Uniqueness of Equilibrium States in Some Models of Interacting Particle Systems

Dissertation zur Erlangung des Doktorgrades

der Fakultät für Mathematik

der Universität Bielefeld

vorgelegt von

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im Juli 2014

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Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706

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Chapter 1

Introduction

For a countable set V, a *random field* on V is a collection of random variables, called *spins*, indexed by the elements $\ell \in \mathsf{V}$. These variables are defined on some probability space and take values in the corresponding *single-spin* spaces Ξ_ℓ. Typically, it is assumed that each Ξ_ℓ is a copy of a Polish space Ξ. In a canonical version, the underlying probability space is $(\Xi^{\mathsf{V}}, \mathcal{B}(\Xi^{\mathsf{V}}), \mu)$, where μ is a probability measure on the Borel σ-field $\mathcal{B}(\Xi^{\mathsf{V}})$. Then μ itself is also called random field. A particular case of such a field is the infinite product measure of some single-spin probability measures σ_ℓ. A particular example is given by *Gibbs random fields.* Due to physical motivation, they are constructed as perturbations of the product measure $\otimes_{\ell\in\mathsf{V}}\sigma_\ell$ by the "densities"

$$\exp\left(-H_{\mathsf{L}}(x_{\mathsf{L}}|y)\right),$$

where $H_{\mathsf{L}}(x_{\mathsf{L}}|y)$ are the *local energies* of the corresponding subsystems in L, subject to some boundary condition y. In other words, these are probability measures on the space $X \ni x$ of infinite volume configurations, which have prescribed conditional probabilities $\mu_{\mathsf{L}}(dx|y)$ with respect to the boundary conditions y fixed outside finite regions. This was the fundamental idea behind the pioneering works of R. Dobrushin ([Dob68], [Dob70]) and O. E. Lanford and D. Ruelle ([LR69], [Rue69]) dated back to 1968-1970.

Gibbs random fields are a tool for modelling the equilibrium states of a system in the presence of interaction between particles. For bounded interaction, the Gibbs measures usually exist. Moreover, there is only one such measure if the interaction small enough and the underlying graph is more-or-less regular. The case of a special interest is where the potentials describing the interaction are unbounded. Then both existence and uniqueness issues turn into serious problems of the theory. Starting from the first successful attempts to construct Gibbs fields with *unbounded spins* [LP76], steps towards elaborating tools for proving their uniqueness were undertaken in [COPP78], [DP83],[MM91], [MN84], [PY95]. However, except for a technique elaborated in [MN84], applicable to potentials and single-spin measures of a special type, and also for methods applicable to 'attractive' potentials, see [AKRP97], [KP07], [Pas], [Roy77], [Yos99] and [Zeg90], there is only one work presenting a kind of general approach to this problem. This work is due to R. L. Dobrushin and E. A. Pechersky [DP83], which was first published in Russian and later translated to English. Since that time, it was cited only few times, cf. [CM12], presumably for the following reasons: (a) the English translation in [DP83] was made with numerous typos and errors, whereas the Russian version was inaccessible for the most of the readers; (b) most of the proofs in [DP83] are very involved and intricate, and essential parts of them are only sketched. In Chapter 2, we present a refined and complete description of the Dobrushin-Pechersky method extended in the following directions: (a) instead of the cubic lattice $\mathbb{Z}^d$ we consider general graphs as underlying sets of the Gibbs fields, the only restriction imposed being a uniform bound on the degree of the graph; (b) we do not employ the compactness arguments crucially used in [DP83]. Due to the latter fact, one can consider singular interaction and the single-spin spaces Ξ being just standard Borel spaces, e.g., infinite dimensional spaces which are not locally compact, see [KP07],[Pas].

The main technical results of Chapter 2 thoroughly describe the reconstruction procedure introduced in [DP83] (see Section 2.2). Moreover, we show that applying the same type of procedure, this time in finite volumes (Section 2.2.3), yields a result for the exponential decay of spatial correlations for the Gibbs measures under consideration (Theorem 2.19). In Appendix 2.B, we briefly discuss the existence of random fields consistent with a specification that satisfies the Dobrushin-Pechersky conditions.

After establishing the ground theoretical results of the thesis, our aim will be to see how they can be applied to several models. In Chapter 3, we start (in the historically correct order) with a *system of classical anharmonic oscillators*, described by the formal potential energy functional

$$H(x) := \sum_{\ell} V_\ell(x_\ell) + \sum_{\{\ell,\ell'\}} W_{\ell\ell'}(x_\ell, x_{\ell'}), \tag{1.1}$$

where the sums run through the lattice $\mathbb{Z}^d$. The potentials V_ℓ and $W_{\ell,\ell'}$ are supposed to obey certain uniform bounds responsible for the stability of the entire system. For fixed inverse temperature $\beta > 0$, the associated Gibbs states

$$\mu(dx) := \frac{1}{Z_\beta} \exp\{-\beta H(x)\} \times_{\ell\in\mathbb{Z}^d} dx_\ell$$

are rigorously defined as those measures on the configuration space $X := (\mathbb{R}^n)^{\mathbb{Z}^d}$, which satisfy the system of DLR equations

$$\mu\pi_{\mathsf{L}} = \mu,$$

indexed by bounded domains $\mathsf{L} \Subset \mathbb{Z}^d$.

The corresponding Gibbs specification $\Pi = \{\pi_{\mathsf{L}}(dx|y), y \in X, \mathsf{L} \Subset \mathbb{Z}^d\}$ is constructed by the means of the local Hamiltonians $H_{\mathsf{L}}(x|y)$. In this setting, we are able to show that the one-point specification corresponding to Π satisfies the revised Dobrushin-Pechersky conditions, hence a uniqueness result can be established even in the case of super-quadratic interactions in the high-temperature, but also in the low-temperature regime. The contents of this chapter is based on some reviewed and essentially improved results from Section 2.3 of [Pas]. A main new issue is that we give computable bounds on the critical parameter and prove the decay of correlations in this type of systems.

In Chapter 4, the uniqueness problem for Gibbs measures corresponding to particle systems in the continuum (e.g. in $\mathbb{R}^d$) is considered. The equilibrium states of classical free gases are modelled by Poisson measures (Poisson point processes) on the configuration space. The states of interacting gases can be defined as Gibbs measures, which are "singular perturbations" of Poisson measures in the framework of the DLR formalism. The main approaches used in the study of equilibrium states of such systems are via Ruelle's superstability estimates ([Kun99], [Rue70], [Rue69]) and via Dobrushin's method ([BP02], [PZ99]). For the reader's convenience, we first present the standard case of a (non-translation invariant, possibly discontinuous) pair interaction $V(x, y)$ assigned to particles in the Euclidian space $\mathbb{R}^d$, $d \geq 1$, for which the existence and uniqueness of Gibbs measures were already studied in [PZ99], [KPR12] and [PZ99], respectively. In this case, the Gibbs states are obtained as perturbations of the Poisson measure on the configuration space $\Gamma(\mathbb{R}^d)$. Here, our aim is not to prove the best possible results, but to illustrate a short analytical proof of the uniqueness based on our criterion. The uniqueness result proved in [PZ99] has an complex combinatorial proof, which requires the use of multiple configurations (i.e., at a point $x \in \mathbb{R}^d$ there can be more than one particles). Such an approach is, however, not physically meaningful and we are able to show, by using the properties of the Lebesgue-Poisson measure, that it is also not necessary. To prove both the existence and uniqueness results, we principally use the exponential integrability of a certain Lyapunov functional, given by the energy $H(\gamma_{Q_k})$ of a configuration γ restricted to a small cube Q_k (cf. Lemma 4.4). Such type of result was established in [KPR12] and is actually the key-point in proving both existence and uniqueness.

Next, we consider systems with strong superstable interactions, i.e. for which

$$H(\gamma) \;\geq\; D \sum_{k\in\mathbb{Z}^d} |\gamma_k|^P - E|\gamma| \quad \text{for all } \gamma \in \mathit{\Gamma}_0, \tag{1.2}$$

where

$$H(\gamma) := \sum_{\{x,y\}\subset\gamma} V(x,y), \quad \text{for any finite configuration } \gamma,$$

hence eliminating any particular assumptions on the interaction potentials. For this types of systems, existence and a-priori bounds for Gibbs measures were established in Section 4.2 of [KPR12]. We are able to prove a uniqueness result due to small chemical activity (cf. Theorem 4.14).

In addition, we also consider a special type of multi-body interaction, the Lebowitz-Mazel-Presutti model, first introduced in [LMP98] and more thoroughly studied in [LMP99] and later in [Pre09]. This model is characterized by a competition between an attractive pair and repulsive four-body potential. It has the following type of Hamiltonian

$$H_\epsilon(\gamma) := -\sum_{\{x_1,x_2\}\subset\gamma} V_\varepsilon^{(2)}(x_1,x_2) + \sum_{\{x_1,x_2,x_3,x_4\}\subset\gamma} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4), \tag{1.3}$$

where both $V_\varepsilon^{(2)}$ and $V_\varepsilon^{(4)}$ are positive. We are able to prove existence of Gibbs measures corresponding to the Hamiltonian given by (1.3). In [LMP99] this model is used to prove a type of liquid-vapor phase transition, which is the only result of such type known so far for particle systems in the continuum. The natural question remaining, is whether under a different choice of system parameters uniqueness of the equilibrium state can be established. The answer to this question will be given in Section 4.4.

The aim of Chapter 5 is to study Gibbs measures (= states of thermal equilibrium) of the so-called amorphous (liquid) crystals, incorporating features both of the unbounded spin systems on graphs (see Chapters 2 and 3) and the classical particle systems in the continuum (see Chapter 4). The model under interest can be described as follows:

Let us consider a countable collection $\gamma \in \Gamma(X)$ of identical point particles chaotically distributed over a Euclidean space X (e.g. $\mathbb{R}^d$), which is modelled by the Poisson process $\pi_z(d\gamma)$ on $\Gamma(X)$. Additionally, we assume that each particle $x \in \gamma$ possesses an internal structure described by a mark (spin) σ_x taking values in a single-spin space S (e.g. $\mathbb{R}^m$) and characterized by a single-spin measure $g(d\sigma_x)$. Each two particles $x, y \in \gamma$ interact via a pair potential given by the sum of two components:

(i) a purely positional (e.g., distance dependent, possibly singular or hard-core) background potential (representing a molecular force)

$$\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}, \quad \Phi(x, y) = \Phi(y, x), \quad x, y \in X;$$

(ii) a spin-spin interaction of the form $\hat{W}((x, \sigma_x), (y, \sigma_y)) := J(x, y)W(\sigma_x, \sigma_y)$, where

$$J : X \times X \to \mathbb{R} \text{ and } W : S \times S \to \mathbb{R}$$

are both symmetric functions.

For technical simplicity we suppose that the interaction has a finite range, i.e., there exists $R > 0$ such that $\Phi(x, y) = 0$ and $J(x, y) = 0$ if $|x - y| > R$.

The whole system is then governed by the heuristic Hamiltonian

$$H(\hat{\gamma}) := \sum_{\{x,y\} \subset \gamma} \Phi(x, y) + \sum_{\{x,y\} \subset \gamma} J(x, y)W(\sigma_x, \sigma_y)$$

on the phase space $\hat{\Gamma}(X) := \Gamma(X, S)$ consisting of marked configurations $\hat{\gamma} = \{(x, \sigma_x)\}$. Given an inverse temperature $\beta > 0$, the corresponding Gibbs states are probability distributions on $\hat{\Gamma}(X)$ having a formal presentation

$$\nu(d\hat{\gamma}) = \frac{1}{Z} \exp\{-\beta H(\hat{\gamma})\} \lambda_z(d\hat{\gamma}),$$

where $\pi_z(d\hat{\gamma})$ is the Poisson point process ("free state") with intensity measure $zdx \otimes g(d\sigma_x)$ on $X \times S$. A rigorous definition to such Gibbs states constituting the set $\mathcal{G}(\hat{X})$ will be given through the standard Dobrushin-Lanford-Ruelle (DLR) approach in Section 5.2.1.

In the particular case of one-dimensional spins ($S = \mathbb{R}^1$, $m = 1$) and “ferromagnetic” spin couplings

$$J(x, y) \leq 0 \quad \text{and} \quad W(\sigma_x, \sigma_y) := \sigma_x \sigma_y,$$

the above model is well known in mathematical physics as a ferromagnetic fluid (see [GG86], [RZ98], [GZ98], [GTZ02]). The importance of this continuum fluid model is related with the phenomenon of the orientational ordering phase transition occurring in it for large chemical potentials ($z \gg 1$) and low temperatures ($\beta \gg 1$); see e.g. Proposition 6.1 in [RZ98]. Such type of phase transitions is typical in lattice ferromagnets. Of the major interest in critical behaviour of continuum models is, however, the positional ordering that relates to a liquid-vapor transition and involves positions of the particles rather than orientation of their spins ([Pre09]). However, it is believed that there is a direct interplay between the positional and the orientational structure of the above system, in so far the ferromagnetic ordering can lead to a strengthening of the indirect attractive forces between the particles and, hence, to a jump increase of the particle density (see a discussion in the Introduction of [GZ98]).

Finally, we note that in [GG86], [RZ98], [GZ98] and [GTZ02] only the case of discrete or bounded spins, attractive spin-spin forces and hard-core pure positional potentials were considered. A general theory of Gibbs measures with the Ruelle-type (super-) stable interactions on marked configuration spaces can be found e.g. in [Kun99], [AKLU00], [KKdS98] and [Mas00], however, it is essentially restricted to bounded spins again and hence does not apply to our model (see Remark 5.17). The case of unbounded spins and non-attractive interactions, including the existence and uniqueness problems for the associated Gibbs states, has not been treated so far in the literature. This is our main objective in Chapter 5. So, under reasonable stability assumptions on the interaction potentials Φ and W, we will prove that the set $\mathcal{G}^t$ of tempered Gibbs measures is not empty (Theorem 5.16) and, moreover, that $\mathcal{G}^t$ is a singleton provided the couplings $J(x, y)$ and the particle density z are small enough (Theorem 5.22). To this end, we will refer to the general results of Chapter 2 and adapt them to the framework of marked configuration spaces. A crucial moment here is the proper choice of the Lyapunov functional $F : \Gamma_0(X, S) \to \mathbb{R}$, defined by $F(\hat{\gamma}) = |\gamma|^p + \sum_{x\in\gamma} |\sigma_x|^q$, where $\hat{\gamma} = (\gamma, \sigma)$, which allows us to control the interaction growth and to check the conditions of Dobrushin-Pechersky theorem. As a by-product of our method we also get a decay of correlations for the (unique) Gibbs measure (Corrolary 5.25), which seems to be entirely new for such systems.

We extend the setting of Chapter 5, by considering systems of particles lying on the cone of discrete measures

$$\mathbb{K}(\mathbb{R}^d) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \middle| s_i \in \mathbb{R}_+^*,\ x_i \in \mathbb{R}^d \right\}.$$

This setting can be used to model complex systems with a non-trivial internal structure of their elements (e.g. ecological systems in the presence of biological diversity) and will be the object of study in Chapter 6. This situation appears to be somehow new in the literature. Such systems were considered recently in [Hag11], [HKPR13] and [HKLV]. In these papers, the role of equilibrium states is attributed to Gamma processes on the corresponding location spaces. We are able to extend the framework to what we will call generalized Lévy processes. To each particle $x \in \mathbb{R}^d$, we attach a positive characteristic (mark) s_x such that (s_x, x) is distributed according to some *generalized Lévy intensity measure* $\tau(ds, dx)$ on $(0, \infty) \times \mathbb{R}^d$ (see Definition 6.2). In this sense, we obtain an extension of some results concerning existence of Gibbs measures from [Hag11] and [HKPR13], where the case $\tau(ds, dx) = \lambda(ds)m(dx)$ was considered, for $m(dx)$ the Lebesgue measure on $\mathbb{R}^d$ and $\lambda(ds) = e^{-s}/sds$ the Gamma measure on $\mathbb{R}_+^* = (0, \infty)$.

The interaction of the system will be described via a bounded pair potential ϕ, in terms of the relative energy

$$H_{\mathsf{U}}(\eta\xi) := \int_{\mathsf{U}}\int_{\mathsf{U}} \phi(x,y)\eta(dx)\eta(dy) + 2\int_{\mathsf{U}^c}\int_{\mathsf{U}} \phi(x,y)\eta(dx)\xi(dy),$$

for η, ξ belonging to the cone of discrete measures $\mathbb{K}(\mathbb{R}^d)$ and for a finite volume $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$.

Two essential cases will be considered. First, for a spatially bounded Lévy intensity measure $\tau(ds, dx)$, i.e. for which

$$\int s^i \tau(ds, Q_k) \le M < \infty, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d, \tag{1.4}$$

we are able to prove the existence (cf. Theorem 6.31) and also uniqueness due to small interaction or first spatial moment of τ (cf. Theorem. 6.38).

Secondly, in a special case of unbounded Lévy intensity measure $\tau(ds, dx)$, where

$$\int s^i \tau(ds, Q_k) \le C_i e^{a_i|k|}, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d, \tag{1.5}$$

an existence result (Theorem 6.44) for the equilibrium states can be established.

Acknowledgements

This thesis would have not been ready without the support of many people who have helped me directly, as well as indirectly. I would like to convey my gratitude to some of them for their efforts and best wishes.

First of all, I would like to express my gratitude to Prof. Dr. Yuri G. Kondratiev, not only for the beautiful mathematical problems he suggested, but also for being an excellent teacher. A special "thank you" goes to Dr. Tetyana Pasurek for her kindness, her patience and her willingness to help me during the most difficult of moments. I was taught and encouraged by her every step of the way and for this I will always be grateful. I am also indebted to Prof. Dr. Juri Kozitski and Dr. Alexei Daletskii for fruitful discussions. Their experience has been invaluable in the process of writing this thesis.

I would like to thank Prof. Dr. Philippe Blanchard, Prof. Dr. Moritz Kassmann, Prof. Dr. Michael Röckner and Prof. Ludwig Streit and all the visiting professors of the IGK for very beautiful and insightful courses, which have helped me broaden my mathematical knowledge.

I am also grateful to Prof. Dr. Lucian Beznea, for his permanent support before, as well as during my time in Bielefeld.

I would like to say "Grazie!" to Prof. Errico Presutti for accepting to be the referee of this thesis.

Moreover, I deeply appreciate all the financial support by the IGK, German Science Foundation (DFG), Graduate University of Chinese Academy of Sciences (GUCAS) and Collaborative Research Center 701 "Spectral Structures and Topological Methods in Mathematics", as well as the support of the Chinese part of the IGK during our stays in China and not in the least, the technical support of Hannelore Litschewsky, Rebecca Reischuk, Karin Zelmer and Stephan Merkes.

Special thanks go to my colleagues and friends in Bielefeld, who have helped me, encouraged me, listened to me and created an enjoyable working atmosphere. Thank you Andrea, Ante, Diana, Geghard, Jesper, Marina, Maya, Milan, Mykola, Thomas, Tolu, Viktor, Yuhua and Zvonimir. Also, thanks to all my friends from the university's Hochschulsport for permanently cheering me up and encouraging me.

Of course, any of this would have not been possible without my parents. They offered me unconditional love and support and they have taught me to enjoy hard work and the accomplishments thereof. They are my role models.

Last, but definitely not least, a special thanks to Stefan, for being patient, understanding and for being there for me at all times, despite the distance. I have no words to express the gratitude for the love and encouragement that I received from you.

Chapter 2

General Theory: Uniqueness Problem for Gibbs Measures

The focus of this chapter will be to present a general uniqueness criterion for Gibbs measures with non-compact spins. The main result is a refinement of Theorem 1 in [DP83], obtained by Dobrushin and Pechersky in 1982. The major improvements, as compared to the above mentioned paper, are as follows:

(1) instead of the lattice $\mathbb{Z}^d$ we consider general graphs as underlying sets for the Gibbs measures;

(2) we essentially simplify the original proof of Dobrushin and Pechersky, especially dropping the compactness arguments crucially used in [DP83];

(3) we give computable bounds on the critical parameters, whose values imply the uniqueness in question;

(4) we show that the Dobrushin-Pechersky conditions imply the exponential decay of spatial correlations for the Gibbs measures as well.

2.1 Formulation of the Uniqueness Problem

We proceed by presenting some general facts on graph theory and random fields on graphs, by introducing specifications and measures consistent with them and then by describing the main result of this chapter.

2.1.1 Basic notions in Graph Theory

For the convenience of the reader, we briefly recall some notions of Graph Theory that are used throughout this chapter.

Definition 2.1. [Die10]

(i) A *graph* is a pair $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ of sets such that $\mathsf{E} \subseteq \mathsf{V}^{(2)}$, where $\mathsf{V}^{(2)}$ is the set of unordered pairs of distinct elements V[1]. E is called the set of *edges* of the graph, while V stands for the set of its *vertices*. The notation $\ell' \sim \ell$ means that $(\ell, \ell') \in \mathsf{E}$. Such vertices are called *adjacent*.

(ii) The set of neighbours, i.e. of adjacent vertices, of a vertex ℓ is denoted by $\partial_{\mathsf{G}}\ell$, or briefly by $\partial\ell$.[2] More generally for $\mathsf{L} \subseteq \mathsf{V}$, the neighbours of vertices from L lying in the complement $\mathsf{L}^c := \mathsf{V} \setminus \mathsf{L}$ are called neighbours of L; their set is denoted by $\partial\mathsf{L}$. The *degree* $d(\ell)$ of a vertex $\ell \in \mathsf{V}$ is the number of edges having ℓ as an endpoint, i.e. the cardinality of the set $\partial\ell$. The number $\delta = \delta_{\mathsf{G}} := \inf\{d(\ell) : \ell \in \mathsf{V}\} \geq 0$ is the *minimum degree* of the graph. Analogously one defines $\Delta := \sup\{d(\ell) : \ell \in \mathsf{V}\}$ to be the *maximum degree* of G.

(iii) A sequence $\vartheta = \{\ell_0, \ell_1, \ldots, \ell_N\}$ such that $\ell_k \sim \ell_{k+1}$ and $\ell_k \neq \ell_j$, when $k \neq j$ for all $k, j = 0, 1, \ldots N-1$ is called a N*-path*. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G.

(iv) A given $\mathsf{L} \subset \mathsf{V}$ is said to be an *independent set* of vertices if

$$\forall \ell \in \mathsf{L} \qquad \partial\ell \subset (\mathsf{L}^c). \tag{2.1}$$

The *chromatic number* $\chi \in \mathbb{N}$ of G is the smallest number such that

$$\mathsf{V} = \bigsqcup_{j=0}^{\chi-1} \mathsf{V}_j, \qquad \mathsf{V}_j - \text{ independent, } j = 0, \ldots, \chi - 1. \tag{2.2}$$

In the following, we consider graphs, the edges of which represent the interaction between particles located at the vertices of the graph. Therefore, we deal with nearest-neighbour interaction. Since our method essentially uses the fact that neighbouring vertices belong to different "classes", we will *partition* the set of vertices into disjoint independent sets. We remark that for any graph with non-empty E, the following holds true

$$2 \leq \chi \leq \Delta + 1.$$

[1]Sometimes in the literature, this notion of a graph may be described as undirected and simple.

[2]Here, as elsewhere, we drop the index referring to the underlying graph if the reference is clear.

2.1.2 Random fields on graphs

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be an infinite countable connected graph, which we set to be the index space throughout this chapter. In what follows we will consider only graphs of bounded degree, i.e. for which $\Delta < \infty$ and hence $\chi < \infty$. Since we presume that the graph is connected, we know a-priori that $\Delta \geq 2$. Also, by Brook's theorem (see e.g. [Lov75]) for our graph we have $\chi \leq \Delta$.

Let $(\Xi, \mathcal{E})$ be a standard Borel space, which will represent the single spin space. The infinite product space $X := \Xi^{\mathsf{V}}$ will be the configuration space for our particle system. Let $\mathcal{F} := \otimes_{\ell \in \mathsf{V}} \mathcal{E}$ be the infinite product algebra. By Georgii [Geo88], Remark (4.A3), one knows that $(X, \mathcal{F})$ is again a standard Borel space. A configuration from X will be denoted by $x = (x_\ell)_{\ell \in \mathsf{V}}$. By writing $\mathsf{L} \Subset \mathsf{V}$ we mean that L is a non-empty finite subset of V. Likewise, $x_{\mathsf{L}} = (x_\ell)_{\ell \in \mathsf{L}}$ is an element of Ξ^{L}. Also, for convenience, when there is no risk for confusion, we will omit the parentheses from $\{\cdot\}$, e.g. sometimes it is more convenient to write ℓ instead of $\{\ell\}$. A related notion is the one of *local events*, described by the algebra $\mathcal{F}_{loc} := \bigcup_{\mathsf{L} \Subset \mathsf{V}} \mathcal{E}^{\mathsf{L}}$, where $\mathcal{E}^{\mathsf{L}}$ is the product σ-algebra on Ξ^{L}. Other notation will be

$$\mathcal{E}_{\mathsf{L}} := \mathcal{E}^{\mathsf{L}^c},$$

$$\mathcal{E}_\ell := \mathcal{E}^{\mathsf{V} \setminus \{\ell\}},$$

$$y_{\mathsf{L}} \times \tilde{y}_{\mathsf{L}^c} =: x \in X \text{ such that } x_{\mathsf{L}} = y_{\mathsf{L}} \text{ and } x_{\mathsf{L}^c} = \tilde{y}_{\mathsf{L}^c}.$$

A function $f : X \to \mathbb{R}$ is said to be *local* if it is $\mathcal{E}^{\mathsf{L}}/\mathcal{B}(\mathbb{R})-$measurable for some $\mathsf{L} \Subset \mathsf{V}$. By $\mathcal{BF}_{loc}$ we denote the set of all bounded local functions. Consider also the set $\mathcal{P}(X)$ of all probability measures on $(X, \mathcal{F})$. For our purposes, we introduce the following topology on $\mathcal{P}(X)$.

Definition 2.2. The local setwise topology $\mathcal{T}_{loc}$ is the weakest topology on $\mathcal{P}(X)$ for which the evaluation maps $\mathcal{P}(X) \ni \mu \mapsto \mu(A)$, $A \in \mathcal{F}_{loc}$, are continuous. A net $\{\mu_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$ is convergent to a $\mu \in \mathcal{P}(X)$ if $\mu_\alpha(A) \to \mu(A)$ for all $A \in \mathcal{F}_{loc}$ or, equivalently if $\mu_\alpha(f) \to \mu(f)$ for all $A \in \mathcal{BF}_{loc}$ where

$$\mu(f) := \int_X f(x)\mu(dx).$$

Remark 2.3. We notice that the topology of local convergence is Hausdorff, but not metrizable for non-compact Ξ, according to Remark (4.3) in [Geo88].

Let us denote by $\mathcal{C}(\mu_1, \mu_2)$ the set of couplings of the measures $\mu_1, \mu_2 \in \mathcal{P}(X)$, i.e. the set of measures $\nu \in \mathcal{P}(X^2)$ whose projections are μ_1 and μ_2, respectively. The proof of the following result is rather obvious and hence omitted.

Lemma 2.4. *Given $\mu_1, \mu_2 \in \mathcal{P}(X)$, let $\{\nu_\alpha\}_{\alpha \in I} \in \mathcal{C}(\mu_1, \mu_2)$ be locally convergent to a $\nu \in \mathcal{P}(X^2)$. Then $\nu \in \mathcal{C}(\mu_1, \mu_2)$.*

2.1.3 Specifications and their corresponding Gibbs states

In Statistical Physics, one describes a system's state by a probability measure on the configuration space X defined above. Of course, this probability measure should be consistent with the available partial knowledge of the system, which is given by the the so-called *specification* with parameter set V and state space $(\Xi, \mathcal{E})$. Before introducing this concept, let us first give some preliminary definitions (for more details, see e.g. the monographs [Geo88] and [Pre76]). Let $(Y, \mathcal{Y})$ be a measurable space.

Definition 2.5. A function $\pi : \mathcal{F} \times Y \to [0, 1]$ is called a *probability kernel* (or stochastic kernel) from $\mathcal{Y}$ to $\mathcal{F}$ if

(i) $\pi(\cdot|y)$ is a measure on $(X, \mathcal{F})$ for all $y \in Y$,

(ii) $\pi(A|\cdot)$ is $\mathcal{Y}$-measurable for each $A \in \mathcal{F}$ and

(iii) $\pi(X|\cdot) = 1$.

A probability kernel π maps each probability measure μ on $(Y, \mathcal{Y})$ to a probability measure $\mu\pi$ on $(X, \mathcal{F})$, which is defined by

$$\mu\pi(A) = \int \pi(A|\cdot) d\mu, \; A \in \mathcal{F}.$$

Also, for each bounded measurable function $f : X \to \mathbb{R}$ we can consider the measurable function $\pi f : Y \to \mathbb{R}$,

$$(\pi f)(y) := \pi(f|y) = \int f(x)\pi(dx|y), \text{ for any } y \in Y.$$

Now, let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. A probability kernel from $\mathcal{B}$ to $\mathcal{F}$ is said to be *proper* if

$$\pi(B|\cdot) = \mathbb{1}_B, \; B \in \mathcal{B}.$$

Definition 2.6. $\Pi = (\pi_{\mathsf{L}})_{\mathsf{L} \Subset \mathsf{V}}$ is said to be a *specification* if it is a family of proper probability kernels π_{L} from $\mathcal{E}_{\mathsf{L}}$ to $\mathcal{F}$ satisfying the following consistency condition

$$\int_X \pi_{\mathsf{L}}(B|x)\pi_{\mathsf{L}'}(dx|y) = \pi_{\mathsf{L}'}(B|y), \; \mathsf{L} \subset \mathsf{L}' \Subset \mathsf{V}, \tag{2.3}$$

holding for all $B \in \mathcal{F}$ and $y \in X$.

The set of all probability measures consistent with the specification Π, called *Gibbs measures*, will be denoted by $\mathcal{G}(\Pi)$. These are measures $\mu \in \mathcal{P}(X)$ that satisfy the DLR (Dobrushin-Lanford-Ruelle) equations

$$\mu\pi_{\mathsf{L}}(A) = \mu(A), \text{ for any } \mathsf{L} \Subset \mathsf{V} \text{ and } A \in \mathcal{F}. \tag{2.4}$$

Definition 2.7. For $\ell \in \mathsf{V}$ and $x \in X$, let $\pi_\ell^x \in \mathcal{P}(\Xi)$ be such that the map $X \ni x \mapsto \pi_\ell^x(A)$ is $\mathcal{E}_\ell$-measurable, for each $A \in \mathcal{E}$. We say that $\pi = (\pi_\ell^x)_{\ell,x}$ is a *family of one-point local states* (or a one-site specification, for short).

A measure $\mu \in \mathcal{P}(X)$ is consistent with the family of one-point local states π if it satisfies the equation

$$\mu(A) = \int_X \left(\int_\Xi \mathbb{1}_A(z_\ell \times x_{\mathsf{V}\setminus\{\ell\}}) \pi_\ell^x(dz_\ell) \right) \mu(dx), \tag{2.5}$$

for every $\ell \in \mathsf{V}$ and $A \in \mathcal{F}$.

We denote by $\mathcal{M}(\pi)$ the class of all such μ. Obviously each specification $\Pi = (\pi_{\mathsf{L}})_{\mathsf{L}\Subset\mathsf{V}}$ determines a family of one point local states π, by

$$\pi_\ell^x := \pi_{\{\ell\}}(\cdot|x) \circ \mathbb{P}_\ell^{-1} \tag{2.6}$$

for $\ell \in \mathsf{V}$ and $x \in X$. Here, $\mathbb{P}_\ell : X \to \Xi$ is the projection on the ℓ-th component, i.e. $X \ni x \mapsto \mathbb{P}_\ell x := x_\ell$. Obviously, each $\mu \in \mathcal{G}(\Pi)$ belongs to $\mathcal{M}(\pi)$.

In the following section, we show that uniqueness of random fields consistent with a given specification Π (provided such random fields exist) can be established by verifying conditions only on the family of one-point local states π, defined as in equation (2.6).

2.1.4 Dobrushin-Pechersky conditions

For intuitive reasons, $\mathcal{M}(\pi)$ should be a singleton whenever π_ℓ^x depends only "weakly" on the boundary condition x. In order to quantify this dependency, we first introduce a distance between probability measures on the state space Ξ. For $\pi^1, \pi^2 \in \mathcal{P}(\Xi)$ define

$$d(\pi^1, \pi^2) := \inf_{\rho \in \mathcal{C}(\pi^1,\pi^2)} \int_{\Xi^2} \mathbb{1}_{\neq}(\xi,\eta)\rho(d\xi, d\eta),$$

where $\mathbb{1}_{\neq}(\cdot,\cdot)$ is just the discrete metric on Ξ, i.e. $\mathbb{1}_{\neq}(\xi,\eta) = 1$ for $\xi \neq \eta$ and 0 otherwise. Propositions 4.2 and 4.4 in [GHM01] yield that d is actually the *total* *variation distance*. For more details on the total variation distance, see Section 2.A.

Lemma 2.8. *There exists a (not necessarily unique) coupling $\rho_\ell^{x,y}$ such that*

$$d(\pi_\ell^x, \pi_\ell^y) = \int_{\Xi^2} \mathbb{1}_{\neq}(\xi, \eta) \rho_\ell^{x,y}(d\xi, d\eta) \tag{2.7}$$

and $(x, y) \mapsto \rho_l^{x,y}(\varphi)$ is measurable for any bounded measurable $\varphi \in L_\infty(\Xi^2)$.

A proof of this lemma will be given in Section 2.A.

In what follows, we make precise the conditions to be imposed on the family of one point local states π in order to obtain the uniqueness of the random field consistent with it.

Let $h : \Xi \to \mathbb{R}_+ := [0, +\infty)$ be a measurable function, K be a positive real constant and $c = (c_{\ell\ell'})_{\ell,\ell' \in \mathsf{V}}$, $\kappa = (\kappa_{\ell\ell'})_{\ell,\ell' \in \mathsf{V}}$ be matrices with positive entries and null diagonal such that

$$\bar{c} := \sup_{\ell \in \mathsf{V}} \sum_{\ell' \in \partial\ell} c_{\ell\ell'} < 1/\varDelta^{\chi} \tag{2.8}$$

and

$$\bar{\kappa} := \sup_{\ell \in \mathsf{V}} \sum_{\ell' \in \partial\ell} \kappa_{\ell\ell'} < 1. \tag{2.9}$$

Then we denote by $\Pi(h, K, \kappa, c)$ the class of one-site specifications π obeying the following two conditions:

(CC) Contraction condition

$$d(\pi_\ell^x, \pi_\ell^y) \leq \sum_{\ell' \in \partial\ell} \kappa_{\ell\ell'} \mathbb{1}_{\neq}(x_{\ell'}, y_{\ell'}) \tag{2.10}$$

holds for all $x, y \in X_\ell(h, K)$, where

$$X_\ell(h, K) = \{x \in X : h(x_{\ell'}) \leq K \text{ for all } \ell' \in \partial\ell\}. \tag{2.11}$$

(IC) Integrability condition

$$\pi_\ell^x(h) \leq 1 + \sum_{\ell' \in \partial\ell} c_{\ell\ell'} h(x_{\ell'}) \tag{2.12}$$

is satisfied for all $\ell \in \mathsf{V}$ and $x \in X$.

The integrability condition (IC) does not a-priori guarantee that h is also integrable with respect to $\mu \in \mathcal{M}(\pi)$, hence we introduce $\mathcal{M}_h(\pi)$ as the set of measures $\mu \in \mathcal{M}(\pi)$ consistent with π for which the following bound holds

$$\sup_{l \in \mathsf{V}} \int_X h(x_\ell)\mu(dx) < \infty. \tag{2.13}$$

The main result of this chapter follows. Set

$$K_* = \max\left\{ \frac{4\varDelta^{\chi+1}}{\bar{c}(1-\bar{\kappa})}; \ \frac{2\varDelta^{\chi+1}(2\varDelta^{\chi-1}+1)}{(1-\bar{\kappa})^2(1-\bar{c}\varDelta^{\chi})} \right\}. \tag{2.14}$$

Theorem 2.9. *For each $K > K_*$ and each $\pi \in \Pi(h, K, \kappa, c)$, the set $\mathcal{M}_h(\pi)$ contains at most one element.*

Remark 2.10. (i) In the original paper of Dobrushin and Pechersky [DP83], in the (IC) condition it was required that function h be *compact*, i.e. its sub-level sets $L_K := \{\xi \in \Xi : h(\xi) \le K\}$ are compact in Ξ. Combined with the classical Dobrushin criterion (see Theorem 1 of [Dob70]), this condition guarantees the existence of *exactly one* Gibbs measure satisfying (2.13). However, in the proof of our uniqueness result such a restriction is not needed. Nevertheless, in applications, it makes sense to consider a function h growing sufficiently fast such that the sub-level sets L_K are bounded for any $K > 0$, which ensures that we have to verify the contraction condition (2.10) only for a "bounded" set of boundary conditions. Without such behaviour of h, the above uniqueness criterion has no advantage to the classical Dobrushin uniqueness result (given by Theorem 4 in [Dob70]), which requires the contraction condition to hold simultaneously for all boundary conditions and cannot be applied to the case of non-compact state space Ξ.

(ii) If we start from a specification $\Pi = (\pi_{\mathsf{L}})_{\mathsf{L} \Subset \mathsf{V}}$ and consider the family of one-point local states π determined by it, it is obvious that $\mathcal{M}_h(\pi) \supset \mathcal{G}_h(\Pi)$, where $\mathcal{G}_h(\Pi)$ is the set of measures $\mu \in \mathcal{G}(\Pi)$ which satisfy (2.13).

(iii) Notice that in [DP83], condition (IC) was stated as

$$\pi_\ell^x(h) \le C + \sum_{\ell' \in \partial\ell} c_{\ell\ell'} h(x_{\ell'}).$$

However, if such an inequality holds, by a rescaling argument we see that conditions (IC) and (CC) are satisfied for $\overline{h} := C^{-1}h$, the constant $\overline{K} := C^{-1}K$ and the same matrices c and κ.

(iv) It can be easily seen that if each π_ℓ^x were independent of x, the unique element of $\mathcal{M}(\pi)$ would be the product measure $\otimes_{l \in \mathsf{V}} \pi_\ell$.

2.2 Reconstruction procedure

We proceed by presenting the full proof of Theorem 2.9 in the form of subsequent lemmas, as follows.

Lemma 2.11. *Let $\mu_1, \mu_2 \in \mathcal{M}_h(\pi)$ and $\nu_* \in \mathcal{C}(\mu_1, \mu_2)$ such that*

$$\int_X \int_X \mathbb{1}_{\neq}(x_\ell, y_\ell)\nu_*(dx, dy) = 0, \qquad \forall l \in \mathsf{V}. \tag{2.15}$$

Then $\mu_1 = \mu_2$.

Proof. The set $\mathcal{F}_{loc}$ of local events constitutes a measure defining class, cf. Corollary (4.A13) in [Geo88]. Let $A \in \mathcal{F}_{loc}$, hence there exists $\mathsf{L} \Subset \mathsf{V}$ such that $A \in \mathcal{E}^{\mathsf{L}}$. For such a set A we have

$$|\mathbb{1}_A(x) - \mathbb{1}_A(y)| \leq \sum_{\ell \in \mathsf{L}} \mathbb{1}_{\neq}(x_\ell, y_\ell). \tag{2.16}$$

Thus

$$\begin{aligned} |\mu_1(A) - \mu_2(A)| &= \frac{1}{2}\left|\int_{X^2} [\mathbb{1}_A(x) - \mathbb{1}_A(y)]\nu_*(dx, dy)\right| \\ &= \frac{1}{2}\int_{X^2} |\mathbb{1}_A(x) - \mathbb{1}_A(y)|\nu_*(dx, dy) \\ &\leq \frac{1}{2}\sum_{\ell \in \mathsf{L}} \int_{X^2} \mathbb{1}_{\neq}(x_\ell, y_\ell)\nu_*(dx, dy) = 0, \end{aligned}$$

which completes the proof. □

In what follows, the main idea will be to show the existence of such a ν_* such that (2.15) holds. To this end we construct a sequence $\{\nu_n\}_{n\in\mathbb{N}_0} \subset \mathcal{C}(\mu_1, \mu_2)$ such that

$$\gamma(\nu_n) := \sup_{\ell \in \mathsf{L}} \int_{X^2} \mathbb{1}_{\neq}(x_\ell^1, x_\ell^2)\nu_n(dx^1, dx^2) \to 0, \qquad n \to +\infty. \tag{2.17}$$

The sequence will be constructed in a step-by-step procedure based on the so-called *reconstruction transformation* $R_\ell : \mathcal{P}(X^2) \to \mathcal{P}(X^2)$ given by the expression

$$(R_\ell \nu)(f) := \int_{X^2} \left(\int_{\Xi^2} f(\xi \times x_{\ell^c}, \eta \times y_{\ell^c}) \rho_\ell^{x,y}(d\xi, d\eta) \right) \nu(dx, dy), \tag{2.18}$$

where $\ell \in \mathsf{V}$ and $f : X^2 \to \mathbb{R}$ is any bounded measurable function. Here, $\rho_\ell^{x,y}$ is as in (2.7). From the above expression, it is easy to see that $R_\ell \nu$ is well-defined as a probability measure on X^2.

Lemma 2.12. *For each $\ell \in \mathsf{V}$, the mapping R_ℓ defined in (2.18) has the following properties:*

(a) If $\nu \in \mathcal{C}(\mu_1, \mu_2)$ for some $\mu_1, \mu_2 \in \mathcal{M}(\pi)$, then also $R_\ell \nu \in \mathcal{C}(\mu_1, \mu_2)$.

(b) If f is $\mathcal{B}_\ell(X^2)$-measurable and ν-integrable, then $(R_\ell \nu)(f) = \nu(f)$.

Proof. Consider an arbitrary set $A \in \mathcal{F}$. Then

$$(R_\ell \nu)(A \times X) = \int_X \int_\Xi \mathbb{1}_A(\xi \times x_{\ell^c}) \pi_\ell^x(d\xi) \mu_1(dx) = \mu_1(A),$$

where we have used that $\rho_\ell^{x,y}$ and ν are couplings of π_ℓ^x, π_ℓ^y and of μ_1, μ_2, respectively. Similarly $(R_\ell \nu)(X \times A) = \mu_2(A)$. Hence, (a) follows.

Claim (b) is immediate from the fact that f from (2.18) is independent of ξ and η and that $\rho_\ell^{x,y}$ is a probability measure.

□

We remark that in the original article of Dobrushin and Pechersky [Dob70] the explicit formula (2.18) for R_ℓ is not given, instead the reconstruction transformation is characterized just by property (b) in Lemma 2.12.

For a given $\ell \in \mathsf{V}$, set

$$Y_\ell := \{(x^1, x^2) \in X^2 : \mathbb{1}_{\neq}(x_\ell^1, x_\ell^2) \leq \sum_{\ell' \in \partial \ell} \mathbb{1}_{\neq}(x_{\ell'}^1, x_{\ell'}^2)\}. \tag{2.19}$$

Lemma 2.13. *For each $\nu \in \mathcal{P}(X^2)$ and $\ell \in \mathsf{V}$, we have that $(R_\ell \nu)(Y_\ell) = 1$.*

Proof. $(x^1, x^2) \in Y_\ell^c$ implies that $\mathbb{1}_{\neq}(x_\ell^1, x_\ell^2) = 1$ and $\mathbb{1}_{\neq}(x_{\ell'}^1, x_{\ell'}^2) = 0$ for all $\ell' \in \partial \ell$. This means that $x_\ell^1 \neq x_\ell^2$ and $x_{\ell'}^1 = x_{\ell'}^2$ for all $\ell' \in \partial \ell$. For such a pair (x^1, x^2), the definition of π implies $\pi_\ell^{x^1} = \pi_\ell^{x^2}$. Then, we have

$$0 = d(\pi_\ell^{x^1}, \pi_\ell^{x^2}) = \int_{\Xi^2} \mathbb{1}_{\neq}(\xi, \eta) \rho_\ell^{x^1, x^2}(d\xi, d\eta),$$

which, by (2.18) yields $(R_\ell \nu)(Y_\ell^c) = 0$.

□

In order to proceed, we introduce a collection of functions $I_\ell, H^i_\ell : X^2 \to \mathbb{R}_+$ indexed by $\ell \in \mathsf{V}$ and $i = 1, 2$, which will be defined as follows. For $\ell \in \mathsf{V}$ and $(x^1, x^2) \in X^2$, set

$$I_\ell(x^1, x^2) := \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell); \qquad H^i_\ell(x^1, x^2) := h(x^i_\ell),\ i = 1, 2. \tag{2.20}$$

From Lemma 2.12(b), we have that

$$(R_\ell \nu)(I_{\ell_1}) = \nu(I_{\ell_1}), \qquad (R_\ell \nu)(I_{\ell_1} H^i_{\ell_2}) = \nu(I_{\ell_1} H^i_{\ell_2}) \quad \text{for } \ell \notin \{\ell_1, \ell_2\}. \tag{2.21}$$

The following result is a more detailed version of Lemma 3 in [DP83].

Lemma 2.14. *For $\mu_1, \mu_2 \in \mathcal{M}_h(\pi)$ and $\nu \in \mathcal{C}(\mu_1, \mu_2)$, $\ell, \ell_1 \in \mathsf{V}$, with $\ell \neq \ell_1$ and $i = 1, 2$, the following estimates hold*

$$(R_\ell \nu)(I_\ell H^i_{\ell_1}) \leq \sum_{\ell_2 \in \partial \ell} \nu(I_{\ell_2} H^i_{\ell_1}), \tag{2.22}$$

$$(R_\ell \nu)(I_{\ell_1} H^i_\ell) \leq \nu(I_{\ell_1}) + \sum_{\ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H^i_{\ell_2}), \tag{2.23}$$

$$(R_\ell \nu)(I_\ell H^i_\ell) \leq \sum_{\ell_1 \in \partial \ell} \nu(I_{\ell_1}) + \sum_{\ell_1, \ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H^i_{\ell_2}), \tag{2.24}$$

$$(R_\ell \nu)(I_\ell) \leq \sum_{\ell' \in \partial \ell} \kappa_{\ell \ell'} \nu(I_{\ell'}) + K^{-1} \sum_{i=1,2} \sum_{\ell_1, \ell_2 \in \partial \ell} \nu(I_{\ell_1} H^i_{\ell_2}), \tag{2.25}$$

where $K, c_{\ell \ell'}$ and $\kappa_{\ell \ell'}$ are the same as in (2.12) and (2.10).

Proof. We give an analytic proof based on the explicit formula of the reconstruction mapping (2.18). One observes that (2.22) immediately follows by applying Lemma 2.13 and Lemma 2.12 (b). Indeed,

$$\begin{aligned}(R_\ell \nu)(I_\ell H^i_{\ell_1}) &= \int_{X^2} \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell) h(x^i_{\ell_1}) (R_\ell \nu)(dx^1, dx^2) \\ &\leq \sum_{\ell_2 \in \partial \ell} \int_{X^2} \mathbb{1}_{\neq}(x^1_{\ell_2}, x^2_{\ell_2}) h(x^i_{\ell_1}) (R_\ell \nu)(dx^1, dx^2) = \sum_{\ell_2 \in \partial l} \nu(I_{\ell_2} H^i_{\ell_1}).\end{aligned}$$

Now, let us prove (2.23). By (2.18) and the fact that $\rho^{x,y}_\ell \in \mathcal{C}(\pi^x_\ell, \pi^y_\ell)$, we have

$$
\begin{aligned}
(R_\ell \nu)(I_{\ell_1} H^i_\ell) &= \int_{X^2} \left(\int_\Xi h(\xi) \pi_\ell^{x^i}(d\xi) \right) \mathbb{1}_{\neq}(x^1_{\ell'}, x^2_{\ell'}) \nu(dx^1, dx^2) \\
&\leq \nu(I_{\ell_1}) + \sum_{\ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H^i_{\ell_2}),
\end{aligned}
$$

where we have used (2.12). To prove (2.24) we employ Lemma 2.13, by which we get

$$
(R_\ell \nu)(I_\ell H^i_\ell) \leq \sum_{\ell_1 \in \partial \ell} (R_\ell \nu)(I_{\ell_1} H^i_l) \leq \sum_{\ell_1 \in \partial \ell} \nu(I_{\ell_1}) + \sum_{\ell_1, \ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H^i_{\ell_2}),
$$

where the latter estimate follows from (2.23).

Let us prove (2.25). By (2.7) and (2.18), we have

$$
\begin{aligned}
(R_\ell \nu)(I_\ell) &= \int_{X^2} \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell)(R_\ell \nu)(dx^1, dx^2) \\
&= \int_{X^2} \mathbb{1}_{X_\ell}(x^1) \mathbb{1}_{X_l}(x^2) \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell)(R_\ell \nu)(dx^1, dx^2) \\
&+ \int_{X^2} [1 - \mathbb{1}_{X_\ell}(x^1) \mathbb{1}_{X_\ell}(x^2)] \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell)(R_\ell \nu)(dx^1, dx^2),
\end{aligned}
\tag{2.26}
$$

where $\mathbb{1}_{X_\ell}$ is the indicator of the set $X_\ell(h, K)$ defined in(2.11). By (2.10), we have

$$
\int_X \mathbb{1}_{X_\ell}(x^1) \mathbb{1}_{X_\ell}(x^2) \mathbb{1}_{\neq}(x^1_\ell, x^2_\ell)(R_\ell \nu)(dx^1, dx^2) \leq \sum_{l' \in \partial l} \kappa_{\ell \ell'} \nu(I_{\ell'}),
$$

which yields the first term of the right-hand side of (2.25). Using the elementary inequality $|1 - \prod_{i=1}^n a_i| \leq \sum_{i=1}^n |1 - a_i|$ for a collection of n real numbers $0 \leq a_i \leq 1$, $\forall 1 \leq i \leq n$, we have

$$
[1 - \mathbb{1}_{X_\ell}(x^1) \mathbb{1}_{X_\ell}(x^2)] \leq \sum_{i=1,2} \sum_{\ell_1 \in \partial \ell} [1 - \mathbb{1}_{h \leq K}(x^i_{\ell_1})] = \sum_{i=1,2} \sum_{\ell_1 \in \partial \ell} \mathbb{1}_{h > K}(x^i_{\ell_1}),
$$

where $\mathbb{1}_{h \leq K}$ and $\mathbb{1}_{h > K}$ are the indicator of the sets $\{\xi \in \Xi : h(\xi) \leq K\}$ and $\{\xi \in \Xi : h(\xi) > K\}$, respectively. Then the second term of the right-hand side of (2.26) cannot exceed the following

$$\sum_{i=1,2}\sum_{\ell_1\in\partial\ell}\int_{X^2}\mathbb{1}_{h>K}(x^i_{\ell_1})\mathbb{1}_{\neq}(x^1_\ell,x^2_\ell)(R_\ell\nu)(dx^1,dx^2)$$
$$\leq K^{-1}\sum_{i=1,2}\sum_{\ell_1\in\partial\ell}\int_{X^2}h(x^i_{\ell_1})\mathbb{1}_{\neq}(x^1_\ell,x^2_\ell)(R_\ell\nu)(dx^1,dx^2)$$
$$\leq K^{-1}\sum_{i=1,2}\sum_{\ell_1,\ell_2\in\partial\ell}\nu(I_{\ell_2}H^i_{\ell_1}).$$

Using (2.22) we get the latter line and hence, the desired result. □

For $\mu_1,\mu_2\in\mathcal{M}_h(\pi)$, let $\nu\in\mathcal{C}(\mu_1,\mu_2)$. As motivated by (2.11), we are interested in finding a coupling ν for which the quantity

$$\gamma(\nu):=\sup_{\ell\in\mathsf{V}}\nu(I_\ell) \tag{2.27}$$

vanishes. Nevertheless, we notice from inequalities (2.22)-(2.25) that along $\gamma(\nu)$, one also has to control

$$\lambda(\nu):=\max_{i=1,2}\sup_{\ell,\ell'\in\mathsf{V}}\nu(I_\ell H^i_{\ell'}), \tag{2.28}$$

which is finite, since $\mu_1,\mu_2\in\mathcal{M}_h(\pi)$. We take advantage of the fact that the estimates in (2.22)-(2.25) are *linear* and apply the reconstruction procedure on V in order to obtain the desired coupling ν_*. The main idea will be to apply to ν the reconstruction transformation R_ℓ for every site $\ell\in\mathsf{V}$, traversing the graph in a specific order, as detailed below.

2.2.1 Reconstruction in the case $\chi=2$

For the convenience of the reader, we consider first the case when G is a bipartite graph, i.e. $\chi=2$, as the proof of the result is more intuitive and less technical. We refer to Subsection 2.2.2 for the general case.

Lemma 2.15. *For $K>K_*$, take $\pi\in\Pi(h,K,\kappa,c)$ and $\mu_1,\mu_2\in\mathcal{M}_h(\pi)$. Then for each $\nu_0\in\mathcal{C}(\mu_1,\mu_2)$ there exists $\nu\in\mathcal{C}(\mu_1,\mu_2)$ for which the following estimates hold*

$$\gamma(\nu)\leq\bar{\kappa}\gamma(\nu_0)+2\Delta K^{-1}\lambda(\nu_0), \tag{2.29}$$

$$\lambda(\nu)\leq\Delta\gamma(\nu_0)+[\bar{c}\Delta+2\Delta^2K^{-1}]\lambda(\nu_0). \tag{2.30}$$

Proof. We consider the partition of V into the disjoint sets V_0 and V_1.

Reconstruction over V_0

Let $\{\ell_1, \ell_2, \ldots, \}$ be any numbering of the elements of V_0. Set

$$\mathsf{V}_0^{(n)} = \{\ell_1, \ldots, \ell_n\}, \qquad \nu_0^{(n)} = R_{\ell_n} R_{\ell_{n-1}} \cdots R_{\ell_1} \nu_0, \quad n \in \mathbb{N}. \tag{2.31}$$

Our first task is to estimate $\nu_0^{(n)}(I_\ell)$. By claim (b) of Lemma 2.11 we have that

$$\nu_0^{(n)}(I_\ell) = \nu_0(I_\ell) \qquad \text{for} \ \ \ell \notin \mathsf{V}_0^{(n)}. \tag{2.32}$$

For $k \le n$, by (2.1) and claim (b) of Lemma 2.11, and then by (2.25) and (2.32), we have

$$\begin{aligned} \nu_0^{(n)}(I_{\ell_k}) = \nu_0^{(k)}(I_{\ell_k}) \ &\le \ \sum_{\ell \in \partial \ell_k} \kappa_{\ell_k \ell} \nu_0(I_\ell) + K^{-1} \sum_{i=1,2} \sum_{\ell, \ell' \in \partial \ell_k} \nu_0(I_\ell H_{\ell'}^i) \\ &\le \ \bar{\kappa} \gamma(\nu_0) + 2\varDelta^2 K^{-1} \lambda(\nu_0). \end{aligned} \tag{2.33}$$

Next we turn to estimating $\nu_0^{(n)}(I_\ell H_{\ell'}^i)$. As in (2.32) we have

$$\nu_0^{(n)}(I_\ell H_{\ell'}^i) = \nu_0(I_\ell H_{\ell'}^i) \qquad \text{for} \ \ \ell, \ell' \notin \mathsf{V}_0^{(n)}. \tag{2.34}$$

For $k < m \le n$, by claim (b) of Lemma 2.11, and then by (2.23), (2.25), (2.33), and (2.22), we have

$$\begin{aligned} \nu_0^{(n)}(I_{\ell_k} H_{\ell_m}^i) = \nu_0^{(m)}(I_{\ell_k} H_{\ell_m}^i) &\le \nu_0^{(k)}(I_{\ell_k}) + \sum_{\ell \in \partial \ell_m} c_{\ell_m \ell} \nu_0^{(k)}(I_{\ell_k} H_\ell^i) \\ &\le \bar{\kappa} \gamma(\nu_0) + 2\varDelta^2 K^{-1} \lambda(\nu_0) + \sum_{\ell \in \partial \ell_m} c_{\ell_m \ell} \sum_{\ell' \in \partial \ell_k} \nu_0(I_{\ell'} H_\ell^i) \\ &\le \bar{\kappa} \gamma(\nu_0) + \left[\varDelta \bar{c} + 2\varDelta^2 K^{-1}\right] \lambda(\nu_0). \end{aligned} \tag{2.35}$$

For $k \le n$, by (2.24) we have

$$\begin{aligned} \nu_0^{(n)}(I_{\ell_k} H_{\ell_k}^i) = \nu_0^{(k)}(I_{\ell_k} H_{\ell_k}^i) \ &\le \ \sum_{\ell \in \partial \ell_k} \nu_0(I_\ell) + \sum_{\ell, \ell' \in \partial \ell_k} c_{\ell_k \ell'} \nu_0(I_\ell H_{\ell'}^i) \\ &\le \ \varDelta \gamma(\nu_0) + \varDelta \bar{c} \lambda(\nu_0). \end{aligned} \tag{2.36}$$

Next, for $m < k \leq n$, by (2.22) and (2.23) we have

$$\begin{aligned} \nu_0^{(n)}(I_{\ell_k}H^i_{\ell_m}) &= \nu_0^{(k)}(I_{\ell_k}H^i_{\ell_m}) \leq \sum_{\ell\in\partial\ell_k} \nu_0^{(m)}(I_\ell H^i_{\ell_m}) \\ &\leq \sum_{\ell\in\partial\ell_k}\left(\nu_0(I_\ell) + \sum_{\ell'\in\partial\ell_m} c_{\ell_m\ell'}\nu_0(I_\ell H^i_{\ell'})\right) \\ &\leq \varDelta\gamma(\nu_0) + \varDelta\bar{c}\lambda(\nu_0). \end{aligned} \tag{2.37}$$

Now we consider the case where $k \leq n$ and $\ell \notin \mathsf{V}_0^{(n)}$. Then by (2.22) we have

$$\nu_0^{(n)}(I_{\ell_k}H^i_\ell) = \nu_0^{(k)}(I_{\ell_k}H^i_\ell) \leq \sum_{\ell'\in\partial\ell_k} \nu_0(I_{\ell'}H^i_\ell) \leq \varDelta\lambda(\nu_0). \tag{2.38}$$

For $k \leq n$ and $\ell \notin \mathsf{V}_0^{(n)}$, we also have by (2.23) that

$$\begin{aligned} \nu_0^{(n)}(I_\ell H^i_{\ell_k}) = \nu_0^{(k)}(I_\ell H^i_{\ell_k}) &\leq \nu_0(I_\ell) + \sum_{\ell'\in\partial\ell_k} c_{\ell_k\ell'}\nu_0(I_\ell H^i_{\ell'}) \\ &\leq \gamma(\nu_0) + \bar{c}\lambda(\nu_0). \end{aligned} \tag{2.39}$$

Now let us consider the sequence $\{\nu_0^{(n)}\}_{n\in\mathbb{N}_0}$ defined in (2.31). By claim (b) of Lemma 2.11 it stabilizes on local sets $B \in \mathcal{B}(X^2)$, and hence is convergent in the $\mathcal{T}_{loc}$-topology. Let ν_1 be its limit. By Lemma 2.4 we have that $\nu_1 \in \mathcal{C}(\mu_1, \mu_2)$. At the same time, by (2.32), and (2.33) it follows that

$$\nu_1(I_\ell) \leq \begin{cases} \bar{\kappa}\gamma(\nu_0) + 2\varDelta^2 K^{-1}\lambda(\nu_0), & \text{for } \ \ell \in \mathsf{V}_0; \\ \gamma(\nu_0), & \text{for } \ \ell \in \mathsf{V}_1. \end{cases} \tag{2.40}$$

Similarly, by (2.33) – (2.39) we obtain

$$\nu_1(I_\ell H^i_{\ell'}) \leq \begin{cases} \varDelta\gamma(\nu_0) + [\varDelta\bar{c} + 2\varDelta^2 K^{-1}]\,\lambda(\nu_0), & \ell, \ell' \in \mathsf{V}_0; \\ \varDelta\lambda(\nu_0), & \ell \in \mathsf{V}_0, \ell' \in \mathsf{V}_1; \\ \gamma(\nu_0) + \bar{c}\lambda(\nu_0), & \ell \in \mathsf{V}_1, \ell' \in \mathsf{V}_0; \\ \lambda(\nu_0), & \ell, \ell' \in \mathsf{V}_1. \end{cases} \tag{2.41}$$

These estimates complete the reconstruction over V_0.

