is already $2^{16} = 65536$.

Algorithm 7.2.4. Decompose $1 \in \mathbb{F}_q[M_n(\mathbb{F}_q)]$ into primitive idempotents.
Since we work in defining characteristic, we can no longer use closed formulas to determine the idempotents. We can however compute the central idempotents, that will divide $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ into the principal and the defect 0 part.

1. Calculate the central idempotents.
2. Pass from $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ to $\mathbb{F}_q[M_n(\mathbb{F}_q)]c_1$ and $\mathbb{F}_q[M_n(\mathbb{F}_q)]c_2$.
3. Take such an ideal and search for an idempotent in it. If we found one divide up into $\mathbb{F}_q[M_n(\mathbb{F}_q)]c_i e$ and $\mathbb{F}_q[M_n(\mathbb{F}_q)]c_i(c_i - e)$
4. Continue recursively until no more idempotents can be found, in which case they are primitive.
5. Calculate the action of such an idempotent on all the elements of $M_n(\mathbb{F}_q)$.
6. The result will be an idempotent in the endomorphism ring of the regular representation of $M_n(\mathbb{F}_q)$.
7. Multiply with the generators of the regular representation to obtain an $\mathbb{F}_q-$basis an indecomposable projective.

Remark 7.2.5. There are approximation algorithms for primitive idempotents that will work faster that this brute force attempt, but there are enough reasons not to follow that path. First of all while the calculations themselves will work faster, the data structure required is very hard to implement in an effective manner. A second reason, which is a real selling argument, is that calculating primitive idempotents will provide us with more information than actually needed and by loosing just this extra information, we can save a lot of time in the computations.

Luckily there is an easier way to determine the decomposition of $\mathbb{F}_q[M_n(\mathbb{F}_q)]$, or more precisely its regular representation, into indecomposable projectives, because that is what we want to look at. It is called the Meataxe [Rin].

Algorithm 7.2.6. The algorithms behind it are explained in [LMR94].

Using this program package it is very easy and relatively fast to decompose $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ into indecomposable projectives and determine the radical series of each of the summands.
The Meataxe makes it already way more efficient to compute indecomposable projectives but we still rely on the regular representation. Doing it like this is equivalent to evaluating P_λ at $\mathbb{F}_q^n$, with $n(\lambda) \leq n$.
Since we only have a finite amount of memory we cannot expect evaluation at all $n \in \mathbb{N}$, but it sure would be nice to look at $P_\lambda(\mathbb{F}_q^m)$ for an m that is not $n(\lambda)$, since simple functors of higher degree can occur in the top of the radical of P_λ.
The first idea would be to look at a decomposition of $\mathbb{F}_q[M_{n+k}(\mathbb{F}_q)]$, since $(\mathbb{F}_q^n, -)$ is a direct summand of $(\mathbb{F}_q^{n+k}, -)$. That surely does give $P_\lambda(\mathbb{F}_q^{n+k})$ but we have to decompose $\mathbb{F}_q[M_{n+k}(\mathbb{F}_q)]$ first. Let us just look at the dimension of this again: $\dim \mathbb{F}_q[M_n(\mathbb{F}_q)] = q^{n^2}$ and therefore $\dim \mathbb{F}_q[M_{n+k}(\mathbb{F}_q)] = q^{(n+k)^2} = q^{n^2} q^{2nk} q^{k^2}$. So this grows just incredibly fast! Soon the computer will refuse work in this case. So we will have to use a different approach. We can make use of the tensor product here.

Lemma 7.2.7. *We have*

$$(\mathbb{F}_q^n, -) = \bigotimes_{i=1}^{n} (\mathbb{F}_q, -) = \bigoplus_{n(\lambda)=1} \bigoplus_{i=1}^{n} P_\lambda^{\otimes \binom{n}{i}} \oplus \mathbb{F}_q.$$

Proof. Follows from $P_V \otimes P_W = P_{V \oplus W}$ and the decomposition of $P_{\mathbb{F}_q}$. □

From this lemma we can see that all that needs to be understood is already encoded in $(\mathbb{F}_q, -)$. But how do we get to a P_λ with $n(\lambda) > 1$ now? The key is to view $(\mathbb{F}_q, \mathbb{F}_q^n)$ as $\mathbb{F}_q$-$M_n(\mathbb{F}_q)$-bimodule.

Algorithm 7.2.8. Leave out the regular representation.

1. Decompose $1 \in \mathbb{F}_q[\mathbb{F}_q]$ into primitive idempotents.

2. For all these idempotents e_i perform right multiplication on $(\mathbb{F}_q, \mathbb{F}_q^n)$ and choose an $\mathbb{F}_q$-basis of the image.

3. Perform left multiplication on all $(\mathbb{F}_q, \mathbb{F}_q^n)e_i$ with the generators of $M_n(\mathbb{F}_q)$ and note this map in matrix form. These matrices will be generators of the indecomposable projectives $P_i(\mathbb{F}_q^n)$.

4. Use lemma 7.2.7 to generate projectives $P_\lambda(\mathbb{F}_q^n)$ with $n(\lambda) > 1$. Computationally this is done by using the Kronecker product of matrices.

5. Use the Meataxe package to decompose this tensor products into indecomposable projectives.

Remark 7.2.9. Decomposition of $1 \in \mathbb{F}_q[\mathbb{F}_q]$ by idempotents is fairly easy, since this is only q-dimensional. We already have done it by hand in subsection 3.

But as it turns out, even this is too much work to be done. It is sufficient to produce matrix generators for P_1. All the other P_i will turn up in the tensor products.

