Volume Growth and Uniqueness of Nonnegative Solutions of Differential Inequalities on Manifolds

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Preface

Partial differential equation and inequalities of elliptic type is a well-established area of Mathematics. In this thesis, we are concerned with inequality

$$\Delta u + u^\sigma \leq 0, \tag{0.0.1}$$

and its far reaching generalizations in $\mathbb{R}^n$ and on geodesically complete Riemannian manifolds. The main question to be discussed here is under what conditions any nonnegative solution of (0.0.1) (and its generalizations) is identical zero.

The question of uniqueness of nonnegative solution of (0.0.1) in $\mathbb{R}^n$ was investigated by Ni and Serrin [53, 54] (and for the exact equation $\Delta u + u^\sigma = 0$ by Gidas and Spruck [18]).

A rich class of differential inequalities in $\mathbb{R}^n$ was systematically studied by Mitidieri and Pokhozhaev [44, 46, 47], who developed a general method for proving such results.

On the other hand, until recently not much was known for (0.0.1) on general Riemannian manifolds. Our study of this problem was motivated by a celebrated result of Cheng and Yau: they proved that, under a quadratic volume growth hypothesis, any positive superharmonic function on the manifold in question is identical zero. An extension of this result to m-Laplace operator is due to Holopainen [33, 34].

A priori it is not obvious that restrictions on the volume growth of a manifold can be used to obtain information about global nonnegative solutions of (0.0.1) and its generalizations. However, it happens to be the case. It is obvious that any solution of (0.0.1) is a superharmonic function, which by Theorem of Cheng and Yau implies that, under quadratic volume growth, any solution of (0.0.1) is zero.

In this thesis we obtain more relaxed volume growth conditions that imply the uniqueness of nonnegative solutions of (0.0.1) and its generalizations. Moreover, we show that these conditions are sharp. Our results cover many known results in $\mathbb{R}^n$.

Note that differential equations and inequalities in $\mathbb{R}^n$ can be studied using such classical tools as Harnack inequalities, estimate of fundamental solutions, a priori estimate. In the setting of Riemannian manifolds, only under the volume growth assumption, none of these tools is available. We have developed a new method, that is a further elaboration of the method of Grigor'yan and Kondratiev [28], which is based on a subtle choice of test functions. This enables us to investigate a variety of differential inequalities on manifolds.

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Bielefeld, May 23, 2014, Yuhua Sun

Contents

Chapter 1

Introduction

1.1 Differential Inequalities in $\mathbb{R}^n$

This thesis is about the uniqueness of nonnegative solutions of certain differential inequalities on geodesically complete Riemannian manifolds.

In the last fifty years, many attentions have been devoted to the study of differential inequalities in $\mathbb{R}^n$ and on manifolds, see [8, 9, 10, 11, 31, 34, 50, 56, 59, 63] and the references therein. Many topics are involved, such as maximum principle and compact support principle [60, 62], comparison theorem [36, 37], Liouville type theorem [13, 49, 65, 66], a priori estimate [14, 47, 48], existence and nonexistence of solutions [15, 16, 28]. Among these topics, the importance of uniqueness results (or called nonexistence results) in the PDEs is well recognized, and has its roots on the fundamental Liouville theorem for positive harmonic functions in $\mathbb{R}^n$. The previously developed methods for investigation of the above question include such advanced tools as Harnack inequalities and estimates of fundamental solutions.

For the first time the question of uniqueness of nonnegative solutions of semilinear elliptic equations was considered by Gidas and Spruck in [18]. Namely, they studied the following equation

$$\Delta u + u^\sigma = 0, \quad \text{in } \mathbb{R}^n, \tag{1.1.1}$$

and proved that if $n > 2$ and

$$1 \leq \sigma < \frac{n+2}{n-2}, \tag{1.1.2}$$

then any nonnegative solution of (1.1.1) is zero. Their proofs were highly non-trivial and used the De Giorgi-Nash-Moser bootstrap arguments. It is worth pointing out that there is no assumption about the behavior of the possible solutions at infinity.

The sharpness of $\frac{n+2}{n-2}$ was proved by Ni, Serrin, and Bidaut-Veron in [3, 54], namely, they showed that if $\sigma \geq \frac{n+2}{n-2}$, then (1.1.1) has a nontrivial positive solution. Note that the initial motivation for considering (1.1.1) was Yamabe problem of finding a Riemannian metric with constant scalar curvature (cf. [7, 39]). However, the subject of uniqueness of nonnegative solutions of semilinear equations and inequalities happens to be rich in mathematical phenomenas and ideas, and now is being developed on its own merit.

A different critical exponent appears, if we consider the inequality

$$\Delta u + u^\sigma \leq 0, \quad \text{in } \mathbb{R}^n. \tag{1.1.3}$$

Ni and Serrin proved that when $n > 2$, if

$$1 \leq \sigma \leq \frac{n}{n-2}, \tag{1.1.4}$$

then any nonnegative solution of (1.1.1) is identical zero (cf. [53, 54]). They used in the proof spherical mean operator and Jensen operator. Namely, denote by $\bar{u}(r)$ the average of u over the sphere $\mathbb{R}^n$ of radius r and observe that $\bar{u}$ satisfies an ODE

$$-(r^{n-1}\bar{u}')' \geq r^{n-1}\bar{u}^\sigma, \quad \bar{u}'(0) = 0.$$

By analysing the ODE, one solves the problem. It is however clear, that such an approach can not work on arbitrary Riemannian manifolds without rotation symmetry.

A number of generalizations of these results to more general differential equations and inequalities in $\mathbb{R}^n$ have been obtained in a series of works of Mitidieri and Pokhozhaev [44, 46, 47] and many others. Let us mention one of their results, Mitidieri and Pokhozhaev considered the following quasilinear inequality

$$\Delta_m u + u^\sigma \leq 0, \quad \text{in } \mathbb{R}^n, \tag{1.1.5}$$

where

$$\Delta_m u := div(|\nabla u|^{m-2}\nabla u),$$

is called the m-Laplace operator. They proved that if $n > m$, and

$$0 < \sigma \leq \frac{n(m-1)}{n-m}, \tag{1.1.6}$$

then the only nonnegative solution of (1.1.5) is identical zero (cf. [43]). The exponent

$\frac{n(m-1)}{n-m}$ is sharp, namely, if $\sigma > \frac{n(m-1)}{n-m}$, one could check that the function u such that

$$u = c_1(c_2 + |x|^{\frac{m}{m-1}})^{-\frac{m-1}{\sigma-m+1}}, \tag{1.1.7}$$

is a positive solution to (1.1.5), where c_1, c_2 are some appropriate constants.

Bidaut-Veron and Pokhozhaev [4] studied (1.1.5) on the exterior domain Ω rather than entire space $\mathbb{R}^n$. They proved that the only nonnegative solution of (1.1.5) is identical zero, provided under (1.1.6), when $n > m$, or $0 < \sigma < \infty$, when $n = m$.

Other results in this direction were obtained in [15, 51, 56, 63, 65, 73].

Mitidieri and Pohozaev developed a powerful technique to investigate the absence of positive solutions in [48]. The underlying idea of the method is that sharp estimates of the capacity type should be obtained. These techniques are also widely used by Caristi, Filippucci, Pucci, D'Ambrosio [9, 10, 11, 12, 13, 14]. These works are based on a method originating from [57] (see also [58] and other parpers) that uses carefully chosen test functions.

In this thesis, we use the technique developed by Kondratiev, Grigor'yan and the author to investigate the uniqueness of nonnegative solutions to different types of differential inequalities on Riemannian manifolds (cf. [28, 30, 67]).

1.2 Volume Growth of Riemannian manifolds

Let M be a geodesically complete non-compact connected Riemannian manifold. Denote by μ the Riemannian measure on M, and by $d(x, y)$ the geodesic distance between $x, y \in M$. Let $B(x, r)$ be the geodesic ball centered at $x \in M$ of radius r. Fix some $x_0 \in M$ and set

$$\mathrm{Vol}(r) = \mu(B(x_0, r)). \tag{1.2.1}$$

Function $\mathrm{Vol}(r)$ is called the volume growth function, and it will play an important role in our results.

Volume is a very fundamental geometric quantity. It can be used not only to investigate geometric property but also to obtain information about the behavior of differential inequalities on manifold, even without any curvature assumption. It is well known that many problems in differential geometry can be reduced to problems in differential equations on Riemannian manifolds. This motivated an extensive study of PDEs on manifolds, initiated by S.-T. Yau.

In particular, Cheng and Yau proved in [8] that if the volume of the geodesic balls of a complete Riemannian grows at most a quadratic polynomial, namely, if

$$\mathrm{Vol}(r) \le Cr^2, \quad \text{for large enough } r, \tag{1.2.2}$$

then any positive superharmonic function on M is constant. Here the exponent 2 in (1.2.2) is sharp and cannot be replaced by $2+\epsilon$ for any $\epsilon > 0$.

Recall that M is called parabolic, if any of the following equivalent conditions holds:

(i) Any positive superharmonic function on M is constant;

(ii) Δ has no positive fundamental solution;

(iii) Brownian motion on M is recurrent.

Hence, (1.2.2) implies the parabolicity of M.

Grigor'yan relaxed the condition (1.2.2) as follows: if

$$\int^{\infty} \frac{r}{\mathrm{Vol}(r)} dr = \infty, \tag{1.2.3}$$

then M is parabolic, or equivalently, any positive superharmonic function on M is constant (cf. [21, 27].

The notion of parabolicity of manifold originated from the type problem for Riemannian surfaces. By the Uniformization theorem of Koebe-Poincaré, every simply connected Riemannian surface M is conformally equivalent to one of the three model surfaces $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{H}^2$. In the first case M is called elliptic, in the second case M is called parabolic, and in the third case M is hyperbolic. Note $\mathbb{S}^2$ is compact, while $\mathbb{R}^2$, $\mathbb{H}^2$ are non-compact. The problem of deciding whether a non-compact M is parabolic or hyperbolic is called a type problem. A number of famous mathematicians contributed to solution of this problem in different terms, such as Ahlfors, Kakutani, Nevanlinna and many others (cf. [2, 20, 35, 52]).

The notion of parabolicity can be extended to deal with the m-Laplace operator Δ_m. A Riemannian manifold M is called m-parabolic, if any positive m-superharmonic function on M is constant. Here a function u is called m-superharmonic, if $\Delta_m u \le 0$. Holopainen proved in [33, 34] the following generalization of (1.2.3): if

$$\int^{\infty} \left(\frac{r}{\mathrm{Vol}(r)}\right)^{\frac{1}{m-1}} dr = \infty, \tag{1.2.4}$$

then M is m-parabolic.

Volume growth condition can be also used to obtain stochastic completeness of M. Manifold M is also stochastic complete if it satisfies any of the equivalent conditions (cf. [27]):

(a) Brownian motion on M has almost surely infinite lifetime;

(b) Any bounded solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $[0, \infty)$ with the initial conidtion $u(0, x) = 0$ is $u \equiv 0$;

(c) Any bounded solution to $\Delta v = v$ on M is identical zero.

In [22], Grigor'yan proved that if

$$\int^{\infty} \frac{r}{\ln \mathrm{Vol}(r)} dr = \infty, \tag{1.2.5}$$

holds, then M is stochastically complete. In particular, if the Ricci curvature of M is bounded below, it follows that $\mathrm{Vol}(r) \leq e^{Cr}$, and hence (1.2.5) holds, and M is stochastically complete (in a different way, this theorem was also proved by Yau [74]). Note that (1.2.5) is also satisfied if $\mathrm{Vol}(r) \leq e^{cr^2}$ or $\mathrm{Vol}(r) \leq e^{cr^2 \ln r}$. Moreover, Grigor'yan also proved if (1.2.5) holds, then every nonnegative superharmonic function $u \in L^1(M)$ is equal to a constant. The condition (1.2.5) is sharp in the following sense: if

$$\int^{\infty} \frac{r}{\ln f(r)} dr < \infty$$

for a positive solution $f(r)$(regular in some sense), then there exists a complete but stochastically incomplete weighted manifold M such that $\ln \mathrm{Vol}(r) = f(r)$ for some $x_0 \in M$ and large enough r. For the detailed proof, see [27, Example 11.11].

Any parabolic manifold is stochastically complete, but the opposite implication does not hold. A simple example is that all the Euclidean spaces $\mathbb{R}^n$ are stochastically complete, but only $\mathbb{R}^1$ and $\mathbb{R}^2$ are parabolic.

From (1.2.3) and (1.2.5), we see that the volume growth can be used to characterize the parabolicity and the stochastic completeness of M. Besides the volume growth, there are other conditons to characterize the two notions, for example, curvature bounds [74], and the Liouville property for Schrödinger operators [24], and the existence of cut-off functions satisfying certain properties [55].

1.3 Structure

In this thesis, we focus on the inequalities of the type

$$Lu + f(x, u, |\nabla u|) \leq 0, \tag{1.3.1}$$

on a geodesically complete non-compact connected Riemannian manifold M. Here L is a second order partial differential operator in the divergence form, and $f \geq 0$ is a nonnegative function.

In the subsequent chapters, we will present many results related to different types of operator L, which are sharp, in terms of volume growth, and cover many known results in $\mathbb{R}^n$. Let us emphasize again, that the only geometric hypothesis about M we use here is a volume growth assumption. Besides, we make no assumption about the behavior of solutions at infinity, assuming only that the solution is nonnegative and is defined on the whole manifold M.

The structure of the thesis is as follows: In Chapter 2, we study the problem

$$\Delta u + u^\sigma \leq 0, \quad \text{on } M. \tag{1.3.2}$$

We prove that, if for some $x_0 \in M$

$$\mu(B(x_0, r)) \leq c r^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r, \quad \text{for all large enough } r. \tag{1.3.3}$$

then any nonnegative solution of (1.3.2) is identical zero. We show the sharpness of parameters $\frac{2\sigma}{\sigma-1}$ and $\frac{1}{\sigma-1}$ in (1.3.3), namely, if any of these numbers is replaced by a larger value, then the statement is not true.

In Chapter 3, we study the following differential inequality

$$div(A(x)\nabla u) + V(x)u^\sigma \leq 0, \quad \text{on } M, \tag{1.3.4}$$

where $\sigma > 1$, A is a nonnegative definite symmetric operator in the tangent space T_xM, and V is a given locally integrable positive measurable function. Define a new measure ν by

$$d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu.$$

Assume A and V satisfy that the following condition: for almost all $x \in M$ the following

$$c r(x)^{-\delta_1} \leq \frac{V(x)}{\|A(x)\|} \leq C r(x)^{\delta_2}, \tag{1.3.5}$$

holds for all large enough $r(x) := d(x, x_0)$, where δ_1, δ_2 are arbitrary nonnegative constants. We prove that if

$$\nu(B(x_0, r)) \leq C r^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r, \quad \text{for all large enough } r, \tag{1.3.6}$$

then the only nonnegative solution of (1.3.4) is identical zero. We also show the sharpness of parameters $\frac{2\sigma}{\sigma-1}$ and $\frac{1}{\sigma-1}$ in (1.3.6) .

In Chapter 4, we deal with quasilinear and mean curvature type inequalities, namely

$$\Delta_m u + u^\sigma \le 0, \quad \text{on } M, \tag{1.3.7}$$

and

$$div\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right) + u^\sigma \le 0, \quad \text{on } M, \tag{1.3.8}$$

where $m > 1$ and $\sigma > m-1$. We obtain that if

$$\mu(B(x_0, r)) \le Cr^{\frac{m\sigma}{\sigma-m+1}} \ln^{\frac{m-1}{\sigma-m+1}} r, \tag{1.3.9}$$

holds for large enough r, then for both problems (1.3.7) and (1.3.8), the only non-negative solution is identical zero. The sharpness of exponents $\frac{m\sigma}{\sigma-m+1}$ and $\frac{m-1}{\sigma-m+1}$ for (1.3.7) is also showed.

In Chapter 5, we investigate the inequality with the gradient term, that is

$$div(A(x)|\nabla u|^{m-2}\nabla u) + V(x)u^{\sigma_1}|\nabla u|^{\sigma_2} \le 0, \quad \text{on } M, \tag{1.3.10}$$

where $A(x), V(x)$ are given positive measurable functions on M, σ_1, $\sigma_2 \ge 0$, and $\sigma_1 + \sigma_2 > m-1$. Introduce the measure ν by

$$d\nu = A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu.$$

Assume A and V satisfy the following condition: for almost all $x \in M$ the following

$$cr(x)^{-\delta_1} \le \frac{V(x)}{A(x)} \le Cr(x)^{\delta_2}, \tag{1.3.11}$$

holds for all large enough $r(x)$, where δ_1, δ_2 are arbitrary nonnegative constants. We obtain that if

$$\nu(B(x_0, r)) \le Cr^{\frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} \ln^{\frac{m-1}{\sigma_1+\sigma_2-m+1}} r \tag{1.3.12}$$

holds for large enough r, then the only nonnegative solution of (1.3.10) is constant. Noting that in the case of $\sigma_2 = 0$, we obtain again (1.3.9) under the condition $A = V = 1$.

In Chapter 6, we investigate the following quasilinear inequality

$$-\Delta_m u + V(x)u^\sigma \le 0, \quad \text{on } M, \tag{1.3.13}$$

which differs from the previous chapters by the sign in front of Δ_m. Here $\sigma > m-1$, and

$$V(x) = \frac{1}{r(x)^\alpha}, \quad \text{for large enough } r.$$

The result for this type of inequality is drastically different. We prove that: if $\alpha < m$, and if for some positive number N, the following inequality

$$\mu(B(x_0, r)) \le Cr^N, \tag{1.3.14}$$

holds for all large enough r, then the only nonnegative solution of (1.3.13) is identical zero.

In the setting of manifolds, the classical tools such as Harnack inequalities, estimate of fundamental solutions and a priori estimate, dealing with differential equations and inequalities, are not available only under the condition of the volume growth. In this thesis, we developed a new method, that is a further elaboration of the method of Grigor'yan and Kondratiev [28], which is based on a subtle choice of test functions. This enables us to investigate a variety of differential inequalities on manifolds with minimal geometric assumption.

1.4 Notations

Throughout the thesis we assume that M is a geodesically complete non-compact connected Riemannian manifold. Denote by $d(x, y)$ the geodesic distance between $x, y \in M$. Let $B(x, r) = \{y : d(x, y) < r\}$ and $r(x) = d(x, x_0)$. A Riemannian metric on M is a family $g = \{g(x)\}_{x\in M}$ such that $g(x)$ is symmetric, positive definite, bilinear form on the tangent space T_xM, smoothly depending on $x \in M$. In the local coordinates $\{x^1, ..., x^n\}$,

$$g(x) = \sum_{i,j} g_{ij}(x)dx^i dx^j, \tag{1.4.1}$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

The Riemannian measure μ on M is defined in any chart by

$$d\mu = \sqrt{\det g}\, d\lambda, \tag{1.4.2}$$

where $g = (g_{ij})$ is the matrix of the Riemannian metric g, and λ is the Lebesgue measure.

Define the Riemannian gradient ∇ by

$$\nabla f = g^{ik} \frac{\partial f}{\partial x^k} \frac{\partial}{\partial x^i}, \tag{1.4.3}$$

where $(g^{ij}) = (g_{ij})^{-1}$.

Denote *div* the Riemannian divergence: for any smooth vector field $v(x)$ on M,

$$div\ v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k} \left(\sqrt{\det g} v^k \right). \tag{1.4.4}$$

The Laplace-Beltrami operator Δ on M is defined by $\Delta = div \circ \nabla$: in the local coordinates $\{x^1, ..., x^n\}$

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right). \tag{1.4.5}$$

The operator

$$\Delta_m u := div(|\nabla u|^{m-2} \nabla u),$$

is called the m-Laplace operator.

The letters c, c_0, C, C', C_0... denote positive constants whose values are unimportant and may vary at different occurrences. Besides the above notations, we also use the following

const	a constant.
$A(r) \approx B(r)$	there exist c, $C > 0$, such that $cB(r) \leq A(r) \leq CB(r)$.

Chapter 2

Special Semilinear Inequalities

This chapter is based on the joint work with Prof. Grigor'yan [30].

2.1 Background and Statement

In this chapter we are concerned with nonnegative solutions of the differential inequality

$$\Delta u + u^{\sigma} \leq 0, \tag{2.1.1}$$

on M, where Δ is Laplace-Beltrami operator on M, and $\sigma > 1$ is a given parameter.

It is clear that (2.1.1) has always a trivial solution $u \equiv 0$. In $\mathbb{R}^n$ with $n \leq 2$ the only nonnegative solution of (2.1.1) is identical zero for any σ. While, in $\mathbb{R}^n$ with $n > 2$ the uniqueness of nonnegative solutions of (2.1.1) takes place if and only if $\sigma \leq \frac{n}{n-2}$ (cf.[11, 53, 54]).

Inspired by Kurta's work [38], Grigor'yan and Kondratiev developed in [28] a new method, that uses only the gradient of the distance function and volume of geodesic balls and, hence, is free from curvature assumptions. Fix some $\sigma > 1$ in (2.1.1) and set

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{2.1.2}$$

Let $B(x, r)$ be the geodesic ball on M of radius r centered at x. It was proved in [28, Theorem 1.3] that if, for some $x_0 \in M$, $C > 0$, $\varepsilon > 0$ and all large enough r,

$$\mu(B(x_0, r)) \leq C r^p \ln^{q-\varepsilon} r, \tag{2.1.3}$$

then the only nonnegative solution of (2.1.1) on M is zero. The sharpness of the exponent p here is clear from the example of $\mathbb{R}^n$ where (2.1.3) holds with $p = n$ that by (2.1.2) corresponds to the critical value $\sigma = \frac{n}{n-2}$. The question of the sharpness of the exponent of $\ln r$ remained unresolved in [28].

In this chapter, we show that in the critical case $\varepsilon = 0$, the uniqueness of nonnegative solutions of (2.1.1) holds as well. We also show that if $\varepsilon < 0$, then under the condition (2.1.3) there may exist a positive solution of (2.1.1).

Solutions of (2.1.1) are understood in a weak sense. Denote by $W^{1,2}_{loc}(M)$ the space of functions $f \in L^2_{loc}(M)$ whose weak gradient ∇f is also in $L^2_{loc}(M)$. Denote by $W^{1,2}_c(M)$ the subspace of $W^{1,2}_{loc}(M)$ of functions with compact support.

Definition 2.1.1. A function u on M is called a weak solution of the inequality (2.1.1) if u is a nonnegative function from $W^{1,2}_{loc}(M)$, and, for any nonnegative function $\psi \in W^{1,2}_c(M)$, the following inequality holds:

$$-\int_M (\nabla u, \nabla \psi) d\mu + \int_M u^\sigma \psi d\mu \leq 0, \tag{2.1.4}$$

where $(\cdot, \cdot)$ is the inner product in $T_x M$ given by Riemannian metric.

Remark 2.1.2. Note that the first integral in (2.1.4) is finite by the compactness of $\operatorname{supp} \psi$. Therefore, the second integral in (2.1.4) is also finite, and hence, $u \in L^\sigma_{loc}(M)$.

Our main result is as follows:

Theorem 2.1.3. Assume that, for some $x_0 \in M$, the following inequality

$$\mu(B(x_0, r)) \leq C r^p \ln^q r, \tag{2.1.5}$$

holds for all large enough r, where p and q are defined by (2.1.2). Then the only nonnegative weak solution of (2.1.1) is identical zero.

Note that if

$$\mu(B(x_0, r)) \leq C r^2 \ln r \tag{2.1.6}$$

holds for all large r, then the manifold M is parabolic, that is, any nonnegative superharmonic function on M is constant (cf. [8], [25]). For example, $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Since any positive solution of (2.1.1) is a superharmonic function, it follows that, on any parabolic manifold, in particular, under the condition (2.1.6), any nonnegative solution of (2.1.1) is zero, for any value of σ. Obviously, our Theorem 2.1.3 is specific to the value of σ, and the value of p is always greater than 2, so that our hypothesis (2.1.5) is weaker than (2.1.6).

2.2 Proof of Theorem 2.1.3

We divide the proof into three parts. In Part 1, we prove that every non-trivial nonnegative solution to (2.1.1) is in fact positive and, moreover, $\frac{1}{u} \in L^\infty_{loc}(M)$.

In Part 2, we obtain the estimates (2.2.10) and (2.2.11) involving a test function and positive parameters. In Part 3, we choose in (2.2.10) and (2.2.11) specific test functions and parameters, which will allow us to conclude that $\int_M u^\sigma d\mu = 0$ and, hence, to finish the proof.

Part 1. We claim that if u is a nonnegative solution to (2.1.1) and $\operatorname{essinf}_U u = 0$ for some non-empty precompact open set U, then $u \equiv 0$ on M. Let us cover U by a finite family $\{\Omega_j\}$ of charts. Then we must have $\operatorname{essinf}_{U\cap\Omega_j} u = 0$ for at least one value of j. Replacing U by $U \cap \Omega_j$, we can assume that U lies in a chart.

Note that by (2.1.1) the function u is (weakly) superharmonic function. Applying in U a strong minimum principle for weak supersolutions (cf. [19, Theorem 8.19]), we obtain $u = 0$ a.e. in U.

In order to prove that $u = 0$ a.e. on M, it suffices to show that $u = 0$ a.e. on any precompact open set V that lies in a chart on M. Let us connect U with V by a sequence of precompact open sets $\{U_i\}_{i=0}^n$ such that each U_i lies in a chart and

$$U_0 = U, \quad U_i \cap U_{i+1} \neq \emptyset, \quad U_n = V.$$

By induction, we obtain that $u = 0$ a.e. on U_i for any $i = 0, ..., n$. Indeed, the induction bases has been proved above. If it is already known that $u = 0$ a.e. on U_i then the condition $U_i \cap U_{i+1} \neq \emptyset$ implies that $\operatorname{essinf}_{U_{i+1}} u = 0$ whence as above we obtain $u = 0$ a.e. on U_{i+1}. In particular, $u = 0$ a.e. on V, which was claimed.

Hence, if u is a non-trivial nonnegative solution to (2.1.1) then $\operatorname{essinf}_U u > 0$ for any non-empty precompact open set $U \subset M$. It follows that $\frac{1}{u}$ is essentially bounded on U, whence $\frac{1}{u} \in L^\infty_{loc}(M)$ follows.

In what follows we assume that u is a positive solutions of (2.1.1) satisfying the condition $\frac{1}{u} \in L^\infty_{loc}(M)$, and show that this assumption leads to contradiction.