Remark 2.16. One should notice that for a bipartite graph this step is sufficient in itself, since the reconstruction procedure is symmetric with respect to the two partitions. However, in the case of 3 or more partitions, one always has to make a distinction between partitions which were already traversed and the ones which were not. One can see this in the following section.

2.2.2 Reconstruction in the case of $\chi \geq 3$

Set

$$A = \frac{2\Delta^{\chi+1}}{1-\bar{\kappa}}. \tag{2.42}$$

Then, for $K > K_*$, see (2.14), the following holds

$$K^{-1} < \frac{\bar{c}(1-\bar{\kappa})}{4\Delta^{\chi+1}}, \qquad AK^{-1} < \bar{c}/2. \tag{2.43}$$

Lemma 2.17. *For $K > K_*$, take $\pi \in \Pi(h, K, \kappa, c)$ and $\mu_1, \mu_2 \in \mathcal{M}_h(\pi)$. Then for each $\nu_0 \in \mathcal{C}(\mu_1, \mu_2)$ there exists $\nu \in \mathcal{C}(\mu_1, \mu_2)$ for which the following estimates hold*

$$\gamma(\nu) \leq \left[\bar{\kappa} + AK^{-1}\right] \gamma(\nu_0) + 2AK^{-1}\lambda(\nu_0), \tag{2.44}$$

$$\lambda(\nu) \leq \Delta^{\chi-1}\gamma(\nu_0) + \bar{c}\Delta^{\chi}\lambda(\nu_0). \tag{2.45}$$

Proof.

We consider the partition of V into the disjoint sets $\mathsf{V}_0, \ldots, \mathsf{V}_{m-1}$.

From now on, we will use the following notation

$$\mathsf{U}_j := \bigsqcup_{i=0}^{j} \mathsf{V}_i \text{ and } \mathsf{W}_j := \mathsf{V} \setminus \mathsf{U}_j \qquad j = 0, \ldots, \chi - 1. \tag{2.46}$$

(i) **Reconstruction over V_0**

The same computations done in Section 2.2.1 (see (2.31)-(2.39)) yield the existence of a $\nu_1 \in \mathcal{C}(\mu_1, \mu_2)$ such that

$$\nu_1(I_\ell) \leq \begin{cases} \bar{\kappa}\gamma(\nu_0) + 2\Delta^2 K^{-1}\lambda(\nu_0), & \text{for } \ \ell \in \mathsf{V}_0; \\ \gamma(\nu_0), & \text{for } \ \ell \in \mathsf{W}_0, \end{cases} \tag{2.47}$$

and

$$\nu_1(I_\ell H^i_{\ell'}) \leq \begin{cases} \Delta\gamma(\nu_0) + \left[\Delta\bar{c} + 2\Delta^2 K^{-1}\right]\lambda(\nu_0), & \ell, \ell' \in \mathsf{V}_0; \\ \Delta\lambda(\nu_0), & \ell \in \mathsf{V}_0, \ell' \in \mathsf{W}_0; \\ \gamma(\nu_0) + \bar{c}\lambda(\nu_0), & \ell \in \mathsf{W}_0, \ell' \in \mathsf{V}_0; \\ \lambda(\nu_0), & \ell, \ell' \in \mathsf{W}_0. \end{cases} \tag{2.48}$$

(ii) **The induction step (for $j \leq m-1$)**

Here we assume that ν_j satisfies the following estimates, cf. (2.47), where A is as in (2.42).

$$\nu_j(I_\ell) \leq \begin{cases} [\bar{\kappa} + AK^{-1}]\,\gamma(\nu_0) + 2AK^{-1}\lambda(\nu_0), & \text{for } \ \ell \in \mathsf{U}_{j-1}; \\ \gamma(\nu_0), & \text{for } \ \ell \in \mathsf{W}_{j-1}. \end{cases} \tag{2.49}$$

And also, cf. (2.48),

$$\nu_j(I_\ell H^i_{\ell'}) \leq \begin{cases} \varDelta^j \gamma(\nu_0) + \bar{c}\varDelta^{j+1}\lambda(\nu_0), & \ell, \ell' \in \mathsf{U}_{j-1}; \\ \varDelta^j \lambda(\nu_0), & \ell \in \mathsf{U}_{j-1}, \ell' \in \mathsf{W}_{j-1}; \\ j\gamma(\nu_0) + \bar{c}\lambda(\nu_0), & \ell \in \mathsf{W}_{j-1}, \ell' \in \mathsf{V}_{j-1}; \\ \lambda(\nu_0), & \ell, \ell' \in \mathsf{W}_{j-1}. \end{cases} \tag{2.50}$$

Since $\mathsf{W}_{\chi-1} = \emptyset$, see (2.46), for $j = \chi - 1$ we have just the first lines in (2.49) and (2.50), which yields (2.29) and (2.30), respectively. Note that (2.47) agrees with (2.49) as $\varDelta^2 < A$, see (2.42). Also (2.48) agrees with (2.50), which follows from the fact that

$$\bar{c}\varDelta + 2\varDelta^2 K^{-1} < \bar{c}\varDelta + AK^{-1} \leq \bar{c}\varDelta + \bar{c}/2 < \bar{c}\varDelta^2 \leq \bar{c}\varDelta^{j+1}, \quad j = 1, \dots \chi - 1,$$

see (2.42) and (2.43).

Thus, our aim now is to prove that the estimates as in (2.49) and (2.50) hold also for $j+1$. Note that the last lines in these estimates follow by claim (b) of Lemma 2.11. As above, we enumerate $\mathsf{V}_j = \{\ell_1, \ell_2, \cdots\}$ and set

$$\nu_j^{(n)} = R_{\ell_n} R_{\ell_{n-1}} \cdots R_{\ell_1} \nu_j.$$

For $k \leq n$, by (2.25) and we have, cf. (2.33),

$$\begin{aligned} \nu_j^{(n)}(I_{\ell_k}) &= \nu_j^{(k)}(I_{\ell_k}) \leq \sum_{\ell \in \partial\ell_k \cap \mathsf{U}_{j-1}} \kappa_{\ell_k \ell} \nu_j(I_\ell) + \sum_{\ell \in \partial\ell_k \cap \mathsf{W}_j} \kappa_{\ell_k \ell} \nu_j(I_\ell) \\ &+ K^{-1} \sum_{i=1,2} \sum_{\ell, \ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} \nu_j(I_\ell H^i_{\ell'}) \\ &+ K^{-1} \sum_{i=1,2} \sum_{\ell \in \partial\ell_k \cap \mathsf{U}_{j-1}} \sum_{\ell' \in \partial\ell_k \cap \mathsf{W}_j} \nu_j(I_\ell H^i_{\ell'}) \\ &+ K^{-1} \sum_{i=1,2} \sum_{\ell \in \partial\ell_k \cap \mathsf{W}_j} \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} \nu_j(I_\ell H^i_{\ell'}) \\ &+ K^{-1} \sum_{i=1,2} \sum_{\ell, \ell' \in \partial\ell_k \cap \mathsf{W}_j} \nu_j(I_\ell H^i_{\ell'}). \end{aligned} \tag{2.51}$$

Now we use here the assumptions in (2.49) and (2.50) and obtain

$$\begin{aligned} \nu_j^{(n)}(I_{\ell_k}) &\leq \left[\bar{\kappa} + K^{-1}\left(\bar{\kappa}A + 2\Delta^j \Delta_j^2 + 2j\Delta_j\widetilde{\Delta}_j\right)\right]\gamma(\nu_0) \\ &+ 2K^{-1}\left[\bar{\kappa}A + \bar{c}\Delta^{j+1}\Delta_j^2 + \Delta^j\Delta_j\widetilde{\Delta}_j\right. \\ &+ \left.\bar{c}\Delta_j\widetilde{\Delta}_j + \widetilde{\Delta}_j^2\right]\lambda(\nu_0), \end{aligned} \tag{2.52}$$

where

$$\Delta_j := |\partial\ell_k \cap \mathsf{U}_{j-1}|, \qquad \widetilde{\Delta}_j := |\partial\ell_k \cap \mathsf{W}_j|.$$

To prove that, see the first line in (2.49),

$$\bar{\kappa}A + 2\Delta^j\Delta_j^2 + 2j\Delta_j\widetilde{\Delta}_j \leq A$$

we use (2.42), take into account that $\Delta \geq 2$ (hence, $j \leq \Delta^j$, $j = 1, 2, \ldots \chi - 1$) and obtain

$$2\Delta^j\Delta_j^2 + 2j\Delta_j\widetilde{\Delta}_j \leq 2\Delta^j\Delta_j\left(\Delta_i + \widetilde{\Delta}_j(j/\Delta^j)\right) \leq 2\Delta^{j+1} \leq A(1-\bar{\kappa}).$$

To prove that the coefficient at $\lambda(\nu_0)$ in (2.52) agrees with that in (2.49) we use the following estimates

$$\begin{aligned} &\bar{c}\Delta^{j+1}\Delta_j^2 + \Delta^j\Delta_j\widetilde{\Delta}_j + \bar{c}\Delta_j\widetilde{\Delta}_j + \widetilde{\Delta}_j^2 \\ &= \bar{c}\Delta^{j+1}\Delta_j\left(\Delta_j + \widetilde{\Delta}_j\Delta^{-j}\right) + \Delta^j\widetilde{\Delta}_j\left(\Delta_j + \widetilde{\Delta}_j\Delta^{-(j+1)}\right) \\ &\leq \Delta^2 + \Delta^{j+2} \leq 2\Delta^{j+2} \leq A(1-\bar{\kappa}), \end{aligned}$$

where we have taken into account that $j+2 \leq \chi + 1$, see (2.42). For $\ell \in \mathsf{U}_{j-1}$, $\nu_j^{(n)}(I_\ell) = \nu_j(I_\ell)$ and hence obeys the first line of (2.49). For $\ell \in \mathsf{W}_j$, again $\nu_j^{(n)}(I_\ell) = \nu_j(I_\ell)$ and hence obeys the second line of (2.49). Here we also used that $\bar{c} < 1/\Delta^\chi$ and $j+1 \leq \chi$, see (2.8). Thus, (2.49) with $j+1$ holds true.

Now we turn to estimating $\nu_j^{(n)}(I_\ell H^i_{\ell'})$. In the situation where $\ell, \ell' \in \mathsf{U}_{j-1} \cup \mathsf{W}_j$, we have that $\nu_j^{(n)}(I_\ell H^i_{\ell'}) = \nu_j(I_\ell H^i_{\ell'})$ and hence obeys (2.50). Let us consider first the cases where only one vertex of ℓ, ℓ' lies in V_j.

For $\ell' \in \mathsf{U}_{j-1}$ and $k \leq n$, by (2.22) and the first and third lines in (2.50) we obtain

$$\begin{aligned} &\nu_j^{(n)}(I_{\ell_k}H^i_{\ell'}) = \nu_j^{(k)}(I_{\ell_k}H^i_{\ell'}) \leq \sum_{\ell \in \partial\ell_k \cap \mathsf{U}_{j-1}} \nu_j(I_\ell H^i_{\ell'}) + \sum_{\ell \in \partial\ell_k \cap \mathsf{W}_j} \nu_j(I_\ell H^i_{\ell'}) \\ &\leq \left[\Delta^j\Delta_j + j\widetilde{\Delta}_j\right]\gamma(\nu_0) + \left[\bar{c}\Delta^{j+1}\Delta_j + \bar{c}\widetilde{\Delta}_j\right]\lambda(\nu_0) \\ &\leq \Delta^{j+1}\gamma(\nu_0) + \bar{c}\Delta^{j+2}\lambda(\nu_0), \end{aligned} \tag{2.53}$$

which yields the first line in (2.50) with $j+1$. To obtain the last line in (2.53) we used the following estimates

$$\Delta^j \left(\Delta_j + \widetilde{\Delta}_j (j/\Delta^j)\right) \leq \Delta^{j+1}; \quad \bar{c}\Delta^{j+1}\left(\Delta_j + \widetilde{\Delta}_j \Delta^{-(j+1)}\right) \leq \bar{c}\Delta^{j+2}.$$

For $\ell' \in \mathsf{W}_j$ and $k \leq n$, by (2.22) and the second and fourth lines in (2.50) it follows that

$$\begin{aligned} \nu_j^{(n)}(I_{\ell_k} H^i_{\ell'}) = \nu_j^{(k)}(I_{\ell_k} H^i_{\ell'}) &\leq \sum_{\ell \in \partial \ell_k \cap \mathsf{U}_{j-1}} \nu_j(I_\ell H^i_{\ell'}) + \sum_{\ell \in \partial\ell_k \cap \mathsf{W}_j} \nu_j (I_\ell H^i_{\ell'}) \\ &\leq \left(\Delta^j \Delta_j + \widetilde{\Delta}_j\right)\lambda(\nu_0) \leq \Delta^{j+1}\lambda(\nu_0), \end{aligned} \tag{2.54}$$

which agrees with the second line in (2.50).

For $\ell \in \mathsf{U}_{j-1}$ and $k \leq n$, by (2.23) and the first and second lines in (2.50) we get

$$\begin{aligned} &\nu_j^{(n)}(I_\ell H^i_{\ell_k}) = \nu_j^{(k)}(I_\ell H^i_{\ell_k}) \leq \nu_j(I_\ell) + \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} \nu_j(I_\ell H^i_{\ell'}) \\ &+ \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k\ell'}\nu_j(I_\ell H^i_{\ell'}) \leq \left[\bar{\kappa} + AK^{-1}\right]\gamma(\nu_0) \\ &+ 2AK^{-1}\lambda(\nu_0) + \left[\Delta^j \gamma(\nu_0) + \bar{c}\Delta^{j+1}\lambda(\nu_0)\right] \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k\ell'} \\ &+ \Delta^j \lambda(\nu_0) \sum_{\ell' \in \partial\ell_k\cap\mathsf{W}_j} c_{\ell_k\ell'}. \end{aligned} \tag{2.55}$$

In order for this to agree with the first line in (2.50), it is enough that the following holds

$$\begin{aligned} \bar{\kappa} + AK^{-1} + \Delta^j \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k\ell'} &\leq \Delta^{j+1}, \\ 2AK^{-1} + \bar{c}\Delta^{j+1} \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k\ell'} + \Delta^j \sum_{\ell' \in \partial\ell_k \cap \mathsf{W}_j} c_{\ell_k\ell'} &\leq \bar{c}\Delta^{j+2}. \end{aligned} \tag{2.56}$$

Recall that we assume $\Delta \geq 2$. By (2.43) and (2.8)-(2.9) we get that the left-hand side of the first line in (2.56) does not exceed

$$\bar{\kappa} + \bar{c}/2 + \Delta^{-1} < 3 < \Delta^{j+1}, \qquad \text{for } \ j = 1, \dots, \chi - 1.$$

Likewise, the left-hand side of the second line in (2.56) does not exceed

$$\bar{c} + \bar{c} + \bar{c}\Delta^j \leq \bar{c}(2 + \Delta^j) < \bar{c}\Delta^{j+2} \qquad \text{for } \ j = 1, \dots, \chi - 1.$$

For $\ell \in \mathsf{W}_j$ and $k \le n$, by (2.23) and the third and fourth lines in (2.50) we get

$$\begin{aligned}
&\nu_j^{(n)}(I_\ell H^i_{\ell_k}) = \nu_j^{(k)}(I_\ell H^i_{\ell_k}) \le \nu_j(I_\ell) \\
&+ \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} \nu_j(I_\ell H^i_{\ell'}) + \sum_{\ell' \in \partial\ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \nu_j(I_\ell H^i_{\ell'}) \qquad (2.57) \\
&\le \gamma(\nu_0) + [j\gamma(\nu_0) + \bar{c}\lambda(\nu_0)] \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \lambda(\nu_0) \sum_{\ell' \in \partial\ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \\
&\le (1 + j\bar{c})\gamma(\nu_0) + \left(\bar{c} \sum_{\ell' \in \partial\ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \sum_{\ell' \in \partial\ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \right) \lambda(\nu_0),
\end{aligned}$$

which clearly agrees with the third line in (2.50).

Now we consider the cases where both ℓ, ℓ' lie in V_j. For $k < m \le n$, by first (2.23) and (2.22), and then by (2.25), we have

$$\begin{aligned}
\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_m}) &= \nu_j^{(m)}(I_{\ell_k} H^i_{\ell_m}) \le \nu_j^{(k)}(I_{\ell_k}) + \sum_{\ell' \in \partial\ell_m} c_{\ell_m \ell'} \nu_j^{(k)}(I_{\ell_k} H^i_{\ell'}) \\
&\le \sum_{\ell \in \partial\ell_k} \kappa_{\ell_k \ell} \nu_j(I_\ell) + K^{-1} \sum_{s=1,2} \sum_{\ell, \ell' \in \partial\ell_k} \nu_j(I_\ell H^s_{\ell'}) \\
&+ \sum_{\ell' \in \partial\ell_m} c_{\ell_m \ell'} \sum_{\ell \in \partial\ell_k} \nu_j(I_\ell H^i_{\ell'}). \qquad (2.58)
\end{aligned}$$

The next step is to split the sums in (2.58) as it has been done in, e.g., (2.57), and then use (2.49) and (2.50). By doing so we get

$$\begin{aligned}
&\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_m}) \le \left[(\bar{\kappa} + AK^{-1})\gamma(\nu_0) + 2AK^{-1}\lambda(\nu_0) \right] \sum_{\ell \in \partial\ell_k \cap \mathsf{U}_{j-1}} \kappa_{\ell_k \ell} \\
&+\gamma(\nu_0) \sum_{\ell \in \partial\ell_k \cap \mathsf{W}_j} \kappa_{\ell_k \ell} + 2K^{-1} \Delta_j^2 \left[\Delta^j \gamma(\nu_0) + \bar{c}\Delta^{j+1}\lambda(\nu_0) \right] \\
&+2K^{-1} \Delta_j \widetilde{\Delta}_j \left[\Delta^j \lambda(\nu_0) + j\gamma(\nu_0) + \bar{c}\lambda(\nu_0) \right] + 2K^{-1} \widetilde{\Delta}_j^2 \lambda(\nu_0) \\
&+\Delta_j \left[\Delta^j \gamma(\nu_0) + \bar{c}\Delta^{j+1}\lambda(\nu_0) \right] \sum_{\ell' \in \partial\ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'} \\
&+\Delta^j \Delta_j \lambda(\nu_0) \sum_{\ell' \in \partial\ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'} + \widetilde{\Delta}_j (j\gamma(\nu_0) + \bar{c}\lambda(\nu_0)) \sum_{\ell' \in \partial\ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'} \\
&+\widetilde{\Delta}_j \lambda(\nu_0) \sum_{\ell' \in \partial\ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'}. \qquad (2.59)
\end{aligned}$$

In order for this to agree with the first line in (2.50), it is enough that the following two estimate hold

$$(\bar{\kappa} + AK^{-1}) \sum_{\ell \in \partial \ell_k \cap \mathsf{U}_{j-1}} \kappa_{\ell_k \ell} + \sum_{\ell \in \partial \ell_k \cap \mathsf{W}_j} \kappa_{\ell_k \ell} + 2K^{-1} \Delta_j^2 \Delta^j \tag{2.60}$$

$$+2K^{-1} j \Delta_j \widetilde{\Delta}_j + \Delta_j \Delta^j \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'} + \widetilde{\Delta}_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'}$$

$$\leq \Delta^{j+1},$$

$$2AK^{-1} \sum_{\ell \in \partial \ell_k \cap \mathsf{U}_{j-1}} \kappa_{\ell_k \ell} + 2K^{-1} \Delta_j^2 \bar{c} \Delta^{j+1} + 2K^{-1} \Delta_j \widetilde{\Delta}_j \Delta^j \tag{2.61}$$

$$+2K^{-1} \bar{c} \Delta_j \widetilde{\Delta}_j + 2K^{-1} \widetilde{\Delta}_j^2 + \bar{c} \Delta_j \Delta^{j+1} \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'}$$

$$+\Delta_j \Delta^j \sum_{\ell' \in \partial \ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'} + \bar{c} \widetilde{\Delta}_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'}$$

$$+ \quad \widetilde{\Delta}_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'} \leq \bar{c} \Delta^{j+2}.$$

Taking into account that $\bar{\kappa} < 1$ and (2.43), one can show that the left-hand side of (2.60) does not exceed

$$1 + \bar{c}/2 + 2K^{-1} \Delta^j \Delta_j \left(\Delta_j + \widetilde{\Delta}_j (j/\Delta^j) \right) + \bar{c} \Delta^{j+1}$$

$$\leq 1 + \bar{c}/2 + \bar{c}/2 + \bar{c} \Delta^{j+1} < 2 + \frac{1}{\Delta^{\chi}} < \Delta^{j+1}.$$

To prove (2.61) we use (2.43), (2.8), (2.9), combined with the inequality $\Delta_j \widetilde{\Delta}_j \leq \Delta^2/4$, and perform the following calculations

$$\text{LHS(2.61)} \leq 2AK^{-1} \bar{\kappa} + \frac{1}{2} K^{-1} \Delta^{j+2} + 2K^{-1} \left(\Delta_j^2 + \bar{c} \Delta_j \widetilde{\Delta}_j + \widetilde{\Delta}_j^2 \right)$$

$$+\Delta_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'} + \Delta^j \Delta_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'}$$

$$+\bar{c} \widetilde{\Delta}_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{U}_{j-1}} c_{\ell_m \ell'} + \widetilde{\Delta}_j \sum_{\ell' \in \partial \ell_m \cap \mathsf{W}_j} c_{\ell_m \ell'}$$

$$\leq \bar{c} + \frac{\bar{c} \Delta^{j+2}}{8 \Delta^{\chi+1}} + \frac{\bar{c} \Delta^2}{2 \Delta^{\chi+1}} + \bar{c} \Delta^{j+1} + \bar{c} \Delta < \bar{c} \Delta^{j+2},$$

which holds even for $j = 1$, $\chi = 2$, and $\Delta = 2$.

Next, for $k \leq n$, by (2.24) we have

$$\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_k}) = \nu_j^{(k)}(I_{\ell_k} H^i_{\ell_k}) \leq \sum_{\ell \in \partial \ell_k} \nu_j(I_\ell) + \sum_{\ell, \ell' \in \partial \ell_k} c_{\ell_k \ell'} \nu_j (I_\ell H^i_{\ell'}) \tag{2.62}$$

As above, we split the sums in (2.62) and then use (2.49) and (2.50), and obtain

$$\begin{aligned}
\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_k}) &\leq \varDelta_j \left[\bar{\kappa} + AK^{-1}\right] \gamma(\nu_0) + \varDelta_j 2AK^{-1} \lambda(\nu_0) \\
&+ \widetilde{\varDelta}_j \gamma(\nu_0) + \varDelta_j \left[\varDelta^j \gamma(\nu_0) + \bar{c} \varDelta^{j+1} \lambda(\nu_0)\right] \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} \\
&+ \varDelta_j \varDelta^j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} + \widetilde{\varDelta}_j (j \gamma(\nu_0) + \bar{c} \lambda(\nu_0)) \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} \\
&+ \widetilde{\varDelta}_j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'}.
\end{aligned} \tag{2.63}$$

In order for this to agree with the first line in (2.50), it is sufficient that the following two inequalities hold

$$\varDelta_j \left[\bar{\kappa} + AK^{-1}\right] + \widetilde{\varDelta}_j + \left(\varDelta^j \varDelta_j + j \widetilde{\varDelta}^j\right) \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} \leq \varDelta^{j+1}, \tag{2.64}$$

$$\begin{aligned}
&2AK^{-1} \varDelta_j + \bar{c} \varDelta^{j+1} \varDelta_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \varDelta^j \varDelta_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \\
&\quad + \bar{c} \widetilde{\varDelta}_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \widetilde{\varDelta}_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \leq \bar{c} \varDelta^{j+2}.
\end{aligned} \tag{2.65}$$

By means of (2.43) we get

$$\mathrm{LHS}(2.64) \leq \varDelta + \varDelta A K^{-1} + \bar{c} \varDelta^{j+1} < \varDelta + \frac{1}{2 \varDelta^{\chi - 1}} + 1 < \varDelta^{j+1}.$$

Similarly,

$$\begin{aligned}
\mathrm{LHS}(2.65) &\leq \bar{c} \varDelta_j + \varDelta_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \varDelta^j \varDelta_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \\
&\quad + \bar{c} \widetilde{\varDelta}_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{U}_{j-1}} c_{\ell_k \ell'} + \widetilde{\varDelta}_j \sum_{\ell' \in \partial \ell_k \cap \mathsf{W}_j} c_{\ell_k \ell'} \\
&\leq \bar{c} \varDelta + \bar{c} \varDelta^j \varDelta_j + \bar{c} \widetilde{\varDelta}_j < \bar{c} \varDelta + \bar{c} \varDelta^{j+1} \leq \bar{c} \varDelta^{j+2}.
\end{aligned}$$

Now we consider the case where $m < k \leq n$. By (2.22), and then by (2.23), we have

$$\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_m}) = \nu_j^{(k)}(I_{\ell_k} H^i_{\ell_m}) \leq \sum_{\ell \in \partial \ell_k} \nu_j^{(m)}(I_\ell H^i_{\ell_m}) \tag{2.66}$$
$$\leq \sum_{\ell \in \partial \ell_k} \nu_j(I_\ell) + \sum_{\ell \in \partial \ell_k} \sum_{\ell' \in \partial \ell_m} c_{\ell_m \ell'} \nu_j(I_\ell H^i_{\ell'}).$$

Again we split the sums in (2.66) and then use (2.49) and (2.50), which yields

$$\nu_j^{(n)}(I_{\ell_k} H^i_{\ell_m}) \leq \mathrm{RHS}(2.63).$$

Thus, we have that (2.50) with $j+1$ holds in this case as well.

□

Proof of Theorem 2.9. Let $\nu_1 \in \mathcal{C}(\mu_1, \mu_2)$ be the measure on the left-hand side of (2.44) and (2.45). We apply to this measure the same reconstruction procedure and obtain $\nu_2 \in \mathcal{C}(\mu_1, \mu_2)$, for which both estimates (2.44), (2.45) hold with ν_1 on the right-hand side. Then we repeat this due times and obtain a sequence $(\nu_n)_{n \in \mathbb{N}} \subset \mathcal{C}(\mu_1, \mu_2)$ such that

$$\begin{pmatrix} \gamma(\nu_n) \\ \lambda(\nu_n) \end{pmatrix} \leq [M(K)]^n \begin{pmatrix} \gamma(\nu_0) \\ \lambda(\nu_0) \end{pmatrix}, \tag{2.67}$$

where $M(K)$ is the matrix defined by the right-hand sides of (2.44) and (2.45). Its spectral radius is

$$r_K = \frac{1}{2} \left[\bar{\kappa} + AK^{-1} + \bar{c}\Delta^\chi + \sqrt{(\bar{\kappa} + AK^{-1} - \bar{c}\Delta^\chi)^2 + 8\Delta^\chi AK^{-1}} \right]. \tag{2.68}$$

For $K > K_*$, see (2.14), we have $r_K < 1$, which by (2.67) yields (2.17) and thereby completes the proof.

□

2.2.3 Reconstruction over a finite volume

In order to establish a result on decay of correlations that will be presented in Section 2.3, we will show that applying the reconstruction procedure over a finite volume will yield similar estimates as above. For $\mathsf{L} \Subset \mathsf{V}$ a finite volume, we introduce the graph $\mathsf{G}_\mathsf{L} = (\mathsf{L}, \mathsf{E}_\mathsf{L})$, where E_L is the restriction of E to edges with both ends in L. Set Δ_L and χ_L to be the maximum degree of G_L and its chromatic number, respectively. We note that $\Delta_{\mathsf{L}} \leq \Delta$ and $\chi_{\mathsf{L}} \leq \chi$. Analogously to (2.27)-(2.28), we introduce

$$\gamma_{\mathsf{L}}(\nu) := \sup_{\ell\in\mathsf{L}} \nu(I_\ell), \tag{2.69}$$

$$\lambda_{\mathsf{L}}(\nu) := \max_{i=1,2} \sup_{\ell,\ell'\in\mathsf{L}} \nu(I_\ell H^i_{\ell'}). \tag{2.70}$$

Fix $\mathsf{L} \Subset \mathsf{V}$ and let c, κ, h and K_* be as in the statement of Theorem 2.9 and $K > K_*$ arbitrary.

Lemma 2.18. *Let $\nu_0 \in \mathcal{C}(\mu_1, \mu_2)$, where μ_1, μ_2 are consistent with $(\pi^x_\ell)_{\ell\in\mathsf{L}\cup\partial\mathsf{L}, x\in X}$ for some $\pi \in \Pi(h, K, \kappa, c)$. Then, there exists $\nu_* \in \mathcal{C}(\mu_1, \mu_2)$ such that*

$$\begin{pmatrix} \gamma_{\mathsf{L}}(\nu_*) \\ \lambda_{\mathsf{L}}(\nu_*) \end{pmatrix} \leq M(K) \begin{pmatrix} \gamma_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) \\ \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) \end{pmatrix}, \tag{2.71}$$

where $M(K)$ is the matrix given by (2.44)-(2.45).

As in subsection 2.2.2, we decompose L into χ_{L} disjoint independent sets $\mathsf{L}_0, \ldots, \mathsf{L}_{\chi_{\mathsf{L}}-1}$ and following the lines of the proof of Lemma (2.17), we consider a numbering of $\mathsf{L}_0 = \{\ell_0, \ldots, \ell_N\}$, where $N \geq 1$ is the number of elements of L_0. Also, for $0 \leq n \leq N$ we define

$$\nu_0^{(n)} := R_{\ell^0_n} \ldots R_{\ell^0_0} \nu_0.$$

Now, applying the same reconstruction procedure as in Lemma 2.17, we get the following estimates

$$\nu_1(I_\ell) \leq \begin{cases} \bar\kappa \gamma_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) + 2\Delta^2 K^{-1} \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0), & \ell \in \mathsf{L}_0 \\ \gamma_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0), & \ell \in \mathsf{L}_1 \cup \ldots \cup \mathsf{L}_{\chi-1}. \end{cases} \tag{2.72}$$

$$\nu_1(I_\ell H^i_{\ell'}) \leq \begin{cases} \Delta \gamma_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) + [\Delta \bar c + 2\Delta^2 K^{-1}] \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) & \ell, \ell' \in \mathsf{L}_0 \\ \Delta \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0), & \ell \in \mathsf{L}_0, \ell' \in \mathsf{L}_1 \cup \ldots \cup \mathsf{L}_{\chi_{\mathsf{L}}-1} \\ \gamma_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0) + \bar c \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0), & \ell \in \mathsf{L}_1 \cup \ldots \cup \mathsf{L}_{\chi_{\mathsf{L}}-1}, \ell' \in \mathsf{L}_0 \\ \lambda_{\mathsf{L}\cup\partial\mathsf{L}}(\nu_0), & \ell, \ell' \in \mathsf{L}_1 \cup \ldots \cup \mathsf{L}_{\chi_{\mathsf{L}}-1}. \end{cases} \tag{2.73}$$

where by ν_1 we have denoted the measure $\nu_0^{N(0)}$. Proceeding by induction, in χ_{L} steps, we are able to find ν_*, satisfying

$$\gamma(\nu_*) \leq \left[\bar\kappa + AK^{-1}\right] \gamma(\nu_0) + 2AK^{-1} \lambda(\nu_0), \tag{2.74}$$

where A is given by (2.42) and

$$\lambda(\nu_*) \leq \Delta^{\chi}\gamma(\nu_0) + \bar{c}\Delta_{\mathsf{L}}^{\chi_{\mathsf{L}}}\lambda(\nu_0). \tag{2.75}$$

□

2.3 Decay of correlations

Using Lemma 2.18, we are able to derive estimates on the decay of correlations of Gibbs measures in finite volume. Let Π be a specification, whose corresponding family of one point local states, see (2.6), belongs to $\mathsf{\Pi}(h, K, \kappa, c)$, for $K > K_*$, where K_* is defined in (2.14). Assuming it exists, let $\mu \in \mathcal{G}_h(\Pi)$ be the unique Gibbs measure consistent with Π. Consider two finite disjoint volumes $\mathsf{L}, \widetilde{\mathsf{L}} \Subset \mathsf{V}$ and define the distance between them $d(\mathsf{L}, \widetilde{\mathsf{L}})$ to be equal to N, where N is the largest integer such that $\partial_N \mathsf{L} \cap \widetilde{\mathsf{L}} = \emptyset$, where $\partial_N \mathsf{L} = \{\ell' \notin \mathsf{L} : \text{for some } \ell \in \mathsf{L}, \text{ there exists a } N-\text{path with end points } \ell \text{ and } \ell'\}$. Let functions f, g be measurable functions such that there exist $\bar{f} : \Xi^{\widetilde{\mathsf{L}}} \to \mathbb{R}$ and $\bar{g} : \Xi^{\mathsf{L}} \to \mathbb{R}$ such that $f(x) = \bar{f}(x_{\widetilde{\mathsf{L}}})$ and $g(x) = \bar{g}(x_{\mathsf{L}})$, respectively, for any $x \in X$. Furthermore, assume that the following bound

$$|g(x)| \leq \sum_{l \in \mathsf{L}} h(x_\ell) \tag{2.76}$$

holds and that $\sup_{\ell \in \mathsf{L}} \int_X f(x) h(x_\ell) \mu(dx) < \infty$.

Before giving the statement of the result, for every $y \in X$ we set

$$\begin{aligned} \tilde{h}_{\mathsf{L}}(y) &:= \sup_{\ell \in \mathsf{L}} \max \left\{ \int_X h(x_\ell) \pi_\ell(dx|y), \int_X h(x_\ell)\mu(dx), 1 \right\} \\ &\leq \max \left\{ M_1, 1 + \bar{c} \sup_{\ell \in \partial \mathsf{L}} h(y_\ell) \right\}, \end{aligned} \tag{2.77}$$

where $M_1 := \sup_{\ell \in \mathsf{L}} \int_X h(x_\ell)\mu(dx)$. One can easily see that $\tilde{h}_{\mathsf{L}}(y)$ is finite for any $\mu \in \mathcal{G}_h(\Pi)$.

Theorem 2.19. *In the setting described above, one can find constants $D \geq 0$ and $\alpha > 0$ for which one has*

$$|Cov_\mu(f; g)| := |\mu(fg) - \mu(f)\mu(g)| \leq D|\mathsf{L}|^2 \exp\left(-\alpha d(\mathsf{L}, \widetilde{\mathsf{L}})\right) \int_X |f(x)| \tilde{h}(x) \mu(dx). \tag{2.78}$$

Proof. Relation (2.76) implies that, given two probability measures ϱ_1 and ϱ_2 on X and $\rho \in \mathcal{C}(\varrho_1, \varrho_2)$, we have

$$\begin{aligned}
&\left|\int_X g(x)\varrho_1(dx) - \int_X g(y)\varrho_2(dy)\right| \\
&\leq \int_{X^2} \mathbb{1}_{\neq}(x,y)(|g(x)| + |g(y)|)\rho(dx,dy) \\
&\leq \sum_{\ell,\ell' \in \mathsf{L}} \int_{X^2} \mathbb{1}_{\neq}(x_\ell, y_\ell)(h(x_{\ell'}) + h(y_{\ell'}))\rho(dx,dy).
\end{aligned} \tag{2.79}$$

We have to estimate $|Cov_\mu(f;g)|$ and we do this by using that f and g depend only on the sites of $\widetilde{\mathsf{L}}$ and L respectively. We also use the consistency property of μ with respect to its projections on finite volume and to the specification Π, respectively, and the inequality given by (2.79). Therefore, we have

$$\begin{aligned}
&\left|\int_X f(x)g(x)\mu(dx) - \int_X f(x)\mu(dx)\cdot\int_X g(x)\mu(dx)\right| \\
&= \left|\int_X\int_X f(y)g(x)\pi_{\mathsf{L}\cup\partial_N\mathsf{L}}(dx|y)\mu(dy) - \int_X f(y)\mu(dy)\int_X g(x)\mu(dx)\right| \\
&\leq \int_X |f(y)|\cdot|\pi_{\mathsf{L}\cup\partial_N\mathsf{L}}(g|y) - \mu(g)|\mu(dy) \\
&\leq \int_X |f(y)| \sum_{\ell,k\in\mathsf{L}} \int_{X^2} \big[h(x_\ell) + h(z_\ell)\big] \times \mathbb{1}_{\neq}(x_k, z_k)\nu^y(dx,dz)\mu(dy) \\
&\leq 2|\mathsf{L}|^2 \int_X |f(y)|\lambda_{\mathsf{L}}(\nu^y)\mu(dy),
\end{aligned} \tag{2.80}$$

where $\nu^y := \pi_{\mathsf{L}\cup\partial_N\mathsf{L}}(\cdot|y)\otimes\mu$ and

$$\lambda_{\mathsf{L}}(\nu^y) := \max_{i=1,2}\sup_{\ell,k\in\mathsf{L}} \nu^y(h(y_\ell^i)\cdot\mathbb{1}_{\neq}(y_k^1, y_k^2)).$$

Note that $y \mapsto \nu^y(B)$ is $\mathcal{E}_{\mathsf{L}\cup\partial_N\mathsf{L}}$-measurable, for any fixed $B \in \mathcal{F}\otimes\mathcal{F}$, hence the integral of the last inequality is well-defined. Moreover, for any $\ell \in \mathsf{L}\cup\partial_N\mathsf{L}$, $y \mapsto R_\ell\nu^y(B)$ is $\mathcal{E}_{\mathsf{L}\cup\partial_N\mathsf{L}}$-measurable, for any fixed $B \in \mathcal{F}\otimes\mathcal{F}$.

Mimicking the proof of Theorem 2.9, (cf. (2.22)-(2.25)) we introduce the functional

$$\gamma_{\mathsf{L}}(\nu^y) := \sup_{\ell\in\mathsf{L}} \nu^y(I_\ell).$$

In order to give an estimate for $\lambda_{\mathsf{L}}(\nu^y)$, we can use the reconstruction procedure over a finite volume, as presented above in Corollary 2.18, since both $\pi_{\mathsf{L}\cup\partial_N\mathsf{L}}(\cdot|y)$ and μ are consistent with $(\pi_\ell^x)_{\ell\in\mathsf{L}\cup\partial\mathsf{L},x\in X}$. It is important to note that we can apply this procedure up to a finite number of times proportional to the distance between L and $\widetilde{\mathsf{L}}$, since by each step it shrinks the domain on which we can control γ_{L} and λ_{L}. We apply Corollary 2.18 exactly $N = d(\mathsf{L}, \widetilde{\mathsf{L}})$ times to obtain a new coupling ν_*^y such that

$$\begin{pmatrix} \gamma_{\mathsf{L}}(\nu_*^y) \\ \lambda_{\mathsf{L}}(\nu_*^y) \end{pmatrix} \leq M(K)^N \begin{pmatrix} \gamma_{\mathsf{L}\cup\partial_N \mathsf{L}}(\nu^y) \\ \lambda_{\mathsf{L}\cup\partial_N \mathsf{L}}(\nu^y) \end{pmatrix}.$$

Let v_s and v_{s-1} denote the column vector on the left-hand and right-hand sides of (2.71), respectively. Set

$$\xi = \frac{\Delta^{\chi-1}}{r_K - \bar{c}\Delta^{\chi}} = \frac{r_K - \bar{\kappa} - AK^{-1}}{2AK^{-1}} > 0, \tag{2.81}$$

and let T be the 2×2 diagonal matrix with $T_{11} = \xi$ and $T_{22} = 1$. Then the matrix

$$\widetilde{M}(K) := TM(K)T^{-1}, \tag{2.82}$$

cf. [BL88, Corollary 2.9.4, page 102], is positive and such that both its rows sum up to r_K. Set $\tilde{v}_s = Tv_s$ and let $\tilde{v}_s^i$, $i = 1, 2$, be the entries of $\tilde{v}_s$. By (2.71) we then get

$$\|\tilde{v}_s\| := \max\{\tilde{v}_s^1; \tilde{v}_s^2\} \leq \|\widetilde{M}(K)\| \|\tilde{v}_{s-1}\| = r_K \max\{\tilde{v}_{s-1}^1; \tilde{v}_{s-1}^2\},$$

which yields

$$\lambda_{\mathsf{L}}(\nu_*^y) \leq r_K^{N-1} \max\{\gamma_{\mathsf{L}\cup\partial_N \mathsf{L}}(\nu^y); \xi^{-1}\lambda_{\mathsf{L}\cup\partial_N \mathsf{L}}(\nu^y)\}. \tag{2.83}$$

Applying this estimate in (2.80) we arrive at (2.78) with, cf. (2.68) and (2.13),

$$\alpha = -\log r_K, \qquad C = 2r_K^{-1} \max\{1; \xi^{-1}\mu(h)\}.$$

□

Remaining in the context above, for a measurable function $f : \Xi \to \mathbb{R}$ define

$$||f||_{h,\infty} := \sup_{s\in\Xi} \frac{|f(s)|}{h(s)}.$$

Denote by $L_{h,\infty}$ the class of all measurable functions f with the finite norm $||f||_{h,\infty}$. As in the context of Theorem 2.19, along with

$$\sup_{\ell'\in\mathsf{V}} \int_X h(x_{\ell'})\mu(dx) =: M_1 < \infty$$

assume also that

$$\sup_{\ell'\in\mathsf{V}} \left[\int_X h^2(x_{\ell'})\mu(dx)\right]^{1/2} =: M_2 < \infty.$$

Corollary 2.20. *Then for any $f, g \in L_{h,\infty}$ and distinct $\ell, \tilde{\ell} \in L$ we have*

$$|Cov_\mu(f(x_{\tilde{\ell}}); g(x_\ell))| \leq\leq DM_3 \exp\{-\alpha d(\tilde{\ell}, \ell)\}||f||_{h,\infty}||g||_{h,\infty}, \tag{2.84}$$

for a positive M_3, depending on the graph and on function h.

Proof.

By Theorem 2.19, we have

$$\begin{aligned}
|Cov(f(x_{\tilde{\ell}}); g(x_\ell))| &\leq D \exp\{-\alpha d(\tilde{\ell}, \ell)\}||f||_{h,\infty}||g||_{h,\infty} \int_X h(x_{\tilde{\ell}})\tilde{h}(x_\ell)\mu(dx) \\
&\leq D \exp\{-\alpha d(\tilde{\ell}, \ell)\}||f||_{h,\infty}||g||_{h,\infty} \left[\int_X h^2(x_{\tilde{\ell}})\mu(dx)\right]^{1/2} \left[\int_X \tilde{h}^2_\ell(x)\mu(dx)\right]^{1/2} \\
&\leq D \exp\{-\alpha d(\tilde{\ell}, \ell)\}||f||_{h,\infty}||g||_{h,\infty} \left[\int_X h^2(x_{\tilde{\ell}})\mu(dx)\right]^{1/2} \left[\int_X \tilde{h}^2_\ell(x)\mu(dx)\right]^{1/2} \\
&\leq DM_3 \exp\{-\alpha d(\tilde{\ell}, \ell)\}||f||_{h,\infty}||g||_{h,\infty}
\end{aligned}$$

where

$$\begin{aligned}
\left[\int_X \tilde{h}^2_\ell(x)\mu(dx)\right]^{1/2} &\leq \left[\int_X \left(M_1 + 1 + \sum_{\ell' \in \partial\ell} c_{\ell\ell'} h(y_{\ell'})\right)^2 \mu(dx)\right]^{1/2} \\
&\leq M_1 + 1 + 1/\Delta^\chi \sum_{\ell' \in \partial\ell} \left[\int_X h^2(y_{\ell'})\mu(dx)\right]^{1/2} \\
&\leq M_1 + 1 + 1/\Delta^{\chi-1} M_2 =: M_3(h, \Delta, \chi) =: M_3.
\end{aligned}$$

□

Remark 2.21. (i) There exists a series of results of decay of correlations for Gibbs measures based on the classical Dobrushin uniqueness criterion, see e.g. [DS85b], [DS85a], [Föl82], [Gro79], [Kün82]. In particular, Proposition 3.1 in [Kün82] gives an exponential bound similar to (2.78), for $h = 1$, with coefficients depending on the volume of L, the sup norm of f and g and on the contraction parameters.

(ii) A preliminary and less rigorous version of Theorem 2.19, but in a completely different context, can be found in Section 6 of [CM12]. As compared with that paper, we give a complete proof and establish precise estimates on the relaxation parameters.

2.4 Bibliographical notes

A first attempt at solving the uniqueness problem for Gibbsian random fields was made by Dobrushin in 1970 in his pioneering paper [Dob70]. His approach is based on the coupling method and the so-called *reconstruction* (or *"surgery"*) argument (see also [dlRFS08] for an abstract setting). This method applies well to interacting particle systems with bounded spins. In the case of unbounded spins (e.g. taking values in $\mathbb{R}^n$) it is commonly known (see [COPP78], [AKRP97], [AKKR09], [KP07], [Roy77], [Yos99], [Zeg90]) that the Dobrushin contraction condition can be checked only for the pair interactions of at most quadratic growth (see the model description in Section 3.1). So, the case of general (non-attractive) pair (or many-particle) interactions of super-quadratic growth remains so far open (except in the case one uses cluster expansion methods, see e.g. the monograph [MM91]).

Further constructive criteria for uniqueness of Gibbs measures (related with the so-called complete analyticity) in the case of compact spins can be found in subsequent work of Dobrushin, see e.g. [DS85b], [DS85a] for a generalization of the Dobrushin classical criterion to larger volumes, which is however restricted to the translation invariant case and does not apply to general graphs. For classical ferromagnetic systems with scalar (possibly unbounded) spins, the uniqueness of the Gibbs states is related to the exponential relaxation of the corresponding Glauber dynamics, which is described by means of the Poincaré and log-Sobolev inequalities ([BH99], [BH00], [Led01], [OR07], [Wu06], [Yos99], [Zeg90], [Zit08]). A dual approach to proving Dobrushin's uniqueness criterion via averaging over observables later appeared in [Föl82] and [BKMP07]. A nice overview of some of these results can be found in [Bet12].

There are further basic approaches for proving uniqueness of Gibbs measures for specific models in the literature (via Ruelle's method in [Rue70] and [LP76], via cluster expansion in [MM91] and [PY95], via correlation inequalities [LP76], [JB82]). However, the article of Dobrushin and Pechersky [DP83] seems to be the only one dealing with the uniqueness problem for unbounded spins, applicable also in the case of rather general *super-quadratic* interactions. So far, this criterion remained poorly recognized (see the comments in [Pas07] and [CM12]). It was only essentially employed in a series of papers for proving uniqueness for some models of systems of classical gasses in $\mathbb{R}^d$ (see [PZ99] and [BP02]). Our aim for the subsequent chapters will be to show its applicability to newer, more interesting and advanced models.

As a final remark, we note that one could be optimistic about obtaining a result for the uniqueness of Gibbs measures even in the case of graphs with unbounded degree, based on a result of Malyshev and Nikolaev [MN84], where the case of graphs with unbounded degree is considered and an uniqueness result is proven by using the method of cluster expansions. An existence result in this setting was obtained in [KKP10].

Appendix 2.A The total variation distance

We briefly recall some general facts concerning the total variation distance between probability distributions. For a more detailed presentation, one can see, for example, [Pol]. Given two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on a sigma-algebra $\mathcal{A}$ on the same sample space Ω, their total variation distance is given by

$$d_{TV}(\mathbb{P}_1, \mathbb{P}_2) := \sup_{A \in \mathcal{A}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)|. \tag{2.85}$$

If λ is any $\sigma-$finite measure on $(\Omega, \mathcal{A})$ such that both $\mathbb{P}_1$ and $\mathbb{P}_2$ are absolutely continuous with respect to λ, and p_1 and p_2 are the respective densities we have the following equality

$$d_{TV}(\mathbb{P}_1, \mathbb{P}_2) := \frac{1}{2} \int_{\Omega} |p_1 - p_2| d\lambda. \tag{2.86}$$

As already mentioned in Section 2.1.4, an equivalent description of the total variation distance is given by

$$d_{TV}(\mathbb{P}_1, \mathbb{P}_2) := \inf_{\hat{\mathbb{P}} \in \mathcal{C}(\mathbb{P}_1, \mathbb{P}_2)} \int_{\Omega^2} \mathbb{1}_{\neq}(z_1, z_2) \hat{\mathbb{P}}(dz_1, dz_2). \tag{2.87}$$

The idea how to construct an *optimal coupling*, i.e. a coupling such that the infimum in (2.87) is attained, can be found in [Pol], but also in Chapter 6 of [Vil09] and Chapter I.5 of [Lin02], where it is called *Gamma coupling*.

The intuition behind the following seemingly complicated formula (2.88) of this optimal coupling is that one puts the common mass of $\mathbb{P}_1$ and $\mathbb{P}_2$ on the diagonal and distributes the rest uniformly. Namely, let $\mathbf{\Delta} := \{(z, z) : z \in \Omega\}$ be the diagonal of $\Omega \times \Omega$ and define $\psi : \Omega \to \Omega \times \Omega$ by $\psi(z) = (z, z)$. Put $\lambda := \mathbb{P}_1 + \mathbb{P}_2$ and let $p_i := \frac{d\mathbb{P}_i}{d\lambda}$, for $i = 1, 2$. Obviously p_1 and p_2 are well-defined, since $\mathbb{P}_1$ and $\mathbb{P}_2$ are absolutely continuous with respect to λ. Let $\mathbb{Q}$ be the measure on $(\Omega, \mathcal{A})$ such that

$$\frac{d\mathbb{Q}}{d\lambda} = \min(p_1, p_2),$$

and let $\hat{\mathbb{Q}} := \mathbb{Q} \circ \psi^{-1}$ to be the lift of $\mathbb{Q}$ on $\mathbf{\Delta}$. Set $\gamma := \hat{\mathbb{Q}}(\mathbf{\Delta})$ and define

$$\hat{\mathbb{P}} := \hat{\mathbb{Q}} + \frac{(\mathbb{P}_1 - \mathbb{Q}) \otimes (\mathbb{P}_2 \otimes \mathbb{Q})}{1-\gamma}. \tag{2.88}$$

Elementary computations show that $\hat{\mathbb{P}}$ is indeed a coupling of $\mathbb{P}_1$ and $\mathbb{P}_2$. We remark that sometimes in the literature, $\mathbb{Q}$ is also denoted by $\mathbb{P}_1 \wedge \mathbb{P}_2$. Also, it is not difficult to show that $1 - \gamma = d_{TV}(\mathbb{P}_1, \mathbb{P}_2)$.

Proof of Lemma 2.8

In the particular case of a family of one point local states π, for any $\ell \in \mathsf{V}$ and $x, y \in X$, there exists a coupling

$$\rho_\ell^{x,y} := (\widehat{\pi_\ell^x \wedge \pi_\ell^y}) + \frac{(\pi_\ell^x - \pi_\ell^x \wedge \pi_\ell^y) \otimes (\pi_\ell^y - \pi_\ell^x \wedge \pi_\ell^y)}{d_{TV}(\pi_\ell^x, \pi_\ell^y)}. \tag{2.89}$$

which is measurable with respect to the boundary conditions (x, y). The measurability follows easily from the explicit expression of $\rho_\ell^{x,y}$.

□

Appendix 2.B Existence of tempered Gibbs measures

The aim of this section is to show that a specification, whose associated family of one-point local states satisfies conditions (IC) and (CC), admits a measure consistent with it.

2.B.1 Tempered measures

Let $V : \Xi \to \mathbb{R}_+$ be a measurable function such that

$$V(\xi) \geq h(\xi), \qquad \xi \in \Xi, \tag{2.90}$$

and define σ as a probability measure on Ξ by

$$\sigma(d\xi) = C \exp\big(-V(\xi)\big) d\xi, \tag{2.91}$$

where $C > 0$ is the normalizing factor. Since V is positive, the measure σ is well-defined. Let $\sigma_{\mathsf{L}}(dx_{\mathsf{L}}) := \bigotimes_{l \in \mathsf{L}} \sigma(dx_\ell)$ be the product measure on $(\Xi^{\mathsf{L}}, \mathcal{B}(\Xi^{\mathsf{L}}))$. We assume that the interaction in our system can be described by the *conditional* *local Hamiltonians* $\{H_{\mathsf{L}}(\cdot|y)\}$ indexed by all finite sets $\mathsf{L} \Subset \mathsf{V}$ and by all boundary conditions $y \in X$, where each $H_{\mathsf{L}}(\cdot|y) : \Xi^{|\mathsf{L}|} \to \mathbb{R}$ is a measurable function invariant under permutations of its coordinates. Furthermore, we assume that there exist constants $J_1, J_2, J_3 > 0$ such that

$$|H_{\mathsf{L}}(x_{\mathsf{L}}|y)| \leq J_1|\mathsf{L} \cup \partial\mathsf{L}| + J_2 \sum_{\ell\in\mathsf{L}} h(x_\ell) + J_3 \sum_{\ell'\in\partial\mathsf{L}} h(y_{\ell'}). \tag{2.92}$$

We introduce the *local Gibbs specification* π by its kernels

$$\pi_{\mathsf{L}}(B|y) := Z_{\mathsf{L}}^{-1}(y) \int_X \mathbb{1}_B(x_{\mathsf{L}} \times y_{\mathsf{L}^c}) \exp\{-H_{\mathsf{L}}(x_{\mathsf{L}}|y)\}\sigma_{\mathsf{L}}(dx_{\mathsf{L}}) \otimes_{\ell'\in\mathsf{L}^c} \delta_{y_{\ell'}}, \quad y \in X^t, \tag{2.93}$$

where

$$Z_{\mathsf{L}}(y) := \int_X \exp\{-H_{\mathsf{L}}(x_{\mathsf{L}}|y)\}\sigma_{\mathsf{L}}(dx_{\mathsf{L}}) \otimes_{\ell'\in\mathsf{L}^c} \delta_{y_{\ell'}}.$$

It is easy to see that $\pi_{\mathsf{L}}(\cdot|y)$ is a probability measure on X. In what follows, we assume that the family $\{\pi_{\mathsf{L}}\}$ is consistent in the sense of (2.3). We are now interested in the existence of probability measures μ satisfying the the DLR equations (2.4). Denote the set of such measures by $\mathcal{G}(\pi)$. We show that an existence result can be proven under restrictions only on the one-point specification π corresponding to $\{\pi_{\mathsf{L}}\}$, namely, we assume that $\pi \in \Pi(h, K, \kappa, c)$.

On V we introduce the path distance $\rho : \mathsf{V} \times \mathsf{V} \to \mathbb{N}$ by $\rho(\ell, \ell') := N$, being the smallest number such that there exists a N-path from ℓ to ℓ'.

It is typical in the case of systems with unbounded spins that we have to restrict ourselves to a certain subset X^t of reasonable configurations and, respectively, to the measures $\mu \in \mathcal{P}(X)$ supported by X^t. Their choice is strongly based on the conditions imposed by the interaction. In our case, set $\bar{\alpha} := \ln(1/\bar{c})$, where $\bar{c}$ is as in (2.8). Now, we define the set of *tempered configurations* to be

$$X^t := \bigcap_{o\in\mathsf{V}, 0<\alpha<\bar{\alpha}} X_{o,\alpha},$$

where

$$X_{o,\alpha} := \left\{ x \in X : ||x||_{o,\alpha} := \sup_\ell \left\{h(x_\ell) \exp\{-\alpha\rho(o,\ell)\}\right\} < \infty \right\},$$

and, respectively, the set of *tempered Gibbs measures*

$$\mathcal{G}_\pi^h := \{\mu \in \mathcal{G}(\pi) : \mu(X^t) = 1\}. \tag{2.94}$$

2.B.2 Moment estimates

As a first step in proving the existence, we establish moment estimates for $\pi_{\mathsf{L}}(dx|y)$.

Lemma 2.22. *For any $0 < \alpha < \bar{\alpha}$, any $\ell \in \mathsf{V}$ and every $y \in X^t$ we find a constant $M(\bar{c}, \alpha, y)$ such that for all $\mathsf{L} \Subset \mathsf{V}$*

$$\int_X h(x_\ell)\pi_{\mathsf{L}}(dx|y) \le M(\bar{c}, \alpha, y). \tag{2.95}$$

Moreover, we find a positive constant $M(\bar{c}, \alpha)$ such that for all $y \in X_\alpha$

$$\limsup_{\mathsf{L}\to\mathsf{V}} \int_X h(x_\ell)\pi_{\mathsf{L}}(dx|y) \le M(\bar{c}, \alpha). \tag{2.96}$$

Proof. From condition (IC), we know that for all $\ell \in \mathsf{V}$ and any $z \in X^t$

$$\int_X h(x_\ell)\pi_\ell(dx|z) \le 1 + \sum_{\ell'\in\partial\ell} c_{\ell\ell'} h(z_{\ell'}).$$

We integrate with respect to $\pi_{\mathsf{L}}(\cdot|x)$ and use the DLR equation with $z = y$ to obtain,

$$\int_X h(x_\ell)\pi_{\mathsf{L}}(dx|y) \le 1 + \sum_{\ell'\in\partial\ell\cap\mathsf{L}^c} c_{\ell\ell'} h(y_{\ell'}) + \sum_{\ell'\in\partial\ell\cap\mathsf{L}} c_{\ell\ell'} \int_X h(x_{\ell'})\pi_{\mathsf{L}}(dx|y). \tag{2.97}$$

We consider now any domain L containing a fixed point $\ell_0 \in \mathsf{V}$. In equation (2.97), we multiply by $e^{-\alpha\rho(\ell_0,\ell)}$ and take the supremum over the sites $\ell \in \mathsf{V}$ to get

$$\begin{aligned}
\int_X h(x_{\ell_0})\pi_{\mathsf{L}}(dx|y) &\le \sup_{\ell\in\mathsf{L}} \left\{ \int_X h(x_\ell) e^{-\alpha\rho(\ell_0,\ell)} \pi_{\mathsf{L}}(dx|y) \right\} \\
&\le 1 + \sup_{\ell\in\mathsf{L}} \left\{ \sum_{\ell'\in\partial\ell\cap\mathsf{L}} c_{\ell\ell'} \int_X h(x_{\ell'}) e^{-\alpha\rho(\ell_0,\ell')} e^{\alpha(|\rho(\ell_0,\ell')-\rho(\ell_0,\ell))} \pi_{\mathsf{L}}(dx|y) \right\} \\
&+ \sup_{\ell\in\mathsf{L}} \left\{ \sum_{\ell'\in\partial\ell\cap\mathsf{L}^c} c_{\ell\ell'} h(y_{\ell'}) e^{-\alpha\rho(\ell_0,\ell')} e^{\alpha(\rho(\ell_0,\ell')-\rho(\ell_0,\ell))} \right\} \\
&\le 1 + e^\alpha \bar{c} \sup_{\ell\in\mathsf{L}} \left\{ \int_X h(x_\ell) e^{-\alpha\rho(\ell_0,\ell)} \pi_{\mathsf{L}}(dx|y) \right\} \\
&+ e^\alpha \bar{c} \sup_{\ell'\in\partial\mathsf{L}} \left\{ h(y_{\ell'}) e^{-\alpha\rho(\ell_0,\ell')} \right\}.
\end{aligned}$$

Elementary computations then yield

$$\sup_{\ell \in \mathsf{L}} \left\{ \int_X h(x_\ell) e^{-\alpha\rho(\ell_0,\ell)} \pi_{\mathsf{L}}(dx|y) \right\} \leq \frac{1}{1-e^{\alpha}\bar{c}} \left[1 + e^{\alpha}\bar{c} \sup_{\ell' \in \partial \mathsf{L}} \left\{ h(y_{\ell'}) e^{-\alpha|\ell'|} \right\} \right]. \tag{2.98}$$

Thus (2.95) holds by setting

$$M(\bar{c},\alpha,y) := \frac{1}{1-e^{\alpha}\bar{c}} \left[1 + e^{\alpha}\bar{c} \sup_{\ell' \in \partial \mathsf{L}} \left\{ h(y_{\ell'}) e^{-\alpha|\ell'|} \right\} \right].$$

Since for $y \in X^t$, $||y||_{\alpha,\ell_0} = \sup_{\ell' \in \partial \mathsf{L}} \left\{ h(y_{\ell'}) e^{-\alpha|\ell'|} \right\}$ tends to zero as $\mathsf{L} \to \mathsf{V}$, passing to the limit, we obtain (2.96) for any $\ell \in \mathsf{V}$

$$\limsup_{\mathsf{L} \to \mathsf{V}} \int_X h(x_{\ell_0}) \pi_{\mathsf{L}}(dx|y) \leq \frac{1}{1-e^{\alpha}\bar{c}} =: M(\bar{c},\alpha).$$

□

2.B.3 Compactness of the local Gibbs specification

Lemma 2.23. *For every $y \in X_\alpha$, the family $\{\pi_{\mathsf{L}}(\cdot|y)\}_{\mathsf{L} \Subset \mathsf{V}} \subset \mathcal{P}(X)$ is relatively $\mathcal{T}_{loc}$-compact.*

Proof. Following the arguments in [Geo88], it is sufficient to prove that the family $\{\pi_{\mathsf{L}}(\cdot|y)\}_{\mathsf{L} \Subset \mathsf{V}}$ is locally equicontinuous. This means that, for every $\mathsf{L} \Subset \mathsf{V}$ and a sequence $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{B}(X_{\mathsf{L}})$ with $A_k \downarrow \emptyset$ as $k \to +\infty$, we have

$$\lim_{k \to \infty} \limsup_{\mathsf{L} \to \mathsf{V}} \pi_{\mathsf{L}}(A_k|y) = 0. \tag{2.99}$$

In order to obtain this property, we adapt the arguments from [Geo88] (similar reasoning can also be found for example in [KKP10]). Let T be a positive number and let L be as above. Also, set

$$B_T := \{x \in X : h(x_\ell) \leq T, \forall \ell \in \mathsf{L} \cup \partial \mathsf{L}\}, \qquad B_T^c = X \setminus B_T.$$

For a fixed $k \in \mathbb{N}$, we have

$$\limsup_{\mathsf{L} \to \mathsf{V}} \pi_{\mathsf{L}}(A_k|y) \leq \limsup_{\mathsf{L} \to \mathsf{V}} \pi_{\mathsf{L}}(A_k \cap B_T|y) + \limsup_{\mathsf{L} \to \mathsf{V}} \pi_{\mathsf{L}}(B_T^c|y).$$

We will now estimate separately each of the terms in the right-hand side. For the second one we have, by Markov's inequality

$$\pi_{\mathsf{L}}(B_T^c|y) \le \sum_{\ell\in\mathsf{L}\cup\partial\mathsf{L}} \pi_{\mathsf{L}}\big(h(x_\ell) > T|y\big) \le \sum_{\ell\in\mathsf{L}\cup\partial\mathsf{L}} \int_X \frac{h(x_\ell)}{T}\pi_{\mathsf{L}}(dx|y).$$

Passing to the limit, we get

$$\limsup_{\mathsf{L}\to\mathsf{V}} \pi_{\mathsf{L}}(B_T^c|y) \le \frac{1}{T}\sum_{\ell\in\mathsf{L}\cup\partial\mathsf{L}} \limsup_{\mathsf{L}\to\mathsf{V}} \int_X h(x_\ell)\pi_{\mathsf{L}}(dx|y) \le \frac{\varepsilon}{2}$$

for T big enough.

