On Quotient Cohomology of Substitution Tiling Spaces

Dissertation zur Erlangung des akademischen Grades eines
Doktors der Mathematik (Dr. Math.)

vorgelegt von

Enrico Paolo C. Bugarin

Fakultät für Mathematik
Universität Bielefeld

Mai 2014

1. Berichterstatter: Dr. Franz Gähler, Universität Bielefeld
2. Berichterstatter: Dr. Gregory Maloney, Newcastle University

Datum der mündlichen Prüfung: 30. Juni 2014
Prüfungsausschuss: Prof. Dr. Michael Baake, Dr. Franz Gähler, Prof. Dr. Sebastian Herr, Prof. Dr. Henning Krause & Dr. Gregory Maloney

Gedruckt auf alterungsbeständigem Papier °°ISO 9706.

Contents

Introduction

The concept of a *tiling* is common and natural even to non-mathematicians. From nature (e.g., honeycombs) to road bricks to fancy kitchen floors (e.g., with chessboard patterns) to beautiful artworks (e.g., Escher tilings [**Esch**]), different kinds of tilings (usually in two dimensions) are apparent. In an abstract setting, a tiling is understood to be unbounded and so a planar tiling extends in all directions, eventually covering the whole plane $\mathbb{R}^2$. Tilings attract attention, mainly due to their design aspects. But a closer inspection reveals a certain level of complexity that keeps researchers interested, particularly mathematicians. Normally, this complexity is associated with symmetry and repetitivity, and repetitivity is usually paired with periodicity. A tiling is periodic if a non-trivial shift or translation of itself produces an exact copy of the untranslated tiling. The *crystallographic restriction* relates the periodicity and symmetry as follows. A *periodic* (and so repetitive) tiling may only admit symmetries of order 1, 2, 3, 4 or 6. For instance, the honeycombs admit a 6-fold rotational symmetry, a chessboard pattern has 2-fold symmetries (e.g., a 180-degree rotation and a reflection), etc. On the other extreme, a tiling pattern may be amorphous and may seem completely disordered (e.g., a shattered glass). It would be impossible to find non-trivial symmetries on such patterns. We are interested on tilings that are somewhere in between, i.e., tilings that possess a certain level of symmetry and complexity, yet are not periodic. The only question is whether such tilings exist.

Indeed in 1974, Roger Penrose discovered the now called Penrose tiling, which is a tiling that is highly symmetrical, but is not periodic since it admits a 5-fold rotational symmetry. The Penrose tiling provides an example of a non-periodic tiling that exhibits repeating or recurring patterns. Figure A shows a patch of the (rhombic) Penrose tiling, where the 5-pointed grey star centre patch recurs regularly all throughout the tiling, in all rotated and reflected versions. Meanwhile, for a good number of decades, it was believed that *long range order* and periodicity were synonymous in the tiling world. Simply put, a tiling has long range order (i.e., the tiling has sharp bright spots called Bragg peaks in its diffraction pattern) if and only if the tiling is periodic. The (non-periodic) Penrose tiling contradicted this belief as it displayed sharp bright

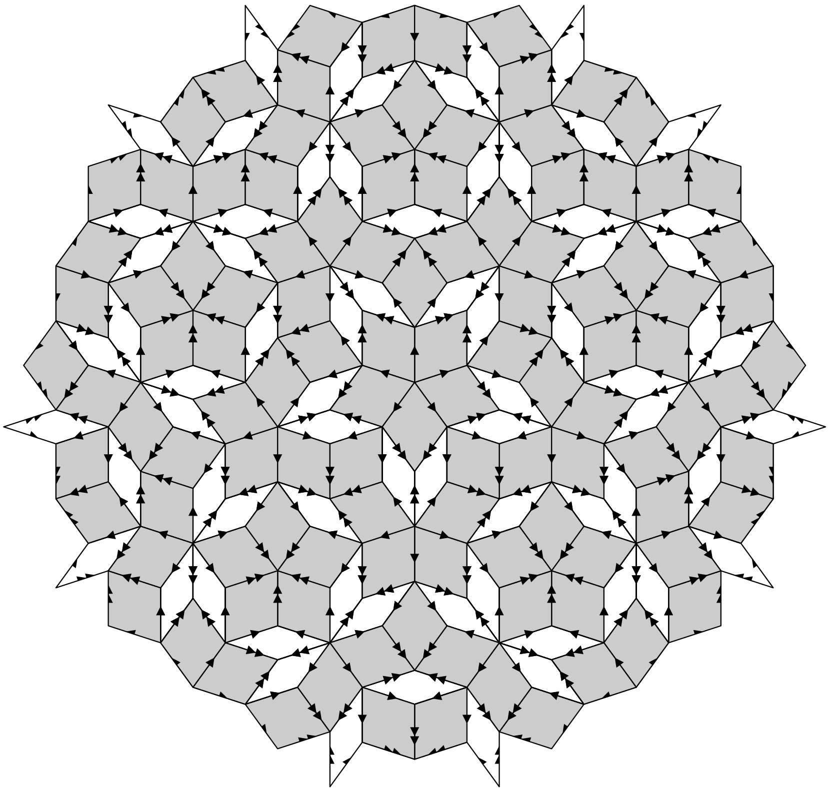


Figure A. A 5-fold symmetric patch of the rhombic Penrose, where five grey rhombi form a regular 5-pointed star patch that 'repeats' regularly all throughout the tiling. Image created by M. Baake & U. Grimm, found in [**BG13a**, Fig. 1.2]; image used with their kind permission.

spots in its diffraction; see [**Sen94**] for instance. This initiated a paradigm shift, and the focus now includes the study of such non-periodic tilings that possess long range order. This paradigm shift was not immediately accepted, and true enough in 1984, when Dan Shechtman announced his discovery of a real world (quasi)crystal whose diffraction pattern exhibited sharp Bragg peaks with 10-fold symmetry, it sparked a controversial discussion. Other (quasi)crystals were later found, which similarly possessed long range order yet admitting non-periodic symmetries. Soon enough, *quasicrystals* become widely accepted giving birth to what is now called *quasicrystallography*, analogous to that of crystallography. In 2011, Schechtman received a Nobel award for his discovery of quasicrystals.

Following these developments, particularly Penrose's lead, many non-periodic tilings have been discovered. There are three main methods in generating non-periodic tilings, namely, via *matching rules*, via the *cut-and-project* method, or via *substitution* [**Sa03**]. In this study, we focus on the third method, and consider tilings produced via substitution called *substitution tilings*. The website Tilings Encyclopedia (http://tilings.math.uni-bielefeld.de) currently provides a collection of some known examples of substitution tilings. In general, we are after the topology of such tilings, and in particular, we will use the topological invariant *cohomology* quite frequently. However, it does not immediately make sense to talk about the cohomology of a tiling, because $\mathbb{R}^d$, which the tiling covers (regardless of the complexity of the tiling), is contractible and so getting the cohomology this way becomes trivial. Instead, we work with a (substitution) *tiling space*, which contains all substitution tilings derived from a particular substitution rule. Note that a point in the tiling space is a tiling.

Classifying tiling spaces through their topology is a systematic yet tedious way of distinguishing tiling spaces. A family of tilings of $\mathbb{R}^d$ defined through a common substitution rule forms a (substitution) tiling space, which can be understood as an inverse limit of branched manifolds called approximants. As an immediate consequence, the Čech cohomology of a substitution tiling space can be computed as the direct limit of the cohomologies of the approximants [**AP98**]. As such, the (Čech) cohomology provides a criterion on classifying tiling spaces, and recent developments involved methods and techniques in computing for the tiling cohomology [**BD08, BDHS09, Sa11, GM13, GHM13**]. Tiling spaces with different cohomologies are necessarily inequivalent. However, the converse is not true in general as can be seen in the case of the Thue-Morse and period doubling sequences. These two substitution sequences form two inequivalent *hulls* (or tiling spaces) that have isomorphic cohomology groups. Hence, the need for additional tools (from the perspective of algebraic topology) is needed in order to classify and distinguish such tiling spaces.

In 2011, Marcy Barge and Lorenzo Sadun introduced the notion of *quotient cohomology*, which is a relative version of the tiling cohomology, aimed at characterising the difference between related substitution tiling spaces [**BS11**]. Further, it can be used to analyse the structure of a family of substitution tiling spaces, related via factor maps. In their paper, the quotient cohomology has been computed for the family of 2-dimensional (generalised) chair tilings, albeit with a few minor errors, which we correct in this text; compare [**BG14**].

As the concept of quotient cohomology is fairly new, much is still unknown and not well understood. It is our objective in this study to provide additional tools to aid in the computation of the quotient cohomology between substitution tiling spaces. More so, we demonstrate how the quotient cohomology can be used to analyse the general structure of a substitution tiling space by considering its factors and then classifying the factor maps through a notion we introduce, called *good matches*. Further, we introduce the concept of the *quotient zeta function*, which is analogous to the typical zeta functions associated with substitution tiling spaces, aimed to have a better grasp of the concept of quotient cohomology.

Outline of the thesis

We divide the writing of this thesis in two parts. The first part, which comprises Chapters 1 to 3, mainly gives an exposition on the background of our study. We begin Chapter 1 by giving a review on some basic notions from algebra and algebraic topology that are essential in the general study of tiling cohomology. In particular, we start with an exposition on direct limits followed by a discussion on CW complexes and their cohomology. We discuss inverse limit spaces and their cohomology, which can be computed as the direct limit of the cohomologies of their approximants. We also give a short review on the Perron-Frobenius theory and the basics of symbolic substitution.

Chapter 2 provides an exposition on substitution tiling spaces and their (Čech) cohomology, based on the main result of Jared Anderson and Ian Putnam in [**AP98**]. As substitution tiling spaces are inverse limit spaces, their cohomology is computed as the direct limit of their approximants now called AP-complexes. Immediately, we get $\check{H}^0(\Omega) \cong \mathbb{Z}$ for any substitution tiling space Ω (see Lemma 2.17). However, this should not be surprising by noting that the tiling space Ω is connected, and the 0th Čech cohomology encodes the number of connected components. We end the chapter on the discussion of the (Artin-Mazur) dynamical zeta functions of substitution systems.

It is our intention to make Chapters 2 and 3 a light but sufficient introduction to the theory of tiling cohomology needed in our study. Two references provide a very good start, namely the book on tiling spaces by Sadun [**Sa08**] and the paper on topological invariants by Anderson and Putnam [**AP98**].

Next in Chapter 3, we recall the definition of the quotient cohomology as introduced in [**BS11**] and provide additional tools in computing the quotient cohomology. As a consequence, determining the quotient cohomology between n-dimensional substitution tiling spaces for $n \leq 2$ becomes more straightforward. For any substitution tiling spaces Ω_X and Ω_Y related via a factor map $f : \Omega_X \longrightarrow \Omega_Y$, we get $H^0_Q(\Omega_X, \Omega_Y) \cong 0$ (see Theorem 3.7). In other words, the 0th quotient cohomology is always trivial between any two substitution tiling spaces related via a factor map.

In the same chapter, we also introduce the notion of the quotient zeta function (between two substitution tiling spaces), which we define as an analogous extension of the dynamical zeta function in the spirit of Proposition 2.21, complementing the quotient cohomology. We show that our definition of the quotient zeta function yields a very nice and convenient result, i.e., the quotient zeta function between two substitution tiling spaces is nothing but the quotient of the respective zeta functions of the tiling spaces. This is given in Theorem 3.18.

The second part of this thesis is dedicated to particular examples of substitution tiling spaces, both in one and two dimensions. In Chapter 4, we consider three families of 1-dimensional substitution tilings and determine their quotient cohomologies. In particular, we consider the twisted Fibonacci substitution, the universal morphism, and the generalised Thue-Morse and period doubling substitutions. We also give a classification of these tiling spaces by considering their factors through the notion of good matches. In Theorem 4.2, we discuss our computation of the quotient cohomology between the generalised Thue-Morse and period doubling sequences. This result is included in our paper [**BG14**], which is to appear in *Acta Physica Polonica A*.

Finally, Chapter 5 is dedicated to the analysis of some substitution tiling spaces in two dimensions, namely the squiral tilings, the Chacon tilings, and the generalised chair tilings. For all three families of 2-dimensional substitution tiling spaces, we give a thorough characterisation of the spaces via their factors, whose factor maps are classified through good matches. Using these tiling spaces, we highlight the direct connection between the quotient zeta function and the quotient cohomology through their respective *degeneracies*, under certain conditions. Particularly for the Chacon tilings, we exhibit a suitable 1-dimensional counterpart to the (2-dimensional) Chacon substitution in order to further appreciate how the degeneracies of the 2-dimensional Chacon tiling space arise. Meanwhile for the generalised chair tilings, we provide a recalculation of the quotient cohomologies to correct the minor errors given in [**BS11**]. These corrections also appear in [**BG14**].

Acknowledgements

I am truly grateful to the German Research Foundation (DFG) via the Collaborative Research Centre (SFB) 701: *"Spectral Structures and Topological Methods in Mathematics"*, for financial support of my doctoral studies. It is also a pleasure to thank a number of people to whom I am overwhelmingly indebted to.

To my supervisor, Prof. Dr. Michael Baake, for giving me the opportunity to join and work in his research group, and for serving as a role model of a mathematician I can only wish to become.

To my thesis adviser and reviewer, Dr. Franz Gähler, for sharing his knowledge and expertise, and for thoroughly reading this thesis which led to many insightful discussions.

To my second reviewer, Dr. Gregory Maloney, for meaningful afternoon discussions and for helping me improve this thesis; and to the other members of my doctoral committee, Prof. Dr. Michael Baake, Prof. Dr. Sebastian Herr and Prof. Dr. Henning Krause, for reading my work despite their busy schedules.

To my colleagues and friends in our research group both former and current (in alphabetical order), Michael Baake, Thomas Blomenkamp, Magnus Dümke, Dirk Frettlöh, Franz Gähler, Svenja Glied, Christian Huck, Tobias Jakobi, Manuel Joseph Loquias, Kai Matzutt, Markus Moll, Natascha Neumärker, Johan Nilsson, Eden Delight Provido, Timo Spindeler, Venta Terauds and Peter Zeiner, for making my stay in Bielefeld extra exciting.

To our former research group secretary, Frau B. Heidrich, and CRC (former and current) secretaries, Frau N. Epp and Herr S. Merkes, for assisting me with administrative matters.

To my family, especially my parents Cora and Rico, for showing love and support that never faltered.

To my dear Sofia, for being gorgeous and loving, and for letting me pursue my dreams even if it meant that we had to be separated for a number of years. Thank you for agreeing to marry me. I love you beyond the stars.

And of course, most especially to God Almighty, for without whom I am nothing.

Maraming salamat (vielen Dank)!

CHAPTER 1

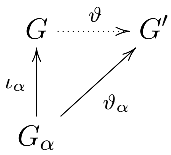
Preliminaries

In this chapter, we summarise some basic notions that we will be using throughout the remainder of this text. We begin with a short exposition on direct limits, covering a family of concrete examples, which usually do not appear in other texts. The discussion on cohomology follows closely from [**Geo08, Hat02, Sa08**] and the discussion on Perron-Frobenius theory follows from [**BG13a**]. We end the chapter with the discussion on the basics of symbolic substitution [**BG13a**], which will be very handy in the following chapters. Most of the definitions and terminologies are taken from these sources. We assume working knowledge on group theory and algebraic topology and a bit on dynamical systems.

1.1. Direct limits

Direct limits can be defined through a collection of objects (of a category) paired with a family of morphisms acting on the objects. In our case, we only need to deal with (abelian) groups and homomorphisms acting between them. Suppose we are given a family of groups $\{G_\alpha\}_{\alpha\in I}$ with I being a directed partially ordered set (or *poset*), i.e., for any pair $\alpha,\beta\in I$, there exists a third element $\gamma\in I$ such that $\alpha<\gamma$ and $\beta<\gamma$. Further, for any pair $\alpha,\gamma\in I$ with $\alpha<\gamma$, there is a homomorphism $\iota_{\gamma\alpha}: G_\alpha \longrightarrow G_\gamma$ so that for any β between α and γ (i.e., $\alpha<\beta<\gamma$), we have $\iota_{\gamma\alpha}=\iota_{\gamma\beta}\circ\iota_{\beta\alpha}$, where $\circ$ denotes the composition. The collection $\{G_\alpha,\iota_{\gamma\alpha},I\}$ (or simply $\{G_\alpha,\iota_{\gamma\alpha}\}$ when the context is clear) forms a *direct system*, also called an *inductive system*.

Let G be an abelian group, and $\{G_\alpha,\iota_{\gamma\alpha},I\}$ a direct system of abelian groups over a directed poset I and assume that there is a homomorphism $\iota_\alpha: G_\alpha\longrightarrow G$ for each $\alpha\in I$. The mappings ι_α are said to be *compatible* if $\iota_\gamma\circ\iota_{\gamma\alpha}=\iota_\alpha$ whenever $\alpha<\gamma$. The abelian group G together with compatible homomorphisms ι_α (for any $\alpha\in I$) is a *direct limit* of the direct system $\{G_\alpha,\iota_{\gamma\alpha},I\}$ if the following universal property is satisfied. For any abelian group G' with a set of compatible homomorphisms $\vartheta_\alpha: G_\alpha\longrightarrow G'$ for all $\alpha\in I$, there exists a unique homomorphism $\vartheta: G\longrightarrow G'$ such that $\vartheta\circ\iota_\alpha=\vartheta_\alpha$ for all $\alpha\in I$.



A direct system $\{G_\alpha, \iota_{\gamma\alpha}, I\}$ always has a direct limit, and this limit is unique up to isomorphism of groups, i.e., if (G, ι_α) and (G', ϑ_α) are two direct limits of a given direct system, then there is a unique isomorphism $\Psi : G \longrightarrow G'$ such that $\vartheta_\alpha = \Psi \circ \iota_\alpha$. The uniqueness easily follows from the universal property condition, whereas the existence follows from the following construction. For any $\alpha, \beta \in I$, we say that $x \in G_\alpha$ is equivalent to $y \in G_\beta$ if there exists a $\gamma \in I$ with $\alpha, \beta < \gamma$, such that $\iota_{\gamma\alpha}(x) = \iota_{\gamma\beta}(y)$. In such case, we write $x \sim y$. With this equivalence relation, we denote the equivalence class of $x \in G_\alpha$ by $\tilde{x}$. For any $x \in G_\alpha$ and $y \in G_\beta$ with $\alpha, \beta < \gamma$, the product $\tilde{x}\tilde{y}$ is defined as the equivalence class of $\iota_{\gamma\alpha}(x)\iota_{\gamma\beta}(y)$. Then we get the direct limit G of the direct system $\{G_\alpha, \iota_\alpha\}$, denoted by $G := \varinjlim(G_\alpha, \iota)$ or simply $G := \varinjlim G_\alpha$, as the disjoint union of all G_α modulo the equivalence $\sim$ defined above, where we let $\iota_\alpha : G_\alpha \longrightarrow G$ to be defined as $\iota_\alpha(x) = \tilde{x}$. Note that the limit G together with the compatible homomorphisms ι_α satisfy the universal property condition. Indeed if $\{\vartheta_\alpha : G_\alpha \longrightarrow G'\}$ is a collection of compatible maps onto another abelian group G' for all $\alpha \in I$, then we define the induced homomorphism $\vartheta : G \longrightarrow G'$ as follows. For $g \in G$ such that $\iota_\alpha(x) = g$ for some $x \in G_\alpha, \alpha \in I$, we let $\vartheta(g) = \vartheta_\alpha(x)$. Note that ϑ is a well-defined homomorphism and $\vartheta \circ \iota_\alpha = \vartheta_\alpha$ for all $\alpha \in I$. Thus, G is the direct limit of the direct system.

We now look at some examples, which demonstrate how the direct limit of a direct system of abelian groups over a poset can be computed. These are the types of examples we usually encounter in our study. For all examples, we take $I = \mathbb{N} := \{1, 2, 3, \ldots\}$ as the set of all natural numbers. As usual, we denote by $\mathbb{Z}$ the set of all integers, and $\mathbb{Q}$ the set of all rational numbers. The set of all real numbers is denoted by $\mathbb{R}$.

Example 1.1. For all $\alpha \in \mathbb{N}$, let $G_\alpha = G$ for some abelian group G. Further let each $\iota_{\gamma\alpha} : G_\alpha \longrightarrow G_\gamma$ be a zero map. The 0s, each in G_α, belong to one equivalence class, say $\tilde{0}$. Since the map ι_{21} sends the whole G_1 to $0 \in G_2$, then $G_1 \in \tilde{0}$. Similarly, $G_2 \in \tilde{0}$, and so on. Thus, direct limit is just the trivial group as everything gets identified to zero. Generally for any $\alpha < \gamma$, if $\iota_{\gamma\alpha} := \iota^{\gamma-\alpha}$ for some nilpotent map ι, then the direct limit is still trivial, as everything gets identified to zero eventually. $\diamondsuit$

Example 1.2. Let $G_\alpha = \mathbb{Z}^2$ for all $\alpha \in \mathbb{N}$, and $\iota_{\gamma\alpha} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be given by the matrix

$$A_{\gamma\alpha} := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{\gamma-\alpha}$$

for any pair $\alpha, \gamma \in \mathbb{N}$ such that $\alpha < \gamma$, i.e., $\iota_{\gamma\alpha}(x) := A_{\gamma\alpha}x$ for $x \in G_\alpha = \mathbb{Z}^2$. Notice that each map is an isomorphism (since $\det(A_{\gamma\alpha}) = (-1)^{\gamma-\alpha}$, where $\det A$ denotes the determinant of A), and so $\iota_{\gamma\alpha}(G_\alpha) = G_\gamma$. Thus, the direct limit is just $\mathbb{Z}^2$, as every element of $G_\alpha = \mathbb{Z}^2$ forms its own equivalence class in the direct limit and nothing more. In general, if $G_\alpha = G$ for all $\alpha \in I$ and each $\iota_{\gamma\alpha}$ is an isomorphism for any $\alpha, \gamma \in I$ with $\alpha < \gamma$, then $\varinjlim G_\alpha = G$. ♢

Remark 1.3. For a positive integer $d \in \mathbb{N}$, let $G_\alpha = \mathbb{Z}^d$ for all $\alpha \in \mathbb{N}$, and let $\iota_{\gamma\alpha} : G_\alpha \longrightarrow G_\gamma$ have the matrix representation $A_{\gamma\alpha}$ that is invertible over $\mathbb{Q}$ for any $\alpha, \gamma \in \mathbb{N}$ with $\alpha < \gamma$. Further, assume that G is the direct limit of the direct system $\{G_\alpha, A_{\gamma\alpha}\}$. The compatible maps $i_\gamma : G_\gamma \longrightarrow G$ are defined as follows. For $\gamma = 1$, ι_γ is the inclusion map, and for $\gamma > 1$, $\iota_\gamma(x) := \iota_\alpha \circ A_{\gamma\alpha}^{-1}$ for any $\alpha < \gamma$. Letting $\alpha = 1$ yields $\iota_\gamma(x) = \iota_1 \circ A_{21}^{-1} \circ A_{32}^{-1} \circ \cdots \circ A_{\gamma\gamma-1}^{-1}$. Thus, G is a subgroup of $\mathbb{Q}^d$, and in particular, if $\det A_{ii-1} = m \neq 0$ for all $i > 1$, then G is a subgroup of $\mathbb{Z}[\frac{1}{m}]^d$. ♢

Example 1.4. Suppose $G_\alpha = \mathbb{Z}$ for all $\alpha \in \mathbb{N}$. For each pair $\alpha, \gamma \in I$ with $\alpha < \gamma$, define $\iota_{\gamma\alpha} : G_\alpha \longrightarrow G_\gamma$ by $\iota_{\gamma\alpha}(x) = 2^{\gamma-\alpha}x$. Thus, we are taking the direct limit of

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \tag{1.1}$$

which is isomorphic to the dyadic rationals, i.e., $\varinjlim(\mathbb{Z}, \times 2) = \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$. To see this, first note that $\mathbb{Z} \cong \tilde{G}_1 \subset \varinjlim G_\alpha$, where $\tilde{G}_1$ is the union of equivalence classes that contains the whole of G_1, twice of G_1 in G_2 (i.e., the even integers in G_2), and in general $2^{k-1}G_1$ in G_k for $k > 0$. Next, the equivalence class $\tilde{G}_2$ contains the odd integers in G_2 (those left out after taking G_2 modulo $\sim$), twice the odd integers in G_3 (equivalently, those integers congruent to 2 mod 4), and so on. Taking the disjoint union of all $\tilde{G}_\alpha$ means the direct limit is isomorphic to the disjoint union of the whole of $G_1 = \mathbb{Z}$, half of G_2 (i.e., $\frac{1}{2}\mathbb{Z}$), fourth of G_3 (i.e., $\frac{1}{4}\mathbb{Z}$), and so on. More precisely, the direct limit of (1.1) is isomorphic to the direct limit of the embedding

$$\mathbb{Z} \hookrightarrow \frac{1}{2}\mathbb{Z} \hookrightarrow \frac{1}{2^2}\mathbb{Z} \hookrightarrow \frac{1}{2^3}\mathbb{Z} \hookrightarrow \frac{1}{2^4}\mathbb{Z} \hookrightarrow \cdots$$

which is just the union $\bigcup_{i=0}^{\infty} \frac{1}{2^i}\mathbb{Z}$. This union is precisely the definition of $\mathbb{Z}[\frac{1}{2}]$.

The direct limit generalises to the ring $\mathbb{Z}[\frac{1}{m}]$ (for any integer m) by replacing 2 with m in the definition above, i.e., $\iota_{\gamma\alpha}(x) = m^{\gamma-\alpha}x$. If $m = 0$, we just regard $\mathbb{Z}[\frac{1}{0}] := \mathbb{Z}[0] = 0$ by an abuse of notation, which is still consistent with Example 1.1. ♢

The previous example extends to higher dimensions as well, the simple case of which is when the homomorphism $\iota_{\gamma\alpha} : \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ ($d \geq 1$, for any $\alpha < \gamma$) has the matrix representation $A^{\gamma-\alpha}$, where A is a $d \times d$ diagonal matrix with integer entries. In this case, the direct limit is computed as $\bigoplus_{i=1}^{d} \mathbb{Z}[\frac{1}{m_i}]$. As before, we take $\mathbb{Z}[\frac{1}{0}] := 0$ by an abuse of notation.

Calculating the direct limit in certain cases may require some manipulation. For instance, for a non-singular matrix A (with integer entries) that may not be diagonalisable over $\mathbb{Q}$, but whose characteristic polynomial has all but the leading coefficient divisible by the determinant of A, the direct limit of the direct system $\{\mathbb{Z}^d, \iota\}$ with $\iota_{\gamma\alpha}(x) = A^{\gamma-\alpha}x$, can be computed as $\bigoplus_{i=1}^{d} \mathbb{Z}[\frac{1}{m}]$, where $0 \neq m = \det(A)$ is the determinant of A. We see this in the next example.

Example 1.5. Let $G_\alpha = \mathbb{Z}^2$ for any $\alpha \in \mathbb{N}$ and let $\iota_{\gamma\alpha}$ (with $\alpha < \gamma$) be given by $\iota_{\gamma\alpha}(x) := A_{\gamma\alpha}x$ for any $x \in \mathbb{Z}^2$, where

$$A_{\gamma\alpha} := A^{\gamma-\alpha} = \begin{pmatrix} 0 & -2 \\ 3 & 6 \end{pmatrix}^{\gamma-\alpha}.$$

Since A is non-singular and has determinant 6, it follows from Remark 1.3 that the direct limit of the system is a subgroup of $\mathbb{Z}[\frac{1}{6}]^2$. Meanwhile, the characteristic polynomial of A, given by $P(\lambda) := \det(A - \lambda I_2) = \lambda^2 - 6\lambda + 6$, implies that $A^2 - 6A + 6I_2 = 0$ or $\frac{1}{6}I_2 = A^{-2}(A - I_2)$, where I_2 denotes the 2×2 identity matrix. Thus, whenever x is in the limit, then so is some $y \sim \frac{1}{6}x$, and so the direct limit contains $\mathbb{Z}[\frac{1}{6}]^2$ as a subgroup, implying that the direct limit is really the whole of $\mathbb{Z}[\frac{1}{6}]^2 = \mathbb{Z}[\frac{1}{6}] \oplus \mathbb{Z}[\frac{1}{6}]$ [**GM13**, cf. Ex. 1.21]. More generally, for a $d \times d$ non-singular matrix A such that $\det(A) = m \neq 0$, if m divides $P(A) - (-1)^d A^d$, where $P(\lambda) := \det(A - \lambda I_d)$ is the characteristic polynomial of A and I_d is the $d \times d$ identity matrix, then we get $\frac{1}{m} I_d = A^{-d} Q(A)$, where $Q(\lambda) = \frac{1}{m} P(\lambda)$ has integer coefficients. By similar reasoning as above, the direct limit is computed as $\bigoplus_{i=1}^{d} \mathbb{Z}[\frac{1}{m}]$.

Loosening the restriction that $m | P(A) - (-1)^d A^d$ to $\pi(m) | P(A) - (-1)^d A^d$, where $\pi(m)$ denotes the product of the distinct prime factors of m, still yields the same direct limit since $\mathbb{Z}[\frac{1}{m}] = \mathbb{Z}[\frac{1}{\pi(m)}]$.

Particularly for 2×2 matrices, m divides $P(A) - A^2$ if and only if $\det A | \operatorname{tr} A$, where $\operatorname{tr} A$ denotes the trace of A. ♢

In some cases, the integer eigenvalue(s) may play a role in the calculation of the direct limit. This is precisely the case in the following examples.

Example 1.6. Suppose $G_\alpha = \mathbb{Z}^d$ for all $\alpha \in \mathbb{N}$ and for some arbitrary but fixed d. Also, suppose that the homomorphism $\iota_{\gamma\alpha}$ is given by $\iota_{\gamma\alpha}(x) = A^{\gamma-\alpha}x$ $(\alpha < \gamma)$ for any $x \in \mathbb{Z}^d$, for some non-singular matrix A with integer entries. Now consider the following commutative diagram of two direct systems.

$$\begin{array}{ccccccccc}
\mathbb{Z}^d & \xrightarrow{A} & \mathbb{Z}^d & \xrightarrow{A} & \mathbb{Z}^d & \xrightarrow{A} & \mathbb{Z}^d & \xrightarrow{A} & \cdots \\
{\scriptstyle A}\uparrow & & {\scriptstyle A^2}\uparrow & & {\scriptstyle A^3}\uparrow & & {\scriptstyle A^4}\uparrow & & \\
A^{-1}\mathbb{Z}^d & \hookrightarrow & A^{-2}\mathbb{Z}^d & \hookrightarrow & A^{-3}\mathbb{Z}^d & \hookrightarrow & A^{-4}\mathbb{Z}^d & \hookrightarrow & \cdots
\end{array}$$

The two direct systems have isomorphic direct limits, and the direct limit of the bottom direct system is easily $\bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^d$.

Now, suppose that the matrix A is diagonalisable (over $\mathbb{Q}$) with integer eigenvalues, and let B be a matrix with integer entries whose columns are the respective eigenvectors of the eigenvalues of A. Then $\varinjlim(G_\alpha, \iota) \cong \bigcup_{n\in\mathbb{N}} BMB^{-1}\mathbb{Z}^d$, where $M := (m_i)_{1\le i\le d}$ is a diagonal matrix and $m_i = \lambda_i^{-n}$ such that $\lambda_i \in \mathbb{Z}$ is the ith eigenvalue of A. In certain cases, we get $\varinjlim(G_\alpha, \iota) \cong \bigoplus_{i=1}^d \mathbb{Z}[\frac{1}{\lambda_i}]$, as in the case when $d = 2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

Indeed,

$$\varinjlim(G_\alpha, \iota) = \bigcup_n \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} (-1)^{-n} & 0 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \mathbb{Z}^2.$$

It can be shown that this direct limit is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$, cf. [**BKS12**]. ♢

Example 1.7. For all $\alpha \in \mathbb{N}$, suppose $G_\alpha = \mathbb{Z}^4$ and each homomorphism $\iota_{\gamma\alpha} : \mathbb{Z}^4 \longrightarrow \mathbb{Z}^4$ has the matrix representation

$$A_{\gamma\alpha} := A^{\gamma-\alpha} = \begin{pmatrix} -2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -4 & 1 & -1 & 3 \\ -5 & 3 & 0 & 3 \end{pmatrix}^{\gamma-\alpha}$$

given that $\alpha < \gamma$. The matrix A is diagonalisable (over $\mathbb{C}$) and has the following eigenvalues: -1, 2, $-\sqrt{2}$ and $\sqrt{2}$. Also, note that $(0, 0, 1, 0)^T$ and $(1, 1, 1, 2)^T$ span the eigenspaces of -1 and 2, respectively, where v^T denotes the transpose of the vector v. To get the direct limit of the direct system

$$\mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{A} \cdots \tag{1.2}$$

we instead consider a re-coordinatisation of $\mathbb{Z}^4$, and study the induced action of $A_{\gamma\alpha}$ on it. Note that the elements of $G_\alpha = \mathbb{Z}^4$ are understood to be expressed using the standard basis vectors e_1, e_2, e_3 and e_4. Now, we consider a new set of basis vectors, given by $f_1 := e_1, f_2 := e_1 + e_2 + e_3 + 2e_4, f_3 := e_3$ and $f_4 := e_4$. The $\mathbb{Z}$-span of this new set of basis vectors is $\mathbb{Z}^4$, isomorphic to each G_α. The induced action of $A_{\gamma\alpha}$ on $\mathbb{Z}^4 = \langle f_1, f_2, f_3, f_4\rangle_\mathbb{Z}$ is deduced as follows. The action on f_3 is multiplication by $(-1)^{\gamma-\alpha}$, and the action on f_2 is multiplication by $2^{\gamma-\alpha}$. The action on the remaining $\mathbb{Z}^2 = \mathbb{Z}^4 \mod \langle f_2, f_3\rangle_\mathbb{Z}$ is computed to have the matrix representation

$$A'_{\gamma\alpha} := A'^{\gamma-\alpha} = \begin{pmatrix} -3 & 1 \\ -7 & 3 \end{pmatrix}^{\gamma-\alpha}.$$

Note that A' has eigenvalues $\pm\sqrt{2}$, and Example 1.5 suggests that $\varinjlim(\mathbb{Z}^2, A'_{\gamma\alpha}) = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}]^2$. Finally, the direct limit of (1.2) is isomorphic to the direct limit of the direct system

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{\times(-1,2,A')} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{\times(-1,2,A')} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{\times(-1,2,A')} \cdots$$

given by $\mathbb{Z}[-1] \oplus \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]^2 = \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]^3$, where the first two summands come from -1 and 2, which are the integer eigenvalues of A. ♢

Generally for a singular matrix, one can look at its eventual range in order to remove a subgroup of G_α that eventually goes to zero under the action of the matrix. The induced action on the remaining subgroup has a non-singular matrix representation and whose direct limit is isomorphic to the original direct system involving the singular matrix. One can then proceed as in the previous examples. (Compare the matrix A_1^* in Example 2.15 and the matrix $A_{TM,1}^*$ in Example 4.1, where $\varinjlim(\mathbb{Z}^3, A_1^*) = \varinjlim(\mathbb{Z}^2, A_{TM,1}^*) = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$. Note that A_1^* has eigenvalues -1, 0 and 2, and $A_{TM,1}^*$ is the matrix representation of the induced action of A_1^* on $\mathbb{Z}^2$ modulo its kernel, left with eigenvalues -1 and 2.)

Determining the direct limit is not always as straightforward as in the previous examples, especially if the direct limit cannot be expressed as a direct sum. We do not wish to explore such cases any more as they do not appear in our study anyway.

1.2. CW complexes

Introduced by J.H.C. Whitehead [**Wh49**] to address needs in homotopy theory, CW complexes soon became a topologist's toy because of their better categorical properties (than the simpler simplicial complexes) without really sacrificing computability. Let us begin with the basic building blocks of a CW complex called its cells.

An *n-cell* is a topological space homeomorphic to the open disk $\text{int}(D^n)$, the interior of the closed disk $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ where $\|\cdot\|$ is the

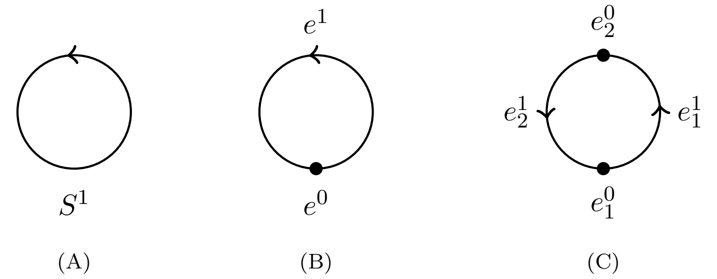


FIGURE 1.1. Two different cell decompositions of the (oriented) circle S^1.

standard norm on $\mathbb{R}^n$. (Equivalently, $\text{int}(D^n) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.) A cell e_α is n-dimensional (i.e., $\dim(e_\alpha) = n$) if and only if it is an n-cell. When necessary, we may also write e_α^n to emphasise the dimension of the cell. A *cell decomposition* of a topological space X is a collection $\mathcal{E} := \{e_\alpha \mid \alpha \in \mathcal{I}\}$, where each cell e_α is a subspace of X and that X is the disjoint union of all cells in $\mathcal{E}$, i.e., $X = \coprod_\alpha e_\alpha = \coprod_{n,\alpha} e_\alpha^n$. Note that the set $\mathcal{I}$ is not necessarily countable. The *n-skeleton* of X is $X^n := \coprod_{m \leq n,\alpha} e_\alpha^m$.

Remark 1.8. A cell decomposition of a space X is obviously not unique. For instance, $\{e^0, e^1\}$ is a cell decomposition of the circle $S^1 = \partial D^2$ (with ∂A denoting the boundary of A), where the 0-cell e^0 is any point on S^1 and the 1-cell is defined as $e^1 := S^1 - \{e^0\}$. Another possible cell decomposition of S^1 is given by $\{e_1^0, e_2^0, e_1^1, e_2^1\}$, where $e_1^0, e_2^0 \in S^1$, and the two arcs in $S^1 - \{e_1^0, e_2^0\}$ are the 1-cells e_1^1 and e_2^1, respectively. (See Figure 1.1.) Clearly, when $e_1^0 = e_2^0$, then we get the earlier cell decomposition of S^1. Further, there is no restriction on the number of cells in a cell decomposition $\mathcal{E}$, and so a collection $\mathcal{E}$ may be uncountable as in the case where $\mathcal{E}$ is the collection of all points in S^1. $\diamondsuit$

Given a Hausdorff space X and a cell decomposition $\mathcal{E}$, the pair $(X, \mathcal{E})$ is called a *CW complex* if the following axioms are satisfied.

(1) Characteristic Maps. For each n-cell $e \in \mathcal{E}$, there is a map $\Phi_e : D^n \longrightarrow X$ restricting to a homeomorphism $\Phi_e \mid_{\text{int}(D^n)}: \text{int}(D^n) \longrightarrow e$ and taking $S^{n-1} := \partial D^n$ into X^{n-1}.

(2) Closure-Finiteness. For any cell $e \in \mathcal{E}$, the closure $\overline{e}$ intersects only a finite number of other cells in $\mathcal{E}$.

(3) Weak Topology. A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e}$ is closed in X for each $e \in \mathcal{E}$.

The term 'CW' comes from the second and third axioms. If $(X, \mathcal{E})$ is a CW complex, then we say that X has the structure of a cell complex, or simply that X itself is a CW complex and write $X := (X, \mathcal{E})$ when the context is clear.

Proposition 1.9 (c.f. [**Geo08**, Props. 1.2.5–1.2.9])**.** *Let $X = (X, \mathcal{E})$ be a CW complex. Then X^n may be obtained from X^{n-1} by attaching of the n-cells in X.* □

As such, a CW complex $X = (X, \mathcal{E})$ may be constructed inductively as follows.

(1) Start with a discrete set X^0, whose points are regarded as the 0-cells.

(2) Inductively construct the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi^n_\alpha : S^{n-1} \longrightarrow X^{n-1}$. This makes X^n the quotient space of the disjoint union X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$, where each e^n_α is an n-cell.

(3) The inductive process may stop at a finite stage, setting $X = X^n$ for some $n < \infty$, or may continue indefinitely, setting $X = \cup_n X^n$. If $X = X^n$, for some n, then X is said to be *finite-dimensional*, and the smallest such n is the dimension of X. Note that the dimension of X is the maximum dimension any cell in X can have.

