Shalika models and p-adic L-functions

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Contents

Introduction

Let F be a totally real number field with group of ideles $\mathbb{I}$, p a prime number and $\pi = \otimes\pi_v$ a unitary cuspidal automorphic representation of GL_{2n} over F. Attached to π and an idele class character $\chi\colon \mathbb{I}/F^* \to \mathbb{C}^*$ is the automorphic L-Function $L(\pi \otimes \chi, s)$, which is an holomorphic function in the complex variable $s \in \mathbb{C}$. The aim of this thesis is to construct for every critical point $s + \frac{1}{2}$ of π - and under certain hypotheses on π and p - a p-adic L-function $L_p(\pi, s, x)$, which is an analytic function in the p-adic variable $x \in \mathbb{Z}_p$.
To be more precise let $\mathcal{G}_p$ be the Galois group of the maximal abelian extension of F unramified outside p and ∞ over F. Under the hypothesis that

- for every prime $\mathfrak{p}$ of F dividing p the local representation $\pi_{\mathfrak{p}}$ is an unramified principal series representations with Satake parameters $\beta_{i,\mathfrak{p}}$, $1 \leq i \leq 2n$
- π has a Shalika model with respect to some character η, such that $\beta_{i,\mathfrak{p}}\beta_{j,\mathfrak{p}} \neq \eta_{\mathfrak{p}}(\varpi)^{\pm 1}$ for all $1 \leq i < j \leq n$ and all $\mathfrak{p}$ dividing p, where ϖ is a local uniformizer at $\mathfrak{p}$
- π is cohomological with respect to some algebraic representation V^{al}

we construct a $\mathbb{C}$-valued distribution $\mu_{\pi,s}$ on $\mathcal{G}_p$ such that for all locally constant characters $\chi\colon \mathcal{G}_p \to \mathbb{C}^*$ we have the following interpolation property:

$$\int_{\mathcal{G}_p} \chi(\gamma)\mu_{\pi,s}(d\gamma) = N(\mathfrak{f}(\chi))^s \tau(\chi)^n \prod_{\mathfrak{p}\in S_p} E(\delta_p, \chi_{\mathfrak{p}}, s)$$
$$\times L_{S_{\mathfrak{p},\infty}}(\pi \otimes \chi, s + \frac{1}{2}) \prod_{v \in S_\infty} c(\pi, \chi_\infty, s).$$

Here $\mathfrak{f}(\chi)$ is the conductor of χ, $\tau(\chi)$ its Gauss sum, $E(\delta_p, \chi_{\mathfrak{p}}, s)$ a certain modified Euler factor at $\mathfrak{p}$ (see Theorem 2.6 for a precise definition) and $c(\pi, \chi_\infty, s)$ a non-vanishing factor at ∞ (see the end of Section 3.1 for its definition.). If we additionally assume that

- the prime p is totally split in F
- the central character of $\pi_{\mathfrak{p}}$, $\mathfrak{p}|p$, takes values in the $(p-1)$-roots of unity
- the weights of V^{al} are p small
- π fullfills a certain weak p-ordinarity condition

we show that the distribution $\mu_{\pi,s}$ is in fact p-adically bounded and define the p-adic L-function as the following integral

$$L_p(\pi, s, x) = \int_{\mathcal{G}_p} \langle\gamma\rangle^x \, \mu_{\pi,s}(d\gamma).$$

See the end of Section 3.3 for more details.

Ash and Ginzburg first constructed p-adic L-functions for cohomological representations having a Shalika model in [AG94]. They only considered the case of trivial weights, i.e. the representation V^{al} is isomorphic to the trivial one-dimensional representation. Although some computations in this thesis are similar to the ones of Ash-Ginzburg, we use a different strategy for proving the boundedness of the distribution and the distribution relation. Whereas Ash-Ginzburg approach is close to the classical construction of p-adic L-function for modular forms by Mazur-Tate-Teitelbaum in [MTT86], our construction is similar to the one of Spieß in [Spi14], which builds on the seminal work of Darmon in [Dar01]. Two main features of our construction are: firstly, the global distribution is build out of local distributions for all places v of F, which makes the interpolation property a purely local computation. Secondly, instead of relating the special L-values to integrals of a fixed cohomology class over different cycles, we will fix the cycle and construct a big cohomology class in the group cohomology of S_p-arithmetic subgroups of $\mathrm{GL}_{2n}(F)$ with coefficients in the module $(\otimes_{\mathfrak{p}\in S_p}\pi_{\mathfrak{p}}^{\vee}) \otimes (V^{al})^{\vee}$.
The structure of this thesis is the following: The first section deals only with the local situation. Given a prime $\mathfrak{p}$ of F lying above p and a principal series representation $\pi_{\mathfrak{p}}$ of $\mathrm{GL}_{2n}(F_{\mathfrak{p}})$ we construct a map

$$\delta\colon C_c^0(GL_n(F_{\mathfrak{p}}), \mathbb{C}) \to \pi_{\mathfrak{p}}$$

and a version of it for locally algebraic representations. The local part at $\mathfrak{p}$ of the global distribution is essentially the pull-back of the Shalika-functional along δ. We proof that the map δ respects p-integral structures on both sides, as long as $\pi_{\mathfrak{p}}$ is "weakly p-ordinary". We want to use the results of Große-Klönne on nice p-integral structures on locally algebraic representations (see [GK14]). Therefore, we have to impose all the other conditions on p, since the results of Große-Klönne are only complete in these cases.
In the second section we use the work of Friedberg-Jacquet in [FJ93] on Shalika models to define the global distribution. An advantage of our construction is that the distribution relation follows immediately.
In the third section we give another description of the distribution in terms of group cohomology. The boundedness then follows from the existence of

p-integral structures on these cohomology groups, the integrality of the map δ and general cohomological properties of arithmetic groups.

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Notations. We will use the following notations throughout the whole thesis. At the beginning of chapter 1 and 2 will be a list of additional notations which may be only valid for that given section.
All rings are commuative and have a unit unless stated otherwise. The group of invertible elements of a ring R will be denoted by R^*. If M is an R-module, we denote the dual module $\mathrm{Hom}_R(M,R)$ by $M^\vee$.
We fix a prime p and embeddings

$$\mathbb{C} \overset{\iota_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \mathbb{C}_p.$$

We let ord_p denote the valuation on $\mathbb{C}_p$ (and on $\overline{\mathbb{Q}}$ via ι_p) normalized such that $\mathrm{ord}_p(p) = 1$. The valuation ring of $\overline{\mathbb{Q}}$ with respect to ord_p will be denoted by $\overline{\mathcal{O}}$.
Given two topological spaces X, Y we will write $C(X,Y)$ for the space of continuous functions from X to Y. If R is a topological ring we define $C_c(X,R) \subset C(X,R)$ as the subspace of continuous functions with compact support. If we consider Y (R resp.) with the discrete topology we will often write $C^0(X,Y)$ ($C_c^0(X,R)$ resp.) instead.
The entry in the i-th row and j-th column of a matrix A is denoted by A_{ij}.
If R is a ring and G a group we will denote the group ring of G over R by $R[G]$. If G is a topological group we write G° for the connected component of the identity.
Let G be a totally disconnected, Hausdorff topological group (e.g. a discrete group, a Galois group of a field or the F-valued points of an algebraic group, where F is a local field) and H a closed subgroup of G. Given a ring R and an R-module M on which H acts R-linearly, we define the (*smooth*) *induction* $\mathrm{Ind}_H^G M$ of M from H to G as the space of all locally constant functions $f\colon G \to M$ such that $f(hg) = hf(g)$ for all $h \in H, g \in G$. The induction $\mathrm{Ind}_H^G M$ is an R-module on which G acts R-linearly via the right regular representation. We define the (*smooth*) *compact induction* $\text{c-ind}_H^G M$ as the $R[G]$-submodule of $\mathrm{Ind}_H^G M$ consisting of functions which have compact support module H. If $M = R$ and the action of H on M is given by the character $\chi\colon H \to R^*$ we will often write $\mathrm{Ind}_H^G \chi$ ($\text{c-ind}_H^G \chi$ resp.) instead of $\mathrm{Ind}_H^G M$ ($\text{c-ind}_H^G M$ resp.). (A word of warning regarding these notations: If H and G are discrete groups, then $\mathrm{Ind}_H^G M$ is usually referred to as the coinduced module and $\text{c-ind}_H^G M$ as the induced module.)
For a set X and a subset $A \subset X$ the characteristic function $\mathbb{1}_A\colon X \to \{0,1\}$ is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else.} \end{cases}$$

1 Local distributions

Throughout this section let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O} = \mathcal{O}_F$ its ring of integers with maximal ideal $\mathfrak{p} = (\varpi)$ and q the number of elements of $\mathcal{O}/\mathfrak{p}$. We denote the group of units of $\mathcal{O}$ by U and put $U^{(m)} = \{u \in U |\ u \equiv 1 \mod \mathfrak{p}^m\}$. We denote by ν the normalized additive valuation on F (i.e. $\nu(\varpi)$=1) and by $|x|$ the modulus of $x \in F^*$ (i.e. $|\varpi| = q^{-1}$).
Let G_r (K_r resp.) denote the group of invertible $(r \times r)$-matrices over F ($\mathcal{O}$ resp.) and let d^*g denote the Haar measure on G_r (normalized such that $\int_{K_r} d^*g = 1$). We denote by $K_r^{(m)}$ the principal congruence subgroup of K_r of level m, i.e. the kernel of the reduction map from K_r to $\mathrm{GL}_r(\mathcal{O}/\mathfrak{p}^m)$. The group of invertible upper triangular matrices in G_r will be denoted by B_r. We fix a character $\psi\colon F \to \overline{\mathbb{Q}}^*$ with conductor $\mathcal{O}$.

1.1 Distribution spaces

Let $\mathcal{X}$ be a totally disconnected σ-compact topological space and R a topological Haussdorff ring. We denote by $C_\diamond(\mathcal{X}, R)$ the subring of $C(\mathcal{X}, R)$ of functions which tend to zero at infinity (i.e. functions $f\colon \mathcal{X} \to R$ which can be continuously extended to the one point compactification of $\mathcal{X}$ by setting $f(\infty) = 0$). Hence we have a chain of inclusions $C_c^0(\mathcal{X}, R) \subset C_c(\mathcal{X}, R) \subset C_\diamond(\mathcal{X}, R) \subset C(\mathcal{X}, R)$. Note that we always have $C_c^0(\mathcal{X}, R) = C_c^0(\mathcal{X}, \mathbb{Z}) \otimes R$.
Let M be an R-module. An M-valued *distribution* on $\mathcal{X}$ is a homomorphism $\mu\colon C_c^0(\mathcal{X}, \mathbb{Z}) \to M$. It extends uniquely to an R-linear homomorphism

$$C_c^0(\mathcal{X}, R) \to M,\ f \mapsto \int_{\mathcal{X}} f\ d\mu.$$

We will denote the R-module of M-valued distributions by $\mathrm{Dist}(\mathcal{X}, M)$. If $\mathcal{G} = \mathcal{X}$ is a topological group, then $\mathcal{G} \times \mathcal{G}$ acts on $C(\mathcal{G}, R)$ ($C_c(\mathcal{G}, R)$ resp.) by $(g_1, g_2)f(h) = f(g_1^{-1} h g_2)$ and on $\mathrm{Dist}(\mathcal{G}, M)$ by $(g_1, g_2)\mu(f) = \mu((g_1, g_2)^{-1} f)$.
Now assume that E is a p-adic field, i.e. E is a field of characteristic 0 which is complete with respect to an absolute value $|\cdot| : E \to \mathbb{R}$ whose restriction to $\mathbb{Q}$ is the usual p-adic absolute value. We will denote the valuation ring of E by R.
Let V be an E-vector space. An R-submodule $L \subset V$ is a *lattice* if the following two conditions are fullfilled: (i) $\bigcup_{a \in E^*} aL = V$ and (ii) $\bigcap_{a \in E^*} aL = \{0\}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying:

- $\|av\| = |a|\, \|v\|$,
- $\|v + w\| \leq \max(\|v\|, \|w\|)$,

- $\|v\| \geq 0$ with equality if and only if $v = 0$

for all $a \in E$, $v, w \in V$. A normed space $(V, \|\cdot\|)$ is a *Banach space* if it is complete with respect to $\|\cdot\|$. For example the E-vector space $C_\diamond(\mathcal{X}, E)$ is a Banach space with respect to the supremum norm $\|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)|$. If $E/\mathbb{Q}_p$ is a finite extension, $C_\diamond(\mathcal{X}, R)$ is an open bounded lattice in $C_\diamond(\mathcal{X}, E)$.
Let $(V, \|\cdot\|)$ be a Banach space. An element $\mu \in \operatorname{Dist}(\mathcal{X}, V)$ is a *bounded distribution* if μ is continuous with respect to the supremum norm, i.e. there exists a constant $C \in \mathbb{R}$ such that $\left\|\int_{\mathcal{X}} f\ d\mu\right\| \leq C \|f\|_\infty$ for all $f \in C_c^0(\mathcal{X}, E)$. We denote the set of E-valued bounded distributions by $\operatorname{Dist}^b(\mathcal{X}, V)$. If $L \subset V$ is an open bounded lattice then $\operatorname{Dist}^b(\mathcal{X}, V)$ is the image of the inclusion $\operatorname{Dist}(\mathcal{X}, L) \otimes E \to \operatorname{Dist}(\mathcal{X}, V)$.
Since $C_c^0(\mathcal{X}, E)$ is dense in the Banach space $C_\diamond(\mathcal{X}, K)$ any bounded V-valued distribution μ can uniquely be extended to a continuous E-linear homomorphism

$$C_\diamond(\mathcal{X}, K) \to V,\ f \mapsto \int_{\mathcal{X}} f\ d\mu.$$

Let $\mathcal{X}$ be compact and $\mu \in \operatorname{Dist}(\mathcal{X}, \mathbb{C})$ a $\mathbb{C}$-valued distribution. We say that μ is a *p-adic measure* if there exists a Dedeking ring $R \subset \overline{\mathcal{O}}$ such that the image of $C^0(\mathcal{X}, \mathbb{Z})$ under μ is contained in a finitely generated R-submodule of $\mathbb{C}$. Let $L_{\mu,R}$ the smallest such R-submodule of $\mathbb{C}$. In this case μ defines a bounded distribution with values in $\widetilde{L_{\mu,R}} := L_\mu \otimes_R \mathbb{C}_p$.
For the rest of this section let E be a finite extension of F with ring of integers R. For a closed subgroup A of $G_r = \mathrm{GL}_r(F)$ and an integer $s \in \mathbb{Z}$ we define $C_c^s(A, E) \subset C_c(A, E)$ to be the subspace of functions f which are locally of the form $f(g) = a \det^s(g)$ with $a \in E$. It is stable under the action of $A \times A$ on $C_c(A, E)$. Further we define $C_c^s(A, R)$ as the intersection $C_c^s(A, E) \cap C_c(A, R)$. Given an E-vector space V we define the space of *V-valued s-distributions* as $\operatorname{Dist}_s(A, V) = \operatorname{Hom}_E(C_c^s(A, E), V)$. If V is a Banach space an s-distribution is *bounded* if it is continuous with respect to the supremum norm. We will denote the space of bounded s-distributions by $\operatorname{Dist}_s^b(A, V)$. Again, bounded s-distribution can be extended uniquely to continuous linear maps $\mu\colon C_\diamond(A, E) \to V$.
Let $\det^{m_1,m_2}\colon G_r \times G_r \to E^*$, $m_1, m_2 \in \mathbb{Z}$, the E-valued character given by $\det^{m_1,m_2}(g_1, g_2) = \det^{m_1}(g_1)\det^{m_2}(g_2)$. Then as a $A \times A$-module $C_c^s(A, E)$ is isomorphic to $C_c^0(A, E) \otimes \det^{-s,s}$.
Let $\mathcal{G}$ be a dense subgroup of G_r and $\Gamma = \mathcal{G} \cap K_r$. For example $\mathcal{G} = \mathrm{GL}_r(\mathcal{F})$, where $\mathcal{F}$ is a number field, whose complection at a prime above p is isomorphic to F. We consider $\mathcal{G}$ and Γ as discrete subgroups.

Lemma 1.1. *The canonical map given by Frobenius reciprocity*

$$\text{c-ind}_{\Gamma\times\Gamma}^{\mathcal{G}\times\Gamma} C_c^s(K_r, R) \to C_c^s(G_r, R)$$

is an isomorphism of $R\,[\mathcal{G}\times\Gamma]$*-modules.*

Proof. Let J be a set of representatives of $(\mathcal{G}\times\Gamma)\,/\,(\Gamma\times\Gamma) \cong \mathcal{G}/\Gamma$. Since $\mathcal{G}$ is dense in G_r we see that G_r is the disjoint union of the jK_r, $j \in J$. Hence we have

$$C_c^s(G_r, R) = \bigoplus_{j\in J} jC_c^s(K_r, R)$$

and the claim follows. □

Of course, the statement remains true when we replace R by E. In view of this lemma we make the following definitions: Let $\mathcal{G}$ and Γ as above and E' a field such that $\det(g) \in E'$ for all $g \in \mathcal{G}$. If E' is a subfield of E, we denote by R' the valuation ring of E' with respect to the valuation induced from E. In this case $\det(\gamma) \in (R')^*$ for all $\gamma \in \Gamma$ and thus we can view $\det^{m,n}$ as an R' valued character of $\Gamma\times\Gamma$. Hence we can define the $R'[\Gamma\times\Gamma]$-module

$$C_c^s(K_r, R') := C_c^0(K_r, R') \otimes_{R'} \det{}^{-s,s}$$

and the $R'[\mathcal{G}\times\Gamma]$-module

$$C_c^s(G_r, R') := \text{c-ind}_{\Gamma\times\Gamma}^{\mathcal{G}\times\Gamma} C_c^s(K_r, R')$$

for all $s \in \mathbb{Z}$. This agrees with the previous definitions in the case $\mathcal{G} = G_r$ and $E' = E$. For general E' (not necessarily contained in E) and $s \in \mathbb{Z}$ we define the $E'[\mathcal{G}\times\Gamma]$-module

$$C_c^s(G_r, E') := C_c^0(G_r, E') \otimes_{E'} \det{}^{-s,s}\,.$$

1.2 The map δ

In this section we construct a map δ (depending on several choices) from the space of locally constant functions on G_n to a principal series representation of G_{2n}. It will be used in Section 2.2 to define the local part of our global distribution. The map δ was first constructed for $n = 1$ in [Spi14].
Let us fix a field E of characteristic 0. Given locally constant characters $\chi_i\colon F^* \to E^*$, $i = 1,\ldots,r$ we let $\chi\colon B_r \to E^*$ be the character given by

$\chi(b) = \prod_{i=1}^n \chi_i(b_{ii})$. Then $\mathrm{Ind}_{B_r}^{G_r}(\chi_1, \dots, \chi_r) := \mathrm{Ind}_{B_r}^{G_r} \chi$ is called a *principal series representation*. It is a smooth, admissible G_r-representations over E.

Now let $r = 2n$ be an even number. For every pair $\rho_1 \in \mathrm{Ind}_{B_n}^{G_n}(\chi_1, \dots, \chi_n)$ and $\rho_2 \in \mathrm{Ind}_{B_n}^{G_n}(\chi_{n+1}, \dots, \chi_{2n})$ we define an E-linear map

$$\delta = \delta_{\rho_1,\rho_2} \colon C_c^0(G_n, E) \to \mathrm{Ind}_{B_{2n}}^{G_{2n}} \chi$$

given by

$$\delta(f)(g) = \begin{cases} \rho_1(g_1)\rho_2(g_2 u) f(u), & \text{if } g = \begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ g_i, u \in G_n \\ 0 & \text{else.} \end{cases}$$

The group $G_n \times G_n$ acts on $C_c^0(G_n, E)$ as in Section 1.1 and on $\mathrm{Ind}_{B_{2n}}^{G_{2n}}$ through the diagonal embedding of $G_n \times G_n$ into G_{2n}.

