The Normal Structure of Hyperbolic Unitary Groups

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
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Mai 2014

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Datum der mündlichen Prüfung: 15.10.2014
Prüfungsausschuss: Prof. Michael Baake, Prof. Anthony Bak, Prof. Michael Röckner, Prof. Markus Rost und Prof. Nikolai Vavilov

Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706

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1 Introduction

The main result of this thesis is the following: If (R, Λ) is a form ring such that R is almost commutative (i.e. finitely generated as module over a subring in its center) and H is a subgroup of the hyperbolic unitary group $U_{2n}(R, \Lambda)$ where $n \geqslant 3$, then

$$\begin{aligned}&H \text{ is normalized by the elementary subgroup } EU_{2n}(R,\Lambda) \text{ of } U_{2n}(R,\Lambda) \Leftrightarrow \\ &\exists! \text{ form ideal } (I,\Gamma) \text{ such that } EU_{2n}((R,\Lambda),(I,\Gamma)) \subseteq H \subseteq CU_{2n}((R,\Lambda),(I,\Gamma))\end{aligned} \tag{1.1}$$

where $EU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the relative elementary subgroup of level (I, Γ) and $CU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the full congruence subgroup of level (I, Γ). This result extends the range of validity of previous results. If R has finite Bass-Serre dimension d (cf.[2]) then the result was proved already in [1] provided $n \geqslant sup(d+2, 3)$ and if R is commutative, it was proved recently in [11]. An incorrect proof, which can be repaired when 2 is invertible in R, was given in [9].

The dissertation is organized as follows.

In section 2 we recall some standard notation which will be used throughout the dissertation.

In section 3 we present a model theoretic approach of A. Bak for studying Chevalley groups, unitary groups, classical-like groups and their generalizations.

In section 4 we recall the definitions of the general linear group and some important subgroups. In section 5 we show how the model theoretic approach given in section 3 can be used to prove the following result (this result is not new, see [8], but its proof is): If R is a ring such that R is almost commutative and H is a subgroup of the general linear group $GL_n(R)$ where $n \geqslant 3$, then

$$\begin{aligned}&H \text{ is normalized by the elementary subgroup } E_n(R) \text{ of } GL_n(R) \Leftrightarrow \\ &\exists! \text{ ideal } I \text{ such that } E_n(R, I) \subseteq H \subseteq C_n(R, I)\end{aligned} \tag{1.2}$$

where $E_n(R, I)$ denotes the relative elementary subgroup of level I and $C_n(R, I)$ denotes the full congruence subgroup of level I.

In section 6 we recall the definitions of the hyperbolic unitary group and some important subgroups. In the last section we use the model theoretic approach of section 3 to prove (1.1).

2 Notation

Let G be a group and H, K be subsets of G. The subgroup of G generated by H is denoted by $\langle H \rangle$. If $g, h \in G$, let ${}^hg := hgh^{-1}$, $g^h := h^{-1}gh$ and $[g, h] := ghg^{-1}h^{-1}$. Set ${}^KH := \langle \{{}^kh | h \in H, k \in K\} \rangle$ and $H^K := \langle \{h^k | h \in H, k \in K\} \rangle$. Analogously define $[H, K]$ and HK. Instead of ${}^K\{g\}$ we write ${}^Kg$ (analogously we write g^K

instead of $\{g\}^K$, ${}^gH$ instead of ${}^{\{g\}}H$, $[g,K]$ instead of $[\{g\},K]$ etc.).

In this thesis, ring will always mean associative ring with 1 such that $1 \neq 0$. Ideal will mean two-sided ideal. By a multiplicative subset of a commutative ring C we mean a subset $S \subseteq C$ such that $0 \notin S$, $1 \in S$ and $xy \in S\ \forall x,y \in S$. If R is a ring and $m,n \in \mathbb{N}$, then the set of all invertible elements in R is denoted by R^* and the set of all $m \times n$ matrices with entries in R is denoted by $M_{m\times n}(R)$. We set $M_n(R) := M_{n\times n}(R)$. The identity matrix in $M_n(R)$ is denoted by e or $e_{n\times n}$ and the matrix with an 1 at position (i,j) and zeros elsewhere is denoted by e_{ij}. If $a = (a_{ij})_{ij} \in M_{m\times n}(R)$, we denote the transpose of a by a^t, the i-th row of a by a_{i*} and the j-th column of a by a_{*j}. If $a = (a_{ij})_{ij} \in M_n(R)$ is invertible, the entry of a^{-1} at position (i,j) is denoted by a'_{ij}, the i-th row of a^{-1} by a'_{i*} and the j-th column of a^{-1} by a'_{*j}. Further we denote by R^n the set of all columns $u = (u_1,\dots,u_n)^t$ with entries in R and by ${}^nR$ the set of all rows $v = (v_1,\dots,v_n)$ with entries in R. In sections 4 and 5, $e_i \in R^n$, where $i \in \{1,\dots,n\}$, denotes the column whose i-th entry is 1 and whose other entries are 0 and $f_i \in {}^nR$, where $i \in \{1,\dots,n\}$, denotes the row whose i-th entry is 1 and whose other entries are 0. In sections 6 and 7, $e_i \in R^{2n}$, where $i \in \{1,\dots,n,-n,\dots,-1\}$, denotes the column whose i-th entry is 1 and whose other entries are 0 if $i \in \{1,\dots,n\}$ and the column whose $(2n+1+i)$-th entry is 1 and whose other entries are 0 if $i \in \{-n,\dots,-1\}$. In sections 6 and 7, $f_i \in {}^{2n}R$, where $i \in \{1,\dots,n,-n,\dots,-1\}$, denotes the row whose i-th entry is 1 and whose other entries are 0 if $i \in \{1,\dots,n\}$ and the row whose $(2n+1+i)$-th entry is 1 and whose other entries are 0 if $i \in \{-n,\dots,-1\}$.

3 Standard groups

The concepts, constructions and results of this section are unpublished work of A. Bak. Their purpose is to provide a model theoretic setting for studying Chevalley groups, unitary groups, classical-like groups and their generalizations. This approach will be applied in the current dissertation to proving sandwich classification results for general linear and unitary groups.

Definition 3.1 Let G denote a group and B a set of subgroups of G such that

(1) for any $U,V \in B$ there is a $W \in B$ such that $W \subseteq U \cap V$ and

(2) for any $g \in G$ and $U \in B$ there is a $V \in B$ such that ${}^gV \subseteq U$.

Then B is called a *base of open subgroups of* $1 \in G$. B is called *discrete* (respectively *nondiscrete*), if it contains (respectively does not contain) the trivial subgroup.

Remark Let B be a base of open subgroups of $1 \in G$. The set of all left cosets of members of B is a base of open sets for a topology on G such that G is a topological group, i.e. such that the operations of taking inverse and multiplication are continuous (cf. [7]). This topology is the discrete one (i.e. any subset of G is open) if and only if B is discrete.

Definition 3.2 Let G be a group, E a subgroup of G and $Gen(E)$ a subset of G containing 1 such that E is generated by $Gen(E)$. Further let $B(E)$ be a set of subgroups of E and $G(\cdot)$ a rule which associates to each $U \in B(E)$ a normal subgroup $G(U)$ of G containing U. The quintuple $(G, E, Gen(E), B(E), G(\cdot))$ is called a *standard group* if the following are satisfied:

(3.2.1) $B(E)$ is a base of open subgroups of $1 \in E$, which contains E as a member.

(3.2.2) A subgroup of E which is generated by members of $B(E)$ is a member of B(E).

(3.2.3) If $U \in B(E)$ then $Gen(U) := Gen(E) \cap U$ generates U. Furthermore, it is assumed that if $g \in Gen(E)$ and $g \in G(U)$, then $g \in U$.

The elements of $Gen(E)$ are called *base generators* and the members of $B(E)$ are called *base subgroups*. For each $U \in B(E)$ the normal closure of U in E is denoted by $E(U)$ and the preimage of $Center(G/G(U))$ under the canonical homomorphism $G \to G/G(U)$ by $C(U)$.

Remark

(1) General linear groups and hyperbolic unitary groups are examples of standard groups. For details see Lemma 5.1 resp. Lemma 7.1.

(2) The following condition is satisfied in many situations including those in (1) above, motivates the key notions below of supplemented base and local map and is inherited by quotients (see 3.4 below), but is needed neither for the results of this section nor their applications in sections 5 and 7.

(3.2.4) Let U be a nontrivial member of $B(E)$. If $g \in Gen(E)$ such that $g \notin U$, then ${}^{E}g$ contains a (nontrivial) member V of $B(E)$, which is not contained in U.

In all of the examples in (1), it turns out that $g \in V$, thus guaranteeing that V is not contained in U.

(3) The following condition is also satisfied in many situations, including those in (1) above, and is inherited by quotients. However, only the weakened form of the condition, which is stated in (3.8.1) in Lemma 3.8, is needed for sandwich classification.

(3.2.5) If $g \in Gen(E)$ and U and V are subgroups of $B(E)$ such that $V \nsubseteq U$ then conjugation by g leaves some elements of $Gen(V) \backslash Gen(U)$ fixed.

Definition 3.3 A *morphism* $\phi : (G, E, Gen(E), B(E), G(\cdot)) \to (G', E', Gen(E'), B(E'), G'(\cdot))$ *of standard groups* is a group homomorphism $\phi : G \to G'$ which maps base generators to base generators and induces a continuous homomorphism $E \to E'$.

Definition 3.4 Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group, $U \in B(E)$ and $\psi : G \to G/G(U)$ the canonical homomorphism. Then $(G, E, Gen(E), B(E), G(\cdot))$ $/U := (G/U, E/U, Gen(E/U), B(E/U), G/U(\cdot))$ where $G/U := G/G(U) = \psi(G)$, $E/U := \psi(E)$, $Gen(E/U) := \psi(Gen(E))$, $B(E/U) := \{\psi(V)|V \in B(E)\}$ and $G/U(\psi(V)) := \psi(G(\langle U, V\rangle))$ is called a *quotient*. (In general, $\psi(G(V))$ is smaller than $\psi(G(\langle U, V\rangle))$.)

Lemma 3.5 *Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group and $U \in B(E)$. Then the quotient $(G, E, Gen(E), B(E), G(\cdot))/U$ is a standard group such that $Gen(\psi(V)) = \psi(Gen(\langle U, V\rangle))$.*

Proof Straightforward. □

There can be nontrivial subgroups $V \in B(E)$, other than E, which have property (3.2.4), namely if $U \in B(E)$ and $g \in Gen(E)$ such that $g \notin U$, then ${}^{V}g$ contains a nontrivial member of $B(E)$ which is not contained in U. In interesting cases, there are usually many such subgroups. The next definition is designed to carve out a useful concept for this situation and use it to define the notion of a local morphism.

Definition 3.6 Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group. A pair (A, B) is called a *supplemented base* for $(G, E, Gen(E), B(E), G(\cdot))$ if A and B and are sets of nontrivial subgroups of E (not necessarily members of $B(E)$) such that A forms a nondiscrete base of open subgroups of $1 \in E$, each member of B is contained in some member of A, and if $U \in A$ and $V \in B$ then $U \cap V$ contains a member of B. A supplemented base (A, B) is called *special*, if $A, B \subseteq B(E)$.

Definition 3.7 Let $\phi : (G, E, Gen(E), B(E), G(\cdot)) \to (G', E', Gen(E'), B(E'), G'(\cdot))$ be a morphism of standard groups. Let (A, B) be a special supplemented base for $(G, E, Gen(E), B(E), G(\cdot))$. Then ϕ is called *local with respect to* (A, B), if the following holds:

(3.7.1) $\phi(A, B) := (\phi(A), \phi(B))$ is a supplemented base (not necessarily a special supplemented base) for $(G', E', Gen(E'), B(E'), G'(\cdot))$.

(3.7.2) ϕ is injective on $G(U)$ for each member U of A.

(3.7.3) If $f' \in E'$, $g' \in Gen(E')\backslash\{1\}$ and $U \in A$ then ${}^{\phi(U)}(f'g')$ contains $\phi(V)$ for some nontrivial member V of B, which we may assume is contained in U.

A morphism ϕ is called *local*, if it is local for some special supplemented base (A, B) for $(G, E, Gen(E), B(E), G(\cdot))$.

Remark If it turns out that $E(U)$ is normal in G, for each $U \in B(E)$ then to prove the results of this section, one can replace in (3.7.2) above $G(U)$ by the smaller group $E(U)$.

In practice, we often find ourselves in the situation that we have a morphism ϕ and a special supplemented base (A, B) for the domain of ϕ such that ϕ satisfies

(3.7.1) and (3.7.2) and we feel that it should also satisfy (3.7.3). The following lemma is a useful tool for verifying the validity (3.7.3).

Lemma 3.8 *Let* $\phi : (G, E, Gen(E), B(E), G(\cdot)) \to (G', E', Gen(E'), B(E'), G'(\cdot))$ *be a morphism of standard groups and* (A, B) *a special supplemented base for* $(G, E, Gen(E), B(E), G(\cdot))$ *such that* (3.7.1) *and* (3.7.2) *hold. Assume the following:*

(3.8.1) If $g' \in Gen(E')\backslash\{1\}$ *and* $U \in A$ *then* ${}^{\phi(U)}g'$ *contains a* $\phi(V)$ *such that* $V \in B$ *and we may assume* $V \subseteq U$.

(3.8.2) If $f' \in Gen(E')$ *and* $V \in B$ *then* ${}^{f'}\phi(V)$ *contains a nontrivial element* $g' \in Gen(E')$.

Then ϕ *satisfies* (3.7.3) *and so is a local morphism.*

Proof Let $f' \in E'$. If $f' = 1$ then we are done, by (3.8.1). Assume $f' \neq 1$ and write f' as a product $f'_k \dots f'_1$ of nontrivial members of $Gen(E')$. We proceed by induction on k.
<u>case 1</u> Assume that $k = 1$. Let $U \in A$. Choose $U_1 \in A$ such that ${}^{f'_1}\phi(U_1) \subseteq \phi(U)$. Then ${}^{\phi(U)}({}^{f'_1}g') \supseteq {}^{\phi(U)}({}^{f'_1}({}^{\phi(U_1)}g')) \overset{(3.8.1)}{\supseteq} {}^{\phi(U)}({}^{f'_1}\phi(V))$ (for some $V \in B$) $\overset{(3.8.2)}{\supseteq} {}^{\phi(U)}g''$ (for some $g'' \in Gen(E')\backslash\{1\}$) $\overset{(3.8.1)}{\supseteq} \phi(V_1)$ (for some $V_1 \in B$) $\supseteq \phi(U \cap V_1) \supseteq$ (by definition of a supplemented base) $\phi(V_2)$ (for some $V_2 \in B$).
<u>case 2</u> Assume that $k > 1$. Let $U \in A$. Let $h' = f'_{k-1} \dots f'_1$. Thus $f' = f'_k \dots f'_1 = f'_k h'$. We can assume by induction on k that given $U_1 \in A$, ${}^{\phi(U_1)}({}^{h'}g') \supseteq \phi(V)$ for some $V \in B$. Now we proceed similarly to case 1, replacing g' by ${}^{h'}g'$ and f'_1 by f'_k. Here are the details. Choose U_1 such that ${}^{f'_k}\phi(U_1) \subseteq \phi(U)$. Then ${}^{\phi(U)}({}^{f'_k h'}g') \supseteq {}^{\phi(U)}({}^{f'_k}({}^{\phi(U_1)}({}^{h'}g'))) \overset{\text{I. A.}}{\supseteq} {}^{\phi(U)}({}^{f'_k}\phi(V)) \overset{(3.8.2)}{\supseteq} {}^{\phi(U)}g''$ (for some $g'' \in Gen(E')$ $\backslash\{1\}$) $\overset{(3.8.1)}{\supseteq} \phi(V_1)$ (for some $V_1 \in B$) $\supseteq \phi(U \cap V_1) \supseteq$ (by definition of a supplemented base) $\phi(V_2)$ (for some $V_2 \in B$). $\square$

Definition 3.9 Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group and let A be a nondiscrete base for E. Then $(G, E, Gen(E), B(E), G(\cdot))$ is called a *solution group for* A and we call the quadruple $(G, E, Gen(E), A))$ a *solution group*, if the following is satisfied: Given a noncentral element $h \in G$ and a member U of A, there are a $k \in \mathbb{N}$, $l_1, \dots, l_k \in \{-1, 1\}$, $\epsilon_0, \dots, \epsilon_k \in E$ and $g_0, \dots, g_k \in G$ such that $g_k \in Gen(E)$, g_k is nontrivial, ${}^{d_i}g_i \in U$ $\forall i \in \{0, \dots, k\}$, where $d_i = (\epsilon_i \cdot \dots \cdot \epsilon_0)^{-1}$ $\forall i \in \{0, \dots, k\}$, and

$$ {}^{\epsilon_k}([{}^{\epsilon_{k-1}}(\dots {}^{\epsilon_2}([{}^{\epsilon_1}([{}^{\epsilon_0}h, g_0]^{l_1}), g_1]^{l_2}) \dots), g_{k-1}]^{l_k}) = g_k. \tag{3.9.1} $$

Clearly (3.9.1) is equivalent to

$$ [\dots[[h, {}^{d_0}g_0]^{l_1}, {}^{d_1}g_1]^{l_2} \dots, {}^{d_{k-1}}g_{k-1}]^{l_k} = {}^{d_k}g_k. \tag{3.9.2} $$

(just conjugate (3.9.1) by $d_k = (\epsilon_k \cdot \dots \cdot \epsilon_0)^{-1}$). A standard group is called a *solution group*, if it is a solution group for some nondiscrete base A of E. The equations (3.9.1) and (3.9.2) are called *solution equations for* h *with respect to* A. In case there is a solution equation for h with respect to A, we shall say that h *satisfies a solution equation with respect to* A.

Remark In practice, to show that a standard group is a solution group for A, one has to supplement A to a supplemented base (A, B).

Definition 3.10 A *covering* of a standard group $(G, E, Gen(E), B(E), G(\cdot))$ is a set of local morphisms ϕ such that the domain of each ϕ is $(G, E, Gen(E), B(E), G(\cdot))$ and such that given a noncentral element $h \in G$ there is a morphism ϕ in the covering such that $\phi(h)$ is noncentral in the codomain of ϕ. For each local morphism ϕ of a covering Cov of $(G, E, Gen(E), B(E), G(\cdot))$, let $(A(\phi), B(\phi))$ denote a special supplemented base such that ϕ is local with respect to $(A(\phi), B(\phi))$. We shall say that a covering Cov is a *covering by solution groups*, if for each ϕ in Cov the codomain of ϕ is a solution group for $\phi(A(\phi))$.

Theorem 3.11 *Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group and h a noncentral element of G. If $(G, E, Gen(E), B(E), G(\cdot))$ has a covering by solution groups then ${}^{E}h$ contains a nontrivial member of $B(E)$. (Compare with* (3.2.4)*)*

Proof By assumption there is a local morphism $\phi : (G, E, Gen(E), B(E), G(\cdot)) \to (G', E', Gen(E'), B(E'), G'(\cdot))$ with respect to a special supplemented base (A, B) (for the domain of ϕ) such that the codomain of ϕ is a solution group for $\phi(A)$ and $\phi(h)$ is a noncentral element of G'. Hence there are a $k \in \mathbb{N}$, $l_1, \dots, l_k \in \{-1, 1\}$, $\epsilon'_0, \dots, \epsilon'_k \in E'$, $g'_0, \dots, g'_k \in G'$ and a $U' \in A' := \phi(A)$ such that $g'_k \in Gen(E')$, g'_k is nontrivial, ${}^{d'_i}g'_i \in U'$ $\forall i \in \{0, \dots, k\}$, where $d'_i = (\epsilon'_i \cdot \dots \cdot \epsilon'_0)^{-1}$ $\forall i \in \{0, \dots, k\}$, and

$$[\dots[[h', {}^{d'_0}g'_0]^{l_1}, {}^{d'_1}g'_1]^{l_2} \dots, {}^{d'_{k-1}}g'_{k-1}]^{l_k} = {}^{d'_k}g'_k. \tag{3.11.1}$$

Let $U \in A(\phi)$ such that $\phi(U) = U'$. Since ${}^{d'_i}g'_i \in U' = \phi(U)$ $\forall i \in \{0, \dots, k\}$, there are $x_0, \dots, x_k \in U$ such that $\phi(x_i) = {}^{d'_i}g'_i$ $\forall i \in \{0, \dots, k\}$. Let $x = [\dots[[h, x_0]^{l_1}, x_1]^{l_2} \dots, x_{k-1}]^{l_k}$. Clearly the l.h.s. of (3.11.1) equals $\phi(x)$ and the r.h.s. of (3.11.1) equals $\phi(x_k)$. Clearly ${}^{E}h \supseteq {}^{E}x \supseteq {}^{U}x$. We shall show that there is a $V \in B$ such that ${}^{U}x \supseteq V$. This will complete the proof, because (A, B) is a special supplemented base for $(G, E, Gen(E), B(E), G(\cdot))$ and therefore V is a nontrivial member of $B(E)$. Since ${}^{U}x \subseteq G(U)$, because $G(U)$ is normal in G, and ϕ is injective on $G(U)$ (at this point in the argument, we could replace $G(U)$ by $E(U)$ and only insist that ϕ be injective on $E(U)$, if $E(U)$ were normal in G), it suffices to show that ${}^{U'}\phi(x) = {}^{U'}({}^{d'_k}g'_k) \supseteq \phi(V)$ for some $V \in B$ which is contained in U. But this follows from the definition of a local morphism with respect to (A, B). □

Definition 3.12 Let $(G, E, Gen(E), B(E), G(\cdot))$ be a standard group such that

(1) $(G, E, Gen(E), B(E), G(\cdot))$ and each of its quotients have a covering by solution groups.

(2) $[C(U), E] = [E(U), E] = E(U)$ holds for any $U \in B(E)$.

Then $(G, E, Gen(E), B(E), G(\cdot))$ is called a *sandwich classification group*.

Theorem 3.13 *Let $(G, E, Gen(E), B(E), G(\cdot))$ be a sandwich classification group and H a subgroup of G. Then H is normalized by E if and only if either H is central, or there is a unique nontrivial $U \in B(E)$ such that $E(U) \subseteq H \subseteq C(U)$.*

Proof
"$\Rightarrow$": Assume that H is normalized by E. If H is central, we are done. Suppose H is noncentral. From Theorem 3.11 it follows that H contains a nontrivial member of $B(E)$. Let U be the largest nontrivial member of $B(E)$ such that $U \subseteq H$. We shall show that $H \subseteq C(U)$. The proof is by contradiction. Suppose H is not contained in $C(U)$. We shall produce a nontrivial $V \in B(E)$ such that V is not contained in U, but $V \subseteq H$. This will contradict the maximality of U. Let $\hat{H}$ denote the image of H in $(G, E, Gen(E), B(E), G(\cdot))/U$. Clearly $\hat{H}$ is normalized by E/U. If $\hat{H}$ is central in G/U, then we are done, because this implies by definition that $H \subseteq C(U)$. Suppose $\hat{H}$ is not contained in $center(G/U)$. Then by Theorem 3.11, $\hat{H}$ contains a nontrivial subgroup $\hat{V}$ of $B(E/U)$. Since $\hat{V} \in B(E/U)$, there is a $V \in B(E)$ such that $\hat{V} = V/(V \cap G(U))$. It follows that $V \subseteq HG(U)$. This implies $E(V) \subseteq HG(U)$, since both H and $G(U)$ are normalized by E. Hence

$$E(V) = [E, E(V)] \subseteq [E, HG(U)] \subseteq [E, H]({}^{H}[E, G(U)]) \subseteq H$$

since $[E, G(U)] = E(U) \subseteq H$. It follows that $V \subseteq H$ which contradicts the maximality of U (Clearly $V \nsubseteq U$ since the image $\hat{V}$ of V in $G/G(U)$ is nontrivial). Thus $E(U) \subseteq H \subseteq C(U)$.

Now we show the uniqueness of U. Let $V \in B(E)$, V nontrivial such that $E(V) \subseteq H \subseteq C(V)$. It follows that $E(U) \subseteq H \subseteq C(V)$ and $E(V) \subseteq H \subseteq C(U)$. Hence

$$E(U) = [E, E(U)] \subseteq [E, C(V)] = E(V)$$

and

$$E(V) = [E, E(V)] \subseteq [E, C(U)] = E(U).$$

By (3.2.3) it follows that $U \subseteq V$ and $V \subseteq U$. Thus $U = V$.

"$\Leftarrow$": If H is central it is clearly normalized by E. If there is a $U \in B(E)$ such that $E(U) \subseteq H \subseteq C(U)$, then

$$[H, E] \subseteq [C(U), E] = E(U) \subseteq H$$

and hence H is normalized by E. □

4 General linear groups

In this section, let R be an associative ring with identity, I an ideal (2-sided) in R and $n \in \mathbb{N}$. We shall recall the definitions of the following subgroups of the general linear group $GL_n(R)$; the preelementary groups $E_n(I)$, the relative elementary groups $E_n(R, I)$, the principal congruence subgroups $GL_n(R, I)$ and the full congruence subgroups $C_n(R, I)$. In the model theoretic setting of section 3, these groups are accounted for respectively by the groups U in $B(E)$, the groups $E(U)$, the groups $G(U)$ and the groups $C(U)$. The elementary group $E_n(R) := E_n(R, R)$ is accounted for by E in the model theoretic setting and the generators of $E_n(R)$, namely the elementary matrices, are accounted for by $Gen(E)$.

Definition 4.1 $GL_n(R) := (M_n(R))^*$ is called the *general linear group.*

Definition 4.2 Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$ and $x \in R$. Then $t_{ij}(x) := e + xe_{ij}$ is called an *elementary transvection.* The subgroup of $GL_n(R)$ generated by all elementary transvections is called the *elementary subgroup* and is denoted by $E_n(R)$. An elementary transvection $t_{ij}(x)$ is called *elementary of level I* or *I-elementary* if $x \in I$. The subgroup of $GL_n(R)$ generated by all I-elementary transvections is called the *preelementary subgroup of level I* and is denoted by $E_n(I)$. Its normal closure in $E_n(R)$ is called the *elementary subgroup of level I* and is denoted by $E_n(R, I)$.

Definition 4.3 Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Define $p_{ij} := e + e_{ij} - e_{ji} - e_{ii} - e_{jj} = t_{ij}(1)t_{ji}(-1)t_{ij}(1) \in E_n(R)$. It is easy show that $(p_{ij})^{-1} = p_{ji}$. If $1 < i < j < n$, p_{ij} has the form

$$\begin{array}{cc}
 & \begin{matrix} 1 & & & i & & & & j & & & n \end{matrix} \\
\begin{matrix} 1 \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ n \end{matrix} &
\begin{pmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 0 & & & & 1 & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
 & & & -1 & & & & 0 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{pmatrix}
\end{array}$$

where all blank entries are zero.

Lemma 4.4 *The relations*

$$t_{ij}(x)t_{ij}(y) = t_{ij}(x + y), \tag{R1}$$
$$[t_{ij}(x), t_{kl}(y)] = e \textit{ and} \tag{R2}$$
$$[t_{ij}(x), t_{jk}(y)] = t_{ik}(xy) \tag{R3}$$

hold where $i \neq l, j \neq k$ in (R2) and $i \neq k$ in (R3).

Proof Straightforward computation. □

Definition 4.5 The kernel of the group homomorphism $GL_n(R) \to GL_n(R/I)$ induced by the canonical map $R \to R/I$ is called the *principal congruence subgroup of level I* and is denoted by $GL_n(R, I)$.

Remark Obviously $GL_n(R, I)$ is a normal subgroup of $GL_n(R)$.

Definition 4.6 The preimage of $Center(GL_n(R)/GL_n(R, I))$ under the canonical homomorphism $GL_n(R) \to GL_n(R)/GL_n(R, I)$ is called the *full congruence subgroup of level I* and is denoted by $C_n(R, I)$.

Remark

(1) Obviously $GL_n(R, I) \subseteq C_n(R, I)$ and $C_n(R, I)$ is a normal subgroup of $GL_n(R)$.

(2) Sometimes $C_n(R, I)$ is defined as the preimage of $Center(GL_n(R/I))$ under the group homomorphism $GL_n(R) \to GL_n(R/I)$ induced by the canonical map $R \to R/I$. One can show, using the fact that $Center(GL_n(R/I))$ equals the set of all matrices in $GL_n(R/I)$ which commute with all elementary transvections, that the two definitions are equivalent.

Recall that R is called *almost commutative* if it is module finite over a subring of $Center(R)$.

Lemma 4.7 *If $n \geqslant 3$ and R is almost commutative, then the equalities*

$$\begin{aligned} &[C_n(R, I), E_n(R)] \\ =&[E_n(R, I), E_n(R)] \\ =&E_n(R, I) \end{aligned}$$

hold.

Proof See [8], Corollary 14. □

5 Sandwich classification for general linear groups

In this section, we construct in the setting of general linear groups, specific supplemented bases, local maps, solution groups and coverings by these solution groups, and show in the Solution Group Lemma 5.7 that any noncentral element in any solution group of any of these coverings satisfies a solution equation. 5.7 is the main technical input of the section. A road map of the proof is provided at its conclusion, in terms of a (long) inverted tree diagram. Then we deduce the sandwich classification theorem (1.2) for subgroups of $GL_n(R)$ normalized by $E_n(R)$ from Theorem 3.13.

In this section let $n \geqslant 3$, R be a ring and C a subring of $Center(R)$. For any ideal I of R and multiplicative subset $S \subseteq C$, set $R_S := S^{-1}R$ and $I_S := S^{-1}I$. Let

$$\phi_S : GL_n(R)/GL_n(R, I) \to GL_n(R_S)/GL_n(R_S, I_S)$$

be the homomorphism induced by F_S where

$$F_S : GL_n(R) \to GL_n(R_S)$$

is the homomorphism induced by the localisation homomorphism

$$f_S : R \to R_S.$$

Let

$$\psi : GL_n(R) \to GL_n(R)/GL_n(R, I)$$

and

$$\rho_S : GL_n(R_S) \to GL_n(R_S)/GL_n(R_S, I_S)$$

be the canonical homomorphisms. Note that the diagram

$$\begin{tikzcd} GL_n(R) \arrow[r, "\psi"] \arrow[d, "F_S"] & GL_n(R)/GL_n(R,I) \arrow[d, "\phi_S"] \\ GL_n(R_S) \arrow[r, "\rho_S"] & GL_n(R_S)/GL_n(R_S, I_S) \end{tikzcd}$$

is commutative for any ideal I of R and multiplicative subset $S \subseteq C$. For any maximal ideal m of C set $S_m := C \backslash m$ and $\phi_m := \phi_{S_m}$ (define F_m, f_m, ρ_m, R_m and I_m similarly).

Lemma 5.1 *Set*

$$\begin{aligned} &G := GL_n(R), \\ &E := E_n(R), \\ &Gen(E) := \{t_{ij}(x) | x \in R, i, j \in \{1, \ldots, n\}, i \neq j\}, \\ &B(E) := \{E_n(I) | I \text{ ideal of } R\} \textit{ and} \\ &G(E_n(I)) := GL_n(R, I)\ \forall I \textit{ ideal of } R. \end{aligned}$$

Then $(G, E, Gen(E), B(E), G(\cdot))$ is a standard group. Further $E(E_n(I)) = E_n(R, I)$ and $C(E_n(I)) = C_n(R, I)$ for any ideal I of R.

Proof We have to show that the conditions (3.2.1) – (3.2.3) in Definition 3.2 are satisfied.

(3.2.1) Obviously $B(E)$ is a base of open subgroups of $1 \in E$, since it contains the identity subgroup $\{1\} = E_n(\{0\})$. Clearly $E = E_n(R) \in B(E)$.

(3.2.2) Let $\{I_j | j \in J\}$ be a family of ideals of R. One checks easily that $\langle \bigcup_{j \in J} E_n(I_j) \rangle = E_n(\langle \bigcup_{j \in J} I_j \rangle)$.

(3.2.3) Let $U \in B(E)$. Then there is an ideal I of R such that $U = E_n(I)$. Clearly $Gen(U) = Gen(E) \cap U$ contains all the elements $t_{ij}(x)$ where $i, j \in \{1, \ldots, n\}$, $i \neq j$ and $x \in I$. But these elements generate U. Hence $U = E_n(I)$ is generated by $Gen(U)$. Now let $g = t_{ij}(x) \in Gen(E)$ and $U = E_n(I) \in B(E)$. Assume that $g \in G(U) = GL_n(R, I)$. Then all nondiagonal entries of g lie in I. It follows that $x \in I$ and hence $g = t_{ij}(x) \in E_n(I) = U$. $\square$

For the rest of this section, we assume that R is a Noetherian C-module.