Lemma 7.2.10. $P_i \subset P_1^{\otimes i}$

Proof. We have to recall theorem 2.2.1 and apply it to $(\mathbb{F}_q, -)$.

$$P_1(V) = \{x \in (\mathbb{F}_q, V) | (\mathbb{F}_q, \lambda \cdot \mathrm{id})(x) = \lambda x \forall \lambda \in \mathbb{F}_q\}$$

$$P_i(V) = \{x \in (\mathbb{F}_q, V) | (\mathbb{F}_q, \lambda \cdot \mathrm{id})(x) = \lambda^i x \forall \lambda \in \mathbb{F}_q\}$$

So it is not hard to see that

$$P_1^{\otimes i} \supset \{x \in (\mathbb{F}_q, V) | (\mathbb{F}_q, \lambda \cdot \mathrm{id})(x) = \lambda^i x \forall \lambda \in \mathbb{F}_q\}.$$

We just use the multi-linearity of the tensor product. Of course this has to stop after q steps, since $\lambda^q = \lambda$ in $\mathbb{F}_q$. □

Algorithm 7.2.11. Simplification of Algorithm 7.2.8

1. Decompose $1 \in \mathbb{F}_q[\mathbb{F}_q]$ into primitive idempotents.

2. For e_1 perform right multiplication on $(\mathbb{F}_q, \mathbb{F}_q^n)$ and choose an $\mathbb{F}_q-$basis of the image.

3. Perform left multiplication on $(\mathbb{F}_q, \mathbb{F}_q^n)e_1$ with the generators of $M_n(\mathbb{F}_q)$ and note this map in matrix form. These matrices will be generators of the indecomposable projectives P_1.

4. Use lemma 7.2.7 to generate projectives P_λ with $n(\lambda) \geq 1$.

5. Use the Meataxe package to decompose this tensor products into indecomposable projectives.

With this we can now start to compute resolutions of simple functors respectively of the radical of indecomposable projectives. We would like to have a knitting algorithm. If possible one that is as simple as the one for the Auslander-Reiten quiver of a finite dimensional algebra. But it is not quite that easy.

Algorithm 7.2.12. Knitting algorithm for the extension quiver

1. Start with $n(\lambda) = 1$.
2. Generate the indecomposable projectives of $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ for $n = n(\lambda)^2(q-1) + n(\lambda) + 1$, for current $n(\lambda)$.
   (a) Look at top(rad P_λ) for one λ with given $n(\lambda)$.
   (b) Find this module(s) in the tops of the other projectives.
   (c) Draw arrows for all of them.
   (d) Repeat for every projective P_λ with current $n(\lambda)$.
3. Increase $n(\lambda)$.

7.3 Properties of the quiver

A nice lemma due to [FFSS99] helps us to understand what the extension quiver will look like:

Lemma 7.3.1 ([FFSS99], Lemma 1.12)**.** *Assume that $F, G \in \mathcal{F}_q$ take only finite dimensional values, then the duality homomorphism*

$$D : \operatorname{Ext}^1(F, G) \to \operatorname{Ext}^1(DG, DF)$$

is an isomorphism.

Corollary 7.3.2.

$$\dim \operatorname{Ext}^1(F, G) = \dim \operatorname{Ext}^1(G, F)$$

if G, F are simple functors in $\mathcal{F}_q$.

Proof. By theorem 2.1.9 we know that simple functors take only finite dimensional values, so we can apply the above lemma here. [Kuh94] now tells us that simple functors are self dual, which yields the result. □

Corollary 7.3.3. *In the extension quiver for each arrow a there is also its reversed $\overline{a}$.*

Lemma 7.3.1 suggests a simplification of the knitting algorithm 7.2.12. The following lemma shows how.

Lemma 7.3.4. *Let $n(\mu) \leq n(\lambda)$ then the following holds for F_μ:*

$$F_\mu \in \operatorname{top}(\operatorname{rad}(P_\lambda)) \text{ then } F_\mu(\mathbb{F}_q^{n(\lambda)}) \in \operatorname{top}(\operatorname{rad}(P_\lambda(\mathbb{F}_q^{n(\lambda)})))$$

Proof. Let F_μ be in top(rad(P_λ)). Since $F_\mu(\mathbb{F}_q^{n(\lambda)}) \neq 0$, it must also appear in top(rad($P_\lambda(\mathbb{F}_q^{n(\lambda)})$)). □

Remark 7.3.5. Keep the following counter example in mind to see that the converse is not true. We have $\mathrm{Ext}^1_{\mathcal{F}_q}(S, S) = 0$ for S simple, since it can occur that $\mathrm{Ext}^1_{\mathcal{M}_{n(\lambda)}}(F_\lambda(\mathbb{F}_q^{n(\lambda)}), F_\lambda(\mathbb{F}_q^{n(\lambda)})) \neq 0$. In this case we just omit the arrow in the Ext-quiver. Such an example is $q = 2$ and Λ^2. For $n = 2$ we have $\mathrm{rad}\, P(\Lambda^2) = \Lambda^2$.

Now we can simplify the knitting algorithm 7.2.12. We use two types of arrows, dotted and bold, dotted arrows are used to indicate possible arrows in the extension quiver. Bold arrows are confirmed arrows.

Algorithm 7.3.6. Knitting algorithm for the extension quiver

1. Start with $n(\lambda) = 1$.
2. Generate the indecomposable projectives of $\mathbb{F}_q[M_n(\mathbb{F}_q)]$ for $n = n(\lambda)$.
   (a) Look at $\mathrm{top}(\mathrm{rad}\, P_\lambda)(\mathbb{F}_q^n)$ for one λ, $n(\lambda) = 1$.
   (b) Find this module(s) in the tops of the other projectives P_μ for $n(\mu) \leq n(\lambda)$.
   (c) Draw dotted arrows for all of them, except $\mu = \lambda$.
   (d) If you draw an arrow from F_λ to F_μ, then check $\mathrm{top}(\mathrm{rad}\, P_\mu)(\mathbb{F}_q^n)$ again to see if this arrow can be confirmed.
   (e) Repeat for every projective $P_\lambda(\mathbb{F}_q^n)$ with the same length.
   (f) Draw bold arrows for $F_\mu \leftrightarrow F_\lambda$ if $n(\lambda) \geq n(\mu)^2(q-1) + n(\mu) + 1$.
3. Increase $n(\lambda)$.

We now turn to $q = 2$ again.