Part 2. Fix some non-empty compact set $K \subset M$ and a Lipschitz function φ on M with compact support, such that $0 \le \varphi \le 1$ on M and $\varphi \equiv 1$ in a neighborhood of K. In particular, we have $\varphi \in W^1_c(M)$. We use the following test function for (2.1.4):

$$\psi(x) = \varphi(x)^s u(x)^{-t}, \tag{2.2.1}$$

where t, s are parameters that will be chosen to satisfy the conditions

$$0 < t < \min\left(1, \frac{\sigma-1}{2}\right) \quad \text{and} \quad s > \frac{4\sigma}{\sigma-1}. \tag{2.2.2}$$

In fact, s can be fixed once and for all as in (2.2.2), while t will be variable and will take all small enough values.

The function ψ has a compact support and is bounded, due to the local boundedness of $\frac{1}{u}$. Since

$$\nabla\psi = -tu^{-t-1}\varphi^s\nabla u + su^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that $\nabla\psi \in L^2(M)$ and, consequently, $\psi \in W_c^1(M)$. We obtain from (2.1.4) that

$$t\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu + \int_M \varphi^s u^{\sigma-t} d\mu \le s\int_M \varphi^{s-1}u^{-t}(\nabla u, \nabla\varphi)d\mu. \tag{2.2.3}$$

Using the Cauchy-Schwarz inequality, let us estimate the right-hand side of (2.2.3) as follows

$$\begin{aligned} s\int_M \varphi^{s-1}u^{-t}(\nabla u,\nabla\varphi)d\mu &= \int_M \left(\sqrt{t}u^{-\frac{t+1}{2}}\varphi^{\frac{s}{2}}\nabla u, \frac{s}{\sqrt{t}}u^{-\frac{t-1}{2}}\varphi^{\frac{s}{2}-1}\nabla\varphi\right)d\mu \\ &\le \frac{t}{2}\int_M u^{-t-1}\varphi^s|\nabla u|^2 d\mu \\ &\quad + \frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}|\nabla\varphi|^2 d\mu. \end{aligned}$$

Substituting this inequality into (2.2.3), and cancelling out the half of the first term in (2.2.3), we obtain

$$\frac{t}{2}\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu + \int_M \varphi^s u^{\sigma-t}d\mu \le \frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}|\nabla\varphi|^2 d\mu. \tag{2.2.4}$$

Applying Young's inequality in the form

$$\int_M fg\,d\mu \le \varepsilon\int_M |f|^{p_1}d\mu + C_\varepsilon\int_M |g|^{p_2}d\mu,$$

where $\varepsilon > 0$ is arbitrary and

$$p_1 = \frac{\sigma-t}{1-t}, \quad \text{and} \quad p_2 = \frac{\sigma-t}{\sigma-1},$$

are the Hölder conjugate, we estimate the right-hand side of (2.2.4) as follows:

$$\begin{aligned} &\frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}|\nabla\varphi|^2 d\mu \\ = &\int_M [u^{1-t}\varphi^{\frac{s}{p_1}}]\cdot[\tfrac{s^2}{2t}\varphi^{\frac{s}{p_2}-2}|\nabla\varphi|^2]d\mu \\ \le &\ \varepsilon\int_M u^{\sigma-t}\varphi^s d\mu + C_\varepsilon\left(\frac{s^2}{2t}\right)^{\frac{\sigma-t}{\sigma-1}}\int_M \varphi^{s-2\frac{\sigma-t}{\sigma-1}}|\nabla\varphi|^{2\frac{\sigma-t}{\sigma-1}}d\mu. \end{aligned} \tag{2.2.5}$$

Choose here $\varepsilon = \frac{1}{2}$ and use in the right-hand side the obvious inequalities

$$\left(\frac{s^2}{t}\right)^{\frac{\sigma-t}{\sigma-1}} \leq \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad \varphi^{s-2\frac{\sigma-t}{\sigma-1}} \leq 1.$$

Combining (2.2.5) with (2.2.4), we obtain that

$$\frac{t}{2}\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu + \frac{1}{2}\int_M \varphi^s u^{\sigma-t} d\mu \leq C t^{\frac{\sigma}{1-\sigma}} \int_M |\nabla \varphi|^{2\frac{\sigma-t}{\sigma-1}} d\mu, \tag{2.2.6}$$

where the value of s is absorbed into constant C.

Let us come back to (2.1.4) and use another test function $\psi = \varphi^s$, which yields

$$\begin{aligned} &\int_M \varphi^s u^\sigma d\mu \\ \leq\ & s\int_M \varphi^{s-1}(\nabla u, \nabla\varphi) d\mu \\ \leq\ & s\left(\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu\right)^{1/2}\left(\int_M \varphi^{s-2}u^{t+1}|\nabla\varphi|^2 d\mu\right)^{1/2}. \end{aligned} \tag{2.2.7}$$

On the other hand, we obtain from (2.2.6) that

$$\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu \leq C t^{-1-\frac{\sigma}{\sigma-1}} \int_M |\nabla\varphi|^{2\frac{\sigma-t}{\sigma-1}} d\mu.$$

Substituting into (2.2.7) yields

$$\begin{aligned} \int_M \varphi^s u^\sigma d\mu \ \leq\ & C\left[t^{-1-\frac{\sigma}{\sigma-1}}\int_M |\nabla\varphi|^{2\frac{\sigma-t}{\sigma-1}} d\mu\right]^{1/2} \\ & \times\left[\int_M \varphi^{s-2}u^{t+1}|\nabla\varphi|^2 d\mu\right]^{1/2}. \end{aligned} \tag{2.2.8}$$

Recall that $\varphi \equiv 1$ in a neighborhood of K so that $\nabla\varphi = 0$ on K. Applying Hölder inequality to the last term in (2.2.8) with the Hölder couple

$$p_3 = \frac{\sigma}{t+1}, \quad p_4 = \frac{\sigma}{\sigma-t-1},$$

we obtain

$$\begin{aligned} &\int_M \varphi^{s-2}u^{t+1}|\nabla\varphi|^2 d\mu \\ =\ & \int_{M\setminus K}\left(\varphi^{\frac{s}{p_3}}u^{t+1}\right)\left(\varphi^{\frac{s}{p_4}-2}|\nabla\varphi|^2\right) d\mu \end{aligned}$$

$$\leq \left(\int_{M\setminus K}\varphi^s u^\sigma d\mu\right)^{\frac{t+1}{\sigma}}\left(\int_{M\setminus K}\varphi^{s-\frac{2\sigma}{\sigma-t-1}}|\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}}d\mu\right)^{\frac{\sigma-t-1}{\sigma}}. \tag{2.2.9}$$

By (2.2.2) we have $s-\frac{2\sigma}{\sigma-t-1}>0$ so that the term $\varphi^{s-\frac{2\sigma}{\sigma-t-1}}$ is bounded by 1. Substituting (2.2.9) into (2.2.8), we obtain

$$\begin{aligned}\int_M \varphi^s u^\sigma d\mu \;\leq\; & C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}}\left(\int_M |\nabla\varphi|^{2\frac{\sigma-t}{\sigma-1}}d\mu\right)^{\frac{1}{2}} \\ & \times\left(\int_{M\setminus K}\varphi^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}\left(\int_M|\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}}d\mu\right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned}\tag{2.2.10}$$

Since $\int_M \varphi^s u^\sigma d\mu$ is finite due to Remark in Introduction, it follows from (2.2.10) that

$$\begin{aligned}\left(\int_M \varphi^s u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \;\leq\; & C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}}\left(\int_M |\nabla\varphi|^{2\frac{\sigma-t}{\sigma-1}}d\mu\right)^{\frac{1}{2}} \\ & \times\left(\int_M|\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}}d\mu\right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned}\tag{2.2.11}$$

Part 3. Set $r(x)=d(x,x_0)$, where x_0 is the point from the hypothesis (2.1.5). Fix some large $R>1$, set

$$t=\frac{1}{\ln R},\qquad K=B_R:=B(x_0,R),$$

and consider the function

$$\varphi(x)=\begin{cases}1, & r(x)<R,\\ \left(\frac{r(x)}{R}\right)^{-t}, & r(x)\geq R.\end{cases}\tag{2.2.12}$$

Note that R will be chosen large enough so that t can be assumed to be sufficiently small, in particular, to satisfy (2.2.2).

We would like to use (2.2.11) with this function $\varphi(x)$. However, since $\operatorname{supp}\varphi$ is not compact, we consider instead a sequence $\{\varphi_n\}$ of functions with compact supports that is constructed as follows. For any $n=1,2,...$ define a cut-off function η_n by

$$\eta_n(x)=\begin{cases}1, & 0\leq r(x)\leq nR,\\ 2-\frac{r(x)}{nR}, & nR\leq r(x)\leq 2nR,\\ 0, & r(x)\geq 2nR.\end{cases}\tag{2.2.13}$$

Consider the function

$$\varphi_n(x)=\varphi(x)\eta_n(x),\tag{2.2.14}$$

so that $\varphi_n(x) \uparrow \varphi(x)$ as $n \to \infty$. Notice that

$$|\nabla \varphi_n|^2 \le 2\left(\eta_n^2 |\nabla \varphi|^2 + \varphi^2 |\nabla \eta_n|^2\right), \tag{2.2.15}$$

which implies that, for any $a \ge 2$,

$$|\nabla \varphi_n|^a \le C_a \left(\eta_n^a |\nabla \varphi|^a + \varphi^a |\nabla \eta_n|^a\right). \tag{2.2.16}$$

We will consider only the values of a of the bounded range $a \le 2p$ so that the constant C_a can be regarded as uniformly bounded.

Let us estimate the integral

$$I_n(a) := \int_M |\nabla \varphi_n|^a d\mu. \tag{2.2.17}$$

By (2.2.16), we have

$$\begin{aligned} I_n(a) &\le C \int_M \eta_n^a |\nabla \varphi|^a d\mu + C \int_M \varphi^a |\nabla \eta_n|^a d\mu \\ &\le C \int_{M \setminus B_R} |\nabla \varphi|^a d\mu + C \int_{B_{2nR} \setminus B_{nR}} \varphi^a |\nabla \eta_n|^a d\mu, \end{aligned} \tag{2.2.18}$$

where we have used that $\nabla \varphi = 0$ in B_R, and $\nabla \eta_n = 0$ outside $B_{2nR} \setminus B_{nR}$. Since $|\nabla \eta_n| \le \frac{1}{nR}$, the second integral in (2.2.18) can be estimated as follows

$$\begin{aligned} \int_{B_{2nR} \setminus B_{nR}} \varphi^a |\nabla \eta_n|^a d\mu &\le \frac{1}{(nR)^a} \int_{B_{2nR} \setminus B_{nR}} \varphi^a d\mu \\ &\le \frac{1}{(nR)^a} \left(\sup_{B_{2nR} \setminus B_{nR}} \varphi^a \right) \mu(B_{2nR}) \\ &\le \frac{C}{(nR)^a} \left(\frac{nR}{R}\right)^{-at} (2nR)^p \ln^q (2nR) \\ &= C' n^{p-a-at} R^{p-a} \ln^q (2nR), \end{aligned} \tag{2.2.19}$$

where we have used the definition (2.2.12) of the function φ and the volume estimate (2.1.5).

Before we estimate the first integral in (2.2.18), observe the following: if f is a nonnegative decreasing function on $\mathbb{R}_+$ then, for large enough R,

$$\int_{M \setminus B_R} f(r(x))\, d\mu(x) \le C \int_{R/2}^{\infty} f(r) r^{p-1} \ln^q r dr, \tag{2.2.20}$$

which follows from (2.1.5) as follows:

$$
\begin{aligned}
\int_{M\setminus B_R} f d\mu &= \sum_{i=0}^{\infty}\int_{B_{2^{i+1}R}\setminus B_{2^iR}} f d\mu \\
&\le \sum_{i=0}^{\infty} f(2^iR)\mu(B_{2^{i+1}R}) \\
&\le C\sum_{i=0}^{\infty} f(2^iR)(2^{i+1}R)^p \ln^q(2^{i+1}R) \\
&\le C'\sum_{i=0}^{\infty} f(2^iR)(2^{i-1}R)^{p-1}(2^{i-1}R)\ln^q(2^{i-1}R) \\
&\le C'\int_{R/2}^{\infty} f(r)r^{p-1}\ln^q r dr.
\end{aligned}
$$

Hence, using $|\nabla\varphi| \le R^t t r^{-t-1}$, (2.2.20), and $R/2 > 1$, we obtain

$$
\begin{aligned}
\int_{M\setminus B_R} |\nabla\varphi|^a d\mu &\le C\int_{R/2}^{\infty} R^{at}t^a r^{-at-a} r^{p-1}\ln^q r dr \\
&\le CR^{at}t^a\int_1^{\infty} r^{-at-a+p}\ln^q r\frac{dr}{r} \\
&= CR^{at}t^a\int_0^{\infty} e^{-b\xi}\xi^q d\xi,
\end{aligned}
$$

where we have made the change $\xi = \ln r$ and set

$$b := at + a - p. \tag{2.2.21}$$

Assuming that $b > 0$ and making one more change $\tau = b\xi$, we obtain

$$\int_{M\setminus B_R} |\nabla\varphi|^a d\mu \le CR^{at}t^a b^{-q-1}\int_0^{\infty} e^{-\tau}\tau^q d\tau = C'R^{at}t^a b^{-q-1}, \tag{2.2.22}$$

where the value $\Gamma(q+1)$ of the integral is absorbed into the constant C'.

Substituting (2.2.19) and (2.2.22) into (2.2.18) yields

$$I_n(a) \le CR^{at}t^a b^{-q-1} + Cn^{-b}R^{p-a}\ln^q(2nR). \tag{2.2.23}$$

We will use (2.2.23) with those values of a for which $b > t$. Noting also that $R^t = \exp(t\ln R) = e$, we obtain

$$I_n(a) \le Ce^a t^{a-q-1} + Cn^{-t}R^{p-a}\ln^q(2nR).$$

As we have remarked above, we will consider only the values of a in the bounded range $a \leq 2p$. Hence, the term e^a in the above inequality can be replaced by a constant. Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} I_n(a) \leq Ct^{a-q-1}. \tag{2.2.24}$$

Let us first use (2.2.24) with $a = \frac{2(\sigma-t)}{\sigma-1}$. Note that $a < p$, and for this value of a and for t as in (2.2.2), we have

$$\begin{aligned} b &= \frac{2(\sigma-t)}{\sigma-1}t + \frac{2(\sigma-t)}{\sigma-1} - \frac{2\sigma}{\sigma-1} \\ &= \frac{2t[(\sigma-1)-t]}{\sigma-1} > t \end{aligned}$$

and

$$a - q - 1 = \frac{2(\sigma-t)}{\sigma-1} - \frac{\sigma}{\sigma-1} = \frac{\sigma-2t}{\sigma-1}.$$

Hence, (2.2.24) yields

$$\limsup_{n\to\infty} I_n\left(\frac{2(\sigma-t)}{\sigma-1}\right) \leq Ct^{\frac{\sigma-2t}{\sigma-1}}. \tag{2.2.25}$$

Similarly, for $a = \frac{2\sigma}{\sigma-t-1}$, we have by (2.2.2) $a < 2p$ and

$$b = \frac{2\sigma}{\sigma-t-1}t + \frac{2\sigma}{\sigma-t-1} - \frac{2\sigma}{\sigma-1} > t,$$

whence

$$\limsup_{n\to\infty} I_n\left(\frac{2\sigma}{\sigma-t-1}\right) \leq Ct^{\frac{2\sigma}{\sigma-t-1}-\frac{\sigma}{\sigma-1}}. \tag{2.2.26}$$

The inequality (2.2.11) with function φ_n implies that

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \leq J_n(t), \tag{2.2.27}$$

where

$$J_n(t) = C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} I_n\left(\frac{2(\sigma-t)}{\sigma-1}\right)^{\frac{1}{2}} I_n\left(\frac{2\sigma}{\sigma-t-1}\right)^{\frac{\sigma-t-1}{2\sigma}}.$$

Letting $n \to \infty$ and substituting the estimates (2.2.25) and (2.2.26), we obtain that

$$\limsup_{n\to\infty} J_n(t) \leq C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} t^{\frac{\sigma-2t}{2(\sigma-1)}} t^{1-\frac{\sigma-t-1}{2(\sigma-1)}} = Ct^{-\frac{t}{2(\sigma-1)}}. \tag{2.2.28}$$

The main point of the above argument is that all the "large" exponents in the power of t have cancelled out, which in the end is a consequence of the estimate (2.2.22) based on the hypothesis (2.1.5). The remaining term $t^{-\frac{t}{2(\sigma-1)}}$ tends to 1 as $t \to 0$, which implies that the right-hand side of (2.2.28) is a bounded function of t. Hence, there is a constant C_1 such that

$$\limsup_{n\to\infty} J_n(t) \le C_1, \tag{2.2.29}$$

for all small enough t. It follows from (2.2.27) that also

$$\int_M \varphi^s u^\sigma d\mu \le C, \tag{2.2.30}$$

for all small enough t. Since $\varphi = 1$ on B_R, it follows that

$$\int_{B_R} u^\sigma d\mu \le C,$$

which implies for $R \to \infty$ that

$$\int_M u^\sigma d\mu \le C. \tag{2.2.31}$$

Inequality (2.2.10) with function φ_n implies that

$$\int_M \varphi_n^s u^\sigma d\mu \le J_n(t) \left(\int_{M\setminus B_R} \varphi_n^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}. \tag{2.2.32}$$

Letting $n \to \infty$ and applying (2.2.29), we obtain

$$\int_M \varphi^s u^\sigma d\mu \le C_1 \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}},$$

whence

$$\int_{B_R} u^\sigma d\mu \le C_1 \left(\int_{M\setminus B_R} u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}, \tag{2.2.33}$$

Since by (2.2.31)

$$\int_{M\setminus B_R} u^\sigma d\mu \to 0 \quad \text{as } R \to \infty,$$

letting in (2.2.33) $R \to \infty$, we obtain

$$\int_M u^\sigma d\mu = 0,$$

which finishes the proof.

2.3 Sharpness of p, q

In this section, we will give an example that shows that the values of the parameters p and q in Theorem 2.1.3 are sharp and cannot be relaxed.

In $\mathbb{R}^n$, when $p = n > \frac{2\sigma}{\sigma-1}$, equivalently, $\sigma > \frac{n}{n-2}$, we know that

$$u(x) := \frac{\epsilon}{(1+|x|^2)^{\frac{1}{\sigma-1}}}, \tag{2.3.1}$$

is a positive solution to (2.1.1), when ϵ is small positive constant. Since σ could be chosen close to $\frac{n}{n-2}$, hence, the parameter p is sharp.

Concerning the sharpness of q, We need the following statement.

Proposition 2.3.1. ([70], [28, Proposition 3.2]) Let $\alpha(r)$ be a positive C^1-function on $(r_0, +\infty)$ satisfying

$$\int_{r_0}^{\infty} \frac{dr}{\alpha(r)} < \infty. \tag{2.3.2}$$

Define the function $\gamma(r)$ on (r_0, ∞) by

$$\gamma(r) = \int_{r}^{\infty} \frac{ds}{\alpha(s)}. \tag{2.3.3}$$

Let $\beta(r)$ be a continuous function on (r_0, ∞) such that

$$\int_{r_0}^{\infty} \gamma(r)^{\sigma} |\beta(r)| dr < \infty. \tag{2.3.4}$$

Then the differential equation

$$(\alpha(r)y')' + \beta(r)y^{\sigma} = 0, \tag{2.3.5}$$

has a positive solution $y(r)$ in an interval $[R_0, +\infty)$ for large enough $R_0 > r_0$, such that

$$y(r) = O(\gamma(r)), \quad \text{as } r \to \infty. \tag{2.3.6}$$

Given $\sigma > 1$, set as before $p = \frac{2\sigma}{\sigma-1}$ and choose some $q > \frac{1}{\sigma-1}$. We will construct an example of a manifold M satisfying the volume growth condition (2.1.5) with these values p, q and admitting a positive solution u of (2.1.1).

The manifold M will be $(\mathbb{R}^n, g)$ with the following Riemannian metric

$$g = dr^2 + \psi(r)^2 d\theta^2, \tag{2.3.7}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$ and $\psi(r)$ is a smooth, positive function on $(0, \infty)$ such that

$$\psi(r) = \begin{cases} r, & \text{for small enough } r, \\ \left(r^{p-1} \ln^q r\right)^{\frac{1}{n-1}}, & \text{for large enough } r. \end{cases} \tag{2.3.8}$$

It follows that, in a neighborhood of 0, the metric g is exactly Euclidean, so that it can be extended smoothly to the origin. Hence, $M = (\mathbb{R}^n, g)$ is a complete Riemannian manifold.

By (2.3.7), the geodesic ball $B_r = B(0, r)$ on M coincides with the Euclidean ball $\{|x| < r\}$. Denote by $S(r)$ the surface area of B_r in M. It follows from (2.3.7) that $S(r) = \omega_n \psi^{n-1}(r)$, that is

$$S(r) = \omega_n \begin{cases} r^{n-1}, & \text{for small enough } r, \\ r^{p-1} \ln^q r, & \text{for large enough } r, \end{cases} \tag{2.3.9}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. The Riemannian volume of the ball B_r can be determined by

$$\mu(B_r) = \int_0^r S(\tau)\, d\tau,$$

whence it follows that, for large enough r,

$$\mu(B_r) \le C r^p \ln^q r. \tag{2.3.10}$$

Hence, the manifold M satisfied the volume growth condition of Theorem 2.1.3.

In what follows we prove the existence of a weak positive solution of $\Delta u + u^\sigma \le 0$ on M. In fact, the solution u will depend only on the polar radius r, so that we can write $u = u(r)$. The construction of u will be done in two steps.

Step I. For a function $u = u(r)$, the inequality (2.1.1) becomes

$$u'' + \frac{S'}{S} u' + u^\sigma \le 0, \tag{2.3.11}$$

(cf. [27, (3.93)]), that is

$$(S u')' + S u^\sigma \le 0. \tag{2.3.12}$$

For $r >> 1$, we have

$$\gamma(r) := \int_r^\infty \frac{d\tau}{S(\tau)} = \int_r^\infty \frac{d\tau}{\tau^{p-1} \ln^q \tau} \approx \frac{1}{r^{p-2} \ln^q r},$$

and

$$\begin{aligned}\int_{r_0}^{\infty} \gamma(\tau)^{\sigma} S(\tau) d\tau &= \int_{r_0} \frac{\tau^{p} \ln^{q} \tau}{\tau^{\sigma(p-2)} \ln^{\sigma q} \tau} \frac{d\tau}{\tau} \\ &= \int_{r_0}^{\infty} \frac{1}{\tau^{\sigma(p-2)-p} \ln^{q(\sigma-1)} \tau} \frac{d\tau}{\tau} \\ &= \int_{r_0}^{\infty} \frac{1}{\ln^{q(\sigma-1)} \tau} \frac{d\tau}{\tau} \\ &< \infty,\end{aligned}$$

where we have used that $q > \frac{1}{\sigma-1}$.

Applying Proposition 2.3.1 with $\alpha(r) = \beta(r) = S(r)$, we obtain that there exists a positive solution u of (2.3.12) on $[R_0, +\infty)$ for some large enough R_0, such that

$$u(r) = O(\gamma(r)) = O(r^{-(p-2)} \ln^{-q} r), \quad \text{as } r \to \infty.$$

In particular, $u(r) \to 0$ as $r \to \infty$. By increasing R_0 if necessary, we can assume that $u^{'}(R_0) < 0$.

Step II. Consider the following eigenvalue problem in a ball B_ρ of M:

$$\begin{cases} \Delta v + \lambda v = 0 \text{ in } B_\rho, \\ v|_{\partial B_\rho} = 0. \end{cases} \tag{2.3.13}$$

Denote by λ_ρ the principal (smallest) eigenvalue of this problem. It is known that $\lambda_\rho > 0$ and the corresponding eigenfunction v_ρ does not change sign in B_ρ (cf. [27, Theorem 10.11, 10.22]). Normalizing v_ρ, we can assume that $v_\rho(0) = 1$ and, hence, $v_\rho > 0$ in B_ρ, while $v_\rho|_{\partial B_\rho} = 0$.

Since the principal eigenvalue λ_ρ is simple (cf. [27, Corollary 10.12]) and the Riemannian metric g is spherically symmetric, the eigenfunction v_ρ must also be spherically symmetric. Therefore, v_ρ can be regarded as a function of the polar radius r only. In terms of r, we can rewrite (2.3.13) as follows

$$v_\rho'' + \frac{S'}{S} v_\rho' + \lambda_\rho v_\rho = 0, \tag{2.3.14}$$

where $v_\rho(\rho) = 0$, $v_\rho(0) = 1$, $v_\rho^{'}(0) = 0$, and $v_\rho > 0$ in $(0, \rho)$.

Multiplying (2.3.14) by S, we obtain

$$(S v_\rho')' + \lambda_\rho S v_\rho = 0.$$

It follows that $(S v_\rho')' \leq 0$, so that the function $S v_\rho'$ is decreasing. Since it vanishes at $r = 0$, it follows that $S v_\rho'(r) \leq 0$ and, hence $v_\rho'(r) \leq 0$ for all $r \in (0, \rho)$. Hence, the function $v_\rho(r)$ is decreasing for $r < \rho$ which together with the boundary conditions implies that $0 \le v_\rho \le 1$. It follows that v_ρ is a positive solution in B_ρ of the inequality

$$\Delta v_\rho + \lambda_\rho v^\sigma \le 0. \tag{2.3.15}$$

Let us show that $\lambda_\rho \to 0$ as $\rho \to \infty$. Indeed, it is known that

$$\lim_{\rho\to\infty} \lambda_\rho = \lambda_{\min}(M)$$

where $\lambda_{\min}(M)$ is the bottom of the spectrum of $-\Delta$ in $L^2(M,\mu)$, while by a theorem of Brooks

$$\lambda_{\min}(M) \le \frac{1}{4}\left(\limsup_{\rho\to\infty} \frac{\ln \mu(B_\rho)}{\rho}\right)^2 \tag{2.3.16}$$

(cf. [6], [27, Theorem 11.19]). The right-hand side of (2.3.16) vanishes by (2.3.10), where we obtain that $\lim_{\rho\to\infty} \lambda_\rho = 0$.

