Microeconomic Theory of Financial Markets under Volatility Uncertainty

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This thesis consists of a general introduction and three independent essays. The summaries are as follows:

1. In the first chapter I give a general overview. Standard concepts and methods are briefly classified. Moreover, I illustrate the meaning and implications of volatility uncertainty. The concrete results are discussed in each essay's respective introduction.

2. The first essay considers a class of general equilibrium economies when the primitive uncertainty model features uncertainty about continuous-time volatility. This requires a set of mutually singular priors, which do not share the same null sets. For this setting we introduce an appropriate commodity space and the dual of linear and continuous price systems.
   All agents in the economy are heterogeneous in their preference for uncertainty. Each utility functional is of variational type. The existence of equilibrium is approached by a generalized excess utility fixed point argument.
   Such Arrow-Debreu allocations can be implemented into a Radner economy with continuous-time trading. Effective completeness of the market spaces alters to an endogenous property. Only mean unambiguous claims equivalently satisfying the classical martingale representation property build the marketed space.

3. I consider fundamental questions of arbitrage pricing arising when the uncertainty model incorporates volatility uncertainty. The resulting ambiguity motivates a new principle of preference-free valuation.
   By establishing a microeconomic foundation of sublinear price systems, the principle of ambiguity-neutral valuation imposes the novel concept of equivalent symmetric martingale measures. Such systems of measures exist when the asset price with uncertain volatility is driven by Peng's G-Brownian motion.

4. This chapter establishes, in the setting of Brownian information, a general equilibrium existence result in a heterogeneous agent economy. The existence is generic among income distributions. Agents differ moreover in their stochastic differential formulation of intertemporal recursive utility. The present class of utility functionals is generated by a recursive integral equation, and incorporates preferences for the local risk of the stochastic utility process.
   The setting contains models in which Knightian uncertainty is represented in terms of maxmin preferences as described by Chen and Epstein (2002). Alternatively, Knightian decision making in terms of an inertia formulation from Bewley (2002) can be modeled as well.

To my parents

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Bielefeld, October 2013 *Patrick Beißner*

Contents

Chapter 1

General Introduction

"The last century's research affirmatively claimed the probabilistic behavior of our universe: God does play dice! Nowadays people believe that everything has its own probability distribution. But a deep research of human behavior shows that for everything involved human or life such, as finance, this may not be true: a person or a community may prepare many different probability distributions for her selection. She changes them, also purposely or randomly, time by time."

-Shige Peng

"It is difficult to argue that economists should have the same faith in a fundamental and reductionist program for a description of financial markets (although such faith does persist in some, a manifestation of physics envy). Markets are tools developed by humans for accomplishing certain tasks -not immutable laws of Nature- and are therefore subject to all the vicissitudes and frailties of human behavior. While behavioral regularities do exist, and can be captured to some degree by quantitative methods, they do not exhibit the same level of certainty and predictability as physical laws."

-Andrew W. Lo and Mark T. Mueller

"On the other hand, perhaps the best case for behavioral finance is that it is nibbling at fundamental neoclassical conundrums and associated phenomena [...], it is hinting at a sort of quantum theory of finance. I hope this will be successful, but, until it is, for now we have developed the Newtonian version of our science."

-Stephen A. Ross

Modern finance has undergone an amazing expansion over the past four decades. As a "crown jewel" of neoclassical economics it was largely responsible for granting legitimacy to the foundation of a derivative exchange.

Things may have changed with the financial crisis of the late 2000s. Nevertheless, the intrinsic property of finance is the indispensable dependence of future events. For an underlying economic model, this implies that it is not certain which state of nature will occur in the future. As a starting point, this approach is based on the formulation of uncertainty by Arrow (1953) through the concept of "states of the environment".

The very recent financial meltdown as a central event in the globalized financial market created new duties for academics. Economics as a social science has been forced to recast its dominant paradigms. On the one hand, new models were to be formulated to explain the emergence of such events. Such considerations in fact had a positive nature. On the other hand, a second class of models seems necessary. This latent normative approach focuses on the prevention of repetitions of former mispricings due to inappropriate models. In this class, the design of (pricing) rules and related institutions are the main objects.

The special nature of economics as a social science is its dialectic role. On the one hand a main, goal is the correct description of an external system - the economy. But an economic theory is also able to bring a part of the economy into being. Here, an insight from economic sociology refers to the assertion by Callon (1998) on the performativity of economics. As worked out by MacKenzie and Millo (2003), the development of the Chicago Board Options Exchange and the theory of option pricing developed by Black and Scholes (1973), is a specific case for the creation of a market (see also the introduction of Ross (2002)).

From this concrete insight, a financial economist should be aware of the responsibility that comes from the ability to influence reality. At the starting point of a model in finance, the representation of uncertainty determines the model. At this stage, the shape of the internal consistency receive its foundation. Here, a more complex uncertainty model increases the scarcity of certainty and the degree of possible internal consistency relations. Consequently, any eschatological ideal in finance depends on the acceptance and propagation of its central paradigm, the uncertainty model.

A very informal outline of the thesis

Financial economics has evolved into many directions. By now, it is considered as an independent field of research. Nevertheless a leading question is always present:

What is today's fair value of a payoff in the future?

Dominant paradigms emerged and their relations were studied. The unified theoretical core of modern finance may be summarized in the three P's *probability*, *preferences* and *prices*, see Lo (1999). Clearly, there is a natural hierarchy for such model ingredients. The fundamental component is the manifestation of uncertainty. Here, the implicit assumption of a known probability law is hidden. One base of the associated probability space is the

axiomatization of Kolmogoroff (1933) and the continuous success in mathematics. With this, one may define a space of future consumption plans. From this point on, the preference structure of a decision maker is faced with the given uncertainty. Depending on the context, this space is associated with the (conceptually different) spaces of net trades. A price system for the underlying economy and equilibrium concept can be formulated on this primitive structure. Otherwise, one has to suppose that prices are given and parametrized by observables.
From this perspective, the main goal of this thesis is to establish a fourth P. This letter corresponds to *possibility*. Possibility refers to the awareness of an imprecise knowledge. The true probability which is a perfect statistical description of observables is removed by a set of probability measures (priors), representing the possibility of different probabilistic scenarios.[1] Moreover, I aim to analyze the relations to the other P's. The effect of possibility changes the underlying concepts and hence the notion of (fair) value.

1.1 Foundation of Modern Finance

Fisher Black and Myron Scholes developed a formula to price financial options like calls or puts in their seminal paper Black and Scholes (1973). The central arguments for observing the explicit pricing formula as a solution to a transformed partial differential equation[2] are based on the principle of replication and an arbitrage-free financial market, modeled by a so called geometric Brownian motion. Shortly thereafter, Ross (1976) formulated this general principle in the arbitrage pricing theory (APT).
In the spirit of Debreu (1954), the notion of a valuation functional receives its economic foundation by the corresponding equilibrium price system, linear and continuous with respect to the topology on the space of contingent claims. Moreover, it is desirable to have a convenient representation of the pertinent pricing operators. Representation in terms of a state price density is proportional to the marginal utility of an agent.
These twin pillars are culminated and formalized by Harrison and Kreps (1979) to deliver a microeconomic foundation of arbitrage-free pricing. The idea of risk-neutral valuation is connected with the concept of equivalent martingale measures. Nowadays these relations are known as the fundamental theorem of asset pricing.

[1] Metaphorically speaking "with a bit of hyperbole", the neoclassic foundation of finance, focusing on the certainty about what is probable and possible, would correspond to the Greek atomic model, (see Sharpe (1993) for the term "nuclear financial economics"). In this allegory, the uncertainty about what is probable and possible would then correspond to a sort of atomic model, where the position of the electron is uncertain, see Pusey, Barrett, and Rudolph (2012).

[2] The solution of the relevant heat equation comes from Physics and is related to the terminal heat distribution of an idealized thin tube, which is given by the payoff structure of the option.

Nevertheless, the principle calculation of the premium for a contract of future cash flow is also an actuarial method. Here, the main idea is to determine the value of a derivative by the self-financed[3] replication of traded assets. The logic is that of a fair game against uncertain nature. In the language of mathematics this concept is called martingale. The risky price process satisfies this property under the risk neutral measure, a virtual probability measure. The relationship to the original probability measure is determined by the mentioned state price density.
Following these lines of arguments, the dogma of probability spaces as the formal uncertainty model affects almost all concepts of modern finance.

Tools from Stochastic and Functional Analysis

Applications of continuous-time stochastic processes to economic modeling is largely focused on financial markets. The mainstream view is to consider the price fluctuation of a liquid asset as an adapted stochastic process $(X_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. The most widely studied evolution of a state variable is the Brownian motion. Explorations of the interconnection between the heat equation and diffusion processes resulted in a fruitful field of research in stochastic analysis. Although the main motivation was to develop a mathematical model for the statistical law of a particle, the available techniques seem to be taylormade for applications in finance.
Another aspect involves the underlying space of contingent claims, which is strongly connected to the commodity space of an associated economy. Unless the states of the world and the points in times are represented by finite sets, the underlying space of contingent claims or consumption profile is an infinite dimensional vector space. Existence of an equilibrium for an exchange economy is one major question in economic modeling. In the language of convex analysis, this problem is related to the existence of a supporting continuous linear price system. In this thesis, the models are based on commodity spaces, which are infinite-dimensional, and the cone of non-negative consumption profiles has an empty interior.[4]

1.2 Uncertainty: Probability and Possibility

The conceptional equalization of risk and uncertainty has a long tradition in modern finance. The opportunity to rely only on probabilities as a representation of uncertainty, opened the door to well developed methods from the mathematical theory of probability. The postulational concept posited by

[3]Again, the idea is based on basic principles from Physics. In this case, the metaphor of an energy preserving operation is used.

[4]In order to guarantee supporting prices, a condition on well-behaved preferences is required. The interpretation is the existence of a desirable bundle which generates an open cone, such that there is no intersection with the preferred set (or the open cone is contained in the preferred set). See Mas-Colell and Zame (1991) for an overview.

Kolmogoroff axiomatization relies on measure theory. This is an analytical approach to a probability theory and allows the application of powerful results from functional analysis. This has a direct influence on the emergence of stochastic calculus. The success of Kolmogoroffs theory can be traced back to the availability of meaningful objects such as the conditional expectation, (continuous-time) stochastic processes and its stochastic calculus.[5]
On the other hand, by possibility we refer to the point of view that many probability priors $\mathbb{P}$ within a set $\mathcal{P}$ are equally possible and it is uncertain which one is the true prior. This generalization of a probability space, denoted by $(\Omega, \mathcal{F}, \mathcal{P})$, is defined as a *possibility space*, where $\mathcal{P}$ is a set of priors on the measurable space $(\Omega, \mathcal{F})$. The expression can be regarded as a formalism for the principle of "Spielräume" motivated by Von Kries (1886) within the rejection of the "orthodox philosophy of Laplace".
The awareness of the difference between risk and uncertainty is not a new idea in economics. This was already marked by Knight (1921), as the following citation indicates:

> *"To preserve the distinction [...] between the measurable uncertainty and an unmeasurable one we may use the term risk to designate the former and the term uncertainty for the latter. [...] The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation a priori or from statistics of past experience), while in the case of uncertainty this is not true."*

In a similar way and directly inspired by Von Kries, Keynes (1937) emphasized the difference between risk and (fundamental) uncertainty:

> *"By uncertain knowledge [...] I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty [...]. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, or the obsolescence of a new invention [...]. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know!"*

In this thesis, I use this distinction for the concept of volatility uncertainty. Here, the uncertainty refers to the natural situation in which the available data might be fragmentary or be considered as a thing of the past, whose connection to volatility in the future is unsettled.

[5] Here I have not touched on the different classification schemes of alternative probability theories. See for instance Weatherford (1982) for such an attempt.

1.2.1 Volatility Uncertainty

The volatility of an underlying asset price process is a fundamental observable or measurant in finance, but not directly observable. The estimation of this parameter, including stochastic volatility structures, contains an intrinsic form of model risk. In a continuous-time setting, the volatility is a function of the quadratic variation of the underlying state process. For instance, in the last two decades a whole branch of stochastic volatility models appeared. The choice of the correct volatility model has a direct and deep influence on the valuation of an involved derivative security. In particular, the parameter sensitivity in two different stochastic volatility models may have a high magnitude effect on the valuation.

One way out is to allow for a time-dependent and non-deterministic confidence interval as a primitive for the volatility. This provision for model risk has direct implications for the underlying uncertainty model. An intrinsic consequence is that the uncertainty model must consist of a set of (possibly) mutually singular probabilistic measures, i.e. the measures do not share the same null sets.[6]

In order to clarify the disparity between the coin tossing view of finance[7] and volatility uncertainty, I illustrate the latter concept in terms of a (trinomial) tree:

> Let a sequence of $n \in \mathbb{N}$ urns describe the uncertainty, where each urn exists independently from the others, and each consists of 50 balls with three types D (down), C (constant) and U (up). We know the number of balls from type D and U are equal in each urn, but we only know that there are less than 10 balls of type C. The time between two draws is given by Δ. The dynamics of the state variable $W = \{W_t\}_{t \in \mathcal{T}}$, with $\mathcal{T} = \{0, \Delta, 2\Delta, \ldots, (n-1)\Delta, n\Delta = T\}$ and $W_0 = 0$, is given by
>
> $$W_{t\Delta} - W_{(t-1)\Delta} = \begin{cases} +\sqrt{\Delta}, & \text{if, } U_t \\ 0, & \text{if, } C_t \\ -\sqrt{\Delta}, & \text{if, } D_t. \end{cases}$$
>
> The variance σ^2 of this increment depends on the number of balls of type C and therefore the possibility of ranges between $\underline{\sigma}^2 = \frac{40}{50}\Delta \leq \sigma^2 \leq \Delta = \overline{\sigma}^2$. This trinomial model converges in a specific sense[8] to a continuous-time limit on the time interval $[0, T]$, again denoted by W. In this context, the volatility refers

[6] This is an important difference compared to the concept of drift uncertainty, where the priors are equivalent and hence share the sets of measure zero.

[7] See Cassidy (2009) for a survey.

[8] In the present setting, weak convergence is shown in Dolinsky, Nutz, and Soner (2012).

to the quadratic variation process of W, given by

$$\langle W\rangle_t = \lim_{\Delta\to 0}\sum_{r\le t}|W_{r\Delta} - W_{(r-1)\Delta}|^2.$$

By construction, the volatility uncertainty persists and can be expressed by means of the volatility constraint $\underline{\sigma}^2 t \le \langle W\rangle_t \le \overline{\sigma}^2 t$. The formulation of such a process forecloses the existence of an underlying probability space. The volatility uncertainty is described in terms of the volatility interval $[\underline{\sigma},\overline{\sigma}]$, where each (adapted) volatility process σ taking values in $[\underline{\sigma},\overline{\sigma}]$ constitutes a possible prior $\mathbb{P}_\sigma$ and corresponds to the shape of $t\mapsto\langle W\rangle_t$ under this prior $\mathbb{P}_\sigma$. Figure 1 gives a schematic illustration. Events

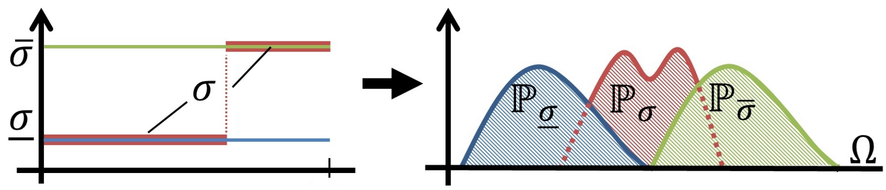

Figure 1: Mutually Singular Priors

related to the quadratic variation $\langle W\rangle$ reveal that the underlying uncertainty model consists of mutually singular probability measures. The process W is a G-Brownian motion, with a given volatility interval $[\underline{\sigma},\overline{\sigma}]$.

The above illustration follows the same lines of a binomial tree without ambiguity in the urns, whose continuous-time limit is the classical Brownian motion.

When presuming uncertainty about volatility, the modeler is forced to give an objective description of the real world in terms of different statistical descriptions with a different event domain of for what is possible or impossible. In Lo and Mueller (2010), a finer taxonomy of uncertainty is proposed. Herein, the classification of volatility uncertainty refers to "partially reducible uncertainty" and can be distinguished with irreducible or ontic uncertainty.

1.2.2 Decision under Uncertainty

A rigorous analysis of preferences under uncertainty is often approached by a consistent set of axioms for its representation on a given set of uncertain outcomes or lotteries. The axiomatization of choice under uncertainty goes back to Von Neumann and Morgenstern (1944). Later, Savage (1954) extended the expected utility to the case of a subjective probability under whose expectation the utility is evaluated.

The preferences for the uncertainty of probabilities, also known as preferences for ambiguity is a classical topic in decision theory, see Ellsberg (1961). In the seminal paper Gilboa and Schmeidler (1989) relax the independence axioms, which is crucial to the representation of expected utility. In a static setting, it was shown that an axiomatic theory of choice with ambiguity aversion is available. This suggests a whole set of probability measures, and considers the minimal expected utility due to ambiguity aversion. A behavioral explanation of ambiguity aversion based on behavioral aspects is discussed in Heath and Tversky (1991). As such, ambiguity (aversion) can also be regarded as a subcategory of behavioral finance, see the survey of Barberis and Thaler (2003). From this perspective, ambiguity about volatility may be directly linked to the notion of excess volatility.[9]

Recently, Maccheroni, Marinacci, and Rustichini (2006) characterize preference in the Anscombe-Aumann setting by extending the worst case evaluation in terms of a penalty term. This increases the flexibility to model ambiguity aversion by giving each probability scenario a different weight of importance. This setting allows for modeling anchored preferences from prospect theory, as well.

Embedding preferences for ambiguity into a dynamic set up, the recursive structure of backward induction emerges. The related concept of dynamic consistency plays a central role in rational decisions. See Epstein and Schneider (2003) for an axiomatization in this setup. Time consistency refers to a rational updating principle, see Riedel (2004) in the case of a dynamic risk measure, which is a risk-neutral version of dynamic multiple-prior preferences.[10]

1.3 Asset Pricing under Volatility Uncertainty

The notion of expectation is a central concept for valuation in financial economics. The *rational expectation hypothesis* (REH) as a collection of assumptions for how agents exploit available information is often modeled in terms of the conditional expectation under a given probability measure. Such an object can be considered as the best predictor with minimal error. In this thesis, the formulation of the REH is affected by the ambiguity about the true probability measure. The magnitude of priors makes it possible to consider a range of reasonable linear conditional expectations. Here, the rational updating of new information is given by a conditional nonlinear expectation. With this modification, an asset pricing principle based on marginal utility changes and depends heavily on the preferences for ambiguity. For instance, Epstein and Wang (1994) introduce and analyze the Lucas model in terms of an ambiguity averse representative agent.

In the presence of volatility uncertainty, I motivate a notion of uncertainty-neutral valuation. This is a canonical generalization of risk-neutral valuation.

[9]This is a phenomena which is often used to claim that the efficient market hypothesis is falsified. In subsection 1.3, I discuss this in more detail.

[10]From a technical point of view, (dynamic) risk measures correspond to (conditional) sublinear expectations.

In essence, I describe a conditional nonlinear expectation under which the uncertain security price process becomes a fair game against a risky and ambiguous nature. Again, this valuation principle is based on a preference-free approach. In the standard case, asset returns are not supposed to be a fair game if agents are risk averse.[11] Similarly, if agents are ambiguity averse (and risk averse) asset returns are risky and uncertain. But, in the multiple prior framework, the notion of a martingale as a representation of a fair game has different degrees of fairness. My notion of fairness refers to the situation where correctly deflated asset prices are fair games under every relevant prior. This principle corresponds to the idea of unambiguous events as introduced in Epstein and Zhang (2001). In this regard the conception of fairness corresponds to the uncertainty-neutral valuation under volatility uncertainty.

For the microeconomic foundation of risk-neutral pricing, continuous and linear price systems are required. In my setting, the uncertainty model induces a nonlinear expectation. The new deflated martingale notion and the REH are now connected in a modified manner. In this regard, a main goal of this thesis is to motivate a new martingale concept, representing the notion of a fair game under volatility uncertainty. Under a nonlinear expectation, the idea of a martingale changes as well and the concept of a fair game should be related to the correct martingale concept. In comparison to the risk neutral pricing, it is questionable whether continuous and linear price systems remain natural types.

As already mentioned, the uncertainty model induces a nonlinear expectation. The new deflated martingale notion may be used to reinterpret the *Efficient Market Hypothesis* (EMH). To illustrate this, I reconsider the relationships between the involved concepts. The nonlinear valuation principle sheds a new light on the EMH. First, I recall the interplay between the falsification of the EMH and the necessity of an asset pricing principle. This is known as the Joint Hypothesis problem. The following quotation in Campbell, Lo, and MacKinlay (1997) indicates this tension:

> *"First, any test of efficiency must assume an equilibrium model that defines normal security returns. If efficiency is rejected, this could be because the market is truly inefficient or because an incorrect equilibrium model has been assumed."*

The importance of the equilibrium concept is directly related to the involved equilibrium price system. But if the price system is based on the probability space via commonly used commodity spaces, then the equilibrium concept also depends on the probability space as well. Under volatility uncertainty, the choice of the implied uncertainty adjusted martingale notion as a convenient representation of the pricing operator has to be modified by taking the given uncertainty model into account.

[11] See Section VI in LeRoy (1989) for a discussion.

Second, the modification of the uncertainty model has implications for a reconsideration of the excess volatility puzzle. The observed excess volatility contradicts the EMH within the standard paradigm of modeling uncertainty via a single probability measure. Here, the notion of present value equals the (linear) conditionally deflated expectation. The variance of this random variable depends on the relevant probability law. As in Shiller (1981) and Shiller (1992), this law is assumed to be known, and induces a linear conditional expectation. The suggested nonlinear valuation principle in Chapter 3 puts this puzzle in a different light, at least from a theoretical point of view. In other words, the claimed falsification of the EMH refers to the case when the uncertainty model is given by a probability space. Hence no disproof affects the present multiple prior asset pricing model for the EMH.[12]

New Tools from Stochastic and Functional Analysis

As mentioned in subsection 1.2.1, when volatility uncertainty is considered, an uncertainty model without an underlying probability space is necessary. However, this circumstance creates several technical difficulties for a mathematical language with powerful tools. In the last six years Shige Peng[13] has, in a series of papers, developed a nonlinear mathematical theory of probability along the same axiomatic lines as in Artzner, Delbaen, Eber, and Heath (1999). Here the degree of nonlinearity is directly connected to the structure of priors, which represents such a risk measure or sublinear expectation. The conditional expectation changes, as an elementary object in the REH. Based on a conditional nonlinear expectation, one may introduce and analyze the concept of a martingale.

In this situation, there is an analogue to the standard Brownian motion, see the example of the trinomial tree in subsection 1.2.1. A fully nonlinear partial differential equation comes into play. The first considerations of Avellaneda, Levy, and Paras (1995) call this object the Black-Scholes-Barrenblatt (BSB) equation. Similarly to the analogy between the Brownian motion and the Laplace operator, as a component of the heat equation, the BSB equation can be associated with G-Brownian motion. As a special case, the related (conditional) G-expectation allows for a stochastic calculus, including stochastic differential equations driven by G-Brownian motion. New types of martingale representation theorems and even a Girsanov type result for G-Brownian motion are available.

The standard pair of the commodity and price space are given by the dual pairing of classical Lebesgue spaces, where the basis is a given measure space. Due to the new uncertainty model, the commodity-price pair is based on a mutually singular set of priors. I suggest a setting which allows the applica-

[12]Suppose a nonlinear expectation operator is considered to compute present discounted value of real dividends. Then small (big) changes of dividends or small (big) arrival of new information may cause big (small) changes of its value, i.e. an *overreaction* (*underreaction*).

[13]Several results are used, which are directly or indirectly influenced by Shige Peng.

tion of an abstract result from the theory of ordered vector spaces. My notion of sublinear prices is built upon the topological dual of continuous and linear price systems. However the explicit representation of the new dual space allows to construct sublinear and continuous price systems in terms of linear functionals from the topological dual space. Here, the book of Aliprantis and Tourky (2007) describes new concepts of lattices for nonlinear functionals.

Chapter 2

Radner Equilibria under Volatility Uncertainty

2.1 Introduction

Ever since the pioneering general theory of competitive markets, the extension to a dynamic equilibrium has served as an initial position for a neoclassical intertemporal asset pricing theory.

Most models of an Arrow-Debreu economy in continuous time assume an underlying and a priori given probabilistic structure. We replace this all-encompassing and basic assumption with a set of pairwise mutually singular probability measures (priors) $\mathcal{P}$. Our main focus is concerned with models where the volatility of the state variable is uncertain or ambiguous. This can only be accomplished through one such set. Furthermore, we aim to analyze the interrelation between volatility uncertainty and incomplete markets. In contrast to the situation of mutually equivalent priors,[1] a new feature emerges about the states of the world:

Certainty about the true prior automatically determines states which cannot occur. A different situation arises when certainty is limited to the knowledge that the true prior is contained in $\mathcal{P}$. This shrinks the set of impossible states and reasonable contingent claims.

The existing literature, when dealing with potentially complete markets, has established a standard way to construct a financial market equilibrium. Here Duffie and Huang (1985) may be regarded as the seminal paper that explores the idea in Kreps (1982), about implementing an Arrow-Debreu allocation into a so called Radner (1972) economy. This is achieved via continuous trading of long-lived securities. A major tool for spanning the complete market

[1] Ambiguity or Knightian uncertainty in continuous time is often modeled by the so called drift uncertainty. Here, the probabilities must be equivalent to each other. Such a description is not appropriate when the volatility is the object which carries the uncertainty. See Chen and Epstein (2002) for a formulation of such preferences via a backward stochastic differential equation (Backward-SDE) and Chapter 4 for the related existence of general equilibrium.

of Arrow-Debreu securities is the concept of a martingale generator, which reduces in a Brownian setting to the classical martingale representation theorem.[2] However, in the present setup the concept of martingale multiplicity as an integer valued measure for the dimension of uncertainty is imprecise. An additional component in the martingale representation suggests, instead, a measure with fraction number values.

This paper establishes the existence of a Radner equilibrium with an endogenously incomplete financial market. The starting point is a heterogeneous agent Arrow-Debreu economy with ambiguity averse agents, where the objective uncertainty is given by the set of priors $\mathcal{P}$. Similarly to representative agent economy in Epstein and Wang (1994) we observe the indeterminacy in the effective equilibrium priors of the price system, as output data of this intermediate economy. As a result, only special Arrow-Debreu equilibrium allocation can be implemented into a Radner economy, and we observe the an incomplete market equilibrium. The endogenous indeterminacy of the Arrow-Debreu equilibrium price system determines the degree and structure of the incompleteness of the implementing financial market.

In the present Radner economy, each agent has to find trading strategies of buying and selling traded claims in order to maximize her utility on net trades when volatility uncertainty of the state variable is present. This is achieved in terms of a suitable dynamic conditional sublinear expectation $X \mapsto \mathbb{E}_t^{\mathcal{Q}_E}[X]$.[3] The set $\mathcal{Q}_E$ refers to all equilibrium price measures, given an equilibrium allocation. In the classical uncertainty model with only one prior, the linear risk-adjusted expectation operator is related to the unique equilibrium price measure.

As demonstrated in the finite state case, Mukerji and Tallon (2001) discuss ambiguity aversion as a source for incompleteness in financial markets. Beyond the related marketed space, a kind of collective portfolio inertia results. In essence, the market-clearing condition in the Radner equilibrium is in action. The role of the financial market as a mechanism to change the shape of income streams is accomplished only partially. Nevertheless, this fits into the arguments by Dow and da Costa Werlang (1992), where inertia for a single agent in a partial equilibrium is detected. In a different setting, De Castro and Chateauneuf (2011) observe similar results on unambiguous trade with unambiguous aggregate endowment.

As argued in Anderson and Raimondo (2008), the candidate equilibrium price process is often assumed to be dynamically complete. Quite frequently

[2]The notion of martingale multiplicity works in a separable framework, so that an orthogonalization procedure counts the dimension of uncertainty.

[3]At this point the assumed weak compactness and stability under pasting of $\mathcal{P}$ play an essential role for the construction of a universal random variable being under each prior simultaneously the conditional expectation. When the set of priors is mutually equivalent, this property is nothing else as the dynamic consistency of conditional expectation. In the volatility uncertainty framework stability under pasting is a stronger condition, see Nutz and Soner (2012). A key feature of this conditional expectation is the semigroup property $\mathbb{E}_s \circ \mathbb{E}_t = \mathbb{E}_s$ for $s \leq t$, which implies the Law of Iterated Expectation.

this assumption is encoded in the exogenous volatility model of the candidate equilibrium price process.[4] In this regard, our model differs in terms of an *intrinsic incompleteness* due to the volatility uncertainty and the appearance of ambiguous net trades. As such, the size and structure of the marketed space is the result of Arrow-Debreu equilibrium.

Martingales and Dynamic Spanning
The relationship between martingale multiplicity and dynamical spanning of the commodity space is an economically meaningful corollary of the martingale representation. In the case of Brownian noise a square integrable random variable X can be represented in terms of a stochastic integral:

$$X = \mathrm{E}^P[X] + \int_0^T \theta_s \mathrm{d}B_s$$

This result is strongly related to the completeness of the financial market. Loosely speaking, in our mutually singular prior framework, a number representing the dimensions of uncertainty does not exist. In essence, this is caused by the more evolved martingale representation theorem. Similarly to the classical Doob-Meyer decomposition for a submartingale, the representation of martingales under a sublinear expectation sustain an additional monotone compensation term:

$$X = \mathbb{E}^{\mathcal{P}}[X] + \int_0^T \theta_s \mathrm{d}B_s - K_T$$

Only a closed subspace of the present commodity space $L^1(\mathcal{P})$ allows for the classical replication of a possible consumption profile $X : \Omega \to \mathbb{R}$. In this case the compensation term (K_t) equals zero. Such random variables are mean unambiguous, i.e. the expectation value of the claim is the same under each prior. At this abstract stage, we can already presume some implications for incompleteness in the involved market structure, see Remark 3.1.

The uncertainty model and the economy
We consider a measurable space $(\Omega, \mathcal{F})$ and fix a set of the probability measures $\mathcal{P}$. In general, three cases of relationships between priors in $\mathcal{P}$ are possible. As described at the beginning of the introduction, two priors maybe mutually singular. This implies a disjoint support of these measures. The second possibility is a mixture. In this case, two priors may be equivalent on a sub σ-field and mutually singular on a complementary sub σ-field. The last case, which does not appear, is mutual equivalence of measures.
In principle, this modeling can describe a set of different probability assessments related to the states of the world $\omega \in \Omega$, so that different possible shapes of the intrinsic volatility may appear. Sure statements concerning random variables in this uncertainty setting cannot be reflected as almost sure events under only one prior $P \in \mathcal{P}$. In this context, arguments are

[4] See for instance Duffie and Zame (1989) and Karatzas, Lehoczky, and Shreve (1990).

based on $\mathcal{P}$-quasi sure analysis, which takes every prior into account simultaneously. Here, a reasonable consumption profile $X : \Omega \to \mathbb{R}$ should have a finite first moment. Thus, our commodity space $L^1(\mathcal{P})$ consists of random variables with a finite expectation for all $P \in \mathcal{P}$.[5] Based on this sublinear expectation, we can define a norm $c_{1,\mathcal{P}}$ such that the space of consumption profiles becomes a Banach space. The positive cone of $L^1(\mathcal{P})$, given by random variables satisfying $X \geq 0$ $\mathcal{P}$-quasi surely, induces an appropriate order structure.

Having the commodity space fixed, we introduce the corresponding topological dual space. This space consists of continuous and linear functionals, which are the candidate price systems. Similarly to the single prior case a generalized Radon-Nikodym density result, representing these price functionals, becomes available. In essence, we can represent every linear and $c_{1,\mathcal{P}}$-continuous functional by a measure μ such that $\mathrm{d}\mu = \psi \mathrm{d}P$, where $P \in \mathcal{P}$ and $\psi \in L^\infty(P)$. This allows us to approach the existence of equilibria via a modified excess utility mapping.

With the given commodity-price duality, we introduce a class of preference relations for the agents in the economy. In the seminal paper by Gilboa and Schmeidler (1989), the well-known maxmin preferences are axiomatized, and account for ambiguity aversion. Later Hansen and Sargent (2001) generalize this concept by introducing an entropy based penalty term for the priors under consideration.[6] In our economy, agents are described by variational preferences. Maccheroni, Marinacci, and Rustichini (2006) introduce and axiomatize variational preferences, a robust version of the expected utility in the form

$$U(X) = \min_{P \in \mathcal{P}} \mathrm{E}^P[u(X)] + \mathfrak{c}(P),$$

where the minimum is taken by a whole class of possible probabilistic views of conceivable scenarios. The functional $\mathfrak{c} : \mathcal{P} \to \mathbb{R}$ penalizes each prior with a different weight. We show that natural properties, such as concavity and upper semicontinuity are imposed when natural conditions on the primitives. When the penalty term is linear even $c_{1,\mathcal{P}}$-continuity can be shown. Moreover, we fully describe the superdifferential of such a utility functional, as in Rigotti and Shannon (2012) for the finite state case.

The economy consists of $I \in \mathbb{N}$ agents, equipped with variational preferences on the positive cone of the commodity space $L^1(\mathcal{P})$. The existence of equilibrium is achieved by a modified Negishi method. In the first step we prove the existence of Pareto optimal allocations.[7] The modification of the excess utility relies on multiple priors, which are now explicit arguments of the excess utility map.[8]

[5]For instance, for each $P \in \mathcal{P}$, the commodity space satisfies $L^1(\mathcal{P}) \subset L^1(\Omega, \mathcal{F}, P)$.

[6]Note that in their model the set of priors are mutually equivalent.

[7]Here, the topological lattice properties of the commodity space ease the proof for the existence of an optimal allocation.

[8]Several technical difficulties motivate this change. A particular problem is that the

In the last part, we implement the net trades of the equilibrium allocation into a Radner type economy. This is achieved via the previously mentioned martingale representation. The implementability of the Arrow-Debreu Equilibrium is limited by the linear price system. An equilibrium with a certain sublinear equilibrium price system can be implemented without further conditions (see Theorem 4).

Related Literature

In the standard single prior Arrow-Debreu setting with expected utility, market prices are directly affected via individual marginal rates of substitution for state contingent commodity bundles (See Martins-da Rocha and Riedel (2010) for a general overview of issues concerning issues the existence of equilibria.) In the simplest version of this model, equilibrium price systems are given by marginal utility weights that can result into risk-neutral probabilities. Continuous-time models and dynamic Arrow-Radner equilibria are treated in Duffie and Huang (1985) and Dana and Pontier (1992). A unique Radner equilibrium is observed in Karatzas, Lehoczky, and Shreve (1990). This approach is based on a representative agent, see Huang (1987). We also refer to Hugonnier, Malamud, and Trubowitz (2012) and Herzberg and Riedel (2013) for a recent discussion of endogenous completeness in continuous-time finance models.

Existence of equilibria in incomplete markets for a finite state space is well developed, starting with the seminal paper by Duffie and Shafer (1985). For an overview we refer the reader to Magill and Quinzii (2002). In Basak and Cuoco (1998), restricted market participation is modeled as a source of market incompleteness. As a consequence, Pareto weights are stochastic.

When the uncertainty is given by an undominated multiple prior setting, considerations of heterogeneous agent economies are treated only for a finite state space, see for instance Dana (2004) and Dana (2002). In Dana and Le Van (2010) no-arbitrage conditions are associated with a risk adjusted set of priors. Rigotti and Shannon (2012) discuss market implications of ambiguity and feature generic determinacy of general equilibrium.

Ravanelli and Svindland (2013) consider efficient allocation with variational preference when the uncertainty is given by a set of equivalent probability measures. In this case, it is possible to start with a reference probability space.

Representative agent economies for the infinite state and discrete time case can be found in Epstein and Wang (1994), where a modification of Lucas' asset pricing model is established in terms of a Choquet expected utility introduced in Chateauneuf (1991). Very recent research by Epstein and Ji (2013a) provide a discussion of the continuous-time case and the notion of sequential trade equilibria with a single agent.

This chapter is organized as follows. Section 2 illustrates the implications

price space is not directly related to a state price density as in the traditional Lebesgue space setting when there is only one prior P.

of the uncertainty model in the case of finitely many states or priors. In Section 3 we introduce the commodity space and the price space. Moreover, we introduce the variational utility functional and discuss its properties. In Section 4, we show the existence of Pareto optimal allocations. Afterwards we establish the existence of equilibrium and the Radner implementation. The appendix collects the details and proofs.

2.2 Simple Economies under Singular Priors

For perspective, we give an outline about the implication of maxmin preferences when there are finitely many states of world $\Omega = \{\omega_1, \ldots, \omega_n\}$. As we will see, the worst case expected utility with a partially disjoint support of possible priors emerges in the form of a Leontief-type utility. In the first subsection, we illustrate the implication in a concrete two agent economy with two priors $\mathcal{P} = \{P_1, P_2\}$ on Ω that are neither singular nor equivalent. Then, we move to the setting with the state space found in Sections 3 and 4 and foreclose some results formulated therein. Two priors $P_1, P_2 \in \mathring{\Delta}_n$, the interior of the simplex of probability measures, are always equivalent. Two priors are singular if their supports are disjoint.

2.2.1 The Finite State Case

In order to illustrate the main point with a concrete example, consider an economy with two agents $i = 1, 2$ and $n = 6$ states of the world at time $T > 0$. The uncertainty is given by two measures represented by $P_1 = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $P_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0)$, see Figure 1.[9] It is unknown which prior is the cor-

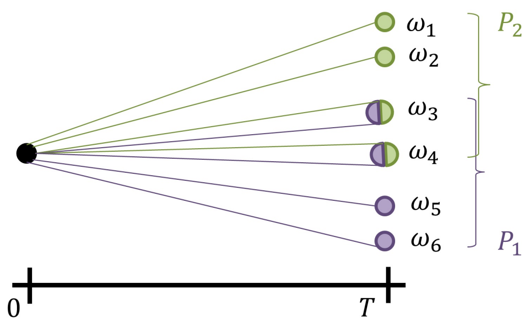

Figure 1: Non-Equivalent and Non-Singular Priors

rect, although each prior determines different states of the world. Each agent is ambiguity averse on $\mathcal{P} = \{P_1, P_2\}$ with maxmin preferences represented in

[9] In the volatility uncertainty setting, such priors occur when the volatility is in agreement up to some time $t > 0$ and then differs.

terms of $U^i : \mathbb{R}^6_+ \to \mathbb{R}$ given by

$$\begin{aligned} U^i(X) &= \min_{P \in \mathcal{P}} E^P[a^i \ln(X)] \\ &= \frac{a_i}{4}\Big(\ln\left(X_{\omega_3} \cdot X_{\omega_4}\right) + \min\left(\ln\left(X_{\omega_1} \cdot X_{\omega_2}\right), \ln\left(X_{\omega_5} \cdot X_{\omega_6}\right)\right)\Big). \end{aligned}$$

The endowments are given by $e^1 = (1,1,2,1,3,3)$ and $e^2 = (2,2,1,2,1,1)$, where the prior-dependent endowment is denoted by $e^i(P)$, for instance we have $e^1(P_1) = (1,1,2,1)$. Due to the singularity in the events $\{\omega_1, \omega_2\}$ and $\{\omega_5, \omega_6\}$, the utility structure has a Leontief flavor in these states. This means for instance, the indifference curve with respect to payoffs in the events $\{\omega_1, \omega_2\}$ and $\{\omega_5, \omega_6\}$ are L-shaped. This is illustrated in the Edgeworth boxes of Figure 2.
After some calculations, we have an equilibrium price system $\Pi(\cdot) = \langle \cdot, p \rangle$, with $p \in \mathbb{R}^6_+$ such that $(p_5, p_6) = 0$ must hold. This follows from the L-shaped indifference curve and $(e_{\omega_1}, e_{\omega_2}) > (e_{\omega_5}, e_{\omega_6})$. The price system has the same support as P_2. This can be infered from the first order conditions, since each agent has P_2 as the minimizing (effective) prior of her maxmin utilities. The non-unique equilibrium allocation $(\bar{X}^1, \bar{X}^2) \in \left[\underline{X}, \overline{X}\right]$ lies on the orange line segment of Figure 2 (b). Arrow securities of state ω_5 and ω_6 are for free, so that a feasible retrade on the order interval $\left[\underline{X}, \overline{X}\right]$ leaves the utility unaffected. On the other hand, consumption in state ω_3 and ω_4 behaves as

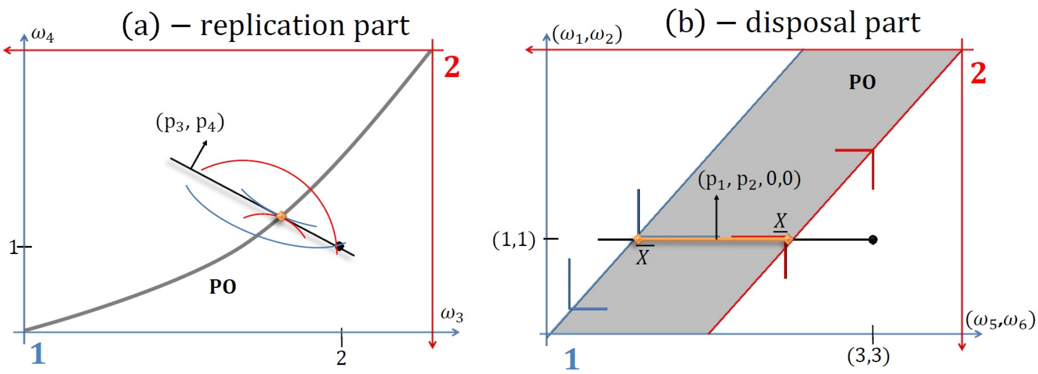

Figure 2: Edgeworth boxes via Leontief-type utility

in the expected utility setting with one prior. Specifically, the consumption is prior independent. This can be seen in the explicit description of $U^i(X)$ above and Figure 2 (a). In Subection 3.1.2 we consider the analog space of *unambiguous* contingent claims denoted by $\mathrm{M}[\mathcal{P}]$. However, trade outside of $\mathrm{M}[\mathcal{P}]$ is possible.

The Modified Negishi-Approach

Continuing with the setting of the last paragraph, we illustrate how the existence of an equilibrium can be shown. To do so, we consider the first order condition

$$\alpha^i \nabla U^i(\bar{X}^i) = \alpha_i \left(\frac{a_i \cdot P_2(\{\omega_1\})}{\bar{X}^i_{\omega_1}}, \dots, \frac{a_i \cdot P_2(\{\omega_4\})}{\bar{X}^i_{\omega_4}}, 0, 0 \right) = \langle p, \cdot \rangle, \quad i = 1, 2.$$

From this characterization of the Pareto optimal (PO) allocation, we denote the set of common effective priors under the efficient allocation by $\mathbb{P}(\alpha)$. The restriction to concentrate on linear prices leads to a price system $\langle p, \cdot \rangle = \mathrm{E}^{P_2}[\psi \cdot]$ having an endogenous support $\{\omega_1, \dots, \omega_4\}$. As such the representation as a sole random variable fails.
We illustrate how the Negishi method applies to show the P_2 almost sure unique equilibrium, so that the indeterminacy of the equilibrium allocation is outside the support of P_2. Let us consider the utility possibility set in Figure 3. The utility possibility set (UPS) for the economy $\mathcal{E}^P$ with expected

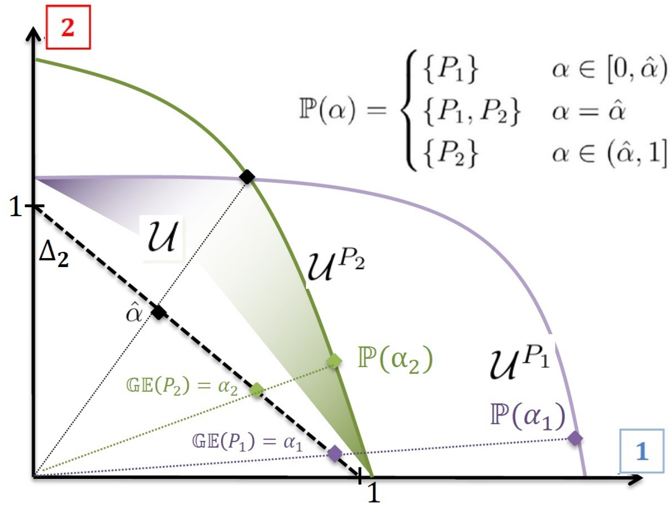

Figure 3: Utility possibility set under $\{P_1, P_2\} = \mathcal{P}$

log utility agents under $P \in \mathcal{P}$ is denoted by $\mathcal{U}^P$. Clearly, each $\mathcal{E}^P$ induces a unique equilibrium weight denoted by $\alpha = \mathbb{GE}(P)$. On the other hand each $\alpha \in \Delta_2$ induces a representative agent U_α, whose effective prior is denoted by $P \in \mathbb{P}(\alpha)$. The UPS of the original economy with multiple priors is then given by $\mathcal{U} = \mathcal{U}^{P_1} \cap \mathcal{U}^{P_2}$. Moreover, we have $P_2 \in \mathbb{P}(\alpha_2)$. While $P_1 \notin \mathbb{P}(\alpha_1)$ is not an effective prior for $\mathcal{U}^{P_1}$ and therefore contradicts the first order conditions with respect to the α_1-efficient allocation. This illustrates how the Negishi approach with a von Neumann-Morgenstern utility still applies under the correct prior of the maxmin preferences, as explained in the following.
An equilibrium has to satisfy two conditions. On the one hand, the prior $P_2 = P^*$ as a component of the price system must be effective for the representative agent, i.e. $P^* \in \mathbb{P}(\alpha^*)$. On the other hand the weighting α^* of the representative agent under P^* must be the correct equilibrium weight denoted by $\mathbb{GE}(P^*) = \alpha^*$. These two conditions can be condensed in a fixed point of a composited correspondence, i.e. $P^* \in \mathbb{P} \circ \mathbb{GE}(P^*)$. This observation will lead to a proof method for the existence of an equilibrium, also

in the volatility uncertainty setting. Moreover, as a byproduct, we observe structural properties more directly.