Lemma 7.3.7 ([Fra96])**.** *Let $q = 2$ and $k, l > 0$ then*

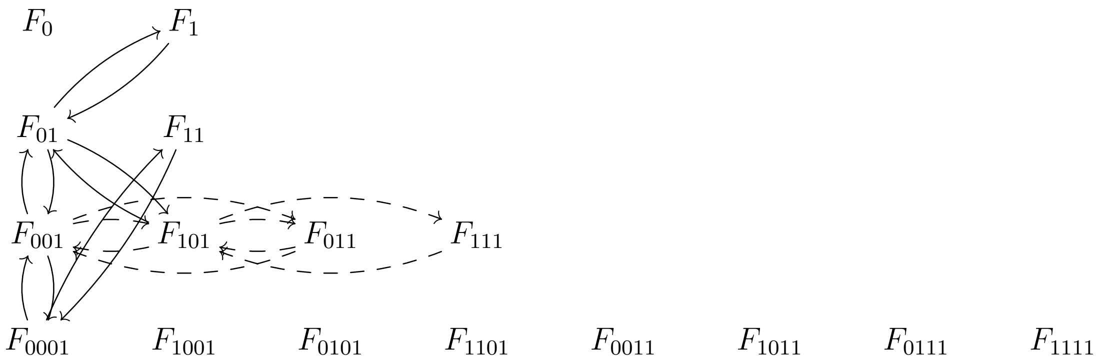
$$\mathrm{Ext}^1_{\mathcal{F}_q}(\Lambda^k, \Lambda^l) \neq 0 \text{ iff } |k - l| = 1.$$

In this case $\mathrm{Ext}^1_{\mathcal{F}_q}(\Lambda^k, \Lambda^{k\pm 1}) \cong \mathbb{F}_2$.

Corollary 7.3.8. *Except for the projective injective simple, the extension quiver of $\mathcal{F}_2$ is connected.*

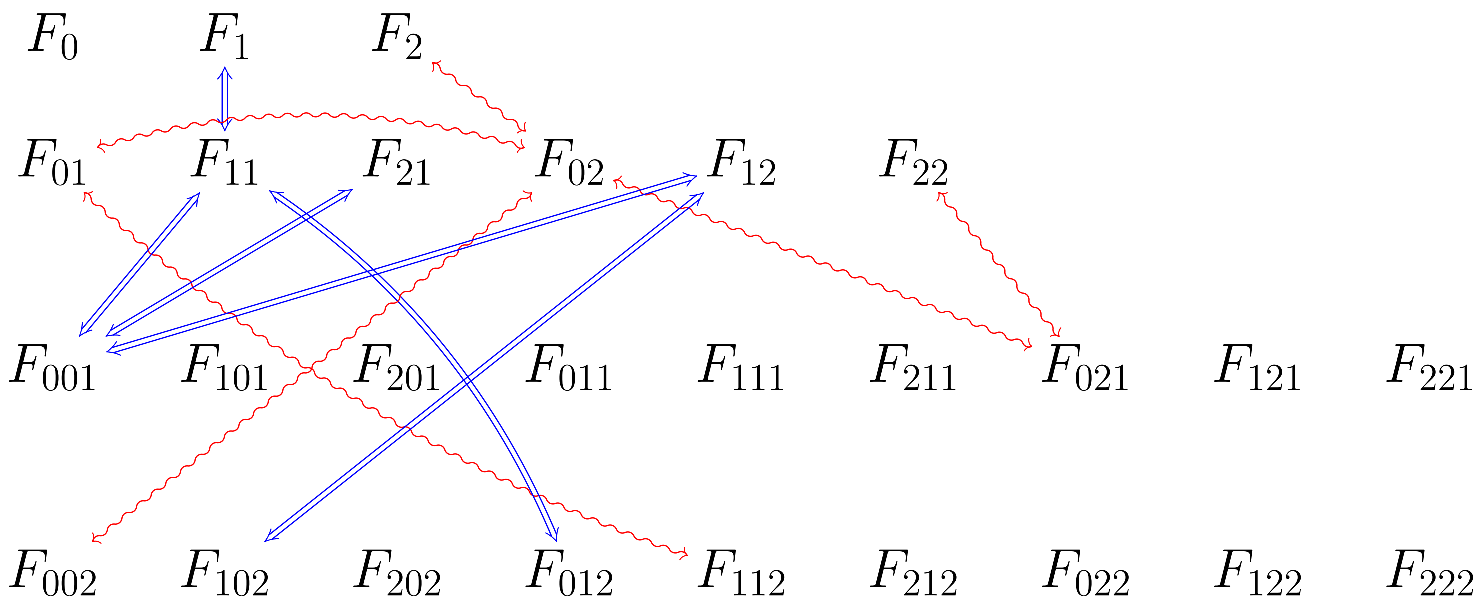
Proof. Let $T^n \in \mathcal{F}_q$ be the n-fold tensor functor for $n > 0$. Then we have $\mathrm{top}(T^n) = \Lambda^n$. Since also every finite and therefore every simple functor occurs as composition functor of some T^n as worked out by N. Kuhn in [Kuh94]. Lemma 7.3.7 then yields the result. □

Example 7.3.9. In the case $p = q = 2$ the following is known:

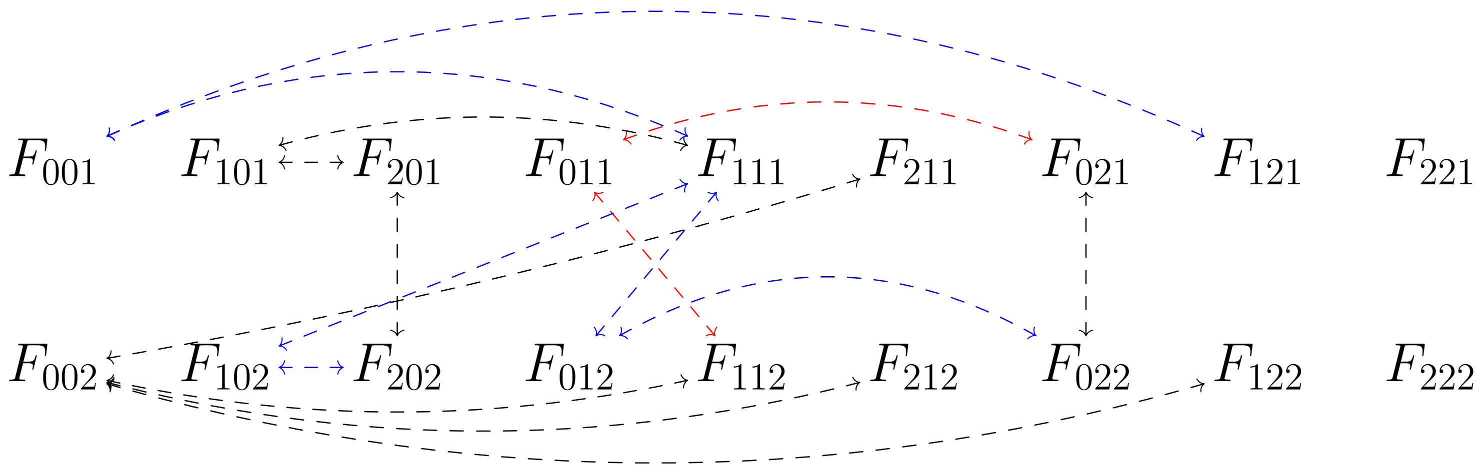


Example 7.3.10. This also provides a good example of the above lemmata about the bounds of the weights of simples at the top of the radical. As of example 2.2.7 we know that there is one more q^n term in the closed form of the dimension of P_{11} than in P_{01}. Therefore we can have an arrow at least one step further in the first case.
We had 2 as the sum of all polynomial degrees plus one in for P_{01} and 3 in P_{11} and both closed forms have a constant, i.e. 2^0 term. Their tops are of polynomial degree 2 and 3, so the upper border for the degree of functors in the top of the radical is 6 in case of P_{01} and 10 in case of P_{11}. And indeed, the top of the radical contains higher degree functors for P_{11}.

Example 7.3.11. In the case $p = q = 3$ we will simplify a pair of reverse arrows $\bullet \rightleftarrows \bullet$ by $\bullet \longleftrightarrow \bullet$, since already for the few cases, where we know about the existence of arrows, we would loose the overview otherwise. We also use two diagrams, one for the confirmed extensions and one for the conjectured ones. We further decorate all the extensions of F_1 as $\bullet \Longleftrightarrow \bullet$ and those of F_2 as $\bullet \leftrightsquigarrow \bullet$.



This diagram now holds all the extensions that are yet to be confirmed. We draw them as $\bullet \leftarrow - \rightarrow \bullet$.



It seems as if this quiver would have 2 connected components, except for the isolated projective injective module everything would be connected. As a consequence of theorem 2.2.1 this is however not true.

Lemma 7.3.12. *There can be no extensions between F_i and F_j for all i, j. Therefore the extension quiver of $\mathcal{F}_q$ has at least q connected components.*

Proof. If $\mathrm{Ext}^n_{\mathcal{F}_q}(F_i, F_j) \neq 0$ for some n then there would have to exist a sequence

$$0 \to F_j \to E_1 \to \cdots \to E_n \to F_i \to 0$$

with all the E_k indecomposable. But for all functors E we have

$$E(V) = E_0(V) \oplus E_1(V) \oplus \cdots \oplus E_{q-1}(V),$$

a direct decomposition of functors. By definition of F_i and F_j, they have to belong to E_i and E_j respectively. But this contradicts the assumption that all the E_k are indecomposable. Therefore F_i and F_j must lie in different connected components. □

Conjecture 7.3.13. *The extension quiver of $\mathcal{F}_q$ has q connected components.*

But why is the quiver in the example connected? Is there maybe something wrong with the algorithm 7.3.6? The situation is a little like the one in remark 7.3.5:

8 An algorithm to compute generators of kernels of representable morphisms and to test them

In this section we will state the source code for an algorithm to explicitly compute kernels of representable morphisms. The language used is the computer algebra system GAP. There is a GUI available for MacOS which is written in JAVA; a jar file for other operating systems is also included. The front end program produces a file named input.g which can also be created by hand. A documentation on how to use it is included separately.
The program has been built to be very modular so that modifications to one part may be made while maintaing compatibility with the other parts. The programs used can be found under the following adress:
http://www.math.uni-bielefeld.de/~plinke/gap/kernelcalc.zip

8.1 input.g

The first script produced by the user interface or by hand. The user can input, modify or review the representable morphism and the standard projectives they wish to investigate.

```
q:=2;
p:=2;
r:=1;
s:=1;
m:=1;
n:=5;
coef:=[1];
f:=[1];
Read("sortf.g");
```

Here p is a prime number and $q = p^r$ the cardinality of the field. We then want to investigate the kernel of $(f, -) : (\mathbb{F}_q^s, -) \to (\mathbb{F}_q^m, -)$. f is given by a list of matrices encoded as integers and a list of coefficients which are also encoded by integers. The functions doing the work will be discussed in connection with the next script.

8.2 sortf.g

The second script calculates sorts the input and removes redundancies. It further displays the input data in a sensible fashion.

```
#Function that generates an a x b integer matrix over a
#prime field from the q-adic decomposition of an integer c.
#Therefore the integer c defines that matrix uniqueliy.
qmat :=function(a,b,c)
# a is row count, b column and c the the number of the
#matrix we want to look at
        local m,mm;
        m:=NullMat(a,b,GF(q));
        #generate a Null matrix
        mm:=CoefficientsQadic( c, q);
        #decompose c q-adicly
        for i in [0..a*b-1] do
        #this loop fills in the elements.
                if Length(mm) = i then break; fi;
                #error break condition
                m[QuoInt(i,b)+1][(i mod b)+1]:=
```

```
                    Elements(GF(q))[mm[i+1]+1];
    od;
    return m;
end;;
```

Example 8.2.1. Let $a = b = 2$ and $c = 9$ then the 2-adic decomposition of c is the vector $[1, 0, 0, 1]$. Therefore we have

$$\text{qmat}(2, 2, 9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

```
##Script that sorts a vector of homomorphisms and adds
#up coefficients
SortParallel(f,coef); #sort both lists

#take out duplicates from the matrices
if IsDuplicateFree(f)=false then
  k:=Length(f);
  while k>1 do
    if f[k]=f[k-1] then
      Remove(f,k);
      coef[k-1]:=coef[k]+coef[k-1];
      Remove(coef,k);
    fi;
    k:=k-1;
  od;
fi;

SortParallel(coef,f); #sort for coef

#remove zero coefficients and matrices that belong to them
if coef[1]=0*Z(q) then
  repeat
    Remove(f,1);
    Remove(coef,1);
  until coef[1]<>0*Z(q);
fi;

SortParallel(f,coef); #sort back for f
```

```
flen:=Length(f); #take length of f

#Output section

Print("testing now in (F_",q,"^",s,",F_",q,"^",n,")-->
(F_",q,"^",m,",F_",q,"^",n,")","\n");
Print("With the Homomorphism","\n");
#Display(f);
#Display(coef);

Print(coef[1],"*",qmat(s,m,f[1]-1));
for i in [2..Length(f)] do
  Print("+",coef[i],"*",qmat(s,m,f[i]-1));
od;
Print("\n");

Read("fkernel.g"); #continue in program
```

The function intmat reverses the action of pmat, such that we will dispense with an example. We will briefly look into the function matp however.