Let us show that there exists a sequence $\{\rho_k\}$ such that $v_{\rho_k} \to 1$, as $k \to \infty$, where the convergence is local in C^1. Indeed, let us first take that $\rho_k = k$. As v_k satisfies the equation $\Delta v_k + \lambda_k v_k = 0$, the sequence $\{v_k\}$ is bounded, and $\lambda_k \to 0$, it follows by local elliptic regularity properties that there exists a subsequence $\{v_{k_i}\}$ that converges in C^∞_{loc} to a function v, and the latter satisfies $\Delta v = 0$ (cf. [27, Theorem 13.14]). The function v depends only on the polar radius and, hence, satisfies the conditions

$$\begin{cases} v'' + \dfrac{S'}{S} v' = 0, \\ v(0) = 1. \end{cases}$$

Solving this ODE, we obtain a general solution

$$v(r) = C \int_0^r \frac{dr}{S(r)} + 1.$$

Since $\int_0^r \frac{dr}{S(r)}$ diverges at 0, so the only bounded solution is $v \equiv 1$. We conclude that

$$v_{k_i} \xrightarrow{C^\infty_{loc}} 1 \quad \text{as } i \to \infty. \tag{2.3.17}$$

Choose ρ large enough so that $\rho > R_0$ and

$$\frac{v_\rho'}{v_\rho}(R_0) > \frac{u'}{u}(R_0), \tag{2.3.18}$$

where u is the function constructed in the first step. Indeed, it is possible to achieve

(2.3.18) by choosing $\rho = k_i$ with large enough i because by (2.3.17)

$$\frac{v'_{k_i}}{v_{k_i}}(R_0) \to 0 \quad \text{as } i \to \infty$$

whereas $\frac{u'}{u}(R_0) < 0$ by construction.

Let us fix $\rho > R_0$ for which (2.3.18) is satisfied, and compare the functions $u(r)$ and $v_\rho(r)$ in the interval $[R_0, \rho)$. Set

$$m = \inf_{r \in [R_0, \rho)} \frac{u(r)}{v_\rho(r)}.$$

Since v_ρ vanishes at ρ and, hence,

$$\frac{u(r)}{v_\rho(r)} \to \infty \quad \text{as} \quad r \to \rho+,$$

the ratio $\frac{u}{v_\rho}$ attains its infimum value m at some point $\xi \in [R_0, \rho)$. We claim that $\xi > R_0$. Indeed, at $r = R_0$, we have by (2.3.18)

$$\left(\frac{u}{v_\rho}\right)'(R_0) = \frac{u'v_\rho - uv'_\rho}{v^2}(R_0) < 0,$$

so that u/v_ρ is strictly decreasing at R_0 and cannot have minimum at R_0. Hence, $\frac{u}{v_\rho}$ attains its minimum at an interior point $\xi \in (R_0, \rho)$, and at this point we have

$$\left(\frac{u}{v_\rho}\right)'(\xi) = 0.$$

It follows that

$$u(\xi) = mv_\rho(\xi) \quad \text{and} \quad u'(\xi) = mv'_\rho(\xi) \tag{2.3.19}$$

The function $u(r)$ has been defined for $r \geq R_0$, in particular, for $r \geq \xi$, whereas $v_\rho(r)$ has been defined for $r \leq \rho$, in particular, for $r \leq \xi$. Now we merge the two definitions by redefining/extending the function $u(r)$ for all $0 < r < \xi$ by setting $u(r) = mv_\rho(r)$.

It follows from (2.3.19) that $u \in C^1(M)$, in particular, $u \in W^{1,2}_{loc}(M)$. By (2.3.15), u satisfies the following inequality in B_ξ:

$$\Delta u + \frac{\lambda_\rho}{m^{\sigma-1}} u^\sigma \leq 0. \tag{2.3.20}$$

By (2.1.1), u satisfies the following inequality in $M \setminus B_{R_0}$:

$$\Delta u + u^\sigma \leq 0. \tag{2.3.21}$$

Combining (2.3.20) and (2.3.21), we obtain that u satisfies on M the following inequality

$$\Delta u + \delta u^\sigma \leq 0, \tag{2.3.22}$$

where $\delta = \min\{\lambda_\rho / m^{\sigma-1}, 1\}$. Finally, changing $u \mapsto cu$ where $c = \delta^{-\frac{1}{\sigma-1}}$ we obtain a positive solution to (2.1.1) on M, which concludes this example.

Chapter 3

Semilinear Differential Inequalities

This chapter is based on the paper [67].

3.1 Background and Statement

Consider a geodesically complete non-compact connected manifold M and the following differential inequality

$$div(A(x)\nabla u) + V(x)u^\sigma \leq 0, \quad \text{on } M, \tag{3.1.1}$$

where $A(x)$ is a nonnegative definite symmetric operator in the tangent space T_xM, such that $x \mapsto A(x)$ is measurable, V is a given locally integrable positive measurable function, and $\sigma > 1$ is a given constant.

Throughout this chapter, assume that $V(x) \in L^1_{loc}(M, \mu)$. Let us set

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{3.1.2}$$

For the case $A = Id$, $V \equiv 1$, the uniqueness result was obtained in Chapter 2. For general A, V in (3.1.1), Grigor'yan and Kondratiev in [28] used measure ν_ϵ defined for any $\epsilon > 0$ by

$$d\nu_\epsilon = \|A\|^{\frac{\sigma}{\sigma-1}-\epsilon} V^{-\frac{1}{\sigma-1}+\epsilon} d\mu,$$

and proved that if

$$\nu_\epsilon(B(x_0, r)) \leq C r^{p+C\epsilon} \ln^\kappa r, \tag{3.1.3}$$

holds for all large enough r, and for some $\kappa < q$ and all small enough $\epsilon > 0$, where p, q are given by (3.1.2), then the only nonnegative solution of (3.1.1) is identical zero. Some conditions for uniqueness of nonnegative solutions in terms of capacities were proved in [28, 38].

Here, we improve the result of [28, Theorem 1.3] by allowing $\epsilon = 0$. Namely, consider the measure ν, defined by

$$d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu. \tag{3.1.4}$$

We also need the following assumption on A, V: there exist δ_1, $\delta_2 \geq 0$ and positive constants c_0, C_0 such that, for almost all $x \in M$,

$$c_0 r(x)^{-\delta_1} \leq \frac{V(x)}{\|A(x)\|} \leq C_0 r(x)^{\delta_2}, \tag{VA}$$

holds for large enough $r(x)$. In particular, we assume $V(x) > 0$ and $\|A(x)\| > 0$ for almost all $x \in M$. Let us emphasize that the operator $A(x)$ is only assumed to be nonnegative definite, so it can be degenerate.

Here is our main result.

Theorem 3.1.1. Assume that (VA) holds with some $\delta_1, \delta_2 \geq 0$. If for some $x_0 \in M$, the following inequality

$$\nu(B(x_0, r)) \leq C r^p \ln^q r, \tag{3.1.5}$$

holds for all large enough r, where ν is defined as in (3.1.4), p and q are defined by (3.1.2), then the only nonnegative weak solution of (3.1.1) is identical zero.

In the following, we will explain the notion of a weak solution of (3.1.1). Let us introduce the following notations. If v, w are the vectors in the tangent space $T_x M$, denote

$$(v, w)_A := (A(x)v, w), \tag{3.1.6}$$

with the corresponding semi-norm defined by

$$|v|_A = (A(x)v, v)^{1/2}.$$

Then for the operator norm $\|A(x)\|$, we have for any $v \in T_x M$

$$|v|_A^2 \leq \|A(x)\| \cdot |v|^2, \tag{3.1.7}$$

where $|v|$ is the Riemannian length of v.

Define

$$d\omega = \|A(x)\| \, d\mu,$$

and denote by

$$W^{1,2}_{loc}(M,\omega) := \left\{ f | f \in L^2_{loc}(M,\omega), \nabla f \in L^2_{loc}(M,\omega) \right\}, \tag{3.1.8}$$

and denote by $W^{1,2}_c(M,\omega)$ the subspace of $W^{1,2}_{loc}(M,\omega)$ of functions with compact support.

Solutions of (3.1.1) are understood in the following weak sense

Definition 3.1.2. A function u on M is called a weak solution of the inequality (3.1.1) if u is a nonnegative function from $W^{1,2}_{loc}(M,\omega)$, and for any nonnegative function $\psi \in W^{1,2}_c(M,\omega)$, the following inequality holds:

$$-\int_M (\nabla u, \nabla \psi)_A d\mu + \int_M V(x) u^\sigma \psi d\mu \le 0, \tag{3.1.9}$$

where $(\cdot,\cdot)_A$ is defined as in (3.1.6).

Remark 3.1.3. Notice here if u is the solution of (3.1.1), the first integral term is bounded. Furthermore, the finiteness of the first integral on the left-hand side will lead to the finiteness of the second one, since this is derived from (3.1.9) automatically.

3.2 First Proof of Theorem 3.1.1

Let u be a nonnegative solution of (3.1.1). Fix some ball $B_R := B(x_0, R)$, where x_0 is the reference point from the hypothesis (3.1.5), and $R > 0$ to be chosen later. Take a Lipschitz function φ on M with compact support, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of $\bar{B}_R$. Particularly, $\varphi \in W^{1,2}_c(M,\omega)$. We use the following test function for (3.1.9):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^{-t}, \tag{3.2.1}$$

where $\rho > 0$ is a parameter near zero, and the constants t, s satisfy the conditions

$$\begin{cases} 0 < t < \min\left(1, \frac{\sigma-1}{2}\right), \\ s > \max\left\{ \frac{4\sigma}{\sigma-1}, 1 + \frac{2+\delta_2}{\sigma-1}, 1 + \frac{2\sigma(\delta_1-2)}{(\sigma-1)^2} \right\}. \end{cases} \tag{3.2.2}$$

In fact, in what follows s will be chosen to be a large enough fixed constant, and t will take arbitrarily small positive values.

Since $\frac{1}{u+\rho}$ is bounded, hence, ψ_ρ has compact support and is bounded. The identity

$$\nabla \psi_\rho = -t\varphi^s (u+\rho)^{-t-1} \nabla u + s\varphi^{s-1}(u+\rho)^{-t} \nabla \varphi,$$

implies that $\nabla\psi_\rho \in L^2(M,\omega)$, hence, $\psi_\rho \in W_c^{1,2}(M,\omega)$. We obtain from (3.1.9) that

$$\begin{aligned} & t\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \\ \leq\ & s\int_M \varphi^{s-1}(u+\rho)^{-t}(\nabla u, \nabla\varphi)_A d\mu. \end{aligned} \tag{3.2.3}$$

Applying Cauchy-Schwarz inequality, let us estimate the right-hand side of (3.2.3) as follows

$$\begin{aligned} & s\int_M \varphi^{s-1}(u+\rho)^{-t}(\nabla u, \nabla\varphi)_A d\mu \\ =\ & \int_M \left(\sqrt{t}\varphi^{\frac{s}{2}}(u+\rho)^{-\frac{t+1}{2}}\nabla u, \frac{s}{\sqrt{t}}\varphi^{\frac{s}{2}-1}(u+\rho)^{-\frac{t-1}{2}}\nabla\varphi\right)_A d\mu \\ \leq\ & \frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|_A^2 d\mu \\ & + \frac{s^2}{2t}\int_M \varphi^{s-2}(u+\rho)^{1-t}|\nabla\varphi|_A^2 d\mu. \end{aligned}$$

Substituting the above into (3.2.3), and cancelling out the half of the first term in (3.2.3), we obtain

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \\ \leq\ & \frac{s^2}{2t}\int_M \varphi^{s-2}(u+\rho)^{1-t}|\nabla\varphi|_A^2 d\mu. \end{aligned} \tag{3.2.4}$$

Using Young's inequality

$$\int_M fg\,d\mu \leq \varepsilon\int_M |f|^{p_1} d\mu + C_\varepsilon\int_M |g|^{p_1'} d\mu,$$

where $\varepsilon > 0$ is arbitrary, and (p_1, p_1') is the Hölder conjugate such that

$$p_1 = \frac{\sigma - t}{1-t}, \quad p_1' = \frac{\sigma - t}{\sigma - 1}.$$

Let us estimate the right-hand side of (3.2.4) as follows

$$\begin{aligned} & \frac{s^2}{2t}\int_M \varphi^{s-2}(u+\rho)^{1-t}|\nabla\varphi|_A^2 d\mu \\ =\ & \int_M [\varphi^{\frac{s}{p_1}} V^{\frac{1}{p_1}}(u+\rho)^{1-t}]\cdot[\tfrac{s^2}{2t}\varphi^{\frac{s}{p_1'}-2}V^{-\frac{1}{p_1}}|\nabla\varphi|_A^2] d\mu \\ \leq\ & \varepsilon\int_M \varphi^s V(u+\rho)^{\sigma-t} d\mu \end{aligned}$$

$$+C_\varepsilon\left(\frac{s^2}{2t}\right)^{\frac{\sigma-t}{\sigma-1}}\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}}V^{-\frac{1-t}{\sigma-1}}|\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}}d\mu. \tag{3.2.5}$$

Choosing $\varepsilon = \frac{1}{2}$ and using in the right-hand side of (3.2.5) the simple inequality

$$\left(\frac{s^2}{t}\right)^{\frac{\sigma-t}{\sigma-1}} \leq \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}}.$$

and combining (3.2.5) with (3.2.4), we obtain that

$$\begin{aligned}&\frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu\\ \leq\ &\frac{1}{2}\int_M \varphi^s V(u+\rho)^{\sigma-t}d\mu + Ct^{-\frac{\sigma}{\sigma-1}}\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}}V^{-\frac{1-t}{\sigma-1}}|\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}}d\mu,\end{aligned} \tag{3.2.6}$$

where the value of s is absorbed into constant C.

Before moving to the next step, let us specify the boundedness of the above integrals. It is easy to obtain from the definition of the solution the boundedness of the following three integral terms

$$\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|_A^2 d\mu,$$

and

$$\int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu,$$

and

$$\int_M \varphi^{s-1}(u+\rho)^{-t}(\nabla u, \nabla\varphi)_A d\mu.$$

The boundedness of $\int_M \varphi^s V(u+\rho)^{\sigma-t}d\mu$ follows by the boundedness of

$$\int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu,$$

and $V \in L^1_{loc}(M,\mu)$.

By Dominated Convergence theorem, we know

$$\lim_{\rho\downarrow 0}\int_M \varphi^s V(u+\rho)^{\sigma-t}d\mu = \int_M \varphi^s V u^{\sigma-t}d\mu,$$

Letting $\rho \downarrow 0$ in (3.2.6), applying Monotone Convergence theorem, we have

$$\begin{aligned} &\frac{t}{2}\int_M \varphi^s u^{-t-1}|\nabla u|_A^2 d\mu + \int_M \varphi^s V u^{\sigma-t} d\mu \\ \leq\ & \lim_{\rho\downarrow 0} \frac{1}{2}\int_M \varphi^s V (u+\rho)^{\sigma-t} d\mu + Ct^{-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu, \end{aligned}$$

that is

$$\begin{aligned} &\frac{t}{2}\int_M \varphi^s u^{-t-1}|\nabla u|_A^2 d\mu + \frac{1}{2}\int_M \varphi^s V u^{\sigma-t} d\mu \\ \leq\ & Ct^{-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu. \end{aligned} \tag{3.2.7}$$

We apply (3.1.9) once more, using another test function $\psi = \varphi^s$, which yields

$$\begin{aligned} &\int_M \varphi^s V u^{\sigma} d\mu \\ \leq\ & s\int_M \varphi^{s-1} (\nabla u, \nabla\varphi)_A d\mu \\ \leq\ & s\left(\int_M \varphi^s u^{-t-1}|\nabla u|_A^2 d\mu\right)^{\frac{1}{2}} \left(\int_M \varphi^{s-2} u^{t+1}|\nabla\varphi|_A^2 d\mu\right)^{\frac{1}{2}}. \end{aligned} \tag{3.2.8}$$

On the other hand, we obtain from (3.2.7) that

$$\int_M \varphi^s u^{-t-1}|\nabla u|_A^2 d\mu \leq Ct^{-1-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu.$$

Substituting into (3.2.8) yields

$$\begin{aligned} \int_M \varphi^s V u^{\sigma} d\mu \ \leq\ & C\left[t^{-1-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu\right]^{\frac{1}{2}} \\ & \times\left[\int_M \varphi^{s-2} u^{t+1}|\nabla\varphi|_A^2 d\mu\right]^{\frac{1}{2}}. \end{aligned} \tag{3.2.9}$$

Recalling that $\nabla\varphi = 0$ on B_R, and applying the Hölder inequality to the last term of (3.2.9) with the Hölder couple

$$p_3 = \frac{\sigma}{t+1}, \quad p_3' = \frac{\sigma}{\sigma-t-1},$$

we obtain

$$\int_M \varphi^{s-2} u^{t+1}|\nabla\varphi|_A^2 d\mu$$

$$
\begin{aligned}
&= \int_{M\setminus B_R} \left(\varphi^{\frac{s}{p_3}} V^{\frac{1}{p_3}} u^{t+1}\right)\left(\varphi^{\frac{s}{p_3'}-2} V^{-\frac{1}{p_3}} |\nabla\varphi|_A^2\right) d\mu \\
&\le \left(\int_{M\setminus B_R} \varphi^s V u^\sigma d\mu\right)^{\frac{t+1}{\sigma}} \left(\int_{M\setminus B_R} \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} |\nabla\varphi|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu\right)^{\frac{\sigma-t-1}{\sigma}} \qquad (3.2.10)
\end{aligned}
$$

Substituting (3.2.10) into (3.2.9), we obtain

$$
\begin{aligned}
&\int_M \varphi^s V u^\sigma d\mu \\
&\le Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} |\nabla\varphi|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu\right)^{\frac{\sigma-t-1}{2\sigma}} \left(\int_{M\setminus B_R} \varphi^s V u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}. \qquad (3.2.11)
\end{aligned}
$$

Since $\int_M \varphi^s V u^\sigma d\mu$ is finite due to Remark 3.1.3, it follows from (3.2.11) that

$$
\begin{aligned}
&\left(\int_M \varphi^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \\
&\le Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} |\nabla\varphi|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu\right)^{\frac{\sigma-t-1}{2\sigma}}. \qquad (3.2.12)
\end{aligned}
$$

Note that the first integral in the right-hand side of (3.2.12) has the following estimate

$$
\begin{aligned}
&\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} |\nabla\varphi|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu \\
&\le \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} |\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}} \|A\|^{\frac{\sigma-t}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} d\mu \\
&= \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} |\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}} d\nu, \qquad (3.2.13)
\end{aligned}
$$

where we have used that $d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu$. Similarly, the second integral in the right-hand side of (3.2.12) can be estimated as follows

$$
\begin{aligned}
&\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} |\nabla\varphi|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu \\
&\le \int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} |\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu. \qquad (3.2.14)
\end{aligned}
$$

Substituting that (3.2.13) and (3.2.14) into (3.2.11), we have

$$\begin{aligned}\int_M \varphi^s V u^\sigma d\mu &\le Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}}\left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}}|\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}}\left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}}d\nu\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}}|\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}}\left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}}d\nu\right)^{\frac{\sigma-t-1}{2\sigma}}\\ &\quad\times\left(\int_{M\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}. \qquad (3.2.15)\end{aligned}$$

Substituting that (3.2.13) and (3.2.14) into (3.2.12), we obtain

$$\begin{aligned}&\left(\int_M \varphi^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}}\\ \le\ & Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}}\left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}}|\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}}\left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}}d\nu\right)^{\frac{1}{2}}\\ &\times\left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}}|\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}}\left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}}d\nu\right)^{\frac{\sigma-t-1}{2\sigma}}. \qquad (3.2.16)\end{aligned}$$

Fix $R > 1$ large enough, and set $t = \frac{1}{\ln R}$ to satisfy (3.2.2). Consider the integral

$$I_n(a,b) := \int_M \varphi_n^{s-a}|\nabla\varphi_n|^a\left(\frac{V}{\|A\|}\right)^b d\nu. \qquad (3.2.17)$$

Here we take the same φ_n as in (2.2.14).

(a,b) are taking values from the couples $\left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1}\right)$ and $\left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}\right)$. By (2.2.16), we have

$$\begin{aligned}I_n(a,b) &\le C\int_{M\setminus B_R}\varphi^{s-a}\eta_n^s|\nabla\varphi|^a\left(\frac{V}{\|A\|}\right)^b d\nu\\ &\quad +C\int_{B_{2nR}\setminus B_{nR}}\varphi^s\eta_n^{s-a}|\nabla\eta_n|^a\left(\frac{V}{\|A\|}\right)^b d\nu\\ &\le C\int_{M\setminus B_R}\varphi^{s-a}|\nabla\varphi|^a\left(\frac{V}{\|A\|}\right)^b d\nu\\ &\quad +C\int_{B_{2nR}\setminus B_{nR}}\varphi^s|\nabla\eta_n|^a\left(\frac{V}{\|A\|}\right)^b d\nu. \qquad (3.2.18)\end{aligned}$$

Here $\nabla\varphi = 0$ in B_R, and $\nabla\eta_n = 0$ outside $B_{2nR}\setminus B_{nR}$ are used, and s will be chosen bigger enough than a.

Since $|\nabla\eta_n| \le \frac{1}{nR}$ by (2.2.13) , and using (VA), when $b > 0$, the last integral in (3.2.18) can be estimated as follows

$$\begin{aligned}
&\int_{B_{2nR}\setminus B_{nR}} \varphi^s|\nabla\eta_n|^a \left(\frac{V}{\|A\|}\right)^b d\nu \\
\le\ & \frac{C}{(nR)^a}\int_{B_{2nR}\setminus B_{nR}} \varphi^s r^{\delta_2 b} d\nu \\
\le\ & \frac{C}{(nR)^a}\left(\sup_{B_{2nR}\setminus B_{nR}} \varphi^s\right)(2nR)^{\delta_2 b}\nu(B_{2nR}) \\
\le\ & \frac{C}{(nR)^a}\left(\frac{nR}{R}\right)^{-st}(2nR)^{\delta_2 b}(2nR)^p \ln^q(2nR) \\
=\ & C' n^{\delta_2 b+p-a-st} R^{\delta_2 b+p-a}\ln^q(2nR), \qquad (3.2.19)
\end{aligned}$$

where we have used the definition (2.2.12) of φ and the volume estimate (3.1.5).

When $b < 0$, the second integral on the right-hand side of (3.2.18) can be estimated as follows

$$\int_{B_{2nR}\setminus B_{nR}} \varphi^s|\nabla\eta_n|^a \left(\frac{V}{\|A\|}\right)^b d\nu \le C n^{-\delta_1 b+p-a-st} R^{-\delta_1 b+p-a}\ln^q(2nR). \qquad (3.2.20)$$

Before we give the estimate of the first integral in (3.2.18), using the following estimate from [30]: if f is a nonnegative decreasing function on $\mathbb{R}_+$, then, for large enough R,

$$\int_{M\setminus B_R} f(r(x))\, d\nu(x) \le C\int_{R/2}^{\infty} f(r) r^{p-1}\ln^q r dr, \qquad (3.2.21)$$

Thus, using $|\nabla\varphi| \le R^t t r^{-t-1}$, (3.2.21), and $R/2 > 1$, when $b > 0$, we obtain

$$\begin{aligned}
\int_{M\setminus B_R} \varphi^{s-a}|\nabla\varphi|^a \left(\frac{V}{\|A\|}\right)^b d\nu &\le C\int_{R/2}^{\infty}\left(\frac{r}{R}\right)^{-t(s-a)} R^{at}t^a r^{-at-a} r^{\delta_2 b} r^{p-1}\ln^q r dr \\
&\le CR^{st}t^a\int_1^{\infty} r^{-st-a+\delta_2 b+p}\ln^q r\frac{dr}{r} \\
&= CR^{st}t^a\int_0^{\infty} e^{-h_1\xi}\xi^q d\xi, \qquad (3.2.22)
\end{aligned}$$

where we have used the change $\xi = \ln r$ and set

$$h_1 := st + a - \delta_2 b - p. \qquad (3.2.23)$$

Assuming that $h_1 > 0$ and making one more change $\tau = h_1\xi$, we obtain

$$\int_{M\backslash B_R} \varphi^{s-a}|\nabla\varphi|^a \left(\frac{V}{||A||}\right)^b d\nu \leq CR^{st}t^a h_1^{-q-1}\int_0^\infty e^{-\tau}\tau^q d\tau \leq C'R^{st}t^a h_1^{-q-1}, \tag{3.2.24}$$

where the value $\Gamma(q+1)$ of the integral is absorbed into the constant C'.

