Spatial Birth-and-Death Markov Processes

Dissertation zur Erlangung des Doktorgrades

der Fakultät für Mathematik

der Universität Bielefeld

vorgelegt von

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September 2014

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Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706.

Contents

Introduction

This thesis deals with spatial birth-and-death processes. Specifically, at each moment of time the system is represented as a collection of motionless points in some metric space X. We interpret the points as particles, or individuals. Existing particles may die and new particles may appear. Each particle is characterized by its location. We consider the cases $X = \mathbb{R}^d$ or $X = \mathbb{Z}^d$. With the exception of Chapter 5, we treat models with finite number of particles in $\mathbb{R}^d$.

The state space of a spatial birth-and-death Markov process on $\mathbb{R}^d$ with finite number of points is the space of finite configurations over $\mathbb{R}^d$,

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η.

Denote by $\mathscr{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$. The evolution of a spatial birth-and-death process in $\mathbb{R}^d$ admits the following description. Two functions characterize the development in time, the birth rate coefficient $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0; \infty)$ and the death rate coefficient $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0; \infty)$. If the system is in state $\eta \in \Gamma_0(\mathbb{R}^d)$ at time t, then the probability that a new particle appears (a "birth") in a bounded set $B \in \mathscr{B}(\mathbb{R}^d)$ over time interval $[t; t + \Delta t]$ is

$$\Delta t \int\limits_B b(x, \eta) dx + o(\Delta t),$$

the probability that a particle $x \in \eta$ is deleted from the configuration (a "death") over time interval $[t; t + \Delta t]$ is

$$d(x, \eta)\Delta t + o(\Delta t),$$

and no two events happen simultaneously. By an event we mean a birth or a death. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x, \eta)dx$, the rate at which a particle $x \in \eta$ dies is $d(x, \eta)$, and no two events happen at the same time.

Such processes, in which the birth and death rates depend on the spatial structure of the system as opposed to classical $\mathbb{Z}_+$-valued birth-and-death processes (see e.g. [KM59], [CG], [Har63, Page 116], [AN72, Page 109], and references therein), were first studied by Preston in [Pre75]. A heuristic description similar to that above appeared already there. Our description resembles the one in [GK06].

The (heuristic) generator of a spatial birth-and-death process should be of the form

$$LF(\eta) = \int_{x \in \mathbb{R}^d} b(x, \eta)[F(\eta \cup x) - F(\eta)]dx + \sum_{x \in \eta} d(x, \eta)(F(\eta \setminus x) - F(\eta)), \tag{1}$$

for F in an appropriate domain, where $\eta \cup x$ and $\eta \setminus x$ are shorthands for $\eta \cup \{x\}$ and $\eta \setminus \{x\}$, respectively.

Spatial point processes have been used in statistics for simulation purposes, see e.g. [MS94], [MW04, chapter 11] and references therein. For application of spatial and stochastic models in biology see e.g. [Lev03], [FOK$^+$14], and references therein.

To construct a spatial birth-and-death process with given birth and death rate coefficients, we consider in Chapter 2 stochastic equations with Poisson type noise

$$
\begin{aligned}
\eta_t(B) = & \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) \\
- & \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v)
\end{aligned}
\tag{2}
$$

where $(\eta_t)_{t\geq 0}$ is a suitable $\Gamma_0(\mathbb{R}^d)$-valued cadlag stochastic process, the "solution" of the equation, I_A is the indicator function of the set A, $B \in \mathscr{B}(\mathbb{R}^d)$ is a Borel set, N_1 is a Poisson point processes on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \times ds \times du$, N_2 is a Poisson point process on $\mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times dr \times dv$, $\#$ is the counting measure on $\mathbb{Z}^d$, η_0 is a (random) initial finite configuration, $b, d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ are functions that are measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\Gamma_0(\mathbb{R}))$ and $\{x_i\}$ is some collection of points satisfying $\eta_s \subset \{x_i\}$ for every moment of time s (the precise definition is given in Section 1.3.1). We require the processes N_1, N_2, η_0 to be independent of each other. Equation (2) is understood in the sense that the equality holds a.s. for all bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq 0$.

Garcia and Kurtz studied in [GK06] equations similar to (2) for infinite systems. In the earlier work [Gar95] of Garcia another approach was used: birth-and-death processes were obtained as projections of Poisson point processes. A further development of the projection method appears in [GK08]. Fournier and Meleard in [FM04] considered a similar equation for the construction of the Bolker-Pacala-Dieckmann-Law process with finitely many particles.

Holley and Stroock [HS78] constructed a spatial birth-and-death process as a Markov family of unique solutions to the corresponding martingale problem. For the most part, they consider a process contained in a bounded volume, with bounded birth and death rate coefficients. They also proved the corresponding result for the nearest neighbor model in $\mathbb{R}^1$ with an infinite number of particles.

Kondratiev and Skorokhod [KS06] constructed a contact process in continuum, with the infinite number of particles. The contact process can be described as a spatial birth-and-death process with

$$b(x, \eta) = \lambda \sum_{y \in \eta} a(x-y), \quad d(x, \eta) \equiv 1,$$

where $\lambda > 0$ and $0 \leq a \in L^1(\mathbb{R}^d)$. Under some additional assumptions, they showed existence of the process for a broad class of initial conditions. Furthermore, if the value of some energy functional on the initial condition is finite, then it stays finite at any point in time.

In the aforementioned references as well as in the present work the evolution of the system in time via Markov process is described. An alternative approach consists in using the concept of statistical dynamics that substitutes the notion of a Markov stochastic process. This approach is based on considering evolutions of measures and their correlation functions. For details see e.g. [FKK12a], [FKK14], and references therein.

There is an enormous amount of literature concerning interacting particle systems on lattices and related topics (e.g., [Lig85], [Lig04], [KL99], [Ald13], [Fra14], [Spi77], etc.) Penrose in [Pen08] gives a general existence result for interacting particle systems on a lattice with local interactions and bounded jump rates (see also [Lig85, Chapter 9]). The spin space is allowed to be non-compact, which gives the opportunity to incorporate spatial birth-and-death processes in continuum. Unfortunately, the assumptions become rather restrictive when applied to continuous space models. More specifically, the birth rate coefficient should be bounded, and for every bounded Borel set B the expression

$$\sum_{x \in \eta \cap B} d(x, \eta)$$

should be bounded uniformly in η, $\eta \in \Gamma(\mathbb{R}^d)$.

Finkelshtein, Kondratiev, Kutoviy and Zhizhina [FKKZ14] consider different aspects of statistical dynamics for the aggregation model. In this model the death rate coefficient is given by

$$d(x,\eta) = \exp\big(-\sum_{y\in\eta\setminus x} \phi(x-y)\big),$$

where ϕ is a positive measurable function. For more details see [FKKZ14] and references therein. In Chapter 4 we consider the corresponding microscopic dynamics. Namely, we show that we can construct the Markov process using Theorem 2.1.16 about the existence and uniqueness of solution to equation (2). We give results about the pathwise, or microscopic, behavior in some bounded region. Also, we estimate the probability of extinction and the speed of growth of the average number of points.

Let us briefly describe the contents of the thesis.

In Chapter 1 we introduce general notions, definitions and results used in other chapters. We start with configuration spaces, which are the state spaces for birth-and-death processes, then we introduce and discuss metrical and topological structures thereof. Also, we present some facts and constructions from probability theory, such as integration with respect to a Poisson point process, or a sufficient condition for a functional transformation of a Markov chain to be a Markov chain again.

In the second chapter we construct a spatial birth-and-death process $(\eta_t)_{t\geq 0}$ as a unique solution to equation (2). We prove strong existence and pathwise uniqueness for (2). A key condition is that we require b to grow not faster than linearly in the sense that

$$\int\limits_{\mathbb{R}^d} b(x,\eta)dx \leq c_1|\eta| + c_2. \tag{3}$$

The equation is solved pathwisely, "from one jump to another". Also, we prove uniqueness in law for equation 2 and the Markov property for the unique solution. Considering (2) with a (non-random) initial condition $\alpha \in \Gamma_0(\mathbb{R}^d)$ and denoting corresponding solution by $(\eta(\alpha,t))_{t\geq 0}$, we see that a unique solution induces a Markov family of probability measures on the Skorokhod space $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)$ (which can be regarded as the canonical space for a solution of (2)).

When birth and death rate coefficients b and d satisfy some continuity assumptions, the solution is expected to have continuous dependence on the initial condition, at least in some proper sense. Realization of this idea and precise formulations are given in Section 2.1.2. The proof is based on considering a coupling of two birth-and-death processes.

The formal relation of a unique solution to (2) and operator L in (1) is given via the martingale problem, in Section 2.1.2, and via some kind of a pointwise convergence, in Section 2.1.5.

If (3) is not fulfilled, we can not rule out the possibility of an explosion. This is the subject of Section 2.2. We show that, indeed, an explosion can occur, and, on the other hand, equation (2) may have a unique solution on $[0;\infty)$ even if (3) is not fulfilled. For that to happen, the death rate coefficient d has to dominate b in some sense.

Several times throughout the thesis we couple a spatial birth-and-death process with another, in some way more convenient for analysis process. The idea to compare a spatial birth-and-death process with some "simpler" process goes back to Preston, [Pre75]. In [FM04] this technique was applied to the study of the probability of extinction. We formulate and prove a theorem about coupling of two birth-and-death processes in Section 2.1.4.

In Chapter 3, we apply the general theory of stochastic stability of Markov chains to $(\eta(\alpha,t))_{t\geq 0}$, the unique solution of (2). The process $(\eta(\alpha,t))_{t\geq 0}$ is of pure jump type, therefore many questions about $(\eta(\alpha,t))_{t\geq 0}$ may be reduced to those about the embedded chain of $(\eta_t)_{t\geq 0}$. In Section 3.1 we list general definitions and facts of stochastic stability we use in the sequel. The main reference to this part is the book by Meyn and Tweedie, [MT93].

The main result of Chapter 3 is the theorem about ψ-irreducibility of the embedded chain of $(\eta(\alpha,t))_{t\geq 0}$. It turns out that $(\eta(\alpha,t))_{t\geq 0}$ will hit set $A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$ with positive probability whichever initial condition we take if, and only if, A is of positive Lebesgue-Poisson measure. Formally,

$$\big(\forall \alpha : P\{(\eta(\alpha,t))_{t\geq 0} \text{ ever enters } A\} > 0\big) \Leftrightarrow \lambda(A) > 0.$$

Based on this theorem and on general recurrence and transience criteria given in [MT93], we give sufficient conditions for a birth-and-death process to be recurrent or transient. In Section 3.3, we discuss recurrence and transience for two specific models: the the Bolker-Pacala process (see e.g. [BP97], [BP99], [DL05], [FOK$^+$14], [FKKK]) and some asymmetric dispersion process.

Chapter 4 is devoted to the aggregation process with finite number of particles. Making certain assumptions on the behavior of the coefficients in some fixed set $\Lambda \in \mathscr{B}(\mathbb{R}^d)$, we obtain results about the asymptotic behavior of the process in this region. In particular, we estimate the probability of extinction, prove that only finitely many deaths may occur and estimate pathwisely the number of points in Λ.

Chapter 5 is devoted to infinite systems. A general result about a cadlag process in $\Gamma(\mathbb{R}^d)$ is given in Section 5.1. In Section 5.2 we consider a birth-and-death process on a lattice. We prove an existence and uniqueness result for the equation

$$\begin{aligned}\omega_t(i) = &\int\limits_{(0;t]\times[0;\infty]} I_{[0;b(i,\omega_{s-})]}(u)dN_1(i,s,u)\\ &- \int\limits_{[0;t]\times[0;\infty]} I_{[0;d(i,\omega_{r-})]}(v)dN_2(i,r,v) + \omega_0(i),\end{aligned} \tag{4}$$

where $i \in \mathbb{Z}^d$, ω_t is a cadlag $\mathbb{Z}_+^{\mathbb{Z}^d}$-valued process, the "solution" of the equation, N_1, N_2 are Poisson point processes on $\mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times ds \times du$, ω_0 is a (random) initial configuration, b, d are birth and death coefficients given below. We require processes N_1, N_2, ω_0 to be independent of each other. Equation (4) is understood in the sense that the equality holds a.s. for all $i \in \mathbb{Z}^d$ and $t \in (0;T]$.

We assume that the initial condition ω_0 satisfies $\sum\limits_{i\in\mathbb{Z}^d} e^{-|i|_1} E\omega_0(i) < \infty$, where $|i|_1 = |i_1| + ... + |i_d|$, $i \in \mathbb{Z}^d$. We consider a special case of (4), $b(i,\omega) = A \sum\limits_{j:j\smile i} \omega(j)$, where $j \smile i$ means that $|i-j| \leq 1$, and $d(i,\omega) = \omega^2(i)$.

The (heuristic) generator of the solution process is

$$L_1 F(\omega) = \sum_{i\in\mathbb{Z}^d} b(i,\omega)[F(\omega_i^+) - F(\omega)] + \sum_{i\in\mathbb{Z}^d} d(i,\omega)[F(\omega_i^-) - F(\omega)], \tag{5}$$

for an appropriate class of functions, where

$$\omega_i^+(j) = \begin{cases} \omega(j), & \text{if } j \neq i, \\ \omega(j)+1, & \text{if } j = i, \end{cases} \qquad \omega_i^-(j) = \begin{cases} \omega(j), & \text{if } j \neq i, \\ \omega(j)-1, & \text{if } j = i. \end{cases}$$

We can regard L_1 in (5) as a generator describing an interacting particle system on $\mathbb{Z}_+^{\mathbb{Z}^d}$. Note that the spin space $\mathbb{Z}_+$ is non-compact. If the system is in state $\omega \in \mathbb{Z}_+^{\mathbb{Z}^d}$ and $\omega(i) = m$, $i \in \mathbb{Z}^d$, then $\omega(i)$ flips to $m+1$ at the rate $b(i,\omega)$ and $\omega(i)$ flips to $m-1$ at the rate $d(i,\omega)$. We see that the flip rates are unbounded. Thus, the unique solution of (4) gives an example of an interacting particle system with a non-compact spin space and unbounded flip rates. To the best of our knowledge, this is the first example of a construction of an interacting particle system of such a class. For an example of a system on $\{0,1\}^{\mathbb{Z}^d}$ with unbounded flip rates see Meester [Mee00]. Boldrighini, De Masi, Pellegrinotti and Presutti [BDMPP87] considered a scaling limit of an interacting particle system with unbounded rates. We note that the system we consider does not belong to the well-studied class of zero-range processes (see e.g. [EH05], [And82], [Spi70], and references therein).

The interaction in the system is produced by the birth rate coefficient b, whose value at $i \in \mathbb{Z}^d$ depends on the values of $\omega \in \mathbb{Z}_+^{\mathbb{Z}^d}$ in the neighboring sites of i. The death rate coefficient is local, but “non-linear”. In Theorem 5.2.3 we prove the existence and uniqueness of solution. We deal with “quadratic” death rate $d(i, \omega) = \omega^2(i)$, but examination of the proof shows that we can take $d(i, \omega) = \varphi(\omega(i))$ for any non-decreasing map $\varphi : \mathbb{Z}_+ \to \mathbb{R}_+$.

Proving the uniqueness of solution represents a serious obstacle in the analysis of equations of type (4). We manage to get uniqueness by combining the “Lipschitz” property of the birth rate coefficient and some kind of monotonicity present in the system. The corresponding result is given in Theorem 5.2.3.

Acknowledgements

I would like to express my appreciation and thanks to my advisor, Prof. Dr. Yuri Kondratiev, for the support and advice he has provided during the last three years. I have been lucky to have an advisor who guided me and gave me enough freedom at the same time, and whose enthusiasm has been a constant source of inspiration for me.

I want to thank Dr. Dmitri Finkelshtein for his interest in my work, for the discussions we have had and for the numerous suggestions which greatly contributed to improve the thesis. I would also like to thank the members of the Collaborative Research Center (CRC) project A5 for close interaction over the last years. Diana and Mykola proofread the manuscript. I am indebted to them for their help. I would like to thank Prof. Dr. Sergey Pirogov for agreeing to be one of the referees of the thesis.

I would like to express my gratitude to the IGK, German Science Foundation (DFG), Graduate University of Chinese Academy of Sciences (GUCAS) and CRC 701 "Spectral Structures and Topological Methods in Mathematics". I deeply appreciate the financial support as well as the opportunity to attend a great many excellent courses and conferences that took place in Bielefeld. I would like to thank Prof. Dr. Philippe Blanchard, Prof. Dr. Moritz Kaßmann, Prof. Dr. Yuri Kondratiev, Prof. Dr. Michael Röckner and Prof. Dr. Ludwig Streit and all the visiting professors for the insightful lectures.

Throughout my PhD scholarship I did not have to focus much on the technical issues because they were quickly resolved with the help of the IGK staff. My thanks go to Stephan Merkes, Rebecca Reischuk and Karin Zelmer.

I would like to thank all my colleagues and friends with whom I had interesting discussions and a good time.

I want to thank my mother for her love, encouragement and unwavering belief in me.

Chapter 1

Preliminaries

In this chapter we list some notions and facts we use in this work.

1.1 Some notations and conventions

Sometimes we write ∞ and $+\infty$ interchangeably, so that $f \to \infty$ and $f \to +\infty$, or $a < \infty$ and $a < +\infty$ may have the same meaning. However, $+\infty$ is reserved for the real line only, whereas ∞ have wider range of applications, e.g. for a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ we may write $x_n \to \infty$, $n \to \infty$, which is equivalent to $|x_n| \to +\infty$. On the other hand, we do not assign any meaning to $x_n \to +\infty$.

In all probabilistic constructions we work on some probability space $(\Omega, \mathscr{F}, P)$, sometimes equipped with a filtration of σ-algebras. Elements of Ω are usually denoted as ω.

The set A^c is the complement of the set $A \subset \Omega$: $A^c = \Omega \setminus A$. We write $[a; b]$, $[a; b)$ etc. for the intervals of real numbers. For example, $(a; b] = \{x \in \mathbb{R} \mid a < x \leq b\}$, $-\infty \leq a < b \leq +\infty$. The half line $\mathbb{R}_+$ includes 0: $\mathbb{R}_+ = [0; \infty)$.

1.2 Configuration spaces

We consider a spatial birth-and-death process as a Markov processes whose state space is some space of configurations. Thus, spaces of configurations play an important role in this thesis. In this section we introduce notions and facts about them that will be used in this thesis, in particular, topological and metrical structures on $\Gamma(\mathbb{R}^d)$ as well as a characterization of compact sets of $\Gamma(\mathbb{R}^d)$. We discuss configurations over Euclidean spaces only.

1.2.1 Definition. For $d \in \mathbb{N}$ and a measurable set $\Lambda \subset \mathbb{R}^d$, the configuration space $\Gamma(\Lambda)$ is defined as

$$\Gamma(\Lambda) = \{\gamma \subset \Lambda : |\gamma \cap K| < +\infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

We recall that $|A|$ denotes the number of elements of A. We also say that $\Gamma(\Lambda)$ is the space of configurations over Λ. Note that $\varnothing \in \Gamma(\Lambda)$.

Let $\mathbb{Z}_+$ be the set $\{0, 1, 2, ...\}$. We say that a Radon measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is a *counting measure* on $\mathbb{R}^d$ if $\mu(A) \in \mathbb{Z}_+$ for all $A \in \mathscr{B}(\mathbb{R}^d)$. When a counting measure ν satisfies additionally $\nu(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$, we call it a *simple counting measure.*

As long as it does not lead to ambiguities, we identify a configuration with a simple counting Radon measures on $\mathbb{R}^d$: as a measure, a configuration $\gamma \in \Gamma(\mathbb{R}^d)$ maps a set $B \in \mathscr{B}$ into $|\gamma \cap B|$. In other words, $\gamma = \sum\limits_{x\in\gamma} \delta_x$.

One equips $\Gamma(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology such that for all $f \in C_c(\mathbb{R}^d)$ (the set of continuous functions on $\mathbb{R}^d$ with compact support) the map

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto \langle \gamma, f \rangle := \sum_{x\in\gamma} f(x) \in \mathbb{R}$$

is continuous.

Equipped with this topology, $\Gamma(\mathbb{R}^d)$ is a Polish space, i.e., there exists a metric on

$\Gamma(\mathbb{R}^d)$ compatible with the vague topology and with respect to which $\Gamma(\mathbb{R}^d)$ is a complete separable metric space, see, e.g., [KK06], and references therein. We say that a metric is compatible with a given topology if the topology induced by the metric coincides with the given topology.

For a bounded $B \subset \mathbb{R}^d$ and $\gamma \in \Gamma(\mathbb{R}^d)$, we denote $\delta(\gamma, B) = min\{|x-y| : x, y \in \gamma \cap B, x \neq y\}$. Let $B_r(x)$ denote the closed ball in $\mathbb{R}^d$ of the radius r centered at x.

A set is said to be *relatively compact* if its closure is compact. The following theorem gives a characterization of compact sets in $\Gamma(\mathbb{R}^d)$, cf. [KK06], [HS78].

1.2.2 Theorem. *A set $F \subset \Gamma(\mathbb{R}^d)$ is relatively compact in the vague topology if and only if*

$$\sup_{\gamma \in F}\{\gamma(B_n(0)) + \delta^{-1}(\gamma, B_n(0))\} < \infty \tag{1.1}$$

holds for all $n \in \mathbb{N}$.

Proof. Assume that (1.1) is satisfied for some $F \subset \Gamma(\mathbb{R}^d)$. In metric spaces compactness is equivalent to sequential compactness, therefore it is sufficient to show that an arbitrary sequence contains a convergent subsequence in $\Gamma(\mathbb{R}^d)$. To this end, consider an arbitrary sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset F$. The supremum $\sup\limits_n \gamma_n(B_1(0))$ is finite, consequently, by the Banach–Alaoglu theorem there exists a measure $\alpha_1 \in C(B_1(0))^*$ $\Big($here $C(B_1(0))^*$ is the dual space of $C(B_1(0))\Big)$ and a subsequence $\{\gamma_n^{(1)}\} \subset \{\gamma_n\}$ such that $\gamma_n^{(1)}|_{B_1(0)} \to \alpha_1$ in $C(B_1(0))^*$. Furthermore, one may see that $\alpha_1 \in \Gamma(B_1(0))$ (it is particularly important here that $\sup\limits_{\gamma\in F}\{\delta^{-1}(\gamma, B_1(0))\} < \infty$). Indeed, arguing by contradiction one may get that $\alpha_1(A) \in \mathbb{Z}_+$ for all Borel sets A, and Lemma 1.2.5 below ensures that α_1 is a simple counting measure.

Similarly, from the sequence $\gamma_n^{(1)}$ we may extract subsequence $\{\gamma_n^{(2)}\} \subset \{\gamma_n^{(1)}\}$ in such a way that $\gamma_n^{(2)}$ converges to some $\alpha_2 \in \Gamma(B_2(0))$. Continuing in the same way, we will find a sequence of sequences $\{\gamma_n^{(m)}\}$ such that $\gamma_n^{(m)} \to \alpha_m \in \Gamma(B_m(0))$ and $\{\gamma_n^{(m+1)}\} \subset \{\gamma_n^{(m)}\}$. Consider now the sequence $\{\gamma_n^{(n)}\}_{n\in\mathbb{N}}$. For any m, restrictions of its elements to $B_m(0)$ converge to α_m in $\Gamma(B_m(0))$, Therefore, $\gamma_n^{(n)} \to \alpha$ in $\Gamma(\mathbb{R}^d)$, where $\alpha = \bigcup\limits_n \alpha_n$.

Conversely, if (1.1) is not fulfilled for some $n_0 \in \mathbb{N}$, then we can construct a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset F$ such that either the first summand in (1.1) tends to infinity:

$$\gamma_j(B_{n_0}(0)) \to \infty, j \to \infty$$

in which case, of course, there is no convergent subsequence, or the second summand in (1.1) tends to infinity. In the latter case, a subsequence of the sequence $\{\gamma_n|_{B_{n_0}(0)}\}_{n\in\mathbb{N}}$ may converge to a counting measure (when all γ_n are considered as measures). However, the limit measure can not be a simple counting measure. Thus, the sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset F$ does not contain a convergent subsequence in $\Gamma(\mathbb{R}^d)$. $\square$

We denote by $CS(\Gamma(\mathbb{R}^d))$ the space of all compact subsets of $\Gamma(\mathbb{R}^d)$.

1.2.3 Proposition. *The topological space $\Gamma(\mathbb{R}^d)$ is not σ - compact.*

Proof. Let $\{K_m\}_{n\in\mathbb{N}}$ be an arbitrary sequence from $CS(\Gamma(\mathbb{R}^d))$. We will show that $\bigcup\limits_n K_n \neq \Gamma(\mathbb{R}^d)$. To each compact K_m we may assign a sequence $q_1^{(m)}, q_2^{(m)}, \ldots$ of positive numbers such that

$$\sup_{\gamma\in K_m} \{\gamma(B_n(0)) + \delta^{-1}(\gamma, B_n(0))\} < q_n^{(m)}.$$

There exists a configuration whose intersection with $B_n(0)$ contains at least $q_n^{(n)} + 1$ points, for each $n \in \mathbb{N}$. This configuration does not belong to any of the sets $\{K_m\}_{m\in\mathbb{N}}$, hence it can not belong to the union $\bigcup\limits_m K_m$. $\square$

Remark. Since $\Gamma(\mathbb{R}^d)$ is a separable metrizable space, Proposition 1.2.3 implies that $\Gamma(\mathbb{R}^d)$ is *not locally compact.*

For another description of all compact sets in $\Gamma(\mathbb{R}^d)$ we will use the set $\Phi \subset C(\mathbb{R}^d)$ of all positive continuous functions ϕ satisfying the following conditions:

1) $\phi(x) = \phi(y)$ whenever $|x| = |y|$, $x, y \in \mathbb{R}^d$,

2) $\lim_{|x|\to\infty} \phi(x) = 0$.

For $\phi \in \Phi$ we denote

$$\Psi = \Psi_\phi(x, y) := \phi(x)\phi(y)\frac{|x-y|+1}{|x-y|} I\{x \neq y\}.$$

1.2.4 Proposition. *(i) For all $c > 0$ and $\phi \in \Phi$*

$$K_c := \left\{\gamma : \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_\phi(x, y)\gamma(dx)\gamma(dy) \leqslant c\right\} \in CS(\Gamma(\mathbb{R}^d));$$

(ii) For all $K \in CS(\Gamma(\mathbb{R}^d))$ there exist $\phi \in \Phi$ such that

$$\sup_{\gamma\in K} \{\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_\phi(x, y)\gamma(dx)\gamma(dy)\} \leqslant 1.$$

Proof. (i) Denote $\theta_n = \min\limits_{x\in B_n(0)} \phi(x) > 0$.

For $\gamma \in K_c$ we have

$$c \geqslant \iint\limits_{B_n(0)\times B_n(0)} \Psi(x, y)\gamma(dx)\gamma(dy)$$

$$\geqslant \iint\limits_{B_n(0)\times B_n(0)} \phi(x)\phi(y)I\{x \neq y\}\gamma(dx)\gamma(dy) \geqslant \theta_n^2\gamma(B_n(0))(\gamma(B_n(0)) - 1)$$

and

$$c \geqslant \iint\limits_{B_n(0)\times B_n(0)} \Psi(x, y)\gamma(dx)\gamma(dy) \geqslant \theta_n^2\frac{\delta^{-1}(\gamma, B_n(0)) + 1}{\delta^{-1}(\gamma, B_n(0))} \geqslant \theta_n^2\delta^{-1}(\gamma, B_n(0)).$$

Consequently,

$$\sup_{\gamma\in K_c} \gamma(B_n(0)) \leqslant \theta_n\sqrt{c} + 1,$$

and

$$\sup_{\gamma\in K_c} \delta^{-1}(\gamma, B_n(0)) \leqslant \frac{c}{\theta_n^2}.$$

It remains to show that K_c is closed, in which case Theorem 1.2.2 will imply compactness of K_c. The space $\Gamma(\mathbb{R}^d)$ is metrizable, therefore sequential closedness will suffice. Take $\gamma_k \in K_c$, $\gamma_k \to \gamma$ in $\Gamma(\mathbb{R}^d)$, $k \to \infty$. For $n \in \mathbb{N}$, let $\Psi_n \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ be an increasing sequence of functions such that $\Psi_n \leqslant \Psi$, $\Psi_n(x, y) = \Psi(x, y)$ for $x, y \in \mathbb{R}^d$ satisfying $|x|, |y| \leq n$, $|x - y| \geq \frac{1}{n}$. For such a sequence we have $\Psi_n(x, y) \uparrow \Psi(x, y)$ for all $x, y \in \mathbb{R}^d$, $x \neq y$. For each $f \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$, the map

$$\eta \mapsto \langle \eta \times \eta, f\rangle := \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f(x, y)\eta(dx)\eta(dy)$$

is continuous in the vague topology. Thus for all $n \in \mathbb{N}$, $\langle\gamma_k \times \gamma_k, \Psi_n\rangle \to \langle\gamma \times \gamma, \Psi_n\rangle$. Consequently, $\langle\gamma \times \gamma, \Psi_n\rangle \leq c$, $n \in \mathbb{N}$, and by Fatou's Lemma

$$\langle\gamma \times \gamma, \Psi\rangle = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x, y)\gamma(dx)\gamma(dy) =$$

$$= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \liminf_n \Psi_n(x, y)\gamma(dx)\gamma(dy) \leq \liminf_n \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_n(x, y)\gamma(dx)\gamma(dy) \leq c.$$

To prove (ii), for a given compact set $K \subset \Gamma(\mathbb{R}^d)$ and a given function $\phi \in \Phi$, denote

$$a_n(K) := \sup_{\gamma\in K}\{\gamma(B_n(0)) + \delta^{-1}(\gamma, B_n(0))\}$$

and

$$b_n(\phi) := \sup_{|x|>n} |\phi(x)|.$$

Theorem 1.2.2 implies $a_n(K) < \infty$, and we can estimate

$$\iint\limits_{\big(B_{n+1}(0)\setminus B_n(0)\big)\times\big(B_{n+1}(0)\setminus B_n(0)\big)} \Psi(x,y)\gamma(dx)\gamma(dy) =$$

$$= \iint\limits_{\big(B_{n+1}(0)\setminus B_n(0)\big)\times\big(B_{n+1}(0)\setminus B_n(0)\big)} \phi(x)\phi(y)\frac{|x-y|+1}{|x-y|}I\{x\neq y\}\gamma(dx)\gamma(dy) \leqslant$$

$$\leqslant \iint\limits_{\big(B_{n+1}(0)\setminus B_n(0)\big)\times\big(B_{n+1}(0)\setminus B_n(0)\big)} b_n^2(a_n+1)\gamma(dx)\gamma(dy) \leqslant b_n^2(a_n+1)^3.$$

Taking a function $\phi \in \Phi$ such that

$$3b_n^2(\phi)(a_n+1)^3 < \frac{6}{\pi^2}\frac{1}{(n+1)^2},$$

we get

$$\sup_{\gamma\in K}\{\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x,y)\gamma(dx)\gamma(dy)\} \leqslant 1.\square$$

1.2.1 The space of finite configurations $\Gamma_0(\mathbb{R}^d)$

For $\Lambda \subset \mathbb{R}^d$, the space $\Gamma_0(\Lambda)$ is defined as

$$\Gamma_0(\Lambda) := \{\eta \subset \Lambda : |\eta| < \infty\}.$$

We see that $\Gamma_0(\Lambda)$ is the collection of all finite subsets of Λ. We denote the space of n-point configurations as $\Gamma_0^{(n)}(\Lambda)$:

$$\Gamma_0^{(n)}(\Lambda) := \{\eta \in \Gamma_0(\Lambda) \mid |\eta| = n\}, \quad n \in \mathbb{N},$$

and $\Gamma_0^{(0)}(\Lambda) := \{\varnothing\}$. Sometimes we will write Γ_0 instead of $\Gamma_0(\mathbb{R}^d)$. Recall that we occasionally write $\eta \setminus x$ instead of $\eta \setminus \{x\}$, $\eta \cup x$ instead of $\eta \cup \{x\}$.

To define a topological structure on $\Gamma_0(\mathbb{R}^d)$, we introduce the following surjections (see, e.g., [KK02] and references therein)

$$
\begin{aligned}
sym : \bigsqcup_{n=0}^{\infty} \widetilde{(\mathbb{R}^d)^n} &\to \Gamma_0(\mathbb{R}^d) \\
sym((x_1, ..., x_n)) &= \{x_1, ..., x_n\},
\end{aligned}
\tag{1.2}
$$

where

$$
\widetilde{(\mathbb{R}^d)^n} := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^n \mid x_j \in \mathbb{R}^d, j = 1, ..., n, x_i \neq x_j, i \neq j\}, \tag{1.3}
$$

and, by convention, $\widetilde{(\mathbb{R}^d)^0} = \{\varnothing\}$.

The map sym produces a one-to-one correspondence between $\Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 1$, and the quotient space $\widetilde{(\mathbb{R}^d)^n}/\sim_n$, where $\sim_n$ is the equivalence relation on $(\mathbb{R}^d)^n$,

$$
(x_1, ..., x_n) \sim_n (y_1, ..., y_n)
$$

when there exist a permutation $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ such that

$$
(x_{\sigma(1)}, ..., x_{\sigma(n)}) = (y_1, ..., y_n).
$$

We endow $\Gamma_0^{(n)}(\mathbb{R}^d)$ with the topology induced by this one-to-one correspondence. Equivalently, a set $A \subset \Gamma_0^{(n)}(\mathbb{R}^d)$ is open iff $sym^{-1}(A)$ is open in $\widetilde{(\mathbb{R}^d)^n}$. The space $\widetilde{(\mathbb{R}^d)^n} \subset (\mathbb{R}^d)^n$ we consider, of course, with the relative, or subspace, topology. As far as $\Gamma_0^{(0)}(\mathbb{R}^d) = \{\varnothing\}$ is concerned, we regard it as an open set.