Lemma 5.2 *Let I be an ideal of R and $S \subseteq C$ a multiplicative subset. Then there is an $s_0 \in S$ with the property that if $x \in s_0R$ and $\exists t \in S : tx \in I$, then $x \in I$. It follows that ϕ_S is injective on $\psi(GL_n(R, I + s_0R))$.*

Proof For any $s \in S$ set $Y(s) := \{x \in R | sx \in I\}$. Then for any $s \in S$, $Y(s)$ is a C-submodule of R. Since R is a Noetherian C-module, the set $\{Y(s)|s \in S\}$ has a maximal element $Y(s_0)$. Clearly all elements $x \in s_0R$ have the property that $tx \in I$ for some $t \in S$ implies $x \in I$. We will show now that ϕ_S is injective on $\psi(GL_n(R, I + s_0R))$. Let $g'_1, g'_2 \in \psi(GL_n(R, I + s_0R))$ such that $\phi_S(g'_1) = \phi_S(g'_2)$. Since $g'_1, g'_2 \in \psi(GL_n(R, I + s_0R))$, there are $g_1, g_2 \in GL_n(R, I + s_0R)$ such that $\psi(g_1) = g'_1$ and $\psi(g_2) = g'_2$. Set $h := (g_1)^{-1}g_2 \in GL_n(R, I + s_0R)$. Clearly $\phi_S(g'_1) = \phi_S(g'_2)$ is equivalent to $F_S(h) \in GL_n(R_S, I_S)$, i.e. $F_S(h) \equiv e(mod\ I_S)$. We want to show that $g'_1 = g'_2$ which is equivalent to $h \in GL_n(R, I)$, i.e. $h \equiv e(mod\ I)$. Let $i, j \in \{1, \dots, n\}$ such that $i \neq j$. Since $f_S(h_{ij}) \in I_S$,

$$\begin{aligned} &\exists x \in I, s \in S : \frac{h_{ij}}{1} = \frac{x}{s} \\ \Rightarrow\ &\exists x \in I, s, t \in S : t(h_{ij}s - x) = 0 \\ \Rightarrow\ &\exists x \in I, s, t \in S : sth_{ij} = tx \in I \\ \Rightarrow\ &\exists u \in S : uh_{ij} \in I. \end{aligned} \tag{5.2.1}$$

Since $h \in GL_n(R, I + s_0R)$, $h_{ij} \in I + s_0R$. Hence there are elements $y \in I$ and $z \in s_0R$ such that $h_{ij} = y + z$. (5.2.1) implies that $uz \in I$. It follows that $z \in I$ since $z \in s_0R$. Thus $h_{ij} \in I$. Analogously one can show that $h_{ii} - 1 \in I$ for all $i \in \{1, \dots, n\}$. Hence $h \equiv e(mod\ I)$. This implies $g'_1 = g'_2$ and thus ϕ_S is injective on $\psi(GL_n(R, I + s_0R))$. □

We construct now a specific supplemented base that we will use to construct specific local morphisms. In the lemma below we use the following convention. If $x \in R$, then RxR denotes the (twosided) ideal of R generated by x.

Lemma 5.3 *Let I be an ideal of R, $S \subseteq C$ a multiplicative subset and $s_0 \in S$ as in the previous lemma. Set $A := \{E_n(ss_0R)|s \in S\}$ and $B := \{E_n(Rxs_0R)|x \in R, xs_0 \notin I\}$. Then (A, B) is a special supplemented base for $GL_n(R)$ and $F_S(A, B)$ is a supplemented base for $GL_n(R_S)$.*

Proof First we show (A, B) is a special supplemented base for $GL_n(R)$. Clearly A and B are sets of nontrivial subgroups of E. We show now that A is a (nondiscrete) base of open subgroups of $1 \in E$. Therefore we must show that A satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let $U = E_n(ss_0R), V = E_n(ts_0R) \in A$. Set $W := E_n(sts_0R) \in A$. Then clearly $W \subseteq U \cap V$.

(2) Let $g \in E$ and $U = E_n(ss_0R) \in A$. There is a $K \in \mathbb{N}$ such that g is the product of K elementary transvections. Set $V := E_n((ss_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}R) \in A$. Then ${}^gV \subseteq U$ (see Lemma 4.6 in [2]).

Hence A is a base of open subgroups of $1 \in E$. Let $E_n(Rxs_0R) \in B$. Then $E_n(Rxs_0R) \subseteq E_n(s_0R) \in A$. It remains to show that if $U \in A$ and $V \in B$ then $U \cap V$ contains a member of B. Let $U = E_n(ss_0R) \in A$ and $V = E_n(Rxs_0R) \in B$. Set $W := E_n(Rxss_0R)$. Clearly $xs_0 \notin I$ implies that $xss_0 \notin I$ (by the definition of s_0, see the previous lemma). Hence $W \in B$. Obviously $W \in U \cap V$. Since $A, B \subseteq B(E)$, (A, B) is a special supplemented base for $GL_n(R)$.
Now we show $F_S(A, B)$ is a supplemented base for $GL_n(R_S)$. Clearly $F_S(A)$ and $F_S(B)$ are sets of nontrivial subgroups of $E' := E_n(R_S)$. We show now that $F_S(A)$ is a (nondiscrete) base of open subgroups of $1 \in E'$. Therefore we show that $F_S(A)$ satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let $U = F_S(E_n(ss_0R)), V = F_S(E_n(ts_0R)) \in F_S(A)$. Set $W := F_S(E_n(sts_0R)) \in F_S(A)$. Then clearly $W \subseteq U \cap V$.

(2) Let $g \in E'$ and $U = F_S(E_n(ts_0R)) \in F_S(A)$. There are a $K \in \mathbb{N}$ and elementary transvections $\tau_1 = t_{i_1j_1}(\frac{x_1}{s_1}), \dots, \tau_K = t_{i_kj_k}(\frac{x_K}{s_K}) \in E'$ such that $g = \tau_1 \dots \tau_K$. Set $s := s_1 \dots s_K$ and $V := F_S(E_n((sts_0)^{2 \cdot 4^K + 4^{K-1} + \dots + 4}R)) \in F_S(A)$. Then ${}^gV \subseteq U$ (see Lemma 4.6 in [2]).

Hence $F_S(A)$ is a base of open subgroups of $1 \in E'$. That each member of $F_S(B)$ is contained in some member of $F_S(A)$ follows from the fact that any member of B is contained in a member of A. That given $U \in F_S(A)$ and $V \in F_S(B)$, $U \cap V$ contains a member of $F_S(B)$ follows from the fact that given $U \in A$ and $V \in B$, $U \cap V$ contains a member of B. Hence $F_S(A, B)$ is a supplemented base for $GL_n(R_S)$. $\square$

Now we construct specific local morphisms which will be used to prove (1.2).

Lemma 5.4 *Let I be an ideal of R and $S \subseteq C$ a multiplicative subset such that $S \cap I = \varnothing$. Then ϕ_S is a local morphism of standard groups.*

Proof First we show that ϕ_S is a morphism of standard groups. Clearly ϕ_S maps a base generator to a base generator. Since $\{1\}$ is base subgroup of $GL_n(R)/GL_n(R, I)$, the topology induced by the base subgroups of $GL_n(R)/GL_n(R, I)$ is the discrete one. It follows that ϕ_S induces a continuous homomorphism $E_n(R)/(E_n(R) \cap GL_n(R, I)) \to E_n(R_S)/(E_n(R_S) \cap GL_n(R_S, I_S))$. Hence ϕ_S is a morphism of standard groups.

Let (A, B) be the special supplemented base for $GL_n(R)$ defined in the previous lemma. Since ψ induces a surjective homomorphism $E_n(R) \to E_n(R)/(E_n(R) \cap GL_n(R, I))$, it follows easily that $\psi(A, B)$ is a special supplemented base for $GL_n(R)/GL_n(R, I)$. We will show now that ϕ_S is local with respect to the special supplemented base $(A(\phi_S), B(\phi_S)) := \psi(A, B)$. Therefore we have to show that the conditions (3.7.1) – (3.7.3) in Definition 3.7 are satisfied.

(3.7.1) By the previous lemma, $F_S(A, B)$ is a supplemented base for $GL_n(R_S)$. Since ρ_S induces a surjective homomorphism $E_n(R_S) \to E_n(R_S)/(E_n\ (R_S) \cap GL_n(R_S, I_S))$, it is easy to deduce that $\rho_S(F_S(A, B))$ is a supplemented base for $GL_n(R_S)/GL_n(R_S,I_S)$. Since $\rho_S \circ F_S = \phi_S \circ \psi$, it follows that $\phi_S(\psi(A,B))$ is a supplemented base for $GL_n(R_S)/GL_n(R_S,I_S)$.

(3.7.2) By Lemma 5.2, ϕ_S is injective on $\psi(GL_n(R, I + s_0R))$. Let $U \in \psi(A)$. Then there is an $s \in S$ such that $U = \psi(E_n(ss_0R))$. Hence $G(U) = G(\psi(E_n(ss_0R))) = \psi(GL_n(R, I + ss_0R)) \subseteq \psi(GL_n(R, I + s_0R))$. It follows that ϕ_S is injective on $G(U)$ for any $U \in \psi(A)$.

(3.7.3) It suffices to show that the conditions (3.8.1) and (3.8.2) in Lemma 3.8 are satisfied.

(3.8.1) Let $g' = \rho_S(t_{ij}(z))$ be a nontrivial base generator in $GL_n(R_S)/GL_n(R_S,I_S)$, and $U = \psi(E_n(ss_0R)) \in \psi(A)$. Choose an $x \in R$ and a $t \in S$ such that $z = \frac{x}{t}$. Since g' is nontrivial, $z \notin I_S$ and hence $xs_0 \notin I$. Set $V := \psi(E_n(R(ss_0)^4xR)) \in \psi(B)$. One can show routinely, using the relations $(R1)-(R3)$ in Lemma 4.4, that ${}^{\phi_S(U)}g'$ contains $\phi_S(V)$. Since $\psi(A,B)$ is a supplemented base, there is a $W \in \psi(B)$ such that $W \subseteq U \cap V$. Clearly ${}^{\phi_S(U)}g'$ contains $\phi_S(W)$ and $W \subseteq U$.

(3.8.2) Let $f' = \rho_S(t_{ij}(\frac{x}{s}))$ be a base generator in $GL_n(R_S)/GL_n(R_S,I_S)$ and $V = \psi(E_n(Rys_0R)) \in \psi(B)$. Choose a nontrivial base generator $g' \in \phi_S(V)$ which commutes with f' (e.g. $g' = \rho_S(t_{ij}(\frac{ys_0}{1})))$. Then ${}^{f'}\phi_S(V)$ clearly contains ${}^{f'}g' = g'$.

Hence ϕ_S is local with respect to $(A(\phi_S), B(\phi_S)) = \psi(A,B)$. □

Next we show that the local morphisms ϕ_m where m is a maximal ideal of C such that $I \cap C \subseteq m$ form a covering.

Lemma 5.5 *Any quotient of the standard group* $(G, E, Gen(E), B(E), G(\cdot))$ *(where* G*,* E*,* $Gen(E)$*,* $B(E)$ *and* $G(\cdot)$ *are defined as in Lemma 5.1) has a covering.*

Proof Let I be an ideal of R. Set $Z := \{\phi_m | m \text{ maximal ideal of C}, I \cap C \subseteq m\}$. We show that Z is a covering of the standard group $GL_n(R)/GL_n(R,I)$. By the previous lemma, for any maximal ideal m of C such that $I \cap C \subseteq m$, ϕ_m is a local morphism (note that $I \cap C \subseteq m$ implies $S_m \cap I = \emptyset$). It remains to show that for any noncentral $g' \in GL_n(R)/GL_n(R,I)$ there is a maximal ideal m of C such that $I \cap C \subseteq m$ and $\phi_m(g')$ is noncentral. Let $g' \in GL_n(R)/GL_n(R,I)$ be noncentral. Then there is an $h' \in GL_n(R)/GL_n(R,I)$ such that $g'h' \neq h'g'$. Let $g, h \in GL_n(R)$ such that $g' = gGL_n(R,I)$ and $h' = hGL_n(R,I)$. Set $\sigma := [g^{-1}, h^{-1}]$. Clearly $g'h' \neq h'g'$ implies $\sigma \notin GL_n(R,I)$. Hence $\sigma_{ij} \notin I$ for some $i,j \in \{1,\dots,n\}$ such that $i \neq j$ or $\sigma_{ii} - 1 \notin I$ for some $i \in \{1,\dots,n\}$.

case 1 Assume that $\sigma_{ij} \notin I$ for some $i,j \in \{1,\dots,n\}$ such that $i \neq j$. Set $Y := \{c \in C | c\sigma_{ij} \in I\}$. Since $\sigma_{ij} \notin I$, Y is a proper ideal of C. Hence it is contained in a maximal ideal m of C. Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \emptyset$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin GL_n(R_m, I_m)$.

Obviously $(F_m(\sigma))_{ij} = f_m(\sigma_{ij})$. Assume $(F_m(\sigma))_{ij} \in I_m$. Then

$$\begin{aligned} &\exists x \in I, s \in S_m : \frac{\sigma_{ij}}{1} = \frac{x}{s} \\ \Rightarrow\ &\exists x \in I, s, t \in S_m : t(\sigma_{ij} s - x) = 0 \\ \Rightarrow\ &\exists x \in I, s, t \in S_m : st\sigma_{ij} = tx \in I \\ \Rightarrow\ &\exists u \in S_m : u\sigma_{ij} \in I. \end{aligned}$$

But this contradicts $S_m \cap Y = \varnothing$. Hence $(F_m(\sigma))_{ij} \notin I_m$ and thus $\phi_m(g')$ is non-central.

case 2 Assume that $\sigma_{ii}-1 \notin I$ for some $i \in \{1, \ldots, n\}$. Set $Y := \{c \in C | c(\sigma_{ii}-1) \in I\}$. Since $\sigma_{ii} - 1 \notin I$, Y is a proper ideal of C. Hence it is contained in a maximal ideal m of C. Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \varnothing$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin GL_n(R_m, I_m)$. Obviously $(F_m(\sigma))_{ii} - 1 = f_m(\sigma_{ii}) - 1 = f_m(\sigma_{ii} - 1)$. Assume $(F_m(\sigma))_{ii} - 1 \in I_m$. Then

$$\begin{aligned} &\exists x \in I, s \in S_m : \frac{\sigma_{ii} - 1}{1} = \frac{x}{s} \\ \Rightarrow\ &\exists x \in I, s, t \in S_m : t((\sigma_{ii} - 1)s - x) = 0 \\ \Rightarrow\ &\exists x \in I, s, t \in S_m : st(\sigma_{ii} - 1) = tx \in I \\ \Rightarrow\ &\exists u \in S_m : u(\sigma_{ii} - 1) \in I. \end{aligned}$$

But this contradicts $S_m \cap Y = \varnothing$. Hence $(F_m(\sigma))_{ii} - 1 \notin I_m$ and thus $\phi_m(g')$ is noncentral. $\square$

The following lemma will be used in the proof of Lemma 5.7.

Lemma 5.6 *Let K be a commutative ring and A a finite K-algebra. Then A is a Dedekind finite ring, i.e. if $x \in A$ is right or left invertible, then x is invertible.*

Proof Let $x, y \in A$ such that $xy = 1$. Define the maps

$$\begin{aligned} \alpha : A &\to A \\ z &\mapsto xz \end{aligned}$$

and

$$\begin{aligned} \beta : A &\to A \\ z &\mapsto yz. \end{aligned}$$

One checks easily that α and β are K-module-homomorphisms, $\alpha \circ \beta = id_A$ and α is surjective. By Nakayama's Lemma, α is a K-module-isomorphism. Hence it has an inverse α^{-1}. Since

$$\begin{aligned} &\beta \\ =&id_A \circ \beta \\ =&(\alpha^{-1} \circ \alpha) \circ \beta \end{aligned}$$

$$
\begin{aligned}
&=\alpha^{-1} \circ (\alpha \circ \beta)\\
&=\alpha^{-1} \circ id_A\\
&=\alpha^{-1},
\end{aligned}
$$

β is an isomorphism. Hence there is a $z \in A$ such that $yz = 1$. It follows that $yx = yxyz = yz = 1$. □

Now we show that the codomains of the local morphisms ϕ_m are solution groups and hence $(G, E, Gen(E), B(E), G(\cdot))$ and each of its quotients have a covering by solution groups.

Solution Group Lemma 5.7 *Let I be an ideal of R and m a maximal ideal of C such that $I \cap C \subseteq m$. Then the codomain of ϕ_m is a solution group for $A' := \phi_m(A(\phi_m))$ where $A(\phi_m)$ is defined as in Lemma 5.4.*

Proof Let $h' \in GL_n(R_m)/GL_n(R_m, I_m)$ be noncentral. We have to show that h' satisfies a solution equation with respect to A'. Let $U' \in A'$. Set $\hat{R}_m := R_m/I_m$. Let $\eta : GL_n(R_m)/GL_n(R_m, I_m) \to GL_n(\hat{R}_m)$ be the homomorphism induced by the canonical homomorphism $R_m \to \hat{R}_m$. One checks easily that η is injective. Hence $h := \eta(h') \in GL_n(\hat{R}_m)$ is noncentral. Set $A := \eta(A')$ and $U' = \eta(U)$. It is easy to show that A is a nondiscrete base of open subgroups of $1 \in E_n(\hat{R}_m)$ (notice that η induces an isomorphism $E_n(R_m)/(GL_n(R_m, I_m) \cap E_n(R_m)) \to E_n(\hat{R}_m)$). Choose $U_0, \dots, U_4 \in A$ such that for all $(k+1)$-tuples $(\epsilon_0, \dots, \epsilon_k)$ used in this proof, ${}^{d_i}U_i \subseteq U \ \forall i \in \{0, \dots, k\}$ (possible since A is a base of open subgroups of $1 \in E_n(\hat{R}_m)$ and there are only finitely many $(k+1)$-tuples $(\epsilon_0, \dots, \epsilon_k)$ which are used in this proof). Since $U_0, \dots, U_4 \in A$, there are $t_0, \dots, t_4 \in S_m$ such that $U_i = \eta(\phi_m(\psi(E_n(t_i s_0 R))) \ \forall i \in \{0, \dots, 4\}$. Set $s_i := f_m(t_i s_0) + I_m$ $(i = 0, \dots, 4)$. Since R is a Noetherian C-module, R_m is semilocal and hence $\hat{R}_m$ has stable rank 1 (see [5]). It follows that there is a matrix $\epsilon_0 \in E_n(\hat{R}_m)$ of the form $\epsilon_0 = \begin{pmatrix} e_{(n-1)\times(n-1)} & * \\ 0 & 1 \end{pmatrix}$ such that $(a_1, \dots, a_{n-1})$ is unimodular where $(a_1, \dots, a_n)^t =: \alpha$ is the first column of ${}^{\epsilon_0}h$. Since $(a_1, \dots, a_{n-1})$ is unimodular, there is a matrix $\epsilon_1 \in E_n(\hat{R}_m)$ of the form

$$\epsilon_1 = \begin{pmatrix} e_{(n-1)\times(n-1)} & 0 \\ * & 1 \end{pmatrix} \text{ such that } \epsilon_1 \alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ 0 \end{pmatrix}.$$

<u>case 1</u> Assume that $\rho := {}^{\epsilon_0}h$ does not commute with $t_{12}(s_1)$.
Set $g_0 := t_{12}(s_0) \in U_0$. We show now that $[{}^{\epsilon_0}h, g_0]$ is noncentral. Suppose that $[{}^{\epsilon_0}h, g_0]$ is central. Then $[{}^{\epsilon_0}h, g_0] = ue$ for some $u \in Center(\hat{R}_m)$. Clearly $\epsilon_1[{}^{\epsilon_0}h, g_0] = \epsilon_1(e + \rho_{*1} s_1 \rho'_{2*}) g_0^{-1} = \epsilon_1 g_0^{-1} + \epsilon_1 \alpha s_1 \rho'_{2*} g_0^{-1}$. Since the last row of $\epsilon_1 \alpha s_1 \rho'_{2*} g_0^{-1}$ is zero, the last row of $\epsilon_1[{}^{\epsilon_0}h, g_0]$ equals the last row of $\epsilon_1 g_0^{-1}$. Hence $(\epsilon_1[{}^{\epsilon_0}h, g_0])_{nn} = 1$. On the other hand $\epsilon_1[{}^{\epsilon_0}h, g_0] = u\epsilon_1$ and hence $(\epsilon_1[{}^{\epsilon_0}h, g_0])_{nn} = (u\epsilon_1)_{nn} = u$ which is a contradiction since ${}^{\epsilon_0}h$ does not commute with g_0 by assumption. Now we show that ${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$ has a zero entry. Clearly ${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0] = \epsilon_1 g_0^{-1} \epsilon_1^{-1} + \epsilon_1 \alpha s_0 \rho'_{2*} g_0^{-1} \epsilon_1^{-1}$. Hence the last row of ${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$ equals the last row of $\epsilon_1 g_0^{-1} (\epsilon_1)^{-1} = e - (\epsilon_1)_{*1} s_0 (\epsilon_1)'_{2*}$. Clearly $((\epsilon_1)_{*1} s_0 (\epsilon_1)'_{2*})_{n1} = 0$. Therefore $(e - (\epsilon_1)_{*1} s_0 (\epsilon_1)'_{2*})_{n1} = 0$ and hence

$({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0])_{n1} = 0$. Since $[{}^{\epsilon_0}h, g_0]$ is noncentral, ${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$ is noncentral.

<u>case 1.1</u> Assume that $\sigma := {}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$ does not commute with $t_{1n}(s_1)$.
Set $g_1 := t_{1n}(s_1) \in U_1$. Clearly $[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1] = [\sigma, g_1] = g_1^{-1} + \sigma_{*1}s_1\sigma'_{n*}g_1^{-1}$. Since $\sigma_{n1} = 0$, the last row of $\xi := [{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1] = [\sigma, g_1]$ equals the last row of g_1^{-1} which equals f_n. Assume that ξ is central. Since the last row of ξ equals f_n, $\xi = e$. But this contradicts the assumption that σ does not commute with g_1. Hence ξ is noncentral. Clearly ξ has the form

$$\xi = \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix}$$

where $x = (x_2, \dots, x_n)^t \in (\hat{R}_m)^{n-1}$ and $A \in M_{n-1}(\hat{R}_m)$.

<u>case 1.1.1</u> Assume $A \neq e_{(n-1)\times(n-1)}$.
For any $l \in \{1, \dots, n-1\}$ set $\omega(l) := [\xi, t_{ln}(s_2)]$. Then for all $l \in \{1, \dots, n-1\}$, $\omega(l)$ has the form

$$\omega(l) = \begin{pmatrix} e_{(n-1)\times(n-1)} & s_2(A - e_{(n-1)\times(n-1)})_{*l} \\ 0 & 1 \end{pmatrix}.$$

Since $A \neq e_{(n-1)\times(n-1)}$ there are $l, j \in \{1, \dots, n-1\}$ such that $(\omega(l))_{jn} \neq 0$. Since $(\omega(l))_{jn} \in \hat{R}$, there are an $a' \in R$ and an $s' \in S$ such that $(\omega(l))_{jn} = \frac{a'}{s'} + I_m$. Set $s := \frac{s'}{1} + I_m$ and $a := \frac{a'}{1} + I_m$. Choose an $i \neq j, n$ and set $g_2 := t_{ln}(s_2) \in U_2$, $g_3 := t_{ij}(s_3s_4s) \in U_3$ and $g_4 := t_{in}(-s_3s_4s(\omega(l))_{jn}) = t_{in}(-s_3s_4a) \in U_4$. Then one checks easily that

$$[[[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1], g_2], g_3] = [\omega(l), g_3] = g_4.$$

Since $(\omega(l))_{jn} \neq 0$ and s_3s_4s is invertible, $-s_3s_4s(\omega(l))_{jn} \neq 0$. Hence $g_4 \neq e$. Let $\eta^{-1} : \eta(GL_n(R_m)/GL_n(R_m, I_m)) \to GL_n(R_m)/GL_n(R_m, I_m)$ be the inverse of η and set $g'_i := \eta^{-1}(g_i)\ \forall i \in \{0, \dots, 4\}$, $\epsilon'_i := \eta^{-1}(\epsilon_i)\ \forall i \in \{0, 1\}$ and $d'_i := \eta^{-1}(d_i)\ \forall i \in \{0, \dots, 4\}$. Then $\epsilon'_0, \epsilon'_1, \epsilon'_2 \in E' := E_n(R_m)/(GL_n(R_m, I_m) \cap E_n(R_m))$, $g'_4 \in Gen(E')\backslash\{e\}$, ${}^{d'_i}g'_i \in U'\ \forall i \in \{0, \dots 4\}$ and

$$[[[{}^{\epsilon'_1}[{}^{\epsilon'_0}h', g'_0], g'_1], g'_2], g'_3] = g'_4.$$

<u>case 1.1.2</u> Assume $A = e_{(n-1)\times(n-1)}$.
Since ξ is noncentral, there is a $j \in \{1, \dots, n-1\}$ such that $x_j \neq 0$. Since $x_j \in \hat{R}$, there are an $a' \in R$ and an $s' \in S$ such that $x_j = \frac{a'}{s'} + I_m$. Set $s := \frac{s'}{1} + I_m$ and $a := \frac{a'}{1} + I_m$. Choose an $i \neq j, n$ and set $g_2 := t_{ij}(s_2s_3s) \in U_2$. Then $[\xi, g_2] = t_{in}(-s_2s_3sx_j) = t_{in}(-s_2s_3a) \in U_3$. As in case 1.1.1, pull this equation back to $GL_n(R_m)/GL_n(R_m, I_m)$ by applying η^{-1}.

<u>case 1.2</u> Assume that $\sigma = {}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$ commutes with $t_{1n}(s_1)$.
Then it follows that the last row of σ equals rf_n for some $r \in \hat{R}_m$. Clearly σ has the form

$$\sigma = \begin{pmatrix} A & x \\ 0 & r \end{pmatrix}$$

where $x = (x_2, \ldots, x_n)^t \in (\hat{R}_m)^{n-1}$ and $A \in M_{n-1}(\hat{R}_m)$. Since $\sigma \in GL_n(\hat{R}_m)$, it follows that r is right invertible. Since R is a Noetherian C-module, R is almost commutative. It follows that $\hat{R}_m$ is almost commutative and hence r is invertible, by Lemma 5.6.

<u>case 1.2.1</u> Assume $A \neq re_{(n-1)\times(n-1)}$.
For any $l \in \{1, \ldots, n-1\}$ set $\omega(l) := [\sigma, t_{ln}(s_1)]$. Then for all $l \in \{1, \ldots, n-1\}$, $\omega(l)$ has the form

$$\omega(l) = \begin{pmatrix} e_{(n-1)\times(n-1)} & s_1(Ar^{-1} - e_{(n-1)\times(n-1)})_{*l} \\ 0 & 1 \end{pmatrix}.$$

Since $A \neq re_{(n-1)\times(n-1)}$ there are $l, j \in \{1, \ldots, n-1\}$ such that $(\omega(l))_{jn} \neq 0$. One can proceed as in case 1.1.1.

<u>case 1.2.2</u> Assume $A = re_{(n-1)\times(n-1)}$.
Since σ is noncentral, $\exists j \in \{1, \ldots, n-1\} : x_j \neq 0$ or $x_j = 0\ \forall j \in \{1, \ldots, n-1\} \wedge r \notin Center(\hat{R}_m)$.

<u>case 1.2.2.1</u> Assume that $\exists j \in \{1, \ldots, n-1\} : x_j \neq 0$.
There are a $a' \in R$ and an $s' \in S$ such that $x_j r^{-1} = \frac{a'}{s'} + I_m$. Set $s := \frac{s'}{1} + I_m$ and $a := \frac{a'}{1} + I_m$. Choose an $i \neq j, n$ and set $g_1 := t_{ij}(s_1 s_2 s) \in U_1$. Then $[\sigma, g_1] = t_{in}(-s_1 s_2 s x_j r^{-1}) = t_{in}(-s_1 s_2 a) \in U_2$. Apply η^{-1} to this equation.

<u>case 1.2.2.2</u> Assume that $x_j = 0\ \forall j \in \{1, \ldots, n\} \wedge r \notin Center(\hat{R}_m)$.
Since $r \notin Center(\hat{R}_m)$, there is an $r' = \frac{a'}{s'} + I_m \in \hat{R}_m$ such that $rr' \neq r'r$. Set $a := \frac{a'}{1} + I_m \in \hat{R}_m$ and $s := \frac{s'}{1} + I_m \in \hat{R}_m$. Since $rr'r^{-1} - r' \in \hat{R}$, there are a $b' \in R$ and a $t' \in S$ such that $rr'r^{-1} - r' = \frac{b'}{t'} + I_m$. Set $t := \frac{t'}{1} + I_m$ and $b := \frac{b'}{1} + I_m$. Set $g_1 := t_{12}(s_1 s_2 s t r') = t_{12}(s_1 s_2 t a) \in U_1$. Then $[\sigma, g_1] = t_{12}(s_1 s_2 s t (rr'r^{-1} - r')) = t_{12}(s_1 s_2 s b) \in U_2$. Since $rr' \neq r'r$, $rr'r^{-1} - r' \neq 0$. Hence $s_2 s_3 s t (rr'r^{-1} - r') \neq 0$ since $s_2 s_3 s t$ is invertible. As in case 1.1.1, pull the result back to $GL_n(R_m)/GL_n(R_m, I_m)$ by applying η^{-1}.

<u>case 2</u> Assume that ${}^{\epsilon_0}h$ commutes with $t_{12}(s_0)$.
Then the second row of ${}^{\epsilon_0}h$ equals rf_2 for some $r \in \hat{R}_m$. Set $\epsilon_{01} := p_{2n} \in E_n(\hat{R}_m)$. Then the last row of ${}^{\epsilon_{01}\epsilon_0}h$ equals rf_n and one can proceed as in case 1.2.

The following diagram language is intended to give an overview of the case analysis above. The overview begins with the second diagram below. It starts with the matrix h. An arrow between two matrices means that one gets the target matrix by applying certain operations to the source matrix. The operations are the following:

(1) Form a commutator with a matrix.

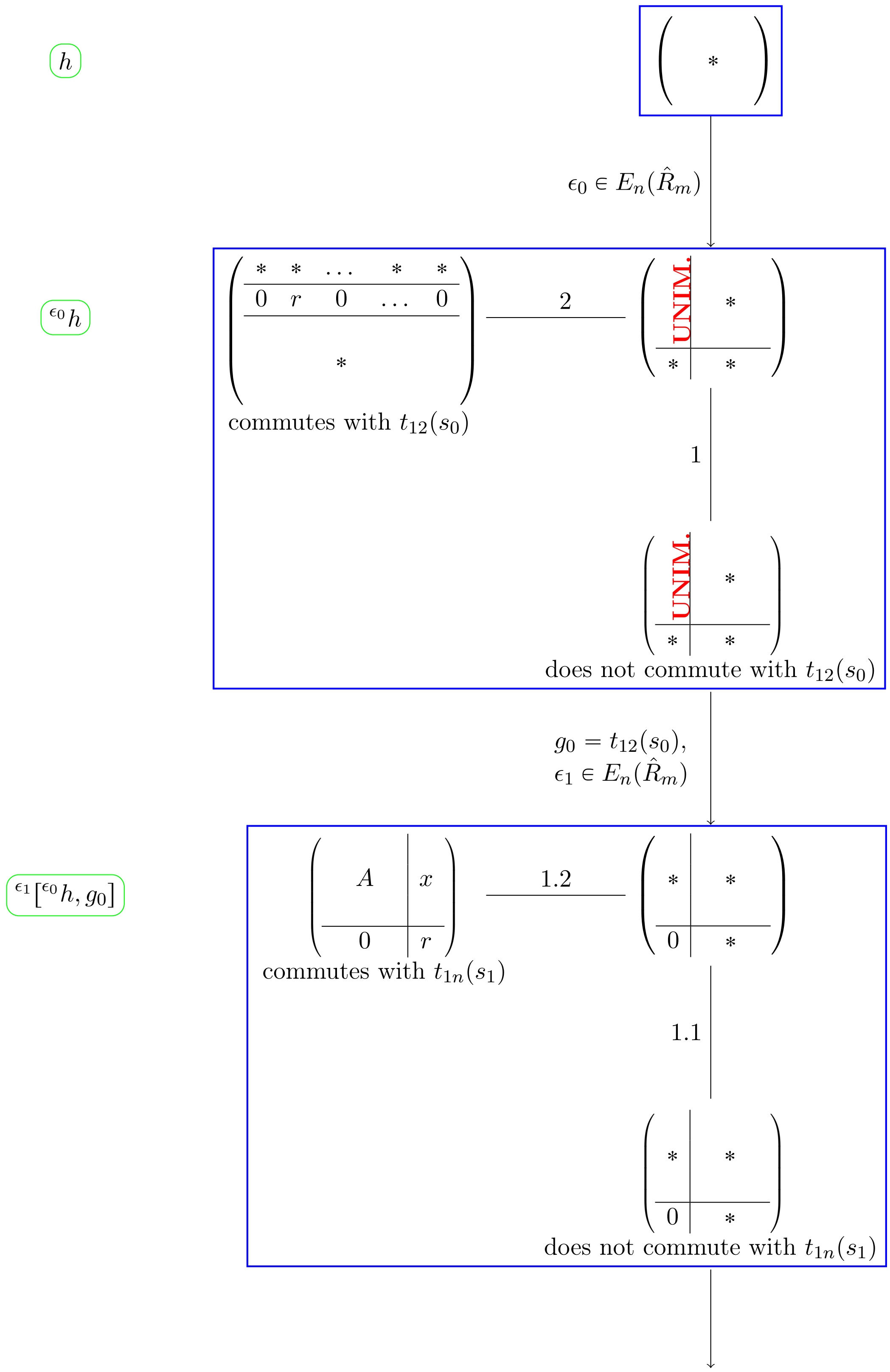
(2) Conjugate by a matrix.