Now, for the first term, we proceed as follows: we estimate $\pi_{\mathsf{L}}(A_K\cap B_T|y)$, $y\in X$, which, in view of its definition is non zero only if $z_{\mathsf{L}}\times y_{\partial\mathsf{L}}\in B_T$. In this case, by (2.92) we have

$$\begin{aligned}
Z_{\mathsf{U}}(y) &= \int_{\Xi^{\mathsf{U}}} \exp\left\{-H_{\mathsf{U}}(x_{\mathsf{U}}|y)\right\}\sigma_{\mathsf{U}}(dz_{\mathsf{L}})\\
&\ge \int_{\Xi^{\mathsf{U}}} \exp\left\{-J_1|\mathsf{U}\cup\partial\mathsf{U}| - J_2\sum_{\ell\in\mathsf{U}} h(x_\ell) - J_3\sum_{\ell'\in\partial\mathsf{U}} h(y_{\ell'})\right\}\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\\
&\ge \exp\left\{-J_1|\mathsf{U}\cup\partial\mathsf{U}| - J_3|\partial\mathsf{U}|T\right\}\int_{\Xi^{\mathsf{U}}} \exp\left\{-J_2\sum_{\ell\in\mathsf{U}} h(z_\ell)\right\}\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\\
&\ge \exp\left\{-J_1|\mathsf{U}\cup\partial\mathsf{U}| - J_3|\partial\mathsf{U}|T - J_2\int_{\Xi^{\mathsf{U}}}\sum_{\ell\in\mathsf{U}} h(z_\ell)\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\right\},
\end{aligned}$$

Hence, again by (2.92) and (2.90)

$$\begin{aligned}
\pi_{\mathsf{U}}(A_k\cap B_T|y) &\le \exp\left\{J_1|\mathsf{U}\cup\partial\mathsf{U}| + J_3|\partial\mathsf{U}|T + J_2\int_{\Xi^{\mathsf{U}}}\sum_{\ell\in\mathsf{U}} h(z_\ell)\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\right\}\\
&\quad\times\int_{\Xi^{\mathsf{U}}} \exp\left\{J_1|\mathsf{U}\cup\partial\mathsf{U}| + J_2\sum_{\ell\in\mathsf{U}} h(x_\ell) + J_3\sum_{\ell'\in\partial\mathsf{U}} h(y_{\ell'})\right\}\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\\
&\le \exp\left\{(2J_1+TJ_2)|\mathsf{U}\cup\partial\mathsf{U}| + J_3|\partial\mathsf{U}|(T+1) + J_2\int_{\Xi^{\mathsf{U}}}\sum_{\ell\in\mathsf{U}} h(z_\ell)\sigma_{\mathsf{U}}(dz_{\mathsf{U}})\right\}.
\end{aligned}$$

Thus $\pi_{\mathsf{U}}(A_k\cap B_T|y) < \varepsilon/2$ for k sufficiently large. Applying the consistency property (2.3), for any $\mathsf{L}\Subset\mathsf{V}$ that contains U, yields

$$\pi_{\mathsf{L}}(A_k\cap B_T|x) < \varepsilon/2,$$

which proves our result. □

Theorem 2.24. *The set $\mathcal{G}_\pi^h$ is non-empty and, for every $0 < \alpha < \bar{\alpha}$, $\mu \in \mathcal{G}_\pi^h$ and $\ell \in \mathsf{V}$, there exists a positive constant $M(\bar{c}, \alpha)$ such that*

$$\int_X h(x_\ell)\mu(dx) \leq M(\bar{c}, \alpha). \tag{2.100}$$

Proof. Fix $y \in X^t$. By Propositions 4.9 and 4.15 in [Geo88] any locally equicontinuous net in $\mathcal{P}(X)$ has at least one $\mathcal{T}_{loc}$-cluster point, which can be obtained as the limit of a certain subsequence. Therefore, by Lemma 2.23 there exists an increasing sequence $\{\mathsf{L}_k\}_{k\in\mathbb{N}}$ which exhausts V and such that the sequence $\{\pi_{\mathsf{L}_k}(\cdot|y)\}_{k\in\mathbb{N}}$ converges to a $\mu \in \mathcal{P}(X)$. We will show now that this μ is a Gibbs measure, i.e. it is consistent with the specification π. For any $\mathsf{L} \Subset \mathsf{V}$, there is an k' such that $\mathsf{L} \subset \mathsf{L}_n$ for any $k \geq k'$. For such k and $A \in \mathcal{B}_{loc}$ local event, by (2.3) we have

$$\int_X \pi_{\mathsf{L}}(A|x)\pi_{\mathsf{L}_k}(dx|y) = \pi_{\mathsf{L}_k}(A|y).$$

Since we know directly from the definition of π_{L} that the function $X \ni x \mapsto \pi_{\mathsf{L}}(A|x)$ is in $\mathcal{F}_{loc}$, we can pass in the relation above to the limit as $k \to \infty$ and obtain that $\mu \in \mathcal{M}(\pi)$. By Levi's monotone convergence theorem and from (2.96) we conclude that for all $0 < \alpha < \bar{\alpha}$

$$\int_X h(x_\ell)\mu(dx) = \lim_{N\to\infty} \int_X \min\{N; h(x_\ell)\}\mu(dx)$$
$$= \lim_{N\to\infty} \lim_{k\to\infty} \int_X \min\{N; h(x_\ell)\}\pi_{\mathsf{L}_k}(dx|y) \leq M(\bar{c}, \alpha),$$

which implies that μ is supported by $\mathcal{G}^t$. $\square$

Chapter 3

Classical lattice systems

From now on, our main purpose will be to show how the uniqueness criterion obtained in Chapter 2 can be applied to several models. We start the series of examples with the interacting systems of classical spins, which is the simplest and most studied model. We show that a uniqueness result holds even in the case of super-quadratic interactions. We shortly review and give essential improvements of some results which can be found in Section 2.3 of [Pas]. A main new issue is that we give computable bounds on the critical parameter and prove the decay of correlations in this type of systems. On the other hand, this chapter can be seen as a preparatory step in considering the so-called amorphous crystals in the annealed approach, combining properties of classical anharmonic crystals and particle systems in the continuum, see Chapter 5.

3.1 Short description of the model

The physical space for the model we consider is given by the lattice $\mathsf{V} = \mathbb{Z}^d$ and the spin space will be $\Xi = \mathbb{R}^n$, for n, d fixed positive integers. The *configuration space* of the system $X := (\mathbb{R}^n)^{\mathbb{Z}^d}$ consists of all sequences $x = (x_\ell)_{\ell \in \mathbb{Z}^d}$. In the framework of statistical mechanics, one can speak about a system of classical particles performing oscillations with vector displacements x_ℓ, around their non-stable equilibrium positions at the sites of $\mathbb{Z}^d$. The energy of a configuration $x \in X$ is represented by the following formal Hamiltonian

$$H(x) := \sum_{\ell} V_\ell(x_\ell) + \sum_{\{\ell,\ell'\}} W_{\ell\ell'}(x_\ell, x_{\ell'}), \tag{3.1}$$

where the interaction potentials are given by the measurable (not necessarily continuous) functions

$$V_\ell : \mathbb{R}^n \to \mathbb{R},\ \ell \in \mathbb{Z}^d,$$

$$W_{\ell\ell'} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ W_{\ell\ell} \equiv 0,\ \ell, \ell' \in \mathbb{Z}^d.$$

satisfying the following basic conditions

(W) There exist constants $R \geq 2$, $I_W \geq 0$ and a symmetric matrix $\mathbb{J} = (J_{\ell\ell'})_{\mathbb{Z}^d \times \mathbb{Z}^d}$ with non-negative entries and zero diagonal, such that for all $x_\ell, x_{\ell'} \in \mathbb{R}^n$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq J_{\ell\ell'}(I_W + |x_\ell|^R + |x_{\ell'}|^R),\ \ell \neq \ell'.$$

Without loss of generality, we assume that the pair potentials $W_{\ell\ell'}$ are invariant with respect to permutations of the coordinates ℓ, ℓ' and variables $x_\ell, x_{\ell'}$, respectively.

(FR) The interaction has finite range, i.e there exists $r > 0$ such that

$$J_{\ell\ell'} = 0 \text{ for any } |\ell - \ell'| > r.$$

Obviously, this yields $W_{\ell\ell'} \equiv 0$ for $|\ell - \ell'| > r$.

We make the following notation

$$||J||_0 := \sup_\ell \sum_{\ell'} J_{\ell\ell'}.$$

Due to the finite range condition, the above quantity is well-defined.

(V) There exist a measurable function $V : \mathbb{R}^n \to \mathbb{R}$ and constants $P \geq R$, $A_V > 0$ and $B_V \in \mathbb{R}$, such that for all $l \in \mathbb{Z}^d$ and $x_\ell \in \mathbb{R}^n$

$$A_V|x_\ell|^P + B_V \leq V_\ell(x_\ell) \leq V(x_\ell).$$

Moreover, the constant A_V can be chosen large enough, so that the following relation holds

$$\frac{2}{3}A_V > ||J||_0.$$

Remark 3.1. One can describe the interaction between particles located on the lattice as nearest neighbour interaction between edges of a graph, by the use of assumption (FR) as follows. We define $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, where $\mathsf{V} = \mathbb{Z}^d$ and E is the set of all pairs $(\ell, \ell') \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $|\ell - \ell'| < r$. We use the same notation as in Chapter 2, namely Δ for the maximum degree of the graph and χ for its chromatic number. For instance, in the case $r = 1$, one has $\Delta = 2n$ and $\chi = 2$. Also, $\ell \sim \ell'$ means that there exists an edge between ℓ and ℓ' (i.e. $|\ell - \ell'| < r$). From now on, whenever we refer to $\mathbb{Z}^d$, we actually mean the associated graph structure.

As it is usual in the case of systems with unbounded spins, we have to restrict ourselves to a certain subset X^t of reasonable configurations and, respectively, to the measures $\mu \in \mathcal{P}(X)$ supported by such $x \in X^t$. The optimal choice for such configurations is strongly determined by the conditions on the interaction. Define

$$X_p := \left\{ x \in X : ||x||_p := \left[\sum_{\ell} (1+|\ell|)^{-p} |x_\ell|^R \right]^{1/R} < \infty \right\}, \qquad p > d, \tag{3.2}$$

where $R \geq 2$ is given by Assumption (W). The restriction $p > d$ is just for technical convenience, for more details see Section 2.1.1 in [Pas]. We introduce the subset of *tempered configurations*

$$X^t := \bigcup_{p>d} X_p = \{x \in X : \exists p = p(x) \text{ s.t. } ||x||_p < \infty\} \tag{3.3}$$

and, respectively, the subset of *tempered measures*

$$\mathcal{P}^t(X) := \{\mu \in \mathcal{P}(X) : \exists p = p(\mu) > d \text{ s.t. } \mu(X_p) = 1\}. \tag{3.4}$$

We follow the standard Dobrushin-Lanford-Ruelle (DLR) approach and define the Gibbsian random fields as probability measures on the space $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra corresponding to the product topology on X. For a finite volume $\mathsf{U} \Subset \mathbb{Z}^d$, let $H_{\mathsf{U}}(\cdot|y)$ be the local interaction energy corresponding to the Hamiltonian (3.1) and to the boundary condition $y \in X$.

$$H_{\mathsf{U}}(x_{\mathsf{U}}|y) = H_{\mathsf{U}}(x_{\mathsf{U}}) + \sum_{\ell \sim \ell' : \ell \in \mathsf{U}, \ell' \in \mathsf{U}^c} W(x_\ell, y_{\ell'}), \tag{3.5}$$

where

$$H_{\mathsf{U}}(x_{\mathsf{U}}) = \sum_{\ell \in \mathsf{U}} V_\ell(x_\ell) + \frac{1}{2} \sum_{\ell \sim \ell' : \ell, \ell' \in \mathsf{U}} W(x_\ell, x_{\ell'}) \tag{3.6}$$

By means of this energy, for U as above, $A \in \mathcal{B}(X)$ and some fixed $\beta > 0$, we can define

$$\pi_{\mathsf{U}}(A|y) = \begin{cases} \frac{1}{Z_{\mathsf{U}}^\beta(y)} \int_{(\mathbb{R}^n)^{\mathsf{U}}} \mathbb{1}_A(x_{\mathsf{U}} \times y_{\mathsf{U}^c}) \exp\left(-\beta H_{\mathsf{U}}(x_{\mathsf{U}}|y)\right) \times_{l \in \mathsf{U}} dx_\ell, & y \in X^t \\ 0, & y \notin X^t. \end{cases}$$
$$Z_{\mathsf{U}}^\beta(y) = \int_{(\mathbb{R}^n)^{\mathsf{U}}} \exp\left(-\beta H_{\mathsf{U}}(x_{\mathsf{U}}|y)\right) \times_{\ell \in \mathsf{U}} dx_\ell, \tag{3.7}$$

as being the *local Gibbs specification* for the model under consideration. It can easily be seen from this definition that $\pi_{\mathsf{U}}(\cdot|y)$ is a probability measure on $(X, \mathcal{B}(X))$ and that this family satisfies the consistency condition

$$\int_X \pi_{\mathsf{W}}(A|y)\pi_{\mathsf{U}}(dy|x) = \pi_{\mathsf{U}}(A|x), \tag{3.8}$$

for all $A \in \mathcal{B}(X)$, all $\mathsf{U} \Subset \mathbb{Z}^d$ and all $\mathsf{W} \subset \mathsf{U}$.

Definition 3.2. A measure $\mu \in \mathcal{P}(X)$ is said to be a Gibbs random field corresponding to Hamiltonian (3.1) if it solves the following DLR equation

$$\mu(A) = \int_{X^t} \pi_{\mathsf{U}}(A|y)\mu(dy), \tag{3.9}$$

for all $A \in \mathcal{B}(X)$ and $\mathsf{U} \Subset \mathbb{Z}^d$.

For fixed inverse temperature β, we denote by $\mathcal{G}(\pi)$ the set of all solutions of (3.9) and by $\mathcal{G}^t(\pi)$ the set of all measures $\mu \in \mathcal{G}(\pi)$ for which

$$\sup_{\ell \in \mathbb{Z}^d} \int_X |x_\ell|^R \mu(dx) < \infty. \tag{3.10}$$

Remark 3.3. (i) The partition function $Z^{\beta}_{\mathsf{U}}(y)$ is well-defined for any $y \in X^t$ and $\mathsf{U} \Subset \mathbb{Z}^d$, cf. Proposition 2.3 in [Pas].

(ii) For results on the existence of random fields satisfying 3.9, see [KKP10] or Section 2.2 in [Pas]. In particular, it was shown that $\mathcal{G}^t$ is not empty under the above assumptions and furthermore, for any $\mu \in \mathcal{G}^t(\pi)$ and $\tau < \beta(A_V - 1/2||J||_0)$, there exists a positive constant C such that

$$\sup_{\ell} \int_X \exp\{\tau |x_\ell|^R\}\mu(dx) \leq C. \tag{3.11}$$

In previous works on classical systems (see [AKRP97], [COPP78], [KP07], [Roy77], [Zeg90]), uniqueness was proved using the Dobrushin uniqueness criterion, but only for quadratic interactions. In what follows, we will show that by a simple application of the Dobrushin-Pechersky criterion, we can obtain an uniqueness result even in the super-quadratic case. We will see that the Dobrushin-Pechersky conditions (CC) and (IC) can be verified with the function

$$h(\xi) := \theta |\xi|^R, \tag{3.12}$$

where $\theta > 0$ will be chosen later such that condition (IC) is satisfied with constant 1 (see Remark 2.10 (iii)).

Before presenting the uniqueness results, we give the following exponential bound for the one-point kernels $\pi_\ell(dx|y)$ subject to the fixed boundary condition $y \in X^t$:

Lemma 3.4. *[Pas] Supposing Assumptions (W) and (V) are satisfied, for every positive* $\tau < \triangle := A_V - \frac{1}{2}||J||_0$ *there exists a corresponding* $\Upsilon_0 := \Upsilon_0(\beta, \tau) > 0$ *such that for all* $\ell \in \mathbb{Z}^d$ *and* $y \in X^t$

$$\int_X \exp\{\beta\tau|x_\ell|^R\}\pi_\ell(dx|y) \leq \Upsilon_0 \exp\left\{\beta \sum_{\ell' \neq \ell} J_{\ell\ell'}|y_{\ell'}|^R\right\}. \tag{3.13}$$

Proof. Let $\ell \in \mathbb{Z}^d$ be fixed. From the Assumption (W) one has that for all $x, y \in X^t$

$$\sum_{\ell \neq \ell'} |W(x_\ell, y_{\ell'})| \leq \frac{||J||_0}{2}|x_\ell|^R + \frac{1}{2}\sum_{\ell' \neq \ell} J_{\ell\ell'}(I_W + |y_{\ell'}|^R).$$

By this estimate and the definition of a Gibbs specification

$$\int_X \exp\{\beta\tau|x_\ell|^R\}\pi_\ell(dx|y) \leq (E_\ell)/(F_\ell) \cdot \exp\left\{\beta\left(I_W||J||_0 + \sum_{\ell' \neq \ell} J_{\ell\ell'}|y_{\ell'}|^R\right)\right\},$$

where

$$E_\ell := \int_{\mathbb{R}^n} \exp\left\{-\beta\left[V(x_\ell) - \left(\tau + \frac{||J||_0}{2}\right)|x_\ell|^R\right]\right\} dx_\ell$$

$$F_\ell := \int_{\mathbb{R}^n} \exp\left\{-\beta\left[V(x_\ell) + \frac{||J||_0}{2}|x_\ell|^R\right]\right\} dx_\ell.$$

Using the bounds from Assumption (V), one observes that

$$E := \sup_\ell E_\ell < \infty \text{ and } F := \inf_\ell F_\ell > 0.$$

This yields the required estimate (3.13) with the constant

$$\Upsilon_0 := \Upsilon_0(\beta, \tau) := \frac{E}{F} \cdot \exp\{\beta I_W ||J||_0\} < \infty. \tag{3.14}$$

□

3.2 High-temperature uniqueness: $\beta \ll 1$

In what follows, uniqueness results due to high-temperature ($\beta \ll 1$) or small strength of interaction ($||J||_0 \ll 1$) will be established. Although such results are to be expected (via *cluster expansions*, see [MM91]), a direct analytical proof in the case of super-quadratic interaction might be unknown. Theorems 3.7-3.9 are the corresponding results. They are are obtained by a direct application of Theorem 2.9.

(i) *Reduction to the $\beta = 1$ case*

Below we analyse the asymptotic behaviour of the constants in condition (IC) as $\beta \to +0$ and show the necessity of the rescaling procedure, which will be detailed afterwards. We also assume the following condition

(V1) Let $V_\ell : \mathbb{R}^n \to \mathbb{R}$ be measurable, with $V_\ell(0) = 0$ for any $\ell \in \mathbb{Z}^d$ and such that, for given $P \geq R$, there exist positive $A_1 \leq A_2$ and real $B_1 \leq B_2$ and the estimate

$$A_1|x_\ell|^P + B_1 \leq V_\ell(x_\ell) \leq A_2|x_\ell|^P + B_2$$

holds for all $x_\ell \in \mathbb{R}^n$.

If $P = R$, the value of $||J||_0$ is small enough so that

$$\triangle_1 := A_1 - \frac{1}{2}||J||_0 > ||J||_0.$$

Lemma 3.5. *Suppose $2 \leq R \leq P$, and assume (V_1) and (W) hold true for all $||J||_0 \leq \mathcal{J}_0 < 2A_1/3$. Then, for any $\beta_0 > 0$ and $\tau < A_1 - \mathcal{J}_0/2$, there exists $\Upsilon_0 := \Upsilon_0(\beta_0, \mathcal{J}_0, \tau) \geq 1$ such that (3.13) holds simultaneously for all $\ell \in \mathbb{Z}^d$, $y \in X^t$, $||J||_0 \leq \mathcal{J}_0$ and $\beta \leq \beta_0$.*

Proof. Lemma 3.4 gives us, for each $||J||_0 \leq \mathcal{J}_0$ and $\beta \leq \beta_0$, the required bound with the constant

$$\Upsilon_0 := \Upsilon_0(\beta, ||J||_0, \tau) := \frac{E_1}{F_1} \cdot \exp\{\beta_0 I_W \mathcal{J}_0 - B_1 + B_2\} < \infty, \tag{3.15}$$

where we have set

$$E_1 := \int_{\mathbb{R}^n} \exp\{-A_1|x_\ell|^P + (\tau + \mathcal{J}_0/2)\beta_0^{1-R/P}|x_\ell|^R\} dx_\ell, \tag{3.16}$$

$$F_1 := \int_{\mathbb{R}^n} \exp\{-A_2|x_\ell|^P - \mathcal{J}_0/2\beta_0^{1-R/P}|x_\ell|^R\} dx_\ell. \tag{3.17}$$

In the above integrals we have already made the change of variables $x_\ell \to \beta^{-1/P} x_\ell$.

□

The asymptotic of the constants in the (IC) condition are given by the following

Corollary 3.6. *There exists a positive* $C_1 := C_1(\beta_0, \mathcal{J}_0, \tau)$ *such that*

$$\int_X |x_\ell|^R \pi_\ell(dx|y) \leq \beta^{-1} C_1 + \tau^{-1} \sum_{l'} J_{\ell\ell'} |y_{\ell'}|^R \tag{3.18}$$

holds for all $l \in Z^d$, $y \in X^t$, $||J||_0 \leq \mathcal{J}_0$ *and* $\beta \leq \beta_0$.

Proof. Applying Jensen's inequality to the exponential estimate obtained above, we get the desired result with $C_1 := \tau^{-1} \log \Upsilon_0$.

□

The above corollary shows us that condition (IC) does not hold for the initial kernels $\pi_\ell^y(dx)$ uniformly as $\beta \to +0$, hence we need to pass to a modified specification to be constructed, as follows. First, set $\alpha := (\beta ||J||_0^\gamma)^{-1/R}$ with $\gamma \in [0,1]$ to be chosen later. Then, consider the sets $\alpha B := \{x \in X : \alpha^{-1} x \in B\}$ for any $B \in \mathcal{B}(X)$. Now we are able to define the local specification $\widetilde{\Pi}_\alpha = \{\widetilde{\pi}_{\alpha,\mathsf{U}}\}_{\mathsf{U} \Subset \mathbb{Z}^d}$ as

$$\widetilde{\pi}_{\alpha,\mathsf{U}}(B|y) := \pi_{\beta,\mathsf{U}}(\alpha B|\alpha y), \tag{3.19}$$

for $B \in \mathcal{B}(X)$ and $y \in X$.

The following potentials correspond to the rescaled specification

$$\widetilde{V}_\ell(x_\ell) := \beta V_\ell(\alpha x_\ell),\ \widetilde{W}_{\ell\ell'}(x_\ell, x'_\ell) := \beta W_{\ell\ell'}(\alpha x_\ell, \alpha x_{\ell'}), \tag{3.20}$$

and they also satisfy the same Assumptions (W), (V_1) and (FR), but with the constants

$$\widetilde{J}_{\ell\ell'} := J_{\ell\ell'}/||J||_0^\gamma,\ \widetilde{I}_W := \beta ||J||_0^\gamma I_W,$$
$$\widetilde{A}_i := \beta^{1-P/R} ||J||_0^{-\gamma P/R} A_i,\ \widetilde{B}_i := \beta B_i,\ i = 1, 2.$$

(ii) *Uniqueness by small* $||J||_0$

In what follows we will show that, for all values $0 < \beta \leq \beta_0$ and $||J||_0 \leq \mathcal{J}(\beta_0) < A_1(\varDelta^\chi + 1/2)$, the modified specification (3.19) satisfies conditions (IC) and (CC) of Theorem 2.9 with $h(x_\ell) := \theta |x_\ell|^R$, namely, we will prove the following

Theorem 3.7. *Suppose that Assumptions (V1), (FR) and (W) hold. Then for every* $\beta_0 > 0$ *one finds* $\mathcal{J} := \mathcal{J}(\beta_0) > 0$, *such that the set* $\mathcal{G}^t(\pi)$ *is a singleton at all values of* $\beta \leq \beta_0$ *and* $||J||_0 \leq \mathcal{J}$.

Lemma 3.8. *The following estimate holds*

$$d_{TV}\big(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)\big) \le \min_{i=1,2} \int_X \Big| 1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{ell'})\big\} \Big| \, \widetilde{\pi}_\ell(dx|y^i), \tag{3.21}$$

where

$$\triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'}) := \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}) - \widetilde{W}_{\ell\ell'}(x_\ell, y^2_{\ell'}). \tag{3.22}$$

Proof.

Direct application of formula (2.86) for the total variation distance yields

$$\begin{aligned}
d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) &= \frac{1}{2}\int_X \Big| 1 - \widetilde{Z}_\ell(y^1)\widetilde{Z}_\ell^{-1}(y^2) \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \Big| \, \widetilde{\pi}_\ell(dx|y^1) \\
&\le \frac{1}{2}\int_X \Big| 1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \Big| \, \widetilde{\pi}_\ell(dx|y^1) \\
&+ \frac{1}{2}\Big| 1 - \widetilde{Z}_\ell(y^1)\widetilde{Z}_\ell^{-1}(y^2) \Big| \int_X \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \widetilde{\pi}_\ell(dx|y^1) \\
&\le \int_X \Big| 1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \Big| \, \widetilde{\pi}_\ell(dx|y^1).
\end{aligned}$$

The last inequality follows from the bound

$$\begin{aligned}
\Big| 1 - \widetilde{Z}_\ell(y^1)\widetilde{Z}_\ell^{-1}(y^2) \Big| \le & \left(\int_X \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \widetilde{\pi}_\ell(dx|y^1) \right)^{-1} \\
& \cdot \int_X \Big| 1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \Big| \, \widetilde{\pi}_\ell(dx|y^1),
\end{aligned} \tag{3.23}$$

which is checked by direct computations

$$\begin{aligned}
& \Big| 1 - \widetilde{Z}_\ell(y^1) Z_\ell^{-1}(y^2) \Big| \\
&= \left| 1 - \left(\widetilde{Z}_\ell^{-1}(y^1) \int_X \exp\Big\{ - \widetilde{V}_\ell(x_\ell) - \sum_{\ell'\neq\ell} \widetilde{W}_{\ell\ell'}(x_\ell, y^2_{\ell'}) \Big\} dx_\ell \otimes_{\ell'\neq\ell} \delta_{y^2_{\ell'}} \right)^{-1} \right| \\
&= \left| 1 - \left(\int_X \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \widetilde{\pi}_\ell(dx|y^1) \right)^{-1} \right| \\
&= \left(\int_X \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \widetilde{\pi}_\ell(dx|y^1) \right)^{-1} \\
&\qquad \left| \int_X \Big(1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\} \Big) \widetilde{\pi}_\ell(dx|y^1) \right|
\end{aligned}$$

$$\leq \Big(\int_X \exp \big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'}) \big\} \widetilde{\pi}_\ell(dx|y^1) \Big)^{-1}$$
$$\int_X \Big| 1 - \exp \big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'}) \big\} \Big| \widetilde{\pi}_\ell(dx|y^1).$$

□

Proof of Theorem 3.7. We begin by rewriting the key estimate (3.13) in the case of the rescaled specification

$$\begin{aligned} \widetilde{\pi}^y_\ell(dx) :&= \pi_\ell(\alpha^{-1}dx|\alpha y) \\ &= \widetilde{Z}_\ell^{-1}(y) \exp\Big\{ -\widetilde{V}_\ell(x_\ell) - \sum_{\ell'\neq\ell} \widetilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) \Big\} dx_\ell \otimes_{\ell'\neq\ell} \delta_{y_{\ell'}}, \end{aligned} \quad (3.24)$$

when $||J||_0$ varies in some bounded interval $[0, \mathcal{J}_0]$. Pick now a $\tau \in (\varDelta^\chi \mathcal{J}_0, A_1 - \mathcal{J}_0/2)$. Under Assumptions (V1) and (W1), Lemma 3.5 gives us the proper estimate in the limit case $\beta \to +0$. For the modified specification, this estimate can be rewritten as

$$\int_X \exp\{\tau ||J||_0^{-\gamma} |x_\ell|^R\} \widetilde{\pi}_\ell(dx_\ell|y) \leq \Upsilon_0 \exp\Big\{ ||J||_0^{-\gamma} \sum_{\ell'\sim\ell} J_{\ell\ell'} |y_{\ell'}|^R \Big\}, \quad (3.25)$$

where the constant $\Upsilon_0 := \Upsilon_0(\beta_0, \mathcal{J}_0, \tau) \geq 1$ is given explicitly by (3.15). By applying Jensen's inequality, we obtain condition (IC) with the constant $\theta := \big(\tau^{-1}\mathcal{J}_0^\gamma \log \Upsilon_0\big)^{-1}$, where θ is as in (3.12), and the contraction matrix $c_{\ell\ell'} := J_{\ell\ell'}\tau^{-1}$. Hence (IC) holds with $\bar{c} \leq \mathcal{J}_0 \tau^{-1} < 1/\varDelta^\chi$.

We proceed by proving condition (CC). Let $\mathtt{k} < 1$ be a positive real number. Knowing the function h and the constant $\bar{c}$, define $K_* := K_*(h, \bar{c}, \mathtt{k})$ by formula (2.14). We are interested in finding the contraction matrix with entries $\kappa_{\ell\ell'}$. So let us fix ℓ and $\ell' \sim \ell$ and consider a pair of boundary conditions $y^1, y^2 \in X$ such that

$$y^1 = y^2 \text{ off } \ell',\ ||y^1||_\ell^R, ||y^2||_\ell^R \leq K_* \text{ where } ||y||_\ell := \sup_{\ell'\sim\ell} |y_{\ell'}|. \quad (3.26)$$

By the above we have the uniform bound

$$\sup_{||y||_\ell^R \leq K_*} \int_X \exp\{\tau ||J||_0^{-\gamma} |x_\ell|^R\} \widetilde{\pi}_\ell(dx|y) \leq \Upsilon_0 \exp\{\mathcal{J}_0^{1-\gamma} K_*\}. \quad (3.27)$$

In this case $\triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})$ obeys

$$|\triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})| \leq J_{\ell\ell'} ||J||_0^{-\gamma} (\beta ||J||_0^{-\gamma} I_W + K_* + |x_\ell|^R). \quad (3.28)$$

By (3.21) we get the following estimate for the total variation distance

$$d_{TV}\big(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)\big) \leq \int_X \Big|1 - \exp\big\{ \triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\big\}\Big| \widetilde{\pi}_\ell(dx|y^1),$$

which, by applying the elementary inequality $|e^x - 1| \leq e^{|x|} - 1$, yields

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \leq \int_X \exp\{|\triangle \widetilde{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})|\}\widetilde{\pi}_\ell(dx|y^1) - 1. \quad (3.29)$$

Plugging in (3.28) and restricting the problem to the case $\gamma = 0$, one easily observes that

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \leq \exp\{J_{\ell\ell'}(\beta I_W + K_*)\} \int_X \exp\{J_{\ell\ell'}|x_\ell|^R\}\widetilde{\pi}_\ell(dx|y^1) - 1.$$

We next apply Hölder's inequality in the form

$$\mathbb{E}\big[|X|^r\big] \leq \big(\mathbb{E}\big[|X|^s\big]\big)^{r/s}, \text{ with } 0 < r < s,$$

for $r = J_{\ell\ell'} < s = \tau$ and use (3.27) to obtain

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \leq \kappa_{\ell\ell'} := \exp\big\{J_{\ell\ell'}[\tau^{-1}\log\Upsilon_0 + \beta_0 I_W + K_* + \tau^{-1}\mathcal{J}_0 K_*]\big\} - 1. \quad (3.30)$$

We notice that by choosing a small enough $||J||_0 \leq \mathcal{J}_0 := \mathcal{J}_0(\beta_0)$ one gets the required contraction condition $\overline{\kappa} \leq \mathtt{k} < 1$.

□

(iii) *Uniqueness by small β*

We are now concerned with proving a uniqueness result for β small enough. For this, we assume the following holds:

(W1) There exists a non-negative C_1 such that for all $\ell, \ell' \in \mathbb{Z}^d$ and $x_\ell, x_{\ell'} \in \mathbb{R}^n$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'}) - W_{\ell\ell'}(0, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}|x_\ell|(C_1 + |x_\ell|^{R-1} + |x_{\ell'}|^{R-1}).$$

Theorem 3.9. *In the situation of Theorem 3.7, suppose additionally that Assumption (W1) holds. Then, one finds β_0 such that, for any $\beta \leq \beta_0$, the set $\mathcal{G}^t(\pi)$ is a singleton.*

Proof. In the following, it will be convenient to analyse the modified specification (3.19) for the particular choice $\gamma = 1$. The validity of condition (IC) has already been checked in the proof of Theorem 3.7, so we will only concern ourselves with checking condition (CC). Again, for $\mathtt{k} > 1$ we obtain $K_* := K_*(h, \overline{c}, \mathtt{k})$ by formula (2.14). Fix $\ell \in \mathbb{Z}^d$ and $\ell' \sim \ell$ and consider a pair of boundary conditions satisfying (3.26). We conventionally rewrite each probability measure (3.24) as

$$\widetilde{\pi}_\ell^y(dx) = \overline{Z}_\ell^{-1}(y)\exp\{-\overline{H}_\ell(x_\ell|y)\}dx_\ell \otimes_{\ell'\neq\ell} \delta_{y_{\ell'}}, \tag{3.31}$$

where

$$\overline{H}_\ell(x_\ell|y) := \widetilde{V}_\ell(x_\ell) + \sum_{\ell'\sim\ell} \overline{W}_{\ell\ell'}(x_\ell, y_{\ell'}),$$

$$\overline{W}_{\ell\ell'}(x_\ell, y_{\ell'}) := \widetilde{W}_{\ell\ell'}(x_\ell, y'_\ell) - \widetilde{W}_{\ell\ell'}(0, y_{\ell'}).$$

Obviously $\overline{W}_{\ell\ell'}(0, y_{\ell'}) = 0$. By assumption (W2) one gets the uniform bound

$$\sup_{|y_{\ell'}|^R \le K_*} |\overline{W}_{\ell\ell'}(x_\ell, y_{\ell'})| \le \frac{1}{2} J_{\ell\ell'} ||J||_0^{-1}(|x_\ell|^R + \mathcal{L}_1|x_\ell|), \tag{3.32}$$

$$\mathcal{L}_1 := (\beta||J||_0)^{1-1/R} + K_*^{1-1/R}.$$

According to (3.29), we only need to get a proper bound for

$$\mathcal{I}_{\ell\ell'} := \sup_{||y^1||_\ell^R, ||y^2||_\ell^R \le K_*} \int_X \exp\{|\triangle \overline{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})|\}\widetilde{\pi}_\ell(dx|y^1), \tag{3.33}$$

where we have set $\triangle\overline{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'}) := \overline{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}) - \overline{W}_{\ell\ell'}(x_\ell, y^2_{\ell'})$. Using the bounds for the rescaled potentials (3.20), the assumptions on the potentials and relations (3.29) and (3.31), we find that

$$\mathcal{I}_{\ell\ell'} \le \exp\{\beta(B_2 - B_1)\}\frac{E_2}{F_2}, \tag{3.34}$$

where

$$E_2 := \int_{\mathbb{R}^n} \exp\left\{\frac{\Delta + 2}{2} J_{\ell\ell'}||J||_0^{-1}\left[|x_\ell|^R + \mathcal{L}_\ell|x_\ell|\right] - \beta^{1-P/R}||J||_0^{-P/R} A_1|x_\ell|^P\right\} dx_\ell,$$

$$F_2 := \int_{\mathbb{R}^n} \exp\left\{\frac{\Delta}{2} J_{\ell\ell'}||J||_0^{-1}\left[|x_\ell|^R + \mathcal{L}_\ell|x_\ell|\right] - \beta^{1-P/R}||J||_0^{-P/R} A_2|x_\ell|^P\right\} dx_\ell,$$

and we recall that Δ is the maximum number of neighbours of a vertex $k \in \mathbb{Z}^d$. Obviously, one can find β_0 such that $\mathcal{I}_{\ell\ell'} < 1 + 1/\Delta^\chi$ and setting $\kappa_{\ell\ell'} := \mathcal{I}_{\ell\ell'} - 1$, we obtain the contraction matrix satisfying condition (CC).

□

3.3 Low-temperature uniqueness: $\beta \gg 1$

In this subsection, we will concern ourselves with Hamiltonians with *unique ground state*, i.e. a unique configuration that minimizes all local Hamiltonians (3.6). We will use again Theorem 2.9 to provide an elementary uniqueness result for $\mu \in \mathcal{G}^t$. The main distinction from the high-temperature situation presented in the previous section is that no reasonable type of interactions (even the ferromagnetic type) can be treated by the original Dobrushin uniqueness theorem as $\beta \to \infty$, for more details, see Section 2.3.5. in [Pas].

Before presenting the main results, we assume that the following conditions are true.

(W2) The pair potentials vanish at the origin, i.e. $W_{\ell\ell'}(0,0) = 0$. Furthermore, they satisfy Assumption (W) with $I_W = 0$, i.e. for all $\ell, \ell' \in \mathbb{Z}^d$ and $x_\ell, x_{\ell'} \in \mathbb{R}^n$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(|x_\ell|^R + |x_{\ell'}|^R).$$

(V2) The one-particle potentials possess the unique global minimum $V_\ell(0) = 0$, so that $V_\ell(x_\ell) > 0$ if $x_\ell \neq 0$. Moreover, there exist $P \geq R \geq 2$, $A_3 \geq A_4 > \frac{a}{2}||J||_0$, $a_3 \geq a_4 > 0$ such that for all $\ell \in \mathbb{Z}^d$ and $x_\ell \in \mathbb{R}^n$

$$A_4|x_\ell|^R + a_4|x_\ell|^2 \leq V_\ell(x_\ell) \leq A_3|x_\ell|^P + a_3|x_\ell|^2.$$

(i) *Asymptotic analysis*

In the following we will give an analogue of Lemma 3.4 in the case $\beta \to \infty$.

Lemma 3.10. *Suppose Assumptions (V2) and (W2) hold for all* $||J||_0 \leq \mathcal{J}_0 < 2A_4/3$. *Then, for any* $\beta^0 > 0$ *there exists a proper* $\Upsilon^0 := \Upsilon^0(\beta^0, \mathcal{J}_0) \geq 1$ *such that*

$$\int_X \exp\{\beta\tau|x_\ell|^R\}\pi_\ell^y(dx) \leq \Upsilon^0 \exp\left\{\beta \sum_{\ell' \sim \ell} J_{\ell\ell'}|y_{\ell'}|^R\right\}, \tag{3.35}$$

for all $\ell \in \mathbb{Z}^d$, $y \in X^t$, $\tau \leq A_4 - \mathcal{J}_0/2$, $||J||_0 \leq \mathcal{J}_0$ *and* $\beta \geq \beta^0$.

Proof. We will use the same reasoning as in Lemma 3.4 and Assumptions (V2) and (W3) to get the required bound (3.35) with the constant

$$\Upsilon^0 := \Upsilon^0(\beta^0, \mathcal{J}_0) := E_3/E_3, \tag{3.36}$$

where we have set

$$E_3 := \int_{\mathbb{R}^n} \exp\left\{-\left[(\beta^0)^{1-R/2}|x_\ell|^R(A_4 - \tau - \mathcal{J}_0/2) + a_4|x_\ell|^2\right]\right\} dx_\ell \tag{3.37}$$

$$F_3 := \int_{\mathbb{R}^n} \exp\left\{ - \left[(\beta^0)^{1-P/2} |x_\ell|^P A_3 + \beta_0^{1-R/2} \mathcal{J}_0/2 |x_\ell|^R + a_3 |x_\ell|^2 \right] \right\} dx_\ell \quad (3.38)$$

□

The next result is an important application to the above lemma, as it concerns the validity of (IC) for large β.

Corollary 3.11. *For each fixed, but arbitrarily small $q > 0$ there exists a proper $\beta^0 := \beta^0(q, \mathcal{J}_0) > 0$ such that*

$$\int_X |x_\ell|^R \pi_\ell^y(dx) \leq q + (A_4 - \mathcal{J}_0/2)^{-1} \sum_{\ell' \sim \ell} J_{\ell\ell'} |y'_\ell|^R, \tag{3.39}$$

for all $\ell \in \mathbb{Z}^d$, $y \in X^t$, $||J||_0 \leq \mathcal{J}_0$ and $\beta \geq \beta^0$.

Proof. Applying Jensen's inequality in (3.35), for $\tau := A_4 - \mathcal{J}_0/2$ yields the wanted result. The constant $q := \log \Upsilon^0 (\beta\tau)^{-1}$ obviously tends to zero as $\beta \to \infty$.

□

(ii) Uniqueness by small $||J||_0$

The following result controls the uniqueness on a temperature interval $\beta \in [\beta^0, \infty)$ by small values of $||J||_0$.

Theorem 3.12. *Suppose that Assumptions (W2), (V2) and (FR) hold. Then for every $\beta^0 > 0$ one finds $\mathcal{J} := \mathcal{J}(\beta^0) < A_4(\Delta^X + 1/2)$ such that the set $\mathcal{G}^t(\pi)$ is a singleton at all values $\beta \geq \beta^0$ and $||J||_0 \leq \mathcal{J}$.*

Proof. For proving condition (IC), we will use the refinement of the exponential bound (3.13) for $\beta \to \infty$, given by Lemma 3.10, but rewritten for the rescaled specification (3.19), namely

$$\int_X \exp\left\{ \tau ||J||_0^\gamma |x_\ell|^R \right\} \widetilde{\pi}_\ell^y(dx) \leq \Upsilon^0 \exp\left\{ \sum_{\ell' \sim \ell} J_{\ell\ell'} ||J||_0^{-\gamma} |y_{\ell'}|^R \right\}.$$

We use again Jensen's inequality to obtain

$$\int_X |x_\ell|^R \widetilde{\pi}_\ell^y(dx) \leq \tau^{-1} \left[||J||_0^\gamma \log \Upsilon^0 + \sum_{\ell' \sim \ell} J_{\ell\ell'} |y_{\ell'}|^R \right].$$

Picking any $\tau \in (\Delta^X \mathcal{J}_0, A_4 - \mathcal{J}_0/2]$, we get condition (IC) with constants $c_{\ell\ell'} := J_{\ell\ell'} \tau^{-1}$, so $\bar{c} \leq \mathcal{J}_0 \tau^{-1} < \frac{1}{\Delta^X}$ and $\theta := \left(\tau^{-1} ||J||_0^\gamma \log \Upsilon^0 \right)^{-1}$. Putting $\gamma = 0$ and choosing a $\mathtt{k} < 1$ we can find $K_* := K_*(h, \bar{c}, \mathtt{k})$, again by formula (2.14).

For proving condition (CC) we proceed as in the case of $\beta \to 0$ and obtain that

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \leq \kappa_{\ell\ell'} := \Upsilon^0 \exp\{2\mathcal{J}_0 K_*\} - 1, \tag{3.40}$$

for all $\beta \geq \beta^0$.

By choosing small enough $\mathcal{J}_0(\beta^0)$ we obtain $\overline{\kappa} \leq \mathtt{k} < 1$, hence we show that (CC) holds.

□

(iii) Uniqueness by large β

We are interested now in proving an existence result for large β, when all other parameters are fixed. For this, we ask also that the following holds.

(W3) For all $\ell, \ell' \in \mathbb{Z}^d$ and $x_\ell, x_{\ell'} \in \mathbb{R}^n$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(|x_\ell|^R + |x_{\ell'}|^R),$$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'}) - W_{\ell\ell'}(0, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}|x_\ell|(|x_\ell|^{R-1} + |x_{\ell'}|^{R-1}).$$

Theorem 3.13. *Suppose that Assumptions (V2), (W3) and (FR) hold with* $P \geq R > 2$. *Then for each* $\beta^0 > 0$ *and* $\mathcal{J}_0 < A_4/(\Delta^{\chi} + 1/2)$ *one finds a proper* $\zeta_0 := \zeta_0(\beta^0, \mathcal{J}_0) > 0$ *such that the corresponding set* $\mathcal{G}^t(\pi)$ *is a singleton at all values of* $\beta \geq \beta^0$ *and* $||J||_0 \leq \mathcal{J}_0$ *related by*

$$\beta^{1-R/2}||J||_0 =: \zeta \leq \zeta_0. \tag{3.41}$$

Proof. Condition (IC) has already been proved in Theorem 3.12, so we will be concerned only with condition (CC). Let $\gamma = 1$. It is clear that, for all $\beta \geq \beta^0$ and $||J||_0 > 0$ satisfying restriction (3.41), we can take the same constant Υ^0 and hence K_* as in the proof of Theorem 3.12. We proceed analogously as in the proof of Theorem 3.9 and we obtain the same estimates for $\overline{W}_{\ell\ell'}(x_\ell, y_{\ell'})$ by replacing I_W by 0.

However, we use another elementary inequality in estimating the total variation distance : $|e^x - 1| \leq |x|e^{|x|}$ and we obtain

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \leq \int_X \left|\triangle\overline{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\right| \exp\left\{\left|\triangle\overline{W}_{\ell\ell'}(x_\ell, y^1_{\ell'}, y^2_{\ell'})\right|\right\} \widetilde{\pi}_\ell(dx|y^1).$$

Using the assumptions on the interacting potentials and making the change of variable $x_\ell \mapsto \zeta^{1/R} x_\ell$ we get

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \le \zeta^{1/R}\frac{E_4}{F_4},$$

where

$$E_4 := \int_{\mathbb{R}^n} (\zeta^{1-1/R}|x_\ell|^R + K_*|x_\ell|) \exp\Big\{ - a_4|x_\ell|^2 + \frac{\Delta+2}{2}K_*\zeta^{1/R}|x_\ell|\Big\} dx_\ell,$$

$$F_4 := \int_{\mathbb{R}^n} \exp\Big\{ - [a_3|x_\ell|^2 + \beta^{1-P/2}A_3|x_\ell|^P + \frac{\Delta}{2}K_*\zeta^{1/R}|x_\ell|\Big\} dx_\ell.$$

This implies that for each $\ell \in \mathbb{Z}^d$ and $y^1, y^2 \in X$ obeying (3.26)

$$d_{TV}(\widetilde{\pi}_\ell(\cdot|y^1), \widetilde{\pi}_\ell(\cdot|y^2)) \le \kappa_{\ell\ell'}(\beta^0, \mathcal{J}_0) = \mathcal{O}(\zeta^{1/R}) \text{ as } \zeta \to 0. \tag{3.42}$$

Moreover, this estimate is uniform with respect to all $\beta \ge \beta^0$ and $||J||_0 < \mathcal{J}_0$ satisfying (3.41). Therefore (CC) holds. □

Remark 3.14. (i) All the uniqueness results obtained in this section for the type of interaction described by the Hamiltonian in (3.1) work for any lattice $\mathsf{L} \subseteq \mathbb{Z}^d$, or even more generally, for any graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ with bounded degree. One needs, however, to impose the following type of *spatial regularity*

(SR) There exists $\delta_0 \ge 0$ such that for all $\delta > \delta_0$

$$\Xi_\delta := \sup_{o \in \mathsf{V}} \sum_\ell \exp\{-\delta\rho(o, \ell)\} < \infty, \tag{3.43}$$

where $\rho(o, \ell)$ is the smallest N such that there exists a N-path between o and ℓ. In the case of the lattice $\mathbb{Z}^d$, (3.43) holds with $\delta_0 = 0$. Also, setting the configuration space to be $X := (\mathbb{R}^n)^{\mathsf{V}}$, one needs to restrict the study of Gibbs measures to the ones supported by the subsets of *(exponentially) tempered configurations*

$$X^{(e)t} := \bigcap_{o \in \mathsf{V}, \delta > \delta_0} X_{o,\delta},$$

$$X_{o,\delta} := \left\{ x \in X : ||x||_{o,\delta} := \Big[\sum_\ell |x_\ell|^R \exp\{-\delta\rho(o,\ell)\}\Big]^{1/R} < \infty \right\}.$$

Under the above assumptions, it can be proved that the set of $\mathcal{G}^{(e)t}$ of Gibbs measures supported by $X^{(e)t}$ is not empty. Furthermore, each tempered μ will satisfy the exp bound (3.13) and hence the a-priori bound (3.11) required in Dobrushin-Pechersky theorem.

(ii) In their paper, Dobrushin and Pechersky consider as an example, a system of scalar spins with the formal Hamiltonian

$$H(x) := \frac{J}{2} \sum_{\ell,\ell':|\ell-\ell'|=1} (x_\ell - x_{\ell'})^R + \sum_{\ell} x_\ell^P,$$

where P and R are even integers such that $P > R$, see Theorem 7 in [DP83]. One can notice that after the change of variables $x_\ell \to \beta^{-1/R} x_\ell$, the problem falls in the context of Theorem 3.9.

(iii) So far, the (high/low temperature) uniqueness in lattice systems with interaction potentials having at most quadratic growth has been studied only by cluster expansions (see e.g. [MM91], [PY94], [PS01]). In particular, the low temperature uniqueness in classical systems with a unique ground state can be established by means of special cluster expansions constituting the Pirogov-Sinai theory of phase diagrams (see e.g. [Sin82],[PS87], [LM87]).

(iv) We notice that if $\mu \in \mathcal{G}^t(\pi)$, the same type of exponential decay of correlations as given in Theorem 2.19 naturally follows, for any two sub-lattices $\mathsf{L}, \widetilde{\mathsf{L}} \Subset \mathbb{Z}^d$ and local functions $f, g \in \mathcal{BF}_{loc}$ depending only on the sites in $\widetilde{\mathsf{L}}$ and L, respectively, such that the following inequality holds

$$|g(x)| \leq \theta \sum_{l \in \mathsf{L}} |x_\ell|^R \tag{3.44}$$

and $\sup_{\ell \in \mathsf{L}} \int_X f(x) h(x_\ell) \mu(dx) < \infty$. Of course, the relaxation parameters D and α in (2.78) depend on the specific coefficients obtained for each of Theorems 3.7-3.13.

(iv) In the high-temperature case, results on the decay of correlations were obtained via Dobrushin's classical uniqueness criterion, in classical lattice systems with attractive pair interactions having at most quadratic growth (i.e, when $R = 2$) can be obtained via the original Dobrushin uniqueness theorem and log-Sobolev inequalities, see e.g. [Wu06], [Yos99],[Zeg90], [BH99], [BH00], [Led01], [Zit08]). In the low temperature case, this was mainly done via two methods: *cluster expansions* (see cluster expansions (see [MM91], [PS01], [BO99]) and *Witten-Laplacian techniques* ([BJS00], [Bl03], [Bl04], [Mat06], [Mat08]). As seen in item (iii) of this remark, decay of correlations in our model follows from (IC) and (CC) by a simple analytic argument.

Chapter 4

Particle Systems in Continuum

Not looking for the best possible results, here we just demonstrate how Theorem 2.9 can be applied in the setting of continuum particle systems, as models of non-ideal classical gas. In Chapter 5 this model will be enriched by considering interacting particle systems with marks.

4.1 Configuration spaces

4.1.1 Spaces of finite configurations

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. By $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all open and Borel sets, respectively. Likewise, $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ consist of all sets in $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, respectively, which are bounded, i.e. have compact closures. The space of n-points configurations is

$$\Gamma_0^{(n)} := \{\eta \subset \mathbb{R}^d \big| |\eta| = n\},\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \qquad \Gamma_0^{(0)} := \emptyset,$$

where $|A|$ denotes the cardinality of the set A. In a similar way one can define the space $\Gamma_{0,\mathsf{L}}^{(n)}$ of n-particle configurations located in a volume $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$. For every $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping

$$N_{\mathsf{L}} : \Gamma_0^{(n)} \to \mathbb{N}_0; \qquad N_{\mathsf{L}}(\eta) := |\eta \cap \mathsf{L}|.$$

For short, we write $\eta_{\mathsf{L}} := \eta \cap \mathsf{L}$.

One can define a topological structure on $\Gamma_0^{(n)}$ by using the mapping

$$sym^n : \widetilde{(\mathbb{R}^d)^n} \to \Gamma_0^{(n)},\ n \in \mathbb{N}$$

$$(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\},$$

where $\widetilde{(\mathbb{R}^d)^n} = \{(x_1, \cdots, x_n) \in (\mathbb{R}^d)^n | x_k \neq x_l \text{ for } k \neq l\}$.

Let $\mathcal{O}(\Gamma_0^{(n)})$ be the topology induced by this map and $\mathcal{B}(\Gamma_0^{(n)})$ the corresponding σ-algebra generated by the maps N_{L}, i.e.

$$\mathcal{B}(\Gamma_0^{(n)}) = \sigma(N_{\mathsf{L}} | \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)).$$

We introduce the *space of finite configurations*

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

and equip it with the topology $\mathcal{O}(\Gamma_0)$ of disjoint union and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$.

4.1.2 The configuration space Γ

Now, let us define the *configuration space* over $\mathbb{R}^d$ as

$$\Gamma := \{\gamma \subset \mathbb{R}^d || \gamma \cap \mathsf{L}| < \infty, \text{ for any } \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)\}.$$

It is easy to see that one can identify a configuration $\gamma \in \Gamma$ with the discrete measure $\sum_{x \in \gamma} \delta_x \in \mathbb{M}(\mathbb{R}^d)$, where $\mathbb{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$. Therefore Γ can be endowed with the vague topology $\mathcal{O}(\Gamma)$ inherited from $\mathbb{M}(\mathbb{R}^d)$, i.e. the weakest topology such that the map

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$$

is continuous for any $f \in C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ is the set of continuous functions with bounded support. A sub-basis of this topology is given by sets of the form

$$\{\gamma \in \Gamma || \gamma_{\mathsf{L}}| = n, \gamma_{\partial \mathsf{L}} = \emptyset\}, \ \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d), \ n \in \mathbb{N}_0,$$

where $\partial \mathsf{L}$ is the topological boundary of L, see [Len75]. Moreover, Γ endowed with this topology is a Polish space, see [KMM78]. The convergence in the $\mathcal{O}(\Gamma)$ topology can be described as follows : $(\gamma^{(n)})_n$ converges to γ iff $N_{\mathsf{L}}(\gamma^{(n)}) \to N_{\mathsf{L}}(\gamma)$ for any $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ with $N_{\partial \mathsf{L}}(\gamma) = 0$.

The Borel σ-algebra $\mathcal{B}(\Gamma)$ corresponding to $\mathcal{O}(\Gamma)$ is the smallest σ-algebra which makes all the mappings $N_{\mathsf{L}} : \Gamma \to \mathbb{N}_0$ measurable, i.e. $\mathcal{B}(\Gamma) = \sigma(N_{\mathsf{L}} | \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$.

Consider also the filtration on Γ given by

$$\mathcal{B}_{\mathsf{L}}(\Gamma) := \sigma(N_{\mathsf{L}'} | \mathsf{L}' \in \mathcal{B}_c(\mathbb{R}^d), \mathsf{L}' \subset \mathsf{L}), \text{ for } \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d).$$

Moreover, one can describe the space Γ in another way. We define the space of configurations restricted to a volume $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$, $\Gamma_{\mathsf{L}} := \{\gamma \in \Gamma | \gamma \subset \mathsf{L}\}$, which is equipped with the topology $\mathcal{O}(\Gamma_{\mathsf{L}})$ induced from $\mathcal{O}(\Gamma)$, under the projection $p_{\mathsf{L}} : \Gamma \to \Gamma_{\mathsf{L}}$ defined by $p_{\mathsf{L}}(\gamma) := \gamma_{\mathsf{L}}$. As was shown in [Len75], a sub-base of open sets for this topology is given by

$$\{\gamma \in \Gamma || \gamma_{\mathsf{L}'}| = n, \gamma_{\partial \mathsf{L}'} = \emptyset\}, \ \mathsf{L}' \in \mathcal{O}_c(\mathbb{R}^d), \text{ with } \bar{\mathsf{L}'} \subset \mathsf{L}.$$

The Borel σ-algebra generated by $\mathcal{O}(\Gamma_{\mathsf{L}})$ will be denoted by $\mathcal{B}(\Gamma_{\mathsf{L}})$. The space Γ can be obtained as the projective limit of the spaces $(\Gamma_{\mathsf{L}})_{\mathsf{L} \in \mathcal{B}_c(X)}$ with respect to the above defined projections p_{L}. One should remark here that the σ-algebras $\mathcal{B}_{\mathsf{L}}(\Gamma)$ and $\mathcal{B}(\Gamma_{\mathsf{L}})$ are σ-isomorphic, i.e. there exists a bijective mapping between them which preserves the operations in each σ-algebra.