When $b > 0$, substituting (3.2.19) and (3.2.24) into (3.2.18) yields

$$I_n(a,b) \leq CR^{st}t^a h_1^{-q-1} + Cn^{-h_1}R^{\delta_2 b+p-a}\ln^q(2nR). \tag{3.2.25}$$

Similarly, when $b < 0$, we have

$$I_n(a,b) \leq CR^{st}t^a h_2^{-q-1} + Cn^{-h_2}R^{-\delta_1 b+p-a}\ln^q(2nR). \tag{3.2.26}$$

where

$$h_2 = st + a + \delta_1 b - p. \tag{3.2.27}$$

We will use (3.2.25) for which $h_1 > t$. Noticing also that $R^t = \exp(t\ln R) = e$, we obtain

$$I_n(a,b) \leq Ce^s t^{a-q-1} + Cn^{-t}R^{\delta_2 b+p-a}\ln^q(2nR).$$

As we have remarked above, we will consider only the values of a in the bounded range $a \leq 3p$. Hence, the term e^a in the above is uniformly bounded. Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} I_n(a,b) \leq Ct^{a-q-1}. \tag{3.2.28}$$

Similarly, when $b < 0$, we have

$$\limsup_{n\to\infty} I_n(a,b) \leq Ct^{a-q-1}. \tag{3.2.29}$$

Let us first use (3.2.28) with $(a,b) = \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1}\right)$. Note that $a < p$, and $b > 0$, for this value of a and for t as in (3.2.2), we should testify that $h_1 > t$, that is

$$\begin{aligned} h_1 &= st + a - \delta_2 b - p \\ &= st + \frac{2(\sigma-t)}{\sigma-1} - \delta_2\frac{t}{\sigma-1} - \frac{2\sigma}{\sigma-1} \\ &= \left(s - \frac{2+\delta_2}{\sigma-1}\right)t > t, \end{aligned}$$

Since

$$a - q - 1 = \frac{2(\sigma-t)}{\sigma-1} - \frac{1}{\sigma-1} - 1 = \frac{\sigma-2t}{\sigma-1}.$$

Hence, we use (3.2.28) to obtain that

$$\limsup_{n\to\infty} I_n\left(\frac{2(\sigma-t)}{\sigma-1},\frac{t}{\sigma-1}\right)\le Ct^{\frac{\sigma-2t}{\sigma-1}}. \tag{3.2.30}$$

While, for $(a,b)=\left(\frac{2\sigma}{\sigma-t-1},-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}\right)$, note that $b<0$, and from (3.2.2) to get that $a<3p$, we should testify that $h_2>t$, that is

$$\begin{aligned} h_2 &= st+a+\delta_1 b-p\\ &= st+\frac{2\sigma}{\sigma-t-1}-\delta_1\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}-\frac{2\sigma}{\sigma-1}\\ &= \left(s-\frac{(\delta_1-2)\sigma}{(\sigma-t-1)(\sigma-1)}\right)t>t. \end{aligned}$$

Since

$$a-q-1=\frac{2\sigma}{\sigma-t-1}-\frac{1}{\sigma-1}-1=\frac{\sigma^2-\sigma+\sigma t}{(\sigma-t-1)(\sigma-1)}.$$

whence (3.2.29) yields

$$\limsup_{n\to\infty} I_n\left(\frac{2\sigma}{\sigma-t-1},-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}\right)\le Ct^{\frac{\sigma^2-\sigma+\sigma t}{(\sigma-t-1)(\sigma-1)}}. \tag{3.2.31}$$

The inequality (3.2.16) with function φ_n implies that

$$\begin{aligned} &\limsup_{n\to\infty}\left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}}\\ \le\ &\limsup_{n\to\infty} Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} I_n\left(\frac{2(\sigma-t)}{\sigma-1},\frac{t}{\sigma-1}\right)^{\frac{1}{2}}\\ &\times I_n\left(\frac{2\sigma}{\sigma-t-1},-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}\right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \tag{3.2.32}$$

Combining with (3.2.30) and (3.2.31), noting that $\varphi_n\uparrow\varphi$, we have

$$\left(\int_M \varphi^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}}\le Ct^{-\frac{t}{2(\sigma-1)}}. \tag{3.2.33}$$

The remaining term $t^{-\frac{t}{2(\sigma-1)}}$ on the right-hand side of (3.2.33) tends to 1 as $t\to 0$, which implies that the right-hand side of (3.2.33) is a bounded function of t. Hence, there is a constant C_1 such that

$$\left(\int_M \varphi^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \le C_1 < \infty, \quad \text{for all small enough } t. \tag{3.2.34}$$

It follows that also

$$\int_M \varphi^s V u^\sigma d\mu \le C < \infty, \tag{3.2.35}$$

Since $\varphi = 1$ on B_R, it follows that

$$\int_{B_R} V u^\sigma d\mu \le C < \infty,$$

which implies for $R \to \infty$ that

$$\int_M V u^\sigma d\mu \le C < \infty. \tag{3.2.36}$$

Applying the same argument, inequality (3.2.15) with function φ_n implies that

$$\int_M \varphi_n^s V u^\sigma d\mu \le C_1 \left(\int_{M\setminus B_R} \varphi_n^s V u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}. \tag{3.2.37}$$

Letting $n \to \infty$ and applying that $\varphi_n \uparrow \varphi$, we obtain

$$\int_M \varphi^s V u^\sigma d\mu \le C_1 \left(\int_{M\setminus B_R} \varphi^s V u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}},$$

whence

$$\int_{B_R} V u^\sigma d\mu \le C_1 \left(\int_{M\setminus B_R} V u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}, \tag{3.2.38}$$

Since by (3.2.36), letting $R \to \infty$, we have

$$\int_{M\setminus B_R} V u^\sigma d\mu \to 0,$$

letting in (3.2.38) $R \to \infty$, we obtain

$$\int_M V u^\sigma d\mu = 0.$$

Since $V > 0$ a.e. on M, thus $u \equiv 0$.

3.3 Second Proof of Theorem 3.1.1

Here we present a modification of the above proof of Theorem 3.1.1. We use the first proof up to (3.2.16). Then letting $s > \frac{4\sigma}{\sigma-1}$, and $t < \frac{\sigma-1}{2}$, and noting that $0 \le \varphi \le 1$, from (3.2.15), we obtain

$$\begin{aligned} \int_M \varphi^s V u^\sigma d\mu \ \le\ & C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M |\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ & \times \left(\int_M |\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}} \\ & \times \left(\int_{M\setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}, \end{aligned} \tag{3.3.1}$$

and from (3.2.16), we obtain

$$\begin{aligned} & \left(\int_M \varphi^s V u^\sigma d\mu \right)^{1-\frac{t+1}{2\sigma}} \\ \le\ & C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M |\nabla\varphi|^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ & \times \left(\int_M |\nabla\varphi|^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \tag{3.3.2}$$

We see, that all integral terms in the right-hand side of (3.3.1) and (3.3.2) have the form

$$\int_M |\nabla\varphi|^a \left(\frac{V}{\|A\|}\right)^b d\nu,$$

with the following two pairs of (a, b)

$$(a,b) = \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right), \quad (a,b) = \left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right). \tag{3.3.3}$$

Consequently, we could write a in the following way

$$a = p + lt, \tag{3.3.4}$$

with the corresponding two values of l

$$l = -\frac{2}{\sigma-1} \quad \text{and} \quad l = \frac{2\sigma}{(\sigma-t-1)(\sigma-1)}. \tag{3.3.5}$$

where $p = \frac{2\sigma}{\sigma-1}$ is defined as before in (3.1.2). It is very clear to obtain that the values of a, b, l are uniformly bounded, when t is near zero. Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ be a sequence satisfying that each $\tilde{\varphi}_k$ is a Lipschitz function such that $supp(\tilde{\varphi}_k) \subset B_{2^k}$, $\tilde{\varphi}_k = 1$ in a neighborhood of $B_{2^{k-1}}$, and

$$|\nabla\tilde{\varphi}_k| \begin{cases} \le \frac{C}{2^{k-1}} & \text{for } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ = 0, & \text{otherwise.} \end{cases} \tag{3.3.6}$$

where C does not depend on k.

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \tag{3.3.7}$$

and

$$\varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}, \tag{3.3.8}$$

Note that $\varphi_n = 1$ on B_{2^n}, $\varphi_n = 0$ outside $B_{2^{2n}}$, $0 \le \varphi_n \le 1$ on M. Note that for any $a \ge 1$, using that $supp(\nabla\tilde{\varphi}_k)$ are disjoint, we have

$$|\nabla\varphi_n|^a = \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a}. \tag{3.3.9}$$

It is easy to see that

$$\varphi_n \in W^{1,2}_{loc}(M, \omega).$$

Consider the integral

$$J_n(a,b) = \int_M |\nabla\varphi_n|^a \left(\frac{V}{\|A\|}\right)^b d\nu, \tag{3.3.10}$$

Assume that $b > 0$. Substituting (3.3.8) into (3.3.10), applying (3.3.9), (3.3.6), and (VA), we obtain

$$\begin{aligned} J_n(a,b) &= \int_M |\nabla\varphi_n|^a \left(\frac{V}{\|A\|}\right)^b d\nu \\ &= \int_M \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a} \left(\frac{V}{\|A\|}\right)^b d\nu \\ &\le \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \frac{|\nabla\tilde{\varphi}_k|^a}{n^a} \left(\frac{V}{\|A\|}\right)^b d\nu \end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=n+1}^{2n}\int_{B_{2^k}\setminus B_{2^{k-1}}}\left(\frac{2^{1-k}}{n}\right)^a r^{\delta_2 b}d\nu\\
&\le C\sum_{k=n+1}^{2n}\left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b}\nu(B_{2^k})\\
&\le C\sum_{k=n+1}^{2n}\left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b}\nu(B_{2^k}). \qquad (3.3.11)
\end{aligned}$$

Noting that $a=p+lt$, $n+1\le k\le 2n$,

$$\begin{aligned}
\left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b} &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt} (2^k)^{\delta_2 b}\\
&\le \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b} \sup_{n+1\le k\le 2n, t=\frac{1}{n}} \left(\frac{2^{-k}}{n}\right)^{lt}\\
&\le C\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}. \qquad (3.3.12)
\end{aligned}$$

Using (3.3.12) and (3.1.5), recalling that by (3.1.2) $p=\frac{2\sigma}{\sigma-1}$, $q=\frac{1}{\sigma-1}$, when $b\ge 0$, we obtain

$$\begin{aligned}
J_n(a,b) &\le C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}\nu(B_{2^k})\\
&\le C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}(2^k)^p\ln^q(2^k)\\
&\le C\frac{1}{n^p}\sum_{k=n+1}^{2n} k^q 2^{k\delta_2 b}\\
&\le Cn^{q+1-p}2^{2n\delta_2 b}\\
&\le Cn^{-\frac{\sigma}{\sigma-1}}2^{2n\delta_2 b}. \qquad (3.3.13)
\end{aligned}$$

Similarly, when $b\le 0$, we have

$$J_n(a,b)\le Cn^{-\frac{\sigma}{\sigma-1}}2^{-2n\delta_1 b}. \qquad (3.3.14)$$

Setting $\varphi=\varphi_n$ in (3.3.2), we obtain

$$\begin{aligned}
\left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} &\le Ct^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}}\left(J_n(\frac{2(\sigma-t)}{\sigma-1},\frac{t}{\sigma-1})\right)^{\frac{1}{2}}\\
&\quad\times\left(J_n(\frac{2\sigma}{\sigma-t-1},-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)})\right)^{\frac{\sigma-t-1}{2\sigma}}.
\end{aligned}$$

(3.3.15)

Substituting (3.3.13) and (3.3.14) into (3.3.15), recalling $t = \frac{1}{n}$, we have

$$\begin{aligned}\left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{\frac{1}{n}+1}{2\sigma}} &\leq Cn^{\frac{1}{2}+\frac{\sigma}{2(\sigma-1)}}\left(n^{-\frac{\sigma}{\sigma-1}}2^{2n\delta_2\frac{\frac{1}{n}}{\sigma-1}}\right)^{\frac{1}{2}} \\ &\quad \times\left(n^{-\frac{\sigma}{\sigma-1}}2^{2n\delta_1\frac{\sigma\frac{1}{n}}{(\sigma-\frac{1}{n}-1)(\sigma-1)}}\right)^{\frac{\sigma-\frac{1}{n}-1}{2\sigma}} \\ &\leq Cn^{\frac{1}{2n(\sigma-1)}}2^{\frac{\delta_1+\delta_2}{\sigma-1}} \\ &\leq C'n^{\frac{1}{2n(\sigma-1)}}. \end{aligned} \tag{3.3.16}$$

Recalling that $\varphi_n = 1$ on B_{2^n}, and taking the limsup of both sides in (3.3.16) as $n \to \infty$, we obtain

$$\begin{aligned}\int_M V u^\sigma d\mu &\leq C \limsup_{n\to\infty} n^{\frac{1}{2n(\sigma-1)}/[1-\frac{\frac{1}{n}+1}{2\sigma}]} \\ &< \infty. \end{aligned} \tag{3.3.17}$$

Applying the same argument as in the first proof, we obtain that $u \equiv 0$.

3.4 Sharpness of p, q

In this section, we will construct examples to show the parameters of p and q in (3.1.5) are sharp and cannot be relaxed.

The sharpness of p is already known in $\mathbb{R}^n$, which is given by Mitidieri and Pokhozhaev in [48]: Let μ be the classical Lebesgue measure, if $\sigma > \frac{n-\gamma_2}{n-2+\gamma_1}$, $\sigma\gamma_1 + \gamma_2 + n(\sigma - 1) > 0$, and $2 - n < \gamma_1 < 2 - \gamma_2$, then the function

$$u(x) := \epsilon[1 + |x|^{2-\gamma_1-\gamma_2}]^{-\frac{1}{\sigma-1}}$$

is a solution to (3.1.1) with $A(x) = |x|^{\gamma_1}, V(x) = |x|^{-\gamma_2}$, where ϵ is a suitable small positive constant. In this case, we know (VA) holds, $V \in L^1_{loc}(\mathbb{R}^n)$, and

$$\begin{aligned}\nu(B(0,r)) &= \int_{B(0,r)} A^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu \\ &= \int_{B(0,r)} |x|^{\frac{\sigma\gamma_1}{\sigma-1}}|x|^{\frac{\gamma_2}{\sigma-1}} d\mu \\ &\approx r^p, \quad \text{as } r \to \infty, \end{aligned} \tag{3.4.1}$$

where $p = \frac{\sigma\gamma_1+\gamma_2}{\sigma-1} + n$. From the assumption $\sigma > \frac{n-\gamma_2}{n-2+\gamma_1}$, we know it is equivalent to

$$p > \frac{2\sigma}{\sigma - 1}, \tag{3.4.2}$$

hence, by carefully choosing parameters of γ_1 and γ_2, p could be close to $\frac{2\sigma}{\sigma-1}$.

Regarding to the sharpness of q, setting as before $p = \frac{2\sigma}{\sigma-1}$, and choosing $q = \frac{1}{\sigma-1} + \epsilon$ for small $\epsilon > 0$, we will construct a positive solution in $\mathbb{R}^n$ with (3.1.5) holding with these values p, q.

Let $M = \mathbb{R}^n$, μ is the classical Lebesgue measure, and x_0 is the origin point, take

$$A = a(r)Id, \quad V = r^{\beta_1} \ln^{\beta_2} r, \quad \text{for large enough } r > 0. \tag{3.4.3}$$

where $a(r) = r^{\alpha_1} \ln^{\alpha_2} r$ for large enough r, and for small r near zero, $a(r)$ and V are constants. Moreover, parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are chosen to satisfy the following condition

$$\begin{cases} \frac{\alpha_1\sigma-\beta_1}{\sigma-1} + n = \frac{2\sigma}{\sigma-1}, \\ \frac{\alpha_2\sigma-\beta_2}{\sigma-1} = \frac{1}{\sigma-1} + \epsilon, \\ \alpha_1 + n > 2. \end{cases} \tag{3.4.4}$$

It is easily to check that (VA) is satisfied with $\delta_1 = \delta_2 = |\beta_1 - \alpha_1| + 1$. Moreover, under the measure $d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu$, for large enough r, the volume $\nu(B(0,r))$ could be estimated from (3.4.3) as follows

$$\begin{aligned} \nu(B(0,r)) &= \int_{B(0,r)} \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu \\ &\le \int^r (s^{\alpha_1} \ln^{\alpha_2} s)^{\frac{\sigma}{\sigma-1}} \left(s^{\beta_1} \ln^{\beta_2} s\right)^{-\frac{1}{\sigma-1}} \omega_n s^{n-1} ds \\ &\le C \int^r s^{\frac{\alpha_1\sigma-\beta_1}{\sigma-1}+n-1} \ln^{\frac{\alpha_2\sigma-\beta_2}{\sigma-1}} s ds \\ &\le C r^{\frac{\alpha_1\sigma-\beta_1}{\sigma-1}+n} \ln^{\frac{\alpha_2\sigma-\beta_2}{\sigma-1}} r, \end{aligned} \tag{3.4.5}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. By (3.4.4), we obtain

$$\nu(B(0,r)) \le C r^p \ln^q r, \quad \text{for large enough } r > 0. \tag{3.4.6}$$

Hence, $\mathbb{R}^n$ satisfies the volume growth condition (3.1.5) with A, V from (3.4.3).

In fact, since A, V are radially defined, thus the solution u to (3.1.1) actually depends on polar radius r, we can write $u = u(r)$. Hence, (3.1.1) could be written in the following form

$$(Sau')' + SVu^\sigma \le 0, \tag{3.4.7}$$

where $S(r) = \omega_n r^{n-1}$.

Applying Proposition 2.3.1 in Chapter 2 with

$$\alpha(r) = S(r)a(r) = \omega_n r^{\alpha_1+n-1} \ln^{\alpha_2} r,$$

and

$$\beta(r) = S(r)V = \omega_n r^{\beta_1+n-1} \ln^{\beta_2} r,$$

we know from (3.4.4)

$$\int_{r_0}^{\infty} \frac{dr}{\alpha(r)} < \infty, \tag{3.4.8}$$

and for $r >> 1$

$$\gamma(r) = \int_r^{\infty} \frac{ds}{\alpha(s)} = \int_r^{\infty} \frac{ds}{\omega_n s^{\alpha_1+n-1} \ln^{\alpha_2} s} \approx \frac{1}{r^{\alpha_1+n-2} \ln^{\alpha_2} r}, \tag{3.4.9}$$

and by (3.4.4), we obtain

$$\begin{aligned}
\int_{r_0}^{\infty} \gamma(r)^{\sigma} \beta(r) dr &\leq C_1 \int_{r_0}^{\infty} \frac{1}{(r^{\alpha_1+n-2} \ln^{\alpha_2} r)^{\sigma}} r^{\beta_1+n-1} \ln^{\beta_2} r dr \\
&\leq C_2 \int_{r_0}^{\infty} \frac{1}{r^{\sigma(\alpha_1+n-2)-(\beta_1+n-1)} \ln^{\sigma\alpha_2-\beta_2} r} dr \\
&= C_2 \int_{r_0}^{\infty} \frac{1}{r^{\sigma\alpha_1+\sigma(n-2)-\beta_1-n} \ln^{\sigma\alpha_2-\beta_2} r} \frac{dr}{r} \\
&= C_2 \int_{r_0}^{\infty} \frac{1}{\ln^{1+(\sigma-1)\epsilon} r} \frac{dr}{r} < \infty.
\end{aligned} \tag{3.4.10}$$

Applying Proposition 2.3.1, we know there exists a solution on $[r_0, \infty)$

$$u(r) = O(\gamma(r)) = O(r^{2-n-\alpha_1} \ln^{-\alpha_2} r), \quad \text{as } r \to \infty, \tag{3.4.11}$$

one could apply similar argument as in [30] to extend u to be a positive solution of (3.1.1) in $\mathbb{R}^n$.

Chapter 4

Quasilinear and Mean Curvature Type Inequalities

This chapter is based on the work [68].

4.1 Background and Statement

In this chapter, we study the uniqueness of nonnegative solutions to the following two differential inequalities

$$\Delta_m u + u^\sigma \leq 0, \quad \text{on } M, \tag{4.1.1}$$

and

$$div\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right) + u^\sigma \leq 0. \tag{4.1.2}$$

where Δ_m is the m-Laplace operator, div and ∇ are the Riemannian divergence and gradient respectively, $m > 1$ and $\sigma > m-1$ are given parameters. We call $div\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right)$ the mean curvature type operator, in particular, when $m = 2$, this is the well-known mean curvature operator.

Mitidieri and Pokhozhaev in [45] obtained the result concerning the nonexistence of positive solution to (4.1.1) in $\mathbb{R}^n$. They proved that if

$$0 < m-1 < \sigma \leq \frac{n(m-1)}{n-m}, \quad \text{and} \quad n > m. \tag{4.1.3}$$

then (4.1.1) has no positive solution. For the mean curvature type inequalities, the problem (4.1.2) in case of $m = 2$ has also been investigated in [45]. They proved that if $n > 2$ and

$$1 < \sigma \leq \frac{n}{n-2}, \tag{4.1.4}$$

then (4.1.2) has no positive solution in $\mathbb{R}^n$.

Let us turn to results on Riemannian manfolds. In [34], Holopainen proved if u is a nonnegative solution to

$$\Delta_m u \leq 0, \tag{4.1.5}$$

and the following condition

$$\int^{\infty} \left(\frac{r}{\mu(B(x_0, r))} \right)^{\frac{1}{m-1}} dr = \infty, \tag{4.1.6}$$

holds for some reference point $x_0 \in M$, then any nonnegative solution to (4.1.5) is identical constant. In particular, this implies that any nonnegative solution to

$$\Delta_m u + u^\sigma \leq 0.$$

is identical zero for any σ. Note that (4.1.6) is satisfied if

$$\mu(B(x_0, r)) \leq Cr^m,$$

or even if

$$\mu(B(x_0, r)) \leq Cr^m \ln^{m-1} r,$$

for all large enough r.

In this part, we concentrate our attention on obtaining optimal volume growth conditions that ensure the uniqueness of nonnegative solutions to (4.1.1) and (4.1.2) for all $m > 1$ and $\sigma > m - 1$.

Before we state the results, let us first give the definitions of solutions to (4.1.1) and (4.1.2). Set

$$W^{1,m}_{loc}(M) = \{f : M \to \mathbb{R} | f \in L^m_{loc}(M), \nabla f \in L^m_{loc}(M)\}, \tag{4.1.7}$$

where ∇f is understood in distributional sense. Denote by $W^{1,m}_c(M)$ the subspace of $W^{1,m}_{loc}(M)$ of functions with compact support.

Definition 4.1.1. A function u on M is called a weak solution of (4.1.1) if $u \in W^{1,m}_{loc}(M)$, $u \geq 0$, and for any nonnegative function $\psi \in W^{1,m}_c(M)$, the following inequality holds:

$$-\int_M |\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu + \int_M u^\sigma \psi d\mu \le 0, \tag{4.1.8}$$

Similarly, we call a function u a weak solution of (4.1.2) if $u \in W^{1,m}_{loc}(M)$, $u \ge 0$ and for any nonnegative function $\psi \in W^{1,m}_c(M)$, the following inequality holds:

$$-\int_M \frac{|\nabla u|^{m-2}(\nabla u, \nabla\psi)}{\sqrt{1+|\nabla u|^m}}d\mu + \int_M u^\sigma \psi d\mu \le 0, \tag{4.1.9}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

Remark 4.1.2. Using the defintion of ψ, we have

$$\begin{aligned}\left|\int_M |\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu\right| &\le \int_{supp(\psi)} |\nabla u|^{m-1}|\nabla\psi|d\mu \\ &\le \left(\int_{supp(\psi)} |\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \left(\int_{supp(\psi)} |\nabla\psi|^m d\mu\right)^{\frac{1}{m}},\end{aligned}$$

Since $u \in W^{1,m}_{loc}(M)$ and $\psi \in W^{1,m}_c(M)$. Therefore, the first term in (4.1.8) is finite, which implies the finiteness of the second term, that is $\int_M u^\sigma\psi d\mu < \infty$.

Similarly, we also can obtain that $\int_M u^\sigma\psi d\mu < \infty$ from (4.1.9).

Assuming always $\sigma > m-1$, let us introduce two paramters

$$p = \frac{m\sigma}{\sigma - m + 1}, \quad q = \frac{m-1}{\sigma - m + 1}. \tag{4.1.10}$$

Here are our main results.

Theorem 4.1.3. If for some $x_0 \in M$, the following inequality

$$\mu(B(x_0, r)) \le Cr^p \ln^q r, \tag{4.1.11}$$

holds for all large enough r, then the only nonnegative solution of (4.1.1) is identical zero.

Theorem 4.1.4. If for some $x_0 \in M$, the following inequality

$$\mu(B(x_0, r)) \le Cr^p \ln^q r, \tag{4.1.12}$$

holds for all large enough r, then the only nonnegative solution of (4.1.2) is identical zero.

Both values of p, q in Theorem 4.1.3 are sharp, that is, the statement of Theorem 4.1.3 is not true for large values of p and q.

4.2 Proof of Theorem 4.1.3

Let u be some nonnegative weak solution to (4.1.1). x_0 is the reference point as before in Theorem 4.1.3. Denote $B_R := B(x_0, R)$, and fix a Lipschitz function φ on M with compact support, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of $\bar{B}_R$. In particular, we have $\varphi \in W_c^{1,m}(M)$. Take the following test function for (4.1.8):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^{-t}, \tag{4.2.1}$$

where ρ is some positive constant near zero, t, s satisfy two conditions

$$0 < t < \min\left\{1, \frac{\sigma - m + 1}{2}\right\}, \quad s > \frac{4m\sigma}{\sigma - m + 1}. \tag{4.2.2}$$

The value of s is fixed, while t is variable and can be chosen arbitrarily close to 0.

Since $\frac{1}{u+\rho}$ is locally bounded, hence, ψ_ρ has compact support and is bounded. Since

$$\nabla\psi_\rho = -t(u+\rho)^{-t-1}\varphi^s \nabla u + s(u+\rho)^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that, $\nabla\psi_\rho \in L^m(M)$. It follows that, $\psi_\rho \in W_c^{1,m}(M)$. We obtain from (4.1.8) that

$$\begin{aligned} & t\int_M \varphi^s (u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \le\ & s\int_M \varphi^{s-1}(u+\rho)^{-t}|\nabla u|^{m-2}(\nabla u, \nabla\varphi) d\mu. \end{aligned}$$

thus

$$\begin{aligned} & t\int_M \varphi^s (u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \le\ & s\int_M \varphi^{s-1}(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi| d\mu. \end{aligned} \tag{4.2.3}$$

In what follows, we use the following Young's inequality

$$\int_M fg d\mu \le \varepsilon\int_M |f|^{p_0} d\mu + C_\varepsilon \int_M |g|^{p_0'} d\mu, \tag{4.2.4}$$

where $\varepsilon > 0$ is arbitrary, and (p_0, p_0') is a Hölder conjugate couple such that

$$p_0 = \frac{m}{m-1}, \quad p_0' = m. \tag{4.2.5}$$

Applying (4.2.4) to the right-hand-side integral of (4.2.3), we obtain

$$\begin{aligned}
& s\int_M \varphi^{s-1}(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi|d\mu \\
= & \int_M \left(t^{\frac{m-1}{m}}\varphi^{\frac{s(m-1)}{m}}(u+\rho)^{-\frac{(t+1)(m-1)}{m}}|\nabla u|^{m-1}\right) \\
& \times\left(\frac{s}{t^{\frac{m-1}{m}}}\varphi^{s-1-\frac{s(m-1)}{m}}(u+\rho)^{-t+\frac{(t+1)(m-1)}{m}}|\nabla\varphi|\right)d\mu \\
\leq & \varepsilon t\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|^m d\mu + C_\varepsilon\frac{s^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu.
\end{aligned}$$

Letting $\varepsilon = \frac{1}{2}$, substituting the above estimate into (4.2.3), and cancelling out the half of the first term in (4.2.3), we obtain

$$\begin{aligned}
& \frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s u^\sigma(u+\rho)^{-t}d\mu \\
\leq & \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu.
\end{aligned} \tag{4.2.6}$$

Using (4.2.4) once more to the right-hand side of (4.2.6) with another Hölder conjugate couple (p_1, p_1') satisfying

$$p_1 = \frac{\sigma-t}{m-t-1}, \quad p_1' = \frac{\sigma-t}{\sigma-m+1}, \tag{4.2.7}$$

we obtain

$$\begin{aligned}
& \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu \\
= & \int_M [\varphi^{\frac{s}{p_1}}(u+\rho)^{m-t-1}]\cdot[\tfrac{Cs^m}{t^{m-1}}\varphi^{\frac{s}{p_1'}-m}|\nabla\varphi|^m]d\mu \\
\leq & \frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu \\
& +C_1\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu.
\end{aligned} \tag{4.2.8}$$

Using in the right-hand side of (4.2.8) the obvious inequality

$$\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}} \leq \left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma}{\sigma-m+1}},$$

and combining (4.2.8) with (4.2.6), we obtain

$$\begin{aligned}
&\frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\
\leq\ &\frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t} d\mu \\
&+C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu, \qquad (4.2.9)
\end{aligned}$$

where the term that contains s is absorbed into constant C_1.