Example 1.10. The sphere S^n has the structure of a CW complex and may be constructed from two cells, namely the point (0-cell) $e^0 \in \mathbb{R}^n$ and the n-cell e^n attached by the constant map $f : S^{n-1} \longrightarrow e^0$. This is equivalent to regarding the sphere S^n as the quotient space $D^n/\partial D^n$. ♢

Remark 1.11. A 1-dimensional CW complex $X = X^1$ is called a *graph*, which consists of vertices (as the 0-cells) to which the edges (as the 1-cells) are attached. When X is an oriented manifold, then its edges are also oriented, and so X is a directed graph. (Also see Figure 1.1.) ♢

Remark 1.12. CW complexes are the generalisation of simplicial complexes. Recall that a *simplex* (also called a hypertetrahedron) generalises the notion of a triangle to higher dimensions. More precisely, an n-simplex is an n-dimensional polytope which is the convex hull of its $n + 1$ vertices, and the convex hull of any non-empty subset of its $n + 1$ vertices is called a *face* of the simplex. In particular, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. A *simplicial complex* $\mathcal{K}$ is a CW complex such that every n-cell is an n-simplex with the following restriction: any face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$, and whenever any two simplices intersect, the intersection forms a face of both the simplices. ♢

1.3. Čech cohomology of CW complexes

The (Čech) cohomology offers certain interesting information about tiling spaces that other topological invariants (e.g., homology and fundamental group) fail to provide [**Sa08**, Ch. 3]. As such, we drop the notion of other usual topological invariants and concentrate mainly on cohomology. There are many types of cohomology that topologists use, depending on the given topological space. It is natural to work with *cellular* cohomology for CW complexes, however, as we will see in this section, it turns out that the Čech, cellular, singular and simplicial cohomology of a CW complex are all isomorphic to one another. Let us review the notion of *cochain complexes* and *singular cohomology* of a topological space before we define the cellular cohomology of a CW complex.

A *graded* R-module (for a commutative ring R) is a sequence $C := \{C_n\}_{n\in\mathbb{Z}}$ of R-modules. Given two graded R-modules C and D, a *graded homomorphism of degree* d from C to D is a sequence $f := \{f_n : C_n \longrightarrow D_{n+d}\}_{n\in\mathbb{Z}}$ of R-module homomorphisms. A *chain complex* over R is a pair (C, ∂), where C is a graded R-module and (the boundary operator) $\partial : C \longrightarrow C$ is a homomorphism of degree -1 such that $\partial \circ \partial = 0$. More precisely, we have the sequence

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

with $\partial_k \circ \partial_{k+1} = 0$.

Of more interest to us is the *cochain complex* over R, which by definition is a pair (C^*, δ), where C^* is a graded R-module and (the coboundary map) $\delta : C^* \longrightarrow C^*$ is a homomorphism of degree 1 such that $\delta \circ \delta = 0$. The cochain complex $(C^*, \delta) = (\{C^n\}, \delta) := (\{C_n^*\}, \delta)$ is dual to the chain complex $(\{C_n\}, \partial)$ by letting $C^n := C_n^* = \mathrm{Hom}_R(C_n, R)$ and δ the (morphism) transpose of ∂, which may also be denoted by $\delta := \partial^*$. In particular, δ_n sends a homomorphism $\varphi : C_n \longrightarrow R$ to the homomorphism $\varphi \circ \partial_{n+1} : C_{n+1} \longrightarrow R$, and similarly as above, we have the sequence

$$\cdots \xrightarrow{\delta_{n-2}} C_{n-1} \xrightarrow{\delta_{n-1}} C_n \xrightarrow{\delta_n} C_{n+1} \xrightarrow{\delta_{n+1}} \cdots$$

The *kth cohomology* is given by $H^k(C; R) = \ker\delta / \operatorname{im}\delta$ at C^k, which is well defined because $\delta_k \circ \delta_{k-1} = 0$. Depending on how the graded R-module C^* is defined determines the type of cohomology being considered, e.g., if $\{C^n\}$ is a collection of simplicial complexes, then the cohomology is defined as the simplicial cohomology, etc. Let us now look at a particular way to define a chain complex, and consequently a cochain complex.

The *standard* n*-simplex* Δ^n is the closed convex hull of the $n+1$ points $\{p_0, p_1, \ldots, p_n\}$ in $\mathbb{R}^{n-1}$, where p_j has $(j+1)$th coordinate 1 and all other coordinates 0, i.e.,

$$\Delta^n := \left\{ \sum_{j=0}^{n} t_j p_j \,|\, 0 \le t_j \le 1, \text{ and } \sum_j t_j = 1 \right\}.$$

A *singular n-simplex* in a topological space X is a continuous map $\sigma : \Delta^n \longrightarrow X$, and we denote by $S_n(X;R)$ the free R-module generated by the set of all singular n-simplexes. The ith *face* of the singular n-simplex σ is the composite map $\Delta^{n-1} \xrightarrow{F_i} \Delta^n \xrightarrow{\sigma} X$, where F_i is the affine map which sends the coordinate p_j to p_j if $j < i$, and p_j to p_{j+1} if $j \ge i$. Note that a 0-simplex has no face, and $S_n(X;R) := 0$ whenever $n < 0$. The boundary operator $\partial_n : S_n(X;R) \longrightarrow S_{n-1}(X;R)$ is given by $\partial_n(\sigma) := \sum_{i=0}^{n}(-1)^i(\sigma \circ F_i)$ for $n > 0$, and $\partial_n := 0$ for $n \le 0$. The graded R-module $S_*(X;R) := \{S_n(X;R)\}_{n\in\mathbb{Z}}$ together with the boundary operator ∂ form a *singular chain*, which can be shown to be a chain complex over R. Shifting our attention to the dual of $(S_*(X;R), \partial)$ gives us the *singular cochain* $(S^*(X;R), \delta)$, where $S^*(X;R) := \{S^n(X;R)\}_{n\in\mathbb{Z}}$ with $S^n := S_n^* = \mathrm{Hom}_R(S_n, R)$ and $\delta_n = \partial_{n+1}^*$. We then define the *singular* cohomology of X to be the cohomology of the singular cochain complex $(S^*(X;R), \delta)$.

When X is a CW complex, the singular cohomology of X can be computed from a cochain complex smaller than $S^*(X;R)$. This can be achieved through the following construction.

Definition 1.13 (Cellular cohomology of a CW complex)**.** Let X be a d-dimensional CW complex. The R-module $C_n := C_n(X;R)$ for $n \in \mathbb{Z}$ is the free R-module generated by all the n-cells in X [**Geo08**, Prop. 2.3.1] (which is also commonly denoted by $C_n := H_n(X^n, X^{n-1}; R)$). In particular, the set of all 0-cells generates $C_0 \cong R^V$, where V denotes the number of 0-cells, all 1-cells generate $C_1 \cong R^E$ with E denoting the number of 1-cells, and so on. Let ∂ be the usual boundary operator $\partial_n : C_n \longrightarrow C_{n-1}$ [**Geo08**, Ch. 12.1], then the *cellular* chain $(C_*(X;R), \partial)$ is a chain complex, where $C_*(X;R) := \{C_n(X;R)\}_{n\in\mathbb{Z}}$ [**Geo08**, Prop. 2.3.3]. Now, to its dual, the cellular cochain $C^*(X;R)$, where $C^*(X;R) := \{C^n(X;R)\}_{n\in\mathbb{Z}}$ with $C^n(X;R) := C_n^*(X;R) = \mathrm{Hom}_R(C_n(X;R), R) =: C^n$, together with the coboundary operator $\delta_n : C^n \longrightarrow C^{n+1}$ with $\delta_n := \partial_{n+1}^*$, is a cochain complex. By definition, $C^n := 0$ for $n < 0$ and the same is true for $n > d$ [**Geo08**, Prop. 12.1.7]. Thus, we have the sequence

$$0 \longrightarrow C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \cdots \longrightarrow C^{d-1} \xrightarrow{\delta_{d-1}} C^d \longrightarrow 0.$$

Representing the boundary and coboundary maps as matrices gives us $\delta_k = \partial_{k+1}^T$, where M^T denotes the transpose of the matrix M. Finally, the kth (cellular) cohomology is defined as $H^k(X;R) := \ker \delta_k / \operatorname{im} \delta_{k-1}$. ♢

For the most part, we let $R = \mathbb{Z}$, and so we may use the term 'abelian group' for the $\mathbb{Z}$-module C^n instead. In such cases, we simply denote the (cellular) cohomology of X as $H^*(X) := H^*(X;\mathbb{Z})$. Further, there is a natural isomorphism between the cellular and singular cohomology of X [**Geo08**, cf. Thm. 2.3.5]. As such, we no longer distinguish the cellular and singular cohomology of X.

Example 1.14 (Cohomology of the circle)**.** Earlier, we discussed how the circle S^1 may be written as a CW complex. Let e^0 and e^1 be the 0- and 1-cells of S^1 as defined in Remark 1.8 (also see Figure 1.1 (B)), and so we have $C^0 = \langle e^{0\prime}\rangle \cong \mathbb{Z}$ and $C^1 = \langle e^{1\prime}\rangle \cong \mathbb{Z}$ as (cellular) cochain groups. Note that e' denotes the dual to the cell e. The coboundary map $\delta_0 : C^0 \longrightarrow C^1$ is the zero map, since $\delta_0(e^{0\prime}) = e^{1\prime} - e^{1\prime} = 0$, where the 1-cell e^1 is considered to be a directed edge. We then form the sequence

$$0 \longrightarrow C^0 \xrightarrow{\delta_0} C^1 \longrightarrow 0,$$

and so $H^0(S^1) = \ker\delta_0 \cong \mathbb{Z}$ and $H^1(S^1) = C^1/\operatorname{im}\delta_0 \cong \mathbb{Z}$. ♢

The Čech cohomology of a topological space is defined through the Čech cohomology of its open covers. Thus, we first define the Čech cohomology of an open cover. A collection of open sets $\mathcal{U} := \{U_\alpha | \alpha \in A\}$ (where A is not necessarily countable) of a topological space is said to be an *open cover* of X if $X \subseteq \bigcup_{\alpha\in A} U_\alpha$. Suppose X is a topological space and $\mathcal{U} := \{U_\alpha\}$ an open cover of X. The *nerve*, denoted by $N(\mathcal{U})$, is a simplicial complex (see Remark 1.12) with the following simplices.

(1) A vertex α for every non-empty open set U_α;

(2) An edge $\alpha\beta$ for each non-empty intersection $U_\alpha \cap U_\beta$;

(3) A face $\alpha\beta\gamma$ for each non-empty intersection $U_\alpha \cap U_\beta \cap U_\gamma$;

(4) An n-simplex for every non-empty intersection of $n+1$ open sets.

The *Čech cohomology* of the cover $\mathcal{U}$ is defined to be the simplicial cohomology of its nerve $N(\mathcal{U})$. However, it can be shown that every singular complex can be subdivided and triangulated to be a simplicial complex [**Hat02**, Prob. 2.1.23]. Since singular (and hence cellular) cohomology is a topological invariant [**Geo08**, Thm. 12.1.9], and every singular complex is homeomorphic to a simplicial complex, the simplicial and singular (and thus cellular) cohomologies are equivalent for spaces that admit simplicial structures. So we just prefer to drop the terms simplicial, singular and cellular altogether and refer to them simply as cohomology. As such, we now arrive at the following definition.

Definition 1.15. The Čech cohomology of an open cover $\mathcal{U}$, denoted by $\check{H}^*(\mathcal{U})$, is the cohomology of its nerve $N(\mathcal{U})$, i.e., $\check{H}^*(\mathcal{U}) = H^*(N(\mathcal{U}))$. ♢

Example 1.16. Different open covers may correspond to different nerves, which then could yield different cohomologies. For instance, consider the three covers of S^1 as shown in Figure 1.2, and call them $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ respectively. The nerves $N(\mathcal{U}_i)$ for $i \in \{1,2,3\}$ are also shown in the figure beneath their respective covers. We get the following non-trivial cohomologies.

$$\begin{aligned}
\check{H}^0(\mathcal{U}_1) &= H^0(N(\mathcal{U}_1)) = \langle e^{0'} \rangle \cong \mathbb{Z};\\
\check{H}^0(\mathcal{U}_2) &= H^0(N(\mathcal{U}_2)) = \langle e_0^{0'} + e_1^{0'} \rangle \cong \mathbb{Z};\\
\check{H}^0(\mathcal{U}_3) &= H^0(N(\mathcal{U}_3)) = \langle e_0^{0'} + e_1^{0'} + e_2^{0'} \rangle \cong \mathbb{Z}, \text{ and}\\
\check{H}^1(\mathcal{U}_3) &= H^1(N(\mathcal{U}_3)) = \langle e_0^{1'}, e_1^{1'}, e_2^{1'} \rangle / \langle e_0^{1'} - e_2^{1'}, e_1^{1'} - e_2^{1'} \rangle \cong \mathbb{Z}.
\end{aligned}$$

Note that $\check{H}^*(\mathcal{U}_3) = H^*(S^1)$ is not a coincidence as a consequence of Proposition 1.19 below. ♢

An open cover $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every open set in $\mathcal{V}$ is contained in some open set in $\mathcal{U}$. As such, there is a simplicial map $\rho_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \longrightarrow N(\mathcal{U})$, which induces a canonical map $\iota_{\mathcal{V}\mathcal{U}} : \check{H}^*(\mathcal{U}) \longrightarrow \check{H}^*(\mathcal{V})$. See [**BT82**, Sec. 10] for details.

Definition 1.17. The Čech cohomology of X, denoted by $\check{H}^*(X)$, is the direct limit of $\check{H}^*(\mathcal{U})$, where the direct limit is taken over all open covers of X together with the canonical map ι from above. ♢

The previous definition requires one to look at all open covers of X, their corresponding Čech cohomology, and the induced maps ι between the cohomologies. Fortunately, we can avoid this difficulty through the notion of good covers. An open cover $\mathcal{U}$ of X is called a *good cover* if every open set in $\mathcal{U}$ is contractible, and every non-empty intersection of a finite number of open sets is also contractible.

Remark 1.18. Consider again the three covers of the circle S^1 in Figure 1.2. The cover $\mathcal{U}_1 := \{U_0\}$ is not a good cover since U_0 is not contractible. The cover $\mathcal{U}_2 := \{U_0, U_1\}$ is also not a good cover as the intersection $U_0 \cap U_1$ is not contractible. On the other hand, the cover $\mathcal{U}_3 := \{U_0, U_1, U_2\}$ is a good cover as every open set U_i for $i \in \{1,2,3\}$ is contractible and so is any of their intersections. ♢

Proposition 1.19 ([**Sa08**, Cor. 3.3])**.** *For a finite CW complex X and a good cover $\mathcal{U}$ of X, we have $\check{H}^*(X) \cong \check{H}^*(\mathcal{U})$.* □

Let us now state formally the Čech cohomology of a finite-dimensional CW complex.

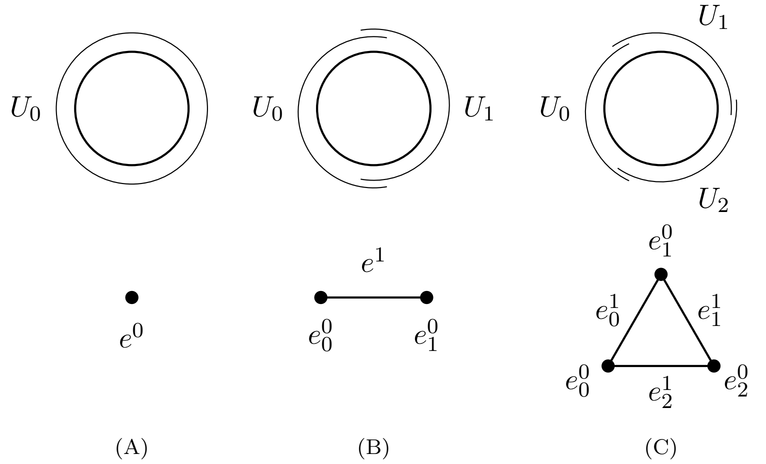


FIGURE 1.2. The circle S^1 together with three open covers (A) $\mathcal{U}_1 := \{U_0\}$, (B) $\mathcal{U}_2 := \{U_0, U_1\}$, and (C) $\mathcal{U}_3 := \{U_0, U_1, U_2\}$. Beneath the open covers are their corresponding nerves; also see [**Sa08**, Ch. 3.3].

Proposition 1.20 ([**Sa08**, Thm. 3.4])**.** *Let X be a finite CW complex. Then the Čech cohomology $\check{H}^*(X)$ is isomorphic to the cohomology $H^*(X)$.* □

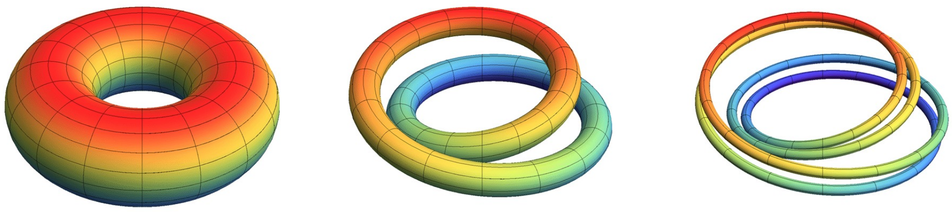
Remark 1.21 (Connectedness vs. Path-connectedness)**.** We could have defined the Čech cohomology of a finite CW complex X simply as its singular or cellular cohomology. However, we do not wish to imply that the two cohomologies are necessarily equivalent, as in the case of X being the *closed topologist's sine curve*, i.e., $X := \{(0,0)\} \cup \{(t, \sin\frac{1}{t}) \mid t \in (0,1]\}$. Indeed, $\check{H}^0(X) \cong \mathbb{Z}$, whereas $H^0(X) \cong \mathbb{Z}^2$ [**SS78**, 118]. The former indicates the number of connected components in X, whereas the latter indicates the number of path-connected components in X. In general though, for any topological space X that is homotopy equivalent to a finite CW complex, the Čech and singular cohomologies are isomorphic. ♢

1.4. Inverse limit spaces and their cohomology

Let $\Upsilon_0, \Upsilon_1, \Upsilon_2, \ldots$ be topological spaces, and $\{f_n : \Upsilon_{n+1} \longrightarrow \Upsilon_n\}_{n \in \mathbb{N}_0}$ be a family of continuous maps. Let us consider the product space $\prod \Upsilon_i$, whose elements are sequences $x := (x_0, x_1, x_2, \ldots)$ with $x_i \in \Upsilon_i$. The *inverse limit space* of Υ_i relative to the (bonding) maps $\{f_n\}$ is defined as

$$\varprojlim(\Upsilon, f) := \{(x_0, x_1, \ldots) \in \prod \Upsilon_i \mid x_n = f_n(x_{n+1}) \, \forall n \in \mathbb{N}_0\}.$$

The spaces Υ_n are called the *approximants* to the inverse limit space because knowing a particular x_N means being able to determine all x_i for $0 \leq i < N$.



(A) (B) (C)

FIGURE 1.3. The first three approximants to the dyadic solenoid, (A) $\Upsilon_0 = S^1$, (B) $\Upsilon_1 = S^1$, and (C) $\Upsilon_3 = S^1$. Images created by Jim Belk, retrieved from http://en.wikipedia.org/wiki/File:Solenoid.gif, used with his kind permission.

The product space $\prod \Upsilon_i$ is equipped with the product topology, which means that two sequences $x := (x_0, x_1, \ldots)$ and $y := (y_0, y_1, \ldots)$ are 'close' if their first N terms are close (in their respective topologies) for a sufficiently large value of N. (One may further say that for any $\varepsilon > 0$, two sequences x and y are 'ε-close' if their first N terms are close, where $N \leq 1/\varepsilon$; and so, x and y are 'close' if for sufficiently small ε, they are ε-close.) If x_N and y_N are close, then x_{N-1} and y_{N-1} are also close by continuity of f_{N-1}, and so are x_i and y_i for any $0 \leq i < N$.

A well known example of an inverse limit space are the solenoids [**Wi74**].

Example 1.22 (Dyadic solenoid)**.** As an inverse limit space, the dyadic solenoid $\mathbb{S}_2$ may be constructed as follows. Let each approximant be a circle, i.e., $\Upsilon_i = \mathbb{R}/\mathbb{Z} = S^1 = \mathbb{T}$ (1-dimensional torus) for all $i \in \mathbb{N}_0$, and let each f_n be multiplication by 2. Then $\mathbb{S}_2 = \varprojlim(S^1, f)$. Geometrically, f_n is a mapping that wraps Υ_{n+1} around Υ_n twice. Figure 1.3 shows the first three approximants Υ_0, Υ_1 and Υ_2 (in 3D for better visualisation) to the dyadic solenoid. Imagine the three tori simply being circles. Then the second torus wraps around the first twice, and the third around the second twice. A point in $\mathbb{S}_2$ is described by the sequence $(x_0, x_1, x_2, \ldots)$, where x_0 is a point on the circle Υ_0, a choice $x_1 \in \Upsilon_1$ from two possible pre-images of x_0, i.e., $x_1 = \frac{1}{2}x_0$ or $x_1 = \frac{1}{2}(1 + x_0)$, a choice $x_2 \in \Upsilon_2$ from two possible pre-images of x_1, and so on. As such, the solenoid is locally a product of an interval and a Cantor set [**Sa08**]. More generally, a p-adic solenoid $\mathbb{S}_p$ is obtained by defining f_n as multiplication by $p > 1$. ♢

Proposition 1.23 ([**Sa08**, Thm. 3.5])**.** *Let X be the inverse limit of a sequence of spaces Υ_n relative to the maps $f_n : \Upsilon_{n+1} \longrightarrow \Upsilon_n$. If each Υ_n is a* *finite CW complex, then* $\check{H}^*(X) = \check{H}^*(\varprojlim(\Upsilon, f)) \cong \varinjlim \check{H}^*(\Upsilon_n) = \varinjlim H^*(\Upsilon_n)$.

$\square$

Example 1.24 (Čech cohomology of the dyadic solenoid)**.** Let us compute the Čech cohomology of the dyadic solenoid $\mathbb{S}_2$, which is the inverse limit of circles $\Upsilon_n = S^1$ relative to the doubling maps f_n as defined in Example 1.22. The continuous map f_n induces a map on the approximants Υ_n as follows. Fix a 0-cell e^0 in $\Upsilon_0 = S^1$ and define the 1-cell e^1 as the complement $S^1 - \{e^0\}$, as in Figure 1.1 (B). The action of f_0 on Υ_0 fixes the 0-cell e^0 but doubles the 1-cell e^1 in length. In fact, for all $n \in \mathbb{N}_0$, f_n fixes e^0 and doubles e^1 $(n+1)$ times. As a consequence (together with Proposition 1.23), we have the following Čech cohomologies:

$$\check{H}^0(\mathbb{S}_2) \cong \varinjlim \check{H}^0(S^1) = \varinjlim(\mathbb{Z}, \times 1), \text{ and}$$
$$\check{H}^1(\mathbb{S}_2) \cong \varinjlim \check{H}^1(S^1) = \varinjlim(\mathbb{Z}, \times 2).$$

As in Examples 1.2 and 1.4, we get

$$\check{H}^0(\mathbb{S}_2) \cong \mathbb{Z}, \quad \text{and} \quad \check{H}^1(\mathbb{S}_2) \cong \mathbb{Z}[\tfrac{1}{2}].$$

For a general $p > 1$, the Čech cohomology of the p-adic solenoid is given by

$$\check{H}^0(\mathbb{S}_p) \cong \mathbb{Z}, \quad \text{and} \quad \check{H}^1(\mathbb{S}_p) \cong \mathbb{Z}[\tfrac{1}{p}]. \tag{1.3}$$

$\diamondsuit$

1.5. Perron-Frobenius theory

This short review on the Perron-Frobenius (PF) theory is intended only to touch a few details that are related to our study. In most cases, we can proceed even without referring to the PF theory. However, for the sake of completeness, we choose to cover the essentials in this section. For a background, see [**BG13a**, Ch. 2.4] and references therein.

A $d \times d$ matrix $M \in \mathrm{Mat}(d, \mathbb{R})$ is called *non-negative* when all its entries are non-negative numbers, and called *strictly positive* if all its entries are positive numbers. A non-negative matrix M is said to be *primitive* if there exists an integer $k \in \mathbb{N}$ such that M^k is strictly positive.

Proposition 1.25 (Perron-Frobenius)**.** *Let $M \in \mathrm{Mat}(d, \mathbb{R})$ be a primitive non-negative matrix. There exists a simple real eigenvalue $\lambda_{PF} > 0$, called the Perron-Frobenius eigenvalue, such that for any eigenvalue $\lambda \neq \lambda_{PF}$ of M, we have $|\lambda| < \lambda_{PF}$. The corresponding eigenvector of λ_{PF} may be chosen so that all its entries are positive numbers.* $\square$

As we will see later on, the PF theory provides the necessary tool for squeezing out the embedded geometric properties of a symbolic substitution (especially in one dimension) allowing for its natural geometric realisation.

1.6. Symbolic substitution

A well studied approach in analysing ordered systems, particularly in one dimension, is provided by substitution rules on finite alphabets; see [**BG13a**, Ch. 4] for a detailed exposition. Let us define the substitution first in one dimension, and then extend it to a particular class of substitutions in higher dimensions, called the block substitutions [**Fra05**].

A finite *alphabet* $\mathcal{A}_n$ consists of a finite number of *letters* $a_i \in \mathcal{A}_n$, i.e., $\mathcal{A}_n := \{a_i \mid 1 \leq i \leq n\}$ for some integer $n > 1$. Letters may be concatenated to create *words*. A (finite) word w consisting of k letters is called a k*-letter* word whose length is denoted by $|w| = k$. Words may be extended semi-infinitely, or bi-infinitely by indefinitely concatenating a letter to the right of the word, or to either ends of the word. Infinite words may be coordinatised as either $w := w_0w_1w_2\cdots$ (semi-infinite) or $w := \cdots w_{-2}w_{-1}w_0w_1w_2\cdots$ (bi-infinite), where each letter w_i is in $\mathcal{A}_n$. Sometimes, the use of a reference marker is also employed, for instance, $w := |w_0w_1w_2\cdots$ or $w := \cdots w_{-2}w_{-1}|w_0w_1w_2\cdots$, where we use a vertical line to serve as a reference marker. By convention, an *empty* word ε may also be introduced. Note that $|\varepsilon| = 0$, and for any word w, we have the following concatenation $w\varepsilon = w = \varepsilon w$.

A *substitution* ϱ on $\mathcal{A}_n$ is a rule that assigns a (finite and non-empty) word to every letter $a_i \in \mathcal{A}_n$. The substitution naturally extends to any word by applying the rule individually to every letter of that word. The process of substitution (of a word) can be repeated indefinitely, and we assume that the lengths of the words go to infinity the further the iteration goes. Thus, a semi- or bi-infinite word may be constructed via substitution. We define the *substitution matrix* M_ϱ (of the substitution ϱ) as $(M_\varrho)_{ij} := \mathrm{card}_{a_i}(\varrho(a_j))$ for $1 \leq i, j \leq n$, where $\mathrm{card}_x(y)$ denotes the number of occurrences of x in y (e.g., $\mathrm{card}_a(aaa) = 3$ and $\mathrm{card}_{aa}(aaa) = 2$). The substitution matrix M_ϱ keeps record of the power counting of a finite word w under k-fold substitution through $(\mathrm{card}_{a_i}(\varrho^k(w)))_i = (M_\varrho^k(\mathrm{card}_{a_i}(w)))_i$.

The substitution ϱ is *primitive* if there is an integer $k \in \mathbb{N}$ such that for any $a_i, a_j \in \mathcal{A}_n$, $\varrho^k(a_i)$ contains a_j. Equivalently, ϱ is primitive if and only if its substitution matrix M_ϱ is primitive. The primitivity of ϱ guarantees that $|\varrho^k(a_i)| \longrightarrow +\infty$ as $k \to +\infty$ for any $a_i \in \mathcal{A}_n$.

A word w' is called a *subword* of w if and only if $\mathrm{card}_{w'}(w) > 0$, and necessarily, $|w'| \leq |w|$. A word w is said to be *legal* (with respect to ϱ) if for each finite subword w' of w, there exist $a_i \in \mathcal{A}_n$ and $k \in \mathbb{N}$ such that w' is a subword of $\varrho^k(a_i)$. When a word w is finite, then it is legal if and only if it is a subword of $\varrho^k(a_i)$. An infinite word w is called a *fixed point* of the substitution ϱ^k (for some $k \in \mathbb{N}$) if $\varrho^k(w) = w$, relative to a reference point. The legal word $u|v$ is called a *seed* of a (two-sided or bi-infinite) fixed point w if and only if $w = \lim_{m\to\infty} \varrho^{km}(u|v)$, where the vertical line $|$ marks the reference point. Analogously, $|u$ is a seed of a (one-sided or semi-infinite) fixed point w if and only if $w = \lim_{m\to\infty} \varrho^{km}(|u)$. Note that when ϱ is primitive, there is always a fixed point of ϱ^k for some $k \in \mathbb{N}$ [**BG13a**, Lem. 4.3].

Example 1.26 (Fibonacci substitution)**.** Perhaps one of the most known examples of a primitive substitution is the Fibonacci substitution [**BG13a**, Ch. 4.6] given by

$$\varrho_F := \begin{cases} a \longmapsto ab \\ b \longmapsto a \end{cases} \tag{1.4}$$

Applying ϱ_F on b a few times gives us

$$b \longmapsto a \longmapsto ab \longmapsto aba \longmapsto abaab \longmapsto abaababa \longmapsto \cdots \tag{1.5}$$

where the number of letters in each iteration forms the Fibonacci numbers [**OEIS**, `A000045`]. After every iteration (except for the very first), the iterated words always begin with the previous iterated words, which means that this sequence of words of increasing length converges to a limit w such that $w = \varrho_F(w)$, where w is a semi-infinite word. In this case we write $w := w^{(|a)}$ to denote that the limit is a one-sided fixed point of ϱ from the initial letter a. However, we are more interested in bi-infinite words as they are a better fit for our purpose when we begin considering substitution tilings later. To obtain a bi-infinite word, we may apply ϱ_F to $a|a$, where the vertical line denotes the reference point. We then get the following iterations

$$\begin{aligned} &\underline{a}|a \xmapsto{\varrho_F} a\underline{b}|ab \xmapsto{\varrho_F} ab\underline{a}|aba \xmapsto{\varrho_F} abaa\underline{b}|abaab \xmapsto{\varrho_F} \\ &\quad abaabab\underline{a}|abaababa \xmapsto{\varrho_F} abaababaabaa\underline{b}|abaababaabaab \xmapsto{\varrho_F} \cdots \end{aligned}$$

In contrast, there is no two-sided fixed point of ϱ_F, since the letter on the underlined position alternates between a and b. However, this also means that we can form two bi-infinite fixed points of ϱ_F^2, namely

$$\begin{aligned} a|a &\xmapsto{\varrho_F^2} aba|aba \xmapsto{\varrho_F^2} abaababa|abaababa \xmapsto{\varrho_F^2} abaababaabaababaababa| \\ &abaababaabaababaababa \xmapsto{\varrho_F^2} \cdots \xmapsto{\varrho_F^2} w := w^{(a|a)} = \varrho_F^2(w) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} b|a &\xmapsto{\varrho_F^2} ab|aba \xmapsto{\varrho_F^2} abaab|abaab \xmapsto{\varrho_F^2} abaababaabaab|abaababaabaab \\ &\xmapsto{\varrho_F^2} \cdots \xmapsto{\varrho_F^2} v := v^{(b|a)} = \varrho_F^2(v) \end{aligned} \tag{1.7}$$

As we see, $a|a$ and $b|a$ are the seeds of the fixed points $w = w^{(a|a)}$ and $v = v^{(b|a)}$. The notion of fixed points under an m-fold substitution is used to define dynamical zeta functions, as we will see later in Chapter 2.5.

The substitution matrix of the Fibonacci substitution is $M_{\varrho_F} = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$, whose PF eigenvalue $\lambda_{PF} := \lambda = \tau := (1+\sqrt{5})/2$ is the golden ratio. This means that asymptotically, the number of letters after every substitution scales by τ at each iteration, a property that is also known for Fibonacci numbers. The left eigenvector of λ, given by $L_\lambda := (\tau, 1)$, may be used to transform a bi-infinite word arising from ϱ_F to a covering (or more precisely a *tiling**) T of $\mathbb{R}$, preserving the natural geometric properties the substitution possesses. This is achieved by transforming the letters a and b to be closed intervals in $\mathbb{R}$ with lengths τ and 1 respectively. Let us emphasise that the lengths are the entries in L_λ. (We generalise the discussion on this in Remark 2.4.)

Meanwhile, the substitution yields $\{aa, ab, ba\}$ as its set of legal 2-letter words, and $\{aab, aba, baa, bab\}$ as the legal 3-letter words. This means that the subwords $\{bb, aaa, \ldots\}$ do not appear anywhere in any word arising from ϱ_F. (Compare Example 2.8.) $\diamondsuit$

Substitution rules on symbolic alphabets may also be used to define substitutions in higher dimensions, say in dimension $d \geq 2$, and one of the most natural ways of doing this would be through a *block substitution*, where a letter is assigned a d-dimensional array of letters, called blocks [**Fra05**]. If $d = 2$, then the blocks are rectangular arrays; for $d = 3$ the blocks are cubes; for $d = 4$ the blocks are hypercubes; and so on. Let us illustrate this with an example in two dimensions.

Example 1.27 (Chair† substitution)**.** Consider the planar block substitution on four letters (or symbols) $\mathcal{A}_4 := \{1, 2, 3, 4\}$ defined by

$$\varrho_{\mathsf{L}} := \left\{ 1 \longmapsto \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad 2 \longmapsto \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad 3 \longmapsto \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \quad 4 \longmapsto \begin{bmatrix} 4 & 3 \\ 1 & 4 \end{bmatrix} \right.$$

whose first few iterations starting with the symbol 1 look like

$$1 \longmapsto \begin{array}{c|c} 4 & 1 \\ \hline 1 & 2 \end{array} \longmapsto \begin{array}{cc|cc} 4 & 3 & 4 & 1 \\ 1 & 4 & 1 & 2 \\ \hline 4 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \end{array} \longmapsto \begin{array}{cccc|cccc} 4 & 3 & 4 & 3 & 4 & 3 & 4 & 1 \\ 1 & 4 & 3 & 2 & 1 & 4 & 1 & 2 \\ 4 & 1 & 4 & 3 & 4 & 1 & 2 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 & 1 & 2 \\ \hline 4 & 3 & 4 & 1 & 2 & 3 & 4 & 3 \\ 1 & 4 & 1 & 2 & 1 & 2 & 3 & 2 \\ 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \end{array} \longmapsto \cdots$$

---

*See definition in Chapter 2.1.

†This substitution is *equivalent* (see Chapter 2.2 for definition) to the chair tiling produced in Figure 2.1 [**Rob99, BS11**].

Except for the very first iteration (i.e., the iteration of 1), notice that the centre patch of every iterated block displays the previous block before the iteration, and so we have found a fixed point of the substitution ϱ_{L}. By having the origin at the very centre of the iterated blocks of letters, and by assigning a unit square (or any square of uniform length) to each letter, and colouring the squares accordingly to distinguish them, we form a covering (or tiling) of $\mathbb{R}^2$ that is faithful to the structure of the actual block substitution. The counterpart of legal subwords in this case would be *legal patches*, which are arrays of a finite number of symbols appearing in $\varrho_{\text{L}}^k(i)$ for some $i \in \mathcal{A}_4$ and integer $k \in \mathbb{N}$. It is not difficult to see that this substitution has an *inflation factor* of 2, i.e., the number of letters along any row or column is doubled after every iteration. More precisely when considering square patches (as geometric representation of the symbols), the side length of the iterated square patch doubles after every substitution. $\Diamond$

CHAPTER 2

Substitution tiling spaces and their cohomology

Perhaps one of the most significant developments in tiling theory in the recent past came when Anderson and Putnam were able to show that a substitution tiling space Ω can be regarded as an inverse limit of branched manifolds, called approximants [**AP98**]. As a result, the Čech cohomology of the substitution tiling space may be computed as the direct limit of the cohomologies of the approximants (also known as AP-complexes) relative to the bonding map induced by the substitution. Recently in [**BD08, BDHS09, GM13**], modified versions of the approximants were introduced, resulting in a simpler computation of the cohomology of the substitution tiling space. However, even though the cohomology is computed correctly, the approximants no longer describe the actual tiling space as their inverse limit, and so, we do not follow these newer methodologies, but rather still base the discussion on substitution tilings mainly from our original source, where most of the terminologies and definitions are taken from.

2.1. Substitution tilings and tiling spaces

A more natural geometric approach to substitution is through tiling theory. In certain scenarios, particularly in one dimension, we will see how the setting used in symbolic substitution parallels that of substitution on tilings of $\mathbb{R}^d$. This provides the necessary connection between the two theories, so that we can use whichever seems more convenient depending on a given situation.

A *tile* t is a subset of $\mathbb{R}^d$ that is homeomorphic to a closed disk in $\mathbb{R}^d$. A *partial tiling* is a set of tiles in $\mathbb{R}^d$, in which any two distinct tiles may only intersect on their boundaries, and whose *support* is the union of its tiles. A *tiling* of $\mathbb{R}^d$ is a partial tiling whose support is $\mathbb{R}^d$. Each tile may carry a label aside from its geometric shape, and so tiles that look alike can be distinguished through their labels. Formally, a tile is an ordered pair consisting of its geometric description and its label. A tiling T is a multi-valued function, such that for any $u \in \mathbb{R}^d$ and $U \subseteq \mathbb{R}^d$, we have

$$\begin{aligned} T(u) &:= \{t \in T \mid u \in t\}, \\ T(U) &:= \bigcup_{u \in U} T(u). \end{aligned}$$

Two tilings T and T' agree on U if $T(U) = T'(U)$. Further, for any partial tiling T, we define an expansion and translation of T by

$$\begin{aligned} \lambda T &:= \{\lambda t \mid t \in T\} \quad \text{for } \lambda \in \mathbb{R}_+, \\ T + u &:= \{t + u \mid t \in T\} \quad \text{for } u \in \mathbb{R}^d. \end{aligned}$$

Note that a tiling T is called *periodic* if $T = T + u$ for some $u \in \mathbb{R}^d - \{0\}$.

We now construct a collection of tilings, denoted by Ω, via a *substitution rule* (which later we denote by ω). Tilings defined through substitution are called *substitution tilings*. Let $\{p_i \mid i = 1, 2, \ldots, n\}$ be a finite[‡] set of (inequivalent) tiles called *prototiles*. Let $\hat{\Omega}$ be the collection of all partial tilings that are constructed using only the prototiles, i.e., each tile in any partial tiling is a translated copy of some prototile. Assume further that a substitution rule comes with an inflation factor $\lambda > 1$, such that whenever a prototile p_i is assigned a partial tiling P_i with support p_i, then λP_i is in $\hat{\Omega}$. (See Figure 2.1 for an illustration.) The substitution rule naturally extends to an inflation map that does not only apply to prototiles, but also to partial tilings. This inflation map $\hat{\omega}$ is defined as

$$\begin{aligned} \hat{\omega} : \hat{\Omega} &\longrightarrow \hat{\Omega} \\ T &\longmapsto \lambda \bigcup_{p_i + u \in T} (P_i + u). \end{aligned}$$

In particular, $\hat{\omega}^2(p_i) = \hat{\omega}(\lambda P_i) \in \hat{\Omega}$ yields a bigger partial tiling than $\hat{\omega}(p_i) = \lambda P_i$, and the partial tiling $\hat{\omega}^k(p_i)$, for any $k \in \mathbb{N}$, is called the level k *supertile* of p_i.

Finally, let Ω be the collection of tilings T in $\hat{\Omega}$ such that for any partial tiling $P \subseteq T$ of bounded support, we have $P \subseteq \hat{\omega}^m(\{p_i + u\})$ for some $m \in \mathbb{N}$, $i \in \{1, 2, \ldots, n\}$ and $u \in \mathbb{R}^d$. The restriction $\omega := \hat{\omega}|_\Omega$ is a mapping from Ω to Ω by applying the substitution rule on each tile (of a tiling in Ω) in a consistent manner as prescribed by $\hat{\omega}$. At times, we may deliberately not distinguish the substitution rule from the inflation map as the difference between the two is only technical in the context of our study. As such, we use the same notation ω whenever we refer to either the substitution rule or the inflation map.

This collection of substitution tilings, denoted by Ω (or Ω_ω when emphasis is needed), is called the *hull* or the *tiling space* associated to the substitution ω. We enumerate three assumptions, which most of the interesting examples satisfy.

[‡]The assumption on finiteness may be relaxed to consider a more general class of substitution tilings such as the Pinwheel tiling (compare [**Sa08**, Ch. 4] and [**BG13a**, Ch. 6.6]). Such examples of (non-simple) tilings are not considered in this text.

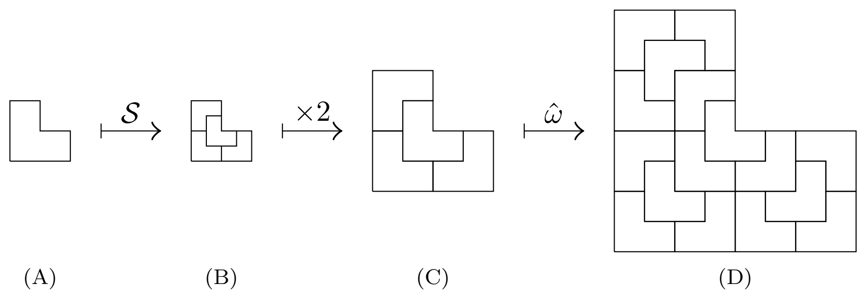


FIGURE 2.1. A substitution rule that produces the classical *chair* (also called *triomino* [**Rob99, AP98**]) tiling of $\mathbb{R}^2$, where a prototile (A) is subdivided into four pieces (B), and then is inflated linearly by a factor of 2 (C). Note that $\hat{\omega} = \times 2 \circ \mathcal{S}$. The right most partial tiling (D) is the level 2 supertile of the prototile in (A).