Having defined topological structures on $\Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 0$, we endow $\Gamma_0(\mathbb{R}^d)$ with the topology of disjoint union,

$$
\Gamma_0(\mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}(\mathbb{R}^d). \tag{1.4}
$$

In this topology, a set $K \subset \Gamma_0(\mathbb{R}^d)$ is compact iff $K \subset \bigsqcup_{n=0}^{N} \Gamma_0^{(n)}(\mathbb{R}^d)$ for some $N \in \mathbb{N}$ and for each $n \le N$ the set $K \cap \Gamma_0^{(n)}(\mathbb{R}^d)$ is compact in $\Gamma_0^{(n)}(\mathbb{R}^d)$. A set $K_n \subset \Gamma_0^{(n)}(\mathbb{R}^d)$ is compact iff $sym^{-1}(K_n)$ is compact in $\widetilde{(\mathbb{R}^d)^n}$. We note that in order for K_n to be compact, the set $sym^{-1}K_n$, regarded as a subset of $(\mathbb{R}^d)^n$, should not have limit points on the diagonals, i.e. limit points from the set $(\mathbb{R}^d)^n \setminus \widetilde{(\mathbb{R}^d)^n}$.

Let us introduce a metric compatible with the described topology on $\Gamma_0(\mathbb{R}^d)$. We set

$$dist(\zeta,\eta) := \begin{cases} 1 \wedge d_{Eucl}(\zeta,\eta), & |\zeta| = |\eta|, \\ 1, & \text{otherwise.} \end{cases}$$

Here $d_{Eucl}(\zeta,\eta)$ is the metric induced by the Euclidean metric and the map sym:

$$d_{Eucl}(\zeta,\eta) = \inf\{|x-y| : x \in sym^{-1}\zeta, y \in sym^{-1}\eta\}, \tag{1.5}$$

where $|x-y|$ is the Euclidean distance between x and y, $sym^{-1}\eta = sym^{-1}(\{\eta\})$. In many aspects, this metric resembles the Wasserstein type distance in [RS99]. The differences are, $dist$ is bounded by 1 and it is defined on $\Gamma_0(\mathbb{R}^d)$ only.

Note that the metric $dist$ satisfies equalities

$$dist(\zeta \cup x, \eta \cup x) = dist(\zeta,\eta) \tag{1.6}$$

for $\zeta,\eta \in \Gamma_0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $x \notin \zeta,\eta$, and

$$dist(\zeta \setminus x, \eta \setminus x) = dist(\zeta,\eta), \tag{1.7}$$

$x \in \zeta,\eta$. We note that the space $\Gamma_0(\mathbb{R}^d)$ equipped with this metric is not complete. Nevertheless, $\Gamma_0(\mathbb{R}^d)$ is a Polish space, i.e., $\Gamma_0(\mathbb{R}^d)$ is separable and there exists a metric $\tilde{\rho}$ which induces the same topology as $dist$ does and such that $\Gamma_0(\mathbb{R}^d)$ equipped with $\tilde{\rho}$

is a complete metric space. To prove this, we embed $\Gamma_0^{(n)}(\mathbb{R}^d)$ into the space $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ of n-point multiple configurations, which we define as the space of all counting measures η on $\mathbb{R}^d$ with $\eta(\mathbb{R}^d) = n$. Abusing notation, we may represent each $\eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ as a set $\{x_1, ..., x_n\}$, where some points among $x_j \in \mathbb{R}^d$ may be equal (recall our convention on identifying a configuration with a measure; as a measure, $\eta = \sum_{j=1}^{n} \delta_{x_j}$). One should keep in mind that $\{x_1, ..., x_n\}$ is not really a set here, since it is possible that $x_i = x_j$ for $i \neq j$, $i, j \in \{1, ..., n\}$. The representation allows us to extend sym to the map

$$\begin{gathered}\overline{sym} : \bigsqcup_{m=0}^{\infty} (\mathbb{R}^d)^n \to \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \\ \overline{sym}((x_1, ..., x_n)) := \{x_1, ..., x_n\},\end{gathered} \tag{1.8}$$

and define a metric on $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$: for $\zeta, \eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ we set $\overline{dist}(\zeta, \eta) = 1 \wedge \overline{d_{Eucl}}(\zeta, \eta)$, $\overline{d_{Eucl}}(\zeta, \eta)$ is the metric induced by the Euclidean metric and the map $\overline{sym}$:

$$\overline{d_{Eucl}}(\zeta, \eta) = \inf\{|x - y| : x \in \overline{sym}^{-1}\zeta, y \in \overline{sym}^{-1}\eta\}, \tag{1.9}$$

The metrics $dist$ and $\overline{dist}$ coincide on $\Gamma_0^{(n)}(\mathbb{R}^d) \times \Gamma_0^{(n)}(\mathbb{R}^d)$ (as functions). Furthermore, one can see that $(\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d), \overline{dist})$ is a complete separable metric space, and thus a Polish space. The next lemma describes convergence in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ (compare with Lemma 3.3 in [KK06]).

1.2.5 Lemma. *Assume that $\eta^m \to \eta$ in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$, and let $\eta = \{x_1, ..., x_n\}$. Then η^m, $m \in \mathbb{N}$, may be numbered, $\eta^m = \{x_1^m, ..., x_n^m\}$, in such a way that*

$$x_i^m \to x_i, \quad m \to \infty$$

in $\mathbb{R}^d$.

Proof. The inequality $\overline{dist}(\eta^m, \eta^m) < \varepsilon$ implies existence of a point from η^m in the ball $B_\varepsilon(x_i)$ of radius ε centered at x_i, $i \in \{1, ..., n\}$. Furthermore, in the case when x_i is a multiple point, i.e., if $x_j = x_i$ for some $j \neq i$, then there are at least as many points from η^m in $B_\varepsilon(x_i)$ as $\eta(\{x_i\})$. Observe that, for $\varepsilon < \frac{1}{2} \inf\{|x-y| : \eta(\{x\}), \eta(\{x\}) \geq 1\} \wedge 1$, we have in the previous sentence "exactly as many" instead of "at least as many", because otherwise there would not be enough points in η^m. The statement of the lemma follows by letting $\varepsilon \to 0$.

1.2.6 Lemma. $\Gamma_0(\mathbb{R}^d)$ *is a Polish space.*

Proof. Since $\Gamma_0(\mathbb{R}^d)$ is a disjoint union of countably many spaces $\Gamma_0^{(n)}(\mathbb{R}^d)$, it suffices to establish that each of them is a Polish space. To prove that $\Gamma_0^{(n)}(\mathbb{R}^d)$ is a Polish space, $n \in \mathbb{N}$, we will show that it is a countable intersection of open sets in a Polish space $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$. Then we may apply Alexandrov's theorem: any G_δ subset of a Polish space is a Polish space, see §33, VI in [Kur66].

To do so, denote by $\mathbf{B}_m$ the closed ball of radius m in $\mathbb{R}^d$, with the center at the origin. Define $F_m := \{\eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \mid \eta(\{x\}) \geq 2 \textit{ for some } x \in \mathbf{B}_m\}$ and note that

$$\Gamma_0^{(n)}(\mathbb{R}^d) = \bigcap_{m=1}^{\infty} [\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \setminus F_m]$$

Since $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ is Polish, it only remains to show that F_m is closed in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$. This is an immediate consequence of the previous lemma.

1.2.2 Lebesgue-Poisson measures

Here we define the Lebesgue-Poisson measure on $\Gamma_0(\mathbb{R}^d)$, corresponding to a non-atomic Radon measure σ on $\mathbb{R}^d$. Our prime example for σ will be the Lebesgue measure on $\mathbb{R}^d$.

For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$. The projection of this measure on $\Gamma_0^{(n)}$ via *sym* we denote by $\sigma^{(n)}$, so that

$$\sigma^{(n)}(A) = \sigma^{\otimes n}(sym^{-1}A), \qquad A \in \mathscr{B}(\Gamma_0^{(n)}).$$

On $\Gamma_0^{(0)}$ the measure $\sigma^{(0)}$ is given by $\sigma^{(0)}(\{\varnothing\}) = 1$. The *Lebesgue-Poisson measure* on $(\Gamma_0(\mathbb{R}^d), \mathscr{B}(\Gamma_0(\mathbb{R}^d)))$ is defined as

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \tag{1.10}$$

The measure λ_σ is finite iff σ is finite. We say that σ is the *intensity measure* of λ_σ.

1.2.3 Spaces $D_{\Gamma(\mathbb{R}^d)}[0;T]$ and $D_{\Gamma_0(\mathbb{R}^d)}[0;T]$

For a complete separable metric space (E, ρ) the space D_E of all cadlag E-valued functions equipped with the Skorokhod topology is a Polish space; for this statement and related definitions, see, e.g., Theorem 5.6, Chapter 3 in [EK86]. Let ρ_D be a metric on D_E compatible with the Skorokhod topology and such that (D_E, ρ_D) is a complete separable metric space. Denote by $(\mathcal{P}(D_E), \rho_p)$ the metric space of probability measures on $\mathscr{B}(D_E)$, the Borel σ - algebra of D_E, with the Prohorov metric, i.e. for $P, Q \in \mathcal{P}(D_E)$

$$\rho_p(P, Q) = \inf\{\varepsilon > 0 : P(F) \leq Q(F^\varepsilon) + \varepsilon \text{ for all } F \in \mathscr{B}(D_E)\} \tag{1.11}$$

where

$$F^\varepsilon = \{x \in D_E : \rho_D(x, F) < \varepsilon\}.$$

Then $(\mathcal{P}(D_E), \rho_p)$ is separable and complete; see, e.g., [EK86], Section 1, Chapter 3, and Theorem 1.7, Chapter 3. The Borel σ-algebra $\mathscr{B}(D_E)$ coincides with the one generated by the coordinate mappings; see Theorem 7.1, Chapter 3 in [EK86]. In this work, we mostly consider $D_{\Gamma_0(\mathbb{R}^d)}[0;T]$ and $D_{\Gamma(\mathbb{R}^d)}[0;T]$ endowed with the Skorokhod topology.

1.3 Integration with respect to Poisson point processes

We give a short introduction to the theory of integration with respect to Poisson point processes. For construction of Poisson point processes with given intensity, see e.g. [Kal02, Chapter 12], [Kin93], [RY05, Chapter 12, § 1] or [IW81, Chapter 1, § 8,9]. All definitions, constructions and statements about integration given here may be found in [IW81, Chapter 2, § 3]. See also [GS79, Chapter 1] for the theory of integration with respect to an orthogonal martingale measure.

On some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P)$, consider a Poisson point process N on $\mathbb{R}_+ \times \mathbf{X} \times \mathbb{R}_+$ with intensity measure $dt \times \beta(dx) \times du$, where $\mathbf{X} = \mathbb{R}^d$ or $\mathbf{X} = \mathbb{Z}^d$. We require the filtration $\{\mathscr{F}\}_{t\geq 0}$ to be increasing and right-continuous, and we assume that $\mathscr{F}_0$ is complete under P. We interpret the argument from the first space $\mathbb{R}_+$ as time. For $\mathbf{X} = \mathbb{R}^d$ the intensity measure β will be the Lebesgue measure on $\mathbb{R}^d$, for $\mathbf{X} = \mathbb{Z}^d$ we set $\beta = \#$, where

$$\#A = |A|, \quad A \in \mathscr{B}(\mathbb{Z}^d).$$

The Borel σ-algebra over $\mathbb{Z}^d$ is the collection of all subsets of $\mathbb{Z}^d$, i.e. $\mathscr{B}(\mathbb{Z}^d) = 2^{\mathbb{Z}^d}$. Again, as is the case with configurations, for $X = \mathbb{R}^d$ we treat a point process as a random collection of points as well as a random measure.

We say that the process N is called *compatible* with $(\mathscr{F}_t, t \geq 0)$ if N is adapted, that is, all random variables of the type $N(\bar{T}_1, U)$, $\bar{T}_1 \in \mathscr{B}([0;t])$, $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$, are $\mathscr{F}_t$-measurable, and all random variables of the type $N(t+h, U) - N(t, U)$, $h \geq 0, U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$, are independent of $\mathscr{F}_t$, $N(t,U) = N([0;t], U)$. For any $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$ with $(\beta \times l)(U) < \infty$, l is the Lebesgue measure on $\mathbb{R}^d$, the process $(N([0;t], U) - t\beta \times l(U), t \geq$

0) is a martingale (with respect to $(\mathscr{F}_t, t \geq 0)$; see [IW81, Lemma 3.1, Page 60]).

1.3.1 Definition. A process $f : \mathbb{R}_+ \times \mathbf{X} \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is *predictable*, if it is measurable with respect to the smallest σ - algebra generated by all g having the following properties:

(i) for each $t > 0$, $(x, u, \omega) \mapsto g(t, x, u, \omega))$ is $\mathscr{B}(\mathbf{X} \times \mathbb{R}_+) \times \mathscr{F}_t$ measurable;

(ii) for each (x, u, ω), the map $t \mapsto g(t, x, u, \omega))$ is left continuous.

For a predictable process $f \in L^1([0;T] \times \mathbf{X} \times \mathbb{R}_+ \times \Omega)$, $t \in [0;T]$ and $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$ we define the integral $I_t(f) = \int\limits_{[0;t]\times U} f(s, x, u, \omega) dN(s, x, u)$ as the Lebesgue-Stieltjes integral with respect to the measure N:

$$\int\limits_{[0;t]\times U} f(s, x, u, \omega) dN(s, x, u) = \sum_{s \leq t, (s,x,u) \in N} f(s, x, u, \omega).$$

This sum is well defined, since

$$E \sum_{s \leq t, (s,x,u) \in N} |f(s, x, u, \omega)| = \int\limits_{[0;t]\times U} |f(s, x, u, \omega)| ds \beta(dx) du < \infty$$

We use $dN(s, x, u)$ and $N(ds, dx, du)$ interchangeably when we integrate over all variables. The process $I_t(f)$ is right-continuous as a function of t, and adapted. Moreover, the process

$$\tilde{I}_t(f) = \int\limits_{[0;t]\times U} f(s, x, u, \omega)[dN(s, x, u) - ds\beta(dx)du]$$

is a martingale with respect to $(\mathscr{F}_t, t \geq 0)$, [IW81, Page 62]. Thus,

$$E \int\limits_{[0;t]\times U} f(s, x, u, \omega) dN(s, x, u) = E \int\limits_{[0;t]\times U} f(s, x, u, \omega) ds \beta(dx) du. \tag{1.12}$$

This equality will be used several times throughout the thesis.

1.3.2 Remark. We can extend the collection of integrands, in particular, we can define

$\int\limits_{[0;t]\times U} f(s,x,u,\omega)dN(s,x,u)$ for f satisfying

$$E \int\limits_{[0;t]\times U} (|f(s,x,u,\omega)| \wedge 1) ds\beta(dx)du < \infty.$$

However, we do not use such integrands in this work.

The Lebesgue-Stieltjes integral is defined ω-wisely and it is a function of an integrand and an integrator. As a result, we have the following statement. The sign $\overset{d}{=}$ means equality in distribution.

1.3.3 Statement. *Let M_k be Poisson point processes defined on some, possibly different, probability spaces, and let α_k be integrands, $k = 1, 2$, such that integrals $\int \alpha_k dM_k$ are well defined. If $(\alpha_1, M_1) \overset{d}{=} (\alpha_2, M_2)$, then*

$$\int \alpha_1 dM_1 \overset{d}{=} \int \alpha_2 dM_2.$$

The proof is straightforward.

1.3.1 An auxiliary construction

Let $\tilde{\#}$ be the counting measure on $[0,1]$, i.e.

$$\tilde{\#}C = |C|, \qquad C \in \mathscr{B}([0;1]).$$

The measure $\tilde{\#}$ is not σ-finite. For a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process $(\eta_t)_{t\in[0;\infty]}$, adapted to $\{\mathscr{F}_t\}_{t\in[0;\infty]}$, we would like to define integrals of the form

$$\int\limits_{\mathbb{R}^d\times[0;\infty]\times[0;\infty)} I_{\{x\in B\cap\eta_{r-}\}} f(x,r,v,\omega) d\tilde{N}_2(x,r,v) \tag{1.13}$$

where B is a bounded Borel subset of $\mathbb{R}^d$, f is a bounded predictable process and $\tilde{N}_2$ is a Poisson point process on $\mathbb{R}^d \times [0;T] \times [0;\infty)$ with intensity $\tilde{\#} \times dr \times dv$, compatible with $\{\mathscr{F}_t\}_{t\in[0;\infty]}$.

We can not hope to give a meaningful definition for an integral of the type (1.13), because of the measurability issues. For example, the map

$$\begin{aligned}
&\Omega \to \mathbb{R}, \\
&\omega \mapsto \tilde{N}_2(u(\omega), [0;1], [0;1]),
\end{aligned}$$

where u is an independent of $\tilde{N}_2$ uniformly distributed on $[0;1]$ random variable, does not have to be a random variable. Even if it were a random variable, some undesirable phenomena would appear, see, e.g., [Pod09].

To avoid this difficulty, we employ another construction. A similar approach was used in [FM04]. If we could give meaningful definition to the integrals of the type (1.13), we would expect

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d\times[0;t]\times[0;\infty)} I_{\{x\in B\cap\eta_{r-}\}} f(x,r,v,\omega) d\tilde{N}_2(x,r,v) - \\
&\int\limits_{\mathbb{R}^d\times[0;t]\times[0;\infty)} I_{\{x\in B\cap\eta_{r-}\}} f(x,r,v,\omega) \tilde{\#}(dx) dr dv
\end{aligned}$$

to be a martingale (under some conditions on f and B).

Having this in mind, consider a Poisson point process N_2 on $\mathbb{Z}\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity $\# \times dr \times dv$, defined on $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P)$ (here $\#$ denotes the counting measure on $\mathbb{Z}$. This measure is σ-finite). We require N_2 to be compatible with $\{\mathscr{F}\}_{t\geq 0}$. Let $(\eta_t)_{t\in[0,\infty]}$ be an adapted cadlag process in $\Gamma_0(\mathbb{R}^d)$, satisfying the following condition: for any $T < \infty$,

$$R_T = |\bigcup_{t\in[0;T]} \eta_t| < \infty \quad \text{a.s.} \tag{1.14}$$

The set $R_\infty := \bigcup_{t\in[0;\infty]} \eta_t$ is at most countable, provided (1.14). Let $\preccurlyeq$ be the lexicographical order on $\mathbb{R}^d$. We can label the points of η_0,

$$\eta_0 = \{x_0, x_{-1}, ..., x_{-q}\}, \quad x_0 \preccurlyeq x_{-1} \preccurlyeq ... \preccurlyeq x_{-q}.$$

There exists an a.s. unique representation

$$R_\infty \setminus \eta_0 = \{x_1, x_2, ...\}$$

such that for any $n, m \in \mathbb{N}$, $n < m$, either $\inf\limits_{s\geq 0}\{s : x_n \in \eta_s\} < \inf\limits_{s\geq 0}\{s : x_m \in \eta_s\}$, or $\inf\limits_{s\geq 0}\{s : x_n \in \eta_s\} = \inf\limits_{s\geq 0}\{s : x_m \in \eta_s\}$ and $x_n \preccurlyeq x_m$. In other words, as time goes on, appearing points are added to $\{x_1, x_2, ...\}$ in the order in which they appear. If several points appear simultaneously, we add them in the lexicographical order.

For the sake of convenience, we set $x_{-i} = \Delta$, $i \leq -q-1$, where $\Delta \notin \mathbb{Z}$. We say that the sequence $\{..., x_{-1}, x_1, x_2, ...\}$ is *related* to $(\eta_t)_{t\in[0;\infty]}$.

For a predictable process $f \in L^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega)$ and $B \in \mathscr{B}(\mathbb{R}^d)$, consider

$$\int\limits_{\mathbb{Z}\times(t_1;t_2]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} f(x_i, r, v, \omega) dN_2(i, r, v). \tag{1.15}$$

Assume that R_T is bounded for some $T > 0$. Then, for a bounded predictable $f \in L^1(R^d \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega)$ and $B \in \mathscr{B}(\mathbb{R}^d)$, the process

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} f(x_i, r, v, \omega) dN_2(i, r, v)$$

$$- \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} f(x_i, r, v, \omega) \#(di) dr dv$$

is a martingale, cf. [IW81, Page 62].

1.3.2 The strong Markov property of a Poisson point process

We will need the strong Markov property of a Poisson point process. To simplify notations, assume that N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $dt \times dx$. Let N be compatible with a right-continuous complete filtration $\{\mathscr{F}_t\}_{t\geq 0}$, and τ be a finite a.s. $\{\mathscr{F}_t\}_{t\geq 0}$-stopping time (stopping time with respect to $\{\mathscr{F}_t\}_{t\geq 0}$). Introduce another Point process $\overline{N}$ on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$\overline{N}([0;s] \times U) = N((\tau;\tau+s] \times U), \quad U \in \mathscr{B}(\mathbb{R}^d).$$

1.3.4 Proposition. *The process $\overline{N}$ is a Poisson point process with intensity $dt \times dx$, independent of $\mathscr{F}_\tau$.*

Proof. To prove the proposition, it is enough to show that

(i) for any $b > a > 0$ and open bounded $U \subset \mathbb{R}^d$, $\overline{N}((a;b), U)$ is a Poisson random variable with mean $(b-a)\beta(U)$, and

(ii) for any $b_k > a_k > 0$, $k = 1, ..., m$, and any open bounded $U_k \subset \mathbb{R}^d$, such that $((a_i;b_i) \times U_i) \cap ((a_j;b_j) \times U_j) = \varnothing$, $i \neq j$, the collection $\{\overline{N}((a_k;b_k) \times U_k)\}_{k=1,m}$ is a sequence of independent random variables, independent of $\mathscr{F}_\tau$.

Indeed, $\overline{N}$ is determined completely by values on sets of type $(b-a)\beta(U)$, a, b, U as in (i), therefore it must be an independent of $\mathscr{F}_\tau$ Poisson point process if (i) and (ii) hold.

Let τ_n be the sequence of $\{\mathscr{F}_t\}_{t\geq 0}$-stopping times, $\tau_n = \frac{k}{2^n}$ on $\{\tau \in (\frac{k-1}{2^n}; \frac{k}{2^n}]\}$, $k \in \mathbb{N}$. Then $\tau_n \downarrow \tau$ and $\tau_n - \tau \leq \frac{1}{2^n}$. The stopping times τ_n take only countably many values. The process N satisfies the strong Markov property for τ_n: the processes $\overline{N}_n$, defined by

$$\overline{N}_n([0;s] \times U) := N((\tau_n;\tau_n+s] \times U),$$

are Poisson point processes, independent of $\mathscr{F}_{\tau_n}$. To prove this, take k with $P\{\tau_n = \frac{k}{2^n}\} > 0$ and note that on $\{\tau_n = \frac{k}{2^n}\}$, $\overline{N}_n$ coincides with process the Poisson point process

$\tilde{N}_{\frac{k}{2^n}}$ given by

$$\tilde{N}_{\frac{k}{2^n}}([0;s] \times U) := N\left((\frac{k}{2^n}; \frac{k}{2^n} + s] \times U) \right), \quad U \in \mathscr{B}(\mathbb{R}^d).$$

Conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is again a Poisson point process, with the same intensity. Furthermore, conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is independent of $\mathscr{F}_{\frac{k}{2^n}}$, hence it is independent of $\mathscr{F}_\tau \subset \mathscr{F}_{\frac{k}{2^n}}$.

To prove (i), note that $\overline{N}_n((a;b) \times U) \to \overline{N}((a;b) \times U)$ a.s. and all random variables $\overline{N}_n((a;b) \times U)$ have the same distribution, therefore $\overline{N}((a;b) \times U)$ is a Poisson random variable with mean $(b-a)\lambda(U)$. The random variables $\overline{N}_n((a;b) \times U)$ are independent of $\mathscr{F}_\tau$, hence $\overline{N}((a;b) \times U)$ is independent of $\mathscr{F}_\tau$, too. Similarly, (ii) follows. $\square$

Analogously, the strong Markov property for a Poisson point process on $\mathbb{R}_+ \times \mathbb{N}$ with intensity $dt \times \#$ may be formulated and proven.

1.3.5 Remark. We assumed in Proposition 1.3.4 that the filtration $\{\mathscr{F}_t\}_{t \geq 0}$, compatible with N, is right-continuous and complete. To be able to apply Proposition 1.3.4, we should show that such filtrations exist.

Introduce the natural filtration of N,

$$\mathscr{F}_t^0 = \sigma\{N_k(C, B), B \in \mathscr{B}(\mathbb{R}^d), C \in \mathscr{B}([0;t])\},$$

and let $\mathscr{F}_t$ be the completion of $\mathscr{F}_t^0$ under P. Then N is compatible with $\{\mathscr{F}_t\}$. We claim that $\{\mathscr{F}_t\}_{t \geq 0}$, defined in such a way, is right-continuous (this may be regarded as an analog of Blumenthal $0-1$ law). Indeed, as in the proof of Proposition 1.3.4, one may check that $\tilde{N}_a$ is independent of $\mathscr{F}_{a+}$. Since $\mathscr{F}_\infty = \sigma(\tilde{N}_a) \vee \mathscr{F}_a$, $\sigma(\tilde{N}_a)$ and $\mathscr{F}_a$ are independent and $\mathscr{F}_{a+} \subset \mathscr{F}_\infty$, one sees that $\mathscr{F}_{a+} \subset \mathscr{F}_a$. Thus, $\mathscr{F}_{a+} = \mathscr{F}_a$.

1.3.6 Remark. We prefer to work with right-continuous complete filtrations, because we want to ensure that there is no problem with conditional probabilities, and that the hitting times we will consider are stopping times.

1.4 Miscellaneous

When we write $\xi \sim Exp(\lambda)$, we mean that the random variable ξ is exponentially distributed with parameter λ.

1.4.1 Lemma. *If α and β are exponentially distributed random variables with parameters a and b respectively (notation: $\alpha \sim Exp(a)$, $\beta \sim Exp(b)$) and they are independent, then*

$$P\{\alpha < \beta\} = \frac{a}{a+b}.$$

Indeed,

$$P\{\alpha < \beta\} = \int_0^\infty aP\{x < \beta\}e^{-ax} = a\int_0^\infty e^{-(a+b)x} = \frac{a}{a+b}.$$

Here are few other properties of exponential distributions. If $\xi_1, \xi_2, ..., \xi_n$ are independent exponentially distributed random variables with parameters $c_1, ..., c_n$ respectively, then $\min\limits_{k\in\{1,...,n\}} \xi_k$ is exponentially distributed with parameter $c_1 + ... + c_n$. Again, the proof may be done by direct computation. If $\xi_1, \xi_2, ...$ are independent exponentially distributed random variables with parameter c and $\alpha_1, \alpha_2, ...$ is an independent sequence of independent Bernoulli random variables with parameter $p \in (0; 1)$, then the random variable

$$\xi = \sum_{i=1}^{\theta} \xi_i, \quad \theta = \min\{k \in \mathbb{N} : \alpha_k = 1\}$$

is exponentially distributed with parameter $\frac{c}{p}$. The random variable ξ is the time of the first jump of a thinned Poisson point process with intensity c. The statement about the distribution of ξ is a consequence of the property that the independent thinning of a

Poisson point process with intensity λ is a Poisson point process with intensity $p\lambda$, see [Kal02, Theorem 12.2,(iv)].

We will also need the result about finiteness of the expectation of the Yule process. A Yule process $(Z_t)_{t\geq 0}$ is a pure birth Markov process in $\mathbb{Z}_+$ with birth rate μn, $\mu > 0$, $n \in \mathbb{Z}_+$. That is, if $Z_t = n$, then a birth occur at rate μn, i.e.

$$P\{Z_{t+\Delta t} - Z_t = 1 \mid Z_t = n\} = \mu n + o(\Delta t).$$

For more details about Yule processes see e.g. [AN72, Chapter 3], [Har63, Chapter 5], [Arn06] and references therein. Let $(Z_t(n))_{t\geq 0}$ be a Yule process started at n. The process $(Z_t(n))_{t\geq 0}$ can be considered as a sum of n independent Yule processes started from 1, see e.g. [Arn06]. The expectation of $Z_t(1)$ is finite and $EZ_t(1) = e^{\mu t}$, see e.g. [AN72, Chapter 3, Section 6] or [Har63, Chapter 5, Sections 6,7]. Consequently, if $(Z_t)_{t\geq 0}$ is a Yule process with $EZ_0 < \infty$, then $EZ_t < \infty$ and $EZ_t = EZ_0 e^{\mu t}$.

Here are some other properties of Poisson point processes which are used throughout the thesis. If N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity $ds \times dx \times du$, then a.s.

$$\forall x \in \mathbb{R}^d : N(\mathbb{R}_+ \times \{x\} \times \mathbb{R}_+) \leq 1. \tag{1.16}$$

Put differently, no plane of the form $\mathbb{R}_+ \times \{x\} \times \mathbb{R}_+$ contains more than 1 point of N. Using the σ-additivity of the probability measure, one can deduce (1.16) from

$$\forall x \in \mathbb{R}^d : N([0;1] \times \{x\} \times [0;1]) \leq 1. \tag{1.17}$$

We can write

$$\Big\{\forall x \in \mathbb{R}^d : N([0;1] \times \{x\} \times [0;1]) \leq 1\Big\}$$
$$\supset \Big\{\forall k \in \{0,1,...,n-1\} : N([0;1] \times [\frac{k}{n};\frac{k+1}{n}] \times [0;1]) \leq 1\Big\},$$

and then we can compute

$$P\left\{\forall k \in \{0,1,...,n-1\} : N([0;1] \times [\frac{k}{n}; \frac{k+1}{n}] \times [0;1]) \leq 1\right\}$$

$$= \Big(P\{N([0;1] \times [0; \frac{1}{n}] \times [0;1]) \leq 1\}\Big)^n = \Big(\exp(-\frac{1}{n})[1 + \frac{1}{n}]\Big)^n = \Big(1 - o(\frac{1}{n})\Big)^n = 1 - o(\frac{1}{n}).$$

Thus, (1.17) holds.

Let $\psi \in L^1(\mathbb{R}^d)$, $\psi \geq 0$. Consider the time until the first arrival

$$\tau = \inf\{t > 0 : \int\limits_{[0;t] \times \mathbb{R}^d \times \mathbb{R}_+} I_{[0;\psi(x)]}(u) N(ds, dx, du) > 0\}. \tag{1.18}$$

The random variable τ is distributed exponentially with the parameter $||\psi||_{L^1}$. From (1.16) we know that a.s.

$$N(\{\tau\} \times \mathbb{R}^d \times \mathbb{R}_+) = N\big(\{(\tau, x, u) \mid x \in \mathbb{R}^d, u \in [0; \psi(x)]\}\big) = 1$$

Let x_τ be the unique element of $\mathbb{R}^d$ defined by

$$N(\{\tau\} \times \{x_\tau\} \times \mathbb{R}_+) = 1.$$

Then

$$P\{x_\tau \in B\} = \frac{\int_B \psi(x) dx}{\int_{\mathbb{R}^d} \psi(x) dx}, \quad B \in \mathscr{B}(\mathbb{R}^d). \tag{1.19}$$

1.5 Pure jump type Markov processes

In this section we give a very concise treatment of pure jump type Markov processes. Most of the definitions and facts given here can be found in [Kal02, Chapter 12]; see also, e.g., [GS75, Chapter 3, § 1].

We say that a process $X = (X_t)_{t \geq 0}$ in some measurable space $(S, \mathcal{S})$ is of *pure jump* *type* if its paths are a.s. right-continuous and constant apart from isolated jumps. In that case we may denote the jump times of X by $\tau_1, \tau_2, ...$, with understanding that $\tau_n = \infty$ if there are fewer that n jumps. The times τ_n are stopping times with respect to the right-continuous filtration induced by X. For convenience we may choose X to be the identity mapping on the canonical path space $(\Omega, \mathscr{F}) = (S^{[0;\infty)}, \mathcal{S}^{[0;\infty)})$. When X is a Markov process, the distribution with initial state x is denoted by P_x, and we note that the mapping $x \mapsto P_x(A)$ is measurable in x, $A \in \Omega$.

Theorem 12.14 [Kal02] *(strong Markov property, Doob) A pure jump type Markov process satisfies strong Markov property at every stopping time.*

We say that a state $x \in S$ is *absorbing* if $P_x\{X \equiv x\} = 1$.

Lemma 12.16 [Kal02] *If x is non-absorbing, then under P_x the time τ_1 until the first jump is exponentially distributed and independent of $\theta_{\tau_1} X$.*

Here θ_t is a shift, and $\theta_{\tau_1} X$ defines a new process,

$$\theta_{\tau_1} X(s) = X(s + \tau_1).$$

For a non-absorbing state x, we may define the *rate function* $c(x)$ and *jump transition kernel* $\mu(x, B)$ by

$$c(x) = (E_x \tau_1)^{-1}, \quad \mu(x, B) = P_x\{X_{\tau_1} \in B\}, \qquad x \in S, \ B \in \mathcal{S}.$$

In the sequel, $c(x)$ will also be referred to as *jump rate.* The kernel $c\mu$ is called a *rate kernel.*

The following theorem gives an explicit representation of the process in terms of a discrete-time Markov chain and a sequence of exponentially distributed random variables. This result shows in particular that the distribution P_x is uniquely determined by the rate kernel $c\mu$. We assume existence of the required randomization variables (so that the underlying probability space is "rich enough").