Example 8.2.2. Let $q = 2$ and $f = [2, 1, 2]$ with $coef = [1, 1, 1]$. Then this input will be sorted to $f = [1]$ and $coef = [1]$.

8.3 fkernel.g

```
#Function that generates an a x b integer matrix over a
#prime field from the q-adic decomposition of an integer c.
#Therefore the integer c defines that matrix uniqueliy.
qmat :=function(a,b,c)
# a is row count, b column and c the the number of the
#matrix we want to look at
        local m,mm;
        m:=NullMat(a,b,GF(q));
        #generate a Null matrix
        mm:=CoefficientsQadic( c, q);
        #decompose c q-adicly
        for i in [0..a*b-1] do
        #this loop fills in the elements.
                if Length(mm) = i then break; fi;
                #error break condition
```

```
                m[QuoInt(i,b)+1][(i mod b)+1]:=
                        Elements(GF(q))[mm[i+1]+1];
        od;
        return m;
end;;


#Function that convertes an a x b Matrix c over a
#finite field back into an integer
intmat := function(c)
        local a,b,m;
        a:=DimensionsMat(c)[1];
        b:=DimensionsMat(c)[2];
        m:=0;
        for i in [0..a*b-1] do
                #analogue to qmat
                m:=m+(Position(Elements(GF(q)),
                        c[QuoInt(i,b)+1][(i mod b)+1])-1)*q^i;
        od;
        return m;
end;;
```

The function intmat reverses the function qmat and converts a matrix over $\mathbb{F}_q$ back into an integer.

```
#function that computes the kernel of a given
#homomorphism between two hom-spaces
#main function of the programm, it needs the
#Dimensions of the start and terminal space, the
#homomorphism togther with its length, i.e. the
#count of non zero entries if applicable, and
#the Matrix that contains all the results of the
#multiplications between the entries in homf and
#the basisvectors in (F_q^dims,F_q^dimn)
kernelcalc := function(dims,dimn,dimm,homf,RR,len,c)
#output matrix for the result of the multiplications,
#or matrix realisation of the homomorphism
#we now compute the result of the multiplication of
#the basis vectors with our given homomorphism from
#the matrix RR
  listout:=NullMat(q^(dims*dimn),q^(dimm*dimn),GF(q));
  for j in [1..len] do
```

```
    if homf[j]<>0 then
      for k in [1..q^(dims*dimn)] do
        listout[k][RR[k][j]+1]:=
        listout[k][RR[k][j]+1]+c[j];
      od;
    fi;
  od;
  #Display(listout);
#result is now the matrix of the homomorphism (f,F_q^n),
#now we need to find its right-kernel, before we do that,
#set the matrix to the finte field that we a currently
#working in
  kernel:=Length(TriangulizedNullspaceMat(listout));
  return kernel;
end;;

## End of Functions
```

The function kernelcalc is the heart of this script. It requires a lot of data, most of which we have already seen. What is new is the matrix of matrices RR. It contains the result of each multiplication of each basis element of $(\mathbb{F}_q^s, \mathbb{F}_q^n)$ as an element of the integers. We will get to this matrix in a bit and also move the example on kernelcalc to the end of this subsection.

```
##This script requires dimensions s,m,n, and a
#homomorphism f with coefficients coef as input. This input
#is generated by the script input.g but can also manually
#be inserted for examples. The script returns the dimension
#of the kernel of f.

#Generates a list of matrices that will contain all the
#results of all multiplications between basis homomorphisms
#and entries in the components of [f]
R:=NullMat(q^(n*s),flen);
for j in [1..flen] do
  fm:=qmat(s,m,f[j]-1);
  for k in [1..q^(n*s)] do
    R[k][j]:=intmat(qmat(n,s,k-1)*fm);
  od;
od;

#Display( R);
```

```
#Print("Multiplications successfully generated! \n");

fkernel:=kernelcalc(s,n,m,f,R,flen ,coef);

#Print("The kernel is generated by: ");
#Display(fkernel);
Print("Dimension of the kernel is: ");
#kk:=Length(fkernel);
Display(fkernel);
#first important step , now we know what we are dealing with
#in terms of dimension. The next step should be to see if
#we got a decomposition of the kernel in direct summands.

#uncomment if used as part of the whole program. In that
#case the path has to be changed manually , relatively to
#the gap bin directory
Read("weakcoker.g");
```

Example 8.3.3. Let $q = 2$, $m = s = 1$, $n = 2$ and $f = [0] + [1]$. Then The matrix R (or RR in the function kernelcalc) is a 4×2 matrix of results of multiplications. These eight multiplications are:

$$\begin{bmatrix}0\\0\end{bmatrix} \circ [0] = \begin{bmatrix}0\\0\end{bmatrix};\begin{bmatrix}1\\0\end{bmatrix} \circ [0] = \begin{bmatrix}0\\0\end{bmatrix}; \qquad \begin{bmatrix}0\\1\end{bmatrix} \circ [0] = \begin{bmatrix}0\\0\end{bmatrix};\begin{bmatrix}1\\1\end{bmatrix} \circ [0] = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}0\\0\end{bmatrix} \circ [1] = \begin{bmatrix}0\\0\end{bmatrix};\begin{bmatrix}1\\0\end{bmatrix} \circ [1] = \begin{bmatrix}1\\0\end{bmatrix}; \qquad \begin{bmatrix}0\\1\end{bmatrix} \circ [1] = \begin{bmatrix}0\\1\end{bmatrix};\begin{bmatrix}1\\1\end{bmatrix} \circ [1] = \begin{bmatrix}1\\1\end{bmatrix}$$

The resulting R is (we recall that we have always shifted the numbers corresponding to the matrices up by one):

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

With this data we can now call the function kernelcalc to obtain the matrix listout which will give us the kernel of $(f, \mathbb{F}_q^n)$. For the matrix listout we just count how many times we will get which element of $(\mathbb{F}_q^m, \mathbb{F}_q^n)$ when we multiply a basis element of $(\mathbb{F}_q^s, \mathbb{F}_q^n)$ with f. With the present data listout will be:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Modulo $q = 2$ it is immediately clear that the (right-)kernel will be generated by the element $[1, 0, 0, 0]$ which corresponds to the element $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $(\mathbb{F}_q^s, \mathbb{F}_q^n)$.