Lemma 1.2. *Let $H \subset G_n$ be the intersection of the stabilizers of ρ_1 and ρ_2. The map δ is $G_n \times H$-equivariant.*

Proof. Let $g = \begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in G_{2n}$ with $g_1, g_2, u \in G_n$, $(h_1, h_2) \in G_n \times H$ and $f \in C_c^0(G_n, E)$. Then we have

$$\begin{aligned} \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \delta(f) \right)(g) &= \delta(f) \left(\begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \right) \\ &= \delta(f) \left(\begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1 & h_1^{-1} u h_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \delta(f) \left(\begin{pmatrix} g_1 h_2 & * \\ 0 & g_2 h_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & h_1^{-1} u h_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \rho_1(g_1 h_2) \rho_2(g h_1 h_1^{-1} u h_2) f(h_1^{-1} u h_2) \\ &= \rho_1(g_1) \rho_2(g u) f(h_1^{-1} u h_2) \\ &= \delta((h_1, h_2) f)(g). \end{aligned}$$

□

By dualizing δ we get an E-linear homomorphism

$$\begin{aligned} \delta^\vee \colon (\mathrm{Ind}_{B_{2n}}^{G_{2n}} \chi)^\vee &\to \mathrm{Dist}(G_n, E) \\ \lambda &\mapsto \mu_\lambda = \lambda \circ \delta. \end{aligned}$$

Given $\lambda \colon \mathrm{Ind}_{B_{2n}}^{G_{2n}} \chi \to E$ and $\varphi \in \mathrm{Ind}_{B_{2n}}^{G_{2n}} \chi$ we let $\xi_\varphi^\lambda \colon G_{2n} \to E$ be the function given by $\xi_\varphi^\lambda(g) = \lambda(g\varphi)$ for $g \in G_{2n}$. If H is an open, compact subgroup of G_n we put $\xi_H^\lambda = \xi_{\delta(\mathbb{1}_H)}^\lambda$.

Proposition 1.3. *Let* $\lambda\colon \operatorname{Ind}_{B_{2n}}^{G_{2n}} \chi \to E$ *be a linear functional and* $H \subset K_n$ *an open, compact subgroup. Then for all* $f \in C_c^0(G_n, E)$*, which are* H*-invariant under right multiplication, we have*

$$\int_{G_n} f(g)\mu_\lambda(dg) = [K_n\colon H] \int_{GL_n} f(g)\xi_H^\lambda\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) d^*g.$$

Proof. It is enough to proof the formula in the case $f = \mathbb{1}_{aH} = (a,1)\mathbb{1}_H$ with $a \in G_n$. We then have

$$\begin{aligned}\int_{G_n} f(g)\mu_\lambda(dg) &= \int_{G_n} (a,1)\mathbb{1}_H(g)\mu_\lambda(dg) \\ &= \xi_H^\lambda\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = [K_n\colon H] \int_{G_n} \mathbb{1}_H(g)\xi_H^\lambda\left(\begin{pmatrix} ag & 0 \\ 0 & 1 \end{pmatrix}\right) d^*g \\ &= [K_n\colon H] \int_{G_n} f(g)\xi_H^\lambda\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) d^*g.\end{aligned}$$

□

1.3 Integrality of δ

In this section we show that under a certain ordinarity condition the map δ (and a variant of it for locally algebraic representations) respects integral structures. This result is crucial for showing that the global distributions, which we will construct in Section 2.2, are p-adic measures.

Since $\operatorname{Ind}_{B_n}^{G_n}(\chi_{n+1}, \dots, \chi_{2n})$ is a smooth representations there exists $m \geq 1$ such that ρ_2 is invariant under right multiplication by $K_n^{(m)}$.

Let $Q \subset G_{2n}$ the subgroup of elements of the form $\begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix}$ with $g \in G_n$ and $u \in M_n(F)$.

Lemma 1.4. *Let* $\alpha = \prod_{i=n+1}^{2n} \chi_i(\varpi)$ *and* $R \subset E$ *a subring such that* $\alpha^{-1} \in R$*. Then the image of* $C_c^0(G_n, R)$ *under* δ *is contained in the* $R[Q]$*-module generated by* $\delta(\mathbb{1}_{K_n^{(m)}})$.

Proof. By Lemma 1.2 it is enough to show that for every $r \geq m$ the element $\delta(\mathbb{1}_{K_n^{(r)}})$ is contained in the $R[Q]$-module generated by $\delta(\mathbb{1}_{K_n^{(m)}})$. We let

$A \in Q$ be the matrix given by

$$A = \begin{pmatrix} \varpi & & & \varpi - 1 & & \\ & \ddots & & & \ddots & \\ & & \varpi & & & \varpi - 1 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

It is the product of the two matrices

$$A_0 = \begin{pmatrix} 1 & & & \varpi - 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & \varpi - 1 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} \varpi & & & & & \\ & \ddots & & & \ddots & \\ & & \varpi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

In the following we will abbreviate a scalar matrix with $a \in F$ on its diagonal simply by a. The claim follows by induction from

$$\begin{aligned} A\delta(\mathbb{1}_{K_n^{(r)}})(g) &= A_0 A_1 \delta(\mathbb{1}_{K_n^{(r)}})(g) \\ &= A_0 \delta(A_1 \mathbb{1}_{K_n^{(r)}})(g) \\ &= A_0 \delta(\mathbb{1}_{\varpi K_n^{(r)}})(g) \\ &= \rho_1(g_1)\rho_2(g_2(u + \varpi - 1))\mathbb{1}_{\varpi K_n^{(r)}}(u + \varpi - 1) \\ &= \alpha\rho_1(g_1)\rho_2(g_2)\mathbb{1}_{\varpi K_n^{(r)}}(u + \varpi - 1) \\ &= \alpha\rho_1(g_1)\rho_2(g_2)\mathbb{1}_{K_n^{(r+1)}}(u) \\ &= \alpha\rho_1(g_1)\rho_2(g_2 u)\mathbb{1}_{K_n^{(r+1)}}(u) \\ &= \alpha\delta(\mathbb{1}_{K_n^{(r+1)}})(g) \end{aligned}$$

for all $g = \begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in G_{2n}$ with $g_i, u \in G_n$. □

Assume for the moment that the characters $\chi_1, \ldots, \chi_{2n}$ are unramified and denote $\beta_i = \chi_i(\varpi)q^{n-i+1/2} = \alpha_i q^{n-i+1/2}$. Then the space $(\operatorname{Ind}_{B_{2n}}^{G_{2n}} \chi)^{K_{2n}}$ is one dimensional, spanned by an element φ such that $\varphi(k) = 1$ for all $k \in K_{2n}$. By Frobenius reciprocity this gives rise to a morphism

$$\theta \colon I_E := \text{c-ind}_{K_{2n}}^{G_{2n}} E \to \operatorname{Ind}_{B_{2n}}^{G_{2n}} \chi.$$

Definition 1.5. *The Hecke algebra $\mathcal{H}(G_{2n}, K_{2n})$ of the pair (G_{2n}, K_{2n}) over E is the algebra of endomorphisms $End_{E[G_{2n}]}(I_E)$.*

The Hecke algebra is isomorphic to the algebra of functions

$$\{\Omega\colon G_{2n}\to E\,|\mathrm{supp}(\Omega) \text{ compact}, \Omega(k_1gk_2)=\Omega(g)\forall k_1,k_2\in K_{2n}, g\in G_{2n}\},$$

where the product is defined via convolution of functions. The isomorphism is given by

$$\Omega(f)(g)=\sum_{x\in K_{2n}\backslash G_{2n}}\Omega(gx^{-1})f(x).$$

The proof is a straightforward calculation (see for example [BL94], Proposition 5).
For $1\leq r\leq 2n$ let $\lambda(r)$ be the diagonal matrix with $\lambda(r)_{ii}=\varpi$ for $1\leq i\leq r$ and $\lambda(r)_{ii}=1$ else. The Hecke operator T_r is given by the function Ω_r with $\mathrm{supp}(\Omega_r)=K_{2n}\lambda_rK_{2n}$ and $\Omega(\lambda_r)=1$. We put $a_r=q^{r(2n-r)/2}\sigma_r$ where σ_r is the rth elementary symmetric polynomial in the β_i. For example, $a_{2n}=\prod_{i=1}^{2n}\beta_i=\prod_{i=1}^{2n}\alpha_i$ and if $n=r=1$, we get $a_1=q^{1/2}(q^{1/2}\alpha_1+q^{-1/2}\alpha_2)=q\alpha_1+\alpha_2$.
By [Kat81], Section 3, the map θ induces an isomorphism

$$\theta\colon I_E/\sum_r(T_r-\sigma_r)I_E\xrightarrow{\cong}\mathrm{Ind}_{B_{2n}}^{G_{2n}}\chi$$

as long as $\mathrm{Ind}_{B_{2n}}^{G_{2n}}\chi$ is irreducible, which we will assume from now on. Let $R\subset E$ be a principal ideal domain with field of fractions E such that $a_r\in R$ for all $r\leq 2n-1$ and $a_{2n}\in R^*$. We define $M_\chi(R)=I_R/\sum_r(T_r-\sigma_r)I_R$ with $I_R=\text{c-ind}_{K_{2n}}^{G_{2n}}R$. By definition we have $M_\chi(R)\otimes_R E=M_\chi(E)$. If $M_\chi(R)$ is R-free, then it is naturally a $R[G_{2n}]$-submodule of $\mathrm{Ind}_{B_{2n}}^{G_{2n}}\chi$ and by Lemma 1.4 there exists $c\in E^*$ such that the image of $C_c^0(G_n,R)$ under $c\delta$ is contained in $M_\chi(R)$ as long as $\alpha^{-1}\in R$.
It is known in many cases that $M_\chi(R)$ is R-free. It is always free if $n=1$ by [BL95] and if $n>1$ the main result of [GK14] implies that $M_\chi(R)$ is R-free if $F=\mathbb{Q}_p$, $p\geq 2n-1$ and R is a discrete valuation ring with residue characteristic p. From this discussion we get the following

Lemma 1.6. *Let $\chi_1,\ldots,\chi_{2n}\colon F^*\to E^*$ be unramified characters such that the principal series representation $\mathrm{Ind}_{B_{2n}}^{G_{2n}}\chi$ is irreducible. Let $R\subset E$ be a principal ideal domain with field of fractions E such that $a_r\in R$ for all $r\leq 2n-1$ and $a_{2n}\in R^*$. Assume either (i) $n=1$ or (ii) $F=\mathbb{Q}_p$, $p\geq 2n-1$ and R is a discrete valuation ring with residue characteristic p. Then $M_\chi(R)$ is R-free. If we further assume that $\alpha^{-1}\in R$ then (up to multiplication by a non-zero constant) we have*

$$\delta\colon C_c^0(G_n,R)\to M_\chi(R)\subset\mathrm{Ind}_{B_{2n}}^{G_{2n}}\chi.$$

Now assume untill the end of the section that E is a finite extension of F with ring of integers R. In the following we will write V^{sm} for the principal series representation $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(\chi)$ we considered before.
We fix a F-rational algebraic representation (V^{al}, π^{al}) of GL_{2n} and write $V_E^{al} = V^{al} \otimes_F E$. As before we embed $\mathrm{GL}_n \times \mathrm{GL}_n$ diagonally into GL_{2n}. Suppose there is a one dimensional $\mathrm{GL}_n \times \mathrm{GL}_n$-stable subrepresentation $V_s^{al} \subset V_E^{al}$ which is isomorphic to the representation given by the character $\det^{-s,s}$ with $s \in \mathbb{Z}$. Then we get a $G_n \times H$-equivariant map

$$\delta_s \colon C_c^s(G_r, E) \cong C_c^0(G_r, E) \otimes V_s^{al} \overset{\delta \otimes \mathrm{id}}{\longrightarrow} V^{sm} \otimes V_s^{al} \hookrightarrow V^{sm} \otimes V^{al}$$

where $H \subset G_n$ is as in Lemma 1.2. Note that the map δ_s depends up to a scalar on a choice of an element $v_0 \in V_s^{al}$.
Let $u \in G_{2n}$ be the following matrix

$$u = \begin{pmatrix} 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

and $\sigma \colon \mathbb{G}_m \to \mathrm{GL}_{2n}$ the cocharacter which sends $t \in \mathbb{G}_m$ to the diagonal matrix $\sigma(t)$ with $\sigma(t)_{ii} = t$ if $1 \leq i \leq n$ and $\sigma(t)_{ii} = 1$ if $n+1 \leq i \leq 2n$. Note that σ is a positive cocharacter with respect to the Borel group of upper triangular matrices. We put $\sigma' = u\sigma u^{-1}$. Then the matrix A we considered in the proof of Lemma 1.4 is nothing but $\sigma'(\varpi)$. Since the $\mathbb{G}_m$-representation $(V^{al}, \pi^{al} \circ \sigma')$ is algebraic it has a weight space decomposition. Thus there exists a basis $(v_1, \ldots, v_d)$ of V^{al} and $e_1, \ldots, e_d \in \mathbb{Z}$ such that

$$\pi^{al}(A)v_l = \varpi^{e_l} v_l.$$

By multiplying the v_l with appropiate scalars we may assume that v_0 is an R-linear combination of the v_l, $1 \leq l \leq d$. With the same notations as in Lemma 1.4 we have

Lemma 1.7. *Let $e = \max(e_1, \ldots, e_d)$ and assume that $\alpha^{-1}\varpi^{-e} \in R$. Then the image of $C_c^s(G_n, R)$ under δ_s is contained in the $R[Q]$-module M generated by $\delta(\mathbb{1}_{K_n^{(m)}}) \otimes v_l$, $1 \leq l \leq d$.*

Proof. As in the previous Lemma it is enough to show that $\delta(\mathbb{1}_{K_n^{(r)}}) \otimes v_0 \in M$ for all $r \geq m$. This is a direct consequence of the following claim: Given a

$R[Q]$-submodule N of $V^{sm} \otimes V^{al}$ such that $\delta(\mathbb{1}_{K_n^{(r)}}) \otimes v_l \in N$ for $l \geq 1$, then $\delta(\mathbb{1}_{K_n^{(r+1)}}) \otimes v_l \in N$ for $l \geq 1$. The claim follows from our assumption and the following formula:

$$A(\delta(\mathbb{1}_{K_n^{(r)}}) \otimes v_l) = \alpha \varpi^{e_l} \delta(\mathbb{1}_{K_n^{(r+1)}}) \otimes v_l.$$

□

Remark 1.8. *Let $V^{al} = V_\mu$ be an irreducible representation given by the highest weight $\mu = (\mu_1, \ldots, \mu_{2n}) \in \mathbb{Z}^{2n}$ with respect to the Borel group of upper triangular matrices. Since σ is a positive cocharacter, we get that $e = e_\mu = \mu_1 + \ldots + \mu_n$.*

In the following we assume that V^{al} is an irreducible self dual representation of highest weight $\mu = (\mu_1, \ldots, \mu_{2n})$ and V^{sm} is an irreducible unramified principal series representation.

Definition 1.9. *The twisted Hecke algebra $\mathcal{H}_{V^{al}}(G_{2n}, K_{2n})$ is the algebra of endomophisms $End_{F[G_{2n}]}(\text{c-ind}_{K_{2n}}^{G_{2n}} V^{al})$.*

As in the untwisted case we can identify $\mathcal{H}_{V^{al}}(G_{2n}, K_{2n})$ with the space of functions $\Omega\colon G_{2n} \to \mathrm{End}_F(V^{al})$ such that Ω has compact support and $\Omega(k_1 g k_2) = k_1 \Omega(g) k_2$ for all $k_1, k_2 \in K_{2n}$, $g \in G_{2n}$.
We have an isomorphism of G_{2n}-representation from $\text{c-ind}_{K_{2n}}^{G_{2n}} F \otimes V^{al}$ to $\text{c-ind}_{K_{2n}}^{G_{2n}} V^{al}$ sending $f \otimes v$ to the function $[g \mapsto f(g)gv]$. Hence we have an isomorphism $\iota\colon \mathcal{H}(G_{2n}, K_{2n}) \to \mathcal{H}_{V^{al}}(G_{2n}, K_{2n})$ sending Ω to the function $[g \mapsto \Omega(g)g]$ (see for example [Her11], Lemma 2.9). We define the twisted Hecke Operators T_r^μ as $\mu(\lambda_r)\iota(T_r)$. Since K_{2n} is compact there exists an K_{2n}-stable $\mathcal{O}$-sublattice L of V^{al}. If $a_r^\mu = \mu(\lambda_r) a_r \in R$ for all r, we define $M_\chi(L) = \text{c-ind}_{K_{2n}}^{G_{2n}}(L \otimes R)/\sum_r (T_r^\mu - a_r^\mu)$. Again it follows by construction that $M_\chi(L) \otimes E$ is isomorphic to $V^{sm} \otimes V_E^{al}$. As before we are interested in the question whether $M_\chi(L)$ is R-free.

Lemma 1.10. *Assume that the representation $L \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ is an irreducible rational $\mathrm{GL}_{2n}(\mathcal{O}/\mathfrak{p})$-representation and $a_r \in R$ for all r. Then $M_\chi(L)$ is R-free provided that (i) $F = \mathbb{Q}_p$, (ii) $a_{2n} \in \mathbb{Z}_p^*$ and (iii) if ρ denotes half of the sum of positive roots we have $\langle \mu + \rho, \check{\beta} \rangle \leq p$ for all positive roots β. If we further assume that $\alpha^{-1}\varpi^{-e_\mu} \in R$ we have (up to a scalar) a factorization*

$$\delta_s\colon C_c^s(G_n, R) \to M_\chi(L) \subset V^{sm} \otimes V_E^{al}$$

Proof. The first part of the Lemma follows from Theorem 1.1 (iii) of [GK14], which gives us the R-freeness of $M_\chi(L)$. The second part then follows from Lemma 1.7 □

Let Z denote the center of G_{2n}. In general, if $a_{2n} \in R^*$ and $L \subset V^{al}$ is a K_{2n}-stable sublattice, we may extend the action of K_{2n} on the R-lattice $L \otimes_{\mathcal{O}} R$ to an action of the bigger group ZK_n as follows: The scalar matrix with ϖ on its diagonal acts via multiplication with a_{2n}. Then we can replace $\text{c-ind}_{K_{2n}}^{G_{2n}}(L \otimes R)$ by $\text{c-ind}_{ZK_{2n}}^{G_{2n}}(L \otimes R)$ in the previous discussion. In this situation Große-Klönne (in [GK14]) constructs a finite Koszul-like complex of R-modules

$$0 \to C_\chi^{2n-1}(L) \to \cdots \to C_\chi^0(L) \to \; M_\chi(L) \to 0 \tag{1}$$

such that each of the $C_\chi^k(L)$ is isomorphic to $\text{c-ind}_{ZK_{2n}}^{G_{2n}}(L \otimes R)^r$ for some $r \in \mathbb{N}$. Under the assumptions of the previous Lemma it is shown in [GK14] that this complex is in fact exact.

Remark 1.11.

- *Actually, we need a variant of the above construction: Let us suppose there is a field extension of number fields $\mathcal{E}/\mathcal{F}$ with primes $\mathfrak{p}_E | \mathfrak{p}_F | p$ such that E is the completion of $\mathcal{E}$ at $\mathfrak{p}_E$, F the completion of $\mathcal{F}$ at $\mathfrak{p}_F$ and V^{sm} (V_F^{al} resp.) is already defined over $\mathcal{E}$ ($\mathcal{F}$ resp.). We let $\mathcal{O}_\mathcal{F}$ ($\mathcal{R}$ resp.) be the valuation ring defined by $\mathfrak{p}_F$ ($\mathfrak{p}_E$ resp.). We can choose an $\mathcal{O}_\mathcal{F}$-lattice $\mathcal{L}$ of $V_\mathcal{F}^{al}$ stable under $GL_{2n}(\mathcal{O}_\mathcal{F})$ and form the modules* $\text{c-ind}_{Z(\mathcal{F})\,\text{GL}_{2n}(\mathcal{O}_\mathcal{F})}^{\text{GL}_{2n}(\mathcal{F})}(\mathcal{L} \otimes \mathcal{R})$ *and* $M_\chi(\mathcal{L})$ *similar to the above construction (provided that $a_i \in \mathcal{R}$ for $1 \leq i \leq 2n-1$ and $a_{2n} \in \mathcal{R}^*$). Since the ring extension $R/\mathcal{R}$ is faithfully flat we see that*
  - *$M_\chi(\mathcal{L})$ is free if and only if $M_\chi(L)$ is free*
  - *the analogue of the resolution (1) for $M_\chi(\mathcal{L})$ is exact if and only if (1) is exact.*

  We get the same statement for the integrality of δ using that $\text{GL}_{2n}(\mathcal{F})$ is dense in $\text{GL}_{2n}(F)$.