One can also introduce on Γ the algebra of *cylinder* (or *local*) Borel sets

$$\mathcal{B}_0(\Gamma) := \bigcup_{\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)} \mathcal{B}_{\mathsf{L}}(\Gamma).$$

Let $L^0(\Gamma, \mathcal{B}(\Gamma))$ denote the set of all measurable functions on Γ. A function $F \in L^0(\Gamma, \mathcal{B}(\Gamma))$ is called *cylinder function* if it is $\mathcal{B}_0(\Gamma)$-measurable, i.e. F is $\mathcal{B}_{\mathsf{L}}(\Gamma)$-measurable for some $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$. The class of these functions will be denoted by $\mathcal{F}L^0(\Gamma, \mathcal{B}(\Gamma))$. Other notation will be $C(\Gamma)$ for the set of functions on Γ which are continuous in the vague topology and $\mathcal{F}C(\Gamma, \mathcal{B}(\Gamma))$ for the set of all continuous cylinder functions.

4.1.3 The Poisson and Lebesgue-Poisson measures

We are now interested in constructing Gibbs measures on Γ, motivated by the fact that, in Statistical Physics, the equilibrium states of a system are described precisely by such measures. In particular, the state of an ideal gas is described by a Poisson random field $\pi_{z\sigma}$ on Γ, the explicit construction of which will be given next. Fix a chemical activity parameter $z > 0$ and an intensity measure $\sigma \geq 0$ on the underlying phase space $\mathbb{R}^d$. We assume that σ is a non-atomic Radon measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$, i.e.

$$\sigma(\mathbb{R}^d) = \infty, \ \sigma(\mathsf{L}) < \infty \text{ for all } \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d) \text{ and } \sigma(\{x\}) = 0, \text{ for any } x \in \mathbb{R}^d.$$

For further considerations, we also assume the following form of spatial regularity

$$\sup_{y\in\mathbb{R}^d} \sigma(B_r(y)) < \infty. \tag{4.1}$$

holds for some, and hence for all $r > 0$, where $B_r(y)$ is the ball of radius r centred at y. One can remark that this condition is fulfilled by any translation invariant measure on $\mathbb{R}^d$, in particular by the Lebesgue measure dx. For any $n \in \mathbb{N}$, the product measure $\sigma^{\otimes n}$ can be considered as a measure on $\widetilde{(\mathbb{R}^d)}^n$. Let $\sigma^{(n)} := \sigma^{\otimes n} \circ (sym^n)^{-1}$ be the corresponding measure on $\Gamma_0^{(n)}$ and set $\sigma^{(0)}(\{\emptyset\}) := 1$. The σ-Poisson measure $\lambda_{z\sigma}$ (or Lebesgue-Poisson, if $\sigma(dx) = dx$) on $\mathcal{B}(\Gamma_0)$ is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

In other words, for each $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$, the measure $\lambda_{z\sigma} := \lambda_{z\sigma}^{\mathsf{L}}$ on $(\Gamma_{\mathsf{L}}, \mathcal{B}(\Gamma_{\mathsf{L}})$ is characterized by the following identity

$$\int_{\Gamma_{\mathsf{L}}} F(\gamma_{\mathsf{L}}) d\lambda_{z\sigma}^{\mathsf{L}}(\gamma_{\mathsf{L}}) = F(\emptyset) + \sum_{n\in\mathbb{N}} \frac{z^n}{n!} \int_{\mathsf{L}^n} F(\{x_1, \ldots, x_n\}) \sigma(dx_1) \ldots \sigma(dx_n),$$

which holds for any bounded measurable function $F \in L^\infty(\Gamma_{\mathsf{L}})$. An easy computation shows that $\lambda_{z\sigma}^{\mathsf{L}}(\Gamma_{\mathsf{L}}) = e^{z\sigma(\mathsf{L})}$. Hence one can define a probability measure $\pi_{z\sigma}^{\mathsf{L}}$ on Γ_{L} by

$$\pi_{z\sigma}^{\mathsf{L}} = e^{-z\sigma(\mathsf{L})} \lambda_{z\sigma}.$$

We note that the family $\{\pi_{z\sigma}^{\mathsf{L}} | \mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)\}$ is consistent, i.e.

$$\pi_{z\sigma}^{\mathsf{L}'} = \pi_{z\sigma}^{\mathsf{L}} \circ p_{\mathsf{L}',\mathsf{L}}^{-1}, \quad \text{whenever } \mathsf{L}' \subset \mathsf{L},$$

where $p_{\mathsf{L}',\mathsf{L}} : \Gamma_{\mathsf{L}} \to \Gamma_{\mathsf{L}'}$ is the projection map acting by $p_{\mathsf{L}',\mathsf{L}}(\gamma_{\mathsf{L}}) = \gamma_{\mathsf{L}'}$. By a version of Kolmogorov's theorem for projective limit spaces (see Chapter V, Theorem 3.2 of [Par67] or Theorem A.5.6 in [Kun99]), this family of measures uniquely determines a probability measure $\pi_{z\sigma}$ on $\mathcal{B}(\Gamma)$ such that $\pi_{z\sigma}^{\mathsf{L}} = \pi_{z\sigma} \circ p_{\mathsf{L}}^{-1}$. The measure $\pi_{z\sigma}$ is called *Poisson measure* with intensity σ and activity z. It is an element of $\mathcal{P}(\Gamma, \mathcal{B}(\Gamma))$, the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$, which are also called simple point processes in [Kal83] or [KMM78].

Another analytic characterization of $\pi_{z\sigma}$ is through its Laplace transform:

$$\int_{\Gamma} \exp\langle f, \gamma\rangle \pi_{z\sigma}(d\gamma) := \exp\left\{ \int_{\mathbb{R}^d} (e^{f(x)} - 1) z\sigma(dx) \right\}, \tag{4.2}$$

for all non-negative $f \in C_0(\mathbb{R}^d)$.

4.2 The case of pair interaction

4.2.1 Specifications and associated Gibbs measures

A symmetric measurable function $V : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ will be called a *pair interaction potential*. For our purposes, neither translation invariance nor continuity of V will be required. We will need however to restrict ourselves to interactions of finite range, i.e. assume the following holds true

(FR) There exists $R > 0$ such that $V(x, y) = 0$ if $|x - y| \geq R$.

For a given pair potential V satisfying the above condition we define the *Hamiltonian* (or *energy functional*) $H : \Gamma_0 \to \overline{\mathbb{R}}$ by

$$H(\gamma) := \sum_{\{x,y\}\subset\gamma} V(x, y), \tag{4.3}$$

where the sum is taken over all unordered pairs of distinct points $x, y \in \gamma$. By convention $H(\emptyset) := 0$ and $H(\{x\}) := 0$ for any $x \in \mathbb{R}^d$. Also, for each $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma, \xi \in \Gamma$, we set the *interaction energy* between $\gamma_\mathsf{L} \in \Gamma_\mathsf{L}$ and ξ_{L^c} to be

$$W_\mathsf{L}(\gamma_\mathsf{L}|\xi) := \sum_{x\in\gamma_\mathsf{L}, y\in\xi_{\mathsf{L}^c}} V(x, y), \tag{4.4}$$

which is well-defined in view of Assumption **(FR)**. We can now introduce the *conditional Hamiltonians* $H_\mathsf{L}(\cdot|\xi) : \Gamma_\mathsf{L} \to \overline{\mathbb{R}}$ by

$$H_\mathsf{L}(\gamma_\mathsf{L}|\xi) := H(\gamma_\mathsf{L}) + W(\gamma_\mathsf{L}|\xi). \tag{4.5}$$

For a fixed parameter $\beta > 0$, called inverse temperature, the *local Gibbs state* with boundary condition ξ is a probability measure on $(\Gamma_\mathsf{L}, \mathcal{B}(\Gamma_\mathsf{L}))$ defined by

$$\mu_\mathsf{L}(d\gamma_\mathsf{L}|\xi) := [Z_\mathsf{L}(\xi)]^{-1} \exp\{-\beta H_\mathsf{L}(\gamma_\mathsf{L}|\xi)\}\lambda_{z\sigma}(d\gamma_\mathsf{L}), \tag{4.6}$$

provided that the corresponding *partition function*

$$\begin{aligned} Z_\mathsf{L}(\xi) :&= \int_{\Gamma_\mathsf{L}} \exp\{-\beta H_\mathsf{L}(\gamma_\mathsf{L}|\xi)\}\lambda_{z\sigma}(d\gamma_\mathsf{L}) \\ &= 1 + \sum_{n\geq1} \frac{z^n}{n!} \int_{\mathsf{L}^n} \exp\{-\beta H_\mathsf{L}(\{x_1, \ldots, x_n\}|\xi)\sigma(dx_1)\ldots\sigma(dx_n) \end{aligned} \tag{4.7}$$

is finite. Otherwise, we set $\mu_\mathsf{L}(d\gamma_\mathsf{L}|\xi) = 0$. Also, note that from the above expression it is obvious that $Z_\mathsf{L}(\xi) \geq 1$.

The family of local Gibbs states determines a family of stochastic kernels $\Pi = \{\pi_\mathsf{L}\}_{\mathsf{L}\in\mathcal{B}_c(\mathbb{R}^d)}$, $\pi_\mathsf{L} : \mathcal{B}(\Gamma)\times\Gamma \to [0,1]$, as follows:

$$\pi_\mathsf{L}(B|\xi) := \mu_\mathsf{L}(B_{\mathsf{L},\xi}|\xi), \text{ where } B_{\mathsf{L},\xi} := \{\gamma_\mathsf{L} \in \Gamma_\mathsf{L} | \gamma_\mathsf{L} \cup \xi_{\mathsf{L}^c} \in B\} \in \mathcal{B}(\Gamma_\mathsf{L}).$$

Π will be called *local specification*. By Proposition 6.3 in [Pre76] or Proposition 2.6 in [Pre05], the family Π obeys the consistency property, meaning that for any $B \in \mathcal{B}(\Gamma)$ and $\xi \in \Gamma$

$$\int_\Gamma \pi_\mathsf{U}(B|\gamma)\pi_\mathsf{L}(d\gamma|\xi) = \pi_\mathsf{L}(B|\xi), \qquad \mathsf{U} \subseteq \mathsf{L}. \tag{4.8}$$

Definition 4.1. A probability measure $\mu \in \mathcal{P}(\Gamma)$ is called a grand canonical *Gibbs measure* (or *state*) with pair potential V, activity z and intensity σ if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equilibrium equation

$$(\pi_\mathsf{L}\mu)(B) := \int_\Gamma \pi_\mathsf{L}(B|\gamma)\mu(d\gamma) = \mu(B), \tag{4.9}$$

for all $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\Gamma)$. For fixed temperature β, the associated set of all Gibbs measures will be denoted by $\mathcal{G}$.

4.2.2 Conditions on the interaction

In what follows, we consider a simple, yet physically realistic model, which allows a precise control of attraction-repulsion effects. Throughout this section we assume the following conditions on the interaction potential.

(**LB**) ***Lower boundedness***: There exist $M \geq 0$ and $r_1, r_2 \in [0, R]$, $r_1 \leq r_2$, such that

$$\begin{aligned} &\inf_{x,y\in\mathbb{R}^d} V(x,y) \geq -M \quad \textit{and} \\ &V(x,y) \geq 0 \quad \textit{if} \quad |x-y| \leq r_1 \quad \textit{or} \quad |x-y| \geq r_2. \end{aligned} \tag{4.10}$$

(**RC**) ***Repulsion condition***: There exists $\delta > 0$ such that

$$\inf_{x,y:\,|x-y|\leq\delta} V(x,y) =: A_\delta > 2Mm_\delta, \tag{4.11}$$

where

$$m_\delta := m_\delta(d, r_1, r_2) := v_d d^{d/2}\left[(r_2/\delta + 1)^d - (r_1/\delta - 1)^d\right] \tag{4.12}$$

and $v_d := \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of a unit ball in $\mathbb{R}^d$.

Remark 4.2. An example of interaction which satisfies the repulsion condition (4.11), with an arbitrarily large but fixed $M > 0$, is any potential V having the following asymptotic behaviour

$$\lim_{|x-y|\to 0} \frac{V(x,y)}{|x-y|^d} = +\infty, \qquad \text{and thus} \qquad \lim_{\delta\to 0} \frac{A_\delta}{\delta^d} = +\infty.$$

Particular members of this class are the so-called *Dobrushin-Fisher-Ruelle* potentials, which are characterized by the following growth at the diagonal: for some $\varkappa, C > 0$

$$V(x,y) \geq C|x-y|^{-(d+\varkappa)} \qquad \text{as } |x-y| \to 0. \tag{4.13}$$

The existence problem for this model was treated in [KPR12], whereas the uniqueness problem was discussed in [PZ99]. However, we present a shorter analytical proof for it. Let us consider a partition of the phase space $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{gk}$ by the cubes

$$Q_{gk} := \left\{ x = (x^{(i)})_{i=1}^d \in \mathbb{R}^d \,\middle|\, g\left(k^{(i)} - 1/2\right) \leq x^{(i)} < g\left(k^{(i)} + 1/2\right) \right\}. \tag{4.14}$$

These cubes have edge length $g > 0$ and are centred at the points gk, $k \in \mathbb{Z}^d$. Recall that $\mathring{Q}_{gk}$ and $\overline{Q}_{gk}$ denote respectively the interior and closure of Q_{gk} in $\left(\mathbb{R}^d, |\cdot|\right)$. For $k \in \mathbb{Z}^d$ and $\gamma \in \Gamma$, we then define

$$\Gamma_k := \Gamma_{Q_{gk}}, \quad \gamma_k := \gamma \cap Q_{gk}, \quad \bar{\gamma}_k := \gamma \cap \overline{Q}_{gk}.$$

In what follows, we pick the parameter $g := \delta/\sqrt{d}$ with some $\delta > 0$ satisfying Assumption (**RC**). By construction

$$\operatorname{diam}(Q_{gk}) := \sup_{x,y\in Q_{gk}} |x-y| = \delta, \tag{4.15}$$

which implies that $V(x,y) \geq A$ for all $x, y \in \gamma_k$. Here and below we shall often drop δ and g in the notation for the corresponding constants A, m in (4.11) and for the cube Q_k in (4.14), respectively.

Technically, only the pairs $\{x,y\} \subset \gamma$ for which $V(x,y) < 0$ need to be controlled. It is clear that $V(x,y)$ may be negative for some $x \in \gamma_k$ and $y \in \gamma_j$ whenever

$$j \in \partial_g^- k := \{k' \neq k \mid \vartheta_1 < |k' - k| < \vartheta_2\}, \tag{4.16}$$

$$\vartheta_1 := (r_1/\delta - 1)\sqrt{d}, \quad \vartheta_2 := (r_2/\delta + 1)\sqrt{d}. \tag{4.17}$$

The total number of such "neighbour" cubes Q_{gj} can be roughly estimated by

$$|\partial_g^- k| \leq m_\delta(r_1, r_2), \tag{4.18}$$

which is the same constant as in (4.12). Note that to each index set $\mathcal{K} \Subset \mathbb{Z}^d$ there corresponds the "cubic" domain

$$\mathsf{L}_{\mathcal{K}} := \bigsqcup_{k\in\mathcal{K}} Q_{gk} \in \mathcal{B}_c(\mathbb{R}^d). \tag{4.19}$$

We denote the family of all such domains by $\mathcal{Q}_c(\mathbb{R}^d)$. On the other hand, for a given volume $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$, we can construct its "minimal" covering

$$\mathsf{L}_g := \bigsqcup_{k\in\mathcal{K}_{\mathsf{L}}} Q_{gk} \in \mathcal{Q}_c(\mathbb{R}^d) \;\text{ with }\; \mathcal{K}_{\mathsf{L}} := \left\{k \in \mathbb{Z}^d \middle|\, \mathsf{L} \cap Q_{gk} \neq \varnothing\right\}, \tag{4.20}$$

where $|\mathcal{K}_{\mathsf{L}}|$ is the number of cubes Q_{gk} having non-void intersection with L.

We cite now a result obtained in [KPR12], which will be useful in the proof of the uniqueness theorem.

Lemma 4.3. *[KPR12] (i) For any partition of $\mathbb{R}^d$ by the cubes (4.14) with edge length $g > 0$, there exist $D_g, E_g > 0$ such that for all $\gamma \in \Gamma_0$ the following holds:*

$$(\mathbf{SS}):\;\; \textit{Ruelle's Superstability:}\;\; H(\gamma) \geq D_g \sum_{k\in\mathbb{Z}^d} |\gamma_k|^2 - E_g|\gamma|. \tag{4.21}$$

(ii) Let $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ be such that $\operatorname{diam}(\mathsf{L}) \leq \delta$, *then for all $\gamma, \xi \in \Gamma$*

$$H_{\mathsf{L}}(\gamma_{\mathsf{L}}|\xi) \geq \frac{A}{2}\left(|\gamma_{\mathsf{L}}|^2 - |\gamma_{\mathsf{L}}|\right) - M|\gamma_{\mathsf{L}}| \cdot |\xi_{\mathsf{L}^c\cap\partial_R\mathsf{L}}|. \tag{4.22}$$

In particular, for any $\varepsilon \in (0, 1]$

$$H(\gamma_{\mathsf{L}}) \geq \frac{A}{2}(1-\varepsilon)|\gamma_{\mathsf{L}}|^2 - \frac{A}{8\varepsilon}. \tag{4.23}$$

The proof of this result follows from direct computations and can be found in the original article. We continue by deriving a one-point estimate, which we use to check that condition (IC) in the criterion (Theorem 2.9) result is satisfied. For $a \geq 0$ let us define

$$\Gamma_0 \ni \gamma \mapsto \Phi(\gamma) := a|\gamma|^2, \tag{4.24}$$

which will play the role of a *Lyapunov functional* in establishing stability properties of our model. According to Hypotheses (4.11) and (4.15),

$$\Phi(\gamma) \geq 0 \quad \text{for any } \gamma \in \Gamma_k,\; k \in \mathbb{Z}^d.$$

The following is a slight modification of Lemma 3.1 in [KPR12], better suited for our purposes.

Lemma 4.4. *Let the parameters* $\epsilon > 0$ *and* $a \geq 0$ *obey the relation*

$$2\epsilon a \leq \beta(\epsilon A - Mm). \tag{4.25}$$

There exists a universal constant $\Upsilon > 0$ *such that for all* $k \in \mathbb{Z}^d$, $\xi \in \Gamma$.

$$\int_{\Gamma_k} \exp\left\{a|\gamma_k|^2\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{\Upsilon + \frac{1}{2}\beta M\epsilon \sum_{j\in\partial_g^- k} |\xi_j|^2\right\}. \tag{4.26}$$

Proof. Direct computations and Young's inequality yield

$$\begin{aligned}
&\int_{\Gamma_k} \exp\left\{\Phi(\gamma_k)\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \leq \int_{\Gamma_k} \exp\left\{\Phi(\gamma_k) - \beta H_k(\gamma_k|\xi)\right\} d\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{\left[a - \frac{A}{2}\beta\right] |\gamma_k|^2 + \left[\frac{A}{2}\beta + \beta M \sum_{j\in\partial_g^- k} |\xi_j|\right] |\gamma_k|\right\} d\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{\left[a - \frac{A}{2}\beta + \frac{1}{2\epsilon}\beta Mm\right] |\gamma_k|^2 + \frac{A}{2}\beta|\gamma_k|\right\} d\lambda_{z\sigma}(\gamma_k) \\
&\quad \times \quad \exp\left\{\frac{1}{2}\beta M\epsilon \sum_{j\in\partial_g^- k} |\xi_j|^2\right\}.
\end{aligned} \tag{4.27}$$

In view of (4.1) and (4.25) the claim holds with

$$\begin{aligned}
\Upsilon := \quad & \sup_k \log \textstyle\int_{\Gamma_k} \exp\left\{\frac{A}{2}\beta|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
& = z \exp\left\{\tfrac{A}{2}\beta\right\} \sup_k \sigma(Q_{gk}) < \infty.
\end{aligned} \tag{4.28}$$

□

Remark 4.5. Based on Lemma 4.4, the existence and a-priori bounds for *tempered Gibbs measures* $\mu \in \mathcal{G}^t$ were proven in [KPR12]. By definition, such measures are supported by the following set of *tempered configurations*

$$\Gamma^t := \bigcap_{\alpha>0} \Gamma_\alpha,$$

where

$$\Gamma_\alpha := \left\{\gamma \in \Gamma : |\gamma|_\alpha := \sup_{k\in\mathbb{Z}} \left[|\gamma_k|^2 \exp\{-\alpha|k|\}\right]^{1/2} < \infty\right\}.$$

Moreover, Theorem 2.2 in [KPR12] gives the following exponential moment estimate, holding simultaneously for all tempered Gibbs measures $\mu \in \mathcal{G}^t$. For θ and a as in (4.25) and $\mu \in \mathcal{G}^t$,

$$\sup_{k \in \mathbb{Z}^d} \int_\Gamma \exp\left\{a|\gamma_k|^2\right\}\mu(d\gamma) \leq \Psi, \tag{4.29}$$

where

$$\Psi := \exp\left\{\frac{A}{A - 2Mm}\left(\Upsilon + \frac{\beta A^2}{4(A-2Mm)}\right)\right\}. \tag{4.30}$$

4.2.3 The associated lattice model

Similarly as in [PZ99] , we start from the chosen partition $(Q_k)_{k\in\mathbb{Z}^d}$ of $\mathbb{R}^d$ (see (4.14)) and construct a lattice system on the space $\check{\Gamma}_{lat} := \left(\Gamma(\overline{Q})\right)^{\mathbb{Z}^d}$, where for simplicity we denoted $Q = Q_0$. Nevertheless, we show that due to specific properties of the Lebesgue-Poisson measure, one does not need to do the analysis on the space of multiple configurations. This idea will considerably simplify the proof of the uniqueness result, as it can be seen in Section 4.2.4

The space $\check{\Gamma}_{lat}$ is endowed with the product topology and with the corresponding Borel σ-algebra $\mathcal{B}(\check{\Gamma}_{lat})$. Then, by Remark 4.A3 in [Geo88], $(\check{\Gamma}_{lat}, \mathcal{B}(\check{\Gamma}_{lat}))$ is a standard Borel space.

Define

$$T : \Gamma \to \check{\Gamma}_{lat},$$

which maps $\gamma \in \Gamma$ into $\check{\gamma} = (\check{\gamma}_k)_{k\in\mathbb{Z}^d} \in \check{\Gamma}_{lat}$, where $\check{\gamma}_k := \gamma \cap \overline{Q}_k - gk$ and by $\gamma - a$ we denote the translated configuration $\{..., x - a, ...\}$ for $\gamma = \{..., x, ...\}$. By T^{-1} we denote the left inverse of T.

Let $B_{k_1} \dots B_{k_L} \in \mathcal{B}(\Gamma_{\overline{Q}})$ for $L \in \mathbb{N}$ and $k_1, \dots k_L \in \mathbb{Z}^d$. Define the cylinder sets

$$A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}} := \{(\check{\gamma}_k)_{k\in\mathbb{Z}^d} \in \check{\Gamma}_{lat} : \check{\gamma}_{k_l} \in B_{k_l}, 1 \leq l \leq L\} \in \mathcal{B}(\check{\Gamma}_{lat})$$

and

$$C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}} := \{\gamma \in \Gamma : \gamma_{\overline{Q}_{k_l}} - k_l \in B_{k_l},\ 1 \leq l \leq L\} \in \mathcal{B}(\Gamma),$$

respectively.

Lemma 4.6. *(i)* $T : \Gamma \to \check{\Gamma}_{lat}$ *is measurable;*

(ii) $T(B) \in \mathcal{B}(\check{\Gamma}_{lat})$ *for any* $B \in \mathcal{B}_0(\Gamma)$.

Proof. (i) One can immediately see that

$$T^{-1}\left(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}\right) = C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}},$$

which proves the statement, since $\mathcal{B}(\check{\Gamma}_{lat})$ is generated by the cylinder sets.

(ii) Assume that $\mathsf{L} \subset \bigcup_{i=1}^{L} Q_{k_i}$. For $\mathsf{B} \in \mathcal{B}(\Gamma(\mathsf{L}))$ we have

$$T\left(\{\gamma \in \Gamma : \gamma_{\mathsf{L}} \in \mathsf{B}\}\right) = \left\{\check{\gamma} \in \check{\Gamma}_{lat} : \cup_{i=1}^{L}(\check{\gamma}_{k_i} + gk_i) \in \mathsf{B}\right\},$$

which is measurable. □

Thus, for any $\mu \in \mathcal{P}(\Gamma)$ we can define its push-forward image $T_*\mu \in \mathcal{P}(\check{\Gamma}_{lat})$, where $\mathcal{P}(\check{\Gamma}_{lat})$ is the set of all probability measures on $\check{\Gamma}_{lat}$.

Lemma 4.7. *The map $T_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\check{\Gamma}_{lat})$ is injective.*

Proof. Let $\mu, \nu \in \mathcal{P}(\Gamma)$ and $\mu \neq \nu$. Then there exists $B \in \mathcal{B}_0(\Gamma)$ such that $\mu(B) \neq \nu(B)$. By Lemma 4.6, $A := T(B) \in \mathcal{B}(\check{\Gamma}_{lat})$. The injectivity of T implies that $T^{-1}(T(B)) = B$. Thus $T_*\mu(A) = \mu(T^{-1}(A)) \neq \nu(T^{-1}(A)) = T_*\nu(A)$, and the statement is proved. □

Let us investigate the correspondence between measures on Γ and $\check{\Gamma}_{lat}$. Let μ be a probability measure on Γ satisfying the following condition:

(A) Consider the sets

$$\begin{aligned} \mathring{\Gamma} &:= \left\{\gamma \in \Gamma \,\middle|\, \gamma \cap \partial Q_k = \varnothing, \quad \forall k \in \mathbb{Z}^d\right\} \in \mathcal{B}(\Gamma), \\ \mathring{\Gamma}_{\mathsf{L}} &:= \left\{\gamma \in \Gamma \,\middle|\, \gamma_{\mathsf{L}} \cap \partial Q_k = \varnothing, \quad \forall k \in \mathbb{Z}^d\right\} \in \mathcal{B}(\Gamma_{\mathsf{L}}), \end{aligned} \tag{4.31}$$

for any $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ and assume $\mu(\mathring{\Gamma}) = 1$. In other words, μ ignores configurations that *touch* the sites of the partition cubes Q_k.

For $B_k \in \mathcal{B}(\Gamma(\overline{Q}))$ with $k \in \mathbb{Z}^d$, we denote $\mathring{B}_k := \{\gamma \in B_k | \gamma \cap \partial\overline{Q} = \emptyset\}$, where $\partial\overline{Q} := \overline{Q} \backslash Q$. Starting from a given μ, probability measure on Γ satisfying condition (4.31) above, we construct a probability measure μ_{lat} on $\check{\Gamma}_{lat}$, as the push-forward of μ. The explicit definition is as follows:

$$\begin{aligned} \mu_{lat}\Big(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}\Big) &:= \mu_{lat}\Big(A_{k_1,\dots,k_L}^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}}\Big) := \mu\Big(T^{-1}\big(A_{k_1,\dots,k_L}^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}}\big)\Big) = \\ &= \mu\Big(\big\{\gamma \in \Gamma | \gamma_{\overline{Q}_{k_l}} - k_l \in \mathring{B}_{k_l}, 1 \le l \le L\big\}\Big) \\ &= \mu\Big(\big\{\gamma \in \Gamma | \gamma_{\overline{Q}_{k_l}} - k_l \in B_{k_l}, 1 \le l \le L\big\}\Big) \\ &= \mu\Big(C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}\Big). \end{aligned} \tag{4.32}$$

Since the cylinder events generate the Borel σ-algebra, μ_{lat} is well-defined on the whole $\mathcal{B}(\breve{\Gamma}_{lat})$. Also, let us denote by $\mathring{\Gamma}_{lat}$ the set $\{\breve{\gamma} \in \breve{\Gamma}_{lat} | \breve{\gamma}_k \cap (\partial \overline{Q}) = \emptyset\}$. We see from the above definition that the corresponding measure on the lattice μ_{lat} puts full mass on $\mathring{\Gamma}_{lat}$. Moreover, $T : \mathring{\Gamma} \to \mathring{\Gamma}_{lat}$ is a bijection.

Remark 4.8. The above construction also extends to σ-finite measures on Γ, i.e, to the Lebesque-Poisson measure λ_z,

$$\breve{\lambda}_{lat,z}(A^{B_{k_1},\ldots,B_{k_L}}_{k_1,\ldots,k_L}) := \prod_{l=1}^{L} \lambda_z(\{\gamma \in \Gamma : \gamma_{\overline{Q}_{k_l}} - gk_l \in B_{k_l}\}. \tag{4.33}$$

We remark that λ_z satisfies condition (4.31) (cf. Lemma 2.2.7 and Proposition 2.2.8 in [Kun99]).

We continue by defining the energy of the new system with the phase space $\breve{\Gamma}_{lat}$. Consider arbitrarily large cubic domains $\mathsf{L}_{\mathcal{K}} := \bigsqcup_{k \in \mathcal{K}} \overline{Q}_k$ indexed by $\mathcal{K} \Subset \mathbb{Z}^d$ and define the local energy as

$$\breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}} | \breve{\eta}) := H\left((T^{-1}\breve{\gamma})_{\mathcal{K}} | (T^{-1}\breve{\eta})\right). \tag{4.34}$$

Using the above definition, we introduce the local one-point Gibbs states as

$$\breve{\mu}_{\mathcal{K}}(d\breve{\gamma}_{\mathcal{K}} | \breve{\eta}) := \begin{cases} [\breve{Z}_{\mathcal{K}}(\breve{\eta})]^{-1} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}} | \breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}_{\mathcal{K}}), & \breve{\eta} \in \mathring{\Gamma}_{lat}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.35}$$

where

$$\breve{Z}_{\mathcal{K}}(\breve{\eta}) := \int_{\breve{\Gamma}_{lat}} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}'_{\mathcal{K}} | \breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}'_{\mathcal{K}}) \tag{4.36}$$

and $\breve{\lambda}_{lat,z}$ is given by (4.33).

We note that elementary computations yield for any $\breve{\eta} \in \mathring{\Gamma}_{lat}$

$$\begin{aligned} \breve{Z}_{\mathcal{K}}(\breve{\eta}) &= \int_{\mathring{\Gamma}_{lat}} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}} | \breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}_{\mathcal{K}}) \\ &= \int_{\Gamma} \exp\{-\beta \breve{H}_{\mathcal{K}}(T(\gamma)_{\mathcal{K}} | \breve{\eta})\} \lambda_z(d\gamma) \\ &= \int_{\Gamma} \exp\{-\beta H_{\mathcal{K}}(T^{-1}(T(\gamma))_{\mathcal{K}} | T^{-1}(\breve{\eta}))\} \lambda_z(d\gamma) = Z_{Q_{\mathcal{K}}}(T^{-1}\breve{\eta}). \end{aligned} \tag{4.37}$$

Similar computations yield that the local Gibbs states for the lattice model are the pushforward measures of the local Gibbs states of the initial model, or more explicitly, $\breve{\mu}_{\mathcal{K}}(d\breve{\gamma}|\breve{\eta}) = (\mu_{\mathcal{K}} \circ T^{-1})(d\gamma|T^{-1}\breve{\eta})$. From here, we go on to define the local Gibbs specification as

$$\breve{\pi}_{\mathcal{K}}(\breve{B}|\breve{\eta}) := \breve{\mu}_{\mathcal{K}}(\breve{B}_{\mathcal{K},\breve{\eta}}|\breve{\eta}), \quad \breve{B}_{\mathcal{K},\breve{\eta}} := \{\breve{\gamma}_{\mathcal{K}}|\breve{\gamma}_{\mathcal{K}} \cup \breve{\eta}_{\mathcal{K}^c} \in \breve{B}\}, \tag{4.38}$$

for any $\breve{B} \in \mathcal{B}(\breve{\Gamma}_{lat})$. The main step is to show that uniqueness of Gibbs measures in the lattice model introduced above implies uniqueness of Gibbs measures in the initial model.

Lemma 4.9. *Let μ be a Gibbs measure on Γ corresponding to the specification $\{\Pi_{\mathsf{L}}\}_{\mathsf{L}}$. Then μ uniquely determines a Gibbs measure μ_{lat} corresponding to $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$. Moreover, if μ_{lat} is unique, then so is μ.*

Proof.

Since μ satisfies condition **(A)** (cf. Section 5.3 of [KPR12]), there exists a measure μ_{lat} as given by (4.32). Let us show that μ_{lat} is a Gibbs measure corresponding to the specification $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$, by checking the DLR equations. Let $\breve{B} \in \mathcal{B}(\mathring{\Gamma}_{lat})$. Applying (4.34)-(4.37) yields

$$\begin{aligned}
&\int_{\breve{\Gamma}_{lat}} \breve{\pi}_{\mathcal{K}}(\breve{B}|\breve{\eta})\mu_{lat}(d\breve{\eta}) \\
&= \int_{\mathring{\Gamma}_{lat}} \int_{\mathring{\Gamma}_{lat}} \breve{Z}_{\mathcal{K}}^{-1}(\breve{\eta}) \exp\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}}|\breve{\eta})\} \mathbb{1}_{\breve{B}}(\breve{\gamma}_{\mathcal{K}} \times \breve{\eta}_{\mathcal{K}^c}) \breve{\lambda}_{lat,z}(d\breve{\gamma})\mu_{lat}(d\breve{\eta}) \\
&= \int_{\Gamma} \int_{\Gamma} Z_{\mathcal{K}}^{-1}(\gamma) \exp\{-\beta \breve{H}_{\mathcal{K}}(T\gamma_{\mathcal{K}}|T\gamma)\} \mathbb{1}_{\breve{B}}(T\gamma_{\mathcal{K}} \times T\gamma_{\mathcal{K}^c}) \lambda_z(d\gamma)\mu(d\gamma) \\
&\overset{\text{DLR}}{=} \mu(T^{-1}\breve{B}) = \mu_{lat}(\breve{B}).
\end{aligned}$$

Uniqueness follows easily by Lemma 4.7. □

4.2.4 The uniqueness result

The aim of this section is to show uniqueness of tempered Gibbs measures in the lattice model introduced in Section 4.2.3. The set of such measures will be denoted by $\mathcal{G}^t_{lat}$ and consists of Gibbs measures μ_{lat}, which are supported by the following set of tempered configurations

$$\Gamma^t_{lat} := \bigcap_{\alpha > 0} \Gamma_{\alpha,lat},$$

where

$$\Gamma_{\alpha,lat} := \left\{ \breve{\gamma} \in \breve{\Gamma}_{lat} : |\breve{\gamma}|_\alpha := \sup_{k\in\mathbb{Z}} \left[|\breve{\gamma}_k|^2 \exp\{-\alpha|k|\} \right]^{1/2} < \infty \right\}.$$

Moreover, by (4.29), any tempered Gibbs measure μ_{lat} satisfies the following exponential moment estimate,

$$\sup_{k\in\mathbb{Z}^d} \int_{\breve{\Gamma}_{lat}} \exp\left\{a|\breve{\gamma}_k|^2\right\} \mu_{lat}(d\breve{\gamma}) \le \Psi, \tag{4.39}$$

where Ψ is given by (4.30). Hence, one can easily see that μ_{lat} satisfies the a-priori bound in Theorem 2.9.

Theorem 4.10. *(Uniqueness due to small activity parameter)*

Under Assumptions (FR), (LB) and (RC), for every $\beta_0 > 0$, one finds $z = z(\beta_0)$ such that $\mathcal{G}^t$ is a singleton at all values of $z \le z_0$ and $\beta < \beta_0$.

We denote the R-vicinity of a point $k \in \mathbb{Z}^d$ by $\partial k := \partial_R k = \{j \in \mathbb{Z}^d | d(Q_k, Q_j) \le R\}$, where R is given by assumption (FR) and by $\varDelta := \sup_{k\in\mathbb{Z}^d} |\partial k|$. Also, let Z_0 be a semigroup of $g\mathbb{Z}^d$ such that $|u - v| > R$ holds for all $u, v \in Z_0$ and define $\chi := \min_{Z_0} |g\mathbb{Z}^d / Z_0|$, the number of elements in the quotient group $g\mathbb{Z}^d/Z_0$. Denote $\breve{\Phi} := T_*\Phi$. More precisely, for $\breve{\gamma} \in \breve{\Gamma}_{lat}$ one has

$$\breve{\Phi}(\breve{\gamma}) = a \left(\sum_{k\in\mathbb{Z}^d} |\breve{\gamma}_k| \right)^2. \tag{4.40}$$

We divide the proof of this result into two technical lemmas, for each of the conditions that need to be checked.

The integrability condition (IC)

Lemma 4.11. *For every $\beta_0 > 0$, there are constants $\theta > 0$ and $0 < \bar{c} < 1/\varDelta^\chi$ such that, for every $k \in \mathbb{Z}^d$, $0 < \beta < \beta_0$ and any boundary condition $\breve{\eta} \in \breve{\Gamma}_{lat}$*

$$\int_{\breve{\Gamma}_{lat}} \theta\breve{\Phi}(\breve{\gamma}_k) \breve{\pi}_{Q_k}(d\breve{\gamma}_k|\breve{\eta}) \le 1 + \bar{c} \sum_{j\in\partial k} \theta\breve{\Phi}(\breve{\eta}_j). \tag{4.41}$$

Proof. We remark that, by a simple change of variables, one has,

$$\int_{\breve{\Gamma}_{lat}} \exp\left\{\breve{\Phi}(\breve{\gamma}_k)\right\} \breve{\pi}_{Q_k}(d\breve{\gamma}_k|\breve{\eta}) = \int_{\Gamma} \exp\left\{\Phi(\gamma_k)\right\} \pi_{Q_k}(d\gamma_k|\eta).$$

By applying Jensen's inequality to the one-point estimate (4.26) we have that, for parameters $a, \epsilon \geq 0$, which obey the relation

$$2a\epsilon \leq \beta(\epsilon A - Mm), \tag{4.42}$$

and for any $k \in \mathbb{Z}^d$, $\xi \in \Gamma$, there exists a constant $\Upsilon = \Upsilon(\beta) > 0$ given by $\Upsilon(\beta) := z \exp\left\{\frac{A}{2}\right\} \sup_k \sigma(Q_{gk})$ such that

$$\int_{\Gamma_k} \Phi(\gamma_k)\mu_k(d\gamma_k|\xi) \leq \Upsilon(\beta) + \frac{\beta M\epsilon}{2a} \sum_{j\in\partial_g^- k} \Phi(\xi_j). \tag{4.43}$$

Hence, also

$$\int_{\Gamma_k} \exp\left\{\breve{\Phi}(\breve{\gamma}_k)\right\}\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}) \leq \Upsilon(\beta) + \frac{\beta M\epsilon}{2a} \sum_{j\in\partial_g^- k} \Phi(\xi_j). \tag{4.44}$$

From here, it is easy to see that condition (IC) holds uniformly for all $\beta < \beta_0$ with constants $\theta = \Upsilon(\beta_0)^{-1} > 0$ and $\bar{c} := \frac{\beta_0 M\epsilon}{2a}$. One can obviously find a proper choice of ϵ and a such that $\bar{c} < 1/\Delta^\chi$.

□

The contraction condition (CC)

Lemma 4.12. *Under Assumptions (FR), (LB) and (RC), for every $\beta_0 > 0$, one finds $z_0 = z_0(\beta_0)$ such that for all values of $z \leq z_0$ and $\beta < \beta_0$,*

$$d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\xi}), \breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta})) \leq \sum_{j\in\partial k} \mathtt{k}\mathbb{1}_{\breve{\eta}_j^1\neq\breve{\eta}_j^2}, \tag{4.45}$$

for some constant $0 < \mathtt{k} < 1$ and boundary conditions $\breve{\xi}, \breve{\eta}$ such that

$$\breve{\Phi}(\breve{\xi}_j), \breve{\Phi}(\breve{\eta}_j) \leq \theta^{-1}K_*, \tag{4.46}$$

where $K_* = K_*(\theta\breve{\Phi}, \bar{c}, \mathtt{k})$ is given by (2.14).

Proof.

Let $\breve{\xi}, \breve{\eta}$ satisfy (4.46), hence $|\breve{\xi}_j|$, $|\breve{\eta}_j| \leq K_0 := \left(a^{-1}\theta^{-1}K_*\right)^{1/2}$, for all $j \in \partial k$. By formula (2.86), $d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^2))$ is equal to

$$\frac{1}{2}\int_{\Gamma_k} \left|\breve{Z}_{Q_k}^{-1}(\breve{\xi})\exp\{-\beta\breve{H}_k(\breve{\gamma}_k|\breve{\xi})\} - \breve{Z}_{Q_k}^{-1}(\breve{\eta})\exp\{-\beta\breve{H}_k(\breve{\gamma}_k|\breve{\eta})\}\right| \breve{\lambda}_{lat,z}(d\breve{\gamma}_k).$$

Notice however, that by a change of variables, the above expression can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_k}\Big|Z_k^{-1}(\xi)\exp\{-\beta H_k(\gamma_k|\xi)\} - Z_k^{-1}(\eta)\exp\{-\beta H_k(\gamma_k|\eta)\}\Big|\lambda_z(d\gamma_k)\\
&\le \frac{1}{2}\int_{\Gamma_k}\Big|Z_k(\eta)\exp\{-\beta H_k(\gamma_k|\xi)\} - Z_k(\xi)\exp\{-\beta H_k(\gamma_k|\eta)\}\Big|\lambda_z(d\gamma_k)\\
&\le \frac{1}{2}\Big[|Z_k(\eta)-Z_k(\xi)|\int_{\Gamma_k}\exp\{-\beta H_k(\gamma_k|\xi)\}\lambda_z(d\gamma_k)\\
&+Z_k(\xi)\int_{\Gamma_k}|\exp\{-\beta H_k(\gamma_k|\xi)\} - \exp\{-\beta H_k(\gamma_k|\eta)\}|\lambda_z(d\gamma_k)\Big]\\
&\le \min(Z_k(\xi),Z_k(\eta))\int_{\Gamma_k}|\exp\{-\beta H_k(\gamma_k|\xi)\} - \exp\{-\beta H_k(\gamma_k|\eta)\}|\lambda_z(d\gamma_k).
\end{aligned} \tag{4.47}$$

Now, let $\xi = \emptyset$ outside Q_j, for a fixed $j \in \partial k$ and $\eta = \emptyset$. In this case the minimum above is equal to 1, so we are left to compute

$$\begin{aligned}
&\int_{\Gamma_k}|\exp\{-\beta H_k(\gamma_k|\xi)\} - \exp\{-\beta H_k(\gamma_k)\}|\lambda_z(d\gamma_k) =\\
&= \int_{\Gamma_k\setminus\{\emptyset\}}\exp\{-\beta H_k(\gamma_k)\}|1-\exp\{-\beta\sum_{\substack{x\in\gamma_k\\ y\in\xi_j}}V(x,y)\}|\lambda_z(d\gamma_k).
\end{aligned} \tag{4.48}$$

Writing $V = V^+ - V^-$ as the sum of its positive $V^+ := V \vee 0$ and negative part $V^- := -(V \wedge 0)$ and using the elementary inequality

$$\big|1-\exp\big\{a-\sum_{i=1}^n a_i\big\}\big| \le (\exp\{a\}-1) + \sum_{i=1}^n\big|1-\exp\{-a_i\}\big|,$$

where $n \in \mathbb{N}$ and $a, a_i \ge 0$, $1 \le i \le n$, we have that

$$\begin{aligned}
&|1-\exp\{-\beta\sum_{\substack{x\in\gamma_k\\ y\in\xi_j}}V(x,y)\}|\\
&\qquad \le \Big|1-\exp\{\beta\sum_{\substack{x\in\gamma_k\\ y\in\xi_j}}V^-(x,y)\}\Big| + \sum_{\substack{x\in\gamma_k\\ y\in\xi_j}}\Big(1-\exp\{-\beta V^+(x,y)\}\Big)\\
&\qquad \le \exp\{\beta M K_0|\gamma_k|\} - 1 + K_0|\gamma_k|\\
&\qquad \le \beta M K_0|\gamma_k|\exp\{\beta M K_0|\gamma_k|\} + K_0|\gamma_k|,
\end{aligned}$$

since $|\xi_j| \le K_0$, by (4.46).

We see now that the integral in (4.48) does not exceed

$$\int_{\Gamma_k \setminus \{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}\beta M K_0 |\gamma_k| \exp\{\beta M K_0 |\gamma_k|\} \lambda_z(d\gamma_k) + \int_{\Gamma_k \setminus \{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\} K_0 |\gamma_k| \lambda_z(d\gamma_k).$$

By (4.23), we have that $MK_0|\gamma_k| - H_k(\gamma_k) \leq (MK_0 + A/2)|\gamma_k| - (A - Mm)/2|\gamma_k|^2$ and by applying Young's inequlity we obtain that there exists a non-negative constant D such that $MK_0|\gamma_k| - H_k(\gamma_k) \leq D$. Hence,

$$\begin{aligned}
&\int_{\Gamma_k \setminus \{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}\beta M K_0 |\gamma_k| \exp\{\beta M K_0 |\gamma_k|\} \lambda_z(d\gamma_k) \\
&\leq \int_{\Gamma_k \setminus \{\emptyset\}} \beta M K_0 |\gamma_k| \exp\{D\} \lambda_z(d\gamma_k) \leq \beta M K_0 \exp\{D\} \int_{\Gamma_k \setminus \{\emptyset\}} |\gamma_k| \lambda(d\gamma_k) \\
&= \beta M K_0 \exp\{D\} \sum_{j=1}^{\infty} j \cdot \frac{z^j |Q_k|^j}{j!} = z\beta M K_0 \exp\{D\} |Q_k| e^{z|Q_k|}.
\end{aligned}$$

Again, by (4.23), we know that $H_k(\gamma_k) \geq A/8$, hence

$$\int_{\Gamma_k \setminus \{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\} K_0 |\gamma_k| \lambda_z(d\gamma_k) \leq z K_0 \exp\{A\beta/8\} |Q_k| e^{z|Q_k|}.$$

Thus

$$\begin{aligned}
&d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^2)) \\
&\qquad \leq z\Big[\beta M K_0 \exp\{D\} |Q_k| e^{z|Q_k|} + K_0 \exp\{A\beta/8\} |Q_k| e^{z|Q_k|}\Big] \\
&\qquad \leq z\Big[\beta_0 M K_0 \exp\{D\} |Q_k| e^{z|Q_k|} + K_0 \exp\{A\beta_0/8\} |Q_k| e^{z|Q_k|}\Big].
\end{aligned}$$

It follows immediately, that there exists $z_0 = z_0(\beta_0)$ such that, for $z < z_0$, $d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^2))$ is smaller than $\mathtt{k}$. Applying the triangle inequality, the result also holds for more general boundary conditions. Hence the conditions of Theorem 2.9 are satisfied and we have obtained the uniqueness of the Gibbs measure.

□

Proof of Theorem 4.10. It follows immediately by Lemmas 4.11 and 4.12 that for the considered lattice system uniqueness holds. By Lemma 4.9, we obtain the desired result.

□

Decay of correlations

We note that uniqueness of $\mu \in \mathcal{G}^t$ yields a result for the decay of correlations, via Theorem 2.19. Let $Q_{\mathcal{K}_1}$ and $Q_{\mathcal{K}_2}$ be two disjoint cubic domains (built with the help of the partition $(Q_k)_k$ given by (4.14)) and let G_1, G_2 be two local functions such that G_i is $\mathcal{B}_{Q_{\mathcal{K}_i}}(\Gamma)$-measurable, for $i = 1, 2$. Also, assume that

$$G_2(\gamma) \leq \sum_{j \in \mathcal{K}_2} \theta \Phi(\gamma_j), \quad \gamma \in \Gamma$$

and

$$\sup_{k \in \mathcal{K}_2} \int_{\Gamma} G_1(\gamma) \Phi(\gamma_k) \mu(d\gamma) < \infty,$$

where Φ is given by (4.24).

Corollary 4.13. *In the setting described above, there exist constants $\alpha, \tau > 0$ such that*

$$|Cov_{\mu}(G_1, G_2)| \leq \tau m(Q_{\mathcal{K}_2})^2 \exp\left(-\alpha d(Q_{\mathcal{K}_1}, Q_{\mathcal{K}_2})\right) \int_{\Gamma} |G_1(\gamma)| \tilde{F}(\gamma) \mu(d\gamma). \quad (4.49)$$

Moreover,

$$\alpha := -\log r_K, \quad (4.50)$$

where r_K is given by (2.68), for the Dobrushin-Pechersky matrix with entries given by Lemmas 4.11 and 4.12.

□

4.3 Systems with strong superstable interaction

4.3.1 Gibbsian formalism

In what follows, we briefly show that the uniqueness results still hold if we replace assumptions **(LB)** and **(RC)** on the behaviour of the pair potential V with the following one

(SSS) ***Strong Superstability***: For a given $P > 2$ and a certain partition $\mathbb{R}^d = \bigsqcup_{k \in \mathbb{Z}^d} Q_{gk}$ with $g > 0$, cf. (4.14), there exist positive D, E such that

$$H(\gamma) \;\geq\; D \sum_{k \in \mathbb{Z}^d} |\gamma_k|^P - E|\gamma| \quad \text{for all } \gamma \in \Gamma_0. \quad (4.51)$$

According to (4.51), the pair potential V is semibounded below, i.e.,

$$\inf_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} V(x,y) := \inf_{\{x,y\}\subset\mathbb{R}^d} H(\{x,y\}) =: -M \geq 2\left(2^{P-1}D - E\right), \tag{4.52}$$

which agrees with the initial Assumption (**LB**). The same is true for the energy in every partition cube Q_{gk}, i.e.,

$$\inf_{\gamma_k\in\Gamma_k} H(\gamma_k) =: -C \geq -D^{\frac{1}{1-P}} E^{\frac{P}{P-1}}. \tag{4.53}$$

We still keep the finite range Assumption (**FR**) from Subsection 4.2.2. Then, for $x \in Q_{gk}$ and $y \in Q_{gj}$, the interaction $V(x,y)$ is zero unless

$$j \in \partial_g k := \left\{ k' \in \mathbb{Z}^d \;\middle|\; |k-k'| < \sqrt{d}\,(1+R/\delta) \right\}. \tag{4.54}$$

Similarly to (4.16) and (4.18), the number of such neighbour cubes Q_{gj} (having the diagonal $\delta := g\sqrt{d}$) does not exceed

$$|\partial_g k| \leq m := v_d d^{d/2}(R/\delta + 3/2)^d. \tag{4.55}$$

Fixing the parameters

$$0 \leq \alpha < \beta \quad \text{and} \quad 0 \leq a < (\beta-\alpha)\,D, \tag{4.56}$$

let us define the Lyapunov functional

$$\Gamma_k \ni \gamma_k \to \Phi(\gamma_k) := \alpha H(\gamma_k) + a|\gamma_k|^P \geq -\beta C, \tag{4.57}$$

where $C \in \mathbb{R}$ is the same as in (4.53). A starting point is the following modification of the exponential bound (4.26)–(4.28) in Lemma 4.4

$$\begin{aligned}
&\int_{\Gamma_k} \exp\left\{\Phi(\gamma_k)\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \\
\leq\; &\int_{\Gamma_k} \exp\left\{ \left[a - (\beta-\alpha)\,D\right]|\gamma_k|^P + \left[(\beta-\alpha)\,E + \beta M \sum_{j\in\partial_g k} |\xi_j|\right] |\gamma_k| \right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
\leq\; &\exp\left\{ \varUpsilon_\varepsilon + \varepsilon \sum_{j\in\partial_g k} |\xi_j|^P \right\},
\end{aligned} \tag{4.58}$$

holding for any

$$0 < \varepsilon < \frac{1}{m}\left((\beta-\alpha)D - a\right)$$

and the corresponding

$$\Upsilon_\varepsilon := m\varepsilon^{\frac{2}{2-P}}(\beta M)^{\frac{P}{P-2}} + z\exp\{\beta E\}\sup_k \sigma(Q_{gk}).$$

4.3.2 Uniqueness for strong superstable interactions

We show that we can obtain the same type of uniqueness result as in the previous section.

Theorem 4.14. *(Uniqueness due to small activity parameter) Under Assumptions (FR) and (SSS) for every $\beta_0 > 0$ one finds $z = z(\beta_0)$ such that $\mathcal{G}^t$ is a singleton at all values of $z \le z_0$ and $\beta < \beta_0$.*

The proof is similar to the one of Theorem 4.10 and in what follows we will only present the main ideas. We consider again the lattice model corresponding to the system, constructed as in Section 4.2.3. Uniqueness in this new model in proven by the following two lemmas.

The integrability condition (IC)

Lemma 4.15. *For every $\beta_0 > 0$, there are constants $\theta > 0$ and $0 < \bar{c} < 1/\Delta^\chi$ such that, for every $k \in \mathbb{Z}^d$, $0 < \beta < \beta_0$ and any boundary condition $\breve{\eta} \in \breve{\Gamma}_{lat}$*

$$\int_{\breve{\Gamma}_{lat}} \theta\breve{\Phi}(\breve{\gamma}_k)\breve{\pi}_{Q_k}(d\breve{\gamma}_k|\breve{\eta}) \le 1 + \bar{c}\sum_{j\in\partial k} \theta\breve{\Phi}(\breve{\eta}_j), \tag{4.59}$$

Proof. Similarly to Lemma 4.11, one can observe that condition (IC) is immediately satisfied by applying Jensen's inequality to the one-point estimate (4.58). We obtain constants $\theta := \Upsilon_\varepsilon^{-1}$ and $\bar{c} := \frac{\varepsilon}{\lambda} < \frac{1}{\Delta^\chi}$, for $\epsilon > 0$ small enough.

□

The contraction condition (CC)

Lemma 4.16. *Under Assumption (SSS), for every $\beta_0 > 0$, one finds $z_0 = z_0(\beta_0)$ such that for all values of $z \le z_0$ and $\beta < \beta_0$,*

$$d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\xi}), \breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta})) \le \sum_{j\in\partial k} \mathtt{k}\mathbb{1}_{\breve{\eta}^1_j \ne \breve{\eta}^2_j}, \tag{4.60}$$

for some constant $0 < \mathtt{k} < 1$ and boundary conditions $\breve{\xi}, \breve{\eta}$ such that

$$\breve{\Phi}(\breve{\xi}_j), \breve{\Phi}(\breve{\eta}_j) \le K_0 := \theta^{-1}K_*, \tag{4.61}$$

where $K_* = K_*(\theta\breve{\Phi}, \bar{c}, \mathtt{k})$ is given by (2.14).

Proof.

Following the lines of the proof of Lemma 4.12, we see that it would be enough to estimate

$$d_{TV}(\mu_k^\xi, \mu_k^\eta) \le \int_{\Gamma_k\setminus\{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}\beta M K_0|\gamma_k| \exp\{\beta M K_0|\gamma_k|\}\lambda_z(d\gamma_k) + \int_{\Gamma_k\setminus\{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}K_0|\gamma_k|\lambda_z(d\gamma_k).$$

From condition (4.51), we have that $MK_0|\gamma_k| - H_k(\gamma_k) \le (MK+E)|\gamma_k| - D|\gamma_k|^P$ and by applying Young's inequlity we obtain that there exists a constant F such that $MK_0|\gamma_k| - H_k(\gamma_k) \le F$. Hence,

$$\begin{aligned}&\int_{\Gamma_k\setminus\{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}\beta M K_0|\gamma_k| \exp\{\beta M K_0|\gamma_k|\}\lambda_z(d\gamma_k)\\ &\le \int_{\Gamma_k\setminus\{\emptyset\}} \beta M K_0|\gamma_k|\exp\{F\}\lambda_z(d\gamma_k) \le \beta M K_0 \exp\{F\}\int_{\Gamma_k\setminus\{\emptyset\}} |\gamma_k|\lambda(d\gamma_k)\\ &= \beta M K_0\exp\{F\}\sum_{j=1}^{\infty} j\cdot\frac{z^j|Q_k|^j}{j!} = z\beta M K_0\exp\{F\}|Q_k|e^{z|Q_k|},\end{aligned}$$

which can be made small for z small enough.

Again by applying Young's inequality in condition (4.51) we get a constant G such that $-H_k(\gamma_k) \le G$, hence

$$\int_{\Gamma_k\setminus\{\emptyset\}} \exp\{-\beta H_k(\gamma_k)\}K_0|\gamma_k|\lambda_z(d\gamma_k) \le zK_0\exp\{\beta G\}|Q_k|e^{z|Q_k|}.$$

Therefore $d_{TV}(\mu_k^\xi, \mu_k^\eta)$ is less than `k` for z small enough. Applying the triangle inequality, the result also holds for more general boundary conditions. Hence the conditions of Theorem 2.9 are satisfied and we have obtained the uniqueness of the Gibbs measure. □

4.4 The Lebowitz-Mazel-Presutti model

In what follows, we study a particular case of multi-body interactions between particles. The model we discuss was first introduced in [LMP98] and more thoroughly discussed in [LMP99] and later in the monograph [Pre09]. We stress the importance of this model, as it is one of the few examples of systems of interacting particles, for which a phase transition could be proven. We consider particles in $\mathbb{R}^d$, $d \geq 2$, interacting via attractive pair and repulsive four-body potentials of Kac type.

4.4.1 A short description of the model

Let $J : \mathbb{R}^2 \to \mathbb{R}$ be such that

(i) it is translation invariant, i.e. $J(x_1+a, x_2+a) = J(x_1, x_2)$ for any $x_1, x_2, a \in \mathbb{R}^d$,

(ii) it is continuous and non-negative,

(iii) $J(0, x)$ is supported by the unit ball, which will give a radius of interaction equal to 2,

(iv) it is normalized: $\int_{\mathbb{R}^d} J(0, x)dx = 1$.

Remark 4.17. From the above assumptions it is immediate that J is bounded, i.e. there exists $M > 0$ such that

$$\max_{x_1, x_2} J(x_1, x_2) \leq M. \tag{4.62}$$

Let us fix a $\delta > 0$ such that

$$\min_{x,y:\,|x-y|\leq\delta} J(x, y) =: A_\delta > 0. \tag{4.63}$$

For convenience, we will give up the δ in the notation of A_δ.

For any $\varepsilon > 0$, define

$$J_\varepsilon(x_1, x_2) := \varepsilon^d J(\varepsilon x_1, \varepsilon x_2) \qquad \text{for all } x_1, x_2 \in \mathbb{R}^d. \tag{4.64}$$

Obviously, J_ε satisfies properties (i), (ii) and (iv), while $J_\varepsilon(0, x)$ is supported by the ball of radius ε^{-1}. Also

$$\max_{x_1, x_2} J(x_1, x_2) \leq \varepsilon^d M. \tag{4.65}$$

Moreover, J_ε satisfies (4.63) with constants $\varepsilon^{-1}\delta$ and $\varepsilon^d A_\delta$, where δ and A_δ are as in (4.63).

We consider again a partition $(Q_k)_{k\in\mathbb{Z}^d}$ of the space $\mathbb{R}^d$ as in (4.14), in cubes of side length $g=\delta/\sqrt{d}$. Additionally, we need the rescaled partition $(Q_k^\varepsilon)_{k\in\mathbb{Z}^d}$, where Q_k^ε are cubes of side length $\varepsilon^{-1}g$.

We set $\Delta := \sup_{k\in\mathbb{Z}^d}|\partial k|$, where $\partial k := \{j\in\mathbb{Z}^d | d(Q_k,Q_j)\le 2\}$ is the vicinity of a point $k\in\mathbb{Z}^d$. Moreover, we say that $\partial^\varepsilon k := \{j\in\mathbb{Z}^d | d(Q_k^\varepsilon,Q_j^\varepsilon)\le 2\varepsilon^{-1}\}$ is the ε-vicinity of k. Notice that $\partial^\varepsilon k=\partial k$, for any $\varepsilon>0$, since the range of interaction is proportional to the rescaling. The volume of the rescaled cube is $|Q_k^\varepsilon|=\varepsilon^{-d}|Q_k|$. By γ_k^ε we denote the restriction of the configuration γ to the cube Q_k^ε, i.e. $\gamma\cap Q_k^\varepsilon$ and $\Gamma_k^\varepsilon := \Gamma_{Q_k^\varepsilon}$.