The operations of type (1) are performed with the g_i's and the operations of type (2) are performed with the ϵ_i's. A box around several matrices means that some case distinction is going on (corresponding to the case analysis above). For example

$\boxed{{}^{\epsilon_1}[A, g_1]}$

$$\begin{tikzcd}
 & A \arrow[d, "{g_1 = t_{i_1j_1}(x_1),\ \epsilon_1 = t_{k_1l_1}(y_1)}"'] & \\
 & B \arrow[dl, dash, "1"'] \arrow[dr, dash, "2"] & \\
C \arrow[d, "g_2 = t_{i_2j_2}(x_2)"'] & & D \arrow[d, "g_2 = t_{i'_2j'_2}(x'_2)"] \\
E & & F
\end{tikzcd}$$

means the following. One gets the matrix B by first forming the commutator $[A, g_1]$ of A and $g_1 = t_{i_1j_1}(x_1)$ and then conjugating this commutator by $\epsilon_1 = t_{k_1l_1}(y_1)$. The matrix ${}^{\epsilon_1}[A, g_1]$ in the left margin reminds us how we got B. Then there are two cases. In case 1, the matrix B looks like C and in case 2, like D. The logic of the situation tells us that B must look like C or D. In case 1 we form the commutator $[C, g_2]$ of $C(= B)$ and $g_2 = t_{i_2j_2}(x_2)$ and get the matrix E. In case 2, we form the commutator $[D, g_2]$ of $D(= B)$ and $g_2 = t_{i'_2j'_2}(x'_2)$ and get the matrix F. It is helpful to keep in mind that all matrices appearing in a diagram are noncentral and the goal is to produce nontrivial elementary matrices which are of course noncentral. When breaking a matrix in several cases, we do not necessarily handle the cases one after the other, but will postpone handling some cases to later. Each case in the entire diagram is given a unique number, so that we can come back to it by referring to its number.


h
$\epsilon_0 \in E_n(\hat{R}_m)$
${}^{\epsilon_0}h$
2
UNIM.
commutes with $t_{12}(s_0)$
1
UNIM.
does not commute with $t_{12}(s_0)$
$g_0 = t_{12}(s_0)$,
$\epsilon_1 \in E_n(\hat{R}_m)$
${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$
1.2
commutes with $t_{1n}(s_1)$
1.1
does not commute with $t_{1n}(s_1)$

$$g_1 = t_{1n}(s_1)$$

$[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_1]$

$$\left(\begin{array}{c|c} A & x \\ \hline 0 & 1 \end{array}\right)$$

1.1.1

$$\left(\begin{array}{c|c} A & x \\ \hline 0 & 1 \end{array}\right)$$

$A \neq e$

1.1.2

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & x \\ & & 1 & \\ \hline & 0 & & 1 \end{array}\right)$$

$x \neq 0$

$$g_2 = t_{ln}(s_2)$$

$[[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_1], g_2]$

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & y \\ & & 1 & \\ \hline & 0 & & 1 \end{array}\right)$$

$y \neq 0$

$$g_3 = t_{ij}(s_3 s_4 s)$$

$[[[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_1], g_2], g_3]$

$\boxed{t_{in}(-s_3 s_4 a)}$

$$g_2 = t_{ij}(s_2 s_3 s)$$

$\boxed{t_{in}(-s_2 s_3 a)}$

1.2

$^{\epsilon_1}[^{\epsilon_0}h, g_0]$

$$\left(\begin{array}{c|c} A & x \\ \hline 0 & r \end{array}\right)$$

1.2.1

$$\left(\begin{array}{c|c} A & x \\ \hline 0 & r \end{array}\right)$$

$A \neq re$

1.2.2

$$\left(\begin{array}{ccc|c} r & & & \\ & \ddots & & x \\ & & r & \\ \hline & 0 & & r \end{array}\right)$$

1.2.2.1 1.2.2.2

$$\left(\begin{array}{c|c} A & x \\ \hline 0 & r \end{array}\right) \qquad \left(\begin{array}{ccc|c} r & & & \\ & \ddots & & x \\ & & r & \\ \hline & 0 & & r \end{array}\right) \qquad \begin{pmatrix} r & & & \\ & \ddots & & \\ & & r & \\ & & & r \end{pmatrix}$$

$A \neq re$ $\qquad$ $x \neq 0$ $\qquad$ r noncentral in $\hat{R}_m$

$g_1 = t_{ln}(s_1)$ $\qquad$ $g_1 = t_{ij}(s_1 s_2 s)$ $\qquad$ $g_1 = t_{12}(s_1 s_2 t a)$

$[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]$

$$\boxed{\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & y \\ & & 1 & \\ \hline & 0 & & 1 \end{array}\right) \; y \neq 0} \qquad \boxed{t_{in}(-s_1 s_2 a)} \qquad \boxed{t_{12}(s_1 s_2 s b)}$$

$g_2 = t_{ij}(s_2 s_3 s)$

$[[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1], g_2]$ $\qquad$ $\boxed{t_{in}(-s_2 s_3 a)}$

2

${}^{\epsilon_0}h$

$$\boxed{\left(\begin{array}{ccccc} * & * & \dots & * & * \\ \hline 0 & r & 0 & \dots & 0 \\ \hline & & * & & \end{array}\right)}$$

$\epsilon_{01} = p_{2n}$

${}^{\epsilon_{01}\epsilon_0}h$

$$\boxed{\left(\begin{array}{c|c} A & x \\ \hline 0 & r \end{array}\right)}$$

see case 1.2

$\square$

Theorem 5.8 *Let H be a subgroup of $GL_n(R)$. Then*

$$\begin{aligned}&H \text{ is normalized by } E_n(R) \Leftrightarrow \\ &\exists! \text{ ideal } I \text{ such that } E_n(R,I) \subseteq H \subseteq C_n(R,I).\end{aligned}$$

Proof It follows from the previous lemmas of this section and from Lemma 4.7 that $(G, E, Gen(E), B(E), G(\cdot))$ (where G, E, $Gen(E)$, $B(E)$ and $G(\cdot)$ are defined as in Lemma 5.1) is a sandwich classification group. Hence we can apply Theorem 3.13 (note that if H is central, then there clearly is a unique ideal I such that $E_n(R,I) \subseteq H \subseteq C_n(R,I)$). □

Remark By [2], p. 377, any almost commutative ring R is the direct limit of subrings R_i of R such that for each i, R_i is a Noetherian C_i-module where $C_i := Center(R_i)$. Hence the theorem above still holds true if we drop the assumption that R is a Noetherian C-module and instead assume that R is almost commutative (note that E_n and C_n commute with direct limits).

6 Bak's hyperbolic unitary groups

In section 5, we saw that the notion of an ideal in a ring is sufficient to classify subgroups of a general linear group normalized by its elementary subgroup. Bak's dissertation [1] showed that the notion of an ideal by itself was not sufficient to solve the analogous classification problem for unitary groups, but that a refinement of the notion an ideal, called a form ideal, was necessary. This led naturally to a more general notion of unitary group, which was defined over a form ring instead of just a ring and generalized all previous concepts. We describe form rings (R, Λ) and form ideals ideals (I, Γ) first, then hyperbolic unitary groups $U_{2n}(R, \Lambda)$ over form rings (R, Λ). For form ideals (I, Γ), we recall the definitions of the following subgroups of $U_{2n}(R, \Lambda)$; the preelementary groups $EU_{2n}(I, \Gamma)$, the relative elementary groups $EU_{2n}((R, \Lambda), (I, \Gamma))$, the principal congruence subgroups $U_{2n}((R, \Lambda), (I, \Gamma))$, and the full congruence subgroups $CU_{2n}((R, \Lambda), (I, \Gamma))$. In the model theoretic setting of section 3, these groups are accounted for respectively by the groups U in $B(E)$, the groups $E(U)$, the groups $G(U)$ and the groups $C(U)$. The elementary group $EU_{2n}(R, \Lambda) := EU_{2n}((R, \Lambda), (R, \Lambda))$ is accounted for by E in the model theoretic situation and the generators of $EU_{2n}(R, \Lambda)$, namely the unitary elementary matrices, are accounted for by $Gen(E)$.

Definition 6.1 Let R be a ring and

$$\begin{aligned}\bar{\ }: R &\to R \\ r &\mapsto \bar{r}\end{aligned}$$

an involution on R, i.e. $\overline{r+s} = \bar{r} + \bar{s}$, $\overline{rs} = \bar{s}\bar{r}$ and $\bar{\bar{r}} = r$ for any $r, s \in R$. Let $\lambda \in Cent(R)$ such that $\lambda\bar{\lambda} = 1$ and set $\Lambda_{min} := \{r - \lambda\bar{r} | r \in R\}$ and $\Lambda_{max} := \{r \in R | r = -\lambda\bar{r}\}$. An additive subgroup Λ of R such that

(1) $\Lambda_{min} \subseteq \Lambda \subseteq \Lambda_{max}$ and

(2) $r\Lambda\overline{r} \subseteq \Lambda \; \forall r \in R$

is called a *form parameter*. If Λ is a form parameter for R, the pair (R,Λ) is called a *form ring*.

Definition 6.2 Let (R,Λ) be a form ring and I an ideal such that $\overline{I} = I$. Set $\Gamma_{max} = I \cap \Lambda$ and $\Gamma_{min} = \{\xi - \lambda\overline{\xi} | \xi \in I\} + \langle\{\zeta\alpha\overline{\zeta} | \zeta \in I, \alpha \in \Lambda\}\rangle$. If we want to stress that Γ_{max} (resp. Γ_{min}) belongs to I, we write Γ^I_{max} (resp. Γ^I_{min}). An additive subgroup Γ of I such that

(1) $\Gamma_{min} \subseteq \Gamma \subseteq \Gamma_{max}$ and

(2) $\alpha\Gamma\overline{\alpha} \subseteq \Gamma \; \forall \alpha \in R$

is called a *relative form parameter of level I*. If Γ is a relative form parameter of level I, then (I,Γ) is called a *form ideal* of (R,Λ).

In the following let $n \in \mathbb{N}$, (R,Λ) be a form ring and (I,Γ) a form ideal of (R,Λ).

Definition 6.3 Let V be a free right R-module of rank $2n$ and $B = (e_1,\ldots,e_n,e_{-n},\ldots,e_{-1})$ an ordered basis of V. Let $\phi_B : V \to R^{2n}$ be the module isomorphism mapping e_i to the column whose i-th coordinate is one and all the other coordinates are zero if $1 \leqslant i \leqslant n$ and the column whose $(2n+1+i)$-th coordinate is one and all the other coordinates are zero if $-n \leqslant i \leqslant -1$. In the following we will identify elements $v \in V$ with their images $\phi_B(v) \in R^{2n}$. Let

$$p = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in M_n(R)$$

be the matrix with ones on the skew diagonal and zeros elsewhere. We define the maps

$$\begin{aligned} \mathbb{f} : V \times V &\to R \\ (v,w) &\mapsto \overline{v}^t \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} w, \end{aligned}$$

$$\begin{aligned} \mathbb{h} : V \times V &\to R \\ (v,w) &\mapsto \overline{v}^t \begin{pmatrix} 0 & p \\ \lambda p & 0 \end{pmatrix} w \end{aligned}$$

and

$$\begin{aligned} \mathbb{q} : V &\to R/\Lambda \\ v &\mapsto \mathbb{f}(v,v) + \Lambda. \end{aligned}$$

The maps $\mathbb{f}$, $\mathbb{h}$ and $\mathbb{q}$ are denoted in [4], page 164, by f, h and q, respectively. It is easy to check that $\mathbb{f}(v,w) = \overline{v}_1 w_{-1} + ... + \overline{v}_n w_{-n}$, $\mathbb{h}(v,w) = \overline{v}_1 w_{-1} + ... + \overline{v}_n w_{-n} + \lambda \overline{v}_{-n} w_n + ... + \lambda \overline{v}_{-1} w_1 = \mathbb{f}(v,w) + \lambda \overline{\mathbb{f}(w,v)}$ and $\mathbb{q}(v) = \overline{v}_1 v_{-1} + ... + \overline{v}_n v_{-n} + \Lambda$ for any $v, w \in V$. For any $v \in V$, $\mathbb{f}(v,v)$ is called the *length* of v and is denoted by $|v|$.

Definition 6.4 The subgroup $U_{2n}(R,\Lambda) := \{\sigma \in GL(V) | (\mathbb{h}(\sigma u, \sigma v) = \mathbb{h}(u,v)) \wedge (\mathbb{q}(\sigma u) = \mathbb{q}(u)) \ \forall u, v \in V\}$ of $GL(V)$ is called the *hyperbolic unitary group.* We will identify $U_{2n}(R,\Lambda)$ with its image in $GL_{2n}(R)$ under the isomorphism $GL(V) \to GL_{2n}(R)$ determined by the ordered base $(e_1, \dots, e_n, e_{-n}, \dots, e_{-1})$.

Definition 6.5 Let $\sigma \in M_n(R)$. By definition σ^* is the matrix in $M_n(R)$ whose entry at position (i,j) equals $\overline{\sigma}_{ji}$. Further we define $AH_n(R,\Lambda) := \{a \in M_n(R) | a = -\lambda a^*, a_{ii} \in \Lambda \ \forall i \in \{1, ..., n\}\}$.

Lemma 6.6 *Let (R,Λ) be a form ring, $n \in \mathbb{N}$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(R)$, where $a, b, c, d \in M_n(R)$. Then $\sigma \in U_{2n}(R,\Lambda)$ if and only if*

(1) $\sigma^{-1} = \begin{pmatrix} pd^*p & \overline{\lambda} pb^*p \\ \lambda pc^*p & pa^*p \end{pmatrix}$ *and*

(2) $a^*pc, b^*pd \in AH_n(R,\Lambda)$.

Proof See [4], p.166.

Remark

(1) If $a \in M_n(R)$, then pa^*p is the matrix one gets by applying the involution to each entry of a and mirroring all entries on the skew diagonal.

(2) In [1], [10] and [11] the ordered basis $(e_1, ..., e_n, e_{-1}, ..., e_{-n})$ is used and hence the matrices may look different. Let $\sigma \in GL(V)$. If the image of σ under the isomorphism $GL(V) \to GL_{2n}(R)$ determined by the ordered base $(e_1, ..., e_n, e_{-1}, ..., e_{-n})$ (which is used in the papers mentioned above) equals $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in M_n(R)$, then the image of σ under the isomorphism $GL(V) \to GL_{2n}(R)$ determined by the ordered base $(e_1, ..., e_n, e_{-n}, ..., e_{-1})$ (which is used in this thesis) equals $\begin{pmatrix} a & bp \\ pc & pdp \end{pmatrix}$.

Definition 6.7 We define $\Omega_+ := \{1, ..., n\}$, $\Omega_- := \{-n, ..., -1\}$, $\Omega := \Omega_+ \cup \Omega_-$ and

$$\begin{aligned} \epsilon : \Omega &\to \{-1, 1\} \\ i &\mapsto \epsilon(i) := \begin{cases} 1, & if\ i \in \Omega_+, \\ -1, & if\ i \in \Omega_-. \end{cases} \end{aligned}$$

Lemma 6.8 *Let $\sigma \in GL_{2n}(R)$. Then $\sigma \in U_{2n}(R,\Lambda)$ if and only if*

(1) $\sigma'_{ij} = \lambda^{(\epsilon(j)-\epsilon(i))/2} \overline{\sigma}_{-j,-i} \ \forall i, j \in \{1, ..., -1\}$ *and*

(2) $|\sigma_{*j}| \in \Lambda \ \forall j \in \{1, ..., -1\}$. *(*$|\sigma_{*j}| = \sum_{i=1}^{n} \bar{\sigma}_{ij}\sigma_{-i,j}$ *is defined just before* 6.4.*)*

Proof See [4], p.167.

Lemma 6.9 *Let* $\sigma \in U_{2n}(R, \Lambda)$*,* $x \in R^*$ *and* $k \in \{1, \ldots, -1\}$*. Then the statements below are true.*

(1) If the k-th column of σ *equals* xe_k *then the* $(-k)$*-th row of* σ *equals* $\overline{x^{-1}}f_{-k}$*.*

(2) If the k-th row of σ *equals* xf_k *then the* $(-k)$*-th column of* σ *equals* $\overline{x^{-1}}e_{-k}$*.*

Proof

(1) Since $\sigma^{-1}\sigma = e$ it follows that

$$(\sigma^{-1}\sigma)_{ij} = \sum_{l=1}^{-1} \sigma'_{il}\sigma_{lj} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

This implies that $1 = \sum_{l=1}^{-1} \sigma'_{kl}\sigma_{lk} = \sigma'_{kk}\sigma_{kk} = \sigma'_{kk}x$. Since any left inverse of an invertible element is the inverse of that element, $\sigma'_{kk} = x^{-1}$. By Lemma 6.8, it follows that $\sigma_{-k,-k} = \overline{x^{-1}}$. On the other hand (6.1) implies that $0 = \sum_{l=1}^{-1} \sigma'_{il}\sigma_{lk} = \sigma'_{ik}\sigma_{kk} = \sigma'_{ik}x \ \forall i \in \{1, \ldots, -1\}\backslash\{k\}$. It follows that $\sigma'_{ik} = 0 \ \forall i \in \{1, \ldots, -1\}\backslash\{k\}$ and hence, by Lemma 6.8, $\sigma_{-k,-i} = 0 \ \forall i \in \{1, \ldots, -1\}\backslash\{k\}$, i.e. $\sigma_{-k,i} = 0 \ \forall i \in \{1, \ldots, -1\}\backslash\{-k\}$.

(2) Since $\sigma\sigma^{-1} = e$ it follows that

$$(\sigma\sigma^{-1})_{ij} = \sum_{l=1}^{-1} \sigma_{il}\sigma'_{lj} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

This implies $1 = \sum_{l=1}^{-1} \sigma_{kl}\sigma'_{lk} = \sigma_{kk}\sigma'_{kk} = x\sigma'_{kk}$. Since any right inverse of an invertible element is the inverse of that element, $\sigma'_{kk} = x^{-1}$. By Lemma 6.8, it follows that $\sigma_{-k,-k} = \overline{x^{-1}}$. On the other hand (6.2) implies that $0 = \sum_{l=1}^{-1} \sigma_{kl}\sigma'_{lj} = \sigma_{kk}\sigma'_{kj} = x\sigma'_{kj} \ \forall j \in \{1, \ldots, -1\}\backslash\{k\}$. It follows that $\sigma'_{kj} = 0 \ \forall j \in \{1, \ldots, -1\}\backslash\{k\}$ and hence, by Lemma 6.8, $\sigma_{-j,-k} = 0 \ \forall j \in \{1, \ldots, -1\}\backslash\{k\}$, i.e. $\sigma_{j,-k} = 0 \ \forall j \in \{1, \ldots, -1\}\backslash\{-k\}$.

$\square$

Definition 6.10 Let $i, j \in \Omega, i \neq j$. If $i \neq -j$ and $\xi \in R$, the matrix

$$T_{ij}(\xi) := e + \xi e_{ij} - \lambda^{(\epsilon(j)-\epsilon(i))/2}\bar{\xi}e_{-j,-i} \in U_{2n}(R, \Lambda)$$

is called an *elementary short root element*. If $i = -j$ and $\alpha \in \lambda^{-(\epsilon(i)+1)/2}\Lambda$, then the matrix

$$T_{i,-i}(\alpha) := e + \alpha e_{i,-i} \in U_{2n}(R,\Lambda)$$

is called an *elementary long root element*. If $\sigma \in U_{2n}(R,\Lambda)$ is an elementary short root element or an elementary long root element, it is called an *elementary unitary transvection*. The subgroup of $U_{2n}(R,\Lambda)$ generated by all elementary unitary transvections is called the *elementary unitary group* and is denoted by $EU_{2n}(R,\Lambda)$. Let $T_{ij}(\xi)$ be an elementary unitary transvection. If $i \neq -j \wedge \xi \in I$ or $i = -j \wedge \xi \in \lambda^{-(\epsilon(i)+1)/2}\Gamma$, then $T_{ij}(\xi)$ is called *elementary of level* (I,Γ) or (I,Γ)-*elementary*. The subgroup of $U_{2n}(R,\Lambda)$ generated by all (I,Γ)-elementary transvections is called the *preelementary subgroup of level* (I,Γ) and is denoted by $EU_{2n}(I,\Gamma)$. Its normal closure in $EU_{2n}(R,\Lambda)$ is called the *elementary subgroup of level* (I,Γ) and is denoted by $EU_{2n}((R,\Lambda),(I,\Gamma))$.

Definition 6.11 Let $i,j \in \{1,\dots,-1\}$ such that $i \neq \pm j$. Define $P_{ij} := e + e_{ij} - e_{ji} + \lambda^{(\epsilon(i)-\epsilon(j))/2}e_{-i,-j} - \lambda^{(\epsilon(j)-\epsilon(i))/2}e_{-j,-i} - e_{ii} - e_{jj} - e_{-i,-i} - e_{-j,-j} = T_{ij}(1)$ $T_{ji}(-1)T_{ij}(1) \in EU_{2n}(R,\Lambda)$. It is easy to show that $(P_{ij})^{-1} = P_{ji}$. If $1 \leqslant i,j \leqslant n$, P_{ij} has the form

$$\begin{pmatrix} p_{ij} & 0 \\ 0 & p_{ji} \end{pmatrix}$$

where $p_{ij}, p_{ji} \in E_n(R)$.

Lemma 6.12 *The relations*

$$T_{ij}(\xi) = T_{-j,-i}(-\lambda^{(\epsilon(j)-\epsilon(i))/2}\bar{\xi}), \tag{R1}$$

$$T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi+\zeta), \tag{R2}$$

$$[T_{ij}(\xi), T_{hk}(\zeta)] = e, \tag{R3}$$

$$[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta), \tag{R4.1}$$

$$[T_{ij}(\xi), T_{h,-j}(\zeta)] = T_{i,-h}(-\lambda^{(\epsilon(-j)-\epsilon(h))/2}\xi\bar{\zeta}), \tag{R4.2}$$

$$[T_{-j,i}(\xi), T_{jh}(\zeta)] = T_{-i,h}(-\lambda^{(\epsilon(i)-\epsilon(-j))/2}\bar{\xi}\zeta), \tag{R4.3}$$

$$[T_{ji}(\xi), T_{hj}(\zeta)] = T_{hi}(-\zeta\xi), \tag{R4.4}$$

$$[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta - \lambda^{-\epsilon(i)}\bar{\zeta}\bar{\xi}), \tag{R5.1}$$

$$[T_{ij}(\xi), T_{i,-j}(\zeta)] = T_{i,-i}(-\lambda^{(\epsilon(-j)-\epsilon(i))/2}\xi\bar{\zeta} + \lambda^{(\epsilon(j)-\epsilon(i))/2}\zeta\bar{\xi}), \tag{R5.2}$$

$$[T_{-j,i}(\xi), T_{ji}(\zeta)] = T_{-i,i}(-\lambda^{(\epsilon(i)-\epsilon(-j))/2}\bar{\xi}\zeta + \lambda^{(\epsilon(i)-\epsilon(j))/2}\bar{\zeta}\xi), \tag{R5.3}$$

$$[T_{ji}(\xi), T_{-i,j}(\zeta)] = T_{-i,i}(-\zeta\xi + \lambda^{\epsilon(i)}\bar{\xi}\bar{\zeta}), \tag{R5.4}$$

$$[T_{i,-i}(\alpha), T_{-i,j}(\xi)] = T_{ij}(\alpha\xi)T_{-j,j}(-\lambda^{(\epsilon(j)-\epsilon(-i))/2}\bar{\xi}\alpha\xi), \tag{R6.1}$$

$$[T_{i,-i}(\alpha), T_{-j,i}(\xi)] = T_{ij}(-\lambda^{(\epsilon(i)-\epsilon(-j))/2}\alpha\bar{\xi})T_{-j,j}(-\lambda^{(\epsilon(i)-\epsilon(-j))/2}\xi\alpha\bar{\xi}) \text{ and} \tag{R6.2}$$

$$[T_{ji}(\xi), T_{i,-i}(\alpha)] = T_{j,-i}(\xi\alpha)T_{j,-j}(\lambda^{(\epsilon(i)-\epsilon(j))/2}\xi\alpha\bar{\xi}) \tag{R6.3}$$

hold where $h \neq j, -i$ and $k \neq i, -j$ in (R3)*, $i, h \neq \pm j$ and $i \neq \pm h$ in* (R4.1)-(R4.4) *and $i \neq \pm j$ in* (R5.1)-(R6.3).

Proof Straightforward calculation.

Definition 6.13 The group consisting of all $\sigma \in U_{2n}(R, \Lambda)$ such that $\sigma \equiv e(mod\ I)$ and $\mathbb{f}(\sigma u, \sigma u) \in \mathbb{f}(u, u) + \Gamma\ \forall u \in V$ is called the *principal congruence subgroup of level* (I, Γ) and is denoted by $U_{2n}((R, \Lambda), (I, \Gamma))$.

Remark One can show that $U_{2n}((R, \Lambda), (I, \Gamma))$ is a normal subgroup of $U_{2n}(R, \Lambda)$ (see [4]).

Lemma 6.14 *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{2n}(R, \Lambda)$*, where* $a, b, c, d \in M_n(R)$*. Then* $\sigma \in U_{2n}((R, \Lambda), (I, \Gamma))$ *if and only if*

(1) $\sigma \equiv e(mod\ I)$ *and*

(2) $|\sigma_{*j}| \in \Gamma\ \forall j \in \{1, ..., -1\}$. *(*$|\sigma_{*j}| = \sum\limits_{i=1}^{n} \bar{\sigma}_{ij}\sigma_{-i,j}$ *is defined just before* 6.4.*)*

Proof See [4], p.174.

Definition 6.15 The preimage of the center of $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ under the canonical homomorphism $U_{2n}(R, \Lambda) \to U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ is called the *full congruence subgroup of level* (I, Γ) and is denoted by $CU_{2n}((R, \Lambda), (I, \Gamma))$.

Remark Obviously $U_{2n}((R, \Lambda), (I, \Gamma)) \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$ and $CU_{2n}((R, \Lambda), (I, \Gamma))$ is a normal subgroup of $U_{2n}(R, \Lambda)$.

Lemma 6.16 *If $n \geqslant 3$ and R is almost commutative, then the equalities*

$$\begin{aligned} &[CU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] \\ =&[EU_{2n}((R, \Lambda), (I, \Gamma)), EU_{2n}(R, \Lambda)] \\ =&EU_{2n}((R, \Lambda), (I, \Gamma)) \end{aligned}$$

hold.

Proof See [4], Theorem 1.1 and Lemma 5.2. □

7 Sandwich classification for hyperbolic unitary groups

In this section, we construct in the setting of hyperbolic unitary groups, specific supplemented bases, local maps, solution groups and coverings by these solution groups, and show in the Solution Group Lemma 7.9 that any noncentral element in any solution group of any of these coverings satisfies a solution equation. 7.9 is the main technical input of the section. Road maps of the proof are provided at the conclusions of Parts I, II, and III of the proof, in terms of (long) inverted tree diagrams. Then we deduce the sandwich classification theorem (1.1) for subgroups of $U_{2n}(R,\Lambda)$ normalized by $EU_{2n}(R,\Lambda)$ from Theorem 3.13.

In the following let $n \geqslant 3$, (R,Λ) be a form ring and C be the subring of R consisting of all finite sums of elements of the form $c\bar{c}$ and $-c\bar{c}$ where c ranges over some subring $C' \subseteq Center(R)$ such that R is module finite over C'. One can check that R is also module finite over C. The reason for replacing C' by C is that any form parameter or form ideal is a C-module. This is not necessarily the case for C'. For any form ideal (I,Γ) of (R,Λ) and multiplicative subset $S \subseteq C$, set $R_S := S^{-1}R$, $\Lambda_S := S^{-1}\Lambda$, $I_S := S^{-1}I$ and $\Gamma_S := S^{-1}\Gamma$. Let

$$\phi_S : U_{2n}(R,\Lambda)/U_{2n}((R,\Lambda),(I,\Gamma)) \to U_{2n}(R_S,\Lambda_S)/U_{2n}((R_S,\Lambda_S),(I_S,\Gamma_S))$$

be the homomorphism induced by F_S where

$$F_S : U_{2n}(R,\Lambda) \to U_{2n}(R_S,\Lambda_S)$$

is the homomorphism induced by the localisation homomorphism

$$f_S : R \to R_S.$$

Let

$$\psi : U_{2n}(R,\Lambda) \to U_{2n}(R,\Lambda)/U_{2n}((R,\Lambda),(I,\Gamma))$$

and

$$\rho_S : U_{2n}(R_S,\Lambda_S) \to U_{2n}(R_S,\Lambda_S)/U_{2n}((R_S,\Lambda_S),(I_S,\Gamma_S))$$

be the canonical homomorphisms. Further set $\lambda_S := f_S(\lambda)$. Note that the diagram

$$\begin{tikzcd}
U_{2n}(R,\Lambda) \arrow[r, "\psi"] \arrow[d, "F_S"] & U_{2n}(R,\Lambda)/U_{2n}((R,\Lambda),(I,\Gamma)) \arrow[d, "\phi_S"] \\
U_{2n}(R_S,\Lambda_S) \arrow[r, "\rho_S"] & U_{2n}(R_S,\Lambda_S)/U_{2n}((R_S,\Lambda_S),(I_S,\Gamma_S))
\end{tikzcd}$$

is commutative for any form ideal (I,Γ) of (R,Λ) and multiplicative subset $S \subseteq C$. For any maximal ideal m of C set $S_m := C \setminus m$ and $\phi_m := \phi_{S_m}$ (define F_m, f_m, ρ_m, R_m, Λ_m, I_m, Γ_m and λ_m similarly).

Lemma 7.1 *Set*

$$
\begin{aligned}
&G := U_{2n}(R, \Lambda),\\
&E := EU_{2n}(R, \Lambda),\\
&Gen(E) := \{T_{ij}(x) | i, j \in \{1, \ldots, -1\}, i \neq j, x \in R \textit{ if } i \neq -j,\\
&\qquad\qquad x \in \lambda^{-(\epsilon(i)+1)/2}\Lambda \textit{ if } i = -j\},\\
&B(E) := \{EU_{2n}(I, \Gamma) | (I, \Gamma) \text{ form ideal of } (R, \Lambda)\} \textit{ and}\\
&G(EU_{2n}(I, \Gamma)) := U_{2n}((R, \Lambda), (I, \Gamma)) \ \forall (I, \Gamma) \textit{ form ideal of } (R, \Lambda).
\end{aligned}
$$

Then $(G, E, Gen(E), B(E), G(\cdot))$ *is a standard group. Further* $E(EU_{2n}(I, \Gamma)) = EU_{2n}((R, \Lambda), (I, \Gamma))$ *and* $C(EU_{2n}(I, \Gamma)) = CU_{2n}((R, \Lambda), (I, \Gamma))$ *for any form ideal* (I, Γ) *of* (R, Λ).

Proof We have to show that the conditions (3.2.1) – (3.2.3) in Definition 3.2 are satisfied.

(3.2.1) Obviously $B(E)$ is a base of open subgroups of $1 \in E$, since it contains the identity subgroup $\{1\} = EU_{2n}(\{0\}, \{0\})$. Clearly $E = EU_{2n}(R, \Lambda) \in B(E)$.

(3.2.2) Let $\{(I_j, \Gamma_j) | j \in J\}$ be a family of form ideals of (R, Λ). One checks easily that $\langle \bigcup_{j \in J} EU_{2n}(I_j, \Gamma_j) \rangle = EU_{2n}(\langle \bigcup_{j \in J} I_j \rangle, \langle \bigcup_{j \in J} \Gamma_j \rangle)$.