- *With our application in mind (see the remark at the beginning of Section 3.3), it would be nice to have results like the above not only for F-rational representations but for $\mathbb{Q}_p$-rational representations, i.e. V^{al} is an irreducible representation of the algebraic group $(\text{Res}_{\mathbb{Q}_p}^F \text{GL}_{2n})_E$. Since the structure of the Hecke-algebra is known in this case by Breuil-Schneider (see [BS07]), one could hope to generalize the results of Große-Klönne to this setting.*

For the sake of clarity let us work out the conditions of Lemma 1.10 in the case $F = \mathbb{Q}_p$, $n = 1$ and a trivial central character. So let $V^{sm} = \text{Ind}_{B_2}^{G_2}(\chi_2^{-1}, \chi_2)$ with $\chi_2 \colon \mathbb{Q}_p^* \to E^*$ an unramified character and $\alpha = \chi_2(p)$.

The irreducibilty of V^{sm} is equivalent to $\alpha \neq \pm 1$. The irreducible self dual $\mathbb{Q}_p$-rational representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are given by $\mathrm{Sym}^k(\mathbb{Q}_p^2) \otimes \det^{-\frac{k}{2}}$ with $k \geq 0$ an even integer. The highest weight of this representation is given by $\mu = \mu_k = (\frac{k}{2}, -\frac{k}{2})$. So we get that $a := a_1^\mu = p^{\frac{k}{2}}(p\alpha^{-1} + \alpha) = p^{\frac{k+2}{2}}\alpha^{-1} + p^{\frac{k}{2}}\alpha$. The condition that a and $\alpha^{-1}p^{-\frac{k}{2}}$ are elements of R is equivalent to $\alpha p^{\frac{k}{2}} \in R^*$. It is well known that $\mathrm{Sym}^k(\overline{\mathbb{F}}_p) \otimes \det^{-\frac{k}{2}}$ is an irreducible $\mathrm{GL}_2(\mathbb{F}_p)$ representation if and only if $k \leq p-1$. Fortunately, we have that $\langle \mu + \rho, \check{\beta} \rangle = k+1$, where β is the unique positive root of GL_2. So condition (iii) of Lemma 1.10 is superfluous in this case.
Moreover for every $s \in \left\{-\frac{k}{2}, \ldots, \frac{k}{2}\right\}$ there exists exactly one subrepresentation $V_s^{al} \subset V_E^{al}$. In fact we have

$$\mathrm{Sym}^k(E^2) \otimes \det^{-\frac{k}{2}} = V_{-k/2}^{al} \oplus \ldots \oplus V_{k/2}^{al}.$$

1.4 Local Shalika models

The Shalika subgroup S of G_{2n} is given by

$$S = \left\{ \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \middle| h \in G_n, X \in M_n(F) \right\}.$$

We fix a unitary character $\eta \colon F^* \to S^1$. It induces a character $\eta\psi \colon S \to S^1$ via

$$\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto \eta(\det(h))\psi(\mathrm{tr}(X)).$$

Definition 1.12. *A smooth irreducible G_{2n}-representation π over $\mathbb{C}$ has a local (η, ψ)-Shalika model, if there exists a nonzero functional $\lambda \colon \pi \to \mathbb{C}$ such that $\lambda(s\varphi) = \eta\psi(s)\lambda(\varphi)$ for all $s \in S$, $\varphi \in \pi$. The functional λ is called a local (η, ψ)-Shalika functional.*

Remark 1.13.

- *Suppose η is the trivial character. It is known in this case that local (η, ψ)-Shalika-functionals of smooth, irreducible representations π of G_{2n} are unique up to scalar (if they exist). This was first proved in [JR96]. An elementary proof can be found in [Nie09]. For a general character η Ash and Ginzburg proved in [AG94] that an unramified, irreducible principal series representation of G_{2n} has at most one (η, ψ)-Shalika-functional up to scalar.*

- *By Frobenius reciprocity a Shalika functional gives an injective intertwining operator $\iota \colon \pi \to \mathrm{Ind}_S^{G_{2n}}(\eta\psi)$. The image of ι is the Shalika model of π.*

Now assume that π is an irreducible unramified principal series representation, i.e. there exist unramified characters $\chi_1, \ldots, \chi_{2n} \colon F^* \to \mathbb{C}^*$ such that $\pi = \mathrm{Ind}_{B_{2n}}^{G_{2n}} \chi$. Assume that π has a Shalika model. Then by Proposition 1.3 of [AG94] we know that η is unramified and we may assume that $\chi_i = \eta \chi_{2n-i+1}^{-1}$ (which we will do in the following) and conversely every such unramified principal series representation has a Shalika model.
More precisely: Put $\beta_i = \chi_i(\varpi) q^{n-i+1/2} = \alpha_i q^{n-i+1/2}$ and let $|\cdot|_\infty$ be the standard norm on $\mathbb{C}$. If we assume that $|\beta_i \beta_j|_\infty < 1$ for all $1 \leq i < j \leq n$, then by [AG94], Lemma 1.4, the following absolutely convergent integral gives the Shalika functional:

$$\lambda(\varphi) = \int_{K_n} \int_{M_n(F)} \varphi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right) \eta^{-1}(\det(g)) \psi^{-1}(\mathrm{tr}(X)) \; dX d^*g.$$

Here dX denotes an additive Haar measure on $M_n(F)$. If $\beta_i \beta_j \neq \eta^{\pm 1}(\varpi)$ for all $1 \leq i < j \leq n$, then the Shalika functional can be defined via analytic continuation of the above integral (see the proof of [AG94], Proposition 1.3).
Let $\rho_1 \in \mathrm{Ind}_{B_n}^{G_n}(\chi_1, \ldots, \chi_n)$ and $\rho_2 \in \mathrm{Ind}_{B_n}^{G_n}(\chi_{n+1}, \ldots, \chi_{2n})$ given by $\rho_i(k) = 1$ for all $k \in K_n$, $i = 1, 2$. We let δ be the corresponding map $\delta \colon C_c^0(G_n, \mathbb{C}) \to \mathrm{Ind}_{B_{2n}}^{G_{2n}}(\chi)$. The Shalika functional λ defines a $\mathbb{C}$-valued distribution μ_λ on G_n.

Proposition 1.14. *Let $\pi = \mathrm{Ind}_{B_{2n}}^{G_{2n}}(\chi)$ be an irreducible unramified principal series as above with Shalika functional λ. Assume that $\beta_i \beta_j \neq \eta^{\pm 1}(\varpi)$ for all $1 \leq i < j \leq n$. Then:*

$$\int_{G_n} f(g) \mu_\lambda(dg) = \int_{M_n(F)} \rho_2(X) f(X) \psi^{-1}(\mathrm{tr}(X)) \; dX$$

for all $f \in C_c^0(G_n, \mathbb{C})$ which are invariant under conjugation by K_n.

Proof. For every $s \in \mathbb{C}$ we put $\rho_1^s \in \mathrm{Ind}_{B_n}^{G_n}(\chi_1 |\cdot|^s, \ldots, \chi_n |\cdot|^s)$ and $\rho_2^s \in \mathrm{Ind}_{B_n}^{G_n}(\chi_{n+1} |\cdot|^{-s}, \ldots, \chi_{2n} |\cdot|^{-s})$ given by $\rho_i^s(k) = 1$ for all $k \in K_n$, $i = 1, 2$. We let δ^s be the corresponding map

$$\delta^s \colon C_c^0(G_n, \mathbb{C}) \to \mathrm{Ind}_{B_{2n}}^{G_{2n}}(\chi_1 |\cdot|^s, \ldots, \chi_n |\cdot|^s, \chi_{n+1} |\cdot|^{-s}, \ldots, \chi_{2n} |\cdot|^{-s}) =: \pi^s.$$

We denote the Shalika functional of π^s by λ^s.
Since the map $s \mapsto \lambda^s(\delta^s(f))$ is analytic we can compute the left hand side as the analytic continuation to $s = 0$ of the integral

$$\int_{K_n}\int_{M_n(F)}\delta^s(f)\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}g&0\\0&g\end{pmatrix}\begin{pmatrix}1&X\\0&1\end{pmatrix}\right)\eta^{-1}(\det(g))\psi^{-1}(\mathrm{tr}(X))\ dXd^*g$$
$$=\int_{K_n}\int_{M_n(F)}\rho_1^s(g)\rho_2^s(gX)f(X)\eta^{-1}(\det(g))\psi^{-1}(\mathrm{tr}(X))\ dXd^*g$$

$$=\int_{K_n}\int_{M_n(F)}\rho_1^s(g)\rho_2^s(Xg)f(g^{-1}Xg)\eta^{-1}(\det(g))\psi^{-1}(\mathrm{tr}(g^{-1}Xg))\ dXd^*g$$
$$=\int_{M_n(F)}\rho_2(X)\left|\det(X)\right|^{-s}f(x)\psi^{-1}(\mathrm{tr}(X))\ dX.$$

□

1.5 Local integrals

We want to study the integrals of Proposition 1.14 in more detail. This is necessary to show that the global distribution has the right interpolation property (see Theorem 2.6). The proof of Lemma 1.20 is heavily inspired by the computations in [AG94].

Definition 1.15. *The order* $\mathrm{ord}(A)$ *of an* $n\times n$*-Matrix* $A\in M_n(F)$ *is the minimum of the* $\nu(A_{ij})$*,* $1\le i,j\le n$*.*

It is a straightforward calculation to show that

$$\mathrm{ord}(AB)\ge\mathrm{ord}(A)+\mathrm{ord}(B)$$

for all $A,B\in M_n(F)$. In particular, we get an equality if one of the matrices is in K_n.

Lemma 1.16. *Let* $\chi\colon F^*\to\mathbb{C}^*$ *be a character of conductor* $\mathfrak{f}(\chi)=\mathfrak{p}^m$*,* $m\ge1$*, and let* $A\in M_n(F)$ *be a matrix with* $\mathrm{ord}(A)>-m$*. Then*

$$\int_{K_n}\chi(\det(g))\psi(\mathrm{tr}(Ag))d^*g=0.$$

Proof. By assumption we can find $b\in F^*$ sucht that $\max(-\mathrm{ord}(A),0)\le\mathrm{ord}(b)<m$, $1+b\in U$ and $\chi(1+b)\ne1$. Let $B\in M_n(F)$ be the matrix defined by

$$B_{ij}=\begin{cases}b, & i=j=1\\0, & \text{else.}\end{cases}$$

Since $1+B \in K_n$ and $AB \in M_n(\mathcal{O})$ we get

$$\begin{aligned}
&\int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag))d^*g \\
&= \int_{K_n} \chi(\det((1+B)g))\psi(\operatorname{tr}(A(1+B)g))d^*g \\
&= \chi(\det(1+B)) \int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag)\psi(\operatorname{tr}(ABg)d^*g \\
&= \chi(1+b) \int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag))d^*g
\end{aligned}$$

hence $\int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag))d^*g = 0$. □

Proposition 1.17. *Let $A \in M_n(F)$ and $m \in \mathbb{Z}$ with $1 \leq m < -\operatorname{ord}(A)$. Then we have*

$$\int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g = 0.$$

Proof. Let $1 \leq k, l \leq n$ such that $\operatorname{ord}(A) = \nu(A_{kl})$. By assumption there exists $b \in F^*$ with $\nu(b) = -\nu(A_{kl}) - 1$ and $\psi(A_{kl}b) \neq 1$. Define the matrix $B \in M_n(F)$ via

$$B_{ij} = \begin{cases} b, & i = l, j = k \\ 0, & \text{else.} \end{cases}$$

Since $1 + B \in K_n^{(m)}$ we have

$$\begin{aligned}
\int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g &= \int_{K_n^{(m)}} \psi(\operatorname{tr}(A(1+B)g))d^*g \\
&= \int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))\psi(\operatorname{tr}(AB))\psi(\operatorname{tr}(AB(g-1))d^*g.
\end{aligned}$$

Since $B(g-1) \in \mathfrak{p}^{-\operatorname{ord}(A)} M_n(\mathcal{O})$, we get $\psi(\operatorname{tr}(AB(g-1))) = 1$ for all $g \in K_n^{(m)}$. Thus, we obtain

$$\begin{aligned}
\int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g &= \psi(\operatorname{tr}(AB)) \int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g \\
&= \psi(A_{kl}b) \int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g
\end{aligned}$$

and hence $\int_{K_n^{(m)}} \psi(\operatorname{tr}(Ag))d^*g = 0$. □

Lemma 1.18. *Let $\chi\colon F^* \to \mathbb{C}^*$ be a character of conductor $\mathfrak{f}(\chi) = \mathfrak{p}^m$, $m \geq 0$ and let $A \in M_n(F)$ with $\operatorname{ord}(A) < -\max(m,1)$. Then*

$$\int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag))d^*g = 0.$$

Proof. Let $m' = \max(m,1)$. We can rewrite the integral as follows

$$\int_{K_n} \chi(\det(g))\psi(\operatorname{tr}(Ag))d^*g = \sum_{k \in K_n/K_n^{(m')}} \chi(\det(k)) \int_{K_n^{(m')}} \psi(\operatorname{tr}(Akg))d^*g.$$

Using the fact that $\operatorname{ord}(Ak) = \operatorname{ord}(A)$ for all $k \in K_n$ the claim follows from Proposition 1.17. □

Definition 1.19. *Let $\chi\colon F^* \to \mathbb{C}^*$ be a quasicharacter of conductor $\mathfrak{f}(\chi) = \mathfrak{p}^m$, $m \geq 0$ and $a \in F^*$ with $\nu(a) = -m$. We define the Gauss sum of χ (with respect to ψ) as*

$$\tau(\chi) = \tau(\chi,\psi) = [U : U^{(m)}] \int_U \chi(ag)\psi(ag)d^*g.$$

By the Iwasawa decomposition every element $g \in G_n$ can be written as $g = Ak$ with $k \in K_n$ and A an upper triangular matrix in G_n. We say that two invertible upper triangular matrices A, B are equivalent if $AK_n = BK_n$. In this case $\operatorname{ord}(A) = \operatorname{ord}(B)$ and moreover, $\nu(A_{ii}) = \nu(B_{ii})$ for all $1 \leq i \leq n$. For $m \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ with $r_i \geq -m$ let $\mathcal{F}_r^m$ be the set of equivalence classes of invertible upper triangular matrices $[A]$ such that $\operatorname{ord} A \geq -m$ and $\nu(A_{ii}) = r_i$ for all i. Note, that for every equivalence class $[A] \in \mathcal{F}_t^m$ we can choose an representative A with $A_{ii} = \varpi^{r_i}$ for all i. Two such matrices A, B are equivalent if there exists an unipotent upper triangular matrix $k \in K_n$ such that $A = Bk$. If we denote by T_r the diagonal matrix with $(T_r)_{ii} = \varpi^{r_i}$, then a complete set of representatives of the equivalence classes in $\mathcal{F}_r^m$ is given by $\{T_r + N\}$, where N are nilpotent upper triangular matrices with entries N_{ij} running through a set of representatives of $\mathfrak{p}^{-m}/\mathfrak{p}^{r_i}$ for all $j > i$. When we let the N_{ij} run through a set of representatives of $\mathfrak{p}^{-m}/\mathfrak{p}^{r_i+m}$ instead, we get for each equivalence class in $\mathcal{F}_r^m$ exactly $q^{m(n^2-n)/2}$ representatives.

Proposition 1.20. *Let $\chi\colon F^* \to \mathbb{C}^*$ be a character of conductor $\mathfrak{f}(\chi) = \mathfrak{p}^m$, $m \geq 1$, and $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ as before, then*

$$\sum_{[A] \in \mathcal{F}_r^m} \int_{K_n} \chi(\det(Ag))\psi(\operatorname{tr}(Ag))d^*g = \begin{cases} c\ q^{-m(n^2+n)/2}\tau(\chi)^n, & r_i = -m\ \forall i \\ 0, & \text{else}, \end{cases}$$

where c is a nonzero rational constant independent of χ.

Proof. By the preceded discussion we can rewrite the sum as follows:

$$\begin{aligned}
&\sum_{[A]\in\mathcal{F}_r^m}\int_{K_n}\chi(\det(Ag))\psi(\operatorname{tr}(Ag))d^*g\\
&=\frac{1}{q^{m(n^2-n)/2}}\sum_N\int_{K_n}\chi(\det(T_rg))\psi(\operatorname{tr}(T_rg))\psi(\operatorname{tr}(Ng))d^*g\\
&=\frac{1}{q^{m(n^2-n)/2}}\int_{K_n}\chi(\det(T_rg))\psi(\operatorname{tr}(T_rg))\sum_N\psi(\operatorname{tr}(Ng))d^*g.
\end{aligned}$$

By orthogonality of characters the sum

$$\begin{aligned}
\sum_N\psi(\operatorname{tr}(Ng)) &= \sum_{N_{ij}\in\mathfrak{p}^{-m}/\mathfrak{p}^{r_i+m}}\prod_{j>i}\psi(N_{ij}g_{ji})\\
&=\prod_{j>i}\sum_{N_{ij}\in\mathfrak{p}^{-m}/\mathfrak{p}^{r_i+m}}\psi(N_{ij}g_{ji})
\end{aligned}$$

is zero unless $\nu(g_{ji})\geq m$ for all $j>i$.
Therefore, denoting $D_m=\{g\in K_n|\nu(g_{ij})\geq m\ \forall i>j\}$, we get

$$\begin{aligned}
&\sum_{[A]\in\mathcal{F}_r^m}\int_{K_n}\chi(\det(Ag))\psi(\operatorname{tr}(Ag))d^*g\\
&=\frac{1}{q^{m(n^2-n)/2}}\prod_{i=1}^n q^{(2m+r_i)(n-i)}\int_{D_m}\chi(\det(T_rg))\psi(\operatorname{tr}(T_rg))d^*g.
\end{aligned}$$

Let $\mathcal{E}$ be the group of invertible diagonal $n\times n$-matrices with entries in $\mathcal{O}/\mathfrak{p}^m$.
Then we have

$$\begin{aligned}
&\int_{D_m}\chi(\det(g))\chi(\det(T_r))\psi(\operatorname{tr}(T_rg))d^*g\\
&=[U\colon U^{(m)}]^{-n}\sum_{E\in\mathcal{E}}\int_{D_m}\chi(\det(T_rEg))\psi(\operatorname{tr}(T_rEg))d^*g\\
&=[U\colon U^{(m)}]^{-n}\int_{D_m}\chi(\det(T_rg))\sum_{E\in\mathcal{E}}\chi(\det(E))\psi(\operatorname{tr}(T_rEg))d^*g.
\end{aligned}$$

Rearranging the sum

$$\begin{aligned}
\sum_{E\in\mathcal{E}}\chi(\det(E))\psi(\operatorname{tr}(T_rEg)) &= \sum_{(\epsilon_i)\in((\mathcal{O}/\mathfrak{p}^m)^*)^n}\prod_{i=1}^n\chi(\epsilon_i)\psi(\varpi^{r_i}\epsilon_ig_{ii})\\
&=\prod_{i=1}^n\sum_{(\epsilon_i)\in(\mathcal{O}/\mathfrak{p}^m)^*}\chi(\epsilon_i)\psi(\varpi^{r_i}\epsilon_ig_{ii})
\end{aligned}$$

we conclude that this is zero unless $r_i = -m$ for all i by Lemma 1.16 (in the case n=1).
In the following assume that $r_i = -m$ holds for all i. Then

$$\prod_{i=1}^{n} \sum_{(\epsilon_i)\in(\mathcal{O}/\mathfrak{p}^m)^*} \chi(\epsilon_i)\psi(\varpi^{r_i}\epsilon_i g_{ii}) = \tau(\chi)^n \prod_{i=1}^{n} \chi(g_{ii}^{-1})$$

and since the conductor of χ is equal to m we obtain that $\chi(g) = \prod_{i=1}^n \chi(g_{ii})$ for all $g \in D_m$. Thus, putting everything together we get

$$\begin{aligned}
&\sum_{[A]\in\mathcal{F}_r^m} \int_{K_n} \chi(\det(Ag))\psi(\mathrm{tr}(Ag))d^*g \\
&= [U : U^{(m)}]^{-n}\tau(\chi)^n \int_{D_m} \chi(\det(g)) \prod_{i=1}^{n} \chi(g_{ii}^{-1})d^*g \\
&= [U : U^{(m)}]^{-n}\tau(\chi)^n \ vol(D_m) \\
&= [U : U^{(1)}]^{-n} q^{-(m-1)n} vol(D_1) q^{-(m-1)(n^2-n)/2} \ \tau(\chi)^n.
\end{aligned}$$

□

For a function $f \in C(G_n, \mathbb{C})$ we define

$$\int_{G_n} f(g) \ d^*g = \lim_{k\to\infty} \sum_{\substack{(r_i)\in\mathbb{Z}^n \\ -k\leq r_i\leq k}} \sum_{A\in\mathcal{F}_r^k} \prod_{i=1}^{n} q^{-r_i} \int_{K_n} f(Ag) \ d^*g$$

if the limit exists. If we are restricting to functions with compact support, this is just the integral over an additive Haar measure on $M_n(F)$. Now we fix unramified characters $\chi_1, \ldots, \chi_n \colon F^* \to \mathbb{C}^*$. We define $\alpha_i = \chi_i(\varpi)$ and $\alpha = \prod_{i=1}^n \alpha_r$. Let $\rho \in \mathrm{Ind}_{B_n}^{G_n}(\chi_1, \ldots, \chi_n)$ be the unique spherical vector with $\rho(1) = 1$. The main result of this section is

Theorem 1.21. *Let $\chi \colon F^* \to \mathbb{C}^*$ be a character of conductor $\mathfrak{f}(\chi) = \mathfrak{p}^m$. If the complex norm $|\chi(\varpi)|_\infty$ is sufficiently small then the integral*

$$E(\chi) := \int_{G_n} \rho(g)\chi(\det(g))\psi(\mathrm{tr}(g)) \ d^*g$$

converges and we have

$$E(\chi) = c \ \tau(\chi)^n \begin{cases} \alpha^{-m} q^{-m(n^2-n)/2}, & \text{if } m \geq 1 \\ \prod_{i=1}^n \frac{1-\alpha_i^{-1}\chi(\varpi)^{-1}q^{i-n}}{1-\alpha_i\chi(\varpi)q^{n-i-1}}, & \text{if } m = 0 \end{cases}$$

where c is a non-zero rational constant independent of χ.