8.4 weakcoker.g

The fourth step is to prepare the search for the a candidate for a weak cokernel of the map f respectively a kernel of the map $(f, -)$.

```
#Function that generates an a x b integer matrix over a
#prime field from the q-adic decomposition of an integer c.
#Therefore the integer c defines that matrix uniqueliy.
qmat :=function(a,b,c)
# a is row count, b column and c the the number of the
#matrix we want to look at
        local m,mm;
        m:=NullMat(a,b,GF(q));
        #generate a Null matrix
        mm:=CoefficientsQadic( c, q);
        #decompose c q-adicly
        for i in [0..a*b-1] do
        #this loop fills in the elements.
                if Length(mm) = i then break; fi;
                #error break condition
                m[QuoInt(i,b)+1][(i mod b)+1]:=
                        Elements(GF(q))[mm[i+1]+1];
        od;
        return m;
end;;


#Function that convertes an a x b Matrix c over a finite
#field back into an integer
intmat := function(c)
        local a,b,m;
        a:=DimensionsMat(c)[1];
        b:=DimensionsMat(c)[2];
        m:=0;
        for i in [0..a*b-1] do
                #analogue to qmat
                m:=m+(Position(Elements(GF(q)),
                        c[QuoInt(i,b)+1][(i mod b)+1])-1)*q^i;
```

```
        od;
        return m;
end;;

#q-adic incrementations function for list of numbers
qinc := function(a)
local i;
        for i in [1..Length(a)] do
                if a[i]=(q-1) mod q then a[i]:=0;
                else a[i]:=a[i]+1; break; fi;
        od;
        return a;
end;;
```

The only new function is qinc. This function helps us to run through all linear combinations of elements in a given vector space over $\mathbb{F}_q$.

Example 8.4.4. Let $q = 5$

$$[4, 0, 1, 0] \mapsto [0, 1, 1, 0]$$

$$[0, 1, 1, 0] \mapsto [1, 1, 1, 0]$$

```
#function that computes the kernel of a given homomorphism
#between two hom-spaces
#main function of the programm, it needs the Dimensions of
#the start and terminal space, the homomorphism togther with
#its length, i.e. the count of non zero entries if applicable,
#and the Matrix that contains all the results of the
#multiplications between the entries in homf and the
#basisvectors in (F_q^dims,F_q^dimn)
kernelcalc := function(dims,dimn,dimm,homf,RR,len,c)
#output matrix for the result of the multiplications,
#or matrix realisation of the homomorphism
#we now compute the result of the multiplication of
#the basis vectors with our given homomorphism from
#the matrix RR
  listout:=NullMat(q^(dims*dimn),q^(dimm*dimn),GF(q));
  for j in [1..len] do
    if homf[j]<>0 then
      for k in [1..q^(dims*dimn)] do
        listout[k][RR[k][j]+1]:=
```

```
        listout[k][RR[k][j]+1]+c[j];
      od;
    fi;
  od;
#result is now the matrix of the homomorphism (f,F_q^n),
#now we need to find its right-kernel, before we do that,
#set the matrix to the finte field that we a currently
#working in
  kernel:=Length(TriangulizedNullspaceMat(listout));
  return kernel;
end;;

##End of Functions, starting main program
#As input we need again dimension vectors s,m, dimension n,
#homomorphism f through f,flen and the kernel given by
#fkernel
# Now we try to find a suiting g:F^h_q-->F^s_q s.t. g*f=0,
#first try smallest h s.t. q^(n*h)>=dim Ker (f,F^n_q), gk is
#the dimension of the kernel of g
Print("Start searching for a weak cokernel.\n");
h:=0;
gk:=0;

while q^(n*h)<fkernel do
```

After making sure that the chosen space $(\mathbb{F}_q^h, \mathbb{F}_q^n)$ is big enough to cover the calculated $\ker(f, \mathbb{F}_q^n)$ in terms of dimension. We calculate the kernel of $(f, -)$ on $(\mathbb{F}_q^s, \mathbb{F}_q^h)$ which gives all the maps g such that $g \circ f = 0$.

Example 8.4.5. Let $q = 2$ and $f = [0] + [1]$, then the space of all g such that $g \circ f = 0$ is generated by $g = [0]$.

Now we take all possible linear combinations of generators of this space and check if we can obtain $\mathrm{Im}(g, \mathbb{F}_q^n) = \ker(f, \mathbb{F}_q^n)$ this is done in the next script. If we are successful, the result is printed out.

```
od;

#use repeat here instead of for for better possible
#manipulation of the count variable bc
repeat
#Generates the matrix that will contain all the results of
#all multiplications between the elements of f and basis
```

```
#homomorphisms that can occur as elements of g
  R:=NullMat(q^(h*s),q^(h*m),GF(q));
  for  k in [1..q^(h*s)] do
    gm:=qmat(h,s,k-1);
    for j in [1..flen] do
      pp:=intmat(gm*qmat(s,m,f[j]-1));
      R[k][pp+1]:=R[k][pp+1]+1*Z(q)^0;
    od;
  od;
  nullgg:=TriangulizedNullspaceMat(R  );
#nullgg now contains a basis of the left annulator of f
  if Length(nullgg)=0 then
    Display("no more morphisms left!");
    break;
  fi;
  Print("Space of [g]*[f]=0 generated.\n");

#Generates the matrix that will contain all the results of
#all multiplications between basis homomorphisms and elements
#of all g, this is only needed if we got a new h since the
#last time we passed this point
  R1:=NullMat(q^(h*n),q^(h*s));
    for  j in [1..q^(h*s)] do
      gm:=qmat(h,s,j-1);
      for k in [1..q^(n*h)] do
        R1[k][j]:=intmat(qmat(n,h,k-1)*gm);
      od;
    od;
  Print("Multiplications successfully generated! \n");

  g:=ListWithIdenticalEntries(q^(h*s),0*Z(p  ));

#Chance to save the workspace to use parallel computing
#SaveWorkspace("P2coker");
#We read the script that does the actual search. This allows
#us to save the workspace and load the search script on
#another computer as well for faster search.
  Read("searchcoker.g");
#end of the repeat
until fkernel=q^(h*n)-gk;
```

```
#output of the result
Print("A weak cokernel is given by ");
Print("(F_",q,"^",h,",-)");
Print("with the homomorphisms g=","\n");
gg:=[];
gc:=[];
for ii in [1..Length(g)] do
  if g[ii]<>0*Z(q) then
    Print(g[ii],"*",qmat(s,m,ii-1),"+");
        Add(gg,ii);
        Add(gc,g[ii]);
  fi;
od;
Print("\n");
Print("Output for test-algorithm","\n");
Print("g:=",gg,";","\n");
Print("coefg:=",gc,";","\n");
```