(3.2.3) Let $U \in B(E)$. By definition there is a form ideal (I, Γ) of (R, Λ) such that $U = EU_{2n}(I, \Gamma)$. Clearly $Gen(U) = Gen(E) \cap U$ contains all the elements $T_{ij}(x)$ where $i, j \in \{1, \ldots, -1\}$, $i \neq j$, $x \in I$ if $i \neq -j$ and $x \in \lambda^{-(\epsilon(i)+1)/2}\Gamma$ if $i = -j$. But these elements generate U. Hence $U = EU_{2n}(I, \Gamma)$ is generated by $Gen(U)$. Now let $g = T_{ij}(x) \in Gen(E)$ and $U = EU_{2n}(I, \Gamma) \in B(E)$. Assume that $g \in G(U) = U_{2n}((R, \Lambda), (I, \Gamma))$.
<u>case 1</u> Asssume $i \neq \pm j$ and $x \in R$. Since $g \in U_{2n}((R, \Lambda), (I, \Gamma))$, all non-diagonal entries of g lie in I. It follows that $x \in I$ and hence $g = T_{ij}(x) \in EU_{2n}(I, \Gamma) = U$.
<u>case 2</u> Asssume that $i = -j$ and $x \in \lambda^{-(\epsilon(i)+1)/2}\Lambda$. Since $g \in U_{2n}((R, \Lambda), (I, \Gamma))$, all lengths of columns of g lie in Γ. It follows that $x \in \lambda^{-(\epsilon(i)+1)/2}\Gamma$ and thus $g = T_{ij}(x) \in EU_{2n}(I, \Gamma) = U$.

□

From now on we assume that R is a Noetherian C-module.

Lemma 7.2 *Let* (I, Γ) *be a form ideal of* (R, Λ) *and* $S \subseteq C$ *a multiplicative subset. Then there is an* $s_0 \in S$ *with the properties*

(1) if $x \in s_0R$ *and* $\exists t \in S : tx \in I$, *then* $x \in I$ *and*

(2) if $x \in s_0R$ *and* $\exists t \in S : tx \in \Gamma$, *then* $x \in \Gamma$.

It follows that ϕ_S *is injective on* $\psi(U_{2n}((R, \Lambda), (I + s_0R, \Gamma + s_0\Lambda)))$.

Proof For any $s \in S$ set $Y(s) := \{x \in R | sx \in I\}$. Then for any $s \in S$, $Y(s)$ is a C-submodule of R. Since R is Noetherian C-module, the set $\{Y(s) | s \in S\}$ has a maximal element $Y(s_1)$. Clearly all elements $x \in s_1R$ have the property that $tx \in I$ for some $t \in S$ implies $x \in I$. For any $s \in S$ set $Z(s) := \{x \in R | sx \in \Gamma\}$. Then for any $s \in S$, $Z(s)$ is a C-submodule of R. Since R is a Noetherian C-module, the set $\{Z(s) | s \in S\}$ has a maximal element $Z(s_2)$. Clearly all elements $x \in s_2R$ have the property that $tx \in \Gamma$ for some $t \in S$ implies $x \in \Gamma$. Set $s_0 := s_1s_2$. Since $s_0R = s_1s_2R \subseteq s_1R \cap s_2R$, s_0 has the properties (1) and (2) above. We will show now that ϕ_S is injective on $\psi(U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda))$. Let $g'_1, g'_2 \in \psi(U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda)))$ such that $\phi_S(g'_1) = \phi_S(g'_2)$. Since $g'_1, g'_2 \in \psi(U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda)))$, there are $g_1, g_2 \in U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda))$ such that $\psi(g_1) = g'_1$ and $\psi(g_2) = g'_2$. Set $h := (g_1)^{-1}g_2 \in U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda))$. Clearly $\phi_S(g'_1) = \phi_S(g'_2)$ is equivalent to $F_S(h) \in U_{2n}((R_S,\Lambda_S),(I_S,\Gamma_S))$, i.e.

(a) $F_S(h) \equiv e(mod\ I_S)$ and

(b) $f_S(|h_{*j}|) \in \Gamma_S\ \forall j \in \{1, \dots, -1\}$.

We want to show that $g'_1 = g'_2$ which is equivalent to $h \in U_{2n}((R,\Lambda),(I,\Gamma))$, i.e.

(a') $h \equiv e(mod\ I)$ and

(b') $x_j := |h_{*j}| \in \Gamma\ \forall j \in \{1, \dots, -1\}$.

First we show (a'). Let $i, j \in \{1, \dots, -1\}$ such that $i \neq j$. Since (a) holds, $f_S(h_{ij}) \in I_S$. Hence

$$\begin{aligned} &\exists x \in I, s \in S : \frac{h_{ij}}{1} = \frac{x}{s} \\ \Rightarrow\ &\exists x \in I, s, t \in S : t(h_{ij}s - x) = 0 \\ \Rightarrow\ &\exists x \in I, s, t \in S : sth_{ij} = tx \in I \\ \Rightarrow\ &\exists u \in S : uh_{ij} \in I. \end{aligned} \tag{7.2.1}$$

Since $h \in U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda))$, $h_{ij} \in I + s_0R$. Hence there are a $y \in I$ and a $z \in s_0R$ such that $h_{ij} = y + z$. (7.2.1) implies that $uz \in I$. Since s_0 has property (1), it follows that $z \in I$. Thus $h_{ij} \in I$. Analogously one can show that $h_{ii} - 1 \in I$ for all $i \in \{1, \dots, -1\}$. Hence $h \equiv e(mod\ I)$. Now we show (b'). Let $j \in \{1, \dots, -1\}$. Since (b) holds, $f_S(x_j) \in \Gamma_S$. Hence

$$\begin{aligned} &\exists y \in \Gamma, s \in S : \frac{x_j}{1} = \frac{y}{s} \\ \Rightarrow\ &\exists y \in \Gamma, s, t \in S : t(x_js - y) = 0 \\ \Rightarrow\ &\exists y \in \Gamma, s, t \in S : stx_j = ty \in \Gamma \\ \Rightarrow\ &\exists u \in S : ux_j \in \Gamma. \end{aligned} \tag{7.2.2}$$

Since $h \in U_{2n}((R,\Lambda),(I + s_0R, \Gamma + s_0\Lambda))$, $x_j \in \Gamma + s_0\Lambda$. Hence there are a $y \in \Gamma$ and a $z \in s_0\Lambda$ such that $x_j = y + z$. (7.2.2) implies that $uz \in \Gamma$. Since s_0 has property (2), it follows that $z \in \Gamma$. Thus $x_j \in \Gamma$. Hence $g'_1 = g'_2$ and thus ϕ_S is injective on

$\psi(U_{2n}((R,\Lambda),(I+s_0R,\Gamma+s_0\Lambda)))$.

□

We construct now a specific supplemented base that we will use to construct specific local morphisms. In the lemma below we use the following conventions. Let $x \in R$. Then RxR denotes the *involution invariant ideal generated by* x, i.e. the ideal of R generated by $\{x,\bar{x}\}$. Now let (I,Γ) be a form ideal of (R,Λ) and assume that $x \in R\backslash I$ or $x \in \Gamma^I_{max}\backslash\Gamma$. Set $\Gamma(x) := \Gamma^{RxR}_{min}$ if $x \in R\backslash I$ and $\Gamma(x) := \Gamma^{RxR}_{min} + \langle\{yx\bar{y}|y \in R\}\rangle$ if $x \in \Gamma^I_{max}\backslash\Gamma$. $\Gamma(x)$ is called the *relative form parameter defined by* x *and* (I,Γ). One checks easily that $(RxR,\Gamma(x))$ is a form ideal of (R,Λ) which is not contained in (I,Γ), i.e. $RxR \nsubseteq I$ or $\Gamma(x) \nsubseteq \Gamma$. It is called the *form ideal defined by* x *and* (I,Γ).

Lemma 7.3 *Let* (I,Γ) *be a form ideal of* (R,Λ)*,* $S \subseteq C$ *a multiplicative subset and* $s_0 \in S$ *as in the previous lemma. Set* $A := \{EU_{2n}(ss_0R, ss_0\Lambda)|s \in S\}$ *and* $B := \{EU_{2n}(Rxs_0R, \Gamma(xs_0))|(x \in R, xs_0 \in R\backslash I) \vee (x \in \Lambda, xs_0 \in I\backslash\Gamma)\}$*. Then* (A,B) *is a special supplemented base for* $U_{2n}(R,\Lambda)$ *and* $F_S(A,B)$ *is a supplemented base for* $U_{2n}(R_S,\Lambda_S)$*.*

Proof First we show (A,B) is a special supplemented base for $U_{2n}(R,\Lambda)$. Clearly A and B are sets of nontrivial subgroups of E. We show now that A is a (nondiscrete) base of open subgroups of $1 \in E$. Therefore we must show that A satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let $U = EU_{2n}(ss_0R, ss_0\Lambda), V = EU_{2n}(ts_0R, ts_0\Lambda) \in A$. Set $W := EU_{2n}(sts_0R, sts_0\Lambda) \in A$. Then clearly $W \subseteq U \cap V$.

(2) Let $g \in E$ and $U = EU_{2n}(ss_0R, ss_0\Lambda) \in A$. There is a $K \in \mathbb{N}$ such that g is the product of K elementary unitary transvections. Set $V := EU_{2n}((ss_0)^{2\cdot4^K+4^{K-1}+\dots+4}R, (ss_0)^{2\cdot4^K+4^{K-1}+\dots+4}\Lambda) \in A$. Then ${}^gV \subseteq U$ (see Lemma 4.1 in [6]).

Hence A is a base of open subgroups of $1 \in E$. Let $EU_{2n}(Rxs_0R,\Gamma(xs_0)) \in B$. Then $EU_{2n}(Rxs_0R,\Gamma(xs_0)) \subseteq EU_{2n}(s_0R, s_0\Lambda) \in A$. It remains to show that if $U \in A$ and $V \in B$ then $U \cap V$ contains a member of B. Let $U = EU_{2n}(ss_0R, ss_0\Lambda) \in A$ and $V = EU_{2n}(Rxs_0R,\Gamma(xs_0)) \in B$. Set $W := EU_{2n}(Rxss_0R,\Gamma(xss_0))$. If $xs_0 \notin I$, then $xss_0 \notin I$ and if $xs_0 \notin \Gamma$, then $xss_0 \notin \Gamma$ (by the definition of s_0, see the previous lemma). Hence $W \in B$. Obviously $W \in U \cap V$. Since $A, B \subseteq B(E)$, (A,B) is a special supplemented base for $U_{2n}(R,\Lambda)$.

Now we show $F_S(A,B)$ is a supplemented base for $U_{2n}(R_S,\Lambda_S)$. Clearly $F_S(A)$ and $F_S(B)$ are sets of nontrivial subgroups of $E' := EU_{2n}(R_S,\Lambda_S)$. We show now that $F_S(A)$ is a (nondiscrete) base of open subgroups of $1 \in E'$. Therefore we must show that $F_S(A)$ satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let $U = F_S(EU_{2n}(ss_0R, ss_0\Lambda)), V = F_S(EU_{2n}(ts_0R, ts_0\Lambda)) \in F_S(A)$. Set $W := F_S(EU_{2n}(sts_0R, sts_0\Lambda)) \in F_S(A)$. Then clearly $W \subseteq U \cap V$.

(2) Let $g \in E'$ and $U = F_S(EU_{2n}(ts_0R, ts_0\Lambda)) \in F_S(A)$. There are a $K \in \mathbb{N}$ and elementary unitary transvections $\tau_1 = T_{i_1j_1}(\frac{x_1}{s_1}), \dots, \tau_K = T_{i_kj_k}(\frac{x_K}{s_K}) \in E'$ such that

$g = \tau_1 \dots \tau_K$. Set $s := s_1 \dots s_K$ and $V := F_S(EU_{2n}((sts_0)^{2\cdot 4^K + 4^{K-1} + \dots + 4}R, (st s_0)^{2\cdot 4^K + 4^{K-1} + \dots + 4}\Lambda)) \in F_S(A)$. Then ${}^gV \subseteq U$ (see Lemma 4.1 in [6]).

Hence $F_S(A)$ is a base of open subgroups of $1 \in E'$. That each member of $F_S(B)$ is contained in some member of $F_S(A)$ follows from the fact that any member of B is contained in a member of A. That given $U \in F_S(A)$ and $V \in F_S(B)$, $U \cap V$ contains a member of $F_S(B)$ follows from the fact that given $U \in A$ and $V \in B$, $U \cap V$ contains a member of B. Hence $F_S(A, B)$ is a supplemented base for $U_{2n}(R_S, \Lambda_S)$. $\square$

Now we construct specific local morphisms which will be used to prove (1.1).

Lemma 7.4 *Let (I, Γ) be a form ideal of (R, Λ) and $S \subseteq C$ a multiplicative subset such that $S \cap I = \varnothing$. Then ϕ_S is a local morphism of standard groups.*

Proof First we show that ϕ_S is a morphism of standard groups. Clearly ϕ_S maps a base generator to a base generator. Since $\{1\}$ is base subgroup of $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$, the topology induced by the base subgroups of $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ is the discrete one. It follows that ϕ_S induces a continuous homomorphism $EU_{2n}(R, \Lambda)/(EU_{2n}(R, \Lambda) \cap U_{2n}((R, \Lambda), (I, \Gamma))) \to EU_{2n}(R_S, \Lambda_S)/(EU_{2n}(R_S, \Lambda_S) \cap U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S)))$. Hence ϕ_S is a morphism of standard groups.

Let (A, B) be the special supplemented base for $U_{2n}(R, \Lambda)$ defined in the previous lemma. Since ψ induces a surjective homomorphism $EU_{2n}(R, \Lambda) \to EU_{2n}(R, \Lambda)/(EU_{2n}(R, \Lambda) \cap U_{2n}((R, \Lambda), (I, \Gamma)))$, it follows easily that $\psi(A, B)$ is a special supplemented base for $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$. We will show now that ϕ_S is local with respect to the special supplemented base $(A(\phi_S), B(\phi_S)) := \psi(A, B)$. Therefore we have to show that conditions (3.7.1) – (3.7.3) in Definition 3.7 are satisfied.

(3.7.1) By the previous lemma, $F_S(A, B)$ is a supplemented base for $U_{2n}(R_S, \Lambda_S)$. Since ρ_S induces a surjective homomorphism $EU_{2n}(R_S, \Lambda_S) \to EU_{2n}(R_S, \Lambda_S)/(EU_{2n}(R_S, \Lambda_S) \cap U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S)))$, it is easy to deduce that $\rho_S(F_S(A, B))$ is a supplemented base for $U_{2n}(R_S, \Lambda_S)/ U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$. Since $\rho_S \circ F_S = \phi_S \circ \psi$, it follows that $\phi_S(\psi(A, B))$ is a supplemented base for $U_{2n}(R_S, \Lambda_S)/U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$.

(3.7.2) By Lemma 7.2, ϕ_S is injective on $\psi(U_{2n}((R, \Lambda), (I + s_0R, \Gamma + s_0\Lambda)))$. Let $U \in \psi(A)$. Then there is an $s \in S$ such that $U = \psi(EU_{2n}(ss_0R, ss_0\Lambda))$. Hence $G(U) = G(\psi(EU_{2n}(ss_0R, ss_0\Lambda))) = \psi(U_{2n}((R, \Lambda), (I + ss_0R, \Gamma + ss_0\Lambda))) \subseteq \psi(U_{2n}((R, \Lambda), (I + s_0R, \Gamma + s_0\Lambda)))$. It follows that ϕ_S is injective on $G(U)$ for any $U \in \psi(A)$.

(3.7.3) It suffices to show that the conditions (3.8.1) and (3.8.2) in Lemma 3.8 are satisfied.

(3.8.1) Let $g' = \rho_S(T_{ij}(z))$ be a nontrivial base generator in $U_{2n}(R_S, \Lambda_S)/U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$, and $U = \psi(EU_{2n}(ss_0R, ss_0\Lambda)) \in \psi(A)$.
case 1 Assume that $i \neq \pm j$.

Choose an $x \in R$ and a $t \in S$ such that $z = \frac{x}{t}$. Since g' is nontrivial, $z \notin I_S$ and hence $xs_0 \notin I$. Set $V := \psi(EU_{2n}(R(ss_0)^5xR, \Gamma_{min}^{R(ss_0)^5xR})) = \psi(EU_{2n}(R(ss_0)^5xR, \Gamma((ss_0)^5x))) \in \psi(B)$. One can show routinely, using the relations $(R1)-(R6)$ in Lemma 6.12, that ${}^{\phi_S(U)}g'$ contains $\phi_S(V)$.

case 2 Assume that $i = -j$.

Choose an $x \in \Lambda$ and a $t \in S$ such that $z = \lambda_S^{-(\epsilon(i)+1)/2}\frac{x}{t}$. Since g' is nontrivial, $z \notin \lambda_S^{-(\epsilon(i)+1)/2}\Gamma_S$ and hence $xs_0 \notin \Gamma$.

case 2.1 Assume that $xs_0 \notin I$.

Set $V := \psi(EU_{2n}(R(ss_0)^6xR, \Gamma_{min}^{R(ss_0)^6xR})) = \psi(EU_{2n}(R(ss_0)^6xR, \Gamma((ss_0)^6x))) \in \psi(B)$. One can show routinely, using the relations $(R1)-(R6)$ in Lemma 6.12, that ${}^{\phi_S(U)}g'$ contains $\phi_S(V)$.

case 2.2 Assume that $xs_0 \in I$.

Set $V := \psi(EU_{2n}(R(ss_0)^7xR, \Gamma_{min}^{R(ss_0)^7xR} + \langle\{y(ss_0)^7x\bar{y}|y \in R\}\rangle)) = \psi(EU_{2n}(R(ss_0)^7xR, \Gamma((ss_0)^7x))) \in \psi(B)$. One can show routinely, using the relations $(R1)-(R6)$ in Lemma 6.12, that ${}^{\phi_S(U)}g'$ contains $\phi_S(V)$.

Since $\psi(A, B)$ is a supplemented base, there is a $W \in \psi(B)$ such that $W \subseteq U \cap V$. Clearly ${}^{\phi_S(U)}g'$ contains $\phi_S(W)$ and $W \subseteq U$.

(3.8.2) Let f' be a base generator in $U_{2n}(R_S, \Lambda_S)/U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$ and $V \in \psi(B)$. Choose a nontrivial base generator $g' \in \phi_S(V)$ which commutes with f'. Such a base generator exists by relation (R3) of Lemma 6.12. Then ${}^{f'}\phi_S(V)$ clearly contains ${}^{f'}g' = g'$.

Hence ϕ_S is local with respect to $(A(\phi_S), B(\phi_S)) = \psi(A, B)$. □

Next we show that the local morphisms ϕ_m where m is a maximal ideal of C such that $I \cap C \subseteq m$ form a covering.

Lemma 7.5 *Any quotient of the standard group $(G, E, Gen(E), B(E), G(\cdot))$ (where G, E, $Gen(E)$, $B(E)$ and $G(\cdot)$ are defined as in Lemma 7.1) has a covering.*

Proof Let (I, Γ) be a form ideal of $U_{2n}(R, \Lambda)$. Set $Z := \{\phi_m | m$ maximal ideal of C, $I \cap C \subseteq m\}$. We show that Z is a covering of the standard group $U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$. By the previous lemma, for any maximal ideal m of C such that $I \cap C \subseteq m$, ϕ_m is a local morphism (note that $I \cap C \subseteq m$ implies $S_m \cap I = \emptyset$). It remains to show that for any noncentral $g' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ there is a maximal ideal m of C such that $I \cap C \subseteq m$ and $\phi_m(g')$ is noncentral. Let $g' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ be noncentral. Then there is an $h' \in U_{2n}(R, \Lambda)/U_{2n}((R, \Lambda), (I, \Gamma))$ such that $g'h' \neq h'g'$. Let $g, h \in U_{2n}(R, \Lambda)$ such that $g' = gU_{2n}((R, \Lambda), (I, \Gamma))$ and $h' = hU_{2n}((R, \Lambda), (I, \Gamma))$. Set $\sigma := [g^{-1}, h^{-1}]$. Clearly $g'h' \neq h'g'$ implies $\sigma \notin U_{2n}((R, \Lambda), (I, \Gamma))$. Hence either $\sigma_{ij} \notin I$ for some $i, j \in \{1, \dots, -1\}$ such that $i \neq j$, or $\sigma_{ii} - 1 \notin I$ for some $i \in \{1, \dots, -1\}$ or $x_j := |\sigma_{*j}| \notin \Gamma$ for some $j \in \{1, \dots, -1\}$.

case 1 Assume that $\sigma_{ij} \notin I$ for some $i, j \in \{1, \dots, -1\}$ such that $i \neq j$. Set $Y := \{c \in$

$C|c\sigma_{ij} \in I\}$. Since $\sigma_{ij} \notin I$, Y is a proper ideal of C. Hence it is contained in a maximal ideal m of C. Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \emptyset$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously $(F_m(\sigma))_{ij} = f_m(\sigma_{ij})$. Assume $(F_m(\sigma))_{ij} \in I_m$. Then

$$\begin{aligned}
&\exists x \in I, s \in S_m : \frac{\sigma_{ij}}{1} = \frac{x}{s}\\
\Rightarrow &\exists x \in I, s, t \in S_m : t(\sigma_{ij}s - x) = 0\\
\Rightarrow &\exists x \in I, s, t \in S_m : st\sigma_{ij} = tx \in I\\
\Rightarrow &\exists u \in S_m : u\sigma_{ij} \in I.
\end{aligned}$$

But this contradicts $S_m \cap Y = \emptyset$. Hence $(F_m(\sigma))_{ij} \notin I_m$ and thus $\phi_m(g')$ is noncentral.

<u>case 2</u> Assume that $\sigma_{ii} - 1 \notin I$ for some $i \in \{1, \ldots, -1\}$. Set $Y := \{c \in C|c(\sigma_{ii} - 1) \in I\}$. Since $\sigma_{ii} - 1 \notin I$, Y is a proper ideal of C. Hence it is contained in a maximal ideal m of C. Clearly $I \cap C \subseteq Y \subseteq m$ and hence $S_m \cap Y = \emptyset$. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously $(F_m(\sigma))_{ii} - 1 = f_m(\sigma_{ii}) - 1 = f_m(\sigma_{ii} - 1)$. Assume $(F_m(\sigma))_{ii} - 1 \in I_m$. Then

$$\begin{aligned}
&\exists x \in I, s \in S_m : \frac{\sigma_{ii} - 1}{1} = \frac{x}{s}\\
\Rightarrow &\exists x \in I, s, t \in S_m : t((\sigma_{ii} - 1)s - x) = 0\\
\Rightarrow &\exists x \in I, s, t \in S_m : st(\sigma_{ii} - 1) = tx \in I\\
\Rightarrow &\exists u \in S_m : u(\sigma_{ii} - 1) \in I.
\end{aligned}$$

But this contradicts $S_m \cap Y = \emptyset$. Hence $(F_m(\sigma))_{ii} - 1 \notin I_m$ and thus $\phi_m(g')$ is noncentral.

<u>case 3</u> Assume that $x_j = |\sigma_{*j}| \notin \Gamma$ for some $j \in \{1, \ldots, -1\}$. Set $Y := \{c \in C|cx_j \in \Gamma\}$. Since $x_j \notin \Gamma$, Y is a proper ideal of C. Hence it is contained in a maximal ideal m of C. Since $x_j \in \Lambda$ and $y^2\Lambda \in \Gamma_{min} \subseteq \Gamma$ for any $y \in I \cap C$, $(I \cap C)^2 \subseteq Y \subseteq m$. This implies $S_m \cap Y = \emptyset$ and $I \cap C \subseteq m$, since m is prime. We show now that $\phi_m(g')$ does not commute with $\phi_m(h')$, i.e. $F_m(\sigma) \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Obviously

$$\begin{aligned}
&|(F_m(\sigma))_{*j}|\\
=&\sum_{i=1}^{n} \overline{(F_m(\sigma))}_{ij}(F_m(\sigma))_{-i,j}\\
=&\sum_{i=1}^{n} \overline{f_m(\sigma_{ij})}f_m(\sigma_{-i,j})\\
=&\sum_{i=1}^{n} f_m(\bar{\sigma}_{ij})f_m(\sigma_{-i,j})\\
=&f_m(\sum_{i=1}^{n} \bar{\sigma}_{ij}\sigma_{-i,j})
\end{aligned}$$

$$
\begin{aligned}
&= f_m(|\sigma_{*j}|) \\
&= f_m(x_j).
\end{aligned}
$$

Assume $f_m(x_j) \in \Gamma_m$. Then

$$
\begin{aligned}
&\exists y \in \Gamma, s \in S_m : \frac{x_j}{1} = \frac{y}{s} \\
\Rightarrow\ &\exists y \in \Gamma, s, t \in S_m : t(x_j s - y) = 0 \\
\Rightarrow\ &\exists y \in \Gamma, s, t \in S : stx_j = ty \in \Gamma \\
\Rightarrow\ &\exists u \in S_m : ux_j \in \Gamma.
\end{aligned}
$$

But this contradicts $S_m \cap Y = \varnothing$. Hence $|(F_m(\sigma))_{*j}| = f_m(x_j) \notin \Gamma_m$ and thus $\phi_m(g')$ is noncentral. □

The lemmas 7.6, 7.7 and 7.8 will be used in the proof of Lemma 7.9.

Lemma 7.6 *Let (I, Γ) be a form ideal of (R, Λ) and $S \subseteq C$ a multiplicative subset. Let $T_{ij}(x) \in EU_{2n}(R_S, \Lambda_S)$ be an elementary short or long root, $\sigma \in U_{2n}(R_S, \Lambda_S)$ and $s \in S$. Then $[\sigma, T_{ij}(x)] \in U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$ if and only if $[\sigma, T_{ij}(f_S(s)x)] \in U_{2n}((R_S, \Lambda_S), (I_S, \Gamma_S))$.*

Proof Straightforward computation. □

Lemma 7.7 *Let m be a maximal ideal of C and $\sigma \in U_{2n}(R_m, \Lambda_m)$. Then there is an $\epsilon \in EU_{2n}(R_m, \Lambda_m)$ such that $({}^{\epsilon}\sigma)_{11}$ is invertible.*

Proof By Lemma 1.4 in [9] and Lemma 3.4 in [3], R_m satisfies the Λ-stable range condition ΛS_1. Hence there is an $\epsilon_1 = \begin{pmatrix} e_{n\times n} & 0 \\ \gamma & e_{n\times n} \end{pmatrix} \in EU_{2n}(R_m, \Lambda_m)$, where $\gamma \in M_n(R_M)$, such that $(x_1, \ldots, x_n)$ is right unimodular where $(x_1, \ldots, x_{-1})$ is the first row of ${}^{\epsilon_1}\sigma$. Since ΛS_1 implies SR_1, there is a matrix $\epsilon_2 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \in EU_{2n}(R_m, \Lambda_m)$, where ω_1 and ω_2 are lower triangular matrices in $M_n(R_M)$ with 1's on the diagonal, such that the entry of $({}^{\epsilon_2\epsilon_1}\sigma)_{11}$ is right invertible. Since R is a Noetherian C-module, R is almost commutative. It follows that R_m is almost commutative and hence $({}^{\epsilon_2\epsilon_1}\sigma)_{11}$ is invertible, by Lemma 5.6. □

The following lemma has been proven by You (see [10], Lemma 3.5).

Lemma 7.8 *Let (I, Γ) be a form ideal of (R, Λ) and $\sigma \in U_{2n}(R, \Lambda)$. Further let $x \in R$ and $i, j \in \{1, \ldots, -1\}$ such that $i \neq \pm j$. Set $\tau := [\sigma, T_{ij}(x)]$. Then*

$$|\tau_{*k}| = \bar{\sigma}'_{jk}\bar{x}|\sigma_{*i}|x\sigma'_{jk} + \bar{\sigma}'_{-i,k}x|\sigma_{*,-j}|\bar{x}\sigma'_{-i,k} + y_k,$$

if $k \neq j, -i$,

$$|\tau_{*j}| = \bar{\sigma}'_{jj}\bar{x}|\sigma_{*i}|x\sigma'_{jj} + \bar{\sigma}'_{-i,j}x|\sigma_{*,-j}|\bar{x}\sigma'_{-i,j} + \bar{x}|\tau_{*i}|x + y_j$$

and

$$|\tau_{*,-i}| = \bar{\sigma}'_{j,-i}\bar{x}|\sigma_{*i}|x\sigma'_{j,-i} + \bar{\sigma}'_{-i,-i}x|\sigma_{*,-j}|\bar{x}\sigma'_{-i,-i} + x|\tau_{*,-j}|\bar{x} + y_{-i}$$

where for each $k \in \{1, \dots, -1\}$, y_k is a finite sum of terms of the form $z - \lambda\bar{z}$ where z lies in the ideal generated by the nondiagonal entries of σ and σ^{-1}. It follows that if $\sigma \in U_{2n}((R,\Lambda),(I, I \cap \Lambda))$, then $|\tau_{*k}| \in \Gamma\ \forall k \neq j, -i$, $|\tau_{*j}| \equiv \bar{x}|\sigma_{*i}|x (\text{mod } \Gamma)$ and $|\tau_{*,-i}| \equiv x|\sigma_{*,-j}|\bar{x}(\text{mod } \Gamma)$.

Proof Straightforward computation. □

Now we show that the codomains of the local morphisms ϕ_m are solution groups and hence $(G, E, Gen(E), B(E), G(\cdot))$ and each of its quotients have a covering by solution groups. In the following lemma we will apply lemmas and corollaries in [1], chapter IV, §3. We are allowed to do this since for any maximal ideal m of C, $C'_m := S_m^{-1}C'$ is semilocal by Lemma 1.4 in [9] and hence the Bass-Serre-dimension of C'_m is 0. Since R is module finite over C', R_m is module finite over C'_m and hence R_m is a finite C'_m-algebra.

Solution Group Lemma 7.9 *Let (I, Γ) be a form ideal of (R, Λ) and m a maximal ideal of C such that $I \cap C \subseteq m$. Then the codomain of ϕ_m is a solution group for $A' := \phi_m(A(\phi_m))$ where $A(\phi_m)$ is defined as in Lemma 7.4.*

Proof Let $h' \in U_{2n}(R_m, \Lambda_m)/U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ be noncentral. We have to show that h' satisfies a solution equation with respect to A'. Let $U' \in A'$. Let $h \in U_{2n}(R_m, \Lambda_m)$ such that $h' = \rho_m(h)$. Since h' is noncentral, $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. The proof is divided into three parts, I, II and III. In Part I we assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n > 3$. In Part II we assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n = 3$. In Part III we assume that $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $A := \{F_m(EU_{2n}(ss_0R, ss_0\Lambda)) | s \in S_m\}$. By Lemma 7.3, A is a base of open subgroups of $1 \in EU_{2n}(R_m, \Lambda_m)$. Let $U \in A$ such that $\rho_m(U) = U'$. In each of the parts I, II and III choose $U_0, \dots, U_9 \in A$ such that for all $(k+1)$-tuples $(\epsilon_0, \dots, \epsilon_k)$ used in that particular part, ${}^{d_i}U_i \subseteq U\ \forall i \in \{0, \dots, k\}$ (possible since A is a base of open subgroups of $1 \in EU_{2n}(R_m, \Lambda_m)$ and in each part there are only finitely many $(k+1)$-tuples $(\epsilon_0, \dots, \epsilon_k)$ which are used). Since $U_0, \dots, U_9 \in A$, there are $t_0, \dots, t_9 \in S_m$ such that $U_i = F_m(EU_{2n}(t_is_0R, t_is_0\Lambda))\ \forall i \in \{0, \dots, 9\}$. Set $s_i := f_m(t_is_0)$ $(i = 0, \dots, 9)$.