Proof. Plugging in the definition we see that

$$E(\chi) = \lim_{k\to\infty} \sum_{\substack{(r_i)\in\mathbb{Z}^n \\ -k\leq r_i\leq k}} \prod_{i=1}^{n} (q^{-r_i}\alpha_i^{r_i}) \sum_{A\in\mathcal{F}_r^k} \int_{K_n} \chi(\det(Ag))\psi(\mathrm{tr}(Ag))\ d^*g.$$

If $m \geq 1$ then only the terms with $\mathrm{ord}(A) = -m$ contribute by Lemma 1.16 and Lemma 1.18. The desired equality then follows from Lemma 1.20.
Now let χ be unramified. Lemma 1.17 gives

$$E(\chi) = \lim_{k\to\infty} \sum_{\substack{(r_i)\in\mathbb{Z}^n \\ -1\leq r_i\leq k}} \prod_{i=1}^{n} q^{-r_i}\alpha_i^{r_i}\chi(\varpi^{r_i}) \sum_{A\in\mathcal{F}_r^1} \int_{K_n} \psi(\mathrm{tr}(Ag))\ d^*g.$$

Like in the proof of the previous proposition we see that

$$\sum_{A\in\mathcal{F}_r^1} \int_{K_n} \psi(\mathrm{tr}(Ag))\ d^*g = \prod_{i=1}^{n} q^{(1+r_i)(n-i)} \int_{D_1} \psi(\mathrm{tr}(T_r g))\ d^*g.$$

The integral on the right is equal to $vol(D_1)$ if $r_i \geq 0$ for all i. Otherwise a computation with diagonal matrices like in the previous Lemma shows that

$$\int_{D_1} \psi(\mathrm{tr}(T_r g)\ d^*g = vol(D_1) \prod_{\substack{i=1 \\ r_i=-1}}^{n} \left(-\frac{1}{q-1}\right).$$

Therefore, we get (switching addition and multiplication) that

$$\begin{aligned}
E(\chi) &= vol(D_1) \prod_{i=1}^{n} -\frac{1}{q-1}\alpha_i^{-1}\chi(\varpi)^{-1}q + q^{n-i}\sum_{k=0}^{\infty}(\alpha_i\chi(\varpi)q^{n-i-1})^k \\
&= vol(D_1) \prod_{i=1}^{n} -\frac{1}{q-1}\alpha_i^{-1}\chi(\varpi)^{-1}q + q^{n-i}\frac{1}{1-\alpha_i\chi(\varpi)q^{n-i-1}} \\
&= vol(D_1) \prod_{i=1}^{n} \frac{q^{n-i+1}}{q-1}\frac{1-\alpha_i^{-1}\chi(\varpi)^{-1}q^{i-n}}{1-\alpha_i\chi(\varpi)q^{n-i-1}} \\
&= vol(D_1)[U:U^{(1)}]^{-n}q^{\frac{n^2+n}{2}} \prod_{i=1}^{n} \frac{1-\alpha_i^{-1}\chi(\varpi)^{-1}q^{i-n}}{1-\alpha_i\chi(\varpi)q^{n-i-1}}.
\end{aligned}$$

□

1.6 The semi local case

All the previous constructions can be easily generalized to the semi-local case. Let $F_1, \ldots, F_l$ be finite extensions of $\mathbb{Q}_p$ with valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_l$ and put $F = F_1 \times \ldots \times F_l$ and $\mathcal{O} = \mathcal{O}_1 \times \ldots \times \mathcal{O}_l$. For every $r \in \mathbb{N}$ we set $G_r = \mathrm{GL}_r(F) = \prod_{k=1}^{l} \mathrm{GL}_r(F_k)$ and $K_r = G(\mathcal{O}) = \prod_{k=1}^{l} \mathrm{GL}_r(\mathcal{O}_k)$.
We fix a finite extension E of $\mathbb{Q}_p$ with valuation ring R and embeddings $F_k \hookrightarrow E$ for all $1 \leq k \leq l$. For every multi index $s = (s_k) \in \mathbb{Z}^l$ and every closed subgroup $A \subset G_r$ we define $C_c^s(A, E) \subset C_c(A, E)$ to be the subspace of functions f which are locally of the form $f(g_1, \ldots, g_l) = a \prod_{k=1}^{l} \det^{s_k}(g_k)$ with $a \in E$. It follows immediately that the canonical map

$$(2) \qquad \bigotimes_{k=1}^{l} C_c^{s_k}(\mathrm{GL}_r(F_k)) \to C_c^s(G_r, E)$$

$$\otimes f_k \mapsto \left[(g_k) \mapsto \prod_{k=1}^{l} f_k(g_k)\right]$$

is an isomorphism. We define $C_c^s(A, R)$ as at the end of Section 1.1. The analogously defined map with R-coefficients is still an isomorphism.
Now fix for every $1 \leq k \leq l$ an irreducible smooth principal series representation $V_k^{sm} = \mathrm{Ind}_{B(F_k)}^{\mathrm{GL}_{2n}(F_k)} \chi_k$ defined over E. Here $B(F_k) \subset \mathrm{GL}_{2n}(F_k)$ denotes the subgroup of upper triangular matrices and we assume for simplicity that the characters $\chi_k = (\chi_{1,k}, \ldots, \chi_{2n,k})$ are unramifed and take values in E^*. Further we fix irreducible algebraic F_k-rational representations V_k^{al} of $\mathrm{GL}_{2n}(F_k)$ of highest weight $\mu_k = (\mu_{1,k}, \ldots, \mu_{2,k})$ with trivial central character. Let $(V_k^{al})_E$ be the base change of V_k^{al} to E. We put $V_k = V_k^{sm} \otimes (V_k^{al})_E$ and $V = \otimes_{k=1}^{l} V_k$. We consider $H = \mathrm{GL}_n \times \mathrm{GL}_n$ as an algebraic subgroup of GL_{2n} via the diagonal embedding. Assume that for every k there exists an $s_k \in \mathbb{Z}$ and an 1-dimensional E-rational H-subrepresentation $V_{k,s_k} \subset (V_k^{al})_E$ which is isomorphic to the character $\det^{-s_k, s_k}$ of H. Let us fix for every k functions $\rho_{1,k}, \rho_{2,k}$ as in the previous sections. The corresponding maps $\delta_{s_k,k} \colon C_c^{s_k}(\mathrm{GL}_n(F_k)) \to V_k$ induce a map

$$(3) \qquad \delta_s \colon C_c^s(G_n, E) \longrightarrow V.$$

via the isomorphism (2).
For $1 \leq k \leq l$ let $a_{1,k}^{\mu_k}, \ldots, a_{2n,k}^{\mu_k}$ be the Hecke eigenvalues of V_k as defined in Section 1.3, $\alpha_k = \prod_{i=n+1}^{2n} \chi_{i,k}(\varpi_k)$, where ϖ_k is a local uniformizer of $\mathcal{O}_k$ and set $e_k = \mu_{1,k} + \ldots + \mu_{n,k}$.

Definition 1.22. *A representation $V = \otimes V_k$ as above is weakly p-ordinary if the following conditions hold*

- $a^{\mu_k}_{1,k}, \ldots, a^{\mu_k}_{2n-1,k} \in R$ *and* $a^{\mu_k}_{2n,k} \in R^*$ *for all* $1 \leq k \leq l$
- $\alpha_k^{-1}\varpi_k^{-e_k} \in R$ *for all* $1 \leq k \leq l$.

Note that the second condition depends not only on the representation V but on a choice of an isomorphism of V_k^{sm} with a principal series representation for all k (which we have fixed in the beginning).
If the first of the above conditions is fullfilled we can choose a $GL_{2n}(\mathcal{O}_k)$-invariant R-lattice L_k of $(V_k^{al})_E$ and define the modules $M_{\chi_k}(L_k)$ for every k as in Section 1.3. We put $M_\chi(L) = \otimes_{k=1}^{l} M_{\chi_k}(L_k)$. The canonical map $M_\chi(L) \to V$ induces an isomorphism of G_{2n}-representations $M_\chi(L) \otimes E \cong V$. Lemma 1.7 immediately implies the following

Lemma 1.23. *Assume that V is weakly p-ordinary and that $M_\chi(L)$ is R-free. Then the canonical map $M_\chi(L) \to V$ is an inclusion and there exists $c \in E^*$ such that*

$$c\delta_s \colon C_c^s(G_n, R) \to M_\chi(L) \subset V.$$

Although Lemma 1.10 gives some sufficient criterions for the R-freeness $M_\chi(L)$ it is not known in general under which conditions $M_\chi(L)$ is R-free. In view of Lemma 1.23 we make the following

Conjecture 1.24. *Assume that V is weakly p-ordinary and that $\otimes_{k=1}^{l} \rho_{i,k}$, $i = 1, 2$, is invariant under the open subgroup $K_n' \subset K_n$. Then there exists $c \in E^*$ and an $G_n \times K_n'$-equivariant map $\delta_{s,int} \colon C_c^s(G_n, R) \to M_\chi(L)$ such that the following diagramm commutes:*

$$\begin{array}{ccc} & & M_\chi(L) \\ & \overset{\delta_{s,int}}{\nearrow} & \downarrow \\ C_c^s(G_n, R) & \xrightarrow{c\delta_s} & V \end{array}$$

We will assume untill the end of the section that V is weakly p-ordinary. For an R-module N and a closed subgroup $A \subset G_n$ we define the group of s-distributions on A with values in N as $\mathrm{Dist}_s(A, N) = \mathrm{Hom}(C_c^s(A, R), N)$. Let Z denote the center of G_{2n}. It acts on V by a character $\omega \colon Z \to R^*$. The $((A \cap G_n) \times (A \cap K_n))Z$-module $\mathrm{Dist}_s(A, \omega, N)$ is defined as follows: As an $(A \cap G_n) \times (A \cap K_n)$-module it is just $\mathrm{Dist}_s(A, N)$ with the usual operation and we let Z act via ω^{-1}. We assume from now on that conjecture 1.24 holds. By multiplying the $\rho_{i,k}$ with appropiate constants we can assume that $c = 1$. Then by dualizing $\delta_{s,int}$ we get a $(G_n \times K_n)Z$-equivariant R-module homomorphism

$$\mathrm{Hom}_R(M_\chi(L), N) \to \mathrm{Dist}_s(G_n, \omega, N)$$

provided that $\rho_{i,k}$, $i = 1, 2$, is spherical for all k. By Frobenius reciprocity we get an $H(F)$-equivariant R-module homomorphism

$$\delta^\vee_{s,N}\colon \mathrm{Hom}_R(M_\chi(L), N) \to \mathrm{Ind}^{H(F)}_{(G_n\times K_n)Z} \mathrm{Dist}_s(G_n, \omega, N). \tag{4}$$

Remark 1.25. *Now assume that $\mathcal{F}$ is a number field such that $F_1, \dots, F_l$ are completions of $\mathcal{F}$ at different places. Denote by $\mathcal{O}'_k$ the intersection of $\mathcal{F}$ with $\mathcal{O}_k$ for $1 \le k \le l$ and put $\mathcal{O}' = \mathcal{O}'_1 \times \ldots \times \mathcal{O}'_l$. Let $\mathcal{E}$ be a number field which contains all embeddings of F into $\bar{\mathbb{Q}}$ and assume that E is the completion of $\mathcal{E}$ at some place. Let $\mathcal{R} \subset \mathcal{E}$ be he valuation ring corresponding to this place. Similarly as at the end of Section 1.1 and Section 1.3 there is a version of the above map with $\mathcal{R}$-coefficients: Assume that the characters χ_k take values in $\mathcal{E}$ and that the V^{al}_k are base changes of algebraic $\mathcal{F}$-rational representations $(V^{al}_k)_\mathcal{F}$ of $\mathrm{GL}_{2n}(\mathcal{F})$. For every k we fix an embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ such that $\mathcal{O}'_k \subset \mathcal{R}$. Then by taking $\mathrm{GL}_n(\mathcal{O}'_k)$-invariant $\mathcal{R}$-lattices $\mathcal{L}_k \subset (V^{al}_k)_\mathcal{E}$ we get an $H(\mathcal{F})$-equivariant $\mathcal{R}$-module homomorphism*

$$\delta^\vee_{s,N}\colon Hom_\mathcal{R}(M_\chi(\mathcal{L}), N) \to \mathrm{Ind}^{H(\mathcal{F})^l}_D \mathrm{Dist}_s(G_n, \omega, N) =: \text{I-Dist}_s(G_n, \omega, N)$$

for every $\mathcal{R}$-module N, which agrees with (4) if N is an R-module. Here D denotes the subgroup $Z(\mathcal{F})^l(\mathrm{GL}_n(\mathcal{F})^l \times \mathrm{GL}_n(\mathcal{O}'))$ of $\mathrm{GL}_{2n}(\mathcal{F})^l$ and $Z(\mathcal{F})$ is the center of $\mathrm{GL}_{2n}(\mathcal{F})$.

2 The global distribution

We will use the following notations throughout the rest of the thesis. We fix a totally real algebraic number field F of degree d with ring of integers $\mathcal{O}_F$. For a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$ we set $N(\mathfrak{a}) = \sharp(\mathcal{O}_F/\mathfrak{a})$. Given a place l of $\mathbb{Q}$ we denote by S_l the set of places of F above l. Let $\sigma_1, \ldots, \sigma_d$ denote the distinct embeddings of F into $\mathbb{R}$ and $\infty_1, \ldots, \infty_d$ the corresponding archimedian places. Via the fixed embedding $\iota_\infty \colon \overline{\mathbb{Q}} \to \mathbb{C}$ we can and will view the σ_i as embeddings into $\bar{\mathbb{Q}}$.

If v is a place of F we denote by F_v the completion of F at v. If $\mathfrak{q}$ is a finite place, we let $\mathcal{O}_\mathfrak{q}$ denote the valuation ring of $F_\mathfrak{q}$ and $\mathrm{ord}_\mathfrak{q}$ the additive valuation such that $\mathrm{ord}_\mathfrak{q}(\varpi) = 1$ for any local uniformizer $\varpi \in \mathcal{O}_\mathfrak{q}$. For an arbitrary place let $|\cdot|_v$ be the normalized multiplicative norm, i.e. $|x|_{\infty_i} = |\sigma_i(x)|$ for $i = 1, \ldots, d$ and $|x|_\mathfrak{q} = N(\mathfrak{q})^{-\mathrm{ord}_q(x)}$ if $\mathfrak{q}$ is a finite place. We denote by U_v the invertible elements of $\mathcal{O}_v$ if v is a finite place and the group of positive elements of F_v if v is a real place. For a finite place $\mathfrak{p}$ we let $U_\mathfrak{p}^{(m)} = \{x \in U_\mathfrak{p} \,|\, x \equiv 1 \mod \mathfrak{p}^m\}$.

Let $\mathbb{A}$ be the ring of adeles of F and $\mathbb{I}$ the idele group of F. We denote by $|\cdot| \colon \mathbb{I} \to \mathbb{R}^*$ the absolute modules, i.e. $|(x_v)_v| = \prod_v |x_v|_v$ for $(x_v)_v \in \mathbb{I}$. For a finite subset S of places of F we define the "S-truncated adeles"' $\mathbb{A}^S$ ("S-truncated ideles" $\mathbb{I}^S$ resp.) as the restricted product of the completions F_v (F_v^* resp.) with $v \notin S$ and put $F_S = \prod_{v \in S} F_v$. We also set $U_S = \prod_{v \in S} U_v$ and $U^S = \prod_{v \notin S} U_v$ and similarly we define $U_S^{(m)}$. If I is a finite set of places of $\mathbb{Q}$ we often write $\mathbb{A}^I$ instead of $\mathbb{A}^{\cup_{l \in I} S_l}$, U_I instead of $\prod_{l \in I} U_{S_l}$ etc.

A (left) Haar measure of a locally compact group G will be denoted by dg. We fix a character $\psi \colon \mathbb{A} \to S^1$ which is trivial on F. For a place v let ψ_v the restriction of ψ to $F_v \subset \mathbb{A}$. We assume that the conductor of $\psi_\mathfrak{p}$ is $\mathcal{O}_\mathfrak{p}$ for all places $\mathfrak{p}$ of F dividing p. Let dx (dx_v resp.) denote the self-dual Haar measure of $M_r(\mathbb{A})$ ($M_r(F_v)$ resp.) associated to the character $\psi \circ \mathrm{tr}$ ($\psi_v \circ \mathrm{tr}$ resp.). It follows that $dx = \prod_v dx_v$. We normalize the multiplicative Haar measure d^*x_v on $\mathrm{GL}_r(F_v)$ by $d^*x_v = m_v \frac{dx_v}{|x_v|_v}$ where $m_v = 1$ if v is real and m_v is chosen such that $\mathrm{GL}(O_v)$ has volume 1 if v is finite. For a linear algebraic group G over F, a character $\chi \colon G(\mathbb{A}) \to \mathbb{C}^*$ and a place v we let $\chi_v \colon G(F_v) \hookrightarrow G(\mathbb{A}) \xrightarrow{\chi} \mathbb{C}^*$ be the local component of χ at v. If $\chi \colon \mathbb{I} \to \mathbb{C}^*$ is a character we define the Gauss sum $\tau(\chi) = \tau(\chi, \psi^{-1})$ of χ as the product of the local Gauss sums, i.e. $\tau(\chi) = \prod_{v < \infty} \tau(\chi_v, \psi_v^{-1})$ and the conductor $\mathfrak{f}(\chi)$ as the product of the local conductors $\mathfrak{f}(\chi_v)$.