Theorem 12.17 [Kal02] *(embedded Markov chain) Let X be a pure jump type Markov process with rate kernel $c\mu$. Then there exists a Markov process Y on $\mathbb{Z}_+$ with transition kernel μ and an independent sequence of i.i.d., exponentially distributed random variables $\gamma_1, \gamma_2, ...$ with mean 1 such that a.s.*

$$X_t = Y_n, \quad t \in [\tau_n, \tau_{n+1}),\ n \in \mathbb{Z}_+, \tag{1.20}$$

where

$$\tau_n = \sum_{k=1}^{n} \frac{\gamma_k}{c(Y_{k-1})}, \quad n \in \mathbb{Z}_+. \tag{1.21}$$

In particular, the differences between the moments of jumps $\tau_{n+1} - \tau_n$ of a pure jump type Markov process are exponentially distributed given the embedded chain Y, with parameter $c(Y_n)$. If $c(Y_k) = 0$ for some (random) k, we set $\tau_n = \infty$ for $n \geq k+1$, while Y_n are not defined, $n \geq k+1$.

Theorem 12.18 [Kal02] *(synthesis) For any rate kernel $c\mu$ on S with $\mu(x, \{x\}) \equiv 0$, consider a Markov chain Y with transition kernel μ and a sequence $\gamma_1, \gamma_2, ...$ of independent exponentially distributed random variables with mean 1, independent of Y. Assume that $\sum_n \frac{\gamma_n}{c(Y_n)} = \infty$ a.s. for every initial distribution for Y. Then* (1.20) *and* (1.21) *define a pure jump type Markov process with rate kernel $c\mu$.*

Next proposition gives a convenient criterion for non-explosion.

Proposition 12.19 [Kal02] *(explosion) For any rate kernel $c\mu$ and initial state x, let (Y_n) and (τ_n) be such as in Theorem 12.17. Then a.s.*

$$\tau_n \to \infty \quad \textit{iff} \quad \sum_n \frac{1}{c(Y_n)} = \infty. \tag{1.22}$$

In particular, $\tau_n \to \infty$ a.s. when x is recurrent for (Y_n).

1.6 Markovian functions of a Markov chain

Let $(S, \mathscr{B}(S))$ be a Polish (state) space. Consider a (homogeneous) Markov chain on $(S, \mathscr{B}(S))$ as a family of probability measures on S^∞. Namely, on the measurable space $(\Omega, \mathscr{F}) = (S^\infty, \mathscr{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s\in S}$ such that for the coordinate mappings

$$X_n : \Omega \to S,$$

$$X_n(s_1, s_2, ...) = s_n$$

the process $X = \{X_n\}_{n\in\mathbb{Z}_+}$ is a Markov chain, and for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$

$$P_s\{X_{n+m_j} \in A_j, j = 1, ..., k_1 \mid \mathscr{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, ..., k_1\}.$$

Here $A_j \in \mathscr{B}(S)$, $m_j \in \mathbb{N}$, $k_1 \in \mathbb{N}$, $\mathscr{F}_n = \sigma\{X_1, ..., X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathscr{B}(S) \to [0; 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S,\ A \in \mathscr{B}(S).$$

Consider a transformation of the chain X, $Y_n = f(X_n)$, where $f : S \to \mathbb{Z}_+$ is a Borel-measurable function, with convention $\mathscr{B}(\mathbb{Z}_+) = 2^{\mathbb{Z}_+}$. In the future we will need to know when the process $Y = \{Y_n\}_{\mathbb{Z}_+}$ is a Markov chain. A similar question appeared for the first time in [BR58].

A sufficient condition for Y to be a Markov chain is given in the next lemma.

1.6.1 Lemma. *Assume that for any bounded Borel function $h : S \to S$*

$$E_s h(X_1) = E_q h(X_1) \ \textit{whenever} \ \ f(s) = f(q), \tag{1.23}$$

Then Y is a Markov chain.

Remark. Condition (1.23) is the equality of distributions of X_1 under two different measures, P_s and P_q.

Proof. For the natural filtrations of the processes X and Y we have an inclusion

$$\mathscr{F}_n^X \supset \mathscr{F}_n^Y, \quad n \in \mathbb{N}, \tag{1.24}$$

since Y is a function of X. For $k \in \mathbb{N}$ and bounded Borel functions $h_j : \mathbb{Z}_+ \to \mathbb{R}$, $j = 1, 2, ..., k$ (any function on $\mathbb{Z}_+$ is a Borel function),

$$\begin{gathered} E_s \left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] = E_{X_n} \prod_{j=1}^k h_j(f(X_j)) = \\ \int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) ... \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) \Big|_{x_0 = X_n} \end{gathered} \tag{1.25}$$

To transform the last integral, we introduce a new kernel: for $y \in f(S)$ chose $x \in S$ with $f(x) = y$, ans then for $B \subset \mathbb{Z}_+$ define

$$\overline{Q}(y, B) = Q(x, f^{-1}(B)); \tag{1.26}$$

The expression on the right-hand side does not depend on the choice of x because of (1.23). To make the kernel $\overline{Q}$ defined on $\mathbb{Z}_+ \times \mathscr{B}(\mathbb{Z}_+)$, we set

$$\overline{Q}(y, B) = I_{\{0 \in B\}}, \ y \notin f(S).$$

Then from the change of variables formula for the Lebesgue integral it follows that the last integral in (1.25) allows the representation

$$\int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) = \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n).$$

Likewise, we set $z_{n-1} = f(x_{n-1})$ in the next to last integral:

$$\int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n) =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-2}), dz_{n-1}) h_n(z_{n-1}) \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n).$$

Further proceeding, we get

$$\int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) ... \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(z_0, dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n),$$

where $z_0 = f(x_0)$.

Thus,

$$E_s \left[\prod_{j=1}^{k} h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(f(X_0), dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n).$$

This equality and (1.24) imply that Y is a Markov chain.

1.6.2 Remark. The kernel $\overline{Q}$ and the chain $f(X_n)$ are related: for all $s \in S$, $n, m \in \mathbb{N}$ and $M \subset \mathbb{N}$,

$$P_s\{f(X_{n+1}) \in M \mid f(X_n) = m\} = \overline{Q}(m, M)$$

whenever $P_s\{f(X_{n+1}) = m\} > 0$. Informally, one may say that $\overline{Q}$ is the transition probability kernel for the chain $\{f(X_n)\}_{n\in\mathbb{Z}_+}$.

1.6.3 Remark. Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space, because the property of a process to be a Markov chain depends on its distribution only.

Chapter 2

Birth-and-death processes in the space of finite configurations

2.1 A birth-and-death process in the space of finite configurations: construction and general theory

We would like to construct a Markov process in the space of finite configurations $\Gamma_0(\mathbb{R}^d)$, with a heuristic generator of the form

$$LF(\eta) = \int_{x\in\mathbb{R}^d} b(x,\eta)[F(\eta\cup x) - F(\eta)]dx + \sum_{x\in\eta} d(x,\eta)(F(\eta\setminus x) - F(\eta)). \tag{2.1}$$

for F in an appropriate domain. We call the functions $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ and $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ the *birth rate coefficient* and the *death rate coefficient*, respectively. Theorem 2.1.16 summarizes the main results obtained in this section.

To construct a spatial birth-and-death process, we consider the stochastic equation with Poisson noise

$$\eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) \\ - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v), \tag{2.2}$$

where $(\eta_t)_{t\geq 0}$ is a suitable cadlag $\Gamma_0(\mathbb{R}^d)$-valued stochastic process, the "solution" of the equation, $B \in \mathscr{B}(\mathbb{R}^d)$ is a Borel set, N_1 is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \times ds \times du$, N_2 is a Poisson point process on $\mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times dr \times dv$; η_0 is a (random) finite initial configuration, $b, d : \mathbb{R}^d \times \Gamma_0(R^d) \to [0;\infty)$ are functions measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\Gamma_0(\mathbb{R}))$, and the sequence $\{x_{-1}, x_0, x_1, ...\}$ is related to $(\eta_t)_{t\in[0;\infty]}$, as described in Section 1.3.1. We require the processes N_1, N_2, η_0 to be independent of each other. Equation (2.2) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq 0$.

As it was said in the preliminaries on Page 15, we identify a finite configuration with a finite simple counting measure, so that a configuration γ acts as a measure in the following way:

$$\gamma(A) = |\gamma \cap A|, \quad A \in \mathscr{B}(\mathbb{R}^d).$$

We will treat an element of $\Gamma_0(\mathbb{R}^d)$ both as a set and as a counting measure, as long as this does not lead to ambiguity. An appearing of a new point will be interpreted as a birth, and a disappearing will be interpreted as a death. We will refer to points of η_t as particles.

Some authors write $\tilde{d}(x, \eta \setminus x)$ where we write $d(x, \eta)$, so that (2.1) translates to

$$LF(\eta) = \int\limits_{x \in \mathbb{R}^d} b(x,\eta)[F(\eta \cup x) - F(\eta)]dx + \sum_{x \in \eta} \tilde{d}(x, \eta \setminus x)(F(\eta \setminus x) - F(\eta)), \tag{2.3}$$

see e.g. [Pre75], [FKK12b].

These settings are formally equivalent: the relation between d and $\tilde{d}$ is given by

$$d(x,\eta) = \tilde{d}(x, \eta \setminus x), \quad \eta \in \Gamma_0(\mathbb{R}^d), x \in \eta,$$

or, equivalently,

$$d(x, \xi \cup x) = \tilde{d}(x, \xi), \quad \xi \in \Gamma_0(\mathbb{R}^d), x \in \mathbb{R}^d \setminus \xi.$$

The settings used in this thesis appeared in [HS78], [GK06], etc.

We define the *cumulative death rate* at ζ by

$$D(\zeta) = \sum_{x \in \zeta} d(x, \zeta), \tag{2.4}$$

and the *cumulative birth rate* by

$$B(\zeta) = \int\limits_{x \in \mathbb{R}^d} b(x, \zeta) dx. \tag{2.5}$$

2.1.1 Definition. A *(weak) solution* of equation (2.2) is a triple $((\eta_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$, where

(i) $(\Omega, \mathscr{F}, P)$ is a probability space, and $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing, right-continuous and complete filtration of sub - σ - algebras of $\mathscr{F}$,

(ii) N_1 is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \times ds \times du$,

(iii) N_2 is a Poisson point process on $\mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times ds \times du$,

(iv) η_0 is a random $\mathscr{F}_0$-measurable element in $\Gamma_0(\mathbb{R}^d)$,

(v) the processes N_1, N_2 and η_0 are independent, the processes N_1 and N_2 are compatible with $\{\mathscr{F}_t\}_{t\geq 0}$,

(vi) $(\eta_t)_{t\geq 0}$ is a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, $\eta_t\big|_{t=0} = \eta_0$,

(vii) all integrals in (2.2) are well-defined, and

(viii) equality (2.2) holds a.s. for all $t \in [0;\infty]$ and all bounded Borel sets B, with $\{x_m\}_{m\in\mathbb{Z}}$ being the sequence related to $(\eta_t)_{t\geq 0}$.

Note that due to Statement 1.3.3 item (viii) of this definition is a statement about the joint distribution of $(\eta_t), N_1, N_2$.

Let

$$\mathscr{C}_t^0 = \sigma\big\{\eta_0, N_1(B,[0;q],C), N_2(i,[0;q],C);$$
$$B \in \mathscr{B}(\mathbb{R}^d), C \in \mathscr{B}(\mathbb{R}_+), q \in [0;t], i \in \mathbb{Z}\big\},$$

and let $\mathscr{C}_t$ be the completion of $\mathscr{C}_t^0$ under P. Note that $\{\mathscr{C}_t\}_{t\geq 0}$ is a right-continuous filtration, see Remark 1.3.5.

2.1.2 Definition. A solution of (2.2) is called *strong* if $(\eta_t)_{t\geq 0}$ is adapted to $(\mathscr{C}_t, t \geq 0)$.

2.1.3 Remark. In the definition above we considered solutions as processes indexed by $t \in [0;\infty)$. The reformulations for the case $t \in [0;T]$, $0 < T < \infty$, are straightforward. This remark applies to the results below, too.

Sometimes only the solution process (that is, $(\eta_t)_{t\geq 0}$) will be referred to as a (strong or weak) solution, when all the other structures are clear from the context.

We will say that the *existence of strong solution* holds, if on any probability space with given N_1, N_2, η_0, satisfying (i)-(v) of Definition (2.1.1), there exists a strong solution.

2.1.4 Definition. We say that pathwise uniqueness holds for equation (2.2) and an initial distribution ν if, whenever the triples $((\eta_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$ and

$((\bar{\eta}_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\bar{\mathscr{F}}_t\}_{t\geq 0})$ are weak solutions of (2.2) with $P\{\eta_0 = \bar{\eta}_0\} = 1$ and $Law(\eta) = \nu$, we have $P\{\eta_t = \bar{\eta}_t, t \in [0;T]\} = 1$ (that is, the processes $\eta, \bar{\eta}$ are indistinguishable).

We assume that the birth rate b satisfies the following conditions: sublinear growth on the second variable in the sense that

$$\int\limits_{\mathbb{R}^d} b(x,\eta)dx \leq c_1|\eta| + c_2, \tag{2.6}$$

and let d satisfy

$$\forall m \in \mathbb{N} : \sup_{x\in\mathbb{R}^d, |\eta|\leq m} d(x,\eta) < \infty. \tag{2.7}$$

We also assume that

$$E|\eta_0| < \infty. \tag{2.8}$$

By a non-random initial condition we understand an initial condition with a distribution, concentrated at one point: for some $\eta' \in \Gamma_0(\mathbb{R}^d)$, $P\{\eta_0 = \eta'\} = 1$.

From now on, we work on some filtered probability space $(\Omega, \mathscr{F}, (\{\mathscr{F}_t\}_{t\geq 0}), P)$. On this probability space, the Poisson point processes N_1, N_2 and η_0 are defined, so that the whole set-up satisfies (i)-(v) of Definition 2.1.1.

Let us now consider the equation

$$\overline{\eta}_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;\bar{b}(x,\overline{\eta}_s)]}dN(x,s,u) + \eta_0(B), \tag{2.9}$$

where $\bar{b}(x,\eta) := \sup\limits_{\xi\subset\eta} b(x,\xi)$. Note that $\bar{b}$ satisfies sublinear growth condition (2.6), if b satisfies it.

This equation is of the type (2.2) (with $\bar{b}$ being the birth rate coefficient, and the zero function being the death rate coefficient), and all definitions of existence and uniqueness of solution are applicable here. Later a unique solution of (2.9) will be used as a majorant of a solution to (2.2).

2.1.5 Proposition. *Under assumptions* (2.6) *and* (2.8)*, strong existence and pathwise uniqueness hold for equation* (2.9)*. The unique solution* $(\bar{\eta}_t)_{t\geq 0}$ *satisfies*

$$E|\bar{\eta}_t| < \infty, \quad t \geq 0. \tag{2.10}$$

Proof. For $\omega \in \{\int\limits_{\mathbb{R}^d} \bar{b}(x,\eta_0)dx = 0\}$, set $\zeta_t \equiv \eta_0$, $\sigma_n = \infty$, $n \in \mathbb{N}$.

For $\omega \in F := \{\int\limits_{\mathbb{R}^d} \bar{b}(x,\eta_0)dx > 0\}$, we define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf\{t > 0 : \int\limits_{\mathbb{R}^d \times (\sigma_n;\sigma_n+t]\times[0;\infty)} I_{[0;\bar{b}(x,\zeta_{\sigma_n})]}(u)dN_1(x,s,u) > 0\} + \sigma_n, \quad \sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d : N_1(x, \sigma_{n+1}, [0; \bar{b}(x, \zeta_{\sigma_n})]) > 0\}$. From (1.16) it follows that the points z_n are uniquely determined almost surely on F. Moreover, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s. on F (particularly because $\bar{b}(x,\zeta_{\sigma_n}) \geq \bar{b}(x,\eta_0)$). For $\omega \in F$, we define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n; \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathscr{F}_{\sigma_n} \cap F$-measurable. By direct substitution we see that $(\zeta_t)_{t\geq 0}$ is a strong solution for (2.9) on the time interval $t \in [0; \lim\limits_{n\to\infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead we are going to show that

$$\lim_{n\to\infty} \sigma_n = \infty \quad \text{a.s.} \tag{2.11}$$

This relation is evidently true on the complement of F. If $P(F) = 0$, then (2.11) is proven.

If $P(F) > 0$, define a probability measure on F, $Q(A) = \frac{P(A)}{P(F)}$, $A \in \mathscr{S} := \mathscr{F} \cap F$, and define $\mathscr{S}_t = \mathscr{F}_t \cap F$.

The process N_1 is independent of F, therefore it is a Poisson point process on $(F, \mathscr{S}, Q)$ with the same intensity, compatible with $\{\mathscr{S}_t\}_{t \geq 0}$. From now on and until other is specified, we work on the filtered probability space $(F, \mathscr{S}, \{\mathscr{S}_t\}_{t \geq 0}, Q)$. We use the same symbols for random processes and random variables, having in mind that we consider their restrictions to F.

The process $(\zeta_t)_{t \in [0; \lim\limits_{n \to \infty} \sigma_n)}$ has the Markov property, because the process N_1 has the strong Markov property and independent increments. Indeed, conditioning on $\mathscr{S}_{\sigma_n}$,

$$E\big[I_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x \in B\}} \mid \mathscr{S}_{\sigma_n}\big] = \frac{\int\limits_B \bar{b}(x, \zeta_{\sigma_n}) dx}{\int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx},$$

thus the chain $\{\zeta_{\sigma_n}\}_{n \in Z_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n \in Z_+}$, $\sigma_{n+1} - \sigma_n$ are distributed exponentially:

$$E\{I_{\{\sigma_{n+1} - \sigma_n > a\}} \mid \{\zeta_{\sigma_n}\}_{n \in Z_+}\} = \exp\{-a \int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1})(\int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx)$ constitute a sequence of independent random variables exponentially distributed with parameter 1, independent of $\{\zeta_{\sigma_n}\}_{n \in Z_+}$. Theorem 12.18 in [Kal02] (see Page 37 of this thesis) implies that $(\zeta_t)_{t \in [0; \lim\limits_{n \to \infty} \sigma_n)}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t \in [0; \lim\limits_{n \to \infty} \sigma_n)}$ is given by

$$c(\alpha) = \int\limits_{\mathbb{R}^d} \bar{b}(x, \alpha) dx.$$

Condition (2.6) implies that $c(\alpha) \leq c_1|\alpha| + c_2$. Consequently,

$$c(\zeta_{\sigma_n}) \leq c_1|\zeta_{\sigma_n}| + c_2 = c_1|\zeta_0| + c_1 n + c_2.$$

We see that $\sum_n \frac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [Kal02] (given in the first chapter of this thesis, Section 1.5) implies that $\sigma_n \to \infty$.

Now, we return again to our initial probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$.

Thus, we have existence of a strong solution. Uniqueness follows by induction on jumps of the process. Indeed, let $(\tilde{\zeta}_t)_{t\geq 0}$ be another solution of (2.9). From (viii) of Definition 2.1.1 and equality

$$\int\limits_{\mathbb{R}^d \times (0;\sigma_1) \times [0;\infty]} I_{[0;\bar{b}(x,\eta_0)]} dN_1(x,s,u) = 0,$$

one can see that $P\{\tilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time, equality

$$\int\limits_{\mathbb{R}^d \times \{\sigma_1\} \times [0;\infty]} I_{[0;\bar{b}(x,\eta_0)]} dN_1(x,s,u) = 1,$$

which holds a.s., yields that $\tilde{\zeta}$ has a birth at the moment σ_1, and in the same point of space at that. Therefore, $\tilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide up to σ_n a.s., and, because $\sigma_n \to \infty$ a.s.,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t \geq 0\} = 1$$

Thus, pathwise uniqueness holds. The constructed solution is strong.

Now we turn our attention to (2.10). We can write

$$|\zeta_t| = |\eta_0| + \sum_{n=1}^{\infty} I\{|\zeta_t| - |\eta_0| \geq n\}$$
$$= |\eta_0| + \sum_{n=1}^{\infty} I\{\sigma_n \leq t\}. \tag{2.12}$$

Since $\sigma_n = \sum\limits_{i=1}^{n} \frac{\gamma_i}{\int\limits_{\mathbb{R}^d} \bar{b}(x,\zeta_{\sigma_i})dx}$, we have

$$\{\sigma_n \leq t\} = \{\sum_{i=1}^{n} \frac{\gamma_i}{\int\limits_{\mathbb{R}^d} \bar{b}(x,\zeta_{\sigma_i})dx} \leq t\} \subset \{\sum_{i=1}^{n} \frac{\gamma_i}{c_1|\zeta_{\sigma_i}| + c_2} \leq t\}$$

$$\subset \{\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t\} = \{Z_t - Z_0 \geq n\},$$

where (Z_t) is the Yule process (see Page 34) with birth rate defined as follows: $Z_t - Z_0 = n$ when

$$\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t < \sum_{i=1}^{n+1} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)},$$

and $Z_0 = |\eta_0|$. Thus, we have $|\zeta_t| \leq Z_t$ a.s., hence $E|\zeta_t| \leq EZ_t < \infty$. $\square$

2.1.6 Theorem. *Under assumptions* (2.6)-(2.8)*, pathwise uniqueness and strong existence hold for equation* (2.2)*. The unique solution* (η_t) *is a pure jump type process satisfying*

$$E|\eta_t| < \infty, \quad t \geq 0. \tag{2.13}$$

Proof. Let us define stopping times with respect to $\{\mathscr{F}_t, t \geq 0\}$, $0 = \theta_0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq ...$, and the sequence of (random) configurations $\{\eta_{\theta_j}\}_{j\in\mathbb{N}}$ as follows: as long as

$$B(\eta_{\theta_n}) + D(\eta_{\theta_n}) > 0,$$

we set

$$\theta_{n+1} = \theta^b_{n+1} \wedge \theta^d_{n+1} + \theta_n,$$

$$\theta^b_{n+1} = \inf\{t > 0 : \int\limits_{\mathbb{R}^d \times (\theta_n;\theta_n+t]\times[0;\infty)} I_{[0;b(x,\eta_{\theta_n})]}(u) dN_1(x,s,u) > 0\},$$

$$\theta^d_{n+1} = \inf\{t > 0 : \int\limits_{(\theta_n;\theta_n+t]\times[0;\infty)} I_{\{x_i \in \eta_{\theta_n}\}} I_{[0;d(x_i,\eta_{\theta_n})]}(v) dN_2(i,r,v) > 0\},$$

$\eta_{\theta_{n+1}} = \eta_{\theta_n} \cup \{z_{n+1}\}$ if $\theta^b_{n+1} \le \theta^d_{n+1}$, where $\{z_{n+1}\} = \{z \in \mathbb{R}^d : N_1(z, \theta_n + \theta^b_{n+1}, \mathbb{R}_+) > 0\}$;

$\eta_{\theta_{n+1}} = \eta_{\theta_n} \setminus \{z_{n+1}\}$ if $\theta^b_{n+1} > \theta^d_{n+1}$, where $\{z_{n+1}\} = \{x_i \in \eta_{\theta_n} : N_2(i, \theta_n + \theta^d_{n+1}, \mathbb{R}_+) > 0\}$;

the configuration $\eta_{\theta_0} = \eta_0$ is the initial condition of (2.2), $\eta_t = \eta_{\theta_n}$ for $t \in [\theta_n; \theta_{n+1})$, $\{x_i\}$ is the sequence related to $(\eta_t)_{t\ge 0}$. Note that

$$P\{\theta^b_{n+1} = \theta^d_{n+1} \text{ for some } \ n \mid B(\eta_{\theta_n}) + D(\eta_{\theta_n}) > 0\} = 0,$$

the points z_n are a.s. uniquely determined, and

$$P\{z_{n+1} \in \eta_{\theta_n} \mid \theta^b_{n+1} \le \theta^d_{n+1}\} = 0.$$

If for some n

$$B(\eta_{\theta_n}) + D(\eta_{\theta_n}) = 0,$$

then we set $\theta_{n+k} = \infty$, $k \in \mathbb{N}$, and $\eta_t = \eta_{\theta_n}$, $t \ge \theta_n$.

As in the proof of Proposition 2.1.5, (η_t) is a strong solution of (2.2), $t \in [0; \lim_n \theta_n)$.

Random variables $\theta_n, n \in \mathbb{N}$, are stopping times with respect to the filtration $\{\mathscr{F}_t, t \ge 0\}$. Using the strong Markov property of a Poisson point process, we see that, on $\{\theta_n < \infty\}$, the conditional distribution of θ^b_{n+1} given $\mathscr{F}_{\theta_n}$ is $\exp(\int\limits_{\mathbb{R}^d} b(x, \eta_{\theta_n})dx)$, and the conditional distribution of θ^d_{n+1} given $\mathscr{F}_{\theta_n}$ is $\exp(\sum\limits_{x \in \eta_{\theta_n}} d(x, \eta_{\theta_n}))$. In particular, $\theta^b_n, \theta^d_n > 0$, $n \in \mathbb{N}$, and the process (η_t) is of pure jump type.

Similarly to the proof of Proposition 2.1.5, one can show by induction on n that equation (2.2) has a unique solution on $[0; \theta_n]$. Namely, each two solutions coincide on $[0; \theta_n]$ a.s. Thus, any solution coincides with (η_t) a.s. for all $t \in [0; \theta_n]$.

Now we will show that $\theta_n \to \infty$ a.s. as $n \to \infty$. Denote by θ'_k the moment of the k-th birth. It is sufficient to show that $\theta'_k \to \infty$, $k \to \infty$, because only finitely many deaths may occur between any two births, since there are only finitely particles. By induction on k' one may see that $\{\theta'_k\}_{k' \in \mathbb{N}} \subset \{\sigma_i\}_{i \in \mathbb{N}}$, where σ_i are the moments of births of $(\overline{\eta}_t)_{t \geq 0}$, the solution of (2.9), and $\eta_t \subset \overline{\eta}_t$ for all $t \in [0; \lim_n \theta_n)$. For instance, let us show that $(\overline{\eta}_t)_{t \geq 0}$ has a birth at θ'_1. We have $\overline{\eta}_{\theta'_1 -} \supset \overline{\eta}_0 = \eta_0$, and $\eta_{\theta'_1 -} \subset \eta_t \mid_{t=0} = \eta_0$, hence for all $x \in \mathbb{R}^d$

$$\overline{b}(x, \overline{\eta}_{\theta'_1 -}) \geq \overline{b}(x, \eta_{\theta'_1 -}) \geq b(x, \eta_{\theta'_1 -})$$

The latter implies that at time moment θ'_1 a birth occurs for the process $(\overline{\eta}_t)_{t \geq 0}$ in the same point. Hence, $\eta_{\theta'_1} \subset \overline{\eta}_{\theta'_1}$, and we can go on. Since $\sigma_k \to \infty$ as $k \to \infty$, we also have $\theta'_k \to \infty$, and therefore $\theta_n \to \infty$, $n \to \infty$.

Since $\eta_t \subset \overline{\eta}_t$ a.s., Proposition 2.1.5 implies (2.13). $\square$

In particular, for any time t the integral

$$\int\limits_{\mathbb{R}^d \times (0;t] \times [0;\infty]} I_{[0; b(x, \eta_{s-})]}(u) dN_1(x, s, u)$$

is finite a.s.

2.1.7 Remark. Let η_0 be a non-random initial condition, $\eta_0 \equiv \alpha$, $\alpha \in \Gamma_0(\mathbb{R}^d)$. The solution of (2.2) with $\eta_0 \equiv \alpha$ will be denoted as $(\eta(\alpha, t))_{t \geq 0}$. Let P_α be the push-forward of P under the mapping

$$\Omega \ni \omega \mapsto (\eta(\alpha, \cdot)) \in D_{\Gamma_0(\mathbb{R}^d)}[0; T]. \tag{2.14}$$

From the proof one may derive that, for fixed $\omega \in \Omega$, constructed unique solution is jointly measurable in (t, α). Thus, the family $\{P_\alpha\}$ of probability measures on $D_{\Gamma_0(\mathbb{R}^d)}[0; T]$ is measurable in α. We will often use formulations related to the probability space $(D_{\Gamma_0(\mathbb{R}^d)}[0;T], \mathscr{B}(D_{\Gamma_0(\mathbb{R}^d)}[0;T]), P_\alpha)$; in this case, coordinate mappings will be denoted by η_t,

$$\eta_t(x) = x(t), \quad x \in D_{\Gamma_0(\mathbb{R}^d)}[0;T].$$

The processes $(\eta_t)_{t\in[0;T]}$ and $(\eta(\alpha,\cdot))_{t\in[0;T]}$ have the same law (under P_α and P, respectively). As one would expect, the family of measures $\{P_\alpha, \alpha \in \Gamma_0(\mathbb{R}^d)\}$ is a Markov process, or a Markov family of probability measures; see Theorem 2.1.15 below. For a measure μ on $\Gamma_0(\mathbb{R}^d)$, we define

$$P_\mu = \int P_\alpha \mu(d\alpha).$$

We denote by E_μ the expectation under P_μ.

2.1.8 Remark. Let b_1, d_1 be another pair of birth and death coefficients, satisfying all conditions imposed on b and d. Consider a unique solution $(\tilde{\eta}_t)$ of (2.2) with coefficients b_1, d_1 instead of b, d, but with the same initial condition η_0 and all the other underlying structures. *If for all* $\zeta \in D$, *where* $D \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$, $b_1(\cdot,\zeta) \equiv b(\cdot,\zeta)$, $d_1(\cdot,\zeta) \equiv d(\cdot,\zeta)$, *then* $\tilde{\eta}_t = \eta_t$ *for all* $t \leq \inf\{s \geq 0 : \eta_s \notin D\} = \inf\{s \geq 0 : \tilde{\eta}_s \notin D\}$. This may be proven in the same way as the theorem above.

2.1.9 Remark. Assume that all the conditions of Theorem 2.1.6 are fulfilled except Condition (2.8). Then we could not claim that (2.13) holds. However, other conclusions of the Theorem would hold. We are mostly interested in the case of a non-random initial condition, therefore we do not discuss the case when (2.13) is not satisfied.

2.1.10 Remark. We solved equation (2.2) ω-wisely. As a consequence, there is a functional dependence of the solution process and the "input": the process $(\eta_t)_{t\geq 0}$ is some function of η_0, N_1 and N_2.

2.1.11 Proposition. *If* $(\eta_t)_{t\geq 0}$ *is a solution to equation* (2.2)*, then the inequality*

$$E|\eta_t| < (c_2 t + E|\eta_0|)e^{c_1 t}$$

holds for all $t > 0$.

Proof. We already know that $E|\eta_t|$ is finite. Since η_t satisfies equation (2.2) we have

$$\eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u)$$

$$- \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v) \leq$$

$$\int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) + \eta_0(B).$$

For $B = \mathbb{R}^d$, taking expectation in the last inequality, we obtain

$$E|\eta_t| = E\eta_t(\mathbb{R}^d) \leq E \int\limits_{\mathbb{R}^d\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) + E\eta_0(\mathbb{R}^d) =$$

$$= E \int\limits_{\mathbb{R}^d\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dxdsdu + E\eta_0(\mathbb{R}^d) = E \int\limits_{\mathbb{R}^d\times(0;t]} b(x,\eta_{s-})dxds + E\eta_0(\mathbb{R}^d).$$

Since η is a solution of (2.2), we have for all $s \in [0;t]$ almost surely $\eta_{s-} = \eta_s$. Consequently, $E|\eta_{s-}| = E|\eta_s|$. Applying this and (2.6), we see that

$$E\eta_t(\mathbb{R}^d) \leq E \int\limits_{(0;t]} (c_1|\eta_{s-}| + c_2)ds + E\eta_0(\mathbb{R}^d) = c_1 \int\limits_{(0;t]} E|\eta_s|ds + c_2 t + E\eta_0(\mathbb{R}^d),$$

so the statement of the lemma follows from (2.8) and Gronwall's inequality. □

2.1.12 Definition. We say that *joint uniqueness in law* holds for equation (2.2) with an initial distribution ν if any two (weak) solutions $((\eta_t), N_1, N_2)$ and $((\eta_t)', N_1', N_2')$ of (2.2), $Law(\eta_0) = Law((\eta_0)') = \nu$, have the same joint distribution:

$$Law((\eta_t), N_1, N_2) = Law((\eta_t)', N_1', N_2').$$

The following corollary is a consequence of Theorem 2.1.6 and Remark 2.1.10 .

2.1.13 Corollary. *Joint uniqueness in law holds for equation* (2.2) *with initial distribution ν satisfying*

$$\int_{\Gamma_0(\mathbb{R}^d)} |\gamma| \nu(d\gamma) < \infty.$$

2.1.14 Remark. We note here that altering the order of the initial configuration does not change the law of the solution. We could replace the lexicographical order with any other. To see this, note that if ς is a permutation of $\mathbb{Z}$ (that is, $\varsigma : \mathbb{Z} \to \mathbb{Z}$ is a bijection), then the process $\tilde{N}_2$ defined by

$$\tilde{N}_2(K, R, V) = N_2(\varsigma K, R, V), \quad K \subset \mathbb{Z}, R, V \in \mathscr{B}(\mathbb{R}_+), \tag{2.15}$$

has the same law as N_2, and is adapted to $\{\mathscr{F}_t\}_{t \geq 0}$, too. Therefore, solutions of (2.2) and of (2.2) with N_2 being replaced by $\tilde{N}_2$ have the same law. But replacing N_2 with $\tilde{N}_2$ in equation (2.2) is equivalent to replacing $\{x_{-|\eta_0|+1}, ..., x_0, x_1, ...\}$ with $\{x_{\varsigma^{-1}(-|\eta_0|+1)}, ..., x_{\varsigma^{-1}(0)}, x_{\varsigma^{-1}(1)}, ...\}$.