<u>Part I</u> Assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n > 3$.
By [1], chapter IV, Corollary 3.10 (applied with $H = {}^{EU_{2n}(R_m,\Lambda_m)}\langle h\rangle$), there is an $\epsilon_0 \in EU_{2n}(R_m, \Lambda_m)$ and an $x = \frac{a}{s} \in R_m$ such that $[{}^{\epsilon_0}h, T_{1,-2}(x)] \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. By [1], chapter IV, Lemma 3.12, Part I, case 7 there is a matrix $\epsilon_1 \in EU_{2n}(R_m, \Lambda_m)$ of the form

$$\epsilon_1 = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where $X, Y, Z \in M_n(R_m)$, such that the first n coordinates of $\epsilon_1({}^{\epsilon_0}h)_{*1}$ equal $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^t$ and the first n coordinates of $\epsilon_1({}^{\epsilon_0}h)_{*2}$ equal $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \end{pmatrix}^t$. Set $f_m(a) := \hat{a}$ and $f_m(s) := \hat{s}$. Set $g_0 := T_{1,-2}(s_0\hat{s}x) = T_{1,-2}(s_0\frac{s}{1}\frac{a}{s}) = T_{1,-2}(s_0\hat{a}) \in U_0$. By Lemma 7.6, $[{}^{\epsilon_0}h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Since $U_{2n}((R_m, \Lambda_m), (I_m,$

$I_m \cap \Lambda_m)$) is normal, it follows that $\sigma := {}^{\epsilon_1}[{}^{\epsilon_0}h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Since

$$
\begin{aligned}
&\sigma\\
=&{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]\\
=&{}^{\epsilon_1}(g_0^{-1} + ({}^{\epsilon_0}h)_{*1}s_0\hat{a}(({}^{\epsilon_0}h)^{-1})_{-2,*}g_0^{-1}\\
&- ({}^{\epsilon_0}h)_{*2}\bar{\lambda}_m\overline{s_0\hat{a}}(({}^{\epsilon_0}h)^{-1})_{-1,*}g_0^{-1})\\
=&{}^{\epsilon_1}(g_0^{-1}) +{}^{\epsilon_1} (({}^{\epsilon_0}h)_{*1}s_0\hat{a}(({}^{\epsilon_0}h)^{-1})_{-2,*}g_0^{-1})\\
&-{}^{\epsilon_1} (({}^{\epsilon_0}h)_{*2}\bar{\lambda}_m\overline{s_0\hat{a}}(({}^{\epsilon_0}h)^{-1})_{-1,*}g_0^{-1})\\
=&{}^{\epsilon_1}(g_0^{-1}) + \epsilon_1({}^{\epsilon_0}h)_{*1}s_0\hat{a}(({}^{\epsilon_0}h)^{-1})_{-2,*}g_0^{-1}(\epsilon_1)^{-1}\\
&- \epsilon_1({}^{\epsilon_0}h)_{*2}\bar{\lambda}_m\overline{s_0\hat{a}}(({}^{\epsilon_0}h)^{-1})_{-1,*})g_0^{-1}(\epsilon_1)^{-1}
\end{aligned}
$$

and ${}^{\epsilon_1}(g_0^{-1}) = e + (\epsilon_1)_{*1}s_0\hat{a}((\epsilon_1)^{-1})_{-2,*} - (\epsilon_1)_{*2}\bar{\lambda}_m\overline{s_0\hat{a}}((\epsilon_1)^{-1})_{-1,*}$ has the form

$$
\begin{pmatrix} e_{n\times n} & * \\ 0 & e_{n\times n} \end{pmatrix},
$$

σ has the form

$$
\left(\begin{array}{cc|cc}
\alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\
0 & e_{(n-2)\times(n-2)} & \beta_3 & \beta_4 \\
\hline
\multicolumn{2}{c|}{\gamma} & \multicolumn{2}{c}{\delta}
\end{array}\right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
$$

where $\alpha_1, \beta_2 \in M_2(R_m)$, $\alpha_2, \beta_1 \in M_{2\times(n-2)}(R_m)$, $\beta_3 \in M_{n-2}(R_m)$, $\beta_4 \in M_{(n-2)\times 2}(R_m)$ and $\alpha = (a_{ij})_{1\leqslant i,j\leqslant n}, \beta = (b_{ij})_{\substack{1\leqslant i\leqslant n\\ -n\leqslant j\leqslant -1}}, \gamma = (c_{ij})_{\substack{-n\leqslant i\leqslant -1\\ 1\leqslant j\leqslant n}}, \delta = (d_{ij})_{-n\leqslant i,j\leqslant -1} \in M_n(R_m)$.

<u>case 1</u> Assume that either $\alpha \not\equiv e_{n\times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$ or $\delta \not\equiv e_{n\times n}(\text{mod } I_m)$.
We will show that it follows that $\alpha \not\equiv e_{n\times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$. Assume that $\alpha \equiv e_{n\times n}(\text{mod } I_m)$ and $\gamma \equiv 0(\text{mod } I_m)$. Let $\kappa : M_n(R_m) \to M_n(R_m/I_m)$ be the homomorphism induced by the canonical homomorphism $R_m \to R_m/I_m$. Since the image of σ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$ equals $\begin{pmatrix} e_{n\times n} & \kappa(\beta) \\ 0 & \kappa(\delta) \end{pmatrix}$, $\kappa(\delta) = e_{n\times n}$ by Lemma 6.9. That is equivalent to $\delta \equiv e_{n\times n}(\text{mod } I_m)$. Since this is a contradiction, $\alpha \not\equiv e_{n\times n}(\text{mod } I_m)$ or $\gamma \not\equiv 0(\text{mod } I_m)$. Hence there is an $i \in \{1, \dots, n\}$ such that $\sigma_{*i} \not\equiv e_i(\text{mod } I_m)$.

<u>case 1.1</u> Assume that $i \in \{1, \dots, n-2\}$.
Clearly the $(n-1)$-th row of

$$
\begin{aligned}
&[\sigma, T_{i,-(n-1)}(1)]\\
=&(e + \sigma_{*i}\sigma'_{-(n-1),*} - \bar{\lambda}_m\sigma_{*(n-1)}\sigma'_{-i,*})T_{i,-(n-1)}(-1)
\end{aligned}
$$

$$
=(e+\begin{array}{c}1\\ \\ \\ \\ \\ i\\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix}a_{1i}\\ a_{2i}\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\\ c_{-n,i}\\ \vdots\\ c_{-1,i}\end{pmatrix}\begin{array}{c}\begin{array}{ccccc}1 & n & -n & & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,n-1}\dots\lambda_m\bar{c}_{-n,n-1} & 0 & 1\,0\dots 0\,\bar{a}_{2(n-1)}\,\bar{a}_{1(n-1)}\end{pmatrix}\end{array}
$$

$$
-\bar{\lambda}_m\begin{array}{c}1\\ \\ \\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix}a_{1(n-1)}\\ a_{2(n-1)}\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ c_{-n,n-1}\\ \vdots\\ c_{-1,n-1}\end{pmatrix}\begin{array}{c}\begin{array}{ccccc}1 & n & -n & -i & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,i}\dots\lambda_m\bar{c}_{-n,i} & 0 & \dots 0\;1\;0\dots 0\,\bar{a}_{2i}\,\bar{a}_{1i}\end{pmatrix}\end{array})
$$

$$
\cdot T_{i,-(n-1)}(-1)
$$

is not congruent to f_{n-1} modulo I_m since $\sigma_{*i} \not\equiv e_i(\text{mod } I_m)$. Hence $[\sigma, T_{i,-(n-1)}(1)] \notin U_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Set $g_2 := T_{i,-(n-1)}(s_1) \in U_1$. By Lemma 7.6, $[\sigma, g_2] \notin U_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Clearly the n-th row of $[\sigma, g_2]$ equals f_n. Set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m,\Lambda_m)$. Then the first row of $\tau :=^{\epsilon_2}[\sigma, g_2]$ equals f_1. Since $U_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Clearly τ has the form

$$
\left(\begin{array}{cc|cc}1 & 0 & 0 & 0\\ A_3 & A_4 & B_3 & 0\\ \hline C_1 & C_2 & D_1 & 0\\ C_3 & C_4 & D_3 & 1\end{array}\right)=\begin{pmatrix}A & B\\ C & D\end{pmatrix}
$$

where $C_3 \in R_m,\ A_3, C_1 \in (R_m)^{n-1},\ C_4, D_3 \in {}^{n-1}(R_m),\ A_4, B_3, C_2, D_1 \in M_{n-1}(R_m)$ and $A = (A_{ij})_{1\leqslant i,j\leqslant n}, B = (B_{ij})_{\substack{1\leqslant i\leqslant n\\ -n\leqslant j\leqslant -1}}, C = (C_{ij})_{\substack{-n\leqslant i\leqslant -1\\ 1\leqslant j\leqslant n}}, D = (D_{ij})_{-n\leqslant i,j\leqslant -1} \in M_n(R_m)$. Set

$$
E = (E_{ij})_{2\leqslant i,j\leqslant -2} := \left(\begin{array}{c|c}A_4 & B_3\\ \hline C_2 & D_1\end{array}\right) \in M_{2n-2}(R_m).
$$

<u>case 1.1.1</u> Assume that $E \not\equiv e_{(2n-2)\times(2n-2)}(\text{mod } I_m)$.

There are $i, j \in \{2, \ldots, -2\}$ such that $(E - e_{(2n-2)\times(2n-2)})_{ij} \notin I_m$. Set $g_2 := T_{-1,i}(s_2) \in U_2$. Then $\omega := [\tau^{-1}, g_2]$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & e_{(2n-2)\times(2n-2)} & 0 \\ * & w & 1 \end{pmatrix}$$

where $w = (w_2, \ldots, w_{-2}) = s_2(E - e_{(2n-2)\times(2n-2)})_{i*}$. Since $(E - e_{(2n-2)\times(2n-2)})_{ij} \notin I_m$, $w_j =: \frac{b'}{t'} \notin I_m$. Set $b := \frac{b'}{1} \in R_m$ and $t := \frac{t'}{1} \in R_m$. Choose an $l \neq \pm 1, \pm j$ and set $g_3 := T_{jl}(s_3) \in U_3$, $g_4 := T_{l,-j}(s_4 s_5 t) \in U_4$ and $g_5 := T_{-1,-j}(s_3 s_4 s_5 t w_j) = T_{-1,-j}(s_3 s_4 s_5 b) \in U_5$. Notice that $g_5 \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, since $w_j \notin I_m$ and $s_3 s_4 s_5 t$ is invertible. One checks easily that

$$\begin{aligned} &[[[^{\epsilon_2}([^{\epsilon_1}[^{\epsilon_0}h, g_0], g_2]^{-1}), g_2], g_3], g_4] \\ =&[[[(^{\epsilon_2}[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_2])^{-1}, g_2], g_3], g_4] \\ =&[[\omega, g_3], g_4] \\ =&g_5. \end{aligned}$$

Set $g'_i := \rho_m(g_i)\ \forall i \in \{0, \ldots, 5\}$, $\epsilon'_i := \rho_m(\epsilon_i)\ \forall i \in \{0, 1, 2\}$ and $d'_i := \rho_m(d_i)\ \forall i \in \{0, \ldots, 5\}$. Then $\epsilon'_0, \epsilon'_1, \epsilon'_2 \in E' := \rho_m(EU_{2n}(R_m, \Lambda_m))$, $g'_5 \in Gen(E')\backslash\{e\}$, ${}^{d'_i}g'_i \in U'\ \forall i \in \{0, \ldots, 5\}$ and

$$[[[^{\epsilon'_2}([^{\epsilon'_1}[^{\epsilon'_0}h', g'_0], g'_1]^{-1}), g'_2], g'_3], g'_4] = g'_5.$$

<u>case 1.1.2</u> Assume that $E \equiv e_{(2n-2)\times(2n-2)} (\text{mod } I_m)$ and $A_3 \equiv 0 (\text{mod } I_m)$.
Set $\xi_1 := \prod\limits_{l=2}^{n} T_{l1}(-A_{l1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Since $\xi_1 \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \subseteq U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)) \subseteq U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\xi_1 \tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly

$$\xi_1 \tau = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array}\right)$$

for some $C'_3 \in R_m$ and $C'_4, D'_3 \in {}^{n-1}(R_m)$ such that $D'_3 \equiv 0 (\text{mod } I_m)$ (consider the image of $\xi_1 \tau$ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$).

<u>case 1.1.2.1</u> Assume that there is an $i \in \{3, \ldots, n\}$ such that $C'_{-1,i} \notin I_m$.
Set $\epsilon_{21} := T_{12}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then ${}^{\epsilon_{21}}(\xi_1 \tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A''_2 & B''_1 & B''_2 \\ 0 & A_4 & B''_3 & B''_4 \\ \hline C''_1 & C''_2 & D''_1 & D''_2 \\ C''_3 & C''_4 & D''_3 & D''_4 \end{array}\right)$$

where $B''_2, C''_3, D''_4 \in R_m$, $B''_4, C''_1 \in (R_m)^{n-1}$, $A''_2, B''_1, C''_4, D''_3 \in {}^{n-1}(R_m)$, $B''_3, C''_2, D''_1 \in M_{n-1}(R_m)$. Furthermore $A''_2 \equiv 0 (\text{mod } I_m)$ and $C'''_{-2,i} \equiv C'_{-1,i} (\text{mod } I_m)$. Set $\xi_2 :=$

$\prod_{l=2}^{n} T_{1l}(-A''_{1l}) \in EU_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$. $\omega :=^{\epsilon_{21}}(\xi_1\tau)\xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B'''_1 & B'''_2 \\ 0 & A_4 & B'''_3 & B'''_4 \\ \hline C'''_1 & C'''_2 & D'''_1 & D'''_2 \\ C'''_3 & C'''_4 & D'''_3 & D'''_4 \end{array}\right)$$

where $B'''_2, C'''_3, D'''_4 \in R_m$, $B'''_4, C'''_1, D'''_2 \in (R_m)^{n-1}$, $B'''_1, C'''_4, D'''_3 \in^{n-1}(R_m)$, $B'''_3, C'''_2, D'''_1 \in M_{n-1}(R_m)$. Further $C'''_{-2,i} \equiv C''_{-2,i} \equiv C'_{-1,i} (\text{mod } I_m)$. Since $C'_{-1,i} \notin I_m$, $C'''_{-2,i} \notin I_m$. Set $g_2 := T_{2,-i}(s_2) \in U_2$. Then

$$\begin{aligned}
&[\omega, g_2]\\
=&(e + \omega_{*2}s_2\omega'_{-i,*} - \omega_{*i}\bar{\lambda}_m s_2\omega'_{-2,*})(g_2)^{-1}\\
=&(e + s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array}\begin{pmatrix} 0 \\ A_{22} \\ \vdots \\ A_{n2} \\ C'''_{-n,2} \\ \vdots \\ C'''_{-1,2} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} 1 & & n & -n & & -1 & \end{array} \\ \begin{pmatrix} \lambda_m\bar{C}'''_{-1,i} & \dots & \lambda_m\bar{C}'''_{-n,i} & \bar{A}_{ni} & \dots & \bar{A}_{2i} & 0 \end{pmatrix} \end{array}\\
&- \bar{\lambda}_m s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array}\begin{pmatrix} 0 \\ A_{2i} \\ \vdots \\ A_{ni} \\ C'''_{-n,i} \\ \vdots \\ C'''_{-1,i} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} 1 & & n & -n & & -1 & \end{array} \\ \begin{pmatrix} \lambda_m\bar{C}'''_{-1,2} & \dots & \lambda_m\bar{C}'''_{-n,2} & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix} \end{array})\\
&\cdot (g_2)^{-1}.
\end{aligned}$$

Clearly $[\omega, g_2]_{1*} = f_1$ and

$$[\omega, g_2]_{22} = 1 + s_2A_{22}\lambda_m\bar{C}'''_{-2,i} - \bar{\lambda}_m s_2A_{2i}\lambda_m\bar{C}'''_{-2,2}.$$

Since $A_{2i} \in I_m$, $-\bar{\lambda}_m s_2A_{2i}\lambda_m\bar{C}'''_{-2,2} \in I_m$. Since $C'''_{-2,i} \notin I_m$, $\bar{C}'''_{-2,i} \notin I_m$. Hence $s_2A_{22}\lambda_m\bar{C}'''{}_{-2,i} \notin I_m$ since $A_{22} \equiv 1 (\text{mod } I_m)$ and s_2 and λ_m are invertible. It follows that $[\omega, g_2]_{22} \not\equiv 1 (\text{mod } I_m)$. One can proceed now as in case 1.1.1 (note that $\rho_m(\xi_1) = \rho_m(\xi_2) = e$ since $\xi_1, \xi_2 \in EU_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m)) \subseteq U_{2n}((R_m,\Lambda_m),(I_m, \Gamma_m)))$.

<u>case 1.1.2.2</u> Assume that $C'_{-1,2} \notin I_m$.
Set $\epsilon_{21} := T_{13}(-1) \in EU_{2n}(R_m, \Lambda_m)$. Then ${}^{\epsilon_{21}}(\xi_1\tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A''_2 & B''_1 & B''_2 \\ 0 & A_4 & B''_3 & B''_4 \\ \hline C''_1 & C''_2 & D''_1 & D''_2 \\ C''_3 & C''_4 & D''_3 & D''_4 \end{array}\right)$$

where $B''_2, C''_3, D''_4 \in R_m$, $B''_4, C''_1 \in (R_m)^{n-1}$, $A''_2, B''_1, C''_4, D''_3 \in {}^{n-1}(R_m)$, $B''_3, C''_2, D''_1 \in M_{n-1}(R_m)$. Furthermore $A''_2 \equiv 0(\text{mod } I_m)$ and $C''_{-3,2} \equiv C'_{-1,2}(\text{mod } I_m)$. Set $\xi_2 := \prod\limits_{l=2}^{n} T_{1l}(-A''_{1l}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. $\omega := {}^{\eta}(\xi_1\tau)\xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B'''_1 & B'''_2 \\ 0 & A_4 & B'''_3 & B'''_4 \\ \hline C'''_1 & C'''_2 & D'''_1 & D'''_2 \\ C'''_3 & C'''_4 & D'''_3 & D'''_4 \end{array}\right)$$

where $B'''_2, C'''_3, D'''_4 \in R_m$, $B'''_4, C'''_1, D'''_2 \in (R_m)^{n-1}$, $B'''_1, C'''_4, D'''_3 \in {}^{n-1}(R_m)$, B'''_3, C'''_2, $D'''_1 \in M_{n-1}(R_m)$. Further $C'''_{-3,2} \equiv C''_{-3,2} \equiv C'_{-1,2}(\text{mod } I_m)$. Since $C'_{-1,2} \notin I_m$, $C'''_{-3,2} \notin I_m$. Set $g_2 := T_{3,-2}(s_2) \in U_2$. Then

$$\begin{aligned} &[\omega, g_2] \\ =&(e + \omega_{*3}s_2\omega'_{-2,*} - \omega_{*2}\bar{\lambda}_m s_2 \omega'_{-3,*})(g_2)^{-1} \end{aligned}$$

$$=(e + s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} 0 \\ A_{23} \\ \vdots \\ A_{n3} \\ C'''_{-n,3} \\ \vdots \\ C'''_{-1,3} \end{pmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & & n & -n & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{C}'''_{-1,2} & \dots & \lambda_m \bar{C}'''_{-n,2} & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix} \end{array}$$

$$- \bar{\lambda}_m s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} 0 \\ A_{22} \\ \vdots \\ A_{n2} \\ C'''_{-n,2} \\ \vdots \\ C'''_{-1,2} \end{pmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & & n & -n & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{C}'''_{-1,3} & \dots & \lambda_m \bar{C}'''_{-n,3} & \bar{A}_{n3} & \dots & \bar{A}_{23} & 0 \end{pmatrix} \end{array})$$

$$\cdot (g_2)^{-1}.$$

Clearly $[\omega,g_2]_{1*}=f_1$ and

$$[\omega,g_2]_{33}=1+s_2A_{33}\lambda_m\bar{C}'''_{-3,2}-\bar{\lambda}_m s_2A_{32}\lambda_m\bar{C}'''_{-3,3}.$$

Since $A_{32}\in I_m$, $-\bar{\lambda}_m s_2A_{32}\lambda_m\bar{C}'''_{-3,3}\in I_m$. Since $C'''_{-3,2}\notin I_m$, $\bar{C}'''_{-3,2}\notin I_m$. Hence $s_2A_{33}\lambda_m\bar{C}'''_{-3,2}\notin I_m$ since $A_{33}\equiv 1(\text{mod } I_m)$ and s_2 and λ_m are invertible. It follows that $[\omega,g_2]_{33}\not\equiv 1(\text{mod } I_m)$. One can proceed now as in case 1.1.1

<u>case 1.1.2.3</u> Assume that $C'_{-1,i}\in I_m\ \forall i\in\{2,\dots,n\}$.
It follows that $C_{i1}\in I_m\ \forall i\in\{-n,\dots,-2\}$. Since $\xi_1\tau\notin U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$, $C'_{-1,1}\notin I_m$. Set $\epsilon_{21}:=T_{12}(-1)\in EU_{2n}(R_m,\Lambda_m)$. Then ${}^{\epsilon_{21}}(\xi_1\tau)$ has the form

$$\left(\begin{array}{cc|cc} 1 & A''_2 & B''_1 & B''_2\\ 0 & A_4 & B''_3 & B''_4\\ \hline C''_1 & C''_2 & D''_1 & D''_2\\ C''_3 & C''_4 & D''_3 & D''_4\end{array}\right)$$

where $B''_2,C''_3,D''_4\in R_m$, $B''_4,C''_1\in(R_m)^{n-1}$, $A''_2,B''_1,C''_4,D''_3\in{}^{n-1}(R_m)$, $B''_3,C''_2,D''_1\in M_{n-1}(R_m)$. Furthermore $A''_2\equiv 0(\text{mod } I_m)$ and $C''_{-2,2}=C_{-2,2}+C'_{-1,2}+C_{-2,1}+C'_{-1,1}\equiv C'_{-1,1}(\text{mod } I_m)$. Set $\xi_2:=\prod_{l=2}^{n}T_{1l}(-A''_{1l})\in EU_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$. $\omega:={}^{\eta}(\xi_1\tau)\xi_2$ has the form

$$\left(\begin{array}{cc|cc} 1 & 0 & B'''_1 & B'''_2\\ 0 & A_4 & B'''_3 & B'''_4\\ \hline C'''_1 & C'''_2 & D'''_1 & D'''_2\\ C'''_3 & C'''_4 & D'''_3 & D'''_4\end{array}\right)$$

where $B'''_2,C'''_3,D'''_4\in R_m$, $B'''_4,C'''_1,D'''_2\in(R_m)^{n-1}$, $B'''_1,C'''_4,D'''_3\in{}^{n-1}(R_m)$, B'''_3,C'''_2, $D'''_1\in M_{n-1}(R_m)$. Further $C'''_{-2,2}\equiv C''_{-2,2}\equiv C'_{-1,1}(\text{mod } I_m)$. Since $C'_{-1,1}\notin I_m$, $C'''_{-2,2}\notin I_m$. Set $g_2:=T_{2,-3}(f_m(t_3s_0))\in U_2$. Then

$$\begin{aligned}&[\omega,g_2]\\ =&(e+\omega_{*2}s_2\omega'_{-3,*}-\omega_{*3}\bar{\lambda}_m s_2\omega'_{-2,*})(g_2)^{-1}\end{aligned}$$

$$=(e+s_2\begin{array}{c}1\\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix}0\\ A_{22}\\ \vdots\\ A_{n2}\\ C'''_{-n,2}\\ \vdots\\ C'''_{-1,2}\end{pmatrix}\begin{array}{cccccc} 1 & & n & -n & & -1\\ \end{array}\begin{pmatrix}\lambda_m\bar{C}'''_{-1,3} & \dots & \lambda_m\bar{C}'''_{-n,3} & \bar{A}_{n3} & \dots & \bar{A}_{23} & 0\end{pmatrix}$$

$$
-\bar{\lambda}_m s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} 0 \\ A_{23} \\ \vdots \\ A_{n3} \\ C''''_{-n,3} \\ \vdots \\ C''''_{-1,3} \end{pmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & & n & -n & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{C}''''_{-1,2} & \dots & \lambda_m \bar{C}''''_{-n,2} & \bar{A}_{n2} & \dots & \bar{A}_{22} & 0 \end{pmatrix} \end{array})
$$

$$
\cdot (g_2)^{-1}.
$$

Clearly $[\omega, g_2]_{1*} = f_1$ and

$$
[\omega, g_2]_{32} = s_2 A_{32} \lambda_m \bar{C}''''_{-2,3} - \bar{\lambda}_m s_2 A_{33} \lambda_m \bar{C}''''_{-2,2}.
$$

Since $A_{32} \in I_m$, $-\bar{\lambda}_m s_2 A_{32} \lambda_m \bar{C}''''_{-2,2} \in I_m$. Since $C''''_{-2,2} \notin I_m$, $\bar{C}''''_{-2,2} \notin I_m$. Hence $s_2 A_{33} \lambda_m \bar{C}''''_{-2,i} \notin I_m$ since $A_{33} \equiv 1 (\text{mod } I_m)$ and s_2 and λ_m are invertible. It follows that $[\omega, g_2]_{32} \notin I_m$. One can proceed now as in case 1.1.1

<u>case 1.1.3</u> Assume that $E \equiv e_{(2n-2)\times(2n-2)} (\text{mod } I_m)$ and $A_3 \not\equiv 0 (\text{mod } I_m)$.
Since $A_3 \not\equiv 0 (\text{mod } I_m)$, $D_3 \not\equiv 0 (\text{mod } I_m)$. Hence there is an $i \in \{-n, \dots, -2\}$ such that $D_{-1,i} \notin I_m$. Choose a $j \in \{2, \dots, n\} \backslash \{-i\}$ and set $g_2 := T_{ij}(s_2) \in U_2$. Then

$$
\begin{aligned}
&[\tau, g_2] \\
=&(e + \tau_{*i} s_2 \tau'_{j*} - \tau_{*,-j} \lambda_m s_2 \tau'_{-i,*})(g_2)^{-1}
\end{aligned}
$$

$$
=(e + s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} 0 \\ B_{2i} \\ \vdots \\ B_{ni} \\ D_{-n,i} \\ \vdots \\ D_{-1,i} \end{pmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & & n & -n & & -1 \end{array} \\ \begin{pmatrix} \bar{D}_{-1,-j} & \dots & \bar{D}_{-n,-j} & \bar{\lambda}_m \bar{B}_{n,-j} & \dots & \bar{\lambda}_m \bar{B}_{2,-j} & 0 \end{pmatrix} \end{array}
$$

$$
-\lambda_m s_2 \begin{array}{c} 1 \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} 0 \\ B_{2,-j} \\ \vdots \\ B_{n,-j} \\ D_{-n,-j} \\ \vdots \\ D_{-1,-j} \end{pmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & & n & -n & & -1 \end{array} \\ \begin{pmatrix} \bar{D}_{-1,i} & \dots & \bar{D}_{-n,i} & \bar{\lambda}_m \bar{B}_{ni} & \dots & \bar{\lambda}_m \bar{B}_{2i} & 0 \end{pmatrix} \end{array})
$$

$\cdot (g_2)^{-1}$.

Clearly $[\tau, g_2]_{1*} = f_1$ and

$$\begin{aligned}&[\tau, g_2]_{-1,j}\\ =&s_2 D_{-1,i}\bar{D}_{-j,-j} - \lambda_m s_2 D_{-1,-j}\bar{D}_{-j,i}\\ &- s_2(s_2 D_{-1,i}\bar{\lambda}_m \bar{B}_{-i,-j} - \lambda_m s_2 D_{-1,-j}\bar{\lambda}_m \bar{B}_{-i,i})\end{aligned}$$

Since $\bar{D}_{-j,i}, \bar{B}_{-i,-j}, \bar{B}_{-i,i} \in I_m$, it follows that $-\lambda_m s_2 D_{-1,-j}\bar{D}_{-j,i} - s_2(s_2 D_{-1,i}\bar{\lambda}_m \bar{B}_{-i,-j} - \lambda_m s_2\ D_{-1,-j}\bar{\lambda}_m \bar{B}_{-i,i}) \in I_m$. On the other hand $s_2 D_{-1,i}\bar{D}_{-j,-j} \notin I_m$ since $D_{1,-i} \notin I_m$, $\bar{D}_{-j,-j} \equiv 1(\text{mod } I_m)$ and s_2 is invertible. It follows that $[\tau, g_2]_{-1,j} \notin I_m$ and hence $[\tau, g_2] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further $[\tau, g_2]_{l1} \in I_m\ \forall l \in \{2, \dots, n\}$ since $B \equiv 0(\text{mod } I_m)$. Thus one can proceed now as in case 1.1.1 or as in case 1.1.2.

<u>case 1.2</u> Assume $\sigma_{*j} \equiv e_j(\text{mod } I_m)\ \forall j \in \{1, \dots, n-2\}$, $\sigma_{*(n-1)} \not\equiv e_{n-1}(\text{mod } I_m)$ and $a_{1(n-1)} \in I_m$.
Consider the first row of

$$\begin{aligned}&[\sigma, T_{1,-(n-1)}(1)]\\ =&(e + \sigma_{*1}\sigma'_{-(n-1),*} - \bar{\lambda}_m \sigma_{*(n-1)}\sigma'_{-1,*})T_{1,-(n-1)}(-1)\end{aligned}$$

$$=(e + \begin{array}{c} 1\\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix} a_{11}\\ a_{21}\\ 0\\ \vdots\\ 0\\ c_{-n,1}\\ \vdots\\ c_{-1,1}\end{pmatrix} \begin{array}{c} \begin{array}{ccccc} 1 & & n & -n & \qquad\qquad -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,n-1} \dots \lambda_m\bar{c}_{-n,n-1} & 0 & 1\,0\dots 0\,\bar{a}_{2(n-1)}\,\bar{a}_{1(n-1)}\end{pmatrix}\end{array}$$

$$- \bar{\lambda}_m \begin{array}{c} 1\\ \\ \\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix} a_{1(n-1)}\\ a_{2(n-1)}\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ c_{-n,n-1}\\ \vdots\\ c_{-1,n-1}\end{pmatrix} \begin{array}{c} \begin{array}{cccc} 1 & n & -n & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,1} \dots \lambda_m\bar{c}_{-n,1} & 0 & \dots 0\,\bar{a}_{21}\,\bar{a}_{11}\end{pmatrix}\end{array})$$

$\cdot T_{1,-(n-1)}(-1)$.

which equals

$$
\begin{array}{ccccccc}
1 & & & n & -n & & -1 \\
\end{array}
$$
$$
\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}
$$

$$
+ a_{11} \begin{array}{cccccccccc}
1 & & n & -n & & & & & & -1 \\
\end{array}
$$
$$
\begin{pmatrix} \lambda_m \bar{c}_{-1,n-1} & \dots & \lambda_m \bar{c}_{-n,n-1} & 0 & 1 & 0 & \dots & 0 & \bar{a}_{2(n-1)} & \bar{a}_{1(n-1)} \end{pmatrix}
$$

$$
- \bar{\lambda}_m a_{1(n-1)} \begin{array}{cccccc}
1 & & n & -n & & -1 \\
\end{array}
$$
$$
\begin{pmatrix} \lambda_m \bar{c}_{-1,1} & \dots & \lambda_m \bar{c}_{-n,1} & 0 & \dots 0 & \bar{a}_{21} & \bar{a}_{11} \end{pmatrix}
$$

$$
+ \begin{array}{ccccccccc}
1 & & n & -n & & & & & -1 \\
\end{array}
$$
$$
\begin{pmatrix} 0 & \dots & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 \end{pmatrix}
$$

where $x_1 = -(1 + a_{11}\lambda_m \bar{c}_{-1,n-1} - \bar{\lambda}_m a_{1(n-1)} \lambda_m \bar{c}_{-1,1})$ and $x_2 = \bar{\lambda}_m (a_{11} \lambda_m \bar{c}_{-(n-1),n-1} - \bar{\lambda}_m a_{1(n-1)} \lambda_m \bar{c}_{-(n-1),1})$. It is clearly not congruent to f_1 modulo I_m since $a_{11} \equiv 1 (\text{mod } I_m)$, $\sigma_{*(n-1)} \not\equiv e_{n-1} (\text{mod } I_m)$ and $a_{1(n-1)} \in I_m$. Hence $[\sigma, T_{1,-(n-1)}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_2 := T_{1,-(n-1)}(s_1) \in U_1$. By Lemma 7.6, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the n-th row of $[\sigma, g_1]$ equals f_n. Set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1. Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

<u>case 1.3</u> Assume $\sigma_{*j} \equiv e_j (\text{mod } I_m)$ $\forall j \in \{1, \dots, n-2\}$ and $a_{1(n-1)} \notin I_m$.
Consider the second row of

$$
\begin{aligned}
&[\sigma, T_{2,-(n-1)}(1)] \\
=&(e + \sigma_{*2} \sigma'_{-(n-1),*} - \bar{\lambda}_m \sigma_{*(n-1)} \sigma'_{-2,*}) T_{1,-(n-1)}(-1)
\end{aligned}
$$

$$
=(e + \begin{array}{c} 1 \\ \\ \\ \\ n \\ -n \\ \\ -1 \end{array}
\begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \\ c_{-n,2} \\ \vdots \\ c_{-1,2} \end{pmatrix}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & & n & -n & & & & & & -1 \end{array} \\
\begin{pmatrix} \lambda_m \bar{c}_{-1,n-1} \dots \lambda_m \bar{c}_{-n,n-1} & 0 & 1 0 \dots 0 \, \bar{a}_{2(n-1)} \, \bar{a}_{1(n-1)} \end{pmatrix}
\end{array}
$$

$$
-\bar{\lambda}_m \begin{array}{c} 1\\ \\ \\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix} a_{1(n-1)}\\ a_{2(n-1)}\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ c_{-n,n-1}\\ \vdots\\ c_{-1,n-1}\end{pmatrix}\begin{array}{c} \begin{array}{cccccccc} 1 & & n & -n & & & & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,2} & \dots & \lambda_m\bar{c}_{-n,2} & 0 & \dots & 0 & \bar{a}_{22} & \bar{a}_{12}\end{pmatrix}\end{array}))
$$

$$
\cdot T_{2,-(n-1)}(-1)
$$

which equals

$$
\begin{array}{c}\begin{array}{ccccccccc} 1 & & & & n & -n & & & -1\end{array}\\ \begin{pmatrix}0 & 1 & 0 & \dots & 0 & 0 & \dots & 0\end{pmatrix}\end{array}
$$

$$
+\,a_{22}\begin{array}{c}\begin{array}{cccccccccc} 1 & & n & -n & & & & & & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,n-1} & \dots & \lambda_m\bar{c}_{-n,n-1} & 0 & 1 & 0 & \dots & 0 & \bar{a}_{2(n-1)} & \bar{a}_{1(n-1)}\end{pmatrix}\end{array}
$$

$$
-\,\bar{\lambda}_m a_{2(n-1)}\begin{array}{c}\begin{array}{ccccccc} 1 & & n & -n & & & -1\end{array}\\ \begin{pmatrix}\lambda_m\bar{c}_{-1,2} & \dots & \lambda_m\bar{c}_{-n,2} & 0 & \dots 0 & \bar{a}_{22} & \bar{a}_{12}\end{pmatrix}\end{array}
$$

$$
+\begin{array}{c}\begin{array}{cccccccccc} 1 & & n & -n & & & & & -1 & \end{array}\\ \begin{pmatrix}0 & \dots & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 & 0\end{pmatrix}\end{array}
$$

where $x_1 = -(1+a_{22}\lambda_m\bar{c}_{-2,n-1}-\bar{\lambda}_m a_{2(n-1)}\lambda_m\bar{c}_{-2,2})$ and $x_2 = \bar{\lambda}_m(a_{22}\lambda_m\bar{c}_{-(n-1),n-1}-\bar{\lambda}_m a_{2(n-1)}\lambda_m\bar{c}_{-(n-1),2})$. Its last entry clearly does not lie in I_m. Hence $[\sigma, T_{2,-(n-1)}(1)] \notin U_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Set $g_1 := T_{2,-(n-1)}(s_1) \in U_1$. By Lemma 7.6, $[\sigma, g_1] \notin U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Clearly the n-th row of $[\sigma, g_1]$ equals f_n. Set $\epsilon_2 := P_{1n} \in EU_{2n}(R_m,\Lambda_m)$. Then the first row of $\tau :={}^{\epsilon_2}[\sigma, g_1]$ equals f_1. Since $U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. One can proceed now as in case 1.1.