(1) The inflation map ω is one-to-one, and so its inverse exists. Substitution rules that admit periodic tilings (under the action of the translation group $\mathbb{R}^d$) do not satisfy this assumption. An example of this is the rule dividing an interval into two equal parts and then doubling their lengths afterwards.
(2) The substitution is *primitive*, i.e., there exists an integer N such that for every pair of prototiles p_i and p_j, the partial tiling $\hat{\omega}^N(\{p_i\})$ contains a translation of p_j. One can also define the substitution matrix associated to ω (which we denote by M_ω) similar to the one in symbolic substitution, but using prototiles instead of letters. Then ω is primitive if and only if M_ω is primitive.
(3) The hull Ω satisfies a *finite pattern condition*, i.e., for each real number r, there are only finitely many partial tilings up to translation $P \subseteq T$ (for any $T \in \Omega$) whose supports have diameter less than r. This property is also referred to as the *finite local complexity (FLC)* property in [**BG13a**, Ch. 2.1].

Under these assumptions, we have the following.

Proposition 2.1 ([**AP98**, Prop. 2.1–2.3])**.** *The tiling space Ω is non-empty and contains no (translation) periodic tilings. Further, $\omega(\Omega) = \Omega$.* □

Remark 2.2. There is an alternative yet equivalent way of defining the substitution tiling space Ω. In general, a tiling space Ω_T associated to a particular tiling T is defined as the closure of the translation orbit of T under a metric, where two tilings are "ε-close" if they agree on a ball of radius $1/\varepsilon$ around the origin, possibly after a translation of at most ε in any direction.

There are a number of equivalent metrics and topologies arising from this notion of tiling distance [**Rob96, Sol97, Sa08**] and the following is the one given in [**AP98**]. For tilings T_1, T_2 of $\mathbb{R}^d$ let

$$\begin{aligned} d(T_1, T_2) &:= \inf(\{1/\sqrt{2}\} \cup \Xi), \text{ where} \\ \Xi &:= \{\varepsilon \,|\, T_1 + u \text{ and } T_2 + v \text{ agree on } B_{1/\varepsilon}(0) \text{ for some } \|u\|, \|v\| < \varepsilon\} \end{aligned} \tag{2.1}$$

with $\|\cdot\|$ denoting the standard norm on $\mathbb{R}^d$ and $B_r(x)$ being the open ball of radius r centred at $x \in \mathbb{R}^d$. Two substitution tilings T and T' arising from the same substitution ω define the same tiling space, and so the hull is instead associated with a substitution rule rather than a particular substitution tiling, i.e., $\Omega_\omega := \Omega_T = \Omega_{T'}$. The collection Ω of substitution tilings defined earlier and the tiling space Ω_ω are equivalent. Compare [**BG13a**, Thm. 4.1]. ♢

With the (metric) topology of Ω induced by d in Equation (2.1), Anderson and Putnam showed that ω is a *topologically mixing* homeomorphism of (Ω, d) and that (Ω, d, ω) is a *Smale* space [**AP98**]. However, we do not pursue these here, but only need the following, which follows directly from the two assertions.

Proposition 2.3 ([**AP98**, Cor. 3.5])**.** *The action of* $\mathbb{R}^d$ *on* Ω *by translation is minimal, i.e., the translation orbit of any tiling* T *in* Ω *is dense.* □

Remark 2.4 (1-dimensional substitution tilings)**.** Clearly, a 1-dimensional tiling substitution ω defines a symbolic substitution ϱ by assigning a letter to each of the prototiles. If ω is primitive, then so is ϱ. Thus, a substitution tiling T of $\mathbb{R}$ defines a bi-infinite word that can be consistently obtained from ϱ. Conversely, we can apply the Perron-Frobenius (PF) theory on primitive symbolic substitutions to derive certain geometric properties associated with them. Given a primitive substitution ϱ on $n \geq 2$ letters with PF eigenvalue λ_{PF} and a left eigenvector $(v_1, v_2, \ldots, v_n)$ of λ_{PF} whose entries are all positive, we can transform a letter in $\mathcal{A}_n$ to a prototile in $\mathbb{R}$ by assigning the interval $[0, v_i]$ to the letter $a_i \in \mathcal{A}_n$. If two different letters are assigned the same interval, we may label or colour them as to keep them distinguished. As such, a bi-infinite word arising from ϱ becomes a substitution tiling of $\mathbb{R}$ with the PF eigenvalue λ_{PF} being the inflation factor associated to the substitution tiling. Therefore, the symbolic primitive substitution ϱ defines a primitive substitution ω.

One can then define a symbolic substitution system, i.e., a (discrete) hull, analogous to a (continuous) tiling space Ω. However, we do not wish to explore this approach here. For a thorough exposition on this, see [**BG13a**, Ch. 4]. In higher dimensions, symbolic block substitutions (of constant length) may define a substitution tiling of $\mathbb{R}^d$ with integer inflation factor λ. An example of this is the chair substitution in Example 1.27, which defines a substitution equivalent to the chair tiling illustrated in Figure 2.1. We will explore more on such examples in the following chapters. ♢

Remark 2.5 (Forcing the border)**.** A substitution is said to *force the border* if there exists a natural number N, such that any two level N supertiles (of the same tile type) have the same pattern of neighbouring tiles. More precisely, there is a fixed positive integer N such that for any tile t and any two tilings T and T' containing t, $\omega^N(T)$ and $\omega^N(T')$ coincide, not just on $\hat{\omega}^N(\{t\})$, but also on all tiles that meet $\hat{\omega}^N(\{t\})$.

The Thue-Morse substitution $\varrho_{TM} := \{1 \longmapsto 1\bar{1},\, \bar{1} \longmapsto \bar{1}1\}$ (Example 2.15) does not force the border while the Fibonacci substitution $\varrho_F := \{a \longmapsto ab,\, b \longmapsto a\}$ (Example 2.14) and the period doubling substitution $\varrho_{pd} := \{a \longmapsto ab,\, b \longmapsto aa\}$ (Example 2.16) force the border, though only on one side, since the substitution of any letter produces a word that begins with a. On the other hand, the substitution $\{a \longmapsto abb,\, b \longmapsto aab\}$ forces its border, since all iterated words begin and end with the same letter.

In two dimensions, the Penrose [**Kel95**] and the half-hex tilings [**Sa08**, Ch. 2.5] force their borders, while the chair tiling [**BS11**] produced in Figure 2.1 does not. ♢

2.2. Equivalence of tiling spaces

The notion of equivalence between tiling spaces can be defined via topological conjugacy or through mutual local derivability. (The former is weaker than the latter though.) We begin by defining topological conjugacy, whose definition is similar to that of other topological spaces.

A *homeomorphism* between tiling spaces is a continuous map $f : \Omega_X \longrightarrow \Omega_Y$ that is 1-to-1 and onto. Since Ω_X is compact and Ω_Y a Hausdorff space [**Sa08**, Ch. 1.3], then the inverse f^{-1} is necessarily continuous, which agrees with the usual topological definition of homeomorphism between topological spaces. A *factor map* between tiling spaces (of $\mathbb{R}^d$) is a map that commutes with the action of the translation group $\mathbb{R}^d$ and a *topological conjugacy* between tiling spaces is a factor map that is also a homeomorphism. Topological conjugacy preserves the structure of tiling spaces as dynamical systems and preserves dynamical invariants such as the dynamical spectrum, and mixing properties, among others.

A stronger notion of equivalence between tiling spaces is given by *mutual local derivability*, or MLD for short [**BSJ91**]. Though, we do not really focus on the study of MLD between tiling spaces, let us just define the notion of MLD nonetheless.

We say that the two tiling spaces Ω_X and Ω_Y are MLD if there is a topological conjugacy between them which is defined locally in either direction.

This means that there exists a radius R such that whenever two tilings T and T' (both in Ω_X) agree on a ball of radius R around x, then $f(T)$ and $f(T')$ (both in Ω_Y) agree on a ball of radius 1 around x. Since f commutes with translation, it is sufficient to check this at $x = 0$ [**Sa08**, Ch. 2.1].

Tilings that are mutual locally derivable are necessarily topologically conjugate, but the converse is not true in general [**Pet99, RS01**]. Also see [**Sa08**, Ch. 3].

2.3. Tiling spaces are inverse limit spaces

As we have seen in the previous chapter, the Čech cohomology of an inverse limit space behaves well under direct limits. However, showing that a space is an inverse limit space is no guarantee for the computability of its Čech cohomology. In this section, we review the construction of the CW complexes, called the AP-complexes, that will serve as approximants to the substitution tiling space that allows for the computation of its cohomology. Let us recall the inverse limit space

$$\varprojlim(\Upsilon, f) := \{(x_0, x_1, \ldots) \in \prod \Upsilon_i \mid x_n = f_n(x_{n+1}) \,\forall n \in \mathbb{N}_0\}, \qquad (2.2)$$

with approximants Υ_n and bonding maps $f_n : \Upsilon_{n+1} \longrightarrow \Upsilon_n$.

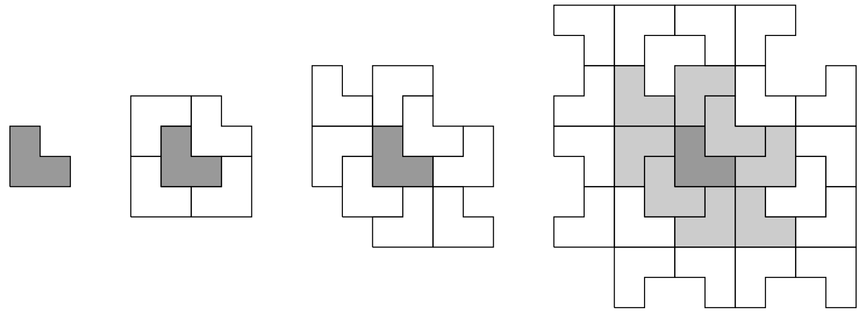
Remark 2.6. A large class of tiling spaces beyond substitution tilings was also shown to be inverse limit spaces. This ranges from simple tiling spaces [**Gäh02, BBG06**], to non-simple substitution tilings including pinwheel-like tilings [**ORS02, BG03**], to certain non-Euclidean spaces [**Sa03**], and even to tiling spaces lacking the finite pattern condition [**FS09**]. All these spaces can now be understood as inverse limit spaces! We do not however pursue these types of tiling spaces in this work, as we only focus on (simple) substitution tiling spaces. ♢

For substitution tiling spaces, the spaces Υ_n and the continuous maps f_n in (2.2) may be defined in such a way that all the approximants are topologically identical to one another and all the bonding maps essentially the same. This simplifies the calculation of the direct limit of the cohomologies. Further, the approximants turn out to be finite-dimensional CW complexes, which we pretty much know how to handle. This construction was cleverly achieved by Anderson and Putnam in [**AP98**], and as such, the (CW complex) approximants are now called *AP-complexes*.

2.3.1. The Anderson-Putnam complexes.

An AP-complex Γ_k for $k \in \{0, 1\}$ is constructed as follows:

(1) For a tile $t \in T$, let $T^{(0)}(t) := \{t\}$ and $T^{(1)}(t) := T(t)$. In general, $T^{(k)}(t)$ is the set of tiles in T that are within k tiles of t, also called the *k-collared*



(A) (B) (C) (D)

FIGURE 2.2. Two distinct 1-collared partial tilings (B and C), and a 2-collared partial tiling (D) of the uncollared chair prototile (A).

partial tiling of t. (See Figure 2.2 and also Table 2.1.) As a consequence of Proposition 2.11 below, there is no need to go beyond $k = 1$, and so explains the restriction on k.

(2) Next, we consider the space $\Omega \times \mathbb{R}^d$ with the product topology, where $\mathbb{R}^d$ has the standard topology and Ω has the discrete topology to ensure that every possible small patch of tiles is represented.

(3) Then let $\sim_k$ be the smallest equivalence relation on $\Omega \times \mathbb{R}^d$ that relates (T_1, u_1) to (T_2, u_2) whenever

$$T_1^{(k)}(t_1) - u_1 = T_2^{(k)}(t_2) - u_2,$$

for some tiles $t_1 \in T_1$, $t_2 \in T_2$ and for some vectors $u_1 \in t_1$, $u_2 \in t_2$. The equivalence class of a point (T, u) is denoted by $(T, u)_k$.

(4) Finally, define $\Gamma_k := \Omega \times \mathbb{R}^d / \sim_k$ with the quotient topology. When needed, we may also write $\Gamma_k(\omega)$ to emphasise that the approximant Γ_k is an AP-complex of Ω_ω.

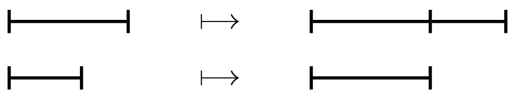
When the prototiles are (topologically) d-dimensional disks, then Γ_k is a d-dimensional CW complex. Equivalently, if the substitution ω yields a tiling of $\mathbb{R}^d$, then Γ_k is d-dimensional as well. In particular, if $d = 1$, then Γ_k is a strongly connected graph and in general, Γ_k is a compact Hausdorff space [**AP98**, Prop. 4.1].

Remark 2.7 (Forgetful map)**.** Let us first discuss a rather simple and intuitive explanation by Gähler and Sadun about how we can view a substitution tiling space Ω as an inverse limit of AP-complexes [**Sa08**, Ch. 2.4]. Proposition 2.11 below formalises this.

The approximants Υ_n in (2.2) may now be defined via the AP-complexes. Let Υ_0 be Γ_k, Υ_1 be the inflated version of Γ_k, i.e., $\Upsilon_1 := \hat{\omega}(\Gamma_k)$ (by an abuse of notation), and in general, let $\Upsilon_n := \hat{\omega}^n(\Gamma_k)$. The approximant Υ_0 tells us how to place a tile around the origin and Υ_1 tells us how to place a 1-collaring of tiles around the origin in such a way that is consistent with the instruction given by Υ_0 (i.e., a 1-collared partial tiling of the tile placed around the origin). In general, Υ_n is a recipe of placing n-collaring of tiles around the tile at the origin that is consistent with that of Υ_{n-1}. There is a *forgetful map* $f_n : \Upsilon_{n+1} \longrightarrow \Upsilon_n$ that restricts attention to the n-collaring around the origin while forgetting about the rest of $(n+1)$-collaring around the origin [**Sa08**, Ch. 2.5]. A point in the inverse limit space is an instruction on how the tiles are laid starting from the origin moving out and eventually covering the whole of $\mathbb{R}^d$, and so defines a tiling. Thus, the set of all points in the inverse limit space is in 1-to-1 correspondence with all the tilings in the hull. However, this simple yet elegant approach does not tell us how the cohomology of the tiling space is computed. $\diamondsuit$

Let us construct both Γ_0 and Γ_1 together with their inflated versions for the 1-dimensional Fibonacci substitution tiling space. Both Γ_0 and Γ_1 are 1-dimensional CW complexes, i.e., they are strongly connected graphs.

Example 2.8 (The Fibonacci substitution tiling space)**.** The 1-dimensional Fibonacci substitution tiling space may be defined via the following inflation map on two prototiles (which are closed intervals in $\mathbb{R}$).



The long prototile has length $\tau = (1+\sqrt{5})/2$ and the short one has length 1, and may be regarded as the closed intervals $[0,\tau]$ and $[0,1]$ respectively. More so, the associated inflation factor $\lambda = \tau$ is the golden ratio. After a substitution, the length of the inflated long prototile is $\tau \cdot \tau = \tau + 1$, which can be consistently divided into a long prototile followed by a short prototile. The inflated short prototile has length $1 \cdot \tau = \tau$, and so may not be further divided. Thus, the rule simply substitutes a long prototile for a short one under each iteration. Starting with any tile, such that the origin is placed anywhere but at the endpoints of the tile, allows us to eventually cover $\mathbb{R}$ after a never ending application of the substitution. A patch of one such Fibonacci tiling of $\mathbb{R}$ (but scaled down) is given in Figure 2.3.

As one may expect, a more compact way of writing this substitution is by assigning a letter to the prototiles and considering the symbolic counterpart of the substitution. Denoting the long and short prototiles by a and b respectively,

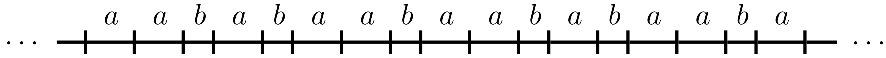


FIGURE 2.3. A Fibonacci tiling of $\mathbb{R}$. Above the tiles are their symbolic representation, where a long tile is labelled a and a short tile b.

gives us precisely the symbolic substitution $\varrho_F = \{a \longmapsto ab,\, b \longmapsto a\}$ defined in (1.4). Let us now construct the AP-complexes that serve as the approximants to the Fibonacci substitution tiling space.

For the approximant Γ_0, the prototiles are taken as the edges (i.e., the interiors of the prototiles are the 1-cells) and their boundaries, which are the vertices (the 0-cells), are identified depending on how the substitution allows the prototiles to be adjacent. As such, Γ_0 has exactly two edges with the long one labelled a and the short one labelled b. Table 2.1 gives a summary of the cell structure of Γ_0. Since a long tile a may be followed by a short tile b and a short tile b may be followed by a long tile a, then all the endpoints of the edges in Γ_0 are identified to a single vertex, as shown in Figure 2.4 (A). The inflated versions of Γ_0 are also illustrated in Figure 2.4 (B and C). The action of the forgetful map defined in Remark 2.7 is as follows. The map f_0 sends the upper loop in Υ_1 to the lower loop in Υ_0; and the lower loop in Υ_1 to the lower loop then followed by the upper loop in Υ_0. Similarly, f_1 sends the upper loop in Υ_2 to the lower loop in Υ_1; and the lower loop in Υ_2 to the lower loop followed by the upper loop in Υ_1.

In most cases, Γ_0 is not a 'good' approximant in the sense that the inverse limit is not homeomorphic to the actual tiling space, and so we need a better one, given by Γ_1. Proposition 2.11 below guarantees that Γ_1 is sufficient. Constructing the approximant Γ_1 means taking all the possible 1-collarings of the prototiles, which naturally makes Γ_1 a more complicated complex than Γ_0. Working with symbols, this means that the 1-cells consist of all the legal 3-letter words (allowed by ϱ_F); and their boundaries, all the legal 2-letter words, constitute the 0-cells. (This simplified construction of the approximant is due to [**GM13**].) As we already saw in Example 1.26, we have $\{aab, aba, baa, bab\}$ and $\{aa, ab, ba\}$ as the legal 3- and 2-letter words, respectively. See Table 2.1 for the cell structure of Γ_1. In Γ_1, the head of the edge $x := x_1x_2x_3$ is identified to the tail of the edge $y := y_1y_2y_3$ if and only if $x_2x_3 = y_1y_2$. The 1-cells of Γ_1 are attached to the 0-cells accordingly, as shown in Figure 2.5 (A). The inflated version of Γ_1 is also shown in Figure 2.5 (B). The forgetful map f_0 acts on the complexes analogously as in the case of Γ_0 above. $\diamondsuit$

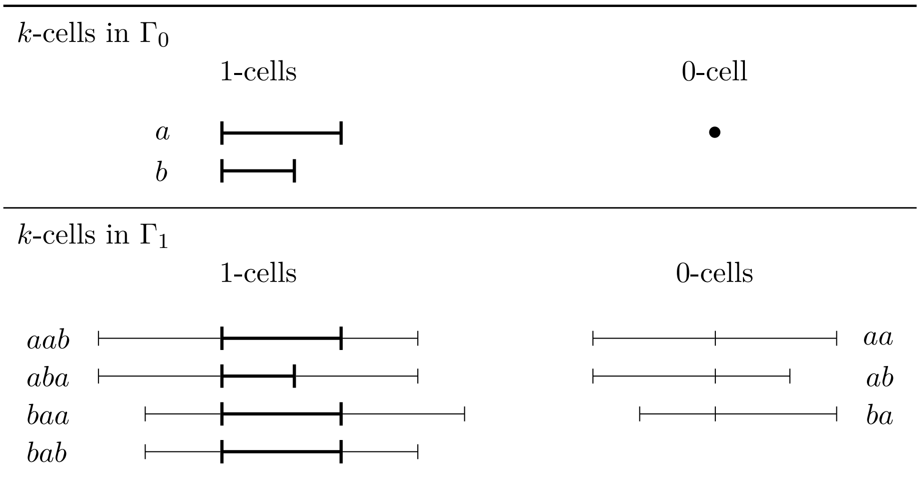
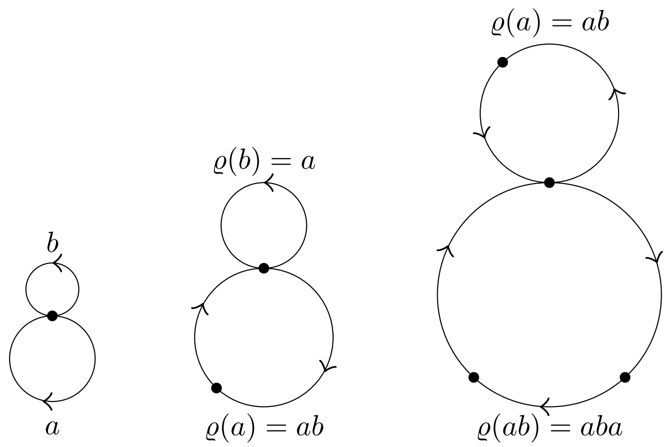


TABLE 2.1. The 0- and 1-cells in the AP-complexes Γ_0 and Γ_1 of the Fibonacci substitution tiling space. Note that the 1-cells of Γ_1 come precisely from all the distinct 1-collared partial tilings of the two prototiles of the substitution.



(A) (B) (C)

FIGURE 2.4. Γ_0 and its inflated versions (under γ_0) as approximants (A) Υ_0, (B) Υ_1 and (C) Υ_2 to the Fibonacci substitution tiling space.

Remark 2.9 (Solenoids via substitution)**.** A p-adic solenoid $\mathbb{S}_p$ ($p > 1$) may also be defined via substitution. The substitution rule that divides an interval into p equal parts and then inflating the subintervals p times as large defines a substitution system whose inverse limit is $\mathbb{S}_p$. This inflation map can be

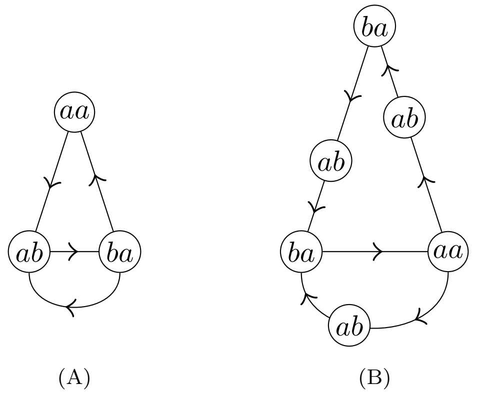


FIGURE 2.5. Γ_1 and its inflated version as approximants to the Fibonacci substitution tiling space. An edge between vertices labelled xy and yz reads xyz, e.g., the edge between ab and ba reads aba.

represented by the symbolic substitution $\{s \longmapsto s^p\}$. The AP-complex of this substitution turns out to be a 1-dimensional torus, i.e., $\Gamma_0 = \Gamma_1 = S^1$, consistent with what we have already seen earlier in Example 1.22. For any $k \in \mathbb{N} - \{1\}$, the substitution $\{s \longmapsto s^k\}$ defines the solenoid $\mathbb{S}_k$ as its inverse limit. This even generalises to higher dimensional solenoids through block substitution. For instance, the 2-dimensional 3-adic solenoid $\mathbb{S}_3 \times \mathbb{S}_3$ may be obtained as the inverse limit of the substitution $\left\{s \longmapsto \begin{smallmatrix} s & s & s \\ s & s & s \\ s & s & s \end{smallmatrix}\right\}$.

Note however that solenoids are not tiling spaces as the tilings obtained from these substitutions are all periodic, and so do not satisfy the assumption on w (the inflation map) being one-to-one. ♢

We now formalise the discussion on how a substitution tiling space can be written as the inverse limit space of the AP-complexes. First, note that the substitution rule induces a continuous surjection acting on the approximants under which the inverse limit is taken to obtain the actual tiling space (up to topological conjugacy).

Proposition 2.10 ([**AP98**, Prop. 4.2])**.** *The substitution rule ω induces a continuous map $\gamma_k : \Gamma_k \longrightarrow \Gamma_k$ defined by $\gamma_k((T,u)_k) = (\omega(T), \lambda u)_k$, where λ is the inflation factor of ω.* □

We then proceed to define the inverse limit of the AP-complexes Γ_k relative to the bonding map γ_k by ${}_k\Omega := \varprojlim(\Gamma_k, \gamma_k)$. By definition in Equation (2.2), the inverse limit space ${}_k\Omega$ consists of all infinite sequences $x := (x_i)_{\mathbb{N}_0}$ of points $x_i \in \Upsilon_i = \Gamma_k$ such that $\gamma_k(x_i) = x_{i-1}$ for $i \in \mathbb{N}$. A basis for its topology is all such cylinder sets

$$B_U^{{}_k\Omega}(n) := \{x \in {}_k\Omega \mid x_i \in \gamma_k^{n-i}(U) \text{ for } i \in I_n\}, \tag{2.3}$$

where $I_n := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$ and $U \subseteq \Gamma_k$ is an open set. We define the *right shift* $\omega_k : {}_k\Omega \longrightarrow {}_k\Omega$ by $\omega_k(x)_i := \gamma_k(x_i) = x_{i-1}$ for $i \in \mathbb{N}_0$ and $x = (x_0, x_1, \ldots)$. Naturally, there is the *left shift*, which is the inverse of ω_k given by $\omega_k^{-1}(x)_i := x_{i+1}$ for $i \in \mathbb{N}_0$. It then follows that ω_k is a homeomorphism and so generates a $\mathbb{Z}$-action on ${}_k\Omega$. We then have a dynamical system $({}_k\Omega, \omega_k)$, which is shown to be topologically conjugate to (Ω, ω) for $k = 1$, and even for $k = 0$ when the substitution forces the border.

Proposition 2.11 ([**AP98**, Thm. 4.3])**.** *The dynamical systems* (Ω, ω) *and* $({}_1\Omega, \omega_1)$ *are topologically conjugate. If the substitution forces the border then* (Ω, ω) *and* $({}_0\Omega, \omega_0)$ *are also topologically conjugate.* □

This proposition essentially says that up to topological conjugacy, the tiling space is the inverse limit of the AP-complexes. It is true that for all $k \geq 1$, $({}_k\Omega, \omega_k)$ and (Ω, ω) are topologically conjugate systems. However, the higher the value of k is, the bigger and more complicated the approximants Γ_k become; see Remark 3.4 for instance. There is no added benefit in considering more complicated approximants, and so we only need to consider $k \leq 1$. Suppose that (Ω', ω') is an induced substitution system, where every 1-collared partial tiling in Ω is considered a prototile in Ω'. This means, that w' is the induced substitution on Ω' acting by applying the same substitution rule ω on each component of a prototile in Ω', which is a 1-collared partial tiling in Ω. Then a restatement of Proposition 2.11 is as follows: The dynamical systems $(\Omega, \omega), ({}_1\Omega, \omega_1), ({}_0\Omega', \omega_0')$ and (Ω', ω') are all topologically conjugate, as the induced substitution rule on the 1-collared partial tilings always forces the border [**Sa08**, Ch. 2.5]. In most cases, a substitution rule does not force the border, and so we cannot use the simplest approximant given by Γ_0. Throughout the text, we always use Γ_1 whenever we need an AP-complex, without any more verifying whether the substitution forces the border or not.

Remark 2.12 (The induced action of γ_k)**.** The action of γ_k on Γ_k induces an action on the d-cells in Γ_k, whose matrix representation we denote by A_d', where we adopt the notation used in [**AP98**]. Suppose we number the d-cells in Γ_k as $1, 2, \ldots, m$. Then, we define the matrix $A_d := (a_{ij})$ by $a_{ij} := \text{card}_i(\hat{\omega}(j))$ for $i, j \in \{1, 2, \ldots, m\}$. The number a_{ij} counts the number of times i occurs in the inflation of j. The transpose of the matrix A_d, denoted by A_d', represents the induced action of γ_k on the d-cells in Γ_k. ♢

2.4. The Čech cohomology of tiling spaces

Let ${}_k\Omega = \varprojlim \Gamma_k$ be the inverse limit of the AP-complexes Γ_k relative to the bonding map γ_k for $k \in \{0, 1\}$ as defined in the previous section. Thus,

$$ {}_k\Omega \;=\; \Gamma_k \xleftarrow{\gamma_k} \Gamma_k \xleftarrow{\gamma_k} \Gamma_k \xleftarrow{\gamma_k} \Gamma_k \xleftarrow{\gamma_k} \cdots \tag{2.4}$$

and so the Čech cohomology of ${}_k\Omega$ reads

$$ \check{H}^d({}_k\Omega) = \check{H}^d(\varprojlim({}_k\Omega, \gamma_k)) \cong \varinjlim(\check{H}^d(\Gamma_k), \gamma_d^*) = \varinjlim H^d(\Gamma_k), \tag{2.5}$$

where the second equality follows from Proposition 1.23 and the last equality follows from Proposition 1.20, and γ_d^* is the induced inflation map acting on the cohomologies $\check{H}^d(\Gamma_k)$. When emphasis for the inflation ω is needed, we may also write $\gamma_{\omega,d}^*$ for γ_d^*, unless the context is already clear.

Proposition 2.13 ([**AP98**, Thm. 6.1])**.** *Let Ω be the tiling space associated to the inflation map ω. Further let the AP-complexes Γ_k be the approximants to the inverse limit space ${}_k\Omega$. Then the Čech cohomology of Ω isomorphic to the Čech cohomology of ${}_k\Omega$ for $k = 1$, i.e.,*

$$ \check{H}^*(\Omega) \cong \check{H}^*({}_1\Omega) = \varinjlim H^*(\Gamma_1). $$

If the substitution forces the border, the same is true for $k = 0$. □

Unlike other types of tiling spaces that can also be expressed as inverse limit spaces (see Remark 2.6), substitution tiling spaces are very special in the sense that their cohomology can be computed rather easily. This is due to the fact that all the approximants in Equation (2.4) have isomorphic cohomology groups, and the induced maps γ_d^* between each of the cohomologies are all the same, which makes Equation (2.5) manageable.

It should be clear that whenever we talk about the cohomology of substitution tiling spaces, we really mean the Čech cohomology and so we may just drop the term Čech. We may also simply write $H^*(\Omega) := \check{H}^*(\Omega)$ as the context is clear. We now demonstrate the method of computing the cohomology of a substitution tiling space through the following examples.

Example 2.14 (Cohomology of Fibonacci substitution)**.** The Fibonacci substitution $\varrho_F = \{a \longmapsto ab,\ b \longmapsto a\}$ does not force the border, and so we cannot use the approximant Γ_0 to correctly compute the cohomology of the Fibonacci substitution tiling space, which we denote by Ω_F (or $\mathbb{Y}_F$). Instead, we consider $\Gamma := \Gamma_1(\varrho_F)$ and its inflated version as illustrated in Figure 2.5. The cochain groups $C^0(\Gamma) = \langle aa', ab', ba' \rangle \cong \mathbb{Z}^3$ and $C^1(\Gamma) = \langle aab', aba', baa', bab' \rangle \cong \mathbb{Z}^4$ are related through the coboundary map $\delta : C^0(\Gamma) \longrightarrow C^1(\Gamma)$, which can be read off directly of Figure 2.5 (A) as

$$\delta = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},$$

and so we have the sequence

$$0 \longrightarrow C^0(\Gamma) \xrightarrow{\delta} C^1(\Gamma) \longrightarrow 0.$$

Thus, $H^0(\Gamma) = \ker \delta = \langle aa' + ab' + ba' \rangle \cong \mathbb{Z}$ and $H^1(\Gamma) = C^1(\Gamma)/\operatorname{im}\delta = \langle aab', aba' \rangle \cong \mathbb{Z}^2$ with $baa' = aab'$ and $bab' = aba' - aab'$ in $H^1(\Gamma)$. The cohomology of Ω_F is computed as the direct limit of the cohomologies of the approximants relative to the induced homomorphism on the cohomologies, i.e.,

$$H^d(\Omega_F) = \varinjlim(H^d(\Gamma), \gamma_d^*),$$

where γ_d^* is the induced inflation map on the cohomologies. To know γ_d^* for $d \in \{0, 1\}$, we first determine the action of γ_d on the cells of Γ (recall Remark 2.12), whose matrix representations we write as A_0' and A_1' respectively. Reading off of Figure 2.5 (B), we get

$$A_0' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_1' = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

We want to know what happens to the generators of the cohomologies under these maps. Since $A_0'(aa' + ab' + ba') = aa' + ab' + ba'$, γ_0^* is just the identity map. Further, $A_1'(aab') = aba'$ and $A_1'(aba') = aab' + aba'$ imply that γ_1^* is a linear map given by

$$A_1^* := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, we compute the cohomology of the Fibonacci tiling space as $H^0(\Omega_F) = \varinjlim(\mathbb{Z}, \gamma_0^*) \cong \mathbb{Z}$ and $H^1(\Omega_F) = \varinjlim(\mathbb{Z}^2, \gamma_1^*) \cong \mathbb{Z}^2$ as in Example 1.2. ♢

The next two examples are substitutions that also do not force the border, hence we cannot work with Γ_0.

Example 2.15 (Cohomology of Thue-Morse substitution)**.** The substitution on two letters given by

$$\varrho_{TM} := \begin{cases} 1 \longmapsto 1\bar{1} \\ \bar{1} \longmapsto \bar{1}1 \end{cases} \tag{2.6}$$

defines the hull of the Thue-Morse substitution, denoted by $\mathbb{Y}^{TM}$ (compare [**BG13a**, Ch. 4.6]). To determine the cohomology of $\mathbb{Y}^{TM}$, we first need to set up the approximant $\Gamma := \Gamma_1(\varrho_{TM})$. The AP-complex Γ is a 1-dimensional complex whose 0-cells consist of the legal 2-letter subwords, namely,

$$11, 1\bar{1}, \bar{1}1, \bar{1}\bar{1},$$

and whose 1-cells consist of the legal 3-letter subwords,

$$11\bar{1}, 1\bar{1}1, 1\bar{1}\bar{1}, \bar{1}11, \bar{1}1\bar{1}, \bar{1}\bar{1}1.$$

Note that the subwords 111 and $\bar{1}\bar{1}\bar{1}$ do not appear anywhere in any bi-infinite word in $\mathbb{Y}^{TM}$ as evident from ϱ_{TM}. Attaching the 1-cells accordingly gives us the approximant Γ, shown in Figure 2.6 (A). The cohomology of Γ is computed via its 0- and 1-cochain groups, i.e.,

$$C^0(\Gamma) = \langle 11', 1\bar{1}', \bar{1}1', \bar{1}\bar{1}' \rangle,$$
$$C^1(\Gamma) = \langle 11\bar{1}', 1\bar{1}1', 1\bar{1}\bar{1}', \bar{1}11', \bar{1}1\bar{1}', \bar{1}\bar{1}1' \rangle.$$

The coboundary operator $\delta : C^0(\Gamma) \longrightarrow C^1(\Gamma)$, is given by

$$\delta = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

By definition, we obtain

$$H^0(\Gamma) = \ker \delta = \langle 11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}' \rangle \cong \mathbb{Z},$$
$$H^1(\Gamma) = C^1(\Gamma)/\operatorname{im} \delta = \langle 11\bar{1}', 1\bar{1}1', 1\bar{1}\bar{1}' \rangle \cong \mathbb{Z}^3,$$

where $\bar{1}11' = 11\bar{1}'$, $\bar{1}1\bar{1}' = -11\bar{1}' + 1\bar{1}1' + 1\bar{1}\bar{1}'$ and $\bar{1}\bar{1}1' = 1\bar{1}\bar{1}'$ in $H^1(\Gamma)$.

Figure 2.6 (B) shows the inflated version of Γ, where the inflation maps A'_0 and A'_1 on the cochains may be read off as

$$A'_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A'_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The induced maps γ_d^* for $d \in \{0, 1\}$ are deduced as follows. For γ_0^*, since $A'_0(11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}') = 11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}'$, γ_0^* is simply the identity map

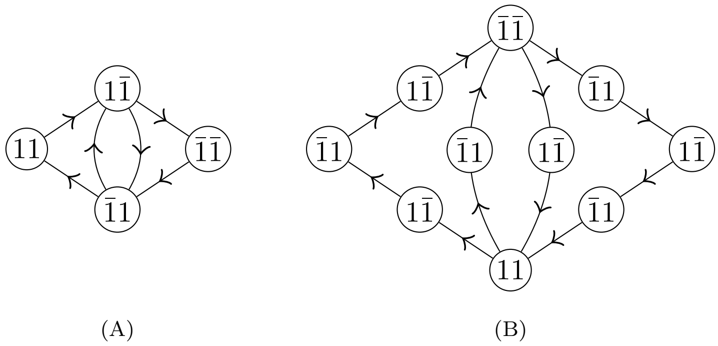


FIGURE 2.6. The AP approximants to the Thue-Morse substitution tiling space.

on $H^0(\Gamma)$. Therefore,

$$\begin{aligned} H^0(\mathbb{Y}^{TM}) &\cong H^0(\Gamma) \xrightarrow{\gamma_0^*} H^0(\Gamma) \xrightarrow{\gamma_0^*} H^0(\Gamma) \xrightarrow{\gamma_0^*} \cdots \\ &\cong \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 1} \cdots \\ &= \mathbb{Z} \cong \langle 11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}' \rangle. \end{aligned}$$

Similarly for γ_1^*, we have

$$\begin{aligned} A_1'(11\bar{1}') &= 11\bar{1}' + \bar{1}1\bar{1}' = 0 \cdot 11\bar{1}' + 1\bar{1}1' + 1\bar{1}\bar{1}', \\ A_1'(1\bar{1}1') &= \bar{1}11' + \bar{1}\bar{1}1' = 11\bar{1}' + 0 \cdot 1\bar{1}1' + 1\bar{1}\bar{1}', \\ A_1'(1\bar{1}\bar{1}') &= 11\bar{1}' + \bar{1}1\bar{1}' = 0 \cdot 11\bar{1}' + 1\bar{1}1' + 1\bar{1}\bar{1}', \end{aligned}$$

which imply that

$$A_1^* = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where A_1^* is the matrix representation of γ_1^*, which is diagonalisable with eigenvalues 2, -1 and 0. Similar to Examples 1.6 and 1.7, we compute the cohomology as

$$\begin{aligned} H^1(\mathbb{Y}^{TM}) &\cong H^1(\Gamma) \xrightarrow{\gamma_1^*} H^1(\Gamma) \xrightarrow{\gamma_1^*} H^1(\Gamma) \xrightarrow{\gamma_1^*} \cdots \\ &\cong \mathbb{Z}^3 \xrightarrow{\times(2,-1,0)} \mathbb{Z}^3 \xrightarrow{\times(2,-1,0)} \mathbb{Z}^3 \xrightarrow{\times(2,-1,0)} \cdots \\ &= \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[-1] \oplus \mathbb{Z}[0] = \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}. \end{aligned}$$

♢

Example 2.16 (Cohomology of period doubling substitution)**.** Related to the Thue-Morse substitution is the period doubling substitution [**BG13a**, Ch.

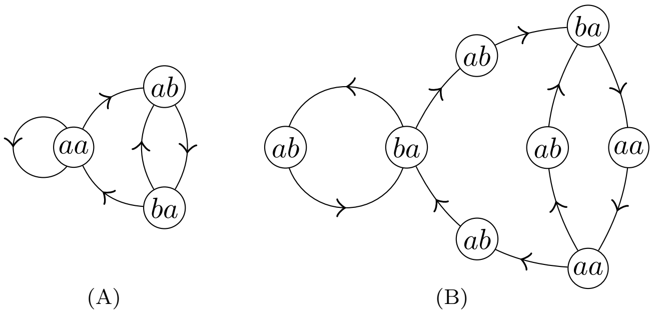


FIGURE 2.7. The AP approximants to the period doubling substitution tiling space.

4.5] whose hull $\mathbb{Y}^{pd}$ is derived from the substitution rule

$$\varrho_{pd} := \begin{cases} a \longmapsto ab \\ b \longmapsto aa \end{cases} \tag{2.7}$$

The hull $\mathbb{Y}^{TM}$ is a uniformly 2-to-1 cover of the hull $\mathbb{Y}^{pd}$ (see Remark 3.4). The cochain groups of $\Gamma := \Gamma_1(\varrho_{pd})$ are given by $C^0(\Gamma) = \langle aa', ab', ba' \rangle \cong \mathbb{Z}^3$ and $C^1(\Gamma) = \langle aaa', aab', aba', baa', bab' \rangle \cong \mathbb{Z}^5$. The approximant Γ and its inflated version are shown in Figure 2.7. The coboundary map $\delta : C^0(\Gamma) \longrightarrow C^1(\Gamma)$ reads

$$\delta = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},$$

and so we get the cohomologies $H^0(\Gamma) = \ker \delta = \langle aa' + ab' + ba' \rangle \cong \mathbb{Z}$ and $H^1(\Gamma) = C^1(\Gamma)/\operatorname{im}\delta = \langle aaa', aab', aba' \rangle \cong \mathbb{Z}^3$ with $baa' = aab'$ and $bab' = aba' - aab'$ in $H^1(\Gamma)$. The induced inflation maps on the cells read

$$A'_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A'_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

One computes γ_0^* to be the identity map, and γ_1^* to be given by the linear map

$$A_1^* := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The last matrix is diagonalisable and has eigenvalues $2, -1$ and 0, and we compute the cohomology of the period doubling tiling space as $H^0(\mathbb{Y}^{pd}) \cong \mathbb{Z}$ and $H^1(\mathbb{Y}^{pd}) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$. ♢

Let us end this section with the following observation.