Let ν be a distribution on $\Gamma_0(\mathbb{R}^d)$, and let $T > 0$. Denote by $\mathscr{L}(\nu, b, d, T)$ the law of the restriction $(\eta_t)_{t \in [0;T]}$ of the unique solution $(\eta_t)_{t \geq 0}$ to (2.2) with an initial condition distributed according to ν. Note that $\mathscr{L}(\nu, b, d, T)$ is a distribution on $D_{\Gamma_0(\mathbb{R}^d)}([0;T])$. As usually, the Markov property of a solution follows from uniqueness.

2.1.15 Theorem. *The unique solution $(\eta_t)_{t \in [0;T]}$ of* (2.2) *is a Markov process.*

Proof. Take arbitrary $t' < t$, $t', t \in [0;T]$. Consider the equation

$$\xi_t(B) = \int\limits_{B\times(t';t]\times[0;\infty]} I_{[0;b(x,\xi_{s-})]}(u)dN_1(x,s,u) - \int\limits_{\mathbb{Z}\times(t';t]\times[0;\infty)} I_{\{x'_i\in\xi_{r-}\cap B\}}I_{[0;d(x'_i,\xi_{r-})]}dN_2(i,r,v) + \eta_{t'}(B), \tag{2.16}$$

where the sequence $\{x'_i\}$ is related to the process $(\xi_s)_{s\in[0;t]}$, $\xi_s = \eta_s$. The unique solution of (2.16) is $(\eta_s)_{s\in[t';t]}$. As in the proof of Theorem 2.1.6 we can see that $(\eta_s)_{s\in[t';t]}$ is measurable with respect to the filtration generated by the random variables $N_1(B,[s;q],U)$, $N_2(i,[s;q],U)$, and $\eta_{t'}(B)$, where $B \in \mathscr{B}(\mathbb{R}^d)$, $i \in \mathbb{Z}$, $t' \le s \le q \le t$, $U \in \mathscr{B}(\mathbb{R}_+)$. Poisson point process have independent increments, hence

$$P\{(\eta_t)_{t\in[s;T]} \in U \mid \mathscr{F}_s\} = P\{(\eta_t)_{t\in[s;T]} \in U \mid \eta_s\}$$

almost surely. Furthermore, using arguments similar to those in Remark 2.1.14, we can conclude that $(\eta_s)_{s\in[t';t]}$ is distributed according to $\mathscr{L}(\nu_{t'}, b, d, t-t')$, where $\nu_{t'}$ is the distribution of $\eta_{t'}$. $\square$

The following theorem sums up the results we have obtained so far.

2.1.16 Theorem. *Under assumptions* (2.6), (2.7), (2.8), *equation* (2.2) *has a unique solution. This solution is a pure jump type Markov process. The family of push-forward measures* $\{P_\alpha, \alpha \in \Gamma_0(\mathbb{R}^d)\}$ *defined in Remark 2.1.7 forms a Markov process, or a Markov family of probability measures, on* $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)$.

Proof. The statement is a consequence of Theorem 2.1.6, Remark 2.1.7 and Theorem 2.1.15. In particular, the Markov property of $\{P_\alpha, \alpha \in \Gamma_0(\mathbb{R}^d)\}$ follows from the statement given in the last sentence of the proof of Theorem 2.1.15. $\square$

We call the unique solution of (2.2) (or, sometimes, the corresponding family of measures on $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty))$ a *(spatial) birth-and-death Markov process.*

2.1.17 Remark. We note that d does not need to be defined on the whole space $\mathbb{R}^d \times \Gamma_0(\mathbb{R}^d)$. The equation makes sense even if $d(x,\eta)$ is defined on $\{(x,\eta) \mid x \in \eta\}$. Of course, any such function may be extended to a function on $\mathbb{R}^d \times \Gamma_0(\mathbb{R}^d)$.

2.1.1 Continuous dependence on initial conditions

In order to prove the continuity of the distribution of the solution of (2.2) with respect to initial conditions, we make the following continuity assumptions on b and d.

2.1.18 Continuity assumptions. *Let b, d be continuous with respect to both arguments. Furthermore, let the map*

$$\Gamma_0(\mathbb{R}^d) \ni \eta \mapsto b(\cdot,\eta) \in L^1(\mathbb{R}^d)$$

be continuous.

In light of Remark 2.1.17, let us explain what we understand by continuity of d when $d(x,\eta)$ is defined only on $\{(x,\eta) \mid x \in \eta\}$. We require that, whenever $\eta_n \to \eta$ and $\eta_n \ni z_n \to x \in \eta$, we also have $d(z_n,\eta_n) \to d(x,\eta)$. Similar condition appeared in [HS78, Theorem 3.1].

2.1.19 Theorem. *Let the birth and death coefficients b and d satisfy the above continuity assumptions 2.1.18. Then for every $T > 0$ the map*

$$\Gamma_0(\mathbb{R}^d) \ni \alpha \mapsto Law\{\eta(\alpha,\cdot).,\cdot \in (0;T]\},$$

which assigns to a non-random initial condition $\eta_0 = \alpha$ the law of the solution of equation (2.2) *stopped at time T, is continuous.*

Remark. We mean continuity in in the space of measures on $D_{\Gamma_0(\mathbb{R}^d)}[0;T]$; see Page 25.

Proof. Denote by $\eta(\alpha,\cdot)$ the solution of (2.2), started from α. Let $\alpha_n \to \alpha$, $\alpha_n, \alpha \in \Gamma_0(\mathbb{R}^d)$, $\alpha = \{x_0, x_{-1}, ..., x_{-|\alpha|+1}\}$, $x_0 \preccurlyeq x_{-1} \preccurlyeq ... \preccurlyeq x_{-|\alpha|+1}$. With no loss in generality we assume that $|\alpha_n| = |\alpha|$, $n \in \mathbb{N}$. By Lemma 1.2.5 we can label elements of α_n, $\alpha_n = \{x_0^{(n)}, x_{-1}^{(n)}, ..., x_{-|\alpha|+1}^{(n)}\}$, so that $x_{-i}^{(n)} \to x_{-i}$, $i = 0, ..., |\alpha| - 1$. Taking into account Remark 2.1.14, we can assume

$$x_0^{(n)} \preccurlyeq x_{-1}^{(n)} \preccurlyeq ... \preccurlyeq x_{-|\alpha|+1}^{(n)} \tag{2.17}$$

without loss of generality (in the sense that we do not have to use lexicographical order; not in the sense that we can make $x_0^{(n)}, x_{-1}^{(n)}, ...$ satisfy (2.17) with the lexicographical order).

We will show that

$$\sup_{t\in[0;T]} dist(\eta(\alpha,t), \eta(\alpha_n,t)) \xrightarrow{p} 0, \quad n \to \infty. \tag{2.18}$$

Let $\{\theta_i\}_{i\in\mathbb{N}}$ be the moments of jumps of process $\eta(\alpha,\cdot)$. Without loss of generality, assume that $d(x,\alpha) > 0$, $x \in \alpha$, and $||b(\cdot,\alpha)||_{L^1} > 0$, $L^1 := L^1(\mathbb{R}^d)$ (if some of these inequalities are not fulfilled, the following reasonings should be changed insignificantly).

Depending on whether a birth or a death occurs at θ_1, we have either

$$N_1(\{x_1\} \times \{\theta_1\} \times [0; b(x_1, \eta_0)]) = 1 \tag{2.19}$$

or for some $x_{-k} \in \alpha$

$$N_2(\{-k\} \times \{\theta_1\} \times [0; d(x_{-k}, \alpha)]) = 1.$$

The probability of last two equalities holding simultaneously is zero, hence we can neglect this event. In both cases $N_1(x_1, \{\theta_1\}, \{b(x_1,\alpha)\}) = 0$, $N_2(-k, \{\theta_1\}, \{d(x_{-k},\alpha)\}) = 0$ a.s. We also have

$$N_1(\mathbb{R}^d \times [0;\theta_1) \times [0; b(x,\alpha)]) = 0,$$

and for all $j \in 0, 1, ..., |\alpha| - 1$

$$N_2(\{-j\} \times [0;\theta_1) \times [0; d(x_{-j},\alpha)]) = 0.$$

Denote

$$m := b(x_1,\alpha) \wedge \min\{d(x,\alpha) : x \in \alpha\} \wedge ||b(\cdot,\alpha)||_{L^1} \wedge 1$$

and fix $\varepsilon > 0$. Let $\delta_1 > 0$ be so small that for $\nu \in \Gamma_0(\mathbb{R}^d)$, $\nu = \{x'_0, x'_{-1}, ..., x'_{-|\alpha|+1}\}$, $|x_{-j} - x'_{-j}| \leq \delta_1$ the inequalities

$$|d(x'_{-j},\nu) - d(x_{-j},\alpha)| < \varepsilon m,$$

and

$$||b(\cdot,\nu) - b(\cdot,\alpha)||_{L^1} < \varepsilon m$$

hold. Then we may estimate

$$P\Big\{ \int\limits_{\mathbb{R}^d \times [0;\theta_1) \times [0;\infty]} I_{[0;b(x,\nu)]}(u) dN_1(x,s,u) \geq 1 \Big\} < \varepsilon. \tag{2.20}$$

and

$$P\Big\{ \int\limits_{\mathbb{Z} \times [0;\theta_1) \times [0;\infty]} I_{\{x'_{-i} \in \nu\}} I_{[0;d(x'_{-i},\nu)]}(v) dN_2(i,r,v) \geq 1 \Big\} < \varepsilon|\alpha|. \tag{2.21}$$

Indeed, the random variable

$$\tilde{\theta} := \inf_{t>0} \{ \int\limits_{\mathbb{R}^d \times [0;t) \times [0;\infty]} I_{[0;0 \vee \{b(x,\nu) - b(x,\alpha)\})]}(u) dN_1(x,s,u) \geq 1\} \tag{2.22}$$

is exponentially distributed with parameter $||(b(\cdot,\nu)-b(\cdot,\alpha))_+||_{L^1} < \varepsilon||b(\cdot,\alpha)||_{L^1}$. By Lemma 1.4.1,

$$P\{\tilde{\theta} < \theta_1\} < \frac{\varepsilon||b(\cdot,\alpha)||_{L^1}}{||b(\cdot,\alpha)||_{L^1}} = \varepsilon, \tag{2.23}$$

which is exactly (2.20). Likewise, (2.21) follows.

Similarly, the probability that the same event as for $\eta(\alpha,\cdot)$ occurs at time θ_1 for $\eta(\nu,\cdot)$ is high. Indeed, assume, for example, that a birth occurs at θ_1, that is to say that (2.19) holds. Once more using Lemma 1.4.1 we get

$$P\{N_1(\{x_1\}\times\{\theta_1\}\times[0;b(x_1,\nu)]) = 0\} \le \frac{||(b(\cdot,\nu)-b(\cdot,\alpha))_+||_{L^1}}{||b(\cdot,\alpha)||_{L^1}} \le \varepsilon.$$

The case of death occurring at θ_1 may be analyzed in the same way.

From inequalities (1.6) and (1.7) we may deduce

$$\sup_{t\in(0;\theta_1]} dist(\eta(\alpha,t),\eta(\alpha_n,t)) \xrightarrow{p} 0, n\to\infty. \tag{2.24}$$

Proceeding in the same manner we may extend this to

$$\sup_{t\in(0;\theta_n]} dist(\eta(\alpha,t),\eta(\alpha_n,t)) \xrightarrow{p} 0, n\to\infty, \tag{2.25}$$

particularly because of the strong Markov property of a Poisson point process. In fact, with high probability the processes $\eta(\alpha_n,\cdot)$ and $\eta(\alpha,\cdot)$ change up to time θ_n in the same way in the following sense: births occur in the same places at the same time moments. Deaths occur at the same time moments, and when a point is deleted from $\eta(\alpha,\cdot)$, then its counterpart is deleted from $\eta(\alpha_n,\cdot)$.

Since $\theta_n\to\infty$, we get (2.18). $\square$

2.1.20 Remark. In fact, we have proved an even stronger statement. Namely, take $\alpha_n\to\alpha$. Then there exist processes $(\xi_t^{(n)})_{t\in[0;T]}$ such that $(\xi_t^{(n)})_{t\in[0;T]} \stackrel{d}{=} (\eta(\alpha_n,t))_{t\in[0;T]}$

and

$$\sup_{t \in [0;T]} dist(\eta(\alpha,t), \xi_t^{(n)}) \xrightarrow{p} 0, \quad n \to \infty.$$

Thus, $Law\{\eta(\alpha,\cdot), \cdot \in (0;T]\}$ and $Law\{\eta(\alpha_n,\cdot), \cdot \in (0;T]\}$ are close in the space of measures over D_{Γ_0}, even when D_{Γ_0} is considered as topological space equipped with the *uniform* topology (induced by metric $dist$), and not with the Skorokhod topology.

2.1.2 The martingale problem

Now we briefly discuss the martingale problem associated with L defined in (2.1). Let $C_b(\Gamma_0(\mathbb{R}^d))$ be the space of all bounded continuous functions on $\Gamma_0(\mathbb{R}^d)$. We equip $C_b(\Gamma_0(\mathbb{R}^d))$ with the supremum norm.

2.1.21 Definition. A probability measure Q on $(D_{\Gamma_0}[0;\infty), \mathscr{B}(D_{\Gamma_0}[0;\infty)))$ is called a solution to the local martingale problem associated with L if

$$M_t^f = f(y(t)) - f(y(0)) - \int_0^t Lf(y(s-))ds, \quad \mathscr{I}_t, \quad 0 \le t < \infty,$$

is a local martingale for every $f \in C_b(\Gamma_0)$. Here y is the coordinate mapping, $y(t)(\omega) = \omega(t)$, $\omega \in D_{\Gamma_0}[0;\infty)$, $\mathscr{I}_t$ is the completion of $\sigma(y(s), 0 \le s \le t)$ under Q.

Thus, we require M^f to be a local martingale under Q with respect to $\{\mathscr{I}_t\}_{t \ge 0}$. Note that L can be considered as a bounded operator on $C_b(\Gamma_0(\mathbb{R}^d))$.

2.1.22 Proposition. *Let $(\eta(\alpha,t))_{t \ge 0}$ be a solution to (2.2). Then for every $f \in C(\Gamma_0)$ the process*

$$M_t^f = f(\eta(\alpha,t)) - f(\eta(\alpha,t)) - \int_0^t Lf(\eta(\alpha,s-))ds \tag{2.26}$$

is a local martingale under P with respect to $\{\mathscr{F}_t\}_{t \ge 0}$.

Proof. In this proof ζ_t will stand for $\eta(\alpha,t)$. Denote $\tau_n = \inf\{t \ge 0 : |\zeta_t| > n$ or $\zeta_t \nsubseteq$

$[-n;n]^d\}$. Clearly, τ_n, $n\in\mathbb{N}$, is a stopping time and $\tau_n\to\infty$ a.s. Let $\zeta_t^n=\zeta_{t\wedge\tau_n}$. We want to show that $({}^{(n)}M_t^f)_{t\geq 0}$ is a martingale, where

$$ {}^{(n)}M_t^f = f(\zeta_t^n) - f(\zeta_t^n) - \int_0^t Lf(\zeta_{s-}^n)ds. \tag{2.27}$$

The process $(\zeta_t)_{t\geq 0}$ satisfies

$$\zeta_t = \sum_{s\leq t, \zeta_s\neq\zeta_{s-}} [\zeta_s - \zeta_{s-}] + \zeta_0. \tag{2.28}$$

In the above equality as well as in few other places throughout this proof we treat elements of $\Gamma_0(\mathbb{R}^d)$ as measures rather than as configurations. Since (ζ_t) is of the pure jump type, the sum on the right-hand side of (2.28) is a.s. finite. Consequently we have

$$f(\zeta_t^n) - f(\zeta_0^n) = \sum_{s\leq t, \zeta_s\neq\zeta_{s-}} [f(\zeta_s^n) - f(\zeta_{s-}^n)]$$

$$= \int_{B\times(0;t]\times[0;\infty]} [f(\zeta_s) - f(\zeta_{s-})] I_{\{s\leq\tau_n\}} I_{[0;b(x,\zeta_{s-})]}(u) dN_1(x,s,u) \tag{2.29}$$

$$- \int_{\mathbb{Z}\times(0;t]\times[0;\infty]} I_{\{x_i\in\zeta_{s-}\}} [f(\zeta_s) - f(\zeta_{s-})] I_{\{s\leq\tau_n\}} I_{[0;d(x_i,\zeta_{s-})]}(v) dN_2(i,s,v).$$

Note that $\zeta_s = \zeta_{s-}\cup x$ a.s. in the first summand on the right-hand side of (2.29), and $\zeta_s=\zeta_{s-}\setminus x_i$ a.s. in the second summand. Now, we may write

$$\int_0^t I_{\{s\leq\tau_n\}} Lf(\zeta_s)ds =$$

$$\int_0^t \int_{x\in\mathbb{R}^d, u\geq 0} I_{\{s\leq\tau_n\}} I_{[0;b(x,\zeta_{s-})]}(u) [f(\zeta_{s-}\cup x) - f(\zeta_{s-})] dx du ds - \tag{2.30}$$

$$\int_0^t \int_{x\in\mathbb{R}^d, u\geq 0} I_{\{s\leq\tau_n\}} I_{[0;d(x,\zeta_{s-}))]}(v) [f(\zeta_{s-}\setminus x) - f(\zeta_{s-})] \zeta_{s-}(dx) dv ds.$$

Functions $b, d(\cdot,\cdot)$ and f are bounded on $\mathbb{R}^d \times \{\alpha : |\alpha| \leq n \text{ and } \alpha \subset [-n;n]^d\}$ and $\{\alpha : |\alpha| \leq n \text{ and } \alpha \subset [-n;n]^d\}$ respectively by a constant $C > 0$. Now, for a predictable bounded processes $(\gamma_s(x,u))_{0\leq s\leq t}$ and $(\beta_s(x,v))_{0\leq s\leq t}$, the processes

$$\int\limits_{B\times(0;t]\times[0;C]} I_{\{s\leq\tau_n\}}\gamma_s(x,u)[dN_1(x,s,u) - dxdsdu],$$

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\leq\tau_n\}}I_{\{x_i\in\zeta_{s-}\}}\beta_s(x_i,v)[dN_2(i,s,v) - \#(di)dsdv].$$

are martingales. Observe that

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\leq\tau_n\}}I_{\{x_i\in\zeta_{s-}\}}\beta_s(x_i,v)\#(di)dsdv = \int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\leq\tau_n\}}\beta_s(x,v)\zeta_{s-}(dx)dsdv$$

Taking

$$\gamma_s(x,u) = I_{[0;b(x,\zeta_{s-})]}(u)[f(\zeta_{s-}\cup x) - f(\zeta_{s-})],$$

$$\beta_s(x,v)) = I_{[0;d(x,\zeta_{s-})]}(v)[f(\zeta_{s-}\setminus x) - f(\zeta_{s-})],$$

we see that the difference on the right hand side of (2.27) is a martingale because of (2.29) and (2.30). $\square$

2.1.23 Corollary. The unique solution of (2.2) induces a solution of the martingale problem 2.1.21.

2.1.24 Remark. Since $y(s) = y(s-)$ P_α - a.s., the process

$$f(y(t)) - f(y(0)) - \int\limits_0^t Lf(y(s))ds,\ 0 \leq t < \infty,$$

is a local martingale, too.

2.1.3 Birth rate without sublinear growth condition

In this section we will consider equation (2.2) with the a birth rate coefficient that does not satisfy the sublinear growth condition (2.6).

Instead, we assume only that

$$\sup_{x\in\mathbb{R}^d, |\eta|\leq m} b(x,\eta) < \infty. \tag{2.31}$$

Under this assumption we can not guarantee existence of solution on the whole line $[0;\infty)$ or even on a finite interval $[0;T]$. It is possible that infinitely many points appear in finite time.

We would like to show that a unique solution exists up to an explosion time, maybe finite. Consider birth and death coefficients

$$\begin{aligned} b_n(x,\eta) &= b(x,\eta)I_{\{|\eta|\leq n\}}, \\ d_n(x,\eta) &= d(x,\eta)I_{\{|\eta|\leq n\}}. \end{aligned} \tag{2.32}$$

Functions b_n, d_n are bounded, so equation (2.2) with birth rate coefficient b_n and death rate coefficient d_n has a unique solution by Theorem 2.1.6. Remark 2.1.8 provides the existence and uniqueness of solution to (2.2) (with birth and death rate coefficients b and d, respectively) up to the (random stopping) time $\tau_n = \inf\{s \geq 0 : |\eta_s| > n\}$. Clearly, $\tau_{n+1} \geq \tau_n$; if $\tau_n \to \infty$ a.s., then we have existence and uniqueness for (2.2); if $\tau_n \uparrow \tau < \infty$ with positive probability, then we have an *explosion*. However, existence and uniqueness hold up to explosion time τ. When we have an explosion we say that the solution blows up. In Section 2.2 we discuss the possibility of an explosion for the Dieckmann-Law model.

2.1.4 Coupling

Here we discuss the coupling of two birth-and-death processes. The theorem we prove here will be used in the sequel. As a matter of fact, we have already used the coupling technique in the proof of Theorem 2.1.6.

Consider two equations of the form (2.2),

$$\begin{aligned}\xi_t^{(k)}(B) = &\int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b_k(x,\xi_{s-}^{(k)})]}(u)dN_1(x,s,u)\\ - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} &I_{\{x_i^{(k)}\in\xi_{r-}^{(k)}\cap B\}} I_{[0;d(x_i^{(k)},\eta_{r-})]}(v)dN_2(i,r,v) + \xi_0^{(k)}(B), \quad k=1,2,\end{aligned} \tag{2.33}$$

where $t \in [0;T]$ and $\{x_i^{(k)}\}$ is the sequence related to $(\xi_t^{(k)})_{t\in[0;T]}$.

Assume that initial conditions $\xi_0^{(k)}$ and coefficients b_k, d_k satisfy the conditions of Theorem 2.1.6. Let $(\xi_t^{(k)})_{t\in[0;T]}$ be the unique strong solutions.

2.1.25 Theorem. *Assume that almost surely $\xi_0^{(1)} \subset \xi_0^{(2)}$, and for any two finite configurations $\eta^1 \subset \eta^2$,*

$$b_1(x,\eta^1) \le b_2(x,\eta^2), \quad x\in\mathbb{R}^d \tag{2.34}$$

and

$$d_1(x,\eta^1) \ge d_2(x,\eta^2), \quad x\in\eta^1.$$

Then there exists a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process $(\eta_t)_{t\in[0;T]}$ such that $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(1)})_{t\in[0;T]}$ have the same law and

$$\eta_t \subset \xi_t^{(2)}, \quad t\in[0;T]. \tag{2.35}$$

Proof. Let $\{...,x_{-1}^{(2)},x_0^{(2)},x_1^{(2)},...\}$ be the sequence related to $(\xi_t^{(2)})_{t\in[0;T]}$. Consider the equation

$$\eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b_k(x,\eta_{s-})]}(u)dN_1(x,s,u)$$
$$- \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i^{(2)}\in\eta_{r-}\cap B\}} I_{[0;d(x_i^{(2)},\eta_{r-})]}(v)dN_2(i,r,v) + \xi_0^{(1)}(B), \quad k=1,2. \tag{2.36}$$

Note that here $\{x_i^{(2)}\}$ is related to $(\xi_t^{(2)})_{t\in[0;T]}$ and not to $(\eta_t)_{t\in[0;T]}$. Thus (2.36) is not an equation of form (2.2). Nonetheless, the existence of a unique solution can be shown in the same way as in the proof of Theorem 2.1.6. Denote the unique strong solution of (2.36) by $(\eta_t)_{t\in[0;T]}$.

Denote by $\{\tau_m\}_{m\in\mathbb{N}}$ the moments of jumps of $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(2)})_{t\in[0;T]}$, $0<\tau_1<\tau_2<\tau_3<....$ More precisely, a time $t\in\{\tau_m\}_{m\in\mathbb{N}}$ iff at least one of the processes $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(2)})_{t\in[0;T]}$ jumps at time t.

We will show by induction that each moment of birth for $(\eta_t)_{t\in[0;T]}$ is a moment of birth for $(\xi_t^{(2)})_{t\in[0;T]}$ too, and each moment of death for $(\xi_t^{(2)})_{t\in[0;T]}$ is a moment of death for $(\eta_t)_{t\in[0;T]}$ if the dying point is in $(\eta_t)_{t\in[0;T]}$. Moreover, in both cases the birth or the death occurs at exactly the same point. Here a moment of birth is a random time at which a new point appears, a moment of death is a random time at which a point disappears from the configuration. The statement formulated above is in fact equivalent to (2.35).

Here we deal only with the base case, the induction step is done in the same way. We have nothing to show if τ_1 is a moment of a birth of $(\xi_t^{(2)})_{t\in[0;T]}$ or a moment of death of $(\eta_t)_{t\in[0;T]}$. Assume that a new point is born for $(\eta_t)_{t\in[0;T]}$ at τ_1,

$$\eta_{\tau_1}\setminus\eta_{\tau_1-}=\{x_1\}.$$

The process $(\eta_t)_{t\in[0;T]}$ satisfies (2.36), therefore $N_1(\{x\},\{\tau_1\},[0;b_k(x_1,\eta_{\tau_1-})])=1$. Since

$$\eta_{\tau_1-}=\xi_0^{(1)}\subset\xi_0^{(2)}=\xi_{\tau_1-}^{(2)},$$

by (2.34)

$$N_1(\{x\}, \{\tau_1\}, [0; b_k(x_1, \xi^{(2)}_{\tau_1 -})]) = 1,$$

hence

$$\xi^{(2)}_{\tau_1} \setminus \xi^{(2)}_{\tau_1 -} = \{x_1\}.$$

The case when τ_2 is a moment of death for $(\xi^{(2)}_t)_{t\in[0;T]}$ is analyzed analogously.

It remains to show that $(\eta_t)_{t\in[0;T]}$ and $(\xi^{(1)}_t)_{t\in[0;T]}$ have the same law. We mentioned above that formally equation (2.36) is not of the form (2.2), so we can not directly apply the uniqueness in law result. However, since $\eta_t \in \xi^{(2)}_t$ a.s., $t \in [0; T]$, we can still consider (2.36) as an equation of the form (2.2). Indeed, let $\{..., y_{-1}, y_0, y_1, ...\}$ be the sequence related to η_t. We have $\{y_{-|\xi^{(1)}_0|+1}, ..., y_{-1}, y_0, y_1, ...\} \subset \{x_{-|\xi^{(2)}_0|+1}, ..., x^{(2)}_{-1}, x^{(2)}_0, x^{(2)}_1, ...\}$. There exists an injection $\varsigma : \{-|\xi^{(1)}_0|+1, ..., 0, 1, ...\} \to \{-|\xi^{(2)}_0|+1, ..., 0, 1, ...\}$ such that $y_{\varsigma(i)} = x_i$. Denote $\theta_i = \inf\{s \geq 0 : y_i \in \eta_s\}$. Note that θ_i is a stopping time with respect to $\{\mathscr{F}_t\}$. Define a Poisson point process $\bar{N}_2$ by

$$\bar{N}_2(\{i\} \times R \times V) = N_2(\{i\} \times R \times V), \quad i \in \mathbb{Z}, R \subset [0; \theta_i], V \subset \mathbb{R}_+,$$

and

$$\bar{N}_2(\{i\} \times R \times V) = N_2(\{\varsigma(i)\} \times R \times V), \quad i \in \mathbb{Z}, R \subset (\theta_i; \infty), V \subset \mathbb{R}_+.$$

The process $\bar{N}_2$ is $\{\mathscr{F}_t\}$-adapted. One can see that $(\eta_t)_{t\in[0;T]}$ is the unique solution of equation (2.2) with N_2 replaced by $\bar{N}_2$. Hence $(\eta_t)_{t\in[0;T]} \overset{d}{=} (\xi^{(1)}_t)_{t\in[0;T]}$.

2.1.5 Related semigroup of operators

We say now a few words about the semigroup of operators related to the unique solution of (2.2). We write $\eta(\alpha, t)$ for a unique solution of (2.2), started from $\alpha \in \Gamma_0(\mathbb{R}^d)$. We want to define an operator S_t by

$$S_t f(\alpha) = Ef(\eta(\alpha, t)) \quad (= E_\alpha f(\eta(t))) \tag{2.37}$$

for an appropriate class of functions. Unfortunately, it seems difficult to make S_t a C_0-semigroup on some functional Banach space for general b, d satisfying (2.6) and (2.7).

We start with the case when the cumulative birth and death rates are bounded. Let $C_b = C_b(\Gamma_0(\mathbb{R}^d))$ be the space of all bounded continuous functions on $\Gamma_0(\mathbb{R}^d)$. It becomes a Banach space once it is equipped with the supremum norm. We assume the existence of a constant $C > 0$ such that for all $\zeta \in \Gamma_0(\mathbb{R}^d)$

$$|B(\zeta)| + |D(\zeta)| < C, \tag{2.38}$$

where B and D are defined in (2.4) and (2.5). Formula (2.1) defines then a bounded operator $L : C_b \to C_b$, and we will show that S_t coincides with e^{tL}. For $f \in C_b$, the function $S_t f$ is bounded and continuous. Boundedness is a consequence of the boundedness of f, and continuity of $S_t f$ follows from Remark 2.1.20. Indeed, let $\alpha_n \to \alpha$, $\xi_t^{(n)} \stackrel{d}{=} \eta(\alpha_n, t)$ and

$$dist(\eta(\alpha, t), \xi_t^{(n)}) \xrightarrow{p} 0, \quad n \to \infty.$$

Unlike $\Gamma(\mathbb{R}^d)$, the space $\Gamma_0(\mathbb{R}^d)$ is a σ-compact space. Consequently, for all $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \Gamma_0(\mathbb{R}^d)$ such that for large enough n

$$P\{\eta(\alpha, t) \in K_\varepsilon, \ \xi_t^{(n)} \in K_\varepsilon\} \geq 1 - \varepsilon.$$

Also, for fixed $\delta > 0$ and for large enough n

$$P\{dist(\eta(\alpha, t), \xi_t^{(n)}) \leq \delta\} \geq 1 - \delta.$$

Fix $\varepsilon > 0$. There exists $\delta_\varepsilon \in (0;\varepsilon)$ such that $|f(\beta)-f(\gamma)| \le \varepsilon$ whenever $dist(\beta,\gamma) \le \delta_\varepsilon$, $\beta,\gamma \in K_\varepsilon$. We have for large enough n

$$|E[f(\eta(\alpha,t)) - f(\xi_t^{(n)})]|$$

$$\le E|f(\eta(\alpha,t)) - f(\xi_t^{(n)})|I\{\eta(\alpha,t) \in K_\varepsilon,\ \xi_t^{(n)} \in K_\varepsilon, dist(\eta(\alpha,t),\xi_t^{(n)}) \le \delta_\varepsilon\}$$

$$+2(\delta_\varepsilon + \varepsilon)||f|| \le \varepsilon + 2(\delta_\varepsilon + \varepsilon)||f||,$$

where $||f|| = \sup_{\zeta \in \Gamma_0(\mathbb{R}^d)} |f(\zeta)|$. Letting $\varepsilon \to 0$, we see that

$$Ef(\eta(\alpha_n,t)) = Ef(\xi_t^{(n)}) \to Ef(\eta(\alpha,t)).$$

Thus, $S_t f$ is continuous (note that the continuity of $S_t f$ does not follow from Theorem 2.1.19 alone, since for a fixed $t \in [0;T]$ the functional $D_{\Gamma_0(\mathbb{R}^d)}[0;T] \ni x \mapsto x(t) \in \mathbb{R}$ is not continuous in the Skorokhod topology). Furthermore, since for small t and for all $A \in \mathscr{B}(\mathbb{R}^d)$,

$$P\{\eta(\alpha,t) = \alpha\} = 1 - t[B(\alpha) + D(\alpha)] + o(t), \tag{2.39}$$

$$P\{\eta(\alpha,t) = \alpha \cup \{y\} \text{ for some } y \in A\} = t \int\limits_{y \in A} b(y,\alpha)dy + o(t), \tag{2.40}$$

and for $x \in \alpha$

$$P\{\eta(\alpha,t) = \alpha \setminus \{x\}\} = td(x,\alpha) + o(t), \tag{2.41}$$

we may estimate

$$|S_t f(\alpha) - f(\alpha)| \le t\,[B(\alpha) + D(\alpha)]\,||f|| + o(t)||f|| \le C||f||t + o(t).$$

Therefore, (2.37) defines a C_0 semigroup on C_b. Its generator

$$\tilde{L}f(\alpha) = \lim_{t\to 0+} \frac{S_t f(\alpha)}{t} =$$

$$\lim_{t\to 0+} \left[\int\limits_{x\in\mathbb{R}^d} b(x,\alpha)[f(\alpha\cup x) - f(\alpha)]dx + \sum_{x\in\alpha} d(x,\alpha)(f(\alpha\setminus x) - f(\alpha)) + o(t) \right] = Lf(\alpha).$$

Thus, $S_t = e^{tL}$, and we have proved the following

2.1.1 Proposition. *Assume that* (2.38) *is fulfilled. Then the family of operators* $(S_t, t \geq 0)$ *on* C_b *defined in* (2.37) *constitutes a* C_0*-semigroup. Its generator coincides with* L *given in* (2.1).