<u>case 1.4</u> Assume $\sigma_{*j} \equiv e_j (\text{mod } I_m)$ $\forall j \in \{1,\dots,n-1\}$, $\sigma_{*n} \not\equiv e_n (\text{mod } I_m)$ and $a_{1n} \in I_m$.
Consider the first row of

$$
\begin{aligned}
&[\sigma, T_{1,-n}(1)]\\
=&(e+\sigma_{*1}\sigma'_{-n,*}-\bar{\lambda}_m\sigma_{*n}\sigma'_{-1,*})T_{1,-n}(-1)
\end{aligned}
$$

$$
=\Bigg(e+\begin{array}{r}1\\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix}a_{11}\\ a_{21}\\ 0\\ \vdots\\ 0\\ c_{-n,1}\\ \vdots\\ c_{-1,1}\end{pmatrix}
\begin{array}{c}
\begin{array}{ccccccccc}1 & & n & -n & & & & & -1\end{array}\\
\begin{pmatrix}\lambda_m\bar{c}_{-1,n} & \dots & \lambda_m\bar{c}_{-n,n} & 1 & 0 & \dots & 0 & \bar{a}_{2n} & \bar{a}_{1n}\end{pmatrix}
\end{array}
$$

$$
-\bar{\lambda}_m\begin{array}{r}1\\ \\ \\ \\ \\ n\\ -n\\ \\ -1\end{array}\begin{pmatrix}a_{1n}\\ a_{2n}\\ 0\\ \vdots\\ 0\\ 1\\ c_{-n,n}\\ \vdots\\ c_{-1,n}\end{pmatrix}
\begin{array}{c}
\begin{array}{ccccccccc}1 & & n & -n & & & & & -1\end{array}\\
\begin{pmatrix}\lambda_m\bar{c}_{-1,1} & \dots & \lambda_m\bar{c}_{-n,1} & 0 & \dots & 0 & \bar{a}_{21} & \bar{a}_{11}\end{pmatrix}
\end{array}\Bigg)
$$

$$
\cdot T_{1,-n}(-1)
$$

which equals

$$
\begin{array}{c}
\begin{array}{ccccccc}1 & & & n & -n & & -1\end{array}\\
\begin{pmatrix}1 & 0 & \dots & 0 & 0 & \dots & 0\end{pmatrix}
\end{array}
$$

$$
+\,a_{11}\begin{array}{c}
\begin{array}{ccccccccc}1 & & n & -n & & & & & -1\end{array}\\
\begin{pmatrix}\lambda_m\bar{c}_{-1,n} & \dots & \lambda_m\bar{c}_{-n,n} & 1 & 0 & \dots & 0 & \bar{a}_{2n} & \bar{a}_{1n}\end{pmatrix}
\end{array}
$$

$$
-\,\bar{\lambda}_m a_{1n}\begin{array}{c}
\begin{array}{cccccccc}1 & & n & -n & & & & -1\end{array}\\
\begin{pmatrix}\lambda_m\bar{c}_{-1,1} & \dots & \lambda_m\bar{c}_{-n,1} & 0 & \dots 0 & \bar{a}_{21} & \bar{a}_{11}\end{pmatrix}
\end{array}
$$

$$
+\begin{array}{c}
\begin{array}{cccccccc}1 & & n & -n & & & & -1\end{array}\\
\begin{pmatrix}0 & \dots & 0 & x_1 & 0 & \dots & 0 & x_2\end{pmatrix}
\end{array}
$$

where $x_1 = -(1 + a_{11}\lambda_m\bar{c}_{-1,n} - \bar{\lambda}_m a_{1n}\lambda_m\bar{c}_{-1,1})$ and $x_2 = \bar{\lambda}_m(a_{11}\lambda_m\bar{c}_{-n,n} - \bar{\lambda}_m a_{1n}\lambda_m\bar{c}_{-n,1})$. It is clearly not congruent to f_1 modulo I_m since $a_{11} \equiv 1(\text{mod } I_m)$, $\sigma_{*n} \not\equiv e_n(\text{mod } I_m)$ and $a_{1n} \in I_m$. Hence $[\sigma, T_{1,-n}(1)] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{1,-n}(s_1) \in U_1$. By Lemma 7.6, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the $(n-1)$-th row of $[\sigma, g_1]$ equals f_{n-1}. Set $\epsilon_2 := P_{1(n-1)} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1. Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

case 1.5 Assume $\sigma_{*j} \equiv e_j (\text{mod } I_m)\ \forall j \in \{1, \dots, n-1\}$ and $a_{1n} \notin I_m$. Consider the second row of

$$\begin{aligned}&[\sigma, T_{2,-n}(1)]\\ =&(e + \sigma_{*2}\sigma'_{-n,*} - \bar{\lambda}_m \sigma_{*n}\sigma'_{-2,*})T_{2,-n}(-1)\end{aligned}$$

$$=(e + \begin{array}{c} 1 \\ \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \\ c_{-n,2} \\ \vdots \\ c_{-1,2} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} 1 & & n & -n & & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{c}_{-1,n} \dots \lambda_m \bar{c}_{-n,n} & 1 & 0 \dots 0 & \bar{a}_{2n} & \bar{a}_{1n} \end{pmatrix} \end{array}$$

$$- \bar{\lambda}_m \begin{array}{c} 1 \\ \\ \\ \\ \\ n \\ -n \\ \\ -1 \end{array} \begin{pmatrix} a_{1n} \\ a_{2n} \\ 0 \\ \vdots \\ 0 \\ 1 \\ c_{-n,n} \\ \vdots \\ c_{-1,n} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} 1 & & n & -n & & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{c}_{-1,2} \dots \lambda_m \bar{c}_{-n,2} & 0 & \dots 0 & \bar{a}_{22} & \bar{a}_{12} \end{pmatrix} \end{array})$$

$$\cdot T_{2,-n}(-1)$$

which equals

$$\begin{array}{c} \begin{array}{cccccccc} 1 & & & & n & -n & & -1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

$$+ a_{22} \begin{array}{c} \begin{array}{ccccccccc} 1 & & n & -n & & & & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{c}_{-1,n} & \dots & \lambda_m \bar{c}_{-n,n} & 1 & 0 & \dots & 0 & \bar{a}_{2n} & \bar{a}_{1n} \end{pmatrix} \end{array}$$

$$- \bar{\lambda}_m a_{2n} \begin{array}{c} \begin{array}{cccccccc} 1 & & n & -n & & & & -1 \end{array} \\ \begin{pmatrix} \lambda_m \bar{c}_{-1,2} & \dots & \lambda_m \bar{c}_{-n,2} & 0 & \dots 0 & \bar{a}_{22} & \bar{a}_{12} \end{pmatrix} \end{array}$$

$$+ \begin{array}{c} \begin{array}{ccccccccc} 1 & & n & -n & & & & -1 & \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & x_1 & 0 & \dots & 0 & x_2 & 0 \end{pmatrix} \end{array}$$

where $x_1 = -(1 + a_{22}\lambda_m \bar{c}_{-2,n} - \bar{\lambda}_m a_{2n} \lambda_m \bar{c}_{-2,2})$ and $x_2 = \bar{\lambda}_m (a_{22}\lambda_m \bar{c}_{-n,n} - \bar{\lambda}_m a_{2n} \lambda_m \bar{c}_{-n,2})$. Its last entry does clearly not lie in I_m and hence $[\sigma, T_{2,-n}(1)] \notin U_{2n}((R_m,$

$\Lambda_m), (I_m, I_m \cap \Lambda_m))$. Set $g_1 := T_{2,-n}(s_1) \in U_1$. By Lemma 7.6, $[\sigma, g_1] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly the $(n-1)$-th row of $[\sigma, g_1]$ equals f_{n-1}. Set $\epsilon_2 := P_{1(n-1)} \in EU_{2n}(R_m, \Lambda_m)$. Then the first row of $\tau := {}^{\epsilon_2}[\sigma, g_1]$ equals f_1. Since $U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ is normal, $\tau \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. One can proceed now as in case 1.1.

<u>case 2</u> Assume that $\alpha, \delta \equiv e_{n\times n} (\text{mod } I_m)$ and $\gamma \equiv 0 (\text{mod } I_m)$.
Recall that $\sigma = {}^{\epsilon_1}[{}^{\epsilon_0}h,\, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ has the form

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e_{(n-2)\times(n-2)} & \beta_3 & \beta_4 \\ \hline \multicolumn{2}{c|}{\gamma} & \multicolumn{2}{c}{\delta} \end{array}\right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha_1, \beta_2 \in M_2(R_m)$, $\alpha_2, \beta_1 \in M_{2\times(n-2)}(R_m)$, $\beta_3 \in M_{n-2}(R_m)$, $\beta_4 \in M_{(n-2)\times 2}(R_m)$ and $\alpha, \beta, \gamma, \delta \in M_n(R_m)$. Clearly $\beta \not\equiv 0(\text{mod } I_m)$ since $\sigma \notin U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$.

<u>case 2.1</u> Assume that $\beta_3 \not\equiv 0(\text{mod } I_m)$ or $\beta_4 \not\equiv 0(\text{mod } I_m)$.
Set $g_1 := T_{-n,n-1}\ (s_1) \in U_1$ and $\omega := [\sigma^{-1}, g_1]$. Then

$$\begin{aligned}
&\omega\\
=&[\sigma^{-1}, g_1]\\
=&(e + \sigma'_{*,-n} s_1 \sigma_{(n-1)*} - \sigma'_{*,-(n-1)} \lambda_m s_1 \sigma_{n*})(g_1)^{-1}
\end{aligned}$$

$$=(e + s_1 \begin{array}{c} 1 \\ \\ n \\ -n \\ \\ \\ -1 \end{array} \begin{pmatrix} \bar\lambda \bar B_{n,-1} \\ \vdots \\ \bar\lambda \bar B_{n,-n} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \begin{array}{cccccccc} 1 & & & n & -n & & & -1 \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & B_{n-1,-n} & \dots & B_{n-1,-1} \end{pmatrix} \end{array}$$

$$-\lambda_m s_1 \begin{array}{c} 1 \\ \\ n \\ -n \\ \\ \\ \\ -1 \end{array} \begin{pmatrix} \bar\lambda \bar B_{n-1,-1} \\ \vdots \\ \bar\lambda \bar B_{n-1,-n} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} 1 & & & n & -n & & -1 \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & 1 & B_{n,-n} & \dots & B_{n,-1} \end{pmatrix} \end{array})$$

$$\cdot (g_1)^{-1}.$$

Since $\beta_3 \not\equiv 0(\text{mod } I_m)$ or $\beta_4 \not\equiv 0(\text{mod } I_m)$, $(\omega_{-n,-n},\dots,\omega_{-n,-1}) \not\equiv (1,0,\dots,0)(\text{mod } I_m)$ or $(\omega_{-(n-1),-n},\dots,\omega_{-(n-1),-1}) \not\equiv (0,1,0,\dots,0)(\text{mod } I_m)$. Hence $\omega \notin U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Further the next to last row of ω equals f_{-2}. Set $\epsilon_2 := P_{1,-2} \in EU_{2n}(R_m,\Lambda_m)$. Then the first row of ${}^{\epsilon_2}\omega$ equals f_1. Since $U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$ is normal, ${}^{\epsilon_2}\omega \notin U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. One can proceed now as in case 1.1 (${}^{\epsilon_2}\omega$ has the same properties as τ in case 1.1).

<u>case 2.2</u> Assume that $\beta_3 \equiv 0(\text{mod } I_m)$ and $\beta_4 \equiv 0(\text{mod } I_m)$.
It follows that $\beta_1 \equiv 0(\text{mod } I_m)$. Since $\beta \not\equiv 0(\text{mod } I_m)$, $\beta_2 \not\equiv 0(\text{mod } I_m)$. Set

$$\xi := \left(\begin{array}{c|cc} & \bar{\lambda}p(\beta_4)^*p & 0 \\ e_{n\times n} & & \\ & 0 & -\beta_4 \\ \hline 0 & & e_{n\times n} \end{array}\right) \in EU_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m)).$$

Then $\omega := \sigma\xi$ has the form

$$\left(\begin{array}{cc|cc} \alpha'_1 & \alpha'_2 & \beta'_1 & \beta'_2 \\ 0 & e_{(n-2)\times(n-2)} & \beta'_3 & 0 \\ \hline & \gamma' & & \delta' \end{array}\right) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

where $\alpha'_1,\beta'_2 \in M_2(R_m)$, $\alpha'_2,\beta'_1 \in M_{2\times(n-2)}(R_m)$, $\beta'_3 \in M_{n-2}(R_m)$ and $\alpha',\beta',\gamma',\delta' \in M_n(R_m)$. Further $\alpha',\delta' \equiv e_{n\times n}(\text{mod } I_m)$, $\gamma' \equiv 0(\text{mod } I_m)$, $\beta'_1 \equiv 0(\text{mod } I_m)$, $\beta'_3 \equiv 0(\text{mod } I_m)$ and $\beta'_2 \not\equiv 0(\text{mod } I_m)$. Since $\beta'_2 \not\equiv 0(\text{mod } I_m)$, there are an $i \in \{1,2\}$ and a $j \in \{-2,-1\}$ such that $\beta'_{ij} \notin I_m$. Let $l = 1$ if $j = -2$ and $l = 2$ if $j = -1$. Set $\epsilon_{11} := T_{jl}(-1) \in EU_{2n}(R_m,\Lambda_m)$. Then ${}^{\epsilon_{11}}\omega$ has the form

$$\left(\begin{array}{cc|cc} \alpha''_1 & \alpha''_2 & \beta''_1 & \beta''_2 \\ 0 & e_{(n-2)\times(n-2)} & \beta''_3 & \beta''_4 \\ \hline & \gamma'' & & \delta'' \end{array}\right) = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

where $\alpha''_1,\beta''_2 \in M_2(R_m)$, $\alpha''_2,\beta''_1 \in M_{2\times(n-2)}(R_m)$, $\beta''_3 \in M_{n-2}(R_m)$, $\beta''_4 \in M_{(n-2)\times 2}(R_m)$ and $\alpha'',\beta'',\gamma'',\delta'' \in M_n(R_m)$. Further $\alpha''_{il} \notin I_m$ if $i \neq l$ and $\alpha''_{il} \not\equiv 1(\text{mod } I_m)$ if $i = l$. Hence $\alpha'' \not\equiv e_{n\times n}(\text{mod } I_m)$ and thus one can proceed as in case 1.1.

The following inverted tree diagram extending over several pages gives an overview of the case by case proof just concluded of Part I. How to read a diagram is explained at the conclusion of the proof of the Solution Group Lemma 5.7.

h

$$\begin{pmatrix} & * & \end{pmatrix}$$
$\notin CU_{2n}((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$

$\epsilon_0 \in EU_{2n}(R_m,\Lambda_m)$,
$g_0 = T_{1,-2}(s_0\hat{a})$,
$\epsilon_1 \in EU_{2n}(R_m,\Lambda_m)$

${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$\alpha,\delta \equiv e_{n\times n}(\text{mod } I_m)$
$\wedge\gamma \equiv 0(\text{mod } I_m)$
$\wedge\beta \not\equiv 0(\text{mod } I_m)$

2

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e_{(n-2)\times(n-2)} & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$\not\equiv e(\text{mod } I_m)$

1

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$\alpha \not\equiv e_{n\times n}(\text{mod } I_m)$
$\vee\gamma \not\equiv 0(\text{mod } I_m)$
$\vee\delta \not\equiv e_{n\times n}(\text{mod } I_m)$

1.3

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$(\alpha_{*(n-1)} \not\equiv e_{n-1}(\text{mod } I_m)$
$\vee\gamma_{*(n-1)} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{22}-1, a_{12} \in I_m$
$\wedge a_{1(n-1)} \notin I_m$

1.4

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$(\alpha_{*n} \not\equiv e_n(\text{mod } I_m)$
$\vee\gamma_{*n} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{11}-1, a_{1n} \in I_m$

1.2

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$(\alpha_{*(n-1)} \not\equiv e_{n-1}(\text{mod } I_m)$
$\vee\gamma_{*(n-1)} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{11}-1, a_{1(n-1)} \in I_m$

1.1

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$\exists i \in \{1,\dots,n-2\}:$
$(\alpha_{*i} \not\equiv e_i(\text{mod } I_m)$
$\vee\gamma_{*i} \not\equiv 0(\text{mod } I_m))$

1.5

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \multirow{2}{*}{β} \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$
$(\alpha_{*n} \not\equiv e_n(\text{mod } I_m)$
$\vee\gamma_{*n} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{22}-1, a_{12} \in I_m$
$\wedge a_{1n} \notin I_m$

$g_1 = T_{i,-(n-1)}(s_1)$,
$\epsilon_2 = P_{1n}$

${}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A_3 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & 1 \end{array}\right)$$
$\not\equiv e(\text{mod } I_m)$,
$A_4, D_1 \equiv e(\text{mod } I_m)$,
$B_3, C_2 \equiv 0(\text{mod } I_m)$,
$A_3 \equiv 0(\text{mod } I_m)$

1.1.2

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$\not\equiv e(\text{mod } I_m)$

1.1.3

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A_3 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & 1 \end{array}\right)$$
$\not\equiv e(\text{mod } I_m)$,
$A_4, D_1 \equiv e(\text{mod } I_m)$,
$B_3, C_2 \equiv 0(\text{mod } I_m)$,
$A_3 \not\equiv 0(\text{mod } I_m)$

1.1.1

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$E \not\equiv e(\text{mod } I_m)$

$$\downarrow g_2 = T_{-1,i}(s_2)$$

$[({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1])^{-1}, g_2]$

$$\boxed{\begin{array}{c}\begin{pmatrix} 1 & 0 & 0 \\ * & e & 0 \\ * & w & 1 \end{pmatrix} \\ w \not\equiv 0(\text{mod } I_m)\end{array}}$$

$$\downarrow g_3 = T_{jl}(s_3)$$

$[[({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1])^{-1}, g_2], g_3]$

$$\boxed{\begin{array}{c}\begin{pmatrix} 1 & 0 & 0 \\ * & e & 0 \\ * & w' & 1 \end{pmatrix} \\ w' \not\equiv 0(\text{mod } I_m), \\ \text{at most two entries} \\ \text{of } w' \text{ are nonzero}\end{array}}$$

$$\downarrow g_4 = T_{l,-j}(s_4 s_5 t)$$

$[[[({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1])^{-1}, g_2], g_3], g_4]$

$$\boxed{T_{-1,-j}(s_3 s_4 s_5 b)}$$

$$\downarrow 1.1.2$$

${}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]$

$$\boxed{\begin{array}{l}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A_3 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & 1 \end{array}\right) \\ \not\equiv e(\text{mod } I_m), \\ A_4, D_1 \equiv e(\text{mod } I_m), \\ B_3, C_2 \equiv 0(\text{mod } I_m), \\ A_3 \equiv 0(\text{mod } I_m)\end{array}}$$

$$\downarrow \xi_1 = \prod_{l=2}^{n} T_{l1}(-A_{l1})$$

$\xi_1({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1])$

$$\begin{array}{l}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3' & C_4' & D_3' & 1 \end{array}\right) \\ \not\equiv e(\text{mod } I_m), \\ A_4, D_1 \equiv e(\text{mod } I_m), \\ B_3, C_2 \equiv 0(\text{mod } I_m), \\ D_3' \equiv 0(\text{mod } I_m)\end{array}$$

1.1.2.1

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array}\right)$$

$A_4, D_1 \equiv e(\mathrm{mod}\ I_m)$,
$B_3, C_2 \equiv 0(\mathrm{mod}\ I_m)$,
$\exists i \in \{3, \dots, n\} : C'_{-1,i} \notin I_m$

$$\downarrow \epsilon_{21} = T_{12}(-1)$$

${}^{\epsilon_{21}}(\xi_1({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]))$

$$\left(\begin{array}{cc|c} 1 & A''_2 & \\ 0 & A_4 & B'' \\ \hline \multicolumn{2}{c|}{C''} & D'' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$A''_2 \equiv 0(\mathrm{mod}\ I_m)$,
$C''_{-2,i} \notin I_m$

$$\downarrow \xi_2 = \prod_{l=2}^{n} T_{1l}(-A''_{1l})$$

${}^{\epsilon_{21}}(\xi_1({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]))\xi_2$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & A_4 & B''' \\ \hline \multicolumn{2}{c|}{C'''} & D''' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$C'''_{-2,i} \notin I_m$

$$\downarrow g_2 = T_{2,-i}(s_2)$$

$[{}^{\epsilon_{21}}(\xi_1({}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]))\xi_2, g_2]$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$

$E_{22} \not\equiv 1(\mathrm{mod}\ I_m)$

$\downarrow$

see case 1.1.1

1.1.2.2

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array}\right)$$

$A_4, D_1 \equiv e(\mathrm{mod}\ I_m)$,
$B_3, C_2 \equiv 0(\mathrm{mod}\ I_m)$,
$C'_{-1,2} \notin I_m$

$$\downarrow \epsilon_{21} = T_{13}(-1)$$

$$\left(\begin{array}{cc|c} 1 & A''_2 & \\ 0 & A_4 & B'' \\ \hline \multicolumn{2}{c|}{C''} & D'' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$A''_2 \equiv 0(\mathrm{mod}\ I_m)$,
$C''_{-3,2} \notin I_m$

$$\downarrow \xi_2 = \prod_{l=2}^{n} T_{1l}(-A''_{1l})$$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & A_4 & B''' \\ \hline \multicolumn{2}{c|}{C'''} & D''' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$C'''_{-3,2} \notin I_m$

$$\downarrow g_2 = T_{3,-2}(s_2)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$

$E_{33} \not\equiv 1(\mathrm{mod}\ I_m)$

$\downarrow$

see case 1.1.1

1.1.2.3

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array}\right)$$

$A_4, D_1 \equiv e(\mathrm{mod}\ I_m)$,
$B_3, C_2 \equiv 0(\mathrm{mod}\ I_m)$,
$C'_{1,i} \in I_m\ \forall i \in \{2, \dots, n\}$,
$C'_{-1,1} \notin I_m(\mathrm{mod}\ I_m)$

$$\downarrow \epsilon_{21} = T_{12}(-1)$$

$$\left(\begin{array}{cc|c} 1 & A''_2 & \\ 0 & A_4 & B'' \\ \hline \multicolumn{2}{c|}{C''} & D'' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$A''_2 \equiv 0(\mathrm{mod}\ I_m)$,
$C''_{-2,2} \notin I_m$

$$\downarrow \xi_2 = \prod_{l=2}^{n} T_{1l}(-A''_{1l})$$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & A_4 & B''' \\ \hline \multicolumn{2}{c|}{C'''} & D''' \end{array}\right)$$

$A_4 \equiv e(\mathrm{mod}\ I_m)$,
$C'''_{-2,2} \notin I_m$

$$\downarrow g_2 = T_{2,-3}(s_2)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$

$E_{32} \notin I_m$

$\downarrow$

see case 1.1.1

1.1.3

$${}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A_3 & A_4 & B_3 & 0 \\ \hline C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & 1 \end{array}\right)$$

$\not\equiv e(\text{mod } I_m)$,
$A_4, D_1 \equiv e(\text{mod } I_m)$,
$B_3, C_2 \equiv 0(\text{mod } I_m)$,
$A_3 \not\equiv 0(\text{mod } I_m)$

$$\xrightarrow{g_2 = T_{ij}(s_2)}$$

$$[{}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1], g_2]$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ A'_3 & A'_4 & B'_3 & 0 \\ \hline C'_1 & C'_2 & D'_1 & 0 \\ C'_3 & C'_4 & D'_3 & 1 \end{array}\right)$$

$\not\equiv e(\text{mod } I_m)$,
$A'_3 \equiv 0(\text{mod } I_m)$

see case 1.1.1 or 1.1.2

$${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$$

1.2

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \beta \\ 0 & e & \\ \hline \gamma & & \delta \end{array}\right)$$

$(\alpha_{*(n-1)} \not\equiv e_{n-1}(\text{mod } I_m)$
$\vee \gamma_{*(n-1)} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{11} - 1, a_{1(n-1)} \in I_m$

$g_1 = T_{1,-(n-1)}(s_1)$,
$\epsilon_2 = P_{1n}$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$\not\equiv e(\text{mod } I_m)$

see case 1.1

1.3

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \beta \\ 0 & e & \\ \hline \gamma & & \delta \end{array}\right)$$

$(\alpha_{*(n-1)} \not\equiv e_{n-1}(\text{mod } I_m)$
$\vee \gamma_{*(n-1)} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{22} - 1, a_{12} \in I_m$
$\wedge a_{1(n-1)} \notin I_m$

$g_1 = T_{2,-(n-1)}(s_1)$,
$\epsilon_2 = P_{1n}$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$\not\equiv e(\text{mod } I_m)$

see case 1.1

1.4

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \beta \\ 0 & e & \\ \hline \gamma & & \delta \end{array}\right)$$

$(\alpha_{*n} \not\equiv e_n(\text{mod } I_m)$
$\vee \gamma_{*n} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{11} - 1, a_{1n} \in I_m$

$g_1 = T_{1,-n}(s_1)$,
$\epsilon_2 = P_{1(n-1)}$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$\not\equiv e(\text{mod } I_m)$

see case 1.1

1.5

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \beta \\ 0 & e & \\ \hline \gamma & & \delta \end{array}\right)$$

$(\alpha_{*n} \not\equiv e_n(\text{mod } I_m)$
$\vee \gamma_{*n} \not\equiv 0(\text{mod } I_m))$
$\wedge a_{22} - 1, a_{12} \in I_m$
$\wedge a_{1n} \notin I_m$

$g_1 = T_{2,-n}(s_1)$,
$\epsilon_2 = P_{1(n-1)}$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$
$\not\equiv e(\text{mod } I_m)$

see case 1.1

$${}^{\epsilon_2}[{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0], g_1]$$

2

$^{\epsilon_1}[^{\epsilon_0}h, g_0]$

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e & \beta_3 & \beta_4 \\ \hline \multicolumn{2}{c|}{\gamma} & \multicolumn{2}{c}{\delta} \end{array}\right)$$

$\alpha, \delta \equiv e_{n\times n}(\text{mod } I_m)$
$\wedge \gamma, \beta_3, \beta_4 \equiv 0(\text{mod } I_m)$
$\wedge(\beta_1 \not\equiv 0(\text{mod } I_m)$
$\vee \beta_2 \not\equiv 0(\text{mod } I_m))$

2.2

$$\left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & \beta \\ 0 & e & \\ \hline \multicolumn{2}{c|}{\gamma} & \delta \end{array}\right)$$

$\alpha, \delta \equiv e_{n\times n}(\text{mod } I_m)$
$\wedge \gamma \equiv 0(\text{mod } I_m)$
$\wedge \beta \not\equiv 0(\text{mod } I_m)$

2.1

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e & \beta_3 & \beta_4 \\ \hline \multicolumn{2}{c|}{\gamma} & \multicolumn{2}{c}{\delta} \end{array}\right)$$

$\alpha, \delta \equiv e_{n\times n}(\text{mod } I_m)$
$\wedge \gamma \equiv 0(\text{mod } I_m)$
$\wedge(\beta_3 \not\equiv 0(\text{mod } I_m)$
$\vee \beta_4 \not\equiv 0(\text{mod } I_m))$

$g_1 = T_{-n,n-1}(s_1)$,
$\epsilon_2 = P_{1,-2}$

$^{\epsilon_2}[(^{\epsilon_1}[^{\epsilon_0}h, g_0])^{-1}, g_1]$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & * & 1 \end{pmatrix}$$

$\not\equiv e(\text{mod } I_m)$

see case 1.1

2.2

$^{\epsilon_1}[^{\epsilon_0}h, g_0]$

$$\left(\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & e & \beta_3 & \beta_4 \\ \hline \multicolumn{2}{c|}{\gamma} & \multicolumn{2}{c}{\delta} \end{array}\right)$$

$\alpha, \delta \equiv e_{n\times n}(\text{mod } I_m)$
$\wedge \gamma, \beta_3, \beta_4 \equiv 0(\text{mod } I_m)$
$\wedge(\beta_1 \not\equiv 0(\text{mod } I_m)$
$\vee \beta_2 \not\equiv 0(\text{mod } I_m))$

$$\xi = \left(\begin{array}{c|cc} e & \bar{\lambda}p(\beta_4)^*p & 0 \\ & 0 & -\beta_4 \\ \hline 0 & \multicolumn{2}{c}{e} \end{array}\right)$$

$\boxed{{}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]\xi}$

$$\boxed{\begin{array}{c} \left(\begin{array}{cc|cc} \alpha_1' & \alpha_2' & \beta_1' & \beta_2' \\ 0 & e & \beta_3' & 0 \\ \hline \multicolumn{2}{c|}{\gamma'} & \multicolumn{2}{c}{\delta'} \end{array}\right) \\ \alpha', \delta' \equiv e_{n\times n}(\text{mod } I_m) \\ \wedge \gamma', \beta_1', \beta_3' \equiv 0(\text{mod } I_m) \\ \beta_2' \not\equiv 0(\text{mod } I_m) \end{array}}$$

$$\Big\downarrow \ \epsilon_{11} = T_{jl}(-1)$$

$\boxed{{}^{\epsilon_{11}}({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]\xi)}$

$$\boxed{\begin{array}{c} \left(\begin{array}{cc|cc} \alpha_1'' & \alpha_2'' & \beta_1'' & \beta_2'' \\ 0 & e & \beta_3'' & \beta_4'' \\ \hline \multicolumn{2}{c|}{\gamma''} & \multicolumn{2}{c}{\delta''} \end{array}\right) \\ \alpha_1'' \not\equiv e_{2\times 2}(\text{mod } I_m) \end{array}}$$

$$\Big\downarrow$$

see case 1.1

Part II Assume that $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and $n = 3$.
There are a $g_0 \in U_0$ and a $\epsilon_0, \epsilon_1 \in EU_6(R_m, \Lambda_m)$ such that $\sigma := {}^{\epsilon_1}[{}^{\epsilon_0}h, g_0] \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ and σ has the form

$$\left(\begin{array}{ccc|c} & * & & \beta \\ 0 & 0 & 1 & \\ \hline & \gamma & & \delta \end{array}\right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha = (\alpha_{ij})_{1 \leqslant i,j \leqslant 3}, \beta = (\beta_{ij})_{\substack{1 \leqslant i \leqslant 3 \\ -3 \leqslant j \leqslant -1}}, \gamma = (\gamma_{ij})_{\substack{-3 \leqslant i \leqslant -1 \\ 1 \leqslant j \leqslant 3}}, \delta = (\delta_{ij})_{-3 \leqslant i,j \leqslant -1} \in M_3(R_m)$ (see Part I above and [1], chapter IV, Lemma 3.12, Part II, general case).

case 1 Assume that there is an $i \in \{-3, -2, -1\}$ such that $\gamma_{i2} \notin I_m$.
Set $g_1 := T_{1,-2}(s_1) \in U_1$ and $\omega := [\sigma, g_1]$. Then

$$\begin{aligned}
&\omega \\
=&[\sigma, g_1] \\
=&(e + \sigma_{*1}s_1\sigma'_{-2,*} - \sigma_{*2}\bar{\lambda}_m s_1 \sigma'_{-1,*})(g_1)^{-1} \\
=&(e + s_1 \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ 0 \\ \gamma_{-3,1} \\ \gamma_{-2,1} \\ \gamma_{-1,1} \end{pmatrix} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,2} & \lambda_m\bar{\gamma}_{-2,2} & \lambda_m\bar{\gamma}_{-3,2} & 0 & \bar{\alpha}_{22} & \bar{\alpha}_{12} \end{pmatrix} \\
&- \bar{\lambda}_m s_1 \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ 0 \\ \gamma_{-3,2} \\ \gamma_{-2,2} \\ \gamma_{-1,2} \end{pmatrix} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,1} & \lambda_m\bar{\gamma}_{-2,1} & \lambda_m\bar{\gamma}_{-3,1} & 0 & \bar{\alpha}_{21} & \bar{\alpha}_{11} \end{pmatrix}) \\
&\cdot (g_1)^{-1}.
\end{aligned}$$

Assume that

$$\begin{aligned}
&s_1\sigma_{*1} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,2} & \lambda_m\bar{\gamma}_{-2,2} & \lambda_m\bar{\gamma}_{-3,2} \end{pmatrix} \\
&- \bar{\lambda}_m s_1 \sigma_{*2} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,1} & \lambda_m\bar{\gamma}_{-2,1} & \lambda_m\bar{\gamma}_{-3,1} \end{pmatrix} \equiv 0 (\text{mod } I_m).
\end{aligned}$$

By multiplying σ'_{1*} from the left we get that $s_1 \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,2} & \lambda_m\bar{\gamma}_{-2,2} & \lambda_m\bar{\gamma}_{-3,2} \end{pmatrix} \equiv 0(\text{mod } I_m)$ which implies $\begin{pmatrix} \gamma_{-1,2} & \gamma_{-2,2} & \gamma_{-3,2} \end{pmatrix} \equiv 0(\text{mod } I_m)$. Since that is a contradiction,

$$\begin{aligned}
&s_1\sigma_{*1} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,2} & \lambda_m\bar{\gamma}_{-2,2} & \lambda_m\bar{\gamma}_{-3,2} \end{pmatrix} \\
&- \bar{\lambda}_m s_1 \sigma_{*2} \begin{pmatrix} \lambda_m\bar{\gamma}_{-1,1} & \lambda_m\bar{\gamma}_{-2,1} & \lambda_m\bar{\gamma}_{-3,1} \end{pmatrix} \not\equiv 0 (\text{mod } I_m)
\end{aligned}$$

and hence $\omega \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Further the third row of ω equals f_3. Set $\epsilon_2 := P_{13} \in EU_6(R_m, \Lambda_m)$. Then the first row of ${}^{\epsilon_2}\omega$ equals f_1. Since

$U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$ is normal, ${}^\theta\omega\notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. One can proceed now as in Part I, case 1 (${}^\theta\omega$ has the same properties as τ in Part I, case 1).