2.2.2 The Infinite State Case

One special property of every finite dimensional commodity space L is the equivalence of every two arbitrary norms $\|\cdot\|_i : L \to \mathbb{R}_+$, $i = 1, 2$, with this in mind, we move to the infinite (and uncountable) state space Ω, consisting of continuous paths $\omega : [0, T] \to \mathbb{R}$, equipped with the usual Borel σ-algebra $\mathcal{B}(\Omega) = \mathcal{F}$. Let us consider two mutually singular priors $\mathcal{P} = \{P_1, P_2\}$ on $(\Omega, \mathcal{F})$ as the uncertainty model. In Section 3 we describe this in more detail. Let the endowment e_i of each agent $i = 1, 2$ depend on the prior. So that we have $e_i = (e_i^{P_1}, e_i^{P_2}) \in L^2(P_1) \times L^2(P_2)$, where $L^2(P) = L^2(\Omega, \mathcal{F}, P)$ is the usual Lebesgue space of integrable random variables equipped with a standard norm $\|x\|_{L^2(P)} = \mathrm{E}^P[|x|^2]^{1/2}$. Since both priors are possible, it is reasonable to consider endowments satisfying $c_{2,\mathcal{P}}(e_i) = \max\{\|e_i^P\|_{L^2(P)}, P \in \mathcal{P}\} < \infty$. The finiteness condition under the $c_{2,\mathcal{P}}$-norm corresponds to the space $L^2(\mathcal{P})$, being with each $P \in \mathcal{P}$ a strict sub space of $L^2(P)$.
A standard price system $\Pi : L^2(\mathcal{P}) \to \mathbb{R}$ for equilibria in infinite dimensional commodity space is linear and continuous in the topology of the underlying commodity space. As we will present in Section 3.1 the related price dual space of $L^2(\mathcal{P})$, denoted by $L^2(\mathcal{P})^*$, is strictly larger than $L^2(P)^*$, $P \in \mathcal{P}$, due to the stronger $c_{2,\mathcal{P}}$-norm. We have the following sequence of inclusions:

$$L^2(\mathcal{P}) \subset L^2(P) \cong L^2(P)^* \subset L^2(\mathcal{P})^*, \quad P \in \mathcal{P}$$

Again, each agent has maxmin utility $U_i(X) = a_i \min(\mathrm{E}^{P_1}[\ln(X)], \mathrm{E}^{P_2}[\ln(X)])$, defined on the positive cone $L^2(\mathcal{P})_+$. Let $(\bar{X}_1, \bar{X}_2)$ be an equilibrium allocation of the economy $\mathcal{E} = \{L^2(\mathcal{P}), U_i, e_i\}_{i=1,2}$ such that $\bar{X}_1^P + \bar{X}_2^P = e^P$ and $\bar{X}_1^P = \bar{X}_1$ P-a.s. holds under every $P \in \mathcal{P}$.
Now, consider the situation when $U_i(\bar{X}_i) = a_i \mathrm{E}^{P_2}[\ln(\bar{X}_i)] \neq a_i \mathrm{E}^{P_1}[\ln(\bar{X}_i)]$ for each $i = 1, 2$. The supergradients of U_i at a consumption bundle in $L^2(\mathcal{P})_+$ lie in the dual $L^2(\mathcal{P})^*$. The first order condition to characterize a Pareto optimal allocation gives us

$$\alpha_1 \cdot \nabla U_1(\bar{X}_1) = \mu_1 = \mu_2 = (1 - \alpha_1) \cdot \nabla U_2(\bar{X}_2), \text{ where } \quad \mathrm{d}\mu_i = \alpha_i \cdot u_i'(\bar{X}_i)\mathrm{d}P_2.$$

In comparison to the traditional general equilibrium theory the equilibrium pricing measure $Q = \frac{\mu}{|\mu|}$ cannot contain all the information about the uncertainty model. Although Q completely represents the linear and continuous price system $\Pi(\cdot) = \mathrm{E}^Q[\cdot]$, it is decoupled from the non-effective prior (relatively to the equilibrium allocation) P_1. Note, that this conceptual observation is consistent with the finite state case in Subsection 2.1. These informal computations are condensed in the following observation:

Let $(p, (\bar{X}_i))$ be an equilibrium in $\{L^2(P_2), U_i, e_i\}_{i=1}^2$ then an equilibrium for $\{L^2(P_1) \times L^2(P_2), U_i, e_i\}_{i=1}^2$ is given by $((0, p), (\bar{X}_1^{P_1}, \bar{X}_1^{P_2}), (\bar{X}_2^{P_1}, \bar{X}_2^{P_2}))$.

In contrast to a standard finance model with an underlying probability space $(\Omega, \mathcal{F}, P)$, the equilibrium price system $\Pi \in L^2(\mathcal{P})^*$ no longer carries the information of all null sets. This has direct implications for the related concepts of asset pricing, such as arbitrage, equivalent martingale measures and stochastic discount factors.

2.3 The Primitives of the Economy

Let us start with the underlying uncertainty model. We consider scenarios, represented by probabilistic priors, which do not share the same null sets. As such, it is not appropriate to assume the existence of a given reference probability measure. Concerning the construction of priors, our method needs some structure on the state space.
Let Ω be the set of all possible states of the world. A state is an exogenous sequence of circumstances from time 0 to time T which are relevant to the economy. We assume $\Omega = \{\omega \in C([0,T];\mathbb{R}) : \omega_0 = 0\}$ to be the canonical space of continuous sample paths starting in zero and endowed with the uniform topology.[10] The σ-field of events is given by the Borel σ-field of Ω, called $\mathcal{F} = \mathcal{B}(\Omega)$. Let $\mathcal{M}_1(\Omega)$ be the set of all probability measure on $(\Omega, \mathcal{F})$. Now, we construct a set of priors on the measurable space $(\Omega, \mathcal{F})$. The canonical process $B_t(\omega) = \omega_t$ is a Brownian motion under the Wiener measure P_0.[11] We denote by $\mathbb{F}^o = \{\mathcal{F}^o_t\}_{t\in[0,T]}$, with $\mathcal{F}^o_t = \sigma(B_s, s \in [0,t])$ the raw filtration of the canonical process B. The strong formulation of volatility uncertainty is based upon martingale laws in terms of stochastic integrals:

$$P^a := P_0 \circ (X^a)^{-1}, \quad \text{where } X^a_t = \int_0^t \sqrt{a_s}\,\mathrm{d}B_s, \quad t \in [0,T].$$

The stochastic integral X^a is the classical Itô integral under P_0. The process $a = (a_t)_{t\in[0,T]}$ is $\mathbb{F}^o$-adapted and has a finite first moment. Probability measures generated in this way are denoted by $\mathcal{P}_S$, referring to the strong formulation of volatility uncertainty.

Assumption 1 *The uncertainty of each agent is generated by a convex set $\mathcal{D}$ of processes, such that the set of priors is weakly compact*[12] *and given by*

$$\mathcal{P} = \{P^a \in \mathcal{P}_S : a \in \mathcal{D}\}.$$

[10]This topology is generated by the supremum norm $\|\omega\|_\infty = \sup_{t\in[0,T]} |\omega_t|$, $\omega \in \Omega$.

[11]Note that P_0 is not a reference measure and its technical purpose is linked to the construction of the uncertainty model. The case $P_0 \notin \mathcal{P}$ is possible and refers to $1 \notin \mathcal{D}$ in Assumption 1.

[12]The set of measures is relatively compact if and only if for each sequence of closed sets $F_n \searrow \emptyset$ implies $\sup_{P\in\mathcal{P}} P(F_n) \searrow 0$. Regularity in terms of monotonic continuity of $\mathbb{E}^{\mathcal{P}}[\cdot]$ is equivalent to weak relative compactness for $\mathcal{P}$. We refer to Huber and Strassen (1973) and Denis, Hu, and Peng (2011).

Recall that, the volatility of a stochastic integral $X^a = \int \sqrt{a}\mathrm{d}B$ is given by the quadratic variation $\langle X^a \rangle_t = \int_0^t a_s \mathrm{d}s$. As such, by construction the volatility uncertainty is encoded in the quadratic variation. The mutual singularity of priors is an intrinsic and natural property in the continuous-time setting. For instance, $P^a(\langle B_T \rangle = T) = 0 \neq 1 = P_0(\langle B_T \rangle = T)$ may appear, for a some constant $a \neq 1$.
In order to address this fact, we need to modify the notion of a sure event. To do so, we say a property holds $\mathcal{P}$-*quasi surely* ($\mathcal{P}$-q.s.) if it holds outside a $\mathcal{P}$-polar set. Such sets have zero probability under every prior $P \in \mathcal{P}$. Next, we illustrate this construction method of priors for Peng's G-expectation.[13]

Example 1 *Let the uncertainty be given by a G-expectation. The volatility is associated with the volatility bounds* $0 < \underline{\sigma} < \overline{\sigma}$. *The associated nonlinear expectation* $\mathbb{E}_G[X]$ *can be represented by* $\max_{P\in\mathcal{P}} \mathrm{E}^P[X] = \mathbb{E}_G[X]$, $\mathcal{P}$ *is induced by* $\mathcal{D} = \{a \in L^2(P \otimes \mathrm{d}t)$ *and* $\mathbb{F}^o$*-adapted* $: a_t(\omega) \in [\underline{\sigma}, \overline{\sigma}]\, P_0$*-a.s.*$\}$. *This is a weakly compact set of probability measure on* $(\Omega, \mathcal{F})$.[14] *The quadratic variation process is no longer deterministic. All the volatility uncertainty for* B *is concentrated in the quadratic variation* $\langle B \rangle$. *Under every prior* P^a *in* $\mathcal{P}$, *the volatility process is given by* $\langle B \rangle_t^{P^a} = \int_0^t a_s \mathrm{d}s$. *This bracket process is absolutely continuous with respect to the Lebesgue measure on* $[0, T]$ *and its density satisfies* $\underline{\sigma}^2 t \leq \langle B \rangle_t \leq \overline{\sigma}^2 t$, $t \in [0, T]$, P_0*-a.s.*

2.3.1 The Commodity Space and the Price Dual

We aim to consider contingent claims having a finite expectation for every possible prior $P \in \mathcal{P}$. In the tradition of Debreu (1959), we present an *axiomatic analysis of economic equilibrium*, when Assumption 1 defines the uncertainty model. We introduce the underlying space of consumption bundles (c, C) consisting of consumption at time 0 and time T. The comprehensive set of priors prevents the consideration of a classical Lebesgue space. Nevertheless, we repeat similar steps and begin with a rather small set of reasonable random variable. Then we introduce a reasonable norm with which we accomplish the (topological) completion.
We begin to describe the state-dependent consumption good at time T, where we consider only claims on consumption with a finite expectation for each prior $P \in \mathcal{P}$. As in Huber and Strassen (1973), for each $\mathcal{F}$-measurable real functions $X : \Omega \to \mathbb{R}$ such that the expectation $\mathrm{E}^P[X]$ exists under every $P \in \mathcal{P}$, we define the upper expectation operator[15]

$$\mathbb{E}^{\mathcal{P}}[X] = \max_{P\in\mathcal{P}} \mathrm{E}^P[X].$$

[13] We refer to Peng (2010) for the analytic construction of G-expectations.

[14] See Proposition 5 in Denis and Kervarec (2013) for the weak compactness and convexity. Alternatively, by Theorem 2.1.20 in

[15] $\mathbb{E}^{\mathcal{P}}[\cdot]$ satisfies the property of a sublinear expectation (see Peng (2006)), i.e. monotonicity, positive homogeneity, constant preserving, sub-additivity. This object builds the basis of our model.

For a general treatment, see Denis, Hu, and Peng (2011) and the references therein. Let $\mathcal{C}_b(\Omega)$ denote the set of all bounded, continuous and $\mathcal{F}$-measurable real functions. The concrete description of our uncertainty model allows us to define an appropriate commodity space which considers every prior in $\mathcal{P}$ as relevant. Consequently, we suggest a norm taking every prior into account, so that we consider the capacity-type norm $c_{1,\mathcal{P}}$ on $\mathcal{C}_b(\Omega)$ by $c_{1,\mathcal{P}}(X) = \mathbb{E}^{\mathcal{P}}[|X|]$.

The Commodity Space

Let the closure of $\mathcal{C}_b(\Omega)$ under $c_{1,\mathcal{P}}$ be denoted by $\mathcal{L}^1(\mathcal{P}) = \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{P})$.[16] Moreover let $L^1(\mathcal{P}) = \mathcal{L}^1(\mathcal{P})_{/\mathcal{N}}$ be the quotient space of $\mathcal{L}^1(\mathcal{P})$ given by the $c_{1,\mathcal{P}}$ null elements denoted by $\mathcal{N}$.[17] We do not distinguish between classes and their representatives. Two random variables $X, Y \in L^1(\mathcal{P})$ can be distinguished if there is a prior in $P \in \mathcal{P}$ such that $P(X \neq Y) > 0$.[18] For the given commodity space we may introduce an order structure $X \leq Y$ if $P(X \leq Y) = 1$ for every prior $P \in \mathcal{P}$. We obtain the following result.[19]

Proposition 1 *The given triplet $(L^1(\mathcal{P}), c_{1,\mathcal{P}}(\cdot), \leq)$ is a Banach lattice with an σ-order continuous norm, that is $X_n \searrow 0$, with $X_n \in L^1(\mathcal{P})$ implies $c_{1,\mathcal{P}}(X_n) \searrow 0$.*

As usual, we define by $L^1(\mathcal{P})_+ = \{X \in L^1(\mathcal{P}) : X \geq 0 \ \mathcal{P}\text{-q.s.}\}$ the positive cone of $L^1(\mathcal{P})$. In Subsection 3.2, the fine quasi sure order structure causes a more involved notion of strict monotonicity.

Cone	**Order**	**Monotonicity**	**Arbitrage**
$L^1(\mathcal{P})_+$	$X \geq Y$ q.s.	standard	–
$L^1(\mathcal{P})_+ \backslash \{0\}$	$X \geq Y$ q.s. & $X \neq Y$	strict	weak
$L^1(\mathcal{P})_\oplus$	$X \geq Y$ & $X \neq Y$ P-a.s. $\forall P \in \mathcal{P}$	semi-strict	semi weak
$L^1(\mathcal{P})_{++}$	$X > Y$ q.s.	weakly strict	strong

Table 1: Order Structures in the Commodity Space $L^1(\mathcal{P})$

Loosely speaking, a strictly desirable consumption bundle must be nonzero under every possible prior. In preparation, let us introduce the cone of *semi-strictly positive* random variables

$$L^1(\mathcal{P})_\oplus = \left\{X \in L^1(\mathcal{P})_+ : P(X > 0) > 0 \ \forall P \in \mathcal{P}\right\}.$$

[16] It is easily verified that $\mathcal{C}_b(\Omega) \subset dom(\mathbb{E}^{\mathcal{P}}[\cdot]) = \{X \in L(\Omega) : \mathbb{E}^{\mathcal{P}}[X] < \infty\}$ holds, where $L(\Omega)$ denotes the set of Borel measurable function $X : \Omega \to \mathbb{R}$.

[17] One can show that these null elements are $\mathcal{P}$-quasi surely zero.

[18] In a setting with equivalent priors, i.e. priors sharing the same sets of mass zero, this implies that $P(X \neq Y) > 0$ for all $P \in \mathcal{P}$.

[19] This is important for the application of an abstract existence result for quasi equilibria. However, we take a different approach to prove existence, see also Remark 2.

Note that this intermediate cone contains $L^1(\mathcal{P})_{++} = \{L^1(\mathcal{P})_+ : X > 0 \ q.s.\}$, the quasi interior of $L^1(\mathcal{P})_+$.[20] Accordingly, we have the following strict inclusions $L^1(\mathcal{P})_{++} \subsetneqq L^1(\mathcal{P})_{\oplus} \subsetneqq L^1(\mathcal{P})_+ \setminus \{0\}$. Table 1 summarizes the different cones and their interrelation to monotonicity and possible arbitrage notions.

Unambiguous Contingent Claims

As illustrated in Figure 2, there are contingent claims which can be perfectly replicated. Such random variables are not affected by the volatility uncertainty. As we will discuss later, especially in Subsection 4.1, there is a subspace of $L^1(\mathcal{P})$ which becomes a natural candidate for the marketed space of perfectly replicable contingent claims, as given by

$$\begin{aligned} \mathrm{M}[\mathcal{P}] &= \{\xi \in L^1(\mathcal{P}) : \ \mathbb{E}^{\mathcal{P}}[\xi] = -\mathbb{E}^{\mathcal{P}}[-\xi]\} \\ &= \left\{\xi \in L^1(\mathcal{P}) : E^P[\xi] = E^{P'}[\xi] \text{ for all } P, P' \in \mathcal{P}\right\}. \end{aligned}$$

Random variables in $\mathrm{M}[\mathcal{P}]$ are called *$\mathcal{P}$-unambiguous*. For another set of priors $\mathcal{Q}$, the notion of $\mathcal{Q}$-unambiguity (in $L^1(\mathcal{P})$) is still meaningful and well-defined. Mean unambiguity is strongly related to ambiguity neutrality. For instance, we may take the viewpoint of Epstein and Zhang (2001) and consider the Dynkin system of unambiguous events $\mathcal{U}(\mathcal{P}) = \{A \in \mathcal{F} : P(A) \text{ is constant for all } P \in \mathcal{P}\}$.

The Price Space

We turn now to the space of price systems on $\mathbb{R} \times L^1(\mathcal{P})$. A model which aims to observe the existence of a general equilibrium should first of all clarify what price system decentralizes an allocation. As it is common, we suppose a linear price system $\Psi : \mathbb{R} \times L^1(\mathcal{P}) \to \mathbb{R}$. Moreover we require continuity under the topology of the $c_{1,\mathcal{P}}$-norm.[21] We discuss the topological dual of $L^1(\mathcal{P})$. For our purposes we need to determine market prices via marginal rates of the agents. For the existence proof for equilibrium, the following result is of importance.

Proposition 2 *Elements in the topological dual of $(L^1(\mathcal{P}), c_{1,\mathcal{P}})$ can be represented by an absolutely continuous measure:*

$$L^1(\mathcal{P})^* \supset \left\{ l(\cdot) = \int \cdot \mathrm{d}\mu = \mathrm{E}^P[\psi \cdot] : P \in \mathcal{P} \text{ and } \psi \in L^\infty(P) \right\} = \tilde{L}^1(\mathcal{P})^*$$

[20] By Proposition 1, $L^1(\mathcal{P})$ is a Banach lattice. The representation follows then by Lemma 4.15 in Abramovich and Aliprantis (2002).

[21] Later on we assume semi-strict monotonicity of preferences. This guarantees semi-strictly positive prices. Since $\mathbb{R} \times L^1(\mathcal{P})$ is a Banach lattice, this implies norm continuity.

The subspace $\tilde{L}^1(\mathcal{P})$ in Proposition 2 is smaller than $L^1(\mathcal{P})$. Same arguments, as in Proposition 1, show that $\left(\tilde{L}^1(\mathcal{P}), c_{1,\mathcal{P}}(\cdot), \leq\right)$ is an order continuous Banach lattice.[22]

Remark 1 *In Lemma 1 we consider a class of utility functionals on $L^1(\mathcal{P})$ such that there are super-gradients even in $\tilde{L}^1(\mathcal{P})^*$. In principle, the dual of $\tilde{L}^1(\mathcal{P})$ seems to be more acceptable. On the other side, it is unclear how to work within $\tilde{L}^1(\mathcal{P})$, when we apply results from the dynamic theory of G-expectation, whose natural domain is $L^1(\mathcal{P})$. Moreover, some convergence results are only available for $L^1(\mathcal{P})$ (see the beginning of the Appendix).*

The representation in Proposition 2 has similarities to the duality of Lebesgue spaces from classical measure theory, when only one prior P describes the uncertainty. Note that the stronger capacity norm $c_{1,\mathcal{P}}(\cdot)$ in comparison to the single prior $L^1(P)$-norm implies a richer dual space, controlled by the set of priors $\mathcal{P}$.[23] Let us introduce the space of semi-strictly positive functionals

$$L^1(\mathcal{P})^*_{\oplus} = \left\{ l \in L^1(\mathcal{P})^* : l(\cdot) = \mathrm{E}^P[\psi \cdot] \text{ with } P \in \mathcal{P} \text{ and } \psi \in L^\infty(P)_{++} \right\}.$$

Suppose $l \in L^1(\mathcal{P})^*_{\oplus}$, then l may not be strictly positive, i.e. $l(Y) = 0$ if $Y \in L^1(\mathcal{P})_+ \backslash \{0\}$.[24] This indicates that we need a weaker notion than strict positivity. In Table 2, we give the different dual cones and their interrelation to the representation property. Similarly to the commodity space we have different order structures with respect to its order dual. Specifically, we compare the representing measure of Proposition 2, is given. Furthermore,

Dual Cone	**Order**	**Positivity**	**Repr.** $\mathrm{d}\mu = \psi \mathrm{d}P$ **of** l
$L^1(\mathcal{P})^*_+$	$l \geq 0$ on $L^1(\mathcal{P})_+$	standard	$\psi \in L^\infty(P)_+ \& P \in \mathcal{P}$
$\tilde{L}^1(\mathcal{P})^*_+ \backslash \{0\}$	$l > 0$ on $\tilde{L}^1(\mathcal{P})_+ \backslash \{0\}$	strict	$\psi \in L^\infty(P)_{++} \& P \in \mathcal{P}_{can}$[25]
$L^1(\mathcal{P})^*_{\oplus}$	$l > 0$ on $L^1(\mathcal{P})_{\oplus}$	semi-strict	$\psi \in L^\infty(P)_{++} \& P \in \mathcal{P}$
$L^1(\mathcal{P})^*_{++}$	$l > 0$ on $L^1(\mathcal{P})_{++}$	weakly strict	$\psi \in L^\infty(P)_+ \backslash \{0\}$

Table 2: Order Structures in the Dual of $L^1(\mathcal{P})$

the following result shows that exactly semi-strictly positive random variables have strictly positive values with respect to functionals in $L^1(\mathcal{P})^*_{\oplus}$.

[22] For more details, we refer to Section 2, Lemma 4.1 and Proposition 4.1 in Bion-Nadal and Kervarec (2012).

[23] With the explicit representation in Proposition 2, the weak topology of the dual pairing is tractable and allows us to apply standard convergence results from measure theory.

[24] This can be seen as follows. Let $\hat{P}(Y > 0) > 0$ and $Y = 0$ P-a.s. for every $P \in \mathcal{P} \backslash \{\hat{P}\}$ and let $X \mapsto l(X) = \mathrm{E}^{P_l}[\psi X]$ such that $P_l, \hat{P}$ are mutually singular and $\psi > 0$ P^l-a.s., hence $l(Y) = 0$.

[25] Note that $\mathcal{P}_{can}$ denotes the canonical equivalence class, which we mention in Example 3 of Subsection 3.2, below. For details we refer to section 4 in Bion-Nadal and Kervarec (2012) and especially Definition 4.3 therein.

Corollary 1 *Let $l : L^1(\mathcal{P}) \to \mathbb{R}$ be a linear and continuous functional, we have: $l \in L^1(\mathcal{P})^*_\oplus \Leftrightarrow l(X) > 0$ for all $X \in L^1(\mathcal{P})_\oplus$.*

Now, the representing measure μ of a linear and $c_{1,\mathcal{P}}$-continuous functional can be decomposed by P and ψ. The P-almost surely strictly positive random variable $\psi \in L^\infty(P)_{++}$ can be seen as a state price density under the cohesive probability model P. This allows us to represent every semi-strictly positive, normalized, continuous and linear price system on $\mathbb{R} \times L^1(\mathcal{P})$ as

$$\Psi(x, X) = \pi(x) + \Pi(X) = \pi \cdot x + \mathrm{E}^Q[X],$$

where $\pi > 0$. The equilibrium price measure given by $Q(A) = \int_A \psi(\omega) \mathrm{d}P(\omega)$, $A \in \mathcal{F}$, describes the value of any claim in terms of an expected payoff, where ψ is normalized to unit expectation under $P \in \mathcal{P}$. To sum up, our commodity-price pair is given by $(\mathbb{R} \times L^1(\mathcal{P}), \mathbb{R} \times L^1(\mathcal{P})^*)$.

2.3.2 Variational Preferences

A priori, each agent is faced with the same objective uncertainty model $(\Omega, \mathcal{F}, \mathcal{P})$. In this situation the heterogeneity is induced by different ambiguity attitudes which we describe below.
Every agent is determined by an initial endowment $(e, E) \in \mathbb{R}_+ \times L^1(\mathcal{P})_+$, and a utility functional V on $\mathbb{R}_+ \times L^1(\mathcal{P})_+$ being additively separable, so that we can write $V(x, X) = u^0(x) + U(X)$. We describe the utility functional U on $L^1(\mathcal{P})_+$ for an arbitrary agent, where the positive cone is induced by the order relation of Section 3.1. As we consider preference relations $\succeq$ on $L^1(\mathcal{P})_+$, convexity and continuity can be defined in a standard way. Monotone preferences are directly related to the order structure and the set of priors $\mathcal{P}$. We state a weak notion of strict monotonicity on $L^1(\mathcal{P})_+$.[26]

Definition 1 *A preference relation is called* semi-strictly monotone at $X \in L^1(\mathcal{P})_+$, *if $X + Z \succ X$ for all $Z \in L^1(\mathcal{P})_\oplus$.*

In Remark 1 below, we discuss in more detail why this modified strict monotonicity condition is more appropriate when mutually singular priors constitute the uncertainty. We allow for variational preferences introduced and axiomatized by Maccheroni, Marinacci, and Rustichini (2006):[27]

$$C \succeq X \Longleftrightarrow \min_{P \in \mathcal{P}} \left(\mathrm{E}^P[u(C)] + \mathfrak{c}(P) \right) \geq \min_{P \in \mathcal{P}} \left(\mathrm{E}^P[u(X)] + \mathfrak{c}(P) \right), \qquad (1)$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a utility index and $\mathfrak{c} : \mathcal{P}_S \to \mathbb{R}$ is an ambiguity index. For each $C \in L^1(\mathcal{P})_+$ define the set of effective probability scenarios

$$M(C) = \operatorname*{argmin}_{P' \in \mathcal{P}} \mathrm{E}^{P'}[u(C)] + \mathfrak{c}(P').$$

[26] An alternative notion of weak strict monotonicity could refer to a cone $L^1(\mathcal{P})_X = \{Y \in L^1(\mathcal{P})_+ : P(Y > 0) > 0 \, \forall P \in M(X)\}$ depending on the effective priors $M(X)$ at $X \in L^1(\mathcal{P})_+$, defined below. In this situation strict utility improving consumption at X refers to the intermediate cone $L^1(\mathcal{P})_X$, heavily depending on X. Another alternative is the local monotonicity concept in Nutz and Soner (2012).

[27] In their setting the domain of the preference relation is the space of simple acts.

As we show in Lemma 1, this set is proportional to the superdifferential of the utility. Such priors minimize the variational utility at C. We assume that variational preferences are defined on a weakly closed set of priors for the uncertainty model $\mathcal{P}$, i.e. $\mathrm{dom}(\mathfrak{c}) = \{P \in \mathcal{P}_S : \mathfrak{c}(P) < \infty\} \subseteq \mathcal{P}$. The case $\mathrm{dom}(\mathfrak{c}) = \{P\}$ corresponds to an expected utility under $P \in \mathcal{P}$. The following lemma gives standard properties of the variational utility functional.

Lemma 1 *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be monotone, strictly concave, continuous and differentiable on $\mathbb{R}_{++}$. Let $\mathfrak{c} : \mathcal{P}_S \to [0, \infty]$ be grounded*[28]*, convex, and weakly lower semi-continuous. Define the utility functional $U : L^1(\mathcal{P})_+ \to \mathbb{R}$ by (1).*
Then we have, $U : L^1(\mathcal{P})_+ \to \mathbb{R}$ is

1. *monotone and semi-strictly monotone if u is strictly monotone.*
2. *concave, not strictly concave on $L^1(\mathcal{P})_+$ and strictly concave on $\mathrm{M}[\mathcal{P}]$.*
3. *$c_{1,\mathcal{P}}$-upper semi-continuous. If the penalty term is linear on $\mathrm{dom}(\mathfrak{c})$, then $U : L^1(\mathcal{P})_+ \to \mathbb{R}$ is $c_{1,\mathcal{P}}$-continuous. In this case, the penalty term is given by $\mathfrak{c}(P) = {}_{L^1(\mathcal{P})}\langle \phi, P\rangle_{L^1(\mathcal{P})^*} = \mathrm{E}^P[\phi]$, for some $\phi \in L^1(\mathcal{P})$.*
4. *The superdifferential of $U : L^1(\mathcal{P})_+ \to \mathbb{R}$ at C is given by $\partial U(C) = \{\mu : \mathcal{F} \to \mathbb{R} : \mathrm{d}\mu = u'(C)\mathrm{d}P,\ P \in M(C)\}$.*

The lemma is an extension to the case of infinite (uncountable) states, see Rigotti and Shannon (2012) for the finite state case. This indicates that the present commodity price duality is tractable. For finite valued measurable functions, the explicit formula of the superdifferential is proven in Maccheroni, Marinacci, and Rustichini (2006). In the following we present examples to illustrate the usefulness and flexibility of variational preferences. The first example refers to the classical maxmin preferences found in Gilboa and Schmeidler (1989). The second example refers to anchored preferences axiomatized by Faro (2009).

Example 2 1. Maxmin Preferences: *An agent with maxmin preferences is modeled by the following criterion*

$$U(C) = -\mathbb{E}^{\mathcal{P}}[-u(C)] = \min_{P \in \mathcal{P}} \mathrm{E}^P[u(C)] + \delta_{\mathcal{P}}(P),$$

where $\mathfrak{c} = \delta_{\mathcal{P}} : \mathcal{P}_S \to [0, \infty]$ is the convex indicator function of $\mathrm{dom}(\mathfrak{c}) = \mathcal{P}$.

2. Anchored maxmin preferences: *Fix an initial endowment $E \in L^1(\mathcal{P})_+$. We can define the following anchored preference representation*

$$U(C) = \min_{P \in \mathrm{dom}(\mathfrak{c})} \mathrm{E}^P[u(C) - u(E)], \quad \mathfrak{c}(P) = \begin{cases} \mathrm{E}^P[u(E)] & \text{if } P \in \mathrm{dom}(\mathfrak{c}) \\ +\infty, & \text{otherwise.} \end{cases}$$

[28]This means that its infimum value is zero.

Here the penalty term is linear, with $\phi = -u(E)$.[29]

In multiple prior models based on a reference measure, i.e. all other priors are absolutely continuous, it is possible to consider the relative entropy in terms of $\mathfrak{c}$. The usage is limited and does not apply to mutually singular priors directly, as the following example illustrates.

Example 3 λ-Relative Entropy: *In this example, we concentrate on the cone* $\tilde{L}^1(\mathcal{P})_+$. *In case of a penalty term associated to the relative entropy we need some preparation for the construction. We introduce some synthetic probability measure. As discussed in Bion-Nadal and Kervarec (2012), there is a canonical equivalence class* $R(c_1, \mathcal{P}) = \mathcal{P}^{can}$ *of probability measures. This class is based on a countable dense subset* $\{P^n\}_{n\in\mathbb{N}}$ *of* $\mathcal{P}$. *Taking a sequence of real number* $\{\lambda_n\}_{n\in\mathbb{N}}$ *with* $\lambda_n > 0$ *such that* $\sum \lambda_n = 1$ *the resulting probability measure* $P^\lambda = \sum \lambda_n P^n$ *is in* $\mathcal{P}^{can}$. *Let a reference measure* P_λ *be fixed. The* λ*-relative entropy* $R^\lambda(\cdot\|\mathcal{P}) : \mathcal{P}_S \to [0, \infty]$ *is defined by* $R^\lambda(P\|\mathcal{P}) = \int_\Omega \log\left(\frac{\mathrm{d}P}{\mathrm{d}P_\lambda}\right)\mathrm{d}P_\lambda$. *Priors closer to the dominating prior* P_λ *have a larger influence on the utility evaluation. Finally we can write the utility functional as*

$$U_{\restriction \tilde{L}^1(\mathcal{P})_+}(C) = \min_{P\in\mathcal{P}} \mathrm{E}^P[u(C)] + \mathfrak{c}(P) = \min_{P\in\mathcal{P}} \mathrm{E}^P[u(C)] + \theta R^\lambda(P\|\mathcal{P}),$$

where $\theta \in \mathbb{R}$ *is an intensity parameter. Note that this relative entropy formulation heavily depends on the parameter* λ *for the synthetic measure* P^λ.

We close the section with a discussion on semi-strict monotonicity.

Remark 2 1. *Strict monotonicity is usually defined by* $U(X+Y) > U(X)$ *for some* $Y \in L^1(\mathcal{P})_+ \setminus \{0\}$. *The singularities of the priors do not allow for such a notion of monotonicity. We illustrate this issue for maxmin preferences in Example 2.1. Let* $Y \in L^1(\mathcal{P})_+$ *has only one effective prior* $P^Y \in \mathcal{P}$. *Then we may have*

$$U(Y) = \min_{P\in\mathcal{P}} \mathrm{E}^P[u(Y)] = \mathrm{E}^P[u(Y)] = \mathrm{E}^P[u(0)] = U(0), \text{ where } P \neq P^Y.$$

This means, the commodity bundle Y *may not be strictly desirable in comparison to zero consumption. From this point of view variational preferences seem to be rather consistent with semi-strictly preferences. Another argument refers to the representation property of the topological dual. As pointed out in Lemma 1, the utility gradients can be decomposed, with the strict positivity of the density part satisfied for semi-strict monotone and concave variational preferences (see also Corollary 1). In this situation, semi-strict positive functionals are compatible with the utility gradients. The equilibrium results in Section 4 are strongly connected to this issue.*

[29]In Dana and Riedel (2013) such anchored preferences are related to Bewley preferences, see Bewley (2002). However, in their discrete time framework the uncertainty is modeled by a set of mutually equivalent priors.

2. *The* $c_{1,\mathcal{P}}$*-continuity of the utility functional is a desirable property. By the same argument as for the classical Lebesgue space, related to some probability space, we have an empty interior of the positive cone* $L^1(\mathcal{P})_+$. *Monotone and concave variational utility defined on the whole space are* $c_{1,\mathcal{P}}$*-continuous. This follows for instance from an application of the extended Namioka-Klee Theorem in Biagini and Frittelli (2010).*

2.4 Equilibria and Implementation

This Section is divided into three parts. In a preliminary step we introduce the martingale theory of the considered conditional sublinear expectation. Then we establish the existence of an equilibrium allocation and discuss some new structural properties. In the last step we achieve the implementation of the equilibrium allocation into a Radner economy.

2.4.1 A Detour: Spanning and Martingales

In order to establish a Radner implementation, we introduce a new sublinear expectation, generated via the supremum of the adjusted priors. Due to the present uncertainty model $\mathcal{P}$, a new well-behaved conditioning principle is needed. We roll out the dynamics of security markets by introducing the concept of conditional sublinear expectations. The involved implementation via a security market accounts for such well-behaved priors.
We proceed similarly to the single prior case, where the Radner implementation in continuous time is based on a certain classical martingale representation property. In the present situation, the multiple prior model enforces a conditional sublinear expectation which spawns an elaborated martingale representation theorem. As we indicate at the end of this subsection an effect on the space of unambiguous claims is apparent. A possible replication of the claim via the security market provoke the appearance of incomplete markets.

Structure of Priors
For the purpose of a martingale representation theorem under a conditional sublinear expectation we need the following notion of stability under pasting for $\mathcal{P}$, also called fork-convexity. In essence, this property refers to a rational updating principle. Before, we define the set of priors with a time depending restriction on the related sub σ-field

$$\mathcal{P}(t,P)^o = \{P' \in \mathcal{P} : P = P' \text{ on } \mathcal{F}_t^o\}, \text{ with } t \in [0,T] \text{ and } P \in \mathcal{P}.$$

This set of priors consists of all extensions of $P : \mathcal{F}_t^o \to [0,1]$ from $\mathcal{F}_t^o$ to $\mathcal{F}$ in $\mathcal{P}$. More precisely, this is the set of all probability measures in $\mathcal{P}$ defined on $\mathcal{F}$ that agree with P in the events up to time t.

Assumption 2 *The set of priors* $\mathcal{P}$ *is* stable under pasting*, i.e. for every* $P \in \mathcal{P}$*, every* $\mathbb{F}^o$*-stopping time* τ*,* $B \in \mathcal{F}_\tau^o$ *and* $P_1, P_2 \in \mathcal{P}(\mathcal{F}_\tau^o, P)$*,* $\mathcal{P}$ *contains* *again the prior P_τ given by*

$$P_\tau(A) = \mathrm{E}^P\big[P_1(A|\mathcal{F}^o_\tau)1_B + P_2(A|\mathcal{F}^o_\tau)1_{B^c}\big], \quad A \in \mathcal{F}^o_\tau.$$

Note, that we use the raw filtration $\mathbb{F}^o$. The stability under pasting property is closely related to dynamic consistency or rectangularity of Epstein and Schneider (2003). However in the present volatility uncertainty setting these notions are not equivalent.[30] For details we refer to Section 3 in Nutz and Soner (2012). For instance, the set of priors which defines a G-expectation, illustrated in Example 1, satisfies automatically Assumption 2.

Information Structure
The usual conditions of a rich σ-field at time 0 is widely used in Mathematical Finance.[31] But the economic meaning is questionable. In our uncertainty model of mutually singular priors we can augment, similarly to the classical case, the right continuous filtration given by $\mathbb{F}^+ = \{\mathcal{F}^+_t\}_{t\in[0,T]}$ where $\mathcal{F}^+_t = \bigcap_{s>t}\mathcal{F}^o_t$ for $t \in [0,T)$. The second step is to augment the minimal right continuous filtration $\mathbb{F}^+$ by all polar sets of $(\mathcal{P},\mathcal{F}^o_T)$, i.e. $\mathcal{F}_t = \mathcal{F}^+_t \vee \mathcal{N}(\mathcal{P},\mathcal{F}^o_T)$, see Appendix A.1 for details. This augmentation is strictly smaller than the universal enlargement procedure.[32] Note that the augmentation does not affect the commodity space of equivalence classes, whose elements are $\mathcal{P}$-q.s. indistinguishable. Additionally we have $\mathcal{B}(\Omega) = \mathcal{F}^o_T$ and $\mathcal{F}^o_T = \mathcal{F}_T$ $\mathcal{P}$-q.s.
This choice is economically reasonable, because the initial σ-field does not contain all 0-1 limit events, see Section 4.1 in Nutz and Soner (2012). In nearly all continuous-time Finance models, such a rich initial σ-field is assumed. This implies a rich knowledge of every decision maker about events in the long run. In Huang (1985) one can find a detailed discussion of information structures for asset prices and trading strategies, when the uncertainty is given by a probability space.

Conditional sublinear expectation
We introduce the dynamics and the different notions of martingales of our uncertainty model $(\Omega,\mathcal{F},\mathcal{P})$. The so called strong formulation of uncertainty in Assumption 1 guarantees the existence of a martingale concept which allow for a martingale representation. The efficient use of information is often formalized by the concept of conditional expectation. Implicitly, this depends on the uncertainty structure and the given filtration. Due to the pasting property of $\mathcal{P}$ we have a *universal* conditional expectation being under every prior almost surely equal to the essential supremum of relevant conditional expectations. This concept is formulated in the following definition.

Definition 2 *A set of priors $\mathcal{P}$ has the* aggregation property in $L^1(\mathcal{P})$ *if for all $X \in L^1(\mathcal{P})$ and $t \in [0,T]$, there exists an $\mathcal{F}_t$-measurable random variable*

[30] Lemma 8 in Appendix B of Riedel (2009) shows the equivalence of these concepts, when the priors are mutually equivalent.

[31] One reason may be, that in this case the full stochastic calculus is applicable.

[32] This means $\mathcal{F}_t \subsetneqq \bigcap_{P\in\mathcal{P}}\sigma(\mathcal{F}^+,\mathcal{N}(P,\mathcal{F}^o_t))$, for $t \in [0,T]$.

$\mathbb{E}_t^{\mathcal{P}}[X] \in L^1(\mathcal{P})$ *such that*

$$\mathbb{E}_t^{\mathcal{P}}[X] = {}^{P}\operatorname*{ess\,sup}_{P' \in \mathcal{P}(t,P)} \mathrm{E}^{P'}[X|\mathcal{F}_t] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P}.$$

Note that in the definition the random variable is defined in the quasi sure sense. The linear conditional expectation under a probability space has strong connections to a positive linear projection operator. In the presence of multiple priors, the conditional updating in an ambiguous environment involves a sublinear projection $\mathbb{E}_t^{\mathcal{P}} : L^1(\mathcal{P}) \to L_t^1(\mathcal{P})$, where $L_t^1(\mathcal{P}) \subset L^1(\mathcal{P})$ denotes the closed subspace of $\mathcal{F}_t$ measurable random variables. In this regard the aggregation property just states that we can find a well-defined sequence of conditional expectations satisfying a rational updating principle. The weak compactness and stability under pasting allows for such a conditional sublinear expectation.

Lemma 2 *Under Assumption 1 and 2, $\mathcal{P}$ satisfies the aggregation property. Moreover, we have $\mathbb{E}_s^{\mathcal{P}} \circ \mathbb{E}_t^{\mathcal{P}} = \mathbb{E}_s^{\mathcal{P}}$, $s \leq t$.*

Without a well-behaved conditional expectation, the introduction of a martingale or its representation seems unreproducible.[33] Now, we introduce martingales under the conditional expectation $\mathbb{E}_t^{\mathcal{P}}$. Fix a random variable $X \in L^1(\mathcal{P})$. The sublinearity of the dynamic conditional expectation defines a martingale similarly to the single prior setting,[34] as being its own estimator.

Definition 3 *An $\mathbb{F}$-adapted process $(X_t)_{t\in[0,T]}$ is a $\mathcal{P}$-*martingale *if*

$$X_s = \mathbb{E}_s^{\mathcal{P}}[X_t] \quad \mathcal{P}\text{-q.s.} \quad \text{for all } s \leq t.$$

We call X a symmetric $\mathcal{P}$-martingale *if X and $-X$ are both $\mathcal{P}$-martingales.*

The nonlinearity of the conditional expectation implies that if a process (X_t) is a martingale under $\left(\mathbb{E}_t^{\mathcal{P}}\right)_{t\in[0,T]}$, then $-X$ is not necessarily a martingale.[35] As we will discuss in detail, a fair game refers to the symmetric martingale property. In this case, the process is equivalently a P-martingale under every $P \in \mathcal{P}$. In subsection 4.3 we discuss the relationship to asset prices under the sublinear expectation generated by $\mathcal{P}$.
In a dynamic trading setting, it is essential if a contingent claim $X \in L^1(\mathcal{P})$

[33]Without the weak compactness of $\mathcal{P}$, a construction of random variables in the quasi sure sense involves more technical difficulties. However, in this situation one can take the separability condition of Soner, Touzi, and Zhang (2012b), see also Example 4.14 therein. An aggregation result, in the sense of Definition 2, can then be observed with the so called Hahn property of Cohen (2011). Here the definition of an ess sup in the quasi sure sense approaches the aggregation property.

[34]For the multiple prior case with equivalent priors we refer to Riedel (2009).

[35]Representations of non symmetric martingales can be formulated via a so called second order backward stochastic differential equation (2BSDE). This concept is introduced in Cheridito, Soner, Touzi, and Victoir (2007) and developed further in Soner, Touzi, and Zhang (2012a).

can be represented in terms of a stochastic integral. As mentioned in the Introduction this corresponds to the mean unambiguity property, introduced in Section 3.1. For the replication of a claim, the following result is central. It can be seen as a generalized martingale representation theorem, when the uncertainty is given by the present mutually singular uncertainty model, see Nutz and Soner (2012) for a proof.

Martingale Representation: *Under Assumption 1 and 2, we have for every $X \in L^1(\mathcal{P})$ a unique pair (θ, K)[36] of $\mathbb{F}$-predictable processes with*

1. *θ such that $\int_0^T |\theta_s|^2 \mathrm{d}\langle B\rangle_s < \infty$ $\mathcal{P}$-q.s.,*
2. *K such that all paths of (K_t) are càdlàg, nondecreasing and $K_T \in L^1(\mathcal{P})$,*

such that $$\mathbb{E}_t^{\mathcal{P}}[X] = \mathbb{E}_0^{\mathcal{P}}[X] + \int_0^t \theta_s \mathrm{d}B_s - K_t \quad \text{for all } t \in [0,T], \quad \mathcal{P}\text{-q.s.}$$

The positive and increasing process K in the representation is new and can be understood as a correction term. The sublinear conditional expectation allows for biased martingales, i.e. we only have $\mathrm{E}^P[-K_T^X] = 0$ if and only if $P \in \operatorname{argmax}_{P\in\mathcal{P}} \mathrm{E}^P[X]$. Here, K^X is the output of the martingale representation theorem applied with respect to $X \in L^1(\mathcal{P})$.

Remark 3 1. *Already at this stage, the interplay between the existence and the structure of a competitive equilibrium and absence of arbitrage opportunities are at work. As illustrated in Vorbrink (2010) in the G-framework (see Example 1) absence of weak arbitrage (see Table 1) does not imply $\mathrm{E}^P[-K_T] = 0$ for every $P \in \mathcal{P}$. Note that this arbitrage notion is consistent with strictly monotone preferences, stated in Table 1 and refers to a robust approach to finance.*
If an exchange economy is in equilibrium, net trades should not admit for arbitrage. But, by Proposition 2 the equilibrium price system perceives only P^*-a.s. events, since the representing measure μ of the equilibrium price system can be decomposed by $\mathrm{d}\mu = \psi \mathrm{d}P^*$. The value of net trades $\xi \in L^1(\mathcal{P})$ should not differ under such equilibrium priors P^*. Therefore, the case $P'(K_T^\xi \neq 0) > 0$ must refer to a non-equilibrium prior P', see Example 5 for an application of this issue.
2. *In the G-framework the compensation part can be written more explicit, when X is contained in a (uncertain) subset of $L^1(\mathcal{P})$:*

$$K_t = \int_0^t \eta_r \mathrm{d}\langle B^G\rangle_r - \int_0^t G(\eta_r)\mathrm{d}r, \quad t \in [0,T],$$

where B^G is the so called G-Brownian motion[37] *and η is an endogenous output of the martingale representation, so that K is now a function of η.*

[36] The pair is unique up to $\{\mathrm{d}s \times P, P \in \mathcal{P}\}$-polar sets. More precisely, the process K is an aggregated object under the Continuum Hypothesis, see Remark 4.17 of Nutz and Soner (2012) and paragraph 8 and 9 of Chapter 0 in Dellacherie and Meyer (1978).

[37] As already mentioned in Example 1, a G expectation can be induced by some volatility bounds. Here, the function G is given by $\eta \mapsto G(\eta) = \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \sigma \cdot |\eta|$.

Example 5 and 6 make use of this fact.
As such it is an open problem, if every $X \in L^1(\mathcal{P})$ can be represented in this complete form. We refer to Peng, Song, and Zhang (2013) for the latest discussion, on the complete representation property.

The following corollary gives an alternative representation and a justification of unambiguous random variables. It illustrates which random variables have the replication property in terms of a stochastic integral. The space of feasible integrands $\Theta(B)$ is given below in Subsection 4.3.

Corollary 2 *The marketed space* $\mathrm{M}[\mathcal{P}]$ *of unambiguous contingent claims is a* $c_{1,\mathcal{P}}$*-closed subspace of* $L^1(\mathcal{P})$. *More precisely, we have*

$$\mathrm{M}[\mathcal{P}] = \left\{ \xi \in L^1(\mathcal{P}) : \ \xi = \mathbb{E}^{\mathcal{P}}[\xi] + \int_0^T \theta_s \mathrm{d}B_s \text{ for some } \theta \in \Theta(B) \right\}.$$

Furthermore, the stochastic integral has continuous paths $\mathcal{P}$*-q.s.*

The notion of perfect replication is associated to the situation when $K \equiv 0$. Exactly at this step the martingale representation comes into play. This space of random variables is strongly related to symmetric martingales. More precisely, elements in $\mathrm{M}[\mathcal{P}]$ generate symmetric martingales, via the successive application of the conditional sublinear expectation along the augmented filtration $\mathbb{F}$. In the lights of Corollary 2, the analogy between unambiguous events and unambiguous random variables becomes apparent.[38] Note that this analogy is already used and indicated in Beißner (2012), where a notion of ambiguity and risk neutral valuation is considered.