8.5 searchcoker.g

The final step is to calculate if the representable map $(g,-)$ such that $(f,\mathbb{F}_q^n)\circ(g,\mathbb{F}_q^n)$ is exact. This script admits to be used in parallel computing for faster speed.

```
##Warning this Script needs a loaded Workspace created by
#weakcoker.g
gs:=ListWithIdenticalEntries(Length(nullgg),0);
close:=false;

#Manipulation chance for g if we were to use this script
#on a different computer then the one we did the original
#computation on
#gs[Length(gs)]:=1; #Example for manipulation
  repeat
    gs:=qinc(gs);
    g:=g*0;
    for jk in [1..Length(gs)] do
      if gs[jk]<>0 then
        g:=g+Z(q)^(gs[jk]-1)*nullgg[jk];
      fi;
```

```
    od;

    gg:=[];
    for jk in [1..Length(g)] do
      if g[jk]<>0*Z(q) then
        Add(gg,1);
      else
        Add(gg,0);
      fi;
    od;

#compute the kernel of g and determine its dimension
    gk:=kernelcalc(h,n,s,g,R1,q^(s*h),g);
    #Display(q^(h*n)-gk);

#test if we got the desired dimension of the image
    if (q^(h*n)-gk=fkernel) then
      #Display(gk);
      close:=true;
      break;
    fi;
    Print(gs,"\r");

#test where we are currently standing; because of the
#break condition before gg=Length(gs) can only occur
#if we don't find anyting
    until gs=ListWithIdenticalEntries(Length(gs),q-1);
#fail condition
if (gs=ListWithIdenticalEntries(Length(gs),q-1))and
        (close=false) then
  Print("No Sucess! ",h,"\n");
  Print("Trying new h","\n");
  h:=h+1;
fi;

#The important Output
#g;
#h;
```

We just loop over all the possible linear combinations for g and calculate the dimension of their image using the function kernelcalc. If we find our desired result, it is printed out.

Example 8.5.6. Let $q = 2$, $m = n = 3$, $n = 3$. As homomorphism we choose

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The function kernelcalc yields a dimension of 148 for the kernel. An example of a g such that $g \circ f = 0$ is

$$g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

But they yield the wrong dimensions of the image. The algorithm computes that the kernel is generated by the map

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} +$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8.6 kertest.g

The program kertest.g computes whether a candidate that we have found in the previous program can actually pass as the kernel as it varies the dimension n of the test-space $\mathbb{F}_q^n$. Most of the functions used already appear in the previous scripts.

```
#Function that generates an a x b integer matrix over a
#prime field from the q-adic decomposition of an integer c.
#Therefore the integer c defines that matrix uniqueliy.
qmat :=function(a,b,c)
# a is row count, b column and c the the number of the
#matrix we want to look at
        local m,mm,i;
        m:=NullMat(a,b,GF(q));
```

```
        #generate a Null matrix
        mm:=CoefficientsQadic( c, q);
        #decompose c q-adicly
        for i in [0..a*b-1] do
        #this loop fills in the elements.
                if Length(mm) = i then break; fi;
                #error break condition
                m[QuoInt(i,b)+1][(i mod b)+1]:=
                        Elements(GF(q))[mm[i+1]+1];
        od;
        return m;
end;;

#Function that convertes an a x b Matrix c over a
#finite field back into an integer
intmat := function(c)
        local a,b,m,i;
        a:=DimensionsMat(c)[1];
        b:=DimensionsMat(c)[2];
        m:=0;
        for i in [0..a*b-1] do
                #analogue to qmat
                m:=m+(Position(Elements(GF(q)),
                        c[QuoInt(i,b)+1][(i mod b)+1])-1)*q^i;
        od;
        return m;
end;;

#Function that checkes if two homomorphisms in the
#K-linearised category compose tirvially
homprod := function(gg,cog,ff,cof,dh,ds,dm)
  local pos,re,jk,ik;
  re:=ListWithIdenticalEntries(q^(dh*dm),Zero(GF(q)));
  for jk in [1..Length(gg)] do
    for ik in [1..Length(ff)] do
      pos:=intmat(qmat(dh,ds,gg[jk]-1)*
        qmat(ds,dm,ff[ik]-1))+1;
      re[pos]:=re[pos]:=re[pos]+cog[jk]*cof[ik];
    od;
  od;
```

```
  if re=ListWithIdenticalEntries(q^(dh*dm),Zero(GF(q))) then
    return true;
  else
    return false;
  fi;
end;;

#function that computes the kernel of a given
#homomorphism between two hom-spaces
#main function of the programm, it needs the
#Dimensions of the start and terminal space, the
#homomorphism togther with its length, i.e. the
#count of non zero entries if applicable, and
#the Matrix that contains all the results of the
#multiplications between the entries in homf and
#the basisvectors in (F_q^dims,F_q^dimn)
kernelcalc := function(dims,dimn,dimm,homf,RR,len,c)
#output matrix for the result of the multiplications,
#or matrix realisation of the homomorphism
#we now compute the result of the multiplication of
#the basis vectors with our given homomorphism from
#the matrix RR
  listout:=NullMat(q^(dims*dimn),q^(dimm*dimn),GF(q));
```