Lemma 2.17. *Suppose* Ω *is a substitution tiling space. Then* $\check{H}^0(\Omega) = \langle \sum e_i' \rangle \cong \mathbb{Z}$*, where the sum runs over all 0-cells of the AP-complex of* Ω *and* e_i' *is the dual to the 0-cell* e_i*.*

Proof. Let $\Gamma := \Gamma_1$ be an AP-complex associated to the tiling space Ω (or $\Gamma := \Gamma_0$ if the substitution forces the border). The 1-skeleton Γ^1, consisting of all 1-cells attached accordingly at their boundaries, is a strongly connected graph, which means that there is a (directed) path from each 0-cell to every other 0-cell in the graph. This is due to the primitivity of the substitution. As such, the kernel of the coboundary map $\delta : C^0(\Gamma) \longrightarrow C^1(\Gamma)$ is always $\sum_i e_i'$, and so $H^0(\Gamma) = \langle \sum_i e_i' \rangle$, where the sum runs over all 0-cells in Γ. Finally, it is not difficult to see that γ_0^* is always the identity map since $\gamma_0^*(\sum_i e_i') = \sum_i e_i'$ as the pre-image of a 0-cell is totally distinct from any other pre-images of the other 0-cells, and so the pre-image of the entire sum is the entire sum itself. Getting the direct limit gives us the result we need. □

Remark 2.18. The previous lemma is rather expected because the Čech cohomology measures the number of connected components and any substitution tiling space Ω is connected. Thus, $\check{H}^0(\Omega) \cong \mathbb{Z}$ [**Sa08**, Ch. 3]. ♢

2.5. Dynamical zeta functions

The (Artin-Mazur) dynamical zeta function $\zeta(z)$ of a substitution tiling system (Ω, ω), is a generating function that encodes the number of points in the tiling space that are invariant under an m-fold substitution, for some $m \in \mathbb{N}$. By definition, we have

$$\zeta_\omega(z) := \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} z^m\right), \tag{2.8}$$

where N_m is the number of fixed points of ω^m in $\Omega = \Omega_\omega$ [**Rue94, AP98**].

Example 2.19. Let us consider the solenoid $\mathbb{S}_2 = \varprojlim(S^1, f)$ as in Example 1.22 and compute its dynamical zeta function using the definition. Recall that a point in $\mathbb{S}_2$ may be represented as a sequence $x := (x_0, x_1, x_2, \ldots)$ such that $2x_i = x_{i-1}$ for $i \in \mathbb{N}$ and $x_i \in S^1$. A point $x \in \mathbb{S}_2$ is invariant under f^m for $m \in \mathbb{N}$ if $x = f^m(x)$, i.e., $x_i = 2^m x_i$ (modulo 1) for all $i \in \mathbb{N}_0$. The values of N_m and the respective fixed points for $m \in \{1, 2, 3\}$ are listed in Table 2.2. In

m	N_m	Fixed point(s)	m	N_m	Fixed point(s)	
1	1	$(0,0,\ldots)$	3	7	$(0,0,0,0,\ldots)$	
2	3	$(0,0,0,\ldots)$			$(\frac{1}{7},\frac{4}{7},\frac{2}{7},\frac{1}{7},\ldots)$	$(\frac{4}{7},\frac{2}{7},\frac{1}{7},\frac{4}{7},\ldots)$
		$(\frac{1}{3},\frac{2}{3},\frac{1}{3},\ldots)$			$(\frac{2}{7},\frac{1}{7},\frac{4}{7},\frac{2}{7},\ldots)$	$(\frac{5}{7},\frac{6}{7},\frac{3}{7},\frac{5}{7},\ldots)$
		$(\frac{2}{3},\frac{1}{3},\frac{2}{3},\ldots)$			$(\frac{3}{7},\frac{5}{7},\frac{6}{7},\frac{3}{7},\ldots)$	$(\frac{6}{7},\frac{3}{7},\frac{5}{7},\frac{6}{7},\ldots)$

TABLE 2.2. The fixed points of the dyadic solenoid $\mathbb{S}_2$ for $m \in \{1,2,3\}$. In general, $N_m = 2^m - 1$.

general, $N_m = 2^m - 1$ and so we compute the zeta function of $\mathbb{S}_2$ as

$$\zeta_{\mathbb{S}_2}(z) = \exp\Big(z + \tfrac{3}{2}z^2 + \tfrac{7}{3}z^3 + \tfrac{15}{4}z^4 + \tfrac{31}{5}z^5 + \cdots\Big) = \frac{1-z}{1-2z}$$

where the last equality is a consequence of Proposition 2.21 below. ♢

Example 2.20. Let us now see how we can compute the zeta function for the Fibonacci substitution tiling space. As the substitution system (Ω_F, ϱ_F) is topologically conjugate to $({}_1\Omega, \omega_1)$, where ${}_1\Omega := \varprojlim(\Gamma_1(\varrho_F), \omega_1)$ with ω_1 being the right shift map (i.e., $\omega_1(x)_i = x_{i-1}$), the fixed points of ϱ_F^m in Ω_F are in 1-to-1 correspondence with the sequences $x := (x_0, x_1, x_2, \ldots) \in {}_1\Omega$ such that $x_i = x_{i+m}$ for $x_i \in \Upsilon_i = \Gamma_1(\varrho_F)$ (i.e., the fixed points of ω_1^m in ${}_1\Omega$). Necessarily, $\gamma_1(x_i) = x_{i-1}$ and recall that $\gamma_1(T,u)_1 = (\varrho_F(T), \tau u)_1$, where $(T,u)_1 \in \Gamma_1(\varrho_F)$ and $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. It can be computed that $N_1 = 0$, $N_2 = 2$, $N_3 = 3$, and in general, $N_m = N_{m-1} + N_{m-2} + 1$.

In particular, we get the following fixed points of ${}_1\Omega$ (corresponding to the fixed points of Ω_F). First, recall in Example 1.26 that ϱ_F has no fixed points, then neither does ω_1 in ${}_1\Omega$. Now, for $m = 2$, the two bi-infinite fixed points of the substitution ϱ_F^2 are given by $w := w^{(a|a)}$ and $v := v^{(b|a)}$, see (1.6) and (1.7). A corresponding fixed point in ${}_1\Omega$ is given by $x := (x_i)_{\mathbb{N}_0}$, where $x_i = (w, 0)$ if i is even, and $x_i = (v, 0)$ if i is odd, i.e., $x := ((w,0),(v,0),(w,0),\ldots)$. The second fixed point is $y := \omega_1(x)$, i.e., $y := (y_i)_{\mathbb{N}_0}$ with $y_i = (w, 0)$ if i is odd, and $y_i = (v, 0)$ otherwise. Thus, $y = ((v,0),(w,0),(v,0),\ldots)$. For $m = 3$, one fixed point is $x := ((v, 1+2\tau),(w, 1+\tau),(v,\tau),(w,1),\ldots)$. It is an easy check that $(w, 1) \sim (v, 1+2\tau)$, and so $x_0 = x_3$, which is a necessary condition for a fixed point of ω_1^3 in ${}_1\Omega$. The other two fixed points are the ones that begin with $(w, 1+\tau)$ and (v,τ) respectively, i.e., $\omega_1(x) = ((w,1+\tau),(v,\tau),(w,1),(v,\tau-1),\ldots)$ and $\omega_1^2(x) = ((v,\tau),(w,1),(v,\tau-1),(w,2-\tau),\ldots)$. Thus, the dynamical zeta function of the Fibonacci substitution is given by

$$\begin{aligned}\zeta_F(z) &= \exp\left(0z + \tfrac{2}{2}z^2 + \tfrac{3}{3}z^3 + \tfrac{6}{4}z^4 + \tfrac{10}{5}z^5 + \tfrac{17}{6}z^6 + \cdots\right)\\ &= 1 + z^2 + z^3 + 2z^4 + 3z^5 + 5z^6 + 8z^7 + \cdots\\ &= 1 + \sum_{n=1}^{\infty} F_{n-1}z^n\end{aligned}$$

where F_n is the nth Fibonacci number [**OEIS**, `A000045`]. A more compact way of writing this zeta function is seen below. ♢

Proposition 2.21 ([**AP98**, Thm. 9.1])**.** *Let (Ω, ω) be a substitution tiling system as above. Further let A'_d be the induced cellular (or inflation) maps acting on the d-cells of an n-dimensional AP-complex Γ_1 (or Γ_0 when ω forces the border) associated to the substitution ω. Then the zeta function is given by*

$$\zeta_\omega(z) = \frac{\prod_{k\ odd} \det(I - zA'_{n-k})}{\prod_{k\ even} \det(I - zA'_{n-k})} = \frac{\prod_{k\ odd}\prod_{i}(1 - z\lambda_{n-k,i})}{\prod_{k\ even}\prod_{i}(1 - z\lambda_{n-k,i})},$$

where the second equality holds if all A'_m are diagonalisable, with eigenvalues $\lambda_{m,i}$. □

Note that instead of using A'_k, one can also get the same zeta function by using the matrix representation of γ_k^* acting on the cohomology groups, but with rational coefficients. (Recall in Definition 1.13 that this can be done by letting $R = \mathbb{Q}$; also see Example 3.17.) If the cohomology groups (with integer coefficients) do not contain any torsion component, then we can replace A'_k with A_k^*, the matrix representation of γ_k^* acting on H^* (with integer coefficients as before).

Example 2.22. The computation of the following zeta functions is straightforward. The first equality uses the eigenvalues of A'_d, while the second uses the eigenvalues of A_d^* as none of the cohomology groups considered possesses any torsion component.

$$\begin{aligned}\zeta_F(z) &= \frac{(1-z)(1+z)(1-0z)}{(1+z)(1-0z)(1-\tau z)(1+\frac{1}{\tau}z)} = \frac{1-z}{(1-\tau z)(1+\frac{1}{\tau}z)} = \frac{1-z}{1-z-z^2}\\ \zeta_{TM}(z) &= \frac{(1+z)^2(1-z)^2}{(1-2z)(1+z)^3(1-z)(1-0z)} = \frac{1-z}{(1-2z)(1+z)}\\ \zeta_{pd}(z) &= \frac{(1-z)(1+z)(1-0z)}{(1-2z)(1+z)^2(1-0z)^2} = \frac{1-z}{(1-2z)(1+z)} = \zeta_{TM}(z)\end{aligned}$$

♢

CHAPTER 3

Quotient cohomology between tiling spaces

Topological invariants can provide a systematic way of classifying topological spaces. Now that tiling cohomology is within reach, distinguishing substitution tiling spaces through their cohomology is a prudent next step. A number of substitution tiling spaces in one and two dimensions have already been examined and classified through their cohomology, see [**AP98, BGG13, Gäh13**] for instance. Recently, Gähler, Hunton and Maloney [**GHM13**] computed the cohomology of the 3-dimensional Danzer tiling, demonstrating a more efficient way of calculating cohomologies especially in higher dimensions.

Tiling spaces with non-isomorphic cohomology groups are necessarily inequivalent. The converse though is not true in general, as in the case of the (classical) Thue-Morse and period doubling substitutions which form two inequivalent tiling spaces yet with isomorphic cohomology groups. For tiling spaces related through a factor map (regarding the spaces as topological dynamical systems), a relative version of the tiling cohomology can be used to tell the spaces apart. Barge and Sadun [**BS11**] introduced the concept of *quotient cohomology*, which is a topological invariant that distinguishes factors of tiling spaces. As such, quotient cohomology can be used to dissect and analyse the structure of a substitution tiling space via its factors.

In this chapter, we review the notion of the quotient cohomology between substitution tiling spaces and offer some results that aid in the computation of the quotient cohomology. In an attempt to have a glimpse of what the quotient cohomology means and can offer, we introduce the notion of the *quotient zeta function*, whose definition is derived from that of the typical dynamical zeta function for substitution tiling spaces, inspired by Proposition 2.21. As we will see, for certain conditions, the quotient cohomology fails to distinguish related yet inequivalent spaces.

3.1. Definition and examples

Suppose X and Y are topological spaces that are related via a quotient map $f : X \longrightarrow Y$, i.e., f is surjective and continuous, and for any $U \subseteq Y$, U is open in Y if and only if $f^{-1}(U)$ is open in X. If f is injective, then the relative cohomology group $H^*(Y, X)$ relates the cohomology groups of X and

Y via the long exact sequence

$$\cdots \longrightarrow H^{k+1}(Y,X) \longrightarrow H^k(Y) \xrightarrow{*f} H^k(X) \longrightarrow H^k(Y,X) \longrightarrow \cdots$$

However, we cannot extend the same mechanism to tiling spaces. Substitution tiling spaces are minimal dynamical systems (Proposition 2.3), and factor maps between these spaces are surjective and generally not injective. A factor map $f : \Omega_X \longrightarrow \Omega_Y$ induces a quotient map on the level of approximants that is also surjective and generally not injective. Thus, the relative cohomology is not immediately available for tiling spaces, but can be used in proving some results [**BS11**].

Typically for substitution tiling spaces, the quotient map $f : \Gamma_k(X) \longrightarrow \Gamma_k(Y)$ (on the approximants of Ω_X and Ω_Y respectively) induces a *pullback* map $f^* : C^*(\Gamma_k(Y)) \longrightarrow C^*(\Gamma_k(X))$ that is injective on the cochain complexes of $\Gamma_k(Y)$ and $\Gamma_k(X)$. Denoting by e' the dual to the cell e, then in particular, a $y' \in C^*(\Gamma_k(Y))$ is said to pull back to $\sum x_i' \in C^*(\Gamma_k(X))$ if and only if $f(x_i) = y$ for all $i \in I$, for some index I such that outside I, we have $f(x) \neq y$. This motivates the following definition of the quotient cohomology [**BS11**].

Definition 3.1. Let $f : X \longrightarrow Y$ be a quotient map between two topological spaces such that the pullback map f^* is injective on cochains. Also, let $C_Q^k(X,Y) := C^k(X)/f^*(C^k(Y))$ be the quotient cochain groups and take $\delta_{XY,k} := \delta_k : C_Q^k(X,Y) \longrightarrow C_Q^{k+1}(X,Y)$ to be the usual coboundary operator. The (kth) *quotient cohomology* is defined as $H_Q^k(X,Y) := \ker \delta_k / \operatorname{im} \delta_{k-1}$. ◇

By the snake lemma, the short exact sequence of cochain complexes

$$0 \longrightarrow C^k(Y) \xrightarrow{f^*} C^k(X) \longrightarrow C_Q^k(X,Y) \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H_Q^{k-1}(X,Y) \longrightarrow H^k(Y) \xrightarrow{f_k^*} H^k(X) \longrightarrow H_Q^k(X,Y) \longrightarrow \cdots \quad (3.1)$$

that relates the cohomology groups of X and Y to $H_Q^*(X,Y)$.

Example 3.2 (Quotient cohomology between AP-complexes)**.** Let us consider the complexes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ in Figure 3.1 together with two quotient maps $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $g : \mathcal{X} \longrightarrow \mathcal{Z}$, such that

$$f(A) = f(B) = 1, \quad f(C) = f(D) = \bar{1},$$
$$g(B) = g(C) = a, \quad g(A) = g(D) = b.$$

Note that the 0- and 1-cells are encoded by two and three adjacent letters, such that the edge $v_1v_2v_3$ has the vertex v_1v_2 as its tail and the vertex v_2v_3 as its head. Since these quotient maps induce a homomorphism on the components

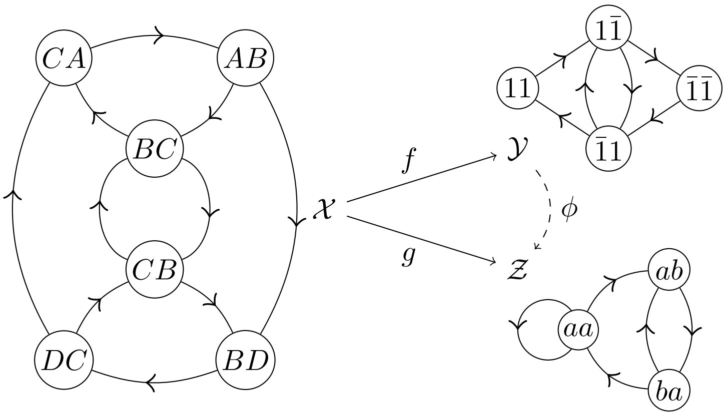


FIGURE 3.1. Quotient maps between AP-complexes. Here, the vertices are labelled such that the edge from the vertex v_1v_2 to the vertex v_2v_3 reads $v_1v_2v_3$.

of the complexes, we have in particular we have $f(AB) = 11$, $f(ABD) = 11\bar{1}$, $g(BCB) = aaa$, etc.

To compute the quotient cohomology groups $H_Q^*(\mathcal{X}, \mathcal{Y})$ between $\mathcal{X}$ and $\mathcal{Y}$, we first determine the quotient cochains $C_Q^*(\mathcal{X}, \mathcal{Y})$, and the coboundary maps between them, i.e., $\delta_{\mathcal{X}\mathcal{Y}}^k : C_Q^k(\mathcal{X}, \mathcal{Y}) \longrightarrow C_Q^{k+1}(\mathcal{X}, \mathcal{Y})$. Note that $f^*(1') = A' + B'$ and $f^*(\bar{1}') = C' + D'$, and so $f^*(1\bar{1}) = BC' + BD'$, $f^*(\bar{1}1) = CA' + CB'$, etc.

From the definition, we immediately see that $H_Q^k(\mathcal{X}, \mathcal{Y}) = 0$ for $k \notin \{0, 1\}$ since $C_Q^k(\mathcal{X}, \mathcal{Y}) = 0$ for $k \notin \{0, 1\}$. Now for $k \in \{0, 1\}$, we get

$$\begin{aligned} C_Q^0(\mathcal{X}, \mathcal{Y}) :=& \, C^0(\mathcal{X})/f^*(C^0(\mathcal{Y})) \\ =& \, \langle BC', BD', CA', CB', DC', AB' \rangle / \\ & \quad \langle BC' + BD', CA' + CB', DC', AB' \rangle \\ =& \, \langle BC', CA' \rangle \cong \mathbb{Z}^2, \end{aligned}$$

and

$$\begin{aligned} C_Q^1(\mathcal{X}, \mathcal{Y}) :=& \, C^1(\mathcal{X})/f^*(C^1(\mathcal{Y})) \\ =& \, \langle ABC', ABD', BCA', BCB', BDC', CAB', CBC', \\ & \quad CBD', DCA', DCB' \rangle / \langle ABC' + ABD', BCA' + BCB', \\ & \quad BDC', CAB', CBC' + CBD', DCA' + DCB' \rangle \\ =& \, \langle ABC', BCA', CBC', DCA' \rangle \cong \mathbb{Z}^4. \end{aligned} \tag{3.2}$$

The induced coboundary map $\delta_{\mathcal{X}\mathcal{Y}} := \delta_{\mathcal{X}\mathcal{Y},0}$ can be read off of Figure 3.1 as $\delta_{\mathcal{X}}$ (modulo f^*), where $\delta_{\mathcal{X}}$ is the usual coboundary operator in $\mathcal{X}$. Thus we get,

$$\delta_{\mathcal{X}\mathcal{Y}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and so

$$\begin{aligned} H^0_Q(\mathcal{X},\mathcal{Y}) &= \ker \delta_{\mathcal{X}\mathcal{Y}} \cong 0, \\ H^1_Q(\mathcal{X},\mathcal{Y}) &= C^1_Q(\mathcal{X},\mathcal{Y})/\operatorname{im}\delta_{\mathcal{X}\mathcal{Y}} \cong \mathbb{Z}^2, \end{aligned} \tag{3.3}$$

where $\operatorname{im}\delta_{\mathcal{X}\mathcal{Y}} = \langle ABC' + CBC', BCA' + DCA'\rangle$. The long exact sequence (3.1) for $\mathcal{X}$ and $\mathcal{Y}$ reads:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \mathbb{Z} & \xrightarrow{f_0^*} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^3 & \xrightarrow{f_1^*} & \mathbb{Z}^5 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow 0. \\ & \| & & \| & & \| & & \| & & \| & & \| & \\ & H^0(\mathcal{Y}) & & H^0(\mathcal{X}) & & H^0_Q(\mathcal{X},\mathcal{Y}) & & H^1(\mathcal{Y}) & & H^1(\mathcal{X}) & & H^1_Q(\mathcal{X},\mathcal{Y}) & \end{array}$$

Meanwhile, for the quotient map $g : \mathcal{X} \longrightarrow \mathcal{Z}$, $H^k_Q(\mathcal{X},\mathcal{Z}) \cong 0$ for $k \notin \{0,1\}$ and for $k \in \{0,1\}$, we obtain the following quotient cochains:

$$\begin{aligned} C^0_Q(\mathcal{X},\mathcal{Z}) &:= C^0(\mathcal{X})/g^*(C^0(\mathcal{Z})) \\ &= \langle BC', BD', CA', CB', DC', AB'\rangle / \\ &\qquad \langle BC' + CB', BD' + CA', DC' + AB'\rangle \\ &= \langle BC', BD', DC'\rangle \cong \mathbb{Z}^3, \end{aligned}$$

and

$$\begin{aligned} C^1_Q(\mathcal{X},\mathcal{Z}) &:= C^1(\mathcal{X})/g^*(C^1(\mathcal{Z})) \\ &= \langle ABC', ABD', BCA', BCB', BDC', CAB', CBC', \\ &\qquad CBD', DCA', DCB'\rangle / \langle ABC' + DCB', ABD' + DCA', \\ &\qquad\quad BCA' + CBD', BCB' + CBC', BDC' + CAB'\rangle \\ &= \langle ABC', ABD', BCA', BCB', BDC'\rangle \cong \mathbb{Z}^5. \end{aligned} \tag{3.4}$$

Also, the induced coboundary map $\delta_{\mathcal{X}\mathcal{Z}} := \delta_{\mathcal{X}\mathcal{Z},0} : C^0_Q(\mathcal{X},\mathcal{Z}) \longrightarrow C^1_Q(\mathcal{X},\mathcal{Z})$, which can be read off from Figure 3.1 as $\delta_{\mathcal{X}}$ (modulo g^*), is given by

$$\delta_{\mathcal{X}\mathcal{Z}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The quotient cohomology groups are computed as

$$\begin{aligned} H_Q^0(\mathcal{X},\mathcal{Z}) &= \ker\delta_{\mathcal{X}\mathcal{Z}} \cong 0, \\ H_Q^1(\mathcal{X},\mathcal{Z}) &= C_Q^1(\mathcal{X},\mathcal{Z})/\operatorname{im}\delta_{\mathcal{X}\mathcal{Z}} \cong \mathbb{Z}^2\oplus\mathbb{Z}_2. \end{aligned} \tag{3.5}$$

The torsion in the quotient cohomology comes in from one of the generators of $H_Q^1(\mathcal{X},\mathcal{Z})$. Writing

$$\begin{aligned} \operatorname{im}\delta_{\mathcal{X}\mathcal{Z}} = \langle & ABC' + BCA' + 2BDC', \\ & ABD' + BCA' + 2BCB' + BDC', \\ & 2BCA' + 2BCB' + 2BDC'\rangle \cong \mathbb{Z}^2\oplus 2\mathbb{Z}, \end{aligned}$$

we have

$$H_Q^1(\mathcal{X},\mathcal{Z}) = \langle BCA', BDC', BCA' + BCB' + BDC'\rangle. \tag{3.6}$$

Since twice the third generator of $H_Q^1(\mathcal{X},\mathcal{Z})$ is 0 (modulo 2), the quotient cohomology contains torsion. The corresponding long exact sequence relating the cohomologies of $\mathcal{X}$ and $\mathcal{Z}$ to their quotient cohomologies is given by

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \mathbb{Z} & \xrightarrow{g_0^*} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^3 & \xrightarrow{g_1^*} & \mathbb{Z}^5 & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}_2 & \longrightarrow 0. \\ & \| & & \| & & \| & & \| & & \| & & \| & \\ & H^0(\mathcal{Z}) & & H^0(\mathcal{X}) & & H_Q^0(\mathcal{X},\mathcal{Z}) & & H^1(\mathcal{Z}) & & H^1(\mathcal{X}) & & H_Q^1(\mathcal{X},\mathcal{Z}) & \end{array}$$

$\diamondsuit$

The pullback on the cohomologies $f_k^* : H^k(Y) \longrightarrow H^k(X)$ naturally extends to the cohomologies of their inverse limit spaces $f_k^* : \varinjlim(H^k(Y),\gamma_{Y,k}^*) \longrightarrow \varinjlim(H^k(X),\gamma_{X,k}^*)$. Further, $\gamma_{X,k}^*$ (modulo f^*) induces $\gamma_{Q,k}^*$ under which the direct limit of the quotient cohomologies of X and Y is computed. If X and Y are the AP-complexes for the tiling spaces Ω_X and Ω_Y, then we have the following diagram, where we write $H_Q^k := H_Q^k(X,Y)$.

$$\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
H_Q^{k-1} & \xrightarrow{\gamma^*_{Q,k-1}} & H_Q^{k-1} & \xrightarrow{\gamma^*_{Q,k-1}} & H_Q^{k-1} & \xrightarrow{\gamma^*_{Q,k-1}} & \cdots \cong & & H_Q^{k-1}(\Omega_X, \Omega_Y) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
H^k(Y) & \xrightarrow{\gamma^*_{Y,k}} & H^k(Y) & \xrightarrow{\gamma^*_{Y,k}} & H^k(Y) & \xrightarrow{\gamma^*_{Y,k}} & \cdots \cong & & \check{H}^k(\Omega_Y) \\
\downarrow f_k^* & & \downarrow f_k^* & & \downarrow f_k^* & & & & \downarrow f_k^* \\
H^k(X) & \xrightarrow{\gamma^*_{X,k}} & H^k(X) & \xrightarrow{\gamma^*_{X,k}} & H^k(X) & \xrightarrow{\gamma^*_{X,k}} & \cdots \cong & & \check{H}^k(\Omega_X) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
H_Q^{k} & \xrightarrow{\gamma^*_{Q,k}} & H_Q^{k} & \xrightarrow{\gamma^*_{Q,k}} & H_Q^{k} & \xrightarrow{\gamma^*_{Q,k}} & \cdots \cong & & H_Q^{k}(\Omega_X, \Omega_Y) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow
\end{array}$$

Along the horizontal direction are the direct limits, whereas the sequences along the vertical direction form long exact sequences. Thus, the long exact sequence in (3.1) extends to the following long exact sequence involving tiling spaces, where we abbreviate the quotient cohomology $H_Q^*(\Omega_X, \Omega_Y)$ by H_Q^*.

$$\cdots \longrightarrow H_Q^{k-1} \longrightarrow H^k(\Omega_Y) \xrightarrow{f_k^*} H^k(\Omega_X) \longrightarrow H_Q^k \longrightarrow \cdots \tag{3.7}$$

For emphasis, we may sometimes write $\gamma^*_{Q,k} := \gamma^*_{XY,k}$ as in the following example.

Example 3.3 (Quotient cohomology between tiling spaces)**.** The three complexes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ in Figure 3.1 are the AP-complexes for a family of one-dimensional tiling spaces, derived from the following primitive substitution rules. Namely,

$$\varrho_{\mathcal{X}} := \begin{cases} A \longmapsto BC \\ B \longmapsto BD \\ C \longmapsto CA \\ D \longmapsto CB \end{cases} \qquad \varrho_{\mathcal{Y}} := \begin{cases} 1 \longmapsto 1\bar{1} \\ \bar{1} \longmapsto \bar{1}1 \end{cases} \qquad \varrho_{\mathcal{Z}} := \begin{cases} a \longmapsto ab \\ b \longmapsto aa \end{cases}$$

Of course, we already know that $\varrho_{\mathcal{Y}} = \varrho_{TM}$ and $\varrho_{\mathcal{Z}} = \varrho_{pd}$ from (2.6) and (2.7). Writing $\Omega_{\mathcal{X}}$, $\Omega_{\mathcal{Y}}$ (or $\mathbb{Y}^{TM}$) and $\Omega_{\mathcal{Z}}$ (or $\mathbb{Y}^{pd}$) for the respective hulls of the substitutions, we compute the quotient cohomology as the direct limit of the quotient cohomology of their approximants relative to the homomorphisms induced by the substitution rule $\varrho_{\mathcal{X}}$ depending on the factor maps, i.e.,

$$H_Q^k(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Y}}) \cong \varinjlim (H_Q^k(\mathcal{X}, \mathcal{Y}), \gamma^*_{\mathcal{X}\mathcal{Y},k})$$
$$H_Q^k(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Z}}) \cong \varinjlim (H_Q^k(\mathcal{X}, \mathcal{Z}), \gamma^*_{\mathcal{X}\mathcal{Z},k})$$

where $H_Q^k(\mathcal{X}, \mathcal{Y})$ and $H_Q^k(\mathcal{X}, \mathcal{Z})$ are the quotient cohomologies of the AP-complexes and $\gamma^*_{\mathcal{X}\mathcal{Y},k}$ and $\gamma^*_{\mathcal{X}\mathcal{Z},k}$ are the induced homomorphisms on the quotient cohomologies of the approximants. Note that $\gamma^*_{\mathcal{X}\mathcal{Y},k} = \gamma^*_{\mathcal{X},k}$ (modulo f^*) and $\gamma^*_{\mathcal{X}\mathcal{Z},k} = \gamma^*_{\mathcal{X},k}$ (modulo g^*).

Since $H_Q^0(\mathcal{X}, \mathcal{Y}) \cong 0$ and $H_Q^0(\mathcal{X}, \mathcal{Z}) \cong 0$ from Equations (3.3) and (3.5), then it follows immediately that

$$H_Q^0(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Y}}) \cong 0, \quad \text{and} \quad H_Q^0(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Z}}) \cong 0.$$

The inflation maps $\gamma_{\mathcal{X},0}, \gamma_{\mathcal{X},1}$ on the cells of $\mathcal{X}$ are given by the linear maps,

$$A'_{\mathcal{X},0} = \begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{bmatrix}, \quad A'_{\mathcal{X},1} = \begin{bmatrix} 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&1\\ 1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0 \end{bmatrix}.$$

Let us write $B'_{\mathcal{X}\mathcal{Y},1}$ for the matrix representation of the inflation map $\gamma_{\mathcal{X}\mathcal{Y},1}$ acting on $C_Q^1(\mathcal{X}, \mathcal{Y}) \cong \mathbb{Z}^4$ of Equation (3.2). Then $B'_{\mathcal{X}\mathcal{Y},1} = A'_{\mathcal{X},1}$ (modulo f^*) is given by

$$B'_{\mathcal{X}\mathcal{Y},1} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}.$$

It then follows that $H_Q^1(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Y}}) \cong 0$ as $\gamma^*_{\mathcal{X}\mathcal{Y},1}$ is just a the zero map. A less trivial computation is carried out for $H_Q^1(\Omega_{\mathcal{X}}, \Omega_{\mathcal{Z}})$.

Similarly, let $B'_{\mathcal{X}\mathcal{Z},1} := A'_{\mathcal{X},1}$ (modulo g^*) be the inflation map acting on $C_Q^1(\mathcal{X}, \mathcal{Z}) \cong \mathbb{Z}^5$ of Equation (3.4), and so we have

$$B'_{\mathcal{X}\mathcal{Z},1} = \begin{bmatrix} 0&0&-1&0&1\\ 0&0&-1&0&1\\ 0&-1&0&0&-1\\ 0&-1&0&0&-1\\ -1&0&0&-1&0 \end{bmatrix}.$$

The map $\gamma^*_{\mathcal{XZ},1}$ that acts on $H^1_Q(\mathcal{X},\mathcal{Z}) = \langle BCA', BDC', BCA' + BCB' + BDC'\rangle$ of Equation (3.6) is $\gamma^*_{\mathcal{XZ},1} = B'_{\mathcal{XZ},1}$ (modulo $\operatorname{im}\delta_{\mathcal{XZ}}$). For any $m > 2$,

$$\begin{aligned}(\varrho^*_{XZ})^m(BCA') &= BCA' + BCB' + BDC',\\ (\varrho^*_{XZ})^m(BDC') &= BCA' + BCB' + BDC',\\ (\varrho^*_{XZ})^m(BCA' + BCB' + BDC') &= BCA' + BCB' + BDC',\end{aligned}$$

which means that the first two components of $H^1_Q(\mathcal{X},\mathcal{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$ eventually vanish under $\gamma^*_{\mathcal{XZ},1}$, and only the third component (which is torsion) eventually remains. Thus, the eventual range of $\gamma^*_{\mathcal{XZ},1}$ is zero on the summand $\mathbb{Z}^2$ and the identity on $\mathbb{Z}_2$. The matrix representation of $\gamma^*_{\mathcal{XZ},1}$ is given by

$$B^*_{\mathcal{XZ},1} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}. \tag{3.8}$$

The long exact sequence (3.7) for $\Omega_\mathcal{X}$ and $\Omega_\mathcal{Z}$ is

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \mathbb{Z} & \xrightarrow{f^*} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\frac{1}{2}]\oplus\mathbb{Z} & \xrightarrow{f^*} & \mathbb{Z}[\frac{1}{2}]\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow 0.\\ & \| & & \| & & \| & & \| & & \| & & \| & \\ & H^0(\Omega_\mathcal{Z}) & & H^0(\Omega_\mathcal{X}) & & H^0_Q & & H^1(\Omega_\mathcal{Z}) & & H^1(\Omega_\mathcal{X}) & & H^1_Q & \end{array}$$

$\diamondsuit$

Remark 3.4. The substitution $\varrho_\mathcal{X}$ in the previous example is the induced substitution rule on the set of 2-letter legal words of the Thue-Morse substitution, i.e., $A := 11, B := 1\bar{1}, C := \bar{1}1$, and $D := \bar{1}\bar{1}$. In particular, we have $A = (1)1 \longmapsto (1\bar{1})1\bar{1} =: BC$, where the parentheses on the first letter indicate a reference marker and we take $(1\bar{1})1\bar{1} =: BC = 1\bar{1}1$ so that the last letter in $B = 1\bar{1}$ coincides with the first letter in $C = \bar{1}1$ (analogous to how two adjacent edges are attached in Figure 3.1). Similarly, $B = (1)\bar{1} \longmapsto (1\bar{1})\bar{1}1 =: BD = 1\bar{1}\bar{1}$, and also we get $C \longmapsto CA$ and $D \longmapsto CB$ to obtain the induced substitution $\varrho_\mathcal{X}$ from $\varrho_\mathcal{Y} = \varrho_{TM}$.

Thus, $\Omega_\mathcal{X}$ and $\Omega_\mathcal{Y}$ basically describe the same hull, which means $H^*(\Omega_\mathcal{X}) = H^*(\Omega_\mathcal{Y})$, and so explains $H^k_Q(\Omega_\mathcal{X},\Omega_\mathcal{Y}) \cong 0$ for all k. By a severe abuse of notation, we actually have $\mathcal{X} = \Gamma_{3/2}(\varrho_{TM})$, the 1.5-collared AP-complex of the Thue-Morse substitution, which is a bigger complex than $\mathcal{Y} = \Gamma_1(\varrho_{TM})$.

The quotient cohomologies we have computed correspond to the following:

$$\begin{aligned} H^0_Q(\Omega_\mathcal{X},\Omega_\mathcal{Z}) &= H^0_Q(\mathbb{Y}^{TM},\mathbb{Y}^{pd}) \cong 0, \text{ and}\\ H^1_Q(\Omega_\mathcal{X},\Omega_\mathcal{Z}) &= H^1_Q(\mathbb{Y}^{TM},\mathbb{Y}^{pd}) \cong \mathbb{Z}_2.\end{aligned}$$

It is not common to define the factor map between the hulls of the Thue-Morse and period doubling substitution as we did with g, as usually, the factor map is defined through a *sliding block map* $\phi : \Omega_\mathcal{Y} \longrightarrow \Omega_\mathcal{Z}$ (equivalently,

$\phi : \mathbb{Y}^{TM} \longrightarrow \mathbb{Y}^{pd}$) given by $\phi(1\bar{1}) = \phi(\bar{1}1) = a$ and $\phi(11) = \phi(\bar{1}\bar{1}) = b$ (also see Figure 3.1), i.e., for every 3-letter subword of a bi-infinite word in $\mathbb{Y}^{TM}$, we have $\phi(xyz) = \phi(xy)\phi(yz)$. Given a bi-infinite word w in $\mathbb{Y}^{pd}$, any 1-letter subword of w (at any arbitrary position) lifts to two possible pre-images in $\mathbb{Y}^{TM}$. After choosing to which pre-image the 1-letter subword lifts, its adjacent letters then lift definitely to a pre-image in $\mathbb{Y}^{TM}$. Thus, the word $w \in \mathbb{Y}^{pd}$ can be lifted in exactly two possible pre-images, say $u, v \in \mathbb{Y}^{TM}$, such that $u \neq v$ and $\phi(u) = \phi(v) = w$, which explains why the factor map ϕ is uniformly 2-to-1. We will encounter this again later in our discussion of the generalised Thue-Morse and period doubling substitutions in Chapter 4.3. $\diamondsuit$

Let us end this section with the following observations.

Lemma 3.5. *Let $f : X \longrightarrow Y$ be a quotient map, whose pullback f^* is injective on cochains. If $H^{n+1}(Y) \cong 0$, then $H^n_Q(X,Y) \cong H^n(X)/f^*_n(H^n(Y))$. For X and Y being approximant spaces for substitution tiling spaces, $H^0_Q(X,Y) \cong 0$ if and only if $f^*_1 : H^1(Y) \longrightarrow H^1(X)$ is injective.*

Proof. If X and Y are approximant spaces for substitution tiling spaces, then $H^0(X) \cong \mathbb{Z} \cong \langle \sum x'_i \rangle = \langle \sum f^*(y'_j) \rangle \cong f^*_0(\mathbb{Z}) \cong f^*_0(H^0(Y))$, where the x'_is and y'_js are the duals to the 0-cells in X and Y respectively. Thus f^*_0 is surjective, and so is an isomorphism. Further, the map from $H^0(X)$ to $H^0_Q(X,Y)$ in (3.1) must be a zero map and so the map from $H^0_Q(X,Y)$ to $H^1(Y)$ must be injective. If f^*_1 is injective, then the map from $H^0_Q(X,Y)$ to $H^1(Y)$ must be a zero map as well, which is already shown to be injective, forcing $H^0_Q(X,Y) \cong 0$. Conversely, if $H^0_Q(X,Y) \cong 0$, then f^*_1 must be injective since the sequence in (3.1) is exact. Meanwhile, $H^{n+1}(Y) \cong 0$ implies that the map $H^n(X)$ to $H^n_Q(X,Y)$ is surjective, and so it follows that $H^n_Q(X,Y) \cong H^n(X)/f^*_n(H^n(Y))$. $\square$

Theorem 3.6. *For n-dimensional substitution tiling spaces, related by a factor map $f : \Omega_X \longrightarrow \Omega_Y$, we have $H^n_Q(\Omega_X, \Omega_Y) \cong H^n(\Omega_X)/f^*_n(H^n(\Omega_Y))$ and for $k < n$, if f^*_k is an isomorphism, then $H^k_Q(\Omega_X, \Omega_Y) \cong \ker f^*_{k+1}$. In particular, $H^0_Q(\Omega_X, \Omega_Y) \cong \ker f^*_1$, since f^*_0 is always an isomorphism.*

Proof. The proof follows similarly as in the proof of the previous lemma working with the long exact sequence in (3.7) instead, and noting that $H^{n+1}(\Omega_Y) \cong 0$. Further, if f^*_k is an isomorphism for some $k < n$, then $H^k(\Omega_X) \longrightarrow H^k_Q(\Omega_X, \Omega_Y)$ is a zero map, and $H^k_Q(\Omega_X, \Omega_Y) \longrightarrow H^{k+1}(\Omega_Y)$ is injective. Thus, $H^k_Q(\Omega_X, \Omega_Y) \cong \ker f^*_{k+1}$. From Lemma 2.17, it follows that $f^*_0 : H^0(\Omega_Y) \longrightarrow H^0(\Omega_X)$ is an isomorphism. $\square$

Theorem 3.7. *For any substitution tiling spaces Ω_X and Ω_Y related via a factor map $f : \Omega_X \longrightarrow \Omega_Y$, $H_Q^0(\Omega_X, \Omega_Y) \cong 0$, and so the induced map f_1^* is injective.*

Proof. The quotient cohomology $H_Q^0(\Omega_X, \Omega_Y)$ is computed as the direct limit $\varinjlim(H_Q^0(X,Y), \gamma_{XY,0}^*)$, where X and Y are the AP-complexes of the tiling spaces Ω_X and Ω_Y respectively, and the induced map $\gamma_{XY,0}^*$ comes from $\gamma_{X,0}^*$(modulo f^*). From Lemma 2.17, we deduce that $\gamma_{X,0}^0$ is the identity acting on the sum of all the 0-cells in X. Taking modulo f^* in particular means taking the modulo under the pullback of the sum of all the 0-cells in Y, which yields $\gamma_{X,0}^*(\text{modulo } f^*) = 0$. As in Example 1.1, we compute $H_Q^0(\Omega_X, \Omega_Y) \cong 0$. From the previous Theorem, it follows that f_1^* is injective. ☐

The last two theorems make the calculation of H_Q^* simpler for 1- and 2-dimensional substitution tiling spaces. More precisely, let Ω_X and Ω_Y be two n-dimensional substitution tiling spaces related via a factor map $f : \Omega_X \longrightarrow \Omega_Y$. For $n = 1$, we have $H_Q^0 \cong 0$ and $H_Q^1 \cong H^1(\Omega_X)/f_1^*(H^1(\Omega_Y))$. For $n = 2$, if f_1^* is an isomorphism, then we get $H_Q^0 \cong 0$, $H_Q^1 \cong \ker f_2^*$ and $H_Q^2 \cong H^2(\Omega_X)/f_2^*(H^2(\Omega_Y))$. In both cases, notice that knowing the action of f_n^* is sufficient to compute the quotient cohomology H_Q^*.