Now we turn out attention to general b, d satisfying (2.6) and (2.7) but not necessarily (2.38). The family of operators $(S_t)_{t\geq 0}$ still constitutes a semigroup, however it does not have to be strongly continuous anymore. Consider truncated birth and death coefficients (2.32) and corresponding process $\eta^n(\alpha,t)$. Remark 2.1.8 implies that $\eta^n(\alpha,t) = \eta(\alpha,t)$ for all $t \in [0;\tau_n]$, where

$$\tau_n = \inf\{s \geq 0 : |\eta(\alpha,s)| > n\}. \tag{2.42}$$

Growth condition (2.6) implies that $\tau_n \to \infty$ for any $\alpha \in \Gamma_0(\mathbb{R}^d)$.

Truncated coefficients b_n, d_n satisfy (2.38) and

$$S_t^{(n)} f(\alpha) = Ef(\eta^{(n)}(\alpha,t)) \tag{2.43}$$

defines a C_0 - semigroup on C_b. In particular, for all $\alpha \in \Gamma_0(\mathbb{R}^d)$

$$L^{(n)} f(\alpha) = \lim_{t\to 0+} \frac{Ef(\eta^{(n)}(\alpha,t)) - f(\alpha)}{t},$$

where $L^{(n)}$ is operator defined as in (2.1) but with b_n, d_n instead of b, d. Letting $n \to \infty$

we get, for fixed α and f,

$$Lf(\alpha) = \lim_{t \to 0+} \frac{Ef(\eta(\alpha, t)) - f(\alpha)}{t} = \lim_{t \to 0+} \frac{S_t f(\alpha) - f(\alpha)}{t}. \tag{2.44}$$

Taking limit by n is possible: for $n \geq |\alpha| + 2$, $\eta^{(n)}(\alpha, t)$ satisfies (2.39), (2.40) and (2.41), therefore $\eta(\alpha, t)$ satisfies (2.39), (2.40) and (2.41), too. Thus, we have

2.1.2 Proposition. *Let* b *and* d *satisfy* (2.6) *and* (2.7) *but not necessarily* (2.38). *Then the family of operators* $(S_t, t \geq 0)$ *constitutes a semigroup on* C_b *which does not have to be strongly continuous. However, for every* $\alpha \in \Gamma_0(\mathbb{R}^d)$ *and* $f \in C_b$ *we have* (2.44).

Formula (2.44) gives us the formal relation of $(\eta(\alpha, t))_{t \geq 0}$ to the operator L. Of course, for fixed f the convergence in (2.44) does not have to be uniform in α.

2.1.3 Remark. The question about the construction of a semigroup acting on some class of probability measures on $\Gamma_0(\mathbb{R}^d)$ is yet to be studied.

2.2 Explosion and non-explosion for birth-and-death processes in $\Gamma_0(\mathbb{R}^d)$

In this section we consider the possibility of an explosion in finite time for some birth-and-death Markov processes in $\Gamma_0(\mathbb{R}^d)$. We are only interested in the cases which are not covered by the theorem about existence and uniqueness of solution to (2.2). The idea of the proof is suggested by Finkelshtein and Kondratiev (private conversation).

2.2.1 Explosion

Consider a generator of the form

$$(LF)(\gamma) = \sum_{y\in\gamma} \int_{\mathbb{R}^d} a_+(x-y)\left(\lambda + \sum_{y'\in\gamma\setminus y} b_+(y-y')\right)\big[F(\gamma\cup x) - F(\gamma)\big]dx, \qquad (2.45)$$

so that the birth and death coefficients are

$$b(x,\gamma) = \sum_{y\in\gamma} a_+(x-y)\left(\lambda + \sum_{y'\in\gamma\setminus y} b_+(y-y')\right), \quad d(x,\gamma)\equiv 0. \qquad (2.46)$$

Here $0 \le a^+, b^+ \in L^1(\mathbb{R}^d)$, $\lambda > 0$ and $\int_{\mathbb{R}^d} a^+(x)dx = 1$.

Let Λ, a_+ and b_+ be such that there exists a constant $c_\Lambda > 0$ for which

$$a_+(x-y),\ a_+(x-y)b_+(y-y') \ge c_\Lambda, \quad x,y,y' \in \Lambda. \qquad (2.47)$$

Under these assumptions, we will prove that explosion happens with positive probability under P_μ , where μ is the distribution of the initial condition satisfying

$$\int |\gamma\cap\Lambda|\mu(d\gamma) = E_\mu|\eta_0| > 0. \qquad (2.48)$$

2.2.1 Proposition. *For a solution (η_t) of* (2.2) *with birth and death coefficients given in* (2.46) *and initial distribution μ satisfying* (2.48)*, an explosion occurs with positive probability.*

It suffices to consider μ concentrated at one point, that is, we may assume that η_0 is not random and $|\eta_0\cap\Lambda| \ge 1$. We are going to use the coupling technique. Let us introduce a new birth rate coefficient, $\tilde{a}_+(x,y) = c_\Lambda I_{\{x,y\in\Lambda\}}$, $\tilde{b}_+(x,y) = I_{\{x,y\in\Lambda\}}$,

$$\tilde{b}(x,\gamma) = \sum_{y\in\gamma} \tilde{a}_+(x,y)\left(\lambda + \sum_{y'\in\gamma\setminus y} \tilde{b}_+(y,y')\right) = I_{\{x\in\Lambda\}}|\gamma\cap\Lambda|(\lambda + |\gamma\cap\Lambda| - 1). \qquad (2.49)$$

Then $\tilde{b}(x,\gamma_1) \le \tilde{b}(x,\gamma_2) \le b(x,\gamma_2)$, $\gamma_1 \subset \gamma_2$. Consider the unique solution $(\xi(\eta_0,t))_{t\in[0;T]}$ of (2.2) with the death coefficient equal identically zero and the birth coefficient as in (2.49) and initial condition η_0. Let $\tau_n = \inf\{t > 0 : |\eta_t(\eta_0,t)| > n\}$. Remark 2.1.8 and a straightforward generalization of Theorem 2.1.25 imply that there exists a process $(\bar{\xi}_t)$ such that $(\bar{\xi}_t) \stackrel{d}{=} (\xi_t)$ and a.s.

$$\bar{\xi}_t \subset \eta(\eta_0,t), \ t \in [0;\tau). \tag{2.50}$$

Thus, it is sufficient to show that $(\xi(\eta_0,t))_{t\in[0;T]}$ blows up. Clearly, the embedded chain $Y = (Y_n)_{n\in\mathbb{N}}$ of $(\xi(\eta_0,t))_{t\in[0;T]}$ satisfies $|Y_n| = n + |\eta_0|$ and, since all new points appear inside Λ, $|Y_n \cap \Lambda| = n + |\eta_0 \cap \Lambda|$. Given Y, the times $\{\varsigma_n\}_{n\in\mathbb{N}}$ between consequent jumps of $(\xi_t)_{t\in[0;T]}$ are exponentially distributed, and the parameter of ς_{n+1} is almost surely equal to

$$p_n = \int_{\mathbb{R}^d} \tilde{b}(x,Y_n)dx = \sum_{y\in Y_n} \int_{\mathbb{R}^d} \tilde{a}_+(x,y)\left(\lambda + \sum_{y'\in\gamma\setminus y} \tilde{b}_+(y,y')\right)dx =$$

$$c_\Lambda \sum_{y\in Y_n\cap\Lambda} l(\Lambda)\big(\lambda + |Y_n\cap\Lambda| - 1\big) = c_\Lambda l(\Lambda)|Y_n\cap\Lambda|\big(\lambda + |Y_n\cap\Lambda| - 1\big) =$$

$$c_\Lambda l(\Lambda)(n + |\eta_0\cap\Lambda|)\big(\lambda + n + |\eta_0\cap\Lambda| - 1\big),$$

where l is the Lebesgue measure on $\mathbb{R}^d$. We see that the parameter p_n is a.s. constant, therefore ς_n is exponentially distributed with parameter $p_n \ge c_\Lambda l(\Lambda)n^2$. Since $\sum_{n\ge1}\frac{1}{p_n} < \infty$ and $\sum_{n\ge1}\frac{1}{p_n^2} < \infty$ (those are the sums of the expectations and the variances of ς_n, respectively), we have $\sum_{n\ge1}\varsigma_n < \infty$ a.s. Thus, the process $(\xi(\eta_0,t))_{t\in[0;T]}$ explodes.

2.2.2 Non-explosion

Let us consider so-called Dieckmann-Law heuristic generator

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(m + \sum_{y\in\gamma\setminus x} a_-(x-y)\right)\big[F(\gamma\setminus x) - F(\gamma)\big] \tag{2.51}$$
$$+ \sum_{y\in\gamma}\int_{\mathbb{R}^d} a_+(x-y)\left(\lambda + \sum_{y'\in\gamma\setminus y} b_+(y-y')\right)\big[F(\gamma\cup x) - F(\gamma)\big]dx.$$

The birth rate coefficient is the same as in (2.46), the death rate coefficient is given by

$$d(x,\gamma) = m + \sum_{y\in\gamma\setminus x} a_-(x-y). \tag{2.52}$$

We saw in the previous section that a solution of (2.2) may explode when the birth rate coefficient b is of such a "quadratic" form. Here we want to demonstrate that the growth of the number of points may be suppressed by including a sufficiently large death rate coefficient.

To do so, we will assume that

$$\begin{gathered} m > \lambda||a_+||_{L^1}, \\ a_-(z) \geq b_+(z)||a_+||_{L^1},\ \ z\in\mathbb{R}^d. \end{gathered} \tag{2.53}$$

Let L_n be the generator of the C_0-semigroup associated with the process $\eta^{(n)}(\cdot,\cdot)$. In fact,

$$L_nF(\zeta) = [LF(\zeta)]I_{|\zeta|\leq n}.$$

Assumptions (2.53) ensure that the function $V_n \in C_b(\Gamma_0(\mathbb{R}^d))$, $V_n(\zeta) = |\zeta|\wedge n$ satisfies

$$L_nV_n(\zeta) \leq 0,\quad \zeta\in\Gamma_0(\mathbb{R}^d). \tag{2.54}$$

Indeed, for $|\zeta| \le n$

$$L_n V_n(\zeta) = -\sum_{y\in\gamma}\left(m + \sum_{y'\in\gamma\setminus y} a_-(y-y')\right)$$

$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} a_+(x-y)\left(\lambda + \sum_{y'\in\gamma\setminus y} b_+(y-y')\right)dx =$$

$$|\gamma|(\lambda||a_+||_{L^1} - m) + \sum_{\{y,y'\}\subset\gamma}\left(||a_+||_{L^1} b_+(y-y') - a_-(y-y')\right) \le 0.$$

We have for all ζ

$$EV_n(\eta^{(n)}(\zeta,t)) = e^{tL_n}V_n(\zeta) = V_n(\zeta) + \int\limits_0^t e^{sL_n}L_nV_n(\zeta)ds,$$

see e.g. [EN00, Chapter 2, Lemma 1.3] or [EN06, Chapter 2, Lemma 1.3]. By the Markov property we obtain an analog of Dynkin's formula: for $t' < t$

$$E[V_n(\eta^{(n)}(\zeta,t)) \mid \mathscr{I}_{t'}] = EV_n(\eta^{(n)}(\zeta,t')) + E\int\limits_{t'}^t e^{sL_n}L_nV_n(\eta^{(n)}(\zeta,t'))ds,$$

where $\mathscr{I}_{t'} = \sigma((\eta^{(n)}(\zeta,s)), s \le t')$. Now, e^{sL_n} preserves the cone of non-positive functions, therefore (2.54) implies that

$$E[V_n(\eta^{(n)}(\zeta,t)) \mid \mathscr{I}_{t'}] \le EV_n(\eta^{(n)}(\zeta,t')).$$

We see that the process $(V_n(\eta^{(n)}(\zeta,t)))_{t\in[0;T]}$ is a bounded non-negative supermartingale. In particular,

$$P\{\sup_{t\in[0;T]} V_n(\eta^{(n)}(\zeta,t)) \ge n\} \le \frac{EV_n(\eta^{(n)}(\zeta,0)}{n} \le \frac{|\zeta|}{n} \to 0$$

as $n\to\infty$. Since $\tau_n = \inf\{t>0 : \eta(\zeta,t) > n\} = \inf\{t > 0 : \eta^{(n+1)}(\zeta,t) > n\}$, we have for all $T > 0$

$$P\{\tau_n \leq T\} \to 0, \quad n \to \infty.$$

Consequently, $\tau_n \to \infty$ a.s., and a solution to equation (2.2) exists on $[0; \infty]$ and is unique in the sense that any two solutions are version of each other. Just, we have

2.2.2 Proposition. *Strong existence and pathwise uniqueness hold for equation* (2.2) *with a non-random initial condition, the birth rate coefficient given in* (2.46) *and the death rate coefficient given in* (2.52).

Remark. Considering equations for the density functions of the processes corresponding to heuristic generators (2.45) and (2.51), one could show that an explosion of densities occurs or does not occur, respectively. Computations are similar to those we use here.

Chapter 3

Stability analysis

The spatial birth-and-death process constructed in the previous chapter is a pure jump type Markov process in $\Gamma_0(\mathbb{R}^d)$. Therefore, asymptotic analysis of the process sometimes comes down to the analysis of the embedded chain. For example, the question when the unique solution $(\eta(\alpha, t))_{t\geq 0}$ of (2.2) hits a Borel set $A \subset \Gamma_0(\mathbb{R}^d)$ with positive probability is discussed in section (3.2). This question admits an equivalent formulation in terms of the embedded Markov chain. Indeed, the process $(\eta(\alpha, t))_{t\geq 0}$ hits A if and only if its embedded chain hits A.

3.1 Stability structures

We give here several definitions and statements from the theory of stochastic stability for (discrete time) Markov chains. The main reference for this section is the book by Meyn and Tweedie [MT93], where one can find all the definitions and statements given here except for Lemma 3.1.2.

3.1.1 Stochastic stability for discrete Markov chains: general notions

Let X be a Polish space. Consider a transition probability kernel $P : X \times \mathscr{B}(X) \to [0;1]$.

Theorem ([MT93, Theorem 3.4.1]) For any initial measure μ on $\mathscr{B}(X)$ and any transition probability kernel $P = \{P(x, A), x \in X, A \in \mathscr{B}(X)\}$, there exists a stochastic process $\Phi = \{\Phi_1, \Phi_2, ...\}$ on $\Omega = X^\infty$, measurable with respect to $\mathscr{F} = \mathscr{B}(X^\infty)$, and a probability measure P_μ on $\mathscr{F}$ such that $P_\mu(B)$ is the probability of the event $\{\Phi \in B\}$ for $B \in \mathscr{F}$; and for measurable $A_i \subset X_i$, $i = 0, 1, ..., n$, and $n \in \mathbb{Z}_+$

$$\begin{aligned} P_\mu(\Phi_0 \in & A_0, \Phi_1 \in A_1, ..., \Phi_n \in A_n) = \\ & \int_{y_0 \in A_0} ... \int_{y_{n-1} \in A_{n-1}} \mu(dy_0) P(y_0, dy_1) ... P(y_{n-1}, dy_n). \end{aligned} \tag{3.1}$$

According to this theorem, for any probability measure μ on X and transition probability kernel P there exists a Markov chain with initial distribution μ and transition probability kernel P. We note that there exists only one such chain, in the sense that any two have the same distribution in X^∞. We will consider a Markov chain defined on the canonical space $\Omega = \prod_{i=0}^{\infty} X$, with Φ_n being the coordinate mappings,

$$\Phi_n((x_0, x_1, ...)) = x_n.$$

Let P_x denote the distribution of Φ in X^∞ when the initial distribution is the Dirac measure at x, $P_x\{\Phi_0 = x\} = 1$.

For any set $A \in \mathscr{B}(X)$, the variables

$$\tau_A = \min\{n \geq 1 : \Phi_n \in A\}, \quad \sigma_A = \min\{n \geq 0 : \Phi_n \in A\}$$

are called the *first return time* and the *first hitting time* respectively, and the variable

$$J_A = \sum_{n=1}^{\infty} I_{\{\Phi_n \in A\}}$$

is called the *occupation time*. Define also the return probabilities

$$\begin{aligned} L(x, A) :&= P_x\{\tau_A < \infty\} \\ &= P_x\{\Phi \text{ ever enters } A\}. \end{aligned} \tag{3.2}$$

3.1.1 Definition. A chain Φ is called *ϕ-irreducible* if there exists a finite non-trivial measure ϕ on $\mathscr{B}(\Gamma_0(\mathbb{R}^d))$ such that $\phi(A) > 0$ implies $L(x, A) > 0$ for all $x \in X$. A Markov chain Φ is called *ψ-irreducible* if there exists a finite non-trivial measure ψ on $\mathscr{B}(\Gamma_0(\mathbb{R}^d))$ such that

$$(\forall x \in X: \ L(x, A) > 0) \Leftrightarrow \psi(A) > 0.$$

The measures ϕ and ψ from the definition above are called an *irreducibility measure* and a *maximal irreducibility measure* for Φ, respectively.

Let $a = \{a(n)\}$ be a probability measure on $\mathbb{Z}_+ = \{0, 1, 2, ...\}$. Define the probability transition kernel

$$K_a(x, A) := \sum_{n=0}^{\infty} P^n(x, A)a(n), \quad x \in X, \ A \in \mathscr{B}(X). \tag{3.3}$$

For $\epsilon \in (0; 1)$, we set a_ϵ to be the geometric distribution with parameter ϵ: $a_\epsilon(n) = (1-\epsilon)\epsilon^n$. The relation between irreducibility and maximal irreducibility is clarified in the next proposition.

Proposition ([MT93, Proposition 4.2.2]) If Φ is ϕ-irreducible for some measure ϕ, then there exists a probability measure ψ on $\mathscr{B}(X)$ such that

(i) Φ is ψ-irreducible;

(ii) for any other measure ϕ', the chain Φ is ϕ'-irreducible if and only if ϕ' is absolutely continuous with respect to ψ;

(iii) if $\psi(A) = 0$, then $\psi\{y \in X : P(y, A) > 0\} = 0$;

(iv) the probability measure ψ is equivalent to

$$\psi' := \int_X \phi'(dy) K_{a_{\frac{1}{2}}}(y, A),$$

for any finite irreducibility measure ϕ'.

The next Lemma provides a sufficient condition for an irreducibility measure to be a maximal irreducibility measure.

3.1.2 Lemma. *If Φ is ϕ-irreducible and the measure ϕ is such that $\phi\{y : P(y, A) > 0\} = 0$ whenever $\phi(A) = 0$, then Φ is ψ-irreducible with $\phi = \psi$.*

Proof. Let ϕ be a measure satisfying conditions of the lemma. We first prove that

$$\phi\{y : L(y, A) > 0\} = 0 \text{ whenever } \phi(A) = 0. \tag{3.4}$$

Note that

$$\{y : L(y, A) > 0\} = \bigcup_{n \in \mathbb{N}} \{y : P^n(y, A) > 0\}. \tag{3.5}$$

For $A \in \mathscr{B}(X)$ and $k \in \mathbb{N}$, denote $A^{(-k)} := \{x \in X : P^k(x, A) > 0\}$. To prove (3.4), we will proceed by induction and show that $\phi\{y : P^n(y, A) > 0\} = 0$ as long as $\phi(A) = 0$, for all $n \in \mathbb{N}$. Assume that $\phi\{y : P^m(y, A) > 0\} = 0$ whenever $\phi(A) = 0$. Then, if $\phi(A) = 0$,

$$\phi\{y : P^{m+1}(y, A) > 0\} = \phi\{y : \int_{x \in X} P(y, dx) P^m(x, A) > 0\} \leq$$

$$\phi\{y : \int_{x\in X} P(y,dx) I_{A^{(-m)}}(x) > 0\} = \phi\{y : P(y, A^{(-m)}) > 0\} = 0.$$

The base case is given in the condition, therefore (3.4) holds.

Assume now that the statement of the lemma does not hold, so that ϕ is not a maximal irreducible measure for Φ. Proposition 4.2.2 from [MT93] (see Page 79 of the present work) implies the existence of a maximal irreducible measure ψ' for Φ. Then there exists a set $C \in \mathscr{B}(X)$ such that $\phi(C) = 0$ whereas $\psi'(C) > 0$. By definition of irreducibility, $L(x, C) > 0$ for all $x \in X$. By (3.4), $\phi\{y : L(y, C) > 0\} = 0$, hence $\phi(X) = 0$, which contradicts to the non-triviality of ϕ. □

3.1.3 Definition. A set $C \in \mathscr{B}(X)$ is called a *small* set if there exists an $m \in \mathbb{N}$, and a non-trivial measure ν_m on $\mathscr{B}(X)$ such that for all $x \in C, B \in \mathscr{B}(X)$,

$$P^m(x, B) \geq \nu_m(B).$$

A set $C \in \mathscr{B}(X)$ is ν_a*-petite* if the transition probability kernel K_a satisfies the bound

$$K_a(x, B) \geq \nu_a(B), \quad B \in \mathscr{B}(X)$$

for all $x \in C$, where ν_a is a non-trivial measure on $\mathscr{B}(X)$. Here a is a distribution on $\mathbb{Z}_+$, K_a is defined in (3.3). C is called *petite* when it is ν_a-petite for some non-trivial measure ν_a and some distribution a.

3.1.2 Transience and recurrence

Here we briefly discuss the concepts of transience and recurrence for a Markov chain on a general state space.

3.1.4 Definition. A set $A \in \mathscr{B}(X)$ is called *uniformly transient* if there exist $M < \infty$ such that $E_x J_A \leq M$ for all $x \in A$.

A set $A \in \mathscr{B}(X)$ is called *recurrent* if $E_x J_A = \infty$ for all $x \in A$.

For a ψ-irreducible Markov chain Φ, denote by $\mathscr{B}^+(X)$ the collection of sets from σ-algebra $\mathscr{B}(X)$ of positive ψ measure: $A \in \mathscr{B}^+(X) \Leftrightarrow \psi(A) > 0$. Note that all maximal irreducible measures are equivalent, therefore $\mathscr{B}^+(X)$ does not depend on the choice of ψ.

Theorem. ([MT93, Theorem 8.0.1]) Suppose that Φ is ψ-irreducible. Then either

(i) every set in $\mathscr{B}^+(X)$ is recurrent, in which case we call Φ recurrent, or

(ii) there is a countable cover of X with uniformly transient sets, in which case we call Φ transient, and every petite set is uniformly transient.

Let us introduce another, stronger concept of recurrence, known as *Harris recurrence.*

Definition([MT93, page 200]). A set $A \in \mathscr{B}(X)$ is called *Harris recurrent* if

$$P_x\{J_A = \infty\} = 1, \quad x \in A.$$

A chain Φ is called *Harris (recurrent)* if it is ψ-irreducible and every set in $\mathscr{B}^+(X)$ is Harris recurrent.

For a measurable function $V : X \to [0; \infty)$, denote $\Delta V(x) = \int\limits_{y \in X} V(y) P(x, dy) - V(x)$. The operator Δ is called the *drift operator*. We say that V is *unbounded off petite sets* for Φ if for any $r < \infty$, the sublevel set

$$C_V(r) := \{x \in X : V(x) \leq r\}$$

is petite.

In the next section we will discuss transience and recurrence for spatial birth-and-death Markov processes. We will use the following criteria, cf. [MT93, Theorems 8.4.2, 8.4.3 and 9.1.8] .

3.1.5 Drift criterion for transience. *Suppose Φ is a ψ-irreducible chain. Then Φ is* *transient if and only if there exists a bounded function $V : X \to [0;\infty)$ and $r \geq 0$ such that*

(i) the set $C_V(r)$ and its complement $C_V(r)^c$ lie in $\mathscr{B}^+(X)$;

(ii) whenever $x \in C_V(r)^c$,

$$\Delta V(x) > 0. \tag{3.6}$$

3.1.6 Drift condition for recurrence. *Suppose Φ is ψ-irreducible. If there exists a petite set $C \subset X$ and a positive function V which is unbounded off petite sets such that*

$$\Delta V(x) \leq 0 \tag{3.7}$$

holds whenever $x \in C^c$, then $L(x, C) \equiv 1$, and Φ is recurrent. Furthermore, Φ is Harris recurrent.

3.2 Stochastic stability for birth-and-death processes in $\Gamma_0(\mathbb{R}^d)$

In this section we give some results about stability of the embedded chain of the unique solution $(\eta(\alpha,t))_{t\geq 0}$ of (2.2). The main result in this section is Theorem 3.2.2.

Denote by $(\xi_n)_{n\in\mathbb{Z}_+}$ the embedded chain of $(\eta(\alpha,t))_{t\geq 0}$.

The transition probabilities of $(\xi_n)_{n\in\mathbb{Z}_+}$ are completely described by

$$\begin{aligned} Q(\eta, \{\eta \setminus \{x\}\}) &= \frac{d(x,\eta)}{(B+D)(\eta)}, \qquad x \in \eta, \quad \eta \in \Gamma_0(\mathbb{R}^d), \\ Q(\eta, \{\eta \cup \{x\}, x \in U\}) &= \frac{\int_{x\in U} b(x,\eta)dx}{(B+D)(\eta)}, \quad U \in \mathscr{B}(\mathbb{R}^d), \eta \in \Gamma_0(\mathbb{R}^d), \end{aligned} \tag{3.8}$$

where $(B+D)(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)dx + \sum\limits_{x\in\eta} d(x,\eta)$, the jump rate at η.

Let λ be the Lebesgue-Poisson measure (see Section 1.2.2) on $(\Gamma_0(\mathbb{R}^d), \mathscr{B}(\Gamma_0(\mathbb{R}^d)))$ whose intensity measure is the Lebesgue measure on $\mathbb{R}^d$. Denote by Q_α the distribution of the embedded chain of $(\eta(\alpha,t))_{t\geq 0}$ on $((\Gamma_0(\mathbb{R}^d))^\infty, \mathcal{B}((\Gamma_0(\mathbb{R}^d))^\infty))$, i.e., the distribution of $(\xi_n)_{n\in\mathbb{Z}_+}$, started from α. Here $\mathcal{B}((\Gamma_0(\mathbb{R}^d))^\infty))$ is the σ-algebra generated by the coordinate mappings. In order not to introduce new notations, we use the same symbol ξ_n for a coordinate mapping on $((\Gamma_0(\mathbb{R}^d))^\infty, \mathcal{B}((\Gamma_0(\mathbb{R}^d))^\infty))$. For example, for $B \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$

$$P\{\xi_n \in B\} = Q_\alpha\{\xi_n \in B\}.$$

Under some assumptions, we will prove that λ is a maximal irreducibility measure for $(\xi_n)_{n\in\mathbb{Z}_+}$. In other words,

$$(\forall \alpha : Q_\alpha\{(\xi_n)_{n\in\mathbb{Z}_+} \text{ ever enters } A\} > 0) \Leftrightarrow \lambda(A) > 0.$$

This is the statement of Theorem 3.2.2 below.

While the thesis is devoted to continuous time spatial birth-and-death processes, here we formulate and prove our results in terms of the embedded chain $(\xi_n)_{n\in\mathbb{Z}_+}$. We do so because formulations are more natural and simpler for the questions we consider here. Besides that, the process $(\xi_n)_{n\in\mathbb{Z}_+}$ can be an object of interest on its own. A Markov chain on $\Gamma_0(\mathbb{R}^d)$ with transition probabilities (3.8) may be considered as a generalized version of the classical discrete time $\mathbb{Z}_+$-valued birth-and-death Markov processes. The generalization consists in taking into account the spatial structure. We may deem $(\xi_n)_{n\in\mathbb{Z}_+}$ a discrete time spatial birth-and-death Markov process. However, we will give reformulations of obtained results in terms of $(\eta(\alpha,t))_{t\geq 0}$, too.

Assumption on b, d. We require that b, d satisfy all condition assumed in Theorem 2.1.6.

Furthermore, assume that b, d are continuous functions of two variables,

$$\inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta) > 0, \tag{3.9}$$

and for some constants $r > 0$ and $c_3 > 0$,

$$\begin{gathered} b(x, \eta) > c_3, \text{if there exists } \ y \in \eta, |x - y| \leq r, \\ \text{and } \ b(x, \varnothing) > c_3 \text{ for } x \in B_\varnothing, B_\varnothing \text{ is some open ball in } R^d. \end{gathered} \tag{3.10}$$

Under the above assumptions we have

$$\sup\{(B + D)(\eta) : |\eta| \leq n + 1\} < \infty.$$

3.2.1 Remark. The second part of (3.10) means that points may come “out of nowhere”. We need such kind of condition in order for $\varnothing$ not to be an absorbing state of the Markov chain $(\xi_n)_{n \in \mathbb{N}}$. Also, each of conditions (3.9) and (3.10) implies that every state $\eta \in \Gamma_0(\mathbb{R}^d)$, $\eta \neq \varnothing$, is non-absorbing.

For two finite configurations η, ζ, $|\eta| = |\zeta| > 0$, we define

$$\rho(\eta, \zeta) = \min_{\varsigma} \max_{x \in \eta} \{|\varsigma(x) - x|\},$$

where minimum is taken over the set of all bijections $\varsigma : \eta \to \zeta$. The function ρ is symmetric. One can check that ρ defines a metric on $\Gamma_0^{(n)}(\mathbb{R}^d)$ which induces the same topology as *dist*.

Define a *path* of configurations as a finite sequence of configurations $\zeta_0, \zeta_1, ..., \zeta_n$ such that $|\zeta_k \bigtriangleup \zeta_{k+1}| = 1$, $k = 0, ..., n-1$, and if $\zeta_{k+1} = \zeta_k \cup z$, then $|z - y| \leq \frac{r}{2}$ for some $y \in \zeta_k$; that is, ζ_{k+1} is obtained from ζ_k either by adding one point to ζ_k or by removing one point from ζ_k; in the case of the adding, it is required that the “new” point appears not further than $\frac{r}{2}$ from an “old” one. If $\zeta_k = \varnothing$, then we require $\zeta_{k+1} = \{x_\varnothing\}$, where $x_\varnothing$ is the center of $B_\varnothing$. We say that such a path has length n, and we call ζ_0 and ζ_n the starting vertex and the final vertex, respectively. Also, we say that $\zeta_0, \zeta_1, ..., \zeta_n$ is a path from ζ_0 to ζ_n.

3.2.1 Lemma. *For all $\eta \in \Gamma_0(\mathbb{R}^d)$ there exists a path from $\varnothing$ to η.*

Proof. We will show that there exists a path from $\varnothing$ to η of length less than

$$2\Big(\sum_{x\in\eta} |x - x_\varnothing|\frac{4}{r} + |\eta|\Big),$$

where $x_\varnothing$ is the center of $B_\varnothing$.

Starting from $\varnothing$ and adding points only, we can see that there exists a path of length

$$\leq \Big(\sum_{x\in\eta} |x - x_\varnothing|\frac{4}{r} + |\eta|\Big),$$

with the starting vertex $\varnothing$ and with the final vertex being some configuration $\eta' \supset \eta$. Indeed, for each $x \in \eta$ there exists a sequence of points $x_\varnothing = x_0, x_1, ..., x_n = x$ such that $|x_i - x_{i+1}| \leq \frac{r}{4}$ and $n \leq |x - x_\varnothing|\frac{4}{r}$. Having reached $\eta' \supset \eta$, we only need to delete some points from η'. $\square$

For a configuration $\eta \in \Gamma_0(\mathbb{R}^d)$ and $a > 0$, let

$$\mathbb{B}_\rho(\eta, a) := \{\zeta \in \Gamma_0^{(|\eta|)} \mid \rho(\eta, \zeta) \leq a\}.$$

3.2.2 Lemma. *Let $\varnothing = \eta_0, \eta_1, ..., \eta_n$ be a path. Then for every $a > 0$*

$$Q^n(\eta_0, \mathbb{B}_\rho(\eta_n, a)) > 0.$$

Proof. Without loss of generality we can assume $a < \frac{r}{4}$. Denote $A_k = \mathbb{B}_\rho(\eta_k, a)$. We will first show that

$$\inf_{\eta\in A_k} Q(\eta, A_{k+1}) \geq \bar{c}_n \tag{3.11}$$

for some positive constant $\bar{c}_n$ that depends on n but does not depend on the path we consider.

We have either $\eta_k \subset \eta_{k+1}$ or $\eta_k \supset \eta_{k+1}$. Consider first the case $\eta_k \subset \eta_{k+1}$. We know that $\eta_{k+1} = \eta_k \cup z$, and $|z-y| \leq \frac{r}{2}$ for some $y \in \eta_k$.

Take arbitrary $\eta \in A_k$. There exists $y' \in \eta$ such that $|y - y'| \leq a$. For $x \in B_a(z)$ we have then $|x-y'| \leq |x-z|+|z-y|+|y-y'| \leq a + \frac{r}{2} + a < r$. Moreover, if $x \in B_a(z) \setminus \eta$, then $\eta \cup \{x\} \in A_{k+1}$.

From (3.10) we obtain

$$Q(\eta, A_{k+1}) \geq \frac{\int\limits_{x \in B_a(z)} b(x,\eta)dx}{(B+D)(\eta)} \geq \frac{\int\limits_{x \in B_a(z)} c_3 dx}{(B+D)(\eta)}$$

$$= \frac{c_3 a^d v_d}{(B+D)(\eta)},$$

where v_d is the volume of a unit ball in $\mathbb{R}^d$. The denumerator of the last fraction is bounded in η, $\eta \in \bigsqcup_{k=0}^{n} \Gamma_0^{(k)}(\mathbb{R}^d)$. Therefore, (3.11) holds.

Now we turn our attention to the case when $\eta_k \supset \eta_{k+1}$. We may write $\eta_{k+1} = \eta_k \setminus y$ for some $y \in \eta_k$, and (3.11) follows from (3.9).