2.4.2 Existence of Arrow-Debreu Equilibrium

With the discussion of the primitives in Section 3, we introduce now the heterogeneous agent economy, consisting of a finite set of individuals $\mathbb{I} = \{1, \ldots, I\}$ consuming at time $t = 0$ and $t = T$. The economy is given by $\mathcal{E}(e, E) = \big(\mathbb{R}_+ \times L^1(\mathcal{P})_+, \{V_i, (e_i, E_i)\}_{i \in \mathbb{I}}\big)$, where the initial endowment of agent i satisfies $(e_i, E_i) \in \mathbb{R}_+ \times L^1(\mathcal{P})_+$. Her utility is given by $V_i : \mathbb{R}_+ \times L^1(\mathcal{P})_+ \to \mathbb{R}$ such that $(c, C) \mapsto u_i^0(c) + U_i(C)$. The utility $U_i : L^1(\mathcal{P})_+ \to \mathbb{R}$ is a variational functional, introduced in Section 3.2, with utility functions $u_i^0, u_i^T : \mathbb{R}_+ \to \mathbb{R}$ and ambiguity indexes $\mathfrak{c}_i$,[39] we have

$$V_i(c, C) = u_i^0(c) + \min_{P \in \mathcal{P}} \mathrm{E}^P[u_i^T(C)] + \mathfrak{c}_i(P). \tag{2}$$

[38] Under one prior or a set of mutually equivalent priors, indicator functions are elements of the related Lebesgue space. In our setting this is not necessarily true, since $\omega \mapsto 1_A(\omega)$, for some $A \in \mathcal{F}$ is not continuous.

[39] We assume that the agents in the economy share the same set of priors, but they do not agree via their ambiguity index. A simple generalization could be a heterogeneity in the ambiguity via a modification of equivalent priors with a bounded density.

The aggregate endowment of the economy is denoted by $(e, E) \in \mathbb{R}_+ \times L^1(\mathcal{P})_+$. Note that we allow for a heterogeneity in the sets of priors. This can be achieved via different domains of the penalty terms $\mathfrak{c}_i$, see also the last part of Assumption 3.

Efficient Allocations and Sharing Rules

We describe the optimal allocation of resources by the following problem. A weighting $\alpha \in \Delta_I$, where $\Delta_I = \{\alpha \in \mathbb{R}^I_+ : \sum \alpha_i = 1\}$ denotes the I-dimensional simplex, induces a representative utility $V_\alpha(c, C) := \sum \alpha_i V_i(c_i, C_i)$. An allocation $(\bar{c}, \bar{C}) = ((c_1, C_1) \dots, (c_I, C_I))$ is α-efficient if the functional $V_\alpha : (\mathbb{R}_+ \times L^1(\mathcal{P})_+)^I \to \mathbb{R}$ achieves the maximum over the set of allocations $\Lambda(e, E) = \big\{(c, C) \in (\mathbb{R}_+ \times L^1(\mathcal{P})_+)^I : \sum (c_i, C_i) \leq (e, E)\ \mathcal{P}\text{-q.s.}\big\}$.
Under concavity of the utility functionals, α-efficiency for some $\alpha \in \Delta_I$ is equivalent to Pareto optimality, while this is related to an equilibrium with transfer payment. As a first step we establish the existence of α-efficient allocations.

Theorem 1 *Suppose $V_i : \mathbb{R}_+ \times L^1(\mathcal{P})_+ \to \mathbb{R}$, $i \in \mathbb{I}$, are utility functionals given by (2) with a concave utility index, then there exists an α-efficient allocation. If each $\mathfrak{c}_i$ is linear, the solution correspondence*

$$\mathbb{C}(\alpha, e, E) = \arg\max_{(x,X) \in \Lambda(e,E)} \sum_{i \in \mathbb{I}} \alpha_i V_i(x_i, X_i)$$

is nonempty, convex and weakly compact valued. Moreover, if for each $(t, i) \in \{0, T\} \times \mathbb{I}$, u^t_i is twice continuously differentiable, i.e. $u^t_i \in C^{2,1}(\mathbb{R}_+; \mathbb{R})$, there is a continuous selection $(c, C) \in \mathbb{C}$, such that $\alpha \mapsto C_i(\alpha, E)$ is continuously differentiable on $\mathring{\Delta}_I$. In particular, we have a

$$\mu \in \bigcap_{i \in \mathbb{I}} \alpha_i \partial U_i\left(C_i(\alpha, E)\right) \neq \emptyset, \quad \text{where } \mathrm{d}\mu = \alpha_i u^{T\prime}_i\left(C_i(\alpha, E)\right) \mathrm{d}P. \tag{3}$$

The result is interesting in its own right, but will play as well a central role in the approach to the existence of an (analytic) equilibrium. From the theorem we immediately infer that there is a fully insured efficient allocation, when the aggregate endowment is certain, i.e $E(\omega)$ is constant $\mathcal{P}$-quasi surely.
If the aggregate endowment is uncertain but unambiguous, i.e. $E \in \mathrm{M}[\mathcal{P}]$, structural properties of optimal allocations depend additionally on preferences. The following example illustrates how Pareto sharing rules determine the insurance properties and the resulting net trades.

Example 4 *Let the uncertainty model be that of Example 1 and the aggregate endowment of the economy be unambiguous, i.e. $E \in \mathrm{M}[\mathcal{P}]$ and by Corollary 2, we have $E = E_T = \mathbb{E}^{\mathcal{P}}[E] + \int_0^T \theta^E_t \mathrm{d}B^G_t$, for some adapted and integrable process θ^E. Note, that the individual endowment is still allowed to be ambiguous. Now suppose for each $i \in \mathbb{I}$ that the functional form of optimal* *consumption* $C_i(\alpha, \cdot) \in C^{2,1}(\mathbb{R}_+)$ *is twice continuously differentiable and not linear, which holds if each* $u_i \in C^{3,1}(\mathbb{R}_+)$ *has a nonlinear risk tolerance, for details see Hara, Huang, and Kuzmics (2007). This implies* $C_i''(\alpha, \cdot) \neq 0$ *and we derive for each* $i \in I$ *by the G-Itô formula*[40]

$$C_i(\alpha, E_T) = C_i(\alpha, \mathbb{E}^{\mathcal{P}}[E]) + \int_0^T C_i'(\alpha, E_t)\theta_t^E \mathrm{d}B_t^G + \frac{1}{2}\int_0^T C_i''(\alpha, E_t)\left(\theta_t^E\right)^2 \mathrm{d}\langle B^G\rangle_t.$$

Due to the nonzero $\mathrm{d}\langle B^G\rangle$*-part, we have* $C_i(\alpha, E) \notin \mathrm{M}[\mathcal{P}]$ *by Corollary 2. This means that the Pareto optimal allocation is ambiguous. In case of linear risk tolerance, i.e.* $C_i''(\alpha, \cdot) = 0$*, the same computation imply an unambiguous Pareto optimal allocation. From this we infer that the absence of idiosyncratic ambiguity does not always leads to unambiguous efficient allocations.*
Concerning the net trades $\xi_i = C_i(\alpha, E) - E_i$*, we have, unless the "pathological" case that the* $\mathrm{d}\langle B^G\rangle$*-part of* E_i *eliminates the* $\mathrm{d}\langle B^G\rangle$*-part of* $C_i(\alpha, E)$*, ambiguous net trades, meaning that* $\xi_i \notin \mathrm{M}[\mathcal{P}]$*.*

In the case of linear risk tolerance a sufficient condition for unambiguous net trades is $E_i \in \mathrm{M}[\mathcal{P}]$*, for each agent* $i \in \mathbb{I}$*.*

Comparing this example with De Castro and Chateauneuf (2011), we see that an unambiguous aggregate endowment is not sufficient to observe an unambiguous Pareto optimal allocation. The missing gap relies on the structure of the sharing rule. Note that the arguments in the present setting are based on results from stochastic analysis under G-expectation.

The General Equilibrium

Now we introduce the classical notion of an Arrow-Debreu equilibrium. Note that, the feasibility holds $\mathcal{P}$-quasi surely and for the price functional we require $c_{1,\mathcal{P}}$-continuity as discussed in Section 2.1. By Proposition 1, $L^1(\mathcal{P})$ is a Banach lattice, hence positive and linear functionals on $L^1(\mathcal{P})$ are automatically $c_{1,\mathcal{P}}$-continuous.
The $I+1$-tuple $((\bar{c}_1, \bar{C}_1), \ldots, (\bar{c}_I, \bar{C}_I); (\pi, \Pi)) \in (\mathbb{R}_+ \times L^1(\mathcal{P})_+)^I \times (\mathbb{R} \times L^1(\mathcal{P})^*)$ consisting of a feasible allocation and a continuous linear price functional, is called a *contingent Arrow-Debreu equilibrium*, if

1. For all i, $(\bar{c}_i, \bar{C}_i)$ maximizes V_i on $\mathbb{R}_+ \times L^1(\mathcal{P})_+$ under $\Psi(c - e_i, C - E_i) \leq 0$.
2. The allocation $(\bar{c}, \bar{C})$ is feasible: $\sum_{i \in \mathbb{I}}(\bar{c}_i, \bar{C}_i) = (e, E)$, $\mathcal{P}$-q.s.

Next, we reconsider the utility gradient of the agent when she faces a maximization problem in terms of a first order condition. As in the single prior setting, the excess utility map encodes the "universal system of equations" of the defined equilibrium. In matters of the utility maximization, the particular form of the gradient causes a modification in the definition of the excess utility map, see Appendix A.2 for the details of the construction

[40] The result can be found in Peng (2010).

method. In general, the gradient is an element of the topological dual. The representation of the dual space, see Proposition 2 and Lemma 1, implies $DU_i(C) \in \partial U_i(C) \subset L^1(\mathcal{P})^*$, where a supergradient can be represented by $DU_i(C)(h) = \mathrm{E}^P[u_i'(C)h]$, for some $P \in M_i(C)$ and direction $h \in L^1(\mathcal{P})$.

Remark 4 *In infinite dimensional commodity spaces, the positive cone may have an empty interior. In this situation, a properness condition is needed to establish the existence of an equilibrium. Note that by Proposition 1, $L^1(\mathcal{P})$ is an order continuous Banach lattice. As we aim to establish an equilibrium allocation with an explicit dependency of the effective priors, we only mention this whole branch of abstract existence result. We refer to Martins-da Rocha and Riedel (2010) and the references therein.*

In order to connect the gradient with the price system, in terms of Theorem 1 and the second fundamental theorem of welfare economics, we have to make an assumption on the integrability of $u'(E)$.

Assumption 3 *Let the aggregate endowment $E \in L^1(\mathcal{P})_+$ be strictly positive $\mathcal{P}$-q.s. and let $e = \sum e_i > 0$. We assume*[41]

$$\max_{\alpha \in \Delta_I} \left({u_\alpha^0}'(e) + {u_\alpha^T}'(E) \right) \in L^\infty(\mathcal{P}) \quad \text{and} \quad \bigcap_{i \in \mathbb{I}} \mathrm{dom}(\mathfrak{c}_i) \neq \emptyset.$$

This assumption is closely related to a cone condition, which is important for the existence of an equilibrium in infinite dimensional commodity spaces, see also Remark 2.2 in Dana (1993). Moreover it guarantees that the price system is an element of the semi-strict order dual $L^1(\mathcal{P})^*_\oplus$, see Subsection 3.1 for details. The proof of the following theorem is based on the gross substitute property of the modified excess utility map $\Phi : \Delta_I \times \mathcal{P} \to \mathbb{R}^I$, see Definition 5 in Appendix A.2. In order to guarantee this property we have to make the following well-known assumption.

Assumption 4 *For each $(i,t) \in \mathbb{I} \times \{0,T\}$, $x \mapsto x \cdot {u_i^t}'(x)$ is non-decreasing.*

The assumption is equivalent to the Arrow-Pratt measure of relative risk-aversion being less or equal than one, when ${u_i^t}'$ is twice differentiable. We are ready to state the first main result of the paper.

Theorem 2 *Suppose each agent satisfies the conditions of Lemma 1, with strictly concave and strictly monotone utility index and a linear penalty term $\mathfrak{c}_i$. Under Assumption 1-4 there is a Pareto optimal Arrow-Debreu equilibrium $(c_1^*, C_1^*, \ldots, c_I^*, C_I^*; (\pi, \Pi))$, with $\Pi \in L^1(\mathcal{P})^*_\oplus$.*

[41] Fix $t \in \{0,T\}$ and $\alpha \in \Delta_I$, $u_\alpha^t : \mathbb{R}_{++} \to \mathbb{R}$ is given by $u_\alpha^t(e) = \max_{x \in \Lambda(e,0)} \sum \alpha_i u_i^t(x_i)$. Here $L^\infty(\mathcal{P})$ is the closure of $\mathcal{C}_b$ under the norm $c_{\infty,\mathcal{P}}(X) = \inf\{M \geq 0 : |X| \leq M, \mathcal{P}-q.s.\}$. See again Denis, Hu, and Peng (2011) for more details.

The Pareto optimal equilibrium allocation is based on an α^*-efficient weighting $\alpha^* \in \Delta_I$, so that we denote the set of *equilibrium priors* by

$$\mathcal{P}_E \subset \mathbb{P}(\alpha^*) \subset \mathcal{P}.$$

This set of common unadjusted priors $\mathbb{P}(\alpha^*)$ is constructed in Appendix A.2, see also Subsection 2.1 for an illustration of the construction idea. One important property is that the representative agent behaves as an agent with variational utility. In the following, we illustrate in the sense in which α-efficient allocations are uniquely specified. Namely, under every equilibrium prior $P \in \mathcal{P}_E$ an equilibrium allocation is determined P-a.s. To illustrate this point in more detail, we define a different allocation resulting in the same utility. As the following example illustrates, that the reasoning is consistent with the finite-dimensional example in Section 2.1, where the Leontief-type utility of the agents created a similar degree of freedom, as illustrated in Figure 2 (b) therein.

Example 5 *Consider an economy with two agents $i = 1, 2$ under the uncertainty model of Example 1 and Remark 3.2. Utilities are given by*

$$U_1(C) = \min_{P\in\mathcal{P}} \mathrm{E}^P[\ln(C)] = -E_G\left[\ln(C)\right] \quad \text{and } U_2(C) = \min_{P\in\mathcal{P}} \mathrm{E}^P[C^{1/2}].$$

The endowment of each agent is a function of the G-Brownian motion at time T, i.e. $E_i = \varphi_i(B_T^G) \in L^1(\mathcal{P})_+$, where $\varphi_i : \mathbb{R} \to \mathbb{R}_+$ is assumed to be convex, so that $\varphi_i = \exp$ is in principle a possible choice. Moreover, let $\varphi = \varphi_1 + \varphi_2$ so that the aggregate endowment can be written as a function of the G-Brownian motion B^G, i.e $E = \varphi(B_T^G)$. After some computation an equilibrium consumption allocation $C_i(\alpha, E) = \Upsilon_i^\alpha(B_T^G)$ is given by

$$\Upsilon_1^\alpha(B_T^G) = \frac{2\cdot\varphi(B_T^G)}{1+\sqrt{1+\varphi(B_T^G)\bar{\alpha}^2}}, \quad \Upsilon_2^\alpha(B_T^G) = \left(\frac{\bar{\alpha}\cdot\varphi(B_T^G)}{1+\sqrt{1+\varphi(B_T^G)\bar{\alpha}^2}}\right)^2,$$

where $\alpha = \alpha_1$, $1-\alpha = \alpha_2$ and $\bar{\alpha} = \frac{\alpha}{1-\alpha}$. Since $\Upsilon_1^\alpha(B_T^G) + \Upsilon_2^\alpha(B_T^G) = \varphi(B_T^G)$ holds $\mathcal{P}$-q.s., this results into a feasible allocation. Since $\Upsilon_i^\alpha = C_i(\alpha,\cdot)\circ\varphi$ and $C_2(\alpha,\cdot)$ is convex and increasing, we have that Υ_α^2 is convex as well. In order to observe the effective prior, note that $u_2(C_2(\alpha,E)) = u_2(\Upsilon_2^\alpha(B_T^G))$ is concave, which implies $M_2(C_2(\alpha,E)) = \{P^{\overline{\sigma}}\} = \mathcal{P}_E$ by the following computation:
We discuss the optimal allocation via tools from stochastic analysis under the G-expectation. Suppose that each optimal consumption has the complete representation property of Remark 3.2,[42] we can write

$$\Upsilon_2^\alpha(B_T^G) = E_G\left[\Upsilon_\alpha^2(B_T^G)\right] + \int_0^T \theta_t^2 \mathrm{d}B_t^G - \int_0^T G\left(\eta_t^2\right)\mathrm{d}t + \frac{1}{2}\int_0^T \eta_t^2 \mathrm{d}\langle B^G\rangle_t, \quad (4)$$

[42] A sufficient condition is the boundedness of $\partial_x\Upsilon_i^\alpha(x)$ on $\mathbb{R}_+$.

where $\theta_t^2 = f_x^2(t, B_t^G)$, $\eta_t^2 = f_{xx}^2(t, B_t^G)$ *and* $f^2(T, B_T^G) = \Upsilon_\alpha^2(B_T^G) = C_2(\alpha, E)$. *As illustrated in the first part of the example* Υ_α^2 *is convex and by Section 1 in Chapter II of Peng (2010) it follows that* $f^2(t,\cdot) : \mathbb{R} \to \mathbb{R}_+$ *is convex for each* $t \in [0,T]$. *Hence* $f_{xx}^2 \geq 0$ *and we deduce that the last two terms of (4) can be written as*

$$\begin{aligned} -K_T^2 &= -\int_0^T G\left(f_{xx}^2(t, B_t^G)\right) \mathrm{d}t + \frac{1}{2}\int_0^T f_{xx}^2(t, B_t^G)\mathrm{d}\langle B^G\rangle_t \\ &= -\frac{1}{2}\int_0^T \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \sigma f_{xx}^2(t, B_t^G) + \hat{a}_t f_{xx}^2(t, B_t^G)\mathrm{d}t \\ &= \frac{1}{2}\int_0^T \left(\hat{a}_t - \overline{\sigma}\right) f_{xx}^2(t, B_t^G)\mathrm{d}t, \end{aligned}$$

The martingale representation theorem also tells us, that the process $(-K_t^2)$ *is a G-martingale. Moreover, we have* $-K_t^2 \equiv 0$ $P^{\overline{\sigma}}$*-a.s. and* $-K_t^2 \neq 0$ *under every other prior in* $\mathcal{P} \setminus \{P^{\overline{\sigma}}\}$.
With this observation, we construct a different allocation having the same utility. Consider $\overline{\eta} = \varepsilon \cdot \eta^2$, $\varepsilon \in (0,1)$. *We show that the allocation* $(C_1(\alpha, E) + \varepsilon K_T^2, C_2(\alpha, E) - \varepsilon K_T^2)$ *is also* α*-efficient and satisfies* $C_1(\alpha, E) \neq \bar{C}_1 := C_1(\alpha, E) + \varepsilon K_T^2$ *P-a.s. for every* $P \in \mathcal{P} \setminus \{P^{\overline{\sigma}}\}$.
Since, $\hat{a}_t \in [\underline{\sigma}, \overline{\sigma}]$, *it follows that* $K_t^2 \geq 0$ $\mathcal{P}$*-q.s. Hence, by the monotonicity of the utility functional, this reallocation does not worsen the utility of agent 1, i.e.* $U_1(C_1(\alpha, E) + \varepsilon K_T^2) \geq U_1(C_1(\alpha, E)$.
For agent 2, the positive homogeneity of G implies $G(\overline{\eta}) = \varepsilon G(\eta)$. *From this we see that* $P^{\overline{\sigma}}$ *is still the only effective prior with respect to* $\bar{C}_1$, *since* $\frac{1+\varepsilon}{2}\int_0^T (\overline{\sigma} - \hat{a}_t)\, \eta_t^2 dt = -(1+\varepsilon)K_T^2 = -\bar{K}_T^2$ *yields*

$$\bar{C}_2 = E_G\left[\Upsilon_\alpha^2(B_T^G)\right] + \int_0^T Z_t^2 dB_t^G - (1+\varepsilon)K_T^2.$$

Specifically, under $P^{\overline{\sigma}}$ *the compensation term satisfies* $\bar{K}^2 \equiv 0$ *and hence the utility of agent 2 remains unaffected, i.e.* $U^2(C_2(\alpha, E)) = U^2(\bar{C}^2)$. *Note that, for* ε *sufficiently small, we have* $P^{\overline{\sigma}} \in M(\bar{C}_2)$, *since* $M(C_2(\alpha, E)) = \{P^{\overline{\sigma}}\}$.
Finally, we state the semi-strictly positive equilibrium price system given by $X \mapsto \Pi(X) = \mathrm{E}^{P^{\overline{\sigma}}}[u'_{\alpha^*}(E) \cdot X]$, *where the effective prior is induced by*

$$\arg\min_{P \in \mathcal{P}} \mathrm{E}^P[u_{\alpha^*}(E)] = \{P^{\overline{\sigma}}\} = \mathbb{P}(\alpha^*) = \mathcal{P}_E.$$

Ambiguity aversion creates the worst case prior $P^{\overline{\sigma}}$ *and the density part in terms of risk attitudes is given by*

$$u'_\alpha(E) = \frac{\alpha}{\varphi(B_T^G)}\left(1 + \sqrt{1 + \varphi(B_T^G)\bar{\alpha}^2}\right).$$

Summing up, we have illustrated how the new martingale representation theorem can be applied to construct many different efficient allocations and analyze their structural properties.

Note that the convexity of $\Upsilon^2_\alpha(\cdot)$ induces the unique effective prior $P^{\overline{\sigma}}$, which can be seen as an extreme case. Different effective priors corresponding to more complex volatility specifications depend in general on the structure of the efficient sharing rules, see again Example 4 for the most simplest case.

2.4.3 The Existence of Incomplete Security Markets

With the dynamics of the uncertainty model of Section 4.1, we are now in the position to formulate trading processes and the Radner equilibrium. Before, we introduce an assumption for the space of consumption profiles at time T. This gives us a certain invariance on the space of net trades.

Assumption 5 *The density part of the equilibrium utility gradients at time T, $u^{T'}_\alpha(E)$ is bounded away from zero, i.e.*

$$\varepsilon < \max_{\alpha \in \Delta_I} u^{T'}_\alpha(E) \quad \mathcal{P}\text{-}q.s., \quad \textit{for some } \varepsilon > 0.$$

This assumption is satisfied when the aggregate endowment is bounded away from zero and the utility functions u_i satisfy the Inada condition at zero. It guarantees the boundedness above and below away from zero of our state price density $\psi = u^{T}_{\alpha^*}{}'(E)$, where α^* is the equilibrium weight of Theorem 2 in Subsection 4.2. It follows that $L^1(\mathcal{Q}) = L^1(\mathcal{P})$, where

$$\mathcal{Q} = \left\{ Q : \mathrm{d}Q = \frac{\psi}{\mathrm{E}^P[\psi]}\mathrm{d}P, \text{ for some } P \in \mathcal{P} \right\}.$$

This invariance is of importance, since the density ψ is derived from the equilibrium and is *not* a primitive.[43] Based on the set of unadjusted equilibrium priors $\mathcal{P}_E$, we denote by $\mathcal{Q}_E = \left\{ Q : \mathrm{d}Q = \frac{\psi}{\mathrm{E}^P[\psi]}\mathrm{d}P, P \in \mathcal{P}_E \right\}$ the set of equilibrium pricing measures.

Now, we introduce a Radner equilibrium of prices, plans and price expectation related to the present mutually singular prior model. The price process $S = (S_t)_{t \in [0,T]}$ for our long lived security is a $\mathcal{P}$-semimartingale[44] on the filtered sublinear expectation space $(\Omega, L^1(\mathcal{P}), \mathbb{E}^{\mathcal{P}}, \mathbb{F})$.

As we have seen in Example 4, we observe unambiguous net trades when strong assumptions on endowments and utilities are imposed. By the martingale representation theorem under $\mathcal{Q}$ applied with respect to $\mathcal{Q}$-unambiguous net trades, i.e. $C_i - E_i \notin \mathrm{M}[\mathcal{Q}]$, a disposal term K^i_T appears. To account for this useless consumption, where the equilibrium price system is zero, we allow for a gain process with a feasible possibility of free disposal of wealth under non equilibrium priors. This is achieved in terms of a security with possibly negative dividend under some $Q \in \mathcal{Q}$. We consider the set of admissible

[43] In Section 3 of Duffie and Huang (1985) a similar assumption can be found.

[44] As defined in Pham and Zhang (2012), the uncertain process S is a $\mathcal{P}$-semimartingale if it is a P-semimartingale for every $P \in \mathcal{P}$. Note that their Assumption 4.1 is in the present setting fulfilled, since we augment the filtration with the $\mathcal{P}$-polar sets.

trading processes already mentioned in Corollary 2, with certain regularity conditions:

1. *Well defined*: θ is $\mathbb{F}$-predictable and $\mathbb{E}^{\mathcal{P}}\left[\int_0^T \theta_t^2 \mathrm{d}\langle S\rangle_t\right] < \infty$.[45]

2. *Gain process*: $\int \theta \mathrm{d}S$ is a $\mathcal{P}$-q.s. well defined stochastic integral.

3. *Self-financing*: The trading process satisfies the accounting identity

$$X_t^\theta = \theta_t S_t = \theta_0 S_0 + \int_0^t \theta_r \mathrm{d}S_r, \quad \mathcal{P}\text{-q.s. for every } t \in [0,T].$$

The space of processes θ satisfying conditions 1.-3. is denoted by $\Theta(S)$. From Corollary 2 it follows directly that the gain process must be a symmetric martingale in order to establish a perfect hedge. If we consider the volatility uncertainty as a robustness constraint, the corollary just characterizes a perfect hedging portfolio with some initial value if and only if the terminal payoff is unambiguous. We come now to the formal definition of a Radner equilibrium under volatility uncertainty.

Definition 4 *A* Radner equilibrium *for $\mathcal{E}(e,E)$ is comprised of $N+1$ long lived security claiming $D = (D^0,\dots,D^N) \in L^1(\mathcal{P})^{N+1}$, with price process $S = (S^0,\dots,S^N)$, a set of trading strategies $\theta^i \in \Theta(S)$, $i \in \mathbb{I}$ and a price $\pi > 0$ for consumption at time zero, which satisfies:*

For each agent $i \in \mathbb{I}$, the consumption $\left(e_i - X_0^{\theta^i}\pi^{-1}, E_i + X_T^{\theta^i}\right)$ maximizes $V_i : \mathbb{R}_+ \times L^1(\mathcal{P})_+ \to \mathbb{R}$ on the budget set

$$B\big(e_i, E_i, \pi, D, S\big) = \left\{\left(e_i - X_0^\theta \pi^{-1}, E_i + X_T^\theta\right) \in \mathbb{R}_+ \times L^1(\mathcal{P})_+ : \theta \in \Theta(S)\right\},$$

so that the market clears, $\sum_{i\in\mathbb{I}} \theta_t^i = 0$ $\mathcal{P}$-q.s., for every $t \in [0,T]$.

A priori, the functional capability of the financial market as a mechanism is reflected by the marketed space $\mathrm{M}[\mathcal{Q}]$. In comparison to the single prior case, market completeness is not an intrinsic property in terms of the simpler all-encompassing martingale representation.

Theorem 3 *Suppose the security-spot economy*

$$\mathcal{E}(e,E) = \left\{\big(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P}\big), D, \left\{V_i, \mathbb{R}_+ \times L^1(\mathcal{P}), (e_i, E_i)\right\}_{i\in\mathbb{I}}\right\}$$

satisfies Assumptions 1 to 5, then there is a Radner equilibrium for $\mathcal{E}(e,E)$, $(\{(\theta^i)\}_{i\in\mathbb{I}}, \pi, D, S)$, which implements the given Arrow-Debreu Equilibrium $\big(\{c_i, C_i\}_{i\in\mathbb{I}}; \mathrm{E}^P[\psi\cdot]\big)$ if and only if

$$\mathbb{E}^{\mathcal{P}}[\psi(C_i - E_i)] = \mathrm{E}^P[\psi(C_i - E_i)], \quad \textit{for every } i \in \mathbb{I}. \tag{5}$$

In this case, we have:

[45] The bracket process is given by $\langle B\rangle = B^2 - \int B\mathrm{d}B$, where the stochastic integral is defined pathwise, see Soner, Touzi, and Zhang (2012b) and the references therein.

1. *The financial market is effectively dynamically complete.*
2. *Trade may be ambiguous, i.e.* $(C_i - E_i) \notin \mathrm{M}[\mathcal{Q}]$.
3. *Under the equilibrium prior, each* (c_i, C_i) *is perfectly hedged.*

The condition in (5) states a relation about endogenous objects. More precisely, (5) can be understood as the existence of a worst case prior Q as an element of $\mathcal{Q}_E$. At the same time, Q must be a maximizing prior with respect to the uncertainty-adjusted sublinear expectation $\mathbb{E}^{\mathcal{Q}}$ evaluated at each net trade $\xi_i = C_i - E_i$, i.e.

$$Q \in \mathcal{Q}(\xi_i) = \arg\max_{Q \in \mathcal{Q}} \mathrm{E}^Q[\xi_i].$$

Note that if the net trades are unambiguous then this condition is automatically satisfied, see Example 4 and 5.

In the presence of volatility uncertainty, the proof of Theorem 3 can be regarded as a canonically extension of Duffie and Huang (1985). In their example with Brownian Noise only two long lived security price processes are required to admit a complete Radner equilibrium. This follows from the two summands in the (Brownian) martingale representation. The present volatility uncertainty setup requires a third component in the martingale representation. This is a compensation part of disposal under non maximizing priors $\mathcal{Q}(\xi_i)$.[46] For this reason, we observe a martingale multiplicity of *three*. But Theorem 3 also tells us, that in contrast to the single prior case, the implementation of efficient Arrow-Debreu equilbria into a Radner equilibrium is not always possible. The Pareto efficiency of the Radner equilibrium is quite surprising, since multiple period incomplete markets are typically only constrained efficient. Nevertheless, the efficiency still depends on which equilibrium allocation is considered.

Under a non-equilibrium prior $P \in \mathcal{P} \setminus \mathcal{P}_E$ the consumption profiles are for some agents superhedged. However, under the priors

$$P' \in M_i(C_i) \cap \left\{ P \in \mathcal{P} : \mathrm{d}P = \psi^{-1} \mathrm{d}Q, Q \in \mathcal{Q}(\xi_i) \right\},$$

the hedge is still perfect, i.e. $K^{P',i} = 0$ $P' \otimes \mathrm{d}t$-a.e. Under such priors, the deflated gain process becomes a martingale and under every other effective prior only a supermartingale. This is still consistent with the "no expected gain from trade" hypothesis of Duffie (1986). The following example illustrates under the G-framework, how the Radner equilibrium incorporates with the new component in the martingale representation theorem. The dynamics of the price process of the new security, obtained from Lemma 3 in Appendix A.2, get a more explicit specification. Again, the price process depends heavily on the net trades of the Arrow-Debreu equilibrium.

[46] Alternatively to this particular and novel security, we could introduce a family of securities being contingent on the prior. However, such a prior-dependent contingency would stand in opposition to the present quasi-sure analysis in aggregated terms.

Example 6 *Apart from the condition in Theorem 3, suppose we are in the G-framework and every net trade $\xi_i = C_i - E_i$, $i \in \mathbb{I}$, has the complete martingale representation property, see Remark 3.2. Denoting by $\hat{a}_t = \frac{\mathrm{d}}{\mathrm{d}t}\langle B^G\rangle_t$ the time derivative of the quadratic variation of the G-Brownian motion, we have*

$$\begin{aligned} K_t^i &= \int_0^t G(\eta_t^i)\mathrm{d}t - \frac{1}{2}\int_0^t \eta_t^i \mathrm{d}\langle B^G\rangle_t \\ &= \int_0^t \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \sigma\eta_t^i \mathrm{d}t - \int_0^t \eta_t^i \hat{a}_t \mathrm{d}t = \frac{1}{2}\int_0^t \sigma_t^i\eta_t^i\mathrm{d}t - \frac{1}{2}\int_0^t \eta_t^i\hat{a}_t\mathrm{d}t \\ &= \frac{1}{2}\int_0^t (\sigma_t^i - \hat{a}_t)\cdot\eta_t^i\mathrm{d}t, \end{aligned}$$

for some η^i, as an output of the complete martingale representation. In the derivation, each process σ^i corresponds to a $P^{\sigma^i} \in \mathcal{P}$ with $P^{\sigma^i} \sim Q_i \in \mathcal{Q}$. Similarly to the proof of Theorem 3, let us consider the following dividend structure

$$D^0 \equiv 1, \quad D^1 = B_T^G, \quad D^2 = \int_0^T -\frac{1}{2}\sum_{i\in\mathbb{I}}\left(\sigma_t^i - \hat{a}_t\right)\mathrm{d}t.$$

The price process $S_t^2 = \mathbb{E}_t^{\mathcal{Q}}[D^2]$ for the asset with dividend D^2 and the portfolio processes of each agent are $\mathbb{E}^{\mathcal{Q}}$-martingales, i.e. a Q-supermatingale under every prior $Q \in \mathcal{Q}$ and a Q-martingale under some $Q \in \mathcal{Q}$ corresponding to the no gain from trade hypothesis. Moreover, the price process is absolutely continuous with respect to $\mathrm{d}t$ *and given by*

$$\mathrm{d}S_t^2 = -\frac{1}{2}\sum_{i\in\mathbb{I}}\left(\sigma_t^i - \hat{a}_t\right)\mathrm{d}t, \quad S_0^2 = 0.$$

For the strategy of this bounded variation security S^2, consider the following partition of unity $\sum_{i\in\mathbb{I}}\kappa_t^i = 1$ $\mathcal{Q}$-q.e. so that

$$\theta_t^{i,2} = \begin{cases} \kappa_t^i\eta_t^i & \text{if } \sum_{i\in\mathbb{I}}\sigma_t^i - \hat{a}_t \neq 0 \\ 0 & \text{else} \end{cases} \qquad \kappa_t^i = \frac{(\sigma_t^i - \hat{a}_t)}{\sum_{i\in\mathbb{I}}\sigma_t^i - \hat{a}_t}, \quad i \in \mathbb{I}\setminus\{I\}.$$

Note that in the uncertainty neutral world $\mathcal{Q}$ the asset price S^2, so that $K^i = \int \theta^{i,2}\mathrm{d}S^2$, may become negative under some non equilibrium measures $\mathcal{Q}\setminus\{Q\}$, where $\mathrm{d}Q = \psi\mathrm{d}P$ and $P \in \mathcal{P}_E$ satisfies (5).[47] In essence this depends on the net trade and the equilibrium expectation. Remember, the compensation (or disposal) part in the martingale representation prefigures this possibility. This can be understood as the interplay between the $\mathcal{P}$-q.s. clearing condition and the disposal parts of the net trades. For instance, this phenomenon is not present, when net trades are unambiguous and induces portfolio processes being symmetric $\mathcal{Q}$-martingales without a disposal part.

[47] In the situation of Example 5 this is possible, since $\sigma^i \equiv \overline{\sigma} > \hat{a}$. However, the same is true for S_1. In each case, a splitting of the positive and negative parts would guarantee positive price processes.

Implementation of Equilibria with Sublinear Price Systems

As we have seen in Theorem 3, an implementation of the Arrow-Debreu equilibrium is not always possible. This can be interpreted as a partially negative result. In essence, this is caused by inconsistencies between the linear price structure and the nonlinear expectation in the martingale representation theorem. In order to illustrate this tension, we show that equilibria with sublinear price systems are at least for implementation reasons more adequate.

Definition 5 *The* $I+1$*-tuple* $(C_1,\dots,C_I;\Psi)\in L^1(\mathcal{P})^I_+\times L^1(\mathcal{P})^{\circledast}_+$[48] *is a* sublinear price equilibrium, *if the following holds*

1. *We have* $U_i(Y)>U_i(C_i)$ *implies* $\Psi(C_i)<\Psi(Y)$, *for all* $i\in\mathbb{I}$ *and* $Y\in L^1(\mathcal{P})_+$.
2. *The allocation is feasible:* $\sum_{i\in\mathbb{I}}C_i=E,\quad \mathcal{P}$*-q.s.*
3. *For all* $i\in\mathbb{I}$, $\Psi(C_i)=\Psi(E_i)$.

Theorem 4 *Suppose there is a sublinear price equilibrium allocation* $\{C_i\}_{i\in\mathbb{I}}$ *and Assumptions 1 to 3 and 5 are satisfied. Then there is a Radner equilibrium* $(\{(\theta^i\}_{i\in\mathbb{I}},D,S)$ *which implements the sublinear price equilibrium allocation* $\{C_i\}_{i\in\mathbb{I}}$.

Note that the Radner equilibrium in the present situation does not allow for consumption at time $t=0$, the budget set is now given by $B(E_i,D,S)=\{E_i+X^\theta_T\in L^1(\mathcal{P})_+:\theta\in\Theta(S)\}$.

Remark 5 *An alternative and reasonable condition could be* $3'$. $\Psi(C_i-E_i)=0$ *for each agent. In a two agent economy, the net trades become unambiguous. Hence the implementation has no compensation part* (K_t) *to consider. Another alternative refers to the no arbitrage condition* $\Psi\left(\sum_{i\in\mathbb{I}}E_i\right)=\sum_{i\in\mathbb{I}}\Psi(C_i)$, *introduced in Aliprantis, Tourky, and Yannelis (2000). In this context see also footnote 26.*

2.5 Appendix A

The first part of the appendix collects the proofs of Section 2. First we review some convergence results used, and which are relevant especially to the proof for Lemma 1.

Convergence properties of sublinear expectations, Denis, Hu, and Peng (2011):

[48] The sub-order dual space $L^1(\mathcal{P})^{\circledast}$ of $L^1(\mathcal{P})$ consists of certain sublinear functionals on $L^1(\mathcal{P})$, see Beißner (2012) for details or Section 2.2. of Chapter 3. Here, we consider the case $\Gamma(\mathcal{P})=\mathcal{P}$. While linear prices correspond to $\Gamma(\mathcal{P})=\{P\}$ for some $P\in\mathcal{P}_E$. An alternative approach could lead to $\Gamma(\mathcal{P})=\mathcal{P}_E$, which induces the endogenous equivalent symmetric martingale measure set $\mathcal{Q}_E$.

1. Let $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$ converges weakly to $P \in \mathcal{P}$. Then, for each $X \in L^1(\mathcal{P})$, we have $\mathrm{E}^{P_n}[X] \to \mathrm{E}^{P}[X]$.

2. Let $\mathcal{P}$ be weakly compact and let $\{X_n\}_{n\in\mathbb{N}} \subset L^1(\mathcal{P})$ be such that $X_n \searrow X$, then $\mathbb{E}^{\mathcal{P}}[X_n] \searrow \mathbb{E}^{\mathcal{P}}[X]$.

3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathcal{P})$ which converges to X in $L^1(\mathcal{P})$. Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ which converges to X quasi-surely in the sense that it converges to X outside a $\mathcal{P}$-polar set.

We say a sequence $(X_n)_{n\in\mathbb{N}}$ *converges in capacity* to X if for each $\varepsilon > 0$ we have $\sup_{P\in\mathcal{P}} P(|X_n - X| > \varepsilon)$ convergeing to zero.

Hierarchy of convergence, Cohen, Ji, and Peng (2011): Quasi sure convergence implies convergence in capacity.

Dominated convergence for sublinear expectation, Xu (2010): Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathcal{P})$ such that $|X_n| \leq Y \in L^1(\mathcal{P})$, for each $n \in \mathbb{N}$. If $X_n \to X$ in capacity, then $\lim_{n\to\infty} \mathbb{E}^{\mathcal{P}}[X_n] = \mathbb{E}^{\mathcal{P}}[X]$.

In our multiple prior setting quasi sure convergence does not imply convergence in capacity, see the Appendix of Xu (2010) for an example. In this case, the limit X is not necessarily an element of $L^1(\mathcal{P})$.

2.5.1 A 1: Details and Proofs of Section 3

As mentioned in Subsection 2.1, two random variables $X, Y \in L^1(\mathcal{P})$ can be distinguished if there is a prior in $P \in \mathcal{P}$ such that $P(X \neq Y) > 0$. Such null elements are characterized by random variables which are $\mathcal{P}$-polar. $\mathcal{P}$-polar sets which are evaluated under every prior are zero or one, although, the value may differ between different priors. A property holds *quasi-surely* (q.s.) if it holds outside a polar set. Furthermore, the space $L^1(\mathcal{P})$ is characterized in Denis, Hu, and Peng (2011) via

$$L^1(\mathcal{P}) = \left\{ X \in L(\Omega) : X \text{ has a q.c. version, } \lim_{n\to\infty} \mathbb{E}^{\mathcal{P}}\left[|X| 1_{\{|X|>n\}}\right] = 0 \right\}. \quad (6)$$

A mapping $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous (q.c.) if for all $\varepsilon > 0$ there exists an open set O with $c(O) = \sup_{P\in\mathcal{P}} P(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. We say that $X : \Omega \to \mathbb{R}$ has a *q.c. version* if there exists a quasi–continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s.

Proof of Proposition 1 *We show* $\inf(X, Y) = X \wedge Y \in L^1(\mathcal{P})$ *for every* $X, Y \in L^1(\mathcal{P})$ *via the representation in (6). Since* $\{|X| > n\} \supset \{|X \wedge Y| > n\}$, *we have by the sublinearity of* $\mathbb{E}^{\mathcal{P}}$

$$\mathbb{E}^{\mathcal{P}}\left[|X \wedge Y| 1_{\{|X\wedge Y|>n\}}\right] \leq \mathbb{E}^{\mathcal{P}}\left[|X| 1_{\{|X|>n\}}\right] + \mathbb{E}^{\mathcal{P}}\left[|Y| 1_{\{|Y|>n\}}\right] \xrightarrow[n\to\infty]{} 0.$$

Since X *and* Y *have a q.c. version, there are* $\bar{\varepsilon}, \varepsilon_X, \varepsilon_Y > 0$ *such that* $\varepsilon_X + \varepsilon_Y < \bar{\varepsilon}$ *with* $c(O_X) < \varepsilon_X$, $c(O_Y) < \varepsilon_Y$ *and hence* $c(O_X \cup O_Y) \leq c(O_X) + c(O_Y) < \bar{\varepsilon}$.

Because $X|_{(O_X \cup O_Y)^c}$ *and* $Y|_{(O_X \cup O_Y)^c}$ *are both continuous, the quasi-continuity of* $X \wedge Y$ *follows. The order relation is indeed a lattice operation.*
That $L^1(\mathcal{P})$ *is a Banach space is shown in Denis, Hu, and Peng (2011).* $L^1(\mathcal{P})$ *is a Banach lattice, since for all* $X, Y \in L^1(\mathcal{P})$ *with* $|X| \leq |Y|$*, i.e.* $|X| \leq |Y|$ *P-a.s. for all* $P \in \mathcal{P}$ *imply*

$$c_{1,\mathcal{P}}(X) = \max_{P \in \mathcal{P}} \mathrm{E}^P[|X|] = \mathrm{E}^{P'}[|X|] \leq \mathrm{E}^{P'}[|Y|] \leq c_{1,\mathcal{P}}(Y),$$

for some maximizing prior P' *for* $c_{1,\mathcal{P}}(X)$*. Fix a sequence of positive random variables* (X_n) *in* $L^1(\mathcal{P})$ *such that* $X_n \searrow 0$ *in* $L^1(\mathcal{P})$*. Hence* X_1 *dominates the sequence and an application of the dominated convergence under sublinear expectation gives us*

$$\lim_{n\to\infty} c_{1,\mathcal{P}}(X_n) = \lim_{n\to\infty} \mathbb{E}^{\mathcal{P}}[|X_n|] = \mathbb{E}^{\mathcal{P}}[|\lim_{n\to\infty} X_n|] = 0.$$

Hence, $L^1(\mathcal{P})$ *is an order continuous Banach lattice.* ■

Proof of Proposition 2 *In our construction, the underlying sublinear expectation space is given by* $(\Omega, \mathcal{C}_b(\Omega), \mathbb{E}^{\mathcal{P}})$*, as given by Theorems 25 and 52 in Denis, Hu, and Peng (2011)* $L^1(\mathcal{P}) = L^1_G(\Omega)$*. Since* Ω *is a polish space and* $\mathcal{P}$ *is a weakly compact by Assumption 1,* $c_{1,\mathcal{P}}$ *is a Prokhorov capacity.*
If $l : \tilde{L}^1(\mathcal{P}) \to \mathbb{R}$ *is a non-negative linear functional, then there is a non-negative measure* μ *with support* Ω *such that*

$$l(X) = \int X \mathrm{d}\mu, \quad \textit{for every } X \in L^1(\mathcal{P}),$$

This is shown in Proposition 11 of Feyel and de La Pradelle (1989). In Theorem 6 by Feyel and De La Pradelle (1977), it is shown that every continuous linear functional is the difference of two non-negative linear functionals.
$\tilde{L}^1(\mathcal{P})$ *is given by the space of* $c_{1,\mathcal{P}}$*-equivalence classes of* $\overline{\mathcal{C}_b(\Omega)}^{c_{1,\mathcal{P}}}$*, so that the domain is modified via the so called Lebesgue prolongation. The explicit representation of the* $c_{1,\mathcal{P}}$*-topological dual of* $\tilde{L}^1(\mathcal{P})$*, can be found in the first chapter of Kervarec (2008), Theorem I.30.* ■

Proof of Corollary 1 *"$\Rightarrow$": By Proposition 2, we have* $l(X) = \mathrm{E}^P[\psi X]$*, since* $\psi > 0$ *P-a.s and* $P(X > 0) > 0$*, therefore* $l(X) > 0$ *follows.*
"$\Leftarrow$": $l \in L^1(\mathcal{P})^*$ *implies again by Proposition 2 that we can write* $l(X) = \mathrm{E}^P[\psi X]$*. Suppose* $\psi \notin L^\infty(P)_{++}$ *then* $P(\psi > 0, X > 0) = 0$ *for some* $P \in \mathcal{P}$ *is possible. We have a contradiction.* ■

Proof of Lemma 1 *1. Monotonicity follows directly from the monotonicity of the utility index u. Let* $P_X \in M(X) \subset \mathcal{P}$ *be a minimizing prior of* $U(X)$*. Semi-strict monotonicity follows from* $u'(X) > 0$ *on a set with a positive measure with respect to* P_X *and*

$$U(X+Z) - U(X) \geq \mathrm{E}^{P_X}[u(X+Z) - u(X)] > \mathrm{E}^{P_X}[u'(X+Z) \cdot Z] \geq 0,$$

where the strict inequality follows from strict concavity of u and $P(Z > 0) > 0$ for every $P \in \mathcal{P}$.