3.2. Topological tools

We now review some results in [**BS11**] leading to a framework (Proposition 3.11 below), which is really helpful in understanding some of the computations on quotient cohomology.

Suppose that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are quotient maps, whose pullbacks on cochains are injective. Then $h := g \circ f : X \longrightarrow Z$ is also a quotient map, and there exists a short exact sequence of the corresponding chain complexes

$$0 \longrightarrow C_Q^k(Y,Z) \xrightarrow{\alpha} C_Q^k(X,Z) \xrightarrow{\beta} C_Q^k(X,Y) \longrightarrow 0$$

that induces the long exact sequence for the triple

$$\cdots \longrightarrow H_Q^k(Y,Z) \xrightarrow{\alpha^*} H_Q^k(X,Z) \xrightarrow{\beta^*} H_Q^k(X,Y) \longrightarrow H_Q^{k+1} \longrightarrow \cdots \tag{3.9}$$

Proposition 3.8 ([**BS11**, Thm. 1])**.** *Suppose $f : X \longrightarrow Y$ is a quotient map that induces an injection on the cohains. Further, suppose $Z \subset X$ is an open set such that the restriction on the closure of Z, denoted by $f|_{\bar{Z}}$, is a homeomorphism onto its image. Then the inclusion induced homomorphism is an isomorphism, i.e., $H_Q^*(X,Y) \cong H_Q^*(X - Z, Y - f(Z))$.* ☐

Proposition 3.9 ([**BS11**, Thm. 2])**.** *Suppose that $X_1, X_2 \subset X$ such that* $\text{int}(X_1) \cup \text{int}(X_2) = X$. *Also, suppose that $f : X \longrightarrow Y$, $f|_{X_1}$, $f|_{X_2}$ and*

$f|_{X_1 \cap X_2}$ *are all quotient maps onto* Y *that induce injections on the cochains. Then there is a long exact sequence (from the Mayer-Vietoris sequence)*

$$\cdots \longrightarrow H_Q^k(X,Y) \longrightarrow H_Q^k(X_1,Y) \oplus H_Q^k(X_2,Y) \longrightarrow$$
$$H_Q^k(X_1 \cap X_2, Y) \longrightarrow H_Q^{k+1}(X,Y) \longrightarrow \cdots$$

□

Corollary 3.10. *With the same assumptions as in the previous proposition and supposing further that* $f|_{X_1 \cap X_2}$ *is a homeomorphism, we have* $H_Q^k(X,Y) \cong H_Q^k(X_1,Y) \oplus H_Q^k(X_2,Y)$.

Proof. As $X_1 \cap X_2$ and Y are homeomorphic, then $H^*(X_1 \cap X_2) \cong H^*(Y)$ and in particular $H_Q^*(X_1 \cap X_2, Y) \cong 0$. The assertion then follows immediately from the previous proposition by having the exact sequence, $0 \longrightarrow H_Q^k(X,Y) \longrightarrow H_Q^k(X_1,Y) \oplus H_Q^k(X_2,Y) \longrightarrow 0$ for all k. □

Often, we encounter a sequence of tiling spaces related through a series of factor maps (compare for instance Figures 4.1 and 5.2), the simplest case of which is given by

$$\Omega_X \xrightarrow{f} \Omega_Y \xrightarrow{g} \Omega_Z, \tag{3.10}$$

and naturally, we are curious as to how f and g relate to their composition $g \circ f$ with respect to their corresponding quotient cohomologies. In particular, the previous corollary becomes useful in our attempt to classify such factor maps (and consequently their respective quotient cohomologies) by introducing a notion we call *good matches*. We say that the two factor maps $f : \Omega_X \longrightarrow \Omega_Y$ and $g : \Omega_Y \longrightarrow \Omega_Z$, as in (3.10), form a good match if and only if

$$H_Q^*(\Omega_X, \Omega_Z) \cong H_Q^*(\Omega_X, \Omega_Y) \oplus H_Q^*(\Omega_Y, \Omega_Z).$$

In other words, determining whether two factor maps (say f and g) form a good match is equivalent to determining whether the quotient cohomology between two tiling spaces (say Ω_X and Ω_Z) may be dissected as the direct sum of the quotient cohomologies involving an intermediate tiling space (say Ω_Y) between them, for which the two factor maps are defined respectively. Further, if f and g are a good match, we write $g \circ f := fg = (f)(g)$ for emphasis. In contrast, when f and g do not form a good match, then we just write $g \circ f := fg = (fg) \neq (f)(g)$. In the following chapters, we will classify factor maps and their compositions through the notion of good matches.

Now, let Ω_X be an n-dimensional tiling space (of $\mathbb{R}^n$), $\Omega_{X'}$ a closed subset of Ω_X and Λ a k-dimensional subspace of $\mathbb{R}^n$. We say that $\Omega_{X'}$ is a *k-dimensional tiling subspace* of Ω_X (along the direction Λ) provided that if $T \in \Omega_{X'}$, then $T - v \in \Omega_{X'}$ if and only if $v \in \Lambda$. If $\Omega_{X'}$ is a k-dimensional tiling subspace of Ω_X along the direction of Λ and $\sim$ is an equivalence relation on $\Omega_{X'}$, then

$\sim$ is *uniformly asymptotic* given that for each $\varepsilon > 0$, there is an $r \in \mathbb{R}$ such that if $T, T' \in \Omega_{X'}$ and $T \sim T'$, then $d(T - u, T' - u) < \varepsilon$ for all $u \in \Lambda^{\perp}$ with $\|u\| \geq r$. (Here, we can use the tiling metric d as defined in Equation (2.1).)

The following result is the heart of the paper [**BS11**] on quotient cohomology.

Proposition 3.11 ([**BS11**, Prop. 4])**.** *Suppose Ω_X and Ω_Y are n-dimensional substitution tiling spaces related via a factor map $f : \Omega_X \longrightarrow \Omega_Y$ such that the pullback f^* induces an injection on the cochains with the following assumptions:*

(1) $\Omega_{X'}$ is a k-dimensional tiling subspace of Ω_X along the direction of Λ;

(2) $f(\Omega_{X'}) = \Omega_{Y'}$;

(3) The equivalence relation $\sim$, defined as $T \sim T'$ if and only if $f(T) = f(T')$, is uniformly asymptotic;

(4) The factor map f is injective outside $\Omega_{X'} - \mathbb{R}^n := \{T - v \mid T \in \Omega_{X'}, v \in \mathbb{R}^n\}$; and,

(5) For $T, T' \in \Omega_{X'}$ and $v \in \mathbb{R}^n$, if $f(T' - v) = f(T)$, then $v \in \Lambda$.

Then, $H^m_Q(\Omega_X, \Omega_Y) \cong H^{m-n+k}_Q(\Omega_{X'}, \Omega_{Y'})$. □

The last proposition may seem very restrictive due to its many assumptions. However, some of the interesting substitution tiling spaces (in a number of instances, depending on the factor map) satisfy this framework. The effectiveness of the last proposition lies on the computability of $H^*_Q(\Omega_{X'}, \Omega_{Y'})$. The pair $\Omega_{X'}$ and $\Omega_{Y'}$ is called a *degeneracy* and the cohomology $H^*_Q(\Omega_{X'}, \Omega_{Y'})$ is called the *degeneration* (of the factor map f) [**BS11**]. The following degenerations will appear in Chapter 5, and so are worthwhile to discuss.

Example 3.12 (Degeneration A)**.** Suppose $\Omega_{X'} = \mathbb{S}_p \times \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$ (i.e., $\Omega_{X'}$ is composed of m copies of the solenoid) and $\Omega_{Y'} = \mathbb{S}_p$, for some $p \in \mathbb{N} - \{1\}$. Further, let $f : \Omega_{X'} \longrightarrow \Omega_{Y'}$ be a projection onto $\mathbb{S}_p$. If $m = 2$, then $H^1_Q(\Omega_{X'}, \Omega_{Y'}) \cong H^1(\Omega_{X'} - \Omega_{Y'}) \cong H^1(\mathbb{S}_p) \cong \mathbb{Z}[\frac{1}{p}]$ and $H^0_Q(\Omega_{X'}, \Omega_{Y'}) \cong H^0(\Omega_{X'} - \Omega_{Y'}) \cong H^0(\mathbb{S}_p) \cong \mathbb{Z}$ by Proposition 3.8. (Also, see Equation (1.3).) In general, for $m \in \mathbb{N}$, we have $H^1_Q(\Omega_{X'}, \Omega_{Y'}) \cong \bigoplus_{k=2}^{m} H^1_Q(\mathbb{S}_p \times \{1, k\}, \mathbb{S}_p) \cong \mathbb{Z}[\frac{1}{p}]^{m-1}$ and $H^0_Q(\Omega_{X'}, \Omega_{Y'}) \cong \mathbb{Z}^{m-1}$ using Corollary 3.10. ♢

Example 3.13 (Degeneration B)**.** Similarly, suppose $\mathbb{Y}$ is a 1-dimensional substitution tiling space and that $f : \Omega_{X'} \longrightarrow \Omega_{Y'}$ is a projection onto $\Omega_{Y'}$, where $\Omega_{X'} = \mathbb{Y} \times \{1, 2, \ldots, m\}$ and $\Omega_{Y'} = \mathbb{Y}$ for some $m \in \mathbb{N}$. Then by Proposition 3.8 and Corollary 3.10, $H^1_Q(\Omega_{X'}, \Omega_{Y'}) \cong H^1(\mathbb{Y})^{m-1}$ and $H^0_Q(\Omega_{X'}, \Omega_{Y'}) \cong$

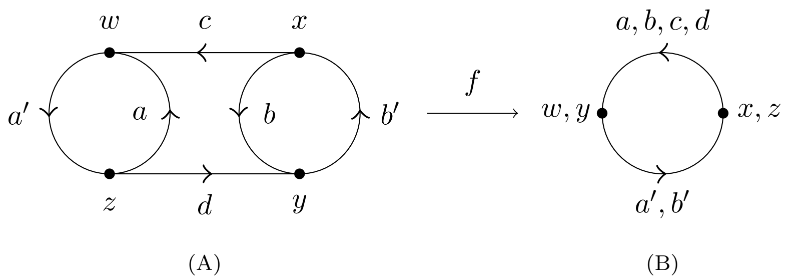


FIGURE 3.2. Quotient map between two approximants, where $f(a') = f(b')$, $f(a) = f(b) = f(c) = f(d)$, $f(w) = f(y)$ and $f(x) = f(z)$.

$H^0(\mathbb{Y})^{m-1}$. In particular, if $\mathbb{Y} = \mathbb{Y}^{pd}_{k,\ell}$ (the generalised period doubling substitution, see Chapter 4.3), then $H^1_Q(\Omega_{X'}, \Omega_{Y'}) \cong \mathbb{Z}[\frac{1}{k+\ell}]^{m-1} \oplus \mathbb{Z}^{m-1}$ and $H^0_Q(\Omega_{X'}, \Omega_{Y'}) \cong \mathbb{Z}^{m-1}$. ♢

Example 3.14 (Degeneration C)**.** Let us define X'_1 as the hull of the tiling AaB, where A is a semi-infinite word composed only of a and similarly, B is a semi-infinite word composed only of b. In the notation AaB, A extends to the left and B extends to the right. Similarly, let us define X'_2 as the hull of the tiling BaA. Further, we define $X' = \overline{X'_1 \cup X'_2}$. The space X' consists of the four translation orbits AA, BB, $BaA = BA$ and $AaB = AB$. Also, we define Y' to be the hull of the bi-infinite word C, which is comprised of only the letter c. We also define the substitution systems $\Omega_{X'} := (X', \varrho)$ and $\Omega_{Y'} := (Y', \varrho)$, where $\varrho(x) = x^p$ for some $p > 1$ and $x \in \{a, b, c\}$, together with the factor map $f : \Omega_{X'} \longrightarrow \Omega_{Y'}$ that identifies a and b, i.e., $f(a) = f(b) = c$. (Thus, $\Omega_{Y'} = \mathbb{S}_p$.) An approximant of $\Omega_{X'}$ is given in Figure 3.2 (A) and an approximant for the solenoid $\Omega_{Y'}$ is given in Figure 3.2 (B). The induced quotient map identifies the 1-cells labelled $a' := a$ and $b' := b$; the 1-cells a, b, c and d; the 0-cells w and y; and finally the 0-cells x and z, as also shown in the figure. The quotient cohomology between the approximants are $H^1_Q(X', Y') \cong \mathbb{Z}^2$ and $H^0_Q(X', Y') \cong 0$. The induced map ϱ_1^* on $H^1_Q(X', Y')$ is the identity on the first component and a multiplication of p on the second. Thus we get $H^1_Q(\Omega_{X'}, \Omega_{Y'}) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{p}]$, and trivially, $H^0_Q(\Omega_{X'}, \Omega_{Y'}) \cong 0$.

Analogous to the previous degenerations, we can extend this further to get $H^1_Q(\Omega_{X'} \times \{1, 2, \ldots, m\}, \Omega_{Y'}) \cong \bigoplus_m (\mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}) \cong \mathbb{Z}[\frac{1}{p}]^m \oplus \mathbb{Z}^m$ for some $m \in \mathbb{N}$ and $H^0_Q(\Omega_{X'} \times \{1, 2, \ldots, m\}, \Omega_{Y'}) \cong 0$ using Corollary 3.10. ♢

Example 3.15 (Degeneration D)**.** Suppose X' is a topological space with only k points for some $k \in \mathbb{N}$, and Y' a single point. If $f : X' \longrightarrow Y'$ is a quotient map such that f is a projection onto the first component, then $H_Q^0(X', Y') \cong \mathbb{Z}^{k-1}$ by Proposition 3.8 and Corollary 3.10. $\diamondsuit$

3.3. Quotient zeta functions

By definition, the dynamical zeta function of a substitution tiling space Ω_ω is a generating function that keeps track of the number of fixed points in Ω_ω that are invariant under an m-fold substitution ω^m, for some integer $m \in \mathbb{N}$. As we saw in the previous chapter, Proposition 2.21 unravels the deeper connection between the zeta function and the induced substitution action on the cohomology of the tiling space. In the spirit of this proposition, we extend the notion of zeta functions to accommodate the study of the quotient cohomology, i.e., we want to be able to define a generating function (between two tiling spaces related via a factor map), which we call the *quotient zeta function*, that is consistent with Proposition 2.21. We do this through the following definition, which is nothing but a restatement of the said proposition. As we will see in Theorem 3.18 below, the quotient zeta function is simply the quotient of the respective zeta functions of the two tiling spaces, i.e., it encodes the difference between the number of fixed points of the two spaces invariant under their respective m-fold substitution, for some integer $m \in \mathbb{N}$.

Definition 3.16. For two n-dimensional substitution tiling spaces Ω_X and Ω_Y related through a factor map $f : \Omega_X \longrightarrow \Omega_Y$, we define the quotient zeta function of Ω_X and Ω_Y as

$$\zeta_{X,Y}(z) := \frac{\prod\limits_{k \text{ odd}} \det(I - zC'_{n-k})}{\prod\limits_{k \text{ even}} \det(I - zC'_{n-k})} = \frac{\prod\limits_{k \text{ odd}} \prod\limits_{i} (1 - z\lambda_{n-k,i})}{\prod\limits_{k \text{ even}} \prod\limits_{i} (1 - z\lambda_{n-k,i})} \tag{3.11}$$

where C'_m is the matrix representation of the induced map γ_m acting on the quotient m-cochain with rational coefficients, and the second equality follows if all C'_m are diagonalisable with eigenvalues $\lambda_{m,i}$. Analogous to the usual dynamical zeta function, we can replace C'_m with C^*_m, the matrix representation of γ^*_m acting on the quotient cohomologies with rational coefficients. When the quotient cohomology (with integer coefficients) contains no torsion component, then $C^*_m = B^*_m$, the matrix representation of γ^*_m acting on H^m_Q (with integer coefficients). $\diamondsuit$

For convenience, we introduce the notation $\zeta_{Q,k}$, which denotes the rational function corresponding to the induced substitution action on H^k_Q in the spirit of the definition above, i.e.,

$$\zeta_{Q,m} := \det(I - zC^*_m) = \prod_i (1 - z\lambda_{m,i}), \tag{3.12}$$

where the last equality follows if C_m^* is diagonalisable. Thus, if n is odd, then

$$\zeta_{X,Y} = \frac{\zeta_{Q,0}\zeta_{Q,2}\cdots\zeta_{Q,n-1}}{\zeta_{Q,1}\zeta_{Q,3}\cdots\zeta_{Q,n}}, \tag{3.13}$$

otherwise when n is even, we get

$$\zeta_{X,Y} = \frac{\zeta_{Q,1}\zeta_{Q,3}\cdots\zeta_{Q,n-1}}{\zeta_{Q,0}\zeta_{Q,2}\cdots\zeta_{Q,n}}. \tag{3.14}$$

Example 3.17. Consider again the Thue-Morse and period doubling substitutions in Examples 2.15 and 2.16 and recall the following cohomologies:

$$H^0(\mathbb{Y}^{TM}) \cong \mathbb{Z} \quad \text{and} \quad H^1(\mathbb{Y}^{TM}) \cong \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{2}];$$
$$H^0(\mathbb{Y}^{pd}) \cong \mathbb{Z} \quad \text{and} \quad H^1(\mathbb{Y}^{pd}) \cong \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{2}];$$
$$H^0_Q(\mathbb{Y}^{TM}, \mathbb{Y}^{pd}) \cong 0 \quad \text{and} \quad H^1_Q(\mathbb{Y}^{TM}, \mathbb{Y}^{pd}) \cong \mathbb{Z}_2.$$

Also, recall from (3.8) that the matrix representation of the action of γ_1^* on the quotient cohomology $H^1_Q(\Gamma_1(\varrho_{TM}), \Gamma_1(\varrho_{pd})) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$ (with integer coefficients) is given by

$$B^*_{TMpd,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

where everything gets sent to the third generator of $H^1_Q(\Gamma_1(\varrho_{TM}), \Gamma_1(\varrho_{pd}))$. With rational coefficients, the action of γ_1^* becomes the zero map, and so $C_1^* = 0$. Trivially, $C_0^* = 0$, and so we get

$$\zeta_{TM,pd}(z) = \frac{\zeta_{Q,0}}{\zeta_{Q,1}} = \frac{1 - 0z}{1 - 0z} = 1 = \zeta_{TM}/\zeta_{pd}.$$

Notice that C_1^* may be directly obtained from $B^*_{TMpd,1}$ by simply making the torsion component zero. Working with rational coefficients, all the torsions become zero. ♢

Theorem 3.18. $\zeta_{X,Y}(z) = \zeta_X(z)/\zeta_Y(z)$.

Proof. Suppose Ω_X and Ω_Y are n-dimensional substitution tiling spaces and X and Y their respective AP-complexes. Considering the cohomologies of X and Y, but using rational coefficients, the long exact sequence in (3.7) reads

$$\cdots \longrightarrow H^{k-1}_Q(X,Y) \xrightarrow{s_k} H^k(Y) \xrightarrow{f_k^*} H^k(X) \xrightarrow{t_k} H^k_Q(X,Y) \longrightarrow \cdots$$

where of course, here $H^*_Q(X,Y) := H^*_Q(X,Y;\mathbb{Q})$, and analogously for $H^*(X;\mathbb{Q})$ and $H^*(Y;\mathbb{Q})$. Throughout the proof, each cohomology must be understood as computed over rational coefficients, and so for brevity, we just omit the rational coefficient in the notation as we did in the exact sequence above.

Note that $\operatorname{im} s_k = \ker f_k^*$ and $\operatorname{im} t_k = \ker s_{k+1}$. The long exact sequence induces a series of short exact sequences given by the following:

$$0 \longrightarrow \operatorname{coker} s_k \xrightarrow{f_k^*} H^k(X) \xrightarrow{t_k} \operatorname{im} t_k \longrightarrow 0$$

where $\operatorname{coker} s_k \cong H^k(Y)/\operatorname{im} s_k = H^k(Y)/\ker f_k^*$. Further, $\ker s_{k+1} = \operatorname{im} t_k \cong H^k(X)/f_k^*(\operatorname{coker} s_k)$. Because the cohomologies are computed over rational integers, it then follows that $H_Q^{k-1} \cong \ker s_k \oplus \operatorname{im} s_k = \ker s_k \oplus \ker f_k^*$. If q' is a generator of H^k such that $q' \in \ker f_k^*$, then $f_k^*(q') = 0$, and so we get $f_k^*(\operatorname{coker} s_k) \cong f_k^*(H^k(Y))$. Thus,

$$H_Q^k \cong H^k(X)/f_k^*(H^k(Y)) \oplus \ker f_{k+1}^*.$$

If f_{k+1}^* is injective, then $H_Q^k \cong H^k(X)/f_k^*(H^k(Y))$ and in particular $H_Q^n \cong H^n(X)/f_k^*(H^n(Y))$. The rational function in Equation (3.12) becomes

$$\zeta_{Q,m} = \frac{\det(I - zC_{X,m}^*)\det(I - zD_{\ker f_{m+1}^*}^*)}{\det(I - zD_{Y,m}^*)}$$

where $C_{X,m}^*$, $D_{Y,m}^*$ and $D_{\ker f_{m+1}^*}^*$ denote the matrix representation of the induced substitution actions on $H^m(X)$, $f_m^*(H^m(Y))$ and on the subspace $\ker f_{m+1}^*$. Further, we get the following,

$$\begin{aligned}
\frac{\zeta_{Q,n-1}}{\zeta_{Q,n}} &= \frac{\det(I - zC_{X,n-1}^*)\det(I - zD_{f_n^*}^*)}{\det(I - zD_{Y,n-1}^*)} \Bigg/ \frac{\det(I - zC_{X,n}^*)}{\det(I - zD_{Y,n}^*)} \\
&= \frac{\det(I - zC_{X,n-1}^*)}{\det(I - zC_{X,n}^*)} \Bigg/ \frac{\det(I - zD_{Y,n-1}^*)}{\det(I - zD_{Y,n}^*)\det(I - zD_{f_n^*}^*)} \\
&= \frac{\det(I - zC_{X,n-1}^*)}{\det(I - zC_{X,n}^*)} \Bigg/ \frac{\det(I - zD_{Y,n-1}^*)}{\det(I - zC_{Y,n}^*)}
\end{aligned}$$

where the last equality follows by noting that $f_n^*(H^k(Y)) \oplus \ker f_n^* \cong H^n(Y)$ and so $\det(I - zC_{Y,n}^*) = \det(I - zD_{Y,n}^*)\det(I - zD_{f_n^*}^*)$, where $C_{Y,m}^*$ is analogous to $C_{X,m}^*$. Similarly, we get the following expression for ζ_{m-3}/ζ_{m-2} for $m \le n$:

$$\frac{\zeta_{m-3}}{\zeta_{m-2}} = \frac{\det(I - zC_{X,m-3}^*)}{\det(I - zC_{X,m-2}^*)} \Bigg/ \frac{\det(I - zD_{Y,m-3}^*)\det(I - zD_{Y,m-1}^*)}{\det(I - zC_{Y,m-2}^*)}$$

For the case when n is odd, Equation (3.13) becomes

$$\begin{aligned}\zeta_{XY} &= \zeta_{Q,0}\frac{\zeta_{Q,2}}{\zeta_{Q,1}}\cdots\frac{\zeta_{Q,n-1}}{\zeta_{Q,n}}\\ &= \frac{\det(I-zC^*_{X,0})\det(I-zC^*_{X,2})\cdots\det(I-zC^*_{X,n-1})}{\det(I-zC^*_{X,1})\det(I-zC^*_{X,3})\cdots\det(I-zC^*_{X,n})}\Bigg/\\ &\qquad\frac{\det(I-zC^*_{Y,0})\det(I-zC^*_{Y,2})\cdots\det(I-zC^*_{Y,n-1})}{\det(I-zC^*_{Y,1})\det(I-zC^*_{Y,3})\cdots\det(I-zC^*_{Y,n})}\\ &= \zeta_X/\zeta_Y\end{aligned}$$

noting that $\ker f_0^* = 0$ (compare proof of Lemma 3.5) and so $f_0^*(H^0(Y)) \cong H^0(Y)$. The last equality follows from Proposition 2.21. The case when n is even follows analogously. □

Definition 3.16 and Theorem 3.18 are equivalent in the sense that one can instead start with the theorem as the definition, which then yields Equation (3.11) as a consequence.

In the case of the Thue-Morse and period doubling substitutions, the quotient zeta function is trivial because the two substitution systems have identical number of fixed points invariant under the m-fold substitution, a rather expected result as a consequence of the previous theorem. More so, Theorem 3.18 highlights two things. The first is the rather simple calculation of the quotient zeta function, and the second, which is more subtle, is a hint as to when the quotient cohomology may fail to distinguish two inequivalent tiling spaces possessing isomorphic cohomology groups. More precisely, given a factor map $f : \Omega_X \longrightarrow \Omega_Y$ that is 1-to-1 almost everywhere, if Ω_X and Ω_Y have the same number of fixed points, then it follows that $H^*_Q \cong 0$. However, there is no requirement for Ω_X and Ω_Y to be equivalent. This is precisely the case for the following example due to Franz Gähler. Let us emphasise though that this is not the case between the Thue-Morse and period doubling substitutions as the factor map between them is not 1-to-1 almost everywhere, but rather is uniformly 2-to-1.

Example 3.19. Consider the hull Ω_G of the substitution $\varrho_G := \{a \longmapsto aba,\ b \longmapsto baa\}$ and let $f : \Omega_G \longrightarrow \mathbb{S}_3$ be a factor map from the hull of ϱ_G to the 3-adic solenoid $\mathbb{S}_3$ by simply identifying all the letters (prototiles) in any sequence (tiling) in Ω_G. It then follows that $H^0(\Omega_G) \cong H^0(\mathbb{S}_3) \cong \mathbb{Z}$ and $H^1(\Omega_G) \cong H^1(\mathbb{S}_3) \cong \mathbb{Z}[\frac{1}{3}]$. More so, $H^0_Q \cong H^1_Q \cong 0$ but clearly $\Omega_G \neq \mathbb{S}_3$, as the former is a substitution tiling space, which the solenoid is not. ♢

CHAPTER 4

One-dimensional substitution tiling spaces

In this chapter, we look at some known examples of substitution tiling spaces and compute their respective quotient cohomologies. In particular, we are interested in the twisted Fibonacci substitution, the 'universal morphism', which is a substitution in one dimension introduced in [**BGG13**], and the generalised Thue-Morse substitution as introduced in [**BGG12**]. As we will see, computing the quotient cohomology among 1-dimensional substitution tiling spaces is rather straightforward, and can be done mostly by just using the definition without much difficulty.

4.1. The twisted Fibonacci substitution

The hull of the Fibonacci substitution $\varrho_F := \{a \longmapsto ab,\, b \longmapsto a\}$ (also see Example 2.8) may be viewed as a factor of the bigger hull $\mathbb{Y}_{tF}$, called the *twisted* Fibonacci substitution tiling space, derived from the substitution rule

$$\varrho_{tF} := \begin{cases} a \longmapsto a\bar{b} \\ b \longmapsto a \\ \bar{a} \longmapsto \bar{a}b \\ \bar{b} \longmapsto \bar{a} \end{cases}$$

whose cohomology is computed as $H^0(\Omega_{tF}) \cong 0$ and $H^1(\Omega_{tF}) \cong \mathbb{Z}^7$.

Let $f : \Omega_{tF} \longrightarrow \Omega_F$ be a factor map where a letter is identified with its barred counterpart, i.e., $f(a) = f(\bar{a})$ and $f(b) = f(\bar{b})$. Equivalently, we may regard f as the map that simply removes the bar of a letter, if any. We now describe the AP-complex of Ω_{tF} and compute the quotient cohomology $H^*_Q(\Omega_{tF}, \Omega_F)$. Let us denote the AP-complex of Ω_{tF} by $\Gamma_{tF} := \Gamma_1(\varrho_{tF})$ and the AP-complex of Ω_F as $\Gamma_F := \Gamma_1(\varrho_F)$. To make it easier to the eyes, we rename the letters in Ω_{tF} as $1 := a$, $2 := b$, $3 := \bar{a}$ and $4 := \bar{b}$. In this way, we can simply write $f(1) = f(3) = a$ and $f(2) = f(4) = b$, where the letters a and b are clearly understood to be the letters in Ω_F.

The complex Γ_{tF} has 10 vertices and 16 edges, and we summarise the action of the (induced) quotient map f in Table 4.1. The quotient cochains are given by $C^0_Q := C^0(\Gamma_{tF})/f^*(C^0(\Gamma_F)) \cong \mathbb{Z}^7$ and $C^1_Q := C^1(\Gamma_{tF})/f^*(C^1(\Gamma_F)) \cong \mathbb{Z}^{12}$.

0-cells	$f(14) = f(32) = ab$
	$f(41) = f(43) = f(21) = f(23) = ba$
	$f(11) = f(13) = f(31) = f(33) = aa$
1-cells	$f(114) = f(132) = f(314) = f(332) = aab$
	$f(141) = f(143) = f(321) = f(323) = aba$
	$f(211) = f(213) = f(431) = f(433) = baa$
	$f(214) = f(232) = f(414) = f(432) = bab$

TABLE 4.1. The induced quotient map on the legal 2- and 3-letter words of the twisted Fibonacci substitution.

The coboundary map $\delta_Q : C^0_Q \longrightarrow C^1_Q$ reads

$$\delta_Q = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The quotient cohomologies are computed as $H^0_Q(\Gamma_{tF}, \Gamma_F) \cong \ker \delta_Q \cong 0$ and $H^1_Q(\Gamma_{tF}, \Gamma_F) \cong \mathbb{Z}^{12}/\operatorname{im} \delta_Q \cong \mathbb{Z}^5$. The substitution ϱ_{tF} induces the matrices B^*_0 and B^*_1, which act on $H^0_Q(\Gamma_{tF}, \Gamma_F)$ and $H^1_Q(\Gamma_{tF}, \Gamma_F)$ respectively. In this case, there is really no need to get B^*_0 since $H^0_Q(\Gamma_{tF}, \Gamma_F) \cong 0$, and so immediately we have $H^0_Q(\Omega_{tF}, \Omega_F) \cong 0$. (Also, see Theorem 3.7.)

On the other hand, the matrix B^*_1 reads

$$B^*_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

and so the direct limit $\mathbb{Z}^5 \xrightarrow{B^*_1} \mathbb{Z}^5 \xrightarrow{B^*_1} \mathbb{Z}^5 \xrightarrow{B^*_1} \cdots$ is equal to $\mathbb{Z}^5$, since the map B^*_1 has determinant -1 and so is an isomorphism. Thus, $H^1_Q(\Omega_{tF}, \Omega_F) \cong \mathbb{Z}^5$.

We get the following long exact sequence for Ω_{tF} and Ω_F:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \mathbb{Z} & \xrightarrow{f_0^*} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{f_1^*} & \mathbb{Z}^7 & \longrightarrow & \mathbb{Z}^5 & \longrightarrow 0. \\ & \| & & \| & & \| & & \| & & \| & & \| & \\ & H^0(\Omega_F) & & H^0(\Omega_{tF}) & & H^0_Q & & H^1(\Omega_F) & & H^1(\Omega_F) & & H^1_Q & \end{array}$$

By Theorem 3.6, we see that f_1^* is injective, i.e., an inclusion map embedding $\mathbb{Z}^2$ isomorphically onto $\mathbb{Z}^7$. Further, the set of eigenvalues of B_1^* is just the set of eigenvalues of $A_{tF,1}^*$ minus the set of eigenvalues of $A_{F,1}^*$, which is also explained by the notion of the quotient zeta function. (Note that $A_{tF,1}^*$ and $A_{F,1}^*$ are the induced substitution on the cohomologies $H^1(\Omega_{tF})$ and $H^1(\Omega_F)$, respectively.) Indeed, the corresponding zeta functions read,

$$\begin{aligned} \zeta_{tF}(z) &= \frac{1-z}{(1-z-z^2)(1+z)(1+z^2)(1-z+z^2)} \\ &= \zeta_F \frac{1}{(1+z)(1+z^2)(1-z+z^2)} = \zeta_F \zeta_{tF,F} \end{aligned}$$

where $\zeta_{tF,F}$ is the associated quotient zeta function.

4.2. The universal morphism

Substitution systems are *dynamical* systems with the tiling space Ω being a compact metric space, together with the (continuous) inflation map $\omega : \Omega \longrightarrow \Omega$. Dynamical systems may be characterised through their dynamical *spectra* [**Koo31, Neu33**]. The dynamical spectrum is then classified into three components, namely the *pure point* part, the *absolutely continuous* part, and the *singular continuous* part. On certain conditions, the calculation of the dynamical spectrum yields the determination of the *diffraction* spectrum, up to certain isomorphisms [**BLvE14**].

One example of a substitution system that has all kinds of dynamical spectra is given by the following primitive substitution called the *universal morphism* [**BGG13**].

$$\varrho_U := \begin{cases} a \longmapsto a\bar{b}, & \bar{a} \longmapsto \bar{a}b \\ b \longmapsto a\bar{d}, & \bar{b} \longmapsto \bar{a}d \\ c \longmapsto c\bar{d}, & \bar{c} \longmapsto \bar{c}d \\ d \longmapsto c\bar{b}, & \bar{d} \longmapsto \bar{c}b \end{cases}$$

By identifying a letter with its barred version, we define a factor map from the hull $\mathbb{Y}_U$ (of ϱ_U) to the hull $\mathbb{Y}_{RS}$ of the well known (quaternary) Rudin-Shapiro substitution [**BG13a**, Ch. 4.7] derived from

$$\varrho_{RS} := \begin{cases} a \longmapsto ab \\ b \longmapsto ad \\ c \longmapsto cd \\ d \longmapsto cb \end{cases}$$

Ω	$H^0(\Omega)$	$H^1(\Omega)$	Dynamical spectrum
U	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}]^5 \oplus \mathbb{Z}^9$	pp + ac + sc
RS	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}$	pp + ac
T	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$	pp
TM	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$	pp + sc
pd	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$	pp
$\mathbb{S}_2$	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{2}]$	pp

TABLE 4.2. The cohomology of the universal morphism and some of its factors; pp, ac, and sc denote the pure point, absolutely continuous and singular continuous parts of the spectrum.

The dynamical spectrum of the hull $\mathbb{Y}_{RS}$ consists of an absolutely continuous part and a pure point part [**BGG13**]. By introducing a sliding block map on the legal 2-letter words of the substitution ϱ_{RS}, given by

$$\chi(ab) = \chi(cd) = A, \quad \chi(ad) = \chi(cb) = B,$$
$$\chi(ba) = \chi(dc) = C, \quad \chi(bc) = \chi(da) = D,$$

we get a factor of $\mathbb{Y}_{RS}$, whose hull $\mathbb{Y}_T$ is obtained from the substitution

$$\varrho_T := \begin{cases} A \longmapsto AC \\ B \longmapsto AD \\ C \longmapsto CB \\ D \longmapsto BC \end{cases}$$

As it turns out, the hull $\mathbb{Y}_T$ defines a dynamical system with pure point dynamical spectrum, which coincides with the pure point part of the dynamical spectrum of $\mathbb{Y}_{RS}$. Finally, by identifying all the letters in ϱ_T, we recover the solenoid $\mathbb{S}_2$ as a factor.

On the other side of the spectrum, we obtain the hull $\mathbb{Y}^{TM}$ of the Thue-Morse substitution ϱ_{TM} as factor of $\mathbb{Y}_U$, through a factor map defined from identifying all the unbarred letters, and at the same time identifying all the barred letters. The Thue-Morse substitution produces a hull, whose dynamical spectrum comprises a singular continuous part and a pure point part [**BGG12**]. As we already know, the sliding block map ϕ (as in Remark 3.4) produces the period doubling hull $\mathbb{Y}^{pd}$, which has pure point dynamical spectrum. Similar to ϱ_T, by identifying all letters in ϱ_{pd}, we obtain the solenoid $\mathbb{S}_2$ as a factor.

Table 4.2 lists the cohomology of the universal morphism together with its factors. In particular, the computation of $H^1(\Omega_{RS}) \cong \mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}$ is demonstrated in Example 1.7, where the matrix A in the example is the induced action on

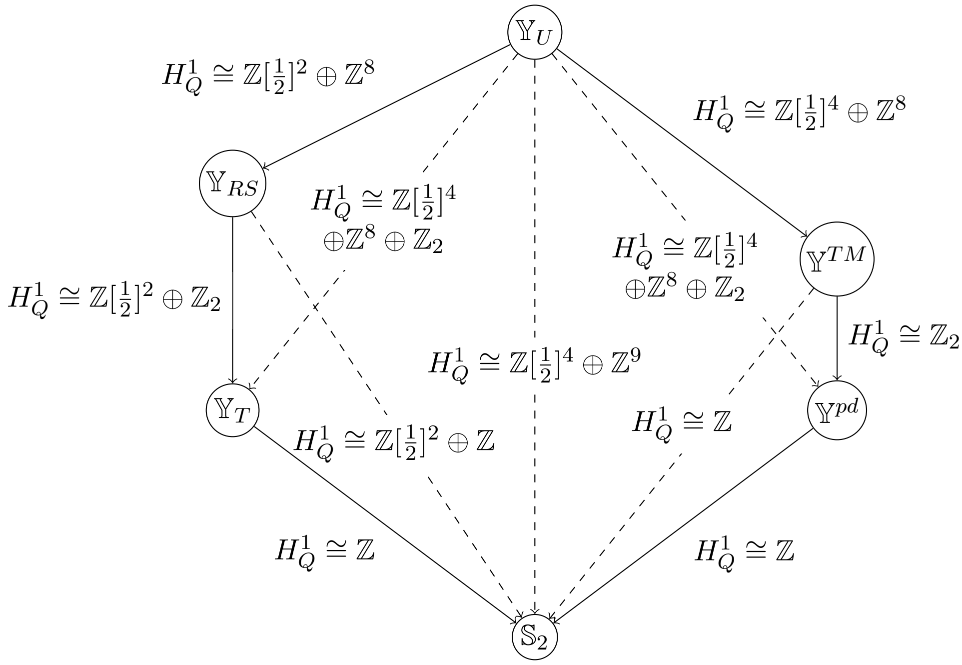


FIGURE 4.1. The universal morphism and its factors together with their pairwise quotient cohomologies. In all cases, $H_Q^0 \cong 0$.

the 1-cells of the AP-complex of Ω_{RS}. Further, we summarise the hierarchy of the factors in Figure 4.1 together with their pairwise quotient cohomologies noting that $H_Q^0 \cong 0$ for all pairs of tiling spaces. All the quotient cohomologies are computed using using Theorems 3.6 and 3.7, though the solution is rather long to include here, albeit pretty straightforward; compare Example 4.1.

Among the factor maps in Figure 4.1, there are only two compositions that form good matches, namely $\mathbb{Y}_U \longrightarrow \mathbb{Y}_{RS} \longrightarrow \mathbb{Y}_T$ and $\mathbb{Y}_U \longrightarrow \mathbb{Y}^{TM} \longrightarrow \mathbb{Y}^{pd}$ since $H_Q^*(\mathbb{Y}_U, \mathbb{Y}_T) \cong H_Q^*(\mathbb{Y}_U, \mathbb{Y}_{RS}) \oplus H_Q^*(\mathbb{Y}_{RS}, \mathbb{Y}_T)$ and $H_Q^*(\mathbb{Y}_U, \mathbb{Y}^{pd}) \cong H_Q^*(\mathbb{Y}_U, \mathbb{Y}^{TM}) \oplus H_Q^*(\mathbb{Y}^{TM}, \mathbb{Y}^{pd})$. The rest of the compositions, particularly $\mathbb{Y}^{TM} \xrightarrow{\phi} \mathbb{Y}^{pd} \xrightarrow{\psi} \mathbb{S}_2$, no longer form good matches.