Since if

$$\begin{aligned}
\int_M \varphi^s(u+\rho)^{\sigma-t} d\mu &= \int_M \varphi^s(u+\rho)^{\sigma}(u+\rho)^{-t} d\mu \\
&\leq C\left(\int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu + \rho^\sigma \int_M \varphi^s (u+\rho)^{-t} d\mu\right),
\end{aligned}$$

By the definition of solution, we know $\int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu$ is bounded. Hence, the term

$$\int_M \varphi^s(u+\rho)^{\sigma-t} d\mu$$

is bounded. By Dominated Convergence theorem, letting $\rho \to 0$, we have

$$\lim_{\rho\to 0}\int_M \varphi^s(u+\rho)^{\sigma-t} d\mu = \int_M \varphi^s u^{\sigma-t} d\mu,$$

Letting $\rho \to 0$, applying Monotone Convergence theorem to the right-hand side integrals in (4.2.9), we obtain

$$\begin{aligned}
&\lim_{\rho\to 0}\frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}|\nabla u|^m d\mu + \lim_{\rho\to 0}\int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\
\leq\ &\lim_{\rho\to 0}\frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t} d\mu \\
&+C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu,
\end{aligned}$$

which is

$$\begin{aligned}
&\frac{t}{2}\int_M \varphi^s u^{-t-1}|\nabla u|^m d\mu + \frac{1}{2}\int_M \varphi^s u^{\sigma-t} d\mu \\
\leq\ &C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu, \qquad (4.2.10)
\end{aligned}$$

Applying (4.1.8) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned}
&\int_M \varphi^s u^\sigma d\mu \\
\leq\ & s\int_M \varphi^{s-1}|\nabla u|^{m-2}(\nabla u, \nabla\varphi)d\mu \\
\leq\ & s\int_M \varphi^{s-1}|\nabla u|^{m-1}|\nabla\varphi|d\mu \\
\leq\ & s\left(\int_M \varphi^s u^{-t-1}|\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \\
& \times\left(\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right)^{\frac{1}{m}}. \qquad (4.2.11)
\end{aligned}$$

On the other hand, we obtain from (4.2.10) that

$$\int_M \varphi^s u^{-t-1}|\nabla u|^m d\mu \leq Ct^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu.$$

Substituting this into (4.2.11) yields

$$\begin{aligned}
\int_M \varphi^s u^\sigma d\mu \ \leq\ & C\left[t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right]^{\frac{m-1}{m}} \\
& \times\left[\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right]^{\frac{1}{m}}. \qquad (4.2.12)
\end{aligned}$$

Recalling that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (4.2.12) with the following Hölder couple (p_2, p_2')

$$p_2 = \frac{\sigma}{(t+1)(m-1)}, \quad p_2' = \frac{\sigma}{\sigma-(t+1)(m-1)}, \qquad (4.2.13)$$

we obtain

$$\begin{aligned}
&\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu \\
=\ & \int_{M\setminus B_R}\left(\varphi^{\frac{s}{p_2}}u^{(t+1)(m-1)}\right)\left(\varphi^{\frac{s}{p_2'}-m}|\nabla\varphi|^m\right)d\mu \\
\leq\ & \left(\int_{M\setminus B_R}\varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{\sigma}} \\
& \times\left(\int_{M\setminus B_R}\varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{1-\frac{(t+1)(m-1)}{\sigma}}. \qquad (4.2.14)
\end{aligned}$$

Substituting (4.2.14) into (4.2.12), we obtain

$$\begin{aligned}
&\int_M \varphi^s u^\sigma d\mu \\
\leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu\right)^{\frac{m-1}{m}} \\
&\times \left(\int_M \varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}} \\
&\times \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \qquad (4.2.15)
\end{aligned}$$

Noting $s > \frac{4m\sigma}{\sigma-m+1}$, and $t < \frac{\sigma-m+1}{2}$ in (4.2.2), and recalling that $0 \leq \varphi \leq 1$, from (4.2.15), we obtain

$$\begin{aligned}
&\int_M \varphi^s u^\sigma d\mu \\
\leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu\right)^{\frac{m-1}{m}} \\
&\times \left(\int_M |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}} \\
&\times \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \qquad (4.2.16)
\end{aligned}$$

Since $\int_M \varphi^s u^\sigma d\mu$ is finite due to the definition of the solution, it follows from (4.2.16) that

$$\begin{aligned}
&\left(\int_M \varphi^s u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}} \\
\leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu\right)^{\frac{m-1}{m}} \\
&\times \left(\int_M |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}. \qquad (4.2.17)
\end{aligned}$$

We see, that all integral terms in the right-hand side of (4.2.17) have the form

$$\int_M |\nabla\varphi|^a d\mu,$$

with the following two values of a such that

$$a = \frac{m(\sigma - t)}{\sigma - m + 1}, \quad a = \frac{m\sigma}{\sigma - (t+1)(m-1)}. \tag{4.2.18}$$

Consequently, a could be written in the form

$$a = p + bt, \tag{4.2.19}$$

with the following two respective values of b

$$b = -\frac{m}{\sigma - m + 1}, \quad b = \frac{m\sigma(m-1)}{[\sigma - (t+1)(m-1)](\sigma - m + 1)}. \tag{4.2.20}$$

where $p = \frac{m\sigma}{\sigma - m + 1}$ is defined as before in (4.1.10). Clearly, the both values of a and b are uniformly bounded, when t is near zero.

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \tag{4.2.21}$$

Consider the integral

$$J_n(a) = \int_M |\nabla \varphi_n|^a d\mu, \tag{4.2.22}$$

where a is as above, and φ_n is the same as defined in (3.3.8).

Substituting (3.3.8) into (4.2.22), applying (3.3.9) and (3.3.6), we obtain

$$\begin{aligned} J_n(a) &= \int_M |\nabla \varphi_n|^a d\nu \\ &= \int_M \frac{\sum_{k=n+1}^{2n} |\nabla \tilde{\varphi}_k|^a}{n^a} d\mu \\ &= \sum_{k=n+1}^{2n} \int_{B_{2^k} \setminus B_{2^{k-1}}} \frac{|\nabla \tilde{\varphi}_k|^a}{n^a} d\mu \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a \mu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^a \mu(B_{2^k}), \end{aligned} \tag{4.2.23}$$

where we have used that a is uniformly bounded. Noting that $a = p + bt$, $n + 1 \leq$

$k \leq 2n$, and substituting $t = \frac{1}{n}$, if $b > 0$, we obtain

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^{a} &= \left(\frac{2^{-k}}{n}\right)^{p}\left(\frac{2^{-k}}{n}\right)^{bt} \\ &\leq \left(\frac{2^{-k}}{n}\right)^{p}. \end{aligned}$$

If $b < 0$, since $|b|$ is uniformly bounded, then

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^{a} &= \left(\frac{2^{-k}}{n}\right)^{p}\left(\frac{2^{-k}}{n}\right)^{bt} \\ &\leq \left(\frac{2^{-k}}{n}\right)^{p}\left(n2^{2n}\right)^{\frac{|b|}{n}} \\ &\leq \left(\frac{2^{-k}}{n}\right)^{p}\sup_{n}\left(n2^{2n}\right)^{\frac{|b|}{n}} \\ &< C\left(\frac{2^{-k}}{n}\right)^{p}. \end{aligned}$$

Thus, in both cases, we obtain

$$\left(\frac{2^{-k}}{n}\right)^{a} \leq C\left(\frac{2^{-k}}{n}\right)^{p}. \tag{4.2.24}$$

Using (4.2.24) and (4.1.11), recalling that by (4.1.10) $p = \frac{m\sigma}{\sigma-m+1}$, $q = \frac{m-1}{\sigma-m+1}$, we obtain

$$\begin{aligned} J_n(a) &\leq C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^{p}\mu(B_{2^k}) \\ &\leq C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^{p}(2^k)^p\ln^q(2^k) \\ &\leq C\frac{1}{n^p}\sum_{k=n+1}^{2n}k^q \\ &\leq Cn^{q+1-p} \\ &\leq Cn^{-\frac{(m-1)\sigma}{\sigma-m+1}}. \end{aligned} \tag{4.2.25}$$

Setting $\varphi = \varphi_n$ in (4.2.17), we obtain

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}}$$

$$\leq \ ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(J_n(\frac{m(\sigma-t)}{\sigma-m+1})\right)^{\frac{m-1}{m}}$$
$$\times\left(J_n(\frac{m\sigma}{\sigma-(t+1)(m-1)})\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}. \qquad (4.2.26)$$

Substituting (4.2.25) into (4.2.26), using $t=\frac{1}{n}$ as before, we obtain

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}$$
$$\leq \ Cn^{\frac{m-1}{m}+\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}\right)^{\frac{m-1}{m}}$$
$$\times\left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}\right)^{\frac{1}{m}-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \qquad (4.2.27)$$

the exponent in the power of n in the right-hand side of (4.2.27) is then equal to

$$\frac{m-1}{m}+\frac{\sigma(m-1)^2}{m(\sigma-m+1)}-\frac{(m-1)\sigma}{\sigma-m+1}\cdot\frac{m-1}{m}-\frac{(m-1)\sigma}{\sigma-m+1}\cdot[\frac{1}{m}-\frac{m-1}{m\sigma}-\frac{m-1}{nm\sigma}]$$

a (careful) computation shows that all the terms here that do not contain n miraculously cancel out, so that the above expression reduces to

$$\frac{(m-1)^2}{nm(\sigma-m+1)}$$

Therefore, we obtain

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}} \leq Cn^{\frac{(m-1)^2}{nm(\sigma-m+1)}}. \qquad (4.2.28)$$

Recalling that $\varphi_n=1$ on B_{2^n}, and taking the lim sup of both sides in (4.2.28) as $n\to\infty$, we obtain

$$\int_M u^\sigma d\mu \ \leq \ C\limsup_{n\to\infty} n^{\frac{(m-1)^2}{nm(\sigma-m+1)}/[1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}]}$$
$$< \ \infty. \qquad (4.2.29)$$

The same computation can be used in (4.2.16), which implies

$$\int_M \varphi_n^s u^\sigma d\mu \leq C\left(\int_{M\setminus B_{2^n}} \varphi_n^s u^\sigma d\mu\right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \qquad (4.2.30)$$

Using that $0 \leq \varphi_n \leq 1$ and $\varphi_n|_{B_{2^n}} = 1$ once more, we obtain

$$\int_{B_{2^n}} u^\sigma d\mu \leq C \left(\int_{M \setminus B_{2^n}} u^\sigma d\mu \right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \tag{4.2.31}$$

Letting $n \to \infty$, and using (4.2.29), we obtain

$$\int_M u^\sigma d\mu = 0,$$

which implies that $u = 0$.

4.3 Proof of Theorem 4.1.4

Let u be some nonnegative weak solution to (4.1.2). Taking the same test function ψ_ρ as in (4.2.1), we obtain from (4.1.9) that

$$\begin{aligned} & t \int_M \varphi^s (u+\rho)^{-t-1} \frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}} d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq \ & s \int_M \varphi^{s-1} (u+\rho)^{-t} |\nabla u|^{m-2} \frac{(\nabla u, \nabla \varphi)}{\sqrt{1+|\nabla u|^m}} d\mu. \end{aligned}$$

thus

$$\begin{aligned} & t \int_M \varphi^s (u+\rho)^{-t-1} \frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}} d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq \ & s \int_M \varphi^{s-1} (u+\rho)^{-t} \frac{|\nabla u|^{m-1} |\nabla \varphi|}{\sqrt{1+|\nabla u|^m}} d\mu. \end{aligned} \tag{4.3.1}$$

Applying (4.2.4) with (4.2.5) to the right-hand-side integral of (4.3.1), we obtain

$$\begin{aligned} & s \int_M \varphi^{s-1} (u+\rho)^{-t} \frac{|\nabla u|^{m-1} |\nabla \varphi|}{\sqrt{1+|\nabla u|^m}} d\mu \\ = \ & \int_M \frac{1}{\sqrt{1+|\nabla u|^m}} \left(t^{\frac{m-1}{m}} \varphi^{\frac{s(m-1)}{m}} (u+\rho)^{-\frac{(t+1)(m-1)}{m}} |\nabla u|^{m-1} \right) \\ & \times \left(\frac{s}{t^{\frac{m-1}{m}}} \varphi^{s-1-\frac{s(m-1)}{m}} (u+\rho)^{-t+\frac{(t+1)(m-1)}{m}} |\nabla \varphi| \right) d\mu \\ \leq \ & \varepsilon t \int_M \varphi^s (u+\rho)^{-t-1} \frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}} d\mu \\ & + C_\varepsilon \frac{s^m}{t^{m-1}} \int_M \varphi^{s-m} (u+\rho)^{m-t-1} \frac{|\nabla \varphi|^m}{\sqrt{1+|\nabla u|^m}} d\mu. \end{aligned}$$

Letting $\varepsilon = \frac{1}{2}$, substituting the above estimate into (4.3.1), and cancelling out the half of the first term in (4.3.1), we obtain

$$
\begin{aligned}
& \frac{t}{2}\int_M \varphi^s (u+\rho)^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\
\leq\ & \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}\frac{|\nabla\varphi|^m}{\sqrt{1+|\nabla u|^m}}d\mu \\
\leq\ & \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu. \qquad (4.3.2)
\end{aligned}
$$

Using (4.2.4) once more to the right-hand side of (4.3.2) with the Hölder conjugate couple (p_1, p_1') of (4.2.7) we obtain

$$
\begin{aligned}
& \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu \\
=\ & \int_M [\varphi^{\frac{s}{p_1}}(u+\rho)^{m-t-1}]\cdot[\tfrac{Cs^m}{t^{m-1}}\varphi^{\frac{s}{p_1'}-m}|\nabla\varphi|^m] d\mu \\
\leq\ & \frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu \\
& + C_1\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu. \qquad (4.3.3)
\end{aligned}
$$

Using in the right-hand side of (4.3.3) the obvious inequality

$$
\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}} \leq \left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma}{\sigma-m+1}},
$$

and combining (4.3.3) with (4.3.2), we obtain

$$
\begin{aligned}
& \frac{t}{2}\int_M \varphi^s (u+\rho)^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\
\leq\ & \frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu \\
& + C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \qquad (4.3.4)
\end{aligned}
$$

where the term that contains s is absorbed into constant C_1.

By the same arguments as in Section 4.2, we can show $\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu$ is bounded. Hence, by Dominated Convergence theorem, letting $\rho \to 0$, we have

$$
\lim_{\rho\to 0}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu = \int_M \varphi^s u^{\sigma-t}d\mu,
$$

Letting $\rho \to 0$ in (4.3.4), applying Monotone Convergence theorem to the right-hand side integrals of (4.3.4), we obtain

$$\begin{aligned} & \lim_{\rho\to 0}\frac{t}{2}\int_M \varphi^s (u+\rho)^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu + \lim_{\rho\to 0}\int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq\; & \lim_{\rho\to 0}\frac{1}{2}\int_M \varphi^s (u+\rho)^{\sigma-t} d\mu \\ & + C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \end{aligned}$$

which is

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s u^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu + \frac{1}{2}\int_M \varphi^s u^{\sigma-t} d\mu \\ \leq\; & C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \end{aligned} \tag{4.3.5}$$

Applying (4.1.9) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned} & \int_M \varphi^s u^\sigma d\mu \\ \leq\; & s\int_M \varphi^{s-1}|\nabla u|^{m-2}\frac{(\nabla u, \nabla\varphi)}{\sqrt{1+|\nabla u|^m}}d\mu \\ \leq\; & s\int_M \varphi^{s-1}\frac{|\nabla u|^{m-1}}{\sqrt{1+|\nabla u|^m}}|\nabla\varphi| d\mu \\ \leq\; & s\left(\int_M \varphi^s u^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu\right)^{\frac{m-1}{m}} \\ & \times\left(\int_M \varphi^{s-m}u^{(t+1)(m-1)}\frac{|\nabla\varphi|^m}{\sqrt{1+|\nabla u|^m}}d\mu\right)^{\frac{1}{m}} \\ \leq\; & s\left(\int_M \varphi^s u^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu\right)^{\frac{m-1}{m}} \\ & \times\left(\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right)^{\frac{1}{m}}. \end{aligned} \tag{4.3.6}$$

On the other hand, we obtain from (4.3.5) that

$$\int_M \varphi^s u^{-t-1}\frac{|\nabla u|^m}{\sqrt{1+|\nabla u|^m}}d\mu \leq Ct^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu.$$

Substituting this into (4.3.6) yields

$$\int_M \varphi^s u^\sigma d\mu \leq C\left[t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right]^{\frac{m-1}{m}} \times\left[\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right]^{\frac{1}{m}}. \quad (4.3.7)$$

Recalling that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (4.3.7) with (p_2, p_2') of (4.2.13), we obtain

$$\begin{aligned}
&\int_M \varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\\
=\ &\int_{M\setminus B_R}\left(\varphi^{\frac{s}{p_2}}u^{(t+1)(m-1)}\right)\left(\varphi^{\frac{s}{p_2'}-m}|\nabla\varphi|^m\right)d\mu\\
\leq\ &\left(\int_{M\setminus B_R}\varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{\sigma}}\\
&\times\left(\int_{M\setminus B_R}\varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{1-\frac{(t+1)(m-1)}{\sigma}}. \quad (4.3.8)
\end{aligned}$$

Substituting (4.3.8) into (4.3.7), we obtain

$$\begin{aligned}
&\int_M \varphi^s u^\sigma d\mu\\
\leq\ &Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}}\\
&\times\left(\int_M \varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}\\
&\times\left(\int_{M\setminus B_R}\varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \quad (4.3.9)
\end{aligned}$$

Noting $s > \frac{4m\sigma}{\sigma-m+1}$, and $t < \frac{\sigma-m+1}{2}$ in (4.2.2), and recalling that $0 \leq \varphi \leq 1$, from (4.3.9), we obtain

$$\begin{aligned}
&\int_M \varphi^s u^\sigma d\mu\\
\leq\ &Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_M |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}}\\
&\times\left(\int_M |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}
\end{aligned}$$

$$\times \left(\int_{M \setminus B_R} \varphi^s u^\sigma d\mu \right)^{\frac{(t+1)(m-1)}{m\sigma}}. \tag{4.3.10}$$

Applying the same arguments as in Section 4.2 from (4.2.16), we obtain that $u \equiv 0$. Thus, we complete the proof of Theorem 4.1.4.

4.4 Sharpness of p, q

In this section, we show the sharpness of parameters p and q in Theorem 4.1.3. Obviously, it suffices to verify the sharpness of q.

Fix $p = \frac{m\sigma}{\sigma - m + 1}$, and choose any $q > \frac{m-1}{\sigma-m+1}$. We will construct an example of a manifold M satisfying the volume growth condition (4.1.11) with these values p,q and admitting a positive solution u of (4.1.1).

We will use the following Proposition (cf. [70])

Proposition 4.4.1. *Let $m > 1$, $\sigma > 0$ be a constant. Let $\beta(r)$ be a positive C^1-function on $[r_0, \infty)$ satisfying*

$$\int_{r_0}^{\infty} (\beta(s))^{-\frac{1}{m-1}} ds < \infty. \tag{4.4.1}$$

Define the function $\gamma(r)$ on $[r_0, \infty)$ by

$$\gamma(r) = \int_r^{\infty} (\beta(s))^{-\frac{1}{m-1}} ds. \tag{4.4.2}$$

If

$$\int_{r_0}^{\infty} \beta(s)\gamma(s)^\sigma ds < \infty, \tag{4.4.3}$$

then the following equation

$$(\beta(r)|y'|^{m-2}y')' + \beta(r)y^\sigma = 0, \tag{4.4.4}$$

has at least one positive solution on $[r_0, \infty)$, which satisfies

$$y(r) = O(\gamma(r)), \quad \text{as } r \to \infty. \tag{4.4.5}$$

By definition in [70], a solution of (4.4.4) is a C^1-function y, such that $|y'|^{m-2}y'$ is also C^1, and (4.4.4) is satisfied.

Let M be $(\mathbb{R}^n, g)$ with the following Riemannian metric

$$g = dr^2 + \psi(r)^2 d\theta^2, \tag{4.4.6}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$ and $\psi(r)$ is a smooth, positive function on $(0, \infty)$ such that

$$\psi(r) = \begin{cases} r, & \text{for small enough } r, \\ \left(r^{p-1} \ln^q r\right)^{\frac{1}{n-1}}, & \text{for large enough } r. \end{cases} \tag{4.4.7}$$

thus, in a neighborhood of 0, the metric g is exactly Euclidean, which can be extended smoothly to the origin. Hence, $M = (\mathbb{R}^n, g)$ is a complete Riemannian manifold.

By (4.4.6), the geodesic ball $B_r = B(0, r)$ on M coincides with the Euclidean ball $\{|x| < r\}$. Denote by $S(r)$ the surface area of B_r in M. It follows from (4.4.6) that $S(r) = \omega_n \psi^{n-1}(r)$, that is

$$S(r) = \omega_n \begin{cases} r^{n-1}, & \text{for small enough } r, \\ r^{p-1} \ln^q r, & \text{for large enough } r, \end{cases} \tag{4.4.8}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. The Riemannian volume of the ball B_r can be determined by

$$V(r) := \mu(B_r) = \int_0^r S(\tau)\, d\tau, \tag{4.4.9}$$

whence it follows that, for large enough r,

$$V(r) \leq C r^p \ln^q r. \tag{4.4.10}$$

Hence, the manifold M satisfies the volume growth condition of Theorem 4.1.3.

In what follows we prove the existence of a weak positive solution of

$$\Delta_m u + u^\sigma \leq 0,$$

on M. This solution u will depend only on the polar radius r, so that we write $u = u(r)$.

The construction of u will be done in two steps.

Step 1. For a function $u = u(r)$, the inequality (4.1.1) becomes

$$[S|u'|^{m-2}u']' + Su^\sigma \leq 0. \tag{4.4.11}$$

Note that for large enough r_0

$$\begin{aligned}\int_{r_0}^{\infty} S(r)^{-\frac{1}{m-1}} dr &= \int_{r_0}^{\infty} \frac{1}{(\omega_n r^{p-1} \ln^q r)^{\frac{1}{m-1}}} dr \\ &= \int_{r_0}^{\infty} \frac{1}{\omega_n^{\frac{1}{m-1}} r^{\frac{p-1}{m-1}} \ln^{\frac{q}{m-1}} r} dr \\ &< \infty, \end{aligned} \tag{4.4.12}$$

this is because $p = \frac{m\sigma}{\sigma-m+1} > m$. For all $r \geq r_0$, we have

$$\gamma(r) := \int_r^{\infty} (S(\tau))^{-\frac{1}{m-1}} d\tau = \int_r^{\infty} \frac{d\tau}{(\omega_n \tau^{p-1} \ln^q \tau)^{\frac{1}{m-1}}} \approx \frac{1}{r^{\frac{p-1}{m-1}-1} \ln^{\frac{q}{m-1}} r}.$$

It follows that

$$\begin{aligned}\int_{r_0}^{\infty} S(\tau)\gamma^{\sigma}(\tau) d\tau &\leq C \int_{r_0}^{\infty} \frac{\tau^p \ln^q \tau}{\tau^{\frac{\sigma(p-1)}{m-1}-\sigma} \ln^{\frac{\sigma q}{m-1}} \tau} \frac{d\tau}{\tau} \\ &\leq C \int_{r_0}^{\infty} \frac{1}{\tau^{\frac{\sigma(p-1)}{m-1}-\sigma-p} \ln^{\frac{\sigma q}{m-1}-q} \tau} \frac{d\tau}{\tau} \\ &\leq C \int_{r_0}^{\infty} \frac{1}{\ln^{\frac{q(\sigma-m+1)}{m-1}} \tau} \frac{d\tau}{\tau} \\ &< \infty, \end{aligned} \tag{4.4.13}$$

where we have used that $p = \frac{m\sigma}{\sigma-m+1}$ and $q > \frac{m-1}{\sigma-m+1}$.

Applying Proposition 4.4.1 with $\beta(r) = S(r)$, we obtain that there exists some C^1 solution u of (4.4.11) on $[r_0, \infty)$, such that

$$u(r) = O(\gamma(r)) = O(r^{-\frac{m}{\sigma-m+1}} \ln^{-\frac{q}{m-1}} r), \quad \text{as } r \to \infty.$$

In particular, $u(r) \to 0$ as $r \to \infty$. By increasing r_0 if necessary, we can assume that $u'(r_0) < 0$.

Step 2. Consider the following eigenvalue problem in a ball B_ρ of M

$$\begin{cases} div(|\nabla v|^{m-2}\nabla v) + \lambda_\rho |v|^{m-2} v = 0, & \text{in } B_\rho, \\ v|_{\partial B_\rho} = 0. \end{cases} \tag{4.4.14}$$

We denote by λ_ρ the principal (smallest) eigenvalue of this problem. It is known from [41] that $\lambda_\rho > 0$ and the corresponding eigenfunction $v_\rho > 0$ in B_ρ. Hence, we rewrite (4.4.14) in the following form

$$\begin{cases} div(|\nabla v|^{m-2}\nabla v) + \lambda_\rho v^{m-1} = 0, & \text{in } B_\rho, \\ v|_{\partial B_\rho} = 0. \end{cases} \tag{4.4.15}$$

Moreover, by [41, Theorem 1.3] and [71], we know the principal eigenvalue λ_ρ is simple, and v_ρ depends only on the polar radius, we have $v_\rho = v_\rho(r)$. From [64] and [72], we know $v_\rho(r)$ is $C^{1,\beta}$ for some $\beta \in (0,1)$. Normalizing v_ρ, we can assume that $v_\rho(0) = 1$, while $v_\rho|_{\partial B_\rho} = 0$.