It is easy to see that

$$\max_{\{x_1,x_2\}} J_\varepsilon(x_1,x_2)\le\varepsilon^d M,$$
$$\min_{|x_1-x_2|\le\varepsilon^{-1}\delta} J_\varepsilon(x_1,x_2)=\varepsilon^d A.$$

For $\gamma\in\Gamma_0$, we define the *LMP Hamiltonian* as

$$H^\varepsilon(\gamma) := -\sum_{\{x_1,x_2\}\subset\gamma} V_\varepsilon^{(2)}(x_1,x_2)+\sum_{\{x_1,x_2,x_3,x_4\}\subset\gamma} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4), \tag{4.66}$$

where

$$V_\varepsilon^{(2)}(x_1,x_2) := \int_{\mathbb{R}^d} J_\varepsilon(x,x_1)J_\varepsilon(x,x_2)dx \tag{4.67}$$

and

$$V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4) := \int_{\mathbb{R}^d} J_\varepsilon(x,x_1)J_\varepsilon(x,x_2)J_\varepsilon(x,x_3)J_\varepsilon(x,x_4)dx. \tag{4.68}$$

For any $\xi\in\Gamma$ and any volume $\mathsf{L}\in\mathcal{B}_c(\mathbb{R}^d)$, the *LMP conditional energy* is given by

$$\begin{aligned}W_{\mathsf{L}}^\varepsilon(\gamma_{\mathsf{L}}|\xi) := &-\sum_{x_1\in\gamma_{\mathsf{L}},x_2\in\xi_{\mathsf{L}^c}} V_\varepsilon^{(2)}(x_1,x_2)+\sum_{\{x_1,x_2,x_3\}\subset\gamma_{\mathsf{L}},x_4\in\xi_{\mathsf{L}^c}} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4)\\ &+\sum_{\{x_1,x_2\}\subset\gamma_{\mathsf{L}},\{x_3,x_4\}\subset\xi_{\mathsf{L}^c}} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4)+\sum_{x_1\in\gamma_{\mathsf{L}},\{x_2,x_3,x_4\}\subset\xi_{\mathsf{L}^c}} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4).\end{aligned} \tag{4.69}$$

and the *LMP conditional Hamiltonian*, respectively, by

$$H^{\varepsilon}_{\mathsf{L}}(\gamma_{\mathsf{L}}|\xi) := H^{\varepsilon}(\gamma_{\mathsf{L}}) + W^{\varepsilon}_{\mathsf{L}}(\gamma_{\mathsf{L}}|\xi). \tag{4.70}$$

For a fixed inverse temperature $\beta > 0$, the *local Gibbs state* with boundary condition ξ is a probability measure on $(\Gamma_{\mathsf{L}}, \mathcal{B}(\Gamma_{\mathsf{L}}))$ defined by

$$\mu^{\varepsilon}_{\mathsf{L}}(d\gamma_{\mathsf{L}}|\xi) := [Z_{\mathsf{L},\varepsilon}(\xi)]^{-1} \exp\{-\beta H^{\varepsilon}_{\mathsf{L}}(\gamma_{\mathsf{L}}|\xi)\}\lambda_z(d\gamma_{\mathsf{L}}), \tag{4.71}$$

provided that the corresponding *partition function*

$$\begin{aligned} Z_{\mathsf{L},\varepsilon}(\xi) : &= \int_{\Gamma_{\mathsf{L}}} \exp\{-\beta H^{\varepsilon}_{\mathsf{L}}(\gamma_{\mathsf{L}}|\xi)\}\lambda_z(d\gamma_{\mathsf{L}}) \\ &= 1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathsf{L}^n} \exp\{-\beta H^{\varepsilon}_{\mathsf{L}}(\{x_1, \dots, x_n\}|\xi) dx_1 \dots dx_n \end{aligned} \tag{4.72}$$

is finite. Otherwise, we set $\mu^{\varepsilon}_{\mathsf{L}}(d\gamma_{\mathsf{L}}|\xi) = 0$. Also, note that from the above expression it is obvious that $Z_{\mathsf{L},\varepsilon}(\xi) \geq 1$. The definitions to be given below are standard and straightforwardly extend the constructions of Section 4.2.1.

The family of local Gibbs states determines a family of stochastic kernels $\Pi = \{\pi_{\mathsf{L}}\}_{\mathsf{L}\in\mathcal{B}_c(\mathbb{R}^d)}$, $\pi_{\mathsf{L}} : \mathcal{B}(\Gamma) \times \Gamma \to [0, 1]$, as follows

$$\pi^{\varepsilon}_{\mathsf{L}}(B|\xi) := \mu^{\varepsilon}_{\mathsf{L}}(B_{\mathsf{L},\xi}|\xi), \text{ where } B_{\mathsf{L},\xi} := \{\gamma_{\mathsf{L}} \in \Gamma_{\mathsf{L}}|\gamma_{\mathsf{L}} \cup \xi_{\mathsf{L}^c} \in B\} \in \mathcal{B}(\Gamma_{\mathsf{L}});$$

this Π will be called *local specification*. By Proposition 6.3 in [Pre76] or Proposition 2.6 in [Pre05], the family Π obeys the consistency property, meaning that for any $B \in \mathcal{B}(\Gamma)$ and $\xi \in \Gamma$

$$\int_{\Gamma} \pi^{\varepsilon}_{\mathsf{U}}(B|\gamma)\pi^{\varepsilon}_{\mathsf{L}}(d\gamma|\xi) = \pi^{\varepsilon}_{\mathsf{L}}(B|\xi), \qquad \mathsf{U} \subseteq \mathsf{L}. \tag{4.73}$$

Definition 4.18. A probability measure $\mu \in \mathcal{P}(\Gamma)$ is called a grand canonical *Gibbs measure* (or *state*) corresponding to the Hamiltonian H^{ε}, with activity z, if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equilibrium equation

$$(\pi^{\varepsilon}_{\mathsf{L}}\mu)(B) := \int_{\Gamma} \pi^{\varepsilon}_{\mathsf{L}}(B|\gamma)\mu(d\gamma) = \mu(B), \tag{4.74}$$

for all $\mathsf{L} \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\Gamma)$. For fixed temperature β and fixed scaling parameter ε, the associated set of all Gibbs measures will be denoted by $\mathcal{G}^{\varepsilon}$.

Note that, for simplicity, from now on, we denote $H^{\varepsilon}_{Q^{\varepsilon}_k}$ by H^{ε}_k, $\mu^{\varepsilon}_{Q^{\varepsilon}_k}$ by μ^{ε}_k and $Z_{Q^{\varepsilon}_k,\varepsilon}$ by $Z_{k,\varepsilon}$, respectively.

Lemma 4.19. *The following bounds on the potentials hold:*

$$-V_\varepsilon^{(2)}(x_1,x_2) \geq -\varepsilon^d M \tag{4.75}$$

and

$$V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4) \geq \varepsilon^{3d} A^4 g^d. \tag{4.76}$$

Proof. Elementary computations yield

$$\begin{aligned} &-V_\varepsilon^{(2)}(x_1,x_2) = -\int_{\mathbb{R}^d} J_\varepsilon(x,x_1)J_\varepsilon(x,x_2)dx \\ &\geq \sup_{y,z} J_\varepsilon(y,z)\int_{\mathbb{R}^d} J_\varepsilon(x,x_2)dx \geq -\varepsilon^d M \end{aligned} \tag{4.77}$$

and

$$\begin{aligned} &V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4) = \int_{\mathbb{R}^d} J_\varepsilon(x,x_1)J_\varepsilon(x,x_2)J_\varepsilon(x,x_3)J_\varepsilon(x,x_4)dx \\ &\geq \int_{Q_k^\varepsilon} J_\varepsilon(x,x_1)J_\varepsilon(x,x_2)J_\varepsilon(x,x_3)J_\varepsilon(x,x_4)dx \\ &\geq \varepsilon^{4d}A^4|Q_k^\varepsilon| = \varepsilon^{3d}A^4|Q_k| = \varepsilon^{3d}A^4 g^d. \end{aligned} \tag{4.78}$$

□

Lemma 4.20. *There exist constants $B, D > 0$ such that for any boundary condition $\xi \in \Gamma$, the following bound for the conditional Hamiltonian holds:*

$$\begin{aligned} H_k^\varepsilon(\gamma_k^\varepsilon|\xi) \geq & \varepsilon^{3d}A^4g^dD|\gamma_k^\varepsilon|^4 - \Big[\frac{\varepsilon^d M}{2}(1+\Delta)\Big]|\gamma_k^\varepsilon|^2 \\ & -\varepsilon^{3d}A^4g^dB + \frac{\varepsilon^d M}{2}|\gamma_k^\varepsilon| - \varepsilon^d M/2\sum_{j\in\partial k}|\xi_j^\varepsilon|^2. \end{aligned} \tag{4.79}$$

Moreover,

$$\begin{aligned} H_k^\varepsilon(\gamma_k^\varepsilon) \geq & \varepsilon^{3d}A^4g^dD|\gamma_k^\varepsilon|^4 - \frac{\varepsilon^d M}{2}|\gamma_k^\varepsilon|^2 \\ & -\varepsilon^{3d}A^4g^dB + \frac{\varepsilon^d M}{2}|\gamma_k^\varepsilon|. \end{aligned} \tag{4.80}$$

Proof. First, let us notice that

$$-\sum_{\{x_1,x_2\}\subset\gamma_k^\varepsilon} V_\varepsilon^{(2)}(x_1,x_2) \geq -\varepsilon^d M\binom{|\gamma_k^\varepsilon|}{2} \geq -\varepsilon^d M/2\big(|\gamma_k^\varepsilon|^2 - |\gamma_k^\varepsilon|\big), \tag{4.81}$$

$$-\sum_{x_1\in\gamma_k^\varepsilon, x_2\in\xi_j^\varepsilon} V_\varepsilon^{(2)}(x_1,x_2) \geq -\varepsilon^d M|\gamma_k^\varepsilon|\cdot|\xi_j^\varepsilon| \geq -\varepsilon^d M/2\big(|\gamma_k^\varepsilon|^2+|\xi_j^\varepsilon|^2\big), \tag{4.82}$$

and

$$\begin{aligned}\sum_{\{x_1,x_2,x_3,x_4\}\subset\gamma_k^\varepsilon} V_\varepsilon^{(4)}(x_1,x_2,x_3,x_4) &\geq \varepsilon^{3d}A^4g^d\binom{|\gamma_k^\varepsilon|}{4}\\ &\geq \varepsilon^{3d}A^4g^d\frac{1}{4!}\big[|\gamma_k^\varepsilon|^4-6|\gamma_k^\varepsilon|^3+11|\gamma_k^\varepsilon|^2-6|\gamma_k^\varepsilon|\big]\\ &\geq \varepsilon^{3d}A^4g^d\big[D|\gamma_k^\varepsilon|^4-B\big],\end{aligned} \tag{4.83}$$

for some positive constants B and D, independent of ε and explicitly computable by Young's inequality.

Summing up (4.81)-(4.83) and taking into account the positivity of the four-body potential one obtains (4.79). Inequality (4.80) follows similarly. □

Classical particle systems with multi-body interactions in the continuum have been treated e.g. in [BP02] and [KR04]. Moreover, in [Ter08], sufficient conditions for the superstability of such interactions were given. However, these results are not applicable in the present setting, due to the negativity of the two-body potential.

4.4.2 Existence of Gibbs measures

A first step in the theory of unbounded spins is the proper notion of *temperedness*. For our purpose, it makes sense to introduce the following subsets of *tempered configurations*(see Remark 4.5, as well as Section 2.4 of [KPR12])

$$\Gamma^{t,\varepsilon} := \bigcap_{\alpha>0}\Gamma_{\alpha,\varepsilon},$$

$$\Gamma_{\alpha,\varepsilon} := \left\{\gamma\in\Gamma : \|\gamma\|_\alpha := \Big[\sum_{k\in\mathbb{Z}^d}|\gamma_k^\varepsilon|^2\exp\{-\alpha|k|\}\Big]^{1/2}<\infty\right\}.$$

We see that these sets indeed coincide for different values of $\varepsilon > 0$.

We claim that under the conditions imposed above, the set of the tempered Gibbs measures (i.e., those supported by $\Gamma^{t,\varepsilon}$ for $\varepsilon = 1$) is not empty. To prove the existence of tempered Gibbs measures $\mu \in \mathcal{G}^{t,\varepsilon}$ we use the analytical method developed in [KPR12].

One-point estimate

Fix some arbitrarily big $a > 0$. Let us introduce the functional

$$\Phi^\varepsilon(\gamma_k^\varepsilon) := a|\gamma_k^\varepsilon|^2,\ \gamma\in\Gamma. \tag{4.84}$$

Lemma 4.21. *There exists a universal constant $\Upsilon_\varepsilon > 0$ such that for all $k \in \mathbb{Z}^d$, $\xi \in \Gamma$.*

$$\int_{\Gamma_k^\varepsilon} \exp\left\{\Phi^\varepsilon(\gamma_k^\varepsilon)\right\} \mu_k^\varepsilon(d\gamma_k^\varepsilon|\xi) \le \exp\left\{\Upsilon_\varepsilon + \frac{\varepsilon^d}{2}\beta M \sum_{j\in\partial k} |\xi_j^\varepsilon|^2\right\}. \tag{4.85}$$

Proof. Direct computations based on Lemma 4.20 yield

$$\begin{aligned}\int_{\Gamma_k^\varepsilon} \exp\left\{a|\gamma_k^\varepsilon|^2\right\} \mu_k^\varepsilon(d\gamma_k^\varepsilon|\xi) &\le \int_{\Gamma_k^\varepsilon} \exp\left\{a|\gamma_k^\varepsilon|^2 - \beta H_k^\varepsilon(\gamma_k^\varepsilon|\xi)\right\} d\lambda_{z\sigma}(\gamma_k^\varepsilon)\\ &\le \int_{\Gamma_k^\varepsilon} \exp\Bigg\{-\beta\varepsilon^{3d}A^4 g^d D|\gamma_k^\varepsilon|^4 + \left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]|\gamma_k^\varepsilon|^2\\ &\quad + \beta\varepsilon^{3d}A^4 g^d B - \frac{\beta\varepsilon^d M}{2}|\gamma_k^\varepsilon| + \frac{\beta M\varepsilon^d}{2}\sum_{j\in\partial k}|\xi_j^\varepsilon|^2\Bigg\} d\lambda_{z\sigma}(\gamma_k^\varepsilon)\end{aligned}$$

By Young's inequality one gets

$$\begin{aligned}&\left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]|\gamma_k^\varepsilon|^2\\ &\quad\le \beta\varepsilon^{3d}A^4 g^d D|\gamma_k^\varepsilon|^4 + \left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]^2 \left(4\beta\varepsilon^{3d}A^4 g^d D\right)^{-1}.\end{aligned}$$

Thus, the claim holds with

$$\begin{aligned}\Upsilon_\varepsilon &= \exp\left\{\beta\varepsilon^{3d}A^4 g^d B + \left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]^2 \left(4\beta\varepsilon^{3d}A^4 g^d D\right)^{-1}\right\} \int_{\Gamma_k^\varepsilon} d\lambda_{z\sigma}(\gamma_k^\varepsilon)\\ &= \exp\left\{\left(\beta\varepsilon^{3d}A^4 g^d B + \left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]^2 \left(4\beta\varepsilon^{3d}A^4 g^d D\right)^{-1}\right) + z|Q_k^\varepsilon|\right\}\\ &= \exp\left\{\left(\beta\varepsilon^{3d}A^4 g^d B + \left[a + \beta\left[\frac{\varepsilon^d M}{2}(1+\Delta)\right]\right]^2 \left(4\beta\varepsilon^{3d}A^4 g^d D\right)^{-1}\right) + zg^d\varepsilon^{-d}\right\} < \infty.\end{aligned} \tag{4.86}$$

□

An important sequel of Lemma 4.21 is the following bound on the exponential moments of the specification kernels holding in large cubic domains $\mathsf{L}_\mathcal{K}$, defined by (4.19). Again, in order to simplify notation, $\pi_\mathcal{K}^\varepsilon(d\gamma|\xi)$ represents the kernel $\pi_{\mathsf{L}_\mathcal{K}}^\varepsilon(d\gamma|\xi)$.

Lemma 4.22. *Under the assumptions of Lemma 4.21, there exists a finite $\Psi > 0$ such that uniformly for all $k \in \mathbb{Z}^d$, $\xi \in \Gamma^{\mathrm{t}}$ and $a > 0$.*

$$\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} \int_\Gamma \exp\left\{\Phi^\varepsilon(\gamma_k^\varepsilon)\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \leq \Psi. \tag{4.87}$$

Proof. For a fixed $\xi \in \Gamma^{\mathrm{t}}$ let us consider the quantities

$$n_k(\mathcal{K}|\xi) := \log\left\{\int_\Gamma \exp\left\{\Phi^\varepsilon(\gamma_k^\varepsilon)\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right\}, \quad k \in \mathbb{Z}^d, \tag{4.88}$$

which are nonnegative and finite by Lemma 4.21. In particular,

$$n_k(\mathcal{K}|\xi) := \Phi(\xi_k) \quad \text{if} \quad k \notin \mathcal{K}.$$

A natural idea is to establish global bounds on the whole sequence $(n_k(\mathcal{K}|\xi))_{k\in\mathbb{Z}^d}$, which then imply the required estimates on each of its components. Set $\vartheta := \sup_k \operatorname{diam}(\partial k)$. Hence $|k - j| \leq \vartheta$, for any $j \in \partial k$.

Let us start from (4.85) written for all specification kernels $\pi_k^\varepsilon(\mathrm{d}\gamma_k|\gamma)$ with boundary conditions $\gamma \in \Gamma$

$$\int_\Gamma \exp\left\{\Phi(\eta_k)\right\} \pi_k^\varepsilon(\mathrm{d}\eta_k|\gamma) \leq \exp\left\{\Upsilon_\varepsilon + \frac{\varepsilon^d}{2}\beta M \sum_{j\in\partial k} |\gamma_j^\varepsilon|^2\right\}. \tag{4.89}$$

Without loss of generality, we can assume that $a > 1/2\beta\varepsilon^d M\Delta$. Integrating both sides of (4.89) with respect to $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)$ and taking into account the consistency property (4.8), we arrive at the following estimate for each $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) &\leq \Upsilon_\varepsilon + \frac{\beta\varepsilon^d M}{2} \sum_{j\in\mathcal{K}^c\cap\partial k} |\xi_j|^2 \\ &+ \log\left\{\int_\Gamma \exp\left(\frac{\beta\varepsilon^d M}{2} \sum_{j\in\mathcal{K}\cap\partial k} |\gamma_j|^2\right) \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right\} \\ &\leq \Upsilon_\varepsilon + \frac{\beta\varepsilon^d M}{2} \sum_{j\in\mathcal{K}^c\cap\partial k} |\xi_j|^2 \\ &+ \frac{\beta\varepsilon^d M}{2a} \sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K}|\xi), \end{aligned} \tag{4.90}$$

where Υ_ε is the same as in (4.86).

Here we have used the multiple Hölder inequality

$$\mu\left(\prod_{j=1}^K f_j^{s_j}\right) \leq \prod_{j=1}^K \mu^{s_j}(f_j), \quad \mu(f_j) := \int f_j \mathrm{d}\mu, \tag{4.91}$$

valid for any probability measure μ, measurable functions $f_j \geq 0$, and numbers $s_j \geq 0$ such that $\sum_{j=1}^{K} s_j \leq 1$. In our context, $f_j := \exp\{\Phi(\gamma_j)\}$, $s_j := \beta\varepsilon^d M(2a)^{-1} < 1/\varDelta$.

Now let us consider any domain $\mathcal{K} \Subset \mathbb{Z}^d$ containing a fixed point $k_0 \in \mathbb{Z}^d$. After taking the upper bound in (4.90) with the weights $\exp\{-\alpha|k-k_0|\}$, we get

$$\begin{aligned}
&\sup_{k\in\mathcal{K}} [n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}] \\
&\qquad \leq \Upsilon_\varepsilon + \frac{\beta\varepsilon^d M}{2} \sup_{k\in\mathcal{K}} \sum_{j\in\mathcal{K}^c\cap\partial k} |\xi_j|^2 \exp\{\alpha\,[|j-k|-|j-k_0|]\} \\
&\qquad + \frac{\beta\varepsilon^d M}{2a} \sup_{k\in\mathcal{K}} \sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K}|\xi)\exp\{\alpha\,[|j-k|-|j-k_0|]\}
\end{aligned}$$

and hence

$$\begin{aligned}
n_{k_0}(\mathcal{K}|\xi) &\leq \sup_{k\in\mathcal{K}} [n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}] \\
&\leq \left[1 - \frac{\beta\varepsilon^d M\varDelta}{2a} e^{\alpha\vartheta}\right]^{-1} \left[\Upsilon_\varepsilon + \frac{\beta\varepsilon^d M}{2} \varDelta e^{\alpha(\vartheta+|k_0|)} ||\xi_{\mathcal{K}^c}||_\alpha^2\right]. \qquad (4.92)
\end{aligned}$$

Since for $\xi \in \Gamma^{\mathrm{t}}$ the seminorm $||\xi_{\mathcal{K}^c}||_\alpha$ tends to zero as $\mathcal{K} \nearrow \mathbb{Z}^d$, we obtain for each $k_0 \in \mathbb{Z}^d$

$$\begin{aligned}
&\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \sup_{k\in\mathcal{K}} [n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}] \\
&\qquad \leq \Upsilon_\varepsilon \left[1 - \frac{\beta\varepsilon^d M\varDelta}{2a} e^{\alpha\vartheta}\right]^{-1} \qquad (4.93)
\end{aligned}$$

and thus, by letting $\alpha \to 0$,

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K}|\xi) \leq \varUpsilon_\varepsilon \left[1 - \frac{\beta\varepsilon^d M\varDelta}{2a}\right]^{-1} := \Psi_\varepsilon \qquad (4.94)$$

The existence result for $\mu \in \mathcal{G}^{t,\varepsilon}$ follows now from standard arguments, as one can see in Section 2.B or in Section 3.2 of [KPR12]. Moreover, the following exponential moment estimate holds for all tempered Gibbs measures $\mu \in \mathcal{G}^{t,\varepsilon}$. For any $a > 0$,

$$\sup_{k\in\mathbb{Z}^d} \int_\Gamma \exp\{a|\gamma_k^\varepsilon|^2\}\mu(d\gamma) \leq \Psi_\varepsilon, \qquad (4.95)$$

where Ψ_ε is given by (4.94).

By Jensen's inequality, one also has that

$$\sup_{k\in\mathbb{Z}^d}\int_\Gamma a|\gamma_k^\varepsilon|^2\mu(d\gamma)\le\log\Psi_\varepsilon<\infty. \tag{4.96}$$

4.4.3 Uniqueness

In what follows, we prove an uniqueness result for this model (see Theorem 4.23), by using the same technique that was already employed in Section 4.2. More precisely, we construct again a lattice system on $\breve{\Gamma}^\varepsilon_{lat}=\left(\Gamma(\overline{Q}^\varepsilon_0)\right)^{\mathbb{Z}^d}$, this time corresponding to the rescaled partition $(Q^\varepsilon_k)_{k\in\mathbb{Z}^d}$. Let T^ε denote the correspondence map between the two systems (compare with Section 4.2.3). Denote by $\breve{\Phi}_\varepsilon=T^\varepsilon_*\Phi$. More precisely, for $\breve{\gamma}\in\breve{\Gamma}^\varepsilon_{lat}$ one has

$$\breve{\Phi}_\varepsilon(\breve{\gamma})=a\left(\sum_{k\in\mathbb{Z}^d}|\breve{\gamma}^\varepsilon_k|\right)^2. \tag{4.97}$$

Evidently, for $\breve{\gamma}=(\breve{\gamma}^\varepsilon_j)_{j\in\mathbb{Z}^d}$ with $\breve{\gamma}^\varepsilon_j=\emptyset$ for $j\neq k$, we have

$$\breve{\Phi}_\varepsilon(\breve{\gamma})=a|\gamma^\varepsilon_k|^2=\Phi_\varepsilon(\gamma^\varepsilon_k).$$

Moreover, by the above and (4.95),

$$\sup_{k\in\mathbb{Z}^d}\int_\Gamma\breve{\Phi}_\varepsilon(\breve{\gamma})\mu_{lat}(d\gamma)\le\log\Psi_\varepsilon<\infty, \tag{4.98}$$

hence all the tempered Gibbs measures μ_{lat} on the lattice model satisfy the a-priori bound (2.13) required in Theorem 2.9.

Also, let Z_0^ε be a semigroup of $\varepsilon\mathbb{Z}^d$ such that $|u-v|>2\varepsilon^{-1}$ holds for all $u,v\in Z_0$, and define $\chi_\varepsilon=\min_{Z_0^\varepsilon}|\mathbb{Z}^d/Z_0^\varepsilon|$, the number of elements in the quotient group $\varepsilon\mathbb{Z}^d/Z_0^\varepsilon$. It is again easy to notice, that, as in the case of $\varDelta$, due to the fact that the range of interaction is proportional to the scaling parameter ε, $\chi_\varepsilon=\chi$, for any $\varepsilon>0$.

Theorem 4.23. *For any fixed $0<\varepsilon_*<\varepsilon^*<+\infty$ and $\beta_0>0$, one can find $z_0=z_0(\varepsilon_*,\varepsilon^*,\beta)$ such that $\mathcal{G}^t$ is a singleton for all values of $\varepsilon\in(\varepsilon_*,\varepsilon^*)$, $\beta<\beta_0$ and $z<z_0$.*

We divide the proof of this result into two technical lemmas, for each of the conditions that need to be checked.

The integrability condition (IC)

Lemma 4.24. *For any fixed* $0 < \varepsilon_* < \varepsilon^* < +\infty$ *and every* $\beta_0 > 0$*, there are constants* $\theta > 0$ *and* $0 < \bar{c} < 1/\Delta^\chi$ *such that, for every* $k \in \mathbb{Z}^d$, $0 < \beta < \beta_0$, $\varepsilon \in (\varepsilon_*, \varepsilon^*)$ *and any boundary condition* $\breve{\eta} \in \breve{\Gamma}_{lat}$

$$\int_{\breve{\Gamma}_{lat}} \theta \breve{\Phi}_\varepsilon(\breve{\gamma}_k) \breve{\pi}_{Q_k}(d\breve{\gamma}_k|\breve{\eta}) \leq 1 + \bar{c} \sum_{j \in \partial k} \theta \breve{\Phi}_\varepsilon(\breve{\eta}_j). \tag{4.99}$$

Proof. We remark that, by a simple change of variables, one has,

$$\int_{\breve{\Gamma}_{lat}} \exp\left\{\breve{\Phi}_\varepsilon(\breve{\gamma}_k)\right\} \breve{\pi}_{Q_k^\varepsilon}(d\breve{\gamma}_k|\breve{\eta}) = \int_\Gamma \exp\left\{\Phi(\gamma_k^\varepsilon)\right\} \pi_{Q_k^\varepsilon}(d\gamma_k^\varepsilon|\eta).$$

By applying Jensen's inequality to the one-point estimate (4.85) we have that, for any value of the parameter $a \geq 0$, there exists a constant $\Upsilon_\varepsilon > 0$ such that, for any $k \in \mathbb{Z}^d$, $\xi \in \Gamma$.

$$\int_\Gamma \Phi_\varepsilon(\gamma_k^\varepsilon) \pi_{Q_k^\varepsilon}(d\gamma_k^\varepsilon|\xi) \leq \Upsilon_\varepsilon + \frac{\beta \varepsilon^d M}{2a} \sum_{j \in \partial k} \Phi_\varepsilon(\xi_j). \tag{4.100}$$

Hence, also

$$\int_\Gamma \exp\left\{\breve{\Phi}_\varepsilon(\breve{\gamma}_k)\right\} \breve{\pi}_{Q_k^\varepsilon}(d\breve{\gamma}_k|\breve{\eta}) \leq \Upsilon_\varepsilon + \frac{\beta \varepsilon^d M}{2a} \sum_{j \in \partial k} \Phi_\varepsilon(\xi_j). \tag{4.101}$$

From here, it is easy to see that condition (IC) holds with constants $\theta = \Upsilon_\varepsilon^{-1} > 0$ and $\bar{c} := \frac{\beta_0 \varepsilon_*^d M \epsilon}{2a}$. Since ε is varying in a bounded interval $(\varepsilon_*, \varepsilon^*)$, we can make a proper choice of a such that $\bar{c} < 1/\Delta^\chi$.

□

The contraction condition (CC)

Lemma 4.25. *For any fixed* $0 < \varepsilon_* < \varepsilon^* < +\infty$ *and* $\beta_0 > 0$*, one can find* $z_0 = z_0(\varepsilon_*, \varepsilon^*, \beta)$ *such that, for all values of* $\varepsilon \in (\varepsilon_*, \varepsilon^*)$, $\beta < \beta_0$ *and* $z < z_0$*, we have*

$$d_{TV}(\breve{\mu}_k^\varepsilon(d\breve{\gamma}_k^\varepsilon|\breve{\xi}), \breve{\mu}_k^\varepsilon(d\breve{\gamma}_k^\varepsilon|\breve{\eta})) \leq \sum_{j \in \partial k} \mathtt{k} \mathbb{1}_{\breve{\xi}_j^\varepsilon \neq \breve{\eta}_j^\varepsilon}, \tag{4.102}$$

for some constant $0 < \mathtt{k} < 1$ *and boundary conditions* $\breve{\xi}, \breve{\eta}$ *such that*

$$\breve{\Phi}_\varepsilon(\breve{\xi}_j^\varepsilon), \breve{\Phi}_\varepsilon(\breve{\eta}_j^\varepsilon) \leq \theta^{-1} K_*, \quad \forall j \in \partial k, \tag{4.103}$$

where $K_* = K_*(\theta\breve{\Phi}_\varepsilon, \bar{c}, \mathtt{k})$ *is given by (2.14).*

Proof.

Let $\breve{\xi}, \breve{\eta}$ satisfy (4.103), which implies that $|\breve{\xi}_j|^2$, $|\breve{\eta}_j|^2 \leq a^{-1}\theta^{-1}K_*$ and hence $|\xi_j^\varepsilon|$, $|\eta_j^\varepsilon| \leq K_0 := \left(a^{-1}\theta K_*\right)^{1/2}$, for all $j \in \partial k$. By formula (2.86), $d_{TV}(\breve{\mu}_k^\varepsilon(d\breve{\gamma}_k^\varepsilon|\breve{\xi}), \breve{\mu}_k^\varepsilon(d\breve{\gamma}_k^\varepsilon|\breve{\eta}))$ is equal to

$$\frac{1}{2}\int_{\Gamma_k^\varepsilon}\left|\breve{Z}_{k,\varepsilon}^{-1}(\breve{\xi})\exp\{-\beta\breve{H}_k^\varepsilon(\breve{\gamma}_k|\breve{\xi})\} - \breve{Z}_{k,\varepsilon}^{-1}(\breve{\eta})\exp\{-\beta\breve{H}_k^\varepsilon(\breve{\gamma}_k|\breve{\eta})\}\right|\breve{\lambda}_{lat,z}(d\breve{\gamma}_k).$$

By a change of variables, the above expression can be rewritten as (see also 4.47)

$$\begin{aligned}&\frac{1}{2}\int_{\Gamma_k^\varepsilon}\left|Z_{k,\varepsilon}^{-1}(\xi)\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon|\xi)\} - Z_{k,\varepsilon}^{-1}(\eta)\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon|\eta)\}\right|\lambda_z(d\gamma_k^\varepsilon)\\ &\leq \min(Z_{k,\varepsilon}(\xi), Z_{k,\varepsilon}(\eta))\int_{\Gamma_k^\varepsilon}\left|\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon|\xi)\} - \exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon|\eta)\}\right|\lambda_z(d\gamma_k^\varepsilon).\end{aligned} \tag{4.104}$$

Now, let $\xi = \emptyset$ outside Q_j^ε, for a fixed $j \in \partial k$ and $\eta = \emptyset$. In this case the minimum above is equal to 1, so we are left to compute

$$\begin{aligned}&\int_{\Gamma_k^\varepsilon}|\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon|\xi)\} - \exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}|\lambda_z(d\gamma_k^\varepsilon) =\\ &= \int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}\Big|1 - \exp\{-\beta W_k^\varepsilon(\gamma_k^\varepsilon, \xi)\}\Big|\lambda_z(d\gamma_k^\varepsilon).\end{aligned} \tag{4.105}$$

Writing $W_k^\varepsilon = (W_k^\varepsilon)^+ - (W_k^\varepsilon)^-$ as the sum of its positive $(W_k^\varepsilon)^+ := W_k^\varepsilon \vee 0$ and negative part $(W_k^\varepsilon)^- := -(W_k^\varepsilon \wedge 0)$ and using the elementary inequality

$$\left|1 - \exp\left\{a - \sum_{i=1}^n a_i\right\}\right| \leq (\exp\{a\} - 1) + \sum_{i=1}^n \left|1 - \exp\{-a_i\}\right|,$$

where $n \in \mathbb{N}$ and $a, a_i \geq 0$, $1 \leq i \leq n$, we have that

$$\begin{aligned}\Big|1-\exp\{-\beta W_k^\varepsilon(x,y)\}\Big| &\le \Big|1-\exp\{\beta\sum_{\substack{x_1\in\gamma_k^\varepsilon\\ x_2\in\xi_j}} V^{(2)}(x_1,x_2)\}\Big|\\
&\quad+\sum_{\{x_1,x_2,x_3\}\subset\gamma_{\mathsf{L}},x_4\in\xi_{\mathsf{L}^c}}\Big(1-\exp\{-\beta V^{(4)}(x_1,x_2,x_3,x_4)\}\Big)\\
&\quad+\sum_{\{x_1,x_2\}\subset\gamma_{\mathsf{L}},\{x_3,x_4\}\subset\xi_{\mathsf{L}^c}}\Big(1-\exp\{-\beta V^{(4)}(x_1,x_2,x_3,x_4)\}\Big)\\
&\quad+\sum_{x_1\in\gamma_{\mathsf{L}},\{x_2,x_3,x_4\}\subset\xi_{\mathsf{L}^c}}\Big(1-\exp\{-\beta V^{(4)}(x_1,x_2,x_3,x_4)\}\Big)\\
&\le \exp\{\beta MK_0|\gamma_k^\varepsilon|\}-1+K_0|\gamma_k^\varepsilon|^3+K_0^2|\gamma_k^\varepsilon|^2+K_0^3|\gamma_k^\varepsilon|\\
&\le \beta MK_0|\gamma_k^\varepsilon|\exp\{\beta MK_0|\gamma_k^\varepsilon|\}+K_0|\gamma_k^\varepsilon|^3+K_0^2|\gamma_k^\varepsilon|^2+K_0^3|\gamma_k^\varepsilon|.\end{aligned}$$

We see now that the integral in (4.105) does not exceed

$$\begin{aligned}&\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}\beta MK_0|\gamma_k^\varepsilon|\exp\{\beta MK_0|\gamma_k^\varepsilon|\}\lambda_z(d\gamma_k^\varepsilon)\\
&+\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}\big(K_0|\gamma_k^\varepsilon|^3+K_0^2|\gamma_k^\varepsilon|^2+K_0^3|\gamma_k^\varepsilon|\big)|\lambda_z(d\gamma_k^\varepsilon).\end{aligned}\tag{4.106}$$

By applying Young's inequlity to (4.80), we obtain that there exists a non-negative constant $D_1=D_1(\varepsilon)$ such that $MK_0|\gamma_k^\varepsilon|-H_k^\varepsilon(\gamma_k^\varepsilon)\le D_1$. Hence,

$$\begin{aligned}&\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}\beta MK_0|\gamma_k^\varepsilon|\exp\{\beta MK_0|\gamma_k^\varepsilon|\}\lambda_z(d\gamma_k^\varepsilon)\\
&\le\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\beta MK_0|\gamma_k^\varepsilon|\exp\{D_1\}\lambda_z(d\gamma_k^\varepsilon)\le\beta MK_0\exp\{D\}\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}|\gamma_k^\varepsilon|\lambda(d\gamma_k^\varepsilon)\\
&=\beta MK_0\exp\{D_1\}\sum_{j=1}^\infty j\cdot\frac{z^j|Q_k^\varepsilon|^j}{j!}=z\beta MK_0\exp\{D_1\}|Q_k^\varepsilon|e^{z|Q_k^\varepsilon|}.\end{aligned}\tag{4.107}$$

Again, by applying Young's inequlity to (4.80), we can that $H_k^\varepsilon(\gamma_k^\varepsilon)\ge D_2$, for some non-negative constant $D_2=D_2(\varepsilon)$. Hence

$$\begin{aligned}&\int_{\Gamma_k^\varepsilon\setminus\{\emptyset\}}\exp\{-\beta H_k^\varepsilon(\gamma_k^\varepsilon)\}\big(K_0|\gamma_k^\varepsilon|^3+K_0^2|\gamma_k^\varepsilon|^2+K_0^3|\gamma_k^\varepsilon|\big)\lambda_z(d\gamma_k^\varepsilon)\\
&\quad\le\exp\{D_2\beta\}z|Q_k^\varepsilon|e^{z|Q_k^\varepsilon|}\Big[K_0\big(1+3z|Q_k^\varepsilon|+z^2|Q_k^\varepsilon|^2\big)+K_0^2(1+z|Q_k^\varepsilon|)+K_0^3\Big].\end{aligned}\tag{4.108}$$

Note that the last inequality is obtained via elementary computations of the moments of $|\gamma_k^\varepsilon|$ with respect to $\lambda_z(d\gamma_k^\varepsilon)$. Also, since ε belongs to the finite interval $(\varepsilon_*, \varepsilon^*)$, the two constants D_1 and D_2 can be chosen uniformly for all such ε.

It follows immediately from (4.106)–(4.108), that there exists $z_0 = z_0(\varepsilon_*, \varepsilon^*, \beta_0)$ such that, for $z < z_0$, $d_{TV}(\breve{\mu}_k^\varepsilon(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\mu}_k^\varepsilon(d\breve{\gamma}_k|\breve{\eta}^2))$ is smaller than $\mathtt{k}$. Applying the triangle inequality, the result also holds for more general boundary conditions. Hence the conditions of Theorem 2.9 are satisfied and we have obtained the uniqueness of the Gibbs measure.

$\square$

Proof of Theorem 4.23. It follows immediately by Lemmas 4.24 and 4.25 that for the considered lattice system the uniqueness holds. By Lemma 4.9, we obtain the desired result.

$\square$

Chapter 5

Gibbs States on Random Configurations: Annealed Approach

The aim of this chapter is to study Gibbs measures of the so-called amorphous (liquid) crystals, incorporating features both of the unbounded spin systems on graphs (see Chapters 2 and 3) and the classical particle systems in the continuum (see Chapter 4). The main results concern the existence (see Theorem 5.16) and the uniqueness (see Theorem 5.22) of such Gibbs measures.

5.1 Description of the Model

5.1.1 Spaces of marked configurations

Consider the product space $X \times S$, where $X = \mathbb{R}^d$ and $S = \mathbb{R}^m$ $(d, m \in \mathbb{N})$ are two Euclidean spaces and denote the configuration space over this product space by $\Gamma(X \times S)$. Observe that for a configuration $\widehat{\gamma} \in \Gamma(X \times S)$ its image $p_X(\widehat{\gamma})$ is a subset of X that, in general, admits accumulation and multiple points. Here, p_X is the natural extension to $\Gamma(X \times S)$ of the canonical projection $X \times S \to X$. The marked configuration space $\Gamma(X, S)$ is defined in the following way:

$$\Gamma(X, S) := \{\widehat{\gamma} \in \Gamma(X \times S) : p_X(\widehat{\gamma}) \in \Gamma(X)\}. \tag{5.1}$$

We also need the space of finite marked configurations

$$\Gamma_0(X, S) := \{\widehat{\gamma} \in \Gamma(X, S) : p_X(\widehat{\gamma}) \in \Gamma_0(X)\}. \tag{5.2}$$

The space $\Gamma(X, S)$ will be endowed with a (completely metrizable) topology defined as the weakest topology that makes the map

$$\Gamma(X, S) \ni \widehat{\gamma} \mapsto \langle f, \widehat{\gamma} \rangle \tag{5.3}$$

continuous for any bounded continuous function $f \in X \times S \to \mathbb{R}$ such that $\mathrm{supp} f \subset \Lambda \times S$, for some $\Lambda \in \mathcal{B}_0(X)$, i.e. with spatially compact support. This topology has been used in e.g. [AKLU00], [CG11] and [Kun99]. In what follows, we will call it τ-topology. Notice that $(\Gamma(X, S), \tau)$ is a Polish space, cf. Section 2 in [CG11], where a concrete metric that generates the topology τ is given. We equip $\Gamma(X, S)$ with the corresponding Borel σ-algebra. One should also note that in the standard vague topology (i.e. the topology generated by functions f from $C_0(X \times S)$), a sequence of configurations could converge to the empty configuration just by the convergence of marks to infinity. This topology is weaker than the τ-topology.

We stress that the space $\Gamma(X, S)$ has the structure of a fibre bundle over $\Gamma(X)$, with fibres $p_X^{-1}(\gamma)$ which can be identified with the product spaces

$$S^\gamma = \prod_{x \in \gamma} S_x, \quad S_x = S.$$

Therefore each $\widehat{\gamma} \in \Gamma(X, S)$ can be represented by the pair

$$\widehat{\gamma} = (\gamma,\ \sigma_\gamma), \text{ where } \gamma = p_X(\widehat{\gamma}) \in \Gamma(X),\ \sigma_\gamma = (\sigma_x)_{x \in \gamma} \in S^\gamma.$$

It follows directly from the definition of the corresponding topologies that the map $p_X : \Gamma(X, S) \to \Gamma(X)$ is continuous. Thus for any configuration $\gamma \in \Gamma(X)$ the space S^γ is a Borel subset of $\Gamma(X, S)$.

The space $(\Gamma(X, S), \mathcal{B}(\Gamma(X, S)))$ can be obtained as a projective limit of spaces $(\Gamma_{\mathsf{W}}(X, S), \mathcal{B}(\Gamma_{\mathsf{W}}(X, S)))$, $\mathsf{W} \in \mathcal{B}_c(X)$, with respect to projection maps

$$p_{\mathsf{W}_2, \mathsf{W}_1} : \Gamma(\mathsf{W}_2, S) \ni \widehat{\gamma} \mapsto \widehat{\gamma}_{\mathsf{W}_1} := (\gamma_{\mathsf{W}_1},\ \sigma_{\gamma_{\mathsf{W}_1}}) \in \Gamma(\mathsf{W}_1, S), \tag{5.4}$$

where $\gamma_{\mathsf{W}} := \gamma \cap \mathsf{W}$ and $\Gamma_{\mathsf{W}}(X, S) := \{\widehat{\gamma} \in \Gamma(X, S) | \widehat{\gamma}_{X \setminus \mathsf{W}} = \emptyset\}$. We remark that $\Gamma(\mathsf{W}, S)$ and $\Gamma_{\mathsf{W}}(X, S)$ coincide as sets, whereas the σ-algebras $\mathcal{B}(\Gamma(\mathsf{W}, S))$ and $\mathcal{B}_{\mathsf{W}}(\Gamma(X, S)) := p_{X, \mathsf{W}}^{-1} \mathcal{B}(\Gamma(X, S))$ are σ-isomorphic. For more details, see Section 3.1 in [Kun99]. Define the algebra $\mathcal{B}_0(\Gamma_0(X, S))$ of local sets by the formula

$$\mathcal{B}_0(\Gamma(X, S)) := \bigcup_{\mathsf{W} \in \mathcal{B}_c(X)} \mathcal{B}(\Gamma(\mathsf{W}, S)).$$

5.1.2 Marked Poisson and Lebesgue-Poisson measures

Let $g(ds)$ be a given probability measure on S. We introduce the Lebesgue-Poisson measure $\widehat{\lambda}_z$ (with intensity $dx \otimes g(ds)$ and activity parameter $z > 0$) on $\mathcal{B}_0(\Gamma_0(X,S))$ by setting

$$\int_{\Gamma_0(X,S)} F(\widehat{\gamma})\widehat{\lambda}_z(d\widehat{\gamma}) = F(\emptyset) + \sum_{k=1}^{\infty} \frac{z^k}{k!} \int_{(X\times S)^k} F((x_1,\sigma_1),...,(x_k,\sigma_k))g(d\sigma_1)dx_1...g(d\sigma_k)dx_k$$

for any non-negative $\mathcal{B}_0(\Gamma_0(X,S))$-measurable ("local") function F. It follows from this definition that

$$\int_{\Gamma_0(X,S)} F(\widehat{\gamma})\widehat{\lambda}_z(d\widehat{\gamma}) = \int_{\Gamma_0(X)} \int_{S^\gamma} F((\gamma,\sigma_\gamma)) \bigotimes_{x\in\gamma} g(d\sigma_x)\ \lambda(d\gamma),$$

where λ is the Lebesgue-Poisson measure (with intensity $m(dx) := dx$ and activity z) on $\mathcal{B}_0(\Gamma(X))$ (see Chapter 4).

In the same manner as in Section 4.1.3, we check that $\widehat{\lambda}_z$ is finite on $\Gamma(\mathsf{U},S)$, for $\mathsf{U} \in \mathcal{B}_c(X)$ and $\widehat{\lambda}_z(\Gamma(\mathsf{U},S)) = e^{zm(\mathsf{U})g(S)} = e^{zm(\mathsf{U})}$. Hence one can define a probability measure $\widehat{\pi}_z^{\mathsf{U}}$ on $\Gamma(\mathsf{U},S)$ by

$$\widehat{\pi}_z^{\mathsf{U}} = e^{-zm(\mathsf{U})}\widehat{\lambda}_z.$$

As in the case of simple (i.e. unmarked) configurations, observe that the family $\{\widehat{\pi}_z^{\mathsf{U}} : \mathsf{U} \in \mathcal{B}_c(X)\}$ is consistent, i.e.

$$\widehat{\pi}_z^{\widehat{\mathsf{W}}} = \widehat{\pi}_z^{\widehat{\mathsf{U}}} \circ p_{\widehat{\mathsf{W}},\widehat{\mathsf{U}}}^{-1}, \quad \text{whenever } \mathsf{W} \subset \mathsf{U},$$

where $p_{\widehat{\mathsf{W}},\widehat{\mathsf{U}}} : \Gamma(\mathsf{U},S) \to \Gamma(\mathsf{W},S)$ is the projection map acting by $p_{\widehat{\mathsf{W}},\widehat{\mathsf{U}}}(\widehat{\gamma}_{\widehat{\mathsf{U}}}) = \widehat{\gamma}_{\widehat{\mathsf{W}}}$. Again, by a version of Kolmogorov's theorem for projective limit spaces (see Chapter V, Theorem 3.2 of [Par67] or Theorem A.5.6 in [Kun99]), this family of distributions uniquely determines a probability measure $\widehat{\pi}_z$ on $\mathcal{B}(\Gamma)$ such that $\widehat{\pi}_z^{\mathsf{U}} = \widehat{\pi}_z \circ p_{\widehat{\mathsf{U}}}^{-1}$. The measure $\widehat{\pi}_z$ is called *Poisson measure* on $\Gamma(X,S)$.

Next we show how local absolute continuity with respect to the marked Lebesgue-Poisson measure $\widehat{\lambda}_z$ implies the negligibility of several events. The following results are modifications of Lemma 2.2.7 and Proposition 2.2.8 in [Kun99] to suit the marked configurations setting.

Lemma 5.1. *Let $\mu \in \mathcal{P}(\Gamma(X,S), \mathcal{B}(\Gamma(X,S)))$ be locally absolutely continuous with respect to $\widehat{\lambda}_z$. Then for all $\widehat{\gamma} \in \Gamma(X,S)$, the set $A_{\widehat{\gamma}} := \{\widehat{\gamma}' \in \Gamma(X,S) | \gamma \cap \gamma' = \emptyset\}$ has μ-measure zero.*

Proof. Consider a sequence of volumes $(\mathsf{W}_k)_{k\in\mathbb{N}}$ from $\mathcal{B}_c(X)$ with $\bigcup_{k\in\mathbb{N}} \mathsf{W}_k = X$. One can therefore decompose the set $A_{\widehat{\gamma}}$ as

$$A_{\widehat{\gamma}} = \bigcup_{k\in\mathbb{N}} p^{-1}_{\mathsf{W}_k}\{\widehat{\gamma}' \in \Gamma_{\mathsf{W}_k}(X,S) | \gamma_{\mathsf{W}_k} \cap \gamma' \neq \emptyset\},$$

hence

$$\mu(A_{\widehat{\gamma}}) \leq \sum_{k\in\mathbb{N}} \mu_{\mathsf{W}_k}(\{\widehat{\gamma}' \in \Gamma_{\mathsf{W}_k}(X,S) | \gamma_{\mathsf{W}_k} \cap \gamma' \neq \emptyset\}).$$

Due to the absolute continuity of μ_{W_k} with respect to $\widehat{\lambda}_z$, it is enough to prove that

$$\widehat{\lambda}_z(\{\widehat{\gamma}' \in \Gamma_{\mathsf{W}_k}(X,S) | \gamma_{\mathsf{W}_k} \cap \gamma' \neq \emptyset\}) = 0,$$

which follows immediately from

$$\begin{aligned}
&\widehat{\lambda}_z(\{\widehat{\gamma}' \in \Gamma_{\mathsf{W}_k}(X,S) | \gamma_{\mathsf{W}_k} \cap \gamma' \neq \emptyset\}) \\
&\leq \sum_{x\in\gamma_{\mathsf{W}_k}} \widehat{\lambda}_z(\{\widehat{\gamma}' \in \Gamma_{\mathsf{W}_k}(X,S) | x \in \gamma'\}) \\
&\leq \sum_{x\in\gamma_{\mathsf{W}_k}} \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} g(S)^n m(\{x\}) m(\mathsf{W}_k)^{n-1} = 0,
\end{aligned}$$

where by m we have denoted the Lebesgue measure dx on X.

□

Proposition 5.2. *Let A be a $\mathcal{B}(X)$-measurable set such that $m(A) = 0$. Then the set of marked configurations, whose spatial projection does not touch A, i.e.*

$$\Gamma(A; X, S) := \{\widehat{\gamma} | \gamma \subset A^c\},$$

has full μ-measure for any $\mu \in \mathcal{P}(\Gamma(X,S), \mathcal{B}(\Gamma(X,S)))$ which is locally absolutely continuous with respect to $\widehat{\lambda}_z$.

Proof. We will prove that the complement of this set has μ-measure zero. As in the previous result, let $(\mathsf{W}_k)_{k\in\mathbb{N}}$ in $\mathcal{B}_c(X)$ with $\bigcup_{k\in\mathbb{N}} \mathsf{W}_k = X$. The we can write

$$\begin{aligned}\Gamma(A;X,S)^c &= \{\widehat{\gamma} \in \Gamma(X,S) | x \in A, \text{ for some } x \in \gamma\} \\ &= \bigcup_{k\in\mathbb{N}} p_{\mathsf{W}_k}^{-1}(\{\widehat{\gamma} \in \Gamma_{\mathsf{W}_k}(X,S) | x \in A, \text{ for some } x \in \gamma\}).\end{aligned}$$

Hence

$$\mu(\Gamma(A;X,S)^c) \leq \sum_{k\in\mathbb{N}} \mu_{\mathsf{W}_k}(\{\widehat{\gamma} \in \Gamma_{\mathsf{W}_k}(X,S) | x \in A, \text{ for some } x \in \gamma\}).$$

From the local absolute continuity of μ with respect to $\widehat{\lambda}_z$ we see it is enough to prove that

$$\widehat{\lambda}_z(\{\widehat{\gamma} \in \Gamma_{\mathsf{W}_k}(X,S) | x \in A, \text{ for some } x \in \gamma\}) = 0. \tag{5.5}$$

This is true since the left hand side term is equal to

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{z^n}{n!}(g \otimes m)^{\otimes n}(\{((x_1,\sigma_{x_1}),\ldots,(x_n,\sigma_{x_n})) \in (\widehat{\mathsf{W}_k})^n | x_i \in A \text{ for some } i\}) \\ &\leq \sum_{n=0}^{\infty} \frac{z^n}{(n-1)!} g(S)^n m(\mathsf{W}_k)^{n-1} m(A) = 0.\end{aligned}$$

□

5.2 Gibbsian formalism

5.2.1 Specifications and their corresponding Gibbs measures

The interaction in our system will be described by the two different components

(i) a *pure positional pair potential* $\Phi : X^2 \to \mathbb{R}$; and

(ii) a *spin-spin pair potential* $\widehat{W} : (X \times S)^2 \to \mathbb{R}$, defined by

$$\widehat{W}\big((x,\sigma),(y,\xi)\big) := J(x,y)W(\sigma,\xi),$$

where $J : X^2 \to \mathbb{R}$ is bounded and $W : S^2 \to \mathbb{R}$.

Then the energy function $H : \Gamma_0(X, S) \to \mathbb{R}$ is given by the formula

$$H(\widehat{\gamma}) = H^\Phi(\gamma) + E_\gamma(\sigma), \qquad \widehat{\gamma} \in \Gamma_0(X, S), \tag{5.6}$$

with

$$H^\Phi(\gamma) = \sum_{\{x,y\}\subset\gamma} \Phi(x, y) \tag{5.7}$$

$$E_\gamma(\sigma) = \sum_{\{x,y\}\subset\gamma} J(x, y)W(\sigma_x, \sigma_y). \tag{5.8}$$

Given $A \subset X$, we will use the notation $\widehat{A} := A\times S$, corresponding to the cylindrical set in $X \times S$.

Next, for any $\mathsf{U} \in \mathcal{B}_c(X)$ we define the relative local interaction energy

$$H_{\widehat{\mathsf{U}}}(\widehat{\gamma}_{\widehat{\mathsf{U}}}\,|\widehat{\eta}) = H^\Phi_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) + E_{\gamma_{\mathsf{U}},\eta}(\sigma\,|\xi), \ \ \widehat{\eta} = (\eta, \xi) \in \Gamma(X, S),$$

where

$$H^\Phi_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) = H^\Phi(\gamma_{\mathsf{U}}) + \sum_{x\in\gamma_{\mathsf{U}}} \sum_{y\in\eta_{\mathsf{U}^c}} \Phi(x, y)$$

and

$$E_{\gamma_{\mathsf{U}},\eta}(\sigma_{\gamma_{\mathsf{U}}}\,|\xi) = E_{\gamma_{\mathsf{U}}}(\sigma_{\gamma_{\mathsf{U}}}) + \sum_{x\in\gamma_{\mathsf{U}}} \sum_{y\in\eta_{\mathsf{U}^c}} J(x, y)W(\sigma_x, \xi_y). \tag{5.9}$$

Let us fix a probability measure on S to be $g(ds) := e^{-V(s)}ds$, where $V : \mathbb{R} \to \mathbb{R}$ is a measurable (bounded from below) function. For $\mathsf{U} \in \mathcal{B}_c(X)$, introduce a measure $\Pi_{\widehat{\mathsf{U}}}\,(d\widehat{\gamma}\,|\widehat{\eta})$ on $\Gamma(X, S)$ via the integral relation

$$\int_{\Gamma(X,S)} F(\widehat{\gamma})\pi_{\widehat{\mathsf{U}}}\,(d\widehat{\gamma}\,|\widehat{\eta}) = Z_{\mathsf{U}}(\widehat{\eta})^{-1} \int_{\Gamma(X,S)} F(\widehat{\gamma}_{\widehat{\mathsf{U}}} \times \widehat{\eta}_{\widehat{\mathsf{U}}^c})$$
$$\times \exp\ \left(-\beta H^\Phi_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) - \beta E_{\gamma_{\mathsf{U}}\cup\eta_{\mathsf{U}^c}}(\sigma_{\gamma_{\mathsf{U}}}\,|\xi)\right) \bigotimes_{x\in\gamma_{\mathsf{U}}} g(d\sigma_x)\ \lambda(d\gamma_{\mathsf{U}}), \tag{5.10}$$

where F is a positive measurable function on $\Gamma(X, S)$, $\widehat{\eta} = (\eta, \xi) \in \Gamma(X, S)$ and

$$Z_{\widehat{\mathsf{U}}}(\widehat{\eta}) := \int \exp\ \left(-\beta H^\Phi_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) - \beta E_{\gamma_{\mathsf{U}}\cup\eta_{\mathsf{U}^c}}(\sigma_{\gamma_{\mathsf{U}}}\,|\xi)\right) \bigotimes_{x\in\gamma_{\mathsf{U}}} g(d\sigma_x)\ \lambda(d\gamma_{\mathsf{U}})$$

is the normalizing factor (called the partition function) making $\Pi_{\widehat{\mathsf{U}}}\,(d\widehat{\gamma}\,|\widehat{\eta})$ a probability measure on $\Gamma(X, S)$ (provided $Z_{\widehat{\mathsf{U}}}(\widehat{\eta}) \neq 0$, which will be the case under certain conditions on the interaction potentials, cf. Proposition 5.10).

The family $\Pi := \left\{\Pi_{\widehat{\mathsf{U}}}\,(d\widehat{\gamma}\,|\widehat{\eta})\right\}_{\mathsf{U}\in\mathcal{B}_c(X),\widehat{\eta}\in\Gamma(X,S)}$ constitutes a Gibbsian specification on $\Gamma(X, S)$ (in the standard sense, see e.g. [Geo88], [Pre76]). In particular, it satisfies the consistency property

$$\int_{\Gamma(X,S)} \Pi_{\widehat{\mathsf{U}}_1}(B\,|\widehat{\gamma})\,\Pi_{\widehat{\mathsf{U}}_2}(d\widehat{\gamma}\,|\widehat{\eta}) = \Pi_{\widehat{\mathsf{U}}_1}(B\,|\widehat{\eta}), \tag{5.11}$$

which holds for any $B \in \mathcal{B}(\Gamma(X,S))$, $\widehat{\eta} \in \Gamma(X,S)$ and $\mathsf{U}_1, \mathsf{U}_2 \in \mathcal{B}_c(X)$ such that $\mathsf{U}_1 \subset \mathsf{U}_2$ (and thus $\widehat{\mathsf{U}}_1 \subset \widehat{\mathsf{U}}_2$).

Definition 5.3. Let ν be a probability measure on $\Gamma(X,S)$. We say that ν is a Gibbs measure associated with the specification π if it satisfies the DLR equation

$$\nu(B) = \int_{\Gamma(X,S)} \Pi_{\widehat{\mathsf{U}}}(B\,|\widehat{\gamma})\,\nu(d\widehat{\gamma}) \tag{5.12}$$

for all $B \in \mathcal{B}(\Gamma(X,S))$ and $\mathsf{U} \in \mathcal{B}_c(X)$. We denote by $\mathcal{G}(\Gamma(X,S))$ the set of all such measures.

5.2.2 Conditions on the interaction

We introduce a partition of X by elementary volumes, similarly to Section 4.2.2. Denote by Q_k a cube in X with side length 1, centred at point $k = (k^{(1)}, ..., k^{(d)}) \in \mathbb{Z}^d$, that is,

$$Q_k := \left\{x = (x^{(1)}, ..., x^{(d)}) \in X :\ x^{(i)} \in \left[k^{(i)} - 1/2, k^{(i)} + 1/2\right)\right\}. \tag{5.13}$$

It is worth noting that, unlike the situation presented in Chapter 4, the length of the edge of the cubes is not important, hence we work with unit cubes.

The following assumptions on the interaction potentials are needed:

(FR) Finite range: $\Phi(x,y) = 0$, $J(x,y) = 0$ if $|x-y| \geq R$, for some $R > 0$.

(LSSS) Local strong super stability of H^Φ: $\exists\ P > 2$ such that

$$H^\Phi(\gamma_k) \geq A_\Phi|\gamma_k|^P - B_\Phi|\gamma_k|,\ \ \gamma_k \in \Gamma(Q_k),$$

for any $k \in \mathbb{Z}^d$ and some constants $A_\Phi > 0$, $B_\Phi \geq 0$ (which may depend on k).