Finally, we write $\mathcal{G}_p$ for the Galois group of the maximal abelian extension of F unramified outside p and ∞ over F.

2.1 Shalika models

In this section we recall the basics on global Shalika models and their connection to L-functions. The main reference is [FJ93]. We denote by G the algebraic group GL_{2n} and by Z its center. Let B be the Borel subgroup of upper triangular matrices in G. We view $H = \mathrm{GL}_n \times \mathrm{GL}_n$ as an algebraic subgroup of G via the diagonal embedding and let $\det^{m_1,m_2}\colon H \to \mathbb{G}_m$, $m_1, m_2 \in \mathbb{Z}$, the morphism of algebraic groups defined by $\det^{m_1,m_2}(g_1, g_2) = \det(g_1)^{m_1}\det(g_2)^{m_2}$. The Shalika subgroup S of G is defined as

$$S = \left\{ \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \middle| \, h \in GL_n, X \in M_n \right\}.$$

For the rest of the chapter we fix a finite order character $\eta\colon \mathbb{I}/F^* \to \overline{\mathbb{Q}}^*$. It induces a character $\eta\psi\colon S(\mathbb{A}) \to S^1$ via

$$\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto \eta(\det(h))\psi(\mathrm{tr}(X)).$$

Let $\pi = \otimes \pi_v$ be a unitary cuspidal automorphic representation of $G(\mathbb{A})$ with central character $\omega = \eta^n$, i.e. π consists of the smooth vectors of an irreducible direct summand of the right regular representation of $G(\mathbb{A})$ on $L^2_{\mathrm{cusp}}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}), \omega)$.

Definition 2.1. *The cuspidal representation π has a (global) (η, ψ)-Shalika-model, if there exist $\Phi \in \pi$ and $g \in G(\mathbb{A})$ such that the following integral does not vanish:*

$$\Xi_\Phi(g) = \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} (\pi(g)\Phi)(s)(\eta\psi(s))^{-1} ds.$$

The integral is well-defined since Φ is a cusp form. The global Shalika functional $\Lambda\colon \pi \to \mathbb{C}$ is defined by $\Lambda(\Phi) = \Xi_\Phi(1)$. Making a change of variables we see that $\Lambda(\pi(s)\Phi) = \eta\psi(s)\Lambda(\Phi)$ holds for all $s \in S(\mathbb{A})$ and $\Phi \in \pi$.
We will assume from now on that π has a (η, ψ)-Shalika-model.

Remark 2.2.

- *If $n = 1$ a Shalika functional is the same as a Whittaker functional. Thus every cuspidal automorphic representation of $GL_2(\mathbb{A})$ has a Shalika-model.*
- *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. Assume that the symmetric cube lift $\Pi = \mathrm{Sym}^3(\pi)$ is cuspidal. Then Π has a Shalika model (see for example [GRG14], Proposition 8.1.1).*

Proposition 2.3. *Let $f\colon \mathbb{I}/F^* \to \mathbb{C}$ be a locally constant function, Φ an element of π and $s \in \mathbb{C}$. Then the following integral converges absolutely*

$$\Psi(\Phi, f, s) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \Phi(h) \left|\det{}^{1,-1}(h)\right|^{s-1/2} f\left(\det{}^{1,-1}(h)\right) \eta^{-1}(\det{}^{0,1}(h)) dh$$

and defines an holomorphic function in s. If $\mathfrak{Re}(s)$ is sufficiently large it is equal to the following absolutely convergent integral:

$$Z(\Phi, f, s) = \int_{GL_n(\mathbb{A})} \Xi_\Phi \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) f(\det(g)) \left|\det(g)\right|^{s-1/2} d^*g.$$

Proof. If $f = \chi\colon \mathbb{I}/F^* \to \mathbb{C}^*$ is a character this is precisely Proposition 2.3 of [FJ93]. Since the locally constant characters form a basis of $C^0(\mathbb{I}/F^*, \mathbb{C})$ the claim follows. □

The global Shalika functional factors as a product of local Shalika functionals in the following sense: There exist non-zero functionals $\lambda_v\colon \pi_v \to \mathbb{C}$ for every place v of F such that $\Lambda(\Phi) = \prod_v \lambda_v(\varphi_v)$ holds for all pure tensors $\Phi = \otimes_v \varphi_v \in \pi = \otimes \pi_v$. Hence we have the equality $\Xi_\Phi = \prod_v \xi_{\varphi_v}^{\lambda_v}$ (see 1.3 for the defintion of $\xi_{\varphi_v}^{\lambda_v}$). Moreover, $\lambda_v(s_v\varphi_v) = \eta_v\psi_v(s_v)\lambda_v(\varphi_v)$ for all $\varphi \in \pi_v$ and $s_v \in S(F_v)$. Thus for v finite λ_v is a local Shalika-functional of π_v as in Definition 1.12.

Proposition 2.4. *Let $\varphi_v \in \pi_v$, $s \in \mathbb{C}$ with $\mathfrak{Re}(s)$ sufficiently large and $\chi_v\colon F_v^* \to \mathbb{C}$ a character. Then the following local zeta integral converges absolutely:*

$$\zeta_v(\varphi_v, \chi_v, s) = \int_{GL_n(F_v)} \xi_{\varphi_v}^{\lambda_v} \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_v(\det(g)) \left|\det(g)\right|_v^{s-1/2} d^*g.$$

There exists $\varphi_v \in \pi_v$ such that $L(\pi_v \otimes \chi_v, s) = \zeta_v(\varphi_v, \chi_v, s)$ holds for $\mathfrak{Re}(s)$ large enough and all unramified characters $\chi_v\colon F^* \to \mathbb{C}^*$. Moreover, if π_v is unramified this equality holds for a spherical vector. By our choice of Haar measure on $\mathrm{GL}_{2n}(F_v)$ we can choose the normalized spherical vector for allmost all v.

Proof. See Proposition 3.1 and Proposition 3.2 of [FJ93]. □

2.2 The global distribution

The goal of this section is to construct our global distribution and show that it fullfills the right interpolation property.

Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ having a (η, ψ)-Shalika model. Further we assume that for all places $\mathfrak{p}$ above p the representation $\pi_\mathfrak{p}$ is spherical, i.e. has a $G(\mathcal{O}_\mathfrak{p})$-fixed vector. Since every local component of a cuspidal automorphic representation of $G(\mathbb{A})$ has a Whittaker model, it follows from [BM94] that $\pi_\mathfrak{p}$ is isomorphic to an irreducible unramified principle series. Hence there exist unramified characters $\chi_{1,\mathfrak{p}}, \dots, \chi_{2n,\mathfrak{p}} \to \mathbb{C}^*$ and isomorphisms

$$\pi_v \cong \mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi_\mathfrak{p} = \mathrm{Ind}_{B(F_v)}^{G(F_v)} (\chi_{1,\mathfrak{p}}, \dots, \chi_{2n,\mathfrak{p}}).$$

Since π_v has a local (η_v, ψ_v)-Shalika model we may assume as in Section 1.4 that η_v is unramified and we can choose the χ_i such that $\chi_i = \eta \chi_{2n-i+1}^{-1}$ for all $1 \leq i \leq n$. We will denote $\beta_{i,\mathfrak{p}} = \alpha_{i,\mathfrak{p}} N(\mathfrak{p})^{n-i+1/2} = \chi_{i,\mathfrak{p}}(\varpi) N(\mathfrak{p})^{n-i+1/2}$, where ϖ is a local uniformizer at $\mathfrak{p}$ and $\alpha_\mathfrak{p} = \prod_{i=n+1}^{2n} \alpha_{i,\mathfrak{p}}$. In the following we will always identify π_v and $\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi_\mathfrak{p}$ via the above isomorphism. Choosing functions $\rho_{1,\mathfrak{p}}, \rho_{2,\mathfrak{p}} \colon \mathrm{GL}_n(F_v) \to \mathbb{C}$ given by

$$\rho_{1,\mathfrak{p}}(bk) = \prod_{i=1}^{n} \chi_{i,\mathfrak{p}}(b_{ii}) \text{ and } \rho_{2,\mathfrak{p}}(bk) = \prod_{i=1}^{n} \chi_{n+i,\mathfrak{p}}(b_{ii})$$

where $k \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ and $b = (b_{ij})_{ij}$ is upper triangular we get maps

$$\delta_\mathfrak{p} = \delta_{\rho_{1,\mathfrak{p}}, \rho_{2,\mathfrak{p}}} \colon C_c^0(\mathrm{GL}_n(F_v), \mathbb{C}) \to \pi_\mathfrak{p}$$

for all places $\mathfrak{p}$ lying above p (see section 1.2).

Remark 2.5.

- *The characters $\chi_{1,\mathfrak{p}}, \dots, \chi_{2n,\mathfrak{p}}$ are not uniquely determinded by the local representation $\pi_\mathfrak{p}$. The map $\delta_\mathfrak{p}$ depends on this choice.*
- *One can define a map similar to $\delta_\mathfrak{p}$ in a more general situation: it is enough that $\pi_\mathfrak{p}$ is a quotient of a principal series representation $\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi_\mathfrak{p}$, e.g. π_v is an unramified twist of the Steinberg representation. Choosing appopriate functions $\rho_{1,\mathfrak{p}}$ and $\rho_{2,\mathfrak{p}}$ one can hope that we still can define a distribution with the right interpolation property (see Theorem 2.6 below).*

We define $\Phi_m^\infty = \otimes_{v \nmid \infty} \varphi_{m,v} \in \otimes_{v \nmid \infty} \pi_v$, $m \geq 1$, to be the following pure tensor:

- If v is not lying over p, $\varphi_{m,v}$ is choosen as in the end of Proposition 2.4. Especially, it is independent of m.

- If $\mathfrak{p}$ is lying over p, we define $\varphi_{m,v} = \left[GL_n(\mathcal{O}_\mathfrak{p}) : K_{n,\mathfrak{p}}^{(m)}\right] \delta_p(\mathbb{1}_{K_{n,\mathfrak{p}}^{(m)}})$, where $K_{n,\mathfrak{p}}^{(m)}$ is the m-th principle congruence subgroup of $GL_n(\mathcal{O}_\mathfrak{p})$.

For now we fix a vector $\Phi_\infty = \otimes\varphi_v \in \otimes_{v|\infty}\pi_v$ and put $\Phi_m = \Phi_m^\infty \otimes \Phi_\infty$. The exact choice of Φ_∞ will be discussed at the end of section 3.1.
If $f\colon \mathbb{I}/F^* \to \mathbb{C}$ is a locally constant function there exists some integer $m \geq 1$ such that f factors through $\mathbb{I}/F^*U_{S_p}^{(m)}$. For every $s \in \mathbb{C}$ the integral

$$\int_{\mathbb{I}/F*} f(x)|x|^s\mu_\pi(dx) := \Psi(\Phi_m, f, s+\frac{1}{2})$$

converges absolutely by Proposition 2.3 and is a holomorphic function in s. It is easy to see that the integral is independent of the choice of m.
Remember that $\mathcal{G}_p$ denotes the Galois group of the maximal abelian extension of F unramified outside p and ∞ over F. By class field theory the Artin map $\mathrm{rec}\colon \mathbb{I}/F^* \to \mathcal{G}_p$ is continuous and surjective. Hence for every $s \in \mathbb{C}$ we can define a distribution $\mu_{\pi,s} \in \mathrm{Dist}(\mathcal{G}_p, \mathbb{C})$ uniquely determined by

$$\int_{\mathcal{G}_p} f(\gamma)\mu_{\pi,s}(d\gamma) = \int_{\mathbb{I}/F*} f(\mathrm{rec}(x))|x|^s\mu_\pi(dx)$$

for all $f \in C^0(\mathcal{G}_p, \mathbb{C})$.

Theorem 2.6. *Assume that $\beta_{i,\mathfrak{p}}\beta_{j,\mathfrak{p}} \neq \eta_\mathfrak{p}^{\pm 1}(\varpi)$ for all $1 \leq i < j \leq n$ and every prime $\mathfrak{p}$ lying over p, where ϖ is a local uniformizer at $\mathfrak{p}$. Then for every finite order character $\chi\colon \mathcal{G}_p \to \mathbb{C}^*$ of conductor $\mathfrak{f}(\chi)$ we have (up to a non-zero scalar)*

$$\begin{aligned}\int_{\mathcal{G}_p} \chi(\gamma)\mu_{\pi,s}(d\gamma) =& N(\mathfrak{f}(\chi))^s\tau(\chi,\psi^{-1})^n \prod_{\mathfrak{p}\in S_p} E(\delta_p, \chi_\mathfrak{p}, s)\\ &\times L_{S_{\mathfrak{p},\infty}}(\pi\otimes\chi, s+\frac{1}{2}) \prod_{v\in S_\infty} \zeta_v(\varphi_v, \chi_v, s+\frac{1}{2})\end{aligned}$$

with

$$E(\delta_p, \chi_\mathfrak{p}, s) = \begin{cases} \prod_{i=1}^n \frac{1-\alpha_{n+i,\mathfrak{p}}^{-1}\chi_\mathfrak{p}(\varpi)^{-1}q^{i-n+s}}{1-\alpha_{n+i,\mathfrak{p}}\chi_\mathfrak{p}(\varpi)q^{n-i-s-1}}, & \text{if } \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0 \\ (q^{\frac{n^2-n}{2}}\alpha_\mathfrak{p})^{-\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi))}, & \text{if } \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) > 0, \end{cases}$$

where $q = N(\mathfrak{p})$.

Proof. We view χ as a character on $\mathbb{I}/F^*$. Since both sides of the equation are holomorphic in s it is enough to show the equality for $\mathfrak{Re}(s)$ large. For m large enough we get

$$\begin{aligned} & \int_{\mathcal{G}_p} \chi(\gamma)\mu_{\pi,s}(d\gamma) = \Psi(\Phi_m, \chi, s+\frac{1}{2}) \\ = \;& Z(\Phi_m, \chi, s+\frac{1}{2}) \\ = \;& \prod_v \zeta_v(\varphi_{m,v}, \chi_v, s+\frac{1}{2}) \\ = \;& \prod_{v \notin S_{p,\infty}} L(\pi_v \otimes \chi_v, s+\frac{1}{2}) \prod_{v \in S_{p,\infty}} \zeta_v(\varphi_{m,v}, \chi_v, s+\frac{1}{2}) \end{aligned}$$

by using Proposition 2.4 and our choice of $\varphi_{m,v}$ if v is finite and does not divide p. Using Proposition 1.3 we get

$$\begin{aligned} \zeta_{\mathfrak{p}}(\varphi_{m,v}, \chi_v, s+\frac{1}{2}) &= \int_{GL_n(F_{\mathfrak{p}})} \xi^{\lambda_{\mathfrak{p}}}_{\varphi_{m,\mathfrak{p}}}\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_{\mathfrak{p}}(\det(g))\, |\det(g)|^s_{\mathfrak{p}}\, d^*g \\ &= \int_{GL_n(F_{\mathfrak{p}})} \chi_{\mathfrak{p}}(\det(g))\, |\det(g)|^s_{\mathfrak{p}}\, \mu_{\lambda_{\mathfrak{p}}}(dg) \end{aligned}$$

for all $\mathfrak{p}$ dividing p.
By our assumptions on the local representation $\pi_{\mathfrak{p}}$ we have by Proposition 1.14

$$\begin{aligned} & \int_{GL_n(F_{\mathfrak{p}})} \chi_{\mathfrak{p}}(\det(g))\, |\det(g)|^s_{\mathfrak{p}}\, \mu_{\lambda_{\mathfrak{p}}}(dg) \\ = & \int_{M_n(F_{\mathfrak{p}})} \rho_{2,\mathfrak{p}}(X)\chi_{\mathfrak{p}}(\det(X))\, |\det(X)|^s_{\mathfrak{p}}\, \psi^{-1}_{\mathfrak{p}}(\mathrm{tr}(X))dX \end{aligned}$$

and so the claim follows from Theorem 1.21. □

3 Boundedness of the distribution

3.1 Cohomological cuspidal representations and the Eichler-Shimura map

Let π be an unitary cuspidal automorphic representation of $G(\mathbb{A})$ with central character $\omega\colon \mathbb{I}/F^* \to \overline{\mathbb{Q}}^*$ and let $\pi_\infty = \otimes_{v|\infty}\pi_v$ be its component at infinity. For every infinite place v we define $K_v \subset G(F_v) = G(\mathbb{R})$ as the product of the maximal compact subgroup $\mathrm{O}(2n)$ and the center $Z(F_v) = Z(\mathbb{R})$ of $G(F_v)$ and we denote by $\mathfrak{g}_v$ the complexification of the Lie algebra of $G(F_v)$ and similarly we write $\mathfrak{k}_v$ for the complexification of the Lie algebra of K_v. We put $K_\infty = \prod_{v|\infty} K_\infty$, $\mathfrak{k}_\infty = \oplus_{v|\infty}\mathfrak{k}_v$ and $\mathfrak{g}_\infty = \oplus_{v|\infty}\mathfrak{g}_v$. The $(\mathfrak{g}_\infty, K_\infty^\circ)$-cohomology of a $(\mathfrak{g}_\infty, K_\infty^\circ)$-module V is defined by

$$H^\bullet((\mathfrak{g}_\infty, K_\infty^\circ), V) = H^\bullet(\mathrm{Hom}_{K_\infty^\circ}(\Lambda^\bullet(\mathfrak{g}_\infty/\mathfrak{k}_\infty), V)).$$

(See [BW00] for the basics on $(\mathfrak{g}_v, K_v^\circ)$-modules.) Note that there is a Künneth rule for $(\mathfrak{g}_\infty, K_\infty^\circ)$-cohomology, i.e. if $V = \otimes_v V_v$, where each V_v is a $(\mathfrak{g}_v, K_v^\circ)$-module, we have

$$H^q((\mathfrak{g}_\infty), K_\infty^\circ, V) = \bigoplus_{\sum q_v = q} \bigotimes_{v|\infty} H^{q_v}((\mathfrak{g}_v, K_v^\circ), V_v).$$

The representation π_v is a $(\mathfrak{g}_v, K_v^\circ)$-module for all $v \in S_\infty$. Given dominant weights $\mu_v = (\mu_{1,v}, \dots, \mu_{2n,v}) \in \mathbb{Z}^{2n}$ for all $v \in S_\infty$ we let V_{μ_v} the complexification of the irreducible F_v-rational representation of $G(F_v)$ of highest weight μ_v and put $V_\mu = \otimes_{v|\infty} V_{\mu_v}$. (As always, highest weight is meant with respect to the Borel group of upper triangular matrices.)

Definition 3.1. *The representation π is cohomological of weight μ if there exists $q \in \mathbb{N}$ such that the $(\mathfrak{g}_\infty, K_\infty^\circ)$-cohomology group $H^q((\mathfrak{g}_\infty, K_\infty^\circ), \pi_\infty \otimes V_\mu^\vee)$ does not vanish.*

We assume from now on that π is cohomological of weight μ.

Remark 3.2.
- *The representation $V = \otimes V_{\mu,v}$ can be seen as a $\mathbb{C}$-rational representation of the group $(\mathrm{Res}_{\mathbb{Q}}^F(G))_{\mathbb{C}} \cong G(\mathbb{C})^d$.*
- *If π is unitary, cohomological and has a (η, ψ)-Shalika model, then it follows from Lemma 3.6.1 of [GRG14] that $\eta_v = 1$ for all $v \in S_\infty$.*
- *Since π is unitary it follows from [Clo90] that V_{μ_v} must be self-dual for all $v \in S_\infty$, hence we know that the highest weights are of the form*

$$\mu_v = (\mu_{1,v}, \dots, \mu_{n,v}, -\mu_{n,v}, \dots, -\mu_{1,v})$$

with $\mu_{1,v} \geq \ldots \geq \mu_{n,v} \geq 0$. If we drop the condition that π is unitary we only get that the V_{μ_v} are essentially self-dual with respect to the so called purity weight $w \in \mathbb{Z}$. If w is even, we can always twist our representation by an integral power of the determinant to obtain a unitary representation.