2. The mapping $C \mapsto \mathrm{E}^P[u(C)] + \mathfrak{c}(P)$ is concave for each $P \in \mathcal{P}$ and the inf *operation preserves concavity. We prove the strict concavity on $\mathrm{M}[\mathcal{P}]$. Let $\alpha \in (0,1)$ and $C, X \in L^1(\mathcal{P})_+ \cap \mathrm{M}[\mathcal{P}]$, with $C \neq X$. We compute*

$$\begin{aligned}\alpha U(C) + (1-\alpha)U(X) &\leq \min_{P \in \mathcal{P}} \mathrm{E}^P[\alpha u(C) + (1-\alpha)u(X)] + \mathfrak{c}(P)\\ &< \min_{P \in \mathcal{P}} \mathrm{E}^P[u(\alpha C + (1-\alpha)X)] + \mathfrak{c}(P)\\ &= U(\alpha C + (1-\alpha)X),\end{aligned}$$

where the first inequality follows from the concavity of $C \mapsto U(C)$.
The functional U is not strictly concave on its whole domain, since $C \neq X$ in $L^1(\mathcal{P})$ does not imply $C \neq X$ under every $P \in \mathcal{P}$, hence one can easily pick two elements C and X which are P-a.s. equal, where $P \in M(\lambda C + (1-\lambda)X)$ and deduce a contradiction, when following the proof of concavity.

3. Let the sequence $(X_n)_{n\in\mathbb{N}} \subset L^1(\mathcal{P})_+$ converges to X in $L^1(\mathcal{P})$. In order to prove the assertion, we show that every subsequence $(Y_{n_k})_{k\in\mathbb{N}}$ of (X_n) has in turn a subsequence $(Z_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} U(Z_n) \leq U(X).$$

Let $P^X \in M(X)$ be a minimizing prior and $(Y_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(X_n)_{n\in\mathbb{N}}$. There is a subsequence $(Z_n)_{n\in\mathbb{N}}$ in $(Y_{n_k})_{k\in\mathbb{N}}$ and some $Z \in L^1_+(\mathcal{P})$ satisfying

$$Z_n(\omega) \to X(\omega) \text{ and } 0 \leq Z_n(\omega) \leq Z(\omega) \quad \text{for } P^X\text{-a.s.}$$

We may take $Z = X + \sum_{n\in\mathbb{N}} |Z_{n+1} - Z_n|$, with $c_{1,\mathcal{P}}(Z_{n+1} - Z_n) \leq 2^{-n}$. Monotonicity of u implies $0 \leq u(Z_n(\omega)) \leq u(Z(\omega))$ and $u(Z_n(\omega)) \to u(X(\omega))$ for P^X almost every $\omega \in \Omega$.
So, by the lim sup*-version of Fatou's lemma under P^X we deduce*

$$\begin{aligned}\limsup_{n\to\infty} U(Z_n) &\leq \limsup_{n\to\infty} \mathrm{E}^{P^X}[u(Z_n)] + \mathfrak{c}(P^X)\\ &\leq \mathrm{E}^{P^X}[u(X)] + \mathfrak{c}(P^X) = U(X).\end{aligned}$$

We prove the norm continuity of U, when $\mathfrak{c}$ is linear. To show $U(X_n) \to U(X)$ for some norm convergent sequence $(X_n)_{n\in\mathbb{N}}$, it suffices again to show that every subsequence of $(X_n)_{n\in\mathbb{N}}$ has a subsequence $(Z_n)_{n\in\mathbb{N}}$ with $U(Z_n) \to U(X)$.
Let $(X_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(X_n)_{n\in\mathbb{N}}$. We have $X_{n_k} \to X$ in $c_{1,\mathcal{P}}$. There is a subsequence $(Z_n)_{n\in\mathbb{N}}$ of $(X_{n_k})_{k\in\mathbb{N}}$ and a $Z \in L^1(\mathcal{P})$ with $0 \leq Z_n \leq Z$ $\mathcal{P}$-q.s. and $Z_n \to X$ $\mathcal{P}$-q.s., which implies convergence in capacity, (see the beginning of Appendix A). We may take Z as before. By the monotonicity and continuity of u, we have $0 \leq u(Z_n) \leq u(Z)$ $\mathcal{P}$-q.s. and $u(Z_n) \to u(X)$

$\mathcal{P}$-q.s. An application of dominated convergence under sublinear expectation, as shown in the beginning of Appendix A, gives us

$$\begin{aligned} \lim_{n\to\infty} U(Z_n) &= -\mathbb{E}^{\mathcal{P}}\left[\phi - u\left(\lim_{n\to\infty} Z_n\right)\right] \\ &= -\mathbb{E}^{\mathcal{P}}[\phi - u(X)] = \min_{P\in\mathcal{P}} \mathrm{E}^P[u(X)] - \langle P, \phi\rangle = U(X). \end{aligned}$$

This implies the $c_{1,\mathcal{P}}$-continuity of the utility functional. Note that the lower semi-continuity and linearity of the penalty term implies continuity, hence we can find a $\phi \in L^1(\mathcal{P})$, to give a representation in terms of a bilinear form.

4. That $\mathcal{P}$ is also $\sigma(L^1(\mathcal{P}), L^1(\mathcal{P})^*)$-weakly compact follows by the same arguments as in the proof of Proposition 2.4 in Bion-Nadal and Kervarec (2012), since it is a closed subset of the nonnegative part of the unit ball of $L^1(\mathcal{P})^*$. Effective priors exist, since $P \mapsto \mathrm{E}^P[X]$ is weakly continuous for every $X \in L^1(\mathcal{P})$, and build a convex weakly compact subset of $\mathcal{P}$.
Let $P^* \in M(C)$ be an effective prior for C and let $X \in L^1(\mathcal{P})_+$ be arbitrary. By the concavity and differentiability of the utiltiy index u, this implies

$$\begin{aligned} U(X) - U(C) &= \min_{P\in\mathcal{P}} \mathrm{E}^P[u(X)] + \mathfrak{c}(P) - \min_{P\in\mathcal{P}} \mathrm{E}^P[u(C)] + \mathfrak{c}(P) \\ &\leq \mathrm{E}^{P^*}[u(X)] + \mathfrak{c}(P^*) - \mathrm{E}^{P^*}[u(C)] + \mathfrak{c}(P^*) \\ &\leq \mathrm{E}^{P^*}[u'(C)(X - C)]. \end{aligned}$$

The characterization of the superdifferential follows from the fact that $\partial U(X) \subset L^1(\mathcal{P})^*$, Proposition 2 and Corollary 2 of Theorem 2.8.2 in Clarke (1990). ■

2.5.2 A 2: Details and Proofs of Section 4

Proof of Lemma 2 *By Assumption 1 the set of priors is convex, weakly compact and stable under pasting. The semigroup property of the conditional expectations can be found in Proposition 3.6 (i) of Nutz and Soner (2012). Alternatively, when the set $\mathcal{D}$ is given by an explicit correspondence process, one can also apply Theorem 2.6 form Epstein and Ji (2013b).* ■

Proof of Corollary 2 *The alternative representation is an application of the martingale representation theorem in Section 4.2. It can be easily verified that $\mathrm{M}[\mathcal{P}]$ is a closed subspace of $L^1(\mathcal{P})$. Unambiguous random variables can be identified as the terminal value of the stochastic integral, which is the image of a linear operator, with preimage $\Theta(B)$.* ■

In order to concentrate on the essential difficulties of the proofs in Subsection 4.2, we do not consider consumption and endowments at time 0, until the proof of Theorem 3. The product structure of the consumption profiles and the additive utility imply that proofs within the two period economy are slight generalizations. To do so we identify $\Lambda(0, E)$ with $\Lambda(E) = \left\{C \in L^1(\mathcal{P})^I_+ : \sum C_i \leq E\right\}$.

Proof of Theorem 1 *The functional* $U_\alpha : \Lambda(E) \to \mathbb{R}$ *is weakly upper semi-continuous, by Lemma 1. By Proposition 1,* $L^1(\mathcal{P})$ *is a Banach lattice with order continuous norm. This implies that the order interval* $[[0, E]] = \{x \in L^1(\mathcal{P}) : 0 \le x \le E\}$ *is* $\sigma(L^1(\mathcal{P}), L^1(\mathcal{P})^*)$*-compact, this result can be found in Theorem 2.3.8 of Aliprantis, Brown, and Burkinshaw (1990) and Section 2 in Yannelis (1991). Hence,* $\Lambda(E)$ *is* $\sigma(L^1(\mathcal{P})^I, (L^1(\mathcal{P})^I)^*)$*-compact, as a closed subset of* $[[0, E]]^I$ *under the same topology. The Weierstrass Theorem (Theorem 2.43 in Aliprantis and Border (2006)) implies the existence of a maximizer.*
The upper hemicontinuity of $\mathbb{C}(\alpha, \cdot) : \Delta_I \times L^1(\mathcal{P})_+ \rightrightarrows L^1(\mathcal{P})^I_+$ *follows from Berge Maximum theorem, where each* U^i *is continuous. We prove the existence of a continuously differentiable selection. The well defined mapping* $C(\alpha, e) : \Delta_I \times \mathbb{R}_+ \to \mathbb{R}^I_+$ *is the unique solution of the pointwise problem*

$$C(\alpha, e) = \operatorname*{argmax}_{x_i \ge 0, \sum x_i \le e} \sum \alpha_i u_i(x_i), \quad (\alpha, e) \in \Delta_I \times \mathbb{R}_+,$$

which is continuously differentiable on $\mathring{\Delta}_I \times \mathbb{R}_{++}$*, the interior of* $\operatorname{dom}(C)$*. For every* $\alpha \in \Delta_I$ *there is a* $P \in \mathcal{P}$ *such that the modified economy with* $\operatorname{dom}(\tilde{\mathfrak{c}}_i) = \{P\}$, $i \in \mathbb{I}$*, satisfies the same first order condition as in the original economy*

$$\mu \in L^1(\mathcal{P})^*, \quad \mathrm{d}\mu = u'_\alpha(E)\mathrm{d}P = \alpha_i u'_i\left(C_i(\alpha, E)\right)\mathrm{d}P,$$

for every $i \in \mathbb{I}$ *such that* $\alpha_i \neq 0$*. This implies the* α*-efficiency of* $\{C_i(\alpha, E)\}_{i \in \mathbb{I}}$ *in the original and* $\tilde{\mathfrak{c}}_i$*-modified economy. Feasibility holds by construction. Hence,* $C \in \mathbb{C}$ *is a continuously differentiable selection in* α*.* ■

The proof of Theorem 2 needs some preparation and is divided into the following four propositions, which make use of the conditions for Theorem 2. The proof strategy of Theorem 2 adapts the ideas of Section 3 in Dana (2004). With the existence of an α-efficient allocation from Theorem 1, we can consider the single-valued solution selection $C : \Delta_I \times L^1(\mathcal{P})_+ \to L^1(\mathcal{P})^I_+$ of the concave program $(U_\alpha, \Lambda(E))$, given by

$$\{C_i(\alpha, E)\}_{i \in \mathbb{I}} \in \operatorname*{argmax}_{(C_i) \in \Lambda(E)} \sum_{i \in \mathbb{I}} \alpha_i U_i(C_i),$$

in the next steps. Now, we introduce our excess utility map. In comparison to the classical case, the mapping has to be modified, since the utility gradient cannot be solely represented by a random variable with a conjugate integrability order. In general for some $X \in L^1(\mathcal{P})_+$, the set of effective priors $M(X)$ is not unique, since it is the minimizer of a convex (and not strictly convex) program, i.e. Gateaux differentiability is in general not true. Hence, we propose a prior dependency in the excess utility to account for this change in the universal system of equations.

Proposition 3 *Under Assumptions 1-4 with* $\operatorname{dom}(\mathfrak{c}_i) = \{P\}$*, for all* $i \in \mathbb{I}$*, there is a* P*-a.s. unique equilibrium.*

As illustrated in Example 5, in general there is no hope for a $\mathcal{P}$-q.s. unique equilibrium.

Proof of Proposition 3 *By Theorem 1, For each $\alpha \in \Delta_I$, a unique α-efficient allocation exists. The proof now follows the lines of Dana (1993), where the present commodity price duality is given by $\langle L^1(\mathcal{P}), L^\infty(P)\rangle$. Here, the continuity of the excess utility map follows by the dominated convergence result at the beginning of Appendix A.* ■

Let us denote by $\mathbb{GE} : \mathcal{P} \to \Delta_I$, the single-valued correspondence which asserts to every prior the unique equilibrium weight α^P of the relevant vNM-economy(P) in Proposition 3. This motivates the following definition.

Definition 6 *Let $C_i(\alpha, E)$ be the* argmax *of an α-efficient allocation with von Neumann Morgenstern utility under $P \in \mathcal{P}$. The* excess utility map *$\Phi : \Delta_I \times \mathcal{P} \to \mathbb{R}^I$ is given by*

$$\Phi_i(\alpha, P) = \alpha_i^{-1}\mathrm{E}^P\left[u'_\alpha(E)\cdot(C_i(\alpha,E) - E_i)\right], \quad i \in \mathbb{I}.$$

The primitives of the economy specify this modified excess utility map. A zero for the standard excess utility map, when only the utility weight α is the variable, guarantees an equilibrium. The modification in the definition is caused by the equilibrium prior, a new component in the universal system of equations. Due to the first order conditions of individual maximization, this object appears beacause of the given structure of the topological dual space. A zero $(\alpha, P) \in \Delta_I \times \mathcal{P}$ of Φ is not sufficient to guarantee an equilibrium, since an arbitrary $P \in \mathcal{P}$ may not lie in the set of common effective priors $\bigcap_{i\in\mathbb{I}} M_i(C_i(\alpha, E))$, where the consumption $C_i(\alpha, E)$ is taken from the α-efficient allocation. To account for this situation, we need the following two Propositions.
First, we prove that at every Pareto optimal allocation, the intersection of the risk adjusted effective prior is not empty. As we will see below, this ensures that the excess utility map can attain a zero in $\mathbb{R}^I$ on the appropriate set of priors. To do so, we reformulate α-efficiency in terms of a supremal convolution from convex analysis. For $E \in L^1(\mathcal{P})$, let

$$\Box_{i\in\mathbb{I}}\alpha_i U_i(E) = \max_{\sum C_i = E} \sum \alpha_i U_i(C_i),$$

and denote the superdifferential of $\Box_{i=1}^I \alpha_i U_i$ by $\partial\Box U_\alpha$, see Laurent (1972) for details. Note that the domain of each U_i equals $L^1(\mathcal{P})_+$.
The following proposition states that for α-efficient allocations the utility supergradients of the agents agree. We also discuss the α-dependency of common effective priors.

Proposition 4 1. *Let $(C_1(\alpha, E), \ldots, C_I(\alpha, E)) \in \Lambda(E)$ be the α-efficient allocation of Theorem 1. We get*

$$\bigcap_{i\in\mathbb{I}} \partial\alpha_i U_i(C_i(\alpha, E)) = \partial\Box U_\alpha(E) \neq \emptyset,$$

for some $\alpha \in \Delta_I$. Moreover, the set $\partial \Box U_\alpha(E)$ is weakly compact and convex.
2. *The set of* common risk unadjusted priors $\mathbb{P}(\alpha) = \cap_{i\in\mathbb{I}} M_i(C_i(\alpha, E))$ *satisfies,*

$$\begin{aligned} \mathbb{P}(\alpha) &= \{P \in \mathcal{P} : \exists \mu \in \partial \Box U_\alpha(E) \text{ with } \mathrm{d}\mu = u'_\alpha(E)\mathrm{d}P\} \\ &= \arg\min_{P\in\mathcal{P}} \mathrm{E}^P \left[u_\alpha(E) - \sum_{i\in\mathbb{I}} \alpha_i \phi_i \right], \end{aligned}$$

where ϕ_i is the representation of the linear penalty term $\mathfrak{c}_i$.
3. *The correspondence $\mathbb{P} : \Delta_I \to \mathcal{P}$ is upper hemicontinuous on Δ_I. Moreover, $\mathbb{P}$ is weakly compact and convex valued.*

Proof of Proposition 4 1. *The allocation $(C_1(\alpha, E), \dots, C_I(\alpha, E)) \in \Lambda(E)$ can be related to an α-weighted program $(U^\alpha, \Lambda(E))$. We formulate this in terms of supremal convolution. By construction we have $\sum_i C_I(\alpha, E) = E$ $\mathcal{P}$-q.s. and*

$$\Box_{i\in\mathbb{I}} \alpha_i U_i(E) = \sum \alpha_i U_i(C_i(\alpha, E)).$$

By Lemma 1, we have $\partial U_i(C_I(\alpha, E)) \neq \emptyset$, for each $i \in \mathbb{I}$. The first part of the proposition follows from Proposition 6.6.4 in Laurent (1972). The convexity of $\partial \Box U^\alpha(E)$ can be found in Theorem 47A in Zeidler (1985). The intersection of compact sets is again compact.
2. *Let $\bar{P} \in \mathbb{P}(\alpha)$, we derive*

$$\begin{aligned} &\max_{(X)\in\Lambda(E)} \sum_{i\in\mathbb{I}} \alpha_i U_i(X_i) = \sum_{i\in\mathbb{I}} \alpha_i \min_{P\in\mathcal{P}} \mathrm{E}^P[u_i(C_i(\alpha, E) - \phi_i] \\ = \quad &\mathrm{E}^{\bar{P}} \left[\sum_{i\in\mathbb{I}} \alpha_i \big(u_i(C_i(\alpha, E)) - \phi_i \big) \right] = \min_{P\in\mathcal{P}} \mathrm{E}^P \left[u_\alpha(E) - \sum_{i\in\mathbb{I}} \alpha_i \phi_i \right], \end{aligned}$$

where the pointwise definition of u_α can be found in the footnote in Assumption 3. The result follows from Lemma 1.4.
3. *The upper hemicontinuity of the correspondence $\mathbb{P}$ follows from Berge's maximum theorem with respect to the α-parametrized and linear problem* $\min_{P\in\mathcal{P}} \mathrm{E}^P[u_\alpha(E) - \sum \alpha_i \phi_i]$. *The values are weakly compact and convex, due to the first part of the proposition.* ■

In the next step, we relate Proposition 3 with our notion of excess utility.

Proposition 5 *The tuple $\big(\{C_i(\alpha^*, E)\}_{i\in\mathbb{I}}, \mathrm{E}^{P^*}[u'_{\alpha^*}(E)\cdot]\big)$ is an Arrow-Debreu equilibrium if and only if*

$$\Phi(\alpha^*, P^*) = 0 \text{ and } P^* \in \mathbb{P}(\alpha^*),$$

which is equivalent to $(\alpha, P) \in \mathrm{gr}(\mathbb{GE}^{-1}) \cap \mathrm{gr}(\mathbb{P})$ or $P \in \mathbb{P} \circ \mathbb{GE}(P)$.

Proof of Proposition 5 *"$\Leftarrow$": Each prior $P \in \mathbb{P}(\alpha)$ is associated to a supergradient $DU_i(C_i)(X) = \mathrm{E}^P[u_i'(C_i)X]$ for each agent $i \in \mathbb{I}$ simultaneously. A possible prior $P \notin \mathbb{P}(\alpha)$ with a zero in the excess demand is not related to at least one agent k's first order condition with a positive weight α_k. Whereas, if Φ is not zero, we have only an equilibrium with transfer payment.*
"$\Rightarrow$": By Proposition 4, $\Phi(\alpha^*, P^*) = 0$ and $P^* \in \mathbb{P}(\alpha^*)$ implies the existence of an equilibrium $\left(\{C_i^{P^*}(\alpha^*, E)\}_{i\in\mathbb{I}}, u'_{\alpha^*}(E)\right)$ under vNM(P^*) utility, i.e. $X \mapsto \mathrm{E}^{P^*}[u_i(X)] - c_i$ under $P^* \in \mathbb{P}(\alpha^*)$, where $c_i = E^{P^*}[\phi_i]$ and $C_i^P(\alpha, \cdot)$ corresponds to the α-efficient consumption of agent i under vNM(P) utility. We get

$$\begin{aligned} & \mathrm{E}^{P^*}[u'_{\alpha^*}(E)(C - E_i)] \le 0 \\ \textit{implies} \qquad & \mathrm{E}^{P^*}[u_i(C)] - c_i \le \mathrm{E}^{P^*}[u_i(C_i^{P^*}(\alpha^*, E))] - c_i. \end{aligned}$$

This implies $U_i(C) \le U_i(C_i^{P^*}(\alpha^*, E))$, due to $U_i(C) \le \mathrm{E}^{P^*}[u_i(C) - \phi_i]$. Hence, $(\{C_i^{P^*}(\alpha^*, E)\}_{i\in\mathbb{I}}, \mathrm{E}^{P^*}[u'_{\alpha^*}(E)\cdot]$ is an equilibrium of the original economy. ■

Agent i's set of effective priors $M_i(C_i(\alpha, E)) \supset \mathbb{P}(\alpha)$ at an optimal consumption forms the basis for the set of equilibrium priors. The first order condition of α-efficient allocations relies on the set of common supergradient $\partial \square U_\alpha$. The risk adjustment via the normalized marginal utility $u'_\alpha(E)$ of the representative agent delivers the correct set of equilibrium priors, see Proposition 4.2. This is consistent with the decomposition of the linear price systems and the modified excess utility map.

Proof of Theorem 2 *Define the functional $\rho : \Delta_I \times \mathcal{P} \to \mathbb{R}$ by*

$$\rho(\alpha, P) = \min_{i\in\mathbb{I}} \Phi_i(\alpha, P).$$

By Proposition 6.2 $\Phi_i(\alpha, \cdot)$ is linear and weakly continuous, hence $\rho(\alpha, \cdot)$ is weakly continuous. From an application of Proposition 6.1 we follow for each $P \in \mathcal{P}$ the continuity of $\rho(\cdot, P)$, since the pointwise infimum of continuous functions is again continuous. Since the maximum of $\rho(\cdot, P)$ over Δ_I is by construction a zero, the solution mapping $\mathbb{GE}$ is also given by

$$\mathbb{GE}(P) = \arg\max_{\alpha\in\Delta_I} \rho(\alpha, P).$$

Therefore by Berge's maximum theorem, $\mathbb{GE}$ is a single-valued and upper-hemicontinuous correspondence and hence continuous when viewed as a function.
Now, $\mathbb{P} \circ \mathbb{GE} : \mathcal{P} \to \mathcal{P}$ is a composition of upper hemicontinuous correspondences and hence again upper hemicontinuous. By Proposition 4.1., $\mathbb{P}$ is convex and weakly compact valued and hence so is $\mathbb{P} \circ \mathbb{GE}$. Since the vector space of signed measure on $(\Omega, \mathcal{F})$ equipped with the topology of weak convergence is a locally convex topological vector space, we apply the Kakutani-Glicksberg-Fan fixed point theorem (Theorem 17.55 in Aliprantis and Border (2006)) with respect to $\mathbb{P} \circ \mathbb{GE}$, and the result follows by Proposition 5. ■

Proposition 6 1. *For each $P \in \mathcal{P}$, the function $\Phi(\cdot, P)$ is continuous in the interior of Δ_I and $\|\Phi(\alpha, P)\|_{\mathbb{R}^I} \to +\infty$ whenever $\alpha_i \to 0$ for some $i \in \mathbb{I}$.*

2. *For each $\alpha \in \Delta_I$, the function $\Phi(\alpha, \cdot)$ is weakly continuous.*

The following result is used in the proof of Theorem 2.

Proof of Proposition 6 1. *This follows from Proposition 3 and the continuous differentiability of each u_i. Since $P \in \mathcal{P}$ is fixed, the limit behavior follows by same argument as in the the standard single prior case.*

2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}$ which converges weakly to some prior P. According to the first result at the beginning of Appendix A,*

$$\lim_{n\to\infty} \mathrm{E}^{P_n}[u'_\alpha(E) \cdot (C_i(\alpha, E) - E_i)] = \mathrm{E}^{P}[u'_\alpha(E) \cdot (C_i(\alpha, E) - E_i)]$$

and which proves continuity in the weak topology. ■

The equilibrium weight α^* relates the residual set of priors by $\mathcal{P}_E = \mathbb{P}(\alpha^*)$.

Proofs of Subsection 4.3

By $\mathcal{Q}$-q.e., we denote $\mathcal{Q} \otimes \mathrm{d}t = \{\mathbb{Q} \otimes \mathrm{d}t, \mathbb{Q} \in \mathcal{Q}\}$-quasi everywhere.

Proof of Theorem 3 *We begin with the only if part of the theorem and denote $\xi_i = C_i - E_i$. Suppose there is an agent $i \in \mathbb{I}$, such that $\mathcal{Q}_E \cap \mathcal{Q}(\xi_i) = \emptyset$.*[49] *This implies $\mathrm{E}^Q[\xi_i] < \mathbb{E}^{\mathcal{Q}}[\xi_i]$, where $Q \in \mathcal{Q}_E$. In order to guarantee the implementation of the Arrow-Debreu equilibrium, the portfolio process X^{θ^i} with $\theta^i \in \Theta(S)$ requires $X_T^{\theta^i} = X_0^{\theta^i} + \int_0^T \theta_t^i \mathrm{d}S_t$ and must satisfy*

$$X_T^{\theta^i} = \xi \quad \text{and} \quad X_0^{\theta^i} = \pi(e_i - c_i).$$

An application of the martingale representation theorem to ξ_i implies $X_0^{\theta^i} = \mathbb{E}^{\mathcal{Q}}[\xi_i]$, so that the only constants in the martingale representation and the self-financing condition must be equal. On the other side, we have by the Arrow-Debreu budget set $X_0^{\theta^i} = \pi(e_i - c_i) = \mathrm{E}^Q[\xi_i]$, which is a contradiction to $\mathcal{Q}_E \cap \mathcal{Q}(\xi_i) \neq \emptyset$ or equivalently to (5) in the formulation Theorem 3.

To proof the other direction, fix the following elements in $L^1(\mathcal{P})$ as the dividend of the first two securities:

$$D^0 \equiv 1, \quad D^1 = B_T, \quad D^2 = K_T,$$

where K_T is specified in step two below by virtue of Lemma 3. We introduce the candidates for the price of consumption at time zero and the price process of the security. Let the price of D at time t be $S_t^1 = \mathbb{E}_t^{\mathcal{Q}}[D^1]$ and $S_t^0 = \mathbb{E}_t^{\mathcal{Q}}[D^0] = 1$. The positive scalar π is the price of time zero consumption. We

[49] Here, $\mathcal{Q}(X) = \arg\max_{Q\in\mathcal{Q}} \mathrm{E}^Q[X]$ denotes the effective (uncertainty adjusted) priors under the sublinear equilibrium expectation.

divide the proof into four steps. In the first step and second, we introduce the candidate trading strategies for agent $i \in \mathbb{I} \setminus \{I\}$ and show market clearing in the third step. The last step shows that the trading strategies are maximal elements in the budget sets.

1. Let $\xi_i \in L^1(\mathcal{P})$, $i \in \mathbb{I} \setminus \{I\}$, be some feasible net trades. The process

$$X_t^i = \mathbb{E}_t^{\mathcal{Q}}[\xi_i] - \mathbb{E}^{\mathcal{Q}}[\xi_i], \quad t \in [0,T]$$

is an integrable $\mathcal{Q}$-martingale and we have by the martingale representation

$$X_t^i = \int_0^t \theta_r^{i,1} \mathrm{d}S_r^1 - K_t^i, \tag{7}$$

$\mathcal{Q}$-q.e. Fix some strategy $\theta^i := (\theta^{i,0}, \theta^{i,1}, \theta^{i,2}) \in \Theta(S^0, S^1, S^2) =: \Theta(S)$, where $\theta^{i,0}$ and $\theta^{i,2}$ are specified in step two.
As a candidate Radner equilibrium allocation at time T, we consider the allocation generated by the Arrow-Debreu equilibrium allocation, i.e. $\xi_i = \big(\bar{C}_i - E_i\big)$, for each $i \in \mathbb{I}$.

2. Applying Lemma 3 to $\{K^i\}_{i \in \mathbb{I} \setminus \{I\}}$ in (7), there is a predictable process of bounded variation S^2 starting in zero with $S^2 \in L^1(\mathcal{P})$ and predictable S^2-integrable processes $\{\theta^{i,2}\}_{i \in \mathbb{I} \setminus \{I\}}$, such that[50]

$$-K_t^i = \int_0^t \theta_r^{i,2} \mathrm{d}S_r^2 \quad \mathcal{Q}\text{-a.e.}$$

From this we can reformulate the bounded variation part in (5) and get $-K_t^i = \int_0^t \theta_r^{i,2} \mathrm{d}S_r^2$, for each $i \in \mathbb{I} \setminus \{I\}$. Fix the following trading process for the riskless security S^0 for agent $i \in \mathbb{I} \setminus \{I\}$:

$$\theta_t^{i,0} = \mathrm{E}^Q[\xi_i] + \int_0^t \theta_r^{i,1} \mathrm{d}S_r^1 + \int_0^t \theta_r^{i,2} \mathrm{d}S_t^2 - \theta_t^{i,1} S_t^1 - \theta_t^{i,2} S_t^2, \quad \mathcal{Q}\text{-q.e.},$$

where $\mathbb{E}^{\mathcal{Q}}[\xi_i] = \mathrm{E}^Q[\xi_i]$ for some Arrow-Debreu equilibrium pricing measure $Q \in \mathcal{Q}(\xi_i) \cap \mathcal{Q}_E \neq \emptyset$. Clearly, $\int \theta^{i,0} \mathrm{d}S^0 \equiv 0$ is a well defined square integrable integral and $\mathrm{E}^Q[\xi_i] = \langle \theta_0^i, S_0 \rangle$. Predictability of $\theta^{i,0}$ can easily be verified.
Substitution of the integral equations yields the self-financing property for θ^i:

$$\langle \theta_t^i, S_t \rangle = \langle \theta_0^i, S_0 \rangle + \int_0^t \langle \theta_r^i, \mathrm{d}S_r \rangle \quad \mathcal{Q}\text{-q.e.}$$

It follows that each trading strategy is admissible, i.e. $\theta^i \in \Theta(S)$, for each $i = \mathbb{I} \setminus \{I\}$. We observe via the self-financing property

$$\begin{aligned} \langle \theta_T^i, S_T \rangle + E_i &= (\theta_T^{i,0}, \theta_T^{i,1}, \theta_T^{i,2})^\top (D^0, D^1, D^2) + E_i = \bar{C}_i, \quad \mathcal{Q}\text{-q.s.} \\ \langle \theta_0^i, S_0 \rangle &= \mathbb{E}^{\mathcal{Q}}[\xi_i] = \mathrm{E}^Q[\xi_i] = \mathrm{E}^Q[\bar{C}_i - E_i] = \pi(e_i - \bar{c}_i). \end{aligned}$$

[50] Note that the asset price S^2 depends heavily on the equilibrium net trades.

Hence, each agent $i = \mathbb{I} \setminus \{I\}$ *consumes* $(\bar{c}_i, \bar{C}_i)$ *via the portfolio strategy* θ^i.

3. In order to meet the market clearing condition in the Radner economy, consider the last agent $I \in \mathbb{I}$, *equipped with* $\theta^I = -\sum_{j \in \mathbb{I}\setminus\{I\}} \theta^j$, *which guarantees market clearing, by the linear structure of* $\Theta(S)$. *The self-financing condition* $\theta_I \in \Theta(S)$ *holds by construction. We derive again by the Arrow-Debreu budget constraint, since* $Q \in \mathcal{Q}_E$

$$\langle \theta_0^I, S_0 \rangle = \mathrm{E}^Q \Big[- \sum_{j \in \mathbb{I}\setminus\{I\}} \xi_j \Big] = \mathrm{E}^Q [\xi_I] = \pi(e_I - \bar{c}_I).$$

By the clearing condition of the Arrow-Debreu equilibrium we derive

$$\xi_I = - \sum_{j \in \mathbb{I}\setminus\{I\}} \xi_j = \left\langle - \sum_{j \in \mathbb{I}\setminus\{I\}} \theta_T^j, S_T \right\rangle = - \sum_{j \in \mathbb{I}\setminus\{I\}} \left(\theta_T^{j,0} S_T^0 + \theta_T^{j,1} S_T^1 + \theta_T^{j,2} S_T^2 \right),$$

which gives us the clearing condition in the Radner economy.

4. In the last step we show the individual optimality of the trading strategies. Suppose there is an agent $k \in \mathbb{I}$ *capable of achieving a strictly preferred bundle* $(c, C) \succ_k (\bar{c}_k, E_k + \xi_k)$ *in terms of a different trading strategy* $\theta^C \in \Theta(S)$. *The Arrow-Debreu price system (at time T) in Theorem 2 satisfies* $\Pi \in L^1(\mathcal{P})^*_\oplus$, *the value of* (c, C) *should be strictly higher in comparison to* $(\bar{c}_k, \bar{C}_k)$, *since preferences are semi-strictly monotone.*[51] *This means*

$$\pi c + \mathrm{E}^Q[C] > \pi \bar{c}_k + \mathrm{E}^Q[E_k + \xi_k], \quad \textit{for some } Q \in \mathcal{Q}_E.$$

Applying the Radner budget constraint for (c, C), *we have*

$$\pi e_k - \langle \theta_0^C, S_0 \rangle + \mathrm{E}^Q \left[E_k + \langle \theta_0^C, S_0 \rangle + \int_0^T \langle \theta_t^C, \mathrm{d}S_t \rangle \right] > \pi \bar{c}_k + \mathrm{E}^Q[E_k + \xi_k],$$

for some $Q \in \mathcal{Q}_E$. *Since* $\int \theta^{C,0} \mathrm{d}S^0 \equiv 0$ *and the stochastic integral* $\int_0^t \theta_r^{C,1} \mathrm{d}S_r^1$ *is a symmetric* $\mathcal{Q}$*-martingale, and hence a* Q*-martingale for every* $Q \in \mathcal{Q}$ *as well. By the market clearing and Lemma 3,* $K_T^C = \int_0^T \theta_r^{C,2} \mathrm{d}S_r^2$ *holds* $\mathcal{Q}$*-q.s. and since* $-K^C$ *is a* $\mathcal{Q}$*-martingale starting in zero, we conclude*

$$\pi e_k + \mathrm{E}^Q [E_k] = \pi e_k + \mathrm{E}^Q \left[E_k - K_T^{Q,C} \right] > \pi \bar{c}_k + \mathrm{E}^Q[E_k + \xi_k].$$

This implies $0 > \pi(\bar{c}_k - e_k) + \mathrm{E}^Q \left[\bar{C}_k - E_k \right]$, *and contradicts the given Arrow-Debreu budget optimality of* $(\bar{c}_k, \bar{C}_k)$.

This proves the existence of the Radner equilibrium. The properties of the equilibrium follow directly from the construction. ■

[51] In this argument, we benefit from the semi-strict positivity of the price system.

Proof of Theorem 4 *We follow a similar proof strategy as in Theorem 3 and introduce four assets $S^0, \dots, S^3$. To each endowment and consumption of agent $i \in \mathbb{I}$ apply the martingale representation theorem under $\mathcal{Q}$:*

$$C_i = \mathbb{E}^{\mathcal{Q}}[C_i] + \int_0^T \theta_r^{C_i} \mathrm{d}S_r^1 - K_T^{C_i} \quad \text{and} \quad E_i = \mathbb{E}^{\mathcal{Q}}[E_i] + \int_0^T \theta_r^{E_i} \mathrm{d}S_r^1 - K_T^{E_i}$$

Since $\mathbb{E}^{\mathcal{Q}}[C_i] = \Psi(C_i) = \Psi(E_i) = \mathbb{E}^{\mathcal{Q}}[E_i]$, the net trades $\xi_i = C_i - E_i$ can be written as, with $\theta^{i,1} = \theta^{C_i} - \theta^{E_i}$,

$$\xi_i = \int_0^T \theta_r^{C_i} - \theta_r^{E_i} \mathrm{d}B_r - K_T^{C_i} + K_T^{E_i} = \int_0^T \theta_r^{i,1} \mathrm{d}S_r^1 + \int_0^T \theta_r^{i,2} \mathrm{d}S_r^2 - \int_0^T \theta_r^{i,3} \mathrm{d}S_r^3,$$

where S^2 and S^3 are induced by $(K^{C_i})_{i \in \mathbb{I}}$, $(K^{E_i})_{i \in \mathbb{I}}$, respectively via the application of Lemma 3.
To meet the store of value condition, set $S_t^0 = 1$ and

$$\theta_t^{i,0} = \int_0^t \langle \theta_r^i, \mathrm{d}S_r \rangle - \sum_{1 \le k \le 3} \theta_t^{i,k} S_t^k.$$

This gives us the self-financing condition $\langle \theta_t^i, S_t \rangle = \int_0^t \langle \theta_r^i, \mathrm{d}S_r \rangle = X_t^{\theta^i}$. Moreover, each agent $i \in \mathbb{I} \setminus \{I\}$ with trading strategy $\theta^i = (\theta^{i,0}, \theta^{i,1}, \theta^{i,2}, \theta^{i,3})$ can consume ξ_i at time T.
The last agent I is equipped with the strategy $\theta^I = -\sum_{j \in \mathbb{I} \setminus \{I\}} \theta^j$. Since $\Theta(S^0, S^1, S^2, S^3) = \Theta(S)$ is a linear space, we have $\theta^I \in \Theta(S)$. The market clearing condition holds by construction, while the Arrow-Debreu clearing condition and the linearity of the stochastic integral and the bounded variation integrals imply that θ^I generates $C_I = E_I + \langle \theta_T^I, S_T \rangle$. This argument follows step three in the proof of Theorem 3.
Finally we check the maximality of the strategy in the Radner budget set. Suppose an agent $j \in \mathbb{I}$ receives a strictly better allocation $\mathfrak{C}$ financed by some $\theta^{\mathfrak{C}} \in \Theta(S)$ such that $U_j(\mathfrak{C}) > U_j(C_j)$. Then the sublinear equilibrium price of $\mathfrak{C}$ must be strictly higher, i.e. $\Psi(\mathfrak{C}) > \Psi(C_j) = \Psi(E_j)$. Since $\mathfrak{C}$ is financed by $\theta^{\mathfrak{C}}$ we have $E_j + X_T^{\theta^{\mathfrak{C}}} = \mathfrak{C}$. Applying the martingale representation theorem with respect to E_j, with $-K^{E_j} = \int \theta^{j,3} \mathrm{d}S^3 = \int \theta^{\mathfrak{C},3} \mathrm{d}S^3$ we derive

$$\begin{aligned}
\mathbb{E}^{\mathcal{Q}}[E_j] &< \Psi(\mathfrak{C}) \\
&= \mathbb{E}^{\mathcal{Q}}\left[\mathbb{E}^{\mathcal{Q}}[E_j] + \int_0^T \theta_t^{E_j} \mathrm{d}S_t^1 + \int_0^T \theta_t^{3,E_j} \mathrm{d}S_t^3 + \int_0^t \langle \theta_r^{\mathfrak{C}}, \mathrm{d}S_r \rangle\right] \\
&= \mathbb{E}^{\mathcal{Q}}[E_j] + \mathbb{E}^{\mathcal{Q}}\left[\int_0^T \left(\theta_t^{\mathfrak{C},1} + \theta_t^{E_j,1}\right) \mathrm{d}S_t^1 - \int_0^T \theta_t^{2,\mathfrak{C}} \mathrm{d}S_t^2\right] \\
&= \mathbb{E}^{\mathcal{Q}}[E_j] + \mathbb{E}^{\mathcal{Q}}\left[-K_T^{\mathfrak{C}}\right] \\
&= \mathbb{E}^{\mathcal{Q}}[E_j],
\end{aligned}$$

where we applied the symmetric martingale property of $\int \theta^{\mathfrak{C},1} + \theta^{E_j,1} \mathrm{d}S^1$ in *terms of the additivity of the sublinear expectation and the martingale property of $-K^{\mathfrak{C}}$ with $-K_0^{\mathfrak{C}} = 0$. The second equality holds by*

$$\int_0^T \langle \theta_t^{\mathfrak{C}}, \mathrm{d}S_r \rangle = \int_0^T \theta_t^{1,\mathfrak{C}} \mathrm{d}S_t^1 + \int_0^T \theta_t^{2,\mathfrak{C}} \mathrm{d}S_t^2 - \int_0^T \theta_t^{3,\mathfrak{C}} \mathrm{d}S_t^3$$

and by the cash translatability.[52] *The contradiction proves the result.* ■

By $\mathcal{P}(\mathbb{F})$, we denote the predictable σ-algebra on $\bar{\Omega} = [0, T] \times \Omega$ with respect to the filtration $\mathcal{F}$ in Subsection 4.1. In the proof of Theorem 3, we applied the the following result.

Lemma 3 *Fix a finite set $\{K^i\}_{i\in\mathbb{I}}$ of predictable, nondecreasing processes, starting in zero with $K_T^i \in L^1(\mathcal{P})$, then there is a predictable, nondecreasing process S, starting in zero with $S_T \in L^1(\mathcal{P})$ and a set $(\eta^i)_{i\in\mathbb{I}}$ of predictable and S-integrable processes such that*

$$K_t^i = \int_0^t \eta_r^i \mathrm{d}S_r \quad \mathcal{P}\text{-q.e.} \quad \text{for every } i \in \mathbb{I}.$$

Proof of Lemma 3 *Set $K^{i,P} = K^i$ as a process on $(\Omega, \mathcal{B}(\Omega), P)$. By the properties of each $K^{i.P}$, there is a positive (random) measure $\mu^{i,P}$ on $(\bar{\Omega}, \mathcal{P}(\mathbb{F}))$ satisfying*

$$A \mapsto \mu^P(A) = \mathrm{E}^P \left[\int_0^T 1_A \mathrm{d}K_t^{i,P} \right], \quad A \in \mathcal{P}(\mathbb{F}).$$

The space of σ-finite signed measures $\mathcal{M}^\sigma(\bar{\Omega}, \mathcal{P}(\mathbb{F}))$ is a Banach lattice[53] *(see section IX.2 of Jacobs and Kurzweil (1978)), and especially a lattice group. By Proposition 5.1.12 of Constantinescu (1984), there is a finite family of strictly positive and σ-finite measures $(\nu_\lambda^P)_{\lambda\in\mathbb{L}} \subset \mathcal{M}^\sigma(\bar{\Omega}, \mathcal{P}(\mathbb{F}))$ such that*

$$\sum_{\lambda\in\mathbb{L}} \nu_\lambda^P = \mu^P = \bigvee_{i\in\mathbb{I}} \mu^{i,P} \in \mathcal{M}^\sigma(\bar{\Omega}, \mathcal{P}(\mathbb{F}))$$

and for every $i \in \mathbb{I}$ there exists a subset $\mathbb{L}_i \subset \mathbb{L}$ with

$$\mu^{i,P} = \sum_{\lambda\in\mathbb{L}_i} \nu_\lambda^P \in \mathcal{M}^\sigma(\bar{\Omega}, \mathcal{P}(\mathbb{F})).$$

Absolute continuity follows, i.e. $\nu_\lambda^P \ll \mu^P$ for every $\lambda \in \mathbb{L}$. Hence, by the Radon-Nykodym theorem applied on $(\bar{\Omega}, \mathcal{P}(\mathbb{F}), \mu^P)$, we have

$$\mathrm{d}\nu_\lambda^P = \frac{\mathrm{d}\nu_\lambda^P}{\mathrm{d}\mu^P} \mathrm{d}\mu^P, \quad \text{for every } \lambda \in \mathbb{L}.$$

[52] This follows from the constant preserving property and the sublinearity of $\mathbb{E}^{\mathcal{Q}}$.

[53] Here, $\mathcal{M}^\sigma(\bar{\Omega}, \mathcal{P}(\mathbb{F}))$ is quipped with the natural ordering and the total variation norm.

The density $\frac{\mathrm{d}\nu_\lambda^P}{\mathrm{d}\mu^P}$ *is in* $L^1(\bar{\Omega}, \mathcal{P}(\mathbb{F}), \mu^P)$ *if and only if* ν_λ^P *is* σ*-finite, and we have*

$$\mathrm{d}\mu^{i,P} = \sum_{\lambda \in \mathbb{L}_i} \mathrm{d}\nu_\lambda^P = \sum_{\lambda \in \mathbb{L}_i} \frac{\mathrm{d}\nu_\lambda^P}{\mathrm{d}\mu^P} \mathrm{d}\mu^P = \eta^{i,P} \mathrm{d}\mu^P.$$

Similarly to the identification of $K^{i,P}$ *via* $\mu^{i,P}$*, there is a predictable process* S^P *with* $S_0^P = 0$ *and increasing paths and a* $\eta^{i,P} \in L^1(\bar{\Omega}, \mathcal{P}(\mathbb{F}), \mu^P)$*, such that*

$$\mathrm{d}K_t^{i,P} = \eta_t^{i,P} \mathrm{d}S_t^P \quad \textit{for every } t \in [0,T] \textit{ and } i \in \mathbb{I}.$$

In order to guarantee aggregating objects, i.e. $S = S^P$ *and* $\eta^i = \eta^{i,P}$ $P \otimes \mathrm{d}t$*-a.e. for every* $P \in \mathcal{P}$*, we use the weak compactness of* $\mathcal{P}$ *in Assumption 1. The aggregation property holds by an application Theorem 5.1 of Soner, Touzi, and Zhang (2012b), (see also Example 4.14 therein). The result follows.* ■

Chapter 3

Ambiguity-Neutral Pricing under Volatility Uncertainty

3.1 Introduction

A fundamental assumption behind models in Finance refers to the modeling of uncertainty via a single probability measure. Instead, we allow for a set of probability measures $\mathcal{P}$, such that we can guarantee awareness of potential model misspecification.[1] We investigate the implications of a related and reasonable arbitrage concept. In this context, we suggest a *fair* pricing principle associated with an appropriate martingale concept. The multiple prior setting influences the price system in terms of the simultaneous control of different null sets. This motivates a pricing theory of possible means.[2]
The pricing of derivatives via arbitrage arguments is fundamental. Before stating an arbitrage concept, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed such that marketed claims or tradeable assets with trading strategies can be defined. The implicit assumption is that the probabilities are known exactly. The Fundamental Theorem of Asset Pricing (FTAP) then asserts equivalence between the absence of $\mathbb{P}$-arbitrage in the market model and the existence of a consistent linear price extension such that the market model can price all contingent claims. The equivalent martingale measure is then an alternative description of this extension via the Riesz representation theorem.
In contrast to this standard setup, we introduce an uncertainty model described as a set of possibly mutually singular probability measures or priors.[3] Our leading motivation is a general form of *volatility uncertainty*. This perspective deviates from models with term structures of volatilities, including stochastic volatility models such as Heston (1993). As argued in Carr and Lee (2009), we question this confidence and avoid formulating the volatil-

[1]The distinction between measurable and unmeasurable uncertainty drawn by Knight (1921) serves as a starting point for modeling the uncertainty in the economy. Keynes (1937) later argued that single prior models cannot represent irreducible uncertainty.
[2]This was originally discussed by de Finetti and Obry (1933).
[3]Two priors are mutually singular if they live on two disjoint supports.

ity process of a continuous-time asset price via another process whose law of motion is exactly known. Instead, the legitimacy of the probability law still depends on an infinite repetition of variable observations, as highlighted by Kolmogoroff (1933). We include this residual uncertainty by giving no concrete model for the stochastics of the volatility process and instead fix a confidence interval for the volatility variable.[4]

A coherent valuation principle changes considerably when the uncertainty is enlarged by the possibility of different probabilistic scenarios having different null sets. In order to illustrate this point, we consider for a moment the uncertainty given by one probability model, i.e. $\mathcal{P} = \{\mathbb{P}\}$. An arbitrage refers to a claim X with zero cost, a $\mathbb{P}$-almost surely positive and with a positive probability a strictly positive payoff. Formally, this can be written as $\pi(X) \leq 0$,

$$\mathbb{P}(X \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(X > 0) > 0.$$

The situation changes in the case of an uncertainty model described by a set of mutually singular priors $\mathcal{P}$. The second and third condition should be carefully reformulated, because every prior $P \in \mathcal{P}$ could be the correct market description. We rewrite an arbitrage as $\pi(X) \leq 0$,

$$\text{for all } \mathbb{P} \in \mathcal{P} \quad \mathbb{P}(X \geq 0) = 1 \quad \text{and} \quad \mathbb{P}'(X > 0) > 0 \text{ for some } \mathbb{P}' \in \mathcal{P}.$$

In accepting this new $\mathcal{P}$-arbitrage notion as a weak dominance principle, we might ask for the structure of the related objects.[5] Suppose we apply the same idea of linear and coherent extensions to the present multiple prior uncertainty model. Coherence corresponds to a strictly positive and continuous price systems on the space of claims L which is consistent with the given data of a possibly incomplete market. Marketed claims $M \subset L$ can be traded frictionless and are priced by a linear functional $\pi : M \to \mathbb{R}$.