The statement of the lemma follows from (3.11), since

$$Q^n(\varnothing, \mathbb{B}_\rho(\eta_n, a)) = \int\limits_{\zeta_1,\zeta_2,\dots,\zeta_n} Q(\varnothing, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{n-1}, d\zeta_n) I_{\{\zeta_n \in \mathbb{B}_\rho(\eta_n, a)\}}$$

$$\geq \int\limits_{\zeta_1,\zeta_2,\dots,\zeta_n} Q(\varnothing, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{n-1}, d\zeta_n) I_{\{\zeta_k \in \mathbb{B}_\rho(\eta_k, a), k=1,\dots,n\}} \geq (\bar{c}_n)^n.$$

3.2.3 Lemma. *Let $A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$, $\beta' \in \Gamma_0^{(n)}$ and $\lambda(A \cap \mathbb{B}_\rho(\beta', \frac{r}{4})) > 0$. Then*

$$Q^{2n}(\beta, A) > 0$$

for any $\beta \in \mathbb{B}_\rho(\beta', \frac{r}{4})$.

The idea of the proof. Let $\beta = \{x_1, ..., x_n\}$. The event R described in the next sentence has positive probability. Let $\xi_1 = \beta \cup y_1$ for some $y_1 \in B_{\frac{r}{4}}(x_1)$, $\xi_2 = \xi_1 \setminus x_1$, $\xi_3 = \xi_2 \cup y_2$ for some $y_2 \in B_{\frac{r}{4}}(x_2)$, $\xi_4 = \xi_3 \setminus x_2$, and so on, so that $\xi_{2n} = \xi_{2n-1} \setminus x_n$. We will see that $Q_\beta\{\xi_{2n} \in A \mid R\} > 0$.

Proof.

Fix $\beta = \{x_1, ..., x_n\}$. Consider a measurable subset Ξ of $(\Gamma_0)^{(2n)}$,

$$\Xi = \Big\{(\zeta_1, ..., \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, ..., y_k, x_k, ..., x_n\}, \zeta_{2k} = \{y_1, ..., y_k, x_{k+1}, ..., x_n\}$$
$$\text{for some } y_j \in \mathbb{R}^d, j = 1, ..., n, |y_j - x_j| \le \frac{r}{4}\Big\}.$$

Define $R = \{(\xi_1, ..., \xi_{2n}) \in \Xi\}$.

By the Markov property,

$$\begin{aligned} Q^{2n}(\beta, A) &= \int\limits_{\zeta_1, \zeta_2, ..., \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times ... \times Q(\zeta_{2n-1}, d\zeta_{2n}) I_{\{\xi_{2n} \in A\}} \\ &\ge \int\limits_{\zeta_1, \zeta_2, ..., \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times ... \\ &\times Q(\zeta_{2n-1}, d\zeta_{2n}) I_{\{(\zeta_1, ..., \zeta_{2n}) \in \Xi\}} I_{\{(\zeta_2 \setminus \zeta_1) \curlyvee (\zeta_4 \setminus \zeta_3) \curlyvee ... \curlyvee (\zeta_{2n} \setminus \zeta_{2n-1}) \in sym^{-1} A\}}. \end{aligned} \tag{3.12}$$

Here for singletons $\mathbb{S}_1 = \{s_1\}, \mathbb{S}_2 = \{s_2\}, ..., \mathbb{S}_n = \{s_n\}$ we define

$$\mathbb{S}_1 \curlyvee \mathbb{S}_2 \curlyvee ... \curlyvee \mathbb{S}_n = (s_1, s_2, ..., s_n).$$

Note that $\zeta_{2n} = (\zeta_2 \setminus \zeta_1) \curlyvee (\zeta_4 \setminus \zeta_3) \curlyvee ... (\zeta_{2n} \setminus \zeta_{2n-1})$, if $(\zeta_1, ..., \zeta_{2n}) \in \Xi$.

From the definition of the Lebesgue Poisson measure we have

$$l(sym^{-1} A) = n! \lambda(A), \tag{3.13}$$

where l is the Lebesgue measure on $(\mathbb{R}^d)^n$.

Define a measure σ on $\big(\prod\limits_{k=1}^{n} B_{\frac{r}{4}}(x_k), \mathscr{B}(\prod\limits_{k=1}^{n} B_{\frac{r}{4}}(x_k))\big)$ by

$$\sigma(D) = \int\limits_{\zeta_1,\zeta_2,\ldots,\zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{2n-1}, d\zeta_{2n})$$

$$\times I_{\{(\zeta_1,\ldots,\zeta_{2n})\in\Xi\}} I_{\{(\zeta_2\setminus\zeta_1)\curlyvee(\zeta_4\setminus\zeta_3)\curlyvee\ldots\curlyvee(\zeta_{2n}\setminus\zeta_{2n-1})\in D\}}, \quad D \in \mathscr{B}\big(\prod_{k=1}^{n} B_{\frac{r}{4}}(x_k)\big).$$

We can rewrite (3.12) as

$$Q^{2n}(\beta, A) \geq \sigma(sym^{-1}A). \tag{3.14}$$

We will show that

$$\sigma(D) \geq \tilde{c}_3 l(D), \quad D \in \mathscr{B}(\prod_{k=1}^{n} B_{\frac{r}{4}}(x_k)) \tag{3.15}$$

for some constant $\tilde{c}_3 > 0$.

The statement of the lemma is a consequence of (3.13), (3.14) and (3.15). To establish (3.15) we only need to consider sets of the form $D_1 \times \ldots \times D_n$, $D_j \in \mathscr{B}(B_{\frac{r}{4}}(x_j))$. Define

$$\Xi_{(D_1,\ldots,D_n)} = \Big\{(\zeta_1, \ldots, \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, \ldots, y_k, x_k, \ldots, x_n\}, \zeta_{2k} = \{y_1, \ldots, y_k, x_{k+1}, \ldots, x_n\}$$
$$\text{for some } y_j \in D_j, j = 1, \ldots, n\Big\}.$$

We have

$$\sigma(D_1 \times \ldots \times D_n) = \int\limits_{\zeta_1,\zeta_2,\ldots,\zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots$$
$$\times Q(\zeta_{2n-1}, d\zeta_{2n}) I\{(\zeta_1, \ldots, \zeta_{2n}) \in \Xi_{(D_1,\ldots,D_n)}\}.$$

Fix $z_j \in D_j$. Using our assumptions on b and d, we see that

$$Q\Big(\{z_1, ..., z_k, x_k, ..., x_n\}, \big\{\{z_1, ..., z_k, x_{k+1}, ..., x_n\}\big\}\Big) = \frac{d(x_k, \{z_1, ..., z_k, x_k, ..., x_n\})}{(B+D)\{z_1, ..., z_k, x_k, ..., x_n\}}$$

$$\geq \frac{d(x_k, \{z_1, ..., z_k, x_k, ..., x_n\})}{\sup\{(B+D)(\eta) \mid |\eta| \leq n+1\}} \geq \frac{\inf\limits_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta)}{\sup\{(B+D)(\eta) \mid |\eta| \leq n+1\}}$$

and

$$Q\Big(\{z_1, ..., z_k, x_{k+1}, ..., x_n\}, \big\{\{z_1, ..., z_k, y_{k+1}, x_{k+1}, ..., x_n\} \mid y_{k+1} \in D_{k+1}\big\}\Big) =$$

$$= \frac{\int\limits_{y \in D_{k+1}} b(y, \{z_1, ..., z_k, x_{k+1}, ..., x_n\}) dy}{(B+D)(\{z_1, ..., z_k, x_{k+1}, ..., x_n\})} \geq \frac{c_3 l_d(D_{k+1})}{(B+D)(\{z_1, ..., z_k, x_{k+1}, ..., x_n\})},$$

where l_d is the Lebesgue measure on $\mathbb{R}^d$. Hence

$$\sigma(D_1 \times ... \times D_n) \geq \Big(\frac{\inf\limits_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta)}{\sup\{(B+D)(\eta) : |\eta| \leq n+1\}}\Big)^n \prod_{j=1}^{n} \frac{c_3 l_d(D_j)}{\sup\{(B+D)(\eta) : |\eta| \leq n+1\}}.$$

It remains to note that $\prod\limits_{j=1}^{n} l_d(D_j) = l(D_1 \times ... \times D_n)$.

3.2.2 Theorem. *The Lebesgue-Poisson measure λ is an irreducibility measure for $(\xi_n)_{n \in \mathbb{N}}$. Furthermore, the Lebesgue-Poisson measure λ is a maximal irreducibility measure for $(\xi_n)_{n \in \mathbb{N}}$.*

Proof. We will first establish ϕ-irreducibility. Starting from any configuration, the process may extinct in finite time: for all $\eta \in \Gamma_0(\mathbb{R}^d)$

$$Q_\eta\{\xi_k = \varnothing \text{ for some } k > 0\} > 0.$$

Therefore, it is sufficient to show that

$$L(\varnothing, A) > 0 \quad \text{whenever } \lambda(A) > 0, A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d)). \tag{3.16}$$

Let us take $A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$ with $\lambda(A) > 0$. There exists $n \in \mathbb{N}$ and $\beta' \in \Gamma_0^{(n)}$ such that

$$\lambda(A \cap \mathbb{B}_\rho(\beta', \frac{r}{4})) > 0. \tag{3.17}$$

By Lemma 3.2.1 there exists a path from $\varnothing$ to β'. Denote by m the length of this path. Applying Lemma 3.2.2 and Lemma 3.2.3 we get

$$Q^{m+2n}(\varnothing, A) \geq \int\limits_{\beta \in \mathbb{B}_\rho(\beta', \frac{r}{4})} Q^m(\varnothing, d\beta) Q^{2n}(\beta, A) > 0,$$

which proves (3.16).

Now let us prove that λ is a maximal irreducibility measure for $(\xi_n)_{n\in\mathbb{N}}$. Taking into account Lemma 3.1.2, we see that it suffices to show that for all $A \subset \Gamma_0(\mathbb{R}^d)$ with $\lambda(A) = 0$ we have

$$\lambda\{\eta : Q(\eta, A) > 0\} = 0. \tag{3.18}$$

With no loss of generality, we assume that $A \subset \Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 2$. We have $sym^{-1}(A) \subset (\mathbb{R}^d)^n$ and $l_{dn}(sym^{-1}(A)) = 0$, l_{dk} is the Lebesgue measure on $(\mathbb{R}^d)^k = \mathbb{R}^{dk}$. Now, $\eta \in \Gamma_0^{(n+1)}(\mathbb{R}^d)$ and $Q(\eta, A) > 0$ if and only if η may be represented as $\xi \cup \{x\}$, where $\xi \in A$, $x \in \mathbb{R}^d \setminus \xi$. Then we also have for any $y = (y_1, ..., y_{n+1}) \in sym^{-1}(\eta)$

$$\check{\Pi}_j y \in sym^{-1}(A)$$

for some $j \in \{1, 2, ..., n+1\}$, where $\check{\Pi}_j y = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_{n+1}) \in (\mathbb{R}^d)^n$. Since

$l_{dn}(sym^{-1}(A)) = 0$, one also has $\lambda_{d(n+1)}(\check{\Pi}(\cdot)_j^{-1}(sym^{-1}(A))) = 0$, and consequently

$$\lambda\{\eta : \eta \in \Gamma_0^{(n+1)}, Q(\eta, A) > 0\} = 0. \tag{3.19}$$

Similarly, if $\eta \in \Gamma_0^{(n-1)}(\mathbb{R}^d)$ and $Q(\eta, A) > 0$, then for $y \in sym^{-1}(\eta)$

$$l_d\{z \in \mathbb{R}^d : (z, y) \in sym^{-1}(A)\} > 0. \tag{3.20}$$

because a "newly born" point has an absolutely continuous distribution with respect to the Lebesgue measure on $\mathbb{R}^d$, in the sense that $Q(\eta, \{\eta \cup z \mid z \in D\}) = 0$ if $l_d(D) = 0$. However, the set of all y satisfying (3.20) has zero Lebesgue measure, otherwise we would have

$$l_{dn}(sym^{-1}(A)) = \int l_{d(n-1)}(dy) l_d\{z : (z, y) \in sym^{-1}(A)\} > 0.$$

Therefore,

$$\lambda\{\eta : \eta \in \Gamma_0^{(n-1)}, Q(\eta, A) > 0\} = 0. \tag{3.21}$$

Note that in cases $n = 0, 1$ some changes should be made in the proofs of (3.19), (3.21), because of the special structure of $\Gamma_0^{(0)}(R^d) = \{\varnothing\}$. Now, we also have

$$\{\eta : \eta \in \Gamma_0^{(k)}, P(\eta, A) > 0\} = \varnothing,$$

$k \neq n-1, n+1$, $n \geq 0$. Consequently, (3.19) and (3.21) imply (3.18). $\square$

3.2.3 Corollary. *The chain $(\xi_n)_{n \in \mathbb{N}}$ is either recurrent or transient.*

3.2.4 Corollary. *Under conditions of Theorem 3.2.2 we have*

$$(\forall \alpha : P\{(\eta(\alpha, t))_{t \geq 0} \textit{ ever enters } A\} > 0) \Leftrightarrow \lambda(A) > 0,$$

or

$$(\forall \alpha : P\{\exists t > 0 : \eta(\alpha, t) \in A, \eta(\alpha, t) \neq \eta(\alpha, t-)\} > 0) \Leftrightarrow \lambda(A) > 0.$$

3.2.5 Remark. Assume that all conditions of Theorem 3.2.2 are satisfied except for the second part of (3.10). If $b(x, \varnothing) \equiv 0$, then $(\xi_n)_{n\in\mathbb{N}}$ is ψ-irreducible with ψ being a multiplier of the Dirac measure at $\varnothing$.

In the next section, we will use recurrence and transience Criteria 3.1.5 and 3.1.6. To be able to apply drift condition for recurrence, we need the following Lemma.

3.2.6 Lemma. *For every $m \in \mathbb{N}$ the set $\bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}$ is petite.*

Proof. Let $a(k) = \frac{1}{m+1} I_{\{k\leq m\}}$. The cumulative birth rate $\int\limits_{x\in\mathbb{R}^d} b(x,\eta)dx$ is bounded uniformly in η, $\eta \in \bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}$, by (2.6), and the cumulative death rate $\sum\limits_{x\in\eta} d(x,\eta)$ is separated from zero by (3.9), $\eta \neq \varnothing$. Therefore,

$$\inf\big\{P_\eta\{\xi_k = \varnothing \text{ for some } k \in \{0, 1, ..., m\}\} : \eta \in \bigsqcup_{k=0}^{k=m} \Gamma_0^{(k)}\big\} > 0.$$

Thus, the set $\bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}$ is ν_a - petite with $\nu_a = s\delta_\varnothing$, for some small enough constant $s > 0$.

3.2.1 Recurrence criteria for birth-and-death processes

Now that we have proved the irreducibility of the chain $(\xi_n)_{n\in\mathbb{N}}$, we turn our attention to transience and recurrence. Let the conditions of Theorem 3.2.2 hold. We will first represent the drift Criterion for recurrence 3.1.6 in some more specific forms. Namely, set $V(\eta) = |\eta|$. Lemma 3.2.6 implies that V is unbounded off petite sets, and

$$(B + D)(\eta)\Delta V(\eta) = LV(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)dx - \sum_{x\in\eta} d(x,\eta).$$

Thus, we have the following

Proposition 1. *If there exist $n_0 \in \mathbb{N}$ such that for all η with $|\eta| \geq n_0$ one has*

$$\int\limits_{x\in\mathbb{R}^d} b(x,\eta)dx - \sum_{x\in\eta} d(x,\eta) \leq 0, \tag{3.22}$$

then the chain $(\xi_n)_{n\in\mathbb{N}}$ is Harris recurrent.

More generally, set $V(\eta) = \langle \eta, \phi \rangle$, where ϕ is a positive measurable function, $\inf\limits_{y\in\mathbb{R}^d} \phi(y) > 0$. Then V is again unbounded off petite sets, and

$$(B+D)(\eta)\Delta V(\eta) = LV(\eta) = \int\limits_{x\in\mathbb{R}^d} \phi(x)b(x,\eta)dx - \sum_{x\in\eta} d(x,\eta)\phi(x). \tag{3.23}$$

Consequently, we may formulate

Proposition 2. *If there exist a positive measurable function ϕ with $\inf\limits_{y\in\mathbb{R}^d} \phi(y) > 0$ and a constant $C > 0$ such that for all η for which $V(\eta) > C$ one has*

$$\int\limits_{x\in\mathbb{R}^d} \phi(x)b(x,\eta)dx - \sum_{x\in\eta} d(x,\eta)\phi(x) \leq 0, \tag{3.24}$$

then $(\xi_n)_{n\in\mathbb{N}}$ is Harris recurrent.

Comment. We describe here what recurrence or transience of the embedded chain $(\xi_n)_{n\in\mathbb{N}}$ means in terms of the process $(\eta(\alpha,t))_{t\geq 0}$. Recall the definitions of recurrence and transience for a discrete time Markov chain given as a part of the theorem on Page 82. Also, recall that $\bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}$ is petite according to Lemma 3.2.6, $m \in \mathbb{N}$.

Transience. If $(\xi_n)_{n\in\mathbb{N}}$ is transient, then every petite set is uniformly transient, and for every $m \in \mathbb{N}$ there exists a constant $M_m > 0$ such that for every $\alpha \in \Gamma_0(\mathbb{R}^d)$,

$$E\#\left\{t > 0 : \eta(\alpha,t) \in \bigsqcup_{k=0}^{k=m} \Gamma_0^{(k)}, \eta(\alpha,t) \neq \eta(\alpha,t-)\right\} < M_m.$$

In particular, $\#\left\{t > 0 : \eta(\alpha,t) \in \bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}, \eta(\alpha,t) \neq \eta(\alpha,t-)\right\}$ is a.s. finite.

Under assumptions of Theorem 3.2.2, the jump rate of the process $\eta(\alpha, t)$ is separated from 0:

$$\inf_{\alpha\in\Gamma_0(\mathbb{R}^d)} (B(\alpha) + D(\alpha)) > 0.$$

Therefore, Theorem 12.17 in [Kal02] (see (1.21) in our work) implies

$$P\Big\{ l\{t : \eta(\alpha, t) \in A\} < \infty \Big\} = 1,$$

and

$$E\Big[l\{t : \eta(\alpha, t) \in A\} \Big] < \infty,$$

where l is the Lebesgue measure on $\mathbb{R}_+$.

Harris recurrence. If $(\xi_n)_{n\in\mathbb{N}}$ is Harris recurrent and $\lambda(A) > 0$, $A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$, then for every $\alpha \in A$,

$$P\Big\{ \eta(\alpha, t) \in A \text{ and } \eta(\alpha, t) \neq \eta(\alpha, t-) \text{ for infinitely many different } t > 0 \Big\} = 1.$$

In case when $A \subset \bigsqcup\limits_{k=0}^{k=m} \Gamma_0^{(k)}$, the jump rate is bounded:

$$\sup_{\alpha\in A} \big(B(\alpha) + D(\alpha)\big) < \infty.$$

Therefore, Theorem 12.17 in [Kal02] (see (1.21) in our work) and Kolmogorov's three-series theorem yield

$$P\Big\{ l\{t : \eta(\alpha, t) \in A\} = \infty \Big\} = 1. \tag{3.25}$$

Actually, (3.25) holds for all $A \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$ with $\lambda(A) > 0$, since there exists $m \in \mathbb{Z}_+$ such that $\lambda(A \cap \Gamma_0^{(m)}) > 0$.

3.3 Examples and applications

Bolker-Pacala model. Let

$$b(x,\eta) = \sum_{y\in\eta} a_+(x-y),$$

$\eta \neq \varnothing$,

$$d(x,\eta) = m + \sum_{y\in\eta\setminus x} a_-(x-y), \tag{3.26}$$

where m is a constant, $a_+, a_- \in C(\mathbb{R}^d; [0;\infty))$, $a_+ \in L^1(\mathbb{R}^d)$, and let $b(\cdot,\varnothing)$ be a bounded non-negative function from $L^1(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ such that b satisfied (3.10).

Such b and d satisfy all the assumptions of Theorems 2.1.6 and 3.2.2. We will use Criteria 3.1.5 and 3.1.6 in the proof of the next theorem.

3.3.1 Proposition. *(i) The chain $(\xi_n)_{n\in\mathbb{N}}$ is Harris recurrent, if $m \geq ||a_+||_{L^1}$;*

(ii) The chain $(\xi_n)_{n\in\mathbb{N}}$ is transient, if $m < ||a_+||_{L^1}$ and $a_- \equiv 0$.

Proof. To prove (i), set $V(\eta) = |\eta|$. Then (3.22) holds for all η with $V(\eta) \geq 1$, and we get recurrence.

To prove (ii), set $V(\eta) = \frac{1-u^{|\eta|+1}}{1-u} = \sum\limits_{j=0}^{j=|\eta|} u^j$, where $u \in (0;1)$, $u > \frac{m}{||a_+||_{L^1}}$. Then for $\eta \neq \varnothing$,

$$(B+D)(\eta)\Delta V(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)[V(\eta\cup x) - V(\eta)]dx - \sum_{x\in\eta} d(x,\eta)[V(\eta\setminus x) - V(\eta)] =$$

$$|\eta| ||a_+||_{L^1} u^{|\eta|+1} - |\eta| m u^{|\eta|} = |\eta| u^{|\eta|}[||a_+||_{L^1} u - m] > 0,$$

and V is bounded: $V(\eta) \leq \frac{1}{1-u}$. Thus, Criterion 3.1.5 implies the transience of $(\xi_n)_{n\in\mathbb{N}}$.

Asymmetric dispersion model. Consider the birth-and-death process with slightly alternated compared with the previous example coefficients. Namely, let the death rate coefficient be as in (3.26), and for a non-empty configuration η let the birth rate coefficient be given by

$$b(x,\eta) = \sum_{y\in\eta} a_+(x-y)[1 - \frac{\langle y, x-y\rangle}{|y|R}],$$

where R is such that $a_+(x) = 0$, $|x| \geq R$. Note that $\frac{\langle y,x-y\rangle}{|y|R} \leq 1$ if $|x-y| \leq R$, hence b is non-negative. We admit the convention $\frac{0}{0} = 0$ in this example, though it is not necessary.

This model is noticeable because we can get recurrence for the embedded chain, even though the inequality $||a_+||_{L^1} \leq m$ may not hold. The multiplier $[1 - \frac{\langle y,x-y\rangle}{|y|R}]$ does not change the rate of appearing of new points, since for all $y \in \mathbb{R}^d$

$$\int\limits_{x\in R^d} a_+(x-y)[1 - \frac{\langle y, x-y\rangle}{|y|R}]dx = \int\limits_{x\in R^d} a_+(x-y)dx = ||a_+||_1.$$

It does however influence the distribution of a new appearing point, so that it tends to be closer to the origin than its "predecessor".

The assumptions of Theorems 2.1.6 and 3.2.2 are fulfilled. We would like to show under some additional assumptions on b and d that the chain $(\xi_n)_{n\in\mathbb{Z}_+}$ is recurrent. Let a_+ be rotationally invariant and such that for some fixed unit vector $\mathbf{e}$ in $\mathbb{R}^d$

$$-\gamma := (1-\epsilon) \int\limits_{z\in R^d} a_+(z)[1 - \frac{\langle \mathbf{e}, z\rangle}{R}](e^{\langle \mathbf{e},z\rangle} - 1)dz < 0, \tag{3.27}$$

where $\epsilon \in (0;1)$ is some (small) number. Of course, γ does not depend on the choice of $\mathbf{e}$. We will require

$$\gamma + m > ||a_+||_1, \tag{3.28}$$

where $||a_+||_1 = ||a_+||_{L^1}$.

Also, as was already mentioned above, we assume that $a_+(x) = 0$, $|x| \geq R$, and that for some $r > 0$

$$a_-(x) > \delta, \quad |x| \le r. \tag{3.29}$$

To simplify computations we assume $\delta < m$. Under these conditions, we will prove recurrence of $(\xi_n)_{n\in\mathbb{Z}_+}$.

3.3.2 Proposition. *Under assumptions listed above, the chain $(\xi_n)_{n\in\mathbb{Z}_+}$ is Harris recurrent.*

We start with an auxiliary lemma. Denote

$$\psi(y) := \int\limits_{\mathbb{R}^d} a_+(x-y)[1 - \frac{\langle y, x-y\rangle}{|y|R}]e^{|x|}dx - ||a_+||_1 e^{|y|}.$$

3.3.3 Lemma. *The function ψ is a rotationally invariant function satisfying*

$$\limsup_{|y|\to\infty} \frac{\psi(y)}{e^{|y|}} < -\gamma. \tag{3.30}$$

Proof. We first prove that ψ is invariant under rotations. Indeed, let A be a rotation in $\mathbb{R}^d$. Then

$$\int\limits_{\mathbb{R}^d} a_+(x-Ay)[1 - \frac{\langle Ay, x-Ay\rangle}{|Ay|R}]e^{|x|}dx = \int\limits_{\mathbb{R}^d} a_+(Av-Ay)[1 - \frac{\langle Ay, Av-Ay\rangle}{|Ay|R}]e^{|v|}dv =$$

$$\int\limits_{\mathbb{R}^d} a_+(v-y)[1 - \frac{\langle y, v-y\rangle}{|y|R}]e^{|v|}dv,$$

where we set $x = Av$ in the first step; the last step is possible particularly due to the rotational invariance of a_+. Hence, it will be sufficient to establish (3.30) for y moving along some fixed direction, say $y = q\mathbf{e}$. Using change of variables $z = x - q\mathbf{e}$, $q > R$, we obtain

$$\psi(q\mathbf{e}) = \int\limits_{\mathbb{R}^d} a_+(x - q\mathbf{e})[1 - \frac{\langle q\mathbf{e}, x - q\mathbf{e}\rangle}{|q\mathbf{e}|R}]e^{|x|}dx - e^q||a_+||_{l^1} =$$

$$\int\limits_{\mathbb{R}^d} a_+(z)[1 - \frac{\langle \mathbf{e}, z\rangle}{R}](e^{|q\mathbf{e}+z|} - e^q)dz = \tag{3.31}$$

$$e^q \int\limits_{z:|z|\leq R} a_+(z)[1 - \frac{\langle \mathbf{e}, z\rangle}{R}](e^{|q\mathbf{e}+z|-q} - 1)dz.$$

Denote $h_z(q) := |q\mathbf{e} + z| - q$. We have $h_z(q) = \frac{2q\langle \mathbf{e},z\rangle + |z|^2}{|q\mathbf{e}+z|+q} = \frac{2q\langle \mathbf{e},z\rangle}{|q\mathbf{e}+z|+q} + \frac{|z|^2}{|q\mathbf{e}+z|+q}$. The second summand is bounded by $\frac{|R|^2}{2q-R}$ for all z, $|z| \leq R$. As for the first one, let us consider the inequality

$$\frac{2q\langle \mathbf{e}, z\rangle}{|q\mathbf{e} + z| + q} \leq \langle \mathbf{e}, z\rangle. \tag{3.32}$$

In case $\langle \mathbf{e}, z\rangle \geq 0$, (3.32) holds, since $|q\mathbf{e} + z| \geq q$ in this case; in case $\langle \mathbf{e}, z\rangle < 0$ inequality (3.32) is equivalent to $q \geq |q\mathbf{e} + z|$, which is in its turn equivalent to

$$\langle \mathbf{e}, z\rangle \leq -\frac{|z|^2}{2q}.$$

The Lebesgue measure of the set $\{z : |z| \leq R, -\frac{|z|^2}{2q} \leq \langle \mathbf{e}, z\rangle \leq 0\}$ tends to 0 as q tends to infinity. Therefore, by the Lebesgue's dominated convergence theorem,

$$\int\limits_{\substack{z:|z|\leq R,\\ -\frac{|z|^2}{2q}\leq\langle \mathbf{e},z\rangle\leq 0}} a_+(z)[1 - \frac{\langle \mathbf{e}, z\rangle}{R}](e^{|q\mathbf{e}+z|-q} - 1)dz \to 0, \quad q \to \infty,$$

as well as

$$\int\limits_{\substack{z:|z|\leq R\\ -\frac{|z|^2}{2q}\leq\langle \mathbf{e},z\rangle\leq 0}} a_+(z)[1 - \frac{\langle \mathbf{e}, z\rangle}{R}](e^{\frac{|R|^2}{2q-R}+\langle \mathbf{e},z\rangle} - 1)dz \to 0, \quad q \to \infty.$$

For z from $B(0,R)\setminus\{z:|z|\le R, -\frac{|z|^2}{2q}\le\langle \mathbf{e},z\rangle\le 0\}$ inequality (3.32) holds, therefore by (3.27)

$$\limsup_{q\to\infty}\int\limits_{z:|z|\le R} a_+(z)[1-\frac{\langle \mathbf{e},z\rangle}{R}](e^{|q\mathbf{e}+z|-q}-1)dz$$

$$\le \limsup_{q\to\infty}\int\limits_{z:|z|\le R} a_+(z)[1-\frac{\langle \mathbf{e},z\rangle}{R}](e^{\frac{|R|^2}{2q-R}+\langle \mathbf{e},z\rangle}-1)dz$$

$$=\int\limits_{z:|z|\le R} a_+(z)[1-\frac{\langle \mathbf{e},z\rangle}{R}](e^{\langle \mathbf{e},z\rangle}-1)dz<-\gamma$$

The statement of the lemma follows from this inequality and (3.31). □

We would like to show now that the chain $(\xi_n)_{n\in\mathbb{Z}_+}$ is transient. Let $\phi(x)=e^{|x|}$, $V(\eta)=\langle\eta,\phi\rangle$. Take $R_1\ge R$ such that

$$\psi(y)<-\gamma e^{|y|}. \tag{3.33}$$

Such R_1 exists by the previous lemma. Let k denote the minimal number of balls of radius r_1 in $\mathbb{R}^d$ needed to cover a ball of radius R_1.

Claim. *There exists $C>0$ such that $LV(\eta)\le 0$ for all η with $V(\eta)\ge C$.*

Proof. One can write

$$LV(\eta)=\int\limits_{v\in\mathbb{R}^d}\phi(v)b(v,\eta)dv-\sum_{u\in\eta}d(u,\eta)\phi(u)$$

$$=\int\limits_{v\in\mathbb{R}^d}\phi(v)\{\sum_{u\in\eta}a_+(v-u)[1-\frac{\langle u,v-u\rangle}{|u|R}]\}dv-\sum_{u\in\eta}d(u,\eta)\phi(u)$$

$$=\sum_{u\in\eta,|u|\le R_1}\{\int\limits_{v\in\mathbb{R}^d}\phi(v)a_+(v-u)[1-\frac{\langle u,v-u\rangle}{|u|R}]dv-d(u,\eta)\phi(u)\}$$

$$+\sum_{u\in\eta,|u|>R_1}\{\int\limits_{v\in\mathbb{R}^d}\phi(v)a_+(v-u)[1-\frac{\langle u,v-u\rangle}{|u|R}]dv-d(u,\eta)\phi(u)\}.$$

Since $d(u,\eta) \geq m$,

$$\int\limits_{v\in\mathbb{R}^d} \phi(v)a_+(v-u)[1-\frac{\langle u, v-u\rangle}{|u|R}]dv - d(u,\eta)\phi(u)$$

$$= \psi(u) + (||a_+||_1 - d(u,\eta))\phi(u) \leq \psi(u) + (||a_+||_1 - m)\phi(u).$$

Thus,

$$LV(\eta) \leq \sum_{u\in\eta, |u|\leq R_1} \{\int\limits_{v\in\mathbb{R}^d} \phi(v)a_+(v-u)[1-\frac{\langle u, v-u\rangle}{|u|R}]dv - d(u,\eta)\phi(u)\}+$$

$$\sum_{u\in\eta, |u|>R_1} [\psi(u) + (||a_+||_1 - m)\phi(u)] := S_1 + S_2.$$

We are going to estimate S_1 and S_2. We begin with S_1. For $|u| \leq R_1$, we have $\phi(v) \leq e^{R_1+R}$ for all v satisfying $a_+(v-u) > 0$, since $a_+(v-u) = 0$ when $|u-v| > R$. The ball $B(0,R_1)$ contains $|\eta \cap B(0,R_1)|$ points from η, therefore there exists a ball $\mathbf{B}$ of radius r_1 containing not less than $\frac{|\eta\cap B(0,R_1)|}{k}$ points. For $u \in \mathbf{B}$ we have then $d(u,\eta) \geq \frac{|\eta\cap B(0,R_1)|}{k}\delta$ because of (3.29). Note that

$$\int\limits_{v\in\mathbb{R}^d} a_+(v-u)\frac{\langle u, v-u\rangle}{|u|R}dv = \int\limits_{z\in\mathbb{R}^d, |z|\leq R} a_+(z)\frac{\langle u, z\rangle}{|u|R}dz = 0.$$

The first summand S_1 can thus be estimated in the following way:

$$\sum_{u\in\eta, |u|\leq R_1} \{\int\limits_{v\in\mathbb{R}^d} \phi(v)a_+(v-u)[1-\frac{\langle u, v-u\rangle}{|u|R}]dv - d(u,\eta)\phi(u)\} \leq$$

$$\sum_{u\in\eta, |u|\leq R_1} \{\int\limits_{v\in\mathbb{R}^d} e^{R_1+R}a_+(v-u)[1-\frac{\langle u, v-u\rangle}{|u|R}]dv - d(u,\eta)\} \leq$$

$$\sum_{u\in\eta, |u|\leq R_1} \{||a_+||_{L^1}e^{R_1+R} - \frac{\delta}{k}|\eta\cap B(0,R_1)|I_{\{u\in\mathbf{B}\}}\}$$

$$= ||a_+||_{L^1} e^{R_1+R} |\eta \cap B(0,R_1)| - \frac{\delta}{k^2} |\eta \cap B(0,R_1)|^2.$$

Therefore,

$$S_1 \leq \frac{k^2}{4\delta} [||a_+||_{L^1} e^{R_1+R}]^2, \tag{3.34}$$

and $S_1 \leq 0$ whenever $|\eta \cap B(0,R_1)| \geq \frac{k^2}{\delta} ||a_+||_{L^1} e^{R_1+R}$.