<u>case 2</u> Assume that there is an $i\in\{-3,-2,-1\}$ such that $\gamma_{i1}\notin I_m$.
This case can be treated similarly.

<u>case 3</u> Assume that $\gamma_{-3,1},\gamma_{-3,2},\gamma_{-2,1},\gamma_{-2,2},\gamma_{-1,1},\gamma_{-1,2}\in I_m$ and one of the entries $\beta_{3,-3}$ and $\beta_{3,-2}$ does not lie in I_m.
By [1], chapter IV, Lemma 3.12, Part II, general case there are $x_1,x_2\in I_m$ such that $\gamma_{-1,1}+x_2(x_1\gamma_{-1,1}+\gamma_{-2,1})\in rad(R_m)\cap I_m$ where $rad(R_m)$ is the Jacobson radical of the ring R_m. Set $\xi_1:=T_{-1,-2}(x_2)T_{-2,-1}(x_1)\in EU_6((R_m,\Lambda_m),(I_m,\Gamma_m))$. Then $\rho:=\xi_1\sigma$ has the form

$$\left(\begin{array}{ccc|c} & * & & \beta' \\ 0 & 0 & 1 & \\ \hline & \gamma' & & \delta' \end{array}\right)=\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$$

where $\alpha'=(\alpha'_{ij})_{1\leqslant i,j\leqslant 3},\beta'=(\beta'_{ij})_{\substack{1\leqslant i\leqslant 3\\ -3\leqslant j\leqslant -1}},\gamma'=(\gamma'_{ij})_{\substack{-3\leqslant i\leqslant -1\\ 1\leqslant j\leqslant 3}},\delta'=(\delta'_{ij})_{-3\leqslant i,j\leqslant -1}\in M_3(R_m)$. Further $(\beta'_{3,-3}\notin I_m\vee\beta'_{3,-2}\notin I_m)\wedge\gamma'_{-1,1}\in rad(R_m)\cap I_m$. Set $g_1:=T_{13}(s_1)\in U_1$ and $\omega:=[\rho^{-1},g_1]$. Then

$$\begin{aligned}
&\omega\\
=&[\rho^{-1},g_1]\\
=&(e+\rho'_{*1}s_1\rho_{3*}-\rho'_{*,-3}s_1\rho_{-1,*})(g_1)^{-1}\\
=&(e+s_1\begin{pmatrix}\bar\delta'_{-1,-1}\\ \bar\delta'_{-1,-2}\\ \bar\delta'_{-1,-3}\\ \lambda\bar\gamma'_{-1,3}\\ \lambda\bar\gamma'_{-1,2}\\ \lambda\bar\gamma'_{-1,1}\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & \beta'_{3,-3} & \beta'_{3,-2} & \beta'_{3,-1}\end{pmatrix}\\
&-s_1\begin{pmatrix}\bar\lambda\bar\beta_{3,-1}\\ \bar\lambda\bar\beta_{3,-2}\\ \bar\lambda\bar\beta_{3,-3}\\ 1\\ 0\\ 0\end{pmatrix}\begin{pmatrix}\gamma'_{-1,1} & \gamma'_{-1,2} & \gamma'_{-1,3} & \delta'_{-1,-3} & \delta'_{-1,-2} & \delta'_{-1,-1}\end{pmatrix})\\
&\cdot(g_1)^{-1}.
\end{aligned}$$

Assume that

$$\begin{aligned}&s_1\rho'_{*1}\begin{pmatrix}\beta'_{3,-3} & \beta'_{3,-2}\end{pmatrix}\\ &-s_1\rho'_{*,-3}\begin{pmatrix}\delta'_{-1,-3} & \delta'_{-1,-2}\end{pmatrix}\equiv 0(\mathrm{mod}\ I_m).\end{aligned}$$

By multiplying ρ_{1*} from the left we get that $s_1\begin{pmatrix}\beta'_{3,-3} & \beta'_{3,-2}\end{pmatrix}\equiv 0(\mathrm{mod}\ I_m)$ which implies $\begin{pmatrix}\beta'_{3,-3} & \beta'_{3,-2}\end{pmatrix}\equiv 0(\mathrm{mod}\ I_m)$. Since that is a contradiction,

$$\begin{aligned}&s_1\rho'_{*1}\begin{pmatrix}\beta'_{3,-3} & \beta'_{3,-2}\end{pmatrix}\\ &-s_1\rho'_{*,-3}\begin{pmatrix}\delta'_{-1,-3} & \delta'_{-1,-2}\end{pmatrix}\not\equiv 0(\mathrm{mod}\ I_m)\end{aligned}$$

and hence $\omega \notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Obviously $\omega_{-1,*}\equiv f_{-1}(\text{mod } I_m)$ and $\omega_{-1,-1}\equiv 1(\text{mod } rad(R_m)\cap I_m)$. Set $\epsilon_2 := P_{3,-1}\in EU_6(R_m,\Lambda_m)$ and $\zeta := {}^{\epsilon_2}\omega$. Then $\zeta \notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Further $\zeta_{3*}\equiv f_3(\text{mod } I_m)$ and $\zeta_{33}\equiv 1(\text{mod } rad(R_m)\cap I_m)$. By Nakayama's lemma ζ_{33} is invertible. Set $\xi_2 := T_{32}(-(\zeta_{33})^{-1}\zeta_{32})T_{31}(-(\zeta_{33})^{-1}\zeta_{31})T_{3,-1}(-(\zeta_{33})^{-1}\zeta_{3,-1})T_{3,-2}(-(\zeta_{33})^{-1}\zeta_{3,-2})\in EU_6((R_m,\Lambda_m),(I_m,\Gamma_m))$ and $\eta := \zeta\xi_2$. Then η has the form

$$\left(\begin{array}{ccc|ccc} & * & & & * & \\ 0 & 0 & \alpha''_{33} & \beta''_{3,-3} & 0 & 0 \\ \hline & \gamma'' & & & \delta'' & \end{array}\right)=\begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix}$$

where $\alpha''=(\alpha''_{ij})_{1\leqslant i,j\leqslant 3},\beta''=(\beta''_{ij})_{\substack{1\leqslant i\leqslant 3\\ -3\leqslant j\leqslant -1}},\gamma''=(\gamma''_{ij})_{\substack{-3\leqslant i\leqslant -1\\ 1\leqslant j\leqslant 3}},\delta''=(\delta''_{ij})_{-3\leqslant i,j\leqslant -1}\in M_3(R_m)$. Since $\zeta\notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$, $\eta\notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Further $\alpha''_{33}\equiv 1(\text{mod } rad(R_m)\cap I_m)$ and $\beta''_{3,-3}\in I_m$. Since $\eta_{3*}\equiv f_3(\text{mod } I_m)$, $\eta_{*,-3}\equiv e_{-3}(\text{mod } I_m)$ (apply Lemma 6.8 to the image of η in $U_{2n}(R_m/I_m,\Lambda_m/(\Lambda_m\cap I_m))$). Hence $\beta''_{1,-3},\beta''_{2,-3},\delta''_{-3,-3}-1,\delta''_{-2,-3},\delta''_{-1,-3}\in I_m$.

<u>case 3.1</u> Assume that there are an $i\in\{-3,-2,-1\}$ and a $j\in\{1,2\}$ such that $\gamma''_{ij}\notin I_m$.
See case 1.

<u>case 3.2</u> Assume that $\gamma''_{-3,1},\gamma''_{-3,2},\gamma''_{-2,1},\gamma''_{-2,2},\gamma''_{-1,1},\gamma''_{-1,2}\in I_m$ and one of the entries $\beta''_{1,-2}$, $\beta''_{1,-1}$, $\beta''_{2,-2}$, $\beta''_{2,-1}$, $\delta''_{-3,-2}$ and $\delta''_{-3,-1}$ does not lie in I_m.
Set $g_2 := T_{-1,2}(s_2)\in U_2$ and $\theta := [\eta,g_2]$. Then

$$\begin{aligned}
&\theta\\
=&[\eta,g_2]\\
=&(e+\eta_{*,-1}s_2\eta'_{2*}-\eta_{*,-2}\lambda_m s_2\eta'_{1*})(g_2)^{-1}\\
=&(e+s_2\begin{pmatrix}\beta''_{1,-1}\\ \beta''_{2,-1}\\ 0\\ \delta''_{-3,-1}\\ \delta''_{-2,-1}\\ \delta''_{-1,-1}\end{pmatrix}\begin{pmatrix}\bar\delta''_{-1,-2} & \bar\delta''_{-2,-2} & \bar\delta''_{-3,-2} & 0 & \bar\lambda_m\bar\beta''_{2,-2} & \bar\lambda_m\bar\beta''_{1,-2}\end{pmatrix}\\
&-\lambda_m s_2\begin{pmatrix}\beta''_{1,-2}\\ \beta''_{2,-2}\\ 0\\ \delta''_{-3,-2}\\ \delta''_{-2,-2}\\ \delta''_{-1,-2}\end{pmatrix}\begin{pmatrix}\bar\delta''_{-1,-1} & \bar\delta''_{-2,-1} & \bar\delta''_{-3,-1} & 0 & \bar\lambda_m\bar\beta''_{2,-1} & \bar\lambda_m\bar\beta''_{1,-1}\end{pmatrix})\\
&\cdot(g_2)^{-1}.
\end{aligned}$$

Assume that

$$\begin{aligned}
&s_2\eta_{*,-1}\begin{pmatrix}\bar\delta''_{-3,-2} & 0 & \bar\lambda_m\bar\beta''_{2,-2} & \bar\lambda_m\bar\beta''_{1,-2}\end{pmatrix}\\
&-\lambda_m s_2\eta_{*,-2}\begin{pmatrix}\bar\delta''_{-3,-1} & 0 & \bar\lambda_m\bar\beta''_{2,-1} & \bar\lambda_m\bar\beta''_{1,-1}\end{pmatrix}\equiv 0(\text{mod } I_m).
\end{aligned}$$

It follows that $\begin{pmatrix}\delta''_{-3,-2} & 0 & \beta''_{2,-2} & \beta''_{1,-2}\end{pmatrix}, \begin{pmatrix}\delta''_{-3,-1} & 0 & \beta''_{2,-1} & \beta''_{1,-1}\end{pmatrix} \equiv 0(\text{mod } I_m)$. Since that is a contradiction,

$$\begin{aligned} &s_2\eta_{*,-1}\begin{pmatrix}\bar{\delta}''_{-3,-2} & 0 & \bar{\lambda}_m\bar{\beta}''_{2,-2} & \bar{\lambda}_m\bar{\beta}''_{1,-2}\end{pmatrix}\\ &-\lambda_m s_2\eta_{*,-2}\begin{pmatrix}\bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m\bar{\beta}''_{2,-1} & \bar{\lambda}_m\bar{\beta}''_{1,-1}\end{pmatrix} \not\equiv 0(\text{mod } I_m).\end{aligned}$$

and hence $\theta \notin U_6((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Clearly $\theta_{3*} = f_3$. Set $\epsilon_3 := P_{13}$. Thus the first row of ${}^{\epsilon_3}\theta$ equals f_1. Since ${}^{\epsilon_3}\theta \notin U_6((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$, one can proceed now as in Part I, case 1.

<u>case 3.3</u> Assume that $\gamma''_{-3,1}, \gamma''_{-3,2}, \gamma''_{-2,1}, \gamma''_{-2,2}, \gamma''_{-1,1}, \gamma''_{-1,2}, \beta''_{1,-2}, \beta''_{1,-1}, \beta''_{2,-2}, \beta''_{2,-1}, \delta''_{-3,-2}, \delta''_{-3,-1} \in I_m$ and one of the elements $\alpha''_{11}-1$, α''_{21}, $\gamma''_{-1,3}$, $\delta''_{-1,-1}-1$ and $\delta''_{-1,-2}$ does not lie in I_m.
Set $g_2 := T_{13}(s_2) \in U_2$ and $\theta := [\eta^{-1}, g_2]$. Then

$$\begin{aligned} &\theta\\ =&[\eta^{-1}, g_2]\\ =&(e+\eta'_{*1}s_2\eta_{3*}-\eta'_{*,-3}s_2\eta_{-1,*})(g_2)^{-1}\\ =&(e+s_2\begin{pmatrix}\bar{\delta}''_{-1,-1}\\ \bar{\delta}''_{-1,-2}\\ \bar{\delta}''_{-1,-3}\\ \lambda_m\bar{\gamma}''_{-1,3}\\ \lambda_m\bar{\gamma}''_{-1,2}\\ \lambda_m\bar{\gamma}''_{-1,1}\end{pmatrix}\begin{pmatrix}0 & 0 & \alpha''_{33} & \beta''_{3,-3} & 0 & 0\end{pmatrix}\\ &-s_2\begin{pmatrix}0\\ 0\\ \bar{\lambda}_m\bar{\beta}''_{3,-3}\\ \bar{\alpha}''_{33}\\ 0\\ 0\end{pmatrix}\begin{pmatrix}\gamma''_{-1,1} & \gamma''_{-1,2} & \gamma''_{-1,3} & \delta''_{-1,-3} & \delta''_{-1,-2} & \delta''_{-1,-1}\end{pmatrix})\\ &\cdot(g_2)^{-1}.\end{aligned}$$

Assume that $\theta \in U_6((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$. Then $s_2\bar{\delta}''_{-1,-1}\alpha''_{33}-s_2 = \theta_{13} \in I_m$ and $s_2\bar{\delta}''_{-1,-2}\alpha''_{33} = \theta_{23} \in I_m$. Since $\alpha''_{33} \equiv 1(\text{mod } I_m)$, it follows that $\delta''_{-1,-1}-1, \delta''_{-1,-2} \in I_m$. Consider the column

$$\eta'_{*1}s_2\alpha''_{33} - \eta'_{*,-3}s_2\gamma''_{-1,3} - \begin{pmatrix}s_2 & 0 & -s_2^2\bar{\lambda}_m\bar{\beta}''_{3,-3}\gamma''_{-1,1} & -s_2^2\bar{\alpha}''_{33}\gamma''_{-1,1} & 0 & 0\end{pmatrix}^t.$$

Since by assumption $\theta \in U_6((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$, the column above is congruent to 0 modulo I_m. By multiplying $\eta_{-3,*}$ from the left we get that $\gamma''_{-1,3} \in I_m$ since $\gamma''_{-3,1}, \gamma''_{-1,1} \in I_m$ ($\gamma''_{-3,1}$ is the first entry of $\eta_{-3,*}$). Hence $\eta_{-1,*} \equiv f_{-1}(\text{mod } I_m)$. It follows that $\eta_{*1} \equiv e_1(\text{mod } I_m)$ (apply Lemma 6.8 to the image of η in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m\cap I_m)))$ and hence $\alpha''_{11}-1, \alpha''_{21} \in I_m$. Since that is a contradiction, $\theta \notin U_6((R_m,\Lambda_m),(I_m, I_m\cap\Lambda_m))$.

case 3.3.1 Assume that $\theta_{13} \notin I_m$ or $\theta_{23} \notin I_m$.
Set $g_3 := T_{-2,1}(s_3) \in U_3$ and $\tau := [\theta^{-1}, g_3]$. Then

$$\begin{aligned}
&\tau\\
=&[\theta^{-1}, g_3]\\
=&(e + \theta'_{*,-2}s_3\theta_{1*} - \theta'_{*,-1}\lambda_m s_3\theta_{2*})(g_3)^{-1}\\
=&(e + s_3 \begin{pmatrix} \bar\lambda_m\bar\theta_{2,-1} \\ 0 \\ \bar\lambda_m\bar\theta_{2,-3} \\ \bar\theta_{23} \\ \bar\theta_{22} \\ \bar\theta_{21} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{1,-3} & 0 & \theta_{1,-1} \end{pmatrix}\\
&- \lambda_m s_3 \begin{pmatrix} \bar\lambda_m\bar\theta_{1,-1} \\ 0 \\ \bar\lambda_m\bar\theta_{1,-3} \\ \bar\theta_{13} \\ \bar\theta_{12} \\ \bar\theta_{11} \end{pmatrix} \begin{pmatrix} \theta_{21} & \theta_{22} & \theta_{23} & \theta_{2,-3} & 0 & \theta_{2,-1} \end{pmatrix})\\
&\cdot (g_3)^{-1}.
\end{aligned}$$

Assume that $\tau \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $\theta'_{*,-2}s_3\theta_{13} - \theta'_{*,-1}\lambda_m s_3\theta_{23} \equiv 0(\text{mod } I_m)$. It follows that $\theta_{13}, \theta_{23} \in I_m$ which is a contradiction. Hence $\tau \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\tau_{2*} = f_2$. Set $\epsilon_4 := P_{12}$. Then the first row of ${}^{\epsilon_4}\theta$ equals f_1. Since ${}^{\epsilon_4}\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed now as in Part I, case 1.

case 3.3.2 Assume that $\theta_{13} \in I_m$ and $\theta_{23} \in I_m$.
Let $\hat\theta$ be the image of θ in $U_{2n}(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$. Clearly $\hat\theta$ has the form

$$\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat\theta_{33} & 0 & 0 & 0 \\
\hline
0 & 0 & \hat\theta_{-3,3} & \hat\theta_{-3,-3} & \hat\theta_{-3,-2} & \hat\theta_{-3,-1} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

It follows that $\hat\theta_{33}$ is invertible. Let $\mathbb{h}$ be the map defined in Definition 6.3. Then

$$\begin{aligned}
&\bar{\hat\theta}_{33}\hat\theta_{-3,-2}\\
=&\mathbb{h}(\hat\theta_{*3}, \hat\theta_{*,-2})\\
=&\mathbb{h}(\hat\theta e_3, \hat\theta e_{-2})\\
=&\mathbb{h}(e_3, e_{-2})\\
=&0.
\end{aligned}$$

Hence $\hat{\theta}_{-3,-2} = 0$ and therefore $\theta_{-3,-2} \in I_m$. Further

$$\begin{aligned}
&\bar{\hat{\theta}}_{33}\hat{\theta}_{-3,-1}\\
=&\mathbb{h}(\hat{\theta}_{*3}, \hat{\theta}_{*,-1})\\
=&\mathbb{h}(\hat{\theta}e_3, \hat{\theta}e_{-1})\\
=&\mathbb{h}(e_3, e_{-1})\\
=&0.
\end{aligned}$$

Hence $\hat{\theta}_{-3,-1} = 0$ and therefore $\theta_{-3,-1} \in I_m$. Clearly $\theta_{22} = 1$. Set $\xi_3 := T_{23}(-\theta_{23})$ $T_{2,-1}(-\theta_{2,-1}) \in U_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\chi := \theta\xi_3$. Then $\chi_{23}, \chi_{2,-1} = 0$ and the image $\hat{\chi}$ of χ in $U_6(R_m/I_m, \Lambda_m/(\Lambda_m \cap I_m))$ has the form

$$\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & \hat{\chi}_{33} & 0 & 0 & 0\\
\hline
0 & 0 & \hat{\chi}_{-3,3} & \hat{\chi}_{-3,-3} & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

Since $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\chi \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ (i.e. $\hat{\chi} \neq e$). Set $g_3 := T_{31}(s_3) \in U_3$ and $\mu := [\chi, g_3]$. Then

$$\begin{aligned}
&\mu\\
=&[\chi, g_3]\\
=&(e + \chi_{*3}s_3\chi'_{1*} - \chi_{*,-1}s_3\chi'_{-3,*})(g_3)^{-1}\\
=&(e + s_3\begin{pmatrix}\chi_{13}\\ 0\\ \chi_{33}\\ \chi_{-3,3}\\ \chi_{-2,3}\\ \chi_{-1,3}\end{pmatrix}\begin{pmatrix}\bar{\chi}_{-1,-1} & \bar{\chi}_{-2,-1} & \bar{\chi}_{-3,-1} & \bar{\lambda}_m\bar{\chi}_{3,-1} & 0 & \bar{\lambda}_m\bar{\chi}_{1,-1}\end{pmatrix}\\
&- s_3\begin{pmatrix}\chi_{1,-1}\\ 0\\ \chi_{3,-1}\\ \chi_{-3,-1}\\ \chi_{-2,-1}\\ \chi_{-1,-1}\end{pmatrix}\begin{pmatrix}\lambda_m\bar{\chi}_{-1,3} & \lambda_m\bar{\chi}_{-2,3} & \lambda_m\bar{\chi}_{-3,3} & \bar{\chi}_{33} & 0 & \bar{\chi}_{13}\end{pmatrix})\\
&\cdot (g_3)^{-1}.
\end{aligned}$$

Assume that $\mu \in U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Then $s_3\chi_{33}\bar{\chi}_{-1,-1} - s_3\chi_{3,-1}\lambda_m\bar{\chi}_{-1,3} -$ $s_3(1 + s_3\chi_{33}\bar{\chi}_{-3,-1} - s_3\chi_{3,-1}\lambda_m\bar{\chi}_{-3,3}) = \mu_{31} \in I_m$ and hence $s_3\chi_{33}\bar{\chi}_{-1,-1} - s_3 \in I_m$. It follows that $\chi_{33} \equiv 1 (\mathrm{mod}\ I_m)$ (i.e. $\hat{\chi}_{33} = 1$) since $s_3 \in (R_m)^*$ and $\chi_{-1,-1} \equiv$

$1(\text{mod } I_m)$. That implies

$$\begin{aligned}&\hat{\chi}_{-3,-3}\\ =&\mathbb{h}(\hat{\chi}_{*3},\hat{\chi}_{*,-3})\\ =&\mathbb{h}(\hat{\chi}e_3,\hat{\chi}e_{-3})\\ =&\mathbb{h}(e_3,e_{-3})\\ =&1.\end{aligned}$$

Further $s_3\chi_{-3,3}\bar{\chi}_{-1,-1}-s_3\chi_{-3,-1}\lambda_m\bar{\chi}_{-1,3}-s_3(s_3\chi_{-3,3}\bar{\chi}_{-3,-1}-s_3\chi_{-3,-1}\lambda_m\bar{\chi}_{-3,3})=\mu_{-3,1}\in I_m$ and hence $s_3\chi_{-3,3}\bar{\chi}_{-1,-1}\in I_m$. It follows that $\chi_{-3,3}\equiv 0(\text{mod } I_m)$ (i.e. $\hat{\chi}_{-3,3}=0$) since $s_3\in(R_m)^*$ and $\chi_{-1,-1}\equiv 1(\text{mod } I_m)$. But that implies the contradiction $\hat{\chi}=e$. Hence $\mu\notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Clearly $\mu_{2*}=f_2$. Set $\epsilon_4:=P_{12}$. Then the first row of ${}^{\epsilon_4}\theta$ equals f_1. Since ${}^{\epsilon_4}\mu\notin U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$, one can proceed as in Part I, case 1.

<u>case 3.4</u> Assume that $\gamma''_{-3,1},\gamma''_{-3,2},\gamma''_{-2,1},\gamma''_{-2,2},\gamma''_{-1,1},\gamma''_{-1,2},\beta''_{1,-2},\ \beta''_{1,-1},\ \beta''_{2,-2},\ \beta''_{2,-1},\ \delta''_{-3,-2},\delta''_{-3,-1},\alpha''_{11}-1,\ \alpha''_{21},\ \gamma''_{-1,3},\ \delta''_{-1,-1}-1,\delta''_{-1,-2}\in I_m$ and one of the elements $\alpha''_{12},\alpha''_{22}-1,\gamma''_{-2,3},\delta''_{-2,-1},\delta''_{-2,-2}-1,\alpha''_{13}$ and α''_{23} does not lie in I_m.
Set $g_2:=T_{21}(s_2)\in U_2$ and $\theta:=[\eta,g_2]$. Then

$$\begin{aligned}&\theta\\ =&[\eta,g_2]\\ =&(e+\eta_{*2}s_2\eta'_{1*}-\eta_{*,-1}s_2\eta'_{-2,*})(g_2)^{-1}\\ =&(e+s_2\begin{pmatrix}\alpha''_{12}\\ \alpha''_{22}\\ 0\\ \gamma''_{-3,2}\\ \gamma''_{-2,2}\\ \gamma''_{-1,2}\end{pmatrix}\begin{pmatrix}\bar{\delta}''_{-1,-1} & \bar{\delta}''_{-2,-1} & \bar{\delta}''_{-3,-1} & 0 & \bar{\lambda}_m\bar{\beta}''_{2,-1} & \bar{\lambda}_m\bar{\beta}''_{1,-1}\end{pmatrix}\\ &-s_2\begin{pmatrix}\beta''_{1,-1}\\ \beta''_{2,-1}\\ 0\\ \delta''_{-3,-1}\\ \delta''_{-2,-1}\\ \delta''_{-1,-1}\end{pmatrix}\begin{pmatrix}\lambda_m\bar{\gamma}''_{-1,2} & \lambda_m\bar{\gamma}''_{-2,2} & \lambda_m\bar{\gamma}''_{-3,2} & 0 & \bar{\alpha}''_{22} & \bar{\alpha}''_{12}\end{pmatrix})\\ &\cdot(g_2)^{-1}.\end{aligned}$$

Assume that $\theta\in U_6((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. Then $s_2\eta_{*2}\bar{\lambda}_m\bar{\beta}''_{1,-1}-s_2\eta_{*,-1}\bar{\alpha}''_{12}\equiv 0(\text{mod } I_m)$. It follows that $\alpha''_{12}\in I_m$. Hence $-s_2\eta_{*,-1}\bar{\alpha}''_{22}+s_2e_{-1}\equiv 0(\text{mod } I_m)$. By multiplying $\eta'_{-1,*}$ from the left we get that $-s_2\bar{\alpha}''_{22}+s_2\bar{\alpha}''_{11}\in I_m$ which implies $\alpha''_{22}\equiv 1(\text{mod } I_m)$ since $\alpha''_{11}\equiv 1(\text{mod } I_m)$. Let $\hat{\eta}$ be the image of η in $U_{2n}(R_m/I_m,\Lambda_m/(\Lambda_m\cap I_m))$. By Lemma 6.8, $\hat{\eta}_{-2,*}=f_{-2}$ since $\hat{\eta}_{*2}=e_2$. Hence $\gamma''_{-2,3},\delta''_{-2,-2}-1,\delta''_{-2,-1}\in I_m$.

Hence $\hat{\eta}$ has the form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \hat{\alpha}''_{13} & 0 & 0 & 0 \\ 0 & 1 & \hat{\alpha}''_{23} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{\gamma}''_{-3,3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly

$$\begin{aligned} &\overline{\hat{\alpha}''_{13}} \\ =&\mathbb{h}(\hat{\eta}_{*3}, \hat{\eta}_{*,-1}) \\ =&\mathbb{h}(\hat{\eta}e_3, \hat{\eta}e_{-1}) \\ =&\mathbb{h}(e_3, e_{-1}) \\ =&0 \end{aligned}$$

and

$$\begin{aligned} &\overline{\hat{\alpha}''_{23}} \\ =&\mathbb{h}(\hat{\eta}_{*3}, \hat{\eta}_{*,-2}) \\ =&\mathbb{h}(\hat{\eta}e_3, \hat{\eta}e_{-2}) \\ =&\mathbb{h}(e_3, e_{-2}) \\ =&0. \end{aligned}$$

Hence $\hat{\alpha}''_{13} = 0 = \hat{\alpha}''_{23}$ and therefore $\alpha''_{13}, \alpha''_{23} \in I_m$. Since that is a contradiction, $\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. Clearly $\theta_{3*} = f_3$. Set $\epsilon_3 := P_{13}$. Then the first row of ${}^{\epsilon_3}\theta$ equals f_1. Since ${}^{\epsilon_3}\theta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, one can proceed as in Part I, case 1.

<u>case 3.5</u> Assume that $\gamma''_{-3,1}, \gamma''_{-3,2}, \gamma''_{-2,1}, \gamma''_{-2,2}, \gamma''_{-1,1}, \gamma''_{-1,2}, \beta''_{1,-2}, \beta''_{1,-1}, \beta''_{2,-2}, \beta''_{2,-1}, \delta''_{-3,-2}, \delta''_{-3,-1}, \alpha''_{11} - 1, \alpha''_{21}, \gamma''_{-1,3}, \delta''_{-1,-1} - 1, \delta''_{-1,-2}, \alpha''_{12}, \alpha''_{22} - 1, \gamma''_{-2,3}, \delta''_{-2,-1}, \delta''_{-2,-2} - 1, \alpha''_{13}, \alpha''_{23} \in I_m$.
Since $\eta \notin U_6((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $\gamma''_{-3,3} \notin I_m$. By [1], chapter IV, Lemma 3.12, Part II, case 4 there are $x_1, x_2 \in I_m$ such that $\gamma''_{-1,3} + x_2(x_1\gamma''_{-1,3} + \gamma''_{-2,3}) \in rad(R_m) \cap I_m$. Set $\xi_3 := T_{-1,-2}(x_2)T_{-2,-1}(x_1) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\theta := \xi_3\eta$. Then $\theta \equiv \eta (\text{mod } I_m)$, $\theta_{33} = \alpha''_{33} \equiv 1 (\text{mod } rad(R_m) \cap I_m)$ and $\theta_{-1,3} \in rad(R_m) \cap I_m$. Set $\epsilon_{21} := T_{3,-1}(1) \in EU_6(R_m, \Lambda_m)$ and $\tau :={}^{\epsilon_3}\theta$. Then $\tau_{33} \equiv 1 (\text{mod } rad(R_m) \cap I_m)$ and $\tau_{-3,-1} = \theta_{-3,-1} - \theta_{-3,3} \equiv \theta_{-3,3} \equiv \gamma''_{-3,3} (\text{mod } I_m)$. Since $\gamma''_{-3,3} \notin I_m$, it follows that $\tau_{-3,-1} \notin I_m$. Set $\xi_4 := T_{32}(-(\tau_{33})^{-1}\tau_{32})T_{31}(-(\tau_{33})^{-1}\tau_{31})T_{3,-1}(-(\tau_{33})^{-1}\tau_{3,-1})T_{3,-2}(-(\tau_{33})^{-1}\tau_{3,-2}) \in EU_6((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\chi := \tau\xi_4$. Then χ has the form

$$\left(\begin{array}{ccc|ccc} & * & & & * & \\ 0 & 0 & \alpha'''_{33} & \beta'''_{3,-3} & 0 & 0 \\ \hline & \gamma''' & & & \delta''' & \end{array}\right) = \begin{pmatrix} \alpha''' & \beta''' \\ \gamma''' & \delta''' \end{pmatrix}$$

where $\alpha''', \beta''', \gamma''', \delta''' \in M_3(R_m)$. Further $\alpha'''_{33} \equiv 1(\text{mod } rad(R_m) \cap I_m)$, $\beta'''_{3,-3} \in I_m$ and $\delta'''_{-3,-1} \notin I_m$. One can proceed now as in case 3.1 or case 3.2.