Another important aspect focuses on the order structure for contingent claims and the underlying topology of similarity for L. This comprises the basis of any financial model that asks for coherent pricing. The representation of linear and continuous price systems[6] indicates inconsistencies between positive linear price systems and the concept of $\mathcal{P}$-arbitrage. As is usual, the easy part of establishing an FTAP is deducing an arbitrage-free market model from the existence of an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P} \in \mathcal{P}$. When seeking a modified FTAP, the following question (and answer) serves to clarify the issue:

[4] For further motivation to consider volatility uncertainty, we refer to Subsection 1.1 of Epstein and Ji (2013a). Very recent developments in stochastic analysis have established a complete theory to model volatility uncertainty in continuous time. A major objective refers to the sublinear expectation operator introduced by Peng (2006).

[5] See Remark 3.14 in Vorbrink (2010) for a discussion of a weaker arbitrage definition and its implication in the G-framework.

[6] We discuss the precise description in Section 2.2.

> Is the existence of a measure $\mathbb{Q}$ equivalent to some $\mathbb{P} \in \mathcal{P}$ such that prices of all traded assets are $\mathbb{Q}$-martingales, and therefore a sufficient condition to prevent a $\mathcal{P}$-arbitrage opportunity?

A short argument gives us a negative answer: Let $X \in M$ be a marketed claim with price $0 = \pi(X)$. We have $E^{\mathbb{Q}}[X] = 0$ since $\mathbb{Q}$ is related to a consistent price system. Suppose $X \in M$ with $X \geq 0$ $\mathbb{P}$-a.s for every $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}'(X > 0) > 0$ for some $\mathbb{P}' \in \mathcal{P}$ exists. The point is now,

> with $\mathcal{P} = \{\mathbb{P}\}$ we would observe a contradiction since $\mathbb{Q} \sim \mathbb{P}$ implies $E^{\mathbb{Q}}[X] > 0$. But $X \in M$ may be such that $\mathbb{P}'(X > 0) > 0$ with $\mathbb{P}' \in \mathcal{P}$ being mutually singular to $\mathbb{Q} \sim \mathbb{P} \in \mathcal{P}$.

This indicates that our *robust* arbitrage notion is, in general, not consistent with a linear theory of valuation. In other words, a single pricing measure $\mathbb{Q}$ is not able to contain all the information about what is possible under $\mathcal{P}$. Since our goal is to suggest a modified framework for a coherent pricing principle, the concept of marketed claim is reformulated by a prior-dependent notion of possible marketed spaces $M_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$. As discussed in Example 3 below, such a step is necessary to address the prior dependency of the asset span $M_{\mathbb{P}}$. The likeness of marketed spaces depends on the similarity of the priors in question. Hence, the possibility of different priors creates a friction caused by the new uncertainty.

A New Commodity-Price Duality

The very basic principle of uncertainty is the assumption of different possible future states of the world Ω, which is equipped with the Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$.[7] In the most general framework, we assume a weakly compact set of priors $\mathcal{P}$.[8] This encourages us to consider the sublinear expectation operator

$$\mathcal{E}^{\mathcal{P}}(X) = \sup_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[X].$$

In our economy, the Banach space of contingent claims $L^2(\mathcal{P})$ consists of all random variables with a finite variance for all $\mathbb{P} \in \mathcal{P}$. The primitives are prior-dependent representative agent economies given by preference relations in $\mathbb{A}(\mathbb{P})$, being convex, continuous, strictly monotone and rational.

In the single prior setting, the expectation under an equivalent martingale measure $\mathbb{Q}$ refers to a normalized, linear and continuous price system in the sense of Arrow-Debreu. The topological dual space of $L^2(\mathcal{P})$,

[7] In order to tackle the mutually singular priors, we need some structure in the state space. See Bion-Nadal and Kervarec (2010) for a discussion of different state spaces. In the most abstract setting, the states of the world $\omega \in \Omega$ build a complete separable metric space, also known as a Polish space. The state space contains all realizable paths of security prices. For the greater part of the paper, we assume $\Omega = C([0, T]; \mathbb{R})$, the Banach space of continuous functions between $[0, T]$ and $\mathbb{R}$, equipped with the supremum norm.

[8] If one accepts a deterministic upper bound on the volatility, i.e. the derivative of every possible quadratic variation, then the (relatively) weak compactness of $\mathcal{P}$ is a sufficient condition.

a first candidate for the space of price systems, does not consist of elements which can be merely represented by a state price density ψ. Rather, in the present volatility uncertainty framework, it is represented by the pairs $(\mathbb{P}, \psi) \in \cup_{\mathbb{P}\in\mathcal{P}}\{\mathbb{P}\} \times L^2(\mathbb{P})$. As explained before, such linear valuations are inconsistent with the fine and robust arbitrage we are interested in. Loosely speaking, such price systems only see the null sets of a particular $\mathbb{P}$ and are blind for the null sets of any mutually singular prior $\mathbb{P}' \in \mathcal{P}$. However, the present space of nonlinear price functionals $L^2(\mathcal{P})^{\circledast}$ built upon this dual space. Proposition 1 lists important properties and indicates a possible axiomatic approach to the price systems inspired by the coherent risk measures of Artzner, Delbaen, Eber, and Heath (1999).
Sublinear prices are constructed by the price systems of partial equilibria, which consist of prior-dependent linear price functionals $\pi_{\mathbb{P}}$ restricted to the prior-dependent marketed spaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. These spaces are joined to a product of marketed spaces. The consolidation operation Γ transforms the extended product of price systems $\{\pi_{\mathbb{P}}\}_{P\in\mathbb{P}}$ to *one* coherent element in the price space $L^2(\mathcal{P})^{\circledast}_{+}$. Scenario-based viability can then model a preference-free valuation concept in terms of consolidation of possibilities. The first main result, Theorem 1, gives an equivalence between our notion of scenario-based viable price systems, and the extension of sublinear functionals. The present viability concept, corresponding to a no trade equilibrium, is based on sublinear prices so that the price functional act linearly under unambiguous contingent claims.

Risk- and Ambiguity-Neutral Valuation

In the second part, we consider the dynamic framework on a time interval $[0, T]$ with an augmented filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ modeling the arrival of new information. Its special feature is its reliance on the initial σ-algebra, which does not contain all null sets. Built upon this information structure, we introduce a dynamic updating principle based on a sequence of conditional sublinear expectations $\mathcal{E}_t(\cdot) = \mathbb{E}^{\mathcal{P}}[\,\cdot\,|\mathcal{F}_t]$, $t \in [0, T]$. These operators are well defined under every $\mathbb{P} \in \mathcal{P}$ and satisfy the Law of Iterated Expectation.
With the conditional sublinear expectation, a martingale theory is available which represents a possibilistic model of a fair game against nature.[9] In this fashion, the multiple prior framework forces us to generalize the concept of equivalent martingale measures. Instead of considering *one* probability measure representing the risk-neutral world, we suggest that the appropriate concept is a set of priors $\mathcal{Q}$. The relation to the statistical set of priors $\mathcal{P}$ is induced through a prior-dependent family of state price densities $\psi_{\mathbb{P}} \in L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. This creates a new sublinear expectation, $\mathcal{E}^{\mathcal{Q}}$, generated by $\mathcal{Q}$. For this rationale, the uncertain asset price (S_t) becomes under $\mathcal{E}^{\mathcal{Q}}$ mean unambiguous, i.e. $E^{\mathbb{Q}}[S_T] = E^{\mathbb{Q}'}[S_T]$, for all $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}$.
The essential renewal is to consider $\mathcal{Q}$ as the appropriate uncertainty-neutral world. At this stage, ambiguity neutrality as a part of uncertainty neutrality

[9] More precisely, a whole hierarchy of different fairness degrees is possible.

comes into play. The central idea follows the same lines as in the classical risk-neutral valuation. Preferences on ambiguity become neutral when we move to the uncertainty neutral world $\mathcal{Q}$.[10] And it is exactly this kind of neutrality which corresponds to the notion of symmetric martingales, i.e $(-S_t)$ is a $\mathcal{E}^{\mathcal{Q}}$-martingale as well. This reasoning motivates the modification of the martingale concept, now based on the idea of a fair game under $\mathcal{Q}$. As such, the condition that the price process S is a symmetric martingale motivates qualifying the valuation principle as *uncertainty neutral*.
The principal idea of our modified notion of $\mathcal{P}$-arbitrage was introduced by Vorbrink (2010) for the G-expectation framework. In Theorem 2 we show that under no $\mathcal{P}$-arbitrage there is a one-to-one correspondence between the extensions of Theorem 1 and (special) *equivalent symmetric martingale measure sets* $\mathcal{Q}$, called EsMM-sets. We thus establish an asset pricing theory based on a (discounted) nonlinear expectation payoff.
Having established the relation between these concepts, we continue in the same fashion as in the classical literature with a single prior. We consider a special class of asset prices driven by G-Brownian motion, related to a G-expectation E_G. This is a zero-mean and stationary process with novel $N(0, [\underline{\sigma}, \overline{\sigma}])$-normally distributed independent increments. Such a normally distributed random variable is the outcome of a robust central limit theorem under the sublinear G-expectation. Moreover, in this uncertainty setup, independence of random variables is no longer a symmetric property.[11] This process can be regarded as a canonical generalization of the standard Brownian motion, in which the quadratic variation may move almost arbitrarily in a positive interval. The related G-heat equation is now a fully nonlinear PDE, see Peng (2006).
We consider a Black-Scholes like market under volatility uncertainty driven by a G-Brownian motion B^G. The uncertain asset price process (S_t) is modeled as a stochastic differential equation[12]

$$dS_t = \mu(t, S_t)d\langle B^G \rangle_t + V(t, S_t)dB^G_t, \quad S_0 = 1.$$

Intuitively, the increment dS_t is divided into the locally certain part[13] and the locally risky and ambiguous part $V(t, S_t)dB^G_t$. An interpretation of this

[10] This symmetry of priors is essential for creating a process via a conditional expectation which satisfies the classical martingale representation property, see Appendix B.3.

[11] In the mathematical literature, the starting point for consideration is a sublinear expectation space, consisting of the triple $(\Omega; \mathcal{H}; \mathcal{E})$, where $\mathcal{H}$ is a given space of random variables. If the sublinear expectation space can be represented via the supremum of a set of priors, see Denis, Hu, and Peng (2011), one can take $(\Omega, \mathcal{B}(\Omega), \mathcal{P})$ as the associated *uncertainty space* or Dynkin space, see Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2013).

[12] This related stochastic calculus comprises a stochastic integral notion, a G-Itô formula and a martingale representation theorem.

[13] For this part one usually has a dt-drift as the inner clock of classical Brownian motion. Since the inner clock or quadratic variation is now given by the ambiguous $\langle B^G \rangle_t$, we relate it to the drift part.

G-Itô differential representation reads as follows:

$$\frac{d}{dr}\mathrm{var}_r^{\mathbb{P}}(S_t)\Big|_{r=t} \in V(t,S_t)\cdot[\underline{\sigma},\overline{\sigma}], \qquad \mathbb{P}\in\mathcal{P},$$

where $\mathrm{var}_r^{\mathbb{P}}(S_t)$ is the $(\mathcal{F}_t,\mathbb{P})$-conditional variance. In abuse of notation we could write this issue as $\mathrm{var}_t(dS_t) = V(t,S_t)^2 d\langle B^G\rangle_t$, $\mathcal{P}$-quasi surely.
In this mutually singular prior setting, the (more evolved) martingale representation property, related to a conditional sublinear expectation, is not equivalent to the completeness of the model because the volatility uncertainty is encoded in the integrator of the price process. For the state price density process we introduce an exponential martingale $(\mathtt{E}_t)_{t\in[0,T]}$[14] under G-Brownian motion and apply a new Girsanov type theorem under E_G. For every contingent claim $X \in L^2(\mathcal{P})$, this yields following robust pricing formula

$$\Psi(X) = \mathcal{E}^{\mathcal{Q}}(X) = E_G[\mathtt{E}_T X].$$

Related Literature
We embed the present paper into the existing literature. In Harrison and Kreps (1979), the arbitrage pricing principle provides an economic foundation by relating the notion of equivalent martingale measures with a linear equilibrium price system.[15] Risk neutral pricing, as a precursor, was discovered by Cox and Ross (1976). Harrison and Pliska (1981), as well as Kreps (1981) and Yan (1980), continued laying the foundation of arbitrage free pricing. Later, Dalang, Morton, and Willinger (1990) presented a fundamental theorem of asset pricing for finite discrete time. In a general semimartingale framework, the notion of no free lunch with vanishing risk Delbaen and Schachermayer (1994) ensured the existence of an equivalent martingale measure in the given (continuous-time) financial market. All these considerations have in common that the uncertainty of the model is given by a single probability measure.
Moving to models with multiple probability measures, the concept of pasting of probability measures models the intrinsic structure of dynamic convexity, see Riedel (2004) and Delbaen (2006). This type of time consistency is related to recursive equations, see Epstein and Schneider (2003); Chen and Epstein (2002), which can result in nonlinear expectation and generates a rational updating principle. Moreover, the backward stochastic differential equations can model drift-uncertainty, a dynamic sublinear expectation, see Peng (1997). However, in these models of uncertainty, all priors are related to a reference probability measure, i.e. all priors are equivalent or absolutely

[14] The precise PDE description of the G-expectation allows the definition of a universal density. Note that in the more general case we have a prior-dependent family of densities.

[15] The efficient market hypothesis by Fama (1970) introduces information efficiency, a concept closely related to Samuelson (1965), where the notion of a martingale reached neoclassic economics for the first time. Bachelier (1900) influenced the course of Samuelson's work.

continuous. Moreover, drift uncertainty does not create a significant change for a valuation principle of contingent claims.[16]
The possible insufficiency of equivalent prior models for an imprecise knowledge of the environment motivates the consideration of mutually singular priors as illustrated at the beginning of this introduction. The mathematical discussion of such frameworks can be found in Peng (2006); Nutz and Soner (2012); Bion-Nadal and Kervarec (2012). Epstein and Ji (2013a) provide a discussion in economic terms. Similarly to the present paper, the volatility uncertainty is encoded in a non-deterministic quadratic variation of the underlying noise process.
Recalling Gilboa and Schmeidler (1989), this axiomatization of uncertainty aversion represents a non-linear expectation via a worst case analysis. Similarly to risk measures, see Artzner, Delbaen, Eber, and Heath (1999),[17] the related set of representing priors may be not equivalent to each other. This important change permits the application of financial markets under volatility uncertainty. We refer to Avellaneda, Levy, and Paras (1995); Denis and Martini (2006) for a pricing principle of claims via a quasi sure stochastic calculus.
Jouini and Kallal (1995) consider a non-linear pricing caused by bid-ask spreads and transaction costs, where the price system is extended to a linear functional. In Araujo, Chateauneuf, and Faro (2012), pricing rules with finitely many state are considered.[18] A price space of sublinear functionals is discussed in Aliprantis and Tourky (2002). We quote the following interpretation of the classical equilibrium concept with linear prices and its meaning (see Aliprantis, Tourky, and Yannelis (2001)):

> *A linear price system summarizes the information concerning relative scarcities and at equilibrium approximates the possibly non-linear primitive data of the economy.*

The chapter is organized as follows. Section 2 introduces the primitives of the economic model and establishes the connection between our notion of viability and extensions of price systems. Section 3 introduces the security market model associated with the marketed space. We also discuss the corresponding G-Samuelson model. Section 4 concludes and discusses the results of the chapter and lists possible extensions. The first part of the appendix presents the details of the model and provides the theorem proofs. In the second part, we discuss mathematical foundations such as the space of price systems and a collection of results of stochastic analysis and G-expectations.

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[16] Cont (2006) notes that this assumption is "*actually quite restrictive: it means that all models agree on the universe of possible scenarios and only differ on their probabilities. For example, if $\mathbb{P}_0$ defines a complete market model, this hypothesis entails that there is no uncertainty on option prices!*"

[17] Markowitz (1952) postulated the importance of diversification, a fundamental principle in finance, which corresponds to sublinearity of risk measures.

[18] They establish a characterization of super-replication pricing rules via an identification of the space of frictionless claims.

3.2 Viability and Sublinear Price Systems

We begin by recapping the case where uncertainty is given by an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as it emphasizes sensible differences with regard to the uncertainty model posited in this chapter. Following, we introduce the uncertainty model as well as the related space of contingent claims. Then we discuss the space of sublinear price functionals. The last subsection introduces the economy, and Theorem 1 states an extension result.

Background: Classical Viability

Let there be two dates $t = 0, T$, claims at T are elements of the classical Hilbert lattice $L^2(\mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$. Price systems are given by linear and $L^2(\mathbb{P})$-continuous functionals. By Riesz representation theorem, elements of the related topological dual can be identified in terms of elements in $L^2(\mathbb{P})$. A strictly positive functional $\Pi : L^2(\mathbb{P}) \to \mathbb{R}$ evaluates a positive random variable X with $\mathbb{P}(X > 0) > 0$, such that $\Pi(X) > 0$.

A *price system* consists of a (closed) subspace $M \subset L^2(\mathbb{P})$ and a linear price functional $\pi : M \to \mathbb{R}$. The marketed space consists of contingent claims achievable in a frictionless manner. $\mathbb{A}(\mathbb{P})$ is the set of rational, convex, strictly monotone and $L^2(\mathbb{P})$-continuous preference relations on $\mathbb{R} \times L^2(\mathbb{P})$. The consistency condition for an economic equilibrium is given by the concept of viability. A price system is *viable* if there exists a preference relation $\succsim \in \mathbb{A}(\mathbb{P})$ and a bundle $(\hat{x}, \hat{X}) \in \mathbb{R} \times M$ with

$$(\hat{x}, \hat{X}) \in B(0, 0, \pi, M) \text{ and } (\hat{x}, \hat{X}) \succsim (x, X) \text{ for all } (x, X) \in B(0, 0, \pi, M),$$

where $B(x, X, \pi, M) = \{(y, Y) \in \mathbb{R} \times M : y + \pi(Y) \leq x + \pi(X)\}$ denotes the budget set. Harrison and Kreps (1979) prove the following fundamental result:

(M, π) *is viable if and only if there is a strictly positive extension* Π *of* π *to* $L^2(\mathbb{P})$.

Note that strict positivity implies $L^2(\mathbb{P})$-continuity. The proof is achieved by a Hahn-Banach argument and the usage of the properties of $\succsim$ such that Π creates a linear utility functional and hence a preference relation in $\mathbb{A}(\mathbb{P})$.

3.2.1 The Uncertainty Model and the Space of Claims

We begin with the underlying uncertainty model by considering possible scenarios which share neither the same probability measure nor the same null sets. Therefore it is not possible to assume the existence of a given reference probability measure when the null sets are not the same. For this reason we need a topological structure to formulate the uncertainty model.

Let Ω, the states of the world, be a complete separable metric space, $\mathcal{B}(\Omega) = \mathcal{F}$ the Borel σ-algebra of Ω and let $\mathcal{C}_b(\Omega)$ denote the set of all bounded continuous real valued functions. The uncertainty of the model is given by a weakly compact set of Borel probability measure $\mathcal{P} \subset \mathcal{M}_1(\Omega)$ on $(\Omega, \mathcal{F})$.[19]

[19]As shown in Denis, Hu, and Peng (2011), the related capacity $c(\cdot) = \sup_{P \in \mathcal{P}} P(\cdot)$ is *regular* if and only if the set of priors is relatively compact. Here, regularity refers to a

In the following example we illustrate a construction for $\mathcal{P}$, applied in the dynamic setting of Section 3.

Example 1 *We consider a time interval* $[0,T]$*, the Wiener measure* P_0 *on the state space of continuous paths* $\Omega = \{\omega : \omega \in C([0,T];\mathbb{R}) : \omega_0 = 0\}$ *and the canonical process* $B_t(\omega) = \omega_t$*. Let* $\mathbb{F}^o = (\mathcal{F}^o_t)_{t\in[0,T]}$*,* $\mathcal{F}^o_t = \sigma(B_s, s \in [0,t])$ *be the raw filtration of* B*. The strong formulation of volatility uncertainty is based upon martingale laws with stochastic integrals:*

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}, \quad X^\alpha_t = \int_0^t \alpha_s^{1/2} dB_s,$$

where the integral is defined $\mathbb{P}_0$ *almost surely. The process* α *is* $\mathbb{F}^o$*-adapted and has a finite first moment. A set* $\mathcal{D}$ *of* α*'s builds* $\mathcal{P}$ *via the associated prior* $\mathbb{P}^\alpha$*, such that* $\{\mathbb{P}^\alpha : \alpha \in \mathcal{D}\} = \mathcal{P}$ *is weakly compact.*[20]

We describe the set of contingent claims. Following Huber and Strassen (1973), for each $\mathcal{F}$-measurable real function X such that $E^{\mathbb{P}}[X]$ exists for every $\mathbb{P} \in \mathcal{P}$, define the upper expectation operator by $\mathcal{E}^{\mathcal{P}}(X) = \sup_{P\in\mathcal{P}} E^{\mathbb{P}}[X]$.[21] We suggest the following norm for the space of contingent claims, given by the capacity norm $c_{2,\mathcal{P}}$, defined on $\mathcal{C}_b(\Omega)$ by

$$c_{2,\mathcal{P}}(X) = \mathcal{E}^{\mathcal{P}}\left(|X|^2\right)^{\frac{1}{2}}.$$

Define the completion of $\mathcal{C}_b(\Omega)$ under the so called "Lebesgue prolongation" of $c_{2,\mathcal{P}}$[22] by $\mathcal{L}^2(\mathcal{P}) = \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$, and let $L^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$ be the quotient space of $\mathcal{L}^2(\mathcal{P})$ by the $c_{2,\mathcal{P}}$ null elements $\mathcal{N}$. We do not distinguish between classes and their representatives. Two random variables $X, Y \in L^2(\mathcal{P})$ can be distinguished if there is a prior in $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(X \neq Y) > 0$.
It is possible to define an order relation $\leq$ on $L^2(\mathcal{P})$. Classical arguments prove that $(L^2(\mathcal{P}), c_{2,\mathcal{P}}, \leq)$ is a Banach lattice, see Appendix A.1 for details. We consider the space of contingent claims $L^2(\mathcal{P})$ so that under every probability model $\mathbb{P} \in \mathcal{P}$, we can evaluate the variance of a contingent claim. Properties of random variables are required to be true $\mathcal{P}$-quasi surely, i.e. $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$. This indicates that in contrast to drift uncertainty, a related stochastic calculus cannot be based only on one probability space.

reasonable continuity property. In Appendix B.2, we recall some related notions and we give a criterion for the weak compactness of $\mathcal{P}$ when it is constructed via the quadratic variation and a canonical process.

[20] In order to define universal objects, we need the pathwise construction of stochastic integrals, (see Föllmer (1981), Karandikar (1995)).

[21] It is easily verified that $\mathcal{C}_b(\Omega) \subset \{X \ \mathcal{F}\text{-measurable} : \mathcal{E}^{\mathcal{P}}(X) < \infty\}$ holds and $\mathcal{E}^{\mathcal{P}}(\cdot)$ satisfies the property of a sublinear expectation. For details, see Appendix A.1.1, Peng (2010) and Appendix B.3.

[22] We refer to Section 2 in Feyel and de La Pradelle (1989), see also Section 48.7-8 in Choquet (1953) and Section A in Dellacherie (1972).

3.2.2 Scenario-Based Viable Price Systems

This subsection is divided into three parts. First, we introduce the dual space where linear and $c_{2,\mathcal{P}}$-continuous functionals are the elements. As discussed in the introduction, we allow sublinear prices as well. This forces us to extend the linear price space where we discuss two operations on the new price space and take a leaf out of Aliprantis and Tourky (2002). We integrate over the set of priors for the linear consolidation of functionals. In Proposition 1, we list standard properties of coherent price functionals. The last part in this subsection focuses on the consolidation of prior-dependent price systems.

Linear and $c_{2,\mathcal{P}}$-Continuous Price Systems on $L^2(\mathcal{P})$

We present the basis for the modified concept of viable price systems. The mutually singular uncertainty generates a different space of contingent claims. This gives us a new topological dual space $L^2(\mathcal{P})^*$. The discussion of the dual space is only the first step to get a reasonable notion of viability which accounts for the present type of uncertainty. In the second part of the Appendix, we give a result which asserts that the topological dual, the space of all linear and $c_{2,\mathcal{P}}$-continuous functionals on $L^2(\mathcal{P})$, is given by

$$L^2(\mathcal{P})^* = \left\{ E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] : \mathbb{P} \in \mathcal{P} \text{ and } \psi_{\mathbb{P}} \in L^2(\mathbb{P}) \right\}.$$

This representation delivers an appropriate form for possible price systems. The random variable $\psi_{\mathbb{P}}$ in the representation matches the classical state price density of the Riesz representation when only one prior $\{\mathbb{P}\} = \mathcal{P}$ is present. The space's description allows for an interpretation of a state price density $\psi_{\mathbb{P}}$ based on some prior $\mathbb{P} \in \mathcal{P}$. The stronger capacity norm $c_{2,\mathcal{P}}(\cdot)$ in comparison to the classical single prior $L^2(\mathbb{P})$-norm implies a richer dual space, controlled by the set of priors $\mathcal{P}$. Moreover, one element in the dual space implicitly selects a prior $\mathbb{P} \in \mathcal{P}$ and ignores all other priors. This foreshadows the insufficiency of a linear pricing principle under the present uncertainty model, as indicated in the introduction.

The Price Space of Nonlinear Expectations

In this paragraph we introduce a set of sublinear functionals defined on $L^2(\mathcal{P})$. The singular prior uncertainty of our model induces the appearance of non-linear price systems.[23] Let $k(\mathcal{P})$ be the convex hull of $\mathcal{P}$. The *coherent price space* of $L^2(\mathcal{P})$ generated by linear $c_{2,\mathcal{P}}$-continuous functionals is given by

$$L^2(\mathcal{P})^{\circledast}_{+} = \left\{ \Psi : L^2(\mathcal{P}) \to \mathbb{R} : \Psi(\cdot) = \sup_{\mathbb{P} \in \mathcal{R}} E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] \text{ with } \mathcal{R} \subset k(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})_+ \right\}.$$

[23] A subcone of the super order dual is considered in Aliprantis and Tourky (2002). They introduce the lattice theoretic framework and consider the notion of a semi lattice. In Aliprantis, Florenzano, and Tourky (2005); Aliprantis, Tourky, and Yannelis (2001) general equilibrium models with a superlinear price systems are considered in order to discuss a non-linear theory of value.

Elements in $L^2(\mathcal{P})^{\circledast}_{+}$ are constructed by a set of $c_{2,\mathcal{P}}$-continuous linear functionals $\{\Pi_{\mathbb{P}} : L^2(\mathcal{P}) \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}}$, which are consolidated by a combination of the point-wise maximum and convex combination. Strictly positive functionals in $L^2(\mathcal{P})^{\circledast}_{++}$ satisfy additionally $\Psi(X) > 0$ for every $X \in L^2(\mathcal{P})_+$ with $\mathbb{P}(X > 0) > 0$ for some $\mathbb{P} \in \mathcal{P}$. The following example illustrates how a sublinear functional in $L^2(\mathcal{P})^{\circledast}_{+}$ can be constructed.

Example 2 *Let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a partition of $\mathcal{P}$. And let $\mu_n : \mathcal{B}(\mathcal{M}_1(\Omega)) \to \mathbb{R}$ be a positive measure with support $\mathcal{P}_n$ and $\mu_n(\mathcal{P}_n) = 1$. The resulting prior $\mathbb{P}_n(\cdot) = \int_{\mathcal{P}_n} \mathbb{P}(\cdot)\mu_n(d\mathbb{P})$ is given by a weighting operation Γ_{μ_n}. When we apply Γ_{μ_n} to the density $\psi_{\mathbb{P}}$ we get $\bar{\psi}_n(\omega) = \int_{\mathcal{P}_n} \psi_{\mathbb{P}}(\omega)\mu_n(d\mathbb{P})$, $\omega \in \Omega$. These new prior density pairs $(\bar{\psi}_n, \mathbb{P}_n)$ can then be consolidated by the supremum operation of the expectations, i.e. $\Gamma(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}})(\cdot) = \sup_{n\in\mathbb{N}} E^{\mathbb{P}_n}\left[\bar{\psi}_n \cdot\right]$.*

For further details of Example 2, see Appendix A.1.1 and Appendix B.1.1. The following proposition discusses properties and the extreme case of functionals in the price space $L^2(\mathcal{P})^{\circledast}_{+}$. A full lattice-theoretical discussion of our price space $L^2(\mathcal{P})^{\circledast}_{+}$ lies beyond the scope of this chapter.[24]

Proposition 1 *Functionals in $L^2(\mathcal{P})^{\circledast}_{+}$ satisfy 1. sub-additivity, 2. positive homogeneity, 3. constant preserving, 4. monotonicity and 5. $c_{2,\mathcal{P}}$-continuity.*[25]
Moreover, for every positive measure μ of $\mathcal{B}(\mathcal{P})$ with $\mu(\mathcal{P}) = 1$, we have the following inequality for every $X \in L^2(\mathcal{P})$

$$E^{\mathbb{P}_\mu}[\psi_\mu X] \leq \sup_{\mathbb{P}\in k(\mathcal{P})} E^{\mathbb{P}}[\psi_{\mathbb{P}} X], \quad \textit{where } \mathbb{P}_\mu(\cdot) = \int_{\mathcal{P}} \mathbb{P}(\cdot)\mu(d\mathbb{P}).$$

Below, we introduce the consolidation operation Γ for the prior-dependent price systems. $\Gamma(\mathcal{P})$ refers to the set of priors in $\mathcal{P}$ which are relevant. In Example 2, we observe $\Gamma_{\mu_n}(\mathcal{P}) = \mathcal{P}_n$.

Remark 1 *Price systems in $L^2(\mathcal{P})^{\circledast}_{+}$ resemble the structure of ask prices. However, the related bid price can then be described by the super order dual $-L^2(\mathcal{P})^{\circledast}_{-}$, since $\sup(\cdot) = -\inf(-\cdot)$. From this perspective, we could also construct a fully nonlinear, monotone and positive homogeneous price systems Ψ as elements in $L^2(\mathcal{P})^{\circledast}_{+} - L^2(\mathcal{P})^{\circledast}_{-}$. For some cover $\mathcal{P}_+ \cup \mathcal{P}_- = \mathcal{P}$ we have*

$$X \mapsto \Psi(X) = \sup_{\mathbb{P}\in\mathcal{P}_+} E^{\mathbb{P}}[\psi_{\mathbb{P}} X] + \inf_{\mathbb{P}'\in\mathcal{P}_-} E^{\mathbb{P}'}[\psi_{\mathbb{P}'} X]. \tag{1}$$

At this stage, the nonlinear price functional can be seen as a fully nonlinear expectation $\mathfrak{E}(\cdot) \leq \mathcal{E}^{\mathcal{P}}(\cdot)$, being dominated by $\mathcal{E}^{\mathcal{P}}$ on $L^2(\mathcal{P})$ (see Remark 3.1. below and Section 8 of Chapter III in Peng (2010) for more details).

[24] However, it is worthwhile to mention that Theorem 12 in Denis, Hu, and Peng (2011) characterizes σ-order continuity of sublinear functionals in $L^2(\mathcal{P})^{\circledast}_{+}$.

[25] Formally this means: 1.$\Psi(X+Y) \leq \Psi(X) + \Psi(Y)$ for all $X, Y \in L^2(\mathcal{P})$, 2.$\Psi(\lambda X) = \lambda\Psi(X)$ for all $\lambda \geq 0$, $X \in L^2(\mathcal{P})$, 3.$\Psi(c) = c$ for all $c \in \mathbb{R}$, 4. If $X \geq Y$ then $\Psi(X) \geq \Psi(Y)$ for all $X, Y \in L^2(\mathcal{P})$ and 5. Let $(X_n)_{n\in\mathbb{N}}$ converge in $c_{2,\mathcal{P}}$ to some X, then we have $\lim_n \Psi(X_n) = \Psi(X)$.

Marketed Spaces and Scenario-Based Price Systems
In the spirit of Aliprantis, Florenzano, and Tourky (2005) our commodity-price duality is given by the following pairing $\langle L^2(\mathcal{P}), L^2(\mathcal{P})^{\circledast}_{+}\rangle$.
For the single prior framework, viability and the extension of the price system are associated with each other. This structure allows only for linear prices. In our framework this corresponds to a consolidation via the Dirac measure $\delta_{\{\mathbb{P}\}}$ for some $\mathbb{P} \in \mathcal{P}$, so that $\Gamma(\mathcal{P}) = \{\mathbb{P}\}$.
We begin by introducing the marketed subspaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. The underlying idea is that any claim in $M_{\mathbb{P}}$ can be achieved, whenever $\mathbb{P} \in \mathcal{P}$ is the true probability measure. This input data resembles a partial equilibrium, depending on the prior under consideration. [26] Claims in the marketed space $M_{\mathbb{P}}$ can be bought and sold whenever the related prior governs the economy. We illustrate this in the following examples.

Example 3 *1. Let us consider the role of marketed spaces in the very simple situation when no prior dependency is present, i.e.* $M_{\mathbb{P}} = M$ *for every* $\mathbb{P} \in \mathcal{P}$. *Specifically, set*

$$M = \left\{X \in L^2(\mathcal{P}) : E^{\mathbb{P}}[X] = const. \quad for\ every\ \mathbb{P} \in \mathcal{P}\right\}.$$

As we show in Corollary 1, this space consists of (unambiguous) contingent claims which do not depend on the prior of the corresponding linear expectation operator. It turns out that this space has a strong connection to symmetric martingales.

2. Suppose the set of priors is constructed by the procedure in Example 1. The marketed spaces differ because of the $\mathbb{P}$*-dependent replication condition. Specifically, this is encoded in an equation which holds only* $\mathbb{P}$*-almost surely. Let the marketed space be generated by the quadratic variation of an uncertain asset with terminal payoff* $\langle B\rangle_T$ *and a riskless asset with payoff* 1. *We have by construction* $\langle B\rangle_T = \int_0^T \alpha_s ds\ \mathbb{P}^{\alpha}$*-a.s., the marketed space under* $\mathbb{P}^{\alpha}$ *as given by*

$$M_{\mathbb{P}^{\alpha}} = \left\{X \in L^2(\mathbb{P}^{\alpha}) : X = a + b \cdot \int_0^T \alpha_s ds\ \mathbb{P}^{\alpha}\text{-a.s.},\ a, b \in \mathbb{R}\right\}.$$

But $\langle B\rangle$ *coincides with the* $\mathbb{P}$*-quadratic variation under every martingale law* $\mathbb{P} \in \mathcal{P}$. *Therefore a different* $\hat{\alpha}$ *builds a different marketed space* $M_{\mathbb{P}^{\hat{\alpha}}}$. *Suppose* $\alpha = \hat{\alpha}\ \mathbb{P}_0$*-a.s. on* $[0, s]$ *for some* $s \in (0, T]$ *then we have* $M_{\mathbb{P}^{\alpha}} \cap M_{\mathbb{P}^{\hat{\alpha}}}$ *consists also of non trivial claims. Note, that* $\mathbb{P}^{\alpha}$ *and* $\mathbb{P}^{\hat{\alpha}}$ *are neither equivalent nor mutually singular.*[27]

[26] One may think that a countable set of scenarios could be sufficient. As in Bion-Nadal and Kervarec (2012), the norm can be represented via different countable dense subsets of priors. However, for the marketed space we allow for a direct prior dependency of all possible scenarios $\mathcal{P}$. This implies that different choices of countable and dense scenarios can deliver different price systems (see Definition 1 below).

[27] The event $\{\omega : \langle B\rangle_r(\omega) = \int_0^r \alpha_t(\omega)dt, r \in [0, s]\}$ has positive mass under both priors, but the priors restricted to the complement are mutually singular. We refer to Example 3.7 in Epstein and Ji (2013a) for a similar example.

We fix linear price systems $\pi_{\mathbb{P}}$ on $M_{\mathbb{P}}$. As illustrated in Example 3, it is possible that the $\pi_{\mathbb{P}_1}, \pi_{\mathbb{P}_2} \in \{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ have a common domain, i.e $M_{\mathbb{P}_1} \cap M_{\mathbb{P}_2} \neq \{0\}$. In this case one may observe different evaluations among different priors, i.e $\pi_{\mathbb{P}_1}(X) \neq \pi_{\mathbb{P}_2}(X)$ with $X \in M_{\mathbb{P}_1} \cap M_{\mathbb{P}_2}$. To account for this possible phenomenon, we associate a linear price system $\pi_P : M_{\mathbb{P}} \to \mathbb{R}$ for each marketed space. In this context, we posit that coherence is based on *sublinear* price systems,[28] as illustrated in the following example (see also Heath and Ku (2006) for a discussion).

Example 4 *Let the uncertainty model consist of two priors $\mathcal{P} = \{\mathbb{P}, \mathbb{P}'\}$. If $\mathbb{P}$ is the true law, the market model is given by the set of marketed claims $M_{\mathbb{P}}$ priced by a linear functional $\pi_{\mathbb{P}}$. If $\mathbb{P}'$ is the true law, we get $M_{\mathbb{P}'}$ and $\pi_{\mathbb{P}'}$. As in Example 3.2, constructing a claim via self-financing strategies implies an equality of portfolio holdings that must be satisfied almost surely only for the particular probability measure. If the trader could choose between the sets $M_{\mathbb{P}'} + M_{\mathbb{P}}$ to create a portfolio, additivity would be a natural requirement with the consistency condition $\pi_{\mathbb{P}'} = \pi_{\mathbb{P}}$ on $M_{\mathbb{P}'} \cap M_{\mathbb{P}}$. However, the trader is neither free to choose a mixture of claims, nor may she choose a scenario, simply because of existing ignorance.*
An equality of prices at the intersection is less intuitive, since the different priors create a different price structure in each scenario. We therefore argue, that $\sup(\pi_{\mathbb{P}'}(X), \pi_{\mathbb{P}}(X))$ is a robust and reasonable price for a claim $X \in M_{\mathbb{P}'} \cap M_{\mathbb{P}}$ in our multiple prior framework. This yields to subadditivity. In contrast to the classical law of one price, linearity of the pricing functional is merely true under a fixed prior.[29]

The set $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ of linear scenario-based price functionals inherit all the information of the underlying financial market. In the single prior setting incompleteness means $M_{\mathbb{P}} \neq L^2(\mathbb{P})$.[30] $M_{\mathbb{P}} \otimes M_{\mathbb{P}'}$ refers to the Cartesian product of the relevant basis elements in $M_{\mathbb{P}}$ and $M_{\mathbb{P}'}$.

Definition 1 *Fix subspaces $\{M_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ with $M_{\mathbb{P}} \subset L^2(\mathbb{P})$ and a set $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ of linear price functionals $\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}$. A* price system *for $(\{\pi_{\mathbb{P}}\}_{P\in\mathbb{P}}, \Gamma)$ is a functional on the Cartesian product of Γ-relevant scenarios*

$$\pi(\otimes\mathcal{P}) : \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} M_{\mathbb{P}} \to \mathbb{R}$$

[28] This price system can be seen as an envelope of the price correspondence $\pi(X) = \{\pi_P(X) : X \in M_{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, as in Clark (1993).

[29] Sublinearity induced by market frictions is conceptually different. For instance, in Jouini and Kallal (1999) one convex set of marketed claims is equipped with a convex pricing functional, in which case, the possibility of different scenarios is not included.

[30] Note that Ω is separable by assumption, hence $L^2(\mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a separable Hilbert space for each $\mathbb{P} \in \mathcal{P}$ and admits a countable orthonormal basis. In terms of Example 2, $\mathbb{P}_0$ is the Wiener measure. In this situation, $L^2(\mathbb{P}_0)$ can be decomposed via the Wiener chaos expansion. A similar procedure could be done for the canonical process X^α related to some $\mathbb{P}^\alpha$. So we can generate an orthonormal basis for each $L^2(\mathbb{P}^\alpha)$, with $\alpha \in \mathcal{D}$. However, we take an infinite product, if $|\Gamma(\mathcal{P})| \not< \infty$, since an infinite orthonormal sum is not in general a Hilbert space.

such that the projection to $M_{\mathbb{P}}$ *is given by the restriction* $\pi(\otimes\mathcal{P})_{\upharpoonright M_{\mathbb{P}}} = \pi_{\mathbb{P}\upharpoonright M_{\mathbb{P}}}$.

Each $\mathbb{P}$-related marketed space $M_{\mathbb{P}}$ consists of contingent claims which can be achieved frictionless, when $\mathbb{P}$ is the true law. We have a set of different price systems $\{\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}}$. When we aim to establish a meaningful consolidation of the scenarios we need an additional ingredient, namely Γ. This consolidation determines the operator which maps an extension of $\pi(\otimes\mathcal{P})$ into the price space $L^2(\mathcal{P})^{\circledast}_{++}$ and therefore influences the whole marketed space.

3.2.3 Preferences and the Economy

Having discussed the commodity price dual and the role of the consolidation of linear price systems, we introduce agents which are characterized by their preference of trades on $\mathbb{R} \times L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. There is a single consumption good, a numeraire, which agents will consume at $t = 0, T$. Thus, bundles (x, X) are elements in $\mathbb{R} \times L^2(\mathbb{P})$, which are the units at time zero and time T with uncertain outcome. We call the set of rational preference relations $\succsim_{\mathbb{P}}$ on $\mathbb{R} \times L^2(\mathbb{P})$, $\mathbb{A}(\mathbb{P})$, which satisfies convexity, strict monotonicity, and $L^2(\mathbb{P})$-continuity. Let

$$B(x, X, \pi_{\mathbb{P}}, M_{\mathbb{P}}) = \{(y, Y) \in \mathbb{R} \times M_{\mathbb{P}} : y + \pi_{\mathbb{P}}(Y) \leq x + \pi_P(X)\}$$

denote the *budget set* for a price functional $\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}$. We are ready to define an appropriate notion of viability. Such a minimal consistency criterion can be regarded as an inverse no trade equilibrium condition.

Definition 2 *A price system is* scenario-based viable*, if for each* $\mathbb{P} \in \Gamma(\mathcal{P})$ *there is a preference relation* $\succsim_{\mathbb{P}} \in \mathbb{A}(\mathbb{P})$ *and a bundle* $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) \in B(0, 0, \pi_{\mathbb{P}}, M_{\mathbb{P}})$ *such that*

$$(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) \text{ is } \succsim_{\mathbb{P}}\text{-maximal on } B(0, 0, \pi_{\mathbb{P}}, M_{\mathbb{P}}).$$

The conditions are necessary and sufficient for a classical economic equilibrium under each scenario $\mathbb{P} \in \Gamma(\mathcal{P})$, when we find such preference relations. Note that this definition has up to some degree the preference flavor of Bewley (2002). In the case of Example 3.1, scenario-based viability is exactly the existence of an agent with Bewley preferences and a maximal consumption bundle $(\hat{x}, \hat{X})$, not depending on the prior.[31]
In the following, we relate the viability of $(\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}, \Gamma)$ with price systems in $L^2(\mathcal{P})^{\circledast}_{+}$. Let $M^{\mathcal{P}}_{\mathbb{P}} = M_{\mathbb{P}} \cap L^2(\mathcal{P})$, with $\mathbb{P} \in \mathcal{P}$.

[31] The fundamental theorem of asset pricing in Dybvig and Ross (2003) contains a third equivalent statement, the existence of an agent (preferring more than less) being in an optimal state. The adequate concept of strict monotone preferences is subtle and important when the uncertainty is given by a set of mutually singular priors. For instance, the classical strict monotonicity ($X \geq Y$ and $X \neq Y$ implies $X \succ Y$) seems to be too strong. For instance, maxmin preferences of Gilboa and Schmeidler (1989) do not satisfy this monotonicity under the $\mathcal{P}$.

Theorem 1 *A price system* $(\{\pi_{\mathbb{P}}\}_{P\in\mathcal{P}},\Gamma)$ *is scenario-based viable if and only if there is an* $\Psi \in L^2(\mathcal{P})^{\circledast}_{++}$ *such that* $\pi_{\mathbb{P}\upharpoonright M^{\mathcal{P}}_{\mathbb{P}}} \leq \Psi_{\upharpoonright M^{\mathcal{P}}_{\mathbb{P}}}$ *for each* $\mathbb{P} \in \Gamma(\mathcal{P})$*.*

This characterization of scenario-based viability takes scenario-based marketed spaces $\{M_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ as given. Moreover, the consolidation operator Γ is a given characteristic of the coherent price system. With this in mind, one should think that in a general equilibrium system the locally given prices $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ should be part of it. The extension we perceive can be seen as a regulated and coherent price system for every claim in $L^2(\mathcal{P})$.
In comparison to the single prior case, the structure of incompleteness depends on the set of relevant priors $\Gamma(\mathcal{P})$. As described in Example 3.2, this is a natural situation. As such, prior-dependent prices $\pi_{\mathbb{P}}$ are also plausible. The expected payoff as a pricing principle depends on the prior under consideration, as well. In this way, the concept of scenario-based prices accounts for every Γ-relevant price system simultaneously.
As indicated in Example 3.1, there is a closed subspace of unambiguous claims where the valuation is unique. In Section 3, we use the related symmetry property for the introduction of a reasonable martingale notion. Let $\mathcal{R} \subset \mathcal{P}$ and define the $\mathcal{R}$-*marketed space* by

$$\mathbb{M}(\mathcal{R}) = \left\{X \in L^2(\mathcal{P}) : E^{\mathbb{P}}[X] \text{ is constant for all } \mathbb{P} \in \mathcal{R}\right\}.$$

Only the continent claims in $\mathbb{M}(\mathcal{R})$ reduce the valuation to a linear pricing, if $\Gamma(\mathcal{P}) = \mathcal{R}$.[32] Claims in $\mathbb{M}(\mathcal{R})$ are unambiguous. This can also be formulated as a property of events $\mathcal{U}(\mathcal{R}) = \{A \in \mathcal{F} : P(A) \text{ is constant for all } \mathbb{P} \in \mathcal{R}\}$.[33] From Theorem 1 we have the following corollary.