Observe that (LSSS) is equivalent to the following (global) strong super stability condition:

(SSS) $\exists\ A'_\Phi > 0, B'_\Phi \geq 0$ such that

$$H^\Phi(\gamma) \geq A'_\Phi \sum_{k\in\mathbb{Z}^d} |\gamma_k|^P - B'_\Phi|\gamma|,\ \ \gamma_k = \gamma_{Q_k} := \gamma \cap Q_k, \tag{5.14}$$

for any $\gamma \in \Gamma_0(X)$. (Indeed, in (SSS) one can take any $A'_\Phi \in (0, A_\Phi)$.)

(PB) Polynomial bound on W, that is, $\exists\, r > 0$ and $C_W \in \mathbb{R}$ such that

$$|W(u,v)| \le |u|^r + |v|^r + C_W,\ \ u, v \in X.$$

(SQG) Super-quadratic growth of V, that is, $\exists\, q_V > 2$ and $a_V > 0,\ b_V \ge 0$ such that

$$V(s) \ge a_V\, |s|^{q_V} - b_V,\ \ s \in S.$$

(Pqr) We assume that P, q_V and r satisfy the constraint

$$(P-2)(q_V/r - 1) > 1. \tag{5.15}$$

Remark 5.4. (i) The constraint (5.15) is crucial in the proof of Theorem 5.16. It means we need either a strong enough growth of the one-particle potential V (i.e. a big $q_V > r$), or a strong enough repulsion at the diagonal of the pure positional potential W (i.e. a big $P > 2$).

(ii) One of the best-understood examples of the strong super stable interaction is given by the potential satisfying the bound $\Phi(x,y) \ge c\,|x-y|^{-d(1+\epsilon)}$ as $|x-y| \to 0$, in which case $P = 2+\epsilon$. For a detailed study and historical comments see [RT08] and [KPR12, Remark 4.1.], as well as Remark 4.2 in Chapter 4 above.

It is obvious that $g(S) < \infty$ under Condition (SQG). Without loss of generality we may assume that g is a probability measure.

Throughout this chapter, we will use the following notations:

$\Gamma_k := \Gamma(Q_k);\ \gamma_k := \gamma_{Q_k};$

$\widehat{\Gamma}_k := \Gamma(Q_k, S);\ \widehat{\gamma}_k := \widehat{\gamma}_{Q_k \times S};$

$\partial k := \left\{ j \in \mathbb{Z}^d : \text{dist}\ (Q_k, Q_j) \le R \right\}$, where dist is the Euclidean distance between two sets in $\mathbb{R}^d$;

$\varDelta := \sup_{k \in \mathbb{Z}^d} |\partial k|$; obviously, $\varDelta < \infty$;

$K_{\mathsf{U}} := \left\{ k \in \mathbb{Z}^d : \text{dist}\ (Q_k, \mathsf{U}) \le R \right\}$, for any $\mathsf{U} \in \mathcal{B}_c(X)$;

$\mathsf{U}_R := \{ x \in X :\ \text{dist}\ (x, \mathsf{U}) \le R \}$;

$\partial \mathsf{U}_R := \mathsf{U}_R \setminus \mathsf{U} = \mathsf{U}^c \cap \mathsf{U}_R \in \mathcal{B}_c(X)$.

5.3 Existence of Gibbs Measures

5.3.1 Exponential moment estimate

The aim of this section is to prove a uniform estimate on exponential moments of specification kernels, which will be then used in the proofs of the existence and the uniqueness results. For a subset $\mathcal{K} \subset \mathbb{Z}^d$, consider the union of elementary cubes $Q_{\mathcal{K}} := \bigcup_{k\in\mathcal{K}} Q_k$ and the corresponding set $\widehat{Q}_{\mathcal{K}} = Q_{\mathcal{K}} \times S$.

Let us fix $p, q \in \mathbb{N}$ such that $p < P$, $q < q_V$ and

$$(p-2)(qr^{-1}-1) \geq 1. \tag{5.16}$$

Observe that such p and q certainly exist because of condition (5.15). Define functions $F : \Gamma_0(X,S) \to \mathbb{R}$ and $F_\alpha : \Gamma_0(X,S) \to \mathbb{R}$ by formulae

$$F(\widehat{\gamma}) = |\gamma|^p + \sum_{x\in\gamma} |\sigma_x|^q, \ \widehat{\gamma} = (\gamma, \sigma), \tag{5.17}$$

and

$$F_\alpha(\widehat{\gamma}) = \sup_{k\in\mathbb{Z}^d} e^{-\alpha|k|} F(\widehat{\gamma}_k),$$

respectively. Introduce the space of tempered configurations

$$\Gamma^t(X,S) := \bigcap_{\alpha>0} \Gamma_\alpha(X,S),$$

where

$$\Gamma_\alpha(X,S) := \{\widehat{\gamma} \in \Gamma(X,S) : \ F_\alpha(\widehat{\gamma}) < \infty\}$$

and the set $\mathcal{G}^t$ of Gibbs measures $\mu \in \mathcal{G}(\Gamma(X,S))$ supported by $\Gamma^t(X,S)$.

To prove the existence result for $\mu \in \mathcal{G}^t$, we will properly extend the method used in Appendix 2.B for classical systems.

Theorem 5.5. *For any $a \in \mathbb{R}$ and any fixed $\beta > 0$ there exists a constant $\Psi = \Psi(a) < \infty$ such that for all $\widehat{\zeta} \in \Gamma^t(X,S)$ and $\mathcal{K} \Subset \mathbb{Z}^d$, $k \in \mathcal{K}$, the following estimate holds:*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\Gamma(X,S)} \exp\left\{aF(\widehat{\gamma}_k)\right\} \Pi_{\widehat{Q}_{\mathcal{K}}}\left(d\widehat{\gamma}\,\middle|\,\widehat{\zeta}\right) \leq \Psi. \tag{5.18}$$

In order to prove the theorem, we need some preparations. Observe first that Condition (SSS) immediately implies the following lower bound:

$$\inf_{x\neq y} \Phi(x,y) \geq 2(2^{P-1}A'_\Phi - B'_\Phi). \tag{5.19}$$

Thus there exists $M \geq 0$ such that

$$\inf_{x \neq y} \Phi(x, y) \geq -M. \tag{5.20}$$

We start with the proof of two auxiliary results.

Lemma 5.6. *For any $\gamma_k \in \Gamma_k$, $k \in \mathbb{Z}^d$, and $\eta \in \Gamma_X$ we have*

$$-H^{\Phi}_{Q_k}(\gamma_k\,|\eta) \leq -A_\Phi|\gamma_k|^P + \frac{M\Delta}{2}|\gamma_k|^2 + B_\Phi|\gamma_k| + \frac{M}{2}\sum_{j\in\partial k}|\eta_j|^2. \tag{5.21}$$

Proof. By the definition of conditional energy $H^{\Phi}_{Q_k}(\gamma_k\,|\eta)$ we have

$$\begin{aligned}
-H^{\Phi}_{Q_k}(\gamma_k\,|\eta) &= -H^{\Phi}(\gamma_k) - \sum_{x\in\gamma_k}\sum_{\substack{j\in\partial k\\ y\in\eta_j}}\Phi(x,y)\\
&\leq -\big(A_\Phi|\gamma_k|^P - B_\Phi|\gamma_k|\big) + M|\gamma_k|\sum_{j\in\partial k}|\eta_j|\\
&\leq -\big(A_\Phi|\gamma_k|^P - B_\Phi|\gamma_k|\big) + \frac{M}{2}|\partial k||\gamma_k|^2 + \frac{M}{2}\sum_{j\in\partial k}|\eta_j|^2\\
&= -A_\Phi|\gamma_k|^P + \frac{M\Delta}{2}|\gamma_k|^2 + B_\Phi|\gamma_k| + \frac{M}{2}\sum_{j\in\partial k}|\eta_j|^2,
\end{aligned}$$

and the proof is complete. □

Remark 5.7. Similarly, inequalities (5.14) and (5.20) imply that for any $\mathsf{U} \in \mathcal{B}_c(\Gamma)$ we have

$$\begin{aligned}
-H^{\Phi}_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) &= -H^{\Phi}(\gamma_{\mathsf{U}}) - \sum_{x\in\gamma_{\mathsf{U}}}\sum_{y\in\eta_{\partial\mathsf{U}_R}}\Phi(x,y)\\
&\leq -A'_\Phi\sum_{k\in K_{\mathsf{U}}}|\gamma_k|^P + B'_\Phi|\gamma_{\mathsf{U}}| + M|\gamma_{\mathsf{U}}||\eta_{\partial\mathsf{U}_R}|,
\end{aligned} \tag{5.22}$$

so that

$$-H^{\Phi}_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) \leq B'_\Phi|\gamma_{\mathsf{U}}| + M|\gamma_{\mathsf{U}}||\eta_{\partial\mathsf{U}_R}|. \tag{5.23}$$

Lemma 5.8. *For any $\varepsilon > 0$, the conditional energy function $E_{\gamma_k,\eta}(\sigma_k\,|\xi)$ satisfies the following estimate:*

$$\begin{aligned}
|E_{\gamma_k,\eta}(\sigma_k\,|\xi)| \leq \|J\|_\infty\Bigg[&C_1|\gamma_k|^{2+\varepsilon^{-1}} + C_2\sum_{j\in\partial k}|\eta_j|^{2+\varepsilon^{-1}} + C_3\sum_{x\in\gamma_k}|\sigma_x|^{r(1+\varepsilon)}\\
&+ C_4\sum_{\substack{j\in\partial k\\ y\in\eta_j}}|\xi_y|^{r(1+\varepsilon)}\Bigg],
\end{aligned}$$

where $C_1, ..., C_4$ are some positive constants (depending on Δ, C_W and ε) and by $\|J\|_\infty$ we denoted the sup norm of J.

Proof. By definition (5.9) of the conditional energy function $E_{\gamma_k,\eta}(\sigma_k\,|\xi)$ we have

$$
\begin{aligned}
|E_{\gamma_k,\eta}(\sigma_k\,|\xi)| &\le \|J\|_\infty \Bigg[\sum_{\{x,y\}\subset\gamma_k} |W(\sigma_x,\sigma_y)| + \sum_{x\in\gamma_k} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |W(\sigma_x,\xi_y)| \Bigg] \\
&\le \|J\|_\infty \Bigg[\sum_{\{x,y\}\subset\gamma_k} (|\sigma_x|^r + |\sigma_y|^r + C_W) + \sum_{x\in\gamma_k} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} (|\sigma_x|^r + |\xi_y|^r + C_W) \Bigg] \\
&\le \|J\|_\infty \Bigg[\Bigg(\sum_{j\in\partial k} |\eta_j| + 2|\gamma_k| \Bigg) \sum_{x\in\gamma_k} |\sigma_x|^r + |\gamma_k| \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^r \\
&\qquad + |\gamma_k| \Bigg(\frac{|\gamma_k|-1}{2} + \sum_{j\in\partial k} |\eta_j| \Bigg) C_W \Bigg]. \qquad (5.24)
\end{aligned}
$$

Observe that $\frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon^{-1}} = 1$. Fix arbitrary $\rho_1, \rho_2 > 0$ and let be ρ_1', ρ_2' such that $\frac{1}{\rho_k} + \frac{1}{\rho_k'} = 1,\ k = 1,2$. In what follows, we will estimate each of the three terms (5.24) by Holder's inequality.

For the first term we obtain:

$$
\begin{aligned}
A_1 := \Bigg(\sum_{j\in\partial k} |\eta_j| + 2|\gamma_k| \Bigg) \sum_{x\in\gamma_k} |\sigma_x|^r &= \sum_{j\in\partial k} |\eta_j| \sum_{x\in\gamma_k} |\sigma_x|^r + 2|\gamma_k| \sum_{x\in\gamma_k} |\sigma_x|^r \\
&\le \frac{1}{1+\varepsilon^{-1}} |\gamma_k| \sum_{j\in\partial k} |\eta_j|^{1+\varepsilon^{-1}} + \frac{|\partial k|}{1+\varepsilon} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
&\quad + \frac{2}{1+\varepsilon^{-1}} |\gamma_k|^{2+\varepsilon^{-1}} + \frac{2}{1+\varepsilon} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
= \frac{1}{1+\varepsilon^{-1}} |\gamma_k| \sum_{j\in\partial k} |\eta_j|^{1+\varepsilon^{-1}} &+ \frac{2}{1+\varepsilon^{-1}} |\gamma_k|^{2+\varepsilon^{-1}} + \frac{|\partial k|+2}{1+\varepsilon} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
&\le \frac{|\partial k|}{(1+\varepsilon^{-1})\,\rho_1} |\gamma_k|^{\rho_1} + \frac{1}{(1+\varepsilon^{-1})\,\rho_1'} \sum_{j\in\partial k} |\eta_j|^{(1+\varepsilon^{-1})\rho_1'} \\
&\quad + \frac{2}{1+\varepsilon^{-1}} |\gamma_k|^{2+\varepsilon^{-1}} + \frac{|\partial k|+2}{1+\varepsilon} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)}.
\end{aligned}
$$

The middle term can be estimated as follows:

$$\begin{aligned}
A_2 := |\gamma_k| \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^r &= \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\gamma_k|\,|\xi_y|^r \\
&\le \frac{1}{1+\varepsilon^{-1}} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\gamma_k|^{1+\varepsilon^{-1}} + \frac{1}{1+\varepsilon} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
&= \frac{1}{1+\varepsilon^{-1}} \sum_{j\in\partial k} |\eta_j||\gamma_k|^{1+\varepsilon^{-1}} + \frac{1}{1+\varepsilon} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
\le \frac{1}{(1+\varepsilon^{-1})\,\rho_2} \sum_{j\in\partial k} |\eta_j|^{\rho_2} &+ \frac{1}{(1+\varepsilon^{-1})\,\rho_2'} \sum_{j\in\partial k} |\gamma_k|^{(1+\varepsilon^{-1})\rho_2'} + \frac{1}{1+\varepsilon} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
= \frac{1}{(1+\varepsilon^{-1})\,\rho_2} \sum_{j\in\partial k} |\eta_j|^{\rho_2} &+ \frac{1}{(1+\varepsilon^{-1})\,\rho_2'} |\partial k||\gamma_k|^{(1+\varepsilon^{-1})\rho_2'} + \frac{1}{1+\varepsilon} \sum_{\substack{j\in\partial k\\ y\in\eta_j}} |\xi_y|^{r(1+\varepsilon)}.
\end{aligned}$$

Finally, for the last term we have the inequality

$$\begin{aligned}
A_3 := |\gamma_k| \left(\frac{|\gamma_k|-1}{2} + \sum_{j\in\partial k} |\eta_j| \right) C_W \\
&= \frac{C_W}{2}\big(|\gamma_k|^2 - |\gamma_k|\big) + C_W|\gamma_k| \sum_{j\in\partial k} |\eta_j| \\
&\le \frac{C_W}{2}\big(|\gamma_k|^2 - |\gamma_k|\big) + \frac{C_W}{\rho_3}|\partial k||\gamma_k|^{\rho_3} + \frac{C_W}{\rho_3'} \sum_{j\in\partial k} |\eta_j|^{\rho_3'}.
\end{aligned}$$

In order to simplify the expressions above, we set

$$\rho_1 = \rho_2 = 2+\varepsilon^{-1},\ \ \rho_3 = 2.$$

Then

$$\rho_1'(1+\varepsilon^{-1}) = \rho_2'(1+\varepsilon^{-1}) = 2+\varepsilon^{-1},\ \ \rho_3' = 2.$$

Using these values, we obtain the following inequalities:

$$\begin{aligned}
A_1 \le \frac{|\partial k|}{(1+\varepsilon^{-1})\,(2+\varepsilon^{-1})}|\gamma_k|^{2+\varepsilon^{-1}} + \frac{1}{2+\varepsilon^{-1}} \sum_{j\in\partial k} |\eta_j|^{2+\varepsilon^{-1}} \\
+ \frac{2}{1+\varepsilon^{-1}}|\gamma_k|^{2+\varepsilon^{-1}} + \frac{|\partial k|+2}{1+\varepsilon} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)}; \quad (5.25)
\end{aligned}$$

$$A_2 \leq \frac{1}{\left(1+\varepsilon^{-1}\right)\left(2+\varepsilon^{-1}\right)} \sum_{j\in\partial k} |\eta_j|^{2+\varepsilon^{-1}} + \frac{1}{2+\varepsilon^{-1}} |\partial k| |\gamma_k|^{2+\varepsilon^{-1}} + \frac{1}{1+\varepsilon} \sum_{\substack{j\in\partial k \\ y\in\eta_j}} |\xi_y|^{r(1+\varepsilon)}; \quad (5.26)$$

$$A_3 \leq \frac{C_W}{2}\left(|\gamma_k|^2 - |\gamma_k|\right) + \frac{C_W}{2}|\partial k||\gamma_k|^2 + \frac{C_W}{2}\sum_{j\in\partial k}|\eta_j|^2 = \frac{C_W}{2}\left(\left(1+|\partial k|\right)|\gamma_k|^2 - |\gamma_k|\right) + \frac{C_W}{2}\sum_{j\in\partial k}|\eta_j|^2. \quad (5.27)$$

Combining all of the above, we have the estimate

$$|E_{\gamma_k,\eta}(\sigma_k\,|\xi)| \leq \|J\|_\infty \Bigg[C_1|\gamma_k|^{2+\varepsilon^{-1}} + C_2\sum_{j\in\partial k}|\eta_j|^{2+\varepsilon^{-1}} + C_3\sum_{x\in\gamma_k}|\sigma_x|^{r(1+\varepsilon)} + C_4\sum_{\substack{j\in\partial k \\ y\in\eta_j}}|\xi_y|^{r(1+\varepsilon)}\Bigg]$$

for constants $C_1, ..., C_4$ explicit form of which can be seen directly from inequalities (5.25)-(5.27): $C_1 = \frac{\Delta}{(1+\varepsilon^{-1})(2+\varepsilon^{-1})} + \frac{\Delta}{2+\varepsilon^{-1}} + \frac{C_W}{2}(2+\Delta)$, $C_2 = \frac{1}{2+\varepsilon^{-1}} + \frac{1}{(1+\varepsilon^{-1})(2+\varepsilon^{-1})} + \frac{C_W}{2}$, $C_3 = \frac{\Delta+2}{1+\varepsilon}$, $C_4 = \frac{1}{1+\varepsilon}$. □

Remark 5.9. For any $\mathsf{U} \in \mathcal{B}_c(\Gamma)$ we have (similarly to (5.24)) the inequality

$$|E_{\gamma_\mathsf{U},\eta}(\sigma_\mathsf{U}\,|\xi)| \leq \|J\|_\infty \Bigg[\left(|\eta_{\partial \mathsf{U}_R}| + 2|\gamma_\mathsf{U}|\right)\sum_{x\in\gamma_\mathsf{U}}|\sigma_x|^r + |\gamma_\mathsf{U}|\sum_{y\in\eta_{\partial\mathsf{U}_R}}|\xi_y|^r + \left(|\gamma_\mathsf{U}|^2 + |\gamma_\mathsf{U}||\eta_{\partial\mathsf{U}_R}|\right)C_W\Bigg]. \quad (5.28)$$

Lemma 5.10. *The partition function* $Z_{\widehat{\mathsf{U}}}$ *satisfies the estimate*

$$1 \leq Z_{\widehat{\mathsf{U}}}(\widehat{\eta}) < \infty \quad (5.29)$$

for all $\mathsf{U} \in \mathcal{B}_c(X)$ *and* $\widehat{\eta} \in \Gamma(X,S)$.

Proof. The first inequality follows from the definition of the Lebesgue-Poisson measure $\lambda(d\gamma)$ and the fact that $H^\Phi_\mathsf{U}(\gamma_\mathsf{U}\,|\eta) = E_{\gamma_\mathsf{U}\cup\eta_{\mathsf{U}^c}}(\sigma_{\gamma_\mathsf{U}}\,|\xi) = 0$ provided $\mathsf{U} = \emptyset$. The second inequality follows from estimates (5.22) and (5.28). □

Proof of Theorem 5.5. The proof is technical and will be split into two steps.

Step 1. One-point estimate.

Let us fix $k \in \mathbb{Z}^d$ and introduce the notation

$$\begin{aligned}\mathcal{I}_k(\widehat{\eta}) &:= \int_{\widehat{\Gamma}_k} \exp\{aF(\widehat{\gamma}_k)\}\, \Pi_{\widehat{Q}_k}(d\widehat{\gamma}_k\,|\widehat{\eta}) \\ &= \int_{\Gamma_k}\int_{S^\gamma} Z^{-1} \exp\left\{ \left(a|\gamma_k|^p - \beta H^{\Phi}_{Q_k}(\gamma_k\,|\eta)\right) + a\sum_{x\in\gamma_k}|\sigma_x|^q - \beta E_{\gamma_k,\eta}(\sigma_k\,|\xi)\right\} \\ &\qquad\qquad \bigotimes_{x\in\gamma_k} g(d\sigma_x)\ \lambda(d\gamma_k),\end{aligned}$$

where $\widehat{\eta} = (\eta,\xi)$,$\widehat{\gamma} = (\gamma,\sigma)$. Observe that

$$\begin{aligned}\int_{S^\gamma} \exp\left(b_1 \sum_{x\in\gamma_k}|\sigma_x|^{b_2}\right) \bigotimes_{x\in\gamma_k} g(d\sigma_x) &= \left(\int_S \exp\left(b_1\,|s|^{b_2}\right) g(ds)\right)^{|\gamma_k|} \\ &= \exp\left(C_{b_1,b_2}|\gamma_k|\right),\end{aligned}$$

for any $b_1 \in \mathbb{R}$ and $b_2 < q_V$, where

$$C_{b_1,b_2} = \ln \int_S \exp\left(b_1\,|s|^{b_2}\right) g(ds) < \infty.$$

Taking into account that $Z_{Q_k}(\widehat{\eta}) \geq 1$ (cf. (5.29)) we see that

$$\begin{aligned}\mathcal{I}_k(\widehat{\eta}) &= \mathcal{I}_k(\eta,\xi) \\ &\leq \int_{\Gamma_k}\int_{S^\gamma} \exp\left\{ \left(a|\gamma_k|^p - \beta H^{\Phi}_{Q_k}(\gamma_k\,|\eta)\right) + a\sum_{x\in\gamma_k}|\sigma_x|^q - \beta E_{\gamma_k,\eta}(\sigma_k\,|\xi)\right\} \\ &\qquad\qquad \bigotimes_{x\in\gamma_k} dg(\sigma_x)\ \lambda(d\gamma_k),\end{aligned} \tag{5.30}$$

which in turn implies (by Lemma 5.8) the inequality

$$\begin{aligned}&\mathcal{I}_k(\eta,\xi) \\ &\leq \int_{\Gamma_k} \exp\left\{\left(a|\gamma_k|^p - \beta H^{\Phi}_{Q_k}(\gamma_k\,|\eta)\right) + \beta\|J\|_\infty C_1|\gamma_k|^{2+\varepsilon^{-1}} + C_{b_1,b_2}|\gamma_k|\right\}\ \lambda(d\gamma_k) \\ &\qquad \times \exp\left\{\beta\|J\|_\infty\left[C_2\sum_{j\in\partial k}|\eta_j|^{2+\varepsilon^{-1}} + C_4\sum_{j\in\partial k}\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]\right\},\end{aligned} \tag{5.31}$$

where $b_1 = \beta\|J\|_\infty C_3 + a$, $b_2 = \max\ \{r(1+\varepsilon), q\}$. Using estimate (5.21) we finally obtain

$$\mathcal{I}_k(\eta,\xi) \le \int_{\Gamma_k} \exp\left(-\beta A_\Phi |\gamma_k|^P + \mathcal{P}(|\gamma_k|)\right)\ \lambda(d\gamma_k)$$
$$\times \exp\left\{\frac{\beta M}{2}\sum_{j\in\partial k}|\eta_j|^2 + \beta\|J\|_\infty\left[C_2\sum_{j\in\partial k}|\eta_j|^{2+\varepsilon^{-1}} + C_4\sum_{j\in\partial k}\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]\right\},$$

where $\mathcal{P}(|\gamma_k|) := a|\gamma_k|^p + \frac{\beta M \varDelta}{2}|\gamma_k|^2 + \beta B_\Phi|\gamma_k| + \beta\|J\|_\infty C_1|\gamma_k|^{2+\varepsilon^{-1}} + C_{b_1,b_2}|\gamma_k|$, so that

$$\mathcal{I}_k(\eta,\xi) \le e^{C_0}\exp\left\{\sum_{j\in\partial k}\left\{C_5|\eta_j|^{2+\varepsilon^{-1}} + \beta\|J\|_\infty C_4\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right\}\right\} \tag{5.32}$$

with $C_5 = \beta\|J\|_\infty C_2 + \frac{\beta M}{2}$ and

$$C_0 = C_0(a,\beta) = \ln\int_{\Gamma_k}\exp\left(-\beta A_\Phi|\gamma_k|^P + \mathcal{P}(|\gamma_k|)\right)\ \lambda(d\gamma_k).$$

Observe that $C_0 < \infty$ because $P >$ degree $\mathcal{P} = \max\ (2+\varepsilon^{-1}, p)$ and $A_\Phi > 0$.

Step 2. Volume estimate.

Introduce the notation

$$n_k(\mathcal{K},\widehat{\zeta}) := \ln\int_{\Gamma(X,S)}\exp\left\{aF(\widehat{\gamma}_k)\right\}\Pi_{\widehat{Q}_\mathcal{K}}\left(d\widehat{\gamma}\,\middle|\,\widehat{\zeta}\right).$$

An application of equation (5.11) shows that

$$n_k(\mathcal{K},\widehat{\zeta}) = \ln\int_{\Gamma(X,S)}\mathcal{I}_k(\widehat{\eta})\ \Pi_{\widehat{Q}_\mathcal{K}}\left(d\widehat{\eta}\,\middle|\,\widehat{\zeta}\right).$$

By inequality (5.32) we have

$$n_k(\mathcal{K},\widehat{\zeta}) \le C_0$$
$$+ \ln\int_{\Gamma(X,S)}\exp\left\{\sum_{j\in\partial k}\left[C_5|\eta_j|^{2+\varepsilon^{-1}} + \beta\|J\|_\infty C_4\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]\right\}\Pi_{\widehat{Q}_\mathcal{K}}(d\widehat{\eta},\widehat{\zeta})$$
$$\le C_0 + \ln\int_{\Gamma(X,S)}\exp\left\{\sum_{j\in\partial k}\left[C_5|\eta_j|^p + \beta\|J\|_\infty C_4\sum_{y\in\eta_j}|\xi_y|^q\right]\right\}\Pi_{\widehat{Q}_\mathcal{K}}(d\widehat{\eta},\widehat{\zeta})$$

for any ε such that $2+\varepsilon^{-1} \leq p$ and $r(1+\varepsilon) \leq q$, that is, $\varepsilon \in [(p-2)^{-1}, r^{-1}q-1]$. Observe that $(p-2)^{-1} \leq r^{-1}q-1$ because of condition (5.15). Then

$$n_k(\mathcal{K},\widehat{\zeta}) \leq C_0 + \ln \int_{\Gamma(X,S)} \exp\left\{ a \sum_{j\in\partial k} D \left\{ |\eta_j|^p + \sum_{y\in\eta_j} |\xi_y|^q \right\} \right\} \Pi_{\widehat{Q}_{\mathcal{K}}}(d\widehat{\eta},\widehat{\zeta}),$$

with $D = \frac{1}{a}\max\ \{C_5, \beta\|J\|_\infty C_4\}$, which yields that

$$n_k(\mathcal{K},\widehat{\zeta}) \leq C_0 + \ln \int_{\Gamma(X,S)} \prod_{j\in\partial k} \left(\exp\left\{aF(\widehat{\eta}_j)\right\}\right)^D\ \Pi_{\widehat{Q}_{\mathcal{K}}}(d\widehat{\eta},\widehat{\zeta}).$$

Observe that the constants C_4 and C_5 are independent of a and assume without loss of generality that $a \geq \max\ \{C_5, \beta\|J\|_\infty C_4\}\, \Delta$. Then $D\Delta \leq 1$, and we can apply the multiple Hölder inequality, which implies that

$$\int_{\Gamma(X,S)} \prod_{j\in\partial k} \left(\exp\left\{aF(\widehat{\eta}_j)\right\}\right)^D\ \widehat{\Pi}_{\mathcal{K}}(d\widehat{\eta},\widehat{\zeta}) \leq \prod_{j\in\partial k} \left(\int_{\Gamma(X,S)} \exp\left\{aF(\widehat{\eta}_j)\right\}\ \Pi_{\widehat{Q}_{\mathcal{K}}}(d\widehat{\eta},\widehat{\zeta}) \right)^D .$$

Therefore

$$\begin{aligned} n_k(\mathcal{K},\widehat{\zeta}) &\leq C_0 + D \sum_{j\in\partial k} n_j(\mathcal{K},\widehat{\zeta}) \\ &= C_0 + aD \sum_{\substack{j\in\partial k \\ j\notin\mathcal{K}}} F(\widehat{\zeta}_j) + D \sum_{\substack{j\in\partial k \\ j\in\mathcal{K}}} n_j(\mathcal{K},\widehat{\zeta}). \end{aligned} \tag{5.33}$$

Fix arbitrary $k_0 \in \mathcal{K}$ and $\alpha > 0$ such that $e^{\alpha\vartheta} D\Delta < 1$, where $\vartheta = \sup_{k\in\mathbb{Z}^d} \max_{j\in\partial k} |j-k|$. Multiplying both sides of inequality (5.33) by $e^{-\delta|k_0-k|}$ and taking into account that

$$-|k_0-k| \leq |j-k| - |j-k_0| \leq \vartheta - |j-k_0|,$$

we obtain the estimate

$$\begin{aligned} n_k(\mathcal{K},\widehat{\zeta})e^{-\alpha|k_0-k|} &\leq C_0 e^{-\alpha|k_0-k|} + e^{\alpha\vartheta} aD \sum_{\substack{j\in\partial k \\ j\notin\mathcal{K}}} F(\widehat{\zeta}_j) e^{-\alpha|j-k_0|} \\ &\quad + e^{\alpha\vartheta} D \sum_{\substack{j\in\partial k \\ j\in\mathcal{K}}} n_j(\mathcal{K},\widehat{\zeta}) e^{-\alpha|j-k_0|}. \end{aligned} \tag{5.34}$$

Observe that

$$\sup_{k\in\mathcal{K}} \sum_{\substack{j\in\partial k\\ j\in\mathcal{K}}} n_j(\mathcal{K},\widehat{\zeta})e^{-\alpha|j-k_0|} \le |\partial k| \sup_{k\in\mathcal{K}} \left(n_k(\mathcal{K},\widehat{\zeta})e^{-\alpha|k_0-k|}\right)$$
$$\le \varDelta \sup_{k\in\mathcal{K}} \left(n_k(\mathcal{K},\widehat{\zeta})e^{-\alpha|k_0-k|}\right).$$

Applying supremum to both sides of inequality (5.34) we can see that

$$\sup_{k\in\mathcal{K}} \left(n_k(\mathcal{K},\widehat{\zeta})e^{-\alpha|k_0-k|}\right) \le C_0 + e^{\alpha\vartheta} aD \sum_{j\notin\mathcal{K}} F(\widehat{\zeta}_j)e^{-\alpha|k_0-j|}$$
$$+ e^{\alpha\vartheta} D\varDelta \sup_{k\in\mathcal{K}} \left(n_k(\mathcal{K},\widehat{\zeta})e^{-\alpha|k_0-k|}\right),$$

so that

$$\left(1-e^{\alpha\vartheta}D\varDelta\right) \sup_{k\in\mathcal{K}} \left(n_k(\mathcal{K},\widehat{\eta})e^{-\alpha|k_0-k|}\right) \le C_0 + e^{\alpha\vartheta} aD \sum_{j\notin\mathcal{K}} e^{-\alpha|j-k_0|}F(\widehat{\eta}_j)$$
$$\le C_0 + e^{\alpha(\vartheta+k_0)} aD F_\alpha\left(\widehat{\eta}_{\widehat{Q}^c_{\mathcal{K}}}\right).$$

Thus

$$n_{k_0}(\mathcal{K},\widehat{\eta}) \le \sup_{k\in\mathcal{K}} \left(n_k(\bar{k},\widehat{\eta})e^{-\alpha|k_0-k|}\right)$$
$$\le e^{\alpha\vartheta}\left(1-e^{\alpha\vartheta}D\varDelta\right)^{-1}\left(C_0 + e^{\alpha(\vartheta+k_0)} aD F_\alpha\left(\widehat{\eta}_{\widehat{Q}^c_{\mathcal{K}}}\right)\right), \quad (5.35)$$

which implies that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K},\widehat{\zeta}) \le e^{\alpha\vartheta}\left(1-e^{\alpha\vartheta}D\varDelta\right)^{-1} C_0,$$

since

$$F_\alpha\left(\widehat{\eta}_{\widehat{Q}^c_{\mathcal{K}}}\right) \to 0,\ \mathcal{K}\nearrow\mathbb{Z}^d.$$

Passage to the limit as $\alpha \to 0$ shows that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K},\widehat{\zeta}) \le (1-D\varDelta)^{-1} C_0(a) =: \log\Psi(a),$$

which completes the proof. □

Corollary 5.11. *For any cubic domain* $\widehat{\mathsf{U}}$ *and* $N\in\mathbb{N}$, *there exists* $\mathtt{C}(\widehat{\mathsf{U}},N)<\infty$ *such that*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\Gamma(X,S)} F^N(\widehat{\gamma}_{\widehat{\mathsf{U}}})\Pi_{\widehat{Q}_{\mathcal{K}}}(d\widehat{\gamma}|\widehat{\zeta}) \le \mathtt{C}(\widehat{\mathsf{U}},N) < \infty,$$

where $\mathtt{C}(\widehat{\mathsf{U}},N)$ *can be chosen uniformly for all* $\widehat{\zeta}\in\Gamma^t(X,S)$.

5.3.2 Existence result

In this section, we use the estimates obtained in Section 5.3.1 in order to prove that, for any $\widehat{\eta} \in \Gamma(X,S)$, the family of Gibbsian specifications $\{\Pi_{\mathsf{U}}(d\widehat{\gamma}\,|\widehat{\eta})\}_{\mathsf{U}\in\mathcal{B}_c(X)}$ contains a cluster point.

We define the set

$$\widehat{\Gamma}_T := \left\{\widehat{\gamma} \in \Gamma(X,S) : F\left(\widehat{\gamma}_{\widehat{\mathsf{U}}_R}\right) \leq T\right\},\ T > 0.$$

Observe that for any set $\mathsf{W} \in \mathcal{B}_c(X)$ there exists a constant c_{W} such that

$$|\widehat{\gamma}_{\mathsf{W}}| \leq c_{\mathsf{W}} T,\ \widehat{\gamma} \in \widehat{\Gamma}_T,\ T > 0. \tag{5.36}$$

Definition 5.12. We say that a family of probability measures $\{\mu_m\}_{m\in\mathbb{N}}$ on $\Gamma(X,S)$ is locally equicontinuous (LEC) if for any $\mathsf{U} \in \mathcal{B}_c(X)$ and any sequence $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{B}(\Gamma(\mathsf{U},S))$, such that $B_n \downarrow \emptyset$, $n \to \infty$, we have

$$\lim_{n\to\infty} \limsup_{m\in\mathbb{N}} \mu_m(B_n) = 0. \tag{5.37}$$

We equip the space $\mathcal{P}(\Gamma(X,S))$ of probability measures on $\Gamma(X,S)$ with the following local set convergence (see also Definition 2.2):

$$\mu_\alpha \overset{loc}{\to} \mu \text{ iff } \mu_\alpha(B) \to \mu(B),\quad B \in \mathcal{B}_0(\Gamma(X,S)).$$

Observe that the local set convergence is equivalent to convergence in the space $[0,1]^{\mathcal{F}_0}$, where $\mathcal{F}_0 := \mathcal{B}_0(\Gamma(X,S))$.

Generalizing Lemma 2.23 to our setting we obtain the following result.

Proposition 5.13. *(cf. [Geo88, Prop. 4.9]) Any LEC family of probability measures $\{\mu_N\}_{N\in\mathbb{N}}$ on $\Gamma(X,S)$ has a cluster point, which is a probability measure on $\Gamma(X,S)$.*

Proof. It is straightforward that the family $\{\mu_N\}_{N\in\mathbb{N}}$ contains a cluster point μ as an element of the compact space $[0,1]^{\mathcal{F}_0}$, and μ is an additive function on $\mathcal{F}_0$. The LEC property (5.37) implies that $\mu_{\widehat{\mathsf{W}}} := p^*_{\widehat{\mathsf{W}}}\mu$ is σ-additive on each $\mathcal{F}_{\widehat{\mathsf{W}}} := \mathcal{B}(\Gamma(\mathsf{W},S))$. Thus $\{\mu_{\widehat{\mathsf{W}}}\}_{\mathsf{W}\in\mathcal{B}_c(X)}$ forms a consistent (w.r.t. projective maps (5.4)) family measures and by the corresponding version of the Kolmogorov theorem (see [Par67, Theorem V.3.2])) generates a probability measure on $(\Gamma(X,S),\mathcal{F})$, $\mathcal{F} = \mathcal{B}(\Gamma(X,S))$ (which obviously coincides with μ). □

Corollary 5.14. *There exists a subsequence $\{\mu_{m_k}\}_{k\in\mathbb{N}}$ such that $\mu_{m_k} \overset{loc}{\to} \mu$, as $k \to \infty$.*

Let $\{\mathsf{W}_m\}_{m\in\mathbb{N}} \subset \mathcal{B}_c(X)$ be any increasing sequence of sets such that $\mathsf{W}_m \nearrow X$, as $m \to \infty$, and introduce notation $\Pi_m\left(d\widehat{\gamma}\,\middle|\,\widehat{\zeta}\right) := \Pi_{\widehat{\mathsf{W}}_m}\left(d\widehat{\gamma}\,\middle|\,\widehat{\zeta}\right)$.

Proposition 5.15. *For any $\widehat{\zeta} \in \Gamma^t(X,S)$ the family $\{\Pi_m\}_{m\in\mathbb{N}}$ is LEC.*

Proof. Fix $\mathsf{U} \in \mathcal{B}_c(X)$ and $\{B_n\}_{n\in\mathbb{N}}$ as in Definition 5.12. It is sufficient to prove that $\forall\varepsilon > 0$ there exists m_0 and n_0 such that

$$\Pi_m\left(B_n|\widehat{\zeta}\right) \le \varepsilon$$

for any $m \ge m_0$ and $n \ge n_0$.

First, we will fix $T > 0$ and estimate the corresponding measures of the sets $B_n \cap \widehat{\Gamma}_T$ and $B_n \cap \left(\widehat{\Gamma}_T\right)^c$. Using bounds (5.23) and (5.28), inequality (5.36) and obvious estimates $\sum_{x\in\eta_{\mathsf{U}}} |\xi_x|^r \le cF(\widehat{\eta})$, $\sum_{x\in\gamma_{\mathsf{U}^c}} |\sigma_x|^r \le cF(\widehat{\gamma})$ that hold for some constant $c > 0$, we obtain the inequalities

$$\mathbb{1}_{B_n\cap\widehat{\Gamma}_T}(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}) \exp\left\{-H^{\Phi}_{\mathsf{U}}(\eta_{\mathsf{U}}|\gamma)\right\} \le \exp\left\{B'_{\Phi}T + MT^2\right\}$$

and

$$\begin{aligned}
&\mathbb{1}_{B_n\cap\widehat{\Gamma}_T}(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}) \exp\left\{-E_{\gamma_{\mathsf{U}},\eta}(\xi_{\gamma_{\mathsf{U}}}|\sigma)\right\} \\
&\quad\le \exp\left\{\|J\|_\infty[3T\sum_{x\in\eta_{\mathsf{U}}}|\xi_x|^r + T\sum_{x\in\gamma_{\mathsf{U}^c}}|\sigma_x|^r + 2T^2C_W]\right\} \\
&\quad\le \exp\left\{\|J\|_\infty[cT^2(4 + 2C_W)]\right\}.
\end{aligned}$$

Thus there exists a constant $a(\mathsf{U},T)$ such that

$$\mathbb{1}_{B_n\cap\widehat{\Gamma}_T}(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}) \exp\left\{-\beta H^{\Phi}_{\mathsf{U}}(\eta_{\mathsf{U}}|\gamma) - \beta E_{\eta_{\mathsf{U}},\gamma}(\xi_{\eta_{\mathsf{U}}}|\sigma)\right\} \le a(\mathsf{U},T) \tag{5.38}$$

for all $\widehat{\eta},\widehat{\gamma} \in \Gamma(X,S)$ and $n \in \mathbb{N}$.

According to Chebyshev's inequality applied to measure Π_m on $\Gamma(X,S)$ we have

$$\Pi_m\left(\{\widehat{\gamma} \in \Gamma(X,S) : f(\widehat{\gamma}) \ge T\}\,\middle|\,\widehat{\zeta}\right) \le T^{-2}\int_{\Gamma(X,S)} |f(\widehat{\gamma})|^2\,\Pi_m\left(d\widehat{\gamma}|\widehat{\zeta}\right)$$

for any $T > 0$ and $f \in L^2(\Gamma(X,S),\Pi_m)$. Setting $f(\widehat{\gamma}) = F\left(\widehat{\gamma}_{\widehat{\mathsf{U}}_R}\right)$ we obtain, cf. Corollary 5.11.

$$\Pi_m\left(\left(\widehat{\Gamma}_T\right)^c\,\middle|\,\widehat{\zeta}\right) \le \varepsilon \tag{5.39}$$

for any $\varepsilon > 0$ and T greater than some $T(\varepsilon, \widehat{\zeta})$. Now we see that

$$\begin{aligned}\Pi_m\left(B_n\middle|\widehat{\zeta}\right) &= \Pi_m\left(B_n \cap \left(\widehat{\Gamma}_T\right)^c\middle|\widehat{\zeta}\right) + \Pi_m\left(B_n \cap \widehat{\Gamma}_T\middle|\widehat{\zeta}\right)\\ &\le \Pi_m\left(\left(\widehat{\Gamma}_T\right)^c\middle|\widehat{\zeta}\right) + \int_{\Gamma(X,S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\gamma}_{\mathsf{U}}\right)\Pi_m\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right).\end{aligned}$$

Observe that there exists m_0 such that $\mathsf{W}_m \supset \mathsf{U}$ for $m \ge m_0$. For all such m, it follows from (5.10) and consistency property (5.11) of the specification Π that

$$\begin{aligned}&\int_{\Gamma(X,S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\gamma}_{\mathsf{U}}\right)\Pi_m\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right)\\ &\quad= \int_{\Gamma(X,S)}\left[\int_{\Gamma(\mathsf{U},S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}\right)\Pi_{\mathsf{U}}\left(d\widehat{\eta}\,|\widehat{\gamma}\right)\right]\Pi_m\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right)\\ &\quad= \int_{\Gamma(X,S)} Z_{\mathsf{U}}(\widehat{\gamma})^{-1}\int_{\Gamma(\mathsf{U},S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}\right)\\ &\quad\times \exp\left(-\beta H^{\Phi}_{\mathsf{U}}(\gamma_{\mathsf{U}}\,|\eta) - \beta E_{\gamma_{\mathsf{U}}\cup\eta_{\mathsf{U}^c}}(\sigma_{\gamma_{\mathsf{U}}}\,|\xi)\right)\bigotimes_{x\in\gamma_{\mathsf{U}}} g(d\sigma_x)\ \lambda(d\eta_{\mathsf{U}})\Pi_m\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right).\end{aligned}$$

Thus by (5.38) and (5.29) we obtain

$$\int_{\Gamma(\mathsf{U},S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\eta}_{\mathsf{U}} \cup \widehat{\gamma}_{\mathsf{U}^c}\right)\Pi_{\mathsf{U}}\left(d\widehat{\eta}\,|\widehat{\gamma}\right) \le a(\mathsf{U}, T)\widehat{\lambda}_z(B_n) < \varepsilon$$

for n greater than some $n(\varepsilon, T)$. Hence,

$$\int_{\Gamma(X,S)} 1\!\!1_{B_n\cap\widehat{\Gamma}_T}\left(\widehat{\gamma}_{\mathsf{U}}\right)\Pi_m\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right) < \varepsilon.$$

Combining this with estimate (5.39) we can see that $\forall\varepsilon > 0$ and $m \ge m_0$, $n \ge n_0 = n(\varepsilon/2, T(\varepsilon/2))$ we have

$$\Pi_m\left(\widehat{B}_n\middle|\widehat{\zeta}\right) \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The proof is complete. □

Now we are in position to prove the first main result of this chapter.

Theorem 5.16. *(Existence)*

(i) The set $\mathcal{G}^t := \mathcal{G}(\Gamma^t(X,S))$ is not empty.

(ii) Any $\mu \in \mathcal{G}^t$ satisfies the estimate

$$\sup_{k\in\mathbb{Z}^d}\int_{\Gamma(X,S)} \exp\{aF(\widehat{\gamma}_k)\}\mu(d\widehat{\gamma}) < \infty \tag{5.40}$$

for all $a \in \mathbb{R}$.

Proof. It follows from Propositions 5.13 and 5.15 that for any $\widehat{\zeta} \in \Gamma^t$ the family $\left\{\Pi_{\mathsf{W}_m}\left(d\widehat{\gamma}\middle|\widehat{\zeta}\right)\right\}_{m\in\mathbb{N}}$ has a cluster point $\mu = \mu(\widehat{\zeta}) \in \mathcal{P}(\Gamma(X,S))$, so that there exists a subsequence W_{m_j}, $j \in \mathbb{N}$, such that

$$\mu(B) = \lim_{j\to\infty} \Pi_{\mathsf{W}_{m_j}}\left(B\middle|\widehat{\zeta}\right) \tag{5.41}$$

for any $B \in \mathcal{B}_0(\Gamma(X,S))$. Standard limit transition arguments (see also Section 2.B) and the consistency property (5.11) of the specification Π show that μ satisfies the DLR equation (5.12) and (5.40), hence the result follows. □

Remark 5.17. In [Kun99], [AKLU00], [KKdS98] and [Mas00], a theory of Gibbs measures on marked configuration spaces that satisfy Ruelle's stability, respectively superstability conditions has been developed. To this end, one has to require the following bounds to hold on the energy

$$\begin{aligned} &H(\widehat{\gamma}) \geq A_1|\gamma| - B_1 \quad \text{resp.} \\ &H(\widehat{\gamma}) \geq A_2 \sum_{k\in Z^d} |\gamma_k|^2 - B_2|\gamma|, \quad \widehat{\gamma} \in \Gamma_0(\widehat{X}). \end{aligned} \tag{5.42}$$

with some $A_1, B_1, A_2, B_2 > 0$. Obviously, this is impossible in the case of unbounded marks $\sigma_x \in R^d$ and the interactions like in (5.6)-(5.8). However, taking the Lyapunov functional $F(\widehat{\gamma}_k)$, cf. (5.17), instead of the squared counting map $|\gamma_k|^2$ in (5.42), we can develop an analogue of Ruelle's superstability estimates and construct the corresponding Gibbs states ν satisfying the regularity condition

$$\sup_{K\in\mathbb{N}} \left\{ K^{-d} \sum_{|k|\leq K} F(\widehat{\gamma}_k) \right\} := C(\widehat{\gamma}) < \infty, \quad \forall \gamma \in \Gamma(\widehat{X}) \pmod{\nu}.$$

As for the uniqueness problem for such Gibbs states, one has to develop a harmonic analysis on the marked configuration spaces and a theory of the Kirkwood-Salsburg equations for the corresponding correlation functions. So far, this was done via cluster expansions in [Kun99], but only under condition (5.42) which, as already mentioned above, does not cover our model. However, these issues are beyond the scope of the present PhD work.

5.4 Uniqueness of Gibbs Measures

5.4.1 The Corresponding Lattice Model

Here we extend the corresponding constructions of Section 4.2.3 to the case of marked configuration spaces. Starting from the chosen partition $(Q_k)_{k\in\mathbb{Z}^d}$ of X

(see (5.13)), we construct a lattice system on the space $\breve{\Gamma}_{lat} := \left(\Gamma(\overline{Q}, S)\right)^{\mathbb{Z}^d}$, where for simplicity we denoted $Q = Q_0$. The space $\breve{\Gamma}_{lat}$ is endowed with the product topology and the corresponding Borel σ-algebra $\mathcal{B}(\breve{\Gamma}_{lat})$. Then, by Section A.5 in [Kun99] and Remark 4.A3 in [Geo88], $(\breve{\Gamma}_{lat}, \mathcal{B}(\breve{\Gamma}_{lat}))$ is a standard Borel space.

Define the map

$$T : \Gamma(X, S) \to \breve{\Gamma}_{lat},$$

which sends $\widehat{\gamma} \in \Gamma(X, S)$ into $\breve{\gamma} = (\breve{\gamma}_k)_{k \in \mathbb{Z}^d} \in \breve{\Gamma}_{lat}$, where $\breve{\gamma}_k := \widehat{\gamma} \cap (\overline{Q}_k \times S) - k$ and by $\widehat{\eta} - k$ we denote the configuration $\{...(x - k, \sigma_x)...\}$, for $\widehat{\gamma} = \{...(x, \sigma_x)...\}$. By T^{-1} we denote the left inverse of T. Let $B_{k_1} \dots B_{k_L} \in \mathcal{B}(\Gamma_{\overline{Q}}(X, S))$ for $L \in \mathbb{N}$ and $k_1, \dots k_L \in \mathbb{Z}^d$ and define the cylinder sets $A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}} := \{(\breve{\gamma}_k)_{k \in \mathbb{Z}^d} \in \breve{\Gamma}_{lat} : \breve{\gamma}_{k_l} \in B_{k_l}, 1 \le l \le L\} \in \mathcal{B}(\breve{\Gamma}_{lat})$ and $C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}} := \{\widehat{\gamma} \in \Gamma(X, S) : \widehat{\gamma}_{\overline{Q}_{k_l}} - k_l \in B_{k_l},\ 1 \le l \le L\} \in \mathcal{B}(\Gamma(X, S))$, respectively.

Lemma 5.18. *(i)* $T : \Gamma(X, S) \to \breve{\Gamma}_{lat}$ *is measurable;*

(ii) $T(B) \in \mathcal{B}(\breve{\Gamma}_{lat})$ *for any* $B \in \mathcal{B}_0(\Gamma(X, S))$.

Proof. (i) One can immediately see that

$$T^{-1}\left(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}\right) = C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}},$$

which proves the statement, since $\mathcal{B}(\breve{\Gamma}_{lat})$ is generated by the cylinder sets.

(ii) Assume that $\mathsf{W} \subset \bigcup_{i=1}^{L} Q_{k_i}$. For $\mathsf{B} \in \mathcal{B}(\Gamma(\mathsf{W}, S))$ we have

$$T\left(\{\widehat{\gamma} \in \Gamma(X, S) : \widehat{\gamma}_{\widehat{\mathsf{W}}} \in \mathsf{B}\}\right) = \left\{\breve{\gamma} \in \breve{\Gamma}_{lat} : \breve{\gamma}_{(k_1,\dots,k_L)} \in \mathsf{B}\right\},$$

which is measurable. Here $\breve{\gamma}_{(k_1,\dots,k_L)}$ denotes the projection of $\breve{\gamma}$ onto the coordinates $k_1, \dots, k_L$ onto the product space $\prod_{i=1}^{L} \Gamma(\overline{Q}, S)$. □

Thus, for any $\mu \in \mathcal{P}(\Gamma(X, S))$ we can define its push-forward image $T_*\mu \in \mathcal{P}(\breve{\Gamma}_{lat})$, where $\mathcal{P}(\breve{\Gamma}_{lat})$ is the set of all probability measures on $\breve{\Gamma}_{lat}$.

Lemma 5.19. *The map* $T_* : \mathcal{P}(\Gamma(X, S)) \to \mathcal{P}(\breve{\Gamma}_{lat})$ *is injective.*

Proof. Let $\mu, \nu \in \mathcal{P}(\Gamma(X, S))$ and $\mu \neq \nu$. Then there exists $B \in \mathcal{B}_0(\Gamma(X, S))$ such that $\mu(B) \neq \nu(B)$. By Lemma 5.18, $A := T(B) \in \mathcal{B}(\breve{\Gamma}_{lat})$. The injectivity of T implies that $T^{-1}(T(B)) = B$. Thus $T_*\mu(A) = \mu(T^{-1}(A)) \neq \nu(T^{-1}(A)) = T_*\nu(A)$, and the statement is proved. □

Let us investigate the correspondence between measures on $\Gamma(X, S)$ and $\breve{\Gamma}_{lat}$. Let μ be a probability measure on $\Gamma(X, S)$ satisfying the following condition:

(A) Consider the sets

$$\mathring{\Gamma}(X,S) := \left\{\widehat{\gamma} \in \Gamma(X,S) \,\middle|\, \gamma \cap \partial Q_k = \varnothing, \quad \forall k \in \mathbb{Z}^d\right\} \in \mathcal{B}(\Gamma(X,S)),$$
$$\mathring{\Gamma}_{\mathsf{W}}(X,S) := \left\{\gamma \in \Gamma(X,S) \,\middle|\, \gamma_{\mathsf{W}} \cap \partial Q_k = \varnothing, \quad \forall k \in \mathbb{Z}^d\right\} \in \mathcal{B}(\Gamma_{\mathsf{W}}(X,S)), \tag{5.43}$$

for any $\mathsf{W} \in \mathcal{B}_c(\mathbb{R}^d)$ and assume $\mu(\mathring{\Gamma}(X,S)) = 1$. In other words, μ ignores configurations whose supports *touch* the sites of the partition cubes Q_k.

For $B_k \in \mathcal{B}(\Gamma(\overline{Q},S))$ with $k \in \mathbb{Z}^d$, we denote $\mathring{B}_k := \{\widehat{\gamma} \in B_k | \gamma \cap \partial\overline{Q} = \emptyset\}$, where $\partial\overline{Q} := \overline{Q} \setminus Q$. Starting from a given μ, probability measure on $\Gamma(X,S)$ satisfying condition (5.43) above, we construct a probability measure μ_{lat} on $\breve{\Gamma}_{lat}$, as the push-forward of μ. The explicit definition is as follows:

$$\begin{aligned}\mu_{lat}(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}) &:= \mu_{lat}(A_{k_1,\dots,k_L}^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}}) := \mu(T^{-1}(A_{k_1,\dots,k_L}^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}})) = \\ &= \mu(\{\widehat{\gamma} \in \Gamma(X,S) | \widehat{\gamma}_{\overline{Q}_{k_l}} - k_l \in \mathring{B}_{k_l}, 1 \le l \le L\}) \\ &= \mu(\{\widehat{\gamma} \in \Gamma(X,S) | \widehat{\gamma}_{\overline{Q}_{k_l}} - k_l \in B_{k_l}, 1 \le l \le L\}) \\ &= \mu(C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}).\end{aligned} \tag{5.44}$$

Since we know that the cylinder events constitute a measure-defining class, μ_{lat} is well-defined on the whole $\mathcal{B}(\breve{\Gamma}_{lat})$. Also, denoting by $\mathring{\Gamma}_{lat}$ the set $\{\breve{\gamma} \in \breve{\Gamma}_{lat} | \breve{\gamma}_k \cap (\partial\overline{Q} \times S) = \emptyset\}$, we see from the above definition that the corresponding measure on the lattice μ_{lat} puts full mass on $\mathring{\Gamma}_{lat}$. Also it is obvious that $T : \mathring{\Gamma}(X,S) \to \mathring{\Gamma}_{lat}$ is a bijection.

Remark 5.20. Obviously, the definition of μ_{lat} by (5.44) extends to any σ-finite distribution μ on $\Gamma(X,S)$. In particular, for $\mu = \widehat{\lambda}_z$, we have that

$$\breve{\lambda}_{lat,z}(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}) := \prod_{l=1}^{L} \widehat{\lambda}_z\big(\{\widehat{\gamma} \in \Gamma(X,S) : \widehat{\gamma}_{\overline{Q}_{k_l}} - k_l \in B_{k_l}\}\big). \tag{5.45}$$

We remark that the construction of $\breve{\lambda}_{lat,z}$ is possible because $\widehat{\lambda}_z$ satisfies condition (5.43) (see Proposition 5.2).

We continue by defining the energy of the new system with the phase space $\breve{\Gamma}_{lat}$. Consider arbitrarily large cubic domains $\mathsf{W}_{\mathcal{K}} := \bigsqcup_{k \in \mathcal{K}} \overline{Q}_k$ indexed by $\mathcal{K} \Subset \mathbb{Z}^d$ and define the local energy as

$$\breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}} | \breve{\eta}) := H\left((T^{-1}\breve{\gamma})_{\mathcal{K}} | (T^{-1}\breve{\eta})\right). \tag{5.46}$$

Using the above definition, we introduce the local one-point Gibbs states as

$$\breve{\mu}_{\mathcal{K}}(d\breve{\gamma}_{\mathcal{K}}|\breve{\eta}) := \begin{cases} [\breve{Z}_{\mathcal{K}}(\breve{\eta})]^{-1} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}}|\breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}_{\mathcal{K}}), & \breve{\eta} \in \mathring{\Gamma}_{lat}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.47}$$

where

$$\breve{Z}_{\mathcal{K}}(\breve{\eta}) := \int_{\breve{\Gamma}_{lat}} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}'_{\mathcal{K}}|\breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}'_{\mathcal{K}}) \tag{5.48}$$

and $\breve{\lambda}_{lat,z}$ is given by (5.45). For simplicity, by index $\mathcal{K}$ we mean the cubic domain $\mathsf{W}_{\mathcal{K}}$.

We note that elementary computations yield for any $\breve{\eta} \in \mathring{\Gamma}_{lat}$

$$\begin{aligned} \breve{Z}_{\mathcal{K}}(\breve{\eta}) &= \int_{\mathring{\Gamma}_{lat}} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}}|\breve{\eta})\right\} \breve{\lambda}_{lat,z}(d\breve{\gamma}_{\mathcal{K}}) \\ &= \int_{\Gamma(X,S)} \exp\{-\beta \breve{H}_{\mathcal{K}}(T(\widehat{\gamma})_{\mathcal{K}}|\breve{\eta})\} \widehat{\lambda}_z(d\widehat{\gamma}) \\ &= \int_{\Gamma(X,S)} \exp\{-\beta H_{\mathcal{K}}(T^{-1}(T(\widehat{\gamma}))_{\mathcal{K}}|T^{-1}(\breve{\eta}))\} \widehat{\lambda}_z(d\widehat{\gamma}) = Z_{\widehat{Q}_{\mathcal{K}}}(T^{-1}\breve{\eta}). \end{aligned} \tag{5.49}$$

Also, it is easy to check that the local Gibbs states for the lattice model are the pushforward measures of the local Gibbs states of the initial model, or more explicitly, $\breve{\mu}_{\mathcal{K}}(d\breve{\gamma}|\breve{\eta}) = (\mu_{\mathcal{K}} \circ T^{-1})(d\widehat{\gamma}|T^{-1}\breve{\eta})$. From here, we go on to define the local Gibbs specification as:

$$\breve{\pi}_{\mathcal{K}}(\breve{B}|\breve{\eta}) := \breve{\mu}_{\mathcal{K}}(\breve{B}_{\mathcal{K},\breve{\eta}}|\breve{\eta}), \quad \breve{B}_{\mathcal{K},\breve{\eta}} := \{\breve{\gamma}_{\mathcal{K}}|\breve{\gamma}_{\mathcal{K}} \cup \breve{\eta}_{\mathcal{K}^c} \in \breve{B}\}, \tag{5.50}$$

for any $\breve{B} \in \mathcal{B}(\breve{\Gamma}_{lat})$. An important step is to show that uniqueness of Gibbs measures in the lattice model we have introduced above implies uniqueness of Gibbs measures in our initial model.