4.3. Generalised Thue-Morse and period doubling substitutions

The Thue-Morse and period doubling substitutions belong to a larger class of tiling spaces of 1-dimensional substitution (on two letters) that admit integer inflation factors. In particular, we consider the so-called generalised Thue-Morse and period doubling substitutions (compare [**BG13a**, Rem. 4.9]) defined by the primitive substitutions

$$\begin{aligned}
\varrho^{TM}_{k,\ell} \quad &:= \begin{cases} 1 & \longmapsto \quad 1^k\,\bar{1}^\ell \\ \bar{1} & \longmapsto \quad \bar{1}^k\,1^\ell \end{cases} \\
\varrho^{pd}_{k,\ell} \quad &:= \begin{cases} a & \longmapsto \quad b^{k-1}\,a\,b^{\ell-1}\,b \\ b & \longmapsto \quad b^{k-1}\,a\,b^{\ell-1}\,a \end{cases}
\end{aligned} \tag{4.1}$$

for any $k,\ell \in \mathbb{N}$ [**BGG12**]. We denote by $\mathbb{Y}^{TM}_{k,\ell}$ and $\mathbb{Y}^{pd}_{k,\ell}$ the hulls of $\varrho^{TM}_{k,\ell}$ and $\varrho^{pd}_{k,\ell}$, respectively. The case $k = \ell = 1$ yields the classical Thue-Morse and period doubling substitutions; see Examples 2.15 and 2.16. Recently in [**BGG12**], Baake, Gähler and Grimm thoroughly analysed the spectral and topological properties of the family of generalised Thue-Morse substitutions. In this section, we show how to compute the quotient cohomology between the generalised Thue-Morse and period doubling tiling spaces for arbitrary $k,\ell \in \mathbb{N}$.

For any $k,\ell \in \mathbb{N}$, let us consider the tiling spaces $\mathbb{Y}^{TM}_{k,\ell}, \mathbb{Y}^{pd}_{k,\ell}$ and also the solenoid $\mathbb{S}_{k+\ell}$. Recall in Remark 2.9 that we may define $\mathbb{S}_{k+\ell}$ as the inverse limit of the substitution $\{s \longmapsto s^{k+\ell}\}$. The three spaces are related via the factor maps ϕ and ψ, namely

$$\mathbb{Y}^{TM}_{k,\ell} \xrightarrow{\ \phi\ } \mathbb{Y}^{pd}_{k,\ell} \xrightarrow{\ \psi\ } \mathbb{S}_{k+\ell}$$

where ϕ is a sliding block map that identifies $1\bar{1}$ and $\bar{1}1$ with a, and, 11 and $\bar{1}\bar{1}$ with b; and the factor map ψ simply identifies a and b with s, i.e., $\phi(1\bar{1}) = \phi(\bar{1}1) = a$, $\phi(11) = \phi(\bar{1}\bar{1}) = b$ and $\psi(a) = \psi(b) = s$. Note that ϕ is uniformly 2-to-1, whereas ψ is a surjection that is 1-to-1 almost everywhere (see Remark 3.4). Each of these factor maps induces a pullback on their respective cohomologies given by

$$H^*(\mathbb{S}_{k+\ell}) \xrightarrow{\ \psi^*\ } H^*(\mathbb{Y}^{pd}_{k,\ell}) \xrightarrow{\ \phi^*\ } H^*(\mathbb{Y}^{TM}_{k,\ell}).$$

As computed in [**BGG12**], the cohomologies of the three spaces are: $H^0 = \mathbb{Z}$ and

$$H^1(\mathbb{S}_{k+\ell}) \cong \mathbb{Z}[\tfrac{1}{k+\ell}], \quad H^1(\mathbb{Y}^{pd}_{k,\ell}) \cong \mathbb{Z}[\tfrac{1}{k+\ell}] \oplus \mathbb{Z},$$
$$H^1(\mathbb{Y}^{TM}_{k,\ell}) \cong \mathbb{Z}[\tfrac{1}{k+\ell}] \oplus \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{k-\ell}],$$

where $\mathbb{Z}[\frac{1}{0}] := 0$ by an abuse of notation. Let us use Theorem 3.6 to compute the quotient cohomology. In particular, we are interested in knowing the action of $\psi_1^* : H^1(\mathbb{S}_{k+\ell}) \longrightarrow H^1(\mathbb{Y}^{pd}_{k,\ell})$ and $\phi_1^* : H^1(\mathbb{Y}^{pd}_{k,\ell}) \longrightarrow H^1(\mathbb{Y}^{TM}_{k,\ell})$ and consequently of $\phi_1^* \circ \psi_1^* : H^1(\mathbb{S}_{k+\ell}) \longrightarrow H^1(\mathbb{Y}^{TM}_{k,\ell})$. Knowing these three actions enables us to determine the quotient cohomology between any of the three spaces.

Example 4.1 (The classical case)**.** Suppose $k = \ell = 1$ and we want to see how ϕ_1^* and ψ_1^* behave. By choosing particular sets of generators (under which the direct limit is obtained), we can write the following.

$$H^0(\mathbb{Y}_{1,1}^{TM}) \cong \mathbb{Z} \cong \langle 11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}' \rangle \xrightarrow{\times 1} \langle 11' + 1\bar{1}' + \bar{1}1' + \bar{1}\bar{1}' \rangle \xrightarrow{\times 1} \cdots$$

$$H^1(\mathbb{Y}_{1,1}^{TM}) \cong \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{2}] \cong \langle \bar{1}11' + \bar{1}\bar{1}1', \bar{1}1\bar{1}' - \bar{1}\bar{1}1' \rangle \xrightarrow{A^*_{TM,1}}$$

$$\langle \bar{1}11' + \bar{1}\bar{1}1', \bar{1}1\bar{1}' - \bar{1}\bar{1}1' \rangle \xrightarrow{A^*_{TM,1}} \cdots$$

$$H^0(\mathbb{Y}_{1,1}^{pd}) \cong \mathbb{Z} \cong \langle a' + b' \rangle \xrightarrow{\times 1} \langle a' + b' \rangle \xrightarrow{\times 1} \cdots$$

$$H^1(\mathbb{Y}_{1,1}^{pd}) \cong \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{2}] \cong \langle aa' + ba', ba' \rangle \xrightarrow{A^*_{pd,1}} \langle aa' + ba', ba' \rangle \xrightarrow{A^*_{pd,1}} \cdots$$

$$H^0(\mathbb{S}_2) \cong \mathbb{Z} \cong \langle s' \rangle \xrightarrow{\times 1} \langle s' \rangle \xrightarrow{\times 1} \cdots$$

$$H^1(\mathbb{S}_2) \cong \mathbb{Z}[\tfrac{1}{2}] \cong \langle ss' \rangle \xrightarrow{A^*_{\mathbb{S}_2,1}} \langle ss' \rangle \xrightarrow{A^*_{\mathbb{S}_2,1}} \cdots$$

The corresponding induced maps on the (1st) cohomologies have the following matrix representations

$$A^*_{TM,1} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}, \quad A^*_{pd,1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^*_{\mathbb{S}_2,1} = \begin{bmatrix} 2 \end{bmatrix}$$

and have left eigenvectors and eigenvalues

$$\begin{pmatrix} 2 & 3 & | & 2 \\ 1 & 0 & | & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & | & 2 \\ 1 & -2 & | & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & | & 2 \end{pmatrix}.$$

Since $\phi(1\bar{1}1) = \phi(\bar{1}1\bar{1}) = aa$ and $\phi(11\bar{1}) = \phi(\bar{1}\bar{1}1) = ba$, we get $\phi^*(aa') = 1\bar{1}1' + \bar{1}1\bar{1}'$ and $\phi^*(ba') = 11\bar{1}' + \bar{1}\bar{1}1'$. The eigenvector $[1, -2]$ of $A^*_{pd,1}$, corresponding to $aa' - ba'$, pulls back to $2 \cdot \bar{1}1\bar{1}' - 2 \cdot \bar{1}\bar{1}1'$ corresponding to the eigenvector $[2, 0]$, twice the eigenvector $[1, 0]$ of $A^*_{TM,1}$. Meanwhile, the eigenvector $[1, 1]$ of $A^*_{pd,1}$, corresponding to $aa' + 2 \cdot ba'$, pulls back to $3 \cdot \bar{1}11' + 2 \cdot \bar{1}\bar{1}1' + \bar{1}\bar{1}1'$, corresponding to the eigenvector $[2, 3]$ of $A^*_{TM,1}$. In summary, ϕ_1^* is an isomorphism on the summand $\mathbb{Z}[\frac{1}{2}]$ in $H^1(\mathbb{Y}_{1,1}^{pd})$ into the same summand in $H^1(\mathbb{Y}_{1,1}^{TM})$, and maps the summand $\mathbb{Z}$ in $H^1(\mathbb{Y}_{1,1}^{pd})$ into $2\mathbb{Z}$ in $H^1(\mathbb{Y}_{1,1}^{TM})$. By Theorem 3.6, since ϕ_1^* is injective, then $H_Q^0(\mathbb{Y}_{1,1}^{TM}, \mathbb{Y}_{1,1}^{pd}) \cong 0$ and $H_Q^1(\mathbb{Y}_{1,1}^{TM}, \mathbb{Y}_{1,1}^{pd}) \cong H^1(\mathbb{Y}_{1,1}^{TM})/\phi_1^*(H^1(\mathbb{Y}_{1,1}^{pd})) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

Meanwhile, the eigenvector $[1]$ of $A^*_{\mathbb{S}_2,1}$ corresponding to ss' pulls back to $aa' + ab' + ba' = aa' + 2 \cdot ba'$ corresponding to the eigenvector $[1, 1]$ of $A^*_{pd,1}$. Thus, the map ψ_1^* sends $H^1(\mathbb{S}_2) \cong \mathbb{Z}[\frac{1}{2}]$ isomorphically into the same summand in $H^1(\mathbb{Y}_{1,1}^{pd})$, and consequently $\phi_1^* \circ \psi_1^*$ sends $H^1(\mathbb{S}_2) \cong \mathbb{Z}[\frac{1}{2}]$ into the same summand in $H^1(\mathbb{Y}_{1,1}^{TM})$. Similar to ϕ_0^*, the maps ψ_0^* and $\phi_0^* \circ \psi_0^*$ are just the identity maps. Thus, $H^0(\mathbb{Y}^{TM}_{1,1}, \mathbb{S}_2) \cong 0$ and $H^0_Q(\mathbb{Y}^{pd}_{1,1}, \mathbb{S}_2) \cong 0$, and, $H^1(\mathbb{Y}^{TM}_{1,1}, \mathbb{S}_2) \cong \mathbb{Z}$ and $H^1_Q(\mathbb{Y}^{pd}_{1,1}, \mathbb{S}_2) \cong \mathbb{Z}$. ♢

For any $k, \ell \in \mathbb{N}$, the behaviour of ϕ^* and ψ^* on the particular cohomologies remains unchanged [**BGG12**], and so we have the following result.

Theorem 4.2. *For any $k, \ell \in \mathbb{N}$, $H^0_Q \cong 0$. Further,*

$$\begin{aligned} H^1_Q(\mathbb{Y}^{pd}_{k,\ell}, \mathbb{S}_{k+\ell}) &\cong \mathbb{Z}, \\ H^1_Q(\mathbb{Y}^{TM}_{k,\ell}, \mathbb{S}_{k+\ell}) &\cong \mathbb{Z}[\tfrac{1}{k-\ell}] \oplus \mathbb{Z}, \\ H^1_Q(\mathbb{Y}^{TM}_{k,\ell}, \mathbb{Y}^{pd}_{k,\ell}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}[\tfrac{1}{k-\ell}], \end{aligned}$$

where $\mathbb{Z}[\frac{1}{0}] := 0$ by an abuse of notation.

Proof. The map ψ^* sends $H^1(\mathbb{S}_{k+\ell}) \cong \mathbb{Z}[\frac{1}{k+\ell}]$ isomorphically into the same summand in $H^1(\mathbb{Y}^{pd}_{k,\ell})$, which then the map ϕ^* also sends isomorphically into the same summand in $H^1(\mathbb{Y}^{TM}_{k,\ell})$. Thus, $\varphi^* := \phi^* \circ \psi^*$ also maps $H^1(\mathbb{S}_{k+\ell}) \cong \mathbb{Z}[\frac{1}{k+\ell}]$ isomorphically into $\mathbb{Z}[\frac{1}{k+\ell}]$ in $H^1(\mathbb{Y}^{TM}_{k,\ell})$. Furthermore, ϕ^* maps the summand $\mathbb{Z}$ in $H^1(\mathbb{Y}^{pd}_{k,\ell})$ into $2\mathbb{Z}$ in $H^1(\mathbb{Y}^{TM}_{k,\ell})$. Thus, the maps ψ^*_1, ϕ^*_1 and φ^*_1 are all injective maps, and so $H^0_Q \cong 0$ for all pairs of spaces using Theorem 3.6. By the same theorem, we get $H^1_Q(\mathbb{Y}^{TM}_{k,\ell}, \mathbb{Y}^{pd}_{k,\ell}) \cong H^1(\mathbb{Y}^{TM}_{k,\ell})/\phi^*_1(H^1(\mathbb{Y}^{pd}_{k,\ell})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}[\frac{1}{k-\ell}]$. The rest of the results follow similarly. □

The three non-trivial quotient cohomologies are related via the short exact sequence (also see (3.9))

$$0 \longrightarrow H^1_Q(\mathbb{Y}^{pd}_{k,\ell}, \mathbb{S}_{k+\ell}) \xhookrightarrow{\times 2} H^1_Q(\mathbb{Y}^{TM}_{k,\ell}, \mathbb{S}_{k+\ell}) \longrightarrow H^1_Q(\mathbb{Y}^{TM}_{k,\ell}, \mathbb{Y}^{pd}_{k,\ell}) \longrightarrow 0.$$

As one may already suspect, the composition $\psi \circ \phi$ does not form a good match.

The zeta function for the generalized period-doubling sequence [**BGG12**] is given by

$$\zeta^{pd}_{k,\ell} = \frac{1-z}{(1+z)(1-(k+\ell)z)},$$

while the zeta function for the generalized Thue-Morse sequence is given by

$$\begin{aligned} \zeta^{TM}_{k,\ell}(z) &= \frac{1-z}{(1+z)(1-(k+\ell)z)(1-(k-\ell)z)} \\ &= \zeta^{pd}_{k,\ell} \frac{1}{(1-(k-\ell)z)} = \zeta^{pd}_{k,\ell} \zeta^{TM/pd}_{k,\ell} \end{aligned}$$

where $\zeta^{TM/pd}_{k,\ell}$ is the quotient zeta function between the generalised Thue-Morse and period doubling substitutions.

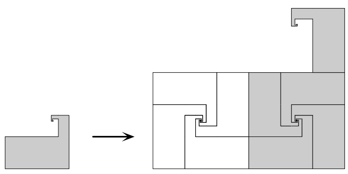
CHAPTER 5

Two-dimensional substitution tiling spaces

Substitution tiling spaces in two dimensions are naturally more complex. However, some are very much related to their 1-dimensional counterparts as we will see in this chapter. This realisation allows for a simpler computation of the quotient cohomology in certain situations. Let us analyse three tiling spaces in two dimensions, namely the squiral and Chacon substitution tilings, which are examples of block substitutions with inflation factor 3, and the generalised chair tilings with inflation factor 2. Although, the family of the generalised chair tilings already appeared in [**BS11**], we still present them here, where we give some corrections to the minor errors present in the original text. For these tiling spaces, we present a breakdown of their factors and classify the factors through our notion of good matches.

5.1. The squiral tiling and its factors

The *squiral tiling* [**GS87**, Fig. 10.1.4] is an example of a 2-dimensional substitution tiling defined using only *one* prototile (together with its three other 90-degree rotated versions and their reflections) with infinitely many edges. It is derived via the following primitive substitution with inflation factor 3, where a reflected prototile is given a different shade for easier identification.§



The dynamical spectrum and other related properties of the squiral tiling have already been analysed in detail [**BG13b**], and recently, the cohomology of the squiral tiling space and its factors were systematically studied [**BGG13**]. However, the work done here would be a first attempt to analyse the structure of the squiral tiling space through quotient cohomology, which complements the study done in [**BGG13**].

§Image created by M. Baake & U. Grimm, which may be found in [**BG13a**, p. 215]; image used with their kind permission.

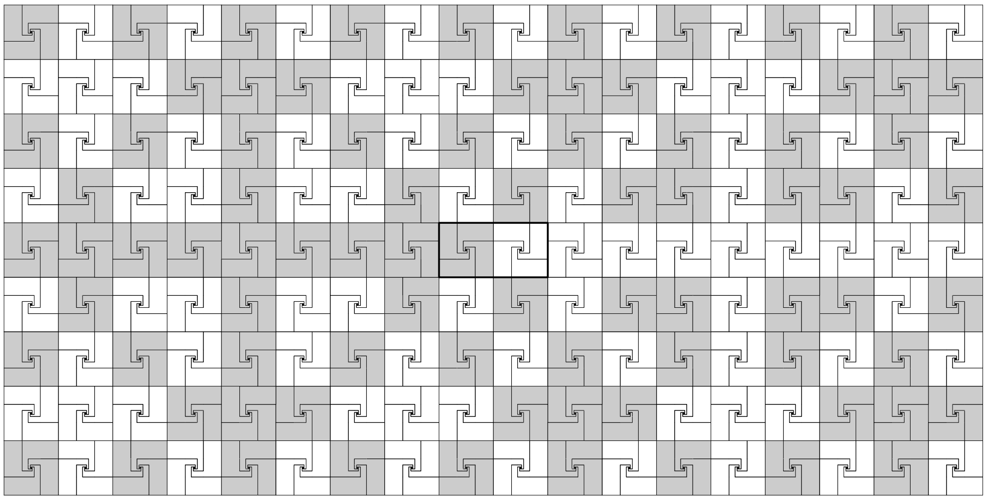
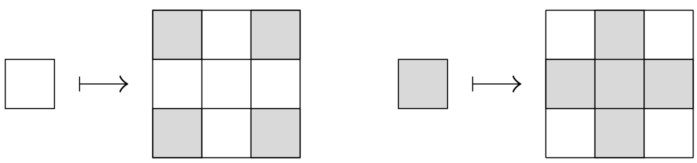


FIGURE 5.1. A rectangular patch of the squiral tiling. The rectangle in the middle highlights how the squiral prototiles group together to form square tiles, which allows us to write the squiral substitution as a planar block substitution. Image created by M. Baake & U. Grimm, found in [**BG13a**, Fig. 6.24]; image used with their kind permission.

At its current form, the squiral substitution does not seem something that fits into the settings of our study because of the irregular shape of its prototiles. However, a simple observation about how these irregularly shaped prototiles clump up together allows us to define the squiral tiling through a planar block substitution that is equivalent to the original form. Figure 5.1 shows a rectangular patch of the squiral tiling, where it is evident that every group of four squiral prototiles forms a square of uniform colour with the prototiles combining perfectly. The marked rectangle in the middle of the tiling is a seed for a tiling obtained via the following planar block substitution



It becomes clear that the two substitution tilings are equivalent (more precisely, they are MLD), and the latter substitution is something that fits nicely into the framework of our study.

The squiral tiling has a special property that the substitution commutes with a 2-fold operation of swapping or inverting the colours analogous to the

Thue-Morse case. Keeping this property in mind, we use the following symbolic planar substitution rule to define the squiral substitution tiling space instead. We have

$$1 \longmapsto \begin{bmatrix} \bar{1} & 1 & \bar{1} \\ 1 & 1 & 1 \\ \bar{1} & 1 & \bar{1} \end{bmatrix} \qquad \bar{1} \longmapsto \begin{bmatrix} 1 & \bar{1} & 1 \\ \bar{1} & \bar{1} & \bar{1} \\ 1 & \bar{1} & 1 \end{bmatrix} \tag{5.1}$$

with $\bar{1} := -1$ so that $\bar{\bar{1}} = -\bar{1} = -(-1) = 1$ and $-(1) = -1 = \bar{1}$. Note that without regard, whenever we say the squiral substitution, we mean to use the symbolic block substitution in (5.1) instead of the original substitution with irregularly shaped prototiles. Also, we denote by Ω_S the hull of the squiral substitution.

5.1.1. The squiral tiling factors and their cohomology. Similar to the sliding block map between the hulls of the Thue-Morse and period doubling substitutions introduced earlier, we can also define sliding block maps on the squiral substitution to obtain some factors of the tiling space. We introduce a sliding block map on the set of legal 2×2 block patches of the squiral substitution by assigning a letter to each of them. The squiral substitution admits $2^4 - 2 = 14$ distinct legal 2×2 patches, with only the two 2×2 blocks of identical entries not included. Analogous to the case of the (generalised) Thue-Morse tiling space, we want a sliding block map that identifies pairs of blocks with inverted colours (or swapped bars). We achieve this through the following.

$$\begin{gathered} \pm \begin{bmatrix} 1 & 1 \\ \bar{1} & 1 \end{bmatrix} \longmapsto a \quad \pm \begin{bmatrix} 1 & 1 \\ 1 & \bar{1} \end{bmatrix} \longmapsto b \quad \pm \begin{bmatrix} 1 & \bar{1} \\ 1 & 1 \end{bmatrix} \longmapsto c \quad \pm \begin{bmatrix} \bar{1} & 1 \\ 1 & 1 \end{bmatrix} \longmapsto d \\ \pm \begin{bmatrix} 1 & 1 \\ \bar{1} & \bar{1} \end{bmatrix} \longmapsto e \quad \pm \begin{bmatrix} 1 & \bar{1} \\ 1 & \bar{1} \end{bmatrix} \longmapsto f \quad \pm \begin{bmatrix} 1 & \bar{1} \\ \bar{1} & 1 \end{bmatrix} \longmapsto g \end{gathered} \tag{5.2}$$

Here, we use the same exact sliding block map introduced in [**BGG13**] for uniformity. Clearly, the factor map defined from this sliding block map is uniformly 2-to-1 on the entire hull of all squiral tilings, analogous to the sliding block map $\phi : \mathbb{Y}_{k,\ell}^{TM} \longrightarrow \mathbb{Y}_{k,\ell}^{pd}$ in Chapter 4.3 (also see Remark 3.4).

The sliding block map in (5.2) then induces a primitive substitution (compare Remark 3.4 for methodology) on seven letters given by

$$a \longmapsto \begin{bmatrix} g & g & a \\ d & c & g \\ a & b & g \end{bmatrix} \quad b \longmapsto \begin{bmatrix} f & f & b \\ d & c & g \\ a & b & g \end{bmatrix} \quad c \longmapsto \begin{bmatrix} f & f & c \\ d & c & e \\ a & b & e \end{bmatrix} \quad d \longmapsto \begin{bmatrix} g & g & d \\ d & c & e \\ a & b & e \end{bmatrix}$$
$$e \longmapsto \begin{bmatrix} g & g & e \\ d & c & e \\ a & b & e \end{bmatrix} \quad f \longmapsto \begin{bmatrix} f & f & f \\ d & c & g \\ a & b & g \end{bmatrix} \quad g \longmapsto \begin{bmatrix} g & g & g \\ d & c & g \\ a & b & g \end{bmatrix} \tag{5.3}$$

which defines the hull of the *maximal model set* factor of the squiral tiling space [**BGG13**]. We denote the hull derived from this substitution by $\Omega_{v,h,4}$. The reasoning behind the label $\{v, h, 4\}$ is explained later.

The tiling space $\Omega_{v,h,4}$ is defined through a planar block substitution of inflation factor 3. Thus, it is easy to see that a factor map (analogous to $\psi : \mathbb{Y}^{pd}_{k,\ell} \longrightarrow \mathbb{S}_{k+\ell}$ in Chapter 4.3) identifying all the letters in the substitution yields the two dimensional 3-adic solenoid $\mathbb{S}_3 \times \mathbb{S}_3$. In fact, the projection of $\Omega_{v,h,4}$ onto the solenoid is 1-to-1 almost everywhere via the torus parametrisation of general model sets [**BLM07**]. Of interest to us are those tiling spaces in between $\Omega_{v,h,4}$ and $\mathbb{S}_3 \times \mathbb{S}_3$, where the projection onto the solenoid fails to be 1-to-1. We need factor maps to find these tiling spaces, and to do so, we define factor maps by simply identifying certain letters in the substitution (5.3) in a consistent and compatible manner. Getting all such factor maps can be done via brute force, i.e., through a computer search. There are 110 factor maps that can be found this way, which yield 110 factor spaces between $\Omega_{v,h,4}$ and the solenoid $\mathbb{S}_3 \times \mathbb{S}_3$. However, many of these factor spaces have isomorphic cohomology groups.

In the analysis of the quotient cohomology between factors of the squiral tiling space, it is helpful to look at the configurations of the different translation orbits in $\Omega_{v,h,4}$, similar to how Barge and Sadun did with the chair tiling in [**BS11**]. This then allows us to see when the framework in Proposition 3.11 is available for use. Closely following the approach done in [**BGG13**], we first consider the form of the supertiles of order n arising from each of the seven letters. We get the following block structures,

$$a \longmapsto \begin{bmatrix} G & a \\ X & G^T \end{bmatrix} \quad b \longmapsto \begin{bmatrix} F & b \\ X & G^T \end{bmatrix} \quad c \longmapsto \begin{bmatrix} F & c \\ X & E^T \end{bmatrix} \quad d \longmapsto \begin{bmatrix} G & d \\ X & E^T \end{bmatrix}$$
$$e \longmapsto \begin{bmatrix} G & e \\ X & E^T \end{bmatrix} \quad f \longmapsto \begin{bmatrix} F & f \\ X & G^T \end{bmatrix} \quad g \longmapsto \begin{bmatrix} G & g \\ X & G^T \end{bmatrix} \tag{5.4}$$

with X being a square block of dimension $3^n - 1$, and E, F and G being rows whose entries are all e, f and g, respectively. Note that X is exactly the same for all supertiles and E^T and G^T denote the transpose of E and G, and so form columns.

Blowing up legal blocks by arranging the infinite level supertiles in (5.4) accordingly yields the following configurations, each of them representing a translation orbit in $\Omega_{v,h,4}$.

$$\begin{matrix} \begin{bmatrix} X & E^T & X \\ G & a & F \\ X & G^T & X \end{bmatrix} & \begin{bmatrix} X & E^T & X \\ F & b & G \\ X & G^T & X \end{bmatrix} & \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix} & \begin{bmatrix} X & G^T & X \\ G & d & F \\ X & E^T & X \end{bmatrix} \\ \begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} & \begin{bmatrix} X & G^T & X \\ F & f & F \\ X & G^T & X \end{bmatrix} & \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix} & \end{matrix} \tag{5.5}$$

Further, these configurations form the seeds of the seven fixed points of the (maximal model set) substitution in (5.3), where the origin or reference point is placed somewhere near the centre.

Baake, Gähler and Grimm [**BGG13**] studied the break down of the factors between $\Omega_{v,h,4}$ and $\mathbb{S}_3 \times \mathbb{S}_3$ and computed their cohomology groups. Let us give a summary on how the factors come about to form the general structure of the hierarchy of the squiral tiling factor spaces. The tiling space $\Omega_{v,h,4}$ consists of one copy of the 2-dimensional 3-adic solenoid $\mathbb{S}_3 \times \mathbb{S}_3$, two copies of the 1-dimensional solenoid $\mathbb{S}_3$, i.e., one vertical and one horizontal (sub)solenoinds, and four extra fixed points near the origin of $\mathbb{S}_3 \times \mathbb{S}_3$. This explains the name given to $\Omega_{v,h,4}$. To get to the other factors, the different components of $\Omega_{v,h,4}$ are either cut or glued and projected down to the solenoid $\mathbb{S}_3 \times \mathbb{S}_3$. Removing the subsolenoid component either horizontally ($f = g$) or vertically ($e = g$) gives us the tiling space $\Omega_{v,4}$ or $\Omega_{h,4}$. Removing the remaining subsolenoid ($e = f = g$) brings us to Ω_4. Eliminating the extra fixed points from this point on one at a time until they are all gone gives us Ω_3 ($a = b$ or $a = d$), Ω_2 ($a = b$, $c = d$; or $a = d$, $b = c$), Ω_1 ($e = f = g = c = d$, $a = b$; or $e = f = g = b = c$, $a = d$) and eventually $\Omega_0 := \mathbb{S}_3 \times \mathbb{S}_3$ (all letters identified). Removing the extra fixed points can also be done before eliminating the second subsolenoid, which would then give us $\Omega_{v,3}$ ($f = g$, $a = b$), $\Omega_{h,3}$ ($e = g$, $a = d$), $\Omega_{v,2}$ ($f = g$, $a = b$, $c = d$), and $\Omega_{h,2}$ ($e = g$, $a = d$, $b = c$). Other cuts are not possible through the type of factor maps we considered that simply identify certain symbols in the substitution. The hierarchy of the factors of the squiral substitution obtained is illustrated in Figure 5.2. Also see [**BGG13**].

Remark 5.1. It is possible to introduce a bigger sliding block map on $\Omega_{v,h,4}$ (say, a sliding block map on its legal 2×2 block patches), which can produce other factors we have not accounted for, such as $\Omega_{v,1}$ or even Ω_v. However, for

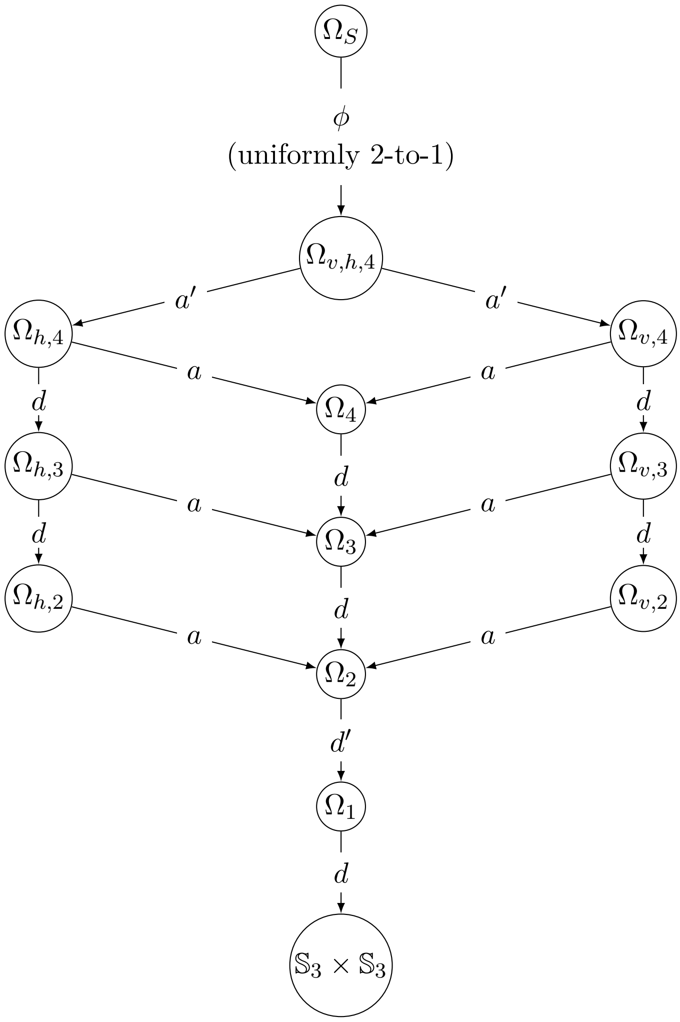


FIGURE 5.2. The squiral tiling space and its factors related via factor maps of type either a or d. As uncomposed maps, $a' = a$ and $d' = d$. The distinction becomes relevant when considering composed maps.

our purpose, there is no need to do this as the 110 factors obtained from simply identifying certain symbols in the substitution is already a good number of tiling factors to consider. ♢

The cohomology of the factor tiling spaces is summarised in Table 5.1. For all (tiling) spaces, we have $H^0 \cong \mathbb{Z}$ and $H^1 \cong \mathbb{Z}[\frac{1}{3}]^2$.

Name	$H^2(\Omega)$	Identifications
Squiral	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^6$	
$v, h, 4$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2$	maximal model set factor
$v, 4$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^3$	$f = g$
$v, 3$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2$	$f = g$, $a = b$
$v, 2$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^1$	$f = g$, $a = b$, $c = d$
4	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^4$	$e = f = g$
3	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^3$	$e = f = g$, $a = b$
2	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^2$	$e = f = g$, $a = b$, $c = d$
1	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^1$	$e = f = g = c = d$, $a = b$
$h, 4$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^3$	$e = g$
$h, 3$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2$	$e = g$, $a = d$
$h, 2$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^1$	$e = g$, $a = d$, $b = c$
4	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^4$	$e = f = g$
3	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^3$	$e = f = g$, $a = d$
2	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^2$	$e = f = g$, $a = d$, $b = c$
1	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^1$	$e = f = g = b = c$, $a = d$
$\mathbb{S}_3 \times \mathbb{S}_3$	$\mathbb{Z}[\frac{1}{9}]$	$a = b = c = d = e = f = g$

TABLE 5.1. The cohomology of the squiral tiling space and some of its factors. For all the (tiling) spaces, we have $H^0 \cong \mathbb{Z}$ and $H^1 \cong \mathbb{Z}[\frac{1}{3}]^2$. Also see [**BGG13**, Tab. 1].

5.1.2. Quotient cohomology between the factors. Knowing the factor maps is only a prerequisite in computing the quotient cohomology between the tiling spaces. Let us categorise the quotient cohomology depending on the type of factor maps and how they are composed. Some computations are rather long and tedious, but are pretty straightforward.

Lemma 5.2. *The quotient cohomology between the squiral tiling space* Ω_S *and a factor in Figure 5.2 is given by the following.*

$\Omega_{v,h,4}:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z}_2,$	$H^2_Q \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^2,$
$\Omega_{v,4}:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z},$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_2,$
$\Omega_{v,3}:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_2,$
$\Omega_{v,2}:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_2,$
$\Omega_{h,4}:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z},$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_2,$
$\Omega_{h,3}:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_2,$
$\Omega_{h,2}:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_2,$
$\Omega_4:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z}^2,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^4,$
$\Omega_3:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z},$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^4,$
$\Omega_2:$	$H^0_Q \cong 0,$	$H^1_Q \cong \mathbb{Z},$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^5,$
$\Omega_1:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^5,$
$\mathbb{S}_3 \times \mathbb{S}_3:$	$H^0_Q \cong 0,$	$H^1_Q \cong 0,$	$H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^6.$

Proof. Let Ω be a (tiling) space in Figure 5.2 together with the factor map $f : \Omega_S \longrightarrow \Omega$. Knowing how the cohomology of Ω maps into the cohomology of Ω_S and using the long exact sequence (3.7) allows us to compute the quotient cohomology. Note that the long exact sequence for Ω_S and Ω is given by

$$\begin{array}{c} 0 \longrightarrow \underset{}{\overset{H^0(\Omega)}{\overset{\|}{\mathbb{Z}}}} \xrightarrow{f_0^*} \overset{H^0(\Omega_S)}{\overset{\|}{\mathbb{Z}}} \longrightarrow H^0_Q \longrightarrow \overset{H^1(\Omega)}{\overset{\|}{\mathbb{Z}[\frac{1}{2}]^2}} \xrightarrow{f_1^*} \overset{H^1(\Omega_S)}{\overset{\|}{\mathbb{Z}[\frac{1}{2}]^2}} \longrightarrow \\ H^1_Q \longrightarrow H^2(\Omega) \xrightarrow{f_2^*} \underset{H^2(\Omega_S)}{\underset{\|}{\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^6}} \longrightarrow H^2_Q \longrightarrow 0 \end{array}$$

Similar to the computation in Example 4.1, we find that the induced maps f_0^* and f_1^* are isomorphisms for any factor map f and factor Ω of the squiral tiling space. Thus by Corollary 3.6, it follows that $H^0_Q \cong 0$ and $H^1_Q \cong \ker f_2^*$ and so knowing the action of f_2^* gives us the quotient cohomologies H^1_Q and $H^2_Q \cong H^2(\Omega_S)/f_2^*(H^2(\Omega))$.

First, let us consider the sliding block map $\phi : \Omega_S \longrightarrow \Omega_{v,h,4}$ as defined above, and recall that $H^2(\Omega_{v,h,4}) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2$. The induced map ϕ_2^* sends the summands $\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2$ in $H^2(\Omega_{v,h,4})$ isomorphically into the same summands in $H^2(\Omega_S)$ and acts by multiplication of 2 on the rest of the summands, i.e., $\mathbb{Z}_2$ gets mapped to $2\mathbb{Z}_2 = 0$, and $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ gets sent to $2\mathbb{Z} \oplus 2\mathbb{Z}$. Thus, $H^1_Q \cong \ker \phi_2^* \cong \mathbb{Z}_2$ and $H^2_Q \cong H^2(\Omega_S)/\phi_2^*(H^2(\Omega_{v,h,4})) \cong$

$\mathbb{Z}^4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^2$. The long exact sequence (of quotient cohomologies) for Ω_S and $\Omega_{v,h,4}$ then reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\phi_0^*} \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}[\tfrac{1}{3}]^2 \xrightarrow{\phi_1^*} \mathbb{Z}[\tfrac{1}{3}]^2 \longrightarrow \mathbb{Z}_2 \longrightarrow$$
$$\mathbb{Z}[\tfrac{1}{9}] \oplus \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2 \xrightarrow{\phi_2^*} \mathbb{Z}[\tfrac{1}{9}] \oplus \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}^6 \longrightarrow \mathbb{Z}^4 \oplus \mathbb{Z}_2^2 \longrightarrow 0.$$

The same analogy applies to the rest of the factor spaces. For $\Omega := \Omega_{v,j+1}$ (or $\Omega := \Omega_{h,j+1}$), $j \in \{1,2,3\}$ with $H^2(\Omega) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^j$, the map f_2^* sends the summands $\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]$ isomorphically onto the same summands in $H^2(\Omega_S)$, while maps exactly one summand of $\mathbb{Z}^j$ to $2\mathbb{Z}$, at most one summand isomorphically into $\mathbb{Z}$ in $H^2(\Omega_S)$, and whatever is left gets mapped to zero. In result, we get $H_Q^1 \cong \mathbb{Z}^{j-2}$ and $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^{6-j} \oplus \mathbb{Z}^{j-2} \oplus \mathbb{Z}_2$, where $\mathbb{Z}^{j-2} := 0$ for $j \neq 3$.

Finally, for $\Omega := \Omega_j$, $j \in \{0,1,2,3,4\}$, recall that $H^2(\Omega) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^j$ and $\Omega_0 = \mathbb{S}_3 \times \mathbb{S}_3$. The map f_2^* sends $\mathbb{Z}[\frac{1}{9}]$ isomorphically onto $\mathbb{Z}[\frac{1}{9}]$ in $H^2(\Omega_S)$, and so (for $j = 0$) we get $H_Q^1(\Omega_S, \mathbb{S}_3 \times \mathbb{S}_3) \cong 0$ and $H_Q^2(\Omega_S, \mathbb{S}_3 \times \mathbb{S}_3) \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^6$. Further, for $j \in \{1,2\}$, f_2^* maps $\mathbb{Z}$ isomorphically onto $\mathbb{Z}$ in $H^2(\Omega_S)$ and whatever is left to zero, and so we have $H_Q^1 \cong \mathbb{Z}^{j-1}$ and $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^5$. Meanwhile, for $j \in \{3,4\}$, f_2^* maps exactly two summands of $\mathbb{Z}^j$ isomorphically into $\mathbb{Z}^2$ in $H^2(\Omega_S)$ and maps the rest to zero. Thus, $H_Q^1 \cong \mathbb{Z} \oplus \mathbb{Z}^{j-3} = \mathbb{Z}^{j-2}$ and $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^4$. □

Lemma 5.3. *The quotient cohomology between the (maximal model set) factor $\Omega_{v,h,4}$ and any one of its factors in Figure 5.2 is given by the following.*

$\Omega_{v,4}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong \mathbb{Z}$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]$,
$\Omega_{v,3}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}_2$,
$\Omega_{v,2}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z} \oplus \mathbb{Z}_2$,
$\Omega_{h,4}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong \mathbb{Z}$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]$,
$\Omega_{h,3}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}_2$,
$\Omega_{h,2}$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z} \oplus \mathbb{Z}_2$,
Ω_4 :	$H_Q^0 \cong 0$,	$H_Q^1 \cong \mathbb{Z}^2$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2$,
Ω_3 :	$H_Q^0 \cong 0$,	$H_Q^1 \cong \mathbb{Z}$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}_2$,
Ω_2 :	$H_Q^0 \cong 0$,	$H_Q^1 \cong \mathbb{Z}$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$,
Ω_1 :	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$,
$\mathbb{S}_3 \times \mathbb{S}_3$:	$H_Q^0 \cong 0$,	$H_Q^1 \cong 0$,	$H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2$.

Proof. The results follow similarly as in the proof of the previous lemma. However, worth looking at are the factor maps $f : \Omega_{v,h,4} \longrightarrow \Omega_{v,4}$ (or $\Omega_{h,4}$)

and $g : \Omega_{v,h,4} \longrightarrow \Omega_4$, as the only cases where there is no torsion in their respective quotient cohomology. As we will see in the following lemmas, these scenarios are not really surprising. Nonetheless, let us see what happens to f_2^* and g_2^* and compute the quotient cohomology in the spirit of Corollary 3.6.

The induced maps f_0^* and f_1^* are isomorphisms, and so we get $H_Q^0 \cong 0$ and $H_Q^1 \cong \ker f_2^*$. As $H^2(\Omega_{v,4}) \cong H^2(\Omega_{h,4}) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^3$, the map f_2^* sends the summands $\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2$ isomorphically into the same summands in $H^2(\Omega_{v,h,4}) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2$. The extra $\mathbb{Z}$ in $H^2(\Omega_{v,4})$ is mapped to $\mathbb{Z}$ (modulo 2) in $\mathbb{Z}_2$, and so we get $\ker f_2^* \cong 2\mathbb{Z}$. This explains $H_Q^1 \cong 2\mathbb{Z} \cong \mathbb{Z}$. Also, we get $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]$.