Corollary 1 *Every* Ψ *in Theorem 1 is linear and* $c_{2,\mathcal{P}}$*-continuous on* $\mathbb{M}(\Gamma(\mathcal{P}))$*.*

We have two operations which constitute the distillation of uncertainty. This consolidation can be seen as a characterization of the Walrasian auctioneer, in which case diversification should be encouraged. But this refers to the sublinearity of Ψ.

Remark 2 *One may ask which* Γ *is appropriate. Such a question is related to the concept of mechanism design. The market planner can choose a consolidation that influences the indirect utility of a reported preference relation. However, the full discussion of these issues lies beyond the scope of this chapter.*[34]

[32] Or unless Γ is given a priori by a linear pricing, e.g. $\Gamma = \delta_{\{\mathbb{P}\}}$ for some $\mathbb{P} \in \mathcal{P}$.

[33] Note, that for the single prior case every closed subspace of $L^2(\mathbb{P})$ can be identified with a sub σ-algebra in terms of a projection via the conditional expectation operator. Although $\mathcal{U}$ is not a σ-algebra, but a Dynkin System, it identifies in a similar way a certain subspace. See also Epstein and Zhang (2001) for a definition of unambiguous events and an axiomatization of preferences on this domain.

[34] A starting point could be Lopomo, Rigotti, and Shannon (2009), who consider a mechanism design problem under Knightian uncertainty.

3.3 Asset Markets and Symmetric Martingales

We extend the primitives with trading dates and trading strategies. A time interval is considered where the market consists of a riskless security and a security under volatility uncertainty. Within the financial market model, we discuss the modified notions of arbitrage and equivalent martingale measures. Theorem 2 associates scenario-based viability with equivalent symmetric martingale measure sets. The last section considers the so called G-framework. Here, the uncertain security process is driven by a G-Itô process, which shows that the concept of symmetric martingale measure sets is far from empty.

Background: Risk-neutral asset pricing with one prior
In order to introduce dynamics and trading dates, we fix a time interval $[0, T]$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Fix an $\mathbb{F}$-adapted risky asset price $(S_t) \in L^2(\mathbb{P} \otimes dt)$ and a riskless bond $S^0 \equiv 1$. We next review some terminology. The portfolio process of a strategy $\eta = (\eta^0, \eta^1)$ is called X^η. Simple self-financing strategies are piecewise constant $\mathbb{F}$-adapted processes η such that $dX^\eta = \eta dS$, which we call $\mathcal{A}(\mathbb{P})$. A $\mathbb{P}$-arbitrage in $\mathcal{A}(\mathbb{P})$ is a strategy (with zero initial capital) such that $X^\eta_T \geq 0$ and $\mathbb{P}\left(X^\eta_T > 0\right) > 0$.
A claim is marketed, i.e. $X \in M$, if there is a $\eta \in \mathcal{A}(\mathbb{P})$ such that $X = \eta_T S_T$ $\mathbb{P}$-a.s., then we have the (by the law of one price) $\pi(X) = \eta_0 S_0$. An equivalent martingale measure (EMM) $\mathbb{Q}$ must satisfy that S is a $\mathbb{Q}$-martingale and $d\mathbb{Q} = \psi d\mathbb{P}$, where $\psi \in L^2(\mathbb{P})_{++}$ is a Radon Nykodym-Density with respect to $\mathbb{P}$. Theorem 2 of Harrison and Kreps (1979) states the following:

Under no $\mathbb{P}$-arbitrage, there is a one to one correspondence between the continuous linear and strictly positive extension of $\pi : M \to \mathbb{R}$ to $L^2(\mathbb{P})$ and a EMM $\mathbb{Q}$. The relation is given by $\mathbb{Q}(B) = \Pi(1_B)$ and $\Pi(X) = E^{\mathbb{Q}}[X]$, $B \in \mathcal{F}_T$ and $X \in L^2(\mathbb{P})$.

This result can be seen as a preliminary version of the first fundamental theorem of asset pricing.

3.3.1 Volatility Uncertainty, Dynamics and Arbitrage

We specify the mathematical framework and the modified notions, such as arbitrage. The present uncertainty model $(\Omega, \mathcal{F}, \mathcal{P})$ is based on the explicit formulation of volatility uncertainty. Afterwards, we introduce the notion of a martingale with respect to a conditional sublinear expectation, the financial market and the robust arbitrage concept.

Dynamics and Martingales under Sublinear Expectation

The principle idea is to transfer the results from Section 2 into a dynamic setup. The specification in Example 1 of Section 2.1 serves as our uncertainty model. We can directly observe the sense in which the quadratic variation creates volatility uncertainty. We introduce the sublinear expectation

$\mathcal{E} : L^2(\mathcal{P}) \to \mathbb{R}$ given by the supremum of expectations of $\mathcal{P} = \{\mathbb{P}^\alpha : \alpha \in \mathcal{D}\}$. It is possible to work within the larger space $\hat{L}^2(\mathcal{P})$. An explicit representation of $\hat{L}^2(\mathcal{P})$ is given in Appendix A.1. Moreover, we assume that $\mathcal{P}$ is stable under pasting (see Appendix A.2. for details).
As we aim to equip the financial market with the dynamics of a sublinear conditional expectation, we introduce the information structure of the financial market given by an augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The setting is based on the dynamic sublinear expectation terminology as instantiated by Nutz and Soner (2012).
We give a generalization of Peng's G-expectation as an example, satisfying the weak compactness of $\mathcal{P}$ when the sublinear expectation is represented in terms of a supremum of linear expectations. In Section 3.3 and in Appendix B.3, we consider the G-expectation as an important special case. That said, a possible association of results in Section 2 depends heavily on the weak compactness of the generated set of priors $\mathcal{P}$.

Example 5 *Suppose a trader is confronted with a pool of models describing volatility, such as the stochastic volatility model in Heston (1993). After a statistical analysis of the data, two models remain plausible $\mathbb{P}^\alpha$ and $\mathbb{P}^{\hat{\alpha}}$. Nevertheless, the implications for the trading decision deviate considerably. Even the asset span on its own depends on each scenario (see Example 3). A mixture of both models does not change this uncertain situation at all. In order to address the possibilistic issue, let us define the universal extreme cases $\underline{\sigma}_t = \inf(\alpha_t, \hat{\alpha}_t)$ and $\overline{\sigma}_t = \sup(\alpha_t, \hat{\alpha}_t)$. When thinking about a reasonable* uncertainty management, *no scenario between $\underline{\sigma}$ and $\overline{\sigma}$ should be ignored. The uncertainty model which accounts for all these cases is given by*

$$\mathcal{P} = \{\mathbb{P}^\alpha : \alpha_t \in [\underline{\sigma}_t, \overline{\sigma}_t] \quad \mathbb{P}_0 \otimes dt \; a.e.\}.$$

A related construction of a sublinear conditional expectation is achieved in Nutz (2012), where the deterministic bounds of the G-expectation are replaced by path dependent bounds.[35]

In the following, we introduce an appropriate concept for the dynamics of the continuous-time multiple-prior uncertainty model. The associated objectives are trading dates, the information structure and the price process (as the carrier of the uncertainty). In order to introduce the price process $S = (S_t)_{t\in[0,T]}$ of an uncertain and long lived security, we have to impose further primitives. Define the time depending set of priors

$$\mathcal{P}(t, \mathbb{P})^o = \{\mathbb{P}' \in \mathcal{P} : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_t^o\}.$$

This set of priors consists of all extensions $\mathbb{P} : \mathcal{F}_t^o \to [0, 1]$ from $\mathcal{F}_t^o$ to $\mathcal{F}$ in $\mathcal{P}$. In other words, $\mathcal{P}(t, \mathbb{P})^o$ contains exactly all probability measures in $\mathcal{P}$ defined

[35] This framework is also included in Epstein and Ji (2013a). In this setting, drift and volatility uncertainty are considered simultaneously. Drift uncertainty or κ-ambiguity are well known terms in financial economics. A coherent and well-developed theory, known as g-expectation, is available under a Brownian filtration.

on $\mathcal{F}$ that agree with $\mathbb{P}$ in the events up to time t. Fix a contingent claim $X \in L^2(\mathcal{P})$. In Nutz and Soner (2012), the unique existence of a sublinear expectation $(\mathcal{E}_t^{\mathcal{P}}(X))_{t\in[0,T]}$ is provided by the following construction[36]

$$\mathcal{E}_t^{\mathcal{P}}(X)^o = \underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})^o}{{}^{\mathbb{P}}\text{ess sup}}\; E^{\mathbb{P}'}[X|\mathcal{F}_t] \quad \mathbb{P}\text{-a.s.} \quad \text{for all } \mathbb{P}\in\mathcal{P}, \quad \lim_{r\downarrow t}\mathcal{E}_r(X)^o = \mathcal{E}_t(X).$$

The conditional expectation operator satisfies the Law of Iterated Expectation, i.e. $\mathcal{E}_s^{\mathcal{P}}(\mathcal{E}_t^{\mathcal{P}}) = \mathcal{E}_s^{\mathcal{P}}$ with $s \leq t$. We can define a martingale similarly to the single prior setting.[37] The nonlinearity implies that if a process $X = (X_t)_{t\in[0,T]}$ is a martingale under $\mathcal{E}_t^{\mathcal{P}}(\cdot)$ then $-X$ is not necessarily a martingale.

Definition 3 *An $\mathbb{F}$-adapted process $X = (X_t)_{t\in[0,T]}$ is a $\mathcal{P}$-*martingale *if*

$$\mathcal{E}_s^{\mathcal{P}}(X_t) = X_s \quad \mathcal{P}\text{-q.s.}, \quad \text{for all } s \leq t.$$

We call X a symmetric $\mathcal{P}$-martingale *if X and $-X$ are both $\mathcal{P}$-martingales.*

In the next subsection we discuss the martingale property of asset prices processes under a modified sublinear expectation. As we will see, the space $\mathbb{M}(\mathcal{P})$ is closely related to symmetric martingales. Conceptually, the symmetry refers to a generalized Put-Call parity and formalizes the uncertainty-neutral valuation in terms of martingales.

The Primitives of the Financial Market and Arbitrage

For the sake of simplicity, we assume that the riskless asset is $S_t^0 = 1$, for every $t \in [0,T]$, i.e. the interest rate is zero. We call the related abstract financial market $\mathcal{M}(1,S)$ on the filtered space uncertainty space $(\Omega,\mathcal{F},\mathcal{P};\mathbb{F})$, whenever the price process of the uncertain asset $S = (S_t)_{t\in[0,T]}$ satisfies $S_t \in L^2(\mathcal{P})$ for every $t\in[0,T]$ and $\mathbb{F}$-adaptedness.
A *simple trading strategy*[38] is an $\mathbb{F}$-adapted stochastic process $(\eta_t)_{t\in[0,T]}$ in $L^2(\mathcal{P}) \times L^2(\mathcal{P})$ when there is a finite sequence of dates $0 < t_0 \leq \cdots \leq t_N = T$ such that $\eta = (\eta^0,\eta^1)$ can be written with $\eta^i \in L^2(\Omega,\mathcal{F}_{t_i},\mathcal{P})$ as $\eta_t = \sum_{i=0}^{N-1} 1_{[t_{i+1},t_i)}(t)\eta^i$. The fraction invested in the riskless asset is denoted by η_t^0, $t\in[0,T]$. A trading strategy is *self-financing* if $\eta_{t_{n-1}}^0 S_{t_n}^0 + \eta_{t_{n-1}}^1 S_{t_n} = \eta_{t_n}^0 S_{t_n}^0 + \eta_{t_n}^1 S_{t_n}$ $\mathcal{P}$-q.s. and for every $n \leq N$. The value of the portfolio satisfies $X_t^\eta \in L^2(\mathcal{P})$ for every $t\in[0,T]$.
The set of simple self-financing trading strategies is denoted by $\mathcal{A}$. This financial market $\mathcal{M}(1,S)$ with trading strategies in $\mathcal{A}$ is called $\mathcal{M}(1,S,\mathcal{A})$.
It is well known that a necessary condition for equilibrium is the absence of

[36] ${}^{\mathbb{P}}\text{ess sup}$ denotes the essential supremum under $\mathbb{P}$. Representations of such martingales can be formulated via a 2BSDE. This concept is introduced in Cheridito, Soner, Touzi, and Victoir (2007), see also Soner, Touzi, and Zhang (2012b).

[37] For the multiple prior case with mutually equivalent priors we refer to Riedel (2009).

[38] As mentioned in Harrison and Pliska (1981) simple strategies rule out the introduction of doubling strategies and hence a notion of admissibility.

arbitrage. Therefore, with regard to the equilibrium consistency condition of the last section, we introduce arbitrage in the financial market of securities. The modeled uncertainty of the financial market motivates us to consider a stronger and robust notion of absence of arbitrage.

Definition 4 *Let $\mathcal{R} \subset \mathcal{P}$. We say there is an* $\mathcal{R}$-arbitrage opportunity in $\mathcal{M}(1, S, \mathcal{A})$ *if there exists an admissible pair $\eta \in \mathcal{A}$ such that $\eta_0 S_0 \leq 0$,*

$$\eta_T S_T \geq 0 \ \ \mathcal{R}\text{-q.s., and} \ \ \hat{\mathbb{P}}\left(\eta_T S_T > 0\right) > 0 \text{ for at least one } \hat{\mathbb{P}} \in \mathcal{R}.$$

The choice of the definition is based on the following observation. This arbitrage strategy is riskless for each $\mathbb{P} \in \mathcal{R}$ and if the prior $\hat{\mathbb{P}}$ constitutes the market one would gain a profit with a strictly positive probability. With this in mind, the $\mathcal{P}$-arbitrage notion can be seen as a weak arbitrage opportunity with the corresponding cone $L^2(\mathcal{P})_+ \setminus \{0\}$. Alternatively, we could argue that absence of $\mathcal{R}$-arbitrage is consistent with a weak dominance principle based on $\mathcal{R}$.

To connect the prior-dependent marketed spaces of Definition 1, we say that a claim $X \in L^2(\mathbb{P})$ is *marketed* in $\mathcal{M}(1, S, \mathcal{A})$ at time zero under $\mathbb{P} \in \mathcal{P}$ if there is an $\eta \in \mathcal{A}$ such that $X = \eta_T S_T$ holds only $\mathbb{P}$-almost surely. In this case we say η hedges X and lies in $M_{\mathbb{P}}$. $\eta_0 S_0 = \pi_{\mathbb{P}}(X)$ is the price of X in $\mathcal{M}(1, S, \mathcal{A})$ under $\mathbb{P} \in \mathcal{P}$.

With Example 3 and 4 in mind, fix the marketed spaces $M_{\mathbb{P}} \subset L^2(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}$. The price of a marketed claim under the prior $\mathbb{P}$ should be well defined. Let $\eta, \eta' \in \mathcal{A}(\mathbb{P})$ generating the same claim $X \in M_{\mathbb{P}}$, i.e. $\eta_T S_T = \eta'_T S_T$ $\mathbb{P}$-a.s. We have $\eta_0 S_0 = \eta'_0 S_0 = \pi_{\mathbb{P}}(X)$ under absence of $\mathbb{P}$-arbitrage. Note, that this may not be true under no $\hat{\mathbb{P}}$-arbitrage, with $\mathbb{P} \neq \hat{\mathbb{P}} \in \mathcal{P}$. This is related to the law of one price under a fixed prior. Now, similarly to the single prior case, we define viability in a financial market. We say that a financial market $\mathcal{M}(1, S, \mathcal{A})$ is *viable* if it is $\Gamma(\mathcal{P})$-arbitrage free and the associated price system $(\{\pi_{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}}, \Gamma)$ is scenario-based viable.

3.3.2 Equivalent Symmetric Martingale Measure Sets

In Section 2 we introduced the price space of sublinear functionals generated by a set of linear $c_{2,\mathcal{P}}$-continuous functionals. The extension of the price functional is strongly related to the involved linear functionals which constitutes the price systems locally. In this fashion, we introduce a modified notion of fair pricing. In essence, we associate a risk-neutral prior to each local and linear extension of a price system. Here, the term local refers to a fixed prior, so that at this stage no volatility uncertainty is present.

In our uncertainty model, the price of a claim equals the (discounted) value under a specific sublinear expectation. Exploration of available information, when multiple priors are present, changes the view of a rational expectation. In economic terms, the notion of symmetric martingales eliminates preferences for ambiguity in the valuation. This is the base to introduce the following rational pricing principle in terms of sublinear expectations with a symmetry condition.

Definition 5 *A set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ is called* equivalent symmetric martingale measure set (EsMM-set) *if the given conditions hold:*

1. *For every $\mathbb{Q} \in \mathcal{Q}$ there is a $\mathbb{P} \in k(\mathcal{P})$ such that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent to each other, so that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{P})$.*

2. *The uncertain asset (S_t) is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale, where $\mathcal{E}^{\mathcal{Q}}$ is the conditional sublinear expectation under $\mathcal{Q}$.*

The first condition formulates a direct relationship between an element $\mathbb{Q}$ in the EsMM-set $\mathcal{Q}$ and the primitive priors $\mathbb{P} \in \mathcal{P}$. The square integrability is a technical condition that guarantees the association to the equilibrium theory of Section 2. The second is the accurately adjusted martingale condition. The idea of a fair gamble should reflect the neutrality of preferences for risk and ambiguity. Under the new sublinear expectation, the asset price and hence the portfolio process, are symmetric martingales. This implies, as discussed in the introduction, that the value of the claim does not depend on the prior. The valuation is mean unambiguous, i.e. preferences for ambiguity under $\mathcal{Q}$ are neutral. One can think of the ambiguity neutral part in the valuation in terms of maxmin preferences from Gilboa and Schmeidler (1989).[39] In this situation, the expected utility is under every prior $\mathbb{Q} \in \mathcal{Q}$ the same. Similarly to pricing under risk, where risk preferences do not matter, analogous reasoning should be true concerning preferences for ambiguity. As such, saying everyone is uncertainty neutral immediately leads one to come up with the uncertainty neutral expectation $\mathcal{E}^{\mathcal{Q}}$.
The case of only one prior is related to the well-known risk-neutral valuation principle. Under volatility uncertainty, this principle needs a new requirement due to the more complex uncertainty model. In this sense the symmetry condition encodes ambiguity neutrality as part of *uncertainty neutrality*.

Remark 3 *1. In the light of Remark 1, let us mention that Definition 3 and 5 can be generalized to the notion nonlinear conditional expectations $(\mathfrak{E}_t)$ satisfying the Law of Iterated Expectation, see Section 9 in Chapter III of Peng (2010). The definition of a $\mathfrak{E}$-martingale is straightforward.*
Concerning the definition of an EsMM-set, the object $\mathcal{Q}$ would refer to the set of priors representing $\mathfrak{E}$. In Remark 1, a possible construction is illustrated. A further weakening of the symmetric martingale property is possible. Instead of that we could merely require the $\mathfrak{E}$-martingale property of (S_t).

2. Note that in the case of a single prior framework, i.e. $\mathcal{P} = \{\mathbb{P}\}$, the notion of EsMM-sets is reduced to accommodate EMM's. In this regard we can think

[39] However, the same argument is applicable to the α-MEU preferences of Ghirardato, Maccheroni, and Marinacci (2004), the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005) and variational preferences of Maccheroni, Marinacci, and Rustichini (2006).

of canonical generalization. On the other hand, classical EMM's and a linear price theory are still present. Every single-valued EsMM-set $\{\mathbb{Q}\}$ *can be seen as an EMM under* $\mathbb{P} \in \mathcal{P}$. *Here, the consolidation is given by* $\Gamma = \delta_{\mathbb{P}}$ *and we have* $\Gamma(\mathcal{P}) = \{\mathbb{P}\}$. *In this situation,* Γ *reveals the ignorance of every other possible prior* $\mathbb{P}' \in \mathcal{P}$.

The following result justifies the discussion involving uncertainty neutrality and the symmetry condition for martingales. The one to one mapping of Theorem 2 and the choice of the price space fall into place. In this manner we show that the existence of an $\mathcal{R}$-arbitrage in $\mathcal{M}(1, S, \mathcal{A})$ with $\Gamma(\mathcal{P}) = \mathcal{R}$ is inconsistent with an economic equilibrium for agents in $\mathbb{A}(\mathbb{P})$, with $\mathbb{P} \in \mathcal{R}$. We fix an associated price system using the procedure described at the end of Subsection 3.1.

Theorem 2 *Suppose the financial market model* $\mathcal{M}(1, S, \mathcal{A})$ *does not allow any* $\mathcal{P}$*-arbitrage opportunity. Then there is a bijection between coherent price systems* $\Psi : L^2(\mathcal{P}) \to \mathbb{R}$ *in* $L^2(\mathcal{P})^{\circledast}_{++}$ *of Theorem 1 and EsMM-sets, satisfying stability under pasting of the induced set* $\Gamma(\mathcal{P})$.[40] *The relationship is given by* $\Psi(X) = \mathcal{E}^{\mathcal{Q}}(X)$, *where*

$$\mathcal{Q} = \left\{\mathbb{Q} \in \mathcal{M}_1(\Omega) : \tfrac{d\mathbb{Q}}{d\mathbb{P}} = \psi_{\mathbb{P}}, \mathbb{P} \in \Gamma(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})_{++}\right\}$$

is the associated EsMM-set.

Let $\mathcal{R} \subset \mathcal{P}$ and $\mathfrak{M}(\mathcal{R})$ be the set of all EsMM-sets $\mathcal{Q}$ such that the related consolidation Γ satisfies $\Gamma(\mathcal{P}) = \mathcal{R}$. Theorem 2 can be seen as the formulation of a one-to-one mapping between a subset of

$$L^2(\mathcal{P})^{\circledast}_{++} \quad \text{and} \quad \bigcup_{\mathcal{R} \subset k(\mathcal{P})} \mathfrak{M}(\mathcal{R}).$$

There is a hierarchy of sublinear expectations, related to the chosen consolidation operator Γ and the EsMM-sets, which are ordered by the inclusion relation. We illustrate the relationship between Γ and an EsMM-set in the following example.

Example 6 *For the sake of simplicity, let us assume that* $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4\}$, *so that any pasting property is ignored. Starting with the sublinear price system, we have four price functionals* $\pi_1, \pi_2, \pi_3, \pi_4$ *and the consolidation operator* Γ. *Let us assume that* $\Gamma = (+, \wedge)$ *and* $\lambda \in (0, 1)$. *This gives us* $\lambda\pi_1 + (1-\lambda)\pi_2 = \pi^\lambda$ *and* $\Gamma(\pi_1, \pi_2, \pi_3, \pi_4) = \pi^\lambda \wedge \pi_3$. *The resulting EsMM-set is given by* $\mathcal{Q} = \{\psi^\lambda \times \mathbb{P}^\lambda, \psi_3 \times \mathbb{P}_3\} \in \mathfrak{M}(\mathcal{P} \backslash \{\mathbb{P}_4\})$, *where* $\mathbb{P}^\lambda = \lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$, $\psi^\lambda = \lambda\psi_1 + (1-\lambda)\psi_2$ *and* $E^{\mathbb{P}^\lambda}[\psi^\lambda] = 1 = E^{\mathbb{P}_3}[\psi_3]$.

We close this consideration with some results analogous to those of the single prior setting where we combine Theorem 2 and Theorem 1.

[40] See Definition 6 in Appendix A for this important concept. In essence, this condition is needed to define a conditional sublinear expectation based on $\mathcal{Q}$, satisfying the iterated law of conditional expectation.

Corollary 2 *Let $\mathcal{R} = \Gamma(\mathcal{P}) \subset \mathcal{P}$ be stable under pasting and given.*

1. *$\mathcal{M}(1, S, \mathcal{A})$ is viable if and only if there is an EsMM-set.*
2. *Market completeness, i.e $M_{\mathbb{P}} = L^2(\mathbb{P})$ for each $\mathbb{P} \in \mathcal{R}$, is equivalent to the existence of exactly one EsMM-set in $\mathfrak{M}(\mathcal{R})$.*
3. *If $\mathfrak{M}(\mathcal{R})$ is nonempty, then there exists no $\mathcal{R}$-arbitrage.*
4. *If there is a strategy $\eta \in \mathcal{A}$ with $\eta_0 S_0 \leq 0$, $\eta_T S_T \geq 0$ $\mathcal{R}$-q.s. and $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) > 0$, for some $\mathcal{Q} \in \mathfrak{M}(\mathcal{R})$, then there is an $\mathcal{R}$-arbitrage opportunity.*

The result does not depend on the preference of the agent. The expected return under the sublinear expectation $\mathcal{E}^{\mathcal{Q}}$ equals the riskless asset. Hence, the value of a claim can be considered as the expected value in the uncertainty-neutral world.[41]

3.3.3 A Special Case: G-Expectation

Now, we select a stronger calculus to model the asset prices as a stochastic differential equation driven by a G-Brownian motion.[42] In this situation the volatility of the process concentrates the uncertainty in terms of the derivative of the quadratic variation. The quadratic variation of a G-Brownian motion creates volatility uncertainty. Again, we review the related result of the single prior framework.

Background: Itô processes in the single prior framework

We specify the asset price in terms of an Itô process $dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t$, $S_0 = 1$, driven by a Brownian motion $B = (B_t)_{t\in[0,T]}$ on the given filtered probability space, $\mu(\cdot, x), \sigma(\cdot, x)$, with $x \in \mathbb{R}_+$ are adapted processes such that (S_t) is a well defined process taking values in $\mathbb{R}_+$. The filtration is generated by B. The interest rate is $r = 0$. Let E^θ be the exponential martingale, given by $d\mathrm{E}^\theta_t = \mathrm{E}^\theta_t \theta_t dB_t$, $\mathrm{E}^\theta_0 = 1$, with a consistent kernel θ we can apply Girsanov theorem. The following result is from Harrison and Kreps (1979):

The set of equivalent martingale measures is not empty if and only if $\rho = \mathrm{E}^\theta_T \in L^2(\mathbb{P})$, $\theta \in L^2(\mathbb{P} \otimes dt)$ and $S^* = \int \sigma dB$ is a $\mathbb{P}$-martingale.

ρ can be interpreted as a state price density. The associated market price of risk $\theta_t = \frac{\mu_t - r}{\sigma_t}$ is the Girsanov or pricing kernel of the state price density.

[41] However, the sublinear expectation depends on Γ.

[42] An illustration of the concept in a discrete time framework is achievable, via the application of the results in Cohen, Ji, and Peng (2011).

Security prices as G-Itô processes and sublinear valuation

An important special case is an uncertainty model given by the G-expectation $E_G : L_G^2(\Omega) \to \mathbb{R}$,[43] where $L_G^2(\Omega) = \hat{L}^2(\mathcal{P})$ is described at the beginning of Appendix A.1. More precisely, the uncertainty model is induced by the following sublinear expectation space $(C([0,T];\mathbb{R}), L_G^2(\Omega), E_G)$ as given.
We select the next rational base, namely an interval $[\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}_{++}$, instead of a constant volatility σ. As indicated in the introduction, volatility uncertainty refers to the awareness that every adapted process (σ_t) taking values in $[\underline{\sigma}, \overline{\sigma}]$ may constitute one possible prior or scenario. We introduce an asset price process driven by a G-Brownian motion $(B_t^G)_{t\in[0,T]}$. In Appendix B.3 we present a small primer of the applied results.
Under the objective description of the real world, given by $\mathcal{P}$ and induced by $[\underline{\sigma}, \overline{\sigma}]$, the asset price is driven by the following G-stochastic differential equation

$$dS_t = \mu(t, S_t)d\langle B^G\rangle_t + V(t, S_t)dB_t^G, \quad t \in [0,T], \quad S_0 = 1.$$

Let $\mu : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $V : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}_+$ be processes such that a unique solution exists.[44] Moreover, let $V(\cdot, x)$ be a strictly positive process for each $x \in \mathbb{R}_+$. The riskless asset has interest rate zero.
The Girsanov theorem for G-Brownian motion is precisely what is needed to verify the symmetric $\mathcal{E}^{\mathcal{Q}}$-martingales property of the price processes S under some sublinear expectation given by an EsMM-set $\mathcal{Q}$. The present uncertainty model enables us to apply the necessary stochastic calculus. As such, we model the financial market in the G-expectation setting, introduced in Peng (2006) and Peng (2010). Central results, such as a martingale representation and a well behaved underlying topology are desired for the foundational grounding of asset pricing. The second condition of Definition 5 highlights how a Girsanov transformation adapts to a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale and thus guarantees the existence of nontrivial EsMM-sets.[45]
For this purpose we define the related sublinear expectation generated by an EsMM-set, $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \psi d\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$:

$$\sup_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}[X] = \mathcal{E}^{\mathcal{Q}}(X) = E_G[\psi X], \quad X \in L_G^2(\Omega).$$

Note that we consider an aggregated family of state price densities $\psi \in L_G^2(\Omega)$ which is defined $\mathcal{P}$-q.s. This means that the density is now a uniform object under our uncertainty model, i.e. $\psi = \psi_{\mathbb{P}}$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$ (see also Remark 4 below). Theorem 3 justifies the choice of this shifted sublinear expectation when the asset price is restrained to a symmetric martingale for an uncertainty-neutral expectation.

[43] It is shown in Theorem 52 by Denis, Hu, and Peng (2011), that this sublinear expectation can be represented by a weakly compact set, when the domain is in $L_G^2(\Omega)$.
[44] We refer to Chapter 5 in Peng (2010) for existence results of G-SDE's.
[45] Trivial EsMM-sets consist of mutually equivalent priors, associated to a single $\mathbb{P} \in \mathcal{P}$.

Let us introduce the universal state price density $\mathtt{E}^\theta$, with $\psi = \mathtt{E}^\theta_T$ $\mathcal{P}$-q.s., being a symmetric martingale of exponential type under the G-expectation, with an integrable pricing kernel $(\theta_t)_{t\in[0,T]}$ (or market price of uncertainty)

$$d\mathtt{E}^\theta_t = \mathtt{E}^\theta_t \theta(t,S_t)dB^G_t, \;\; \mathtt{E}^\theta_0 = 1.$$

Applying the results in Appendix B.3 allows us to write $\mathtt{E}^\theta$ explicitly as

$$\mathtt{E}^\theta_t = \exp\bigg(-\frac{1}{2}\int_0^t \theta(r,S_r)^2 d\langle B^G\rangle_r - \int_0^t \theta(r,S_r)dB^G_r\bigg), \quad t\in[0,T].$$

Let the pricing kernel solve $V(t,S_t)\theta(t,S_t) = \mu(t,S_t)$ $\mathcal{P}$-quasi surely, for every $t\in[0,T]$. Before we formulate the last result we define $S^*_t = S^*_0 + \int_0^t V(r,S^*_r)dB^G_r$, $t\in[0,T]$ and assume that a unique solution on $(\Omega, L^2_G(\Omega), E_G)$ exists for some state-dependent process V, see Peng (2010).

Theorem 3 *The set of EsMM-sets contains a $\mathcal{Q}\in\mathfrak{M}(\mathcal{P})$ if and only if S^* is an E_G-martingale and*

$$E_G\bigg[\exp\bigg(\delta\cdot\int_0^T \theta(r,S_r)^2 d\langle B^G\rangle_r\bigg)\bigg] < \infty, \;\; \textit{for some } \delta > \frac{1}{2}.$$

With Theorem 2 in mind we can associate the concept of scenario-based viability. Let $X\in L^2_G(\Omega)$ be a contingent claim such that it is priced by $\mathcal{P}$-arbitrage then the fair value is given by $\Psi(X) = E_G[\mathtt{E}^\theta_T X]$, whenever Γ consists only of a consolidation via the maximum operation.
Moreover, one can define a new $\hat{G}$-expectation related to a volatility uncertainty of a closed subinterval $[\hat\sigma_1,\hat\sigma_2]\subset[\underline{\sigma},\overline{\sigma}]$. We can identify a consolidation operator by $\Gamma_{\hat G}(\mathcal{P}) = \{\mathbb{P}^\alpha : \alpha\in[\hat\sigma_1,\hat\sigma_2]\}$. In this case Theorem 3 can be reformulated in terms of the existence of an EsMM-set $\mathcal{Q}_{\hat G}\in\mathfrak{M}\left(\Gamma_{\hat G}(\mathcal{P})\right)$.

Remark 4 *The more precise calculus of the G-expectation is based on an analytic description of nonlinear partial differential equations. This allows us to create a uniform state price density process in terms of an exponential martingale, based on a G-martingale representation theorem (see Appendix B 3). With this in mind, a more elaborated notion of EsMM-sets can be formulated by requiring that the family of densities $\{\psi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ create a uniform process as a symmetric martingale under the sublinear expectation $\mathcal{E}^{\mathcal{P}} = E_G$.*

Remark 5 *Under the assumption of G-Brownian motion, Epstein and Ji (2013a) obtain an analogous state price density process. Without applying a Girsanov-type theorem, they use a density process in the important case $\mu(t,S_t) = \mu_t S_t$ and $V(t,S_t) = V_t S_t$. In comparison to our flexible local functional form, their asset price is apparently governed by $dS_t = S_t(b_t dt + V_t dB^G_t)$.[46] In this case, the relationship between the asset price processes, with the special pricing kernel $\left(\frac{\mu_t}{V_t}\right) = (\theta_t)\in M^2_G(0,T)$, is given by*

$$\int_0^t \mu_s d\langle B^G\rangle_s = \int_0^t \mu_s \hat a_s ds = \int_0^t b_s ds, \qquad \textit{where } \hat a_s = \frac{d}{ds}\langle B^G\rangle_s.$$

[46] This allows us to model local volatility structures and volatility uncertainty at the same time.

Extensions to continuous trading strategies seem to be next natural step. Nevertheless, an admissibility condition should be requested, in order to exclude doubling strategies. Considering markets with more than one uncertain security requires a multidimensional Girsanov theorem.[47] We close this section with an example of the connection between super-replication of claims and EsMM-sets.

Example 7 *Under one prior $\mathcal{P} = \{\mathbb{P}\}$, Delbaen (1992) obtained the super-replication price in terms of martingale measures in $\mathfrak{M}(\{\mathbb{P}\})$:*

$$\Lambda^{\mathbb{P}}(X) = \inf\{y \geq 0 | \exists\ \theta \in \mathcal{A}(\mathbb{P}) : y + \theta_T S_T \geq X\ \mathbb{P}\text{-}a.s.\} = \sup_{\mathbb{Q} \in \mathfrak{M}(\{\mathbb{P}\})} E^{\mathbb{Q}}[X]$$

When the uncertainty is given by a set of mutually singular priors, a super-replication price can be derived, see Denis and Martini (2006), in terms of a set of martingale laws $\mathcal{M}$ such that $\Lambda^{\mathcal{P}}(X) = \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[X]$. In the G-framework with simple trading strategies, this set $\mathcal{M}$ is an EsMM-set. When applying our theory to this problem, we get

$$\begin{aligned} \Lambda^{\mathcal{P}}(X) &= \inf\{y \geq 0 | \exists\ \theta \in \mathcal{A} :\ y + \theta_T S_T \geq X \quad \mathcal{P}\text{-}q.s.\} \\ &= \sup_{\mathbb{P} \in \mathcal{P}} \sup_{\mathbb{Q} \in \mathfrak{M}(\{\mathbb{P}\})} E^{\mathbb{Q}}[X] \\ &= E_G[\mathtt{E}_T X], \end{aligned}$$

upon applying our Theorem 3 as well as Theorem 3.6 form Vorbrink (2010). This is associated to the maximal EsMM-set in $\mathfrak{M}(\mathcal{P})$. However, with Proposition 1 every EsMM-set delivers a price below this super-hedging price.

3.4 Discussion and Conclusion

We present a framework and a theory of derivative security pricing where the uncertainty model is given by a set of mutually singular probability measures which incorporates volatility uncertainty. The notion of equivalent martingale measures changes, and the related linear expectation principle becomes a nolinear theory of valuation. The associated arbitrage principle should consider all remaining uncertainty in the consolidation.
The results of this chapter establishes preliminary version of the fundamental theorem of asset pricing (FTAP) under mutually singular uncertainty. In Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998), a definitive FTAP is achieved for the single prior uncertainty model in great generality. The notion of arbitrage is in principle a separation property of convex sets in a topological space. In this regard, the choice of the underlying topological structure is essential for observing a FTAP. For instance, Leventhal and Skorohod (1995), establish a FTAP with an *approximate arbitrage* based on a different notion of convergence.

[47] See Osuka (2011).

In our setting, two aspects must be kept in mind for deriving a FTAP with mutually singular uncertainty. Firstly, the spaces of claims and portfolio processes are based on a capacity norm, and thus forces one to argue for the quasi sure analysis, a fact implied in our definition of arbitrage (see Definition 4). A corresponding notion of free lunch with vanishing *uncertainty* will have to incorporate this more sensitive notion of random variables.

Secondly, the sublinear structure of the price system allows for a nonlinear separation of convex sets. With one prior, the equivalent martingale measure separates achievable claims with arbitrage strategies. In our small meshed structure of random variables this separation is guided by the consolidation operator Γ. The preference-free pricing principle gives us a valuation via expected payoffs of different adjusted priors. In comparison to the preference and distribution free results in a perfectly competitive market, see Ross (1976), the implicit assumption is the common knowledge of uncertainty, described by a single probability measure. The design of uncertainty prescribes the consequences for pricing without a consumption-based utility gradient approach.

The valuation of claims, determined by $\mathcal{P}$-arbitrage, contains a new object Γ, which may inspire skepticism. However, note that the consolidation operator Γ could be seen as a tool to regulate financial markets. The valuation of claims in the balance sheet of a bank should depend on Γ. For instance, this may affect fluctuations of opinion in the market as a consequence of uncertainty. In Remark 2 of Section 2 we describe how a good consolidation may be found via consideration of mechanism design. Such considerations may provide a base for the choice of the valuation principle under multiple priors.[48]

Nonlinear Expectations and Market Efficiency

In Remark 1 of Section 2 and Remark 3.1 of Section 3 we indicate how a fully nonlinear price system can be accomplished. In fact this is an approach hinting at a positive theory of nonlinear expectations, where the observable aggregated market sentiment could be captured by the partition of optimists and pessimists.

Such an attempt is a possible starting point to measure the degree of market efficiency. In fact, if markets are efficient in the weak form, deflated asset prices would by symmetric martingales and reveal all information. An *approximately efficient market* could be detected by observing the martingale property under a nonlinear conditional expectation. In this case the market prices can be regarded as the *best linear* approximation of the nonlinear market expectation of the economy.

Preferences and Asset pricing

The uncertainty model in this chapter is closely related to Epstein and Wang

[48] As a first heuristic, it is possible that utilitarian (convex combination) and Rawlsian (supremum operation) welfare functions may constitute a principle of fair pricing. Here, the prior is chosen behind the "veil of ignorance". See also Section 4 of Wilson (1996).

(1994) and especially to Epstein and Ji (2013a) as they consider equilibria with linear prices in their economy. This leads to an indeterminacy in terms of a continuum of linear price systems. The relationship between uncertainty and indeterminacy is caused by the constraint to pick one *effective* prior. The Lucas critique[49] applies insofar as it describes the unsuitable usage of a pessimistic investor to fix an effective prior in reduced form.

Our valuation principle is based on a *preference-free* approach. We value contingent claims in terms of mean unambiguous asset price processes. In other words, the priors of the uncertainty neutral model yield expectations of the security price that do *not* merely depend on the chosen "risk-neutral" prior. Nevertheless, the idea of a risk-neutral valuation principle is not appropriate, as different mutually singular priors deliver different expectations, that cannot be related via a single density.

From this point of view, we disarrange the indeterminacy of linear prices, and allow for the appearance of a planner to configure the sublinearity. In this sense, the regulator as a policy maker is now able to confront unmeasurable sudden fluctuations in volatility. A single prior, as a part of the equilibrium output, can create an invisible threat of convention, which may be used to create the illusion of security when faced with an uncertain future. In a model with mutually singular priors, the focus on a single prior creates a hazard. Events with a positive probability under an ignored prior may be a null set under an effective prior in a consumption-based approach.

Sublinear prices and regulation via consolidation

In this context, sublinearity is associated with the principle of diversification. In these terms, an equilibrium with a sublinear price system covers the concept of Walrasian prices which decentralize with the coincidental awareness of different scenarios. A priori, the instructed Walrasian auctioneer has no knowledge of which prior $\mathbb{P}$ in $\mathcal{P}$ occurs. The degree of discrimination is related to the intensity of nonlinearity. Note that this is a normative category and opens the door to the economic basis of regulation. Each prior is a probabilistic scenario. The auctioneer *consolidates* the price for each possible scenario into one certain and robust valuation. This is also true for an agent in the model, hence the auctioneer should be able to *discriminate under-diversification* in terms of ignorance of priors in this uncertainty model. Further, a von Neumann-Morgenstern utility assumption results in an overconfidence of certainty in the associated agent.

Since we want to generalize fundamental theorems of asset pricing, we are concerned with the relationship between equivalent martingale measures, viable price systems and arbitrage. In this setting, these concepts must be recast because of the multiple prior uncertainty. In contrast, with one prior an equivalent martingale measure is associated with a linear price system. The underlying neoclassical equilibrium concept is a fully positive theory. In the multiple prior setting such a price extension can be regarded as a

[49] See Section 3.2 in Epstein and Schneider (2010).

diversification-neutral valuation principle. Here, diversification is focused on a given set of priors $\mathcal{P}$. Should the unlucky situation arise that an unconsidered prior governs the market, it is the task of the regulator to robustify these possibilities via an appropriate price system. For instance, uniting two valuations of contingent claims cannot be worse than adding the uncertain outcomes separately. This is the diversification principle under $\mathcal{P}$.

Recalling the quotation of Aliprantis, Tourky, and Yannelis (2001) in the introduction, the degree of sublinearity in our approximation is regulated by the type of consolidation of scenario-dependent linear price systems.

3.5 Appendix A: Details and Proofs

3.5.1 Section 2

Let $L^0(\Omega)$ denote the space of all measurable real-valued functions on Ω. $\hat{L}^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$ be the quotient of $\mathcal{L}^2(\mathcal{P})$, the closure of $\mathcal{C}_b(\Omega)$ by $c_{2,\mathcal{P}}$ in $L^0(\Omega)$. $\mathcal{N}$ denotes the ideal of $c_{2,\mathcal{P}}$ in $L^0(\Omega)$ null elements. Such null elements are characterized by random variables which are $\mathcal{P}$-polar. $\mathcal{P}$-polar sets evaluated under every prior are zero or one, although, the value may differ between different priors. A property holds *quasi-surely* (q.s.) if it holds outside a polar set. Furthermore, the space $\hat{L}^2(\mathcal{P})$ is characterized by

$$\hat{L}^2(\mathcal{P}) = \left\{ X \in L^0(\Omega) : X \text{ has a q.c. version, } \lim_{n\to\infty} \mathcal{E}^{\mathcal{P}}(|X|^2 1_{\{|X|>n\}}) = 0 \right\},$$

A mapping $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous if $\forall \varepsilon > 0$ there exists an open set O with $\sup_{P\in\mathcal{P}} P(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. We say that $X : \Omega \to \mathbb{R}$ has a *quasi-continuous version* (q.c.) if there exists a quasi–continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s. The mathematical framework provided enables the analysis of stochastic processes for several mutually singular probability measures simultaneously. All equations are understood in the quasi-sure sense. This means that a property holds almost-surely for all scenarios $P \in \mathcal{P}$. Since, for all $X, Y \in L^2(\mathcal{P})$ with $|X| \le |Y|$ imply $c_{2,\mathcal{P}}(X) \le c_{2,\mathcal{P}}(Y)$, we have that $L^2(\mathcal{P})$ is a Banach lattice.[50]
In the following we discuss the different operations for consolidation. Let $\Pi_{\mathbb{P}} = E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot] \in L^2(\mathcal{P})^*$, with $\mathbb{P} \in \mathcal{P}$. Let μ be a measure on the Borel measurable space $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$ with $\mu(\mathcal{P}) = 1$ and full support on $\mathcal{P}$. In this context we can consider the additive case in $L^2(\mathcal{P})^{\circledast}_{+}$, where a new prior is generated:[51]

$$\Gamma_\mu : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^* \to L^2(\mathcal{P})^{\circledast}_{+}, \quad \Gamma_\mu(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \int_{\mathcal{P}} \psi_{\mathbb{P}} \cdot \mu(d\mathbb{P}) = E^{\mathbb{P}_\mu}[\psi_{\mathbb{P}_\mu}\cdot],$$

[50]This is of interest for existence results from general equilibrium theory.
[51]The related operation of convex functionals would corresponds to the convolution operation. Without convexity of $\mathcal{P}$, the prior $\mathbb{P}_\mu$ may only lie in the convex hull of $k(\mathcal{P})$.

where ψ_{P_μ} is constructed as in Example 2. We can consider the Dirac measure $\delta_{\mathbb{P}}$ as an example for μ. The related consideration of only one special prior in $\mathcal{P}$ is in essence the uncertainty model in Harrison and Kreps (1979). The operation in question is given by $(\Pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}} \mapsto E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$. The second operation in $L^2(\mathcal{P})^{\circledast}_+$ is a point-wise maximum:

$$\Gamma_{\sup} : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^* \to L^2(\mathcal{P})^{\circledast}_+, \quad \Gamma_{\sup}(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot].$$

This is an extreme form of consolidation and can be considered as the highest awareness of all priors. Note that combinations between the maximum and an addition operation are possible as indicated in Example 2 and Proposition 1.

Proof of Proposition 1 *Since $L^2(\mathcal{P})$ is a Banach lattice, the 5th claim follows from Theorem 1 in Biagini and Frittelli (2010), whereas the other claims follow directly from the construction of the functionals in $L^2(\mathcal{P})^{\circledast}_+$.* ■

For the proof of Theorem 1, we define the shifted preference relationship $\succsim^0_{\mathbb{P}}$ such that every feasible net trade is worse off than $(0,0) \in B(0,0,\pi_{\mathbb{P}}, M_{\mathbb{P}})$. Obviously, an agent given by $\succsim^0_P$ does not trade. Hence, an initial endowment constitutes a no trade equilibrium.