Lemma 5.21. *Let μ be a Gibbs measure on $\Gamma(X,S)$ corresponding to the specification $\{\Pi_{\mathsf{W}}\}_{\mathsf{W}}$. Then μ uniquely determines a Gibbs measure μ_{lat} corresponding to $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$. Moreover, if μ_{lat} is the unique Gibbs measure of the system $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$, then also μ is unique as a Gibbs measure corresponding to $\{\Pi_{\mathsf{W}}\}_{\mathsf{W}}$.*

Proof. First, we show that any Gibbs measure μ, which is consistent to the specification (5.10) satisfies condition **(A)**. The proof is simple and in the case of non-marked configurations can be found in Section 5.3 of [KPR12]. For the reader's convenience, we briefly give here the argument for marked configurations. Let $\mathsf{W} \in \mathcal{B}_c(X)$. Since we already know that $\widehat{\lambda}_z$ satisfies **(A)**, using the definition of Π_{W}, it is easy to see that $\Pi_{\mathsf{W}}(\mathring{\Gamma}_{\mathsf{U}}(X,S)|\widehat{\eta}) = 1$ for any $\mathsf{U} \subseteq \mathsf{W}$. The setwise convergence (5.41) yields

$$\mu(\mathring{\Gamma}_{\mathsf{U}}(X,S)) = \lim_{N\to\infty} \Pi_{\mathsf{W}_N}(\mathring{\Gamma}_{\mathsf{U}}(X,S)|\widehat{\eta}) = 1 \text{ for all } \mathsf{U} \in \mathcal{B}_c(X),$$

therefore

$$\mu(\mathring{\Gamma}(X,S)) = \mu\left(\bigcap_{K\in\mathbb{N}} \mathring{\Gamma}_{\mathsf{U}_K}(X,S)\right) = \lim_{K\to\infty} \mu(\mathring{\Gamma}_{\mathsf{U}_K}(X,S)) = 1 \text{ as } \mathsf{U}_K \nearrow X.$$

Hence, μ satisfies condition **(A)**, which implies the existence of a measure μ_{lat} as given by (5.44). Let us show that μ_{lat} is a Gibbs measure corresponding to the specification $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$, by checking the DLR equations. Let $\breve{B} \in \mathcal{B}(\mathring{\Gamma}_{lat})$. Applying (5.46)-(5.49) yields

$$\begin{aligned}
&\int_{\breve{\Gamma}_{lat}} \breve{\pi}_{\mathcal{K}}(\breve{B}|\breve{\eta})\mu_{lat}(d\breve{\eta}) \\
&= \int_{\mathring{\Gamma}_{lat}} \int_{\mathring{\Gamma}_{lat}} \breve{Z}_{\mathcal{K}}^{-1}(\breve{\eta}) \exp\{-\beta \breve{H}_{\mathcal{K}}(\breve{\gamma}_{\mathcal{K}}|\breve{\eta})\} \mathbb{1}_{\breve{B}}(\breve{\gamma}_{\mathcal{K}} \times \breve{\eta}_{\mathcal{K}^c}) \breve{\lambda}_{lat,z}(d\breve{\gamma})\mu_{lat}(d\breve{\eta}) \\
&= \int_{\Gamma(X,S)} \int_{\Gamma(X,S)} Z_{\mathcal{K}}^{-1}(\widehat{\eta}) \exp\{-\beta \breve{H}_{\mathcal{K}}(T\widehat{\gamma}_{\mathcal{K}}|T\widehat{\eta})\} \mathbb{1}_{\breve{B}}(T\widehat{\gamma}_{\mathcal{K}} \times T\widehat{\eta}_{\mathcal{K}^c}) \widehat{\lambda}_z(d\widehat{\gamma})\mu(d\widehat{\eta}) \\
&\overset{\text{DLR}}{=} \mu(T^{-1}\breve{B}) = \mu_{lat}(\breve{B}).
\end{aligned}$$

Uniqueness follows easily by Lemma 5.19. □

5.4.2 Uniqueness result

In what follows, our aim is to show uniqueness of tempered Gibbs measures in the lattice model introduced in Section 5.4.1. The set of such measures will be denoted by $\mathcal{G}^t_{lat}$ and consists of Gibbs measures μ_{lat}, which are supported by the following set of tempered configurations

$$\Gamma^t_{lat} := \bigcap_{\alpha>0} \Gamma_{\alpha,lat},$$

where

$$\Gamma_{\alpha,lat} := \left\{ \breve{\gamma} \in \breve{\Gamma}_{lat} : |\breve{\gamma}_k|_\alpha := \sup_{k\in\mathbb{Z}} \exp\{-\alpha|k|\}\breve{F}(\breve{\gamma}) < \infty \right\},$$

where $\breve{F} := T_* F$, or, in other words, we have

$$\breve{F}(\breve{\gamma}_k) := |\gamma_k|^p + \sum_{x\in\gamma_k} |\sigma_x|^q, \qquad \breve{\gamma} \in \breve{\Gamma}_{lat}.$$

Moreover, by (5.40), any tempered Gibbs measure μ_{lat} satisfies the following exponential moment estimate,

$$\sup_{k\in\mathbb{Z}^d} \int_{\breve{\Gamma}_{lat}} \exp\left\{a\breve{F}(\breve{\gamma}_k)\right\}\mu_{lat}(d\breve{\gamma}) < \infty. \tag{5.51}$$

Hence, μ_{lat} satisfies the a-priori bound in Theorem 2.9.

Theorem 5.22. *(Uniqueness due to small interaction and small activity parameter)*

For any fixed $\beta_0 > 0$, one finds $\mathcal{J}_0 = \mathcal{J}_0(\beta_0)$ and $z_0 = z_0(\beta_0)$ such that $\mathcal{G}(\Gamma^t(X,S))$ is a singleton at all values of $\beta < \beta_0$, $\|J\|_\infty \le \mathcal{J}_0$ and $z \le z_0$.

In what follows, we properly extend the idea used to prove uniqueness in the continuous model in Chapter 4. Recall that we denote the R-vicinity of a point $k \in \mathbb{Z}^d$ by $\partial k := \partial_R k = \{j \in \mathbb{Z}^d | d(Q_k, Q_j) \le R\}$. Also, let Z_0 be a semigroup of $\mathbb{Z}^d$ such that $|u - v| > R$ holds for all $u, v \in Z_0$, and define $\chi := \min_{Z_0} |\mathbb{Z}^d / Z_0|$, the number of elements in the quotient group $\mathbb{Z}^d/Z_0$. The proof of the theorem will be based on the following two lemmas, stated below.

Integrability condition (IC)

Lemma 5.23. *Let $0 < \beta < \beta_0$. There are constants $\theta > 0$ and $0 < \bar{c} < 1/\Delta^\chi$ such that, for every $k \in \mathbb{Z}^d$ and any boundary condition $\breve{\eta} \in \breve{\Gamma}_{lat}$,*

$$\int_{\breve{\Gamma}_{lat}} \theta \breve{F}(\breve{\gamma}_k)\breve{\pi}_{\widehat{Q}_k}(d\breve{\gamma}_k|\breve{\eta}) \le 1 + \frac{\bar{c}}{\Delta^\chi} \sum_{j\in\partial k} \theta \breve{F}(\breve{\eta}_j). \tag{5.52}$$

Proof. Let us first observe that, by a simple change of variables,

$$\int_{\breve{\Gamma}_{lat}} \exp\left\{a\breve{F}(\breve{\gamma}_k)\right\}\breve{\pi}_{\widehat{Q}_k}(d\breve{\gamma}_k|\breve{\eta}) = \int_{\Gamma(X,S)} \exp\left\{aF(\widehat{\gamma}_k)\right\}\Pi_{\widehat{Q}_k}(d\widehat{\gamma}_k|\widehat{\eta}).$$

In order to prove the (IC) condition, we will use the one-point estimate obtained in the proof of Theorem 5.16, given by relation (5.32) , i.e.

$$\int_{\widehat{\Gamma}_k} \exp\{aF(\widehat{\gamma}_k)\}\Pi_{\widehat{Q}_k}(d\widehat{\gamma}_k|\widehat{\eta}) \leq e^{C_0(\beta,a)} \exp\left\{\sum_{j\in\partial k}\left[C_5|\eta_j|^{2+\varepsilon^{-1}} + \beta\|J\|_\infty C_4 \sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]\right\}.$$

Hence

$$\int_{\widehat{\Gamma}_k} \exp\{a\breve{F}(\breve{\gamma}_k)\}\breve{\pi}_{\widehat{Q}_k}(d\breve{\gamma}_k|\breve{\eta}) \leq e^{C_0(\beta,a)} \exp\left\{\sum_{j\in\partial k}\left[C_5|\eta_j|^{2+\varepsilon^{-1}} + \beta\|J\|_\infty C_4 \sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]\right\},$$

where $C_0(\beta,a) = \ln\int_{\Gamma_k}\exp\{\beta(-A_\Phi|\gamma_k|^P + \mathcal{P}(|\gamma_k|))\}\lambda_z(d\gamma_k)$, for any $a > 0$ and $C_5 = C_5(\beta) = \beta(\|J\|_\infty C_2 + M/2)$. Applying Jensen's inequality for the exponential, we have

$$\int_{\widehat{\Gamma}_k} \breve{F}(\breve{\gamma}_k)\breve{\pi}_{\widehat{Q}_k}(d\breve{\gamma}_k|\breve{\eta}) \leq a^{-1}C_0(\beta,a)+a^{-1}\sum_{j\in\partial k}\left[C_5|\eta_j|^{2+\varepsilon^{-1}}+\beta\|J\|_\infty C_4\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\right]$$

For p and q as in (5.16) and $\varepsilon \in [(p-2)^{-1}, r^{-1}q - 1]$, this, in turn, implies

$$\int_{\widehat{\Gamma}_k} \breve{F}(\breve{\gamma}_k)\breve{\pi}_{\widehat{Q}_k}(d\breve{\gamma}_k|\breve{\eta}) \leq a^{-1}C_0(\beta,a) + a^{-1}\max\{C_5, \beta\|J\|_\infty C_4\}\sum_{j\in\partial k}\breve{F}(\breve{\eta}_j).$$

By Young's inequality, one can see that $C_0(\beta,a)$ is an increasing function of β. Relation (IC) is satisfied with constants

$$\theta := aC_0(\beta_0,a)^{-1} \tag{5.53}$$

and

$$\bar{c} := a^{-1}\Delta\max\{C_5(\beta_0), \beta_0\|J\|_\infty C_4\}. \tag{5.54}$$

It is obvious to see that $\bar{c} < 1/\Delta^\chi$ for a big enough $a > 0$. □

Contraction Condition (CC)

Lemma 5.24. *For a fixed $k \in \mathbb{Z}^d$ and any $\beta_0 > 0$, one can find $\mathcal{J}_0 = \mathcal{J}_0(\beta_0)$ and $z_0 = z_0(\beta_0)$ such that at all values of $\beta < \beta_0$, $\|J\|_\infty \leq \mathcal{J}_0$ and $z \leq z_0$*

$$d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\mu}^1), \breve{\pi}_k(d\breve{\gamma}_k|\breve{\eta}^2)) \leq \sum_{j\in\partial k}\mathtt{k}\mathbb{1}_{\breve{\eta}_j^1\neq\breve{\eta}_j^2}, \tag{5.55}$$

for some constant $0 < \mathtt{k} < 1$ *and boundary conditions* $\breve{\eta}^1, \breve{\eta}^2$ *such that*

$$\theta \breve{F}(\breve{\eta}^i_j) \leq K_*, \quad i = 1, 2 \tag{5.56}$$

and $K_* = K_*(\theta \breve{F}, \bar{c}, \mathtt{k})$ *is given by (2.14).*

Proof. Let $\breve{\eta}^1, \breve{\eta}^2$ be such boundary conditions, satisfying (5.56). For simplicity however, we will just care that $|\eta^i_j| \leq K_0 := \theta^{-1} K_*$ and $\sum_{x \in \eta^i_j} |\xi^i_x|^r \leq K_0$, for $i = 1, 2$. By formula (2.86) for computing the total variation distance between two probability measures, knowing their densities with respect to a given measure $d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\mu}^1), \breve{\pi}_k(d\breve{\gamma}_k|\breve{\eta}^2))$, is equal to

$$\frac{1}{2} \int_{\widehat{\Gamma}_k} \left| \breve{Z}^{-1}_{\widehat{Q}_k}(\breve{\eta}^1) \exp\{-\beta \breve{H}_k(\breve{\gamma}_k|\breve{\eta}^1)\} - \breve{Z}^{-1}_{\widehat{Q}_k}(\breve{\eta}^2) \exp\{-\beta \breve{H}_k(\breve{\gamma}_k|\breve{\eta}^2)\} \right| \breve{\lambda}_{lat,z}(d\breve{\gamma}_k).$$

Notice however, that by a change of variables, the above expression can be rewritten as

$$\frac{1}{2} \int_{\widehat{\Gamma}_k} \left| Z^{-1}_{\widehat{Q}_k}(\widehat{\eta}^1) \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - Z^{-1}_{\widehat{Q}_k}(\widehat{\eta}^2) \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\} \right| \widehat{\lambda}_z(d\widehat{\gamma}_k),$$

where $\widehat{\eta}^i := T^{-1}\breve{\eta}^i$, for $i = 1, 2$. Note also that in this case $F(\widehat{\eta}^i) \leq K_0$, for $i = 1, 2$.

Since the partition function is greater or equal to 1, the above expression has an upper bound given by

$$\begin{aligned}
&\frac{1}{2} \int_{\widehat{\Gamma}_k} \left| Z_{\widehat{Q}_k}(\widehat{\eta}^2) \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - Z_{\widehat{Q}_k}(\widehat{\eta}^1) \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\} \right| \otimes_{x \in \gamma_k} g(d\sigma_x) \lambda(d\gamma_k) \\
&\leq \frac{1}{2} \left[\int_{\widehat{\Gamma}_k} \left| Z_{\widehat{Q}_k}(\widehat{\eta}^2) - Z_{\widehat{Q}_k}(\widehat{\eta}_1) \right| \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} \otimes_{x \in \gamma_k} g(d\sigma_x) \lambda(d\gamma_k) + \right. \\
&\left. + \int_{\widehat{\Gamma}_k} Z_{\widehat{Q}_k}(\widehat{\eta}^1) \left| \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\} \right| \otimes_{x \in \gamma_k} g(d\sigma_x) \lambda(d\gamma_k) \right] \\
&\leq Z_{\widehat{Q}_k}(\widehat{\eta}^1) \int_{\widehat{\Gamma}_k} \left| \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\} \right| \otimes_{x \in \gamma_k} g(d\sigma_x) \lambda(d\gamma_k)
\end{aligned}$$

The above computations work when interchanging $\widehat{\eta}_1$ by $\widehat{\eta}_2$, hence the total variation distance is less or equal that

$$\min\Big(Z_{\widehat{Q}_k}(\widehat{\eta}^1), Z_{\widehat{Q}_k}(\widehat{\eta}^2)\Big) \tag{5.57}$$

$$\times \int_{\widehat{\Gamma}_k} \Big| \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\}\Big| \otimes_{x\in\gamma_k} g(d\sigma_x)\lambda(d\gamma_k). \tag{5.58}$$

In order to simplify further computations let $\widehat{\eta}^1 = \emptyset$ outside $\widehat{Q}_j$ for a $j \in \partial k$ and $\widehat{\eta}^2 = \emptyset$. Since $Z_{\widehat{Q}_k}(\emptyset) = 1$, we will only be interested in the integral factor from the expression above. Expanding its terms yields the following expression

$$\begin{aligned}
&\int_{\widehat{\Gamma}_k} \big|\exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^1)\} - \exp\{-\beta H_k(\widehat{\gamma}_k|\widehat{\eta}^2)\}\big| \otimes_{x\in\gamma_k} g(d\sigma_x)\lambda(d\gamma_k) \\
&= \int_{\Gamma_k\setminus\{\emptyset\}} \int_{S^\gamma} \exp\{-\beta[H_k^\Phi(\gamma_k) + E_{\gamma_k}(\sigma_k)]\}\cdot \\
&\times \left|1 - \exp\Big\{ -\beta\Big[\sum_{\substack{x\in\gamma_k \\ y\in\eta_j}} (\Phi(x,y) + J(x,y)W(\sigma_x,\xi_y))\Big]\Big\}\right| \otimes_{x\in\gamma_k} g(d\sigma_x)\lambda(d\gamma_k).
\end{aligned}$$

In the following, we give separate estimates for the factors in the above product.

From assumption $(LSSS)$ on the interaction potentials we know that

$$\exp\{-\beta H_k^\Phi(\gamma_k)\} \le \exp\{-\beta A_\Phi|\gamma_k|^P + \beta B_\Phi|\gamma_k|\}, \tag{5.59}$$

while Lemma 5.8 implies

$$\exp\{-\beta E_{\gamma_k}(\sigma_k)\} \le \exp\{\beta\|J\|_\infty C_1|\gamma_k|^{2+\varepsilon^{-1}} + \beta\|J\|_\infty C_3 \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)}\}. \tag{5.60}$$

For the last factor, we use a few computation tricks. Write

$$\Phi(x,y) = \Phi^+(x,y) - \Phi^-(x,y),$$

where $\Phi^+(x,y) = \max\{\Phi(x,y),0\}$ and $\Phi^-(x,y) = \max\{-\Phi(x,y),0\}$. In the same way,

$$W(\sigma_x,\xi_y) = W^+(\sigma_x,\xi_y) - W^-(\sigma_x,\xi_y),$$

where W^+ and W^- are, as above, the positive and negative parts of W. Applying the elementary inequality

$$|1-\exp\{a-\sum_{i=1}^{n}a_i\}|\leq(\exp\{a\}-1)+\sum_{i=1}^{n}|1-\exp\{-a_i\}|,$$

where $n\in\mathbb{N}$ and $a,a_i\geq 0, 1\leq i\leq n$, we obtain

$$\begin{aligned}
&\Big|1-\exp\Big\{-\beta\Big[\sum_{\substack{x\in\gamma_k\\ y\in\eta_j}}(\Phi(x,y)+J(x,y)W(\sigma_x,\xi_y))\Big]\Big\}\Big|\\
&\leq\Big(\exp\{\beta\sum_{\substack{x\in\gamma_k\\ y\in\eta_j}}[\Phi^-(x,y)+\|J\|_\infty W^-(\sigma_x,\xi_y)]\}-1\Big)\\
&+\sum_{\substack{x\in\gamma_k\\ y\in\eta_j}}\Big|1-\exp\Big\{-\beta(\Phi^+(x,y)+\|J\|_\infty(\sigma_x,\xi_y))\Big\}\Big|\\
&\leq(\exp\{\beta[MK_0|\gamma_k|+\|J\|_\infty K_0\sum_{x\in\gamma_k}|\sigma_x|^r+\|J\|_\infty K_0(K_0+C_W)|\gamma_k|]\}-1)+K_0|\gamma_k|,
\end{aligned}\tag{5.61}$$

where the last inequality is obtained knowing that $\Phi^-\leq M$ and using bounds on the boundary conditions, i.e. $|\eta_j^i|\leq K_0:=\theta^{-1}K_*$ and $\sum_{x\in\eta_j^i}|\xi_x^i|^r\leq K_0$.

We also know from assumption (SQG), that

$$\int_{S^\gamma}\otimes_{x\in\gamma_k}g(d\sigma_x)\leq\int_{S^\gamma}\exp\big\{-a_V\sum_{x\in\gamma_k}|\sigma_k|^{q_V}+b_V|\gamma_k|\big\}d\sigma_x.\tag{5.62}$$

Putting together relations (5.59)-(5.62) we get that

$$d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^1),\breve{\mu}_k(d\breve{\gamma}_k|\emptyset))\leq I_1+I_2,\tag{5.63}$$

where

$$\begin{aligned}
I_1:=&\int_{\Gamma_k\setminus\{\emptyset\}}\int_{S^\gamma}\exp\Big\{-\beta\Big[H_k^\Phi(\gamma_k)+E_{\gamma_k}(\sigma_k)\Big]-a_V\sum_{x\in\gamma_k}|\sigma_k|^{q_V}+b_V|\gamma_k|\Big\}\\
&\times\Bigg(\exp\Big\{\beta\Big[MK_0|\gamma_k|+\|J\|_\infty K_0\sum_{x\in\gamma_k}|\sigma_x|^r\\
&+\|J\|_\infty K_0(K_0+C_W)|\gamma_k|\Big]\Big\}-1\Bigg)\otimes_{x\in\gamma_k}d\sigma_x\lambda(d\gamma_k)
\end{aligned}\tag{5.64}$$

and

$$I_2 := \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} K_0 |\gamma_k| \exp\Big\{ -\beta [H_k^\Phi(\gamma_k) + E_{\gamma_k}(\sigma_k)] - a_V \sum_{x\in\gamma_k} |\sigma_k|^{q_V} + b_V |\gamma_k| \Big\} \otimes_{x\in\gamma_k} d\sigma_x \lambda(d\gamma_k). \tag{5.65}$$

We start by estimating I_2. We employ again relations (5.59) and (5.60).

$$I_2 \le \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} K_0 |\gamma_k| \exp\{-\beta A_\Phi |\gamma_k|^P + \beta B_\Phi |\gamma_k|\} \cdot \exp\{\beta \|J\|_\infty C_1 |\gamma_k|^{2+\varepsilon^{-1}} + \beta \|J\|_\infty C_3 \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)}\} \cdot \exp\{-a_V \sum_{x\in\gamma_k} |\sigma_k|^{q_V} + b_V |\gamma_k|\} \otimes_{x\in\gamma_k} d\sigma_x \lambda(d\gamma_k). \tag{5.66}$$

In what follows, we will use Young's inequality in the following form

$$xy \le \frac{x^a}{a\epsilon^a} + \frac{y^b \epsilon^b}{b}, \text{ for any } \epsilon > 0 \text{ and } a, b > 1 \text{ s.t. } \frac{1}{a} + \frac{1}{b} = 1. \tag{5.67}$$

Consider now only the factors depending on σ_x:

$$\exp\{\beta \|J\|_\infty C_3 \sum x \in \gamma_k |\sigma_x|^{r(1+\varepsilon)} - a_V \sum_{x\in\gamma_k} |\sigma_x|^{q_V}\}.$$

One can apply Young's inequality in the form (5.67) to obtain that

$$\beta \|J\|_\infty C_3 \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} \le cst + cst_1 \sum_{x\in\gamma_k} |\sigma_x|^{q_V}.$$

Choosing ϵ properly, one can make $cst_1 < a_V$. Then there exist constants D_1 and $\widetilde{D}_1$, with $\widetilde{D}_1 > 0$, such that

$$\exp\{\beta \|J\|_\infty C_3 \sum x \in \gamma_k |\sigma_x|^{r(1+\varepsilon)} - a_V \sum_{x\in\gamma_k} |\sigma_x|^{q_V}\} \le \exp\{\beta \|J\|_\infty |\gamma_k| D_1\} \cdot \exp\{-\widetilde{D}_1 \sum_{x\in\gamma_k} |\sigma_x|^{q_V}\}.$$

Take now the factors depending on $|\gamma_k|$, including the one obtained in the right hand-side of the inequality above:

$$\exp\{-\beta A_\Phi|\gamma_k|^P + |\gamma_k|(\beta B_\Phi + b_V + \beta\|J\|_\infty D_1) + \beta\|J\|_\infty c_1|\gamma_k|^{2+\varepsilon^{-1}}\}.$$

In the same way as above, one can apply twice Young's inequality with properly chosen ϵ, to obtain

$$\exp\{-\beta A_\Phi|\gamma_k|^P + |\gamma_k|(\beta B_\Phi + b_V + \beta\|J\|_\infty D_1) + \beta\|J\|_\infty c_1|\gamma_k|^{2+\varepsilon^{-1}}\} \le \exp D_2,$$

for some positive constant D_2. Hence

$$I_2 \le K_0 \exp\{D_2\} \int_{\Gamma_k\setminus\{\emptyset\}} |\gamma_k| \int_{S^\gamma} \exp\{-\widetilde{D}_1 \sum_{x\in\gamma_k} |\sigma_x|^{q_V}\} \otimes_{x\in\gamma_k} dx\lambda(d\gamma_k).$$

Since $\exp\{-\widetilde{D}_1 \sum_{x\in\gamma_k} |\sigma_x|^{q_V}\}$ is integrable with respect to the Lebesgue measure, there exists a constant D_3 such that

$$I_2 \le D_4 \int_{\Gamma_k\setminus\{\emptyset\}} |\gamma_k|\lambda(d\gamma_k) = D_4 \sum_{j=1}^{\infty} j \cdot \frac{z^j m(Q_k)^j}{j!} = zD_4 m(Q_k)e^{zm(Q_k)}. \qquad (5.68)$$

We proceed in a similar way to estimate I_1, but first notice that

$$\begin{aligned}
&\exp\{\beta[MK_0|\gamma_k| + \|J\|_\infty K_0 \sum_{x\in\gamma_k} |\sigma_x|^r + \|J\|_\infty K_0(K_0 + C_W)|\gamma_k|]\} - 1 \\
&\le \beta[MK_0|\gamma_k| + \|J\|_\infty K_0 \sum_{x\in\gamma_k} |\sigma_x|^r + \|J\|_\infty K_0(K_0 + C_W)|\gamma_k|]\cdot \\
&\times \exp\{\beta[MK_0|\gamma_k| + \|J\|_\infty K_0 \sum_{x\in\gamma_k} |\sigma_x|^r + \|J\|_\infty K_0(K_0 + C_W)|\gamma_k|]\} \\
&\le \|J\|_\infty \exp\{K_0 \sum_{x\in\gamma_k} |\sigma_x|^r\} \le \mathtt{N}_1 + \mathtt{N}_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathtt{N}_1 :=& \beta|\gamma_k|(MK_0 + \|J\|_\infty K_0(K_0 + C_W)) \exp\{\beta[MK_0|\gamma_k| \\
&+ \|J\|_\infty K_0 \sum_{x\in\gamma_k} |\sigma_x|^r + \|J\|_\infty K_0(K_0 + C_W)|\gamma_k|]\},
\end{aligned}$$

$$\mathtt{N}_2 := \beta\|J\|_\infty \exp\{\beta[MK_0|\gamma_k| + \|J\|_\infty K_0 \sum_{x\in\gamma_k} |\sigma_x|^r + \|J\|_\infty K_0(K_0 + C_W)|\gamma_k|]\}.$$

Therefore

$$I_1 \le \int_{\Gamma_k\setminus\{\emptyset\}} \int_{S^\gamma} \exp\{-\beta[H_k^\Phi(\gamma_k) + E_{\gamma_k}(\sigma_k)]\}\mathtt{N}_1 \otimes_{x\in\gamma_k} d\sigma_x \lambda(d\gamma_k)$$
$$+ \int_{\Gamma_k\setminus\{\emptyset\}} \int_{S^\gamma} \exp\{-\beta[H_k^\Phi(\gamma_k) + E_{\gamma_k}(\sigma_k)]\}\mathtt{N}_2 \otimes_{x\in\gamma_k} d\sigma_x \lambda(d\gamma_k)$$

Consider now the first integral, let us denote it by $\mathcal{I}_1$. By separating again the factors depending on σ_x, applying Young's inequality (5.67), then doing the same for the factors depending on $|\gamma_k|$ we get that there exists a positive constant D_5 such that

$$\mathcal{I}_1 \le zD_5 m(Q_k)e^{zm(Q_k)}. \tag{5.69}$$

We do the same for the second integral, denote it by $\mathcal{I}_2$, and obtain that there exists a constant D_6 such that

$$\mathcal{I}_2 \le D_6\|J\|_\infty \int_{\Gamma_k\setminus\{\emptyset\}} \lambda(d\gamma_k) = \|J\|_\infty D_6 m(Q_k)(e^{zm(Q_k)} - 1). \tag{5.70}$$

Putting together (5.63), (5.68), (5.69) and (5.70) yields

$$d_{TV}(\breve{\mu}_k(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\mu}_k(d\breve{\gamma}_k|\emptyset))$$
$$\le zD_4 m(Q_k)e^{zm(Q_k)} + zD_5 m(Q_k)e^{zm(Q_k)} + \|J\|_\infty D_6 m(Q_k)(e^{zm(Q_k)} - 1).$$

It is obvious that controlling the activity parameter z and the interaction $\|J\|_\infty$, we obtain $d_{TV}(\breve{\pi}_k(d\breve{\gamma}_k|\breve{\eta}^1), \breve{\pi}_k(d\breve{\gamma}_k|\emptyset))$ as small as we want, i.e. smaller than $\mathtt{k}$. Applying the triangle inequality, the result also holds for more general boundary conditions.

□

Proof of Theorem 5.22. It follows easily from Lemmas 5.23 and 5.24. □

Decay of correlations

We shortly remark that uniqueness of $\mu \in \mathcal{G}(\Gamma^t(X,S))$ yields a result for the decay of correlations, via Theorem 2.19. Let $\widehat{Q}_{\mathcal{K}_1}$ and $\widehat{Q}_{\mathcal{K}_2}$ be two disjoint cubic spacial domains and let G_1, G_2 be two local functions such that G_i is $\mathcal{B}_{\widehat{Q}_{\mathcal{K}_i}}(\Gamma(X,S))$-measurable, for $i = 1, 2$. Also, assume that

$$G_2(\widehat{\gamma}) \leq \theta \sum_{j \in \mathcal{K}_2} F(\widehat{\gamma}_j), \quad \widehat{\gamma} \in \Gamma(X,S)$$

and

$$\sup_{k \in \mathcal{K}_2} \int_{\Gamma(X,S)} G_1(\widehat{\gamma}) F(\widehat{\gamma}_k) \mu(d\widehat{\gamma}) < \infty,$$

where F is given by (5.17).

Corollary 5.25. *In the setting described above, there exist constants $\alpha, \tau > 0$ such that*

$$|Cov_\mu(G_1, G_2)| \leq \tau m(Q_{\mathcal{K}_2})^2 \exp\left(-\alpha d(\widehat{Q}_{\mathcal{K}_1}, \widehat{Q}_{\mathcal{K}_2})\right) \int_{\Gamma(X,S)} |G_1(\widehat{\gamma})| \tilde{F}(\widehat{\gamma}) \mu(d\gamma). \tag{5.71}$$

Moreover,

$$\alpha := -\log r_K, \tag{5.72}$$

where r_K is given by (2.68), for the Dobrushin-Pechersky matrix with entries given by Lemmas 5.23 and 5.24.

□

Chapter 6

Equilibrium States on the Cone of Discrete Measures

The aim of this chapter is to study the existence and uniqueness of equilibrium states for a class of interacting particle systems in the continuum $\mathbb{R}^d$, $d \in \mathbb{N}$, in which to each particle $x \in \mathbb{R}^d$, one attaches a positive characteristic (called mark or mass) s_x such that (s_x, x) is distributed according to some *generalized Lévy intensity measure* $\tau(ds, dx)$ on $(0, \infty) \times \mathbb{R}^d$ (see Definition 6.2). The microscopic states of this system are locally finite, positive discrete measures on the location space $\mathbb{R}^d$. The set of such measures form the convex cone $\mathbb{K}(\mathbb{R}^d)$ in the space $\mathbb{M}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$. The main difference in this setting, as compared to Chapter 5, is that the positions of the particles $x \in \mathbb{R}^d$, typically form a *dense* countable subset in $\mathbb{R}^d$, i.e. in each open $\mathsf{U} \subset \mathbb{R}^d$ there are a.s. *infinitely* many x_i's.

6.1 Description of the model

6.1.1 The cone of discrete measures

As a location space, we fix the d-dimensional Euclidean space $(\mathbb{R}^d, |\cdot|)$, endowed with the Lebesgue measure $m(dx)$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. By $\mathcal{B}_c(\mathbb{R}^d)$ we denote the ring of all bounded (i.e., those with compact closure) sets from $\mathcal{B}(\mathbb{R}^d)$. The continuous and compactly supported functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ form a locally convex vector space $C_0(\mathbb{R}^d)$, which is given a natural topology of uniform convergence on sets from $\mathcal{B}_c(\mathbb{R}^d)$. By the Riesz representation theorem, the dual space of $C_0(\mathbb{R}^d)$ can be identified with the space $\mathbb{M}(\mathbb{R}^d)$ of all *signed Radon* (i.e., locally finite) measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. By definition, each $\nu \in \mathbb{M}(\mathbb{R}^d)$ is finite on all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. The space $\mathbb{M}(\mathbb{R}^d)$ will be equipped with the *vague topology*, which is the coarsest topology making all maps

$$\mathbb{M}(\mathbb{R}^d) \ni \nu \mapsto \langle a, \nu \rangle := \int_{\mathbb{R}^d} a(x)\nu(dx), \quad a \in C_0(\mathbb{R}^d), \tag{6.1}$$

continuous. It is well known (see e.g. [Kal83, 15.7.7]) that $\mathbb{M}(\mathbb{R}^d)$ is *Polish*, i.e., there exists some separable and complete metric on $\mathbb{M}(\mathbb{R}^d)$ generating the vague topology. By $\mathcal{B}(\mathbb{M}(\mathbb{R}^d))$ we denote the corresponding Borel σ-algebra on $\mathbb{M}(\mathbb{R}^d)$; the one-point sets (e.g., $\{\nu = 0\}$) clearly belong to $\mathcal{B}(\mathbb{M}(\mathbb{R}^d))$. Let us abbreviate $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_+^* := (0, +\infty)$. By $C_0^+(\mathbb{R}^d)$ respectively $\mathbb{M}_+(\mathbb{R}^d)$ we denote the cone of all nonnegative functions $\varphi \in C_0(\mathbb{R}^d)$ resp. the dual cone of all nonnegative measures $\nu \in \mathbb{M}(\mathbb{R}^d)$.

The *cone of (nonnegative) discrete Radon measures* over $\mathbb{R}^d$ is defined as

$$\mathbb{K}(\mathbb{R}^d) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \middle| s_i \in \mathbb{R}_+^*,\, x_i \in \mathbb{R}^d \right\}. \tag{6.2}$$

Here, δ_{x_i} are Dirac measures, the atoms x_i are assumed to be distinct and their total number is at most countable. By convention, the cone $\mathbb{K}(\mathbb{R}^d)$ contains the null mass $\eta = 0$, which is represented by the sum over the empty set of indices i. We refer to each s_i as a *mark* and to each x_i as a *position*. This terminology is motivated by marked configuration spaces (see Chapter 5 and [KdSSU98]). However, the current setting does not fit in that framework because the set of all positions of an arbitrarily chosen $\eta \in \mathbb{K}(\mathbb{R}^d)$, i.e., its *support*

$$\mathfrak{S}(\eta) := \{x \in \mathbb{R}^d \,|\, 0 < \eta(\{x\}) =: s_x(\eta)\}, \tag{6.3}$$

is typically not a (locally finite) configuration in $\mathbb{R}^d$. Whenever it is clear which discrete measure $\eta \in \mathbb{K}(\mathbb{R}^d)$ is meant, we write for short just s_x instead of $s_x(\eta)$.

6.1.2 Measures on the cone $\mathbb{K}(\mathbb{R}^d)$

So far, the most studied measure on the cone was the Gamma measure. For a detailed description of its properties, see [TVY01]. Gibbs measures as perturbations of the Gamma measure were studied only in [HKPR13] and [Hag11]. Also, in [HKLV], dynamics associated to Gibbs measures on the cone $\mathbb{K}(\mathbb{R}^d)$ is considered. We aim to generalize the existence results and to give conditions under which the uniqueness of Gibbs measures holds for a more general class of random measures, containing the Gamma measure.

Definition 6.1. A Radon measure Λ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ satisfying

$$\Lambda(\mathbb{R}_+) = \infty \quad \text{and} \quad \int_{\mathbb{R}_+} s\Lambda(ds) < \infty$$

is called a Lévy measure on $\mathbb{R}_+$.

Let us denote by $m(dx)$ the Lebesgue measure on $\mathbb{R}^d$.

Definition 6.2. Let $\tau(ds, dx)$ be a measure on $\mathbb{R}_+^* \times \mathbb{R}^d$. We say τ is a *generalized Lévy intensity measure* if τ is absolutely continuous with respect to $\Lambda(ds)m(dx)$ (i.e. there exists a measurable function $p : (0, +\infty) \times \mathbb{R}^d \to \mathbb{R}_+$, such that $\tau(ds, dx) = p(x, s)\Lambda(ds)m(dx)$), where Λ is a Lévy measure on $\mathbb{R}_+$, and for any non-empty, open $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ the following conditions hold:

(i) $\int_{\mathbb{R}_+^*} \tau(ds, \mathsf{U}) = \infty$,

(ii) $q_i(x) := \int_{\mathbb{R}_+^*} s^i p(s, x)\Lambda(ds) \in L^1_{loc}(m)$, for $i = 1, 2$.

Let us introduce a special class of examples of measures on the cone $\mathbb{K}(\mathbb{R}^d)$, which will be considered further in our study.

Definition 6.3. Let $\tau(ds, dx)$ be a generalized Lévy intensity measure on $(0, \infty) \times \mathbb{R}^d$. We say $\mathscr{L}_\tau$ is a *generalized Lévy random measure* on $\mathbb{K}(\mathbb{R}^d)$ if its Laplace transform satisfies

$$\mathbb{E}_{\mathscr{L}_\tau}\Big[\exp\big(-\int_{\mathbb{R}^d} a(x)d\eta(x)\big)\Big] = \exp\Big(\int_{(0,\infty)\times\mathbb{R}^d} \big(e^{-sa(x)} - 1\big)\tau(ds, dx)\Big), \quad a \in C_0^+(\mathbb{R}^d). \tag{6.4}$$

In general, the Laplace transform (6.4) uniquely defines an infinitely divisible probability distribution on the cone of positive Radon measures on $\mathbb{R}^d$ (see e.g. Theorem 7.2 in [Kal83]). It is a non-trivial issue to show that the above μ will be supported by $\mathbb{K}(\mathbb{R}^d)$. Below we will give an explicit construction of μ clarifying this and its further properties.

Explicit construction of the generalized Lévy random measure

Similarly as it was done for the Gamma measure on $\mathbb{K}(\mathbb{R}^d)$ (see Section 6 of [HKPR13] or Section 2.2 of [Hag11]), one can give an explicit construction for $\mathscr{L}_\tau$ via the Poisson measure π_τ on the configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ with intensity measure $\tau(ds, dx)$. To this end, we introduce the set of *pinpointing configurations*

$$\Gamma_p(\mathbb{R}_+^* \times \mathbb{R}^d) := \{\gamma \in \Gamma(\mathbb{R}_+^* \times \mathbb{R}^d) : \forall (s_1, x_1), (s_2, x_2) \in \gamma \text{ one has } x_1 = x_2 \Rightarrow s_1 = s_2\}.$$

Proposition 6.4. *The Poisson measure π_τ is supported by $\Gamma_p(\mathbb{R}_+^* \times \mathbb{R}^d)$*

We refer to Proposition 2.2.4 of [Hag11] for a proof of this result in the case $\tau = \Lambda \otimes m$. In our case, due to the absolute continuity of τ with respect to $\Lambda \otimes m$, the proof follows the same way.

Definition 6.5. For each $\gamma \in \Gamma(\mathbb{R}_+ \times \mathbb{R}^d)$, we define its local mass on $\mathsf{U} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\mathfrak{m}_{\mathsf{U}}(\gamma) := \int_{\mathbb{R}_+ \times \mathsf{U}} s\gamma(ds, dx) \in [0, +\infty]. \tag{6.5}$$

In particular, for a pinpointing configuration $\gamma \in \Gamma_p(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\mathsf{U} \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathfrak{m}_{\mathsf{U}}(\gamma) = \sum_{x \in \mathfrak{S}(\gamma) \cap \mathsf{U}} s_x \in [0, \infty]. \tag{6.6}$$

Definition 6.6. The set of pinpointing configurations with finite local mass $\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d)$ is defined by

$$\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d) := \big\{\gamma \in \Gamma_p(\mathbb{R}_+ \times \mathbb{R}^d) : \forall \mathsf{U} \in \mathcal{B}_c(X),\ \mathfrak{m}_{\mathsf{U}}(\gamma) < \infty\big\}. \tag{6.7}$$

Remark 6.7. [Hag11, Remark 2.2.8] We note that the map

$$\Gamma(\mathbb{R}_+ \times \mathbb{R}^d) \ni \gamma \to \mathfrak{m}_{\mathsf{U}}(\gamma) < \infty \in \mathbb{R}$$

is $\mathcal{B}(\Gamma(\mathbb{R}_+ \times \mathbb{R}^d))$-measurable for all $\mathsf{U} \in \mathcal{B}(\mathbb{R}^d)$ and, furthermore, $\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d) \in \mathcal{B}(\Gamma(\mathbb{R}_+ \times \mathbb{R}^d))$.

Theorem 6.8. *The Poisson measure π_τ is supported by $\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d)$*

Proof. Fix $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$. Then

$$\begin{aligned}\int_{\Gamma(\mathbb{R}_+ \times \mathbb{R}^d)} \mathfrak{m}_{\mathsf{U}}(\gamma)\pi_\tau(d\gamma) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} s\mathbb{1}_{\mathsf{U}}(x)\tau(ds, dx) \\ = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} s\mathbb{1}_{\mathsf{U}}(x)p(s,x)\Lambda(ds)m(dx) &= \int_{\mathsf{U}} q_1(x)m(dx) < \infty.\end{aligned} \tag{6.8}$$

The last integral in (6.7) is finite due to the assumption about the local integrability of q_1 with respect to $m(dx)$. Thus, for any $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$

$$\mathfrak{m}_{\mathsf{U}}(\gamma) < \infty, \qquad \text{for } \gamma \in \Gamma(\mathbb{R}_+ \times \mathbb{R}^d)\ (\pi_\tau - \text{a.e.}).$$

□

We equip $\mathbb{K}(\mathbb{R}^d)$ with the strongest topology $\mathcal{O}(\mathbb{K}(\mathbb{R}^d))$ such that the following bijective map is continuous

$$\begin{aligned} T : &\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d) &\to &\mathbb{K}(\mathbb{R}^d) \\ &\widehat{\gamma} = \{(s_x, x)\} &\mapsto &\eta := \sum_{(s_x,x)\in\widehat{\gamma}} s_x \delta_x. \end{aligned} \tag{6.9}$$

We denote by $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ the corresponding Borel σ-algebra.

Remark 6.9. We note that $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$ is a *standard Borel space*. However, it is an open problem whether one can introduce a metric on $\mathbb{K}(\mathbb{R}^d)$ making it a Polish space and being compatible with the vague topology inherited from $\mathbb{M}(\mathbb{R}^d)$ (see Remark 2.1 in [HKPR13]).

By $\mathcal{P}(\mathbb{M}(\mathbb{R}^d))$ and $\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$, we denote the space of all probability measures on $\mathbb{M}(\mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$, respectively. Such measures are sometimes referred to in the literature as *random measures* (see [Kal83]).

Theorem 6.10. *Let τ be a generalized Lévy intensity measure on $\mathbb{R}_+^* \times \mathbb{R}^d$. Then, there exists a corresponding generalized Lévy random measure $\mathscr{L}_\tau$ on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$ which has τ as intensity measure.*

Proof. Since the map defined by (6.9) is bijective, we can consider the image of π_τ under T:

$$\mathscr{L}_\tau := T_* \pi_\tau. \tag{6.10}$$

This is equivalent to

$$\int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) \mathscr{L}_\tau(d\eta) = \int_{\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d)} F\big(T(\gamma)\big) \pi_\tau(d\gamma), \tag{6.11}$$

for all bounded measurable $F : \mathbb{K}(\mathbb{R}^d)\mathbb{R}$.

Obviously, since π_τ is a probability measure on $\Gamma_f(\mathbb{R}_+ \times \mathbb{R}^d)$, then $\mathscr{L}_\tau$ will be a probability measure on $\mathbb{K}(\mathbb{R}^d)$. Moreover, using the Laplace transform formula (4.2) for the Poisson measure π_τ, one can easily check that the measure $\mathscr{L}_\tau$ we have constructed has indeed the Laplace transform given by (6.4).

□

Remark 6.11. (a) For $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$, consider the cone $\mathbb{K}(\Delta) \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$, consisting of those discrete measures $\eta \in \mathbb{K}(\mathbb{R}^d)$ which are supported by U. Introduce the canonical projection

$$p_{\mathsf{U}} : \mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto \eta_{\mathsf{U}} := \sum_{x \in \mathfrak{S}(\eta) \cap \mathsf{U}} s_x \delta_x \in \mathbb{K}(\mathsf{U}). \tag{6.12}$$

We also consider the measure $\mathscr{L}_{\mathsf{U},\tau} := \mathscr{L}_\tau \circ p_{\mathsf{U}}^{-1}$, which has full support on $\mathbb{K}(\mathsf{U})$.

(b) For each non-empty, open set $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ such that $\tau(\mathbb{R}_+^\infty, \mathsf{U}) > 0$, by taking into account the constructive definition of L_τ and property (i) of τ, we have

$$\int_{\mathbb{K}(\mathbb{R}^d)} |\mathfrak{S}(\eta) \cap \mathsf{U}| \mathscr{L}_\tau(d\eta) = \mathbb{E}_{\pi_\tau} \Big| \gamma \cap \mathbb{R}_+ \times \mathsf{U} \Big| = \int_{\mathbb{R}_+ \times \mathsf{U}} \tau(ds, dx) = +\infty.$$

Also, in view of [AB81] and [Ken00], the support

$$\mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto \mathfrak{S}(\eta) \in \mathcal{B}(\mathbb{R}^d)$$

can be seen as a *countable dense random set* in $\mathbb{R}^d$.

(c) We remark that the random measure $\mathscr{L}_\tau$ has *independent increments* (or the *locality property*), in the sense that $\eta(\mathsf{U}_1), \ldots, \eta(\mathsf{U}_N)$ are independent for any $N \in \mathbb{N}$ and disjoint $\mathsf{U}_1, \ldots, \mathsf{U}_N \in \mathcal{B}_c(\mathbb{R}^d)$. That is,

$$\int_{\mathbb{K}(\mathbb{R}^d)} \prod_{i=1}^N \varphi_i(\eta(\mathsf{U}_i)) \mathscr{L}_\tau(d\eta) = \prod_{i=1}^N \int_{\mathbb{K}(\mathbb{R}^d)} \varphi_i(\eta(\mathsf{U}_i)) \mathscr{L}_\tau(d\eta) \tag{6.13}$$

for any collection of $\varphi_i \in L^\infty(\mathbb{R})$, $1 \le i \le N$.

This property is essential in constructing Gibbs perturbations of generalized Lévy random measures $\mathscr{L}_\tau$. Its proof follows from property (ii) in the definition of τ and Theorem 7.2 in [Kal83].

(d) The most studied case in the literature is the case of $\tau(ds, dx) = \frac{e^{-s}}{s} ds \otimes dm(x)$. Then $\mathscr{L}_\tau$ is the well-known Gamma measures, whose properties were thoroughly discussed in [TVY01].

Moments of the $\mathscr{L}_\tau$ measure

Definition 6.12. Let μ be a measure on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$, $a : \mathbb{R}^d \to \mathbb{R}$ be a bounded, compactly supported Borel function and $n \in \mathbb{N}$. Then

$$\mathbb{E}_\mu[\langle a, \cdot \rangle^n]$$

is called the n^{th} moment of μ.

Lemma 6.13. *(see [GR00, eq. 0.430 2]) For $n \in \mathbb{N}_0$, $f, g \in C^n(\mathbb{R})$*

$$\frac{d^n}{dt^n}(g \circ f(t)) = \sum_{\substack{\sum l i_l = n \\ i_1, \cdots, i_k \in \mathbb{N}_0}} \frac{n!}{i_1! \cdots i_k!} \left(\frac{f^{(1)}(t)}{1!} \right)^{i_1} \cdots \left(\frac{f^{(k)}(t)}{k!} \right)^{i_k} \left(\frac{d^n}{ds^n} g \right) \circ f(t).$$

Theorem 6.14. *Let $a : \mathbb{R}^d \to \mathbb{R}$ be a bounded, compactly supported Borel function. Then for all $n \in \mathbb{N}$*

$$\mathbb{E}_{\mathscr{L}_\tau}[\langle a, \cdot\rangle^n] = \sum_{\substack{\sum l i_l = n \\ i_1, \cdots, i_k \in \mathbb{N}_0}} \frac{n!}{i_1! \cdots i_k!} \left(\frac{-\int sa(x)\tau(ds, dx)}{1!} \right)^{i_1} \cdots \left(\frac{(-1)^k \int s^k a^k(x)\tau(ds, dx)}{k!} \right)^{i_k} \tag{6.14}$$

Proof. The result follows easily, by derivation of the Laplace transform of ta w.r.t. $t \in \mathbb{R}$ and evaluating it at 0. For sufficiently small t we have no integrability problems and

$$\begin{aligned} \mathbb{E}_{\mathscr{L}_\tau}[\langle a, \cdot\rangle^n] &= \left(\frac{d}{dt} \right)^n \mathbb{E}_{\mathscr{L}_\tau}[t\langle a, \cdot\rangle^n] \Big|_{t=0} \\ &= \left(\frac{d}{dt} \right)^n \exp\left\{ \int (e^{-tsa(x)} - 1)\tau(ds, dx) \right\} \Big|_{t=0}. \end{aligned}$$

Applying Lemma 6.13 and then multiplying by $(-1)^n$, we obtain the desired formula. □

Remark 6.15. In particular, for $a = \mathbb{1}_{\mathsf{U}}$, with $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$, we have that

$$\mathbb{E}_{\mathscr{L}_\tau}[\langle \mathbb{1}_{\mathsf{U}}, \cdot\rangle^i] = \int s^i \tau(ds, \mathsf{U}) = \int_{\mathsf{U}} q_i(x)dx < \infty, \text{ for } i = 1, 2. \tag{6.15}$$

6.2 Gibbsian formalism

6.2.1 Specifications and the corresponding Gibbs measures

Fixing a proper pair potential, we introduce the notion of related Gibbs measures via a local Gibbs specification. As in the previous chapters, we proceed in the spirit of the Dobrushin-Lanford-Ruelle (***DLR***) approach to Gibbs states in statistical physics).

In what follows, we adopt the basic notations and definitions from [HKPR13].

Assumption (ϕ) Consider a symmetric pair potential

$$\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \tag{6.16}$$

to be a *bounded* and $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$-measurable function. Denote

$$0 \le \|\phi^-\|_\infty := \sup_{x,y\in\mathbb{R}^d} -\phi(x,y)\vee 0 \le \sup_{x,y\in\mathbb{R}^d} |\phi(x,y)| =: \|\phi\|_\infty < \infty.$$

Let ϕ be such that the following conditions hold:

(**FR**) ***Finite range:*** *There exists $R \in (0,\infty)$ such that*

$$\phi(x,y) = 0, \quad \text{if } |x-y| > R.$$

(**RC**) ***Repulsion condition****: There exists $\delta > 0$ such that*

$$A_\delta := \inf_{\substack{x,y\in\mathbb{R}^d \\ |x-y|\le\delta}} \phi(x,y) > 2m_\delta^\phi \|\phi^-\|_\infty, \tag{6.17}$$

with interaction parameter (cf. (6.21) below)

$$m_\delta^\phi := \nu_d d^{d/2}\left[R/\delta + 1\right]^d, \tag{6.18}$$

where $\nu_d := \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of a unit ball in $\mathbb{R}^d$.

Intuitively speaking, relation (6.17) translates by the fact that the repulsion part $\phi^+ := \phi \vee 0$ of ϕ dominates the attraction part $\phi^- := -\phi \vee 0$. We emphasise that neither *translation invariance*, nor *continuity* of ϕ need to be assumed.

Let $\delta > 0$ be such that the repulsion condition (**RC**) holds and define the parameter $g := \delta/\sqrt{d}$. Consider the cubes indexed by $k \in \mathbb{Z}^d$

$$Q_k := \big[-1/2g, 1/2g\big)^d + gk \subset \mathbb{R}^d \tag{6.19}$$

constituting a partition of $\mathbb{R}^d$. Each cube Q_k is centred at the point gk and has edge length $g > 0$, Lebesgue volume $m(Q_k) = g^d$ and diameter

$$\operatorname{diam}(Q_k) := \sup_{x,y\in Q_k} |x-y|_{\mathbb{R}^d} = \delta.$$

The latter implies that $\phi(x,y) \ge A_\delta$ for all $x,y \in Q_k$. To explain the choice of the constant m_δ^ϕ in (6.18), we introduce some more concepts and notation. For each $k \in \mathbb{Z}^d$, the family of «*neighbor*» cubes of Q_k (i.e., those Q_j, $j \ne k$, having a point $y \in Q_j$ that interacts with a point $x \in Q_k$) is indexed by

$$\partial_\delta^\phi k := \left\{ j \in \mathbb{Z}^d \setminus \{k\} \mid \exists x \in Q_k, \exists y \in Q_j : \quad \phi(x,y) \ne 0 \right\}. \tag{6.20}$$

The number of such «*neighbor*» cubes for every Q_k, $k \in \mathbb{Z}^d$, can be roughly estimated by

$$\sup_{k\in\mathbb{Z}^d} |\partial_\delta^\phi k| \le m_\delta^\phi, \tag{6.21}$$

where m_δ^ϕ was defined in (6.18).

To each index set $\mathcal{K} \Subset \mathbb{Z}^d$ (this notation means that $\mathcal{K}$ is a non-void *finite* subset of $\mathbb{Z}^d$) there corresponds

$$\mathsf{U}_{\mathcal{K}} := \bigsqcup_{k\in\mathcal{K}} Q_k \in \mathcal{B}(\mathbb{R}^d); \tag{6.22}$$

the family of all such domains is denoted by $\mathcal{Q}_c(\mathbb{R}^d)$. Respectively, for $\mathsf{U} \in \mathcal{B}(\mathbb{R}^d)$ we define

$$\mathcal{K}_{\mathsf{U}} := \{j \in \mathbb{Z}^d \mid Q_j \cap \mathsf{U} \neq \varnothing\}; \tag{6.23}$$

then $|\mathcal{K}_{\mathsf{U}}|$ is the number of cubes Q_k having non-void intersection with U. Note that

$$|\mathcal{K}_{\mathsf{U}}| < \infty, \qquad \forall \mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d).$$

Remark 6.16. Similar conditions on the potential ϕ were imposed in [HKPR13]. However, that paper deals with the Gibbsian modifications of the canonical Gamma measure on the cone $\mathbb{K}(\mathbb{R}^d)$.

6.2.2 Local Gibbs specification

For each $\eta = \sum_{x\in\tau(\eta)} s_x\delta_x$, $\xi = \sum_{y\in\tau(\xi)} s_y\delta_y \in \mathbb{K}(\mathbb{R}^d)$ and $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$, we define the *relative energy (Hamiltonian)*

$$H_{\mathsf{U}}(\eta|\xi) := \int_{\mathsf{U}}\int_{\mathsf{U}} \phi(x,y)\eta(dx)\eta(dy) + 2\int_{\mathsf{U}^c}\int_{\mathsf{U}} \phi(x,y)\eta(dx)\xi(dy). \tag{6.24}$$

In the particle picture, the Hamiltonian can be written as

$$H_{\mathsf{U}}(\eta|\xi) = \sum_{x,x'\in\tau(\eta)\cap\mathsf{U}} \phi(x,x')s_x s_{x'} + 2\sum_{\substack{x\in\tau(\eta)\cap\mathsf{U}\\ y\in\tau(\xi)\cap\mathsf{U}^c}} \phi(x,y)s_x s_y.$$

Lemma 6.17. *The relative energy is finite, i.e.,*

$$|H_{\mathsf{U}}(\eta|\xi)| < \infty, \quad \text{for all } \eta,\xi \in \mathbb{K}(\mathbb{R}^d) \text{ and } \mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d).$$

Lemma 6.18. *Let Assumption (ϕ) hold. Then for each $\eta, \xi \in \mathbb{K}(\mathbb{R}^d)$ and $\mathsf{U} \in \mathcal{B}_c(\mathbb{K}(\mathbb{R}^d))$*

$$H_{\mathsf{U}}(\eta|\xi) \geq \left[A - 2m\|\phi^-\|_\infty\right] \sum_{j\in\mathcal{K}_{\mathsf{U}}} \eta_{\mathsf{U}}(Q_j)^2 - m\|\phi^-\|_\infty \sum_{l\in\mathcal{K}_{\mathcal{U}_{\mathsf{U}}}} \xi_{\mathsf{U}^c}(Q_l)^2. \tag{6.25}$$

More precisely, we have for each $k \in \mathbb{Z}^d$

$$H_{Q_k}(\eta|\xi) \geq \left[A - m\|\phi^-\|_\infty\right] \eta(Q_k)^2 - \|\phi^-\|_\infty \sum_{j\in\partial k} \xi_{Q_k^c}(Q_j)^2 \tag{6.26}$$

and, choosing $\xi = 0$,

$$H_{Q_k}(\eta_k) := H_{Q_k}(\eta_k|0) \geq \left[A - 2m\|\phi^-\|_\infty\right] \eta(Q_k)^2 \tag{6.27}$$

The proofs of Lemmas 6.17 and 6.18 follow by elementary computations and can be found in [HKPR13].

For each $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$, $\xi \in \mathbb{K}(\mathbb{R}^d)$ and $\beta > 0$, we define the partition function

$$Z_{\mathsf{U}}^{\beta}(\xi) := \int_{\mathbb{K}(\mathsf{U})} \exp\{-\beta H_{\mathsf{U}}(\eta_{\mathsf{U}}|\xi)\}\mathscr{L}_{\mathsf{U},\tau}(d\eta_{\mathsf{U}}).$$

Lemma 6.19. *Let Assumption* (ϕ) *hold. For any* $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$, $\xi \in \mathbb{K}(\mathbb{R}^d)$ *and* $\beta > 0$

$$0 < Z_{\mathsf{U}}^{\beta}(\xi) < \infty.$$

If $\phi \geq 0$, *then obviously* $Z_{\mathsf{U}}^{\beta}(\xi) \leq 1$.

Proof. For each $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi, \eta \in \mathbb{K}(\mathbb{R}^d)$, we define

$$H_{\mathsf{U}}^{+}(\eta|\xi) := \int_{\mathsf{U}}\int_{\mathsf{U}} \phi^+(x,y)\eta(dx)\eta(dy) + 2\int_{\mathsf{U}^c}\int_{\mathsf{U}} \phi^+(x,y)\eta(dx)\xi(dy).$$

Jensen's inequality yields

$$\begin{aligned} Z_{\mathsf{U}}(\xi) &\geq \int_{\mathbb{K}(\mathsf{U})} \exp\{-\beta H_{\mathsf{U}}^{+}(\eta|\xi)\}\mathscr{L}_{\mathsf{U},\tau}(d\eta) \geq \exp\left\{-\int_{\mathbb{K}(\mathsf{U})} \beta H_{\mathsf{U}}^{+}(\eta|\xi)\mathscr{L}_{\mathsf{U},\tau}(d\eta)\right\} \\ &\geq \exp\left\{-\|\phi\|_\infty \beta \int_{\mathbb{K}(\mathsf{U})} \left[\eta(\mathsf{U})^2 + 2\eta(\mathsf{U})\xi_{\mathsf{U}^c}(\mathcal{U}_{\mathsf{U}})\right]\mathscr{L}_{\mathsf{U},\tau}(d\eta)\right\}. \end{aligned}$$

By (6.15), we get that

$$Z_{\mathsf{U}}(\xi) \geq \exp\left\{-\|\phi\|_\infty \beta \left[\mathfrak{m}_2(\mathsf{U}) + 2\xi_{\mathsf{U}^c}(\mathcal{U}_{\mathsf{U}})\mathfrak{m}_1(\mathsf{U})\right]\right\} > 0, \tag{6.28}$$

where $\mathfrak{m}_1(\mathsf{U}) := \int s\tau(ds, \mathsf{U})$ and $\mathfrak{m}_2(\mathsf{U}) := \int s^2\tau(ds, \mathsf{U})$.