Therefore, for a radial function v_ρ, the equation (4.4.15) becomes

$$[S|v_\rho'|^{m-2}v_\rho']' + \lambda_\rho S v_\rho^{m-1} = 0, \tag{4.4.16}$$

where also $v_\rho(\rho) = 0$, $v_\rho(0) = 1$, $v_\rho^{'}(0) = 0$, and $v_\rho > 0$ in $(0,\rho)$.

From (4.4.16), we obtain $[S|v_\rho'|^{m-2}v_\rho']' \leq 0$, so that the function $S|v_\rho'|^{m-2}v_\rho'$ is decreasing. Since $S|v_\rho'|^{m-2}v_\rho'$ vanishes at $r = 0$, it follows that $S|v_\rho'|^{m-2}v_\rho'(r) \leq 0$ and, hence $v_\rho'(r) \leq 0$ for all $r \in (0,\rho)$. Hence, the function $v_\rho(r)$ is decreasing for $r < \rho$ which together with the boundary conditions implies that $0 \leq v_\rho \leq 1$. Since $\sigma > m-1$, it follows that v_ρ is a positive solution in B_ρ of the inequality

$$div(|\nabla v_\rho|^{m-2}\nabla v_\rho) + \lambda_\rho v_\rho^\sigma \leq 0. \tag{4.4.17}$$

Let us show that $\lambda_\rho \to 0$ as $\rho \to \infty$. Indeed, it is known that

$$\lim_{\rho\to\infty} \lambda_\rho = \lambda_{\min}(M)$$

where $\lambda_{\min}(M)$ is the essential m-first eigenvalue of $-\Delta_m$ in $W^{1,m}(M)$ (cf. [40]).

We know from [40, Theorem 1.4] (when $m = 2$, one also could see [6])

$$\lambda_{\min}(M) \leq \left(\limsup_{\rho\to\infty} \frac{\ln V(\rho)}{m\rho}\right)^m, \tag{4.4.18}$$

It follows from (4.4.10) that $\lim_{\rho\to\infty} \lambda_\rho = \lambda_{\min}(M) = 0$.

In what follows we consider only integer values of ρ, and consider the sequence $\{v_k\}_{k=1}^\infty$. Let us show that the sequence $\{v_k\}$ satisfy that $v_k \to 1$ and $v_k' \to 0$ locally uniformly as $k \to \infty$, By the above analysis, we know v_k is decreasing. It follows that $v_k' \leq 0$. Integrating (4.4.16), noting that $v_k'(0) = 0$, we obtain

$$|v_k'|^{m-1}(r) = \frac{\lambda_k \int_0^r S(t)v_k^{m-1}(t)dt}{S(r)}. \tag{4.4.19}$$

Note that $0 \leq v_k \leq 1$ and (4.4.9), it follows that

$$|v_k'|^{m-1}(r) \leq \lambda_k \frac{V(r)}{S(r)},$$

since $\lambda_k \to 0$ as $k \to \infty$, we obtain that

$$v_k' \to 0, \tag{4.4.20}$$

uniformly on any bounded range of r as $k \to \infty$. The identity

$$v_k(r) = 1 + \int_0^r v_k'(t)dt, \tag{4.4.21}$$

implies that

$$v_k \to 1, \tag{4.4.22}$$

uniformly on any bounded range of r as $k \to \infty$.

Choose ρ large enough so that $\rho > r_0$ and

$$\frac{v_\rho'}{v_\rho}(r_0) > \frac{u'}{u}(r_0), \tag{4.4.23}$$

where u is the function constructed in the first step. Indeed, it is possible to achieve (4.4.23) by choosing $\rho = k$ with large enough i because by (4.4.20) and (4.4.22)

$$\frac{v_k'}{v_k}(r_0) \to 0, \quad \text{as } k \to \infty,$$

whereas $\frac{u'}{u}(r_0) < 0$ by construction in Step 1.

Let us fix $\rho > r_0$ for which (4.4.23) is satisfied, and compare the functions $u(r)$ and $v_\rho(r)$ in the interval $[r_0, \rho)$. Set

$$\theta = \inf_{r \in [r_0, \rho)} \frac{u(r)}{v_\rho(r)}.$$

Since v_ρ vanishes at ρ and, hence,

$$\frac{u(r)}{v_\rho(r)} \to +\infty, \quad \text{as } r \to \rho_-,$$

and, at $r = r_0$, by (4.4.23)

$$\left(\frac{u}{v_\rho}\right)'(r_0) = \frac{u' v_\rho - u v_\rho'}{v^2}(r_0) < 0,$$

so that u/v_ρ is strictly decreasing at r_0 and cannot have minimum at r_0. Hence, $\frac{u}{v_\rho}$

attains its minimum at an interior point $\xi \in (r_0, \rho)$, and at this point we have

$$\left(\frac{u}{v_\rho}\right)'(\xi) = 0.$$

It follows that

$$u(\xi) = \theta v_\rho(\xi) \quad \text{and} \quad u'(\xi) = \theta v_\rho'(\xi) \tag{4.4.24}$$

The function $u(r)$ has been defined for $r \geq r_0$, in particular, for $r \geq \xi$, whereas $v_\rho(r)$ has been defined for $r \leq \rho$, in particular, for $r \leq \xi$. Now we merge the two definitions by redefining/extending the function $u(r)$ for all $0 < r < \xi$ by setting $u(r) = \theta v_\rho(r)$.

It follows from (4.4.24) that $u \in C^1(M)$, in particular, $u \in W^{1,m}_{loc}(M)$. By (4.4.17), u satisfies the following inequality in B_ξ:

$$div(|\nabla u|^{m-2}\nabla u) + \frac{\lambda_\rho}{\theta^{\sigma-m+1}}u^\sigma \leq 0. \tag{4.4.25}$$

By (4.1.1), u satisfies the following inequality in $M \setminus B_{r_0}$:

$$div(|\nabla u|^{m-2}\nabla u) + u^\sigma \leq 0. \tag{4.4.26}$$

Combining (4.4.25) and (4.4.26), we obtain that u satisfies on M the following inequality

$$div(|\nabla u|^{m-2}\nabla u) + \delta u^\sigma \leq 0, \tag{4.4.27}$$

where $\delta = \min\{\lambda_\rho/\theta^{\sigma-m+1}, 1\}$. Finally, changing $u \mapsto cu$ where $c = \delta^{-\frac{1}{\sigma-m+1}}$ we obtain a positive solution to (4.1.1) on M, which concludes this example.

Chapter 5

Differential Inequalities with Gradient Terms

This chapter is based on the work [69].

5.1 Background and Statement

In this chapter, we study the uniqueness of nonnegative solutions of the following differential inequality

$$div(A(x)|\nabla u|^{m-2}\nabla u) + V(x)u^{\sigma_1}|\nabla u|^{\sigma_2} \le 0, \quad \text{on } M, \tag{5.1.1}$$

where div and ∇ are the Riemannian divergence and gradient respectively, A, V are positive measurable functions, $m > 1$, and σ_1, $\sigma_2 \ge 0$ are given parameters such that $\sigma_1 + \sigma_2 > m - 1$.

We say that A, V satisfy (VA′) condition, if there exist δ_1, $\delta_2 \ge 0$, and positive constants c_0, C_0 such that for all most $x \in M$, the following

$$c_0 r(x)^{-\delta_1} \le \frac{V(x)}{A(x)} \le C_0 r(x)^{\delta_2}, \tag{VA$'$}$$

holds for all large enough $r(x)$. We can see that the condition (VA′) is a special case of condition (VA) in Chapter 3. We emphasize here that $A(x)$, $V(x) > 0$ for almost all $x \in M$.

Let us introduce two paramters

$$p = \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1}, \quad q = \frac{m - 1}{\sigma_1 + \sigma_2 - m + 1}. \tag{5.1.2}$$

and a new measure ν by

$$d\nu = A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu, \tag{5.1.3}$$

Here are our main results.

Theorem 5.1.1. Assume that (VA′) holds with some $\delta_1, \delta_2 \geq 0$. If for some $x_0 \in M$, the following inequality

$$\nu(B(x_0, r)) \leq Cr^p \ln^q r, \tag{5.1.4}$$

holds for all large enough r, then the only nonnegative solution of (5.1.1) is identical constant.

When $A = V = 1$, we have the following corollary

Corollary 5.1.2. If for some $x_0 \in M$, the following inequality

$$\mu(B(x_0, r)) \leq Cr^p \ln^q r, \tag{5.1.5}$$

holds for all large enough r, then the only nonnegative solution of (5.1.1) is identical constant.

Mitidieri and Pokhozhaev [45] obtained the nonexistence result for problem (5.1.1) in $\mathbb{R}^n$ with $n > m > 1$, σ_1, $\sigma_2 \geq 0$, and $\sigma_1 + \sigma_2 > m - 1$. They proved for the case $A(x) = V(x) = 1$ that if

$$\sigma_1 + \sigma_2 \frac{n-1}{n-m} \leq \frac{n(m-1)}{n-m}, \tag{5.1.6}$$

then (5.1.1) has no positive solutions except constants. By Corollary 5.1.2, we see that if for large enough r (5.1.5) holds, then the only nonnegative solution of (5.1.1) is constant. Note that in $\mathbb{R}^n$

$$\mu(B(0, r)) = cr^n,$$

so that the condition (5.1.5) is equivalent to

$$n \leq p = \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1},$$

which in turn is equivalent to (5.1.6). Therefore, the result of [45] is covered by Corollary 5.1.2.

For the case $V(x) = |x|^{-\gamma_2}$, $A(x) = |x|^{\gamma_1}$ for $|x| \geq 1$, the problem (5.1.1) in $\mathbb{R}^n$

with $n > m > 1$ was studied by Filippucci [16], who proved that if

$$\begin{cases} 0 \le \sigma_2 < m-1, \quad m-n < \gamma_1 < m-\gamma_2-\sigma_2, \\ m-1-\sigma_2 < \sigma_1 \le \frac{(n-\gamma_2)(m-1)}{n-m+\gamma_1} - \sigma_2 \frac{n-1+\gamma_1}{n-m+\gamma_1}. \end{cases} \tag{5.1.7}$$

then (5.1.1) has no positive solutions except constants. Let us compare the result of [16] with our Theorem 5.1.1. Using the measure ν in (5.1.3), we obtain in $\mathbb{R}^n$ for large enough r

$$\begin{aligned} \nu(B(0,r)) &= \int_{B(0,r)} A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu \\ &= C\int_1^r r^{\frac{\gamma_1(\sigma_1+\sigma_2)}{\sigma_1+\sigma_2-m+1}} r^{\frac{(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1}} r^{n-1} dr + C_1 \\ &\approx r^{\frac{\gamma_1(\sigma_1+\sigma_2)+(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1}+n}, \end{aligned}$$

where μ is the Lebesgue measure. The condition (5.1.4) is then equivalent to

$$\frac{\gamma_1(\sigma_1+\sigma_2)+(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1} + n \le p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}, \tag{5.1.8}$$

which in turn is equivalent to (5.1.7). Under (5.1.7), we obtain that (5.1.1) has no positive solutions except constants. Thus, our Theorem 5.1.1 covers the aforementioned results in $\mathbb{R}^n$.

Now we will explain in what sense the solutions of (5.1.1) are defined. Define the measure ω by

$$d\omega = Ad\mu,$$

and set

$$W^{1,m}_{loc}(M,\omega) = \{f : M \to \mathbb{R} | f \in L^m_{loc}(M,\omega), \nabla f \in L^m_{loc}(M,\omega)\}, \tag{5.1.9}$$

where ∇f is understood in distributional sense. Denote by $W^{1,m}_c(M,\omega)$ the subspace of $W^{1,m}_{loc}(M,\omega)$ of functions with compact support.

Definition 5.1.3. A function u on M is called a weak solution of the inequality (5.1.1), if $u \ge 0$, $u \in W^{1,m}_{loc}(M,\omega)$, $V|\nabla u|^{\sigma_2} \in L^1_{loc}(M,\mu)$, and for any nonnegative function $\psi \in W^{1,m}_c(M,\omega)$, the following inequality holds:

$$-\int_M A(x)|\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu + \int_M V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}\psi d\mu \le 0, \tag{5.1.10}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

Remark 5.1.4. Using the defintion of ψ, we have

$$\begin{aligned}
&\left|\int_M A(x)|\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu\right| \\
\leq\ & \int_{supp(\psi)} A|\nabla u|^{m-1}|\nabla\psi|d\mu \\
\leq\ & \left(\int_{supp(\psi)} A|\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \left(\int_{supp(\psi)} A|\nabla\psi|^m d\mu\right)^{\frac{1}{m}},
\end{aligned}$$

Since $u \in W^{1,m}_{loc}(M,\omega)$ and $\psi \in W^{1,m}_c(M,\omega)$. Therefore, the first term in (5.1.10) is finite, which implies the finiteness of the second term, that is

$$\int_M V u^{\sigma_1}|\nabla u|^{\sigma_2}\psi d\mu < \infty.$$

5.2 Proof of Theorem 5.1.1

Let u be some nonnegative weak solution to (5.1.1). x_0 is the reference point as before in Theorem 5.1.1. Denote $B_R := B(x_0, R)$, and fix a Lipschitz function φ on M with compact support, such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of B_R. In particular, we have $\varphi \in W^{1,m}_c(M,\omega)$. Take the following test function for (5.1.10):

$$\psi_\rho(x) = \varphi(x)^s(u+\rho)^{-t}, \tag{5.2.1}$$

where the value of ρ is some positive constant near zero, s is some fixed bigger enough constant, and t is variable and can be chosen arbitrarily close to 0. Hence ψ_ρ has compact support and is locally bounded. Since

$$\nabla\psi_\rho = -t(u+\rho)^{-t-1}\varphi^s\nabla u + s(u+\rho)^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that, $\nabla\psi_\rho \in L^m(M,\omega)$. It follows that, $\psi_\rho \in W^{1,m}_c(M,\omega)$. We obtain from (5.1.10) that

$$\begin{aligned}
& t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu \\
\leq\ & s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-2}(\nabla u, \nabla\varphi)d\mu. \qquad (5.2.2)
\end{aligned}$$

thus

$$t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu$$

$$\leq \ s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi|d\mu. \tag{5.2.3}$$

In what follows, we use the following Young's inequality

$$\int_M fgd\mu \leq \varepsilon\int_M |f|^{p_0}d\mu + C_\varepsilon\int_M |g|^{p_0'}d\mu, \tag{5.2.4}$$

where $\varepsilon > 0$ is arbitrary, and when t is small enough, (p_0, p_0') is a Hölder conjugate couple such that

$$p_0 = \frac{m\sigma_1 - t(m-\sigma_2)+\sigma_2}{m\sigma_1 - \sigma_1 + t - t(m-\sigma_2)} > 1, \quad p_0' = \frac{m\sigma_1 - t(m-\sigma_2)+\sigma_2}{\sigma_1+\sigma_2-t} > 1.$$

Applying (5.2.4) to the right-hand side integral of (5.2.3), we obtain

$$\begin{aligned}
& s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi|d\mu \\
= & \int_M \left(t^{\frac{1}{p_0}}\varphi^{\frac{s}{p_0}}A(x)^{\frac{1}{p_0}}(u+\rho)^{-\frac{t+1}{p_0}}|\nabla u|^{\frac{m}{p_0}}\right) \\
& \times\left(\frac{s}{t^{\frac{1}{p_0}}}\varphi^{s-1-\frac{s}{p_0}}A(x)^{\frac{1}{p_0'}}(u+\rho)^{-t+\frac{t+1}{p_0}}|\nabla u|^{m-1-\frac{m}{p_0}}|\nabla\varphi|\right)d\mu \\
\leq & \ \varepsilon t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu \\
& + C_\varepsilon\frac{s^{p_0'}}{t^{\frac{p_0'}{p_0}}}\int_M \varphi^{p_0'(s-1)-\frac{sp_0'}{p_0}}A(x)(u+\rho)^{-p_0't+\frac{p_0'(t+1)}{p_0}}|\nabla u|^{(m-1)p_0'-\frac{mp_0'}{p_0}}|\nabla\varphi|^{p_0'}d\mu \\
\leq & \ \varepsilon t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu \\
& + C_\varepsilon\frac{s^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'}d\mu.
\end{aligned}$$

Letting $\varepsilon = \frac{1}{2}$, substituting the above into (5.2.3), and cancelling out the half of the first term in (5.2.3), we obtain

$$\begin{aligned}
& \frac{t}{2}\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu \\
\leq & \ C\frac{s^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'}d\mu.
\end{aligned} \tag{5.2.5}$$

Using (5.2.4) once more to the right-hand side of (5.2.5) with another Hölder conjugate couple (p_1, p_1') satisfying

$$p_1 = \frac{\sigma_2}{m-p_0'} = \frac{\sigma_1+\sigma_2-t}{m-t-1}, \quad p_1' = \frac{\sigma_2}{\sigma_2-m+p_0'} = \frac{\sigma_1+\sigma_2-t}{\sigma_1+\sigma_2-m+1},$$

We obtain

$$
\begin{aligned}
&\frac{Cs^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'}d\mu\\
= &\int_M [\varphi^{\frac{s}{p_1}}V(x)^{\frac{1}{p_1}}(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}]\\
&\times[\frac{Cs^{p_0'}}{t^{p_0'-1}}\varphi^{\frac{s}{p_1'}-p_0'}V(x)^{-\frac{1}{p_1}}A(x)|\nabla\varphi|^{p_0'}]d\mu\\
\le &\frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{(p_0'-t-1)p_1}|\nabla u|^{(m-p_0')p_1}d\mu\\
&+C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{-\frac{p_1'}{p_1}}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\\
= &\frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu\\
&+C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu. \qquad (5.2.6)
\end{aligned}
$$

Combining with (5.2.5), we obtain

$$
\begin{aligned}
&\frac{t}{2}\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu+\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu\\
\le &\frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu\\
&+C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu. \qquad (5.2.7)
\end{aligned}
$$

We know that

$$
\begin{aligned}
\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \le\; &C\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu\\
&+C\rho^{\sigma_1-t}\int_M \varphi^s V(x)|\nabla u|^{\sigma_2}d\mu,
\end{aligned}
$$

From (5.2.2) by definition of the solution, we know

$$
\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu,
$$

is bounded, and noting that by definition of the solution $V|\nabla u|^{\sigma_2}\in L^1_{loc}(M,\mu)$, we obtain

$$
\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu
$$

is bounded.

Taking $\rho \to 0$ in (5.2.7), applying Monotone Convergence theorem to left-hand side integrals, and Dominated Convergence theorem to the right-hand side integrals, we obtain

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\ \leq\ & \frac{1}{2}\int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\ & +C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu, \end{aligned}$$

which is

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu + \frac{1}{2}\int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\ \leq\ & C_1 t^{-(p_0'-1)p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu, \end{aligned} \tag{5.2.8}$$

where the term that contains s is absorbed into constant C_1.

Applying (5.1.10) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned} & \int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu \\ \leq\ & s\int_M \varphi^{s-1}A(x)|\nabla u|^{m-2}(\nabla u,\nabla\varphi)d\mu \\ \leq\ & s\int_M \varphi^{s-1}A(x)|\nabla u|^{m-1}|\nabla\varphi|d\mu \\ =\ & s\int_M [\varphi^{\frac{s}{p_2}}A(x)^{\frac{1}{p_2}}u^{-\frac{t+1}{p_2}}|\nabla u|^{\frac{m}{p_2}}]\cdot[\varphi^{(s-1)-\frac{s}{p_2}}A(x)^{\frac{1}{p_2'}}u^{\frac{t+1}{p_2}}|\nabla u|^{m-1-\frac{m}{p_2}}|\nabla\varphi|]d\mu \\ \leq\ & s\left(\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu\right)^{\frac{1}{p_2}} \\ & \times\left(\int_M \varphi^{(s-1)p_2'-\frac{sp_2'}{p_2}}A(x)u^{\frac{(t+1)p_2'}{p_2}}|\nabla u|^{(m-1)p_2'-\frac{mp_2'}{p_2}}|\nabla\varphi|^{p_2'}d\mu\right)^{\frac{1}{p_2'}} \\ =\ & s\left(\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu\right)^{\frac{1}{p_2}} \\ & \times\left(\int_M \varphi^{s-p_2'}A(x)u^{(t+1)(p_2'-1)}|\nabla u|^{m-p_2'}|\nabla\varphi|^{p_2'}d\mu\right)^{\frac{1}{p_2'}}. \end{aligned} \tag{5.2.9}$$

where we have used the following conjugate pair

$$p_2 = \frac{m\sigma_1 + \sigma_2(t+1)}{m\sigma_1 - \sigma_1}, \quad p_2' = \frac{m\sigma_1 + \sigma_2(t+1)}{\sigma_1 + \sigma_2(t+1)}. \tag{5.2.10}$$

From (5.2.8), we have

$$\int_M \varphi^s A(x) u^{-t-1} |\nabla u|^m d\mu \le C t^{-1-(p_0'-1)p_1'} \int_M \varphi^{s-p_0'p_1'} V(x)^{1-p_1'} A(x)^{p_1'} |\nabla \varphi|^{p_0'p_1'} d\mu.$$

Substituting this into (5.2.9) yields

$$\begin{aligned}
&\int_M \varphi^s V(x) u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \\
\le\ & C \left[t^{-1-(p_0'-1)p_1'} \int_M \varphi^{s-p_0'p_1'} V(x)^{1-p_1'} A(x)^{p_1'} |\nabla \varphi|^{p_0'p_1'} d\mu \right]^{\frac{1}{p_2}} \\
&\times \left[\int_M \varphi^{s-p_2'} A(x) u^{(t+1)(p_2'-1)} |\nabla u|^{m-p_2'} |\nabla \varphi|^{p_2'} d\mu \right]^{\frac{1}{p_2'}}.
\end{aligned} \tag{5.2.11}$$

Noting that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (5.2.11) with the following couple (p_3, p_3')

$$p_3 = \frac{\sigma_1 + \sigma_2(t+1)}{(t+1)(m-1)}, \quad p_3' = \frac{\sigma_1 + \sigma_2(t+1)}{\sigma_1 + \sigma_2(t+1) - (t+1)(m-1)},$$

It is easy to check that

$$\begin{aligned}
(t+1)(p_2'-1) &= \frac{(t+1)(m-1)}{\sigma_1 + \sigma_2(t+1)} \sigma_1 = \frac{\sigma_1}{p_3}, \\
m - p_2' &= \frac{(t+1)(m-1)}{\sigma_1 + \sigma_2(t+1)} \sigma_2 = \frac{\sigma_2}{p_3}.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\int_M \varphi^{s-p_2'} A(x) u^{(t+1)(p_2'-1)} |\nabla u|^{m-p_2'} |\nabla \varphi|^{p_2'} d\mu \\
=\ & \int_{M\setminus B_R} \left(\varphi^{\frac{s}{p_3}} V(x)^{\frac{1}{p_3}} u^{(t+1)(p_2'-1)} |\nabla u|^{m-p_2'} \right) \left(\varphi^{\frac{s}{p_3'}-p_2'} V(x)^{-\frac{1}{p_3}} A(x) |\nabla \varphi|^{p_2'} \right) d\mu \\
=\ & \int_{M\setminus B_R} \left(\varphi^{\frac{s}{p_3}} V(x)^{\frac{1}{p_3}} u^{\frac{\sigma_1}{p_3}} |\nabla u|^{\frac{\sigma_2}{p_3}} \right) \left(\varphi^{\frac{s}{p_3'}-p_2'} V(x)^{-\frac{1}{p_3}} A(x) |\nabla \varphi|^{p_2'} \right) d\mu \\
\le\ & \left(\int_{M\setminus B_R} \varphi^s V(x) u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \right)^{\frac{1}{p_3}}
\end{aligned}$$

$$\times\left(\int_{M\setminus B_R}\varphi^{\frac{sp_3'}{p_3}-p_2'p_3'}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_3'}}. \tag{5.2.12}$$

Substituting (5.2.12) into (5.2.11), we obtain

$$\begin{aligned}
&\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\
\le\ & C\left[t^{-1-(p_0'-1)p_1'}\int_{M\setminus B_R}\varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\right]^{\frac{1}{p_2}}\\
&\times\left(\int_{M\setminus B_R}\varphi^{\frac{sp_3'}{p_3}-p_2'p_3'}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_2'p_3'}}\\
&\times\left(\int_{M\setminus B_R}\varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}.
\end{aligned} \tag{5.2.13}$$

Choosing s large enough, and recalling that $0\le\varphi\le 1$, from (5.2.13), we obtain

$$\begin{aligned}
&\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\
\le\ & C\left[t^{-1-(p_0'-1)p_1'}\int_{M\setminus B_R}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\right]^{\frac{1}{p_2}}\\
&\times\left(\int_{M\setminus B_R}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_2'p_3'}}\\
&\times\left(\int_{M\setminus B_R}\varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}.
\end{aligned}$$

Using the new measure $d\nu = A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}}V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}}d\mu$, we have

$$\begin{aligned}
&\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\
\le\ & C\left[t^{-1-(p_0'-1)p_1'}\int_{M\setminus B_R}\left(\frac{V}{A}\right)^{\frac{t}{\sigma_1+\sigma_2-m+1}}|\nabla\varphi|^{p_0'p_1'}d\nu\right]^{\frac{1}{p_2}}\\
&\times\left(\int_{M\setminus B_R}\left(\frac{V}{A}\right)^{-\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}}|\nabla\varphi|^{p_2'p_3'}d\nu\right)^{\frac{1}{p_2'p_3'}}\\
&\times\left(\int_{M\setminus B_R}V(x)\varphi^s u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}.
\end{aligned} \tag{5.2.14}$$

We know $\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu$ is finite in Introduction, it follows from (5.2.14) that

$$\begin{aligned}
&\left(\int_M \varphi^s V(x) u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu\right)^{1-\frac{1}{p_2' p_3}} \\
\leq\ & C\left[t^{-1-(p_0'-1)p_1'} \int_{M\setminus B_R} \left(\frac{V}{A}\right)^{\frac{t}{\sigma_1+\sigma_2-m+1}} |\nabla\varphi|^{p_0' p_1'} d\nu\right]^{\frac{1}{p_2}} \\
&\times \left(\int_{M\setminus B_R} \left(\frac{V}{A}\right)^{-\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}} |\nabla\varphi|^{p_2' p_3'} d\nu\right)^{\frac{1}{p_2' p_3'}} \qquad (5.2.15)
\end{aligned}$$

We notice, that all integral terms in the right-hand side of (5.2.15) have the form

$$\int_M |\nabla\varphi|^a \left(\frac{V}{A}\right)^b d\nu,$$

with the following two pairs of (a, b) such that

$$\begin{cases} a_1 = p_0' p_1' = \frac{m\sigma_1 - t(m-\sigma_2)+\sigma_2}{\sigma_1+\sigma_2-m+1}, \\ b_1 = \frac{t}{\sigma_1+\sigma_2-m+1}. \end{cases} \qquad (5.2.16)$$

and

$$\begin{cases} a_2 = p_2' p_3' = \frac{m\sigma_1+\sigma_2(t+1)}{\sigma_1+\sigma_2(t+1)-(t+1)(m-1)}, \\ b_2 = -\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}. \end{cases} \qquad (5.2.17)$$

Besides, a could be written in the form

$$a = p + lt, \qquad (5.2.18)$$

with the following two respective values of l

$$\begin{cases} l_1 = \frac{\sigma_2-m}{\sigma_1+\sigma_2-m+1}, \\ l_2 = \frac{\sigma_1(m-\sigma_2)(m-1)}{[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)](\sigma_1+\sigma_2-m+1)}. \end{cases} \qquad (5.2.19)$$

where $p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$ is defined as before in (5.1.2). Clearly, it is obvious that all the values of a and l are uniformly bounded, when t is small enough near zero.