Proof of Theorem 1 *Let the price system $(\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}, \Gamma)$ be given and we have a $\Psi \in L^2(\mathcal{P})^{\circledast}_+$ on $L^2(\mathcal{P})$ such that $\pi_{\mathbb{P}\restriction M^{\mathcal{P}}_{\mathbb{P}}} \leq \Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}}$ for each $\mathbb{P} \in \Gamma(\mathcal{P})$, where $M^{\mathcal{P}}_{\mathbb{P}} = M_{\mathbb{P}} \cap L^2(\mathcal{P})$. The preference relation on $\mathbb{R} \times L^2(P)$, given by*

$$(x, X) \succcurlyeq^0_{\mathbb{P}} (x', X') \quad if \;\; x + -\Pi_{\mathbb{P}}(-X) \geq x' + -\Pi_{\mathbb{P}}(-X'),$$

is in $\mathbb{A}(\mathbb{P})$. For each $\mathbb{P} \in \Gamma(\mathcal{P})$, the bundle $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) = (0,0)$ satisfies the viability condition of Definition 2, hence $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\Gamma(\mathcal{P})}$ is scenario-based viable. In the other direction, let $\pi(\otimes\mathcal{P}) : \otimes M_{\mathbb{P}} \to \mathbb{R}$ be a price system. The preference relation $\succcurlyeq^0_{\mathbb{P}} \in \mathbb{A}(\mathbb{P})$ satisfies for each $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}})$, $\mathbb{P} \in \Gamma(\mathcal{P})$, the viability condition. We may assume for each $\mathbb{P}$, $(\hat{x}_{\mathbb{P}}, \hat{X}_{\mathbb{P}}) = (0,0)$, since it is only a geometric deferment. Consider the following sets

$$\begin{aligned} \succ^0_{\mathcal{P}} &= \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} \{(x,X) \in \mathbb{R} \times L^2(\mathbb{P}) : (x,X) \succ_{\mathbb{P}} (0,0)\}, \\ B(\otimes\mathcal{P}) &= \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} B(0,0,\pi_{\mathbb{P}}, M_{\mathbb{P}}). \end{aligned}$$

We have that $B(\otimes\mathcal{P})$ and $\succ^0_{\mathcal{P}}$ are convex sets. The Riesz space product $\otimes L^2(\mathbb{P}) = \otimes_{\mathbb{P}\in\Gamma(\mathcal{P})} L^2(\mathbb{P})$ (see paragraph 352 K in Fremlin (2000)), is under the norm $c_{2,\mathcal{P}}$ again a Banach lattice (see paragraph 354 X (b) in Fremlin (2000)). By the $L^2(\mathbb{P})$-continuity of each $\succsim^0_{\mathbb{P}}$, the set $\succ^0_{\mathcal{P}}$ is $c_{2,\mathcal{P}}$-open in $\otimes L^2(\mathbb{P})$.
According to the separation theorem for a topological vector space, for each $\mathbb{P} \in \Gamma(\mathcal{P})$ there is a non zero linear and $c_{2,\mathcal{P}}$-continuous functional $\phi_{\mathbb{P}}$ on $\otimes_{\mathbb{P}\in\Gamma(\mathcal{P})}(\mathbb{R} \times L^2(\mathbb{P}))$ with

1. $\phi_{\mathbb{P}}(x,X) \geq 0$ *for all* $(x,X) \in \succ^0_{\mathcal{P}}$

2. $\phi(x,X) \leq 0$ *for all* $(x,X) \in B(\otimes\mathcal{P})$

3. $\{(y_{\mathbb{P}},Y_{\mathbb{P}})\}_{\mathbb{P}\in\Gamma(\mathcal{P})} = (y,Y)$ *with* $\mathrm{pr}_{\mathbb{R}\times L^2(\mathbb{P})}(\phi_{\mathbb{P}})(y,Y) =: \phi_{\restriction\mathbb{P}}(y_{\mathbb{P}},Y_{\mathbb{P}}) < 0$,

since $\phi_{\mathbb{P}}$ *is non-trivial. Note that condition 3. depends on the chosen* $\mathbb{P}$. *Strict monotonicity of* $\succsim^0_{\mathbb{P}}$ *implies* $(1,0) \succ^0_{\mathbb{P}} (0,0)$. *The* $L^2(\mathbb{P})$*-continuity of each* $\succcurlyeq^0_{\mathbb{P}}$ *gives us* $(1+\varepsilon y, \varepsilon Y) \succ^0_{\mathbb{P}} (0,0)$, *for some* $\varepsilon > 0$, *hence*

$$\begin{aligned} \phi_{\restriction\mathbb{P}}(1+\varepsilon y_{\mathbb{P}}, \varepsilon Y_{\mathbb{P}}) &= -\phi_{\restriction\mathbb{P}}(1,0) + \varepsilon\phi_{\restriction\mathbb{P}}(y_{\mathbb{P}},Y_{\mathbb{P}}) \leq 0 \\ \textit{and} \quad \phi_{\restriction\mathbb{P}}(1,0) &\geq -\varepsilon\phi_{\restriction\mathbb{P}}(y_{\mathbb{P}},Y_{\mathbb{P}}) > 0. \end{aligned}$$

We have $\phi_{\restriction\mathbb{P}}(1,0) > 0$ *and after a renormalization let* $\phi_{\restriction\mathbb{P}}(1,0) = 1$. *Moreover, we can write* $\phi_{\restriction\mathbb{P}}(x_{\mathbb{P}},X_{\mathbb{P}}) = x_{\mathbb{P}} + \Pi_{\mathbb{P}}(X_{\mathbb{P}})$, *where* $\Pi_{\mathbb{P}} : L^2(\mathbb{P}) \to \mathbb{R}$ *can be identified as an element in the topological dual* $L^2(\mathbb{P})^*$.
We show strict positivity of $\Pi_{\mathbb{P}}$ *on* $L^2(\mathbb{P})$. *Let* $X \in L^2(\mathbb{P})_+ \setminus \{0\}$ *we have* $(0,X) \succ^0_{\mathbb{P}} (0,0)$, *hence* $(-\varepsilon, X) \succ^0_{\mathbb{P}} (0,0)$, *and therefore* $\Pi_P(X) - \varepsilon \geq 0$.
Moreover we have $L^2(\mathcal{P})$*-positivity of* $\Pi_{\mathbb{P}\restriction L^2(\mathcal{P})}$ *on* $L^2(\mathcal{P})$, *i.e.* $X \geq 0$ $\mathcal{P}$*-q.s. implies* $\Pi_{\mathbb{P}\restriction L^2(\mathcal{P})} \geq 0$. *Since* $L^2(\mathcal{P})$ *is a Banach lattice,* $\Pi_{\mathbb{P}} \in L^2(\mathcal{P})^*$ *follows.*
Let $X \in M^{\mathcal{P}}_{\mathbb{P}}$, *since* $(-\pi_{\mathbb{P}}(X), X), (\pi_{\mathbb{P}}(X), -X) \in B(0,0,\pi_{\mathbb{P}}, M^{\mathcal{P}}_{\mathbb{P}})$ *we have* $0 = \phi(\pi_{\mathbb{P}}(X), X) = \pi_P(X) - \Pi_P(X)$ *and* $\Pi_{\mathbb{P}\restriction M^{\mathcal{P}}_{\mathbb{P}}} = \pi_{\mathbb{P}}$ *follows.*
$\Gamma(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\Gamma(\mathcal{P})}) = \Psi$ *is by construction in* $L^2(\mathcal{P})^{\circledast}_{+}$. *The strict positivity of* Ψ *follows from the strict positivity of each* $\Pi_{\mathbb{P}}$. $\Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$ *follows from an inequality in the last part of Proposition 1 and* $\Pi_{\mathbb{P}\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}}$. ∎

We illustrate the construction in the following diagram:

$$\begin{array}{ccc} \{\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}} \longmapsto \pi(\otimes\mathcal{P}) : \bigotimes_{\mathbb{P}\in\Gamma(\mathcal{P})} M_{\mathbb{P}} \to \mathbb{R} & & \\ \quad\quad\quad\quad \mathit{Hahn} \downarrow \mathit{Banach} & & \\ \left\{\Pi_{\mathbb{P}} : L^2(\mathbb{P}) \to \mathbb{R}\right\}_{\mathbb{P}\in\Gamma(\mathcal{P})} \overset{\Gamma}{\longmapsto} \Psi : L^2(\mathcal{P}) \to \mathbb{R} & & \end{array}$$

Proof of Corollary 1 *By construction every functional* Ψ *can be represented as the supremum of priors, which are given by convex combinations. Since* $X \in \mathbb{M}(\Gamma(\mathcal{P}))$, *the supremum operation has no effect on* X *and the assertion follows.* ∎

3.5.2 Section 3

Next, we discuss the augmentation of our information structure. The unaugmented filtration is given by $\mathbb{F}^o$. As mentioned in Subsection 3.1, the set of priors have to be stable under pasting in order to apply the framework of Nutz and Soner (2012). For the sake of completeness, we recall this notion.

Definition 6 *The set of priors is* stable under pasting *if for every* $\mathbb{P} \in \mathcal{P}$, *every* $\mathbb{F}^o$*-stopping time* τ, $B \in \mathcal{F}^o_\tau$ *and* $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\mathcal{F}^o_\tau, \mathbb{P})$, *We have* $\mathbb{P}_\tau \in \mathcal{P}$, *where*

$$\mathbb{P}_\tau(A) = E^P\left[\mathbb{P}_1(A|\mathcal{F}^o_\tau)1_B + \mathbb{P}_2(A|\mathcal{F}^o_\tau)1_{B^c}\right], \quad A \in \mathcal{F}^o_\tau.$$

In the multiple prior setting, with a given reference measure this property is equivalent to the well-known notion of *time consistency*. However, this is not true if there is no dominant prior.[52]
The usual condition of a "rich" σ-algebra at time 0 is widely used in mathematical finance. But the economic meaning is questionable. Our uncertainty model of mutually singular priors can be augmented, similarly to the classical case, using the right continuous filtration given by $\mathbb{F}^+ = \{\mathcal{F}_t^+\}_{t\in[0,T]}$ where

$$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_t^o, \text{ for } t \in [0,T).$$

The second step is to augment the minimal right continuous filtration $\mathbb{F}^+$ by all polar sets of $(\mathcal{P}, \mathcal{F}_T^o)$, i.e. $\mathcal{F}_t = \mathcal{F}_t^+ \vee \mathcal{N}(\mathcal{P}, \mathcal{F}_T^o)$. This augmentation is strictly smaller than the universal augmentation $\bigcap_{P\in\mathcal{P}} \overline{\mathbb{F}^o}^P$. This choice is economically reasonable since the initial σ-field does not contain all 0-1 limit events. An agent considers this exogenously specified information structure. It describes what information the agent *can* know at each date. This is the analogue to a filtration in the single prior framework satisfying the usual conditions. For the proof below, we need results from Appendix B.1.

Proof of Theorem 2 *We fix an EsMM-set $\mathcal{Q}$. The related consolidation Γ gives us the set of relevant priors $\Gamma(\mathcal{P}) \subset \mathcal{P}$. Let $\psi_{\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}$, for each $\mathbb{Q} \in \mathcal{Q}$ and the related $\mathbb{P} \in \mathcal{P}$. We have $\psi_{\mathbb{P}} \in L^2(\mathbb{P})$. Let the associated strictly positive $\Psi \in L^2(\mathcal{P})^{\circledast}_{++}$ be given.*
Take a marketed claim $X^m \in M_{\mathbb{P}}^{\mathcal{P}}$ with $\mathbb{P} \in \Gamma(\mathcal{P})$ and let $\eta \in \mathcal{A}$ be a self-financing trading strategy that hedges X^m. Since $\eta \in \mathcal{A}$, by the decomposition rule for conditional $\mathcal{E}^{\mathcal{Q}}$-expectation, see for instance Theorem 2.6 (iv) in Epstein and Ji (2013b), and since S is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale, the following equalities

$$\mathcal{E}_t^{\mathcal{Q}}(\eta_u S_u) = \eta_t^+ \mathcal{E}_t^{\mathcal{Q}}(S_u) + \eta_t^- \mathcal{E}_t^{\mathcal{Q}}(-S_u) = \eta_t^+ S_t - \eta_t^- S_t = \eta_t S_t,$$

hold, where $\eta = \eta^+ - \eta^-$ with $\eta^+, \eta^- \geq 0$ $\mathcal{P}$-quasi surely and $0 \leq t \leq u \leq T$. Therefore we achieve

$$\Psi(X^m) = \mathcal{E}_0^{\mathcal{Q}}(\eta_T S_T) = \eta_0 S_0 \geq \pi_{\mathbb{P}}(X^m), \quad \mathbb{P} \in \Gamma(\mathcal{P}).$$

For the other direction, let $\Psi \in L^2(\mathcal{P})^{\circledast}_{++}$ with $\Psi_{\restriction M_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$, related to a set of linear functionals $\{\pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}$ and $\{\Pi_{\mathbb{P}}\}_{P\in\mathcal{P}}$, such that $\Pi_{\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}}$. This can be inferred from Ψ and the construction in the proof of the second part of Theorem 1. Now, we define $\mathcal{Q}$ in terms of Γ.
We illustrate the possible cases which can appear. For simplicity we assume $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\}$. Let $\mathbb{P}^{k,j} = \frac{1}{2}\mathbb{P}_k + \frac{1}{2}\mathbb{P}_j$ and $\psi^{k,j} = \frac{1}{2}\psi^k + \frac{1}{2}\psi^j$, recall that we can represent each functional $\Pi_{\mathbb{P}}(\cdot)$ by $E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$. We have

$$\frac{1}{2}(\Pi_1 + \Pi_2) \wedge \Pi_3 \textit{ becomes } \left\{\psi^{1,2} \times \mathbb{P}^{1,2}, \psi_3 \times \mathbb{P}_3\right\} = \mathcal{Q}.$$

[52] Additionally, the set of priors must be chosen maximally. For further consideration, we refer the reader to Section 3 in Nutz and Soner (2012).

Consequently, $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \psi_{\mathbb{P}} d\mathbb{P},\ \mathbb{P} \in \Gamma(\mathcal{P}) \subset k(\mathcal{P}), \psi_{\mathbb{P}} \in L^2(\mathbb{P})\}$, *where* $\psi_{\mathbb{P}}$, *with* $\mathbb{P} \notin \mathcal{P}$, *is constructed by the procedure of Example 2. The first condition of Definition 5 follows, since the square integrability of each* $\psi_{\mathbb{P}}$ *follows from the* $c_{2,\mathcal{P}}$*-continuity of linear functionals which generate* Ψ.
We prove the symmetric $\mathcal{Q}$*-martingale property of the asset price process. Let* $B \in \mathcal{F}_t$, $\eta \in \mathcal{A}$ *be a self-financing trading strategy and*

$$\eta_s^1 = \begin{cases} 1 & s \in [t,u) \text{ and } \omega \in B \\ 0 & \text{else}\ , \end{cases} \qquad \eta_s^0 = \begin{cases} S_t, & s \in [t,u) \text{ and } \omega \in B \\ S_u - S_t, & s \in [u,T) \text{ and } \omega \in B \\ 0 & \text{else.} \end{cases}$$

This strategy yields a portfolio value

$$\eta_T S_T = (S_u - S_t) \cdot 1_B,$$

the claim $\eta_T S_T$ *is marketed at price zero. In terms of the modified conditional sublinear expectation* $(\mathcal{E}_t^{\mathcal{Q}})_{t\in[0,T]}$, *we have with* $t \leq u$

$$\mathcal{E}_t^{\mathcal{Q}}((S_t - S_u)1_B) = 0.$$

By Theorem 4.7 Xu and Zhang (2010), it follows that $S_t = \mathcal{E}_t^{\mathcal{Q}}(S_u)$.[53] *But this means that* $(S_t)_{t\in[0,T]}$ *is an* $\mathcal{E}^{\mathcal{Q}}$*-martingale. The same argumentation holds for* $-S$, *hence the asset price* S *is a symmetric* $\mathcal{E}^{\mathcal{Q}}$*-martingale.* ■

Proof of Corollary 2 *1. Suppose there is a* $\mathcal{Q} \in \mathfrak{M}(\mathcal{P})$ *and let* $\eta \in \mathcal{A}$ *such that* $\eta_T S_T \geq 0$ $\mathcal{P}$*-q.s. and* $\mathbb{P}'(\eta_T S_T > 0) > 0$ *for some* $\mathbb{P}' \in \mathcal{P}$. *Since for all* $\mathbb{Q} \in \mathcal{Q}$ *there is a* $\mathbb{P} \in k(\mathcal{P})$ *such that* $\mathbb{Q} \sim \mathbb{P}$, *there is a* $\mathbb{Q}' \in \mathcal{Q}$ *with* $\mathbb{Q}'(\eta_T S_T > 0) > 0$. *Hence,* $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) > 0$ *and by Theorem 2 we observe* $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) = \eta_0 S_0$. *This implies that no* $\mathcal{P}$*-arbitrage exists.*

2. *In terms of Theorem 1, each* $P \in \mathcal{R}$ *admits exactly one extension. With Theorem 2 the result follows.*

3. *By Theorem 2, there is a related* Ψ *in* $L^2(\mathcal{P})_+^{\circledast}$, *wit* $\Gamma(\mathbb{P}) = \mathcal{R}$. *Fix a costless strategy* $\eta \in \mathcal{A}$ *such that* $\eta_0 S_0 = 0$ *hence* $\Psi(\eta_T S_T) = 0$. *The viability of* Ψ *implies* $\eta_T S_T = 0$ $\mathcal{R}$*-q.s. Hence, no* $\mathcal{R}$*-arbitrage exists.*

4. *This then follows by the same argument as in Harrison and Pliska (1981) (see the Lemma on p.228), since* $\mathcal{E}^{\mathcal{Q}}$ *is strictly positive, by Theorem 2.* ■

For the proof of Theorem 3, we apply results from stochastic analysis in the G-framework. The results are collected in Appendix B.3.

[53] The result is proven for the G-framework. However the assertion in our setting holds true as well by an application of the martingale representation in Proposition 4.10 by Nutz and Soner (2012).

Proof of Theorem 3 *Let $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \rho d\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$ be an EsMM-set, where the density ρ satisfies $\rho \in L^2_G(\Omega)$ and $E_G[\rho] = -E_G[-\rho]$. Next define the stochastic process $(\rho_t)_{t\in[0,T]}$ by $\rho_t = E_G[\rho|\mathcal{F}_t]$ resulting in a symmetric G-martingale to which we apply the martingale representation theorem for G-expectation, stated in Appendix B.3. Hence, there is a $\gamma \in M^2_G(0,T)$ such that we can write*

$$\rho_t = 1 + \int_0^t \gamma_s dB^G_s, \quad t \in [0,T], \quad \mathcal{P}\text{-q.s.}$$

By the G-Itô formula, stated in the Appendix B.3, we have

$$ln(\rho_t) = \int_0^t \phi_s dB^G_s + \frac{1}{2}\int_0^t \phi_s^2 d\langle B^G\rangle_s, \quad \mathcal{P}\text{-q.s}$$

for every $t \in [0,T]$ in $L^2_G(\Omega_t)$ and hence

$$\rho = \mathtt{E}^\phi_T = \exp\Big(-\frac{1}{2}\int_0^T \theta_s^2 d\langle B^G\rangle_s - \int_0^T \theta_s dB^G_s\Big), \quad \mathcal{P}\text{-q.s.}$$

With this representation of the density process we can apply the Girsanov theorem, stated in Appendix B.3. Set $\phi_t = \frac{\rho_t}{\gamma_t}$ and consider the process

$$B^\phi_t = B^G_t - \int_0^t \phi_s ds, \quad t \in [0,T].$$

We deduce that B^ϕ is a G-Brownian motion under $\mathcal{E}^\phi(\cdot) = E_G[\phi\cdot]$ and S satisfies

$$S_t = S_0 + \int_0^t V_s dB^\phi_s + \int_0^t (\mu_s + V_s\phi_s) d\langle B^\phi\rangle_s \quad t \in [0,T]$$

on $(\Omega, L^2_G(\Omega)), \mathcal{E}^\phi)$. Since V is a bounded process, the stochastic integral is a symmetric martingale under $\mathcal{E}^\phi$. S is a symmetric $\mathcal{E}^\phi$-martingale if and only if $\mu_t + V_t\phi_t = 0$ $\mathcal{P}$-q.s. We have shown that ρ is a simultaneous Radon-Nikodym type density of the EsMM-set $\mathcal{Q}$. Hence, there is a nontrivial EsMM-set in $\mathfrak{M}(\mathcal{P})$, since $\phi_t = \theta_t$ $\mathcal{P}$-q.s for every $t \in [0,T]$. ■

3.6 Appendix B: Required results

In this Appendix we introduce the mathematical framework more carefully. We also collect all the results applied in Sections 2 and 3. First, we state the mentioned criterion for the weak compactness of $\mathcal{P}$. Let $\sigma^1, \sigma^2 : [0,T] \to \mathbb{R}_+$ be two measures with a Holder continuous distribution function $t \mapsto \sigma^i([0,t]) = \sigma^i(t)$.
As introduced in Section 2.1, a measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a martingale probability measure if the coordinate process is a martingale with regard to the

canonical (raw) filtration.

Criterion for weak compactness of priors, Denis and Kervarec (2013): Let $\mathcal{P}(\sigma^1, \sigma^2)$ be the set of martingale probability measures with

$$d\sigma^1(t) \leq d\langle B\rangle_t^{\mathbb{P}} \leq d\sigma^2(t),$$

where $\langle B\rangle^P$ is the quadratic variation of B under $\mathbb{P}$. Then the set $\mathcal{P}(\sigma^1, \sigma^2)$ is weakly compact.

3.6.1 The sub-order dual

In this subsection we discuss the mathematical preliminaries for the price space of sublinear functionals for Section 3.
The topological dual space:

1. Let $c_{2,\mathcal{P}}$ be a capacity norm, as defined in Section 2.2. Every continuous linear form l on $L^2(\mathcal{P})$ admits a representation:

$$l(X) = \int X d\mu \quad \forall X \in L^2(\mathcal{P}),$$

where μ is a bounded signed measure defined on a σ-algebra containing the Borel σ-algebra of Ω. If l is a non-negative linear form, the measure μ is non-negative finite.

2. We have $L^2(\mathcal{P})^* = \left\{\mu = \int \psi_{\mathbb{P}} d\mathbb{P} : \mathbb{P} \in \mathcal{P} \textit{ and } \psi_{\mathbb{P}} \in L^2(\mathbb{P})\right\}$.

The first claim is stated in Proposition 11 of Feyel and de La Pradelle (1989). The second assertion can be proven via a modification of Lemma I.28 and Theorem I.30 in Kervarec (2008), where the case of $L^1(\mathcal{P})^*$ is treated.

Semi lattices and their intrinsic structure

The space of coherent price systems $L^2(\mathcal{P})^{\circledast}_{++}$ plays a central role in Theorem 1 and 2. Every consolidation operator has a domain in $\bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^*$ and maps to $L^2(\mathcal{P})^{\circledast}$. We begin with the most simple operation of consolidation, ignoring a subset of priors and giving a weight to the others.
Let $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$ be the positive measure μ such that $\mu(\mathcal{P}) \leq 1$. The underlying space is $\bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^*$, when considering simultaneously the representations of continuous and linear functionals on $L^2(\mathcal{P})$. So let $N \subset \mathcal{B}(\mathcal{P})$ be a Borel measurable set and $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$. The consolidation via convex combination is given by

$$\Gamma(\mu, N) : \bigotimes_{\mathbb{P}\in\mathcal{P}} L^2(\mathbb{P})^{\circledast} \to L^2(\mathcal{P})^*,\ \{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}} \mapsto \int_N \psi_{\mathbb{P}} d\mu(d\mathbb{P}).$$

The size of N determines the degree of ignorance related to the exclusion of the prior in the countable reduction. A measure with a mass strictly less

than one implies an ignorance. Here, a Dirac measure on $\mathcal{P}$ is a projection to one certain probability model.
Next, we consider the supremum operation of functionals. The operation of point-wise maximum preserves the convexity. We review a result which gives an iterated application of the Hahn-Banach Theorem.

Representation of sublinear functionals Frittelli (2000): Let ψ be a sublinear functional on a topological vector space V, then

$$\psi(X) = \max_{x^* \in P_\psi} x^*(X),$$

where $P_\psi = \{x^* \in X^* : x^*(X) \leq \psi(X) \text{ for all } X \in V\} \neq \emptyset$

The maximum operation can also be associated to a lattice structure. In economic terms, this is related to a normative choice of the super hedging intensity. The diversification valuation operator consolidation is set to *one nonlinear* valuation functional. Note that the operation preserves monotonicity.

3.6.2 Stochastic analysis with G-Brownian motion

We introduce the notion of sublinear expectation for the G-Brownian motion. This includes the concept of G-expectation, the Itô calculus with G-Brownian motion and related results concerning the representation of G-expectation and (symmetric) G-martingales. For a more detailed detour we refer to the Appendix in Vorbrink (2010) and to references therein. At the end of this section we present a Girsanov theorem for G-Brownian motion, which we apply in Theorem 3 of Subsection 3.3. Let $\Omega \neq \emptyset$ be a given set. Let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. Note that in our model we consider the completion of $\mathcal{C}_b(\Omega) = \mathcal{H}$ and $\Omega = \Omega_T = C_0([0,T])$.
A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying monotonicity, constant preserving, sub-additivity and positive homogeneity. The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a *sublinear expectation space*. For the construction of the G-expectation, the notion of independence and G-normal distributions we refer to Peng (2010).
A process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a *G–Brownian motion* if the following properties are satisfied:

(i) $B_0 = 0$.

(ii) For each $t, s \geq 0$: $B_{t+s} - B_t \sim B_t$ and $\hat{E}[|B_t|^3] \to 0$ as $t \to 0$.

(iii) The increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

(iv) $\hat{E}[B_t] = -\hat{E}[-B_t] = 0 \quad \forall t \geq 0$.

The following observation is important for the characterization of G–martingales. The space $C_{l,Lip}(\mathbb{R}^n)$, where $n \geq 1$ is the space of all real-valued continuous functions φ defined on $\mathbb{R}^n$ such that $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \;\; \forall x, y \in \mathbb{R}^n$. We define $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \cdots, t_n \in [0,T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}$. The Itô integral can also be defined for the following processes: Let $H_G^0(0,T)$ be the collection of processes η having the following form: For a partition $\{t_0, t_1, \cdots, t_N\}$ of $[0,T]$, $N \in \mathbb{N}$, and $\xi_i \in L_{ip}(\Omega_{t_i}) \; \forall i = 0, 1, \cdots, N-1$, let η be given by

$$\eta_t(\omega) := \sum_{0 \leq j \leq N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t) \quad \textit{for all } t \in [0,T].$$

For $\eta \in H_G^0(0,T)$ let $\|\eta\|_{M_G^2} := \left(E_G\left[\int_0^T |\eta_s|^2 ds\right]\right)^{\frac{1}{2}}$ and denote by $M_G^2(0,T)$ the completion of $H_G^0(0,T)$ under this norm. We can construct Itô's integral I on $H_G^0(0,T)$ and extend it to $M_G^2(0,T)$ continuously, by $I : M_G^2(0,T) \to L^2(\mathcal{P})$. The next result is an Itô formula. The presentation of basic notions on stochastic calculus with respect to G-Brownian motion lies beyond the scope of this appendix.

Itô-formula, Li and Peng (2011): Let $\Phi \in C^2(\mathbb{R})$ and $dX_t = \mu_t d\langle B^G \rangle_t + V_t dB_T^G$, $t \in [0,T]$, $\mu, V \in M_G^2(0,T)$ are bounded processes. Then we have for every $t \geq 0$:

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial\Phi(X_u) V_u dB_u^G + \frac{1}{2}\int_s^t \partial\Phi(X_u)\mu_u + \partial^2\Phi(X_u) V_u^2 d\langle B^G \rangle_u.$$

Next, we discuss martingales in the G-framework. In Song (2011), this identity declares that a G-martingale M can be seen as a multiple prior martingale which is a supermartingale for any $P \in \mathcal{P}$ and a martingale for an optimal measure.

Characterization for G-martingales, Soner, Touzi, and Zhang (2011): Let $x \in \mathbb{R}, z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$. Then the process

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \leq T,$$

is a G–martingale.
In particular, the non-symmetric part $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$, $t \in [0,T]$, is a G-martingale which is different compared to classical probability theory since $\{-K_t\}_{t \in [0,T]}$ is continuous, and non-increasing with a quadratic variation equal to zero. M is a symmetric G–martingale if and only if $K \equiv 0$.

Martingale representation, Song (2011): Let $\xi \in L_G^2(\Omega_T)$. Then the G–martingale X with $X_t := E_G[\xi | \mathcal{F}_t], t \in [0,T]$, has the following unique representation

$$X_t = X_0 + \int_0^t z_s dB_s - K_t,$$

where K is a continuous, increasing process with $K_0 = 0, K_T \in L^\alpha_G(\Omega_T), z \in H^\alpha_G(0,T), \forall\alpha \in [1,2)$, and $-K$ a G–martingale. Here, $H^\alpha_G(0,T)$ is the completion of $H^0_G(0,T)$ under the norm $\|\eta\|_{H^\alpha_G} := \big(E_G\big[\int_0^T |\eta_s|^2 ds\big]^{\frac{\alpha}{2}}\big)^{\frac{1}{\alpha}}$. If ξ is bounded from above we have $z \in M^2_G(0,T)$ and $K_T \in L^2_G(\Omega_T)$ (see Song (2011)).
Finally we state a Girsanov type theorem with G-Brownian motion. In Subsection 3.3 we discussed some heuristics in terms of a G-Doleans-Dade exponential. Define the density process by $\mathtt{E}^\theta$ as the unique solution of $d\mathtt{E}^\theta_t = \mathtt{E}^\theta_t \theta_t dB^G_t$, $\mathtt{E}^\theta_0 = 1$. The proof of the Girsanov theorem is based on a Levy martingale characterization theorem for G-Brownian motion.

Girsanov for G-expectation, Xu, Shang, and Zhang (2011): Assume the following Novikov type condition: There is an $\varepsilon > \frac{1}{2}$ such that

$$E_G\left[\exp\left(\varepsilon \cdot \int_0^T \theta_s^2 d\langle B^G\rangle_s\right)\right] < \infty$$

Then $B^\theta_t = B^G_t - \int_0^t \theta_s \langle B^G\rangle_s$ is a G-Brownian motion under the sublinear expectation $\mathcal{E}^\theta(\cdot)$ given by, $\mathcal{E}^\theta(X) = E_G[\mathtt{E}^\theta_T \cdot X]$, $\mathcal{P}^\theta = \mathtt{E}^\theta_T \cdot \mathcal{P}$ with $X \in L^2(\mathcal{P}^\theta)$.

Chapter 4

Brownian Equilibria under Drift Uncertainty

4.1 Introduction

This work enlarges the class of dynamic utility specification which ensures an equilibrium in continuous time and under uncertainty. We are interested in recursive preference structures which allow for multiple prior uncertainty. The standard model in economics assumes an additive utility structure, i.e. the utility of an uncertain consumption stream $c = (c_t)$ is given by $U(c) = \mathbb{E}[\int_0^T u(c_t)dt]$. Applications to intertemporal asset pricing are based on equilibrium state price densities. Tackling the drawbacks of the standard model, especially the strong functional dependency between aggregate income and equilibrium prices, necessitates broadening the preference specifications. This chapter concentrates on the expansion of utilities, guaranteeing Arrow-Debreu equilibria on the commodity space of adapted consumption rates processes.

In Duffie and Epstein (1992), a recursive utility specification is introduced, called stochastic differential utility (SDU)

$$U_t(c) = \mathbb{E}\left[\int_t^T f(c_s, U_s)ds|\mathcal{F}_t\right],$$

evaluated at $t = 0$. Such a specification factors the future utility of the remaining consumption stream and the evaluation of the present consumption. This makes it possible to model an agent able to distinguish between the different concepts of risk aversion and preferences for intertemporal substitution, whereby in the additive case the systemic relationship of these concepts is unavoidable. With this utility specification, Duffie, Geoffard, and Skiadas (1994) prove the existence of equilibria and discuss the dynamics of efficiency via a system of forward backward integral equations.

Going one step further, in this chapter we consider the generalized SDU (GSDU) from Lazrak and Quenez (2003) and Schroder and Skiadas (2003) under a Brownian filtration $(\mathcal{F}_t)$. The process of the intertemporal utility is a solution of a so called backward stochastic differential equation.[1] The initial value of the solution (U, Z) of this recursive integral equation,

$$U_t(c) = \mathbb{E}\left[\int_t^T f(s, c_s, U_s, Z_s)ds \big| \mathcal{F}_t\right], \quad t \in [0, T] \tag{1}$$

is the utility of an agent with aggregator f and given consumption rate process $c : [0, T] \times \Omega \to \mathbb{R}$. The endogenous process (Z_t), as a part of the solution, refers to the volatility of (U_t). In this family of utility functionals source-dependent risk aversion, a kind of asymmetric risk aversion, can be modeled. Furthermore, the notion of preference for information, introduced and axiomatized by Skiadas (1998), is contained in the present GSDU-class, see also Lazrak (2004).

A main goal is to consider models where each agent is faced with different imprecise knowledge about the probability distribution describing the primitive uncertainty. For instance, in Chen and Epstein (2002) Knightian uncertainty is formalized via drift uncertainty, a continuous-time version of maxmin utility from Gilboa and Schmeidler (1989). A subclass of the utility functionals, for instance κ-ignorance, lies in the GSDU class. The implications of ambiguity aversion for financial markets and asset pricing are studied in Epstein and Miao (2003) for a two-agent equilibrium setting and in Epstein and Wang (1994) for the discrete time case with a single representative agent.

Another class of continuous-time economies which incorporates Knightian uncertainty is related to incomplete preferences motivated and axiomatized by Bewley (2002). Here, the agents foreclose unmotivated gambles by an inertia principle. For the static and finite state case, an equilibrium existence result is established by Rigotti and Shannon (2005). In Dana and Riedel (2013), such economies are considered in a discrete time setting. They relate them to variational preference anchored at the initial endowment (e_t), a special case of variational preferences that have been axiomatized by Maccheroni, Marinacci, and Rustichini (2006).

In order to apply our results to the above examples, we cannot assume differentiability on the whole domain of the aggregator f. This leads to the non differentiability of the utility functional, and forces us to consider supergradient densities.

For the existence of an equilibrium, we follow the classical approach dealing with infinite-dimensional commodity and price spaces by introducing the concept of properness (or cone condition).[2] The existence proof is an application of the abstract existence result of Podczeck (1996). The empty interior of the positive cone of the commodity space requests a pointwise (forward-) properness condition which must hold at each consumption plan of a Pareto optimal allocation.

The chapter is organized as follows. Section 2 introduces the model, recasts

[1]We refer to El Karoui, Peng, and Quenez (1997) for a detailed survey.

[2]For a first overview we refer to Mas-Colell and Zame (1991).

the notion of GSDU utility functionals and discusses the supergradient. Section 3 considers efficient allocations, proves the boundedness away from zero of the components and the existence of general equilibrium. Proofs of auxiliary results are collected in the Appendix.

4.2 The Economy

Fix a time interval $[0, T]$, for some $T \in (0, \infty)$. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by an n-dimensional Brownian motion $(B_t)_{t\in[0,T]}$, and satisfying the usual conditions.
For simplicity, the case of $l = 1$ commodity is examined.[3] We introduce the Hilbert lattice of progressively measurable processes $c : \Omega \times [0, T] \to \mathbb{R}$ with finite norm $\|c\|_2 = \mathbb{E}[\int_0^T |c_t|^2 dt]^{\frac{1}{2}}$, denoted by $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{P} \otimes dt)$. The consumption set is given by the positive cone L^2_+.[4] The Hilbert space structure of the commodity space allows us to consider the commodity-price duality $\langle \mathrm{L}^2, (\mathrm{L}^2)^* \rangle$ given by the scalar product of L^2.

4.2.1 Main Result

The economy consists of $m \in \mathbb{N}$ agents. The utility of each agent $i \in \{1, \dots, m\}$ is described by a utility functional $U^i : \mathrm{L}^2_+ \to \mathbb{R}$ which is given by the initial value $U^i_0 = \mathbb{E}[\int_0^T f^i(t, c_t, U^i_t, Z^i_t)dt]$ of the solution (U^i, Z^i) of the following backward stochastic differential equation (Backward-SDE)

$$dU^i_t = -f^i(t, c_t, U^i_t, Z^i_t)dt + {Z^i_t}^\top dB_t \quad U^i_T = 0. \tag{2}$$

By *generalized stochastic differential utility*, GSDU for short, we denote the functional $c \mapsto U^i_0$, where U_0 is the $\mathbb{P}$-a.s initial value of (1). This is a rigorous formulation of the utility backward recursion principle, considered first by Duffie and Epstein (1992) in the continuous-time case under uncertainty. In Proposition 3.5 of El Karoui, Peng, and Quenez (1997) time consistency of GSDU is shown. This principle can be seen as a benchmark for a rational updating of intertemporal preferences. From an economic point of view, when this Backward-SDE is used to define generalized stochastic differential utility, it increases the modeling degree of freedom when having a Z component in the aggregator. The quadratic variation of the utility process is given by $\langle U \rangle_t = \int_0^t \|Z_s\|^2_{\mathbb{R}^n} ds$. This *intensity process* appears in the intertemporal aggregator, so that a direct effect of the n-dimensional adapted stochastic process Z can explicitly express preferences on "local risk". The following conditions ensure the usual properties of the utility functional.

[3] The case of finite commodities can be treated by the same argumentation, we refer to Duffie, Geoffard, and Skiadas (1994).

[4] Measures on $\Omega \times [0, T]$ which allow considerations of terminal consumption are possible. In this case the BSDE in (2) has a non-trivial terminal condition.

Assumption 1 *The aggregator f is uniform Lipschitz continuous in u and z with constant $k > 0$,*[5] *satisfies a linear growth condition in c and*

1. *For all $(t, u, z) \in [0, T] \times \mathbb{R}^{n+1}$, $f(t, \cdot, u, z)$ is strictly increasing and differentiable on $(0, \infty)$.*
2. $\overline{\delta}_f(c) := \sup_{(t,u,z)} |\partial_c f(t, c, u, z)| < \infty$ *for all $c > 0$*
3. *Each sequence $(c_n) \searrow 0$ implies $\underline{\delta}_f(c_n) := \inf_{(t,u,z)} |\partial_c f(t, c_n, u, z)| \to \infty$.*
4. *For all $t \in [0, T]$, $f(t, \cdot, \cdot, \cdot)$ is a concave and continuous function.*

The Lipschitz-growth assumption on the aggregator guarantees unique existence of (2) for all $c \in \mathrm{L}^2$, see El Karoui, Peng, and Quenez (1997).

Proposition 1 *Under Assumption 1, the GSDU-utility functional is concave, strictly increasing and $\|\cdot\|_2$-continuous.*

In the following, we state a concrete preference specification such that Assumption 1 applies. We pick up this example in Subsection 2.2 and 2.3, where we discuss economies with ambiguity averse agents.

Example 1 (Habit formation with subjective beliefs) *The following aggregator $f^k(t, c, z, u) = u_k(c) - \beta_k(t, c)u - \theta z$ induces a GSDU given by*

$$U^k(c) = \mathbb{E}^{P_\theta}\left[\int_0^T e^{\int_0^t \beta_k(r,c_r)dr} u_k(c_t)dt\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \mathcal{E}_t^{\beta_k,\theta} u_k(c_t)dt\right], \quad c \in \mathrm{L}^2_+.$$

The subjective prior is given by $dP_\theta = \mathcal{E}_T^{0,\theta} d\mathbb{P}$, where $\mathcal{E}^{0,\theta}$ and $\mathcal{E}^{\beta_k,\theta}$ solve (3) respectively. The process θ is strictly positive and bounded.

Next, we discuss the differential of the utility functional. For any process $(\mathfrak{u}, \mathfrak{z}) \in \mathrm{L}^2(\mathbb{P} \otimes dt; \mathbb{R}^{1+n})$, we introduce the *stochastic exponential process* $\mathcal{E}^{\mathfrak{u},\mathfrak{z}}$ which is defined as the solution of the SDE

$$d\mathcal{E}_t^{\mathfrak{u},\mathfrak{z}} = \mathcal{E}_t^{\mathfrak{u},\mathfrak{z}} \mathfrak{u}_t dt + \mathcal{E}_t^{\mathfrak{u},\mathfrak{z}} \mathfrak{z}_t^\top dB_t, \quad \mathcal{E}_0^{\mathfrak{u},\mathfrak{z}} = 1. \tag{3}$$

We do not assume differentiability on the whole domain of the aggregator since concavity allows us to consider the super-differential and give a characterization in terms of super gradient densities (see the Appendix for details).

Proposition 2 *Let Assumption 1 hold, $c \in \mathrm{L}^2_+$ be bounded away from zero and (U, Z) be a solution of the Backward-SDE (2). Define the time-dependent correspondence $\partial_{u,z} f(c)_t : \Omega \rightrightarrows \mathbb{R}^{1+n}$ given by $\partial_{u,z} f(c)_t(\omega) = \partial_{U,Z} f(t, c_t, U_t, Z_t)(\omega)$. Then, the differential of U_0 at c of super-gradient densities is given by*

$$\partial U(c) = \left\{\mathcal{E}_t^{\mathfrak{u},\mathfrak{z}} \cdot \partial_c f(t, c_t, U_t, Z_t) : (\mathfrak{u}_t, \mathfrak{z}_t) \in \partial_{u,z} f(c)_t,\ t \in [0, T]\right\}.$$

[5] This means $|f(t, c, u, z) - f(t, c, u', z')| \leq k|(u, z) - (u', z')|$ for all $u, u' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

The pure exchange economy is given by $\mathtt{E} = \left\{\mathrm{L}^2_+, U^i, e^i\right\}_{1\leq i\leq m}$, where each U^i is a GSDU-utility functional. An element $(\bar{c}^1, \ldots, \bar{c}^m; \Pi) \in (\mathrm{L}^2)^m \times (\mathrm{L}^2)^*$, consisting of a feasible allocation and a non-zero linear price functional, is called an *Arrow-Debreu equilibrium* if for each i, $\bar{c}^i$ maximizes agent i's utility over all $c \in \mathrm{L}^2_+$ satisfying $\Pi(c^i - e^i) \leq 0$ and $\sum \bar{c}^i = e$. The main result is the following.

Theorem 1 *Suppose the endowment $e \in \mathrm{L}^2_+$ of the economy is bounded away from zero. For each agent i Assumption 1 holds and $e^i \in \mathrm{L}^2_+ \setminus \{0\}$.*
Then there exists a contingent Arrow-Debreu equilibrium $(\bar{c}^1, \ldots, \bar{c}^m; \Pi)$ for $\mathtt{E}$, such that for every i, $\bar{c}^i$ is bounded away from zero.
Moreover, the price system Π has a Riesz Representation $\pi \in \mathrm{L}^2_+$. For every i there is a $\mu_i > 0$ such that $\pi = \mu_i \pi^i(\bar{c}^i)$ where $\pi^i(\bar{c}^i)$ is a super gradient density.

The proof is an application of the abstract existence result in Podczeck (1996), while the properties of the equilibrium are based on a priori estimates, stochastic Gronwall inequalities and the full characterization of the superdifferential. The main step is to prove that optimal allocations are bounded away from zero.
The framework of the present economy opens the question on generic existence of equilibria. In finite-dimensional commodity spaces, the usual notion of generic sets corresponds to full Lebesgue measure. In an infinite-dimensional framework, one way out is to consider the concept of prevalence.[6] The principle is discussed in Anderson and Zame (2001) and, as shown in their Theorem 3.2, the set $\left\{(e^1, \ldots, e^m) \in \mathrm{L}^{2,m}_+ : \sum e^k > \varepsilon \, \mathbb{P} \otimes dt - a.e.\right\}$ is finite prevalent in $\mathrm{L}^{2,m}_+$, where $\varepsilon > 0$.[7]

4.2.2 Maxmin Preferences in Continuous Time

An agent may not know the real world probability measure and is confronted with a set of prior probability measures. This uncertainty or unmeasurable risk is referred to as ambiguity. In Chen and Epstein (2002), a continuous-time model is introduced which models the set of priors in terms of the density kernel related to each prior. Let $\kappa^k = (\kappa^k_1, \ldots, \kappa^k_n) \in \mathbb{R}^n_+$[8] and define the set of densities by $\Theta^k = \left\{\theta \in \mathrm{L}^{2,n} : \theta^i_t \in [-\kappa^k_i, \kappa^k_i], \quad 1 \leq i \leq n\right\}$.
We construct the set of priors via Θ^k by considering, for each prior $\theta \in \Theta^k$, a single SDU model $U^{k,\theta}_t = \mathbb{E}^{P^\theta}[\int_t^T g^k(c_t, U_t)dt|\mathcal{F}_t]$ such that risk aversion and

[6] An alternative would be related to Baire's Category theorem. A set of first category is contained in a countable union of closed sets with an empty interior. However it detects an empty interior for first category sets. This notion has little measure theoretic connection. As mentioned in Mas-Colell (1990) on page 318, a topological generic set "has to be thought of much less sharp than measure-theoretic concept available in the finite-dimensional case".

[7] This indicates, that the condition on the aggregate endowment in Theorem 1 is less strong than suspect at first glance.

[8] In principle, each κ^k_i can also be a bounded $\mathbb{F}$-adapted process.

intertemporal preferences can be encoded in g^k, see Example 1. The utility process of a pessimistic or ambiguity averse agent satisfies $U_t^k = \min_{\theta \in \Theta^k} U_t^{k,\theta}$, $t \in [0,T]$, where the process (U_t^k, Z_t^k) solves the Backward-SDE

$$\begin{aligned} dU_t^k &= -\left(g^k(t, c_t^k, U_t^k) + \min_{\theta \in \Theta^k} \langle \theta_t, Z_t^k \rangle \right) dt + Z_t^{k\top} dB_t \\ &= -g^k(t, c_t^k, U_t^k) + \kappa^k \cdot |Z_t^k| dt + Z_t^{k\top} dB_t, \quad U_T^k = 0. \end{aligned}$$

Ambiguity aversion is referred to in consideration of the worst case utility. Heuristically speaking, the bigger κ_i^k is, the more ambiguity aversion is assigned to the agents. Put $K^k = [-\kappa_1^k, \kappa_1^k] \times \ldots \times [-\kappa_n^k, \kappa_n^k]$. Since $z \mapsto \max_{\theta \in K^k} \theta \cdot z$ is the convex conjugate of the indicator function 1_K^k, $f^k(t,c,u,z) = g^k(c,u) + \kappa^k z$ is concave in z. Lipschitz continuity in z is implied by the boundedness of each κ_i^k. The differentiability of the aggregator f^k in z is not satisfied.
In order to apply Theorem 1, we briefly check if the conditions in Assumption 1 hold. We may take an SDU aggregator $g^k : [0,T] \times \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ which is consistent with Assumption 1. To mention a concrete functional form, take a specification of Kreps and Porteus (1978) with time dependent parameters

$$g^k(t,c,u) = \frac{c^p - \beta^k(t)(au)^{p/a}}{p(au)^{(p-a)/a}},$$

with $\beta^k(t) \geq 0$ for all $(t,k) \in [0,T] \times \{1,\ldots,m\}$ and $a, p \leq 1$. In order to keep the exposition simple, we have no heterogeneity or time dependence with respect to the parameter a and p. The conditions of Theorem 1 are satisfied.

Corollary 1 *Equilibria with heterogeneous Maxmin utility exist generically.*

We compute the super-differential of U^k at c explicitly. To do so, define the worst case priors having density kernels given by

$$\Theta_c^k = \left\{ \theta \in \Theta^k : \theta_t \in \arg\max_{y \in \Theta_t} y \cdot \mathcal{E}_t^{0,\theta} \textit{ for all } t \in [0,T] \right\}.$$

According to Theorem 1 the equilibrium allocation components are bounded away from zero. Then by Proposition 2, the super-differential of each agent k can be written as

$$\partial U^k(c) \;\; = \left\{ (\pi_t) : \exists \theta \in \Theta_c^k, \pi_t = \partial_c g^k(c_t, U_t^k) \cdot \mathcal{E}_t^{\partial_u g^k, \theta}, \textit{ for each } t \in [0,T] \right\},$$

In comparison with Chen and Epstein (2002), our Inada condition on $f(t,\cdot,u,z)$, instead of a growth condition on $\partial f_c^k(t,\cdot,u,z)$, allows for a full characterization of the superdifferential $\partial U^k(c^k)$ at the equilibrium consumption c^k.