Let us now turn our attention to S_2. For $|u| > R_1$ we have $\psi(u) < -\gamma\phi(u)$ by (3.33), and

$$\psi(u) + (||a_+||_1 - m)\phi(u) < (-\gamma + ||a_+||_1 - m)\phi(u) = -\theta\phi(u),$$

where $\theta = \gamma - ||a_+||_1 + m > 0$, see (3.28).

Consequently,

$$S_2 \leq -\theta \sum_{u \in \eta, |u| > R_1} \phi(u).$$

In particular, $S_2 \leq 0$.

Take $C > 0$ so large that $\theta C \geq \frac{k^2}{4\delta}[||a_+||_{L^1} e^{R_1+R}]^2$ and $\frac{C}{e^{R_1}} \geq \frac{k^2}{\delta} ||a_+||_{L^1} e^{R_1+R}$, and let η be such that $V(\eta) \geq 2C$. Then at least one of the following two inequalities hold,

$$\sum_{u \in \eta, |u| \leq R_1} \phi(u) \geq C, \tag{3.35}$$

$$\sum_{u \in \eta, |u| > R_1} \phi(u) \geq C. \tag{3.36}$$

If (3.35) holds, then we can write

$$\sum_{u \in \eta, |u| \leq R_1} e^{|u|} \geq C \Rightarrow |\eta \cap B(0,R_1)| \geq \frac{C}{e^{R_1}} \geq \frac{k^2}{\delta} ||a_+||_{L^1} e^{R_1+R} \Rightarrow$$

$$S_1 \leq 0 \Rightarrow LV(\eta) \leq 0.$$

If (3.36) holds, then

$$\sum_{u\in\eta,|u|>R_1}\phi(u)\geq C\Rightarrow-\theta\sum_{u\in\eta,|u|>R_1}\phi(u)\leq-\theta C\leq-\frac{k^2}{4\delta}[||a_+||_{L^1}e^{R_1+R}]^2\Rightarrow$$

$$S_2\leq-\frac{k^2}{4\delta}[||a_+||_{L^1}e^{R_1+R}]^2$$

and the last inequality together with (3.34) implies $LV(\eta)\leq 0$.

Our claim is proven. The drift Criterion 3.1.6 and equality (3.23) imply the recurrence of $(\xi_n)_{n\in\mathbb{Z}_+}$. Proposition 3.3.2 is proven.

3.3.4 Remark. In the case when we have

$$b(x,\varnothing)\equiv 0 \tag{3.37}$$

instead of the second part of (3.10), $\varnothing$ is an absorbing state for $(\xi_n)_{n\in\mathbb{N}}$. The Lebesgue-Poisson measure λ on $\Gamma_0(\mathbb{R}^d)$ is no longer an irreducibility measure for $(\xi_n)_{n\in\mathbb{N}}$. We may get the following information from the results about transience and recurrence. If $(\xi_n)_{n\in\mathbb{N}}$ was Harris recurrent under the assumptions of Theorem 3.2.2, then it ends up at $\varnothing$ (the process "extincts") with probability 1 for any initial condition:

$$Q_\alpha\{\xi_n=\varnothing\text{ for some }n\in\mathbb{N}\}=1.$$

If $(\xi_n)_{n\in\mathbb{N}}$ was transient, then $L(\alpha,\{\varnothing\})<1$ at least for some $\alpha\in\Gamma_0(\mathbb{R}^d)$. That is, and some α

$$Q_\alpha\{\xi_n\neq\varnothing\text{ for all }n\in\mathbb{N}\}>0.$$

Chapter 4

The aggregation process in $\Gamma_0(\mathbb{R}^d)$

4.1 The aggregation process in the space of finite configuration: set-up

The model we discuss here has a pathological property that the death rate coefficient declines as the number of neighbors grows. We treat here the death rate coefficient given in (4.1), and we require the birth rate coefficient to grow linearly on the number of points in configuration in the sense (4.2). We prove in Proposition 4.2.1 that the probability of extinction is small if the initial configuration has many points in some fixed Borel set $\Lambda \subset \mathbb{R}^d$. Propositions 4.2.2, 4.2.3 and Theorem 4.2.4 describe the pathwise behavior of the process.

Let us consider a particular case of equation (2.2), with

$$d(x, \eta) = \exp\{-\sum_{y \in \eta} \varphi(x - y)\}, \tag{4.1}$$

where φ is a nonnegative measurable function. Under assumptions of Theorem 2.1.6 we have existence and uniqueness of solution, and this solution is a pure jump type Markov process.

For the sake of convenience, we work in this section on the probability space

$$\big(D_{\Gamma_0(\mathbb{R}^d)}[0;\infty), \mathscr{B}(D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)), P_\alpha\big),$$

where P_α is the push-forward of the measure P under the map

$$\Omega \ni \omega \mapsto (\eta(\alpha,\cdot)) \in D_{\Gamma_0(\mathbb{R}^d)}[0;\infty),$$

The process $(\eta_t)_{t\geq 0}$ denotes the canonical process. All processes in this section are adapted to the right-continuous filtration $\{\mathscr{B}_t\}_{t\geq 0}$,

$$\mathscr{B}_t = \sigma(\eta_s, s \geq t),$$

see also Subsection 1.2.3. Notions such as stopping times or the strong Markov property are considered with respect to this filtration.

We want to show that, if the initial configuration has m points in some bounded region, then, under some assumption on b and φ, the probability of extinction declines faster than exponentially by m. Also, we would like to give a few statements describing the pace of growth of the number of points of the system.

The main idea behind our analysis in this section is to couple the process $(\eta_t)_{t\geq 0}$ with another, in a way "simpler" birth-and-death process (using Theorem 2.1.25). The "simplicity" consists in the possibility to apply Theorem 1.6.1.

4.2 Asymptotic behavior and qualitative analysis

More specifically, let Λ be a measurable subset of $\mathbb{R}^n$, the birth rate coefficient and the initial condition η_0 satisfy the same condition as Theorem 2.1.6, and, besides that, the inequalities

$$\int_{\Lambda} b(x,\eta)dx \geq c|\eta|, \quad \eta \in \Gamma_0(\mathbb{R}^d) \tag{4.2}$$

and

$$b(x,\eta^1) \leq b(x,\eta^2), \quad \eta^1,\eta^2 \in \Gamma_0(\mathbb{R}^d), \eta^1 \subset \eta^2 \tag{4.3}$$

hold for some positive c. We assume also that

$$\inf_{x,y\in\Lambda} \varphi(x-y) \geq \log a, \tag{4.4}$$

where $a > 1$.

Let us introduce another pair of the birth and death rate coefficients, b_1, d_1, and an initial condition $\xi_0 = \eta_0 \cap \Lambda$, such that $b_1(x,\eta) = d_1(x,\eta) = 0$ for $x \notin \Lambda$, $d_1(x,\eta) = a^{-|\eta|}$ for $x \in \Lambda$, $b_1(x,\eta) \leq b(x,\eta)$ for all x, η, and for some constant $c > 0$

$$\int_{\Lambda} b_1(x,\eta)dx = c|\eta \cap \Lambda|, \quad \eta \in \Gamma_0(\mathbb{R}^d).$$

There exists a function b_1 satisfying these assumptions.

Functions b_1, d_1 satisfy conditions of Theorem 2.1.6. Furthermore, the conditions of Theorem 2.1.25 are fulfilled here: for $\eta^1,\eta^2 \in \Gamma_0(\mathbb{R}^d)$, $\eta^1 \subset \eta^2$ we have

$$b_1(x,\eta^1) \leq b(x,\eta^1) \leq b(x,\eta^2)$$

as well as

$$d_1(x,\eta^1) \geq d(x,\eta^1) \geq d(x,\eta^2).$$

Using Theorem 2.1.25, we consider an auxiliary process $(\xi_t)_{t\geq 0}$ satisfying the following two properties. First, let $(\xi_t)_{t\geq 0}$ have the same law as the unique solution $(\eta^{(1)}(\alpha,t))_{t\geq 0}$ of equation (2.2) with the birth and death coefficients b_1, d_1 and initial condition ξ_0. Second, let $\xi_t \subset \eta_t$ hold P_α-a.s. for all $t \geq 0$. Note that $\xi_t \in \Lambda$ for all $t \geq 0$, P_α -a.s.

Remark. We agreed above to work on the canonical probability space. In general, we may not be able to define $(\xi_t)_{t\geq 0}$ on it, since the process $(\xi_t)_{t\geq 0}$ need not be measurable with respect to the σ-algebra generated by $(\eta_t)_{t\geq 0}$. In this case, we should extend the canonical probability space to $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty) \times D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)$ with the corresponding σ-algebra, and for P_α we should take the push-forward of the measure P under

$$\Omega \ni \omega \mapsto (\eta(\alpha, \cdot), (\xi(\alpha, \cdot)) \in D_{\Gamma_0(\mathbb{R}^d)}[0;\infty) \times D_{\Gamma_0(\mathbb{R}^d)}[0;\infty),$$

where $(\xi(\alpha, t))_{t\geq 0} \stackrel{d}{=} (\eta^{(1)}(\alpha, t))_{t\geq 0}$ and $\xi(\alpha, \cdot) \subset \eta(\alpha, \cdot)$.

The canonical process becomes then

$$(\eta_t(x), \xi_t(x)) = (x_1(t), x_2(t)), \quad x = (x_1, x_2) \in D_{\Gamma_0(\mathbb{R}^d)}[0;\infty) \times D_{\Gamma_0(\mathbb{R}^d)}[0;\infty).$$

The thus defined family of measures is a Markov family with respect to

$$\mathscr{B}_t = \sigma\{\eta_s, \xi_s, s \leq t\}.$$

We say that the process *extincts*, if $\inf\{t \geq 0 : \eta_t = \varnothing\} < \infty$. This infimum is called the *time of extinction.*

Consider the embedded Markov chain of the process $(\xi_t)_{t\geq 0}$, $Y_k := \xi_{\tau_k}$, where τ_k are the moments of jumps of (ξ_t). It turns out that the process $u = \{u_k\}_{k\in\mathbb{N}}$, where $u_k := |Y_k|$, is a Markov chain too. Indeed, the equality

$$P_{\alpha_1}\{|Y_1| = k\} = P_{\alpha_2}\{|Y_1| = k\}, \quad k \in \mathbb{N}, \alpha \in \Gamma_0(\mathbb{R}^d).$$

holds when $|\alpha_1 \cap \Lambda| = |\alpha_2 \cap \Lambda|$, since both sides are equal to

$$\begin{cases} \frac{c}{c+a^{-|\alpha_1\cap\Lambda|}} & \text{if} \quad k = |\alpha_1 \cap \Lambda| + 1, \\ \frac{a^{-|\alpha_1\cap\Lambda|}}{c+a^{-|\alpha_1\cap\Lambda|}} & \text{if} \quad k = |\alpha_1 \cap \Lambda| - 1, \\ 0 & \text{in other cases.} \end{cases}$$

Therefore, Theorem 1.6.1 is applicable here, with $f(\cdot) = |\cdot|$.

4.2.1 Proposition. *Under the assumptions above, the probability of the extinction of the process $(\eta_t)_{t\geq 0}$ declines at least exponentially fast by the number of points of the initial configuration in Λ. More precisely, for any constant $\tilde{C} > 0$ the probability of the extinction is less than $\tilde{C}^{-m}$ for large enough m, where m is the number of points of the intersection of the initial configuration with Λ.*

Proof. Having in mind the inclusion $\xi_t \subset \eta_t$ (P_α-a.s.), we will prove this Lemma for the auxiliary process ξ_t.

The transition probabilities for the Markov chain $\{u_k\}_{k\in\mathbb{Z}_+}$ are given by the formulas

$$p_{i,j} = P_\alpha\{u_k = j \mid |u_{k-1}| = i\} = \begin{cases} \frac{c}{c+a^{-i}} & \text{if} \quad j = i+1, \\ \frac{a^{-i}}{c+a^{-i}} & \text{if} \quad j = i-1, \\ 0 & \text{in other cases,} \end{cases} \tag{4.5}$$

for $i \in \mathbb{N}, j \in \mathbb{Z}_+$, and $p_{0,j} = I_{\{j=0\}}$, see (1.26) and (3.8).

Since the zero is an absorbing state and it is accessible from all other states, there are no recurrent states except zero, and the process u has only two possible types of behavior on infinity:

$$P_\alpha\{\exists l \in \mathbb{N} \text{ s.t. } u_l = \varnothing \text{ or } \lim_{m\to\infty} u_m = \infty\} = 1.$$

We will now use results of the theory of discrete time Markov chains with a countable state space, see e.g. [Chu67, § 12, chapter 1]. Chung considers there Markov chain with a reflecting barrier at 0, but we may still apply those results, adapting them correspondingly. Denote $\varrho_m = \prod_{k=1}^{m} \frac{p_{k,k-1}}{p_{k,k+1}}$. Then the probability $P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\}$ equals to 1 if and only if $\sum_{j=1}^{\infty} \varrho_j = \infty$, whichever initial condition α, $|\alpha \cap \Lambda| > 0$, we have. Moreover, if $\sum_{j=1}^{\infty} \varrho_j < \infty$ and $P_\alpha\{u_0 = q\} = 1$ (or, equivalently, $|\alpha \cap \Lambda| = q$), then $p_q := P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\} = \frac{\sum_{j=q}^{\infty} \varrho_j}{1+\sum_{j=1}^{\infty} \varrho_j}$. From (4.5) we see that in our case $\varrho_j = c^{-j} a^{-\frac{j(j+1)}{2}}$, and

$$p_q = \frac{\sum_{j=q}^{\infty} c^{-j} a^{-\frac{j(j+1)}{2}}}{1 + \sum_{j=1}^{\infty} c^{-j} a^{-\frac{j(j+1)}{2}}} \leq \frac{\sum_{j=q}^{\infty} c^{-j} a^{-\frac{j^2}{2}}}{1 + \sum_{j=1}^{\infty} c^{-j} a^{-\frac{j^2}{2}}}. \tag{4.6}$$

Now, for arbitrary $C > 1$ chose $q \in \mathbb{N}$ for which $c^{-1} a^{-\frac{q}{2}} < C^{-1}$. For $j > q$ we have $c^{-j} a^{-\frac{j^2}{2}} < c^{-j} a^{-\frac{jq}{2}} = (c^{-1} a^{-\frac{-q}{2}})^j < C^{-j}$, and

$$\sum_{j=q}^{\infty} c^{-j} a^{-\frac{j^2}{2}} < \sum_{j=q}^{\infty} C^{-j} = \frac{C^{-q}}{1 - C^{-1}},$$

so that the statement of the lemma for $(\xi_t)_{t \geq 0}$ follows from (4.6). $\square$

Note that under assumptions of Proposition 4.2.1 the number of particles of the process will go to infinity with probability 1 even though the probability of extinction is positive, unless $b(\cdot, \varnothing) = 0$ almost everywhere with respect to the Lebesgue measure. However, if $b(\cdot, \varnothing) \equiv 0$, then

$$P\bigg(\{|\xi_t| = 0 \text{ for large } t\ \} \cup \{|\xi_t| \to \infty\ , t \to \infty\}\bigg) = 1$$

and

$$P\bigg(\{|\xi_t| = 0 \text{ for large } t\ \} \cap \{|\xi_t| \to \infty\ , t \to \infty\}\bigg) = 0.$$

The following equality is also taken from [Chu67, § 12, chapter 1]; for $q > s$ and all β

with $|\beta \cap \Lambda| = q$,

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum\limits_{j=q}^{\infty} \varrho_j(s)}{1 + \sum\limits_{j=s+1}^{\infty} \varrho_j(s)},$$

where $\varrho_m(s) = \prod\limits_{k=s+1}^{m} \frac{p_{k,k-1}}{p_{k,k+1}} = c^{-(m-s)} a^{-\frac{1}{2}(m-s)(m+s+1)}$; in our case

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum\limits_{j=q}^{\infty} c^{-(j-s)} a^{-\frac{1}{2}(j-s)(j+s+1)}}{1 + \sum\limits_{j=s+1}^{\infty} c^{-(j-s)} a^{-\frac{1}{2}(j-s)(j+s+1)}} := c_{q,s} < 1. \tag{4.7}$$

Note that

$$c_{q+1,1} \to 0, \qquad q \to \infty \tag{4.8}$$

4.2.2 Proposition. *For all* $\alpha \in \Gamma_0(\mathbb{R}^d)$,

$$P_\alpha\Bigg(\{|\eta_t \cap \Lambda| \to \infty\} \cup \{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \varnothing\} \Bigg) = 1. \tag{4.9}$$

Remark. Note that we do not require $b(\cdot, \varnothing) \equiv 0$; if $\int_\Lambda b(x, \varnothing)dx > 0$, then (4.9) implies

$$P_\alpha\{|\eta_t \cap \Lambda| \to \infty\} = 1.$$

Proof. Let $(X_k)_{k \in \mathbb{Z}_+}$ be the embedded chain of $(\eta_t)_{t \geq 0}$. Firstly, we will show that for all $m \in \mathbb{N}$ and $\alpha \in \Gamma_0(\mathbb{R}^d)$,

$$P_\alpha\{|X_k \cap \Lambda| = m \text{ infinitely often }\} = 0. \tag{4.10}$$

Let $\beta \in \Gamma_0(\mathbb{R}^d)$, $|\beta \cap \Lambda| = m$, $m \in \mathbb{N}$ (the case of $m = 0$ is similar, and we do not write it down). Denote $\tilde{k} = \min\{k \in \mathbb{N} : X_k \cap \Lambda \neq X_0 \cap \Lambda\}$. Since $\xi_t \subset \eta_t$ holds P_β - a.s.,

$$P_\beta\big\{|X_k \cap \Lambda| > m, \forall k \geq \tilde{k}\big\} \geq P_\beta\big\{|Y_k \cap \Lambda| > m, \forall k \geq 1\big\}$$
$$= P_\beta\big\{u_k > m,\, \forall k \geq 1\big\}. \tag{4.11}$$

By (4.7), the probability $P_\beta\{u_k > m, \forall k \geq 1\}$ is positive and does not depend on β, $|\beta \cap \Lambda| = m$:

$$s_m := P_\beta\{u_k > m, \forall k \geq 1\} \geq p_{m,m+1}(1 - c_{m+1,m}) > 0. \tag{4.12}$$

Define k_i^m, $i \in \mathbb{N}$, subsequently by $k_{j+1}^m = \min\{k > k_j^m : |X_k \cap \Lambda| = m \text{ and } \exists \bar{k} < k : |X_{\bar{k}} \cap \Lambda| \neq m\}$, $k_0^m = 0$. Note that for all β

$$P_\beta\Big\{\exists n_0 : |X_n \cap \Lambda| = m \text{ for all } n \geq n_0\Big\} = 0.$$

By the strong Markov property,

$$P_\alpha\Big\{|X_k \cap \Lambda| = m \text{ infinitely often }\Big\} \leq P_\alpha\Big\{k_j^m < \infty, \forall j \in \mathbb{N}\Big\}$$
$$= \prod_{j=1}^{\infty} P_\alpha\big\{k_{j+1}^m < \infty \mid k_j^m < \infty\big\} = 0,$$

by (4.11) and (4.12). Indeed, if $P_\alpha\{k_j^m < \infty\} > 0$, then

$$P_\alpha\{k_{j+1}^m < \infty \mid k_j^m < \infty\} = \frac{E_\alpha I_{\{k_j^m<\infty\}} P_{X_{k_j^m}}\{k_1^m < \infty\}}{E_\alpha I_{\{k_j^m<\infty\}}}$$
$$\leq \frac{E_\alpha I_{\{k_j^m<\infty\}}\big(1 - P_{X_{k_j^m}}\{|X_k \cap \Lambda| > m, \forall k \geq \tilde{k}\}\big)}{E_\alpha I_{\{k_j^m<\infty\}}}$$
$$\leq \frac{E_\alpha I_{\{k_j^m<\infty\}}\big(1 - P_{X_{k_j^m}}\{u_k > m, \forall k \geq 1\}\big)}{E_\alpha I_{\{k_j^m<\infty\}}} = 1 - s_m < 1.$$

Having proved (4.10), we note that

$$
\begin{aligned}
&\left\{|\eta_t \cap \Lambda| \to \infty\right\} \cup \left\{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \varnothing\right\} \\
&= \left(\bigcup_{m=1}^{\infty} \{|X_k \cap \Lambda| = m \text{ infinitely often}\}\right)^c .
\end{aligned}
\tag{4.13}
$$

Note that if for some element of probability space $\omega \in \Omega$ the process $(\eta_t)_{t \geq 0}$ is stuck in an absorbing state γ, $\gamma \cap \Lambda = \varnothing$, then ω belongs to the set on the left-hand side of (4.13) and does not belong to the set $\left\{|X_k \cap \Lambda| = m \text{ infinitely often}\right\}$, $m \in \mathbb{N}$.

The statement of the lemma follows from (4.10) and (4.13). $\square$

The next lemma is a consequence of the exponentially fast decay of the death rate coefficient.

4.2.3 Proposition. *With probability* 1 *only a finite number of deaths inside* Λ *occur:*

$$
P_\alpha \left\{ |\eta_t \cap \Lambda| - |\eta_{t-} \cap \Lambda| = -1 \textit{ for infinitely many different } \ t \geq 0 \right\} = 0, \ \alpha \in \Gamma_0(\mathbb{R}^d).
$$

Proof. Let $\{\widetilde{X}_k\}_{k \in \mathbb{Z}_+}$ be the process with values in $\Gamma_0(\Lambda)$, which is the "embedded chain" for the process $\widetilde{\eta}_t := \eta_t \cap \Lambda$. More precisely, let $\widetilde{X}_k = \widetilde{\eta}_{\varsigma_k}$, where ς_k is the ordered sequence of jumps of $(\widetilde{\eta}_t)_{t \geq 0}$. Of course, the process $\{\widetilde{\eta}_t\}_{t \geq 0}$ is not Markov in general, and neither is $\{\widetilde{X}_k\}_{k \in \mathbb{N}}$. However, for all $\alpha \in \Gamma_0(\mathbb{R}^d)$ the inequality

$$
P_\alpha\{|\widetilde{X}_1| - |\widetilde{X}_0| = 1\} \geq p_{|\alpha \cap \Lambda|, |\alpha \cap \Lambda| + 1}
$$

holds, because for every $\zeta \in \Gamma_0(\mathbb{R}^d)$, $\zeta \cap \Lambda = m$, the integral of the birth rate coefficient $b(\cdot, \zeta)$ over Λ is larger than cm, and the cumulative death rate in Λ, $\sum\limits_{x \in \zeta \cap \Lambda} d(x, \zeta)$, is less than ma^{-m}.

The probability of the event that absolutely no death occurs is positive, even when the initial configuration contains only one point inside Λ:

$$P_\alpha\Big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\Big\} = P_\alpha\Big\{|\widetilde{X}_{k+1}| - |\widetilde{X}_k| = 1 \text{ for all } \ k \in \mathbb{N}\Big\}$$

$$= \prod_{k\in\mathbb{N}} P_\alpha\left\{|\widetilde{X}_{k+1}| - |\widetilde{X}_k| = 1 \Big| |\widetilde{X}_k| - |\widetilde{X}_{k-1}| = 1, ..., |\widetilde{X}_1| - |\widetilde{X}_0| = 1\right\}$$

$$\geq \prod_{k\in\mathbb{N}} \inf_{\substack{\zeta\in\Gamma^0(\mathbb{R}^d),\\ |\zeta\cap\Lambda| = |\alpha\cap\Lambda| + k}} P_\zeta\{|\widetilde{X}_1| - |\widetilde{X}_0| = 1\}$$

$$\geq \prod_{i=|\alpha|}^{\infty} p_{i,i+1} = \prod_{i=|\alpha|}^{\infty} \frac{c}{c + a^{-i}} = \prod_{i=|\alpha|}^{\infty} (1 - \frac{a^{-i}}{c + a^{-i}}) > 0,$$

because the series $\sum\limits_{i=|\alpha|}^{\infty} \frac{a^{-i}}{c+a^{-i}}$ converges. In particular, $\prod\limits_{i=m}^{\infty} p_{i,i+1} \to 1$ as m goes to ∞. Also,

$$P_{\alpha_n}\big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\big\} \to 1, \quad |\alpha_n \cap \Lambda| \to \infty. \tag{4.14}$$

It is clear only an a.s. finite number of deaths inside Λ occurs on $\{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \varnothing\}$.

By Proposition 4.2.2, it remains to show that only an a.s. finite number of deaths inside Λ occurs on $\{|\eta_t \cap \Lambda| \to \infty\} = \{|\widetilde{\eta}_t| \to \infty\}$. Let us introduce the stopping times $\sigma_n = \inf\{s \in \mathbb{R} : |\widetilde{\eta}_s| \geq n\}$, which are finite on $\{|\widetilde{\eta}_t| \to \infty\}$. Only a finite number of events (births and deaths) occur until arbitrary finite time P_β-a.s. for all $\beta \in \Gamma_0(\mathbb{R}^d)$, hence for $n \in \mathbb{N}$

$$P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } \ t \geq 0\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big)$$

$$\geq P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq \sigma_n\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big)$$

$$= P_\alpha I_{\{|\widetilde{\eta}_t| \to \infty\}} P_{\eta_{\sigma_n}}\big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\big\}.$$

From $|\eta_{\sigma_n}| \geq n$ we have by (4.14)

$$P_{\eta_{\sigma_n}}\Big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\Big\} \to 1, \quad n \to \infty.$$

Therefore,

$$P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } \ t \geq 0\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big) = P_\alpha\{|\widetilde{\eta}_t| \to \infty\}. \ \square$$

Proposition 4.2.3 is also applicable to $(\xi)_{t\geq 0}$, since b_1, d_1 satisfy all the conditions imposed on b, d.

4.2.4 Theorem. *Let $\alpha \in \Gamma_0(\mathbb{R}^d)$. For P_α-almost all $\omega \in F := \{\lim_{t\to\infty} |\eta_t \cap \Lambda| = \infty\}$ the relation*

$$\liminf_{t\to\infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} > 0$$

is fulfilled.

Proof. First we prove the Lemma for $(\xi)_{t\geq 0}$: we prove that for P_α-almost all $\omega \in F_1 := \{\lim_{t\to\infty} |\xi_t \cap \Lambda| = \infty\}$,

$$\liminf_{t\to\infty} \frac{|\xi_t \cap \Lambda|}{e^{ct}} > 0. \tag{4.15}$$

There is no loss in generality in assuming $u_0 = |\alpha \cap \Lambda| > 0$. Let $0 = \tau_0 < \tau_1 < \tau_2 < ...$ be the moments of jumps of $(\xi_t)_{t\geq 0}$, so that $\xi_{\tau_k} = Y_k$. We recall that the random variables $u_n = |Y_n|$ constitute a Markov chain. Denote $\psi(n) = cn + na^{-n}$. Note that

$$\int_\Lambda b_1(x, Y_k)dx + \sum_{x\in Y_k} d_1(x, Y_k) = c|Y_k| + |Y_k|a^{-|Y_k|} = \psi(u_k),$$

By Theorem 12.17 in [Kal02] (see Page 37 of this thesis), random variables $\gamma_k := \psi(u_k)(\tau_{k+1} - \tau_k)$, $k \in \mathbb{Z}_+$ are independent and exponentially distributed with parameter

1. Furthermore, the sequence $\{\gamma_k\}$ is independent of Y. In particular, it is independent of $\{u_k\}_{k\in Z_+}$.

From Proposition 4.2.3 we know that only a finite number of deaths inside Λ occur a.s. on F_1. Particularly, there exists a positive finite random variable $\mathbf{m}$ such that the inequalities

$$u_0 + n \geq u_n \geq u_0 + n - \mathbf{m}(\omega), \quad n \in \mathbb{N} \tag{4.16}$$

hold with probability 1.

We can write

$$\tau_n = \sum_{k=1}^{n-1}(\tau_{k+1} - \tau_k) = \sum_{k=1}^{n-1} \frac{\gamma_k}{\psi(u_k)} \geq \sum_{k=1}^{n-1} \frac{\gamma_k}{u_0 + k}.$$

Due to Kolmogorov's two-series theorem, the series $\sum\limits_{k=1}^{\infty} \frac{\gamma_k}{u_0+k}$ is divergent (we recall that $E\gamma_k = D\gamma_k = 1$). Hence $\tau_n \to \infty$ a.s.

We will show below that

$$c\tau_n \leq \ln(n + u_0) + c\tilde{\gamma}, \quad n \in \mathbb{N}, \tag{4.17}$$

where $\tilde{\gamma}$ is some finite random variable. Using (4.17), we obtain

$$P_\alpha\big\{|\xi_t| \geq \frac{e^{ct}}{2e^{c\tilde{\gamma}}}, t \geq 0\big\} = P_\alpha\big\{|\xi_{\tau_n}| \geq \frac{e^{c\tau_{n+1}}}{2e^{c\tilde{\gamma}}}, n \in \mathbb{N}\big\}$$

$$= P_\alpha\big\{u_n \geq \frac{1}{2}e^{c\tau_{n+1} - c\tilde{\gamma}}, n \in \mathbb{N}\big\} = P_\alpha\big\{\ln(u_n) + \ln 2 \geq c\tau_{n+1} - c\tilde{\gamma}, n \in \mathbb{N}\big\} = 1.$$

Therefore, (4.15) holds.

Inequality (4.17) follows from the convergence of the series

$$\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\psi(u_k)} - \frac{1}{ck}\Big). \tag{4.18}$$

To establish the convergence of (4.18), we note that

$$\sum_{k=1}^{\infty}\left(\frac{\gamma_k}{\psi(u_k)}-\frac{\gamma_k}{cu_k}\right) \tag{4.19}$$

converges by Kolmogorov's theorem:

$$-\sum_{k=1}^{\infty}\left(\frac{\gamma_k}{\psi(u_k)}-\frac{\gamma_k}{cu_k}\right)=\sum_{k=1}^{\infty}\gamma_k\frac{u_k a^{-u_k}}{cu_k\psi(u_k)}\leq\frac{1}{c^2}\sum_{k=1}^{\infty}\gamma_k\frac{a^{-u_k}}{u_k}$$

$$=\frac{1}{c^2}\sum_{k=1}^{\mathbf{m}}+\frac{1}{c^2}\sum_{k=\mathbf{m}+1}^{\infty}\leq\frac{1}{c^2}\sum_{k=1}^{\mathbf{m}}\gamma_k\frac{a^{-u_k}}{u_k}+\frac{1}{c^2}\sum_{j=1}^{\infty}\gamma_k\frac{a^{-j}}{j},$$

and

$$\sum_{k=1}^{\infty}\left(\frac{1}{ck}-\frac{1}{cu_k}\right) \tag{4.20}$$

converges by (4.16).

The convergence of the series in (4.18) follows from the fact that (4.19) and (4.20) converge.

We have thus proved (4.15). To establish the statement of the theorem, note that $\tilde{\sigma}_n=\inf\{t>0:|\eta_t|\geq n\}$ is finite on F and

$$\Big\{\liminf\frac{|\eta_t\cap\Lambda|}{e^{ct}}=0,|\eta_t|\to\infty\Big\}\subset\Big\{\liminf\frac{|\xi_t|}{e^{ct}}=0\Big\}.$$

It follows from what we have already proved that

$$P_\beta\Big\{\liminf\frac{|\xi_t|}{e^{ct}}=0\Big\}=P_\beta\Big\{(\xi_t)_{t\geq0}\ \text{ extincts}\Big\},\qquad\beta\in\Gamma_0(\mathbb{R}^d).$$

Therefore, by Proposition 4.2.1 and the strong Markov property

$$P_\alpha\Big\{\liminf\frac{|\eta_t\cap\Lambda|}{e^{ct}}=0,|\eta_t|\to\infty\Big\}=P_\alpha P_{\eta_{\tilde{\sigma}_n}}\Big\{\liminf\frac{|\eta_t\cap\Lambda|}{e^{ct}}=0,|\eta_t|\to\infty\Big\}$$

$$\leq P_\alpha P_{\eta_{\tilde{\sigma}_n}}\Big\{\liminf \frac{|\xi_t|}{e^{ct}} = 0\Big\} \leq \tilde{C}^{-n},$$

where $\tilde{C}$ is the constant that appeared in Proposition 4.2.1. Since n is arbitrary,

$$P_\alpha\Big\{\liminf \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \to \infty\Big\} = 0.\square$$

4.2.5 Corollary. *For all configurations α with $\alpha \cap \Lambda \neq \varnothing$,*

$$\inf_{t>0} E_\alpha \frac{|\eta_t \cap \Lambda|}{e^{ct}} > 0. \tag{4.21}$$

Proof. Let us fix a configuration α, $\alpha \cap \Lambda \neq \varnothing$. We saw in the proof of Theorem 4.2.4 that for almost all $\omega \in F = \{\omega : \liminf\limits_{t\to\infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} > 0\}$ we have

$$P_\alpha\{|\xi_t| \geq \frac{1}{2e^{c\tilde{\gamma}}}e^{ct}, t \geq 0\} = 1.$$

Let F_k be the set $\{\omega : \frac{1}{2e^{c\tilde{\gamma}}} \geq \frac{1}{k}\}$. Then $\bigcup\limits_{k\in\mathbb{N}} F_k = F$, and, since $P_\alpha(F) > 0$,

$$P_\alpha(F_k) > 0$$

for some $k \in \mathbb{N}$. Hence

$$E_\alpha|\eta_t \cap \Lambda| \geq E_\alpha|\eta_t \cap \Lambda| I_{F_k} \geq \frac{1}{k}e^{ct}P_\alpha(F_k). \quad \square$$

Together with Proposition (2.1.11) the corollary above describe behavior of the average of the process.