Using the lower bound on the Hamiltonian (6.25) we deduce

$$Z_{\mathsf{U}}(\xi) \le \int_{\mathbb{K}(\mathbb{R}^d)} \exp\left\{-[A-2m\|\phi^-\|_\infty]\sum_{j\in\mathcal{K}_{\mathsf{U}}}\eta_{\mathsf{U}}(Q_j)^2\right\}\mathscr{L}_{\mathsf{U},\tau}(d\eta_{\mathsf{U}})$$
$$\times\exp\left\{m\|\phi^-\|_\infty\sum_{l\in\mathcal{K}_{\mathcal{U}_{\mathsf{U}}}}\xi_{\mathsf{U}^c}(Q_l)^2\right\}<\infty,$$

where the last inequality is obtained again using (6.15) and knowing that the first two moments of τ are finite.

□

For each $\mathsf{U}\in\mathcal{B}_c(\mathbb{R}^d)$ and $\beta>0$, the *local Gibbs measures* with boundary conditions $\xi\in\mathbb{K}(\mathbb{R}^d)$ are given by

$$\mu_{\mathsf{U}}(d\eta|\xi) := \frac{1}{Z_{\mathsf{U}}(\xi)}e^{-\beta H_{\mathsf{U}}(\eta|\xi)}\mathscr{L}_{\mathsf{U},\tau}(d\eta).$$

Lemma 6.19 guarantees that each $\mu_{\mathsf{U}}(d\eta|\xi)$ is well-defined as a probability measure on $\mathbb{K}(\mathsf{U})$.

Definition 6.20. The *local specification* $\mathit{\Pi}=\{\pi_{\mathsf{U}}\}_{\mathsf{U}\in\mathcal{B}_c(\mathbb{R}^d)}$ on $\mathbb{K}(\mathbb{R}^d)$ is a family of stochastic kernels

$$\mathcal{B}(\mathbb{K}(\mathbb{R}^d))\times\mathbb{K}(\mathbb{R}^d)\ni(B,\xi)\mapsto\pi_{\mathsf{U}}(B|\xi)\in[0,1] \tag{6.29}$$

given by $\pi_{\mathsf{U}}(B|\xi):=\mu_{\mathsf{U}}(B_{\mathsf{U},\xi}|\xi)$, where

$$B_{\mathsf{U},\xi}:=\{\eta_{\mathsf{U}}\in K(\mathsf{U})\,|\,\eta_{\mathsf{U}}+\xi_{\mathsf{U}^c}\in B\}\in\mathcal{B}(K(\mathsf{U})).$$

Remark 6.21. The family (6.29) obeys the *consistency* (or *Markovian*) *property*, which means that for all $\mathsf{U},\widetilde{\mathsf{U}}\in\mathcal{B}_c(\mathbb{R}^d)$ with $\widetilde{\mathsf{U}}\subseteq\mathsf{U}$

$$\int_{\mathbb{K}(\mathbb{R}^d)}\pi_{\widetilde{\mathsf{U}}}(B|\eta)\pi_{\mathsf{U}}(\mathrm{d}\eta|\xi)=\pi_{\mathsf{U}}(B|\xi), \tag{6.30}$$

for all $B\in\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ and $\xi\in\mathbb{K}(\mathbb{R}^d)$. By the additive structure of the relative energy (cf. Eq. (6.24)) and the independency property $\mathscr{L}_\tau$ (6.13), this property immediately follows by the construction of the family Π (cf. [Pre76, Proposition 6.3] or [Pre05, Proposition 2.6]).

Definition 6.22. A probability measure μ on $\mathbb{K}(\mathbb{R}^d)$ is called a *Gibbs measure* (or *state*) with pair potential ϕ if it satisfies the *Dobrushin-Lanford-Ruelle* (*DLR*) equilibrium equation

$$\int_{\mathbb{K}(\mathbb{R}^d)}\pi_{\mathsf{U}}(B|\eta)\mu(\mathrm{d}\eta)=\mu(B) \tag{6.31}$$

for all $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$. The associated set of all Gibbs states will be denoted by $\mathcal{G}(\mathbb{K}(\mathbb{R}^d)$.

We will mainly be interested in the subset $\mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$ of *tempered* Gibbs measures which are supported by the set of tempered discrete Radon measures $\mathbb{K}^t(\mathbb{R}^d)$, which will be defined depending on the properties of $\mathscr{L}_\tau$, by (6.33) and (6.71), respectively.

6.3 Spatially bounded Lévy intensity measure

In this section we assume that the first two "spatial" moments of τ are uniformly bounded, i.e. there exists $M > 0$ such that

$$\int s^i \tau(ds, Q_k) \leq M < \infty, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d. \tag{6.32}$$

6.3.1 Exponential moment estimate

We deal with the following class of *tempered* discrete Radon measures

$$\mathbb{K}^t(\mathbb{R}^d) := \bigcap_{\alpha > 0} \mathbb{K}_\alpha(\mathbb{R}^d), \tag{6.33}$$

where

$$\mathbb{K}_\alpha(\mathbb{R}^d) := \left\{ \eta \in \mathbb{K}(\mathbb{R}^d) : M_\alpha(\eta) := \left(\sum_{z \in \mathbb{Z}^d} \eta(Q_k)^2 e^{-\alpha|k|} \right)^{1/2} < \infty \right\}. \tag{6.34}$$

Lemma 6.23. *For $k \in \mathbb{Z}^d$, $\xi \in \mathbb{K}(\mathbb{R}^d)$, $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ and $a \in [0, \beta(A - 2m\|\phi^-\|_\infty)]$, the following holds true*

$$\int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\} \pi_{\mathsf{U}}(d\eta|\xi) \leq \exp\left\{ \beta \left[\Upsilon_{\mathsf{U},\varepsilon} + (\frac{\varepsilon}{2}\|\phi\|_\infty + m\|\phi^-\|) \sum_{j \in \mathcal{K}_{\mathcal{U}_{\mathsf{U}}}} \xi(Q_j)^2 \right] \right\}, \tag{6.35}$$

where $\varepsilon > 0$ is arbitrary and

$$\Upsilon_{\mathsf{U},\varepsilon} := \|\phi\|_\infty(\mathfrak{m}_2(\mathsf{U}) + \frac{1}{2\varepsilon}|\mathcal{K}_{\mathcal{U}_{\mathsf{U}}}| m_1^2(\mathsf{U})) < \infty.$$

Proof. Using the lower bound on the conditional Hamiltonian, we have that

$$\begin{aligned}
&\frac{1}{Z_{\mathsf{U}}(\xi)}\int_{\mathbb{K}(\mathbb{R}^d)} \exp\left\{a\eta(Q_k)^2 - \beta H_{\mathsf{U}}(\eta_{\mathsf{U}}|\xi)\right\} \mathscr{L}_{\mathsf{U},\tau}(d\eta_{\mathsf{U}}) \\
&\quad\le \frac{1}{Z_{\mathsf{U}}(\xi)}\int_{\mathbb{K}(\mathsf{U})} \Big\{[\beta A - 2\beta m\|\phi^-\|_\infty - a]\eta_{\mathsf{U}}(Q_k)^2 \\
&\quad - \beta[A - 2m\|\phi^-\|_\infty] \sum_{j\in\mathcal{K}_{\mathsf{U}}, j\neq k} \eta_{\mathsf{U}}(Q_j)^2\Big\}\mathscr{L}_{\mathsf{U},\tau}(d\eta_{\mathsf{U}}) \\
&\quad \times \exp\left\{\beta m\|\phi^-\|_\infty \sum_{l\in\mathcal{K}_{\mathcal{U}_{\mathsf{U}}}} \xi_{\mathsf{U}^c}(Q_l)^2\right\}.
\end{aligned}$$

From (6.28), $a \in [0, \beta(A - 2m\|\phi^-\|_\infty)]$ and Young's inequality we get our conclusion.

□

From here on, it is easy to deduce the following bound in elementary cubes Q_k, $k \in \mathbb{Z}^d$, which represents the weak dependence on boundary conditions.

Lemma 6.24. *For $k \in \mathbb{Z}^d$, $\xi \in \mathbb{K}(\mathbb{R}^d)$, $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ and $a \in [0, \beta(A - m\|\phi^-\|_\infty)]$, the following holds true*

$$\int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_k(d\eta|\xi) \le \exp\left\{\beta\left[\Upsilon_\varepsilon + (\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty \mathfrak{m}_1(Q_k)) \sum_{j\in\partial k} \xi(Q_j)^2\right]\right\}, \tag{6.36}$$

where $\varepsilon > 0$ is arbitrary and

$$\Upsilon_\varepsilon := \|\phi\|_\infty M(1 + m/\varepsilon) < \infty.$$

Proof. The claim follows immediately from the lower bound of the conditional Hamiltonian and the proof of Lemma 6.23.

□

Note that by applying Jensen's inequality to both sides of (6.36), one gets a Dobrushin-type estimate, for strictly positive $a \in [0, \beta(A - 2m\|\phi^-\|_\infty)]$

$$\int_{\mathbb{K}(\mathbb{R}^d)} \eta(Q_k)^2\pi_k(d\eta|\xi) \le \frac{1}{a}\left\{\Upsilon_\varepsilon + \left(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M\right) \sum_{j\in\partial k} \xi(Q_j)^2\right\}. \tag{6.37}$$

Consider now arbitrary large domains $\mathsf{U}_{\mathcal{K}} = \bigsqcup_{k\in\mathcal{K}} Q_k$ indexed by $\mathcal{K} \Subset \mathbb{Z}^d$. Note that $\mathsf{U}_{\mathcal{K}} \nearrow \mathbb{R}^d$ as $\mathcal{K} \nearrow \mathbb{Z}^d$.

Proposition 6.25. *Let $0 \le a \le \beta(A - m\|\phi^-\|_\infty)$. Then there exists $C_a < \infty$ such that for all $k \in \mathbb{Z}^d$ and $\xi \in \mathbb{K}^t(\mathbb{R}^d)$*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_{\mathcal{K}}(d\eta|\xi) \le C_a. \tag{6.38}$$

Proof. Without loss of generality, we assume that $a \in \big(\beta m\|\phi^-\|_\infty, \beta(A - m\|\phi^-\|_\infty)\big]$. We define

$$0 \le n_k(\mathcal{K}|\xi) := \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_{\mathcal{K}}(d\eta|\xi)\right\},\ k \in \mathbb{Z}^d, \tag{6.39}$$

which are finite by Lemma 6.23.

Integrating (6.36) with respect to $\pi_{\mathcal{K}}(d\eta|\xi)$, we get for every $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) &\le \beta\Upsilon_\varepsilon + \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp\left[\beta(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M)\sum_{j\in\partial k}\eta(Q_j)^2\right]\pi_{\mathcal{K}}(d\eta|\xi)\right\} \\ &= \Upsilon_\varepsilon + \left[\beta\Big(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M\Big)\sum_{j\in\mathcal{K}^c\cap\partial k}\xi(Q_j)^2\right] \\ &+ \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp\left[\beta(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M)\sum_{j\in\mathcal{K}\cap\partial k}\eta(Q_j)^2\right]\pi_{\mathcal{K}}(d\eta|\xi)\right\}. \end{aligned} \tag{6.40}$$

For shorthand, we denote $0 < B_\varepsilon := \beta m(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M)$ and choose appropriate $\delta, \varepsilon > 0$ such that $B_\varepsilon < \delta a < a \le a_0 := \beta(A - m\|\phi^-\|_\infty)$. For the logarithmic term in the last inequality of (6.40) we can apply the multiple Hölder inequality to deduce that it is dominated by

$$\sum_{j\in\mathcal{K}\cap\partial l} \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp(a\eta(Q_j)^2)\pi_{\mathcal{K}}(d\eta|\xi)\right\}^{1/a\left(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty \mathfrak{m}_1(Q_k)\right)} \tag{6.41}$$

$$= 1/a(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M)\sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K}|\xi). \tag{6.42}$$

Let $\mathcal{K} \Subset \mathbb{Z}^d$ contain a fixed point $k_0 \in \mathbb{Z}^d$. Let $\vartheta := R/g + \sqrt{d}$ be such that $|j - k_0| \le \vartheta$ for all $j \in \partial^\phi k_0$. Let us pick an $\alpha > 0$ small enough, so that

$B_\varepsilon e^{\alpha\vartheta} < a$. We plug (6.41) in (6.40) and then multiply the obtained inequality by $\exp\{-\alpha|k|\}$, which yields

$$n_k(\mathcal{K}|\xi)e^{-\alpha|k|} \leq \beta\Upsilon_\varepsilon e^{-\alpha|k|} + B_\varepsilon e^{\alpha\vartheta} M_\alpha^2(\xi_{\mathcal{K}^c}) + \frac{B_\varepsilon}{a} e^{\alpha\vartheta} \sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K}|\xi)e^{-\alpha|j|}.$$

Taking supremum after indices in $\mathcal{K}$ we get that

$$n_k(\mathcal{K}|\xi)e^{-\alpha|k|} \leq [\beta\Upsilon_\varepsilon + B_\varepsilon e^{\alpha\vartheta} M_\alpha^2(\xi_{\mathcal{K}^c})] \left(\frac{1}{1-\delta e^{\alpha\vartheta}}\right).$$

We let $\mathcal{K} \nearrow \mathbb{Z}^d$

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} [n_k(\mathcal{K}|\xi)\exp\{-\alpha|k|\}] \leq \Upsilon_\varepsilon \left(\frac{1}{1-\delta e^{\alpha\vartheta}}\right),$$

and hence, by letting $\alpha \searrow 0$ we get

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_k(\mathcal{K}|\xi) \leq \frac{1}{1-\delta}\Upsilon_\varepsilon =: \log C_a.$$

From here we have (6.38). □

An important corollary of the above proposition claims the uniform bound for local Gibbs states.

Corollary 6.26. *Let Assumption (ϕ) hold. Then for all $\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$ there exists $\mathtt{C}(\mathsf{U}, N) < \infty$ such that*

$$\limsup_{\substack{\mathsf{W}\nearrow\mathbb{R}^d \\ \mathsf{W}\in\mathcal{Q}_c(\mathbb{R}^d)}} \int_{\mathbb{K}(\mathbb{R}^d)} \eta(\mathsf{U})^N \pi_{\mathsf{W}}(d\eta|\xi) \leq \mathtt{C}(\mathsf{U}, N) < \infty,$$

where $\mathtt{C}(\Delta, N)$ can be chosen uniformly for all $\xi \in \mathbb{K}^t(\mathbb{R}^d)$.

6.3.2 Existence of Gibbs measures

Similarly as in the previous chapters, a Gibbs measure $\mu \in \mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$ will be constructed as a cluster point of the net of specification kernels $\{\pi_{\mathsf{U}}\}_{\mathsf{U}\in\mathcal{B}_c(\mathbb{R}^d)}$. To this end, an important step is to establish the equicontinuity of the local specification, which yields the existence of limit points in a proper topology.

Definition 6.27. [see also Definition 4.8 in [HKPR13]] On the space of all probability measures $\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$ we introduce the topology of $\mathcal{Q}$-*local convergence*. This topology, which we denote by $\mathcal{T}_{\mathcal{Q}}$, is defined as the coarsest topology making the maps $\mathcal{P}(\mathbb{K}(\mathbb{R}^d)) \ni \mu \mapsto \mu(B)$ continuous for all sets $B \in \mathcal{B}_{\mathcal{Q}}(\mathbb{K}(\mathbb{R}^d))$. Here,

$$\mathcal{B}_{\mathcal{Q}}(\mathbb{K}(\mathbb{R}^d)) := \bigcup_{\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)} \mathcal{B}_{\mathsf{U}}(\mathbb{K}(\mathbb{R}^d)) \tag{6.43}$$

denotes the *algebra of all local events* associated with the partition cubes, where $\mathcal{B}_{\mathsf{U}}(\mathbb{K}(\mathbb{R}^d)) := p_{\mathsf{U}}^{-1}\mathcal{B}(\mathbb{K}(\mathbb{R}^d)a\mathsf{U})$ and the canonical projections p_{U} were defined by (6.12).

Note that the topology of local convergence is *not metrizable* (see [Geo79, p. 57]. So, to describe the corresponding convergence one has to consider nets instead of sequences. We recall that a subset of measures from $\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$ is relatively compact iff each of its net has a cluster point in $\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$; furthermore, every cluster point can be obtained as a limit of a certain subnet. A sufficient condition for the existence of cluster points is the so-called *equicontinuity* property.

More precisely, we modify [Geo88, Definition 4.6] to fit our setting:

Definition 6.28. Fix $\xi \in \mathbb{K}^t(\mathbb{R}^d)$. The net $\{\pi_{\mathsf{U}}(d\eta|\xi)|\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)\}$ is called $\mathcal{Q}$-*locally equicontinuous* if for all $\widetilde{\mathsf{U}} \in \mathcal{Q}_c(\mathbb{R}^d)$ and for each sequence $\{B_N\}_{N\in\mathbb{N}} \subset \mathcal{B}_{\widetilde{\mathsf{U}}}(\mathbb{K}(\mathbb{R}^d))$ with $B_N \downarrow \varnothing$

$$\lim_{N\to\infty} \limsup_{\substack{\mathsf{U}\nearrow\mathbb{R}^d \\ \mathsf{U}\in\mathcal{Q}_c(\mathbb{R}^d)}} \pi_{\mathsf{U}}(B_N|\xi) = 0. \tag{6.44}$$

A crucial issue (resulting from [Geo88, Proposition 4.9]) is that each $\mathcal{Q}$-local equicontinuous net has at least one $\nu_{\mathcal{Q}}$-cluster point in $\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$ (cf. [Pre05, Proposition 5.3]).

Proposition 6.29. *Let Assumption* (ϕ) *hold. Then for each fixed* $\xi \in \mathbb{K}^t(\mathbb{R}^d)$ *the net* $\{\pi_{\mathsf{U}}(\mathrm{d}\eta|\xi)|\ \mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)\}$ *is locally equicontinuous.*

Proof. We will split the set B_N and then use the support property (cf. Corollary 6.26), the consistency property (6.30) and the lower bound for a partition function (cf. 6.28) to estimate the two summands. (The basic idea is given by an adaptation of the arguments used for proving [Geo88, Theorem 4.12 and Corollary 4.13] to the configuration space setting, cf. also [KPR12].)

Let us fix any $\widetilde{\mathsf{U}} \in \mathcal{Q}_c(\mathbb{R}^d)$, and let $\{B_N\}_{N\in\mathbb{N}}$ be any sequence of sets from $\mathcal{B}_{\widetilde{\mathsf{U}}}(\mathbb{K}(\mathbb{R}^d))$ such that $B_N \downarrow \varnothing$ as $N \to \infty$. Consider the following Borel subsets in

$\mathbb{K}(\mathbb{R}^d)$ consisting of those measures η whose local masses over $\mathcal{U}$ are bounded by a given $T > 0$,

$$\mathbb{K}[\mathcal{U}, T] := \left\{\eta \in \mathbb{K}(\mathbb{R}^d) \ \ |\eta(\mathcal{U}) \leq T\right\}, \quad T > 0, \tag{6.45}$$

where we define, using (FR),

$$\mathcal{U} := \bigsqcup_j \left\{Q_j \middle| \exists\, x \in Q_j,\, \exists y \in \widetilde{\mathsf{U}} : |x-y| \leq R\right\} \in \mathcal{Q}_c(\mathbb{R}^d). \tag{6.46}$$

For each $\xi \in \mathbb{K}(\mathbb{R}^d)$ and $\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)$ which contains $\widetilde{\mathsf{U}}$, we have by the consistency property

$$\begin{aligned} \pi_{\mathsf{U}}(B_N|\xi) &= \pi_{\mathsf{U}}(B_N \cap \mathbb{K}[\mathcal{U},T]^c|\xi) + \int_{\mathbb{K}(\mathbb{R}^d)} \pi_{\widetilde{\mathsf{U}}}(B_N \cap \mathbb{K}[\mathcal{U},T]|\eta)\pi_{\mathsf{U}}(d\eta|\xi) \\ &= \pi_{\mathsf{U}}(B_N \cap \mathbb{K}[\mathcal{U},T]^c|\xi) \\ &\quad + \int_{\mathbb{K}(\mathbb{R}^d)} \frac{1}{Z_{\widetilde{\mathsf{U}}}(\eta)} \int_{\mathbb{K}(\widetilde{\mathsf{U}})} \mathbb{1}_{B_N \cap \mathbb{K}[\mathcal{U},T]}(\rho_{\widetilde{\mathsf{U}}} \cup \eta_{\widetilde{\mathsf{U}}^c}) \\ &\quad\quad \times \exp\left\{-\beta H_{\widetilde{\mathsf{U}}}(\rho_{\widetilde{\mathsf{U}}}|\eta)\right\} \mathscr{L}_{\widetilde{\mathsf{U}},\tau}(\mathrm{d}\rho_{\widetilde{\mathsf{U}}})\pi_{\mathsf{U}}(\mathrm{d}\eta|\xi). \end{aligned} \tag{6.47}$$

By Chebyshev's inequality and Corollary 6.26 we get

$$\begin{aligned} &\limsup_{\substack{\mathsf{U}\nearrow\mathbb{R}^d \\ \mathsf{U}\in\mathcal{Q}_c(\mathbb{R}^d)}} \pi_{\mathsf{U}}(\mathbb{K}[\mathcal{U},T]^c) = \limsup_{\substack{\mathsf{U}\nearrow\mathbb{R}^d \\ \mathsf{U}\in\mathcal{Q}_c(\mathbb{R}^d)}} \pi_{\mathsf{U}}\left(\left\{\eta \in \mathbb{K}(\mathbb{R}^d) : \eta(\mathcal{U}) > T\right\}\middle|\, \xi\right) \\ &\leq \limsup_{\substack{\mathsf{U}\nearrow\mathbb{R}^d \\ \mathsf{U}\in\mathcal{Q}_c(\mathbb{R}^d)}} \int_{\mathbb{K}(\mathbb{R}^d)} \frac{\eta(\mathcal{U}^2)}{T^2}\pi_{\mathsf{U}}(d\eta|\xi) < \infty \end{aligned} \tag{6.48}$$

which vanishes as $T \nearrow \infty$. On the other hand, for all $\eta \in \mathbb{K}(\mathbb{R}^d)$ and $\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)$ containing $\widetilde{\mathsf{U}}$, we estimate the outer integrand (using (6.28), the lower bound on the potential and the elementary inequality $|ab| \leq 1/2(a^2 + b^2)$ for $a, b \in \mathbb{R}$) as follows

$$
\begin{aligned}
&\frac{1}{Z_{\widetilde{\mathsf{U}}}(\eta)}\int_{\mathbb{K}(\widetilde{\mathsf{U}})} \mathbb{1}_{B_N\cap\mathbb{K}[\mathcal{U},T]}(\rho_{\widetilde{\mathsf{U}}}\cup\eta_{\widetilde{\mathsf{U}}^c})\exp\left\{-\beta H_{\widetilde{\mathsf{U}}}(\rho_{\widetilde{\mathsf{U}}}|\eta)\right\}\mathscr{L}_{\widetilde{\mathsf{U}},\tau}(d\rho_{\widetilde{\mathsf{U}}})\\
\leq&\exp\left\{\beta\|\phi\|_\infty\Big(\mathfrak{m}_2(\widetilde{\mathsf{U}})+2\mathfrak{m}_1(\widetilde{\mathsf{U}})\cdot\eta_{\widetilde{\mathsf{U}}^c}(\mathcal{U})\Big)\right\}\int_{\mathbb{K}\widetilde{\mathsf{U}}}\mathbb{1}_{B_N\cap\mathbb{K}[\mathcal{U},T]}(\rho_{\widetilde{\mathsf{U}}}\cup\eta_{\widetilde{\mathsf{U}}^c})\\
&\times\exp\left\{\beta\|\phi\|_\infty\left(\rho(\widetilde{\mathsf{U}})^2+2\rho(\widetilde{\mathsf{U}})\eta_{\widetilde{\mathsf{U}}^c}(\mathcal{U})\right)\right\}\mathscr{L}_{\widetilde{\mathsf{U}},\tau}(d\rho)\\
\leq&\exp\left\{\|\phi\|_\infty\Big(\mathfrak{m}_2(\widetilde{\mathsf{U}})+T^2+\mathfrak{m}_1(\widetilde{\mathsf{U}})^2\Big)\right\}e^{3\beta\|\phi\|_\infty T^2}\mathscr{L}_{\widetilde{\mathsf{U}},\tau}\left(B_N\cap\mathbb{K}[\mathcal{U},T]\right)\\
\leq&Ce^{4\beta\|\phi\|_\infty T^2}\mathscr{L}_{\widetilde{\mathsf{U}},\tau}\left(B_N\right)<\infty,
\end{aligned}
$$

where C is an appropriate constant. Summing up, we get

$$
\pi_{\mathsf{W}}(B_N\cap\mathbb{K}[\mathcal{U},T]|\xi)\leq Ce^{4\beta\|\phi\|_\infty T^2}\mathscr{L}_{\widetilde{\mathsf{U}},\tau}\left(B_N\right) \tag{6.49}
$$

which tends to zero for $B_N\searrow\varnothing$. Plugging (6.48) and (6.49) back into (6.47), we get the equicontinuity of the family $\{\pi_{\mathsf{W}}(\mathrm{d}\eta|\xi)|,\ \mathsf{W}\in\mathcal{Q}_c(\mathbb{R}^d)\}$ required in (6.44).

□

Corollary 6.30. *Let Assumption (ϕ) hold and fix some order generating sequence $\{\mathsf{W}_N\}_{N\in\mathbb{N}}\subset\mathcal{Q}_c(\mathbb{R}^d)$. Then, for each boundary condition $\xi\in\mathbb{K}^t(\mathbb{R}^d)$, (a subsequence of) $\{\pi_{\mathsf{W}_N}(\cdot|\xi)\}_{N\in\mathbb{N}}$ converges $\mathcal{Q}$-locally to a probability measure $\mu\in\mathcal{P}(\mathbb{K}(\mathbb{R}^d))$, that means for all $B\in\mathcal{B}_{\mathcal{Q}}(\mathbb{K}(\mathbb{R}^d)$*

$$
\pi_{\mathsf{W}_N}(B|\xi)\to\mu(B)\quad \textit{as } N\to\infty.
$$

Proof. The claim follows by combining Proposition 6.29 with [Geo88, Propositions 4.9 and 4.15] and [Par67, Theorem V.3.2].

□

Theorem 6.31. *Let $\phi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be such that Assumption (ϕ) holds. Then there exists a Gibbs measure corresponding to the potential ϕ and the measure $\mathscr{L}_\tau$, which is supported by $\mathbb{K}^t(\mathbb{R}^d)$.*

Moreover, the set $\mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$ is relatively compact in the topology $\mathcal{T}_{\mathcal{Q}}$.

Theorem 6.32. *Let Assumption (ϕ) hold. Then, for each $a\in\left[0,A-m^\phi\|\phi^-\|_\infty\right]$ there exists an (explicitly computable) $\mathtt{C}_a<\infty$ such that uniformly for all $\mu\in\mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$*

$$
\sup_{k\in\mathbb{Z}^d}\int_{\mathbb{K}(\mathbb{R}^d)}\exp\left\{a\eta(Q_k)^2\right\}\mu(d\eta)\leq\mathtt{C}_a. \tag{6.50}
$$

Corollary 6.33. *Let Assumption* (ϕ) *be fulfilled. For each* $\mathsf{U} \in \mathcal{Q}_c(\mathbb{R}^d)$ *and* $a > 0$*, there exists* $C_a(\mathsf{U}) > 0$ *such that for all* $\mu \in \mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$

$$\int_{\mathbb{K}(\mathbb{R}^d)} e^{a\eta(\mathsf{U})}\mu(d\eta) < C_a(\mathsf{U}). \tag{6.51}$$

The proofs of Theorems 6.31 and 6.32, as well as that of Corollary 6.33, follow by standard arguments, which can be found in [HKPR13].

6.3.3 Uniqueness of Gibbs measures

The corresponding lattice system

Following the lines of Sections 4.2.3 and 5.4.1 we define a lattice system corresponding to our system.

Starting from the chosen partition $(Q_k)_{k\in\mathbb{Z}^d}$ of $\mathbb{R}^d$ (see (6.19)), we construct a lattice system on the space $\mathbb{K}_{lat} := \big(\mathbb{K}(\overline{Q})\big)^{\mathbb{Z}^d}$, where for simplicity we denote $Q = Q_0$. This space is endowed with the product topology and the corresponding Borel σ-algebra $\mathcal{B}(\mathbb{K}_{lat})$. Then, by Remark 4.A3 in [Geo88], $(\mathbb{K}_{lat}, \mathcal{B}(\mathbb{K}_{lat}))$ is a standard Borel space.

Define

$$T : \mathbb{K}(\mathbb{R}^d) \to \mathbb{K}_{lat},$$

which maps $\eta \in \mathbb{K}(\mathbb{R}^d)$ into $\check{\eta} = (\check{\eta}_k)_{k\in\mathbb{Z}^d} \in \mathbb{K}_{lat}$, where

$$\check{\eta}_k := \sum_{x\in\mathfrak{S}(\eta)\cap\overline{Q}_k} s_x\delta_{x-gk},$$

for $\eta = \sum s_x\delta_x$.

By T^{-1} we denote the left inverse of T. Let $B_{k_1}\dots B_{k_L} \in \mathcal{B}(\mathbb{K}(\overline{Q})$ for $L \in \mathbb{N}$ and $k_1, \dots k_L \in \mathbb{Z}^d$ and define the cylinder sets

$$A^{B_{k_1},\dots,B_{k_L}}_{k_1,\dots,k_L} := \{(\check{\eta}_k)_{k\in\mathbb{Z}^d} \in \mathbb{K}_{lat} : \check{\eta}_{k_l} \in B_{k_l}, 1 \le l \le L\} \in \mathcal{B}(\mathbb{K}_{lat})$$

and

$$C^{B_{k_1},\dots,B_{k_L}}_{k_1,\dots,k_L} := \{\eta \in \mathbb{K}(\mathbb{R}^d) : \sum_{x\in\mathfrak{S}(\eta)\cap\overline{Q}_{k_l}} s_x\delta_{x-gk_l} \in B_{k_l},\ 1 \le l \le L\} \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d)),$$

respectively.

Lemma 6.34. *(i)* $T : \mathbb{K}(\mathbb{R}^d) \to \mathbb{K}_{lat}$ *is measurable;*

(ii) $T(B) \in \mathcal{B}(\mathbb{K}_{lat})$ *for any* $B \in \mathcal{B}_0(\mathbb{K}(\mathbb{R}^d))$.

Proof. (i) One can immediately see that

$$T^{-1}\left(A_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}}\right) = C_{k_1,\dots,k_L}^{B_{k_1},\dots,B_{k_L}},$$

which proves the statement, since $\mathcal{B}(\mathbb{K}_{lat})$ is generated by the cylinder sets.

(ii) Assume that $\mathsf{W} \subset \bigcup_{i=1}^{L} Q_{k_i}$. For $\mathtt{B} \in \mathcal{B}(\mathbb{K}(\mathsf{W}))$ we have

$$T\left(\{\eta \in \mathbb{K}(\mathbb{R}^d) : \eta_{\mathsf{W}} \in \mathtt{B}\}\right) = \left\{\breve{\eta} \in \mathbb{K}_{lat} : \breve{\eta}_{(k_1,\dots,k_L)} \in \mathtt{B}\right\},$$

which is measurable. Here $\breve{\eta}_{(k_1,\dots,k_L)}$ denotes the projection of $\breve{\eta}$ onto the coordinates $k_1, \dots, k_L$ on the product space $\prod_{i=1}^{L} \mathbb{K}(\overline{Q})$. □

Thus, for any $\mu \in \mathcal{P}(\mathbb{K}(\mathbb{R}^d))$ we can define its push-forward image $T_*\mu \in \mathcal{P}(\mathbb{K}_{lat})$, where $\mathcal{P}(\mathbb{K}_{lat})$ is the set of all probability measures on $\mathbb{K}_{lat}$.

Lemma 6.35. *The map* $T_* : \mathcal{P}(\mathbb{K}(\mathbb{R}^d)) \to \mathcal{P}(\mathbb{K}_{lat})$ *is injective.*

Proof. Let $\mu, \nu \in \mathcal{P}(\mathbb{K}(\mathbb{R}^d))$ and $\mu \neq \nu$. Then there exists $B \in \mathcal{B}_0(\mathbb{K}(\mathbb{R}^d))$ such that $\mu(B) \neq \nu(B)$. By Lemma 6.34, $A := T(B) \in \mathcal{B}(\mathbb{K}_{lat})$. The injectivity of T implies that $T^{-1}(T(B)) = B$. Thus $T_*\mu(A) = \mu(T^{-1}(A)) \neq \nu(T^{-1}(A)) = T_*\nu(A)$, and the statement is proved. □

Let us investigate the correspondence between measures on $\mathbb{K}(\mathbb{R}^d)$ and $\mathbb{K}_{lat}$. Let μ be a probability measure on $\mathbb{K}(\mathbb{R}^d)$ satisfying the following condition:

(A) Consider the set

$$\mathring{\mathbb{K}} := \left\{\eta \in \mathbb{K}(\mathbb{R}^d) : \ \mathfrak{S}(\eta) \cap \partial Q_k = \varnothing, \quad \forall k \in \mathbb{Z}^d\right\} \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d)), \tag{6.52}$$

and assume $\mu(\mathring{\mathbb{K}}) = 1$. In other words, μ ignores measures whose support *touch* the sites of the partition cubes Q_k.

For $B_k \in \mathcal{B}(\mathbb{K}(\overline{Q}))$ with $k \in \mathbb{Z}^d$, we denote $\mathring{B}_k := \{\eta \in B_k | \mathfrak{S}(\eta) \cap \partial\overline{Q} = \emptyset\}$, where $\partial\overline{Q} := \overline{Q} \setminus Q$. Starting from a given μ, probability measure on $\mathbb{K}(\mathbb{R}^d)$ satisfying condition (6.52) above, we construct a probability measure μ_{lat} on $\mathbb{K}_{lat}$, as the push-forward of μ. The explicit definition is as follows:

$$\begin{aligned}\mu_{lat}(A^{B_{k_1},\dots,B_{k_L}}_{k_1,\dots,k_L}) &:= \mu_{lat}(A^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}}_{k_1,\dots,k_L}) := \mu(T^{-1}(A^{\mathring{B}_{k_1},\dots,\mathring{B}_{k_L}}_{k_1,\dots,k_L})) = \\ &= \mu(\{\eta\in\mathbb{K}(\mathbb{R}^d)| \sum_{x\in\mathfrak{S}(\eta)\cap\overline{Q}_{k_l}} s_x\delta_{x-gk_l}\in\mathring{B}_{k_l}, 1\le l\le L\}) \\ &= \mu(\{\eta\in\mathbb{K}(\mathbb{R}^d)| \sum_{x\in\mathfrak{S}(\eta)\cap\overline{Q}_{k_l}} s_x\delta_{x-gk_l}\in B_{k_l}, 1\le l\le L\}) \\ &= \mu(C^{B_{k_1},\dots,B_{k_L}}_{k_1,\dots,k_L}).\end{aligned} \tag{6.53}$$

Since we know that the cylinder events constitute a measure-defining class, μ_{lat} is well-defined on the whole $\mathcal{B}(\mathbb{K}_{lat})$. Also, denoting by $\mathring{\mathbb{K}}_{lat}$ the set $\{\breve{\eta}\in\mathbb{K}_{lat}|\mathfrak{S}(\eta)\cap\partial\overline{Q}=\emptyset\}$, we see from the above definition that the corresponding measure on the lattice μ_{lat} puts full mass on $\mathring{\mathbb{K}}_{lat}$. Also obvious is that $T:\mathring{\mathbb{K}}\to\mathring{\mathbb{K}}_{lat}$ is a bijection.

Remark 6.36. In particular, for $\mu=\mathscr{L}_\tau$, we have that

$$\mathscr{L}_{lat,\tau}\big(A^{B_{k_1},\dots,B_{k_L}}_{k_1,\dots,k_L}\big) := \prod_{l=1}^{L}\mathscr{L}_\tau\Big(\big\{\eta\in\mathbb{K}(\mathbb{R}^d): \sum_{x\in\mathfrak{S}(\eta)\cap\overline{Q}_{k_l}} s_x\delta_{x-gk_l}\in B_{k_l}\big\}\Big). \tag{6.54}$$

We remark that the construction of $\mathscr{L}_{lat,\tau}$ is possible because $\mathscr{L}_\tau$ satisfies condition (6.52) (by the properties of the Poisson measure π_τ).

We continue by defining the energy of the new system with the phase space $\mathbb{K}_{lat}$. Consider (arbitrarily large) cubic domains $\mathsf{W}_\mathcal{K}:=\bigsqcup_{k\in\mathcal{K}}\overline{Q}_k$ indexed by $\mathcal{K}\Subset\mathbb{Z}^d$ and define the local energy as

$$\breve{H}_\mathcal{K}(\breve{\eta}_\mathcal{K}|\breve{\xi}) := H\left((T^{-1}\breve{\eta})_\mathcal{K}|(T^{-1}\breve{\xi})\right). \tag{6.55}$$

Using the above definition, we introduce the local one-point Gibbs states as

$$\breve{\mu}_\mathcal{K}(d\breve{\eta}_\mathcal{K}|\breve{\xi}) := \begin{cases} [\breve{Z}_\mathcal{K}(\breve{\xi})]^{-1}\exp\left\{-\beta\breve{H}_\mathcal{K}(\breve{\eta}_\mathcal{K}|\breve{\xi})\right\}\mathscr{L}_{lat,\tau}(d\breve{\eta}_\mathcal{K}), & \breve{\xi}\in\mathring{\mathbb{K}}_{lat}, \\ 0, & \text{otherwise,}\end{cases} \tag{6.56}$$

where

$$\breve{Z}_\mathcal{K}(\breve{\xi}) := \int_{\mathbb{K}_{lat}}\exp\left\{-\beta\breve{H}_\mathcal{K}(\breve{\eta}'_\mathcal{K}|\breve{\xi})\right\}\mathscr{L}_{lat,\tau}(d\breve{\eta}'_\mathcal{K}) \tag{6.57}$$

and $\mathscr{L}_{lat,\tau}$ is given by (6.54).

We note that elementary computations yield for any $\breve{\xi} \in \mathring{\mathbb{K}}_{lat}$

$$\begin{aligned}\breve{Z}_{\mathcal{K}}(\breve{\xi}) &= \int_{\mathring{\mathbb{K}}_{lat}} \exp\left\{-\beta \breve{H}_{\mathcal{K}}(\breve{\eta}_{\mathcal{K}}|\breve{\xi})\right\} \mathscr{L}_{lat,\tau}(d\breve{\eta}_{\mathcal{K}}) \\ &= \int_{\mathbb{K}(\mathbb{R}^d)} \exp\{-\beta \breve{H}_{\mathcal{K}}(T(\eta)_{\mathcal{K}}|\breve{\xi})\} \mathscr{L}_{\tau}(d\eta) \\ &= \int_{\mathbb{K}(\mathbb{R}^d)} \exp\{-\beta H_{\mathcal{K}}(T^{-1}(T(\eta))_{\mathcal{K}}|T^{-1}(\breve{\xi}))\} \mathscr{L}_{\tau}(d\eta) = Z_{Q_{\mathcal{K}}}(T^{-1}\breve{\xi}).\end{aligned} \tag{6.58}$$

Also, it is easy to check that the local Gibbs states for the lattice model are the pushforward measures of the local Gibbs states of the initial model, or more explicitly, $\breve{\mu}_{\mathcal{K}}(d\breve{\eta}|\breve{\xi}) = (\mu_{\mathcal{K}} \circ T^{-1})(d\eta|T^{-1}\breve{\xi})$. From here, we go on to define the local Gibbs specification as

$$\breve{\pi}_{\mathcal{K}}(\breve{B}|\breve{\xi}) := \breve{\mu}_{\mathcal{K}}(\breve{B}_{\mathcal{K},\breve{\xi}}|\breve{\xi}), \quad \breve{B}_{\mathcal{K},\breve{\xi}} := \{\breve{\eta}_{\mathcal{K}}|\breve{\eta}_{\mathcal{K}} + \breve{\xi}_{\mathcal{K}^c} \in \breve{B}\}, \tag{6.59}$$

for any $\breve{B} \in \mathcal{B}(\mathbb{K}_{lat})$. An important step is to show that uniqueness of Gibbs measures in the lattice model we have introduced above implies uniqueness of Gibbs measures in our initial model.

Lemma 6.37. *Let μ be a Gibbs measure on $\mathbb{K}(\mathbb{R}^d)$ corresponding to the specification $\{\pi_{\mathsf{W}}\}_{\mathsf{W}}$. Then μ uniquely determines a Gibbs measure μ_{lat} corresponding to $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$. Moreover, if μ_{lat} is the unique Gibbs measure of the system $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$, then also μ is unique as a Gibbs measure corresponding to $\{\pi_{\mathsf{W}}\}_{\mathsf{W}}$.*

Proof. A similar procedure as in Section 5.4.1 can be used to verify that μ satisfies condition **(A)**, which implies the existence of a measure μ_{lat} as given by (6.53). Elementary computations show that μ_{lat} satisfies the DLR relations and hence is a Gibbs measure corresponding to the specification $\{\breve{\pi}_{\mathcal{K}}\}_{\mathcal{K}}$. Uniqueness follows easily by Lemma 6.35. □

Uniqueness by small first two moments of τ

Our aim now is to show uniqueness of tempered Gibbs measures in the lattice model introduced above. The set of such measures will be denoted by $\mathcal{G}^t_{lat}$ and consists of Gibbs measures μ_{lat}, which are supported by the following set of tempered configurations

$$\mathbb{K}^t_{lat} := \bigcap_{\alpha>0} \Gamma_{\alpha,lat},$$

where

$$\mathbb{K}_{\alpha,lat} := \left\{ \breve{\eta} \in \mathbb{K}_{lat} : \left(\sum_{k \in \mathbb{Z}^d} \breve{\eta}(Q_k)^2 e^{-\alpha|k|} \right)^{1/2} < \infty \right\}.$$

Moreover, by (6.51), any tempered Gibbs measure μ_{lat} satisfies the following exponential moment estimate, for any $a \in \left[0, A - m^{\phi}\|\phi^{-}\|_{\infty}\right]$

$$\sup_{k \in \mathbb{Z}^d} \int_{\mathbb{K}_{lat}} \exp\left\{a\breve{\eta}(Q_k)^2\right\} \mu_{lat}(d\breve{\gamma}) \leq \mathtt{C}_a, \tag{6.60}$$

where $\mathtt{C}_\alpha$ is given by (6.51). Hence, one can easily see that μ_{lat} satisfies the a-priori bound in Theorem 2.9.

Theorem 6.38. *Let $\beta > 0$ be fixed. For any $M_0 > 0$ and $\varphi_0 > 0$, there exists a $\zeta_0 = \zeta_0(M_0, \varphi_0)$ such that $\mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$ is a singleton at all values of $\mathfrak{m}_1(Q_k) < M_0$ and $\|\phi\|_\infty < \varphi_0$ related by the constraint*

$$\mathfrak{m}_1(Q_k)\|\phi\|_\infty =: \zeta < \zeta_0. \tag{6.61}$$

In what follows, we properly extend the idea used to prove uniqueness in Chapters 4 and 5. Recall that we denote the R-vicinity of a point $k \in \mathbb{Z}^d$ by $\partial k := \partial_R k = \{j \in \mathbb{Z}^d | d(Q_k, Q_j) \leq R\}$. Also, let Z_0 be a semigroup of $g\mathbb{Z}^d$ such that $|u - v| > R$ holds for all $u, v \in Z_0$, and define $\chi := \min_{Z_0} |g\mathbb{Z}^d / Z_0|$, the number of elements in the quotient group $g\mathbb{Z}^d / Z_0$. The proof of the theorem will be based on two lemmas, as follows.

Integrability condition (IC)

Lemma 6.39. *Let $\beta > 0$ be fixed. There are constants $\theta > 0$ and $0 < \bar{c} < 1/\Delta^\chi$ such that, for every $k \in \mathbb{Z}^d$, and any boundary condition $\breve{\xi} \in \mathbb{K}_{lat}$*

$$\int_{\mathbb{K}_{lat}} \theta \breve{\eta}_k(Q)^2 \breve{\pi}_{Q_k}(d\breve{\gamma}_k | \breve{\eta}) \leq 1 + \frac{\bar{c}}{\Delta^\chi} \sum_{j \in \partial k} \theta \breve{\eta}_j(Q)^2. \tag{6.62}$$

Proof. Let us first notice that, by a simple change of variables,

$$\int_{\mathbb{K}_{lat}} \exp\left\{a\breve{\eta}_k(Q)\right\} \breve{\pi}_{Q_k}(d\breve{\eta}_k | \breve{\xi}) = \int_{\mathbb{K}(\mathbb{R}^d)} \exp\left\{a\eta(Q_k)\right\} \pi_{Q_k}(d\eta | \xi).$$

In order to prove the (IC) condition, we will use the Dobrushin-type estimate obtained in (6.37) , i.e.

$$\int_{\mathbb{K}(\mathbb{R}^d)} \eta(Q_k)^2 \pi_k(d\eta|\xi) \le \frac{1}{a}\left\{\Upsilon_\varepsilon + \big(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M\big) \sum_{j\in\partial k} \xi(Q_j)^2\right\}.$$

Hence, also

$$\int_{\mathbb{K}_{lat}} \breve{\eta}_k(Q)^2 \pi_k(d\breve{\eta}|\breve{\xi}) \le \frac{\beta}{a}\left\{\Upsilon_\varepsilon + \big(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M\big) \sum_{j\in\partial k} \breve{\xi}_j(Q)^2\right\}.$$

Relation (IC) is satisfied with constants

$$\theta := \frac{a}{\beta\Upsilon_\varepsilon} \tag{6.63}$$

and

$$\bar{c} := \frac{\beta\big(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty M\big)}{a}. \tag{6.64}$$

It is obvious to see that $\bar{c} < 1/\Delta^\chi$, for a big enough. □

Denote by $\mathfrak{h}(\breve{\eta}_j) := \breve{\eta}_j(Q)^2$.

Contraction Condition (CC)

Lemma 6.40. *For fixed $k \in \mathbb{Z}^d$, $\beta > 0$, $M_0 > 0$ and $\varphi_0 > 0$, there exists a $\zeta_0 = \zeta_0(M_0, \varphi_0)$ such that at all values of $\mathfrak{m}_1(Q_k) < M_0$ and $\|\phi\|_\infty < \varphi_0$ related by*

$$\mathfrak{m}_1(Q_k)\|\phi\|_\infty =: \zeta < \zeta_0, \tag{6.65}$$

one has

$$d_{TV}(\breve{\mu}_k(d\breve{\eta}_k|\breve{\xi}^1), \breve{\mu}_k(d\breve{\eta}_k|\breve{\xi}^2)) \le \sum_{j\in\partial k} \mathtt{k}\mathbb{1}_{\breve{\xi}^1_j \neq \breve{\xi}^2_j}, \tag{6.66}$$

for some constant $0 < \mathtt{k} < 1$ and boundary conditions $\breve{\xi}^1, \breve{\xi}^2$ such that

$$\theta\mathfrak{h}(\breve{\xi}^i_j) \le K_*, \quad i = 1, 2 \tag{6.67}$$

and $K_* = K_*(\theta\mathfrak{h}, \bar{c}, \mathtt{k})$ is given by (2.14).

Proof. Similarly as in Lemma 5.24, we notice that by a change of variables it is enough to compute the distance $d_{TV}(\mu_k(d\eta|\xi^1), \mu_k(d\eta|\xi^2))$. Set $K_0 := \big(\theta^{-1}K_*\big)^{1/2}$, hence $\xi^i(Q_j) \le K_0$, for all $j \in \partial k$, $i = 1, 2$.

Elementary computations yield

$$\begin{aligned} d_{TV}(\mu_k^{\xi^1}, \mu_k^{\xi^2}) &= \frac{1}{2}\int |Z_k^{-1}(\xi^1)\exp\{-\beta H_k(\eta|\xi^1)\} - Z_k^{-1}(\xi_2)\exp\{-\beta H_k(\eta|\xi^2)\}|\mathcal{G}_{k,\chi}(d\eta) \\ &\leq \int |1-\exp\{\beta|\triangle H_k(\eta,\xi^1,\xi^2)|\}\mu_k(d\eta|\xi^1) \\ &\leq \int \exp\{|\beta \triangle H_k(\eta,\xi^1,\xi^2)|\}\mu_k(d\eta|\xi^1) - 1, \end{aligned}$$

where $\triangle H_k(\eta,\xi^1,\xi^2) := H_k(\eta|\xi^1) - H_k(\eta|\xi^2)$.

Let us fix $\xi^1 = \emptyset$. Then the total variation distance is less or equal than

$$\begin{aligned} &\int |1-\exp\{\beta \triangle H_k(\eta,\emptyset,\xi)\}|\exp\{-\beta H_k(\eta|\emptyset)\}\mathscr{L}_{k,\tau}(d\eta) \\ &\leq \beta|\triangle H_k(\eta,\emptyset,\xi)|\exp\{\beta|\triangle H_k(\eta,\emptyset,\xi)| - \beta H_k(\eta|\emptyset)\}\mathscr{L}_{k,\tau}(d\eta) \\ &\leq \int \beta\|\phi\|_\infty \eta(Q_k)\sum_{j\in\partial k}\xi(Q_j)\exp\{4\beta\|\phi\|_\infty\eta(Q_k)\sum_{j\in\partial k}\xi(Q_j) - \beta A\eta(Q_k)^2\}\mathscr{L}_{k,\tau}(d\eta) \\ &\leq \int \beta\|\phi\|_\infty mK_0\eta(Q_k)\exp\{4\beta\|\phi\|_\infty mK_0\eta(Q_k) - \beta A\eta(Q_k)^2\}\mathscr{L}_{k,\tau}(d\eta), \end{aligned}$$

By applying Young's inequality to the last exponential factor we get that

$$\exp\{4\beta\|\phi\|_\infty mK_0\eta(Q_k) - \beta A\eta(Q_k)^2\} \leq \exp\{4A^{-1}\|\phi\|_\infty m^2K_0\},$$

which implies that the total variation distance is smaller than

$$\begin{aligned} &\beta\|\phi\|_\infty mK_0\exp\{4A^{-1}\|\phi\|_\infty m^2K_0\}\int \eta(Q_k)\mathscr{L}_{k,\tau}(d\eta) \\ &\leq \beta\|\phi\|_\infty mK_0\exp\{4A^{-1}\|\phi\|_\infty m^2K_0\}\mathfrak{m}_1(Q_k). \end{aligned}$$

We know that m_1 is uniformly bounded by a constant M, cf. (6.32). By choosing M small enough, we can also make the total variation distance as small as we want. By applying the triangle inequality, the statement holds also for more general boundary conditions.

□

Proof of Theorem 6.38. It follows immediately from Lemmas 6.39 and 6.40. □

Decay of correlations

We shortly remark that uniqueness of $\mu \in \mathcal{G}^t(\mathbb{K})$ yields a result for the decay of correlations, via Theorem 2.19. Let $Q_{\mathcal{K}_1}$ and $Q_{\mathcal{K}_2}$ be two disjoint cubic domains and let G_1, G_2 be two local functions such that G_i is $\mathcal{B}(\mathbb{K}(Q_{\mathcal{K}_i}))$-measurable, for $i = 1, 2$. Also, assume that

$$G_2(\eta) \leq \sum_{j \in \mathcal{K}_2} \theta \eta(Q_j)^2, \quad \eta \in \mathbb{K}(\mathbb{R}^d)$$

and

$$\sup_{k \in \mathcal{K}_2} \int_{\mathbb{K}(\mathbb{R}^d)} G_1(\eta) \eta(Q_k)^2 \mu(d\eta) < \infty.$$

Corollary 6.41. *In the setting described above, there exist constants $\alpha, \vartheta > 0$ such that*

$$|Cov_\mu(G_1, G_2)| \leq \vartheta m(Q_{\mathcal{K}_2})^2 \exp\left(-\alpha d(Q_{\mathcal{K}_1}, Q_{\mathcal{K}_2})\right) \int_{\mathbb{K}(\mathbb{R}^d)} |G_1(\eta)| \tilde{F}(\eta) \mu(d\eta). \tag{6.68}$$

Moreover,

$$\alpha := -\log r_K, \tag{6.69}$$

where r_K is given by (2.68), for the Dobrushin-Pechersky matrix with entries given by Lemmas 6.39 and 6.40.

□

6.4 Unbounded Lévy intensity measure

6.4.1 Assumptions on the interaction

In this section we assume the first two "spatial" moments of τ have the following exponential bound: for some $a_1, a_2, C_1, C_2 > 0$

$$\int s^i \tau(ds, Q_k) \leq C_i e^{a_i |k|}, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d. \tag{6.70}$$

Define $\alpha_0 := \max\{2a_1, a_2\}$. We will use the following class of *tempered* discrete Radon measures

$$\mathbb{K}^t(\mathbb{R}^d) := \bigcap_{\alpha>\alpha_0} \mathbb{K}_\alpha(\mathbb{R}^d), \tag{6.71}$$

where

$$\mathbb{K}_\alpha(\mathbb{R}^d) := \left\{\eta \in \mathbb{K}(\mathbb{R}^d) : M_\alpha(\eta) := \left(\sum_{z\in\mathbb{Z}^d} \eta(Q_k)^2 e^{-\alpha|k|}\right)^{1/2} < \infty\right\}. \tag{6.72}$$

We now assume $A > m\|\phi^-\|_\infty(1 + e^{\alpha_0\vartheta})$, where A is given by (6.17).

6.4.2 Existence of Gibbs measures

From here on, it is easy to deduce the following bound in elementary cubes Q_k, $k \in \mathbb{Z}^d$, which represents the weak dependence on boundary conditions.

Lemma 6.42. *For $k \in \mathbb{Z}^d$, $\xi \in \mathbb{K}(\mathbb{R}^d)$, $\mathsf{U} \in \mathcal{B}_c(\mathbb{R}^d)$ and $a \in [0, \beta(A - 2m\|\phi^-\|_\infty)]$, the following holds true*

$$\int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_k(d\eta|\xi) \le \exp\left\{\beta\left[\Upsilon_{\varepsilon,k} + (\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)\sum_{j\in\partial k}\xi(Q_j)^2\right]\right\}, \tag{6.73}$$

where $\varepsilon > 0$ is arbitrary and

$$\Upsilon_{\varepsilon,k} := \|\phi\|_\infty(C_2 e^{a_2|k|} + \frac{1}{\varepsilon} m C_1^2 e^{2a_1|k|}),$$

which is finite for any fixed $k \in \mathbb{Z}^d$.

Proof. The claim follows immediately from the lower bound of the conditional Hamiltonian and the proof of Lemma 6.23.

□

Proposition 6.43. *Let $0 \le a \le \beta(A - m\|\phi^-\|_\infty)$. Then there exists $C_{a,k} < \infty$ such that for all $k \in \mathbb{Z}^d$ and $\xi \in \mathbb{K}^t(\mathbb{R}^d)$*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_{\mathcal{K}}(d\eta|\xi) \le C_{a,k}. \tag{6.74}$$

Proof. Without loss of generality, we assume that $a \in (\beta m\|\phi^-\|_\infty e^{\alpha_0\vartheta}, \beta(A - m\|\phi^-\|_\infty)]$. We define

$$0 \leq n_k(\mathcal{K}|\xi) := \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp\{a\eta(Q_k)^2\}\pi_{\mathcal{K}}(d\eta|\xi)\right\},\ k \in \mathbb{Z}^d, \tag{6.75}$$

which are finite by Lemma 6.23.

Integrating (6.73) with respect to $\pi_{\mathcal{K}}(d\eta|\xi)$, we get for every $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) &\leq \beta\Upsilon_{\varepsilon,k} + \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp[\beta(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)\sum_{j\in\partial k}\eta(Q_j)^2]\pi_{\mathcal{K}}(d\eta|\xi)\right\} \\ &= \Upsilon_{\varepsilon,k} + \left[\beta(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)\sum_{j\in\mathcal{K}^c\cap\partial k}\xi(Q_j)^2\right] \\ &+ \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp[\beta(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)\sum_{j\in\mathcal{K}\cap\partial k}\eta(Q_j)^2]\pi_{\mathcal{K}}(d\eta|\xi)\right\}. \end{aligned} \tag{6.76}$$

For shorthand, we denote $0 < B_\varepsilon := \beta m(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)$ and choose appropriate $\delta, \varepsilon > 0$ such that $B_\varepsilon < \delta a < a \leq a_0 := \beta(A - m\|\phi^-\|_\infty)$. For the logarithmic term in the last inequality of (6.40) we can apply the multiple Hölder inequality to deduce that it is dominated by

$$\sum_{j\in\mathcal{K}\cap\partial l} \log\left\{\int_{\mathbb{K}(\mathbb{R}^d)} \exp(a\eta(Q_j)^2)\pi_{\mathcal{K}}(d\eta|\xi)\right\}^{1/a(\|\phi^-\|_\infty+\varepsilon\|\phi\|_\infty)} \tag{6.77}$$

$$= 1/a(\|\phi^-\|_\infty + \varepsilon\|\phi\|_\infty)\sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K}|\xi). \tag{6.78}$$

Let $\mathcal{K} \Subset \mathbb{Z}^d$ contain a fixed point $k_0 \in \mathbb{Z}^d$. Let $\vartheta := R/g + \sqrt{d}$ be such that $|j - k_0| \leq \vartheta$ for all $j \in \partial^\phi k_0$. Let us pick an $\alpha > 0$ small enough, so that $B_\varepsilon e^{\alpha\vartheta} < a$. We plug (6.77) in (6.76) and multiply by $e^{-\alpha|k_0|}$ to obtain

$$n_{k_0}(\mathcal{K}|\xi)e^{-\alpha|k_0|} \leq \beta\Upsilon_{\varepsilon,k_0}e^{-\alpha|k_0|} + B_\varepsilon e^{\alpha\vartheta}||\xi_{\mathcal{K}^c}||_\alpha^2 + \frac{B_{\varepsilon,k}}{a}e^{\alpha\vartheta}\sum_{j\in\cap\partial k} n_j(\mathcal{K}|\xi)e^{-\alpha|j|}.$$

Taking supremum after indices in $\mathcal{K}$ we get that

$$n_k(\mathcal{K}|\xi)e^{-\alpha|k|} \leq [\beta\Upsilon_{\varepsilon,k}e^{-\alpha|k|} + B_{\varepsilon,k}e^{\alpha|k|}||\xi_{\mathcal{K}^c}||_\alpha^2]\left(\frac{1}{1-\delta e^{\alpha\vartheta}}\right).$$

We let $\mathcal{K} \nearrow \mathbb{Z}^d$ to obtain that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \; [n_k(\mathcal{K}|\xi)\exp\{-\alpha|k|\}] \leq \Upsilon_{\varepsilon,k}\left(\frac{1}{1-\delta e^{\alpha\vartheta}}\right),$$

and hence, by letting $\alpha \searrow 0$ we get

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_k(\mathcal{K}|\xi) \leq \frac{1}{1-\delta}\Upsilon_{\varepsilon,k} =: \log C_{a,k}.$$

From here we have (6.74). □

The proof of local equicontinuity follows exactly as in Proposition 6.29. The existence result now follows.

Theorem 6.44. *Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be such that Assumption (ϕ) holds. Then there exists a Gibbs measure corresponding to the potential ϕ and the measure $\mathscr{L}_\tau$, which is supported by $\mathbb{K}^t(\mathbb{R}^d)$.*

Moreover, the set $\mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$ is relatively compact in the topology $\mathcal{T}_\mathcal{Q}$.

Theorem 6.45. *Let Assumption (ϕ) hold. Then, for each $a \in \left[0, A - m^\phi\|\phi^-\|_\infty\right]$ there exists an (explicitly computable) $\mathtt{C}_{\alpha,k} < \infty$ such that uniformly for all $\mu \in \mathcal{G}^t(\mathbb{K}(\mathbb{R}^d))$*

$$\sup_{k\in\mathbb{Z}^d} \int_{\mathbb{K}(\mathbb{R}^d)} \exp\left\{\alpha\eta(Q_k)^2\right\} \mu(d\eta) \leq \mathtt{C}_{\alpha,k}. \tag{6.79}$$

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