4.2.3 Bewley Preferences in Continuous Time

This example studies an auxiliary economy with variational preferences which can be linked to heterogeneous Bewley preferences. We show that Theorem 1 also covers the existence of equilibrium in such an economy. In Dana and Riedel (2013), this concept of a discrete time Bewley economy is considered, where preferences are incomplete by construction.
The set of priors of agent k is given $\mathcal{P}^k = \{P^k : dP^k = \mathcal{E}_t^{0,\theta} dP, \theta \in \Theta^k\}$, where Θ^k is already introduced in Subsection 2.2. Agent k prefers consumption plan $x \in \mathrm{L}^2_+$ in comparison to $y \in \mathrm{L}^2_+$ if and only if for all priors $\mathbb{P}' \in \mathcal{P}^k$

$$\mathbb{E}^{\mathbb{P}'}\left[\int_0^T u^k(t, x_t)dt\right] \geq \mathbb{E}^{\mathbb{P}'}\left[\int_0^T u^k(t, y_t)dt\right]$$

holds. This induces an incomplete preference relation. The existence of an equilibrium with agents having such preferences is established by considering an auxiliary economy with complete static variational preferences. Fix a strictly increasing, concave and continuous utility index $u^k : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada conditions in its second variable, hence Assumption 1.1-3 are satisfied.
We define for each agent $k \in \{1, \dots, m\}$ a variational utility functional anchored at the initial income $e^k \in \mathrm{L}^2_+$ by

$$V^k(x) = \min_{\mathbb{P} \in \mathcal{P}^k} \mathbb{E}^{\mathbb{P}}\left[\int_0^T u^k(t, x_t) - u^k(t, e_t^k)dt\right]. \tag{4}$$

Using the same Backward-SDE-arguments as the in proof of Theorem 2.2 in Chen and Epstein (2002), one obtains that $V^k(x)$ is the initial value of the solution (V^k, Z^k) of the following Backward-SDE

$$dV_t^k = \left(-(u^k(t, x_t) - u^k(t, e_t^k)) + \max_{\theta \in \Theta}\langle \theta_t, Z_t^k\rangle\right) dt + Z_t^{k\top} dB_t, \quad V_T^k = 0.$$

Specifically, the existence of the Backward-SDE follows by the same arguments as in Subsection 2.2., since each $e^k \in \mathrm{L}^2_+$. Moreover, Assumption 1 is satisfied.

Corollary 2 *Equilibria with Bewley preferences exist generically.*

With the existence of an equilibrium in this auxiliary economy, the proof follows by the saw arguments as in Theorem 2.6 of Dana and Riedel (2013).

4.2.4 Radner Equilibria and Asset Pricing

To illustrate the generality of Theorem 1, we formulate the pricing kernel of the underlying heterogeneous agent economy, when Itô processes describe the primitives. Let the aggregate endowment e of the economy and the cumulative dividend process D of a long lived security be given by

$$de_t = \mu_t^e dt + \sigma_t^{e\top} dB_t, \quad dD_t = \mu_t^D dt + \sigma_t^{D\top} dB_t,$$

with initial condition $(e_0, D_0) \in \mathbb{R}^{1+n}_{++}$ and adapted integrable processes μ^e_t, μ^D_t and σ^e_t, σ^D_t. The process (σ^e_t) is n-dimensional and (σ^D_t) is $n \times n$-dimensional with $\mathbb{P} \otimes dt$-a.e. full rank. As a first step to establishing Theorem 1 as an equilibrium foundation for mathematical finance, one has to discuss the structure of the state price density. Under an additional assumption, the appearance of the intensity process Z in the supergradients still ensures that the equilibrium state price density $\pi^i(c^i)_t = \partial_c f^i(t, c^i_t, U^i_t, Z^i_t)\mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}_t$ is indeed an Itô process. This can be seen as follows. $\mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}$ is again an Itô process, as a solution of (3). Now, assume that the partial derivative $\partial_c f^i(t, c, u, z)$ does not depend on z and is three times continuously differentiable in c, u. Note that this assumption holds true in our multiple prior economies of Corollary 1 and 2. Now, following the implicit function argument in Section 2.5 of Duffie, Geoffard, and Skiadas (1994) there is a twice continuously differentiable function K_i, depending on $(t, e, \mathcal{E}, U) = (t, e, \{\mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}, U^i\}^m_{i=1})$, such that the α-efficient allocation can be written as $\{K^i(\cdot, e_\cdot, \mathcal{E}_\cdot, U^i_\cdot)\}^m_{i=1}$, where $\mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}_0 = \alpha^i$.
Since the process U^i is an Itô process, π^i is by the Itô's formula an Itô process as well. The absolute continuity of the bounded variation component allows for an interpretation of a money market captured by an interest rate process.

Corollary 3 *Under Assumption 1, the differentiability assumption and the assumed Itô structure on e and D made before, there exists a Rander equilibrium which implements the equilibrium allocation of Theorem 1. Let $k \in \{1, \dots, n\}$, the price process of the k'th long lived security satisfies*

$$S^k_t = \frac{1}{\mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}_t \cdot \partial_c f^i(t, c^i_t, U^i_t)} \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}_s \cdot \partial_c f^i(t, c^i_t, U^i_t) dD^k_s \middle| \mathcal{F}_t \right], \quad t \in [0, T),$$

where $c^i_t = K^i(t, e_t, \mathcal{E}_t, U_t)$ for some agent $i \in \{1, \dots, m\}$.

Such a Radner implementation procedure can be used to observe a consumption based capital asset pricing model, see Duffie and Zame (1989). For simplicity, we set the consumption spot price to 1. For a direct construction of the Radner equilibrium in a two-agent economy under Knightian uncertainty we refer to Epstein and Miao (2003). In the κ-ignorance case this can be found in Section 5.4 in Chen and Epstein (2002). An ambiguity premium comes into play. This can be used to tackle the so called equity premium puzzle.
The Corollary also has implications for the "pricing kernel puzzle". As in the static and finite state case considered by Hens and Reichlin (2013), the pricing kernel of the state price density at time t is in the present GSDU setting $\pi^i(e_t) = \partial_c f^i(t, K^i(t, e_t, \mathcal{E}_t, U_t), U^i_t, Z^i_t) \cdot \mathcal{E}^{\mathfrak{u}^i,\mathfrak{z}^i}_t$. An application of Itô's formula to the first factor, with respect to the Itô process (e_t), and Itô's

product rule gives the following explicit state price density

$$\begin{aligned}\frac{d\pi^i(e_t)}{\mathcal{E}_t^{\mathfrak{u}^i,\mathfrak{z}^i}} &= \left(f^i_{cc}(e_t)\sigma^e_t - f^i_c(e_t)\cdot f^i_Z(e_t)\right)^\top dB_t \\ &\quad + f^i_{cc}(e_t)\left(\mu^e_t + \langle\sigma^e_t, f^i_Z(e_t)\rangle\right) + \frac{\sigma^{e\top}_t\cdot\sigma^e_t}{2} f^i_{ccc}(e_t) + f^i_c(e_t) f^i_U(e_t)dt,\end{aligned} \tag{E}$$

where $f^i_x(e_t) = \partial_x f^i(t, K^i(t, e_t, \mathcal{E}_t, U_t), U^i_t)$, with $x = c, U, Z$. Note that by (3), $\mathcal{E}_t^{\mathfrak{u}^i,\mathfrak{z}^i}$ can be written in an explicit exponential form.
The very general Euler equation (E) covers many different heterogeneous agent economies and is a testable implication of the present GSDU model. However, we mention that the dynamics of the efficient allocation are given by the solution $(\mathcal{E}_t, U_t, Z_t)$ of a fully coupled system of Forward-Backward-SDE's. In Dumas, Uppal, and Wang (2000), the system is discussed in the case of Stochastic Differential Utility.

4.3 Existence of Equilibria

The objective of this section is to identify efficient and equilibrium allocations for GSDU preferences. We begin by characterizing Pareto optimal allocations with the solution of a social planning problem, and to prove the existence of this solution. Afterwards we introduce the first order conditions. Moreover, we show that Assumption 1 is sufficient to guarantee that the components of the efficient allocation are bounded away from zero.

4.3.1 Efficient Allocations

We define the usual norm on the underlying space for allocations $\mathrm{L}^{2,m} = (\mathrm{L}^2)^m$, $\|c\|^{2,m} := \left(\sum_{i=1}^m \mathbb{E}[\int_0^T |c^i_t|^2 dt]\right)^{\frac{1}{2}}$, where $c = (c^1, \ldots, c^m)$. By $\mathrm{L}^{2,m}_+$, we denote the positive cone of $\mathrm{L}^{2,m}$ and by $\mathrm{L}^{2,m}_{++}$ the quasi interior. The set of *feasible allocations* is defined by $\Lambda(e) := \left\{(c^1, ..., c^m) \in \mathrm{L}^{2,m}_+ : e \geq \sum c^i\right\}$. A weighting $\alpha \in \mathbb{R}^m_+$ induces a *representative agent* $U^\alpha : \mathrm{L}^{2,m}_+ \to \mathbb{R}$, given by $U^\alpha(c^1, \ldots, c^m) := \sum_{i=1}^m \alpha_i U^i(c^i)$.
An allocation $(\hat{c}^1, \ldots, \hat{c}^m) \in \mathrm{L}^{2,m}_+$ is *α-efficient* if it achieves the maximum over $\Lambda(e)$, i.e. $U^\alpha(\hat{c}) = \max_{c\in\Lambda(e)} U^\alpha(c)$. In the following, we state the relation to α-efficiency when utility functionals are concave.

Proposition 3 *Suppose the utility functionals are of GSDU type and satisfy Assumption 1. Then there is an α-efficient allocation. Pareto optimal allocations exist and*

1. *If* $\bigcap_{i=1}^m \partial\alpha_i U^i(\hat{c}^i) \neq \emptyset$, *then* $\hat{c} = (\hat{c}^1, \ldots, \hat{c}^m)$ *is α-efficient.*

2. *If* $(\hat{c}_1, \ldots, \hat{c}_m)$ *is α-efficient and, for all* i, $\hat{c}^i$ *is bounded away from zero, then* $\bigcap_{i=1}^m \partial\alpha_i U^i(\hat{c}^i) \neq \emptyset$ *holds.*

In order to apply the previous result to GSDU we have to establish a criterion which ensures that the components of the efficient allocation are bounded away from zero.

Lemma 1 *Suppose Assumption 1 holds for each $i \in \{1, \dots, m\}$ and is e bounded away from zero. Fix an α-efficient allocation $c = (c^1, \dots, c^m) \in \mathrm{L}^{2,m}_{++}$. Then, for each i, c^i is bounded away from zero.*

Proof of Lemma 1 *Let $\nu = \mathbb{P} \otimes dt$ and take a $c \in \mathrm{L}^{2,m}_{++}$. For every i we have $U^i(c^i) > U^i(0)$ since each U^i is strictly increasing.*
Suppose some c^j is not bounded away from zero. Then for every $h > 0$ there is an $\hat{H} = \hat{H}(h) \in \mathcal{O}$ such that $\nu(\hat{H}) > 0$ and $c^j \leq h$ on $\hat{H}$. Since e is bounded away from zero, we have $e > C$ ν-a.e. for some constant $C > 0$. This gives us, if C is small enough, that there is an agent k such that $c^k \geq \frac{C}{m}$ on $H' \subset \hat{H}$. We choose $H = \{c^j < h\} \cap \{\frac{C}{m} \leq c^k \leq C^k\}$ which has a positive measure.
On the other hand, since $c = (c_1, \dots, c_m)$ is in the quasi interior of $\mathrm{L}^{2,m}_{+}$, for every i, there is a set $A^i \in \mathcal{O}$ with $\nu(A^i) > 0$ and a number $a^i > 0$ such that $c^i \geq a^i$ on A^i.
We show a Pareto improvement when multiples of H and A^j are traded between agent j and k. Let $\lambda^k \in (0, h)$ and $\lambda^j \in (0, \frac{a_j}{2})$. Define the following Backward-SDE's:

$$c^j \mapsto (U, Z),\ c^j - \lambda^j 1_{A^j} \mapsto (U^A, Z^A) \text{ and } c^j - \lambda^j 1_{A^j} + \lambda^k 1_H \mapsto (U^{AH}, Z^{AH}),$$

where $U_0 = U^j(c^j)$, $U^A_0 = U^j(c^j - \lambda^j 1_{A^j})$ and $U^{AH}_0 = U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H)$ are the corresponding evaluated utility functionals. We derive

$$\begin{aligned}
&U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H) - U^j(c^j) \\
&\geq e^{k^j T} \mathbb{E}\left[\int_0^T \underline{\delta}_{f^j}(2h)\lambda^k 1_H(t) - k^j\left(|Z^{AH}_t - Z^A_t| + |Z_t - Z^A_t|\right) + \overline{\delta}_{f^j}\left(\frac{a^j}{2}\right)\lambda^j 1_{A^j}(t)dt\right].
\end{aligned}$$

The inequality employed the estimates in Lemma 4 and Lemma 5 (see the Appendix). Next, we compute appropriate estimates for the Z parts. By the Cauchy-Schwartz inequality and the a priori estimates in El Karoui, Peng, and Quenez (1997), with $\lambda^2 = 2k$, $\mu = 1$ and $\beta \geq 2k(1+k) + 1$, we derive:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |Z_s - Z^A_s| ds\right] \\
\leq\ &\left(\frac{T\lambda^2}{\mu^2(\lambda^2 - k)} \mathbb{E}\left[\int_0^T e^{\beta s} |f^j(s, c^j_s, U_s, Z_s) - f^j(s, c^j_s - \lambda^j 1_{A^j}, U_s, Z_s)|^2 ds\right]\right)^{\frac{1}{2}} \\
\leq\ &(2Te^{\beta T})^{1/2} \mathbb{E}\left[\int_0^T \overline{\delta}_{f^j}(a^j/2)\lambda^j 1_{A^j}(s) ds\right]
\end{aligned}$$

The second inquality is a pointwise application of the mean value theorem, the usage of $\lambda^j < \frac{a^j}{2}$ and $c^j \geq a_j$ on A^j and because $\partial_c f^j$ is decreasing.

Analogous arguments yield

$$\mathbb{E}\left[\int_0^T |Z_s^{AH} - Z_s^A| ds\right] \leq (2Te^{\beta T})^{1/2}\mathbb{E}\left[\int_0^T \overline{\delta}_{f^j}\left(\frac{a^j}{2}\right)\lambda^k 1_H(s)ds\right].$$

Since h *can be taken to be arbitrarily small,* $\underline{\delta}_f(2h)$ *becomes arbitrarily large and by the last two derivations with* $e^{\circ j} = e^{\circ k_j T}$ *and* $\hat{e}^{\circ j} = e^{\circ k_j T} \cdot (2Te^{\beta T})^{1/2}$ *,* $\circ \in \{+,-\}$*:*

$$\begin{aligned}
& U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H) - U^j(c^j) \\
\geq\ & b^j e^{-j}\underline{\delta}_{f^j}(2h)\mathbb{E}\left[\int_0^T \lambda^k 1_H(t)dt\right] - e^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})\mathbb{E}\left[\int_0^T \lambda^j 1_{A^j}(t)dt\right] \\
& -e^{-j}\mathbb{E}\left[\int_0^T k_j|Z_t^{AH} - Z_t^A|dt\right] - e^{+j}\mathbb{E}\left[\int_0^T k_j|Z_t - Z_t^A|dt\right] \\
\geq\ & \lambda^k\Big(e^{-j}\underline{\delta}_{f^j}(2h)\nu(H) - \hat{e}^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j)\Big) \\
& -\lambda^j\Big(e^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j) + \hat{e}^{-j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(H)\Big).
\end{aligned}$$

A utility improvement of agent j *is related to the strict positivity of the last term. An analogous derivation and a modification of Lemma 4 and Lemma 5 yield the corresponding inequality for agent* k*. Hence, in order to achieve a Pareto improvement*

$$1 \quad > \quad \frac{e^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j) + \hat{e}^{-j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(H)}{e^{-j}\underline{\delta}_{f^j}(2h)\nu(H) - \hat{e}^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j)} \cdot \frac{e^{+k}\overline{\delta}_{f^k}(\frac{C}{2m})\nu(A^j) + \hat{e}^{-k}\overline{\delta}_{f^k}(\frac{a^k}{2})\nu(H)}{e^{-k}\underline{\delta}_{f^k}(2C^k)\nu(H)\hat{e}^{+k}\overline{\delta}_{f^k}(\frac{a^j}{2})\nu(A^j)}$$

must hold. If we take a sufficiently small h*, then, by the Inada style condition,* $\underline{\delta}_f(2h)$ *becomes arbitrarily large. Consequently* $\nu(\hat{H})$ *and hence* $\nu(H)$ *becomes arbitrary small. We may choose* A^j *such that* $\nu(H) = \nu(A^j) > 0$*, this gives us*

$$1 \quad > \quad \frac{e^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2}) + \hat{e}^{-j}\overline{\delta}_{f^j}(\frac{a^j}{2})}{e^{-j}\underline{\delta}_{f^j}(2h) - \hat{e}^{+j}\overline{\delta}_{f^j}(\frac{a^j}{2})} \cdot \frac{e^{+k}\overline{\delta}_{f^k}(\frac{C}{2m}) + \hat{e}^{-k}\overline{\delta}_{f^k}(\frac{a^k}{2})}{e^{-k}\underline{\delta}_{f^k}(2C^k)\hat{e}^{+k}\overline{\delta}_{f^k}(\frac{a^j}{2})}$$

by choosing appropriate multiples $\lambda^k \in (0,h)$ *and* $\lambda^j \in (0, \frac{a_j}{2})$ *we finally get:*

$$U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H) > U^j(c^j) \text{ and } U^k(c^k + \lambda^j 1_{A^j} - \lambda^k 1_H) > U^k(c^k).$$

This yields a Pareto improvement, contradicting that $(c^1, \ldots, c^m)$ *is a Pareto optimal allocation. Therefore, each* c^j *of the efficient allocation is bounded away from zero.* ■

4.3.2 Properness and the Proof of Theorem 1

In this section we deal with the existence of an Arrow-Debreu equilibrium. In the end of the Appendix, we consider an economy defined on an abstract

lattice and state the existence of a quasi equilibrium.
The notion of F-properness at x, see Definition 1 in the Appendix, can be written as:

> There is a $v \in \mathrm{L}^2_+$ and an $\varepsilon > 0$ such that for all z with $\|z\|_{\mathrm{L}^2} < \varepsilon$, $U(x + \lambda(v - z)) > U(x)$, for small $\lambda > 0$ with $x + \lambda(v - z) \in \mathrm{L}^2_+$.

Now, we establish the existence of an equilibrium when the utility functional $U^i : \mathrm{L}^2_+ \to \mathbb{R}$ is given by a GSDU. Lemma 1 ensures that every component of the efficient allocation is bounded away from zero.
We prove the existence of the equilibrium by an application of Theorem 2, stated at the end of the Appendix. Therefore, we need F-properness to hold at certain points. This is proven in the following lemma. The principle goes back to Le Van (1996) where the case of separable utilities was treated. Apart from Lemma 1, the main work was already done in Proposition 2, where the square integrability of the super gradient density $\pi(c) = \mathcal{E}\partial_c f^i(\cdot, c., U., Z.)$ was proven. In this case, $\langle \pi(c), \cdot \rangle$ is the supporting linear functional at c.

Lemma 2 *Suppose that $c = (c^1, \ldots, c^m)$ is a Pareto optimal allocation with $U^i(c^i) \geq U^i(e^i)$. Under the assumptions of Theorem 1, the F-properness at each c^i holds.*

Proof of Lemma 2 *By a modification to Lemma 1, each c^i is bounded away from zero. The assumption of a quasi interior allocation may be substituted by individual rationality.*
Fix $v \equiv 1$ as the properness vector. According to Proposition 2, a super gradient density $\pi(D_{U,Z}f^i) \in \mathrm{L}^2_{++}$ at c^i is given by

$$\pi(D_{U,Z}f^i)_t = \mathcal{E}_t^{D_U f^i, D_Z f^i} \cdot \partial_c f^i(t, c^i_t, U^i_t, Z^i_t).$$

The parametrization is related to the super-differential $\partial_{U,Z} f^i$ of the aggregator f^i. For later use define

$$V := \bigcap_{D_{U,Z}f^i \in \partial_{U,Z}f^i} V(D_{U,Z}f^i).$$

where $V(D_{U,Z}f^i) = \{z \in \mathrm{L}^2 : \langle \pi(D_{U,Z}f^i), (1 - z)\rangle_{\mathrm{L}^2} > 0\}$. We show that V is a neighborhood of 0 in L^2. For each $D_{U,Z}f^i$ there exists an open ball around zero which is contained in $V(D_{U,Z}f^i)$. Choose an arbitrary

$$z \in \left\{ y \in \mathrm{L}^2 : \|y\|_{\mathrm{L}^2} < \frac{\|\pi(D_{U,Z}f^i)\|_{\mathrm{L}^1}}{\|\pi(D_{U,Z}f^i)\|_{\mathrm{L}^2}} \right\}.$$

The positivity of π implies $\langle \pi(D_{U,Z}f^i), z\rangle_{\mathrm{L}^2} < \|\pi\|_{\mathrm{L}^1} \langle \pi(D_{U,Z}f^i), 1\rangle_{\mathrm{L}^2}$. Hence, there is an open ball which is contained in V.
Let $c^i + \lambda(1 - z) \in \mathrm{L}^2_+$, where $z \in V$ is arbitrary and $\lambda > 0$ is sufficiently small. When $\lambda > 0$ tends to zero, the term $\lambda^{-1}U^i(c^i + \lambda(1 - z)) - U^i(c^i)$

increases, due to the concavity of U^i. *Fix some* $z \in V$, *whenever* $\lambda \searrow 0$, *the limit of the quotient exists by Giles (1982)*[9] *and we have*

$$\lim_{\lambda \searrow 0} \frac{U^i(c^i + \lambda(1-z)) - U^i(c^i)}{\lambda} \geq \langle \pi(D_{U,Z} f^i)_t (1-z) \rangle_{\mathrm{L}^2} > 0.$$

The first inequality holds by Theorem 3 (p.122) in Giles (1982). The second inequality is valid since $z \in V \supset B_\varepsilon(0)$. *Now, consider a sufficiently small* λ *with* $U^i(c^i + \lambda(1-z)) > U^i(c^i)$. *In other words,* U^i *is F-proper at* c^i . ■

Proof of Theorem 1 *Assumption 1 implies strict monotonicity, concavity and norm continuity for each utility functional* U^i. *The F-properness at each* c^i *is the content of Lemma 2, where* $(c^1, \ldots, c^m)$ *is an* α*-efficient allocation with* $U^i(c^i) \geq U^i(e^i)$ *for all* i. *The existence of the Pareto optimal equilibrium follows from Theorem 2. This implies* α*-efficiency for some* $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ *and by Lemma 1, each* $\bar{c}^i$ *is bounded away from zero.*
The linear functional Π *is* L^2*-continuous. Since* $\mathbb{P} \otimes dt$ *is a finite measure, we conclude that there is a* $\pi \in \mathrm{L}^2$ *such that* $\Pi(\cdot) = \langle \cdot, \pi \rangle_{\mathrm{L}^2}$. *Each* $\bar{c}^i$ *is bounded away from zero. Therefore the set of feasible directions* $F(\bar{c}^i)$ *is norm-dense in* L^2. *The equilibrium allocation maximizes the utility of each agent:*

$$U^i(\bar{c}^i) = \max_{c \in \mathrm{L}^2_+ : \Pi(c^i - e^i) \leq 0} U^i(c^i)$$

Each $\bar{c}^i$ *is bounded away from zero and hence the Slater condition is satisfied with* $\Pi(\frac{\bar{c}^i}{2} - e^i) = g(\frac{\bar{c}^i}{2}) < 0$. *By the Kuhn-Tucker Theorem,*[10] *for concave functionals, it is necessary and sufficient for the optimality of* $\bar{c}^i$ *that there is a* $\mu_i \geq 0$ *such that* $0 \in \partial - U^i(\bar{c}^i) + \mu_i \nabla \Pi(\bar{c}^i)$, $\mu_i g(\bar{c}^i) = 0$, *on the set of feasible directions* $F(\bar{c}^i)$.
U^i *is strictly monotone, consequently* $g(\bar{c}^i) = 0. \mu_i = 0$ *would then imply* $0 \in \partial U^i(\bar{c}^i)$ *and this contradicts the strict monotonicity of* U^i. *This proves the strict positivity of* μ_i *and we have, for all* $h \in F(\bar{c}^i)$, $DU^i(\bar{c}^i)(h) = \mu_i \nabla \Pi(h)$, *for some* $DU^i(\bar{c}^i) \in \partial U^i(\bar{c}^i)$. *Each* $\bar{c}^i$ *is bounded away from zero. By Proposition 2 each supergradient has the stated form.* ■

4.4 Appendix

Proof of Proposition 1 *The first two assertions can be found in El Karoui, Peng, and Quenez (1997), the third is a modification of Proposition 1 in Duffie and Epstein (1992).* ■

The partial super-gradient of the aggregator with respect to the corresponding component x is denoted by $D_x f(t, \cdot, x, \cdot)$. The partial super-differential in u and z, namely $\partial_{z,z} f$ at (t, c, u, z), consists of all pointwise supergradients

[9] See Theorem 1 on page 117.
[10] We refer to Theorem 3.1.4 in Barbu and Precupanu (1986)

$(D_U f(t,c,u,z), D_Z f(t,c,u,z)) = (\mathfrak{u},\mathfrak{z}) \in \mathbb{R}\times\mathbb{R}^n$ such that $f(t,c+x,u+y_1,z+y_2) \leq f(t,c,u,z) + \partial_c f(t,c,u,z)x + ay_1 + by_2$. For $k = U, Z$, the stochastic process $(D_k f(t,c_t,U_t,Z_t))_{t\in[0,T]}$ is denoted by $D_k f$.

Proof of Proposition 2 *The density representation of the supergradient follows from Lemma 3, with $t = 0$. Following the proof of Theorem 4.3 in Aliprantis (1997) and applying the concave alternative of Fan, Glicksberg, and Hoffman (1957), we can show that the right- and left-hand derivatives represent the superdifferential in terms of the order interval:*

$$\partial U(c) = \{g \in (\mathrm{L}^2)^* : \nabla^+ U(c)(\omega)_t \leq g_t(\omega) \leq \nabla^- U(c)(\omega)_t \ \mathbb{P}\otimes dt\text{-}a.e.\}$$

An application of results on Backward-SDE's depending on parameters (see Proposition 2.4 El Karoui, Peng, and Quenez (1997)), proofs that

$$\lim_{\alpha\searrow 0}\frac{U(c)-U(c-\alpha h)}{\alpha} = \langle\nabla^+ U(c), h\rangle = \langle\mathcal{E}^{D_U^+ f, D_Z^+ f}\cdot\partial_c f, h\rangle.$$

In this case, the closed formula of the adjoint process is given by $\mathcal{E}^{D_U^+ f, D_Z^+ f}$. The superdifferential can be written as a specific order interval in L^2, i.e. $\partial U(c) = [\mathcal{E}^{D_U^+ f, D_Z^+ f}\cdot\partial_c f, \mathcal{E}^{D_U^- f, D_Z^- f}\cdot\partial_c f]$ and the assertion follows. ■

Lemma 3 *Fix $t \in [0,T]$ and suppose the conditions of Proposition 2 hold, then for any direction $h \in \mathrm{L}^2$ such that $c + h \in \mathrm{L}^2_{++}$ we have*

$$U_t(c+h) - U_t(c) \leq \mathbb{E}\left[\int_t^T \frac{\mathcal{E}_s}{\mathcal{E}_t}\partial_c f(s,c_s,U_s,Z_s)h_s ds|\mathcal{F}_t\right].$$

Proof of Lemma 3 *Take a c and h as stated. The related utility processes U and U^h are given by*

$$dU_t = -f(t,c_t,U_t,Z_t)dt + Z_t dB_t \text{ and } dU_t^h = -f(t,c_t+h_t,U_t^h,Z_t^h)dt + Z_t^h dB_t,$$

with terminal conditions $U_T = 0 = U_T^h$. We define $\mathcal{E}_t := \mathcal{E}_t^{D_U f, D_Z f}$ and prove the following

claim: *We have $\mathbb{E}[\sup_{t\in[0,T]}\mathcal{E}_t^2] < \infty$.*
proof: *The process $\mathcal{E}$ admits a decomposition $\mathcal{E}_t = \lambda_t\cdot\Gamma_t$ and hence by the boundedness of the super-gradient w.r.t. the aggregator in utility*

$$\lambda_t = \exp(\int_0^t D_U f(s,c_s,U_s,Z_s)ds) \leq \exp(kt). \tag{5}$$

Boundedness of the super-gradient w.r.t. aggregator in the intensity component z implies

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |D_Z f(s,c_s,U_s,Z_s)|^2 ds\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t k^2 ds\right)\right] < \infty,$$

the Novikov criterion is satisfied, hence the process Γ, *given by*

$$\Gamma_t = \exp\left(-\frac{1}{2}\int_0^t |D_Z f(s,c_s,U_s,Z_s)|^2 ds + \int_0^t D_Z f(s,c_s,U_s,Z_s)' dB_s\right), \quad (6)$$

is indeed a martingale. With regard to the local martingale $\int_0^\cdot \Gamma_s d\Gamma_s$, *we take a localizing sequence of stopping times* $(\tau_n)_{n\in\mathbb{N}} \subseteq [0,T]$ *such that* $\tau_n \overset{n\to\infty}{\longrightarrow} T$ $\mathbb{P}$*-a.s., and we see that for each* n, $\left(\int_0^{t\wedge\tau_n} \Gamma_s d\Gamma_s\right)_{t\in[0,T]}$ *is a martingale.*
By Itô's formula, the quadratic variation of Γ, *the boundedness of the super-gradient in the intensity component* z *and Fubini's theorem, we get*

$$\begin{aligned}\mathbb{E}[\Gamma^2_{t\wedge\tau_n}] &= \mathbb{E}\left[1 + 2\int_0^{t\wedge\tau_n} \Gamma_s d\Gamma_s + \frac{1}{2}\int_0^{t\wedge\tau_n} 2d\langle\Gamma\rangle_s\right] \\ &= \mathbb{E}\left[1 + \int_0^{t\wedge\tau_n} \Gamma_s^2 D_Z f(s,c_s,U_s,Z_s)^2 ds\right] \leq 1 + \int_0^t \mathbb{E}[\Gamma^2_{s\wedge\tau_n}] k^2 ds.\end{aligned}$$

Applying the Gronwall lemma with $g(s) = \mathbb{E}[\Gamma^2_{s\wedge\tau_n}]$, *we conclude that* $g(T) \leq \exp(Tk^2) < \infty$ *and by the dominated convergence,* $\mathbb{E}[\Gamma_T^2] \leq \exp(Tk^2)$. *Since* Γ *is a martingale,* Γ^2 *is a submartingale. By virtue of Doob's maximal inequality, (6) and (7), we deduce*

$$\mathbb{E}\left[\sup_t \mathcal{E}_t^2\right] \leq \mathbb{E}\left[\sup_t \lambda_t^2 \sup_t \Gamma_t^2\right] \leq e^{2kT} 4\mathbb{E}\left[\Gamma_t^2\right] < \infty. \qquad \square$$

To see that $\mathcal{E}\partial_c f \in \mathrm{L}^2$, *we argue that there is a constant* $C > 0$ *with* $c > C$ $\mathbb{P}\otimes dt$*-a.e. and, since* f *is a regular aggregator, the process* $t \mapsto \partial_c f(t, c_t, U_t, Z_t)$ *takes values in* $[0, K]$ $\mathbb{P}\otimes dt$*-a.e., where* $K = \sup_{(t,u,z)} \partial_c f(t, C, u, z)$. *Since* c *is bounded away from zero, we have* $\partial_c f \in \mathrm{L}^\infty(\mathbb{P}\otimes dt)$ *and* $\mathcal{E}\partial_c f \in \mathrm{L}^2$ *follows by the previous claim.*
The remaining part follows from Lemma A.5 in Schroder and Skiadas (2003).■

We begin with the first order conditions of optimality for concave and not necessarily Gateaux differentiable functionals. Define the set of *feasible directions* at c^i given by $F(c^i) = \{h \in \mathrm{L}^2 : \exists \mu > 0 \quad c^i + \mu h \in \mathrm{L}^2_+\}$ and the set of *feasible transfers* $H(c) = \{h \in \mathrm{L}^{2,m} : \sum h^i = 0, h^i \in F(c^i), 1 \leq i \leq m\}$. By $\partial_{\mathrm{L}^{2,m}} U$ we denote the super-differential of a functional U on $\mathrm{L}^{2,m}$. We write $\langle DU(c), h\rangle$ for $DU(c)(h)$, where $DU(c) \in \partial U(c)$ is a super-gradient.

Proof of Proposition 3 *The properties of the aggregator imply the norm continuity and concavity of the utility functionals. Alaouglu's theorem implies the weak compactness of* $\Lambda(e)$. *Under concavity and upper semicontinuity, weak upper semicontinuity of the utility functionals follows.* α*-efficient allocation exists by an abstract Weierstrass argument. The equivalence between* α*-efficiency and Pareto optimality is standard in economic theory.*
1. Let $(h^1, \ldots, h^m) = h \in H(\hat{c})$. *By assuming there is a* $DU \in \bigcap_{i=1}^m \partial\alpha_i U^i(\hat{c}^i)$,

with Riesz representation π. *This means for each* i, *there is a* $D\alpha_i U^i(\hat{c}^i) \in \partial\alpha_i U^i(\hat{c}^i)$ *such that* $D\alpha_i U^i(\hat{c}^i) = \langle \pi, \cdot \rangle$ *and therefore*

$$\sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle = \sum \langle \pi, h^i \rangle = \langle \pi, \sum h^i \rangle = \langle \pi, 0 \rangle = 0.$$

Since each U^i *satisfies the conditions of Proposition 4,* $(\hat{c}_1, \dots, \hat{c}_m)$ *is an* α*-efficient allocation.*
2. For each i, *the consumption process* $\hat{c}^i$ *is bounded away from zero. This implies* $\mathrm{L}^\infty(\mathbb{P} \otimes dt) \subseteq F(\hat{c}^i)$. *Suppose the converse, there are two agents* i *and* j *such that* $\partial\alpha_i U^i(\hat{c}^i) \cap \partial\alpha_j U^j(\hat{c}^j) = \emptyset$. *Then there is an* $h_i \in F(\hat{c}^i) \setminus \{0\}$, *an* $h_j \in F(\hat{c}^j) \setminus \{0\}$ *and an* $h \in H(\hat{c})$ *with* $h^k = 0$ *if* $k \notin \{i, j\}$ *such that, for all* $D\alpha_i U^i(c^i) \in \partial\alpha_i U^i(c^i)$ *and* $D\alpha_j U^j(\hat{c}^j) \in \partial\alpha_j U^j(\hat{c}^j)$, *we have*

$$\begin{aligned} 0 &< \mathbb{E}\left[\int_0^T h_t^i \pi^i(\hat{c}^i)_t - h_t^i \pi^j(\hat{c}^j)_t dt\right] = \mathbb{E}\left[\int_0^T h_t^i \pi^i(\hat{c}^i)_t + h_t^j \pi^j(\hat{c}^j)_t dt\right] \\ &= \sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle_{\mathrm{L}^2}, \end{aligned}$$

where $\pi^j(\hat{c}^j)$ *is the Riesz representation of* $D\alpha_j U^j(\hat{c}^j)$, *a contradiction to Proposition 4.* ■

For the proof of Proposition 3 we applied the following result.

Proposition 4 *Assume that for each* i, *the utility functional* U^i *is upper semicontinuous, strictly increasing, concave and let the aggregate endowment* e *be bounded away from zero.*
Then α*-efficiency of* $\hat{c} \in \Lambda(e)$ *is equivalent to the existence of a* $DU^i(\hat{c}^i) \in \partial U^i(\hat{c}^i)$, *for each* i, *such that* $0 \geq \sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle$, $h \in H(\hat{c})$.

Proof of Proposition 4 *Let* $g(c^1, \dots, c^m) = \sum c^i - e$ *and* $g_i(c^1, \dots, c^m) = -c^i$. *Then* α*-efficiency for* $\hat{c} = (\hat{c}^1, \dots, \hat{c}^m)$ *can be written as*

$$U^\alpha(\hat{c}) = \max_{c' \in \Lambda(e)} U^\alpha(c') = \min_{c' \in \mathrm{L}^{2,m} : g_i(c'), g(c') \leq 0} -U^\alpha(c').$$

Since e *is bounded away from zero, the Slater condition holds. We apply the Kuhn-Tucker theorem (see Theorem 3.1.4 in Barbu and Precupanu (1986)), to* $-U^\alpha$. *Hence,* $\hat{c}$ *is* α*-efficient if and only if there are constants* $\mu_i, \mu \geq 0$ *such that* $0 \in (\partial_{\mathrm{L}^{2,m}} - U^\alpha)(\hat{c}) + \mu \nabla_{\mathrm{L}^{2,m}} g(\hat{c}) + \sum \mu_i \nabla_{\mathrm{L}^{2,m}} g_i(\hat{c})$ *and* $\mu g(\hat{c}) = 0$, $\mu_i g_i(\hat{c}) = 0$, $i = 1, \dots, m$. *Taking the non-negativity constraints into account and the existence of a* $-D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}) \in (\partial_{\mathrm{L}^{2,m}} - U^\alpha)(\hat{c})$, *this is equivalent to*

$$0 \leq -D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}) + \mu \nabla_{\mathrm{L}^{2,m}} g(\hat{c}) \text{ and } \quad \mu g(\hat{c}) = 0.$$

Taking the feasible transfers $h \in H(\hat{c})$ *into account, we have*

$$\begin{aligned} 0 &\leq \langle -D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}), h \rangle_{\mathrm{L}^{2,m}} + \langle \mu \cdot \nabla_{\mathrm{L}^{2,m}} g(\hat{c}), h \rangle_{\mathrm{L}^{2,m}} \\ &= -\sum \alpha_i DU^i(\hat{c}^i) h_i + \mu \sum h_i. \end{aligned}$$

Since the U^i*'s are strictly increasing,* $g(\hat{c}) = 0$ *follows.* ■

The following two results are used in Lemma 1 and in the proof of the F-properness in Section 3.3. The approach goes back to Duffie and Zame (1989). The aggregator is not differentiable in u and z (but concave) and hence we need a mean value theorem for convex functions, see Wegge (1974). Lemma 4 and Lemma 5 are formulated so that an application to the contradiction argument in Lemma 1 fits the agent j.

Lemma 4 *Assume that U is a generalized stochastic differential utility generated by an aggregator f that satisfies Assumption 1. Let $A \in \mathcal{O}$ and $a > 0$ be arbitrary. If $y, x \in \mathrm{L}^2_+$ with $y \geq a$ on A, $x = 0$ on A^c and $x \leq \frac{a}{2}$, then*

$$U(y) - U(y-x) \leq e^{kT}\mathbb{E}\left[\int_0^T \overline{\delta}_f(\frac{a}{2})x_t + k|Z_s - \bar{Z}_s|dt\right].$$

Proof of Lemma 4 *Let $(U_t, Z_t)_{t\in[0,T]} = (U, Z)$ be the solution of the utility process related to y and $(\bar{U}, \bar{Z})$ the solution of the utility process related to $y-x$ where x is chosen as above. By assumption, f is differentiable in c. We apply the classical mean value theorem to the consumption component. Since f is uniformly Lipschitz continuous in u and z, upper semicontinuity follows, we apply the mean value theorem for convex functions of Wegge (1974) to $-f(t, c, \cdot, \cdot)$. Hence, there is an $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ valued process (ξ^c, ξ^U, ξ^Z) such that*

$$\begin{aligned} U_t - \bar{U}_t &= \mathbb{E}\left[\int_t^T f(s, y_s, U_s, Z_s) - f(s, y_s - x_s, \bar{U}_s, \bar{Z}_s)ds|\mathcal{F}_t\right] \\ &= \mathbb{E}\Big[\int_t^T \partial_c f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s)x_s \\ &\quad + D_U f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s)(U_s - \bar{U}_s) \\ &\quad + \langle D_Z f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s), (Z_s - \bar{Z}_s)\rangle ds|\mathcal{F}_t\Big]. \end{aligned}$$

Observe $U_t - \bar{U}_t \geq 0$, for all $t \in [0, T]$, by Proposition 1 since $x \geq 0$ and f is increasing in consumption. Combined with the boundedness of the supergradients, we derive:

$$\begin{aligned} U_t - \bar{U}_t &\leq \mathbb{E}[\int_t^T \partial_c f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s)x_s + k(U_s - \bar{U}_s) \\ &\quad + \langle D_Z f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s), (Z_s - \bar{Z}_s)\rangle ds|\mathcal{F}_t] \\ &\leq \mathbb{E}[\int_t^T \overline{\delta}_f(\frac{a}{2})x_s + k(U_s - \bar{U}_s) + k|Z_s - \bar{Z}_s|ds|\mathcal{F}_t] \end{aligned}$$

The last inequality holds because $x \mapsto \partial_c f(s, x, v, z)$ is decreasing and using the estimate $\overline{\delta}_f(\frac{a}{2})$, since $y_s(\omega) + \xi^c_s(\omega) \geq \frac{a}{2}$ on A. Finally, the first Stochastic Gronwall inequality (see Corollary B in the Appendix of Duffie and Epstein (1992)), evaluated at time zero yields

$$U(y) - U(y-x) = U_0 - \bar{U}_0 \leq e^{kT}\mathbb{E}\left[\int_0^T \overline{\delta}_f(\frac{a}{2})x_s + k|Z_s - \bar{Z}_s|ds\right]. \quad \blacksquare$$

Lemma 5 *Assume that U is a generalized stochastic differential utility generated by an aggregator f that satisfies Assumption 1. Let $H \in \mathcal{O}$, $h > 0$ and $y \in \mathrm{L}^2_+$ with $y \leq h$ on H. Then for every $\lambda \in [0, h]$*

$$U(y + \lambda 1_H) - U(y) \geq e^{-kT}\mathbb{E}\left[\int_0^T \underline{\delta}_f(2h)\lambda 1_H(t) - k|\bar{Z}_s - Z_s|dt\right].$$

Proof of Lemma 5 *Let $(U_t, Z_t)_{t\in[0,T]} = (U, Z)$ be the solution of the utility process of the process y and $(\bar{U}, \bar{Z})$ the solution of the utility process of $y + \lambda 1_H$. f is differentiable in consumption and concave in the other components. Applying the mean value theorem for c, there is a $\mathbb{R}^{2+n}$ valued process (ξ^c, ξ^V, ξ^Z) and we have*

$$\bar{U}_t - U_t \geq \mathbb{E}\left[\int_t^T \underline{\delta}_f(2h)\lambda 1_H(t) - k(\bar{U}_s - U_s) - k|\bar{Z}_s - Z_s|ds|\mathcal{F}_t\right].$$

The inequality follows from the application of the estimates $\underline{\delta}_f(2h)$ (since $y_s(\omega) + \xi^c_s(\omega) \leq 2h$ on H) and arguments similar to Lemma 4. We have $U_s - \bar{U}_s \geq 0$ since $\lambda 1_H \geq 0$ and f is increasing. Finally, the second Stochastic Gronwall inequality (see again Corollary B in the Appendix of Duffie and Epstein (1992)), evaluated at time zero gives us

$$U(y) - U(y - x) = U_0 - \bar{U}_0 \geq e^{-kT}\mathbb{E}\left[\int_0^T \underline{\delta}_f(2h)\lambda 1_H(t) - k|\bar{Z}_s - Z_s|ds\right]. \quad \blacksquare$$

Quasi-Equilibrium in Normed Lattices

Let (L, τ) be the commodity space, a vector lattice with a Hausdorff, locally convex topology τ. We fix a pure exchange economy with $m \in \mathbb{N}$ agents $\mathtt{E} = \{L_+, P_i, e^i\}_{1\leq i\leq m}$ in L such that $P_i : L_+ \to 2^{L_+}$ are the preference relations on the consumption set L_+ and $e^i \in L_+$ is the initial endowment of each agent.

An allocation $(x^1, \ldots, x^m)$ is individually rational if $e^i \notin P_i(x^i)$ for every i.

A *quasi-equilibrium* for $\mathtt{E}$ consists of a feasible allocation $(x^1, \ldots, x^m) \in L^m_+$, i.e. $\sum x^i = e$, and a linear functional $\pi : L \to \mathbb{R}$ with $\pi \neq 0$ such that, for all i $\pi(x^i) \leq \pi(e^i)$ and for any i, $y \in K_+$ with $y \in P(x^i)$ implies $\pi(y) \geq \pi(x^i)$. The quasi-equilibrium is an *equilibrium* if $y \in P(x^i)$ implies $\pi(y) > \pi(x^i)$.

Forward properness is a modification of a cone condition (see Yannelis and Zame (1986)).

Definition 1 *A preference relation $P : L_+ \to 2^{L_+}$ is* F-proper at $x \in L_+$ *if: There is a $v \in L_+$, some constant $\rho > 0$ and a τ-neighborhood U satisfying, with $\lambda \in (0, \rho)$:*

If $z \in U$, then $x + \lambda v - z \in L_+$ implies $x + \lambda v - \lambda z \in P(x)$

The following standard assumptions are needed to establish the existence of a quasi-equilibrium.

Assumption 2 *The economy satisfies the following conditions:*

1. $y \notin P_i(y)$ *and* $P_i(y)$ *is for all* $y \in L_+$ *and every* $i \in \{1, \ldots, m\}$
2. *There is a Hausdorff topology* η *on* L *such that* $[0, e]$ *is* η*-compact and for every* i *the graph* $gr(P_i) = \{(x, y) \in L \times L : x \in L_+, y \in P_i(x)\}$ *is a relatively open subset of* $L_+ \times L_+$ *in the product topology* η-τ.
3. $P_i(y) \cap L(e) \neq \emptyset$ *for all* $y \in [0, e]$ *and every* i.
4. $L(e)$[11] *is* τ*-dense in* L *and if* $(x_1, \ldots, x_m) \in L_+^m$ *is an individually rational and Pareto-optimal allocation, then, for every* i, P_i *is F-proper at* x_i.

Theorem 2 *Suppose the economy* E *satisfies Assumption 2. Then there is an* $x \in L_+^m$ *and a* $p \in L^*$ *such that* (x, p) *is a non-trivial quasi-equilibrium.*

This result is proved in Podczeck (1996). If preferences are strictly monotone and continuous and the total endowment is strictly positive, the notions of equilibrium and quasi-equilibrium coincide, see Corollary 8.37 in Aliprantis and Burkinshaw (2003), where it is requested that L^* is a sublattice of the order dual $L^\star$.

[11] $L(e)$ denotes the order ideal L. Details can be found in Aliprantis and Burkinshaw (2003).

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