Our aim is to show that the distribution $\mu_{\pi,s}$ definied in the previous chapter is a p-adic measure provided that $s + \frac{1}{2} \in \mathbb{C}$ is a critical point of π. We do not want to recall the definition of criticallity of a point here. It is enough to know the following two facts: Firstly, the set of critical points of π is given by

$$\mathrm{Crit}(\pi) = \left\{ s + \frac{1}{2} \in \frac{1}{2} + \mathbb{Z} \middle| \mu_{n+1,v} \leq s \leq \mu_{n,v} \ \forall v \in S_\infty \right\}.$$

Especially, $\frac{1}{2}$ is always critical (see [GRG14], Proposition 6.1.1).
Secondly, if $s+\frac{1}{2}$ is critical, then for all $v \in S_\infty$ there is a unique 1-dimensional $H(\mathbb{C})$-stable subrepresentation $V_{s,v}$ of V_{μ_v} which is isomorphic to the representation given by the character $\det^{-s,s}$. This is proven in [GRG14] Proposition 6.3.1. in the case $s = 0$. The other cases follow by twisting the representation with an integral power of the determinant.
In the following we want to explain the (adelic) Eichler-Shimura map: For every $v \in S_\infty$ let $G(F_v)^+ \subset G(F_v)$ be the subgroup of elements with positive determinant and $X_v = G(F_v)^+/K_v^\circ$ the associated symmetric space. We put $X = \prod_{v \in S_\infty} X_v$ and denote by e the image of the unit element under the canonical projection $\prod_{v \in S_\infty} G(F_v)^+ \to X$. We can naturally identify the tangent space $T_{X,e}$ of X at e with $\mathfrak{g}_\infty/\mathfrak{k}_\infty$. The Eichler-Shimura map for $q \in \mathbb{N}$ is a $G(\mathbb{A}^\infty)$-equivariant homomorphism

$$\mathrm{Hom}_{K_\infty^\circ}(\Lambda^q(\mathfrak{g}_\infty/\mathfrak{k}_\infty), \pi \otimes V_\mu^\vee) \to H^0(G(F)^+, C^0(G(\mathbb{A}^\infty), \Omega^q_{fd,har}(V_\mu^v))),$$

where $\Omega^q_{fd,har}(V_\mu^\vee)$ is the space of fast decreasing harmonic q-differential forms on X with values in $V_\mu^\vee$ (see [Bor81]) and $G(F)^+ \subset G(F)$ is the subgroup of elements with totally positive determinant. It is given as follows: By definition π is a subrepresentation of the right regular representation on $C^\infty(G(F)\backslash G(\mathbb{A}))$. Given $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/\mathfrak{h}_\infty), \pi \otimes V_\mu^\vee)$ we can evaluate it on a q-tuple $(Y_1, \ldots Y_p)$ of tangent vectors at e and we get

$$\omega(Y_1, \ldots, Y_p) \in C^\infty(G(F)\backslash G(\mathbb{A}), V_\mu^\vee).$$

For an element $(x, g^\infty) \in X \times G(\mathbb{A}^\infty)$ choose $g_\infty \in \prod_{v \in S_\infty} G(F_v)$ such that $g_\infty e = x$. Let Dg_∞ the differential of the action of g_∞ on X. We put

$$\tilde{\omega}(g^\infty)_x(Y_1, \ldots, Y_p) = g_\infty^{-1}(\omega((Dg_\infty)^{-1}Y_1, \ldots, (Dg_\infty)^{-1}Y_p)(g_\infty, g^\infty))$$

for $Y_1, \ldots, Y_p$ tangent vectors at x. Then $\tilde{\omega}(g^\infty)$ is a differential form on X with values in $V_\mu^\vee$. Since cusp forms are fast decreasing we see that we get in fact a fast decreasing differential form. Now let q be such that $H^q(\mathfrak{g}_\infty, K_\infty^\circ, \pi_\infty \otimes V_\mu^\vee) \neq 0$. Then it follows from Section II.3 of [BW00] that $H^q(\mathfrak{g}_\infty, K_\infty^\circ, \pi_\infty \otimes V_\mu^\vee) = \mathrm{Hom}_{K_\infty^\circ}(\Lambda^q(\mathfrak{g}_\infty/\mathfrak{k}_\infty), \pi \otimes V_\mu^\vee)$ and every differential form in the image of the Eichler-Shimura map is closed and harmonic.
We set $q_0 = n^2 + n - 1$. By the discussion in [GRG14], Section 3.4, the cohomology group $H^{q_v}(\mathfrak{g}_v, K_v^\circ, \pi_v \otimes V_{\mu_v}^\vee)$ is 2-dimensional if $q_v = q_0$ and vanishes if $q_v > q_0$ for every $v \in S_\infty$. Hence we see that

$$\begin{aligned} H^{dq_0}(\mathfrak{g}_\infty, K_\infty^\circ, \pi_\infty \otimes V_\mu^\vee) &= \bigoplus_{q_v = q_0} \bigotimes_{v|\infty} H^{q_v}(\mathfrak{g}_v, K_v^\circ, \pi_v \otimes V_{\mu_v}^\vee) \\ &\cong \bigoplus_{q_v = q_0} \bigotimes_{v|\infty} \mathbb{C}^2. \end{aligned}$$

Moreover, K_∞/K_∞° acts on $H^{dq_0}(\mathfrak{g}_\infty, K_\infty^\circ, \pi_\infty \otimes V_\mu^\vee)$ and the eigenspace of every character $\varepsilon\colon K_\infty/K_\infty^\circ \to \{1, -1\}$ is one-dimensional. We fix generators $[\pi_\infty]^\varepsilon$ of these eigenspaces. They are of the form $[\pi_\infty]^\varepsilon = \otimes_{v \in S_\infty} [\pi_v]^\varepsilon$ with

$$[\pi_v]^\varepsilon = \sum_{i=(i_1,\ldots i_{q_0})} \sum_{d=1}^{\dim V_{\mu_v}^\vee} X_i^* \otimes \varphi_{v,i,d} \otimes b_{v,d}^\vee.$$

Here (X_i^*) is a basis of $(\mathfrak{g}_\mathfrak{v}/\mathfrak{k}_v)^\vee$, $(b_{v,d})^\vee$ a basis of $V_{\mu_v}^\vee$ and $\varphi_{v,i,d} \in \pi_v$, $v \in S_\infty$. Each of these generators determines a non-zero Eichler-Shimura map

$$ES_\varepsilon\colon \bigotimes_{v\nmid\infty} \pi_v \longrightarrow H^0(G(F)^+, C^0(G(\mathbb{A}^\infty), \Omega_{fd,har}^{dq_0}(V_\mu^v))).$$

Given a ring R which contains the image of ω, M an R-module, S a finite set of places containing the archimedean ones and $K \subset G(\mathbb{A}^S)$ an open compact subgroup we let $C(G(\mathbb{A}^S)/KZ(\mathbb{A}^S), \omega, M)$ be the R-module of functions $f\colon G(\mathbb{A}^S) \to M$ such that $f(gkz) = w(z)f(g)$ for all $g \in G(\mathbb{A}^S)$, $k \in K$ and $z \in Z(\mathbb{A}^S)$. If $\Phi \in \otimes_{v\nmid\infty}\pi_v$ is invariant under some open compact subgroup $K \subset G(\mathbb{A}^\infty)$ we get that

$$ES_\varepsilon(\Phi) \in H^0(PG(F)^+, C(G(\mathbb{A}^\infty)/KZ(\mathbb{A}^\infty), \omega, \Omega_{fd,har}^{dq_0}(V_\mu^v))),$$

where $PG(F)^+$ is defined as the quotient $G(F)^+/Z(F)$. If $A \subset G(F)$ is any subgroup containing the center $Z(F)$, the intersection of $A/Z(F)$ with $PG(F)^+$ will be denoted by PA^+.
In fact, we need a slight variant of the above construction. Let π_p be the

tensor product of the local representations $\pi_{\mathfrak{p}}$ over all primes $\mathfrak{p} \in S_p$. Given $\Phi^p \in \otimes_{v\nmid p,\infty}\pi_v$ invariant under some open compact subgroup $K^p \subset G(\mathbb{A}^{p,\infty})$ we define

$$ES^p_\varepsilon(\Phi^p) \in H^0(PG(F)^+, C(G(\mathbb{A}^{p,\infty})/K^pZ(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \Omega^{dq_0}_{fd,har}(V^v_\mu))))$$

by

$$ES^p_\varepsilon(\Phi^p)(g^p, \varphi_p) = ES_\varepsilon(\Phi^p \otimes \varphi_p)(g^p, 1)$$

for φ_p in π_p.

Evaluation at an element φ_p of π_p, which is invariant under some open compact subgroup $K_p \subset \prod_{\mathfrak{p}|p} G(F_{\mathfrak{p}})$, induces a $PG(F)^+$-equivariant map

$$\begin{aligned} &C(G(\mathbb{A}^{p,\infty})/K^pZ(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \Omega^{dq_0}_{fd,har}(V^v_\mu))) \\ &\xrightarrow{ev(\varphi_p)} C(G(\mathbb{A}^\infty)/K^pK_pZ(\mathbb{A}^\infty), \omega, \Omega^{dq_0}_{fd,har}(V^v_\mu)) \end{aligned}$$

such that $ev(\varphi_p)(ES^p_\varepsilon(\Phi^p)) = ES_\varepsilon(\Phi^p \otimes \varphi_p)$.

Let $\bar{X}$ the Borel-Serre bordification of X with boundary ∂X as constructed in [BS73]. It is a smooth manifold with corners, which contains X as an open submanifold. The embedding $X \subset \bar{X}$ is a homotopy equivalence. The opearation of $G(F)^+$ can be naturally extended to $\bar{X}$. If M is a smooth manifold with corners we let $C^{sing}_\bullet(M)$ be the complex of singular chains in M and $C^{sm}_\bullet(M)$ the subcomplex of smooth chains. By Lemma 5 of [Whi34] continuous chains can be approximated by smooth chains. Hence by a standard argument the inclusion $C^{sm}_\bullet(M) \subset C^{sing}_\bullet(M)$ is a quasi-isomorphism (see chapter 16 of [Lee03] for a detailed proof in the case of smooth manifolds without corners). Using this fact for both $\bar{X}$ and its boundary ∂X, we see that the complex $C^{sm}_\bullet(\bar{X}, \partial X) := C^{sm}_\bullet(\bar{X})/C^{sm}_\bullet(\partial X)$ is quasi-isomorphic to the complex of singular relative chains $C^{sing}_\bullet(\bar{X}, \partial X)$. Note that these are in fact quasi-isomorphisms of complexes of $G(F)^+$-modules.

For every $q \in \mathbb{N}$ there is a $PG(F)^+$-equivariant pairing

$$\Omega^q_{fg}(V^\vee_\mu) \times C^{sm}_q(\bar{X}, \partial X) \to V^\vee_\mu$$

given as follows: We denote by Δ_q the standard simplex of dimension q. If $f\colon \Delta_q \to \bar{X}$ is a smooth chain and ω a fast decreasing differential form, we take the integral of the pullback $f^*\omega$ over the preimage of X under f. If the differential form is closed, it vanishes on the image of the boundary map $C^{sm}_{q+1}(\bar{X}, \partial X) \to C^{sm}_q(\bar{X}, \partial X)$ by Stokes' Theorem. Therefore we get a $G(F)^+$-equivariant morphism of cocomplexes

$$\Omega^q_{fd,har}(V^\vee_\mu)[-q] \to \mathrm{Hom}(C^{sm}_\bullet(\bar{X}, \partial X), V^\vee_\mu),$$

which induces the following maps in (hyper-) group cohomology:

$$
\begin{aligned}
&H^0(PG(F)^+, C(G(\mathbb{A}^\infty)/KZ(\mathbb{A}^\infty), \omega, \Omega^q_{fd,har}(V^v_\mu)))\\
\longrightarrow&\mathbb{H}^q(PG(F)^+, C(G(\mathbb{A}^\infty)/KZ(\mathbb{A}^\infty), \omega, \mathrm{Hom}(C^{sm}_\bullet(\bar{X}, \partial X), V^\vee_\mu)))
\end{aligned}
$$

and

$$
\begin{aligned}
&H^0(PG(F)^+, C(G(\mathbb{A}^{p,\infty})/K^pZ(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \Omega^q_{fd,har}(V^v_\mu))))\\
\longrightarrow&\mathbb{H}^q(PG(F)^+, C(G(\mathbb{A}^{p,\infty})/K^pZ(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \mathrm{Hom}(C^{sm}_\bullet(\bar{X}, \partial X), V^\vee_\mu)))).
\end{aligned}
$$

We will denote the image of $ES_\varepsilon(\Phi)$ ($ES^p_\varepsilon(\Phi^p)$ resp.) under the above map for $q = dq_0$ by $ES_\varepsilon(\Phi)_{coh}$ ($ES^p_\varepsilon(\Phi^p)_{coh}$ resp.).
When we defined the distribution in Section 2.2, we had not specified the vector at infinity. We will do it now. For this let $\varepsilon_+ \colon K_\infty/K^\circ_\infty \to \{1, -1\}$ be the trivial character. Let $\mathfrak{h}_\mathbb{Q}$ be the Lie-Algebra of the algebraic group H over $\mathbb{Q}$ and $\mathfrak{k}'_\mathbb{Q}$ the Lie subalgebra of the $\mathbb{Q}$-rational algebraic subgroup $H \cap Z\,\mathrm{SO}_{2n}$. The dimension of $\mathfrak{h}_Q/\mathfrak{k}'_\mathbb{Q}$ is exactly q_0. We fix a $\mathbb{Q}$-basis $T_1, \ldots, T_{q_0}$ and denote by $T_{i,v}$ the image of T_i in $\mathfrak{g}_v/\mathfrak{k}_v$ for every infinite place v. Additionally, we fix a generator $x_{s,v}$ of the subspace $V_{s,v} \subset V_{\mu_v}$ for every critical point $s + \frac{1}{2}$ of π and every infinite place v. Evaluation of

$$[\pi_v]^{\varepsilon_+} = \sum_{i=(i_1,\ldots i_{q_0})} \sum_{d=1}^{\dim V^\vee_{\mu_v}} X^*_i \otimes \varphi_{v,i,d} \otimes b^\vee_{v,d}$$

at $(T_{1,v}, \ldots, T_{q_0,v}, x_{s,v})$ yields an element $\varphi_{s,v} \in \pi_v$. We will choose $\Phi_{s,\infty} = \otimes_{v|\infty}\varphi_{s,v}$ as the vector at infinity in the definition of the distribution $\mu_{\pi,s}$. It is important to know that the complex numbers

$$c(\pi, \chi_\infty, s) = \prod_{v|\infty} \zeta(\varphi_{s,v}, \chi_v, s + \frac{1}{2})$$

do not vanish for given characters $\chi_v \colon \mathbb{R}^* \to \{1, -1\}$. Otherwise, the distribution $\mu_{\pi,s}$ would trivially be zero. Using results of Sun (see [Sun11]) it is shown by Grobner-Rhaguram (in [GRG14], Theorem 6.2.2) that $c(\pi, 1, s)$ is in fact non-zero.

3.2 The Steinberg module

In this chapter we recall some standard facts about the Steinberg module of an algebraic group and about Borel-Serre duality. Let $\underline{G}$ be a connected split reductive group over F of semisimple F-rank $l \geq 1$ and I the set of proper maximal F-rational parabolic subgroups of $\underline{G}(F)$. For $\tau \in I$ let $\underline{P}_\tau$ the corresponding parabolic group. A subset $S = \{\tau_1, \ldots, \tau_n\} \subset I$ is called a $k-1$-simplex if $\sharp S = k$ and $\underline{P}_{\tau_1} \cap \ldots \cap \underline{P}_{\tau_k}$ is a parabolic subgroup. Let St_k be the free abelian group generated by the k-simplices on I. Taking the associated simplicial complex we get a sequence of $\underline{G}(F)$-modules

$$\mathrm{St}_{l-1} \to \mathrm{St}_{-2} \to \cdots \to \mathrm{St}_0 \to \mathbb{Z} \to 0. \tag{5}$$

Definition 3.3. *The Steinberg module* $\mathrm{St}_{\underline{G}}$ *of* $\underline{G}(F)$ *is the kernel of the map* $St_{l-1} \to St_{l-2}$ *(where we set* $\mathrm{St}_{-1} = \mathbb{Z}$ *in the case* $l = 1$*).*

Let $\mathcal{P}_k$ be the set of proper F-rational parabolic subgroups of semisimple F-rank $l-1-k$ containing a fixed Borel subgroup $\underline{B}(F)$ of $\underline{G}(F)$. Then for $0 \leq k \leq l-1$ there is a natural isomorphism of $\underline{G}(F)$-modules

$$\bigoplus_{\underline{P} \in \mathcal{P}_k} \text{c-ind}_{\underline{P}(F)}^{\underline{G}(F)} \mathbb{Z} \xrightarrow{\cong} \mathrm{St}_k\,.$$

The homology of the complex (5) can be identified with the reduced homology of the spherical building associated to $\underline{G}(F)$. Since the reduced homology of the building vanishes outside the top degree (see for example [BS76]) we see that the complex of $\underline{G}(F)$-modules

$$0 \to \mathrm{St}_{\underline{G}} \to \mathrm{St}_{l-1} \to \cdots \to \mathrm{St}_0 \to \mathbb{Z} \to 0$$

is exact.