Similarly, g_0^* and g_1^* are isomorphisms, and g_2^* sends the summands $\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^2$ in $H^2(\Omega_4) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^4$ isomorphically into the same summands in $H^2(\Omega_{v,h,4})$. The remaining $\mathbb{Z}^2$ are then mapped both to the summand $\mathbb{Z}_2$ similar to f_2^* above. Thus, $H_Q^1 \cong \ker g_2^* \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \cong \mathbb{Z}^2$ and $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]^2$. □

Lemma 5.4. *The quotient cohomology between any two adjacent (tiling) spaces in Figure 5.2 is $H_Q^2 \cong \mathbb{Z}[\frac{1}{3}]$, $H_Q^1 \cong \mathbb{Z}$ and $H_Q^0 \cong 0$ if the type of factor map relating the two spaces is labelled a (or a′); otherwise, we get $H_Q^2 \cong \mathbb{Z}$ and $H_Q^1 \cong H_Q^0 \cong 0$ if the factor map type is labelled d (or d′).*

Proof. For any factor map of type labelled a (or a'), the translation orbits (among adjacent tiling spaces in the diagram) are identified through a 1-dimensional tiling subspace either along the horizontal or vertical direction. This 1-dimensional subspace is none other than the 3-adic solenoid $\mathbb{S}_3$. By Proposition 3.11, it follows that $H_Q^2 \cong H_Q^1(\mathbb{S}_3 \times \mathbb{S}_3, \mathbb{S}_3) \cong H^1(\mathbb{S}_3) \cong \mathbb{Z}[\frac{1}{3}]$, $H_Q^1 \cong H_Q^0(\mathbb{S}_3 \times \mathbb{S}_3, \mathbb{S}_3) \cong H^0(\mathbb{S}_3) \cong \mathbb{Z}$ and $H_Q^0 \cong 0$. This is precisely degeneration A (of Example 3.12).

On the other hand for the factor maps of type d (or d'), the translation orbits are identified through a single point near the origin (the single letter at the very centre), and so by Proposition 3.11, we have $H_Q^2 \cong \mathbb{Z}^{2-1} = \mathbb{Z}$ and $H_Q^1 \cong H_Q^0 \cong 0$ (degeneration D of Example 3.15). □

Example 5.5. Let us see in much more detail how the previous proof works for the particular case where we want to get the quotient cohomology between the tiling spaces $\Omega_{v,4}$ and Ω_4. Recall that the tiling space $\Omega_{v,4}$ is obtained from the substitution (5.3) with the letters f and g identified. On the other hand, the tiling space Ω_4 is obtained from (5.3) by identifying e, f and g. The translation orbits in (5.5) become

$$\begin{bmatrix} X & E^T & X \\ G & a & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ G & b & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & c & G \\ X & E^T & X \end{bmatrix}$$

$$\begin{bmatrix} X & G^T & X \\ G & d & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

in $\Omega_{v,4}$. To get Ω_4 as a factor of $\Omega_{v,4}$, we just need to identify e and g in the configurations above. As a result, only the last two translation orbits merge and are identified in Ω_4. By doing so, only 1-dimensional vertical strips need to get identified, as the rest of the letters outside these vertical strips are already the same even before the identification of e and g. The vertical strips correspond to E^T and G^T identified. These 1-dimensional tiling subspaces can be viewed as the inverse limit of the substitutions $\{e \longmapsto eee\}$ and $\{g \longmapsto ggg\}$, which are both $\mathbb{S}_3$. Hence, we get $H^2_Q(\Omega_{v,4}, \Omega_4) \cong H^1(\mathbb{S}_3) \cong \mathbb{Z}[\frac{1}{3}]$, $H^1_Q(\Omega_{v,4}, \Omega_4) \cong H^0(\mathbb{S}_3) \cong \mathbb{Z}$ and $H^0_Q(\Omega_{v,4}, \Omega_4) \cong 0$ by Proposition 3.11. $\diamondsuit$

Recall that two factor maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ form a good match if and only if $H^*_Q(X, Z) \cong H^*_Q(X, Y) \oplus H^*_Q(Y, Z)$. If so, we write $g \circ f := fg = (f)(g)$, otherwise, we write $fg = (fg) \neq (f)(g)$.

Lemma 5.6. *The compositions $a'a$ and d^j for $j \in \{2, 3, 4\}$ (with or without d') in Figure 5.2 are good matches, i.e., $a'a = (a')(a)$ and $d^j = (d)^j$ for $j \in \{2, 3, 4\}$.*

Proof. The case $a'a = (a')(a)$ is very particular to the factor map between $\Omega_{v,h,4}$ and Ω_4. (Compare proof of Lemma 5.3.) This can not be computed using the framework in Proposition 3.11 alone, but by incorporating Corollary 3.10, it is still possible to compute the quotient cohomology without using a longer and more rigorous albeit straightforward approach. Let $X_1 \subset \Omega_{v,h,4}$ be obtained from the closure of the union of the following translation orbits in $\Omega_{v,h,4}$.

$$\begin{bmatrix} X & E^T & X \\ G & a & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ F & b & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix}$$

$$\begin{bmatrix} X & G^T & X \\ G & d & F \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

Similarly we define $X_2 \subset \Omega_{v,h,4}$ from the following translation orbits.

$$\begin{bmatrix} X & E^T & X \\ G & a & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ F & b & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix}$$

$$\begin{bmatrix} X & G^T & X \\ G & d & F \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & f & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

It then follows that Ω_4 is homeomorphic to $X_1 \cap X_2$, and by Corollary 3.10, it follows that $H_Q^*(\Omega_{v,h,4}, \Omega_4) \cong H_Q^*(X_1 \cap \Omega_4) \oplus H_Q^*(X_2 \cap \Omega_4)$. By Proposition 3.8, we have $H_Q^*(\Omega_{v,h,4}, \Omega_{v,4}) \cong H_Q^*(X_1, \Omega_4) \cong H^*(\mathbb{S}_3)$ and $H_Q^*(\Omega_{v,4}, \Omega_4) \cong H_Q^*(X_2, \Omega_4) \cong H^*(\mathbb{S}_3)$ (degeneration A). Then the assertion $a'a = (a)(a)$ follows from Lemma 5.4.

On the other hand, identifying tiling spaces through the composition d^j means identifying some or all of the letters a, b, c and d in (5.3). Note that necessarily, some if not all of the letters in $\{e, f, g\}$ must also be identified for compatibility. In terms of the translation orbits and their configurations, only the single letters around the origin are getting identified. By Proposition 3.11, we have $H_Q^2 \cong \mathbb{Z}^j = \bigoplus_j \mathbb{Z}$ and $H_Q^1 \cong H_Q^0 \cong 0$, for $j \in \{1, 2, 3, 4\}$ (degeneration D). The index j indicates how many of the letters in $\{a, b, c, d\}$ are removed after identification. □

We now analyse the decomposition of the factor maps into what are considered good matches. In this way, we see how the quotient cohomology groups break down and combine together depending on the factor maps between the tiling spaces. For factor maps that are good matches, the general idea is the same as what we saw in the proof of the previous lemmas. Good matches depend on the factor maps specifying how the letters in the substitution are identified.

Lemma 5.7. *The quotient cohomology between any two (tiling) spaces in Figure 5.2 is given by how the types of factor maps between them are composed.*

We have the following decomposition of maps into good matches.

$$
\begin{array}{rlll}
ad = (a)(d): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}, \\
add = (a)(d)(d): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2, \\
\hline
ad' = (ad'): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}], \\
add' = (ad')(d): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}, \\
addd' = (ad')(d)(d): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2, \\
\hline
a'd = (a'd): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}_2, \\
a'd^2 = (a'd)(d): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z} \oplus \mathbb{Z}_2, \\
aa' = (a)(a'): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2, \\
aa'd = (a)(a'd): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}_2, \\
aa'd^2 = (a)(a'd)(d): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2, \\
\hline
aa'd^2d' = (a'd)(ad')(d): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2, \\
a'ad^3d' = (a'd)(ad')(d)^2: & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2.
\end{array}
$$

Proof. For good matches, the proof follows similarly as in the proofs of the previous lemmas. Worth mentioning are those factor maps that when composed, can no longer be broken into good matches. In particular, we have two compositions, namely $ad' \neq (a)(d')$ and $a'd \neq (a')(d)$, which cannot be written as a composition of good matches. For any of these two, none of the translation orbits in (5.5) can be used to define subspaces X_1 and X_2 (similar as in the previous proof) to which Corollary 3.10 may be applied. The framework in Proposition 3.11 also does not apply in these cases.

The composition $a'd$ already appears in the proof above for computing $H^*_Q(\Omega_{v,h,4}, \Omega_{v,4})$ (or $H^*_Q(\Omega_{v,h,4}, \Omega_{h,4})$) and it is clear that $a'd \neq (a')(d)$ as per definition. So let us just look at what happens to the composition ad'. This factor map is very particular to the case $f : \Omega_{v,2} \longrightarrow \Omega_1$ (or $f : \Omega_{h,2} \longrightarrow \Omega_1$). Similar to what we had above, it turns out that f^*_0 and f^*_1 are isomorphisms, and so $H^1_Q \cong \ker f^*_2$. Futher, the map f^*_2 sends $H^2(\Omega_1) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}$ isomorphically into $H^2(\Omega_{v,2}) \cong \mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}$. Thus, we get $H^1_Q \cong 0$ and $H^2_Q \cong H^2(\Omega_{v,2})/f^*_2(H^2(\Omega_1)) \cong \mathbb{Z}[\frac{1}{3}]$. It then follows that $ad' \neq (a)(d')$. □

Remark 5.8. Let X be a factor of $\Omega_{v,h,4}$ obtained by identifying a and g. Now let Y be a factor of X by further identifying d and e. Finally, suppose Z is a factor of Y, where the letters b and f are also identified. Hence we form a sequence of factor maps that reads $\Omega_{v,h,4} \xrightarrow{\alpha} X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z$. In particular, Z is a tiling space whose substitution is defined on four letters. As it turns out, the number of fixed points of all these spaces are the same, even after the identifications. This is evident in the following, where no translation orbit vanishes under any of the factor maps. For X $(a = g)$, we have the following configuration of the translation orbits:

$$\begin{bmatrix} X & E^T & X \\ G & g & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ F & b & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & d & F \\ X & E^T & X \end{bmatrix}$$
$$\begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & f & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

For Y $(a = g, d = e)$, we get:

$$\begin{bmatrix} X & E^T & X \\ G & g & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ F & b & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & e & F \\ X & E^T & X \end{bmatrix}$$
$$\begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & f & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

And finally for Z $(a = g, d = e, b = f)$:

$$\begin{bmatrix} X & E^T & X \\ G & g & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & E^T & X \\ F & f & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & c & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & e & F \\ X & E^T & X \end{bmatrix}$$
$$\begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ F & f & F \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

It follows that $H_Q^* \cong 0$ for any of these spaces. ♢

5.2. The Chacon tiling space and its factors

The Chacon tiling space denoted by Ω_C is the 2-dimensional hull of the following primitive (block) substitution [**Fra08**, Fig. 38]:

$$1 \longmapsto \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad 2 \longmapsto \begin{bmatrix} 5 & 1 & 4 \\ 2 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad 3 \longmapsto \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 1 \\ 3 & 3 & 5 \end{bmatrix}$$
$$4 \longmapsto \begin{bmatrix} 5 & 1 & 4 \\ 2 & 5 & 1 \\ 4 & 3 & 5 \end{bmatrix} \quad 5 \longmapsto \begin{bmatrix} 5 & 1 & 4 \\ 2 & 5 & 1 \\ 5 & 3 & 5 \end{bmatrix} \tag{5.6}$$

As we will see in this section, the Chacon tiling space has some strong resemblances with the maximal model set factor $\Omega_{v,h,4}$ of the squiral tiling space, but we can immediately tell that the two spaces are inequivalent by simply looking at their cohomology.

A closer inspection of the Chacon substitution reveals an embedded 1-dimensional primitive substitution that defines a 1-dimensional substitution tiling subspace. In particular, looking at the last row in the block substitution of the symbols 1 and 2 shows the embedded substitution $\{1 \longmapsto 112,\, 2 \longmapsto 212\}$, which forms a horizontal 1-dimensional tiling subspace of Ω_C. The symbols 3 and 5 yield an equivalent 1-dimensional tiling subspace. Also, one can look at the first column of the block substitution of the symbols 1 and 3, from which the embedded vertical substitution $\{1 \longmapsto 113,\, 3 \longmapsto 313\}$ is revealed forming a vertical 1-dimensional tiling subspace. The same can be observed with the symbols 2 and 5. This embedded substitution defines a primitive substitution on two symbols, namely $\{a \longmapsto aab,\, b \longmapsto bab\}$, which is related to the classical (1-dimensional) Chacon sequences as we discuss next. Also, we will see that the embedded substitution is just a generalised period doubling in disguise!

5.2.1. The Chacon sequences. The (classical) 1-dimensional Chacon sequence can be constructed from two letters, say s and t, beginning with s and whose succeeding terms are obtained via the following non-primitive substitution, namely

$$\{s \longmapsto ssts,\, t \longmapsto t\}. \tag{5.7}$$

The first few terms of the sequence can be read immediately after a finite number of iterations:

$$s \longmapsto ssts \longmapsto ssts\; ssts\; t\; ssts \longmapsto ssts\; ssts\; t\; ssts\; ssts\; ssts\; t\; \ldots$$

The limit of the infinite iteration above, i.e., the (one-sided) fixed point of the substitution, defines a Chacon sequence. It is possible to define a primitive version of the previous substitution, which yields a fixed point that is locally derivable from that above. Let us consider the primitive substitution

$$\{0 \longmapsto 0012,\, 1 \longmapsto 12,\, 2 \longmapsto 012\}. \tag{5.8}$$

which is equivalent to the previous one. That is, the respective (one-sided) fixed points of (5.7) and (5.8) (from the seeds $|s$ and $|0$) are mutually locally derivable (MLD). To see this, consider the following rules with local support:

$$0 \mapsto s, \quad 1 \mapsto t \quad \text{and} \quad 2 \mapsto s;$$

and for the opposite direction,

$$t \mapsto 1, \quad \text{and} \quad s \mapsto \begin{cases} 2, & \text{if there is a } t \text{ before } s; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we get 0012 0012 12 012... from *ssts ssts t ssts*..., which we also obtain after three iterations of 0 under (5.8). As dynamical systems, the two systems derived from the two fixed points are topologically conjugate, and are thus equivalent.

A *replace-and-resize* version of the last substitution can be obtained by introducing the following invertible local rule: $a \leftrightarrow 0$ and $b \leftrightarrow 12$, which is well-defined by noting that the symbol 2 never appears without a 1 before it in (5.8). In this case, the three original symbols are replaced by two new letters, and more so, the size of the adjacent symbols 12 is halved to accommodate the new letter b. This combinatorial process yields the substitution

$$\varrho_C := \begin{cases} a \longmapsto aab \\ b \longmapsto bab \end{cases} \tag{5.9}$$

which we formally define as the (1-dimensional) Chacon substitution. Clearly, this replace-and-resize version of the classical Chacon substitution is primitive with inflation factor 3. More importantly, it defines a tiling space that is embedded as a tiling subspace in the 2-dimensional Chacon tiling space. It is interesting to note that the cohomology of the 1-dimensional Chacon substitution is computed as

$$H^0 \cong \mathbb{Z}, \qquad \text{and} \qquad H^1 \cong \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z},$$

which is the same as the cohomology of the generalised period doubling substitution for the case $k + \ell = 3$. Indeed, we have the following.

Remark 5.9 (Chacon is a generalised period doubling)**.** The substitution rules ϱ_C and $\varrho^{pd}_{1,2}$ (or $\varrho^{pd}_{2,1}$) define the same hull. Every word of length n that appears in the hull of the Chacon substitution also appears in $\mathbb{Y}^{pd}_{1,2}$ (or $\mathbb{Y}^{pd}_{2,1}$) and vice versa. This can be seen by using induction on n. Note that the two hulls have identical sets of legal 2-, 3-, and 4-letter words, namely

$$\{aa, ab, ba, bb\}, \quad \{aab, aba, abb, baa, bab, bba\},$$
$$\{aaba, aabb, abaa, abba, baab, baba, babb, bbab\}.$$

There is a way to construct the set of legal 4-letter words from the set of legal 3-letter words, and that is by adding a letter to the right (or to the left) of the legal 3-letter words, but avoiding the following forbidden subwords:

$$aaa, bbb, abab, bbaa$$

This set of forbidden subwords is the same for both hulls as evident from their substitution rules in (5.8) and (5.9). It becomes clear then by induction, that starting with identical sets of legal k-words, one can construct the set of legal $(k+1)$-words for both the hulls using the method described above, by just avoiding the forbidden subwords, which in turn are again identical sets of legal

$(k+1)$-words. This shows that for any n, the set of legal n-words are identical for the two hulls, and hence establishes the equality of the two hulls. $\diamondsuit$

Remark 5.10 (Tile length matters)**.** There are important differences in the dynamical properties of the classical Chacon sequences (5.8) and the Chacon substitution (5.9). For the classical Chacon sequences, the dynamical properties have been studied in detail in [**Pyth02**], where it is shown to be *weakly mixing*. Let us briefly define mixing properties of dynamical systems associated with substitution systems, namely being *strongly* or *weakly* mixing.

A substitution system (Ω, ω) may be given an invariant measure μ, see [**BG13a**, Eq. (4.3)] for instance, and as such, the substitution system becomes a *measure-preserving* dynamical system $(\Omega_\omega, \mathcal{B}, \mu, T)$ from which certain dynamical properties arise. Here, $\mathcal{B}$ is the σ-algebra generated by all the Borel sets (i.e., the σ-algebra generated by all the cyclinder sets as in (2.3)), μ is an invariant measure, and T is a shift operator (e.g., the right shift ω_k considered in Chapter 2.3). For any $A, B \in \mathcal{B}$, the system is *strongly mixing* (under the T-action) if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B),$$

and is *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |\mu(A \cap T^{-k}B) - \mu(A)\mu(B)| = 0.$$

Strongly mixing systems are weakly mixing but not necessarily the converse.

The hull of the Chacon sequences in (5.8) gives the first example of a weakly mixing system (under the $\mathbb{Z}$-action) that is not strongly mixing [**Pyth02**, Lem. 5.5.1]. On the other hand, the generalised period doubling (tiling space) $\mathbb{Y}^{pd}_{1,2}$ or $\mathbb{Y}^{pd}_{2,1}$ has pure point spectrum $\mathbb{Z}[\frac{1}{3}]$ [**BGG12**] and so cannot be a weakly mixing system under the $\mathbb{R}$-action [**Sol99**], and consequently cannot be weakly mixing under an embedded $\mathbb{Z}$-action. The 1-dimensional Chacon substitution ϱ_C is MLD to the generalised period doubling when $k + \ell = 3$, and so has the same dynamical properties. Let us briefly explain the difference between the dynamical systems of the classical Chacon sequence and our 1-dimensional Chacon substitution.

We emphasised in Remark 2.4 how a given symbolic substitution (in one dimension) may be transformed into a (1-dimensional) tiling substitution in such a way that the embedded natural geometric properties of the symbolic substitution are preserved. Recall that this is done primarily via the Perron-Frobenious (PF) theory. Clark and Sadun in [**CS03**] studied how modifying the length of the prototile associated to a symbolic letter alters the dynamical spectrum of the substitution system. This is precisely the case between the varying results given in [**Pyth02**] and [**BGG12, Sol99**]. When the symbolic letters 0, 1 and 2 in (5.8) are given equal tile lengths, then a tiling space $\mathcal{T}_{1,1,1}$ with a weakly mixing spectrum under the $\mathbb{Z}$-action is achieved (c.f., [**CS03**]), as is the context of the result in [**Pyth02**]. This is opposed to the (naturally associated) tiling space $\mathcal{T}_{3,1,2}$ achieved through the PF theory, where the symbols 0, 1 and 2 are given different tile lengths, in the ratio $3:1:2$. Such tiling space has pure point spectrum and is not weakly mixing under the $\mathbb{Z}$-action [**BGG12, Sol99**]. The two tiling spaces (associated to the classical Chacon substitution) $\mathcal{T}_{1,1,1}$ and $\mathcal{T}_{3,1,2}$ are homeomorphic spaces but are not topologically conjugate [**CS03**], whereas the spaces $\mathcal{T}_{3,1,2}$ and $\mathbb{Y}^{pd}_{1,3}$ (or $\mathbb{Y}^{pd}_{3,1}$) are MLD, and this is the context we consider in our study. ♢

In the following, we show how to derive the 2-dimensional Chacon substitution given in (5.6).

5.2.2. The 2-dimensional Chacon substitution. A natural way to extend 1-dimensional tilings to two dimensions is by considering the direct product $\mathcal{A} \times \mathcal{A}$, where $\mathcal{A}$ is the set of prototiles or letters. In the case of the (1-dimensional) classical Chacon substitution (5.7) on two letters $\mathcal{A} := \{s, t\}$, we consider the alphabet $\{s,t\} \times \{s,t\}$, with the following notations,

$$1 := s \times s, \qquad 2 := s \times t, \qquad 3 := t \times s, \qquad 4 := t \times t.$$

The induced substitution on the four new symbols is given by

$$1 \longmapsto ssts \times ssts, \qquad 2 \longmapsto ssts \times t, \qquad 3 \longmapsto t \times ssts, \qquad 4 \longmapsto t \times t.$$

We interpret $ssts \times ssts$ as a four-by-four array of letters, namely

$$\begin{array}{c|cccc} s & (s,s) & (s,s) & (t,s) & (s,s) \\ t & (s,t) & (s,t) & (t,t) & (s,t) \\ s & (s,s) & (s,s) & (t,s) & (s,s) \\ s & (s,s) & (s,s) & (t,s) & (s,s) \\ \hline & s & s & t & s \end{array} \qquad = \qquad \begin{array}{c|cccc} & 1 & 1 & 3 & 1 \\ & 2 & 2 & 4 & 2 \\ & 1 & 1 & 3 & 1 \\ & 1 & 1 & 3 & 1 \\ \hline & & & & \end{array}$$

where we write (x, y) for $x \times y$ with $x, y \in \mathcal{A}$. Similarly for the remaining three, we get

$$\begin{array}{c|cccc} t & (s,t) & (s,t) & (t,t) & (s,t) \\ \hline & s & s & t & s \end{array} \qquad = \qquad \begin{array}{c|cccc} & 2 & 2 & 4 & 2 \\ \hline & & & & \end{array}$$

$$\begin{array}{c|c} s & (t,s) \\ t & (t,t) \\ s & (t,s) \\ s & (t,s) \\ \hline & t \end{array} \quad = \quad \begin{array}{c|c} & 3 \\ & 4 \\ & 3 \\ & 3 \\ \hline & \end{array} \qquad\qquad \begin{array}{c|c} t & (t,t) \\ \hline & t \end{array} \quad = \quad \begin{array}{c|c} & 4 \\ \hline & \end{array}$$

Writing the induced substitution in a more compact manner gives us the following (2-dimensional) direct product substitution.

$$1 \longmapsto \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 4 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}, \quad 2 \longmapsto \begin{bmatrix} 2 & 2 & 4 & 2 \end{bmatrix}, \quad 3 \longmapsto \begin{bmatrix} 3 \\ 4 \\ 3 \\ 3 \end{bmatrix}, \quad 4 \longmapsto \begin{bmatrix} 4 \end{bmatrix}.$$

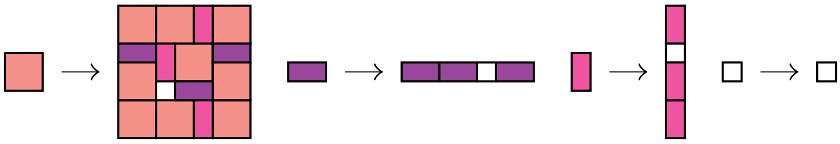
Meanwhile, a *direct product variation* (DPV) substitution may be obtained by rearranging certain entries in the previous substitution to break up the direct product structure in such a way that it still yields a consistent substitution rule. This process should be done with care, as the rearranged version of the substitution must still consistently tile the plane. A DPV of the previous substitution is attained by simply permuting the centre square patch of the four symbols appearing in the substitution of the first symbol and keeping the rest the same to get

$$1 \longmapsto \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & 4 & 2 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}, \quad 2 \longmapsto \begin{bmatrix} 2 & 2 & 4 & 2 \end{bmatrix}, \quad 3 \longmapsto \begin{bmatrix} 3 \\ 4 \\ 3 \\ 3 \end{bmatrix}, \quad 4 \longmapsto \begin{bmatrix} 4 \end{bmatrix}.$$

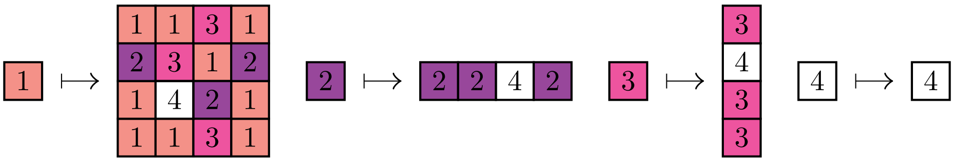
Note that this 2-dimensional Chacon DPV substitution tiles the plane consistently. In particular, there exists a fixed point.

Interestingly, this Chacon DPV substitution has striking similarities with the 2-dimensional Chacon *cut-and-stack* substitution derived separately and differently in [**PR91**], which is shown in Figure 5.3 (A). The similarity becomes apparent when each symbol in the DPV substitution is assigned a unit square and coloured similarly as the tiles in the cut-and-stack substitution, see Figure 5.3 (B). Allowing for some degrees of freedom, the DPV substitution may be obtained from the cut-and-stack substitution, and vice versa, through what is called a *replace-and-rescale* method; also see [**Fra08**].

The substitution rules in Figure 5.3 are not primitive, analogous to the classical Chacon substitution given in (5.7). Without the big square prototile (or the symbol 1), the tiles cannot cover the plane, even after an infinite application of the substitutions. But as in the classical case, the cut-and-stack prototiles can be recoded through the following invertible local rules [**Fra08**], which then induces a primitive substitution.

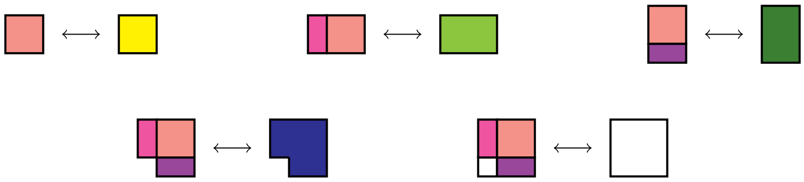


(A)



(B)

Figure 5.3. (A) A 2-dimensional cut-and-stack substitution [**PR91**], and (B) a DPV substitution [**Fra08**] of the classical Chacon substitution.



The local rules applied above induces the following primitive substitution, which is MLD to the cut-and-stack substitution.

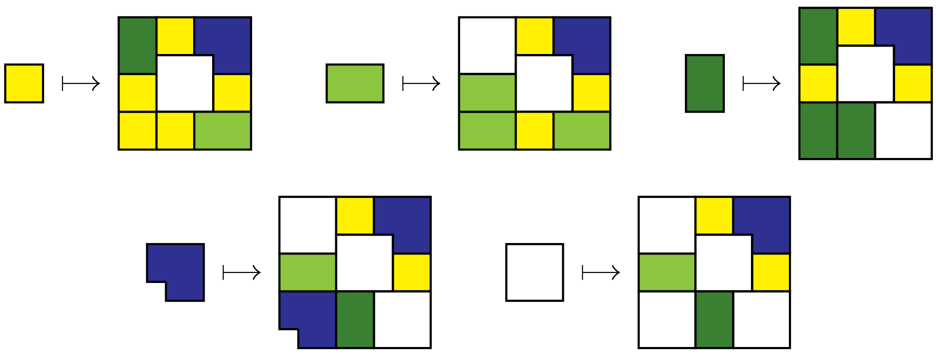
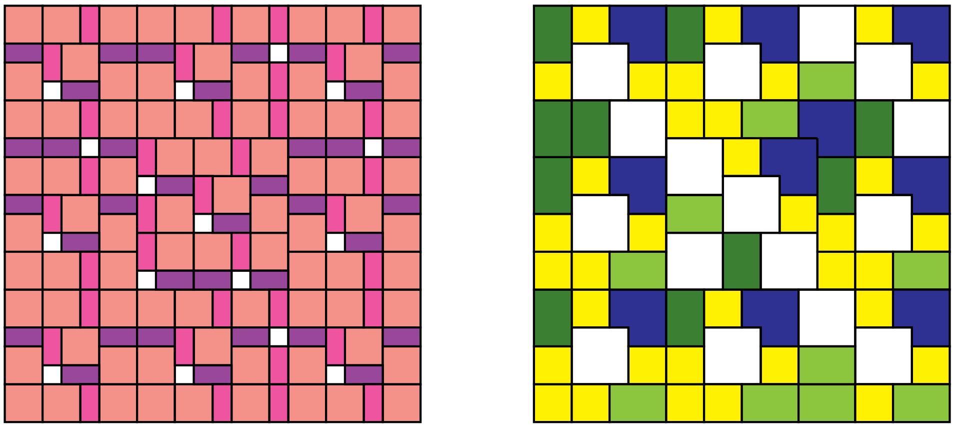


Figure 5.4 shows a comparison between the level 2 supertiles of the pink and yellow prototiles. Finally, a replace-and-scale method applied on the induced primitive cut-and-stack substitution produces a substitution on five congruent square prototiles, distinguished only by their colours, see Figure 5.5. This planar block substitution is the 2-dimensional Chacon substitution provided in (5.6), which will be the focus of our study in the rest of the section.



(A) (B)

FIGURE 5.4. The Chacon level 2 supertiles of the (A) pink, and (B) yellow square prototiles.

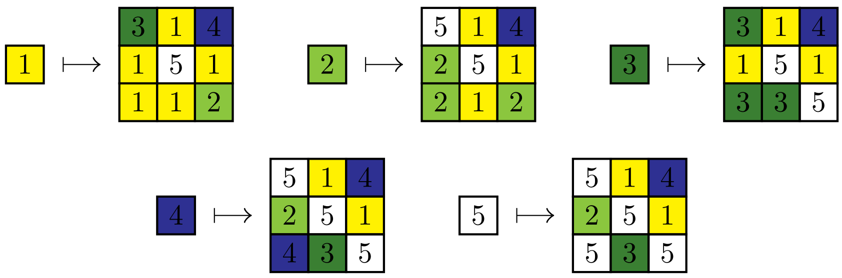


FIGURE 5.5. The 2-dimensional Chacon (planar block) substitution with inflation factor 3.

Remark 5.11. Analogous to their 1-dimensional counterparts, the cut-and-stack substitution (which is the 2-dimensional counterpart of the classical Chacon substitution) induces a dynamical system that is weakly mixing under the $\mathbb{Z}^2$-action [**PR91**], in contrast to the induced dynamical system of the planar block substitution (which is the 2-dimensional counterpart of ϱ_C) that cannot be weakly mixing under the $\mathbb{Z}^2$-action [**Sol99, Fra08**] as the planar block subsitution has a pure point spectrum [**LMS03, FS07**]. Further, the tiling spaces of the two substitutions are not topologically conjugate but are homeomorphic spaces [**CS06**]. ♢

5.2.3. The cohomology of the Chacon tiling space and its factors. Since the 2-dimensional Chacon substitution tiling space Ω_C is defined through a planar block substitution with inflation factor 3, the Chacon tiling space has the 2-dimensional solenoid $\mathbb{S}_3 \times \mathbb{S}_3$ as a factor, onto which it projects 1-to-1 almost everywhere.

Similar to how we proceeded in the squiral case, we want to determine the configuration of the different translation orbits in Ω_C to help compute for the quotient cohomologies. The level-n supertiles of the Chacon substitution have the following block structure,

$$\begin{gathered} 1 \longmapsto \begin{bmatrix} \binom{3}{1} & X \\ 1 & (12) \end{bmatrix} \quad 2 \longmapsto \begin{bmatrix} \binom{5}{2} & X \\ 2 & (12) \end{bmatrix} \quad 3 \longmapsto \begin{bmatrix} \binom{3}{1} & X \\ 3 & (35) \end{bmatrix} \\ 4 \longmapsto \begin{bmatrix} \binom{5}{2} & X \\ 4 & (35) \end{bmatrix} \quad 5 \longmapsto \begin{bmatrix} \binom{5}{2} & X \\ 5 & (35) \end{bmatrix} \end{gathered} \tag{5.10}$$

where X is a square block of dimension $3^n - 1$ that is the same for all supertiles. The horizontal and vertical sequences (ab) and $\binom{b}{a}$ form the Chacon sequences coming from the substitution $\varrho_C = \{a \longmapsto aab,\, b \longmapsto bab\}$ with seeds $|ab$ and $\begin{smallmatrix} b \\ \underline{a} \end{smallmatrix}$. In particular, (12) is a Chacon sequence starting with 1, and similarly $\binom{5}{3}$ is another Chacon sequence starting with 3, but this time in a vertical manner going up, in contrast to the former, which is going to the right.

Arranging two supertiles such that the separating line between them passes near the origin gives us four block arrangements, two with a horizontal and two with a vertical separation line. Note that the separation lines form (1-dimensional) Chacon sequences, where in this case, (ab) and $\binom{b}{a}$ are bi-infinite Chacon sequences:

$$\begin{bmatrix} X \\ (12) \\ X \end{bmatrix}, \quad \begin{bmatrix} X \\ (35) \\ X \end{bmatrix}, \quad \begin{bmatrix} X & \binom{3}{1} & X \end{bmatrix}, \quad \begin{bmatrix} X & \binom{5}{2} & X \end{bmatrix}.$$

Projected to $\mathbb{S}_3 \times \mathbb{S}_3$, the translation orbits of the horizontal pair project to a single translation orbit of a 1-dimensional subsolenoid $\mathbb{S}_3$, and similarly with the vertical pair. By arranging four supertiles of infinite order in such a way that the common corner remains near the origin, gives us the following configurations, where in particular, $(wx)\, a\, (yz)$ means (wx) is a one-sided Chacon fixed point of $wx|$ and (yz) is a one-sided Chacon fixed point of $|yz$.

$$\begin{gathered} \begin{bmatrix} X & \binom{3}{1} & X \\ (12) & 1 & (12) \\ X & \binom{3}{1} & X \end{bmatrix} \begin{bmatrix} X & \binom{3}{1} & X \\ (35) & 1 & (12) \\ X & \binom{5}{2} & X \end{bmatrix} \begin{bmatrix} X & \binom{5}{2} & X \\ (12) & 2 & (12) \\ X & \binom{5}{2} & X \end{bmatrix} \begin{bmatrix} X & \binom{3}{1} & X \\ (35) & 3 & (35) \\ X & \binom{3}{1} & X \end{bmatrix} \\ \begin{bmatrix} X & \binom{5}{2} & X \\ (12) & 4 & (35) \\ X & \binom{3}{1} & X \end{bmatrix} \quad \begin{bmatrix} X & \binom{5}{2} & X \\ (12) & 5 & (35) \\ X & \binom{3}{1} & X \end{bmatrix} \quad \begin{bmatrix} X & \binom{5}{2} & X \\ (35) & 5 & (35) \\ X & \binom{5}{2} & X \end{bmatrix} \end{gathered} \tag{5.11}$$

Each of these configurations represents one of the seeds of the fixed points of the (2-dimensional) Chacon substitution. Analogous to the case of the maximal model set factor $\Omega_{v,h,4}$, the hull of the Chacon substitution also consists of a copy of the 2-dimensional solenoid $\mathbb{S}_3 \times \mathbb{S}_3$, two copies of the subsolenoid $\mathbb{S}_3$, and four extra fixed points above the origin of the solenoid. For this, we denote the Chacon tiling space as $\Omega_C := \Omega_{V,H,4'}$. The Chacon tiling space $\Omega_{V,H,4'}$ is not equivalent to $\Omega_{v,h,4}$ though, as the cohomology of the former reads

$$H^2 \cong \mathbb{Z}[\tfrac{1}{9}] \oplus \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}^3, \quad H^1 \cong \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}, \quad H^0 \cong \mathbb{Z},$$

which is different from that of the tiling space $\Omega_{v,h,4}$. Also, recall that the cohomology of $\Omega_{v,h,4}$ contains torsion. However, the two tiling spaces share the same dynamical zeta function

$$\zeta_C(z) = \frac{(1-z)(1-3z)^2}{(1-z)(1-z)^3(1-3z)^2(1-9z)} = \frac{1}{(1-z)^3(1-9z)} = \zeta_{v,h,4}.$$

Employing the same technique as in the squiral case in searching for factor maps, does not yield a lot of factor spaces. Defining factor maps from simply identifying certain symbols (but maintaining consistency in the substitution) only yields 6 factors between the Chacon tiling space $\Omega_{V,H,4'}$ and the solenoid $\mathbb{S}_3 \times \mathbb{S}_3$. By identifying $1 = 2$ and $3 = 5$ (or $1 = 3$ and $2 = 5$), we are removing a subsolenoid and an extra fixed point in the components of $\Omega_{V,H,4'}$, and so we get $\Omega_{V,3'}$ (or $\Omega_{H,3'}$). Further by identifying $1 = 2 = 3 = 5$, we remove the last subsolenoid together with two extra fixed points in the process to get $\Omega_{1'}$. On the other hand, starting from $\Omega_{V,H,4'}$ again, identifying 4 and 5 removes a fixed point to get $\Omega_{V,H,3'}$. From here, identifying 2, 4 and 5 (or 3, 4 and 5) further removes a subsolenoid, which consequently removes another fixed point as well, and so gives us $\Omega_{H,2'}$ or $\Omega_{V,2'}$. Finally, identifying all the letters produces the solenoid $\mathbb{S}_3 \times \mathbb{S}_3$. We summarise the hierarchy of the Chacon factors in Figure 5.6. Their cohomologies are summarised in Table 5.2.

The general structure of the Chacon substitution in Figure 5.6 shares some resemblance to that of the squiral tiling in Figure 5.2. In particular, $\Omega_{V,3'}$, $\Omega_{V,2'}$ and $\Omega_{1'}$ have the same configuration as $\Omega_{v,3}$, $\Omega_{v,2}$ and Ω_1 respectively. However, it is clear that they are not necessarily equivalent spaces.

The spaces $\Omega_{V,3'}$ and $\Omega_{v,3}$ are not equivalent. To see this, let us compare the structure of their translation orbits. For the former, we have the following configuration of translation orbits:

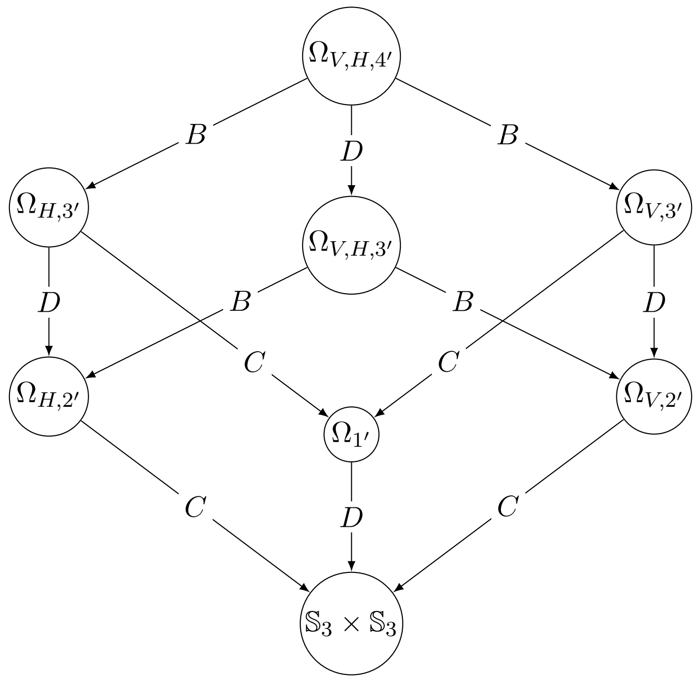


FIGURE 5.6. The Chacon tiling space and its factors related via factor maps of type either B, C or D. Any composition of the factor maps is always a good match.

Name	H^2	H^1	Identifications
$V, H, 4'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^3$	$\mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}$	Chacon tiling
$V, H, 3'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^2$	$\mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}$	$4 = 5$
$V, 3'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 2$, $3 = 5$
$V, 2'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 2$, $3 = 4 = 5$
$H, 3'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^2$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 3$, $2 = 5$
$H, 2'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}^1$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 3$, $2 = 4 = 5$
$1'$	$\mathbb{Z}[\frac{1}{9}] \oplus \mathbb{Z}^4$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 2 = 3 = 5$
$\mathbb{S}_3 \times \mathbb{S}_3$	$\mathbb{Z}[\frac{1}{9}]$	$\mathbb{Z}[\frac{1}{3}]^2$	$1 = 2 = 3 = 4 = 5$

TABLE 5.2. The cohomology of the Chacon tiling and some of its substitution factors. In all cases, $H^0 \cong \mathbb{Z}$.

$$\begin{bmatrix} X & \binom{3}{1} & X \\ (11) & 1 & (11) \\ X & \binom{3}{1} & X \end{bmatrix} \quad \begin{bmatrix} X & \binom{3}{1} & X \\ (33) & 1 & (11) \\ X & \binom{3}{1} & X \end{bmatrix} \quad \begin{bmatrix} X & \binom{3}{1} & X \\ (33) & 3 & (33) \\ X & \binom{3}{1} & X \end{bmatrix}$$

$$\begin{bmatrix} X & \binom{3}{1} & X \\ (11) & 4 & (33) \\ X & \binom{3}{1} & X \end{bmatrix} \quad \begin{bmatrix} X & \binom{3}{1} & X \\ (11) & 3 & (33) \\ X & \binom{3}{1} & X \end{bmatrix}$$

whereas the latter has the following configuration of translation orbits:

$$\begin{bmatrix} X & E^T & X \\ G & a & G \\ X & G^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & c & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & d & G \\ X & E^T & X \end{bmatrix}$$
$$\begin{bmatrix} X & E^T & X \\ G & e & G \\ X & E^T & X \end{bmatrix} \quad \begin{bmatrix} X & G^T & X \\ G & g & G \\ X & G^T & X \end{bmatrix}$$

Clearly, the space $\Omega_{V,3'}$ has the 1-dimensional Chacon substitution as a tiling subspace, which is not locally derivable from any 1-dimensional tiling subspace of $\Omega_{v,3}$. (In particular, a 1-dimensional subsolenoid constitutes the 1-dimensional tiling subspace of $\Omega_{v,3}$). Analogously, the spaces $\Omega_{V,2'}$ and $\Omega_{v,2}$ cannot be equivalent for the same reason. In contrast, the spaces $\Omega_{1'}$ and Ω_1 are equivalent.