Consider the integral

$$J_n(a,b) = \int_M |\nabla\varphi_n|^a \left(\frac{V}{A}\right)^b d\nu, \qquad (5.2.20)$$

where φ_n is the same as defined in (3.3.8), and (a, b) take values from (5.2.16) and (5.2.17).

Substituting (3.3.8) into (5.2.20), applying (3.3.6), (3.3.9) and (VA′), when $b > 0$, we obtain

$$\begin{aligned} J_n(a,b) &= \int_M |\nabla \varphi_n|^a \left(\frac{V}{A}\right)^b d\nu \\ &= C\int_M \frac{\sum_{k=n+1}^{2n} |\nabla \tilde{\varphi}_k|^a}{n^a} \left(\frac{V}{A}\right)^b d\nu \\ &\leq \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \frac{|\nabla \tilde{\varphi}_k|^a}{n^a} r^{b\delta_2} d\nu \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a (2^k)^{b\delta_2} \nu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^a (2^k)^{b\delta_2} \nu(B_{2^k}), \end{aligned} \tag{5.2.21}$$

where we have used that a is uniformly bounded. Noting that $a = p + bt$, $n + 1 \leq k \leq 2n$, and substituting $t = \frac{1}{n}$, if $b > 0$, we obtain

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^a (2^k)^{b\delta_2} &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt} (2^k)^{b\delta_2} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p \sup_{n\leq k\leq 2n, t=\frac{1}{n}} \left(\frac{2^{-k}}{n}\right)^{\frac{l}{n}} (2^k)^{b\delta_2} \\ &\leq C\left(\frac{2^{-k}}{n}\right)^p. \end{aligned}$$

Thus, using (5.1.4), recalling that by (5.1.2) $p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$, $q = \frac{m-1}{\sigma_1+\sigma_2-m+1}$, we obtain

$$\begin{aligned} J_n(a,b) &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p \nu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p (2^k)^p \ln^q(2^k) \\ &\leq C\frac{1}{n^p} \sum_{k=n+1}^{2n} k^q \\ &\leq Cn^{q+1-p} \\ &\leq Cn^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}. \end{aligned} \tag{5.2.22}$$

Similarly, if $b < 0$, we also have

$$J_n(a,b) \le Cn^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}. \tag{5.2.23}$$

Setting $\varphi = \varphi_n$ in (5.2.15), we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu\right)^{1-\frac{1}{p_2' p_3}} \\ \le\ & ct^{-\frac{1}{p_2}-\frac{(p_0'-1)p_1'}{p_2}} \left(J_n(a_1,b_1)\right)^{\frac{1}{p_2}} \left(J_n(a_2,b_2)\right)^{\frac{1}{p_2' p_3'}}. \end{aligned} \tag{5.2.24}$$

where $(a_i, b_i)_{i=1,2}$ are defined in (5.2.16) and (5.2.17).

Substituting (5.2.22), (5.2.23) into (5.2.24), using $t = \frac{1}{n}$ as before, we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu\right)^{1-\frac{1}{p_2' p_3}} \\ \le\ & Cn^{\frac{1}{p_2}+\frac{(p_0'-1)p_1'}{p_2}} \left(n^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}\right)^{\frac{1}{p_2}} \\ & \times \left(n^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}\right)^{\frac{1}{p_2' p_3'}}. \end{aligned} \tag{5.2.25}$$

Substituting the values of p_0', p_2, p_2' and p_3', we obtain that the exponents in the power of n in the right-hand side of (5.2.25) is equal to

$$\frac{1}{p_2} + \frac{(p_0'-1)p_1'}{p_2} - \frac{(m-1)\sigma_1}{p_2(\sigma_1+\sigma_2-m+1)} - \frac{(m-1)\sigma_1}{p_2' p_3'(\sigma_1+\sigma_2-m+1)} = 0.$$

Therefore, we obtain

$$\left(\int_M \varphi_n V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu\right)^{1-\frac{1}{p_2' p_3}} \le C < \infty. \tag{5.2.26}$$

Recalling that $\varphi_n = 1$ on B_{2^n}, and taking the lim sup of both sides in (5.2.26) as $n \to \infty$, we obtain

$$\int_M V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \ \le\ C < \infty. \tag{5.2.27}$$

The same argument can be used in (5.2.14), which implies

$$\int_M \varphi_n^s V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \le C \left(\int_{M\setminus B_{2n}} \varphi_n^s V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu\right)^{\frac{1}{p_2' p_3}}, \tag{5.2.28}$$

Using that $0 \le \varphi_n \le 1$ and $\varphi_n|_{B_{2^n}} = 1$ once more, we obtain

$$\int_{B_{2^n}} Vu^{\sigma_1}|\nabla u|^{\sigma_2} d\mu \le C\left(\int_{M\setminus B_{2^n}} Vu^{\sigma_1}|\nabla u|^{\sigma_2} d\mu\right)^{\frac{1}{p_2' p_3}}, \tag{5.2.29}$$

Letting $n \to \infty$, and using (5.2.27), we obtain

$$\int_M Vu^{\sigma_1}|\nabla u|^{\sigma_2} d\mu = 0,$$

since V is a positive function, which implies that $u \equiv const$.

5.3 Applications

Theorem 5.1.1 could be applied to get the uniqueness of nonnegative m-superharmonic function, namely, the uniqueness of the following problem

$$\Delta_m v \le 0, \quad \text{on } M, \tag{5.3.1}$$

where M is the same as before, i.e. a geodesically complete connected Riemannian manifold.

Let $u = \ln(v+1)$. Since v is nonnegative, hence, u is also nonnegative. Moreover, an easy calculation shows that u satisfies the following inequality

$$e^{(m-1)u}\left(\Delta_m u + (m-1)|\nabla u|^m\right) \le 0, \tag{5.3.2}$$

which simplifies to

$$\Delta_m u + (m-1)|\nabla u|^m \le 0, \tag{5.3.3}$$

By changing $u \to cu$, we can get rid of the factor $m-1$ in (5.3.3). By Theorem 5.1.1, we obtain that if

$$\mu(B(x_0, r)) \le Cr^m \ln^{m-1} r, \tag{5.3.4}$$

then the only nonnegative solution of (5.3.3) is constant, and hence the only nonnegative solution of (5.3.1) is constant.

Let us recall the celebrated result of Holopainen [33]: if

$$\int^{\infty}\left(\frac{r}{\mu(B(x_0, r))}\right)^{\frac{1}{m-1}} dr = \infty, \tag{5.3.5}$$

then any positive m-superharmonic function is constant.

Obviously, (5.3.4) implies (5.3.5). However, the function $r \mapsto r^m \ln^{m-1} r$ is right on the borderline of divergence of the integral in (5.3.5), so that the condition cannot be significantly improved.

Another application of Theorem 5.1.1 is to investigate the following inequality

$$\Delta_m u + |\nabla u|^{m-2} \nabla B \cdot \nabla u + u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0, \quad \text{on } M, \tag{5.3.6}$$

where B is a given measurable function on M, and ∇B does not have any singular point, σ_1, σ_2 are defined as in Section 5.1. One can rewrite (5.3.6) as the following

$$e^{-B} div(e^B |\nabla u|^{m-2} \nabla u) + u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0,$$

which is equivalent to

$$div(e^B |\nabla u|^{m-2} \nabla u) + e^B u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0.$$

Thus, applying Theorem 5.1.1, we obtain the following result.

Corollary 5.3.1. If for some $x_0 \in M$, the following inequality

$$\nu(B(x_0, r)) \leq C r^p \ln^q r, \tag{5.3.7}$$

holds for all large enough r, where ν is defined by $d\nu = e^B d\mu$, p and q are defined by (5.1.2), then the only nonnegative solution of (5.3.6) is constant.

5.4 Sharpness of p, q

In this section, we show the sharpness of parameters p and q in Theorem 5.1.1.

The sharpness of p is already known in $\mathbb{R}^n$ with $n > m > 1$. The following example was given by Mitidieri and Pokhozhaev in [48]: If

$$\begin{cases} \sigma_1 > \frac{(n-\gamma_2)(m-1)}{n-m+\gamma_1} - \sigma_2 \frac{n-1+\gamma_1}{n-m+\gamma_1}, \\ m - n < \gamma_1 < m - \gamma_2 - \sigma_2, \\ \gamma_1(\sigma_1 + \sigma_2) + \gamma_2(m-1) + n(\sigma_1 + \sigma_2 - m + 1) > 0, \\ 0 \leq \sigma_2 < m - 1. \end{cases} \tag{5.4.1}$$

then the function

$$u(x) := \epsilon [1 + |x|^{\frac{m-\sigma_2-\gamma_1-\gamma_2}{m-1-\sigma_2}}]^{-\frac{m-1-\sigma_2}{\sigma_1+\sigma_2-m+1}}$$

is a solution to (5.1.1) with $A(x) = |x|^{\gamma_1}$, $V(x) = |x|^{-\gamma_2}$, where ϵ is a suitable small constant. Actually, using the measure ν of (5.1.4)

$$\begin{aligned}\nu(B(o,r)) &= \int_{B(0,r)} A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu \\ &= \int_{B(0,r)} |x|^{\frac{\gamma_1(\sigma_1+\sigma_2)}{\sigma_1+\sigma_2-m+1}} |x|^{\frac{\gamma_2(m-1)}{\sigma_1+\sigma_2-m+1}} d\mu \\ &\approx r^p, \end{aligned} \tag{5.4.2}$$

where $p = \frac{\gamma_1(\sigma_1+\sigma_2)+\gamma_2(m-1)}{\sigma_1+\sigma_2-m+1} + n$. From the assumption (5.4.1), we know it is equivalent to

$$p > \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1}, \tag{5.4.3}$$

One could let p be close to $\frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$ from above, by carefully choosing γ_1, γ_2.

In what follows we will show the sharpness of q in case of $A = 1$, $V = m - 1$, $\sigma_1 = 0$ and $\sigma_2 = m$. Fix here

$$p = \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1} = m,$$

and for arbitrary $\epsilon > 0$, choose

$$q = \frac{m-1}{\sigma_1 + \sigma_2 - m + 1} + \epsilon = m - 1 + \epsilon.$$

Recall that M is called m-parabolic, if any positive m-superharmonic function v on M is constant, namely $\Delta_m v \leq 0$. Holopainen proved in [33] that M is m-parabolic if and only if (5.3.5) holds. Thus, if

$$\mu(B(x_0, r)) \leq C r^m \ln^{m-1+\epsilon} r, \quad \text{for large enough } r, \tag{5.4.4}$$

we know (5.3.5) does not hold any more. Thus, there exists a positive function v such that $\Delta_m v \leq 0$. Letting $u = \ln(v + 1)$, we know u is a positive solution of (5.3.3). Hence, the exponent $m - 1$ is sharp here.

Chapter 6

Opposite Quasilinear Inequalities

6.1 Background and Statement

In this chapter, we obtain the analogous uniqueness results for inequalities in which the operator Δ_m is replaced by $-\Delta_m$, that is

$$-\Delta_m u + V(x)u^\sigma \leq 0, \quad \text{on } M, \tag{6.1.1}$$

where $\sigma > m-1$, $m > 1$, and $V(x)$ is a nonnegative regular function such that $V \in L^1_{loc}(M)$, and for large enough $r(x)$

$$V(x) = \frac{1}{r(x)^\alpha}, \tag{6.1.2}$$

where α is any real number such that $\alpha \leq m$.

The importance of such inequalities has been widely recognized recently, see [15, 51, 56, 63, 73] and the references therein. Existence and nonexistence results, and asymptotic theory about the positive solution have also been obtained.

Birindelli proved in [5] that in $\mathbb{R}^n$, if $\alpha < m$ and $\sigma > m-1$, then the only nonnegative weak solution to (6.1.1) is identical zero.

Naito and Usami obtained in [50] similar results for the positive entire solutions to (6.1.1) in $\mathbb{R}^n$. Here they call u an entire solution of (6.1.1) if u is a positive function $u \in C^1(\mathbb{R}^n)$ such that $|\nabla u|^{m-2}\nabla u \in C^1(\mathbb{R}^n)$ and satisfies (6.1.1) at each $x \in \mathbb{R}^n$. They proved that if

$$\liminf_{|x|\to\infty} |x|^m V(x) > 0, \tag{6.1.3}$$

then (6.1.1) has no positive entire solutions. They used the ODE method, which strongly depends on the rotation symmetry of $\mathbb{R}^n$.

The aforementioned papers dealt with the nonexistence of the solution. Let us cite one of the papers concerning the existence results: let $M = \mathbb{R}^n$, and let V be radially symmetric. If for some $\epsilon > 0$, either of the following conditions holds

$$\begin{cases} \limsup_{|x|\to\infty} |x|^{m+\epsilon}V(x) < \infty, \quad \text{when } m < n; \\ \limsup_{|x|\to\infty} |x|^{m+\epsilon}\ln^{\sigma+1+\epsilon}|x|V(x) < \infty, \quad \text{when } m = n; \\ \limsup_{|x|\to\infty} |x|^{m+\frac{\sigma(m-n)}{m-1}+\epsilon}V(x) < \infty, \quad \text{when } m > n. \end{cases} \tag{6.1.4}$$

then (6.1.1) has a positive radial entire solution (cf. [17]).

In this chapter we apply for investigation of (6.1.1) a modification of the method that we used in the previous chapters.

First, we should explain in what sense we define the solution to (6.1.1). Denote by

$$W^{1,m}_{loc}(M) := \{f | f \in L^m_{loc}(M), \nabla f \in L^m_{loc}(M)\}, \tag{6.1.5}$$

and denote by $W^{1,m}_c(M)$ the subspace of $W^{1,m}_{loc}(M)$ of functions with compact support.

Solutions of (6.1.1) are understood in the following weak sense

Definition 6.1.1. A function u on M is called a weak solution of the inequality (6.1.1) if u is a nonnegative function from $W^{1,m}_{loc}(M) \cap L^\infty_{loc}(M)$ and for any nonnegative function $\psi \in W^{1,m}_c(M)$, the following inequality holds:

$$\int_M |\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu + \int_M Vu^\sigma\psi d\mu \le 0, \tag{6.1.6}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

Remark 6.1.2. Using the defintion of ψ, we have

$$\begin{aligned} -\int_M |\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu &\le \int_{supp(\psi)} |\nabla u|^{m-1}|\nabla\psi|d\mu \\ &\le \left(\int_{supp(\psi)} |\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \left(\int_{supp(\psi)} |\nabla\psi|^m d\mu\right)^{\frac{1}{m}}, \end{aligned}$$

thus the finiteness of the first term on the left-hand side of (6.1.6) will lead to the finiteness of the second term, that is $\int_M u^\sigma V\psi d\mu < \infty$.

Introduce two paramters denoted by

$$p = \frac{m\sigma}{\sigma - m + 1}, \quad q = \frac{m-1}{\sigma - m + 1}, \tag{6.1.7}$$

and another measure ν by

$$d\nu = V^{-\frac{m-1}{\sigma-m+1}} d\mu. \tag{6.1.8}$$

First, we need the following lemma.

Lemma 6.1.3. If

$$\nu(B(x_0, r)) \leq Cr^{\frac{m\sigma}{\sigma-m+1}} \ln^{\frac{m-1}{\sigma-m+1}} r, \tag{6.1.9}$$

holds for all large enough r, then the only nonnegative solution of (6.1.1) is identical zero.

Here are our main results.

Theorem 6.1.4. (i) Let $\alpha < m$. If for some $N > 1$, the following inequality

$$\mu(B(x_0, r)) \leq Cr^N \tag{6.1.10}$$

holds for all large enough r, then the only nonnegative solution of (6.1.1) is identical zero.

(ii) Let $\alpha = m$. If for some $N > 1$, the following inequality

$$\mu(B(x_0, r)) \leq Cr^m \ln^N r, \tag{6.1.11}$$

holds for all large enough r, then the only nonnegative solution of (6.1.1) is identical zero.

Remark 6.1.5. Restriction of the volume growth (6.1.11) seems to be technical. We conjecture in the case $\alpha = m$, the condition (6.1.10) implies that the only nonnegative solution of (6.1.1) is identical zero.

Remark 6.1.6. The uniqueness results do not hold in case of $0 < \sigma \leq 1$, $m = 2$. Consider

$$-\Delta u + u^\sigma \leq 0, \quad \text{in } \mathbb{R}^n, \tag{6.1.12}$$

where $0 < \sigma \leq 1$. One easily could check that the function

$$u(x) = e^{x_1^2},$$

is a regular solution of (6.1.12).

6.2 Proof of Theorem 6.1.4

Before we give the proof of Theorem 6.1.4, we first show the proof of Lemma 6.1.3.

Proof of Lemma 6.1.3. : Let u be some nonnegative solution to (6.1.1). x_0 is the reference point. Denote $B_R := B(x_0, R)$, and fix a Lipschitz function φ on M with compact support, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of B_R. Particularly, $\varphi \in W_c^{1,m}(M)$. Take the following test function for (6.1.6):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^t, \tag{6.2.1}$$

where ρ is a small positive constant, t, s satisfy that

$$0 < t < \min\left\{1, \frac{\sigma - m + 1}{2}\right\}, \quad s > \frac{4\sigma}{\sigma - m + 1}. \tag{6.2.2}$$

Actually, here t will take arbitrarily small positive constants, and s is a fixed large enough constant.

From Definition 6.1.1, we know ψ_ρ has compact support. Since

$$\nabla \psi_\rho = t(u+\rho)^{t-1}\varphi^s \nabla u + s(u+\rho)^t \varphi^{s-1} \nabla\varphi,$$

noting that for $t < 1$, $(u+\rho)^{t-1}$ is locally bounded, and

$$(u+\rho)^t \le \max\{u+\rho, 1\},$$

and recalling $u \in W_{loc}^{1,m}(M)$, then $(u+\rho)^t \in L_{loc}^m(M)$. Hence, $\psi_\rho \in W_c^{1,m}(M)$. We obtain from (6.1.6) that

$$\begin{aligned} & t\int_M \varphi^s (u+\rho)^{t-1} |\nabla u|^m d\mu + s\int_M \varphi^{s-1}(u+\rho)^t |\nabla u|^{m-2}(\nabla u, \nabla\varphi) d\mu \\ & + \int_M \varphi^s V u^\sigma (u+\rho)^t d\mu \le 0. \end{aligned} \tag{6.2.3}$$

Applying Young's inequality to the second integral of (6.2.3) as follows

$$\begin{aligned} & -s\int_M \varphi^{s-1}(u+\rho)^t |\nabla u|^{m-2}(\nabla u, \nabla\varphi) d\mu \\ \le & \int_M [t^{\frac{m-1}{m}} \varphi^{\frac{s(m-1)}{m}} (u+\rho)^{\frac{(t-1)(m-1)}{m}} |\nabla u|^{m-1}] \\ & \times [s t^{-\frac{m-1}{m}} \varphi^{\frac{s-m}{m}} (u+\rho)^{\frac{t+m-1}{m}} |\nabla\varphi|] d\mu \end{aligned}$$

$$\begin{aligned} \leq\ & \frac{t}{2}\int_{M\setminus B_R}\varphi^s(u+\rho)^{t-1}|\nabla u|^m d\mu \\ & +C_1\frac{s^m}{t^{m-1}}\int_{M\setminus B_R}\varphi^{s-m}(u+\rho)^{t+m-1}|\nabla\varphi|^m d\mu, \end{aligned} \tag{6.2.4}$$

Substituting into (6.2.3), we obtain

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s(u+\rho)^{t-1}|\nabla u|^m d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^t d\mu \\ \leq\ & C_1\frac{s^m}{t^{m-1}}\int_{M\setminus B_R}\varphi^{s-m}(u+\rho)^{t+m-1}|\nabla\varphi|^m d\mu. \end{aligned} \tag{6.2.5}$$

Using Young's inequality once more to the right-hand side of (6.2.5) with the conjugate pair

$$(p_1,p_1') = (\frac{\sigma+t}{t+m-1},\frac{\sigma+t}{\sigma-m+1}),$$

we obtain

$$\begin{aligned} & \frac{s^m}{t^{m-1}}\int_{M\setminus B_R}\varphi^{s-m}(u+\rho)^{t+m-1}|\nabla\varphi|^m d\mu \\ =\ & \int_{M\setminus B_R}[\varphi^{\frac{s(t+m-1)}{\sigma+t}}V^{\frac{1}{p_1}}(u+\rho)^{t+m-1}]\cdot[\frac{s^m}{t^{m-1}}\varphi^{s-m-\frac{s(t+m-1)}{\sigma+t}}V^{-\frac{1}{p_1}}|\nabla\varphi|^m]d\mu \\ \leq\ & \frac{1}{2}\int_{M\setminus B_R}\varphi^s V(u+\rho)^{\sigma+t}d\mu \\ & +C\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma+t}{\sigma-m+1}}\int_{M\setminus B_R}\varphi^{s-\frac{m(\sigma+t)}{\sigma-m+1}}V^{-\frac{t+m-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}}d\mu. \end{aligned} \tag{6.2.6}$$

Combining with (6.2.5), we obtain

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s(u+\rho)^{t-1}|\nabla u|^m d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^t d\mu \\ \leq\ & \frac{1}{2}\int_{M\setminus B_R}\varphi^s V(u+\rho)^{\sigma+t}d\mu \\ & +C\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma+t}{\sigma-m+1}}\int_{M\setminus B_R}\varphi^{s-\frac{m(\sigma+t)}{\sigma-m+1}}V^{-\frac{t+m-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}}d\mu. \end{aligned} \tag{6.2.7}$$

From the definition of the solution, we know the term

$$\int_M \varphi^s V u^\sigma (u+\rho)^t d\mu \tag{6.2.8}$$

is bounded. Noting the term

$$\int_M \varphi^s V(u+\rho)^{\sigma+t} d\mu \tag{6.2.9}$$

could be controlled by the terms of (6.2.8) and $\int_M \varphi^s V(u+\rho)^t d\mu$. Since

$$\begin{aligned}
\int_M \varphi^s V(u+\rho)^t d\mu \ &\leq \ \int_{\{u+\rho\geq 1\}\cap \mathrm{supp}(\varphi)} \varphi^s V(u+\rho)^\sigma d\mu \\
&\quad + \int_{\{u+\rho\leq 1\}\cap \mathrm{supp}(\varphi)} \varphi^s V d\mu \\
&\leq \ C \int_{\{u+\rho\geq 1\}\cap \mathrm{supp}(\varphi)} \varphi^s V u^\sigma d\mu \\
&\quad + C\rho^\sigma \int_{\{u+\rho\geq 1\}\cap \mathrm{supp}(\varphi)} \varphi^s V d\mu \\
&\quad + \int_{\{u+\rho\leq 1\}\cap \mathrm{supp}(\varphi)} \varphi^s V d\mu,
\end{aligned} \tag{6.2.10}$$

we obtain that the term $\int_M \varphi^s V(u+\rho)^t d\mu$ is bounded. Hence, the term of (6.2.9) is bounded. Applying Dominated Convergence theorem, we obtain that

$$\lim_{\rho\to 0} \int_M \varphi^s V(u+\rho)^{\sigma+t} d\mu = \int_M \varphi^s V u^{\sigma+t} d\mu. \tag{6.2.11}$$

Letting $\rho \to 0$ in (6.2.7), applying Monotone Convergence theorem to the left-hand side integrals, combining with (6.2.11), we obtain

$$\begin{aligned}
&\frac{t}{2}\int_M \varphi^s u^{t-1} |\nabla u|^m d\mu + \frac{1}{2}\int_M \varphi^s V u^{\sigma+t} d\mu \\
\leq \ & C\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma+t}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma+t)}{\sigma-m+1}} V^{-\frac{t+m-1}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}} d\mu.
\end{aligned} \tag{6.2.12}$$

Applying (6.1.6) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned}
&\int_M \varphi^s V u^\sigma d\mu \\
\leq \ & -s\int_M \varphi^{s-1} |\nabla u|^{m-2} (\nabla u, \nabla\varphi) d\mu \\
\leq \ & s\int_M \varphi^{s-1} |\nabla u|^{m-1} |\nabla \varphi| d\mu \\
\leq \ & s\left(\int_M \varphi^s u^{t-1}|\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \left(\int_M \varphi^{s-m} u^{(1-t)(m-1)} |\nabla\varphi|^m d\mu\right)^{\frac{1}{m}}.
\end{aligned} \tag{6.2.13}$$