<u>case 4</u> Assume that $\gamma_{-3,1}, \gamma_{-3,2}, \gamma_{-2,1}, \gamma_{-2,2}, \gamma_{-1,1}, \gamma_{-1,2}, \beta_{3,-3}, \beta_{3,-2} \in I_m$ and $\beta_{3,-1} \notin I_m$.
Set $\epsilon_{11} := T_{12}(1) \in EU_6(R_m, \Lambda_m)$ and $\rho :={}^{\epsilon_{12}}\sigma$. Clearly $\rho_{-3,1}, \rho_{-3,2}, \rho_{-2,1}, \rho_{-2,2}, \rho_{-1,1}, \rho_{-1,2} \in I_m$. Further

$$\rho_{3*} = \begin{pmatrix} 0 & 0 & 1 & \beta_{3,-3} & \beta_{-3,-2} + \beta_{-3,-1} & \beta_{-3,-1} \end{pmatrix}.$$

Since $\beta_{3,-2} \in I_m$ and $\beta_{-3,-1} \notin I_m$, $\beta_{-3,-2} + \beta_{-3,-1} \notin I_m$. One can proceed now as in case 3.

<u>case 5</u> Assume that $\gamma_{-3,1}, \gamma_{-3,2}, \gamma_{-2,1}, \gamma_{-2,2}, \gamma_{-1,1}, \gamma_{-1,2}, \beta_{3,-3}, \beta_{3,-2}, \beta_{3,-1} \in I_m$.
One can proceed as in case 3 (σ has the same properties as ζ in case 3).

The following inverted tree diagram extending over several pages gives an overview of the case by case proof just concluded of Part II. In the diagram a "+" at a position (i,j) of a matrix σ means that $\sigma_{ij} \in I_m$ if $i \neq j$ and $\sigma_{ij} \equiv 1(\text{mod } I_m)$ if $i = j$. If positions in a matrix σ are marked by a "$-$", it means that there is one position (i,j) among all the positions marked by a "$-$" such that $\sigma_{ij} \notin I_m$ if $i \neq j$ and $\sigma_{ij} \not\equiv 1(\text{mod } I_m)$ if $i = j$. A "$-$" does not mean that the entry at this position does not lie in I_m (resp. is not congruent to 1 modulo I_m). A "$*$" stands for an arbitrary entry. If we write $T_{ij}(+)$ we mean an elementary transvection $T_{ij}(x)$ where $x \in I_m$.

h

$$\boxed{\begin{gathered}\left(\begin{array}{ccc|ccc} - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ \hline - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{array}\right) \\ \notin CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))\end{gathered}}$$

$$\Big\downarrow \; \begin{gathered}\epsilon_0, \epsilon_1 \in EU_6(R_m, \Lambda_m), \\ g_0 \in U_0\end{gathered}$$

${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]$

$$\begin{tikzcd}
\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & - & - & * \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ + & + & * & * & * & * \end{array}\right) \arrow[r, dash, "3"] & \left(\begin{array}{ccc|ccc} - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & 1 & - & - & - \\ \hline - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{array}\right) \arrow[r, dash, "4"] \arrow[dl, dash, "2"'] \arrow[d, dash, "1"'] \arrow[dr, dash, "5"] & \left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & + & + & - \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ + & + & * & * & * & * \end{array}\right) \\
\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ \hline - & * & * & * & * & * \\ - & * & * & * & * & * \\ - & * & * & * & * & * \end{array}\right) \arrow[d] & \left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ \hline * & - & * & * & * & * \\ * & - & * & * & * & * \\ * & - & * & * & * & * \end{array}\right) \arrow[d] & \left(\begin{array}{ccc|ccc} - & - & - & + & - & - \\ - & - & - & + & - & - \\ 0 & 0 & 1 & + & + & + \\ \hline + & + & - & + & - & - \\ + & + & - & + & - & - \\ + & + & - & + & - & - \end{array}\right) \\
{} & {} &
\end{tikzcd}$$

$$\downarrow$$

$[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_1]$ analog to case 1

$$\Big\downarrow{\scriptstyle g_1 = T_{1,-2}(s_1)}$$

$$\boxed{\left(\begin{array}{ccc|ccc} - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline - & - & - & 1 & - & - \\ - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \end{array}\right)}$$

$$\Big\downarrow{\scriptstyle \epsilon_2 = P_{13}}$$

$^{\epsilon_2}[^{\epsilon_1}[^{\epsilon_0}h, g_0], g_1]$

$$\boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ \hline - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 1 \end{array}\right)}$$

$$\Big\downarrow$$

see Part I, case 1

$$\Big|{\scriptstyle 3}$$

$^{\epsilon_1}[^{\epsilon_0}h, g_0]$

$$\boxed{\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & - & - & * \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ + & + & * & * & * & * \end{array}\right)}$$

$$\Big\downarrow$$

$$\Big\downarrow\ \xi_1 = T_{-1,2}(+)T_{-2,-1}(+)$$

$\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0])$

$$\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & - & - & * \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ x & + & * & * & * & * \end{array}\right)$$
$$x \in rad(R_m) \cap I_m$$

$$\Big\downarrow\ g_1 = T_{13}(s_1)$$

$[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1]$

$$\left(\begin{array}{ccc|ccc} + & - & - & - & - & - \\ + & - & - & - & - & - \\ + & - & - & - & - & - \\ \hline + & - & - & - & - & - \\ + & - & - & - & - & - \\ + & + & + & + & + & y \end{array}\right)$$
$$y \equiv 1(\mathrm{mod}\ rad(R_m) \cap I_m)$$

$$\Big\downarrow\ \epsilon_2 = P_{3,-1}$$

${}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1]$

$$\left(\begin{array}{ccc|ccc} - & - & - & + & - & - \\ - & - & - & + & - & - \\ + & + & y & + & + & + \\ \hline - & - & - & + & - & - \\ - & - & - & + & - & - \\ - & - & - & + & - & - \end{array}\right)$$
$$y \equiv 1(\mathrm{mod}\ rad(R_m) \cap I_m),$$

$$\Big\downarrow\ \begin{array}{l} \xi_2 = T_{32}(+)T_{31}(+) \\ \cdot T_{3,-1}(+)T_{3,-2}(+) \end{array}$$

$({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2$

$$\left(\begin{array}{ccc|ccc} - & * & * & + & + & + \\ - & * & * & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & * & + & + & + \\ + & + & * & + & * & * \\ + & + & - & + & - & - \end{array}\right) \overset{3.3}{=} \left(\begin{array}{ccc|ccc} - & - & - & + & - & - \\ - & - & - & + & - & - \\ 0 & 0 & y & + & 0 & 0 \\ \hline - & - & - & + & - & - \\ - & - & - & + & - & - \\ - & - & - & + & - & - \end{array}\right) \overset{3.4}{=} \left(\begin{array}{ccc|ccc} + & - & - & + & + & + \\ + & - & - & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & * & + & + & + \\ + & + & - & + & - & - \\ + & + & + & + & + & + \end{array}\right)$$
$$y \equiv 1(\mathrm{mod}\ rad(R_m) \cap I_m) \quad y \equiv 1(\mathrm{mod}\ rad(R_m) \cap I_m) \quad y \equiv 1(\mathrm{mod}\ rad(R_m) \cap I_m)$$

3.2 3.1 3.5

$$\left(\begin{array}{ccc|ccc} \ast & \ast & \ast & + & - & - \\ \ast & \ast & \ast & + & - & - \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & \ast & + & - & - \\ + & + & \ast & + & \ast & \ast \\ + & + & \ast & + & \ast & \ast \end{array}\right) \quad \left(\begin{array}{ccc|ccc} \ast & \ast & \ast & + & \ast & \ast \\ \ast & \ast & \ast & + & \ast & \ast \\ 0 & 0 & y & + & 0 & 0 \\ \hline - & - & \ast & + & \ast & \ast \\ - & - & \ast & + & \ast & \ast \\ - & - & \ast & + & \ast & \ast \end{array}\right) \quad \left(\begin{array}{ccc|ccc} + & + & + & + & + & + \\ + & + & + & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & - & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \end{array}\right)$$

$y \equiv 1(\text{mod } rad(R_m) \cap I_m)$ $y \equiv 1(\text{mod } rad(R_m) \cap I_m)$ $y \equiv 1(\text{mod } rad(R_m) \cap I_m)$

see case 1

3.2 3.4

$({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2$

$$\left(\begin{array}{ccc|ccc} \ast & \ast & \ast & + & - & - \\ \ast & \ast & \ast & + & - & - \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & \ast & + & - & - \\ + & + & \ast & + & \ast & \ast \\ + & + & \ast & + & \ast & \ast \end{array}\right) \quad \left(\begin{array}{ccc|ccc} + & - & - & + & + & + \\ + & - & - & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & \ast & + & + & + \\ + & + & - & + & - & - \\ + & + & + & + & + & + \end{array}\right)$$

$y \equiv 1(\text{mod } rad(R_m) \cap I_m)$ $y \equiv 1(\text{mod } rad(R_m) \cap I_m)$

$g_2 = T_{-1,2}(s_2)$ $g_2 = T_{21}(s_2)$

$[({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2, g_2]$

$$\left(\begin{array}{ccc|ccc} - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline - & - & - & 1 & - & - \\ - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \end{array}\right) \quad \left(\begin{array}{ccc|ccc} - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline - & - & - & 1 & - & - \\ - & - & - & 0 & - & - \\ - & - & - & 0 & - & - \end{array}\right)$$

$${}^{\epsilon_3}[({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2, g_2]$$

$$\xrightarrow{\epsilon_3 = P_{13}} \boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ \hline - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 1 \end{array}\right)} \longrightarrow \text{see Part I, case 1}$$

$$\xrightarrow{\epsilon_3 = P_{13}} \boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ \hline - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 1 \end{array}\right)} \longrightarrow \text{see Part I, case 1}$$

$$({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2$$

$$\xrightarrow{3.3} \boxed{\begin{array}{c}\left(\begin{array}{ccc|ccc} - & * & * & + & + & + \\ - & * & * & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & * & + & + & + \\ + & + & * & + & * & * \\ + & + & - & + & - & - \end{array}\right) \\ y \equiv 1(\text{mod } rad(R_m) \cap I_m)\end{array}}$$

$$[(({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2)^{-1}, g_2]$$

$$\xrightarrow{g_2 = T_{13}(s_2)} \left(\begin{array}{ccc|ccc} 1 & 0 & - & + & 0 & + \\ 0 & 1 & - & + & 0 & + \\ + & + & - & + & + & + \\ \hline + & + & - & - & - & - \\ 0 & 0 & + & + & 1 & + \\ 0 & 0 & + & + & 0 & + \end{array}\right)$$

3.3.1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & - & + & 0 & + \\ 0 & 1 & - & + & 0 & + \\ + & + & * & + & + & + \\ \hline + & + & * & * & * & * \\ 0 & 0 & + & + & 1 & + \\ 0 & 0 & + & + & 0 & + \end{array}\right)$$

$\xi_3 = e$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & - & + & 0 & + \\ 0 & 1 & - & + & 0 & + \\ + & + & * & + & + & + \\ \hline + & + & * & * & * & * \\ 0 & 0 & + & + & 1 & + \\ 0 & 0 & + & + & 0 & + \end{array}\right)$$

$l_3 = -1,$
$g_3 = T_{-2,1}(s_3)$

$$\left(\begin{array}{ccc|ccc} - & - & - & - & 0 & - \\ 0 & 1 & 0 & 0 & 0 & 0 \\ - & - & - & - & 0 & - \\ \hline - & - & - & - & 0 & - \\ - & - & - & - & 1 & - \\ - & - & - & - & 0 & - \end{array}\right)$$

$\epsilon_4 = P_{12}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ \hline - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 1 \end{array}\right)$$

see Part I, case 1

3.3.2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & + & + & 0 & + \\ 0 & 1 & + & + & 0 & + \\ + & + & - & + & + & + \\ \hline + & + & - & - & + & + \\ 0 & 0 & + & + & 1 & + \\ 0 & 0 & + & + & 0 & + \end{array}\right)$$

$\xi_3 = T_{23}(+)T_{2,-1}(+)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & + & + & + & + \\ 0 & 1 & 0 & + & + & 0 \\ + & + & - & + & + & + \\ \hline + & + & - & - & + & + \\ 0 & 0 & + & + & + & + \\ 0 & 0 & + & + & + & + \end{array}\right)$$

$l_3 = 1,$
$g_3 = T_{31}(s_3)$

$$\left(\begin{array}{ccc|ccc} - & - & - & - & 0 & - \\ 0 & 1 & 0 & 0 & 0 & 0 \\ - & - & - & - & 0 & - \\ \hline - & - & - & - & 0 & - \\ - & - & - & - & 1 & - \\ - & - & - & - & 0 & - \end{array}\right)$$

$\epsilon_4 = P_{12}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ \hline - & - & - & - & - & 0 \\ - & - & - & - & - & 0 \\ - & - & - & - & - & 1 \end{array}\right)$$

see Part I, case 1

$$([(({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2)^{-1}, g_2])\xi_3$$

$$[(([(({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2)^{-1}, g_2])\xi_3)^{l_3}, g_3]$$

$${}^{\epsilon_4}[(([(({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2)^{-1}, g_2])\xi_3)^{l_3}, g_3]$$

$$\big\downarrow{\scriptstyle 3.5}$$

$$\boxed{({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2} \qquad \boxed{\begin{array}{c}\left(\begin{array}{ccc|ccc} + & + & + & + & + & + \\ + & + & + & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & - & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \end{array}\right) \\ y \equiv 1(\text{mod } rad(R_m) \cap I_m) \end{array}}$$

$$\big\downarrow{\scriptstyle \xi_3 = T_{-1,-2}(+)T_{-2,-1}(+)}$$

$$\boxed{\xi_3({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2} \qquad \boxed{\begin{array}{c}\left(\begin{array}{ccc|ccc} + & + & + & + & + & + \\ + & + & + & + & + & + \\ 0 & 0 & y & + & 0 & 0 \\ \hline + & + & - & + & + & + \\ + & + & + & + & + & + \\ + & + & z & + & + & + \end{array}\right) \\ y-1, z \in rad(R_m) \cap I_m \end{array}}$$

$$\big\downarrow{\scriptstyle \epsilon_{21} = T_{3,-1}(1)}$$

$$\boxed{{}^{\epsilon_{21}}(\xi_3({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2)} \qquad \boxed{\begin{array}{c}\left(\begin{array}{ccc|ccc} + & + & * & + & + & * \\ + & + & + & + & + & + \\ + & + & w & + & + & + \\ \hline + & + & * & + & + & - \\ + & + & + & + & + & + \\ + & + & + & + & + & + \end{array}\right) \\ w \equiv 1(\text{mod } rad(R_m) \cap I_m) \end{array}}$$

$$\big\downarrow{\scriptstyle \begin{array}{l}\xi_4 = T_{32}(+)T_{31}(+) \\ \cdot T_{3,-1}(+)T_{3,-2}(+)\end{array}}$$

$$\boxed{({}^{\epsilon_{21}}(\xi_3({}^{\epsilon_2}[(\xi_1({}^{\epsilon_1}[{}^{\epsilon_0}h, g_0]))^{-1}, g_1])\xi_2))\xi_4} \qquad \boxed{\begin{array}{c}\left(\begin{array}{ccc|ccc} + & + & * & + & + & * \\ + & + & + & + & + & + \\ 0 & 0 & w & + & 0 & 0 \\ \hline + & + & * & + & + & - \\ + & + & + & + & + & + \\ + & + & + & + & + & + \end{array}\right) \\ w \equiv 1(\text{mod } rad(R_m) \cap I_m) \end{array}}$$

$$\big\downarrow$$

see case 3.2

$$\downarrow 4$$

$${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0] \qquad \boxed{\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & + & + & - \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ + & + & * & * & * & * \end{array}\right)}$$

$$\Big\downarrow \epsilon_{11} = T_{21}(1)$$

$${}^{\epsilon_{11}\epsilon_1}[{}^{\epsilon_0}h, g_0] \qquad \boxed{\left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & + & - & * \\ \hline + & + & * & * & * & * \\ + & + & * & * & * & * \\ + & + & * & * & * & * \end{array}\right)}$$

$$\Big\downarrow$$

see case 3

$$\downarrow 5$$

$${}^{\epsilon_1}[{}^{\epsilon_0}h, g_0] \qquad \boxed{\left(\begin{array}{ccc|ccc} - & - & - & + & - & - \\ - & - & - & + & - & - \\ 0 & 0 & 1 & + & + & + \\ \hline + & + & - & + & - & - \\ + & + & - & + & - & - \\ + & + & - & + & - & - \end{array}\right)}$$

$$\Big\downarrow$$

see case 3

Part III Assume that $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$.
This part corresponds to Proposition 3.3 in [1], chapter IV. By [1], chapter IV, Corollary 3.4, there is an elementary transvection $T_{ij}(x) \in EU_{2n}(R_m, \Lambda_m)$ such that $[h, T_{ij}(x)] \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ since $h \notin CU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Since $h \in CU_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$, $[h, T_{ij}(x)] \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$. There are an $a \in R$ and an $s \in S_m$ such that $x = \frac{a}{s}$. Set $\hat{a} := f_m(a)$ and $\hat{s} := f_m(s)$. Set $g_0 := T_{ij}(s_0\hat{s}x) = T_{ij}(s_0\hat{a}) \in U_0$. By Lemma 7.6, $[h, g_0] \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ but $[h, g_0] \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Set $\sigma := [h, g_0]$. By Lemma 7.7, there is an $\epsilon_1 \in EU_{2n}(R_m, \Lambda_m)$ such that $y_1 := ({}^{\epsilon_1}\sigma)_{11}$ is invertible. Clearly ${}^{\epsilon_1}\sigma \in U_{2n}((R_m, \Lambda_m), (I_m, I_m \cap \Lambda_m))$ but ${}^{\epsilon_1}\sigma \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Set $\omega := {}^{\epsilon_1}\sigma$, $\xi_1 := T_{-2,1}(-\omega_{-2,1}(y_1)^{-1}) \dots\ T_{21}(-\omega_{21}(y_1)^{-1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\xi_2 := T_{12}(-(y_1)^{-1}\omega_{12}) \dots T_{1,-2}(-(y_1)^{-1}\omega_{1,-2}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\tau := \xi_1\omega\xi_2$ has the form

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix}$$

where $y_2, y_3, y_4 \in R_m$ and $A \in M_{2n-2}(R_m)$.

case 1 Assume that $y_3(y_1)^{-1} \in \Gamma_m$ and $(y_1)^{-1}y_2 \in \bar{\lambda}_m\Gamma_m$.
Set $\xi_3 := T_{-1,1}(y_3(y_1)^{-1}) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $\xi_4 := T_{1,-1}((y_1)^{-1}y_2) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Then $\zeta := \xi_3\tau\xi_4$ has the form

$$\begin{pmatrix} y_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix}$$

where $y_5 \in R_m$. Clearly $\zeta \in U_{2n}((R_m,\ \Lambda_m), (I_m, I_m \cap \Lambda_m))$ but $\zeta \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. Hence there is an $l \in \{2, \dots, -2\}$ such that $|\zeta_{*l}| \notin \Gamma_m$.

case 1.1 Assume that $\epsilon(l) = 1$.
There are a $b' \in R$ and a $t' \in S_m$ such that $y_1|\zeta_{*l}|\bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m\overline{\zeta_{-l,l}\bar{y}_1} = \frac{b'}{t'}$. Set $t := \frac{t'}{1} \in R_m$ and $g_1 := T_{l,-1}(s_1s_2t) \in U_1$. One can show that $[\zeta, g_1]$ equals

$$\begin{aligned}
&T_{2,-1}(s_1s_2t\zeta_{2l}\bar{y}_1)\cdot \\
&\vdots \\
&T_{l-1,-1}(s_1s_2t\zeta_{(l-1)l}\bar{y}_1)\cdot \\
&T_{l,-1}(s_1s_2t\zeta_{ll}\bar{y}_1 - s_1s_2t)\cdot \\
&T_{l+1,-1}(s_1s_2t\zeta_{(l+1)l}\bar{y}_1)\cdot \\
&\vdots \\
&T_{-2,-1}(s_1s_2t\zeta_{-2,l}\bar{y}_1)\cdot \\
&T_{1,-1}(z)
\end{aligned}$$

where $z = \bar{\lambda}_m \overline{s_1 s_2 t}(y_1 |\zeta_{*l}| \bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1}) s_1 s_2 t$. Since $|\zeta_{*l}| \notin \Gamma_m$ and y_1 is invertible, $y_1|\zeta_{*l}|\bar{y}_1 \notin \Gamma_m$. Since $\zeta_{-l,l} \in I_m$, $-\zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1} \in (\Gamma_m)_{min} \subseteq \Gamma_m$. Since $s_1 s_2 t$ is invertible, it follows that $\overline{s_1 s_2 t}(y_1|\zeta_{*l}|\bar{y}_1 - \zeta_{-l,l}\bar{y}_1 + \lambda_m \overline{\zeta_{-l,l}\bar{y}_1}) s_1 s_2 t \notin \Gamma_m$. Hence $z \notin \bar{\lambda}_m \Gamma_m$. Set

$$\begin{aligned}
&\xi_5 := \\
&T_{-2,-1}(-s_1 s_2 t \zeta_{-2,l}\bar{y}_1)\cdot \\
&\vdots \\
&T_{l+1,-1}(-s_1 s_2 t \zeta_{(l+1)l}\bar{y}_1)\cdot \\
&T_{l,-1}(-(s_1 s_2 t \zeta_{ll}\bar{y}_1 - s_1 s_2 t))\cdot \\
&T_{l-1,-1}(-s_1 s_2 t \zeta_{(l-1)l}\bar{y}_1)\cdot \\
&\vdots \\
&T_{2,-1}(-s_1 s_2 t \zeta_{2l}\bar{y}_1) \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))
\end{aligned}$$

and $g_2 := T_{1,-1}(z) \in U_2$. Then

$$(\xi_5[\xi_3 \xi_1({}^{\epsilon_1}[h, g_0])\xi_2 \xi_4, g_1]) = g_2.$$

Note that $g_2 \notin U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ since $z \notin \bar{\lambda}_m \Gamma_m$. Set $g'_i := \rho_m(g_i)$ $\forall i \in \{0, 1, 2\}$, $\epsilon'_1 := \rho_m(\epsilon_1)$ and $d'_i := \rho_m(d_i)$ $\forall i \in \{0, 1, 2\}$. Then $\epsilon'_1 \in E' = \rho_m(EU_{2n}(R_m, \Lambda_m))$, $g'_2 \in Gen(E')\backslash\{e\}$, ${}^{d'_i}g'_i \in U'$ $\forall i \in \{0, 1, 2\}$ and

$$[{}^{\epsilon'_1}[h', g'_0], g'_1] = g'_2.$$

case 1.2 Assume that $\epsilon(l) = -1$.
This case can be treated similarly.

case 2 Assume that $y_3(y_1)^{-1} \notin \Gamma_m$.

case 2.1 Assume that $|\tau_{*l}| \in \Gamma_m$ $\forall l \in \{2, \dots, -2\}$.
Set $\chi := T_{-1,1}(-y_3(y_1)^{-1}) \in EU_{2n}(R_m, \Lambda_m)$ (one checks easily that $-y_3(y_1)^{-1} \in \Lambda_m$). Then $\zeta := \chi\tau$ has the form

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix}$$

where $y_5 \in R_m$. There are an $b' \in R$ and a $t' \in S_m$ such that $y_3(y_1)^{-1} = \frac{b'}{t'}$. Set $t := \frac{t'}{1} \in R_m$ and $g_1 := T_{12}(s_1 s_2 t) \in U_1$. Using the equality $[\alpha\beta, \gamma] = {}^{\alpha}[\beta, \gamma][\alpha, \gamma]$ one gets that $[\tau, g_1] = [\chi^{-1}\zeta, g_1] = {}^{\chi^{-1}}[\zeta, g_1][\chi^{-1}, g_1]$. It is easy to show that $[\zeta, g_1] \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and hence ${}^{\chi^{-1}}[\zeta, g_1] \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$. On the other hand $[\chi^{-1}, g_1] = T_{-1,2}(y_3(y_1)^{-1}s_1 s_2 t)T_{-2,2}(-\overline{s_1 s_2 t}y_3(y_1)^{-1}s_1 s_2 t)$, by ($R6.1$). Set $\xi_3 := T_{-1,2}(-y_3(y_1)^{-1}s_1 s_2 t)({}^{\chi^{-1}}[\zeta, g_1])^{-1} \in EU_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m))$ and $g_2 :=$

$T_{-2,2}(-\overline{s_1s_2t}y_3\,(y_1)^{-1}s_1s_2t) \in U_2$. Since $y_3(y_1)^{-1} \notin \Gamma_m$, $g_2 \notin U_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$. Clearly

$$(\xi_3[\xi_1({}^{\epsilon_1}[h,g_0])\xi_2,g_1]) = g_2.$$

As above, push this equation into $U_{2n}(R_m,\Lambda_m)/U_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$ by applying ρ_m.

<u>case 2.2</u> Assume that there is an $l \in \{2,\dots,-2\}$ such that $|\tau_{*l}| \notin \Gamma_m$.
Choose a $p \in \{2,\dots,-2\}$ such that $p \neq \pm l$ and set $g_1 := T_{lp}(s_1) \in U_1$. Then $\zeta := [\tau,g_1]$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $B \in M_{2n-2}(R_m)$. Since $\tau \in U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$, $\zeta \in U_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$. By Lemma 7.8, $|\zeta_{*p}| \notin \Gamma_m$. Hence $\zeta \notin U_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$ and thus one can proceed as in case 1.

<u>case 3</u> Assume that $(y_1)^{-1}y_2 \notin \bar{\lambda}_m\Gamma_m$.
See case 2.

The inverted tree diagram below extending over several pages gives an overview of the proof just concluded of Part III.

h

$$\begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix}$$
$\notin CU_{2n}((R_m,\Lambda_m),(I_m,\Gamma_m))$,
$\in CU_{2n}((R_m,\Lambda_m),(I_m,I_m\cap\Lambda_m))$

$$\Big\downarrow \; g_0 = T_{ij}(s_0\hat{a})$$

$[h, g_0]$

$$\begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix}$$
$\equiv e(\text{mod } I_m)$,
length of one column $\notin \Gamma_m$

$$\Big\downarrow \; \epsilon_1 \in EU_{2n}(R_m,\Lambda_m)$$

${}^{\epsilon_1}[h, g_0]$

$$\left(\begin{array}{c|c} y_1 & u \\ \hline v & * \end{array}\right)$$
$\equiv e(\text{mod } I_m)$,
length of one column $\notin \Gamma_m$,
y_1 invertible

$$\Big\downarrow \; \xi_1 = \prod_{k=-2}^{2} T_{k1}(-v_k(y_1)^{-1}),\quad \xi_2 = \prod_{k=2}^{-2} T_{1k}(-(y_1)^{-1}u_k)$$

$\xi_1({}^{\epsilon_1}[h, g_0])\xi_2$

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e(\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$y_3(y_1)^{-1} \notin \Gamma_m$

2

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e(\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible

3

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e(\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$(y_1)^{-1}y_2 \notin \bar{\lambda}_m\Gamma_m$

1

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e(\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$y_3(y_1)^{-1} \in \Gamma_m, (y_1)^{-1}y_2 \in \bar{\lambda}_m\Gamma_m$

$\xi_3 = T_{-1,1}(y_3(y_1)^{-1})$,
$\xi_4 = T_{1,-1}((y_1)^{-1}y_2)$

$\xi_3\xi_1({}^{\epsilon_1}[h, g_0])\xi_2\xi_4$

$$\begin{pmatrix} y_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix} \equiv e(\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible

1.1

$$\begin{pmatrix} y_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix} \equiv e(\text{mod } I_m),$$
y_1 invertible,
$\exists l \in \{2, \ldots, n\}$:
length of l-th column $\notin \Gamma_m$

1.2

$$\begin{pmatrix} y_1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & y_5 \end{pmatrix} \equiv e(\text{mod } I_m),$$
y_1 invertible,
$\exists l \in \{-n, \ldots, -2\}$:
length of l-th column $\notin \Gamma_m$

$$g_1 = T_{l,-1}(s_1 s_2 t)$$

$[\xi_3\xi_1({}^{\epsilon_1}[h,g_0])\xi_2\xi_4, g_1]$

$$\prod_{k=2}^{-2} T_{k,-1}(x_k) T_{1,-1}(z)$$
$x_k \in I_m \ \forall k \in \{2,\dots,-2\}$,
$z \notin \bar{\lambda}_m \Gamma_m$

analog to case 1.1

$$\xi_5 = \Big(\prod_{k=2}^{-2} T_{k,-1}(x_k)\Big)^{-1}$$

$\xi_5[\xi_3\xi_1({}^{\epsilon_1}[h,g_0])\xi_2\xi_4, g_1]$

$T_{1,-1}(z)$
$z \notin \bar{\lambda}_m \Gamma_m$

2

$\xi_1({}^{\epsilon_1}[h,g_0])\xi_2$

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e (\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$y_3(y_1)^{-1} \notin \Gamma_m$

2.1

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e (\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$y_3(y_1)^{-1} \notin \Gamma_m$,
length of l-th column $\in \Gamma_m$
$\forall l \in \{2,\dots,-2\}$

2.2

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix} \equiv e (\text{mod } I_m),$$
length of one column $\notin \Gamma_m$,
y_1 invertible,
$y_3(y_1)^{-1} \notin \Gamma_m$,
$\exists l \in \{2\dots,-2\}$:
length of l-th column $\notin \Gamma_m$

$g_1 = T_{12}(s_1 s_2 t)$

$g_1 = T_{lp}(s_1)$

$[\xi_1({}^{\epsilon_1}[h, g_0])\xi_2, g_1]$

$\sigma T_{-2,2}(z)$
$\sigma \in U_{2n}((R_m, \Lambda_m), (I_m, \Gamma_m)),$
$z \notin \Gamma_m$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$\equiv e (\text{mod } I_m),$
length of p-th column $\notin \Gamma_m$

$\xi_3 = \sigma^{-1}$

$\xi_3[\xi_1({}^{\epsilon_1}[h, g_0])\xi_2, g_1]$

$T_{-2,2}(z)$
$z \notin \Gamma_m$

see case 1

3

$\xi_1({}^{\epsilon_1}[h, g_0])\xi_2$

$$\begin{pmatrix} y_1 & 0 & y_2 \\ 0 & A & 0 \\ y_3 & 0 & y_4 \end{pmatrix}$$
$\equiv e (\text{mod } I_m),$
length of one column $\notin \Gamma_m,$
y_1 invertible,
$(y_1)^{-1} y_2 \notin \bar{\lambda}_m \Gamma_m$

analog to case 2

$\square$

Theorem 7.10 *Let H be a subgroup of $U_{2n}(R, \Lambda)$. Then*

H *is normalized by* $EU_{2n}(R, \Lambda) \Leftrightarrow$
$\exists!$ *form ideal* (I, Γ) *such that* $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$.

Proof It follows from the previous lemmas of this section and from Lemma 6.16 that $(G, E, Gen(E), B(E), G(\cdot))$ (where G, E, $Gen(E)$, $B(E)$ and $G(\cdot)$ are defined as in Lemma 7.1) is a sandwich classification group. Hence we can apply Theorem 3.13 (note that if H is central, then there clearly is a unique form ideal (I, Γ), namely $(I, \Gamma) = (0, 0)$, such that $EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma))$). $\square$

Remark By [6], Corollary 3.8, any form ring (R, Λ) where R is almost commutative is the direct limit of form subrings (R_i, Λ_i) of (R, Λ) where for any i, R_i is a Noetherian C_i-module (where C_i is the subring of R_i consisting of all finite sums of elements of the form $c\bar{c}$ and $-c\bar{c}$ where $c \in Center(R_i)$). Hence the theorem above is still true if we drop the assumption that R is a Noetherian C-module and instead assume only that R is almost commutative (note that EU_{2n} and CU_{2n} commute with direct limits).

Acknowledgement

I would like to thank my supervisor Prof. Anthony Bak for his patience and continuing support. I would also like to express my gratitude to my family for their encouragement.

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