Remark 3.4. *Because every parabolic subgroup of* $\underline{G}$ *contains the center* $\underline{Z}$ *of* $\underline{G}$*, we see that* $\mathrm{St}_{\underline{G}}$ *and* $\mathrm{St}_{\underline{G}/\underline{Z}}$ *are canonically isomorphic.*

The choice of a Borel subgroup $\underline{B}$ with maximal torus $\underline{T}$ gives us the element

$$\tau_{\underline{G}} = \sum_{w \in W_{\underline{G}}} w \otimes \epsilon(w) \in \mathrm{St}_{\underline{G}} \subset \mathbb{Z}[\underline{G}(F)] \otimes_{\mathbb{Z}[\underline{B}(F)]} \mathbb{Z},$$

where $W_{\underline{G}}$ denotes the Weyl group of $\underline{G}$ with respect to $\underline{T}$ and $\epsilon\colon W_{\underline{G}} \to \mathbb{Z}^*$ is the sign character corresponding to $\underline{B}$. Now let $\underline{P}$ be a parabolic subgroup containing $\underline{B}$ and let $\underline{L}$ be the Levi-factor containing the torus $\underline{T}$. There exists an $\underline{L}(F)$ equivariant map $\mathrm{St}_{\underline{L}} \to \mathrm{St}_{\underline{G}}$ which maps $\tau_{\underline{L}}$ to $\tau_{\underline{G}}$ (see Proposition 1.1 of [Ree90]).
By [BS73] every arithmetic subgroup $\Gamma \subset \underline{G}(F)$ is a virtual duality group with duality module $\mathrm{St}_{\underline{G}}$. Let $\nu = \nu(\Gamma)$ be the virtual cohomological dimension of Γ. It is independent of the choice of the arithmetic subgroup. Since $St_{\underline{G}}$ is $\mathbb{Z}$-free it follows from [Bro82], chapter VIII.10, that the map

$$BS_\Gamma\colon H^q(\Gamma, \mathrm{Hom}(\mathrm{St}_{\underline{G}}, M)) \xrightarrow{\cap e} H_{\nu-q}(\Gamma, \mathrm{St}_{\underline{G}} \otimes \mathrm{Hom}(\mathrm{St}_{\underline{G}}, M)) \xrightarrow{ev} H_{\nu-q}(\Gamma, M)$$

is an isomophism for every Γ-module M as long as Γ is torsion-free. Here $e \in H_\nu(\Gamma, \mathrm{St}_{\underline{G}}) \cong \mathbb{Z}$ is a fundamental class (see [Bro82] VIII.6) and ev is the map induced by the evaluation map $\mathrm{St}_{\underline{G}} \otimes \mathrm{Hom}(St_{\underline{G}}, M) \to M$. Now let $K \subset \underline{G}(\mathbb{A}^\infty)$ be a compact open subgroup. After passing to a subgroup of finite index we may assume that $\underline{G}(F) \cap gKg{-}1$ is torsion-free for all $g \in \underline{G}(\mathbb{A}^\infty)$. For every K-module M and every subgroup $\underline{G}' \subset \underline{G}(F)$ of finite index we get an isomorphism

$$(6) \qquad BS_{\underline{G}}\colon H^q(\underline{G}', \mathrm{Ind}_K^{\underline{G}(\mathbb{A}^\infty)} \mathrm{Hom}(\mathrm{St}_{\underline{G}}, M)) \xrightarrow{\cong} H_{\nu-q}(\underline{G}', \text{c-ind}_K^{\underline{G}(\mathbb{A}^\infty)} M)$$

as follows: By strong approximation the quotient $\underline{G}' \backslash \underline{G}(\mathbb{A}^\infty)/K$ is finite. Let $g_1, \ldots, g_r$ be a set of representatives of this double quotient and consider the torsion-free arithmetic subgroups $\Gamma_i = \underline{G}' \cap g_i K g_i^{-1}$. By Shapiro's Lemma we get an isomorphism

$$H^q(\underline{G}', \mathrm{Ind}_K^{\underline{G}(\mathbb{A}^\infty)} \mathrm{Hom}(\mathrm{St}_{\underline{G}}, M)) \xrightarrow{\cong} \bigoplus_{i=1}^r H^q(\Gamma_i, \mathrm{Hom}(\mathrm{St}_{\underline{G}}, M)).$$

Using Borel-Serre duality for every Γ_i and Shapiro's Lemma for homology afterwards we get the isomorphism (6).
We are mostly interested in the case $\underline{G} = H/Z$. In this case Theorem 11.4. of [BS73] gives us $\nu(\Gamma) = d(n^2 + n - 1) - 2n + 1 = dq_0 - 2n + 1$ for every arithmetic subgroup Γ of $H(F)/Z(F)$.
There is another description of the Steinberg module. By Corollary 8.4.2 of [BS73] there is a homotopy equivalence between the boundary ∂X of the Borel-Serre bordification of the symmetric space X and the Bruhat-Tits building of $G(F)$ which gives a $PG(F)^+$-equivariant ismorphism of the singular homology groups. Since $\bar{X}$ is contractible the long exact sequence for relative homology shows that $H_{q+1}(\bar{X}, \partial X)$ is isomorphic to the reduced homology $\tilde{H}_q(\partial X)$. Hence for every coefficient system as in the previous section the complex $\mathrm{Hom}(C_\bullet^{sm}(\bar{X}, \partial X), V_\mu^\vee)$ is quasi-isomorphic to the complex $\mathrm{Hom}(\mathrm{St}_G, V_\mu^\vee)[-2n+1]$. We therefore have isomorphisms

$$\begin{aligned} &\mathbb{H}^q(PG(F)^+, C(G(\mathbb{A}^\infty)/KZ(\mathbb{A}^\infty), \omega, \mathrm{Hom}(C_\bullet^{sm}(\bar{X}, \partial X), V_\mu^\vee))) \\ \xrightarrow{\cong} &H^{q-2n+1}(PG(F)^+, C(G(\mathbb{A}^\infty)/KZ(\mathbb{A}^\infty), \omega, \mathrm{Hom}(\mathrm{St}_G, V_\mu^\vee))) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{H}^q(PG(F)^+, C(G(\mathbb{A}^{p,\infty})/K^p Z(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \mathrm{Hom}(C_\bullet^{sm}(\bar{X}, \partial X), V_\mu^\vee)))) \\ \xrightarrow{\cong} &H^{q-2n+1}(PG(F)^+, C(G(\mathbb{A}^{p,\infty})/K^p Z(\mathbb{A}^{p,\infty}), \omega, \mathrm{Hom}(\pi_p, \mathrm{Hom}(\mathrm{St}_G, V_\mu^\vee)))). \end{aligned}$$

We will identify $ES_\varepsilon(\Phi)_{coh}$ ($ES_\varepsilon^p(\Phi^p)_{coh}$ resp.) with its image under the above isomorphism.

3.3 The Cohomology of PGL_{2n}

For every critical point $s+\frac{1}{2}$ we want to give a cohomological construction of the distribution of Section 2.2. We use the results of Große-Klönne (see [GK14]) on nice resolutions of lattices in locally algebraic representations to show that a certain cohomology group commutes with flat base change. Here we follow closely an argument of Spiess (in [Spi14]) which deals with the case $n=1$. That the distribution is in fact a p-adic measure then follows from the integrality properties of the map δ.
From now on let π be an unitary cuspidal automorphic representaion with finite order central character ω, which is cohomological with respect to a weight μ. As in Section 2.2 we assume that the local representation $\pi_{\mathfrak{p}}$ is spherical for all $\mathfrak{p}\in S_p$ and fix isomorphisms with unramified principal series representations. Since π is cohomological there exists a number field $E\subset\overline{\mathbb{Q}}$ such that the characters that are used to define $\pi_{\mathfrak{p}}$ have values in E for all $\mathfrak{p}$ lying over p (see [Clo90]). By abuse of notation we let $\pi_{\mathfrak{p}}$ denote the representation defined over E. Further we can assume after enlarging E that all embeddings of F into $\mathbb{R}$ factor over E. Hence we can and will assume that the representations V_{μ_v} for $v\in S_\infty$ are E-rational representation. We let R be the valuation ring of E with respect to the fixed p-adic valuation on $\bar{\mathbb{Q}}$ and E_p the completion of E at this valuation. By enlarging E once more we can assume that the image of η lies in R^*.
We will from now on assume that the prime p is split in F. On the one hand this makes the exposition easier, on the other hand we want to use the results of [GK14], which depend on the condition that the local field is $\mathbb{Q}_p$.
In this case we have a one to one correpondence between real and p-adic places as follows: Every real embedding σ gives a map

$$F\xrightarrow{\sigma}E\subset\bar{\mathbb{Q}}\xrightarrow{\iota_p}\mathbb{C}_p$$

and thus a p-adic place $\mathfrak{p}_\sigma$ via pull back of the p-adic valuation on $\mathbb{C}_p$. We will write $\mathcal{O}_\sigma$ for the valuation ring of F with respect to $\mathfrak{p}_\sigma$ and put $\mathcal{O}_\infty=\mathcal{O}_{\sigma_1}\times\ldots\times\mathcal{O}_{\sigma_d}$. It is important that we we do not consider the completions in this case. The completion of $\mathcal{O}_\sigma$ is given by $\mathcal{O}_{\mathfrak{p}_\sigma}$. We define $\mathcal{O}_p=\prod_{\mathfrak{p}|p}\mathcal{O}_{\mathfrak{p}}$ and $F_p=\prod_{\mathfrak{p}|p}F_{\mathfrak{p}}$. Further we put $V_\sigma^{sm}=V_{\mathfrak{p}_\sigma}^{sm}=\pi_{\mathfrak{p}_\sigma}$ and $V_\sigma^{al}=V_{\mu_v}$, where v is the real place corresponding to σ.
We want to study the cohomology of the $G(F)$-representation $V=\otimes_\sigma(V_\sigma^{sm}\otimes V_\sigma^{al})$.

Definition 3.5. *A cuspidal automorphic representation as above is called weakly p-ordinary if the locally algebraic representation V_{E_p} is weakly p-ordinary in the sense of Definition 1.22.*

We will assume from now on that π is weakly p-ordinary. Hence, after choosing a suitable $G(\mathcal{O}_\sigma)$-stable lattice L_σ in $(V_\sigma^{al})_F$ we can define the R-modules $M_{\chi_{\mathfrak{p}_\sigma}}(L_\sigma)$ and a Koszul complex

$$0 \to C_\chi^{2n-1}(L_\sigma) \to \cdots \to C_\chi^0(L_\sigma) \to \ M_{\chi_{\mathfrak{p}_\sigma}}(L_\sigma) \to 0$$

as in Section 1.3. As before we have $M_{\chi_{\mathfrak{p}_\sigma}}(L_\sigma) \otimes_R E = V_\sigma^{sm} \otimes V_\sigma^{al}$. We will always assume in the following that the conditions of Lemma 1.10 are fullfilled, i.e. the reduction of the lattices L_σ are irreducible rational representation, the weights are p-small and $\omega_{\mathfrak{p}}$ takes values in $\mathbb{Z}_p^*$ for all primes $\mathfrak{p}$ of F lying above p. Especially, by Große-Klönne (see [GK14], Theorem 1.1) we now that the above Koszul complex is exact and $M_{\chi_p}(L) = \otimes_\sigma M_{\chi_{\mathfrak{p}_\sigma}}(L_\sigma)$ is R-free.

Remark 3.6. *If the prime p does not split, several embeddings $F \hookrightarrow E$ give the same p-adic place $\mathfrak{p}$. If $\sigma_1, \ldots, \sigma_r$ are the embeddings giving $\mathfrak{p}$, we would have to consider representation of the form $V_{\mathfrak{p}}^{sm} \otimes (\otimes_{i=1}^r V_{\sigma_i}^{al})$. If E_p is the completion of E at the fixed p-adic valuation, then $(\otimes_\sigma V_\sigma^{al})_{E_p}$ is a E_p-rational representation of the group $(\mathrm{Res}_{\mathbb{Q}_p}^{F_\mathfrak{p}} G)_{E_p}$.*

Let $\underline{G}$ be an F-rational linear algebraic subgroup of G containing Z, $\underline{G}' \subset \underline{G}(F)$ any subgroup, M a left $\underline{G}'$-module and N an $R[\underline{G}']$-module. For a compact open subgroup K^p of $G(\mathbb{A}^{p,\infty})$ we let $\mathcal{A}_{\underline{G}}(K^p, M; N)$ be the set of all functions $\Phi\colon \underline{G}(\mathbb{A}^{p,\infty}) \times M \to N$ such that $\Phi(gzk, m) = \omega(z)\Phi(g, m)$ for all $g \in \underline{G}(\mathbb{A}^{p,\infty})$, $k \in K^p \cap \underline{G}(\mathbb{A}^{p,\infty})$, $z \in Z(\mathbb{A}^{p,\infty})$ and $m \in M$, which are linear in M. The group $\underline{G}'$ acts on $\mathcal{A}(K^p, M; N)$ via $(\gamma \cdot \Phi)(g, m) = \gamma\Phi(\gamma^{-1}g, \gamma^{-1}M)$. For an R-module N with trivial $\underline{G}'$-action we set

$$\mathcal{A}_{\underline{G}}(\chi_p, K^p, M; N) = \mathcal{A}_{\underline{G}}(K^p, M; \mathrm{Hom}_R(M_{\chi_p}(L), N)).$$

If $\underline{G}' \cap Z(F)$ acts trivially on M, then the action of $\underline{G}'$ on $\mathcal{A}_{\underline{G}}(\chi_p, K^p, M; N)$ factors through $\underline{G}'/(\underline{G}' \cap Z(F))$. If $\underline{G} = G$ we omit the index $\underline{G}$ to ease the notation.

Proposition 3.7. *(a) Let N be a flat R-module. Then the canonical map*

$$H^q(PG(F)^+, \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; R)) \otimes_R N \to H^q(PG(F)^+, \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; N))$$

is an isomorphism for all $q \geq 0$.
(b) The R-module $H^q(PG(F)^+, \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; R))$ is finitely generated.

Proof. (a) The sequence $0 \to \mathrm{St}_G \to \mathrm{St}_{2n-2} \to \cdots \to \mathrm{St}_0 \to \mathbb{Z} \to 0$ is exact. If we break this exact sequence into short exact sequences $0 \to A \to$

$B \to C \to 0$ and consider the associated long exact sequences $H^q(\cdot, N)$ and $H^q(\cdot, R) \otimes N$ (which is exact since N is flat), we see by induction that it is enough to proof (a) with St_G replaced by St_i, $-1 \leq i \leq 2n-2$. Since the St_i are direct sums of modules of the form $\text{c-ind}_{P(F)}^{\mathrm{PGL}_{2n}(F)} \mathbb{Z}$ with $P \subset \mathrm{PGL}_{2n}$ (not necessarily proper) parabolic subgroups, it is enough to show that

$$\begin{aligned}
&H^q(PG(F)^+, \mathcal{A}(\chi_p, K^p, \text{c-ind}_{P(F)}^{\mathrm{PGL}_{2n}(F)} \mathbb{Z}; N)) \\
=&H^q(PG(F)^+, \mathrm{Ind}_{P(F)}^{\mathrm{PGL}_{2n}(F)} \mathcal{A}(\chi_p, K^p, \mathbb{Z}; N)) \\
=&H^q(P(F)^+, \mathcal{A}(\chi_p, K^p, \mathbb{Z}; N))
\end{aligned}$$

commutes with flat base change. Since by our assumptions the Koszul-complex for $M_{\chi_{\mathfrak{p}_\sigma}}(L_\sigma)$ is exact for all σ a similar argument as above shows that it is enough to proof that the cohomology groups

$$\begin{aligned}
&H^q(P(F)^+, \mathcal{A}(K^p, \mathbb{Z}; \mathrm{Ind}_{Z(F)^d G(\mathcal{O}_\infty)}^{G(F)^d}(\otimes_\sigma L_\sigma^\vee \otimes N))) \\
=&H^q(P(F)^+, \mathrm{Ind}_{Z(\mathbb{A}^{p,\infty})K^p}^{G(\mathbb{A}^{p,\infty})} \omega^{-1} \otimes \mathrm{Ind}_{Z(F)^d G(\mathcal{O}_\infty)}^{G(F)^d}(\otimes_\sigma L_\sigma^\vee \otimes N))
\end{aligned}$$

commute with flat base change.
By strong approximation (and the Iwasawa decomposition) the quotient $P(F)^+ \backslash G(\mathbb{A}^\infty)/Z(\mathbb{A}^\infty)K^p \prod_\sigma GL_{2n}(\mathcal{O}_{\mathfrak{p}_\sigma})$ is finite. We choose a system of representatives $g_1, \ldots, g_r$ of the above double quotient and define the arithmetic subgroups $\Gamma_i = P(F)^+ \cap g_i K^p \prod_\sigma G(\mathcal{O}_{\mathfrak{p}_\sigma}) g_i^{-1} Z(\mathbb{A}^\infty)/Z(\mathbb{A}^\infty)$. From Shapiro's Lemma we get the equality

$$\begin{aligned}
&H^q(P(F)^+, \mathrm{Ind}_{Z(\mathbb{A}^{p,\infty})K^p}^{G(\mathbb{A}^{p,\infty})} \omega^{-1} \otimes \mathrm{Ind}_{Z(F)^d G(\mathcal{O}_\infty)}^{G(F)^d}(\otimes_\sigma L_\sigma^\vee \otimes N)) \\
=&\bigoplus_{i=1}^{r} H^q(\Gamma_i, \otimes_\sigma L_\sigma^\vee \otimes N).
\end{aligned}$$

Since the groups Γ_i are arithmetic groups, they are of type (VFL). It follows that the functor $N \mapsto H^q(\Gamma_i, \otimes_\sigma (L_\sigma^\vee) \otimes N)$ commutes with direct limits (see [Ser72]) and hence the claim follows because every flat module is a direct limit of free modules of finite rank.
(b) can be proven in exactly the same manner using that R is noetherian. □

Let $F_+^* \subset F^*$ be the subgroup of totally positive elements. The Artin reciproctiy map induces a surjective map $\mathbb{I}^\infty/U^{p,\infty} \to \mathcal{G}_p$, which yields an isomorphism $H^0(F_+^*, \mathrm{Ind}_{U^\infty}^{\mathbb{I}^\infty}(C^0(U_p, R))) \to C^0(\mathcal{G}_p, R)$. Therefore, for every subgroup $U' \subset U^\infty$ of finite index the cap product

$$\begin{aligned}
&H^0(F_+^*, \mathrm{Ind}_{U^\infty}^{\mathbb{I}^\infty}(C^0(U_p, R))) \times H_0(F_+^*, \text{c-ind}_{U'}^{\mathbb{I}^\infty}(\mathrm{Dist}(U_p, N))) \\
\xrightarrow{\cap}&H_0(F_+^*, \text{c-ind}_{U'}^{\mathbb{I}^\infty} N) \cong \bigoplus N \xrightarrow{\Sigma} N
\end{aligned}$$

induces a map

$$\partial\colon H_0(F_+^*, \text{c-ind}_{U'}^{\mathbb{I}^\infty}(\mathrm{Dist}(U_p, N))) \to \mathrm{Dist}(\mathcal{G}_p, N)$$

for every R-module N. The direct sum decomposition $H_0(F_+^*, \text{c-ind}_U^{\mathbb{I}^\infty} N) \cong \oplus N$ follows from Shapiro's Lemma and a strong approximation argument as before.
We assume from now on that π has an (η, ψ)-Shalika model and that $s + \frac{1}{2}$ is critical for π. The character η induces a character

$$\begin{aligned} 1 \times \eta\colon\ H(\mathbb{A}) &\to R^* \\ (h_1, h_2) &\mapsto \eta(\det(h_2)). \end{aligned}$$

Let $\Phi^p = \Phi_m^p \in \otimes_{v \nmid p,\infty} \pi_v$ be chosen as in Section 2.2 and $K^p \subset G(\mathbb{A}^{p,\infty})$ an open compact subgroup such that

- Φ^p is invariant under K^p
- $K^p G(\mathcal{O}_p)$ is neat, i.e. $G(F) \cap gK^p G(\mathcal{O}_p) g^{-1}$ is torsion-free for every $g \in G(\mathbb{A}^\infty)$
- $(1 \times \eta)(K_H^p) = 1$ where K_H^p is the intersection $K^p \cap H(\mathbb{A}^{p,\infty})$.