Let us now classify the quotient cohomologies between factors of the Chacon tiling space through the following lemmas, whose proofs are similar to those in the previous section.

Lemma 5.12. *The quotient cohomology between adjacent tiling spaces in Figure 5.6 is given by the following, depending on the type of the factor map between them.*

$$\begin{aligned} B: &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong \mathbb{Z}, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z}, \\ C: &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong 0, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z}, \\ D: &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong 0, \quad H_Q^2 \cong \mathbb{Z}. \end{aligned}$$

□

Lemma 5.13. *Any composition of factor maps in Figure 5.6 forms a good match, i.e., $BC = (B)(C)$, $BD = (B)(D)$, $CD = (C)(D)$ and $DBC = (D)(B)(C)$. Their respective quotient cohomologies are given as follows:*

$$\begin{aligned} BC = (B)(C): &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong \mathbb{Z}, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}^2, \\ BD = (B)(D): &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong \mathbb{Z}, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z}^2, \\ CD = (C)(D): &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong 0, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z}^2, \\ DBC = (D)(B)(C): &\quad H_Q^0 \cong 0, \quad H_Q^1 \cong \mathbb{Z}, \quad H_Q^2 \cong \mathbb{Z}[\tfrac{1}{3}]^2 \oplus \mathbb{Z}^3. \end{aligned}$$

□

5.2.4. Quotient zeta functions. Given a tiling space Ω consisting of several components for which the fixed points can be counted separately, we can write its dynamical zeta function ζ_Ω as the product of the partial zeta functions corresponding to its components [**BGG13**]. Such is the case for the Chacon tiling space $\Omega_{V,H,4'}$, along with its factors that we have enumerated. (This is also the case for the maximal model set $\Omega_{v,h,4}$ and its factors.) Recall that the hull $\Omega_{V,H,4'}$ has several components, namely a copy of $\mathbb{S}_3 \times \mathbb{S}_3$, two copies of the subsolenoid $\mathbb{S}_3$ and four extra fixed points. As such, its zeta function reads

$$\zeta_{\Omega_{V,H,4'}} = \frac{(1-3z)^2}{(1-z)(1-9z)} \cdot \left(\frac{1-z}{1-3z}\right)^2 \cdot \left(\frac{1}{1-z}\right)^4 = \zeta_{\mathbb{S}_3\times\mathbb{S}_3}\zeta^2_{\mathbb{S}_3}\zeta^4_p$$

where $\zeta_p = \frac{1}{1-z}$ is the zeta function of an extra fixed point and as usual, $\zeta_{\mathbb{S}_3\times\mathbb{S}_3}$ and $\zeta_{\mathbb{S}_3}$ are the zeta functions of the solenoid $\mathbb{S}_3 \times \mathbb{S}_3$ and subsolenoid $\mathbb{S}_3$.

A consequence of Theorem 3.18 together with the observation above provides a very quick tool in getting the quotient zeta function between any two factors of the Chacon tiling space. All we have to take note of is the difference between the components of the two tiling spaces. For instance, to get the quotient zeta function between $\Omega_{V,H,4'}$ and $\Omega_{V,2'}$, we simply note that the difference among their components is a copy of a subsolenoid and two extra fixed points. Thus, we get the quotient zeta function

$$\zeta_{\Omega_{V,H,4'},\Omega_{V,2'}} = \zeta_{\mathbb{S}_3} \cdot \zeta^2_p = \frac{1-z}{1-3z} \cdot \left(\frac{1}{1-z}\right)^2 = \frac{1}{(1-z)(1-3z)}$$

which is of course consistent with the quotient cohomology between the two tiling spaces. The same can also be done with any two factors of $\Omega_{v,h,4}$

More so, the connection between the degeneracies and the tiling factors becomes more apparent, through the notion of the quotient zeta function. Recall the degeneracies we considered in Chapter 3.2 and let $f : \Omega \longrightarrow \Omega'$ be a factor map associated to some degeneracy, if any. Then the effect of that degeneracy on the components of the hull Ω corresponds to the components of $f(\Omega) = \Omega'$. For instance in the Chacon case, we encounter degenerations B, C and D associated with the factor maps B, C and D. As one may have already noticed, the effect of degeneration B is the removal of a subsolenoid in the hull together with an extra fixed point, while degeneration C removes a subsolenoid and two more extra fixed points. Meanwhile, the effect of degeneration D is simply the removal of an extra fixed point. In particular, the factor map between $\Omega_{V,H,4'}$ and $\Omega_{V,2'}$ is a composition of maps corresponding to the combination of degenerations B and D. Hence, the components of $\Omega_{V,2'}$ is obtained by removing (which more precisely involves some identifications of) one subsolenoid $\mathbb{S}_3$ and two extra fixed points from the hull $\Omega_{V,H,4'}$. (This effect is the same as degeneration C, which should not be surprising because it

Degeneration	Effects on the hull
A	One subsolenoid is removed
B	One subsolenoid and one fixed point are removed
C	One subsolenoid and two fixed points are removed
D	One fixed point is removed
F	Three subsolenoids are removed
G	Two subsolenoids and one fixed point are removed

TABLE 5.3. The effects of the degeneracies on the hull Ω corresponding to its image under a factor map associated to that degeneracy. The degenerations F and G appear in the generalised chair tilings.

is possible to define a sliding block map¶ $c : \Omega_{V,H,4'} \longrightarrow \Omega_{V,2'}$ that is associated to degeneration C.)

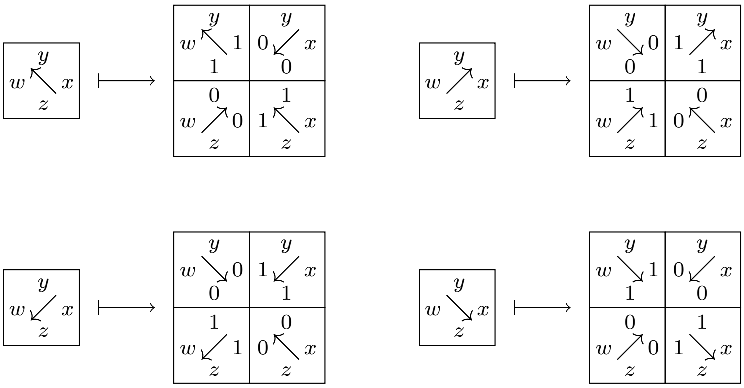
In the squiral case, we encounter a similar effect with degeneration A (associated to the factor maps a or a') by removing a subsolenoid in the component of the hull, in addition to degeneration D (associated to the factor maps b or b') as in the Chacon case.

As we highlight at the end of the next section, the effects of the degeneracies (e.g., A, B, C and D) on the hull is the same for any tiling space (of a constant length planar substitution) that admits such degeneracies. For quick reference, we summarise the effects of the degeneracies in Table 5.3.

5.3. The generalised chair tilings

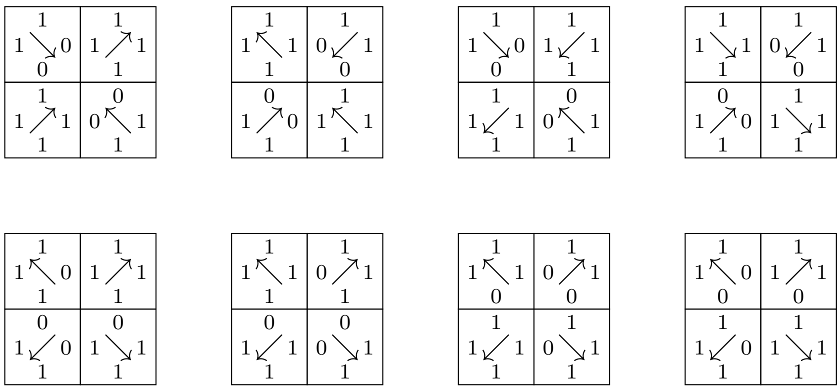
The hull of the classic chair tiling (illustrated in Figure 2.1) belongs to a family of substitution tiling spaces called the *generalised chair tilings*, which Barge and Sadun introduced and analysed in [**BS11**]. Using decorated square tiles as prototiles, the most intricate of them defines the tiling space $\Omega_{X,+}$ through the substitution rule

¶We do not explore such sliding block maps in this study though, as there are too many, around half a million such factor maps.



(5.12)

where $w, x, y, z \in \{0, 1\}$ and with the two labels adjacent to the head of an arrow being the same. Note that the substitution above acts on 32 prototiles. The following 2×2 legal patches are seeds to the eight fixed points of (5.12) representing the different translation orbits in the tiling space $\Omega_{X,+}$ [**BS11**].



(5.13)

Factors of $\Omega_{X,+}$ can be defined by removing and/or identifying certain decorations on the square tiles to which the general substitution rule (5.12) applies. Tiling spaces $\Omega_{a,b}$, with $a \in \{X, /, 0\}$ and $b \in \{+, -, 0\}$, are defined through the identification rules given in Table 5.4. In particular, $\Omega_{X,0}$ is the chair tiling space (i.e., the substitution rule in (5.12) becomes ϱ_{L} in Example 1.27 after the identifications) and $\Omega_{0,0}$ is the 2-dimensional dyadic solenoid $\mathbb{S}_2 \times \mathbb{S}_2$ (obtained as the inverse limit of the block substitution $\{s \longmapsto {}^{s\,s}_{s\,s}\}$). This scheme yields nine (tiling) spaces, whose cohomologies are summarised in Proposition 5.17. The nine spaces are related as follows.

Index		Description
a	X	The four arrows on the square tiles remain.
	$/$	Only the arrows pointing northeast or southwest remain, i.e., arrowheads pointing to other directions are identified.
	0	All arrows are identified/removed.
b	$+$	All four labels remain.
	$-$	Only the labels to the left or to the right remain, i.e., the top and bottom labels are identified.
	0	All labels are identified/removed.

TABLE 5.4. Identification rules on derividing some factors of the generalised chair tiling.

$$\begin{array}{ccccc} \Omega_{X,+} & \xrightarrow{A} & \Omega_{/,+} & \xrightarrow{A} & \Omega_{0,+} \\ \big\downarrow B & & \big\downarrow B & & \big\downarrow B \\ \Omega_{X,-} & \xrightarrow{A} & \Omega_{/,-} & \xrightarrow{A} & \Omega_{0,-} \\ \big\downarrow A & & \big\downarrow A & & \big\downarrow C \\ \Omega_{X,0} & \xrightarrow{A} & \Omega_{/,0} & \xrightarrow{C} & \Omega_{0,0} \end{array} \tag{5.14}$$

Barge and Sadun beautifully computed the quotient cohomology between adjacent tiling spaces appearing in (5.14), using a framework (see Proposition 3.11) discussed in [**BS11**]. The quotient cohomologies, depending on the type of factor maps (i.e., degeneracies), are given by:

$$\begin{array}{llll} A: & H_Q^0 \cong 0, & H_Q^1 \cong \mathbb{Z}, & H_Q^2 \cong \mathbb{Z}[\tfrac{1}{2}], \\ B: & H_Q^0 \cong 0, & H_Q^1 \cong \mathbb{Z}, & H_Q^2 \cong \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}, \\ C: & H_Q^0 \cong 0, & H_Q^1 \cong 0, & H_Q^2 \cong \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}. \end{array} \tag{5.15}$$

Tracing a path in (5.14) pertains to a factor map from one tiling space (starting point) to another tiling space (ending point). As such, paths having identical starting and ending points pertain to equivalent (composition of) factor maps. In this sense, we say that the diagram commutes. We generalise the results in (5.15) by giving the quotient cohomology between tiling spaces in (5.14), depending on the factor map between them as obtained by tracing an arbitrary path. We formalise this as the following lemma, whose calculation is similar to the ones in the previous sections.

Lemma 5.14. *The quotient cohomologies between adjacent tiling spaces in* (5.14) *are given in* (5.15)*; for the remaining pairs of tiling spaces, we have the following decomposition into good matches.*

$$
\begin{array}{rlll}
AA = (A)(A): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}[\tfrac{1}{2}]^2, \\
AB = (A)(B): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}[\tfrac{1}{2}]^2 \oplus \mathbb{Z}, \\
AAB = (A)(A)(B): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^3, & H^2_Q \cong \mathbb{Z}[\tfrac{1}{2}]^3 \oplus \mathbb{Z}, \\
BC = (B)(C): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\tfrac{1}{2}]^2 \oplus \mathbb{Z}^2, \\
\hline
AC = (AC): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}[\tfrac{1}{2}]^2, \\
AAC = (A)(AC): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}[\tfrac{1}{2}]^3, \\
BAC = (B)(AC): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}[\tfrac{1}{2}]^3 \oplus \mathbb{Z}, \\
ABAC = (A)(B)(AC): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}[\tfrac{1}{2}]^4 \oplus \mathbb{Z}.
\end{array}
$$

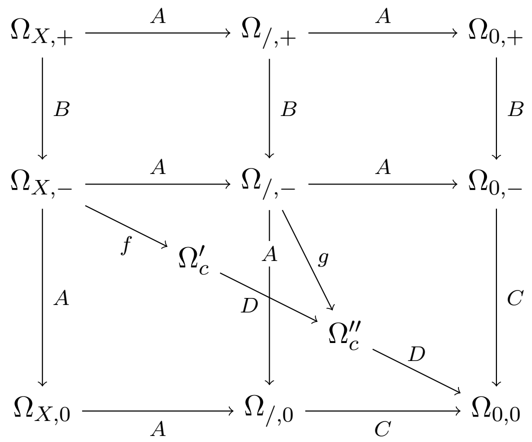
□

Note that the quotient cohomology depends only on the type of path, and not necessarily on particular tiling spaces. Also, the quotient cohomology groups sum up whenever factor maps are composed. The only exception is when composing A and C which produces the torsion component $\mathbb{Z}_3$. For the rest of the compositions, the operation is associative and commutative.

Remark 5.15 (Two more factors)**.** The substitution in (5.12), which produces the tiling space $\Omega_{X,+}$, acts on 32 prototiles. Identifying certain prototiles in a consistent and compatible manner, analogous to identifying certain symbols in the squiral and Chacon cases, produces some factors of $\Omega_{X,+}$, including all the factors in (5.14). Interestingly, under this scheme, we find two more tiling spaces that the identification rules in Table 5.4 do not cover, and so are not accounted for in (5.14). Denoting the two (extra) tiling spaces by Ω'_c and Ω''_c, we get the following sequences of factor maps.

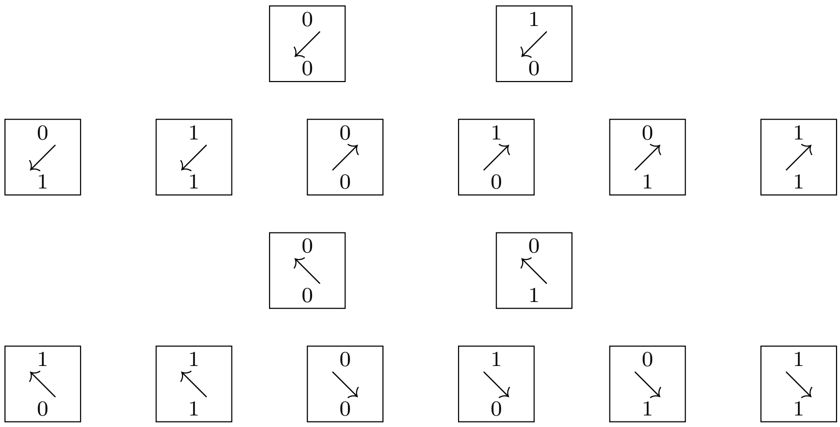
$$
\Omega_{X,-} \xrightarrow{f} \Omega'_c \xrightarrow{D} \Omega''_c \xrightarrow{D} \Omega_{0,0} = \mathbb{S}_2 \times \mathbb{S}_2
$$
$$
\Omega_{X,-} \xrightarrow{A} \Omega_{/,-} \xrightarrow{g} \Omega''_c \xrightarrow{D} \mathbb{S}_2 \times \mathbb{S}_2
$$

Incorporating the two extra tiling spaces in (5.14) gives us the following diagram.



(5.16)

The 16 prototiles of $\Omega_{X,-}$ are enumerated in the following,



where prototiles that belong in the same row are identified. This identification yields four new prototiles to which the general substitution (5.12) yields the tiling space Ω'_c.

Identifying further the last two rows of prototiles enumerated above yields the tiling space Ω''_c, which has three prototiles. The cohomologies of Ω'_c and Ω''_c read

$$H^0(\Omega'_c) \cong \mathbb{Z}, \quad H^1(\Omega'_c) \cong \mathbb{Z}[\tfrac{1}{2}]^2, \quad \text{and} \quad H^2(\Omega'_c) \cong \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}^2;$$
$$H^0(\Omega''_c) \cong \mathbb{Z}, \quad H^1(\Omega''_c) \cong \mathbb{Z}[\tfrac{1}{2}]^2, \quad \text{and} \quad H^2(\Omega''_c) \cong \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}.$$

We list the quotient cohomologies associated to f and g together with their compositions in the following lemma. ♢

Lemma 5.16. *The quotient cohomologies between the tiling spaces in* (5.16) *involving the factor maps f or g are given as follows, where the composition of* *factor maps are also decomposed into good matches. Note that* $fD = (fD) = (A)(g) = Ag$ *and* $fD^2 = (fD^2) = (A)(gD) = AgD$.

$$
\begin{array}{rlll}
D: & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}, \\
D^2 = (D)^2: & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}^2, \\
\hline
f: & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^3, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^3, \\
fD = (fD): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^3, \\
fD^2 = (fD^2): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}_3, \\
\hline
g: & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^2, \\
gD = (gD): & H^0_Q \cong 0, & H^1_Q \cong 0, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}_3, \\
\hline
Ag = (A)(g): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}^2, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^3, \\
AgD = (A)(gD): & H^0_Q \cong 0, & H^1_Q \cong \mathbb{Z}, & H^2_Q \cong \mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}_3.
\end{array}
$$

□

The following propositions already appear in [**BS11**] as Theorems 6 and 7, although with some errors. The absolute and quotient cohomologies have been recalculated and the corrected results appear below. The particular corrections are boxed for easier identification. Proposition 5.18 may also be read off of Lemma 5.14.

Proposition 5.17 (cf. [**BS11**, Theorem 6])**.** *The absolute cohomologies of the nine tiling spaces in* (5.14) *are given as follows. All spaces have* $H^0 \cong \mathbb{Z}$. *The first cohomology is given by*

$$
\begin{array}{ccccc}
\mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}^2 & \xleftarrow{A^*} & \boxed{\mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z} \\
\uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} \\
\mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z} & \xleftarrow{A^*} & \boxed{\mathbb{Z}[\frac{1}{2}]^2} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}]^2 \\
\uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle C^*} \\
\mathbb{Z}[\frac{1}{2}]^2 & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}]^2 & \xleftarrow{C^*} & \mathbb{Z}[\frac{1}{2}]^2
\end{array}
$$

The second cohomology is given by

$$\begin{array}{ccccc}
\frac{1}{3}\mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z} & \xleftarrow{A^*} & \boxed{\frac{1}{3}\mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}^2 \\
\uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} \\
\frac{1}{3}\mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^3 & \xleftarrow{A^*} & \boxed{\frac{1}{3}\mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^2} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} \\
\uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle C^*} \\
\frac{1}{3}\mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^2 & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} & \xleftarrow{C^*} & \mathbb{Z}[\frac{1}{4}]
\end{array}$$

□

Proposition 5.18 (cf. [**BS11**, Theorem 7])**.** *The quotient cohomologies of the nine tiling spaces in* (5.14)*, relative to the solenoid* $\Omega_{0,0}$*, are given as follows. For all spaces,* $H^0_Q \cong 0$*. The first quotient cohomology is given by*

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \xleftarrow{A^*} & \boxed{\mathbb{Z}} & \xleftarrow{A^*} & \mathbb{Z} \\
\uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} \\
\mathbb{Z} & \xleftarrow{A^*} & \boxed{0} & \xleftarrow{A^*} & 0 \\
\uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle C^*} \\
0 & \xleftarrow{A^*} & 0 & \xleftarrow{C^*} & 0
\end{array}$$

The second quotient cohomology is given by

$$\begin{array}{ccccc}
\mathbb{Z}_3 \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z} & \xleftarrow{A^*} & \boxed{\mathbb{Z}_3 \oplus \mathbb{Z}[\frac{1}{2}]^3 \oplus \mathbb{Z}} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}^2 \\
\uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} & & \uparrow{\scriptstyle B^*} \\
\mathbb{Z}_3 \oplus \mathbb{Z}[\frac{1}{2}]^3 & \xleftarrow{A^*} & \boxed{\mathbb{Z}_3 \oplus \mathbb{Z}[\frac{1}{2}]^2} & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} \\
\uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle C^*} \\
\mathbb{Z}_3 \oplus \mathbb{Z}[\frac{1}{2}]^2 & \xleftarrow{A^*} & \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} & \xleftarrow{C^*} & 0
\end{array}$$

□

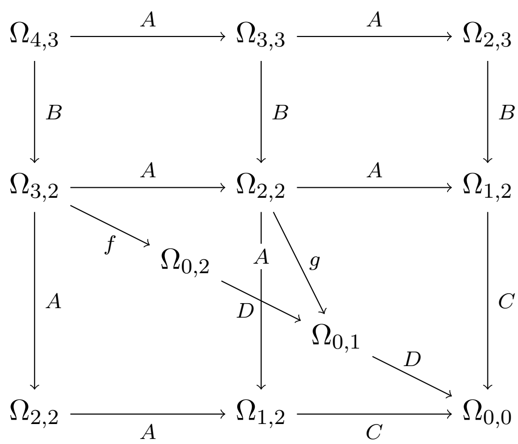
5.3.1. Translation orbits and the components of the hull. The zeta function of the tiling space $\Omega_{X,+}$ is computed as

$$\begin{aligned}
\zeta_{\Omega_{X,+}} = \frac{1}{(1-2z)^2(1-4z)} &= \frac{(1-2z)^2}{(1-z)(1-4z)} \cdot \left(\frac{1-z}{1-2z}\right)^4 \cdot \left(\frac{1}{1-z}\right)^3 \\
&= \zeta_{\mathbb{S}_2\times\mathbb{S}_2} \cdot \zeta_{\mathbb{S}_2}^4 \cdot \zeta_p^3,
\end{aligned}$$

where $\zeta_{\mathbb{S}_2\times\mathbb{S}_2}, \zeta_{\mathbb{S}_2}$ and ζ_p denote the zeta function of the 2-dimensional dyadic solenoid $\mathbb{S}_2\times\mathbb{S}_2$, the dyadic subsolenoid $\mathbb{S}_2$ and an extra fixed point, respectively.

The factorisation of $\zeta_{\Omega_{X,+}}$ is not surprising because by employing the same analysis as in the squiral and Chacon cases, we see that the translation orbits represented in (5.13) yield a hull with precisely a copy of $\mathbb{S}_2 \times \mathbb{S}_2$ onto which everything projects 1-to-1 almost everywhere, four copies of the subsolenoid $\mathbb{S}_2$ (along the vertical, horizontal and the two diagonals) and three extra fixed points.

The 11 models in (5.16) all project to the solenoid $\mathbb{S}_2 \times \mathbb{S}_2$ 1-to-1 almost everywhere. Taking into account the extra components in their hull, we may write $\Omega_{X,+}$ as $\Omega_{4,3}$, denoting that aside from a copy of $\mathbb{S}_2 \times \mathbb{S}_2$, it also has four copies of the subsolenoid $\mathbb{S}_2$ and three extra fixed points. This notation is uniform with the notation we used for the squiral and Chacon tiling spaces. In general, writing $\Omega_{\alpha,\beta}$ for a factor of $\Omega_{X,+} = \Omega_{4,3}$, which has α copies of the subsolenoid $\mathbb{S}_2$ and β extra fixed points, allows us to rewrite the diagram in (5.16) as follows:



(5.17)

This way, the effect of the degeneracies on the respective tiling spaces becomes very visible by just inspecting the notation of $\Omega_{\alpha,\beta}$. Also, note that the factor maps f and g are associated to the degenerations F and G, and as usual, the factor maps A, B, C and D are associated to the degenerations A, B, C and D. The effects of such degeneracies on the hull are enumerated in Table 5.3.

Let us also emphasise, that considering this notation, the quotient zeta function becomes very easy to compute, as it would only involve powers of the factors $\zeta_{\mathbb{S}_2}$ and ζ_p. For instance, the quotient zeta function between $\Omega_{3,3}$ and $\Omega_{0,1}$ is simply $\zeta_{\Omega_{3,3},\Omega_{0,1}} = \zeta_{\mathbb{S}_2}^3 \zeta_p^2$ as the their components differ by three copies of the subsolenoid $\mathbb{S}_3$ and two extra fixed points.

Outlook

In general, a trivial quotient cohomology does not imply equivalence, as in the case between the tiling space of the substitution $\varrho_G = \{a \longrightarrow aba, b \longrightarrow baa\}$ and the 3-adic solenoid $\mathbb{S}_3$ in Example 3.19. However, the inequivalence of the two spaces only becomes apparent after noting that the solenoid is not a tiling space, in particular, it is periodic in contrast to the tiling space of ϱ_G, which is non-periodic. The question whether a trivial quotient cohomology between tiling spaces, under certain conditions, can imply equivalence remains unsolved. A consequence to this question could easily determine whether the maximal model set factor $\Omega_{v,h,4}$ (of the squiral tiling space) is equivalent to the tiling spaces X, Y and Z in Remark 5.8. Recall that the quotient cohomology is trivial among these spaces. We are inclined to believe that these spaces are equivalent spaces, but do not have a valid proof or argument ready yet.

We have seen that the squiral tiling and the Chacon tiling share interesting topological properties. In particular, the tiling space $\Omega_{v,h,4}$ and the Chacon tiling space $\Omega_{V,H,4'}$ have identical set of translation orbits, and as a consequence share the same dynamical zeta function, albeit being inequivalent tiling spaces, evident from their different cohomologies. Nonetheless, they might share several common tiling factors. The search for such factors may be done by introducing larger sliding block maps on the letters of the substitutions in (5.3) and (5.6). This would result into a richer breakdown of the tiling spaces $\Omega_{v,h,4}$ and $\Omega_{V,H,4'}$, i.e., the diagrams in Figures 5.2 and 5.6 would contain more factors, such as $\Omega_{v,h,2}$ or $\Omega_{V,1'}$ for instance. The number of such factors is too great and would be a computational nightmare, but would certainly give more description on the said tiling spaces.

A natural next step is to go one dimension higher. A good candidate to begin with is the 3-dimensional chair substitution shown in Figure B. Further, the 3-dimensional generalised chair substitution may be obtained analogous to the (2-dimensional) generalised chair substitution in (5.12). Once the factors of the 3-dimensional tiling space are enumerated, the quotient cohomology between them can then be computed. Of particular interest would be the degeneracies in this case, which should be related to the 2-dimensional generalised chair tilings we have already considered.

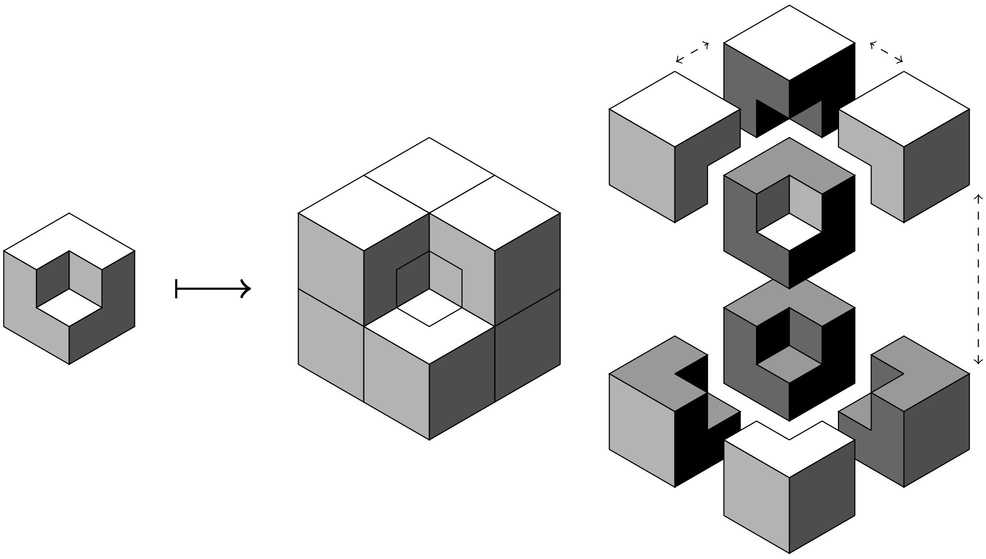


FIGURE B. The 3-dimensional chair substitution, where in each iteration, the volume of the cube is multipied by $2^3 = 8$. The inflated cube comprises 8 smaller cubes, as shown.

Finally, we are intrigued by the quotient zeta function, and its possible role in studying more general dynamical systems, beyond the realm of substitution systems. Of course, this generating function counts the difference between the fixed points between a dynamical system and its factor, but beyond this is still unexplored.

Bibliography

[AP98] Anderson, J., & Putnam, I. (1998). Topological invariants for substitution tilings and their associated C^*-algebras. *Ergodic Theory and Dynamical Systems*, *18*(3), 509–537. doi:10.1017/S0143385798100457

[BGG12] Baake, M., Gähler, F., & Grimm, U. (2012). Spectral and topological properties of a family of generalised Thue-Morse sequences. *Journal of Mathematical Physics*, *53*(3), 032701. doi:10.1063/1.3688337

[BGG13] Baake, M., Gähler, F., & Grimm, U. (2013). Examples of substitution systems and their factors. *Journal of Integer Sequences*, 16, Article 13.2.14. https://cs.uwaterloo.ca/journals/JIS/VOL16/Baake/baake3.html

[BG13a] Baake, M., & Grimm, U. (2013). *Encyclopedia of Mathematics and Its Applications - 149: Aperiodic Order Volume 1: A Mathematical Invitation*. New York, NY: Cambridge University Press.

[BG13b] Baake, M., & Grimm, U. (2013). Squirals and beyond: substitution tilings with singular continuous spectrum. *Ergodic Theory and Dynamical Systems*, FirstView, 1–26. doi:10.1017/etds.2012.191

[BLM07] Baake, M., Lenz, D., & Moody, R.V. (2007). Characterization of model sets by dynamical systems. *Ergodic Theory and Dynamical Systems*, *27*(2), 341–382. doi:10.1017/S0143385706000800

[BLvE14] Baake, M., Lenz, D., & van Enter, A. (2014). Dynamical versus diffraction spectrum for structures with finite local complexity. *Ergodic Theory and Dynamical Systems*, to appear. arXiv:1307.7518

[BSJ91] Baake, M., & Schlottman, M., & Jarvis, P.D. (1991). Quasicperiodic patterns with tenfold symmetry and equivalence with respect to local derivability. *Journal of Physics A: Mathematical and General*, *24*(19), 4637–4654. doi:10.1088/0305-4470/24/19/025

[BD08] Barge, M., & Diamond, B. (2008). Cohomology in one-dimensional substitution tiling spaces. *Proceedings of the American Mathematical Society*, *136*, 2183–2191. doi:10.1090/S0002-9939-08-09225-3

[BDHS09] Barge, M., Diamond, B., Hunton, J., & Sadun, L. (2010). Cohomology of substitution tiling spaces. *Ergodic Theory and Dynamical Systems* *30*(6), 1607–1627. doi:10.1017/S0143385709000777

[BKS12] Barge, M., Kellendonk, J., & Schmieding, S. (2012). Maximal equicontinuous factors and cohomology for tiling spaces. *Fundamenta Mathematicae 218*(3), 243–267. doi:10.4064/fm218-3-3

[BS11] Barge, M., & Sadun, L. (2011). Quotient cohomology for tiling spaces. *New York Journal of Mathematics*, 17, 579–599. http://nyjm.albany.edu/j/2011/17-25.html

[BBG06] Bellissard, J., Benedetti, R., & Gambaudo, J.-M. (2006). Spaces of tilings, finite telescopic approximants and gap-labelling. *Communications in Mathematical Physics*, *261*(1), 1–41. doi:10.1007/s00220-005-1445-z

[BG03] Benedetti, R., & Gambaudo, J.-M. (2003). On the dynamics of $\mathbb{G}$-solenoids: applications to Delone sets. *Ergodic Theory and Dynamical Systems*, *23*(3), 673–691. doi:10.1017/S0143385702001578

[BT82] Bott, R., & Tu, L. (1982). Differential Forms in Algebraic Topology. *Graduate Texts in Mathematics*, *Vol. 82.* New York, NY: Springer-Verlag. doi:10.1007/978-1-4757-3951-0

[BG14] Bugarin, E.P., & Gähler, F. (2014). Quotient cohomology of certain 1- and 2-dimensional substitution tiling spaces. *Acta Physica Polonica A*, to appear. arXiv:1401.1583

[CS03] Clark, A., & Sadun, L. (2003). When size matters: subshifts and their related tiling spaces. *Ergodic Theory and Dynamical Systems*, *23*(4), 1043–1057. doi:10.1017/S0143385702001633

[CS06] Clark, A., & Sadun, L. (2006). When shape matters: deformations of tiling spaces. *Ergodic Theory and Dynamical Systems*, *26*(1), 69–86. doi:10.1017/S0143385705000623

[Esch] Escher, M.C. [Online content]. Retrieved March 3, 2014, from http://www.mcescher.com/gallery/symmetry/

[Fra05] Frank, N.P. (2005). Multi-dimensional constant length substitution sequences. *Topology and its Applications*, *152*(1-2), 44–69. doi:10.1016/j.topol.2004.08.014

[Fra08] Frank, N.P. (2008). A primer of substitution tilings of the euclidean plane, *Expositiones Mathematicae*, *26*(4), 295–326. doi:10.1016/j.exmath.2008.02.001

[FS09] Frank, N.P., & Sadun, L. (2009). Topology of (some) tiling spaces without finite local complexity. *Discrete and Continuous Dynamical Systems - Series A*, *23*(3), 847–865. doi:10.3934/dcds.2009.23.847

[FS07] Frettlöh, D., & Sing, B. (2007). Computing Modular Coincidences for substitution tilings and point sets. *Discrete & Computational Geometry*, *37*(3), 381–407. doi:10.1007/s00454-006-1280-9

[Gäh02] Gähler, F. (2002). Cohomology of quasiperiodic tilings. In *Aperiodic Order, Dynamical Systems, Operator Algebras and Topology* [Conference]. University of Victoria, Victoria, Canada. August 2002.

[Gäh13] Gähler, F. (2013). Substitution rules and topological properties of the Robinson tilings. In *Aperiodic Crystals*, S. Schmid, R.L. Withers and R. Lifshitz (Eds.), Springer Netherlands, 67–73. doi:10.1007/978-94-007-6431-6_9

[GM13] Gähler, F., & Maloney, G. (2013). Cohomology of one-dimensional mixed substitution tiling spaces. *Topology and its Applications*, *160*(5), 703–719. doi:10.1016/j.topol.2013.01.019

[GHM13] Gähler, F., Hunton, J., & Maloney, G. (2013). *In preparation*.

[Geo08] Geoghegan, R. (2008). Topological Methods in Group Theory. *Graduate Texts in Mathematics, Vol. 243*. New York, NY: Springer-Verlag. doi:10.1007/978-0-387-74614-2

[GS87] Grünbaum S. and Shephard G. (1987). *Tilings and Patterns*. New York, NY: W. H. Freeman and Company.

[Hat02] Hatcher, A. (2002). *Algebraic Topology*. New York, NY: Cambridge University Press. http://www.math.cornell.edu/~hatcher/AT/ATpage.html

[Kel95] Kellendonk, J. (1995). Pattern-equivariant functions and cohomology. *Journal of Physics A: Mathematical and General*, *36*(21), 5765–5772. doi:10.1088/0305-4470/36/21/306

[Koo31] Koopman, B.O. (1931). Hamiltonian systems and transformations in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, *17*(5), 315–318. http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1076052/

[LMS03] Lee, J.-Y., Moody, R.V., & Solomyak, B. (2003). Consequences of pure point diffraction spectra for multiset substitution systems. *Discrete & Computational Geometry*, *29*(4), 525–560. doi:10.1007/s00454-003-0781-z

[Neu33] von Neumann, J. (1933). Zur Operatorenmethode in der klassischen Mechanik. *Annals of Mathematics*, *33*(3), 587–642. http://www.jstor.org/stable/1968537

[OEIS] The On-Line Encyclopedia of Integer Sequences™. http://oeis.org

[ORS02] Ormes, N., Radin, C., & Sadun, L. (2002). A homeomorphism invariant for substitution tiling spaces. *Geometriae Dedicata*, *90*(1), 153–182. doi:10.1023/A:1014942402919

[PR91] Park, K., & Robinson Jr., E.A. (1991). The joinings within a class of $\mathbb{Z}^2$ actions. *Journal d'Analyse Mathématique*, *57*(1), 1–36. http://link.springer.com/article/10.1007/BF03041064

[Pet99] Petersen, K. (1999). Factor maps between tiling dynamical systems. *Forum Mathematicum*, *11*(4), 503–512. doi:10.1515/form.1999.011

[Pyth02] Pytheas Fogg, N. (2002). *Lecture Notes in Mathematics - 1794*: *Substitutions in Dynamics, Arithmetics, and Combinatorics*. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel (Eds.). Berlin Heidelberg: Springer-Verlag. doi:10.1007/b13861

[RS01] Radin, C., & Sadun, L. (2001). Isomorphism of hierarchical structures. *Ergodic Theory and Dynamical Systems*, *21*(4), 1239–1248. doi:10.1017/S0143385701001572

[Rob96] Robinson Jr., E.A. (1996). The dynamical theory of tilings and Quasicrystallography. *London Mathematical Society Lecture Note Series (No. 228)*: *Ergodic theory of $\mathbb{Z}^d$ actions*, 451–474. New York, NY: Cambridge University Press. doi:10.1017/CBO9780511662812.020

[Rob99] Robinson Jr., E.A. (1999). On the table and chair. *Indagationes Mathematicae*, *10*(4), 581–599. doi:10.1016/S0019-3577(00)87911-2

[Rue94] Ruelle, D. (1994). Dynamical zeta functions for piecewise monotone maps of the interval. *CRM Monograph Series*, *Vol. 4*. Providence, RI: American Mathematical Society.

[Sa03] Sadun, L. (2003). Tiling spaces are inverse limits. *Journal of Mathematical Physics*, *44*(11), 5410–5414. doi:10.1063/1.1613041

[Sa08] Sadun, L. (2008). *University Lecture Series - 46*: *Topology of Tiling Spaces*. J.L. Bona, E.M. Friedlander, N.D. Higson and J.T. Stafford (Eds.). Providence, RI: American Mathematical Society.

[Sa11] Sadun, L. (2011). Exact regularity and the cohomology of tiling spaces. *Ergodic Theory and Dynamical Systems*, *31*(6), 1819–1834. doi:10.1017/S0143385710000611

[Sen94] Senechal, M. (1994). Graphics Gallery: Tilings, Diffraction, and Quasicrystals. *The Mathematica Journal*, *4*(2), 10–15. http://www.mathematica-journal.com/issue/v4i2/graphics/10-15_senechal42.mj.pdf

[SS78] Seebach Jr., J.A., & Steen, L.A. (1978). Counterexamples in Topology (2nd ed.). New York, NY: Springer-Verlag. doi:10.1007/978-1-4612-6290-9

[Sol97] Solomyak, B. (1997). Dynamics of self-similar tilings. *Ergodic Theory and Dynamical Systems*, *17*(3), 695–738. http://journals.cambridge.org/article_S0143385797084988

[Sol99] Solomyak, B. (1999). Dynamics of self-similar tilings (Errata). *Ergodic Theory and Dynamical Systems*, *19*(6), 1685. http://journals.cambridge.org/article_S014338579917161X

[Wh49] Whitehead, J.H.C. (1949). Combinatorial homotopy: I. *Bulletin of the Mathematical American Society*, *55*(5), 213–245. doi:10.1090/S0002-9904-1949-09175-9

[Wi74] Williams, R.F. (1974). Expanding attractors. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, *43*(1), 169–203. doi:10.1007/BF02684369