Chapter 5

Infinite systems

5.1 Markov processes on $\Gamma(\mathbb{R}^d)$

In this section, we prove one general result about a right-continuous stochastic process in $\Gamma(\mathbb{R}^d)$.

Denote by $\Sigma C(\Gamma(\mathbb{R}^d)$ the collection of those subsets of $\Gamma(\mathbb{R}^d)$ which can be represented as a union of countably many compact sets, i.e., $U \in \Sigma C(\Gamma(\mathbb{R}^d)$ if

$$U = \bigcup_n K_n, \quad K_n \in CS(\Gamma(\mathbb{R}^d)).$$

We can prove the following statement.

5.1.1 Lemma. *Let $\gamma(\omega, t)$ be a right continuous (ω-wisely) stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$. Then for some set $U \in \Sigma C(\Gamma(\mathbb{R}^d)$*

$$P\{\omega : \gamma(\omega, t) \in U, t \geq 0\} = 1. \tag{5.1}$$

Proof. Since the collection of sets $\Sigma C(\Gamma(\mathbb{R}^d)$ is closed under countable unions, it is sufficient to show that there exists $U \in \Sigma C(\Gamma(\mathbb{R}^d)$ such that

$$P\{\omega : \gamma(\omega, t) \in U, 0 \leq t \leq 1\} = 1.$$

The space $\Gamma(\mathbb{R}^d)$ is a Polish space. Therefore, the Skorohod space $\mathscr{D}_{\Gamma(\mathbb{R}^d)}[0; 1]$ is a Polish space too, see Section 1.2.3. Every probability measure is tight on a Polish space, hence for all $\varepsilon > 0$ there exists a compact set $\mathscr{K}_\varepsilon \subset \mathscr{D}_{\Gamma(\mathbb{R}^d)}[0; 1]$ such that $P\{\gamma(\omega, \cdot\,) \in \mathscr{K}_\varepsilon\} > 1 - \varepsilon$. All elements of $\mathscr{K}_\varepsilon$ take values in some compact set $K_\varepsilon \subset \Gamma(\mathbb{R}^d)$. Consequently,

$$P\{\omega : \gamma(\omega, t) \in K_\varepsilon\} > 1 - \varepsilon.$$

The statement follows by taking a sequence $\varepsilon_n \to 0$.

5.2 Spatial birth-and-death processes with infinitely many particles

In this section, we discuss a stochastic equation of the form (5.2) below. A solution of the equation is a $\mathbb{Z}_+^{\mathbb{Z}^d}$-valued cadlag stochastic process. The space $\mathbb{Z}_+^{\mathbb{Z}^d}$ is the space of all maps from $\mathbb{Z}^d$ to $\mathbb{Z}_+$.

Informally, the space $\mathbb{Z}_+^{\mathbb{Z}^d}$ may be regarded as a discrete analog of the configuration space $\Gamma(\mathbb{R}^d)$. Assume that we do not distinguish points of a configuration η in each cube of the form $\prod_{i=1}^d (n_i - \frac{1}{2}; n_i + \frac{1}{2}]$. Then the configuration "becomes" an element of $\mathbb{Z}_+^{\mathbb{Z}^d}$. For $\beta \in \mathbb{Z}_+^{\mathbb{Z}^d}$, we interpret $\beta(i)$ as the number of points at i, $i \in \mathbb{Z}^d$.

Let us introduce the vague topology on $\mathbb{Z}_+^{\mathbb{Z}^d}$ as the minimal topology such that for every function $f : \mathbb{Z}^d \to \mathbb{R}$ with compact support the map

$$\mathbb{Z}_+^{\mathbb{Z}^d} \ni \beta \mapsto \sum_{i \in \mathbb{Z}^d} \beta(i) f(i)$$

is continuous. A function $f : \mathbb{Z}^d \to \mathbb{R}$ has a compact support if and only if $f(i) = 0$ for all but finitely many $i \in \mathbb{Z}^d$. We note that if $(\eta_t, t \in [0;T])$ is a cadlag $\mathbb{Z}_+^{\mathbb{Z}^d}$-valued function, then $(\eta_t(i), t \in [0;T])$ is a cadlag $\mathbb{Z}_+$-valued function for every $i \in \mathbb{Z}^d$.

The generator of a solution to (5.2) should be of the form

$$LF(\chi) = \sum_{i \in \mathbb{Z}^d} b(i,\chi)[F(\chi_i^+) - F(\chi)] + \sum_{i \in \mathbb{Z}^d} d(i,\chi)[F(\chi_i^-) - F(\chi)],$$

where $b : \mathbb{Z}^d \times \mathbb{Z}_+^{\mathbb{Z}^d} \to \mathbb{R}_+$ and $d : \mathbb{Z}^d \times \mathbb{Z}_+^{\mathbb{Z}^d} \to \mathbb{R}_+$ are the birth rate coefficient and the death rate coefficient, respectively. We recall that

$$\chi_i^+(j) = \begin{cases} \chi(j), & \text{if } j \neq i, \\ \chi(j) + 1, & \text{if } j = i, \end{cases} \qquad \chi_i^-(j) = \begin{cases} \chi(j), & \text{if } j \neq i, \\ \chi(j) - 1, & \text{if } j = i. \end{cases}$$

A solution to (5.2) should evolve in time as follows. If the system is in state $\chi \in \mathbb{Z}_+^{\mathbb{Z}^d}$ at time t, then the probability that the number of points at a site $i \in \mathbb{Z}^d$ is increased by 1 ("birth") in the next time interval of length Δt is

$$b(i,\chi)\Delta t + o(\Delta t),$$

the probability that the number of points at the site i is decreased by 1 ("death") in the next time interval of length Δt is

$$d(i,\chi)\Delta t + o(\Delta t),$$

and no two changes occur at the same time. Put differently, a birth at the site i occurs at the rate $b(i,\chi)$, a death at the site i occurs at the rate $d(i,\chi)$. No two events happen simultaneously.

Consider the equation

$$\chi_t(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\chi_{s-})]}(u) N_1(i, ds, du) - \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-})]}(v) N_2(i, dr, dv) + \chi_0(i), \tag{5.2}$$

where $(\chi_t)_{t\in[0;T]}$ is a cadlag $\mathbb{Z}_+^{\mathbb{Z}^d}$-valued solution process, $i \in \mathbb{Z}^d$, N_1, N_2 are Poisson point processes on $\mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times ds \times du$, $\#$ is the counting measure on $\mathbb{Z}^d$, χ_0 is a (random) initial configuration, b, d are birth and death coefficients given below. We require processes N_1, N_2, χ_0 to be independent of each other. Equation (5.2) is understood in the sense that the equality holds a.s. for every $i \in \mathbb{Z}^d$ and $t \in (0; T]$.

Here we consider a special case of (5.2), $b(i, \chi) = A \sum\limits_{j: j \smile i} \chi(j)$, where $j \smile i$ means $|i - j| \le 1$, and $d(i, \chi) = \chi^2(i)$. For $i \in \mathbb{Z}^d$ we define $|i|_1 = |i_1| + ... + |i_d|$.

5.2.1 Definition. A *(weak) solution* of equation (5.2) is a triple $((\chi_t)_{t\in[0;T]}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\in[0;T]})$, where

(i) $(\Omega, \mathscr{F}, P)$ is a probability space, $\{\mathscr{F}_t\}_{t\in[0;T]}$ is an increasing, right-continuous and complete filtration of sub - σ - algebras of $\mathscr{F}$,

(ii) $(\chi_t)_{t\in[0;T]}$ is a cadlag adapted to $\{\mathscr{F}_t\}_{t\in[0;T]}$ process in $\mathbb{Z}_+^{\mathbb{Z}^d}$, $E \sum\limits_{i\in\mathbb{Z}^d} e^{-|i|_1}\chi_t(i) < \infty$, N_1, N_2 are independent Poisson point processes with intensity $\# \times ds \times du$, compatible with $\{\mathscr{F}_t\}_{t\in[0;T]}$,

(iii) all integrals in (5.2) are well-defined, and

(iv) equality (5.2) holds a.s. for all $t \in [0; T]$ and all $i \in \mathbb{Z}^d$.

5.2.2 Definition. A solution is called *strong* if $(\chi_t)_{t\in[0;T]}$ is adapted to the completion under P of the filtration

$$\mathscr{S}_t = \sigma\{\chi_0, N_k(i, [0; q], C), i \in \mathbb{Z}^d, C \in \mathscr{B}(\mathbb{R}_+), q \in [0; t], k = 1, 2\}.$$

We say that pathwise uniqueness holds for equation (5.2) and an initial distribution ν if any two solutions of the form $((\chi_t)_{t\in[0;T]}, N_1, N_2)$ $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\in[0;T]})$ and $((\tilde{\chi}_t)_{t\in[0;T]}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\tilde{\mathscr{F}}_t\}_{t\in[0;T]})$ such that $P\{\chi_0 = \tilde{\chi}_0\} = 1$ and $Law(\chi_0) = \nu$ satisfy $P\{\chi_t = \tilde{\chi}_t \text{ for all } t \geq 0\} = 1$.

We assume that the initial condition χ_0 satisfies

$$\sup_{i\in\mathbb{Z}^d} E\chi_0(i) < \infty. \tag{5.3}$$

The functions b, d posses the properties

$$b(i, \chi_1 \vee \chi_2) \geq b(i, \chi_1) \vee b(i, \chi_2), \tag{5.4}$$

and

$$d(i, \chi_1 \vee \chi_2) = d(i, \chi_1) \vee d(i, \chi_2). \tag{5.5}$$

For a cadlag process $(\chi_t)_{t\in[0;T]}$, we define

$$F_t(\chi)(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\chi_{s-})]}(u) N_1(i, ds, du)$$

$$- \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-})]}(v) N_2(i, dr, dv) + \chi_0(i), \quad t \in (0;T].$$

The process $\{F_t(\chi), t \in (0;t]\}$ is an adapted process with values in $\mathbb{Z}^{\mathbb{Z}^d}$, provided that $\int\limits_0^t E|b(i, \chi_{s-}) \vee d(i, \chi_{s-})|ds$ is finite for all i. Note that if α is a solution of equation (5.2), then $F(\alpha) = \alpha$ in the sense that $F_t(\alpha) = \alpha_t$ a.s. for all $t \in (0;t]$.

Let α, β be adapted processes with values in $(\mathbb{Z}_+)^{\mathbb{Z}^d}$. Using (5.4) and (5.5), we see that

$$\int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\alpha_{s-}\vee\beta_{s-})]}(u)N_1(i,ds,du) \geq \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\alpha_{s-})\vee b(i,\beta_{s-})]}(u)N_1(i,ds,du)$$

$$\geq \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\alpha_{s-})]}(u)N_1(i,ds,du) \vee \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,b(\beta_{s-})]}(u)N_1(i,ds,du),$$

and

$$\int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\alpha_{s-}\vee\beta_{s-})]}(u)N_2(i,ds,du) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\alpha_{s-})\vee d(i,\beta_{s-})]}(u)N_2(i,ds,du).$$

5.2.3 Theorem. *Assume that* (5.3) *is fulfilled. Then pathwise uniqueness and strong existence hold for equation* (5.2).

Proof. *Uniqueness.* Let ξ, ζ be two solutions to (5.2), and let $\xi \vee \zeta$ be the cadlag process defined by

$$(\xi \vee \zeta)_t(i) = \xi_t(i) \vee \zeta_t(i).$$

Denote by $d_t(i)$ the number of deaths for the process $\xi \vee \zeta$ at site i occurred before time t. Then

$$d_t(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\xi_{s-})\vee d(i,\zeta_{s-})]}(u)N_2(i,ds,du) =$$

$$= \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\xi_{s-}\vee\zeta_{s-})]}(u)N_2(i,ds,du).$$

Indeed, if at some moment, say τ, a death for $\xi \vee \zeta$ occurs (that is, $\xi_\tau(i) \vee \zeta_\tau(i) - \xi_{\tau-}(i) \vee \zeta_{\tau-}(i) = -1$), then a death also occurs for the process (ξ or ζ) whose value at i before τ was larger. Consequently, $N_2(i, \{\tau\}, [0; d(\xi_{\tau-} \vee \zeta_{\tau-})]) = 1$. Conversely, if $N_2(i, \{\tau\}, [0; d(\xi_{\tau-} \vee \zeta_{\tau-})]) = 1$, then a death occurs at the moment τ for a process with the largest value at i, therefore a death occurs for $\xi \vee \zeta$, too.

On the other hand, one can see that the number $b_t(i)$ of births of the process $\xi \vee \zeta$ on the interval $(0;t]$ satisfies

$$b_t(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\xi_{s-})\vee b(i,\zeta_{s-})]}(u) N_1(i,ds,du) \leq$$

$$\leq \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i,ds,du).$$

We used here (5.4).

Thus,

$$F_t(\xi\vee\zeta)(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i,ds,du) - \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\xi_{r-}\vee\zeta_{r-})]}(v) N_2(i,dr,dv)$$

$$+\chi_0(i) \geq b_t(i) - d_t(i) + \chi_0(i) = \xi_t \vee \zeta_t.$$

Now we may write

$$0 \leq \xi_t(i) \vee \zeta_t(i) - \xi_t(i) \leq F_t(\xi_t \vee \zeta_t)(i) - F_t(\xi_t)(i) =$$

$$\int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i,ds,du) - \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\xi_{r-}\vee\zeta_{r-})]}(v) N_2(i,dr,dv)$$

$$-\int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\xi_{s-})]}(u) N_1(i,ds,du) + \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\xi_{r-})]}(v) N_2(i,dr,dv) =$$

$$\int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\xi_{s-});b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i,ds,du) - \int\limits_{(0;t]\times[0;\infty)} I_{[d(i,\xi_{r-});d(i,\xi_{r-}\vee\zeta_{r-})]}(v) N_2(i,dr,dv) \leq$$

$$\int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\xi_{s-});b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i,ds,du).$$

Taking expectation, we obtain

$$E(\xi_t(i) \vee \zeta_t(i) - \xi_t(i)) \leq E \int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\xi_{s-});b(i,\xi_{s-}\vee\zeta_{s-})]}(u) N_1(i, ds, du)$$

$$= E \int\limits_{(0;t]} [b(i, \xi_{s-} \vee \zeta_{s-}) - b(i, \xi_{s-})]ds = A \int\limits_{(0;t]\times[0;\infty)} \sum_{j\smile i} E[\xi_{s-}(j) \vee \zeta_{s-}(j) - \xi_{s-}(j)]ds.$$

Denote $\phi(i,t) := E(\xi_t(i) \vee \zeta_t(i) - \xi_t(i))$. Then the inequality above becomes

$$\phi(i,t) \leq A \int\limits_0^t \sum_{j\smile i} \phi(j, s-)ds. \tag{5.6}$$

Denote $\varphi(t) = \sum\limits_{i\in\mathbb{Z}^d} e^{-|i|_1}\phi(i,t)$. Multiplying (5.6) by $e^{-|i|_1}$ and summing over $\mathbb{Z}^d$, we obtain

$$\varphi(t) \leq (2d+1)A \int\limits_0^t \varphi(s-)ds.$$

The function φ is finite by item (iii) of Definition 5.2.1. Consequently, the Gronwall's inequality implies that $\varphi \equiv 0$. Therefore, for a fixed t,

$$E(\xi_t(i) \vee \zeta_t(i) - \xi_t(i)) = 0.$$

for all i, hence $\zeta_t(i) \leq \xi_t(i)$ a.s. Since $\zeta_t(i), \xi_t(i)$ are cadlag processes, it follows that $\zeta_t(i) \leq \xi_t(i)$ a.s. for *all* $t \in (0;T]$.

Swapping the roles of ζ and ξ, we see that $\zeta_t(i) = \xi_t(i)$ a.s. for all $t \in (0;T]$.

Now we turn our attention to the existence.

Existence. Let us consider equation 5.2 with a 'truncated' initial condition, that is, with the initial condition $\chi_0^{(n)}(i) = I_{\{|i|_1\leq n\}}\chi_0^{(n)}(i)$, $n \in \mathbb{N}$. Then the initial configuration is "finite" in the sense that $\sum_i \chi_0^{(n)}(i) < \infty$, therefore one can show that equation (5.2) has a unique solution $\chi_t^{(n)}$, which stays finite. Furthermore, some kind of monotonicity can be shown; namely, if η_0 and ζ_0 are finite initial conditions with $\eta_0(i) \leq \zeta_0(i)$ a.s. for all $i \in \mathbb{Z}^d$, then the inequality is preserved for the solutions: for all $t \in (0;T]$ and $i \in \mathbb{Z}^d$

$$\eta_t(i) \leq \zeta_t(i) \quad \text{a.s.}$$

In particular, $\chi_t^{(m)} \leq \chi_t^{(n)}$ a.s. provided $m \leq n$. We have

$$\chi_t^{(n)}(i) - \chi_t^{(m)}(i) = \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\chi_{s-}^{(n)})]}(u) N_1(i,ds,du) - \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-}^{(n)})]}(v) N_2(i,dr,dv)$$

$$+ I_{\{|i|_1 \leq n\}}\chi_0(i) - \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\chi_{s-}^{(m)})]}(u) N_1(i,ds,du)$$

$$- \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-}^{(m)})]}(v) N_2(i,dr,dv) + I_{\{|i|_1 \leq m\}}\chi_0(i)$$

$$= \int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\chi_{s-}^{(m)});b(i,\chi_{s-}^{(n)})]}(u) N_1(i,ds,du)$$

$$- \int\limits_{(0;t]\times[0;\infty)} I_{[d(i,\chi_{r-}^{(m)});d(i,\chi_{r-}^{(n)})]}(v) N_2(i,dr,dv) + I_{\{m<|i|_1 \leq n\}}\chi_0(i)$$

$$\leq \int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\chi_{s-}^{(m)});b(i,\chi_{s-}^{(n)})]}(u) N_1(i,ds,du) + I_{\{m<|i|_1 \leq n\}}.$$

The sum $\sum\limits_{k\in\mathbb{Z}^d} e^{-|k|_1}\chi_0(k)$ converges a.s. and in $L^1(\Omega, P)$, therefore

$$\sum_{k\in\mathbb{Z}^d} I_{\{m<|k|_1\leq n\}} e^{-|k|_1}\chi_0(k) \to 0$$

a.s. and in $L^1(\Omega, P)$ as $n, m \to \infty$. Let us note that for $\zeta \leq \eta$

$$\begin{aligned}\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[b(i,\eta)-b(i,\zeta)] = A\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}\sum_{j\smile i}[\eta(j)-\zeta(j)] \\ \le A(2d+1)e\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[\eta(i)-\zeta(i)]\end{aligned} \tag{5.7}$$

Consequently we have

$$E\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[\chi_t^{(n)}(i)-\chi_t^{(m)}(i)] \le$$

$$E\Big[\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}\int\limits_{(0;t]\times[0;\infty)} I_{[b(i,\chi_{s-}^{(m)};b(i,\chi_{s-}^{(n)})]}(u)N_1(i,ds,du)+\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}I_{\{m<|i|_1\le n\}}\chi_0(i)\Big] =$$

$$E\Big[\int\limits_{(0;t]}\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[b(i,\chi_{s-}^{(n)}-b(i,\chi_{s-}^{(m)})]ds+\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}I_{\{m<|i|_1\le n\}}\chi_0(i)\Big] \le$$

$$\int\limits_{(0;t]} A(2d+1)eE\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[\chi_{s-}^{(n)}(i)-\chi_{s-}^{(m)}(i)]ds+E\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}I_{\{m<|i|_1\le n\}}\chi_0(i).$$

Denote $f(t) := E\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}[\chi_t^{(n)}(i)-\chi_t^{(m)}(i)]$. We can show that f is finite in the same way as we showed the finiteness of $E|\zeta_t|$ in the proof of Proposition 2.1.5.

Taking into account continuity of f, we see that

$$f(t)\le A(2d+1)e\int\limits_{(0;t]} f(s)ds+E\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}I_{\{m<|i|_1\le n\}}\chi_0(i).$$

Hence

$$f(t)\le e^{A(2d+1)e}E\sum_{i\in\mathbb{Z}^d} e^{-|i|_1}I_{\{m<|i|_1\le n\}}\chi_0(i). \tag{5.8}$$

Define

$$Y_t^{(k)}(i) := \chi_t^{(k)}(i) + \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-}^{(k)})]}(v) N_2(i,dr,dv)$$
$$\left(= \int\limits_{(0;t]\times[0;\infty)} I_{[0;b(i,\chi_{s-}^{(k)})]}(u) N_1(i,ds,du) + \chi_0^{(k)}(i) \right),$$

and let $X_k(t) = \sum\limits_{i\in\mathbb{Z}^d} e^{-|i|_1} Y_t^{(k)}(i)$. For $m < n$ the process $X_n(t) - X_m(t)$ is an (cadlag) increasing process, hence

$$P\{\sup_{s\in(0;T]} (X_n(s) - X_m(s)) \geq \varepsilon\} \leq \frac{E(X_n(T) - X_m(T))}{\varepsilon} = \frac{A(2d+1)e\int\limits_0^T f(s)ds}{\varepsilon}.$$

By (5.8), the last fraction goes to zero as n, m go to infinity. The inclusion

$$\left\{ \sup_{s\in(0;T]} (X_n(t) - X_m(t)) < \varepsilon \right\} \subset \left\{ \text{for all } \ t \in (0;T], k \text{ with } \ e^{-|k|_1} \geq \varepsilon : Y_k^n(t) = Y_k^m(t) \right\}$$

implies that for arbitrary $R > 0$

$$P\left\{ Y_t^{(n)}(k) = Y_t^{(m)}(k) \ \text{ for all } \ t \in (0;T], k \in \mathbb{Z}^d, |k|_1 \leq R \right\} \to 1, \tag{5.9}$$

$m, n \to \infty$.

Monotonicity of d and the inequality $\chi_t^{(n)} \geq \chi_t^{(m)}$, $t \in (0;T]$ give us

$$\int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-}^{(n)})]}(v) N_2(i,dr,dv) \geq \int\limits_{(0;t]\times[0;\infty)} I_{[0;d(i,\chi_{r-}^{(m)})]}(v) N_2(i,dr,dv),$$

hence $\chi_t^{(n)}(k) - \chi_t^{(m)}(k) \leq Y_t^{(n)}(k) - Y_t^{(m)}(k)$. Together with (5.9) this implies

$$P\Big\{\chi_t^{(n)}(k)=\chi_t^{(n)}(k)\ \text{ for all }\ t\in(0;T],k\in\mathbb{Z}^d,|k|_1\le R\Big\}\to 1, m,n\to\infty. \qquad (5.10)$$

One can construct a subsequence $\{n_m\}$ of $\mathbb{N}$ along which the convergence in probability in (5.10) is replaced by convergence almost surely; for some finite random variable $n_0(\omega)$,

$$P\Big\{\chi_t^{(n_{m+1})}(k)=\chi_t^{(n_m)}(k)\ \text{ for all }\ t\in(0;T],k\in\mathbb{Z}^d,|k|_1\le R,n_m\ge n_0(\omega)\Big\}=1. \quad (5.11)$$

Because the functions b,d are local (in the sense that their values at (i,η) depend only on $\{\eta(j),j\smile i\}$), we see by the direct substitution that the limit $\chi_t=\lim\limits_{m\to\infty}\chi_t^{(n_m)}$ satisfies (5.2).

It remains to check that

$$E\sum_{i\in\mathbb{Z}^d}e^{-|i|_1}\chi_t(i)<\infty \qquad (5.12)$$

to conclude that $(\chi_t)_{t\in[0;T]}$ is a solution to (5.2). Arguments similar to those we used to prove (5.8) give

$$E\sum_{i\in\mathbb{Z}^d}e^{-|i|_1}\chi_t^{(n)}(i)\le e^{A(2d+1)e}E\sum_{i\in\mathbb{Z}^d}e^{-|i|_1}I_{\{|i|_1\le n\}}\chi_0(i)\le e^{A(2d+1)e}E\sum_{i\in\mathbb{Z}^d}e^{-|i|_1}\chi_0(i).$$

Letting $n\to\infty$, we get (5.12).

The solution $(\chi_t)_{t\in[0;T]}$ is strong, since $\chi_t^{(n)}(k)$ is adapted to $\{\mathscr{S}_t\}_{t\in[0;T]}$ for every $n\in\mathbb{N}$.
□

5.2.4 Remark. An examination of the proof shows that Theorem (5.2.3) holds for general $b,d:\mathbb{Z}^d\times\mathbb{Z}_+^{\mathbb{Z}^d}\to\mathbb{R}_+$ satisfying certain conditions. Namely, we need d to be local and monotone in the sense that $d(i,\xi)=g(\xi(i))$ for some non-decreasing function $g:\mathbb{Z}_+\to$

$\mathbb{R}_+$, and we need b to be monotone and to have a Lipschitz property and a finite range property. By the monotonicity we mean

$$b(i,\xi) \leq b(i,\zeta), \quad \xi \leq \zeta,$$

by the finite range property we mean existence of $R \in \mathbb{N}$ such that

$$b(i,\xi) = b(i,\zeta) \qquad \text{whenever} \;\; \xi(i+j) = \zeta(i+j),\;\; |j|_1 \leq R,$$

and the Lipschitz property we understand in the sense that

$$|b(i,\xi) - b(i,\zeta)| \leq \sum_{j \in \mathbb{Z}^d} \varphi(j) |\xi(i+j) - \zeta(i+j)|$$

holds for some summable function $\varphi : \mathbb{Z}^d \to \mathbb{R}_+$. Furthermore, we expect that some of these conditions may be relaxed.

Bibliography

[Ald13] D. Aldous. Interacting particle systems as stochastic social dynamics. *Bernoulli*, 19:1122–1149, 2013.

[AN72] K.B. Athreya and P.E Ney. *Branching processes.* Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1972.

[And82] E. D. Andjel. Invariant measures for the zero range processes. *Ann. Probab.*, 10(3):525–547, 1982.

[Arn06] d.L.F. Arnaud. Yule process sample path asymptotics. *Electron. Comm. Probab.*, 11:193–199, 2006.

[BDMPP87] C. Boldrighini, A. De Masi, A. Pellegrinotti, and E. Presutti. Collective phenomena in interacting particle systems. *Stochastic Process. Appl.*, 25(1):137–152, 1987.

[BP97] B.M. Bolker and S.W. Pacala. Using moment equations to understand stochastically driven spatial pattern formation in ecological systems. *Theoretical Population Biology*, 52:179–197, 1997.

[BP99] B.M. Bolker and S.W. Pacala. Spatial moment equations for plant competitions: Understanding spatial strategies and the advantages of short dispersal. *The American Naturalist*, 153(6):575–602, 1999.

[BR58] C. J. Burke and M. Rosenblatt. A Markovian function of a markov chain. *E Ann. Math. Statist*, 29:1112–1122, 1958.

[CG] M. M. Castro and F. A. Grünbaum. On a seminal paper by karlin and mcgregor. *Symmetry Integrability Geom. Methods Appl.*, 9.

[Chu67] K. L. Chung. *Markov chains with stationary transition probabilities.* Springer, 1967.

[DL05] U. Dieckmann and R. Law. Relaxation projections and the method of moments ulf dieckmann. page 412–455. Cambridge University Press, 2005.

[EH05] M. R. Evans and T. Hanney. Nonequilibrium statistical mechanics of the zero-range process and related models. *J. Phys. A*, 38(19):R195–R240, 2005.

[EK86] S. N. Ethier and T. G. Kurtz. *Markov Processes. Characterization and convergence.* Wiley-Interscience, New Jersey, 1986.

[EN00] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Mathematics. Springer, 2000.

[EN06] K.-J. Engel and R. Nagel. *A Short Course on Operator Semigroups.* Springer, 2006.

[FKK12a] D. Finkelshtein, O. Kutovyi, and Yu. Kondratiev. Semigroup approach to birth-and-death stochastic dynamics in continuum. *Journal of Functional Analysis*, 262(3):1274–1308, 2012.

[FKK12b] D. Finkilstein, Yu. Kondratiev, and O. Kutoviy. Semigroup approach to birth-and-death stochastic dynamics in continuum. *J. Funct. Anal.*, 262(3):1274–1308, 2012.

[FKK14] D. Finkelshtein, O. Kutovyi, and Yu. Kondratiev. Statistical dynamics of continuous systems: perturbative and approximative approaches. *Arabian Journal of Mathematics*, 2014. doi:10.1007/s40065-014-0111-8.

[FKKK] D. Finkelshtein, O. Kutovyi, Yu. Kondratiev, and Yu. Kozitsky. The statistical dynamics of a spatial logistic model and the related kinetic equation. *Mathematical Models and Methods in Applied Sciences*. to appear.

[FKKZ14] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and E. Zhizhina. On an aggregation in birth-and-death stochastic dynamics. *Nonlinearity*, 27(6):1105–1133, 2014.

[FM04] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab*, 14(4):1880–1919, 2004.

[FOK+14] D. Finkelshtein, O. Ovaskainen, O. Kutovyi, S. Cornell, B. Bolker, and Yu. Kondratiev. A mathematical framework for the analysis of spatial-temporal point processes. *Theoretical Ecology*, 7:101–113, 2014.

[Fra14] T. Franco. Interacting particle systems: hydrodynamic limit versus high density limit. 2014. preprint; arXiv:1401.3622 [math.PR].

[Gar95] N. L. Garcia. Birth and death processes as projections of higher-dimensional poisson processes. *Adv. in Appl. Probab.*, 27(4):911–930, 1995.

[GK06] N. L. Garcia and T. G. Kurtz. Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat. Am. J. Probab. Math. Stat.*, (1):281–303, 2006.

[GK08] N. L. Garcia and T. G. Kurtz. Spatial point processes and the projection method. *Progr. Probab. In and out of equilibrium. 2,*, 60(2):271–298, 2008.

[GS75] I. I. Gikhman and A. V. Skorokhod. *The Theory of Stochastic Processes*, volume 2. Springer, 1975.

[GS79] I. I. Gikhman and A. V. Skorokhod. *The Theory of Stochastic Processes*, volume 3. Springer, 1979.

[Har63] T.E. Harris. *The theory of branching processes.* Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1963.

[HS78] R. A. Holley and D. W. Stroock. Nearest neighbor birth and death processes on the real line. *Acta Math*, 140(1-2):103–154, 1978.

[IW81] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes.* Nord-Holland publiching company, 1981.

[Kal02] O. Kallenberg. *Foundations of modern probability.* Springer, 2 edition, 2002.

[Kin93] J. F. C. Kingman. *Poisson Processes.* Oxford University Press, 1993.

[KK02] Y. G. Kondratiev and T. Kuna. Harmonic analysis on configuration space. i. general theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(2):201–233, 2002.

[KK06] Yu. G. Kondratiev and O. V. Kutoviy. On the metrical properties of the configuration space. *Math. Nachr.*, 279:774–783, 2006.

[KL99] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems.* Springer, 1999.

[KM59] S. Karlin and J. McGregor. Random walks. *Illinois J. Math.*, 3:66–81, 1959.

[KS06] Yu. Kondratiev and A. Skorokhod. On contact processes in continuum. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 9(2):187–198, 2006.

[Kur66] K. Kuratowski. *Topology*, volume 1. Academic Press, New York and London, 1966.

[Lev03] S.A. Levin. Complex adaptive systems:exploring the known, the unknown and the unknowable. *Bulletin of the AMS*, 40(1):3–19, 2003.

[Lig85] T. M. Liggett. *Interacting particle systems.* Grundlehren der Mathematischen Wissenschaften. Springer, 1985.

[Lig04] T. M. Liggett. *Interacting particle systems—an introduction.* 2004. ICTP Lect. Notes, XVII.

[Mee00] R. Meester. On a long range particle system with unbounded flip rates. *Markov Process. Related Fields*, 9(1):59–84, 2000.

[MS94] J. Møller and M. Sørensen. Statistical analysis of a spatial birth-and-death process model with a view to modelling linear dune fields. *Scand. J. Statist.*, 21(1):1–19, 1994.

[MT93] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability.* Springer, 1993.

[MW04] J. Møller and R. P. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes.* Chapman and Hall/CRC, 2004.

[Pen08] M. D. Penrose. Existence and spatial limit theorems for lattice and continuum particle systems. *Probab. Surv.*, 5:1–36, 2008.

[Pod09] K. Podczeck. On existence of rich fubini extensions. *Econom. Theory*, 45(1-2):1–22, 2009.

[Pre75] C. Preston. Spatial birth-and-death processes. In *Proceedings of the 40th Session of the International Statistical Institute*, volume 46 of *Bull. Inst. Internat. Statist*, pages 371–391, 405–408, 1975.

[RS99] M. Röckner and A. Schied. Rademacher's theorem on configuration spaces and applications. *J. Funct. Anal.*, 169(2):325–356, 1999.

[RY05] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer, 3 edition, 2005.

[Spi70] F. Spitzer. Interaction of markov processes. *Advances in Math.*, 5:246–290, 1970.

[Spi77] F. Spitzer. Stochastic time evolution of one dimensional infinite particle systems. *Bull. Amer. Math. Soc.*, 83(5):880–890, 1977.