The functions homprod and kfprod are used to calculate the product $g \circ f$ in the category $\mathbb{F}_q[\mathrm{mod}-\mathbb{F}_q]$ and check if it is trivial. If $g \circ f \neq 0$ we do not need to do further computations.

```
      for k in [1..q^(dims*dimn)] do
        listout[k][RR[k][j]+1]:=
        listout[k][RR[k][j]+1]+c[j];
      od;
    fi;
  od;
  #Display(listout);
#result is now the matrix of the homomorphism (f,F_q^n),
#now we need to find its right-kernel, before we do that,
#set the matrix to the finte field that we a currently
#working in
  kernel:=Length(TriangulizedNullspaceMat(listout));
  return kernel;
end;;
```

```
## End of Functions, starting main program

## Input
q:=2;
Print("Using the prime power q=" , q ,"\n");
#and so far no support for in time read
Print("Set now h,s and m to check if a given g gives
        a weak cokernel in (F_q^(h_1,...,h_l),F_q^n) -->
        (F_q^(s_1,...,s_r),F_q^n) -->
        (F_q^(m_1,     ,m_s),F_q^n)" ,"\n");
Print("reading h ");
h:=3;
Print("reading s ");
s:=3;
Print("reading m ");
m:=3;
Print("testing now in (F_",q,"^",h,",F_",q,"^n) -->
        (F_",q,"^",s,",F_",q,"^n) -->
        (F_",q,"^",m,",F_",q,"^n)","\n");
g:=[257,273,274,278,305,308,312];
coefg:=[Z(q)^0,Z(q)^0,Z(q)^0,Z(q)^0,Z(q)^0,Z(q)^0,Z(q)^0];
f:=[274,308];
coeff:=[Z(q)^0,Z(q)^0];
Print("With the homomorphisms g=","\n");
Print(coefg[1],"*",qmat(s,m,g[1]-1));
for i in [2..Length(g)] do
  Print("+",coefg[i],"*",qmat(s,m,g[i]-1));
od;
Print("\n");
Print("and f=","\n");
Print(coeff[1],"*",qmat(s,m,f[1]-1));
for i in [2..Length(f)] do
  Print("+",coeff[i],"*",qmat(s,m,f[i]-1));
od;
Print("\n");

teststart:=1;
testdepth:=4;

if homprod(g,coefg,f,coeff,h,s,m)=true then
```

```
  Print("Success!","\n");
  flen:=Length(f);
  glen:=Length(g);

for n in [teststart..testdepth] do
  Rf:=NullMat(q^(n*s),flen);
    for jk in [1..flen] do
      for k in [1..q^(n*s)] do
        Rf[k][jk]:=intmat(qmat(n,s,k-1)*qmat(s,m,f[jk]-1));
      od;
    od;
#Multiplications for f generated
   Rg:=NullMat(q^(n*h),glen);
    for jk in [1..glen] do
      for k in [1..q^(n*h)] do
        Rg[k][jk]:=intmat(qmat(n,h,k-1)*qmat(h,s,g[jk]-1));
      od;
    od;
#Multiplications for g generated

  kerf:=kernelcalc(s,n,m,f,Rf,flen,coeff);
  img:=q^(n*s)-kernelcalc(h,n,s,g,Rg,glen,coefg);

  #Display(CoefficientsQadic( kerf, q^n));

  Print("Dimension of the Kernel: ",kerf,"
        Dimension of the Image: ",img,"\n");
  succ:=true;
  if kerf<>img then
    Print("This composition is not exact at ",n,"\n");
    succ:=fail;
    break;
  fi;
  od;   #end of testdepth-loop
  if succ=true then Print("The testdepth ",testdepth,
        " was sucessful!\n"); fi;
else
  Print("The composition of f and g does not seem to
        be trivial. Check the homomorphisms.\n");
fi;
```

After inputting we the homomorphisms together with their initial and terminal spaces we need to specify the lowest dimension we want to test, teststart, and the highest dimension, testdepth. For each dimension in that interval the dimensions of $\mathrm{Im}(g, \mathbb{F}_q^n)$ and $\ker(f, \mathbb{F}_q^n)$ are computed. If they match up the test was successful. As we know that $\mathrm{Im}(g, \mathbb{F}_q^n) = \ker(f, \mathbb{F}_q^n)$ holds for any n smaller or equal to the n where we calculated g it is of particular interest what happens if we increase testdepth beyond that point. In the case that is implanted in the script, which is the same case that was discussed in the previous subsection we have the following output, where GAP regards $0 * Z(2)$ as $0 \in \mathbb{F}_2$ and $Z(2)^0$ as $1 \in \mathbb{F}_2$:

```
Using the prime power q=2
Set now h,s and m to check if a given g gives a weak cokernel in
(F_q^(h_1,...,h_l),F_q^n) -->(F_q^(s_1,...,s_r),F_q^n) -->
(F_q^(m_1,,m_s),F_q^n)
reading h reading s reading m testing now in (F_2^3,F_2^n) -->
(F_2^3,F_2^n) -->(F_2^3,F_2^n)
With the homomorphisms g=
Z(2)^0*[ [ 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), 0*Z(2), 0*Z(2) ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2) ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2) ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ Z(2)^0, 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2) ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ Z(2)^0, Z(2)^0, 0*Z(2) ], [ 0*Z(2), Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ Z(2)^0, Z(2)^0, Z(2)^0 ], [ 0*Z(2), Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]
and f=
Z(2)^0*[ [ Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2) ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]+Z(2)^0*
[ [ Z(2)^0, Z(2)^0, 0*Z(2) ], [ 0*Z(2), Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]
Success!
Dimension of the Kernel: 4 Dimension of the Image: 4
Dimension of the Kernel: 22 Dimension of the Image: 22
Dimension of the Kernel: 148 Dimension of the Image: 148
Dimension of the Kernel: 1096 Dimension of the Image: 1096
```

```
The testdepth 4 was sucessful!
```

As g was calculated usind $n = 3$, exactness at $n = 4$ gives a pretty strong motivation that g might actually generate $\ker(f, -)$.

9 References

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