From (6.2.12), we have

$$\begin{aligned}&\int_M \varphi^s u^{t-1}|\nabla u|^m d\mu\\ \le\ & Ct^{-\frac{(\sigma+t)(m-1)}{\sigma-m+1}-1}\int_M \varphi^{s-\frac{m(\sigma+t)}{\sigma-m+1}}V^{-\frac{t+m-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}}d\mu. \qquad (6.2.14)\end{aligned}$$

where the term that contains s is absorbed into the constant C. Noting that $\nabla\varphi = 0$ on B_R. Applying Hölder inequality to the last term of (6.2.13) with the Hölder couple

$$(p_2, p_2') = \left(\frac{\sigma}{(1-t)(m-1)}, \frac{\sigma}{\sigma-(1-t)(m-1)}\right),$$

we obtain

$$\begin{aligned}&\int_M \varphi^{s-m}u^{(1-t)(m-1)}|\nabla\varphi|^m d\mu\\ =\ & \int_{M\setminus B_R}[\varphi^{\frac{s}{p_2}}V^{\frac{1}{p_2}}u^{(1-t)(m-1)}]\cdot[\varphi^{\frac{s}{p_2'}-m}V^{-\frac{1}{p_2}}|\nabla\varphi|^m]d\mu\\ \le\ & \left(\int_{M\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{(1-t)(m-1)}{\sigma}}\\ &\times\left(\int_M \varphi^{s-\frac{m\sigma}{\sigma-(1-t)(m-1)}}V^{-\frac{(1-t)(m-1)}{\sigma-(1-t)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}}d\mu\right)^{1-\frac{(1-t)(m-1)}{\sigma}} \qquad (6.2.15)\end{aligned}$$

Substituting (6.2.14) and (6.2.15) into (6.2.13), we obtain

$$\begin{aligned}&\int_M \varphi^s V u^\sigma d\mu\\ \le\ & C\left(t^{-\frac{(\sigma+t)(m-1)}{\sigma-m+1}-1}\int_M \varphi^{s-\frac{m(\sigma+t)}{\sigma-m+1}}V^{-\frac{t+m-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}}\\ &\times\left(\int_M \varphi^{s-\frac{m\sigma}{\sigma-(1-t)(m-1)}}V^{-\frac{(1-t)(m-1)}{\sigma-(1-t)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}}d\mu\right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}\\ &\times\left(\int_{M\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{(1-t)(m-1)}{m\sigma}}. \qquad (6.2.16)\end{aligned}$$

where the term that contains s is absorbed into the constant C. Taking $s > \frac{4m\sigma}{\sigma-m+1}$, when t is small enough, we obtain

$$\int_M \varphi^s V u^\sigma d\mu \ \le\ Ct^{-\frac{(\sigma+t)(m-1)^2}{m(\sigma-m+1)}-\frac{m-1}{m}}\left(\int_M V^{-\frac{t+m-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}}$$

$$\times \left(\int_M V^{-\frac{(1-t)(m-1)}{\sigma-(1-t)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}} d\mu \right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}$$
$$\times \left(\int_{M\setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{(1-t)(m-1)}{m\sigma}}. \tag{6.2.17}$$

Since $\int_M \varphi^s u^\sigma d\mu$ is finite by the Remark 6.1.2, we obtain

$$\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1-\frac{(1-t)(m-1)}{m\sigma}}$$
$$\leq Ct^{-\frac{(\sigma+t)(m-1)^2}{m(\sigma-m+1)}-\frac{m-1}{m}} \left(\int_M V^{-\frac{t+m-1}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}} d\mu \right)^{\frac{m-1}{m}}$$
$$\times \left(\int_M V^{-\frac{(1-t)(m-1)}{\sigma-(1-t)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}} d\mu \right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}. \tag{6.2.18}$$

Taking $s > \frac{4m\sigma}{\sigma-m+1}$, and using another measure $d\nu = V^{-\frac{m-1}{\sigma-m+1}} d\mu$, the above becomes

$$\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1-\frac{(1-t)(m-1)}{m\sigma}-\frac{(t+m-1)(m-1)}{m(\sigma+t)}}$$
$$\leq Ct^{-\frac{(\sigma+t)(m-1)^2}{m(\sigma-m+1)}-\frac{m-1}{m}} \left(\int_M V^{-\frac{t}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}} d\nu \right)^{\frac{m-1}{m}}$$
$$\times \left(\int_M V^{\frac{(m-1)\sigma t}{[\sigma-(1-t)(m-1)](\sigma-m+1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}} d\nu \right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}. \tag{6.2.19}$$

and

$$\int_M \varphi^s V u^\sigma d\mu$$
$$\leq Ct^{-\frac{(\sigma+t)(m-1)^2}{m(\sigma-m+1)}-\frac{m-1}{m}} \left(\int_M V^{-\frac{t}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma+t)}{\sigma-m+1}} d\nu \right)^{\frac{m-1}{m}}$$
$$\times \left(\int_M V^{\frac{(m-1)\sigma t}{[\sigma-(1-t)(m-1)](\sigma-m+1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(1-t)(m-1)}} d\nu \right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}$$
$$\times \left(\int_M \varphi^s V u^\sigma d\mu \right)^{\frac{(1-t)(m-1)}{m\sigma}}. \tag{6.2.20}$$

We see, that the first two integral terms in the right-hand side of (6.2.19) have the form

$$\int_M V^b |\nabla\varphi|^a d\nu,$$

with various (a,b) such that

$$(a,b) = \left(\frac{m(\sigma+t)}{\sigma-m+1}, -\frac{t}{\sigma-m+1}\right), \tag{6.2.21}$$

$$(a,b) = \left(\frac{m\sigma}{\sigma-(1-t)(m-1)}, \frac{(m-1)\sigma t}{[\sigma-(1-t)(m-1)](\sigma-m+1)}\right). \tag{6.2.22}$$

Consequently, a could be written in the form

$$a = p + lt, \tag{6.2.23}$$

with various l

$$l = \frac{m}{\sigma-m+1} \quad \text{and} \quad l = -\frac{m\sigma(m-1)}{[\sigma-(1-t)(m-1)](\sigma-m+1)}. \tag{6.2.24}$$

where $p = \frac{m\sigma}{\sigma-m+1}$ is defined as before in (6.1.7). Here both a, l and b are uniformly bounded, when t is small enough.

Consider the integral

$$J_n(a,b) = \int_M V^b |\nabla\varphi_n|^a d\nu, \tag{6.2.25}$$

where φ_n is the same as defined in (3.3.8).

Substituting (3.3.8) into (6.2.25), applying (3.3.9) and (3.3.6), when $b \le 0$, we obtain

$$\begin{aligned} J_n(a,b) &= \int_M V^b |\nabla\varphi_n|^a d\nu \\ &\le \int_M V^b \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a} d\nu \\ &\le C \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} (2^k)^{-b\alpha} \frac{|\nabla\tilde{\varphi}_k|^a}{n^a} d\nu \\ &\le C \sum_{k=n+1}^{2n} (2^k)^{-b\alpha} \left(\frac{2^{1-k}}{n}\right)^a \nu(B_{2^k}) \\ &\le C \sum_{k=n+1}^{2n} (2^k)^{-b\alpha} \left(\frac{2^{-k}}{n}\right)^a \nu(B_{2^k}), \end{aligned} \tag{6.2.26}$$

Noting that $n+1 \le k \le 2n$, $a = p + lt$, and taking $t = \frac{1}{n}$

$$(2^k)^{-b\alpha} \left(\frac{2^{-k}}{n}\right)^a = (2^k)^{-b\alpha} \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt}$$

$$\leq \quad C(2^k)^{-b\alpha}\left(\frac{2^{-k}}{n}\right)^p. \tag{6.2.27}$$

thus, we obtain

$$\begin{aligned}
J_n(a,b) &\leq C\sum_{k=n+1}^{2n}(2^k)^{-b\alpha}\left(\frac{2^{-k}}{n}\right)^p(2^k)^p\ln^q(2^k) \\
&\leq C\frac{1}{n^p}\sum_{k=n+1}^{2n}(2^k)^{-b\alpha}k^q \\
&\leq Cn^{q+1-p}2^{-2nb\alpha},
\end{aligned} \tag{6.2.28}$$

where we have used the volume assumption (6.1.9).

Similarly, when $b \geq 0$, we have

$$J_n(a,b) \leq Cn^{q+1-p}2^{-nb\alpha}. \tag{6.2.29}$$

Setting $\varphi = \varphi_n$ in (6.2.20), we obtain

$$\begin{aligned}
&\left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{(1-t)(m-1)}{m\sigma}} \\
\leq\ & ct^{-\frac{(\sigma+t)(m-1)^2}{m(\sigma-m+1)}-\frac{m-1}{m}}\left(J_n(\frac{m(\sigma+t)}{\sigma-m+1},-\frac{t}{\sigma-m+1})\right)^{\frac{m-1}{m}} \\
&\times\left(J_n(\frac{m\sigma}{\sigma-(1-t)(m-1)},-\frac{(m-1)\sigma t}{[\sigma-(1-t)(m-1)](\sigma-m+1)})\right)^{\frac{1}{m}-\frac{(1-t)(m-1)}{m\sigma}}.
\end{aligned} \tag{6.2.30}$$

Substituting (6.2.29),(6.2.28) into (6.2.30), and letting $t = \frac{1}{n}$, we obtain

$$\begin{aligned}
&\left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{(1-\frac{1}{n})(m-1)}{m\sigma}} \\
\leq\ & Cn^{\frac{(\sigma+\frac{1}{n})(m-1)^2}{m(\sigma-m+1)}+\frac{m-1}{m}}\left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}2^{2n\alpha\frac{\frac{1}{n}}{\sigma-m+1}}\right)^{\frac{m-1}{m}} \\
&\times\left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}2^{-n\alpha\frac{(m-1)\sigma\frac{1}{n}}{[\sigma-(1-\frac{1}{n})(m-1)](\sigma-m+1)}}\right)^{\frac{1}{m}-\frac{(1-\frac{1}{n})(m-1)}{m\sigma}} \\
\leq\ & C2^{\frac{\alpha(m-1)}{m(\sigma-m+1)}}.
\end{aligned} \tag{6.2.31}$$

Hence

$$\int_M Vu^\sigma d\mu \ < \ \infty. \tag{6.2.32}$$

Repeating the same procedures in (6.2.20), we obtain

$$\int_M \varphi_n^s Vu^\sigma d\mu \le C\left(\int_{M\setminus B_{2^n}} \varphi_n^s Vu^\sigma d\mu\right)^{\frac{(1-\frac{1}{n})(m-1)}{m\sigma}}, \tag{6.2.33}$$

Using that $0 \le \varphi_n \le 1$ and $\varphi_n|_{B_{2^n}} = 1$ once more, we obtain

$$\int_{B_{2^n}} Vu^\sigma d\mu \le C\left(\int_{M\setminus B_{2^n}} Vu^\sigma d\mu\right)^{\frac{(1-\frac{1}{n})(m-1)}{m\sigma}}, \tag{6.2.34}$$

Letting $n \to \infty$, and using (6.2.32), we obtain

$$\int_M Vu^\sigma d\mu = 0,$$

by the positiveness of V, which implies that $u \equiv 0$. Thus, we complete the proof of Lemma 6.1.3. □

Now we give the proof of Theorem 6.1.4:

Proof of Theorem 6.1.4. Notice that if u satisfies (6.1.1), then $v = u^\beta$ with $\beta \ge 1$ satisfies the following inequality

$$-\Delta_m v + \beta^{m-1} V v^{\frac{\sigma+(\beta-1)(m-1)}{\beta}} \le -\beta^{m-1}(\beta-1)(m-1)u^{(\beta-1)(m-1)-1}|\nabla u|^m. \tag{6.2.35}$$

Consequently, since u is a nonnegative weak solution to (6.1.1), we obtain $v \in W^{1,m}_{loc}(M) \cap L^\infty_{loc}(M)$, satisfying

$$-\Delta_m v + \beta^{m-1} V v^{\frac{\sigma+(\beta-1)(m-1)}{\beta}} \le 0, \tag{6.2.36}$$

By Lemma 6.1.3, we know under the measure

$$\begin{aligned} d\nu_1 &= \left(\beta^{m-1}V\right)^{-\frac{m-1}{\frac{\sigma+(\beta-1)(m-1)}{\beta}-m+1}} d\mu \\ &= \beta^{-\frac{\beta(m-1)^2}{\sigma-m+1}} V^{-\frac{\beta(m-1)}{\sigma-m+1}} d\mu, \end{aligned} \tag{6.2.37}$$

if for large enough r, the following inequality

$$\nu_1(B(x_0, r)) \le Cr^{\gamma_1} \ln^{\gamma_2} r, \tag{6.2.38}$$

holds, then the only nonnegative solution of (6.2.36) is identical zero. Here

$$\gamma_1 = \frac{m \frac{\sigma+(\beta-1)(m-1)}{\beta}}{\frac{\sigma+(\beta-1)(m-1)}{\beta} - m + 1} = \frac{m[\sigma + (\beta-1)(m-1)]}{\sigma - m + 1}, \tag{6.2.39}$$

and

$$\gamma_2 = \frac{m-1}{\frac{\sigma+(\beta-1)(m-1)}{\beta} - m + 1} = \frac{\beta(m-1)}{\sigma - m + 1}, \tag{6.2.40}$$

We know that for large enough r, (6.2.38) is implied by

$$\begin{aligned} \mu(B(x_0, r)) &\le C\beta^{\frac{\beta(m-1)^2}{\sigma-m+1}} V^{\frac{\beta(m-1)}{\sigma-m+1}} r^{\frac{m[\sigma+\beta(m-1)]}{\sigma-m+1}} \ln^{\frac{\beta(m-1)}{\sigma-m+1}} r \\ &\le C_1 r^{-\frac{\alpha\beta(m-1)}{\sigma-m+1}} r^{\frac{m[\sigma+(\beta-1)(m-1)]}{\sigma-m+1}} \ln^{\frac{\beta(m-1)}{\sigma-m+1}} r \\ &= C_1 r^{\frac{m\sigma+[(m-\alpha)\beta-m](m-1)}{\sigma-m+1}} \ln^{\frac{\beta(m-1)}{\sigma-m+1}} r, \end{aligned} \tag{6.2.41}$$

In the case $\alpha < m$ choose β so large that

$$\frac{m\sigma + [(m-\alpha)\beta - m](m-1)}{\sigma - m + 1} \ge N, \tag{6.2.42}$$

then we know (6.1.10) implies that (6.2.41), and also implies that (6.2.38). By Lemma 6.1.3, we conclude that $v \equiv 0$, and hence $u \equiv 0$.

In the case $\alpha = m$, choose β to satisfy $\frac{\beta(m-1)}{\sigma-m+1} \ge N$, then the condition (6.1.11) implies that (6.2.41), and also implies that (6.2.38). By Lemma 6.1.3, we obtain that $u \equiv 0$. □

6.3 Model Manifolds

In this section, we want to investigate the uniqueness of nonnegative solution of the problem

$$-\Delta_m u + V(x)u^\sigma \le 0, \tag{6.3.1}$$

on the model manifold M. Here $\sigma > m-1$, $m > 1$, and $V(x)$ is a nonnegative regular function such that for large enough $r(x)$

$$V(x) = \frac{1}{r(x)^m}. \tag{6.3.2}$$

Let M be the model manifold with the following Riemannian metric

$$g = dr^2 + \phi(r)^2 d\theta^2, \tag{6.3.3}$$

where (r,θ) are the polar coordinates in $\mathbb{R}^n$, and $\phi(r)$ is a smooth positive increasing function on $(0,\infty)$. Let B_r be the geodesic ball centered at 0. Denote by $S(r)$ the surface area of B_r in M. It follows from (6.3.3) that

$$S(r) = \omega_n \phi(r)^{n-1}. \tag{6.3.4}$$

For more information about the Model manifolds, one can refer to [27].

In this section we call that u is the entire solution of (6.3.1), if $u \in C^1(M)$, and $|\nabla u|^{m-2}\nabla u \in C^1(M)$ and satisfies (6.3.1) at each $x \in M$.

We have the following results analogous to Theorem 6.1.4

Theorem 6.3.1. If $S(r)$ is increasing, and $S(2r) \le cS(r)$ holds for all $r \ge 0$, then the only nonnegative entire solution of (6.3.1) is identical zero.

Remark 6.3.2. Theorem 6.3.1 improves in the case of model manifolds the result of (ii) of Theorem 6.1.4 as the latter required the restriction (6.1.11), whereas the hypothesis of Theorem 6.3.1 is satisfied for the case

$$\mu(B_r) = cr^N, \tag{6.3.5}$$

for any $N > 1$.

Consider the following ODE

$$(S(r)|v'|^{m-2}v')' = S(r)V(r)v^\sigma, \tag{6.3.6}$$

with $v'(0) = 0$.

Assume that the maximal interval for v is $[0,R)$, and remains positive in $[0,R)$, then we get that $v'(r) > 0$ for $0 < r < R$. Integrating (6.3.6), we obtain

$$|v'|^{m-2}v'(r) = \frac{1}{S(r)}\int_0^r S(t)V(t)v^\sigma(t)dt, \quad 0 < r < R. \tag{6.3.7}$$

Moreover, if $R < \infty$, we have

$$\lim_{r \to R_-} v(r) = \infty.$$

If $R = \infty$, we have

$$v'(r) = \left(\frac{1}{S(r)} \int_0^r S(t)V(t)v^\sigma(t)dt \right)^{\frac{1}{m-1}}, \quad r \geq 0, \tag{6.3.8}$$

and

$$v(r) = v(0) + \int_0^r \left(\frac{1}{S(s)} \int_0^s S(t)V(t)v^\sigma(t)dt \right)^{\frac{1}{m-1}} ds, \quad r \geq 0. \tag{6.3.9}$$

Similar to [50, Lemma 2.1, 2.2], we introduce the following lemmas.

Lemma 6.3.3. Let Ω be a bounded domain in M with smooth boundary $\partial\Omega$. Let u be a nonnegative entire solution of (6.3.1) and let $v \in C(\bar{\Omega}) \cap C^1(\Omega)$ be a nonnegative function satisfying $|\nabla v|^{m-2}\nabla v \in C^1(\Omega)$. If $\Delta_m v \leq V(x)v^\sigma$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Let $\varphi : \mathbb{R} \to [0, \infty)$ be a C^1 function which vanishes on $(-\infty, 0]$, and is strictly increasing on $(0, \infty)$. Since

$$(\Delta_m u - \Delta_m v)\varphi(u - v) \geq V(x)(u^\sigma - v^\sigma)\varphi(u - v) \quad \text{in } \Omega, \tag{6.3.10}$$

thus

$$\begin{aligned} & -\int_\Omega (|\nabla u|^{m-2}\nabla u - |\nabla v|^{m-2}\nabla v) \cdot (\nabla u - \nabla v)\varphi'(u - v)d\mu \\ \geq & \int_\Omega V(x)(u^\sigma - v^\sigma)\varphi(u - v)d\mu, \end{aligned} \tag{6.3.11}$$

Noting that

$$\begin{aligned} & (|\nabla u|^{m-2}\nabla u - |\nabla v|^{m-2}\nabla v) \cdot (\nabla u - \nabla v) \\ = & (|\nabla u|^{m-1} - |\nabla v|^{m-1}) \cdot (|\nabla u| - |\nabla v|) \\ & + (|\nabla u|^{m-2} + |\nabla v|^{m-2})(|\nabla u||\nabla v| - \nabla u \cdot \nabla v) \\ \geq & 0, \end{aligned} \tag{6.3.12}$$

we obtain

$$\int_\Omega V(x)(u^\sigma - v^\sigma)\varphi(u - v)d\mu \leq 0. \tag{6.3.13}$$

which implies that $u \leq v$ in Ω. □

Lemma 6.3.4. If (6.3.1) has a non-trivial nonnegative entire solution u, then there exists a positive solution v of (6.3.6) defined on the interval $[0, \infty)$.

Proof. Suppose that there exists no such function v. Assume to the contrary, that there exists a non-trivial nonnegative entire solution u, by considering a suitable parallel transformation, we may assume that $u(0) > 0$. Let v be a solution of (6.3.6) with initial values $v(0), v'(0)$ such that $0 < v(0) < u(0)$ and $v'(0) = 0$. Since that v is is not globally defined, we assume the maximal existence interval of v is $[0, R)$ with $R < \infty$. We know that $v'(r) > 0$ for $r \in (0, R)$, and $\lim_{r \to R_-} v(r) = \infty$. Hence, there exists an $R_1 \in (0, R)$ so that

$$v(R_1) \geq \max_{|x|=R_1} u(x).$$

Let $\Omega = B_{R_1}$. We have $\Delta_m v \leq V(x)v^\sigma$ in Ω, and $v \geq u$ on $\partial\Omega$. By Lemma 6.3.3, we have $u \leq v$ in Ω. But this contradicts $v(0) < u(0)$. Thus, we complete the proof. □

Now we give the proof of Theorem 6.3.1

Proof of Theorem 6.3.1. Suppose that (6.3.1) has a non-trivial nonnegative entire solution. Then, by Lemma 6.3.4 that (6.3.6) has a positive entire solution $v(r)$. If

$$\int_0^\infty \left(\frac{1}{S(s)} \int_0^s S(t)V(t)dt \right)^{\frac{1}{m-1}} ds = \infty, \tag{6.3.14}$$

we claim that

$$\lim_{r \to \infty} v(r) = \infty. \tag{6.3.15}$$

By (6.3.9) and noting that v is increasing, we obtain

$$v(r) \geq v(0) + v(0)^{\frac{\sigma}{m-1}} \int_0^r \left(\frac{1}{S(s)} \int_0^s S(t)V(t)dt \right)^{\frac{1}{m-1}} ds. \tag{6.3.16}$$

If

$$\int_0^\infty \left(\frac{1}{S(s)} \int_0^s S(t)V(t)dt \right)^{\frac{1}{m-1}} ds = \infty, \tag{6.3.17}$$

hence

$$\lim_{r \to \infty} v(r) = \infty. \tag{6.3.18}$$

Integrating (6.3.6) twice over $[R, r]$, we have

$$v(r) \geq v(R) + \int_R^r \left(\frac{1}{S(s)} \int_R^s S(t) V(t) v^\sigma(t) dt \right)^{\frac{1}{m-1}} ds, \tag{6.3.19}$$

For $R \leq r \leq 2R$, note that

$$\frac{S(R)}{S(r)} \geq \frac{S(R)}{S(2R)} \geq c_1, \tag{6.3.20}$$

we have

$$v(r) \geq v(R) + \int_R^r \left(\frac{1}{S(s)} \int_R^s S(t) V(t) v^\sigma(t) dt \right)^{\frac{1}{m-1}} ds, \tag{6.3.21}$$

Recalling that $S(r)$ is an increasing function and (6.3.20), for $R \leq r \leq 2R$, we obtain

$$v(r) \geq v(R) + c_1^{\frac{1}{m-1}} \int_R^r \left(\int_R^s V(t) v^\sigma(t) dt \right)^{\frac{1}{m-1}} ds. \tag{6.3.22}$$

Let w satisfy $w(R) = v(R)$, $w(r) \leq v(r)$, and

$$w'(r) = c_1^{\frac{1}{m-1}} \left(\int_R^r V(t) v^\sigma(t) dt \right)^{\frac{1}{m-1}} \geq 0, \tag{6.3.23}$$

consequently

$$(|w'|^{m-2} w')' = c_1 V(r) v^\sigma(r) \geq c_1 V(r) w^\sigma(r), \tag{6.3.24}$$

Multiplying by $w' \geq 0$ and integrating on $[R, r)$ for $R \leq r \leq 2R$, we obtain

$$\begin{aligned} \frac{m-1}{m} (w')^m &\geq c_1 \int_R^r V(s) w^\sigma(s) w'(s) ds \\ &\geq c_1 V(r) \int_R^r w^\sigma(s) w'(s) ds = \frac{c_1}{\sigma+1} V(r) [w^{\sigma+1}(r) - w^{\sigma+1}(R)], \end{aligned}$$

It follows that

$$[w^{\sigma+1}(r) - w^{\sigma+1}(R)]^{-\frac{1}{m}} w'(r) \geq c_2 V(r)^{-\frac{1}{m}}, \quad R < r < 2R,$$

where $c_2 = \left(\frac{mc_1}{(\sigma+1)(m-1)} \right)^{\frac{1}{m}} > 0$. Making once more integration over $[R, 2R]$, we obtain

$$\begin{aligned}\int_{v(R)}^{\infty}[s^{\sigma+1}-w^{\sigma+1}(R)]^{-\frac{1}{m}}ds &\geq \int_{w(R)}^{w(2R)}[s^{\sigma+1}-w^{\sigma+1}(R)]^{-\frac{1}{m}}ds \\ &\geq c_2\int_{R}^{2R}V(r)^{-\frac{1}{m}}dr \geq c_3\ln 2.\end{aligned}$$

Here we have used that $V(r)=\frac{c}{r^m}$ for large enough r. Making the variable change of $s=w(R)t$ to the first integral above, we obtain

$$[v(R)]^{-\frac{\sigma+1-m}{m}}\int_{1}^{\infty}(t^{\sigma+1}-1)^{-\frac{1}{m}}dt \geq c_3\ln 2. \tag{6.3.25}$$

On the other hand, from (6.3.18) and $\sigma+1-m>0$, we obtain

$$\lim_{R\to\infty}[v(R)]^{-\frac{\sigma+1-m}{m}}\int_{1}^{\infty}(t^{\sigma+1}-1)^{-\frac{1}{m}}dt = 0,$$

which is a contradiction with (6.3.25).

The only thing left here is to verify (6.3.14). Actually, the condition (6.3.14) could be derived from the double property $S(2r)\leq cS(r)$, since

$$\begin{aligned}&\int_{0}^{\infty}\left(\frac{1}{S(s)}\int_{0}^{s}S(t)V(t)dt\right)^{\frac{1}{m-1}}ds \\ \geq\ &\int_{1}^{\infty}\left(\frac{1}{S(s)}\int_{\frac{s}{2}}^{s}S(t)V(t)dt\right)^{\frac{1}{m-1}}ds \\ \geq\ &\int_{1}^{\infty}\left(\frac{S(\frac{s}{2})}{S(s)}\int_{\frac{s}{2}}^{s}V(t)dt\right)^{\frac{1}{m-1}}ds \\ \geq\ &C\int_{1}^{\infty}\left(\frac{S(\frac{s}{2})V(\frac{s}{2})s}{S(s)}\right)^{\frac{1}{m-1}}ds \\ \geq\ &C\int_{1}^{\infty}\frac{1}{s}d\mu=\infty.\end{aligned} \tag{6.3.26}$$

where we have used that $V(r)=Cr^{-m}$ for large enough r. Hence

$$\int_{0}^{\infty}\left(\frac{1}{S(s)}\int_{0}^{s}S(t)V(t)dt\right)^{\frac{1}{m-1}}ds=\infty.$$

Thus, we complete the proof. □

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