The image of $K_H^p H(O_p)$ under the character $\det^{1,-1}$ will be denoted by U'.
We want to construct maps

$$\Delta_N^s\colon H^{dq_0 - 2n+1}(PG(F)^+, \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; N)) \to H_0(F_+^*, \text{c-ind}_{U'}^{\mathbb{I}^\infty}(\mathrm{Dist}(U_p), N)),$$

functorial in N, such that $\partial(\Delta_{\mathbb{C}}^s(ES_{\varepsilon_+}^p(\Phi^p)_{coh})) = \mu_{\pi,s}$ (up to a non-zero constant). It then follows from Proposition 3.7 that $\mu_{\pi,s}$ is in fact a p-adic measure.
The homomorphism Δ_N^s is constructed as a composition of several maps, which are functorial in N. Firstly, the map $\mathrm{St}_H \to \mathrm{St}_G$ of Section 3.2 together with the restriction of functions yields the $H(F)$-equivariant map

$$Res_H\colon \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; N) \to \mathcal{A}_H(\chi_p, K^p, \mathrm{St}_H; N).$$

Secondly, since $s + \frac{1}{2}$ is critical, there is for every σ a unique 1-dimensional $H(F)$-subrepresentation $V_{s,\sigma}$ of V_σ^{al}, which is isomorphic to the character $\det^{-s,s}$ (see the discussion in Section 3.1). Let

$$\delta_{s,N}^\vee\colon \mathrm{Hom}_{\mathcal{R}}(M_\chi(\mathcal{L}), N) \to \text{I-Dist}_{(s)}(\mathrm{GL}_n(F_p), \omega_p, N),$$

be the corresponding semi-local map (see Section 1.6), where we have chosen δ as in Section 2.2. It induces a map

$$\mathcal{A}_H(\chi_p, K^p, \mathrm{St}_H; N) \to \mathcal{A}_H(K^p, \mathrm{St}_H; \text{I-Dist}_{(s)}(\mathrm{GL}_n(F_p), \omega_p, N))),$$

which by abuse of notation we will also call $\delta_{s,N}^\vee$. Here and in the rest of this section we abbreviate the constant multi index $(s, \ldots, s)$ by (s). Unravelling the definitions we see that

$$\begin{aligned}&\mathcal{A}_H(K^p, \mathrm{St}_H; \text{I-Dist}_{(s)}(\mathrm{GL}_n(F_p), \omega_p, N))\\ =&\,\mathrm{Ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d H(\mathcal{O}_\infty)}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \omega^{-1} \otimes \mathrm{Hom}(\mathrm{St}_H, \mathrm{Dist}_{(s)}(\mathrm{GL}_n(\mathcal{O}_p), N)).\end{aligned}$$

Thus, by Borel-Serre duality (see Section 3.2), there is an isomorphism

$$\begin{aligned}&H^{dq_0-2n+1}(PH(F)^+, \mathcal{A}_H(K^p, \mathrm{St}_H; \text{I-Dist}_{(s)}(\mathrm{GL}_n(F_p), \omega_p, N)))\\ \xrightarrow{BS}&H_0(PH(F)^+, \text{c-ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d H(\mathcal{O}_\infty)}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \omega^{-1} \otimes \mathrm{Dist}_{(s)}(\mathrm{GL}_n(\mathcal{O}_p), N)).\end{aligned}$$

In accordance with our previous abbreviation we define

$$\begin{aligned}\det{}^{-(s),(s)} \colon Z(F)^d H(\mathcal{O}_\infty) &\to R^* \text{ via}\\ (h_{1,\sigma}, h_{2,\sigma})_\sigma &\mapsto \prod_\sigma \frac{\det(\sigma(h_{2,\sigma}))^s}{\det(\sigma(h_{1,\sigma}))^s}.\end{aligned}$$

The $H(F)$-invariant character

$$\begin{aligned}\eta_s \colon H(\mathbb{A}) &\to \mathbb{C}, \text{ where}\\ (h_1, h_2) &\mapsto \frac{|\det(h_2)|^s \eta(\det h_2)}{|\det(h_1)|}\end{aligned}$$

yields a cohomology class

$$[\eta_s] \in H^0(PH(F)^+, \mathrm{Ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \omega \otimes \det{}^{-(s),(s)})$$

by embedding $F^d \hookrightarrow \prod_\sigma F_{\mathfrak{p}_\sigma} \hookrightarrow \mathbb{A}$ and defining

$$\begin{aligned}&[\eta_s] \colon H(\mathbb{A}^{p,\infty}) \times H(F)^d \to R\\ &\quad (h_1^p, h_2^p) \times (h_{1,p}, h_{2,p}) \mapsto \eta_s(h_1^p h_{1,p}, h_2^p h_{2,p}) \det{}^{-(s),(s)}(h_{1,p}, h_{2,p}).\end{aligned}$$

Taking cap product with $[\eta_s]$ yields a homomorphism

$$\begin{aligned}&H_0(PH(F)^+, \text{c-ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d H(\mathcal{O}_\infty)}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \omega^{-1} \otimes \mathrm{Dist}_{(s)}(\mathrm{GL}_n(\mathcal{O}_p), N))\\ \xrightarrow{\cap[\eta_s]}\ &H_0(PH(F)^+, \text{c-ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d H(\mathcal{O}_\infty)}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \mathrm{Dist}(\mathrm{GL}_n(\mathcal{O}_p), N)).\end{aligned}$$

At last, since $\mathrm{GL}_n(\mathcal{O}_p)$ is compact, pullback of functions along the determinant $\det\colon \mathrm{GL}_n(\mathcal{O}_p) \to U_p$ induces a $Z(F)^d G(\mathcal{O}_\infty)$-equivariant map

$$\det\colon \mathrm{Dist}(\mathrm{GL}_n(\mathcal{O}_p), N) \to \mathrm{Dist}(U_p, N).$$

Here we let $Z(F)^d H(\mathcal{O}_\infty)$ act on $\mathrm{Dist}(U_p, N)$ via the group homomorphism $\det^{-1,1}\colon Z(F)^d H(\mathcal{O}_\infty) \to \mathcal{O}_\infty^*$. By applying $\det^{-1,1}$ to $H(F)^d$, $Z(\mathbb{A}^{p,\infty})K_H^p$, $PH(F)^+$ etc. as well we get the following map on cohomology groups:

$$\begin{aligned} &H_0(PH(F)^+, \text{c-ind}_{Z(\mathbb{A}^{p,\infty})K_H^p \times Z(F)^d H(\mathcal{O}_\infty)}^{H(\mathbb{A}^{p,\infty})\times H(F)^d} \mathrm{Dist}(\mathrm{GL}_n(\mathcal{O}_p), N)) \\ \xrightarrow{H_0(\det)} &H_0(F_+^*, \text{c-ind}_{U'}^{\mathbb{I}^\infty}(\mathrm{Dist}(U_p, N))). \end{aligned}$$

Now we can define Δ_N^s as the followng composition: First, take the restriction of $PG(F)^+$-cohomology to $PH(F)^+$-cohomology, then apply $H^{dq_0-2n+1}(\delta_{s,N}^\vee \circ Res_H)$, followed by the Borel-Serre duality map BS. After that take cap product with the class $[\eta_s]$ and finally compose the result with $H_0(\det)$.

Theorem 3.8. *There exists a constant $c \in \mathbb{C}^*$ such that*

$$\partial(\Delta_{\mathbb{C}}^s(ES_{\varepsilon_+}^p(\Phi^p)_{coh})) = c\ \mu_{\pi,s}.$$

Therefore, $\mu_{\pi,s}$ is a p-adic measure.

Proof. The second claim follows from the first one since the diagramm

$$\begin{array}{ccc} \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; R) \otimes \mathbb{C} & \longrightarrow & \mathcal{A}(\chi_p, K^p, \mathrm{St}_G; \mathbb{C}) \\ \downarrow{\scriptstyle \partial\circ\Delta_R^s\otimes 1} & & \downarrow{\scriptstyle \partial\circ\Delta_{\mathbb{C}}^s} \\ \mathrm{Dist}(\mathcal{G}_p, R) \otimes \mathbb{C} & \longrightarrow & \mathrm{Dist}(\mathcal{G}_p, \mathbb{C}) \end{array}$$

is commutative and the top row is an isomorphism by Lemma 3.7.
To proof the first assertion let $f\colon \mathcal{G}_p \to \mathbb{C}$ be a locally constant function. We will view f as an F_+^*-invariant function on $\mathbb{I}^\infty/U^{p,\infty}$. Since $\mathbb{I}^\infty/F_+^*$ is compact, there exists an $m \in \mathbb{N}$ such that f factors over the quotient $\mathbb{I}^\infty/U^{p,\infty}U_p^{(m)}$, or in other words: There exists $m \in \mathbb{N}$ such that f is in the image of the canonical inclusion

$$\tau_m\colon \mathrm{Ind}_{U^\infty}^{\mathbb{I}^\infty} C(U_p/U_p^{(m)}, \mathbb{C}) \to \mathrm{Ind}_{U^\infty}^{\mathbb{I}^\infty} C^0(U_p, \mathbb{C}).$$

Let $K_p^{(m)}$ be the m-th principal congruence subgroup of $\mathrm{GL}_n(\mathcal{O}_p)$. As before, the determinant $\det\colon \mathrm{GL}_n(\mathcal{O}_p) \to U_p$ induces a map

$$C(U_p/U_p^{(m)}, \mathbb{C}) \to C(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)}, \mathbb{C}),$$

whose image we denote by $C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C})$. Together with the group homomorphism $\det^{-1,1}\colon H(\mathbb{A}^\infty)\to\mathbb{I}^\infty$ it induces the map

$$\det\colon \operatorname{Ind}_{U^\infty}^{\mathbb{I}^\infty} C(U_p/U_p^{(m)},\mathbb{C}) \to \operatorname{Ind}_{Z(\mathbb{A}^\infty)K_H^pH(\mathcal{O}_p)}^{H(\mathbb{A}^\infty)} C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C}).$$

By Frobenius reciprocity the element $\delta(\mathbb{1}_{K_p^{(m)}})\in\pi_p=\otimes_{\mathfrak{p}\in S_p}\pi_{\mathfrak{p}}$ determines a morphism of smooth $G(F_p)$-representations

$$\theta_m\colon \text{c-ind}_{K_{p,m}}^{G(F_p)}\mathbb{C}\to\pi_p,$$

where $K_{p,m}$ is the stabilizer of $\delta(\mathbb{1}_{K_p^{(m)}})$ in $G(\mathcal{O}_p)$. It is easy to that the intersection of $K_{p,m}$ with $H(F_p)$ is given by the group

$$K_{p,m,H}=\left\{(k_1,k_2)\in H(\mathcal{O}_p)\mid k_1k_2^{-1}\in K_p^{(m)}\right\}.$$

The elements of $C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C})$ are invariant under the action of $K_{p,m,H}$. Hence the following $H(\mathcal{O}_p)$-equivariant map is well-defined:

$$\begin{aligned}\delta^m\colon C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C}) &\to \text{c-ind}_{K_{p,m,H}}^{H(\mathcal{O}_p)}\mathbb{C}\\ \delta^m(g)(k_1,k_2)&=g(k_1^{-1}k_2).\end{aligned}$$

This is a lift of the restriction of δ to $C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)})$, i.e. the diagram

$$\begin{array}{ccccc} C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C}) & \xrightarrow{\delta^m} & \text{c-ind}_{K_{p,m,H}}^{H(\mathcal{O}_p)}\mathbb{C} & \longrightarrow & \text{c-ind}_{K_{p,m}}^{G(F_p)}\mathbb{C}\\ \big\downarrow & & \big\downarrow{\scriptstyle\theta_m} & & \big\downarrow{\scriptstyle\theta_m}\\ C^0(\mathrm{GL}_n(F_p),\mathbb{C}) & \xrightarrow{\quad\delta\quad} & \pi_p & \xrightarrow{\quad=\quad} & \pi_p \end{array}$$

is commutative. Using the transitivity of induction we get a map

$$\operatorname{Ind}_{Z(\mathbb{A}^\infty)K_H^pH(\mathcal{O}_p)}^{H(\mathbb{A}^\infty)} C_{\det}(\mathrm{GL}_n(\mathcal{O}_p)/K_p^{(m)},\mathbb{C}) \to \operatorname{Ind}_{Z(\mathbb{A}^\infty)K_H^pK_{p,m,H}}^{H(\mathbb{A}^\infty)}\mathbb{C},$$

which by abuse of notation we also call δ^m. The image under $\delta^m\circ\det$ of the function f we started with is simply given by

$$f\circ\det^{-1,1}\colon (k_1,k_2)\mapsto f\left(\frac{\det k_2}{\det k_1}\right).$$

Since this function is $PH(F)^+$-invariant it defines a cohomology class

$$[f_{\det}]\in H^0(PH(F)^+,\operatorname{Ind}_{Z(\mathbb{A}^\infty)K_H^pK_{p,m,H}}^{H(\mathbb{A}^\infty)}\mathbb{C}).$$

Chasing through the definitions and using the commutativity of the above diagram we see that

$$\begin{aligned}(7)\qquad &\partial(\Delta^s_{\mathbb{C}}(ES^p_{\varepsilon_+}(\Phi^p)_{coh}))(f)\\ &=\sum(\mathrm{Res}_{H,s}(ES_{\varepsilon_+}(\Phi^p\otimes\delta(\mathbb{1}_{K_p^{(m)}}))_{coh})\cup[\eta_s]\cup[f_{\det}]),\end{aligned}$$

where $Res_{H,s}$ denotes the natural restriction map

$$\begin{aligned}Res_{H,s}\colon &H^\bullet(PG(F)^+,C(G(\mathbb{A}^\infty)/K^pK_{p,m}Z(\mathbb{A}^\infty),\omega,\mathrm{Hom}(\mathrm{St}_G,V_\mu^\vee)))\\ \to &H^\bullet(PH(F)^+,C(H(\mathbb{A}^\infty)/K_H^pK_{p,m,H}Z(\mathbb{A}^\infty),\omega,\mathrm{Hom}(\mathrm{St}_H,(\otimes_{i=1}^d V^{al}_{s,\sigma_i})^\vee)))\end{aligned}$$

and the map $\sum$ is defined via a strong approximation argument, i.e. there exists arithmetic subgroups $\Gamma_i\subset PH(F)^+$ such that

$$\begin{aligned}&H^{dq_0-2n+1}(PH(F)^+,\mathrm{Ind}^{H(\mathbb{A}^\infty)}_{K_H^pK_{p,m,H}Z(\mathbb{A}^\infty)}\mathrm{Hom}(St_H,\mathbb{C}))\\ =&\bigoplus_i H^{dq_0-2n+1}(\Gamma_i,\mathrm{Hom}(St_H,\mathbb{C}))=\bigoplus_i\mathbb{C}\xrightarrow{\Sigma}\mathbb{C}.\end{aligned}$$

The right hand side of (7) can be written purely in terms of the (deRham) cohomology of the associated symmetric spaces. By standard computations (see for example [Har87], Section 5.3) we see that it equals the integral

$$\begin{aligned}&[H(\mathcal{O}_p):K_{p,m,H}]\int_{H(F)\backslash H(\mathbb{A})}(\Phi_{s,\infty}\otimes\Phi^p\otimes\delta(\mathbb{1}_{K_p^m}))(h)\eta_s(h^{-1})f(\det{}^{-1,1}(h^{-1}))dh\\ =&\Psi(\Phi_m,f,s+\frac{1}{2})\end{aligned}$$

up to a constant $c\in\mathbb{C}^*$, which is independet of f and m. □

The character $\mathcal{N}\colon\mathcal{G}_p\to\mathbb{Z}_p^*$ is defined by $\gamma\zeta=\zeta^{\mathcal{N}(\gamma)}$ for all p-power roots of unity. For $x\in\mathbb{Z}_p$ and $\gamma\in\mathcal{G}_p$ we put $\langle\gamma\rangle^x=\exp_p(x\log_p(\mathcal{N}(\gamma)))$, where $\exp_p$ ($\log_p$ resp.) is the p-adic exponential map (logarithm map resp.).

Definition 3.9. *For an automorphic representation π as above and a critical point $s+\frac{1}{2}$ of π we define the p-adic L-function*

$$L_p(\pi,s,x)=\int_{\mathcal{G}_p}\langle\gamma\rangle^x\,\mu_{\pi,s}(d\gamma).$$

It is an analytic function on $\mathbb{Z}_p$ with values in $\widetilde{L_{\mu_{\pi,s}}}$.

4 Odds and ends

4.1 Examples

We want to give some examples to our construction. The natural source for these are odd symmetric powers of p-ordinary Hilbert modular forms over totally real fields F, in which p is totally split. To keep the notation simple we only deal with the case of a trivial central character and $F = \mathbb{Q}$. Let f be a cuspidal newform of level $\Gamma_0(N)$, $p \nmid N$, and even weight k. The associated unitary cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ is cohomological with respect to the representation $\mathrm{Sym}^{k-2}\,\mathbb{C}^2 \otimes \det^{-(k-2)/2}$. The local component π_p is an unramified principal series representation of the form $\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(Q_p)}(\chi^{-1}, \chi)$. As always, we put $\alpha = \chi(p)$. As explained at the end of section 1.3 the condition that the Hecke eigenvalue $p^{\frac{k}{2}}\alpha^{-1} + p^{\frac{k-2}{2}}\alpha$ and $\alpha^{-1}p^{\frac{-(k-2)}{2}}$ are p-integral is equivalent to $\alpha^{-1}p^{\frac{-(k-2)}{2}} \in \overline{\mathcal{O}}^*$, hence $p^{\frac{k}{2}}\alpha^{-1} + p^{\frac{k-2}{2}}\alpha \in \overline{\mathcal{O}}^*$. Thus the notion of weakly p-ordinarity coincides with the usual ordinarity condition at p. In this case our constuction gives the classical Mazur-Tate-Teilbaum p-adic L-function constructed in [MTT86].

The symmetric cube $\mathrm{Sym}^3\,\pi$ of π is known to be a cuspidal automophic representation of $\mathrm{GL}_4(\mathbb{Q})$ if f is not a CM-form by Kim-Shahidi (see [KS02]). The representation $\mathrm{Sym}^3\,\pi$ is cohomological with respect to the algebraic representation of highest weight $(\frac{3(k-2)}{2}, \frac{(k-2)}{2}, \frac{-(k-2)}{2}, \frac{-3(k-2)}{2})$ (see [RS08]) and the local representation at p is given by $\mathrm{Ind}_{B_4(\mathbb{Q}_p)}^{GL_4(Q_p)}(\chi^{-3}, \chi^{-1}, \chi, \chi^3)$. Hence if f is p-ordinary, we get that the eigenvalues of the Hecke operators at p and the number

$$\alpha^{-3}\alpha^{-1}p^{\frac{-3(k-2)}{2}}p^{\frac{-1(k-2)}{2}} = (\alpha^{-1}p^{\frac{-(k-2)}{2}})^4$$

are p-integral, so $\mathrm{Sym}^3\,\pi$ is weakly p-ordinary. One can combine the results of Kim (see [Kim03]) and Jacquet-Shalika (see [JS90]) to show that $\mathrm{Sym}^3\,\pi$ has a Shalika model (see [GRG14], Section 8, for a detailed discussion). Thus our construction yields a p-adic L-function for the symmetric cube of a p-ordinary modular form of level $\Gamma_0(N)$, $p \nmid N$ and even weight which is not of CM-type.

The construction can be applied to higher odd symmetric powers as well. We assume that $\Pi = \mathrm{Sym}^{2r+1}\,\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_{2(r+1)}(\mathbb{Q})$. The representation Π is cohomological with respect to the representation of highest weight $(k-2)\rho_{2(r+1)}$, where $\rho_{2(r+1)}$ denotes half of the sum over all positve roots of $\mathrm{GL}_{2(r+1)}$ (see for example [RS08]). This can again be used to deduce that that Π is weakly p-ordinary if f is ordinary at p. If we would know that Π has a Shalika-model our construction would yield a p-adic L-function for every critical point of Π. By Proposition 8.1.4 of [GRG14] Π has a Shalika model if $\mathrm{Sym}^{4(r+a)}\,\pi$, $0 \leq a \leq r$, is an isobaric sum of unitary cuspidal automorphic representations.

4.2 Further directions

It seems likely that the machinery developed in this thesis can be adapted to construct p-adic L-functions in more general situations. First of all, the condition that the representation π is unitary seems unnecessary. The whole construction should easily be extended to cohomological representations with arbitrary purity weight. Secondly, by extending the results of Große-Klönne to more general locally algebraic extensions and other fields than $\mathbb{Q}_p$, one can hope to get rid of the condition that the prime p is split in the totally real field F.
Since the distributions $\mu_{\pi,s}$ for different critical points $s+\frac{1}{2}$ all come from the same cohomology class, one can hope to construct a distribution μ_π which interpolates the L-values at all critical points, as in the classical case of modular forms. As a consequence, by using the theory of Amice-Velu one could construct p-adic L-Functions as long as π has "weakly small p-slopes". Another application would be non-vanishing results for critical values of L-functions as long as π has enough critical points: If π has a critical point inside the region of convergence of the Euler-product, it follows that the L-function at this point does not vanish. Hence the distribution μ_π is non-zero. But this implies that for any critical point $s+\frac{1}{2}$ we can find a character χ, which is unramified outside p, such that $L(\pi,\chi,s+\frac{1}{2})$ is non-zero.
Further, it should be possible to extend the results to representations π such that the local components $\pi_\mathfrak{p}$ for primes $\mathfrak{p}$ lying above p have only an Iwahori invariant vector by realizing those local representations as quotients of unramified principal series representations. A natural example is the following: Let us start with a cuspidal newform of level $\Gamma_0(pN)$, $p \nmid N$, with corresponding cuspidal representation π. The local component at π_p is an unramified twist of the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Therefore, the local component of $\mathrm{Sym}^{2r+1}\,\pi$ is an unramified twist of the Steinberg representation of $\mathrm{GL}_{2(r+1)}(\mathbb{Q}_p)$ for all $r \in \mathbb{N}$, and thus has an Iwahori fixed vector. Since the construction of the distribution in this article is very similar to the construction of Spieß in [Spi14], there might be a chance to generalize his results on vanishing orders of p-adic L-functions to our setting.

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